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THE DIAMETER OF THE NON-NILPOTENT GRAPH OF A FINITE GROUP

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Abstract. We prove that the graph obtained from the non-nilpotent graph of a finite group by deleting the isolated vertices is connected with diameter at most 3. This bound is the best possible.

1. **Introduction**

Suppose that G is a group, and define the non-nilpotent graph $\mathcal{N}_G$ of G as follows: the vertices of $\mathcal{N}_G$ are the elements of G, and two vertices are joined whenever they do not generate a nilpotent subgroup. Let $\mathcal{R}_G$ be the subgraph of $\mathcal{N}_G$ induced by $G \setminus \mathrm{nil}(G)$, where $\mathrm{nil}(G) = \{x \in G \mid \langle x, y\rangle$ is nilpotent for all $y \in G\}$. A. Abdollahi and M. Zarrin [1] proved that if G is finite, then $\mathrm{nil}(G)$ coincides with the hypercenter $Z_\infty(G)$ of G and that $\mathcal{R}_G$ is connected, with $\mathrm{diam}(\mathcal{R}_G) \leq 6$. They proved that $\mathrm{diam}(\mathcal{R}_G) = 2$ in several cases. This could lead to conjecture that $\mathrm{diam}(\mathcal{R}_G) = 2$ for every finite group G, but this is false. Andrew Davis, Julie Kent and Emily McGovern, three students of the Missouri State University, investigated the non-nilpotent graph of the semidirect product $\langle a\rangle \rtimes \mathrm{Sym}(4)$, where $|a|$ is odd and $a^\sigma = a^{\mathrm{sgn}(\sigma)}$ for every $\sigma \in \mathrm{Sym}(4)$. Let $g = a^i\sigma \in G$. If $\langle a, g\rangle$ is not nilpotent, then $\sigma \notin \mathrm{Alt}(4)$, while if $\langle (1,2)(3,4), g\rangle$ is not nilpotent then σ is a 3-cycle. This implies that the vertices a and $(1,2)(3,4)$ do not have a common neighbor in the graph $\mathcal{R}_G$, so $\mathrm{dist}_{\mathcal{R}_G}(a, (1,2)(3,4)) \geq 3$. However this is the worst possible situation. Indeed our main result is the following.

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Theorem 1.1. *If G is a finite group, then* $\mathrm{diam}(\mathcal{R}_G) \leq 3$.

Our second result says that if $\mathrm{dist}_{\mathcal{R}_G}(x, y) = 3$, then at least one of the two elements x and y belong to the Fitting subgroup $F(G)$ of G.

Theorem 1.2. *If G is a finite group and $x, y \notin F(G)$, then* $\mathrm{dist}_{\mathcal{R}_G}(x, y) \leq 2$.

2. Proofs of Theorems 1.1 and 1.2

Throughout this section, we will say that g is a p-element, where p is a prime, meaning that the order of g is a power of p.

Lemma 2.1. *Let G be a finite group and let $g \in G$. If H is a subgroup of G and $g \notin H$, then there exist a prime p and a positive integer n such that g^n is a p-element and $g^n \notin H$.*

Proof. Let $|g| = p_1^{n_1} \cdots p_r^{n_r}$, with $p_1, \ldots, p_r$ distinct primes. For $1 \leq i \leq r$, set $m_i = \prod_{j \neq i} p_j^{n_j}$. Since $\langle g^{m_1}, \ldots, g^{m_r} \rangle = \langle g \rangle \not\leq H$, there exists $i \in \{1, \ldots, r\}$ such that the p_i-elements $g_i^{m_i}$ does not belong to H. □

Lemma 2.2. *Let p be a prime and x a p-element of a finite group G. If $x \notin Z_\infty(G)$, then there exist a prime $q \neq p$ and a q-element y such that $\langle x, y \rangle$ is not nilpotent.*

Proof. Suppose, by contradiction, that $\langle x, y \rangle$ is nilpotent for every q-element y and every prime $q \neq p$. Let $K := \langle Q \mid Q \in \mathrm{Syl}_q(G), q \neq p \rangle$. Then K is a normal subgroup of G and $K \leq C_G(x)$. Moreover $|G/K|$ is a p-group, so if P is a Sylow subgroup of G containing x, then $G = KP$. Let g be an arbitrary element of G and write $g = ab$, with $a \in K$ and $b \in P$. Then $\langle x, x^g \rangle = \langle x, x^{ab} \rangle = \langle x, x^b \rangle \leq P$. By a theorem of R. Baer (see for example [2, 2.12]) $x \in O_p(G)$. In particular, if z is a p-element of G, then $\langle x, z \rangle$ is a p-group. Now let g be an arbitrary element of G and write $g = \alpha\beta$, where α is a p-element, β a p'-element and $[\alpha, \beta] = 1$. We have $\langle x, g \rangle = \langle x, \alpha\beta \rangle = \langle x, \alpha, \beta \rangle = \langle x, \alpha \rangle \langle \beta \rangle \cong \langle x, \alpha \rangle \times \langle \beta \rangle$, since $\beta \in K \leq C_G(x)$. But, as we noticed before, $\langle x, \alpha \rangle$ is a p-group, and so $\langle x, g \rangle \cong \langle x, \alpha \rangle \times \langle \beta \rangle$ is nilpotent. This implies $x \in \mathrm{nil}(G) = Z_\infty(G)$, against our assumption. □

Proof of Theorem 1.1. Let x_1, x_2 be two distinct elements of $G \setminus Z_\infty(G)$. By Lemma 2.1, there exist two positive integers m_1, m_2 and two primes p_1, p_2 such that $x_1^{m_1}$ is a p_1-element, $x_2^{m_2}$ is a p_2-element and $x_1^{m_1}, x_2^{m_2} \notin Z_\infty(G)$. By Lemma 2, there exist two primes $q_1 \neq p_1$ and $q_2 \neq p_2$, a q_1-element z_1 and a q_2-element z_2 such that $\langle x_1^{m_1}, z_1 \rangle$ and $\langle x_2^{m_2}, z_2 \rangle$ are not nilpotent. If $\langle z_1, z_2 \rangle$ is not nilpotent, then (x_1, z_1, z_2, x_2) is a path in the graph $\mathcal{R}_G$ joining x_1 and x_2 and $\mathrm{dist}_{\mathcal{R}_G}(x_1, x_2) \leq 3$. So we may assume that $\langle z_1, z_2 \rangle$ is nilpotent. If $q_1 \neq q_2$, then $\langle z_1, z_2 \rangle = \langle z_1 z_2 \rangle$. This implies that $\langle x_1, z_1 z_2 \rangle = \langle x_1, z_1, z_2 \rangle$ and $\langle x_2, z_1 z_2 \rangle = \langle x_2, z_1, z_2 \rangle$ are not nilpotent, and $(x_1, z_1 z_2, x_2)$ is a path in $\mathcal{R}_G$. If $q_1 = q_2$, then $q_1 \neq p_2$. If $\langle x_1, z_2 \rangle$ is not nilpotent, then (x_1, z_2, x_2) is a path in $\mathcal{R}_G$. Otherwise $\langle x_1^{m_1}, z_2 \rangle \leq \langle x_1, z_2 \rangle$ is nilpotent, hence $\langle x_1^{m_1}, z_2 \rangle = \langle x_1^{m_1} z_2 \rangle$ and $(x_1, z_1, x_1^{m_1} z_2, x_2)$ is a path in $\mathcal{R}_G$. □

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Lemma 2.3. *Let G be a finite group. If $x, y \notin F(G)$ and $\gcd(|x|, |y|) = 1$, then $\mathrm{dist}_{\mathcal{R}_G}(x, y) \leq 2$.*

Proof. Assume, by contradiction, $\mathrm{dist}_{\mathcal{R}_G}(x, y) > 2$. Since $x, y \notin F$ by [2, 2.12] there exist g and h in G such that $\langle x, x^g\rangle$ and $\langle y, y^h\rangle$ (and consequently also $\langle x, x^{g^{-1}}\rangle$ and $\langle y, y^{h^{-1}}\rangle$ are not nilpotent). If $\langle x^g, y^{h^{-1}}\rangle$ were nilpotent, then $[x^g, y^{-h}] = 1$ and $(x, x^g y^{h^{-1}}, y)$ would a path in $\mathcal{R}_G$. So $\langle x^g, y^{h^{-1}}\rangle$ (and consequently also $\langle x, y^{h^{-1}g^{-1}}\rangle$ and $\langle x^{gh}, y\rangle$) is not nilpotent. We prove, by induction on n, that $\langle x^{(gh)^n}, y\rangle$ is not nilpotent, for every $n \in \mathbb{N}$. Indeed, assuming that $\langle x^{(gh)^n}, y\rangle$ is not nilpotent, then $\langle x^{(gh)^n}, y^{(gh)^{-1}}\rangle$ is also non nilpotent, otherwise $[x^{(gh)^n}, y^{(gh)^{-1}}] = 1$ and $(x, x^{(gh)^n} y^{(gh)^{-1}}, y)$ would be a path in $\mathcal{R}_G$. But then, taking $n = |gh|$, we get that $\langle x, y\rangle$ is not nilpotent and $\mathrm{dist}_{\mathcal{R}_G}(x, y) = 1$, against our assumption. □

Lemma 2.4. *Let G be a finite soluble group and let p be a prime. If $g_1, g_2 \in G \setminus \mathrm{nil}(G)$ are p-elements such that $\mathrm{dist}_{\mathcal{R}_G}(g_1, g_2) > 2$, then $g_1, g_2 \in O_p(G)$.*

Proof. Let $C_1 := C_G(g_1)$ and $C_2 := C_G(g_2)$. By Lemma 2.2, there exist a prime $q \neq p$ and a q-element x such that $\langle g_1, x\rangle$ is not nilpotent. Let K be a p-complement in G containing x. It must be $K \subseteq C_1 \cup C_2$ (indeed if $y \in K \setminus (C_1 \cup C_2)$, then (g_1, y, g_2) would be a path in $\mathcal{R}_G$). Hence either $K \leq C_1$ or $K \leq C_2$. However $x \in K \setminus C_1$, so we must exclude the first possibility and conclude $K \leq C_2$. In particular $|G : C_2|$ is a p-power and therefore $G = C_2 P$, being P a Sylow p-subgroup of G containing g_2. As in the proof of Lemma 2.2, applying Baer's theorem we conclude $g_2 \in O_p(G)$. With the same argument we can prove $g_1 \in O_p(G)$. □

Proof of Theorem 2.2. By Lemma 2.1, we may assume that there exists two primes p and q such that x is a p-element and y is a q-element. By Lemma 2.3, we may assume $p = q$. If $x, y \notin R(G)$ (where $R(G)$ denotes the soluble radical of G), then, by [3, Theorem 6.4], there exists $z \in G$ such that $\langle x, z\rangle$ and $\langle y, z\rangle$ are not soluble. Hence (x, z, y) is a path in $\mathcal{R}_G$ and $\mathrm{dist}_{\mathcal{R}_G}(x, y) \leq 2$. So it is not restrictive to assume $x \in R(G)$. In particular $H = R(G)\langle y\rangle$ is a soluble group containing x, so by Lemma 2.4, either $\mathrm{dist}_{\mathcal{R}_G}(x, y) \leq \mathrm{dist}_{\mathcal{R}_H}(x, y) \leq 2$ or $x, y \in F(H)$. However in the second case, we would have $x \in F(H) \cap R(G) \leq F(R(G)) \leq F(G)$. □

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