

Eigenvalues of the Finsler p -Laplacian on varying domains.

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Abstract

We study the dependence of the first eigenvalue of the Finsler p -Laplacian and the corresponding eigenfunctions upon perturbation of the domain and we generalize a few results known for the standard p -Laplacian. In particular, we prove a Fréchet differentiability result for the eigenvalues, we compute the corresponding Hadamard formulas and we prove a continuity result for the eigenfunctions. Finally, we briefly discuss a well-known overdetermined problem and we show how to deduce the Rellich-Pohozaev identity for the Finsler p -Laplacian from the Hadamard formula.

Keywords: Finsler, anisotropic p -Laplacian, stability of eigenvalues, domain perturbation.

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1 Introduction

Let F be a positive, one-homogeneous, convex function in \mathbb{R}^N and $p \in]1, +\infty[$. In this paper we deal with a class of operators of the form

$$\mathcal{Q}_p u := \operatorname{div}(F^{p-1}(Du)F_\xi(Du)), \quad (1)$$

acting on real-valued functions u defined on an open connected subset Ω of \mathbb{R}^N , (by F_ξ we denote the gradient of F). This class includes the standard p -Laplacian

$$\sum_{i=1}^N \frac{\partial}{\partial x_i} \left(|Du|^{p-2} \frac{\partial u}{\partial x_i} \right)$$

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which is obtained when F is the Euclidean modulus, as well as the pseudo- p -Laplacian

$$\sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right)$$

which is obtained when $F(\xi) = (\sum_i^N |\xi_i|^p)^{1/p}$ for all $\xi \in \mathbb{R}^N$. The operator in (1) is often called anisotropic p -Laplacian or Finsler p -Laplacian and has been studied in several papers, see e.g., [1, 6, 7, 12, 13, 18, 28]. Clearly, operator (1) is of interest mainly for $N \geq 2$ since for $N = 1$ one obtains only the standard p -Laplacian. Thus, also for the purposes of the paper itself, we shall tacitly assume throughout the paper that $N \geq 2$, although, in principle, this is not really required. Here we consider the eigenvalue problem associated with \mathcal{Q}_p

$$\begin{cases} -\mathcal{Q}_p u = \lambda |u|^{p-2} u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (2)$$

where Ω is assumed to be of class $C^{1,\alpha}$, $\alpha \in (0, 1]$. It is well known that the eigenvalue problem (2) admits a unique eigenvalue $\lambda_p(\Omega)$ associated with a positive eigenfunction, which is simple and isolated, see e.g., [6, 15].

Our aim is to analyze the dependence of $\lambda_p(\Omega)$ on Ω . To do so, we adopt the approach developed in [21] for the standard p -Laplacian and we extend the corresponding results to the general case. Namely, we consider a suitable class of domains which are represented as diffeomorphic images of Ω via a diffeomorphism $\phi : \Omega \rightarrow \phi(\Omega)$ of class $C^{1,\alpha}$, we study the map which takes ϕ to the eigenvalue $\lambda_p(\phi(\Omega))$, we prove a Frechét differentiability result for dependence of $\lambda_p(\phi(\Omega))$ on ϕ , and we compute the corresponding Hadamard formula for the Frechét differential, see Theorems 4.1 and 4.2. We also prove that the uniquely determined positive normalized eigenfunction $u_{p,\phi}$ associated with $\lambda_p(\phi(\Omega))$ depends with continuity on ϕ in the sense that its pull-back $u_{p,\phi} \circ \phi$, which is a function defined in the fixed domain Ω , varies continuously in $C^1(\overline{\Omega})$ upon variation of ϕ , see Theorem 3.1.

Finally, we briefly discuss how our results provide a motivation for the study of the well-known overdetermined problem (28) associated with the Faber-Krahn inequality for the Finsler p -Laplacian and we show how to deduce the Rellich-Pohozahev identity (32) from the Hadamard formula.

We note that domain perturbation problems have been extensively studied in the Euclidean case for the Dirichlet Laplacian as well as for more general elliptic operators and there is a wide related literature, see e.g., [2, 3, 10, 8, 21, 25, 26] and the references therein. In particular, we refer to the monograph [17] where the method of domain transplantation via diffeomorphisms is extensively applied and to [9] for a collection of Hadamard-type formulas. For the case of quasi-linear PDEs, much less is available in the literature and we refer to [11, 16, 21] for the analysis of the standard p -Laplacian and to [4] for a p -Laplacian problem involving a Hardy potential.

2 Preliminaries and notation

Throughout the paper we will consider a convex even 1-homogeneous function

$$\mathbb{R}^N \ni \xi \mapsto F(\xi) \in [0, +\infty[,$$

that is, a convex function such that

$$F(t\xi) = |t|F(\xi), \quad (3)$$

for all $t \in \mathbb{R}$, $\xi \in \mathbb{R}^N$ and we assume that

$$a|\xi| \leq F(\xi), \quad (4)$$

for all $\xi \in \mathbb{R}^N$, for some constant $a > 0$. The hypotheses on F imply there exists $b \geq a$ such that

$$F(\xi) \leq b|\xi|,$$

for all $\xi \in \mathbb{R}^N$. Moreover, throughout the paper we will assume that $p \in]1, \infty[$ and that $F \in C^2(\mathbb{R}^N \setminus \{0\})$, and

$$[F^p]_{\xi\xi}(\xi) \text{ is positive definite in } \mathbb{R}^N \setminus \{0\}. \quad (5)$$

The hypothesis (5) ensures that the operator \mathcal{Q}_p defined in (1) is elliptic, hence there exists a positive constant γ such that

$$\sum_{i,j=1}^n \frac{\partial}{\partial \xi_j} (F^{p-1}(\eta) F_{\xi_i}(\eta)) \xi_i \xi_j \geq \gamma |\eta|^{p-2} |\xi|^2, \quad (6)$$

for all $\eta \in \mathbb{R}^n \setminus \{0\}$ and for all $\xi \in \mathbb{R}^n$. We recall that the *polar* function $F^o: \mathbb{R}^N \rightarrow [0, +\infty[$ of F is defined by

$$F^o(v) = \sup_{\xi \in \mathbb{R}^N \setminus \{0\}} \frac{\langle \xi, v \rangle}{F(\xi)},$$

for all $v \in \mathbb{R}^N$, where $\langle \xi, v \rangle$ denotes the scalar product of ξ and v . It is easy to verify that also F^o is a convex function which satisfies properties (3) and (4). Furthermore,

$$F(v) = \sup_{\xi \in \mathbb{R}^N \setminus \{0\}} \frac{\langle \xi, v \rangle}{F^o(\xi)},$$

for all $v \in \mathbb{R}^N$. From the above property it holds that

$$|\langle \xi, \eta \rangle| \leq F(\xi) F^o(\eta), \quad \forall \xi, \eta \in \mathbb{R}^N.$$

The set

$$\mathcal{W} = \{\xi \in \mathbb{R}^N : F^o(\xi) < 1\}$$

is called *Wulff shape* centered at the origin. We put $\kappa_N = |\mathcal{W}|$, where $|\mathcal{W}|$ denotes the Lebesgue measure of \mathcal{W} . More generally, we denote with $\mathcal{W}_r(x_0)$ the set $x_0 + r\mathcal{W}$, that is the Wulff shape centered at x_0 with measure $\kappa_N r^N$, and $\mathcal{W}_r(0) = \mathcal{W}_r$.

We also recall the following properties of F and F^o (see e.g. [5]):

$$\begin{aligned} \langle F_\xi(\xi), \xi \rangle &= F(\xi), & \langle F_\xi^o(\xi), \xi \rangle &= F^o(\xi), & \forall \xi \in \mathbb{R}^N \setminus \{0\} \\ F(F_\xi^o(\xi)) &= F^o(F_\xi(\xi)) = 1, & \forall \xi \in \mathbb{R}^N \setminus \{0\}, \\ F^o(\xi) F_\xi(F_\xi^o(\xi)) &= F(\xi) F_\xi^o(F_\xi(\xi)) = \xi & \forall \xi \in \mathbb{R}^N \setminus \{0\}. \end{aligned}$$

Let Ω be a bounded open subset of \mathbb{R}^N and $p \in]1, \infty[$. We recall that the eigenvalue problem (2) has to be considered in the weak form, which means that the unknown u belongs to the standard Sobolev space $W_0^{1,p}(\Omega)$ and satisfies the equation

$$\int_{\Omega} F^{p-1}(Du) F_{\xi}(Du) \cdot D\varphi^T dy = \lambda \int_{\Omega} |u|^{p-2} u \varphi dy, \quad (7)$$

for all $\varphi \in W_0^{1,p}(\Omega)$. We observe that in this paper the elements of \mathbb{R}^N are thought as row vectors and that the inverse and transpose of a matrix M are denoted by M^{-1} and M^T respectively.

Note that if $u \in W_0^{1,p}(\Omega)$ is a non-trivial solution to the previous equation, then choosing $\varphi = u$ in (7) and using Euler's homogeneous function Theorem, yield the equality $\lambda = \int_{\Omega} F^p(Du) dx / \int_{\Omega} |u|^p dx$ which suggests that the smallest eigenvalue, denote by $\lambda_p(\Omega)$, is positive and has the following variational characterisation

$$\lambda_p(\Omega) = \min_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} F^p(Du) dx}{\int_{\Omega} |u|^p dx}. \quad (8)$$

In fact, the following known results hold, see e.g., [6, 15].

Theorem 2.1. *If $p > 1$ and Ω is a bounded open set in \mathbb{R}^N there exists a function $u_1 \in W_0^{1,p}(\Omega)$ which achieves the minimum in (8), and satisfies the problem (7) with $\lambda = \lambda_p(\Omega)$. Moreover, if Ω is connected, then $\lambda_p(\Omega)$ is simple, that is, the corresponding eigenfunction is unique up to a multiplicative constant, and the first eigenfunction has constant sign in Ω . Finally, if in addition Ω is of class $C^{1,\alpha}$ with $\alpha \in]0, 1]$ then u_1 is of class $C^{1,\beta}(\bar{\Omega})$ for some $\beta \in]0, \alpha]$.*

We recall the scaling and monotonicity properties of $\lambda_p(\Omega)$ in the following lemma the proof of which can be carried out as in the case of the classical p -Laplacian.

Lemma 2.2. *Let Ω be a bounded open set in \mathbb{R}^N . Then the following properties hold.*

1. *For $t > 0$ it holds $\lambda_p(t\Omega) = t^{-p}\lambda_p(\Omega)$.*
2. *If $\tilde{\Omega}$ is an open set contained in Ω then $\lambda_p(\tilde{\Omega}) \geq \lambda_p(\Omega)$.*
3. *If $1 < p < s < +\infty$ then $p[\lambda_p(\Omega)]^{1/p} < s[\lambda_s(\Omega)]^{1/s}$.*

3 Domain perturbation and stability of eigenfunctions

In order to study the dependence of $\lambda_p(\Omega)$ with respect to the variation of the domain Ω , we consider problem (7) in a class of domains which are parameterized by means of diffeomorphisms defined on a domain Ω . More precisely, we fix a bounded domain (i.e., a bounded connected open set) Ω of class $C^{1,\alpha}$ with $\alpha \in (0, 1]$ and we set

$$\Phi_{1,\alpha}(\Omega) \equiv \left\{ \phi \in C^{1,\alpha}(\bar{\Omega}, \mathbb{R}^N) : \phi \text{ is injective and } \det D\phi(x) \neq 0, \text{ for all } x \in \bar{\Omega} \right\}.$$

The space $C^{1,\alpha}(\bar{\Omega}, \mathbb{R}^N)$ is endowed with its standard norm defined by

$$\|\Phi\|_{C^{1,\alpha}(\bar{\Omega}, \mathbb{R}^N)} = \max\{\|\phi\|_\infty, \|\nabla\phi\|_\infty, |\nabla\phi|_\alpha\},$$

where $|\cdot|_\alpha$ denotes the usual Hölder semi-norm.

The set $\Phi_{1,\alpha}(\Omega)$ is an open subset of $C^{1,\alpha}(\bar{\Omega}, \mathbb{R}^N)$; moreover, also $\phi(\Omega)$ is a bounded connected open subset of \mathbb{R}^N of class $C^{1,\alpha}$ and $\partial\phi(\Omega) = \phi(\partial\Omega)$ for all $\phi \in \Phi_{1,\alpha}(\Omega)$, see [22] and [23] for details.

For $\phi \in \Phi_{1,\alpha}(\Omega)$, we consider the smallest eigenvalue $\lambda_p(\phi(\Omega))$ of problem (7) in the domain $\phi(\Omega)$ and we set

$$\lambda_{p,\phi} = \lambda_p(\phi(\Omega)).$$

We observe that since the open set $\phi(\Omega)$ has a regular boundary, the corresponding eigenfunctions attain the boundary value zero in the classical sense. It follows that there exists a uniquely determined eigenfunction of $\lambda_{p,\phi}$, denoted by $u_{p,\phi}$, such that

$$\int_{\phi(\Omega)} |u_{p,\phi}|^p dy = 1 \quad \text{and} \quad u_{p,\phi}(y) > 0 \quad \forall y \in \phi(\Omega), \quad (9)$$

and such that $u_{p,\phi} \in C^{1,\beta}(\phi(\tilde{\Omega}))$ for some $\beta \in]0, \alpha]$. In particular we have that

$$\lambda_{p,\phi} = \min_{\varphi \in C_0^1(\tilde{\Omega}) \setminus \{0\}} \frac{\int_{\phi(\Omega)} F^p(D\varphi) dy}{\int_{\phi(\Omega)} |\varphi|^p dy} = \int_{\phi(\Omega)} F^p(Du_{p,\phi}) dy, \quad (10)$$

where $C_0^1(\tilde{\Omega})$ denotes the space of functions in $C^1(\tilde{\Omega})$ which vanish at the boundary of Ω .

We plan to study the differentiability of the map which takes $\phi \in \Phi_{1,\alpha}(\Omega)$ to $\lambda_{p,\phi} \in \mathbb{R}$. To do so, we first analyze the dependence of the eigenfunction corresponding to $\lambda_{p,\phi}$, with respect to ϕ . We stress the fact that the domain of $u_{p,\phi}$ changes when ϕ is perturbed, hence we will not consider $u_{p,\phi}$ itself but its pull-back

$$v_{p,\phi} \equiv u_{p,\phi} \circ \phi,$$

which is defined in the fixed domain Ω .

Using a change of variable we rewrite problem (7) on $\phi(\Omega)$ into an eigenvalue problem defined in Ω which is solved by $v_{p,\phi}$. More precisely, for $\phi \in \Phi_{1,\alpha}(\Omega)$ and $\lambda \in (0, +\infty)$, it is easy to see that a function $u \in W_0^{1,p}(\phi(\Omega))$ satisfies the eigenvalue problem (7) on $\phi(\Omega)$ if and only if the function $v := u \circ \phi$ belongs to $W_0^{1,p}(\Omega)$ and satisfies the equation

$$\begin{aligned} \int_{\Omega} F^{p-1}(Dv \cdot (D\phi)^{-1}) F_\xi(Dv \cdot (D\phi)^{-1}) \cdot [(D\phi)^{-1}]^T \cdot D\varphi^T |\det D\phi| dx \\ = \lambda \int_{\Omega} |v|^{p-2} v \varphi |\det D\phi| dx, \end{aligned} \quad (11)$$

for all $\varphi \in W_0^{1,p}(\Omega)$. Moreover,

$$\lambda_{p,\phi} = \int_{\Omega} F^p (Dv_{p,\phi} \cdot (D\phi)^{-1}) |\det D\phi| dx.$$

In what follows we denote the inverse matrix of $D\phi$ with J , that is $J = (D\phi)^{-1}$.

The following result on the continuous dependence of the function $v_{p,\phi}$ on $\phi \in \Phi_{1,\alpha}(\Omega)$ holds.

Theorem 3.1. *Let $p > 1$ and Ω be a bounded domain in \mathbb{R}^N of class $C^{1,\alpha}$ with $\alpha \in (0, 1]$. The function which maps $\phi \in \Phi_{1,\alpha}(\Omega)$ to $v_{p,\phi} \in C^1(\bar{\Omega})$ is continuous.*

Proof. Let $\tilde{\phi} \in \Phi_{1,\alpha}(\Omega)$ be fixed, and let us consider a sequence ϕ_n , $n \in \mathbb{N}$, which converges to $\tilde{\phi}$ in $\Phi_{1,\alpha}(\Omega)$. We want to prove that there exists a subsequence ϕ_{n_m} , $m \in \mathbb{N}$, of ϕ_n such that $v_{p,\phi_{n_m}}$ converges to $v_{p,\tilde{\phi}}$ in $C^1(\bar{\Omega})$. First of all we observe that u_{p,ϕ_n} is bounded. Indeed, see [15], there exists a constant $C = C(N, p, F)$ such that

$$\|u_{p,\phi_n}\|_{\infty} \leq C \lambda_{p,\phi_n} \|u_{p,\phi_n}\|_1. \quad (12)$$

Let W be bounded open set (for example, a ball or a Wulff shape) with closure strictly contained in $\tilde{\phi}(\Omega)$. As in [21, Prop. 2.1], by the strong convergence of ϕ_n to $\tilde{\phi}$ one can see that W is contained in $\phi_n(\Omega)$ for all n sufficiently large. Recalling that the eigenvalues are monotone with respect to domain inclusion, we get

$$\lambda_{p,\phi_n} \leq \lambda_p(W), \quad (13)$$

for all n sufficiently large. By (12), (13), Hölder inequality and (9) we get

$$\|u_{p,\phi_n}\|_{\infty} \leq C_1,$$

with $C_1 > 0$. In particular

$$\sup_{n \in \mathbb{N}} \|v_{p,\phi_n}\|_{\infty} \neq \infty. \quad (14)$$

By rewriting (11) in a simpler form, we see that functions v_{p,ϕ_n} satisfy the equation

$$\operatorname{div} A(\phi_n, x, Dv_{p,\phi_n}) + B(\phi_n, x, v_{p,\phi_n}) = 0$$

in the weak sense, where A, B are defined by

$$\begin{aligned} A(\phi, x, \zeta) &= F^{p-1}(\zeta J) F_{\xi}(\zeta J) J^T |\det D\phi(x)| \\ B(\phi, x, z) &= \lambda_{p,\phi} |\det D\phi(x)| |z|^{p-2} z \end{aligned} \quad (15)$$

for all $x \in \bar{\Omega}$, $\zeta \in \mathbb{R}^N \setminus \{0\}$, $\phi \in \Phi_{1,\alpha}(\Omega)$ and $z \in \mathbb{R}$.

Let $M_0 > 0$ be fixed. We claim that there exists an open neighborhood \mathcal{V} of $\tilde{\phi}$ and $c_1, c_2 \in (0, +\infty)$ such that

$$\sum_{i,j=1}^n \frac{\partial A_i(\phi, x, \zeta)}{\partial \zeta_j} \eta_i \eta_j \geq c_1 |\zeta|^{p-2} |\eta|^2, \quad (16)$$

$$\left| \frac{\partial A_i(\phi, x, \zeta)}{\partial \zeta_j} \right| \leq c_2 |\zeta|^{p-2}, \quad (17)$$

$$|A(\phi, x, \zeta) - A(\phi, y, \zeta)| \leq c_2 (1 + |\zeta|)^{p-1} |x - y|^\alpha \quad (18)$$

$$|B(\phi, x, z)| \leq c_2 (1 + |\zeta|)^p, \quad (19)$$

for all $\phi \in \mathcal{V}$, $x, y \in \overline{\Omega}$, $\zeta \in \mathbb{R}^n \setminus \{0\}$ and $z \in (-M_0, M_0)$.

Indeed, condition (16) follows by condition (6) and a continuity argument. Condition (17) follows observing that F_ξ is zero homogenous and using a continuity argument. Condition (18) and (19) follow using (3) and reasoning as Lemma 3.1 of [21].

Clearly, $\phi_n \in \mathcal{V}$ for all $n \in \mathbb{N}$ sufficiently large, say for $n \geq \bar{n}$, hence conditions (16)-(19) are satisfied with ϕ replaced by ϕ_n . Thus, by (14) we are in position to apply the boundary regularity result in [24], which assures that there exists a constant $K > 0$, such that

$$v_{p, \phi_n} \in C^{1, \beta}(\overline{\Omega}) \quad \text{and} \quad \|v_{p, \phi_n}\|_{1, \beta} \leq K,$$

for all $n \geq \bar{n}$. The thesis follows recalling that $C^{1, \beta}(\overline{\Omega})$ is compactly imbedded in $C^1(\overline{\Omega})$ and using standard regularity results. \square

4 Differentiability result and Hadamard formula

Theorem 4.1. *Let $p > 1$ and Ω be a bounded domain in \mathbb{R}^N of class $C^{1, \alpha}$, with $\alpha \in (0, 1]$. Then the function which maps $\phi \in \Phi_{1, \alpha}(\Omega)$ to $\lambda_{p, \phi}$ is of class C^1 . Moreover,*

$$\begin{aligned} d\lambda_{p, \phi}(\psi) &= \int_{\phi(\Omega)} (F^p(Du_{p, \phi}) - \lambda_{p, \phi} |u_{p, \phi}|^p) \operatorname{div}(\psi \circ \phi^{-1}) \, dy \\ &\quad - p \int_{\phi(\Omega)} F^{p-1}(Du_{p, \phi}) F_\xi(Du_{p, \phi}) \cdot (D(\psi \circ \phi^{-1}))^T \cdot (Du_{p, \phi})^T \, dy, \end{aligned} \quad (20)$$

for all $\phi \in \Phi_{1, \alpha}(\Omega)$ and $\psi \in C^{1, \alpha}(\overline{\Omega}, \mathbb{R}^N)$.

Proof. We consider the following real-valued functional

$$R_\phi(v) = \frac{\int_{\Omega} F^p(Dv \cdot (D\phi)^{-1}) |\det D\phi| \, dx}{\int_{\Omega} |v|^p |\det D\phi| \, dx},$$

defined for all $(v, \phi) \in C_0^1(\overline{\Omega}) \setminus \{0\} \times \Phi_{1, \alpha}(\Omega)$. Since the map

$$\phi \in \Phi_{1, \alpha}(\Omega) \rightarrow (D\phi)^{-1} \in C(\overline{\Omega}, \mathbb{R}^{N \times N})$$

is real-analytic, by exploiting the continuity of the functions $\phi \mapsto \partial_\phi R(v)$ and $\phi \mapsto v_{p, \phi}$ (see Theorem 3.1), we can use the same argument in [21] to prove that the map $\phi \mapsto \lambda_{p, \phi}$ is continuously Frechét differentiable and that its Frechét differential at any point $\phi \in \Phi_{1, \alpha}(\Omega)$ is given by the formula

$$d\lambda_{p, \tilde{\phi}}(\psi) = \partial_\phi R_{\tilde{\phi}}(v_{p, \tilde{\phi}})(\psi)$$

which is valid for all $\psi \in C^{1, \alpha}(\overline{\Omega}, \mathbb{R}^N)$.

Now we fix $\psi \in C^{1,\alpha}(\bar{\Omega}, \mathbb{R}^N)$ and we compute $\partial_\phi R_{\tilde{\phi}}(v_{p,\tilde{\phi}})(\psi)$. To this end let $\delta > 0$ be such that $\phi_t := \tilde{\phi} + t\psi \in \Phi_{1,\alpha}(\Omega)$, for all $t \in (-\delta, \delta)$. Putting $R_{\phi_t}(v_{p,\tilde{\phi}}) = \frac{N}{D}$ where

$$N = \int_{\Omega} F^p(Dv_{p,\tilde{\phi}} \cdot (D\phi_t)^{-1}) |\det D\phi_t| dx,$$

and

$$D = \int_{\Omega} |v_{p,\tilde{\phi}}|^p |\det D\phi_t| dx,$$

by standard calculus we obtain

$$\left. \frac{dD}{dt} \right|_{t=0} = \int_{\tilde{\phi}(\Omega)} |u_{p,\tilde{\phi}}|^p \operatorname{div} \chi dy,$$

and

$$\begin{aligned} \left. \frac{dN}{dt} \right|_{t=0} &= -p \int_{\tilde{\phi}(\Omega)} F^{p-1}(Du_{p,\tilde{\phi}}) F_\xi(Du_{p,\tilde{\phi}}) \cdot (D\chi)^T \cdot (Du_{p,\tilde{\phi}})^T dy \\ &\quad + \int_{\tilde{\phi}(\Omega)} F^p(Du_{p,\tilde{\phi}}) \operatorname{div} \chi dy, \end{aligned}$$

where $\chi = \psi \circ (\tilde{\phi})^{-1}$. Putting all together and recalling (9) and (10), formula (20) follows. \square

For any vector $n \in \mathbb{R}^N \setminus \{0\}$, we set

$$n_F = F_\xi(n), \quad \frac{\partial u}{\partial n_F} = Du \cdot n_F.$$

A consequence of the previous result is the following generalization of the Hadamard formula to the anisotropic setting.

Theorem 4.2. *Let the same assumptions of Theorem 4.1 hold. Let $\phi \in \Phi_{1,\alpha}(\Omega)$ be such that $\phi(\Omega)$ is of class $C^{2,\delta}$, for some $\delta \in (0, 1]$. Then*

$$\begin{aligned} d\lambda_{p,\phi}(\psi) &= (1-p) \int_{\partial\phi(\Omega)} |F(Du_{p,\phi})|^p (\psi \circ \phi^{-1}) \cdot \nu d\sigma \\ &= (1-p) \int_{\partial\phi(\Omega)} \left| \frac{\partial u_{p,\phi}}{\partial \nu_F} \right|^p (\psi \circ \phi^{-1}) \cdot \nu d\sigma, \end{aligned} \tag{21}$$

for all $\psi \in C^{1,\alpha}(\bar{\Omega}, \mathbb{R}^N)$, where ν is the unit outer normal to $\partial\phi(\Omega)$.

Proof. To shorten our notation, we set $u = u_{p,\phi}$. Recall that by classical boundary regularity results, there exists $\beta \in (0, 1)$ such that $u \in C^{1,\beta}(\phi(\bar{\Omega}))$, hence the right hand side of (21) is well defined, and $u = 0$ on $\partial\phi(\Omega)$. The possible lack of C^2 regularity of u leads us to consider a family of approximating problems, as often done in the literature (we refer to [16, 20] for the case of the p -Laplacian, to [30] for the Finsler Laplacian and to [8] for the

Finsler p -Laplacian). As done in [8, Lemma 3.2] and [30, Theorem 2.6], consider a family of sufficiently regular real-valued convex functions $V^{(\varepsilon)}$, with $\varepsilon > 0$, defined in \mathbb{R}^N such that $V^{(\varepsilon)}$ converges to F^p/p together with the first order derivatives, as $\varepsilon \rightarrow 0$. To be more specific, following [8, Lemma 3.2], we take $V^{(\varepsilon)} = (F^2 + \varepsilon)^{\frac{p}{2}}/p$, with $\varepsilon > 0$. Then we consider the boundary value problem

$$\begin{cases} -\operatorname{div} \left(V_{\xi}^{(\varepsilon)} (Du_{\varepsilon}) \right) = \lambda_{p,\phi} |u|^{p-2} u, & \text{in } \phi(\Omega), \\ u_{\varepsilon} = 0, & \text{on } \partial\phi(\Omega), \end{cases} \quad (22)$$

where we recall that $V_{\xi}^{(\varepsilon)}$ denotes the gradient of $V^{(\varepsilon)}$. By the same proof in [8], classical regularity results (see e.g. [19]) assure that there exists a solution u_{ε} to problem (22) such that $u_{\varepsilon} \in W^{2,2}(\phi(\Omega)) \cap C^{1,\gamma}(\phi(\bar{\Omega}))$ for some $\gamma \in (0, 1)$, and $|u_{\varepsilon}| \leq M_0$, with M_0 a constant independent of ε . Moreover, the $C^{1,\gamma}$ estimates in [24] imply that $u_{\varepsilon} \rightarrow u$ in $C^1(\phi(\bar{\Omega}))$ up to subsequences.

Now we approximate the first term of the right-hand side of (20). In order to shorten our notation we set $\chi = \psi \circ (\tilde{\phi})^{-1}$. Using the Divergence Theorem and recalling that $u = 0$ on $\partial\phi(\Omega)$, we get

$$\begin{aligned} & \int_{\phi(\Omega)} (pV^{(\varepsilon)}(Du_{\varepsilon}) - \lambda_{p,\phi}|u|^p) \operatorname{div} \chi dy \\ &= \int_{\partial\phi(\Omega)} pV^{(\varepsilon)}(Du_{\varepsilon}) \chi_j \nu_j d\sigma - \int_{\phi(\Omega)} \frac{\partial}{\partial y_j} (pV^{(\varepsilon)}(Du_{\varepsilon}) - \lambda_{p,\phi}|u|^p) \chi_j dy. \end{aligned}$$

Note that here and in the sequel the summation symbol is omitted.

Now we take into account the second term of the right hand side of (20). Reasoning as before, by the divergence Theorem, we get

$$\begin{aligned} -p \int_{\phi(\Omega)} V_{\xi}^{(\varepsilon)}(Du_{\varepsilon}) \cdot (D\chi)^T \cdot (Du_{\varepsilon})^T dy &= -p \int_{\partial\phi(\Omega)} V_{\xi_i}^{(\varepsilon)}(Du_{\varepsilon}) \frac{\partial u_{\varepsilon}}{\partial y_j} \chi_j \nu_i d\sigma \\ &+ p \int_{\phi(\Omega)} \frac{\partial}{\partial y_i} \left(V_{\xi_i}^{(\varepsilon)}(Du_{\varepsilon}) \frac{\partial u_{\varepsilon}}{\partial y_j} \right) \chi_j dy. \end{aligned} \quad (23)$$

We first consider the second integral of right hand side of (23). Recalling that u_{ε} are solutions to problem (22), we obtain

$$\begin{aligned} & \int_{\phi(\Omega)} \frac{\partial}{\partial y_i} \left(V_{\xi_i}^{(\varepsilon)}(Du_{\varepsilon}) \frac{\partial u_{\varepsilon}}{\partial y_j} \right) \chi_j dy \\ &= \int_{\phi(\Omega)} \operatorname{div} \left(V_{\xi}^{(\varepsilon)}(Du_{\varepsilon}) \right) \frac{\partial u_{\varepsilon}}{\partial y_j} \chi_j dy + \int_{\phi(\Omega)} V_{\xi_i}^{(\varepsilon)}(Du_{\varepsilon}) \frac{\partial^2 u_{\varepsilon}}{\partial y_j \partial y_i} \chi_j dy \\ &= -\lambda_{p,\phi} \int_{\phi(\Omega)} |u|^{p-2} u \frac{\partial u_{\varepsilon}}{\partial y_j} \chi_j dy + \int_{\phi(\Omega)} \frac{\partial V^{(\varepsilon)}(Du_{\varepsilon})}{\partial y_j} \chi_j dy. \end{aligned}$$

Substituting in (23), we get

$$\begin{aligned} -p \int_{\phi(\Omega)} V_{\xi}^{(\varepsilon)}(Du_{\varepsilon}) \cdot (D\chi)^T \cdot (Du_{\varepsilon})^T dy &= -p \int_{\partial\phi(\Omega)} V_{\xi}^{(\varepsilon)}(Du_{\varepsilon}) \frac{\partial u_{\varepsilon}}{\partial y_j} \chi_j \nu_i d\sigma \\ &- p \lambda_{p,\phi} \int_{\phi(\Omega)} |u|^{p-2} u \frac{\partial u_{\varepsilon}}{\partial y_j} \chi_j dy + p \int_{\phi(\Omega)} \frac{\partial V^{(\varepsilon)}(Du_{\varepsilon})}{\partial y_j} \chi_j dy. \end{aligned}$$

It follows that

$$\begin{aligned}
& \int_{\phi(\Omega)} (pV^{(\varepsilon)}(Du_\varepsilon) - \lambda_{p,\phi}|u|^p) \operatorname{div}\chi dy \\
& - p \int_{\phi(\Omega)} V_\xi^{(\varepsilon)}(Du_\varepsilon) \cdot (D\chi)^T \cdot (Du_\varepsilon)^T dy \\
& = \int_{\partial\phi(\Omega)} pV^{(\varepsilon)}(Du_\varepsilon)\chi_j\nu_j d\sigma + \lambda_{p,\phi} \int_{\phi(\Omega)} \frac{\partial}{\partial y_j} (|u|^p) \chi_j dy \\
& - p \int_{\partial\phi(\Omega)} V_{\xi_i}^{(\varepsilon)}(Du_\varepsilon) \frac{\partial u_\varepsilon}{\partial y_j} \chi_j \nu_i d\sigma - p\lambda_{p,\phi} \int_{\phi(\Omega)} |u|^{p-2} u \frac{\partial u_\varepsilon}{\partial y_j} \chi_j dy. \tag{24}
\end{aligned}$$

Taking $\varepsilon \rightarrow 0$ in (24), we have

$$\begin{aligned}
& \int_{\phi(\Omega)} (F^p(Du) - \lambda_{p,\phi}|u|^p) \operatorname{div}\chi dy \\
& - p \int_{\phi(\Omega)} F^{p-1}(Du) F_\xi(Du) \cdot (D\chi)^T \cdot (Du)^T dy \\
& = \int_{\partial\phi(\Omega)} F^p(Du) \chi_j \nu_j d\sigma + \lambda_{p,\phi} \int_{\phi(\Omega)} \frac{\partial}{\partial y_j} (|u|^p) \chi_j dy \\
& - p \int_{\partial\phi(\Omega)} F^{p-1}(Du) F_{\xi_i}(Du) \frac{\partial u}{\partial y_j} \chi_j \nu_i d\sigma - p\lambda_{p,\phi} \int_{\phi(\Omega)} |u|^{p-2} u \frac{\partial u}{\partial y_j} \chi_j dy \\
& = \int_{\partial\phi(\Omega)} F^p(Du) \chi_j \nu_j d\sigma - p \int_{\partial\phi(\Omega)} F^{p-1}(Du) F_{\xi_i}(Du) \frac{\partial u}{\partial y_j} \chi_j \nu_i d\sigma. \tag{25}
\end{aligned}$$

Now since $u = 0$ on $\partial\phi(\Omega)$ and $u \in C^1(\phi(\overline{\Omega}))$, it follows that $Du|_{\partial\phi(\Omega)} = \frac{\partial u}{\partial \nu} \nu$, so by the homogeneity of F it follows that

$$-p \int_{\partial\phi(\Omega)} F^{p-1}(Du) F_{\xi_i}(Du) \frac{\partial u}{\partial y_j} \chi_j \nu_i d\sigma = -p \int_{\partial\phi(\Omega)} F^p(Du) \chi \cdot \nu d\sigma. \tag{26}$$

Finally, by Theorem 4.1, using equalities (25) and (26), we obtain

$$d\lambda_{p,\phi}(\psi) = (1-p) \int_{\partial\phi(\Omega)} F^p(Du) \chi \cdot \nu d\sigma \tag{27}$$

which is the first equality in (21). Recalling that $Du = \frac{\partial u}{\partial \nu} \nu$, it follows that

$$F^p(Du) = F^p\left(\frac{\partial u}{\partial \nu} \nu\right) = \left|\frac{\partial u}{\partial \nu} F(\nu)\right|^p = \left|\frac{\partial u}{\partial \nu} F_\xi(\nu) \cdot \nu\right|^p = \left|\frac{\partial u}{\partial \nu_F}\right|^p,$$

which combined with (27) provides the validity of the second equality in (21). \square

5 Corollaries

In this section we briefly discuss two immediate corollaries of the previous section.

5.1 Overdetermined problem

As in the Euclidean case, formula (21) naturally leads to the formulation of an overdetermined problem. Indeed, let us assume that the assumptions of Theorem 4.2 hold and let us consider the functionals J_1 and J_2 defined on $\Phi_{1,\alpha}(\Omega)$ by $J_1(\phi) = \lambda_{p,\phi}$ and $J_2(\phi) = |\phi(\Omega)|$ for all $\phi \in \Phi_{1,\alpha}(\Omega)$. If ϕ is a critical point for J_1 under the volume constraint $|\phi(\Omega)| = c$ for a fixed constant $c > 0$, then $\text{Ker}dJ_2(\phi) \subset \text{Ker}dJ_1(\phi)$, hence $dJ_1(\phi) = \kappa dJ_2(\phi)$ for some $\kappa \in \mathbb{R}$ (the Lagrange multiplier). This means that

$$(1-p) \int_{\partial\phi(\Omega)} \left| \frac{\partial u_{p,\phi}}{\partial \nu_F} \right|^p \psi \circ (\tilde{\phi})^{-1} \cdot \nu d\sigma = \kappa \int_{\partial\phi(\Omega)} \psi \circ (\tilde{\phi})^{-1} \cdot \nu d\sigma,$$

for all $\psi \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^N)$.

Thus, $\left| \frac{\partial u_{p,\phi}}{\partial \nu_F} \right|^p = \kappa/(1-p)$ on $\partial\phi(\Omega)$, and this implies that $\frac{\partial u_{p,\phi}}{\partial \nu_F}$ is also constant because the function $u_{p,\phi}$ doesn't change sign.

In conclusion, ϕ is a critical point for J_1 under the volume constraint above if and only if $\frac{\partial u_{p,\phi}}{\partial \nu_F}$ is constant on $\partial\phi(\Omega)$. In other words, $u_{p,\phi}$ solves the following overdetermined problem

$$\begin{cases} -\text{div}(F^{p-1}(Du_{p,\phi})F_\xi(Du_{p,\phi})) = \lambda_{p,\phi}|u_{p,\phi}|^{p-2}u_{p,\phi}, & \text{in } \phi(\Omega), \\ u_{p,\phi} = 0, & \text{on } \partial\phi(\Omega), \\ \frac{\partial}{\partial \nu_F}u_{p,\phi} = \text{constant}, & \text{on } \partial\phi(\Omega). \end{cases} \quad (28)$$

System (28) is satisfied when $\phi(\Omega)$ is homotetic to a Wulff shape. This can be deduced either by the symmetry result in [14, Theorem 2.4] or by the Faber-Krahn inequality which assures that if \mathcal{W} is the Wulff shape centered in the origin such that $|\mathcal{W}| = |\phi(\Omega)|$, then $\lambda_p(\mathcal{W}) \leq \lambda_p(\phi(\Omega))$, see e.g. [6]. On the other hand, it is proved in [29] that if system (28) is satisfied then Ω is homotetic to a Wulff shape. We note that the same conclusion holds for the overdetermined system

$$\begin{cases} -\text{div}(F^{p-1}(Du)F_\xi(Du)) = 1, & \text{in } \phi(\Omega), \\ u = 0, & \text{on } \partial\phi(\Omega), \\ \frac{\partial}{\partial \nu_F}u = \text{constant}, & \text{on } \partial\phi(\Omega), \end{cases} \quad (29)$$

as it has been proved in [12].

5.2 Rellich-Pohozaev identity

Given a bounded domain Ω be in \mathbb{R}^N of class $C^{1,\alpha}$, with $\alpha \in (0, 1]$, we consider a family of dilations $(1+t)\Omega$ of Ω which can be viewed as a family of diffeomorphisms $\phi_t = I + tI$, $t \in \mathbb{R}$.

By differentiating with respect to t and applying formula (21) with $\phi = \psi = I$, we obtain

$$\frac{d}{dt}\lambda_{p,\phi_t} \Big|_{t=0} = (1-p) \int_{\partial\Omega} \left| \frac{\partial u_{p,\phi}}{\partial \nu_F} \right|^p x \cdot \nu d\sigma. \quad (30)$$

By Lemma 2.2 we get that $\lambda_{p,\phi_t} = \lambda_p((1+t)\Omega) = (1+t)^{-p}\lambda_p(\Omega)$. If we derive this last expression with respect to t , we obtain

$$\frac{d}{dt}\lambda_{p,\phi_t} \Big|_{t=0} = \frac{d}{dt}((1+t)^{-p}\lambda_p(\Omega)) \Big|_{t=0} = -p\lambda_p(\Omega). \quad (31)$$

Combining (30) and (31), we obtain the following Rellich-Pohozaev identity

$$\lambda_p(\Omega) = \frac{p-1}{p} \int_{\partial\Omega} \left| \frac{\partial u_{p,\phi}}{\partial \nu_F} \right|^p x \cdot \nu d\sigma. \quad (32)$$

Formula (32) holds not only for the first eigenvalue but for any eigenvalue. In fact, the following theorem holds.

Theorem 5.1. *Let $p > 1$ and Ω be a bounded domain in \mathbb{R}^N of class $C^{2,\alpha}$, with $\alpha \in (0, 1]$. Let λ be an eigenvalue of equation (2) and u be a corresponding eigenfunction normalized by $\|u\|_{L^p(\Omega)} = 1$. Then*

$$\lambda = \frac{p-1}{p} \int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu_F} \right|^p x \cdot \nu d\sigma.$$

Since the Hadamard formula for the Finsler Laplacian is currently proved only for the first eigenvalue, the proof of the previous theorem in the case of an arbitrary eigenvalue cannot be performed as above. However, one can adapt the same argument used in [20] for the p -Laplacian and based on the original approach of F. Rellich [27].

Proof. As done in the proof of Theorem 4.2 one may follow the approximation argument of [8, Lemma 3.2] or [30, Theorem 2.6] and consider the approximating family of boundary value problems (22) defined in Ω . Reasoning as [8], classical regularity results (see e.g. [19]) assure that there exists a solution u_ε to problem (22) such that $u_\varepsilon \in W^{2,2}(\Omega) \cap C^{1,\gamma}(\bar{\Omega})$ for some $\gamma \in (0, 1)$, and $|u_\varepsilon| \leq M_0$, with M_0 a constant independent of ε . Moreover, the $C^{1,\gamma}$ estimates in [24] imply that $u_\varepsilon \rightarrow u$ in $C^1(\bar{\Omega})$ up to subsequences. Reasoning as in [20], one can multiply both sides of the first equality in (22) by $\sum_{k=1}^N x_k \frac{\partial u_\varepsilon}{\partial x_k}$ and integrating, it follows that

$$- \int_{\Omega} x_k \frac{\partial}{\partial x_h} \left(V_{\xi_h}^{(\varepsilon)}(Du_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_k} \right) + \int_{\Omega} x_k \frac{\partial}{\partial x_k} (V^{(\varepsilon)}(Du_\varepsilon)) = \lambda \int_{\Omega} |u|^{p-2} u x_k \frac{\partial u_\varepsilon}{\partial x_k} dx. \quad (33)$$

Here and in the sequel the summation symbol is omitted. By applying the Divergence Theorem to both sides of the previous equality and letting $\varepsilon \rightarrow 0$, we get

$$\begin{aligned} & - \int_{\partial\Omega} x_k F^{p-1}(Du) F_{\xi_h}(Du) \frac{\partial u}{\partial x_k} \nu_h d\sigma + \int_{\Omega} F^{p-1}(Du) F_{\xi_k}(Du) \frac{\partial u}{\partial x_k} \\ & + \frac{1}{p} \int_{\partial\Omega} x_k F^p(Du) \nu_k d\sigma - \frac{N}{p} \int_{\Omega} F^p(Du) dx = -\frac{N\lambda}{p}, \end{aligned} \quad (34)$$

where the last equality follows since $\|u\|_{L^p(\Omega)} = 1$. Finally, the statement follows using the properties of F and recalling that $F^p(Du) = \left| \frac{\partial u}{\partial \nu_F} \right|^p$. □

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