

Words of Engel type are concise in residually finite groups. Part II

Eloisa Detomi, Marta Morigi and Pavel Shumyatsky¹

Abstract. This work is a natural follow-up of the article [5]. Given a group-word w and a group G , the verbal subgroup $w(G)$ is the one generated by all w -values in G . The word w is called concise if $w(G)$ is finite whenever the set of w -values in G is finite. It is an open question whether every word is concise in residually finite groups. Let $w = w(x_1, \dots, x_k)$ be a multilinear commutator word, n a positive integer and q a prime power. In the present article we show that the word $[w^q, {}_n y]$ is concise in residually finite groups (Theorem 1.2) while the word $[w, {}_n y]$ is boundedly concise in residually finite groups (Theorem 1.1).

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1. Introduction

Let $w = w(x_1, \dots, x_k)$ be a group-word. Given a group G , we denote by $w(G)$ the verbal subgroup corresponding to the word w , that is, the subgroup generated by the set G_w of all values $w(g_1, \dots, g_k)$, where g_1, \dots, g_k are elements of G . The word w is called concise if $w(G)$ is finite whenever the set of w -values in G is finite. More generally, a word w is called concise in a class of groups \mathcal{X} if $w(G)$ is finite whenever the set of w -values in G is finite for a group $G \in \mathcal{X}$. In the sixties Hall raised the problem whether all words are concise. In 1989 S. Ivanov [12] (see also [15, p. 439]) solved the problem in the negative. On the other hand, the problem for residually finite groups remains open (cf. Segal [18, p. 15] or Jaikin-Zapirain [13]). In recent years a number of new results with respect to this problem were obtained (see [1, 10, 8, 5, 6]). The work [5] deals with conciseness of words of Engel type in residually finite groups. Recall that multilinear commutator words, also known under the name of outer commutator words, are

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precisely the words that can be written in the form of multilinear Lie monomials. The multilinear commutator words were shown to be concise (in the class of all groups) by J. C. R. Wilson [24]. Set $[x, {}_1y] = [x, y] = x^{-1}y^{-1}xy$ and $[x, {}_{i+1}y] = [[x, {}_iy], y]$ for $i \geq 1$. The word $[x, {}_ny]$ is called the n th Engel word. One of the results obtained in [5] says that for any multilinear commutator word $w = w(x_1, \dots, x_k)$ and any positive integer n the word $[w, {}_ny]$ is concise in residually finite groups.

We say that a word w is boundedly concise in a class of groups \mathcal{X} if for every integer m there exists a number $\nu = \nu(\mathcal{X}, w, m)$ such that whenever $|G_w| \leq m$ for a group $G \in \mathcal{X}$ it always follows that $|w(G)| \leq \nu$. Fernandez-Alcober and Morigi showed that every word which is concise in the class of all groups is actually boundedly concise [7]. It was conjectured in [8] that every word which is concise in the class of residually finite groups is boundedly concise.

One purpose of the present article is to show that the words $[w, {}_ny]$, where $w = w(x_1, \dots, x_k)$ is a multilinear commutator word, are boundedly concise in residually finite groups.

Theorem 1.1. *Suppose that $w = w(x_1, \dots, x_k)$ is a multilinear commutator word and n is a positive integer. The word $[w, {}_ny]$ is boundedly concise in residually finite groups.*

The other purpose of this paper is to establish conciseness of somewhat more complicated words.

Theorem 1.2. *Suppose that $w = w(x_1, \dots, x_k)$ is a multilinear commutator word. For any prime-power q and any positive integer $n \geq 1$ the word $[w^q, {}_ny]$ is concise in residually finite groups.*

We do not know if the word $[w^q, {}_ny]$ in the above theorem is boundedly concise in residually finite groups. Note that at present, among all words that are known to be concise in residually finite groups, the only ones whose bounded conciseness remains unconfirmed are the words $[w^q, {}_ny]$ dealt with in Theorem 1.2.

Another open question related to Theorem 1.2 is whether the theorem remains valid if q is allowed to be an arbitrary positive integer rather than a prime-power. It seems that at present we do not have sufficient tools to handle words of that kind.

The proofs of most of the known results on conciseness of words in residually finite groups use Zelmanov's theorem on nilpotency of finitely generated Lie algebras with a polynomial identity [25]. This is also the case with both Theorem 1.1 and Theorem 1.2 of this article.

2. Preliminaries

Throughout the paper we denote by G' the commutator subgroup of a group G and by $\langle M \rangle$ the subgroup generated by a subset $M \subseteq G$. In this section we collect some preliminary results which will be needed for the proofs of the main theorems.

A proof of the following lemma can for example be found in [5].

Lemma 2.1. *Let v be a word and G a group such that the set of v -values in G is finite with at most m elements. Then the order of the commutator subgroup $v(G)'$ is m -bounded.*

An important family of multilinear commutator words is formed by the derived words δ_k , on 2^k variables, which are defined recursively by

$$\delta_0 = x_1, \quad \delta_k = [\delta_{k-1}(x_1, \dots, x_{2^{k-1}}), \delta_{k-1}(x_{2^{k-1}+1}, \dots, x_{2^k})].$$

Of course $\delta_k(G) = G^{(k)}$, the k -th term of the derived series of G . We will need the following well-known results.

Lemma 2.2. [19, Lemma 4.1] *Let G be a group and let w be a multilinear commutator word on k variables. Then each δ_k -value is a w -value.*

Lemma 2.3. [7, Lemma 2.2] *Let w be a multilinear commutator word. Then G_w is symmetric, that is, $x \in G_w$ implies that $x^{-1} \in G_w$.*

An element g of a group G is called a (left) Engel element if for any $x \in G$ there exists $n = n(g, x) \geq 1$ such that $[x, {}_n g] = 1$. If n can be chosen independently of x , then g is a (left) n -Engel element. The next lemma is a combination of a well-known theorem of Baer [11, Satz III.6.15] and a result due to Gruenberg [9].

Lemma 2.4. *Let G be a group generated by (left) Engel elements.*

- (1) *If G is finite, then it is nilpotent.*
- (2) *If G is soluble, then it is locally nilpotent.*

An element $g \in G$ is a right Engel element if for each $x \in G$ there exists a positive integer n such that $[g, {}_n x] = 1$. If n can be chosen independently of x , then g is a right n -Engel element. The next observation is due to Heineken (see [17, 12.3.1]).

Lemma 2.5. *Let g be a right n -Engel element in a group G . Then g^{-1} is a left $(n+1)$ -Engel element.*

A proof of the next two results can be found in [5].

Lemma 2.6. *Let $G = \langle g, t \rangle$ be a group such that $[g, {}_n t] = 1$. Then the normal closure of the subgroup $\langle g \rangle$ in the group G is generated by the set $\{g^{t^i} \mid i = 0, \dots, n-1\}$.*

Lemma 2.7. *Let $G = U \langle t \rangle$ be a group that is a product of a normal subgroup U and a cyclic subgroup $\langle t \rangle$. Assume that U is nilpotent of class c and there exists a generating set A of U such that $[a, {}_n t] = 1$ for every $a \in A$. Then G is nilpotent of (c, n) -bounded class.*

For a subgroup A of a group G , an element $x \in G$ and a positive integer n , we write $[A, {}_n x]$ to denote the subgroup generated by all elements $[a, {}_n x]$, with $a \in A$. The following lemma is due to Casolo.

Lemma 2.8. [3, Lemma 6] *Let A be an abelian group, and let x be an automorphism of A such that $[A, {}_n x] = 1$ for some $n \geq 1$. If x has finite order q , then $[A^{q^{n-1}}, x] = 1$.*

The following proposition is a consequence of a result from [23] (see [5, Proposition 1]).

Proposition 2.9. *Given positive integers d, q, n and a multilinear commutator word w , let G be a finite group in which the w^q -values are n -Engel. Suppose that a subgroup H can be generated by d elements which are w^q -values. Then H is nilpotent with (d, q, n, w) -bounded class.*

We will also require the next variation on the same theme.

Proposition 2.10. [2, Proposition 14] *Given positive integers q, n and a multilinear commutator word w , let G be a residually finite group in which the w^q -values are n -Engel. Suppose that a subgroup H can be generated by finitely many Engel elements. Then H is nilpotent.*

The following lemma is taken from [21].

Lemma 2.11. [21, Lemma 4.1] *Let G be a group generated by d elements which are n -Engel. Suppose that G is soluble with derived length s . Then G is nilpotent with (d, n, s) -bounded class.*

Lemma 2.12. *Let n, d, q be positive integers. Suppose that $w = w(x_1, \dots, x_k)$ is a multilinear commutator word and $v = [w^q, {}_n y]$. Let G be a residually finite group such that $v(G)$ is abelian. Let g_1, \dots, g_d be w^q -values which are right $(n+1)$ -Engel. Then for every $t \in G$ the subgroup $\langle g_1, \dots, g_d, t \rangle$ is nilpotent of (d, n, w, q) -bounded class.*

Proof. It is sufficient to prove the result in the case where G is finite. Let H be the normal closure of the subgroup $\langle g_1, \dots, g_d \rangle$ in $\langle g_1, \dots, g_d, t \rangle$. By Lemma 2.6, H is generated by at most $(n+1)d$ conjugates of the elements g_1, \dots, g_d , which are all right $(n+1)$ -Engel. Note that, by Lemma 2.5 and Lemma 2.3, all w^q -values are left $(n+1)$ -Engel in $G/v(G)$. As $Hv(G)/v(G)$ is generated by $(n+1)d$ elements which are w^q -values, by Proposition 2.9 it is nilpotent with (d, n, w, q) -bounded class. Thus H is soluble with (d, n, w, q) -bounded derived length. As H is generated by at most $(n+1)d$ element which are $(n+2)$ -Engel, by applying Lemma 2.11, we obtain that H is nilpotent with (d, n, w, q) -bounded class. Now the result follows from Lemma 2.7. \square

The next lemma can be seen as an analogue of Theorem 1.1 for soluble groups. The result holds for any $q \geq 1$ and G here is not required to be residually finite.

Lemma 2.13. [5, Lemma 11] *Let m, n, q, s be positive integers. Suppose that $w = w(x_1, \dots, x_k)$ is a multilinear commutator word and $v = [w^q, {}_n y]$. Assume that G is a soluble group of derived length s such that v has at most m values in G . Then the order of $v(G)$ is (v, m, s) -bounded.*

A proof of the next lemma can be found in [5].

Lemma 2.14. [5, Lemma 10] *Let $w = w(x_1, x_2, \dots, x_k)$ be a word. Let G be a nilpotent group of class c generated by k elements a_1, a_2, \dots, a_k . Denote by X the set of all conjugates in G of elements of the form $w(a_1^i, a_2^i, \dots, a_k^i)$, where i ranges over the set of all integers, and assume that X is finite with at most m elements. Then $|\langle X \rangle|$ is (c, m) -bounded.*

Corollary 2.15. *Let m, n, c be positive integers and K a nilpotent group of class at most c . Let $g \in K$ and denote by Y the set of conjugates of elements of the form $[g, {}_n x]$, where $x \in K$. Assume that Y is finite with at most m elements. Then each element in Y has finite order bounded by a function of c and m .*

Proof. Choose $t \in K$. Note that

$$[g, {}_n t] = [(t^{-1})^{gt}, {}_{n-1} t] = [t^{-g}, {}_{n-1} t]^t,$$

and so $[g, {}_n t]$ is a value of the word $u = [x_{1, n-1} x_2]$ in the subgroup $\langle t^{-g}, t \rangle$ of K . As $u((t^{-g})^i, t^i) = [g, {}_n t^i]^{t^{-i}} \in Y$, the set of all conjugates in $\langle t^{-g}, t \rangle$ of elements of the form $u((t^{-g})^i, t^i)$, where i is any integer, has at most m elements. It follows from Lemma 2.14 that the element $[g, {}_n t]$ has (c, m) -bounded order, as required. \square

3. Combinatorics of multilinear commutators

Let k, n be positive integers. Throughout this section, $w = w(x_1, \dots, x_k)$ denotes an arbitrary but fixed multilinear commutator on k variables while v denotes the word $v = [w, {}_n y]$.

If \mathcal{U} is a set of words and G is a group, then $\mathcal{U}(G)$ will denote the subgroup generated by all values in G of the words $u \in \mathcal{U}$.

We denote by \mathbf{I} the set of all k -tuples (i_1, \dots, i_k) , where all entries i_s are non-negative integers. We will view \mathbf{I} as a partially ordered set with the partial order defined by the rule that

$$(i_1, \dots, i_k) \leq (j_1, \dots, j_k)$$

if and only if $i_1 \leq j_1, \dots, i_k \leq j_k$.

Given $\mathbf{i} = (i_1, \dots, i_k) \in \mathbf{I}$, we write

$$w_{\mathbf{i}} = w(\delta_{i_1}, \dots, \delta_{i_k})$$

where the variables appearing in the δ_{i_j} 's are renamed in such a way that they are all different. Further, let

$$\mathcal{W}_{\mathbf{i}+} = \{w_{\mathbf{j}} \mid \mathbf{j} \in \mathbf{I}; \mathbf{j} > \mathbf{i}\}.$$

The following lemmas are Corollary 6 and Proposition 7 in [4].

Lemma 3.1. *Let G be a group and $\mathbf{i} \in \mathbf{I}$. If $\mathcal{W}_{\mathbf{i}+}(G) = 1$, then $w_{\mathbf{i}}(G)$ is abelian.*

Lemma 3.2. *Let G be a group and $\mathbf{i} \in \mathbf{I}$. Suppose that $a_j \in G^{(i_j)}$ for $j = 1, \dots, k$, and let $b_s \in G^{(i_s)}$. If $\mathcal{W}_{\mathbf{i}+}(G) = 1$, then*

$$w(a_1, \dots, a_{s-1}, b_s a_s, a_{s+1}, \dots, a_k)$$

$$= w(\bar{a}_1, \dots, \bar{a}_{s-1}, b_s, \bar{a}_{s+1}, \dots, \bar{a}_k) w(a_1, \dots, a_{s-1}, a_s, a_{s+1}, \dots, a_k),$$

where \bar{a}_j is a conjugate of a_j and moreover $\bar{a}_j = a_j$ if $i_j \leq i_s$.

Corollary 3.3. *Let G be a group and $\mathbf{i} \in \mathbf{I}$. If $\mathcal{W}_{\mathbf{i}+}(G) = 1$, then every power of a $w_{\mathbf{i}}$ -value is a w -value.*

Proof. Let $w(a_1, \dots, a_k)$ be a $w_{\mathbf{i}}$ -value, where each a_j is a δ_{i_j} -value. In particular $a_j \in G^{(i_j)}$. Let i_s be the maximum among all i_j with $j = 1, \dots, k$ and let l be any integer. Then Lemma 3.2 and induction on l show that:

$$w(a_1, \dots, a_{s-1}, a_s, a_{s+1}, \dots, a_k)^l = w(a_1, \dots, a_{s-1}, a_s^l, a_{s+1}, \dots, a_k).$$

This proves the result. □

We say that a word u is an *extension* of w if $u = w(u_1, \dots, u_k)$, for some words u_1, \dots, u_k . The series constructed in the following proposition is similar to those appearing in [7, 4, 8]

Proposition 3.4. *Let F be the free group on countably many free generators. Then for each $s \geq k$ there exist a (k, s) -bounded integer r and a series of length r*

$$F^{(s)} = \mathcal{U}_0(F) \leq \mathcal{U}_1(F) \leq \dots \leq \mathcal{U}_r(F) = w(F)$$

such that:

- (1) Each \mathcal{U}_j is a finite set of multilinear commutators which are extensions of w .
- (2) Every section $\mathcal{U}_{j+1}(F)/\mathcal{U}_j(F)$ is abelian.
- (3) For every $j = 1, \dots, r$ and $u \in \mathcal{U}_j$ the inclusion $(F_u)^i \subseteq F_w \mathcal{U}_{j-1}(F)$ holds for all integers i , that is, i -th powers of u -values are w -values modulo $\mathcal{U}_{j-1}(F)$.

Proof. We set $\mathcal{U}_0 = \{\delta_s\}$. Note that there are only (k, s) -boundedly many k -tuples \mathbf{i} such that $w_{\mathbf{i}}(F) \not\leq \mathcal{U}_0(F)$. Choose \mathbf{i} such that $w_{\mathbf{i}}(F) \not\leq \mathcal{U}_0(F)$ but $\mathcal{W}_{\mathbf{i}^+}(F) \leq \mathcal{U}_0(F)$ and set $\mathcal{U}_1 = \{w_{\mathbf{i}}\} \cup \mathcal{U}_0$. More generally, once we have defined \mathcal{U}_{j-1} , we choose \mathbf{i} such that $w_{\mathbf{i}}(F) \not\leq \mathcal{U}_{j-1}(F)$ but $\mathcal{W}_{\mathbf{i}^+}(F) \leq \mathcal{U}_{j-1}(F)$ and set $\mathcal{U}_j = \{w_{\mathbf{i}}\} \cup \mathcal{U}_{j-1}$. After a (k, s) -bounded number of steps r we get $\mathbf{i} = (0, \dots, 0)$. So $\mathcal{U}_r = \{w\} \cup \mathcal{U}_{r-1}$. As all the words in \mathcal{U}_{r-1} are extensions of w we obtain that $\mathcal{U}_r(F) = w(F)$.

As $\mathcal{W}_{\mathbf{i}^+}(F) \leq \mathcal{U}_{j-1}(F)$, it follows that

$$\mathcal{U}_j(F)/\mathcal{U}_{j-1}(F) = w_{\mathbf{i}}(F)\mathcal{U}_{j-1}(F)/\mathcal{U}_{j-1}(F)$$

is a homomorphic image of $w_{\mathbf{i}}(F)/\mathcal{W}_{\mathbf{i}^+}(F)$. So the quotient group $\mathcal{U}_j(F)/\mathcal{U}_{j-1}(F)$ is abelian by Lemma 3.1. Moreover, by Corollary 3.3, powers of $w_{\mathbf{i}}$ -values are again w -values modulo $\mathcal{W}_{\mathbf{i}^+}(F)$ so the same holds modulo $\mathcal{U}_{j-1}(F)$. \square

Proposition 3.5. *Let $\mathcal{U}_0, \dots, \mathcal{U}_r$ be sets of words as in Proposition 3.4. Then for every positive integer i there exists an integer r_i such that for every group G , for every $j = 1, \dots, r$, for every \mathcal{U}_j -value g in G and every $t \in G$ we have*

$$[g, {}_n t]^i = \tilde{v}\tau,$$

where \tilde{v} is a v -value and τ is the product of r_i \mathcal{U}_{j-1} -values.

Proof. Let F be the free group of countable rank with free generators $\{y, x_\eta \mid \eta = 1, 2, \dots\}$ and let

$$F^{(s)} = \mathcal{U}_0(F) \leq \mathcal{U}_1(F) \leq \dots \leq \mathcal{U}_r(F) = w(F),$$

be a series as in Proposition 3.4. Consider an arbitrary j and let $u \in \mathcal{U}_j$. Recall that u -values are w -values and their powers are w -values modulo $\mathcal{U}_{j-1}(F)$. So for every integer i we have

$$u^i \mathcal{U}_{j-1}(F) = \tilde{u}_i \mathcal{U}_{j-1}(F),$$

for some w -value \tilde{u}_i in F (which depends on i). As $\mathcal{U}_j(F)/\mathcal{U}_{j-1}(F)$ is abelian, it follows that

$$[u, ny]^i \mathcal{U}_{j-1}(F) = [u^i, ny] \mathcal{U}_{j-1}(F) = [\tilde{u}_i, ny] \mathcal{U}_{j-1}(F).$$

So

$$[u, ny]^i = [\tilde{u}_i, ny] \tau_i,$$

where $[\tilde{u}_i, ny]$ is a v -value in F and τ_i is the product of a finite number $r(i, u)$ of \mathcal{U}_{j-1} -values. Let r_i be the maximum of all $r(i, u)$'s, for all words $u \in \mathcal{U}_j$ and all $j = 0, \dots, r$. Note that r_i depends only on w and i .

Now let G be any group and let g be a u -value for some $u \in \mathcal{U}_j$, say $u = u(x_1, \dots, x_l)$, so that $g = u(g_1, \dots, g_l)$ where each g_η belongs to G . Consider the epimorphism $\varphi : F \rightarrow \langle g_1, \dots, g_l, t \rangle$ defined by $\varphi(y) = t$, $\varphi(x_\eta) = g_\eta$ if $\eta \in \{1, \dots, l\}$, $\varphi(x_\eta) = 1$ otherwise. Then $[g, nt]^i = \varphi([u, ny]^i)$ has the desired properties, where $\tilde{v} = \varphi([\tilde{u}_i, ny])$ and $\tau = \varphi(\tau_i)$. \square

4. Proof of Theorem 1.1

In this section we will prove the first of our main results.

Proof of Theorem 1.1. Let $v = [w, ny]$, where w is a multilinear commutator word in k variables and $n \geq 1$. We need to show that if G is a residually finite group with at most m values of the word v , then the order of $v(G)$ is (v, m) -bounded. Evidently, it suffices to establish this result for finite quotients of G . Therefore, without loss of generality, we assume that G is finite.

In view of Lemma 2.1 we may pass to the quotient $G/v(G)'$ and assume that $v(G)$ is abelian. The image of $w(G)$ in $G/v(G)$ is generated by right Engel elements. Hence $w(G)/v(G)$ is nilpotent. Since w is a multilinear commutator on k variables, Lemma 2.2 tells us that every δ_k -value is a w -value. It follows that the group G is soluble. Taking into account that $C_G(v(G))$ has m -bounded index in G , we deduce that $G/C_G(v(G))$ has m -bounded derived length. Let l be the smallest integer greater than k such that $G^{(l)}$ centralizes $v(G)$. Note that l is (m, k) -bounded and every δ_l -value is also a w -value.

Let $x, y \in G$ where y is a δ_l -value. Using the formula $[x, y, y] = [y^{-xy}, y]$ and taking into account that y^{-1} is a w -value, we deduce that

the commutator $[x, {}_{n+1}y]$ is a v -value and therefore, since $[v(G), y] = 1$, we have $[x, {}_{n+2}y] = 1$. So each δ_l -value is $(n+2)$ -Engel in G .

Note that any δ_{2l} -value can naturally be written as a δ_l -value whose entries are also δ_l -values. Let $g = \delta_l(g_1, \dots, g_{2l})$ be a δ_{2l} -value, where each g_i is a δ_l -value. Further, choose an arbitrary element $t \in G$ and let H be the minimal t -invariant subgroup of G containing all these elements g_i . Since the image of t in $G/v(G)$ acts on each g_i as an Engel element, Lemma 2.6 tells us that the image of H in $G/v(G)$ is generated by a (v, m) -bounded number of δ_l -values, which are $(n+2)$ -Engel. By Proposition 2.9, $Hv(G)/v(G)$ is nilpotent of (v, m) -bounded class. Since $v(G)$ is abelian, we conclude that $H\langle t \rangle$ is soluble of (v, m) -bounded derived length. Lemma 2.13 tells us that $v(H\langle t \rangle)$ is finite with order bounded by an integer $R = R(v, m)$ depending only on v and m . Since $v(G)$ is abelian of rank at most m , the order of the subgroup M_0 of $v(G)$ generated by all elements of order at most R is (v, m) -bounded. In particular, as $[g, {}_nt] \in v(H\langle t \rangle)$, it follows that $[g, {}_nt] \in M_0$. Thus we have found a normal subgroup M_0 of (v, m) -bounded order such that for every δ_{2l} -value g , and every $t \in G$ we have that $[g, {}_nt] = 1$ in the quotient group G/M_0 .

Consider now the sets of words $\mathcal{U}_0, \dots, \mathcal{U}_r$ defined in Proposition 3.4 and the integers r_i defined in Proposition 3.5 with $s = 2l$. Then for every integer i , for every $j = 1, \dots, r$, for every \mathcal{U}_j -value $g \in G$ and every $t \in G$ we have

$$[g, {}_nt]^i = \tilde{v}\tau, \quad (1)$$

where \tilde{v} is a v -value and τ is the product of r_i \mathcal{U}_{j-1} -values.

We will prove by induction on j that

- (*) G has a normal subgroup $M_j \leq v(G)$ of (v, m, j) -bounded order such that $[g, {}_{n+1}t] \in M_j$ for every \mathcal{U}_j -value g and every $t \in G$.

In the case $j = 0$ we have $\mathcal{U}_0 = \{\delta_{2l}\}$, and the result has already been proved.

Assume that $j > 0$. Without loss of generality we can assume that M_{j-1} is trivial and so every \mathcal{U}_{j-1} -value is right $(n+1)$ -Engel in G .

Let g be an arbitrary \mathcal{U}_j -value in G and let $t \in G$. As in G there are at most m v -values, equality (1) implies that for some $0 \leq i_1 < i_2 \leq m$ we have

$$[g, {}_nt]^{i_1} = \tilde{v}\tau_1,$$

$$[g, {}_nt]^{i_2} = \tilde{v}\tau_2,$$

where τ_1 and τ_2 are products of at most r_1 and r_2 \mathcal{U}_{j-1} -values, respectively. So $[g, {}_nt]^{i_2-i_1}$ is the product of f (possibly trivial) \mathcal{U}_{j-1} -values z_1, \dots, z_f where

$$f = 2 \cdot \max\{r_i \mid 0 \leq i \leq m\}.$$

By induction, $[z_i,_{n+1}t] = 1$ for each i . Lemma 2.12 guarantees that the subgroup $Z = \langle z_1, \dots, z_f, t \rangle$ is nilpotent of (v, m) -bounded class. So there exists a (v, m) -bounded integer s such that $[Z,_{s}t] = 1$. Since the index of $C_G(v(G))$ in G is finite, conjugation by t is an automorphism of $v(G) \cap Z$ of bounded order, say e . In view of Lemma 2.8 we obtain that $[(v(G) \cap Z)^{e^{s-1}}, t] = 1$. As $v(G) \cap Z$ is abelian, it follows that $[v(G) \cap Z, t]$ has finite exponent at most e^{s-1} , which is (v, m) -bounded. Let N be the subgroup of $v(G)$ generated by all elements of order at most e^{s-1} . Note that N has (v, m) -bounded order.

If $N = 1$, then $[a, t] = 1$ for every $a \in v(G) \cap Z$ and $t \in G$, that is, $v(G) \cap Z$ is contained in the center of the group G . Moreover, as $[g,_{n}t]^{i_2-i_1} \in Z \cap v(G)$ and $v(G)$ is abelian, we deduce that

$$1 = [[g,_{n}t]^{i_2-i_1}, t] = [[g,_{n}t], t]^{i_2-i_1} = [g,_{n+1}t]^{i_2-i_1}.$$

Let M_j be the subgroup generated by all elements of $v(G)$ whose $m!$ -th power belongs to N . The above equality shows that $[g,_{n+1}t] \in M_j$ for an arbitrary \mathcal{U}_j -value g and an arbitrary $t \in G$. Obviously, the order of M_j is bounded. This concludes the proof of (*).

Therefore there exists a normal subgroup M_r of G of (v, m) -bounded order such that in the quotient group G/M_r the equality $[g,_{n+1}t] = 1$ holds for every w -value g and for every element t .

Passing to the quotient G/M_r , we can assume that $M_r = 1$. By Lemma 2.12 the subgroup $\langle g, t \rangle$ is nilpotent of (v, m) -bounded class for every $g \in G_w$ and every $t \in G$. In view of Corollary 2.15 we obtain that for every $g \in G_w$ and every $t \in G$ the element $[g,_{n}t]$ has (v, m) -bounded order.

Thus, $v(G)$ is an abelian group of rank at most m generated by elements of (v, m) -bounded order. Hence, the order of $v(G)$ is (v, m) -bounded. The proof is complete. \square

5. Theorem 1.2

In the present section Theorem 1.2 will be proved. We need the following proposition, which is the main result in [20].

Proposition 5.1. *Let G be a residually finite group satisfying some identity. Suppose G is generated by a normal commutator-closed set of p -elements. Then G is locally finite.*

Recall that a group is locally graded if every non-trivial finitely generated subgroup has a proper subgroup of finite index. The class of locally graded groups is fairly large and in particular it contains all residually finite

groups. Proposition 5.1 can be easily extended to the case of locally graded groups.

Lemma 5.2. *Let G be a locally graded group satisfying some identity. Suppose G is generated by a normal commutator closed set of p -elements for some prime p . Then G is locally finite.*

Proof. Let X be a normal commutator closed set of p -elements which generate G . It is sufficient to prove that any subgroup generated by finitely many elements from X is finite. Let $K = \langle x_1, \dots, x_s \rangle$, where $x_1, \dots, x_s \in X$. Let R be the finite residual of K . It follows from Proposition 5.1 that K/R is finite. In particular R is finitely generated. Now it is enough to prove that $R = 1$. Assume by contradiction that R is nontrivial. As G is locally graded, R has a proper subgroup N of finite index. Clearly, N has also finite index in K , and as R is the finite residual of K it follows that $R \leq N$, a contradiction. \square

Important properties of the verbal subgroup corresponding to a multilinear commutator word in a soluble-by-finite group are described in the following proposition.

Proposition 5.3. [1, Proposition 2.6] *Let w be a multilinear commutator word, and let G be a group having a normal soluble subgroup of finite index e and derived length s . Then G has a series of subgroups*

$$1 = T_1 \leq T_2 \leq \dots \leq T_l = w(G)$$

such that:

- (1) All subgroups T_i are normal in G .
- (2) The length l of the series is (w, s, e) -bounded.
- (3) Every section T_{i+1}/T_i is abelian and can be generated by w -values in G/T_i all of whose powers are also w -values, except possibly one section whose order is finite and (s, e) -bounded.

The next lemma is a version of Theorem 1.2 in the case when the group G is soluble-by-finite. Note that G here is not required to be residually finite and q is an arbitrary integer.

Proposition 5.4. *Let m, n, q, s, e be positive integers. Suppose that $w = w(x_1, \dots, x_k)$ is a multilinear commutator word and $v = [w^q, {}_n y]$. Assume that G has a normal soluble subgroup of finite index e and derived length s and suppose that v has at most m values in G . Then the order of $v(G)$ is (v, m, s, e) -bounded.*

Proof. In view of Lemma 2.1 we may assume that $v(G)$ is abelian. Consider a series as in Proposition 5.3. We will use induction on the length of this series, the case $w(G) = 1$ being trivial. Let L be the last nontrivial term of the series. By induction we assume that the image of $v(G)$ in G/L has finite (v, m, s, e) -bounded order. Set $K = v(G) \cap L$. It follows that the index of K in $v(G)$ is (v, m, s, e) -bounded.

Without loss of generality, we may assume that the subgroup L is abelian and can be generated by w -values in G all of whose powers are also w -values. Let $g \in L$ be one of those w -values and let $t \in G$. Then for every positive integer i the element $[g^{iq}, {}_n t]$ is a v -value. As L is abelian, it follows that $[g^{iq}, {}_n t] = [g^q, {}_n t]^i$. Since v has at most m values in G , there are two different integers i_1, i_2 with $0 \leq i_1 < i_2 \leq m$ such that $[g^q, {}_n t]^{i_1} = [g^q, {}_n t]^{i_2}$. It follows that $[g^q, {}_n t]$ has order at most m , and consequently $[g, {}_n t]$ has order at most mq . Let T_1 be the subgroup of $v(G)$ generated by all elements of order at most mq . As $v(G)$ is abelian with at most m generators, T_1 has (m, q) -bounded order. Thus, we can pass to the quotient G/T_1 and assume that $[g, {}_n t] = 1$ for every generator g of L chosen as above and for every $t \in G$. Since L is abelian, it follows that $[L, {}_n t] = 1$ for every $t \in G$. In particular, $[K, {}_n t] = 1$ for every $t \in G$.

Since the index of $C_G(v(G))$ in G is m -bounded, the conjugation by an arbitrary element $t \in G$ induces an automorphism of $v(G)$ of m -bounded order, say r . Lemma 2.8 tells us that $[K^{r^{n-1}}, t] = 1$. As K is abelian, it follows that $[K, t]$ has exponent dividing r^{n-1} , which is (m, n) -bounded. Let T_2 be the subgroup of $v(G)$ generated by all elements of order at most r^{n-1} . We can pass to the quotient G/T_2 and without loss of generality assume that $[K, t] = 1$ for every $t \in G$. Therefore K is contained in the center of the group G . Further, note that $K\langle t^r \rangle$ is a central subgroup of $v(G)\langle t \rangle$ and has (v, m, s, e) -bounded index in $v(G)\langle t \rangle$. So by Schur's Theorem [17, 10.1.4] the derived subgroup of $v(G)\langle t \rangle$ has (v, m, s, e) -bounded order, and the bound does not depend on the choice of t . Arguing as before and factoring out an appropriate subgroup of $v(G)$ of bounded order, we may assume that $[v(G), t] = 1$ for every $t \in G$, that is, $v(G)$ is contained in the center of G .

In particular, $[g^q, {}_{n+1} t] = 1$ for every $g \in G_w$ and every $t \in G$. So every w^q -value is right $(n+1)$ -Engel in G . Thus, by Lemma 2.5 combined with Lemma 2.3, every w^q -value is left $(n+2)$ -Engel.

It follows from Lemma 2.6 that the normal closure U of the subgroup $\langle g^q \rangle$ in the group $\langle g^q, t \rangle$ is generated by the set $A = \{(g^q)^{t^i} \mid i = 0, \dots, n\}$ whose elements are left $(n+2)$ -Engel. The subgroup U has a normal soluble subgroup N of index at most e and derived length at most s . It follows from Lemma 2.4 that U/N is nilpotent, and as it has bounded order, in particular it is soluble of e -bounded derived length. Lemma 2.11 now tells us that U

is nilpotent with (n, s, e) -bounded class. As $[a,_{n+1}t] = 1$ for every $a \in A$, Lemma 2.7 shows that $U\langle t \rangle = \langle g^q, t \rangle$ is nilpotent of (v, m, s, e) -bounded class.

By Corollary 2.15 the element $[g^q,_{n}t]$ has (v, m, s, e) -bounded order. Thus, we have shown that $v(G)$ is an abelian group of rank at most m generated by elements of (v, m, s, e) -bounded order. Hence, the order of $v(G)$ is (v, m, s, e) -bounded. The proof is complete. \square

Proof of Theorem 1.2. Let G be a residually finite group with finitely many v -values, where $v = [w^q,_{n}y]$, w is multilinear commutator word and q is a prime power. In view of Lemma 2.1 we may pass to the quotient $G/v(G)'$ and assume that $v(G)$ is abelian. Since $v(G)$ is finitely generated, it is clear that elements of finite order in $v(G)$ form a finite normal subgroup. We pass to the quotient over this subgroup and without loss of generality assume that $v(G)$ is torsion-free.

Since G is residually finite, we can choose a normal subgroup H of finite index such that H intersects G_v trivially. Observe that every w^q -value in H is right n -Engel. It follows from Lemmas 2.5 and 2.3 that every w^q -value in H is left $(n+1)$ -Engel. Let $K = w^q(H)$. It follows from Proposition 2.10 that K is locally nilpotent. Moreover, Lemma 2.7 implies that for each $t \in G$ the subgroup $\langle K, t \rangle$ is locally nilpotent. Note that G/K need not be residually finite but by the result in [14] the quotient G/K is locally graded. We deduce from Lemma 5.2 that $w(H)/K$ is locally finite.

Note that in the group $\bar{G} = G/w(H)$ the subgroup \bar{H} is a soluble normal subgroup of finite index. Proposition 5.4 guarantees that $v(\bar{G})$ is finite. Thus, $v(G) \cap w(H)$ has finite index in $v(G)$.

Using the fact that $v(G)$ is finitely generated and the local finiteness of $w(H)/K$ we deduce that $v(G) \cap K$ has finite index in $v(G) \cap w(H)$. Thus $v(G) \cap K$ has finite index in $v(G)$. In particular, $v(G) \cap K$ is finitely generated.

Choose $t \in G$. Since the subgroup $\langle K, t \rangle$ is locally nilpotent and $v(G) \cap K$ is finitely generated, there exists an integer s such that $[v(G) \cap K,_{s}t] = 1$.

Since the index of $C_G(v(G))$ in G is finite, the conjugation by t induces an automorphism of $v(G)$ of finite order, say r . Lemma 2.8 tells us that $[(v(G) \cap K)^{r^{s-1}}, t] = 1$. As $v(G) \cap K$ is abelian, it follows that $[v(G) \cap K, t]$ has finite exponent dividing r^{s-1} . Since $v(G)$ is torsion-free, $[v(G) \cap K, t] = 1$.

We see that $(v(G) \cap K)\langle t^r \rangle$ is a central subgroup of finite index in $v(G)\langle t \rangle$. So by Schur's Theorem the commutator subgroup of $v(G)\langle t \rangle$ is finite. Since the commutator subgroup is contained in $v(G)$, which is torsion-free, it follows that $[v(G), t] = 1$. Since t is arbitrary, $v(G)$ is contained in the center of G . In particular, if $g \in G_w$ and $t \in G$, we have $[g^q,_{n+1}t] = 1$.

Hence, by Lemma 2.12 $\langle g^q, t \rangle$ is nilpotent and so by Corollary 2.15 we obtain that $[g^q,_{n}t]$ has finite order.

Since $[g^a, {}_n t]$ is an arbitrary v -value and $v(G)$ is torsion-free, we conclude that $v(G) = 1$. The theorem is established. \square

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Eloisa Detomi, Dipartimento di Matematica “Tullio Levi-Civita”, Università di Padova, Via Trieste 63, 35121 Padova, Italy

e-mail: detomi@math.unipd.it

Marta Morigi, Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato 5, 40126 Bologna, Italy

e-mail: marta.morigi@unibo.it

Pavel Shumyatsky, Department of Mathematics, University of Brasilia, Brasilia-DF, 70910-900 Brazil

e-mail: pavel@unb.br