Flow Equations for Generalized $T\bar{T}$ Deformations

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We consider the most general set of integrable deformations extending the $T\bar{T}$ deformation of two-dimensional relativistic QFTs. They are CDD deformations of the theory's factorized S matrix related to the higher-spin conserved charges. Using a mirror version of the generalized Gibbs ensemble, we write down the finite-volume expectation value of the higher-spin charges, and derive a generalized flow equation that every charge must obey under a generalized $T\bar{T}$ deformation. This also reproduces the known flow equations on the nose.

Introduction.—Our understanding of physics has been unfailingly advanced by the study of exactly solvable models—from the Kepler problem to the latest advances in interacting quantum field theories (QFTs). A powerful illustration of this approach is given by integrable QFTs (IQFTs) in two spacetime dimensions; see, e.g., Refs. [1,2] for reviews. Physically, we can think of IQFTs as arising from deforming a two-dimensional conformal field theory (CFT) by carefully chosen relevant operators, inducing a renormalization group flow. The resulting theory is not conformal, but is nonetheless endowed with infinitely many independent mutually commuting conserved quantities—a remnant of conformal symmetry [3–6]. This constrains the dynamics to the point that it allows one to efficiently compute a wealth of observables—something very remarkable for an interacting QFT.

Given an exactly solvable theory it is natural to ask how much we may modify it while preserving its solvability. Recently we started to realize that deforming CFTs (or QFTs) by *irrelevant* operators might be as physically interesting as the better-understood relevant deformations. Moreover, this paves the way to quantitatively describe a new class of theories. The prime example of irrelevant deformations is the "$T\bar{T}$" deformation [7,8], built out of the stress-energy tensor $T_{\mu\nu}$ [9]. This arises by infinitesimally deforming the Hamiltonian density H by the composite operator $O_{T\bar{T}} = T^{0\mu}T^{1\nu}\epsilon_{\mu\nu}$ and integrating the resulting

flow, $\partial_\alpha H = O_{T\bar{T}}$. $T\bar{T}$-deformed theories are remarkable, and despite intensive study still mysterious: on the one hand, they can be related to two-dimensional gravity [10–13], or to random geometries [14]. On the other hand, they can be also reformulated in terms of string theory [8,15–20] (see also Refs. [21,22] for earlier observations of the relation between strings and $T\bar{T}$) and holography [23–25] and even defined for spin chains [26,27]. $T\bar{T}$ deformations are special as they preserve many symmetries: supersymmetry [28–31], modular invariance [32], and most remarkably *integrability* [7,8]. By this we mean that if the original theory is a CFT, or an IQFT, its infinitely many conserved charges are preserved by the deformation. More is true: even if the original theory is not integrable, the deformation is exactly solvable: the finite-volume spectrum $\{H_n\}$ of $T\bar{T}$-deformed theories obeys the Burgers equation,

$$\partial_\alpha H_n(R,\alpha) = H_n(R,\alpha)\partial_R H_n(R,\alpha) + \frac{1}{R}P_n{}^2, \quad (1)$$

where $P_n = 2\pi N_n/R$, $N_n \in \mathbb{Z}$, is the momentum and $H(R,0)$ the original Hamiltonian. Similar equations may be written for the $T\bar{T}$ deformation of more general charges [33]. Still, $T\bar{T}$ is just one of infinitely many similar integrable deformations of relativistic QFTs [7]. This Letter investigates such arbitrary deformations and derives the analog of the flow equation (1) for generic observables—not just energy and momentum.

To do so, we first review how $T\bar{T}$ deformations and their generalizations are defined in terms of the S matrix of any IQFT [34]. This particular formulation makes it possible to employ integrability techniques such as the thermodynamic Bethe ansatz (TBA) [35,36] to derive Eq. (1) and to study

the theory [8]. For generalized deformations the TBA will not suffice. Ordinarily, Eq. (1) tells us that tuning α corresponds to changing R; we will see that more general deformations correspond to changing new parameters, cousins of R, that may be interpreted as *twists* of the fields' boundary conditions. Periodic boundary conditions $\Phi(0) = \Phi(R)$ will then be modified conjugating the right-hand side by an additional unitary operator $e^{iJ\eta}$, where $\eta \in \mathbb{R}$ is the twist. We will see that such twists may be described using the generalized Gibbs ensemble (GGE) [37] for a *mirror* theory, which we will introduce in the sense of Refs. [38,39]; then η plays the role of a chemical potential. This *mirror* GGE construction is to our knowledge new, though work in this direction appeared earlier in Refs. [40–43]. With this machinery we derive the analog of Eq. (1) for an infinite family of integrable deformations—our main result (19).

The factorized S matrix.—Because of the existence of IQFT conservation laws, scattering is heavily constrained: the only allowed processes are sequences of elastic two-particle collisions. Hence all scattering amplitudes may be written in terms of the two-to-two particle S matrix S_{12}, whose matrix structure must satisfy a consistency condition, the celebrated Yang-Baxter equation [44]. This, along with global symmetries, unitarity, analyticity, and crossing symmetry, constrains S_{12}. Often S_{12} is almost entirely determined by these requirements—it can be *bootstrapped* [44]. The solution is not unique, however: it is only defined up to a Castillejo-Dalitz-Dyson (CDD) factor [45].

CDD deformations.—A two-dimensional relativistic S matrix is most easily described by introducing the *rapidity* θ, related to energy and momentum as $H = m \cosh \theta$ and $p = m \sinh \theta$, where m is the mass. Then S_{12} depends on the difference of rapidities $S_{12} = S(\theta_1 - \theta_2)$, and each S-matrix entry is meromorphic on the θ plane. Linearly realized symmetries and the Yang-Baxter equation leave an overall prefactor $\chi(\theta)$ undetermined. If S_{12} is appropriately normalized, $\chi(\theta)$ is a meromorphic function on the complex plane satisfying Hermitian analyticity [46], $\chi(\theta^*)^* = \chi(\theta)$, unitarity, $\chi(\theta)\chi(-\theta) = 1$, and crossing symmetry, $\chi(\theta) = \chi(i\pi - \theta)$. This means that we may set $\chi(\theta) = e^{i\Sigma(\theta)}$, where $\Sigma(\theta)$ is a $2\pi i$-periodic meromorphic, real-analytic function. The space of such χ's defines a family of integrable theories, at least in terms of their S matrix. There are two natural ways of parametrizing $\chi(\theta)$. We can define it by its poles and resonances, $\chi(\theta|\mathbf{a}) = \prod_j \tanh(\theta - ia_j)/2$: such singularities have a clear physical interpretation in terms of the infrared properties of theory. Otherwise we can write a Fourier series,

$$\Sigma(\theta|\boldsymbol{\alpha}) = \sum_{j \, \text{odd}} \alpha_j e^{-j\theta}, \qquad \alpha_j = -\alpha_{-j}, \qquad \alpha_j \in \mathbb{R}, \quad (2)$$

where we restricted the coefficients using unitarity, real analyticity, and crossing. Each α_j in Eq. (2) affects the

large-θ asymptotic of $S(\theta)$, corresponding to an integrable *irrelevant* deformation [7,47,48]. In particular, $\Sigma(\theta) = \alpha m^2 \sinh \theta$ yields the $T\bar{T}$ flow (1) [8,10]. More general deformations correspond to composite operators of the form $J^\mu_{(j)} J^\nu_{(-j)} \epsilon_{\mu\nu}$ [7], where $J^\mu_{(j)}$ are the infinitely many conserved currents of the integrable theory—the higher-spin currents [49]. The charges J_j commute among themselves and act diagonally on multiparticle scattering states, i.e., on states where particles are well separated:

$$J_j |\theta_1, ..., \theta_N\rangle = \sum_{k=1}^{N} J_j(\theta_k) |\theta_1, ..., \theta_N\rangle, \qquad (3)$$

with $J_j(\theta) = e^{j\theta}$. We shall describe the finite-volume properties of such CDD deformations.

Finite-volume (and finite-temperature) theories.—Consider a two-dimensional Euclidean theory, defined on a torus like in Fig. 1. We take one radius to be very large, $L \gg R$, and eventually $L \to \infty$. There are two ways to obtain Minkowski theories. First, we may Wick rotate and define

$$\text{direct theory:} \quad \sigma \equiv r, \qquad \tau \equiv il. \qquad (4)$$

We call this the direct theory; it lives in finite-volume R but at almost-zero temperature $1/L$. Conversely, we may set

$$\text{mirror theory:} \quad \tilde{\sigma} \equiv l, \qquad \tilde{\tau} \equiv ir. \qquad (5)$$

This is the mirror theory, at finite temperature $1/R$ but in large volume; we denote mirror quantities with tildes. For simplicity we consider relativistic integrable theories with one particle flavor, so that the two-particle S matrix is a *function* $S(\theta)$. Given Eqs. (4)–(5), we may go from the direct theory to its mirror by

$$H \to i\tilde{p}, \qquad p \to -i\tilde{H}. \qquad (6)$$

Hence, up to a parity transformation, the mirror theory is the analytic continuation of the direct one by $\tilde{\theta} \equiv \theta - i\pi/2$ (half of a crossing transformation). This leaves S_{12} and the dispersion unchanged. This construction yields an equality between the *thermal* partition function of the mirror model $\text{Tr}[e^{-R\tilde{H}}]$ and finite-volume spectrum of the direct one $\sum_n e^{-LH^{(n)}}$: when $L \to \infty$ the mirror free-energy density $\tilde{F}(R)$ and direct-theory ground state energy are related as $R\tilde{F}(R) = H^{(0)}(R)$, see Ref. [50] for a recent review. We will consider such quantities in the presence of boundary conditions *twisted* by charges $\mathcal{X}$ and $\mathcal{Y}$ as in Fig. 1. These twists break Poincaré invariance, but leave local properties such as dispersion and S matrix unaffected.

Mirror generalized Gibbs ensemble (GGE).—To study the finite-volume direct theory, we compute the twisted (generalized) free energy of the *mirror* theory.
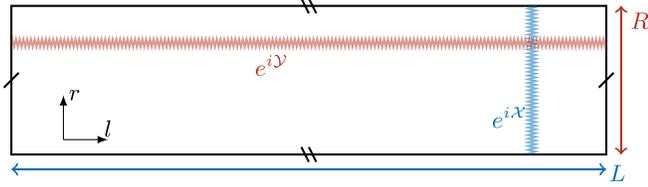
FIG. 1. A Euclidean theory on a torus. The Cartesian coordinates (l, r) are periodically identified, $l \cong l + L$ and $r \cong r + R$. Later we shall twist the boundary conditions of fields $\Phi(l, r)$ by $\mathcal{X}$ and $\mathcal{Y}$. In the direct theory, $\sigma \equiv r$ so that $\mathcal{X}$ is related to a charge (integrated over space) while $\mathcal{Y}$ leads a twist along the spatial direction, which may be interpreted as a defect. In the mirror, this is reversed. The twisted partition function is given in Eq. (9).

Since $mL \gg 1$, the mirror Bethe-Yang equations are approximately correct [51,52]. The L-cycle twist affects the spatial boundary conditions for the mirror theory (Fig. 1). This means replacing the monodromy $e^{i\tilde{p}(\tilde{\theta})L} \equiv e^{H(\tilde{\theta}+i\pi/2)L}$ by $e^{X(\tilde{\theta}+i\pi/2|\boldsymbol{\xi})L}$, which depends on a set of parameters $\boldsymbol{\xi} = \{\xi_j\}$ that identify the charges appearing in $\mathcal{X}$: $X(\theta|\boldsymbol{\xi}) = \sum_j \xi_j J_j(\theta)$ [53]. Then the Bethe-Yang equations are

$$X\left(\tilde{\theta}_k + i\frac{\pi}{2}\Big|\boldsymbol{\xi}\right)L + \sum_{l \neq k}^{N} \log S(\tilde{\theta}_k - \tilde{\theta}_l) = 2\pi i n_k, \quad (7)$$

with $n_k \in \mathbb{Z}$. As $X(\theta)$ is defined in the direct theory, in Eq. (7) we need to analytically continue it to the real-mirror line; all terms in the equation are purely imaginary for $\tilde{\theta}_k \in \mathbb{R}$. In the thermodynamic limit $N \sim mL \gg 1$,

$$\varrho_p(\tilde{\theta}) + \varrho_h(\tilde{\theta}) = \frac{1}{2\pi i}\partial_{\tilde{\theta}}X\left(\tilde{\theta} + i\frac{\pi}{2}\Big|\boldsymbol{\xi}\right) + [\varphi * \varrho_p](\tilde{\theta}), \quad (8)$$

in terms of the densities of particles and holes ϱ_p, ϱ_h and of the Kernel $\varphi_{12} = \partial_1 \log S_{12}/(2\pi i)$, all on the real-mirror line [54]. To twist the R cycle we introduce chemical potentials in

$$Z(L, R|\boldsymbol{\xi}, \boldsymbol{\eta}) = \mathrm{Tr}\, \exp[-R\tilde{Y}(\boldsymbol{\eta})]\big|_{\Phi_{\mathcal{X}(\boldsymbol{\xi})}}. \quad (9)$$

Here $\tilde{Y}$ is the operator in the mirror theory corresponding to the charge $\mathcal{Y}$, parametrized as $R\tilde{Y} = \sum_j \eta_j \tilde{J}_j$. Each $\tilde{J}_j$ acts diagonally on *mirror* states $|\tilde{\theta}_1, \ldots, \tilde{\theta}_N\rangle$, cf. Eq. (3). Here $\tilde{Y}$ is an operator on the mirror-theory Hilbert space. We extremize the free energy in terms of ϱ_p, ϱ_h. Standard manipulations (see, e.g., Ref. [50]) yield the mirror GGE equations,

$$\varepsilon(\tilde{\theta}|\boldsymbol{\eta}) = R\tilde{Y}(\tilde{\theta}|\boldsymbol{\eta}) + [\Lambda(\boldsymbol{\eta}) * \varphi](\tilde{\theta}), \quad (10)$$

where $\Lambda(\tilde{\theta}|\boldsymbol{\eta}) = \pm \log(1 \mp e^{-\varepsilon(\tilde{\theta}|\boldsymbol{\eta})})$ in terms of the mirror-pseudoenergy $\varepsilon(\tilde{\theta}|\boldsymbol{\eta})$ for bosons and fermions, respectively

[55]. This equation is real on the real-mirror line. Remarkably, this equation is sensitive to $\boldsymbol{\eta}$ but *not to* $\boldsymbol{\xi}$. The mirror free-energy density depends on both parameters:

$$R\tilde{F}(R|\boldsymbol{\xi}, \boldsymbol{\eta}) = \frac{1}{2\pi i}\int d\tilde{\theta}\partial_{\tilde{\theta}}X\left(\tilde{\theta} + i\frac{\pi}{2}\Big|\boldsymbol{\xi}\right)\Lambda(\tilde{\theta}|\boldsymbol{\eta}). \quad (11)$$

Here X is the direct-theory operator corresponding to the Euclidean $\mathcal{X}$, parametrized as $X = \sum_j \xi_j J_j$, cf. Eq. (3). In Eq. (11) X is on the *mirror line* [56]. For the direct theory, we have

$$Z(L, R|\boldsymbol{\xi}, \boldsymbol{\eta}) = \mathrm{Tr}\, \exp[-LX(\boldsymbol{\xi})]\big|_{\Phi_{\mathcal{Y}(\boldsymbol{\eta})}}. \quad (12)$$

Thus for $L \to \infty$ we relate Eq. (11) to the direct theory as

$$R\tilde{F}(R|\boldsymbol{\xi}, \boldsymbol{\eta}) = \sum_j \xi_j J_j^{(0)}(R|\boldsymbol{\eta}). \quad (13)$$

By comparing the ξ_j dependence of (11)–(13) we find

$$J_j^{(0)}(R|\boldsymbol{\eta}) = \frac{1}{2\pi i}\int d\tilde{\theta}\partial_{\tilde{\theta}}J_j\left(\tilde{\theta} + i\frac{\pi}{2}\right)\Lambda(\tilde{\theta}|\boldsymbol{\eta}). \quad (14)$$

Note that we expressed the vacuum value $J_j^{(0)}$ of the direct-theory charge J_j (on a state with spatial boundary conditions twisted by Y) through a mirror-theory integral. The relations between defects and chemical potentials in the direct and mirror theories were previously investigated in Refs. [40,42].

Excited states.—Our derivation may also be extended to excited states of the direct theory. They should be described by the same equations with integrals on some state-dependent contour Γ [57] rather than on the real mirror line. (See Refs. [43,58,59] for recent investigations of excited-state expectation values.) The equations then differ by residues picked up between Γ and the real-mirror line at points θ_k, where $e^{-\varepsilon(\theta_k)} = \mp 1$, as the log becomes singular. This may happen when θ_k is on (or around) the real line *of the direct theory* (hence the lack of tilde). Integrating by parts the GGE equations (10), (14) we find residues of the form $\log S(\tilde{\theta} - \theta_k)$ and $J_j(\theta_k)$, respectively. This modifies the vacuum equations by driving terms. In particular in Eq. (14) the driving term is $\sum_k J_j(\theta_k)$, where the charge J_j is evaluated *in the direct theory* owing to analytic continuation to θ_k. Remark than when $mR \gg 1$ the GGE should reduce to the asymptotic result; indeed in this limit $\sum_k J_j(\theta_k)$ dominates and reproduces the asymptotic eigenvalue of J_j on a well-separated direct-theory state, cf. Eq. (3).

CDD deformations in the GGE.—Knowing the finite-volume spectrum with twisted boundary conditions, we can study general CDD deformations of the form (2). Such modifications shift linearly the Kernel $\varphi(\theta_1 - \theta_2) = \varphi(\tilde{\theta}_1 - \tilde{\theta}_2)$. We get $\varphi(\tilde{\theta}_{12}) \to \varphi(\tilde{\theta}_{12}) + (1/2\pi)\sum_j j\alpha_{-j}e^{j\tilde{\theta}_{12}}$. Then the GGE equation (10) becomes

$$\varepsilon(\tilde{\theta}|\boldsymbol{\eta},\boldsymbol{\alpha}) = R\tilde{Y}(\tilde{\theta}|\boldsymbol{\eta}) + [\Lambda(\boldsymbol{\eta},\boldsymbol{\alpha}) * \varphi](\tilde{\theta})$$
$$+ \sum_{j\text{odd}} \frac{\alpha_j}{i^{j+1}} e^{j\tilde{\theta}} J_{-j}(\boldsymbol{\eta},\boldsymbol{\alpha}),$$

$$J_j(\boldsymbol{\eta},\boldsymbol{\alpha}) = \frac{1}{2\pi i} \int d\tilde{\theta} \partial_{\tilde{\theta}} e^{j(\tilde{\theta}+i(\pi/2))} \Lambda(\tilde{\theta}|\boldsymbol{\eta},\boldsymbol{\alpha}). \quad (15)$$

By comparing this with Eq. (14), we see that $J_j(\boldsymbol{\eta},\boldsymbol{\alpha})$ is the ground-state value of a direct-theory charge with density $J_j(\theta) = e^{j\theta}$, i.e., of the direct-theory higher-spin charges, Eq. (3). We can simplify the GGE by setting all $\xi_j = 0$, as this identification also works for infinitesimal ξ_j's. We see that the new term in Eq. (15) can be reabsorbed into $\tilde{Y}(\tilde{\theta}|\boldsymbol{\eta})$ by a constant (but charge-dependent) shift of the parameters η_j, namely, $\eta_j \to \eta_j + i^{j+1}\alpha_j J_{-j}$, implying

$$\varepsilon(\tilde{\theta}|\boldsymbol{\eta},\boldsymbol{\alpha}) = \varepsilon(\tilde{\theta}|\boldsymbol{\eta} + \boldsymbol{\alpha}\check{\mathbf{J}}, \mathbf{0}), \quad (16)$$

where $\check{\mathbf{J}}$ is the ordered set $\{i^{j+1}J_{-j}\}$. Hence, all physical quantities derived from the GGE will depend on $(\boldsymbol{\eta} + \boldsymbol{\alpha}\check{\mathbf{J}})$ only.

The generalized flow equation.—A consequence of Eq. (16) is that every conserved charge satisfies a flow equation. A charge J_j, like all quantities computed from the pseudoenergy, obeys $J_j(\boldsymbol{\eta},\boldsymbol{\alpha}) = J_j[\boldsymbol{\eta} + \boldsymbol{\alpha}\check{\mathbf{J}}(\boldsymbol{\eta},\boldsymbol{\alpha}), \mathbf{0}]$. Hence, defining the differential operator

$$\mathsf{D}_n = i^{n-1}\frac{\partial}{\partial\alpha_n} + J_{-n}(\boldsymbol{\eta},\boldsymbol{\alpha})\frac{\partial}{\partial\eta_n} + J_n(\boldsymbol{\eta},\boldsymbol{\alpha})\frac{\partial}{\partial\eta_{-n}}, \quad (17)$$

for any *positive* odd integer n, we obtain by a direct computation

$$\mathsf{D}_n J_j(\boldsymbol{\xi},\boldsymbol{\alpha}) = \sum_{\ell\text{odd}} M_{j\ell}(\boldsymbol{\eta} + \boldsymbol{\alpha}\check{\mathbf{J}})\mathsf{D}_n J_{-\ell}(\boldsymbol{\xi},\boldsymbol{\alpha}), \quad (18)$$

where $M_{j\ell}(\mathbf{z}) = (\ell/|\ell|)i^{|\ell|+1}\alpha_{|\ell|}(\partial/\partial z_\ell)J_j(\mathbf{z},\mathbf{0})$. Therefore, as long as the operator with matrix elements $M_{j\ell} + \delta_{j\ell}$ is nonsingular (which is the case for small deformations), the only solution to Eq. (18) is

$$\mathsf{D}_n J_j(\boldsymbol{\xi},\boldsymbol{\alpha}) = 0, \quad (19)$$

for all positive odd integers n and all odd integers j. This gives an infinite set of flow equations obeyed by every physical observable $\mathcal{O}(\boldsymbol{\eta},\boldsymbol{\alpha})$ derived from the GGE: $\mathsf{D}_k \mathcal{O}(\boldsymbol{\eta},\boldsymbol{\alpha}) = 0$, since J_j is a basis for any such $\mathcal{O}$.

Recovering the Burgers equation.—The flow equation (1) of ordinary $T\bar{T}$ follows from Eq. (19) when we have only $\alpha_1 \equiv \frac{1}{2}\alpha$. The only nonvanishing chemical potentials are $\eta_{\pm 1} \equiv \frac{1}{2}me^{\pm\zeta}$. Here ζ is an auxiliary parameter (a chemical potential for the direct-theory energy), useful [8] to derive the inhomogenous Burgers equation; we will eventually set $\zeta = 0$. Note that ζ enters the GGE (10) as a rapidity shift,

$$\varepsilon(\tilde{\theta}|R,\zeta;\alpha) = \varepsilon(\tilde{\theta} + \zeta|R,0;\alpha). \quad (20)$$

The physical chemical potential is the mass m. As the GGE depends on m through the dimensionless combination mR, we may trade ∂_m for ∂_R. There is only one flow operator, $\mathsf{D} \equiv \frac{1}{2}\mathsf{D}_1$,

$$\mathsf{D} = \frac{\partial}{\partial\alpha} + H(R,0;\alpha)\frac{\partial}{\partial R} - \frac{1}{R}P(R,0;\alpha)\frac{\partial}{\partial\zeta}. \quad (21)$$

Here we expressed $\partial_{\boldsymbol{\eta}}$ as $\partial_R = \partial_m$ and ∂_ζ, and we introduced the total energy and momentum $2H = J_1 + J_{-1}$ and $2P = J_1 - J_{-1}$. Let us now compute the ∂_ζ derivatives. Note first that using Eq. (20) and shifting the integration measure in Eq. (14), we find that $J_j(R,\zeta;\alpha) = e^{-j\zeta}J_j(R,0;\alpha)$. Therefore, omitting the arguments for convenience, $\partial_\zeta H = -P$ and $\partial_\zeta P = -H$. Hence our flow equation $\mathsf{D}H|_{\zeta=0} = 0$ is precisely the Burgers equation (1). The other equation, $\mathsf{D}P|_{\zeta=0} = 0$, gives that $\partial_\alpha P = 0$ if we also use that $\partial_R P = -P/R$ (which can be derived from the GGE equations). This is expected from the quantization of P. Repeating this argument for $\mathsf{D}J_j = 0$ reproduces the $T\bar{T}$ flow equations for J_j proposed in Ref. [33].

Conclusions and outlook.—We derived flow equations (19) for generalized $T\bar{T}$ deformations that constrain all the GGE observables. We argued this for the vacuum, but clearly our starting point (16) holds for excited states too—these are governed by the same GGE equations (10), (14) up to changing the integration contour. Hence, Eq. (19) is completely general. Our construction uses relativistic invariance sparingly, so that it should be possible to extend it to nonrelativistic setups like those of Refs. [60,61] and Refs. [33,62,63]. Finally, this mirror GGE might be useful beyond the present case to study twists in relativistic and nonrelativistic integrable models.

It would be interesting to study the generalized flows for some simple systems. For a supersymmetric free theory (with Neveu-Schwarz conditions) the GGE trivializes (much like in Refs. [15,64]) and we only have to deal with algebraic equations. The ground-state GGE of a single-flavor theory can also be studied relatively easily. Either case would require numerical investigations, though. A preliminary analysis points to qualitative difference to $T\bar{T}$. This is expected as in Eq. (15) even a tiny generalized deformation yields the dominant contribution to the pseudoenergy at large $|\tilde{\theta}|$, and dramatically affects the convergence properties of the GGE integrals. We plan to report on this elsewhere [65].

In Ref. [32] it was found that Eq. (1) is the only flow equation for finite-volume energy levels preserving modular invariance. We should investigate whether generalized deformations preserve modular covariance of GGE partition functions and whether this requirement uniquely

defines them. Another important question is whether these deformations can be obtained by introducing gauge fields coupled to the higher-spin currents of the IQFTs, similarly to how $T\bar{T}$ may be obtained by coupling the undeformed theory to a gravitational sector.

Finally, in Ref. [66] (see also Refs. [67,68]), the GGE was proposed as a tool to access, in the specific case of sinh-Gordon model, the finite-volume expectation values of local operators. We expect this perspective to be useful to investigate the expectation values of the deformed theories presented in this Letter.

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