EULER INTEGRAL AND PERIHELION LIBRATIONS

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Abstract. We discuss dynamical aspects of an analysis of the two–centre problem started in [15]. The perturbative nature of our approach allows us to foresee applications to the three–body problem.

1. **Introduction.** The two–centres (or *Euler–*) problem is the 3–degrees of freedom (2 in the plane) system of one particle interacting with two fixed masses via Newton Law. If $\pm\mathbf{v}_0 \in \mathbf{R}^3$ are the position coordinates of the centres, $\mathrm{m}_\pm$ their masses; $\mathbf{v}$, with $\mathbf{v} \neq \pm\mathbf{v}_0$, the position coordinate of the moving particle; $\mathbf{u} = \dot{\mathbf{v}}$ its velocity, and 1 its mass, the Hamiltonian of the system (*Euler Hamiltonian*) is

$$\mathrm{J} = \frac{\|\mathbf{u}\|^2}{2} - \frac{\mathrm{m}_+}{\|\mathbf{v} + \mathbf{v}_0\|} - \frac{\mathrm{m}_-}{\|\mathbf{v} - \mathbf{v}_0\|} \ , \tag{1}$$

with $\| \cdot \|$ being the Euclidean distance in $\mathbf{R}^3$. Euler showed [11] that J exhibits 2 independent first integrals, in involution. One of these first integrals is the projection

$$\Theta = \mathbf{M} \cdot \frac{\mathbf{v}_0}{\|\mathbf{v}_0\|} \tag{2}$$

of the angular momentum $\mathbf{M} = \mathbf{v} \times \mathbf{u}$ of the particle along the direction $\mathbf{v}_0$. It is not specifically due to the Newtonian potential, but, rather, to its invariance by rotations around the axis $\mathbf{v}_0$. For example, it persists if the Newtonian potential is replaced with a α–homogeneous one. The existence of the following constant of motion, which we shall refer to as *Euler integral*:

$$\mathrm{E} = \|\mathbf{v} \times \mathbf{u}\|^2 + (\mathbf{v}_0 \cdot \mathbf{u})^2 + 2\mathbf{v} \cdot \mathbf{v}_0 \left(\frac{\mathrm{m}_+}{\|\mathbf{v} + \mathbf{v}_0\|} - \frac{\mathrm{m}_-}{\|\mathbf{v} - \mathbf{v}_0\|} \right) \tag{3}$$

is pretty specific of J. As observed in [2], in the limit of merging centres, i.e., $\mathbf{v}_0 = \mathbf{0}$, J reduces to the Kepler Hamiltonian, and E to the squared length of the angular momentum of the moving particle. Note however that J reduces to the Kepler Hamiltonian also when m_- (or m_+) vanishes. The limiting value of E for this case is precisely what is studied in the present paper.

The formula in (3) is not easy[1] to be found in the literature. There is a classical argument of separation of variables (which we shall recall in Section 1.1) which, besides showing the integrability of (1), also can be used to derive (3). Such argument, however, does not provide a complete outline of the problem, since, as a matter of fact, leaves important questions unanswered, like, as an example, the existence of action–angle coordinates, of periodic orbits, the complete picture of the bifurcation diagram. Because of this, the problem has received, in the last decades, a renewed interest and noticeable papers appeared [16, 3, 8, 4]. A common ingredient of the mentioned literature is a separation–like change of coordinates, possibly combined with a "regularising" change of time, which allows, following Euler's ideas, to decouple the Hamiltonian.

Our approach to the problem is, in a sense, affected by methods of perturbation theory, and goes as follows. We do not use decoupling coordinates and, for conveniency, begin with a situation where the attracting centres are in a "asymmetric" position. Namely, in place of (1), we write

$$\mathrm{J} = \frac{\|\mathbf{y}\|^2}{2\mathrm{m}} - \frac{\mathrm{m}\mathcal{M}}{\|\mathbf{x}\|} - \frac{\mathrm{m}\mathcal{M}'}{\|\mathbf{x}' - \mathbf{x}\|} \tag{4}$$

Here, m is the mass of the moving particle, $(\mathbf{y}, \mathbf{x})$, with $\mathbf{y} = \mathrm{m}\dot{\mathbf{x}}$ are its impulse–position coordinates, and $\mathcal{M}$, $\mathcal{M}'$ are the masses of the two attracting centres, posed at $\mathbf{0}$, $\mathbf{x}'$, respectively.

In this case, as we shall show below, apart for a negligible additive term, its Euler integral takes the expression

$$\mathrm{E} = \|\mathbf{M}\|^2 - \mathbf{x}' \cdot \mathbf{L} + \mathrm{m}^2 \mathcal{M}' \frac{(\mathbf{x}' - \mathbf{x}) \cdot \mathbf{x}'}{\|\mathbf{x}' - \mathbf{x}\|} \tag{5}$$

where

$$\mathbf{M} := \mathbf{x} \times \mathbf{y} , \qquad \mathbf{L} := \mathbf{y} \times \mathbf{M} - \mathrm{m}^2 \mathcal{M} \frac{\mathbf{x}}{\|\mathbf{x}\|} = \mathrm{m}^2 \mathcal{M} \, \mathrm{e}\mathbf{P} \tag{6}$$

are the *angular momentum* and the *eccentricity vector* associated to the Kepler Hamiltonian

$$\mathrm{J}_0 := \frac{\|\mathbf{y}\|^2}{2\mathrm{m}} - \frac{\mathrm{m}\mathcal{M}}{\|\mathbf{x}\|} \tag{7}$$

with e and $\mathbf{P}$ being the eccentricity and the perihelion direction ($\|\mathbf{P}\| = 1$). With these notations, J reduces to a Kepler Hamiltonian either for $\mathbf{x}' = \mathbf{0}$, in which case, as in the symmetric case above, E reduces to $\|\mathbf{M}\|^2$; or for $\mathcal{M}' = 0$. In the latter case, J and E become, respectively, J_0 in (7) and

$$\mathrm{E}_0 \quad = \quad \|\mathbf{M}\|^2 - \mathbf{x}' \cdot \mathbf{L} \tag{8}$$

which, as expected, is a combination of first integrals of J_0.

The second (and main) difference with the traditional approach to the problem is that, as mentioned, we *do not use elliptic coordinates*. More closely to a perturbative point of view, we use a special *partial Kepler map* which reduces J to a two–degrees of freedom Hamiltonian. We call *partial Kepler map* any canonical map

$$\mathcal{C} : \qquad (\Lambda, \ell, u, v) \in \mathcal{A} \times \mathbf{T} \times V \to (\underline{\mathbf{y}}, \underline{\mathbf{x}}) = (\mathbf{y}', \mathbf{y}, \mathbf{x}', \mathbf{x}) \in (\mathbf{R}^3)^4 \tag{9}$$

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[1]See however [8] for a formula related to (3).

where $\mathcal{A}$ is a domain[2] in $\mathbf{R}$, V is a domain in $\mathbf{R}^{10}$, $\mathbf{T} := \mathbf{R}/(2\pi\mathbf{Z})$ is the standard torus, $(u, v) = \big((u_1, u_2, u_3, u_4, u_5), (v_1, v_2, v_3, v_4, v_5)\big)$, which "preserves the standard two–form":

$$d\mathbf{y}' \wedge d\mathbf{x}' + d\mathbf{y} \wedge d\mathbf{x} = d\Lambda \wedge d\ell + du \wedge dv$$

and "integrates the Keplerian motions of $(\mathbf{y}, \mathbf{x})$":

$$\left(\frac{\|\mathbf{y}\|^2}{2\mathrm{m}} - \frac{\mathrm{m}\mathcal{M}}{\|\mathbf{x}\|} \right) \circ \mathcal{C} = -\frac{\mathrm{m}^3 \mathcal{M}^2}{2\Lambda^2} \ , \tag{10}$$

where m, $\mathcal{M}$ are fixed "mass parameters". Of course, we have assumed that the image of $\mathcal{C}$ in (9) is a domain where the left hand side of (10) takes negative values. We consider the Lagrange average of the Newtonian potential in (4) written in terms of $\mathcal{C}$, namely, the function

$$\mathrm{U}(\Lambda, u, v) := -\frac{\mathrm{m}\mathcal{M}'}{2\pi} \int_{\mathbf{T}} \frac{d\ell}{\|\mathbf{x}'(\Lambda, \ell, u, v) - \mathbf{x}(\Lambda, \ell, u, v)\|} \tag{11}$$

We call such function *partially averaged Newtonian potential*. This function has been investigated in [15]. We recall the main results of that analysis.

(i) As a six degrees of freedom Hamiltonian, U is integrable by quadratures for possessing, besides itself, five independent and commuting first integrals which Poisson–commute with it. These are:

$\mathrm{I}_1 :=$ the semi–major axis action $\Lambda := \mathrm{m}\sqrt{\mathcal{M}a}$;

$\mathrm{I}_2 :=$ the Euclidean length $\mathrm{r} := \|\mathbf{x}'\|$;

$\mathrm{I}_3 :=$ the Euclidean length of the total angular momentum $\mathbf{C} := \mathbf{x}' \times \mathbf{y}' + \mathbf{x} \times \mathbf{y}$, with "$\times$" denoting skew–product;

$\mathrm{I}_4 :=$ its third component $\mathrm{Z} := \mathbf{C} \cdot \mathbf{k}$, where $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ is a prefixed orthonormal frame;

$\mathrm{I}_5 :=$ the projection of $\mathbf{M} = \mathbf{x} \times \mathbf{y}$ along the direction $\mathbf{x}'$, defined as in (2), with $\mathbf{v}_0$ replaced by $\mathbf{x}'$.

(ii) Besides with $\mathrm{I}_1, \cdots, \mathrm{I}_5$, U also Poisson–commutes with the function E_0 in (8), which turns to be independent of, and commuting with, $\mathrm{I}_1, \cdots, \mathrm{I}_5$.

(iii) There is a special partial Kepler map, which we denote as $\mathcal{K}$, which includes $\mathrm{I}_1, \cdots, \mathrm{I}_5$ among its coordinates. Written in terms of $\mathcal{K}$, U and E_0 depend only on *one* coordinate couple of canonical coordinates, which we denote (G, g). Here, $\mathrm{G} = \|\mathbf{x} \times \mathbf{y}\|$ and g defines the direction of $\mathbf{P}$ in a suitable reference frame.

(iv) The most remarkable property (which [15] has been called *renormalizable integrability*) is that U depends on the coordinates (G, g) only as a function of the function E_0 in (8).

(v) As a consequence of (iv), apart for certain particular initial data that can be described in closed form, the motions of the only coordinate couple of $\mathcal{K}$ that moves are the same, whether under U or E_0, up to an unessential change of time. In other words, the phase portraits of the functions E_0 and U expressed in terms of $\mathcal{K}$ coincide. The utility of this assertion relies on the fact that, while U is defined after a quadrature, the expression of E_0 in terms of $\mathcal{K}$ is very simple.

The purpose of this paper is to discuss the dynamical aspects. More precisely, this paper is organised as follows. In Sections 1.1 and 1.2, we recall the classical integrability argument of the Hamiltonian (1) and derive the formulae in (3), (5) and (8). In Section 2, we define the $\mathcal{K}$–coordinates and provide the expressions

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[2]By "domain" we mean an open and connected set in $\mathbb{K} = \mathbf{R}^m, \mathbf{C}^m$.

of J, U and E in their terms. In particular, the function U, expressed in terms of $\mathcal{K}$, is one–dimensional. In Section 3 we review the concept of the mentioned renormalizable integrability. In Section 4 we discuss the dynamical consequences in the particular case of the planar problem. In Section 5 we outline a possible application of the results of the paper to the three–body problem, deferring the complete analysis to a next paper. For definiteness, and by the author's tastes, we just look at the "full" problem. The author is not aware whether addressing the same question to the "restricted" problem would be simpler.

1.1. **The classical integration of the two–centre problem.** Let J be as in (1). After fixing a reference frame with the third axis in the direction of $\mathbf{v}_0$ and denoting as (v_1, v_2, v_3) the coordinates of $\mathbf{v}$ with respect to such frame, one introduces the so–called "elliptic coordinates"

$$\lambda = \frac{1}{2}\left(\frac{r_+}{r_0} + \frac{r_-}{r_0}\right), \quad \beta = \frac{1}{2}\left(\frac{r_+}{r_0} - \frac{r_-}{r_0}\right), \quad \omega := \arg\left(-v_2, v_1\right) \tag{12}$$

where we have let, for short,

$$r_0 := \|\mathbf{v}_0\|, \quad r_\pm := \|\mathbf{v} \pm \mathbf{v}_0\|.$$

Regarding r_0 as a fixed external parameter and calling p_λ, p_β, p_ω the generalized momenta associated to λ, β and ω, it turns out that the Hamiltonian (1), written in the coordinates $(p_\lambda, p_\beta, \lambda, \beta)$ is independent of ω and has the expression

$$
\begin{aligned}
J(p_\lambda, p_\beta, p_\omega, \lambda, \beta, r_0) = \ & \frac{1}{\lambda^2 - \beta^2}\Big[\frac{p_\lambda^2(\lambda^2 - 1)}{2r_0^2} + \frac{p_\beta^2(1 - \beta^2)}{2r_0^2} \\
& + \frac{p_\omega^2}{2r_0^2}\Big(\frac{1}{1 - \beta^2} + \frac{1}{\lambda^2 - 1}\Big) - \frac{(m_+ + m_-)\lambda}{r_0^2} \\
& + \frac{(m_+ - m_-)\beta}{r_0^2}\Big].
\end{aligned}
\tag{13}
$$

It follows that the solution of "Hamilton–Jacobi" equation

$$J(W_\lambda, W_\beta, p_\omega, \lambda, \beta, r_0) = h \tag{14}$$

can be searched of the form

$$W(\lambda, \beta, p_\omega, r_0, h) = W^{(1)}(\lambda, p_\omega, r_0, h) + W^{(2)}(\beta, p_\omega, r_0, h)$$

as (14) separates completely as

$$\mathcal{F}_1(W^{(1)}_\lambda, \lambda, p_\omega, r_0, h) + \mathcal{F}_2(W^{(2)}_\beta, \beta, p_\omega, r_0, h) = 0 \tag{15}$$

with

$$\mathcal{F}_1(p_\lambda, \lambda, p_\omega, r_0, h) = p_\lambda^2(\lambda^2 - 1) + \frac{p_\omega^2}{\lambda^2 - 1} - 2(m_+ + m_-)\lambda - 2r_0^2\lambda^2 h$$

$$\mathcal{F}_2(p_\beta, \beta, p_\omega, r_0, h) = p_\beta^2(1 - \beta^2) + \frac{p_\omega^2}{1 - \beta^2} + 2(m_+ - m_-)\beta + 2r_0^2\beta^2 h.$$

The identity (15) implies that there must exist a function E, which we call *Euler integral*, depending on (p_ω, r_0, h) only, such that

$$\mathcal{F}_\lambda(p_\lambda, \lambda, p_\omega, r_0, h) = -\mathcal{F}_\beta(p_\lambda, \lambda, p_\omega, r_0, h) = E(p_\omega, r_0, h) \quad \forall \ (p_\lambda, p_\beta, \lambda, \beta).$$

It is given by

$$E = \frac{1}{2}(\mathcal{F}_\beta - \mathcal{F}_\lambda) = \frac{p_\beta^2}{2}(1 - \beta^2) - \frac{p_\lambda^2}{2}(\lambda^2 - 1) + \frac{p_\omega^2}{2}\Big(\frac{1}{1 - \beta^2} - \frac{1}{\lambda^2 - 1}\Big)$$
$$+ \quad m_+(\lambda + \beta) + m_-(\lambda - \beta) + r_0^2(\lambda^2 + \beta^2)h . \tag{16}$$

We now check that the function E, written in the initial coordinates $(\mathbf{u}, \mathbf{v})$, coincides with (3). To this end, we replace the coordinates $(\mathbf{u}, \mathbf{v})$ with the "canonical spherical coordinates relatively to $\mathbf{v}_0$", which we denote as

$$\mathcal{D}_{\mathbf{v}_0} = (\Theta, M, R, \vartheta, m, r) .$$

Their definition is as follows. Given three vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{b} \in \mathbf{R}^3$, with $\mathbf{n}_1$, $\mathbf{n}_2 \perp$, $\mathbf{b}$, let $\alpha_{\mathbf{b}}(\mathbf{n}_1, \mathbf{n}_2)$ denotes the oriented angle defined by the ordered couple $(\mathbf{n}_1, \mathbf{n}_2)$, relatively to the positive verse established by $\mathbf{b}$. Let

$$\mathbf{M} := \mathbf{v} \times \mathbf{u} , \quad \mathbf{n}_0 := \mathbf{v}_0 \times \mathbf{M} , \quad \mathbf{n} := \mathbf{M} \times \mathbf{v} .$$

Then we define the coordinates $\mathcal{D}_{\mathbf{v}_0}$ via the formulae

$$\begin{cases} \Theta := \frac{\mathbf{M} \cdot \mathbf{v}_0}{\|\mathbf{v}_0\|} \\ M := \|\mathbf{M}\| \\ R := \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|} \end{cases} \qquad \begin{cases} \vartheta := \alpha_{\mathbf{v}_0}(\mathbf{i}, \mathbf{n}_0) \\ m := \alpha_{\mathbf{M}}(\mathbf{n}_0, \mathbf{v}) \\ r := \|\mathbf{v}\| \end{cases} \tag{17}$$

As it is well known, the coordinates $\mathcal{D}_{\mathbf{v}_0}$ are homogeneous–canonical[3].

Since Θ is a first integral to J, this Hamiltonian will depend only on the four coordinates (M, R, m, r) while the action Θ will play the rôle of an "external parameter", together with r_0. Using such coordinates, J becomes

$$J = \frac{R^2}{2} + \frac{M^2}{2r^2} - \frac{m_+}{r_+} - \frac{m_-}{r_-} \tag{18}$$

with

$$r_\pm := \sqrt{r_0^2 \mp 2r_0 r \sqrt{1 - \frac{\Theta^2}{M^2}} \cos m + r^2} .$$

Combining this and (12), one obtains

$$r = r_0\sqrt{\lambda^2 + \beta^2 - 1} \qquad m = \cos^{-1}\Big(-\frac{\lambda\beta}{\sqrt{\lambda^2 + \beta^2 - 1}\sqrt{1 - \frac{\Theta^2}{M^2}}}\Big) . \tag{19}$$

The use of the associated generating function

$$S(M, \Theta, \lambda, \beta) = Rr_0\sqrt{\lambda^2 + \beta^2 - 1}$$
$$+ \int^M \cos^{-1}\Big(-\frac{\lambda\beta}{\sqrt{\lambda^2 + \beta^2 - 1}\sqrt{1 - \frac{\Theta^2}{M'^2}}}\Big) dM'$$

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[3]Namely, they leave the standard 1–form unvaried:

$$\mathbf{u} \cdot d\mathbf{v} := \sum_{i=1}^{3} u_i dv_i = \Theta d\vartheta + M dm + R dr .$$

allows to find the generalized impulses $\bar{p}_\lambda$, $\bar{p}_\beta$ associated to λ, β as

$$
\begin{cases}
\bar{p}_\lambda = \dfrac{r_0 \lambda R}{\sqrt{\lambda^2 + \beta^2 - 1}} - \dfrac{\beta\sqrt{(1-\beta^2)(\lambda^2-1)M^2 - (\lambda^2+\beta^2-1)\Theta^2}}{(\lambda^2+\beta^2-1)(\lambda^2-1)} \\[4mm]
\bar{p}_\beta = \dfrac{r_0 \beta R}{\sqrt{\lambda^2 + \beta^2 - 1}} + \dfrac{\lambda\sqrt{(1-\beta^2)(\lambda^2-1)M^2 - (\lambda^2+\beta^2-1)\Theta^2}}{(\lambda^2+\beta^2-1)(1-\beta^2)}
\end{cases}
$$

We solve for R, M^2:

$$
\begin{cases}
R = \dfrac{\lambda(\lambda^2-1)\bar{p}_\lambda + \beta(1-\beta^2)\bar{p}_\beta}{r_0(\lambda^2-\beta^2)\sqrt{\lambda^2+\beta^2-1}} \\[4mm]
M^2 = \dfrac{(\lambda\bar{p}_\beta - \beta\bar{p}_\lambda)^2(\lambda^2-1)(1-\beta^2)}{(\lambda^2-\beta^2)} + \dfrac{\lambda^2+\beta^2-1}{(1-\beta^2)(\lambda^2-1)}\Theta^2
\end{cases}
$$

Using these formulae and the (19) inside the Hamiltonian (18), we find exactly the expression in (13), with p_λ, p_β, p_ω replaced by $\bar{p}_\lambda$, $\bar{p}_\beta$, Θ. Therefore, the Euler integral will be as in (16), with the same substitutions. After some elementary computation, we find that the E has, in terms of $\mathcal{D}_{\mathbf{v}_0}$, the expression

$$
\begin{aligned}
E = {} & M^2 + r_0^2\left(1 - \frac{\Theta^2}{M^2}\right)\left(-R\cos m + \frac{M}{r}\sin m\right)^2 \\
& -2rr_0\cos m\sqrt{1 - \frac{\Theta^2}{M^2}}\left(\frac{m_+}{r_+} - \frac{m_-}{r_-}\right)
\end{aligned}
$$

with $r_\pm$ as in (12). Turning back to the coordinates $\mathbf{u}$, $\mathbf{v}$ via (17), one sees that E has the expression in (3).

1.2. The "asymmetric" case.

We prove that, if the two attracting centres are posed in "asymmetric" positions with respect to a prefixed reference frame, namely, the Euler Hamiltonian is written in the form (4), then its Euler integral takes the expression in Eqs. (5), (6), (8), apart for a negligible additive term. To this end, we let

$$
\widehat{J}(\widehat{\mathbf{y}}, \widehat{\mathbf{x}}, \widehat{\mathbf{x}}') := \frac{1}{m}J(m\widehat{\mathbf{y}}, \widehat{\mathbf{x}}, \widehat{\mathbf{x}}') = \frac{\|\widehat{\mathbf{y}}\|^2}{2} - \frac{\mathcal{M}}{\|\widehat{\mathbf{x}}\|} - \frac{\mathcal{M}'}{\|\widehat{\mathbf{x}} - \widehat{\mathbf{x}}'\|}
$$

and then we change, canonically,

$$
\widehat{\mathbf{x}}' = 2\mathbf{v}_0 , \quad \widehat{\mathbf{x}} = \mathbf{v}_0 + \mathbf{v} , \quad \widehat{\mathbf{y}}' = \frac{1}{2}(\mathbf{u}_0 - \mathbf{u}) , \quad \widehat{\mathbf{y}} = \mathbf{u}
$$

(where $\widehat{\mathbf{y}}'$, $\widehat{\mathbf{u}}_0$ denote the generalized impulses conjugated to $\widehat{\mathbf{x}}'$, $\widehat{\mathbf{v}}_0$, respectively) we reach the Hamiltonian J in (1), with $m_+ = \mathcal{M}$, $m_- = \mathcal{M}'$. Turning back with the transformations, one sees that the function E in (3) takes the expression

$$
\begin{aligned}
\frac{E}{m} := {} & \frac{1}{m}\left\|\left(\mathbf{x} - \frac{\mathbf{x}'}{2}\right) \times \mathbf{y}\right\|^2 + \frac{1}{4m}(\mathbf{x}' \cdot \mathbf{y})^2 \\
& + m\mathbf{x}' \cdot \left(\mathbf{x} - \frac{\mathbf{x}'}{2}\right)\left(\frac{\mathcal{M}}{\|\mathbf{x}\|} - \frac{\mathcal{M}'}{\|\mathbf{x}' - \mathbf{x}\|}\right) .
\end{aligned}
$$

After multiplying by m, we rewrite the latter integral as

$$
E = E_0 + E_1 + E_2
$$

with

$$E_0 := \|\mathbf{M}\|^2 - \mathbf{x}' \cdot \mathbf{L} , \quad E_1 := \mathrm{m}^2 \mathcal{M}' \frac{(\mathbf{x}' - \mathbf{x}) \cdot \mathbf{x}'}{\|\mathbf{x}' - \mathbf{x}\|}$$

$$E_2 := \mathrm{m} \frac{\|\mathbf{x}'\|^2}{2} \left(\frac{\|\mathbf{y}\|^2}{2\mathrm{m}} - \frac{\mathrm{m}\mathcal{M}}{\|\mathbf{x}\|} - \frac{\mathrm{m}\mathcal{M}'}{\|\mathbf{x}' - \mathbf{x}\|} \right)$$

where $\mathbf{M}$, $\mathbf{L}$ are as in (6). Since E_2 is itself an integral for J, we can neglect it and rename

$$E := E_0 + E_1 \tag{20}$$

the Euler integral to J.

2. $\mathcal{K}$ coordinates. We describe a set of canonical coordinates, which we denote as $\mathcal{K}$, which we shall use for our analysis of the Euler Hamiltonian (4).

We consider, in the region of phase space where J_0 in (7) takes negative values, the ellipse with initial datum $(\mathbf{y}, \mathbf{x})$. Denote as:

- a the *semi–major axis*;
- $\mathbf{P}$, with $\|\mathbf{P}\| = 1$, the direction of perihelion, assuming the ellipse is not a circle;
- ℓ: the mean anomaly, defined, mod 2π, as the area of the elliptic sector spanned by $\mathbf{x}$ from $\mathbf{P}$, normalized to 2π.
- Given $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in \mathbf{R}^3$, with $\mathbf{u}$, $\mathbf{v}$, $\perp \mathbf{w}$, we denote as $\alpha_w(u, v)$ the oriented angle $\mathbf{u}$ to $\mathbf{v}$, with respect to the counterclockwise orientation established by $\mathbf{w}$.

We fix an arbitrary ("inertial") frame

$$F_0 : \quad \mathbf{i}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad \mathbf{j}_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \quad \mathbf{k}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

in $\mathbf{R}^3$, and denote as

$$\mathbf{M} = \mathbf{x} \times \mathbf{y} , \quad \mathbf{M}' = \mathbf{x}' \times \mathbf{y}' , \quad \mathbf{C} = \mathbf{M}' + \mathbf{M} ,$$

where "$\times$" denotes skew–product in $\mathbf{R}^3$. Observe the following relations

$$\mathbf{x}' \cdot \mathbf{C} = \mathbf{x}' \cdot (\mathbf{M} + \mathbf{M}') = \mathbf{x}' \cdot \mathbf{M} , \quad \mathbf{P} \cdot \mathbf{M} = 0 , \quad \|\mathbf{P}\| = 1 . \tag{21}$$

Assume that the "nodes"

$$\mathbf{i}_1 := \mathbf{k}_0 \times \mathbf{C} , \quad \mathbf{i}_2 := \mathbf{C} \times \mathbf{x}' , \quad \mathbf{i}_3 := \mathbf{x}' \times \mathbf{M} \tag{22}$$

do not vanish. We define the coordinates

$$\mathcal{K} = (Z, C, \Theta, G, R, \Lambda, \zeta, g, \vartheta, \mathrm{g}, \mathrm{r}, \ell)$$

via the following formulae:

$$\begin{cases} Z := \mathbf{C} \cdot \mathrm{k} \\[2mm] C := \|\mathbf{C}\| \\[2mm] R := \dfrac{\mathbf{y}' \cdot \mathbf{x}'}{\|\mathbf{x}'\|} \\[3mm] \Lambda = \mathrm{m}\sqrt{\mathrm{M}a} \\[2mm] G := \|\mathbf{M}\| \\[2mm] \Theta := \dfrac{\mathbf{M} \cdot \mathbf{x}'}{\|\mathbf{x}'\|} \end{cases} \qquad \begin{cases} \mathrm{z} := \alpha_{\mathrm{k}}(\mathbf{i}, \mathbf{i}_1) \\[2mm] g := \alpha_{\mathbf{C}}(\mathbf{i}_1, \mathbf{i}_2) \\[2mm] \mathrm{r} := \|\mathbf{x}'\| \\[2mm] \ell := \text{mean anomaly of } \mathbf{x} \text{ on } \mathbb{E} \\[2mm] \mathrm{g} := \alpha_{\mathbf{M}}(\mathbf{i}_3, \mathbf{M} \times \mathbf{P}) \\[2mm] \vartheta := \alpha_{\mathbf{x}'}(\mathbf{i}_2, \mathbf{i}_3) \end{cases} \tag{23}$$

The canonical character of $\mathcal{K}$ follows from [13]. Indeed, in [13], we considered a set of coordinates for the three–body problem[4], that here[5] we denote as $\mathcal{P} = (Z, C, R, \overline{R}, \Theta, \Phi, \mathrm{z}, g, \mathrm{r}, \overline{\mathrm{r}}, \vartheta, \varphi)$, that are related to $\mathcal{K}$ above via the canonical change

$$\mathcal{D}_{e\ell,\mathrm{pl}} : \quad (\Lambda, G, \ell, \mathrm{g}) \to (\overline{R}, \overline{\Phi}, \overline{\mathrm{r}}, \overline{\varphi}) \tag{24}$$

usually referred to as *planar Delaunay map*, defined as

$$\begin{cases} \overline{R} = \dfrac{\mathrm{m}^2 \mathcal{M}}{\Lambda} \dfrac{\mathrm{e}\sin\xi}{1 - \mathrm{e}\cos\xi} \\[3mm] \overline{\Phi} = G \end{cases} \qquad \begin{cases} \overline{\mathrm{r}} = a(1 - \mathrm{e}\cos\xi) \\[2mm] \overline{\varphi} = \nu + \mathrm{g} - \dfrac{\pi}{2} \end{cases} \tag{25}$$

where $\xi = \xi(\Lambda, G, \ell)$, $\nu = \nu(\Lambda, G, \ell)$ are, respectively, the eccentric and the true anomaly, defined below (see (27), (29)). Since the map $\mathcal{D}_{e\ell,\mathrm{pl}}$ in (24) and the coordinates $\mathcal{P}$ of [13] are canonical, so is $\mathcal{K}$. Observe, incidentally, the unusual $\frac{\pi}{2}$–shift in (25), due to the fact that, according to the definitions in (23), g is the longitude of $\mathbf{M} \times \mathbf{P}$ in the plane of $\mathbf{i}_3, \mathbf{j}_3$, relatively to $\mathbf{i}_3$.

Remark 1. (i) We briefly discuss the geometrical meaning of the coordinates $\mathcal{K}$, deferring to [13] or [14, Chapter II and Appendix E] for more details. The definitions (23) are based on a multiple change of reference frames. Chains of reference frames have been firstly used by A. Deprit, in order to extend to an arbitrary number n of particles the classical "reduction of the nodes" discovered by Jacobi in the case $n = 2$ [7, 5, 10]. In the case of the coordinates $\mathcal{K}$, we define three orthogonal (not necessarily orthonormal) frames $\mathrm{F}_i = (\mathbf{i}_i, \mathbf{j}_i, \mathbf{k}_i)$, $i = 1, 2, 3$, where $\mathbf{i}_j$ are as in (22), while

$$\mathbf{k}_1 := \mathbf{C} , \quad \mathbf{k}_2 := \mathbf{x}' , \quad \mathbf{k}_3 := \mathbf{M} , \quad \mathbf{j}_i := \mathbf{k}_i \times \mathbf{i}_i$$

The frame F_1 is also used in [7] and is often referred to as the "invariable frame", since it does not move under the motions of a SO(3)–invariant Hamiltonian. F_2 and F_3 are quite specific of $\mathcal{K}$. The triples (Z, C, z), $(\mathrm{r}, \Theta, \mathrm{g})$, (G, Θ, ϑ) have the meaning of "spherical coordinates" of $\mathbf{C}$, $\mathbf{x}$, $\mathbf{M}$ relatively to F_0, F_1, F_2, respectively. While the triple (Z, C, z) also appears[6] in [7], Θ and ϑ are specific of $\mathcal{K}$. As it can be seen

---

[4]An extension to the case of an arbitrary number of planets has been successively worked out in [14].

[5]$(Z, C, R, \overline{R}, \Theta, \Phi, \mathrm{z}, g, \mathrm{r}, \overline{\mathrm{r}}, \vartheta, \varphi)$ are called $(C_3, G, R_1, R_2, \Theta, \Phi_2, \zeta, \eth, \mathrm{r}_1, \mathrm{r}_2, \vartheta, \varphi_2)$ in [13].

[6]The reader should beware that, even though some groups of coordinates have been already separately used in the literature, similarly to [7], the mix $\mathcal{K}$ is not a Cartesian product of canonical coordinates (namely, the standard two–form is not the sum of forms associated to groups of coordinates, as it happens, for example, in the case of Delaunay coordinates).

from the definitions (23), Θ measures the convex angle between $\mathbf{x}'$ and $\mathbf{C}$ (or $\mathbf{M}$, by the first identity in (21)) and vanishes when the two vectors are orthogonal (in the planar case). The angle ϑ measures the rotation of $\mathbf{M}$ with respect to $\mathbf{x}$. The quadruplet $(\Lambda, \mathrm{G}, \ell, \mathrm{g})$ is the Delaunay set of coordinates associated to $(\mathbf{y}, \mathbf{x})$ in F_3. The coordinate R measures the radial velocity of $\mathbf{x}'$ (as well known, the normal velocity is measured by $\frac{\mathrm{G}}{\mathrm{r}}$).

(ii) Contrary to the Jacobi reduction of the nodes, $\mathcal{K}$ are well defined also in the planar case. In such case, they reduce to $\mathcal{K}_{\mathrm{pl}} = (\mathrm{C}, \mathrm{R}, \Lambda, \mathrm{G}, g, \mathrm{r}, \ell, \mathrm{g})$, and correspond to take the Delaunay coordinates $(\Lambda, \mathrm{G}, \ell, \overline{\mathrm{g}})$ for $(\mathbf{y}, \mathbf{x})$, the "symplectic polar coordinates" $(\mathrm{R}, \Phi', \mathrm{r}, \varphi')$ (also used in [7]) for $(\mathbf{y}', \mathbf{x}')$ and next reduce the angular momentum via the relations $\Phi' = \mathrm{C} - \mathrm{G}$, $\varphi' = g$, $\overline{\mathrm{g}} = \varphi' - \nu(\Lambda, \mathrm{G}, \ell) + \mathrm{g}$.

2.1. Expression of J and E in terms of $\mathcal{K}$.
Using the formulae in the previous section, we provide the expressions of J in (4) and and E in (8) in terms of $\mathcal{K}$:

$$\mathrm{J}(\Lambda, \mathrm{G}, \Theta, \mathrm{r}, \ell, \mathrm{g}) = -\frac{\mathrm{m}^3 \mathcal{M}^2}{2\Lambda^2} - \frac{\mathrm{m}\mathcal{M}'}{\sqrt{\mathrm{r}^2 + 2\mathrm{r}a\sqrt{1 - \frac{\Theta^2}{\mathrm{G}^2}}\mathrm{p} + a^2\varrho^2}}$$

$$=: \ \mathrm{J}_0 + \mathrm{J}_1$$

$$\mathrm{E}(\Lambda, \mathrm{G}, \Theta, \mathrm{r}, \ell, \mathrm{g}) = \mathrm{G}^2 + \mathrm{m}^2\mathcal{M}'\mathrm{r}\sqrt{1 - \frac{\Theta^2}{\mathrm{G}^2}}\sqrt{1 - \frac{\mathrm{G}^2}{\Lambda^2}}\cos\mathrm{g}$$

$$+ \ \mathrm{m}^2\mathcal{M}'\mathrm{r}\frac{\mathrm{r} + a\sqrt{1 - \frac{\Theta^2}{\mathrm{G}^2}}\mathrm{p}}{\sqrt{\mathrm{r}^2 + 2\mathrm{r}a\sqrt{1 - \frac{\Theta^2}{\mathrm{G}^2}}\mathrm{p} + a^2\varrho^2}}$$

$$=: \ \mathrm{E}_0 + \mathrm{E}_1 \tag{26}$$

and, if $\xi = \xi(\Lambda, \mathrm{G}, \ell)$ is the *eccentric anomaly*, defined as the solution of *Kepler equation*

$$\xi - \mathrm{e}(\Lambda, \mathrm{G})\sin\xi = \ell \tag{27}$$

and $a = a(\Lambda)$ the *semi–major axis*; $\mathrm{e} = \mathrm{e}(\Lambda, \mathrm{G})$, the *eccentricity* of the ellipse, $\varrho = \varrho(\Lambda, \mathrm{G}, \ell)$, $\mathrm{p} = \mathrm{p}(\Lambda, \mathrm{G}, \ell, \mathrm{g})$ are defined as

$$a(\Lambda) = \frac{\Lambda^2}{\mathrm{m}^2\mathcal{M}}$$

$$\mathrm{e}(\Lambda, \mathrm{G}) := \sqrt{1 - \frac{\mathrm{G}^2}{\Lambda^2}}$$

$$\varrho(\Lambda, \mathrm{G}, \ell) := 1 - \mathrm{e}(\Lambda, \mathrm{G})\cos\xi(\Lambda, \mathrm{G}, \ell)$$

$$\mathrm{p}(\Lambda, \mathrm{G}, \ell, \mathrm{g}) := (\cos\xi(\Lambda, \mathrm{G}, \ell) - \mathrm{e}(\Lambda, \mathrm{G}))\cos\mathrm{g} - \frac{\mathrm{G}}{\Lambda}\sin\xi(\Lambda, \mathrm{G}, \ell)\sin\mathrm{g} \ . \tag{28}$$

The angle

$$\nu(\Lambda, \mathrm{G}, \ell) := \arg\left(\cos\xi(\Lambda, \mathrm{G}, \ell) - \mathrm{e}(\Lambda, \mathrm{G}), \frac{\mathrm{G}}{\Lambda}\sin\xi(\Lambda, \mathrm{G}, \ell)\right) \tag{29}$$

is usually referred to as *true anomaly*, so one recognises that $\mathrm{p}(\Lambda, \mathrm{G}, \ell, \mathrm{g}) = \varrho\cos(\nu + \mathrm{g})$. Observe that E and J are both independent of $\mathrm{C}, \mathrm{Z}, \zeta, \gamma, \mathrm{R}, \vartheta$, because the conjugated coordinates to these variables are first integrals of the motion. Remark, at this respect, that: (i) the couples (Z, ζ) and (C, γ) are, simultaneously, couples of first integrals to J and E and couples of cyclic coordinates, determined by the

conservation of $\mathbf{C}$ and $\mathbf{x}'$; (ii) the independence on ϑ corresponds to the invariance of J and E under the one–parameter group of rotations around $\mathbf{x}'$.

The details on the derivation of the formulae in (26) may be found in Appendix A.2.

3. **Renormalizable integrability.** In this section we review the property of *renormalizable integrability* pointed out in [15].

Definition 3.1. Let h, g be two functions of the form

$$h(p,q,y,x) = \widehat{h}(\mathrm{I}(p,q),y,x) , \qquad g(p,q,y,x) = \widehat{g}(\mathrm{I}(p,q),y,x) \tag{30}$$

where

$$(p,q,y,x) \in \mathcal{D} := \mathcal{B} \times U \tag{31}$$

with $U \subset \mathbf{R}^2$, $\mathcal{B} \subset \mathbf{R}^{2n}$ open and connected, $(p,q) = (p_1, \cdots, p_n, q_1, \cdots, q_n)$ conjugate coordinates with respect to the two–form $\omega = dy \wedge dx + \sum_{i=1}^{n} dp_i \wedge dq_i$ and $\mathrm{I}(p,q) = (\mathrm{I}_1(p,q), \cdots, \mathrm{I}_n(p,q))$, with

$$\mathrm{I}_i : \ \mathcal{B} \to \mathbf{R} , \qquad i = 1, \cdots, n$$

pairwise Poisson commuting:

$$\left\{ \mathrm{I}_i, \mathrm{I}_j \right\} = 0 \qquad \forall \ 1 \le i < j \le n \qquad i = 1, \cdots, n . \tag{32}$$

We say that h is *renormalizably integrable via g* if there exists a function

$$\widetilde{h} : \qquad \mathrm{I}(\mathcal{B}) \times g(U) \to \mathbf{R} ,$$

such that

$$h(p,q,y,x) = \widetilde{h}(\mathrm{I}(p,q), \widehat{g}(\mathrm{I}(p,q),y,x)) \tag{33}$$

for all $(p,q,y,x) \in \mathcal{D}$.

Proposition 1. *If h is renormalizably integrable via g, then: (i) $\mathrm{I}_1, \cdots, \mathrm{I}_n$ are first integrals to h and g; (ii) h and g Poisson commute.*

Observe that, if h is renormalizably integrable via g, then, generically, their respective time laws for the coordinates (y,x) are the same, up to rescale the time:

Proposition 2. *Let h be renormalizably integrable via g. Fix a value I_0 for the integrals I and look at the motion of (y,x) under h and g, on the manifold $\mathrm{I} = \mathrm{I}_0$. For any fixed initial datum (y_0, x_0), let $g_0 := g(\mathrm{I}_0, y_0, x_0)$. If $\omega(\mathrm{I}_0, g_0) := \partial_g \widetilde{h}(\mathrm{I}, g)|_{(\mathrm{I}_0, g_0)} \ne 0$, the motion $(y^h(t), x^h(t))$ with initial datum(y_0, x_0) under h is related to the corresponding motion $(y^g(t), x^g(t))$ under g via*

$$y^h(t) = y^g(\omega(\mathrm{I}_0, g_0)t) , \quad x^h(t) = x^g(\omega(\mathrm{I}_0, g_0)t)$$

In particular, under this condition, all the fixed points of g in the plane (y,x) are fixed point to h. Values of (I_0, g_0) for which $\omega(\mathrm{I}_0, g_0) = 0$ provide, in the plane (y,x), curves of fixed points for h (which are not necessarily curves of fixed points to g).

We consider the ℓ–average of the of the function J_1 in (26):

$$\mathrm{U}(\mathrm{r}, \Lambda, \Theta, \mathrm{G}, \mathrm{g}) := -\frac{\mathrm{m}\mathcal{M}'}{2\pi} \int_0^{2\pi} \frac{d\ell}{\sqrt{\mathrm{r}^2 + 2\mathrm{r}a\sqrt{1 - \frac{\Theta^2}{\mathrm{G}^2}}\mathrm{p} + a^2\varrho^2}} \tag{34}$$

We observe that U and E_0 have the form in (30), with $I = (I_1, I_2, I_3) = (r, \Lambda, \Theta)$ verifying (32) and $(y, x) = (G, g)$.

Proposition 3. U *is renormalizably integrable via* E_0. *Namely, there exists a function* F *such that*

$$U(r, \Lambda, \Theta, G, g) = F(r, \Lambda, \Theta, E_0(r, \Lambda, \Theta, G, g)) .$$

A^7 *function* F *is given by*

$$F(r, \Lambda, \Theta, E_0) = \widetilde{F}(r, a(\Lambda), \mathcal{E}(\Lambda, E_0), \mathcal{I}(\Lambda, \Theta, E_0))$$

where

$$\widetilde{F}(r, a, \mathcal{E}, \mathcal{I}) = \frac{1}{2\pi} \int_{\mathbf{T}} \frac{(1 - \mathcal{E} \cos w) dw}{\sqrt{r^2 + a^2 - 2a(r\mathcal{I} \sin w + a\mathcal{E} \cos w) + a^2 \mathcal{E}^2 \cos^2 w}} ; \quad (35)$$

$$\mathcal{E}(\Lambda, E_0) = \frac{\sqrt{\Lambda^2 - E_0}}{\Lambda} \qquad \mathcal{I}(\Lambda, \Theta, E_0) = \frac{\sqrt{E_0 - \Theta^2}}{\Lambda_2} . \quad (36)$$

The results of the present section may be summarised as follows. Proposition 3 implies

$$\left\{ U, \ E_0 \right\} = 0 , \quad (37)$$

but, actually, combining it with Proposition 2, we have that much more is true:

  (i) If $F_{E_0} \neq 0$, the time laws of (G, g) under U or E_0 are basically (i.e., up to a change of time) the same;

  (i) Motions of E_0 corresponding to level sets for which $F_{E_0} = 0$ are fixed points curves to U ("frozen orbits"). In [15] we provided an example of frozen orbit of U in the case $\delta := \frac{r}{a} \ll 1$;

 (iii) U and E_0 have the same action–angle coordinates.

In the next section, we investigate the dynamical properties of E_0 for the planar case ($\Theta = 0$). The study of the spatial case appears less explicit, so it is deferred to a next publication.

4. **Dynamical properties of** E_0 **in the case** $\Theta = 0$. In this section we focus on the dynamical consequences of renormalizable integrability.

4.1. **Phase portrait.** We study the phase portrait of E_0 in (26) setting $\Theta = 0$. In this case, we have to study the curves

$$E_0(\Lambda, G, g; r) = G^2 + m^2 \mathcal{M}' r \sqrt{1 - \frac{G^2}{\Lambda^2}} \cos g = \mathcal{E} \quad (38)$$

in the plane (g, G). To simplify notations, we divide this equation by Λ^2, and we rewrite it as

$$\widehat{E}_0(g, \widehat{G}) = \widehat{G}^2 + \delta \sqrt{1 - \widehat{G}^2} \cos g = \widehat{\mathcal{E}} , \quad (39)$$

where

$$\widehat{\mathcal{E}} := \frac{\mathcal{E}}{\Lambda^2} , \quad \widehat{G} := \frac{G}{\Lambda} , \quad \delta := m^2 \mathcal{M}' \frac{r}{\Lambda^2} = \frac{r}{a} \quad (40)$$

and we study the rescaled level sets (39) in the plane $(g, \widehat{G})$. Observe, incidentally, that the level sets (39) extend also for $\delta < 0$, due to the symmetry of E_0 for

$$(\delta, g) \to (-\delta, \pi - g) .$$

---

[7] We remark that F may have several expressions, as well as U, which is defined via a quadrature.

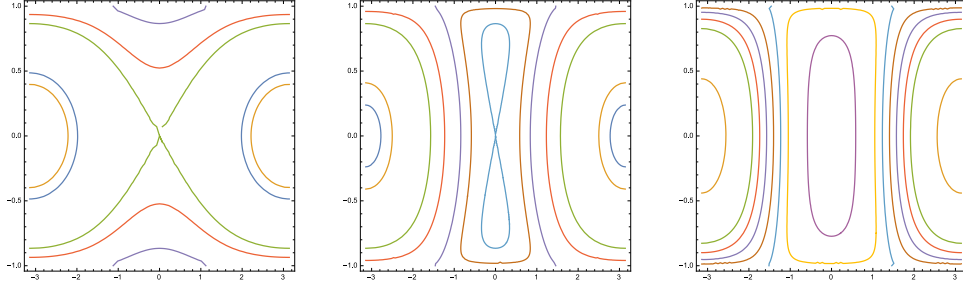FIGURE 1. The phase portrait of E_0 in the plane (g, G). Left: $0 < \delta < 1$; Center: $0 < \delta < 1$; Right: $\delta > 2$.

We limit to study them in the case $\delta > 0$.

For $\delta \in (0, 2)$, the function $\widehat{E}_0(g, \widehat{G})$ has a minimum, a saddle and a maximum, respectively at

$$\hat{\mathbf{P}}_- = (\pi, 0) \ , \qquad \hat{\mathbf{P}}_0 = (0, 0) \ , \qquad \hat{\mathbf{P}}_+ = \left(0, \sqrt{1 - \frac{\delta^2}{4}}\right)$$

where it takes the values, respectively,

$$\widehat{\mathcal{E}}_- = -\delta \ , \qquad \widehat{\mathcal{E}}_0 = \delta \ , \quad \widehat{\mathcal{E}}_+ = 1 + \frac{\delta^2}{4} \ .$$

Thus, the level sets in (39) are non–empty only for

$$\widehat{\mathcal{E}} \in \left[-\delta, 1 + \frac{\delta^2}{4}\right] \ . \tag{41}$$

We denote as $\mathcal{S}_0$, the level set through the saddle $\mathbf{P}_0$. When $\widehat{G} = 1$, $\widehat{E}_0$ takes the value 1 for all g and we denote as $\mathcal{S}_1$ the level curve with $\widehat{\mathcal{E}} = 1$. The equations of $\mathcal{S}_0$, $\mathcal{S}_1$ are, respectively:

$$\mathcal{S}_0(\delta) = \left\{(g, \widehat{G}) : \ \widehat{G}^2 + \delta\sqrt{1 - \widehat{G}^2}\cos g = \delta\right\}$$

$$\mathcal{S}_1(\delta) = \left\{\widehat{G} = \pm 1\right\} \cup \left\{\widehat{G}_1 = \pm\sqrt{1 - \delta^2\cos^2 g}\right\} \tag{42}$$

$\mathcal{S}_1$ is composed of two branches, which will be referred to as "horizontal", "vertical", respectively, glue smoothly at $(\pm\frac{\pi}{2}, 1)$, with g mod 2π. Note that, when $0 \le \delta \le 1$, the vertical branch is defined for all $g \in \mathbf{T}$; when $\delta > 1$, its domain in g is made of two disjoint neighborhoods of $\pm\frac{\pi}{2}$.

When $\delta > 2$, the saddle $\hat{\mathbf{P}}_0$ and its manifold $\mathcal{S}_0$ do not exist, $\hat{\mathbf{P}}_- = (\pi, 0)$ is still a minimum, while $\hat{\mathbf{P}}_+ = (0, 0)$ becomes a maximum. The manifold $\mathcal{S}_1$ still exists, with the vertical branch closer and closer, as $\delta \to +\infty$, to the portion of straight $g = \pm\frac{\pi}{2}$ in the strip $-1 \le \widehat{G} \le 1$. In this case the admissible values for $\widehat{\mathcal{E}}$ are

$$\widehat{\mathcal{E}} \in [-\delta, \delta] \ .$$

We now turn to the phase portrait induced by the level curves (39) in the plane (g, G). According to the value of δ, the scenario changes, as detailed below. See also Figure 1.

Phase portrait.

1) $0 < \delta \le 1$

1_1) $-\delta \le \widehat{\mathcal{E}} < \delta$: the level curves encircle the minimum $\mathbf{P}_-$;

1_2) $\widehat{\mathcal{E}} = \delta$ is the manifold $\mathcal{S}_0$;

1_3) $\delta < \widehat{\mathcal{E}} < 1$: the level curves are defined for all $g \in \mathbf{T}$;

1_4) $\widehat{\mathcal{E}} = 1$ is the manifold $\mathcal{S}_0$;

1_5) $1 < \widehat{\mathcal{E}} \le 1 + \frac{\delta^2}{4}$: the level curves encircle the maximum $\mathbf{P}_+$;
Here, with $\delta = 1$ the items 1_2), 1_3) and 1_4), and hence the two manifolds $\mathcal{S}_0$ and $\mathcal{S}_1$, merge. With $\delta = 0$, 1_1) merges with 1_2); 1_5) merges with 1_4). $\mathcal{S}_0$ is contracted to the g–axis; $\mathcal{S}_1$ is contracted to the axis $\widehat{G} = 1$; the level curves are straight lines, parallel to the g–axis. The level sets reduce to $G = $ const.

2) $1 < \delta \le 2$

2_1) $-\delta \le \widehat{\mathcal{E}} < 1$: the level curves encircle the minimum $\mathbf{P}_-$;

2_2) $\widehat{\mathcal{E}} = 1$ is the manifold $\mathcal{S}_1$:

2_3) $1 < \widehat{\mathcal{E}} < \delta$: the level curves encircle the saddle $\mathbf{P}_0$;

2_4) $\widehat{\mathcal{E}} = \delta$ is the manifold $\mathcal{S}_0$;

2_5) $\delta < \widehat{\mathcal{E}} \le 1 + \frac{\delta^2}{4}$; the level curves encircle the maximum $\mathbf{P}_+$.
Here, when $\delta = 2$, items 2_4) and 2_5) merge and the manifold $\mathcal{S}_0$ with the librations inside are contracted to the point $\mathbf{P}_0$.

3) $\delta > 2$

3_1) $-\delta \le \widehat{\mathcal{E}} < 1$: the level curves encircle the minimum $\mathbf{P}_-$;

3_2) $\widehat{\mathcal{E}} = 1$ corresponds to the manifold $\mathcal{S}_1$;

3_3) $1 < \widehat{\mathcal{E}} < \delta$: the level curves encircle the maximum $\mathbf{P}_0$.

Remark 2 (*perihelion librations*)**.** Figure 1 shows that librational motions of (g, G) about $\mathbf{P}_- = (\pi, 0)$ are stable for any value of δ. Physically, such librations correspond to small periodic oscillations of the perihelion of the instantaneous ellipse of $(\mathbf{y}, \mathbf{x})$ along[8] the direction $\mathbf{j}_3$, while the ellipse squeezes to a segment before reversing its direction and again decreasing its eccentricity. Other kind of librations of course occur in the three cases, as Figure 1 shows. It is quite astonishing to see that rotational motions of the perihelion do exist only when $0 < \delta < 1$ and only between $\mathcal{S}_0$ and $\mathcal{S}_1$. The singularity of this fact is even more evident if one recalls that, for arbitrarily large values of δ, in the planetary problem, rotational motions of the perihelia occupy a positive measure set in phase space [1, 12, 9, 6].

The computations leading to the phase portrait above are elementary. We report here the complete discussion for $\delta \in (0, 2)$. The case $\delta > 2$ is similar.

Solving equation (39) for g, we find two branches

$$g = g_\pm = \pm \cos^{-1}\left(\frac{\widehat{\mathcal{E}} - \widehat{G}^2}{\delta\sqrt{1 - \widehat{G}^2}}\right) \qquad \mathrm{mod} \ \ 2\pi \ . \tag{43}$$

Using

$$1 - \left(\frac{\widehat{\mathcal{E}} - \widehat{G}^2}{\delta\sqrt{1 - \widehat{G}^2}}\right)^2 = \frac{\delta^2 - \widehat{\mathcal{E}}^2 - 2(\frac{\delta^2}{2} - \widehat{\mathcal{E}})\widehat{G}^2 - \widehat{G}^4}{\delta^2(1 - \widehat{G}^2)} = \frac{(\widehat{G}^2 - \widehat{G}_-^2)(\widehat{G}_+^2 - \widehat{G}^2)}{\delta^2(1 - \widehat{G}^2)} \tag{44}$$

---

[8]Recall that, in the planar case, the perihelion anomaly is $g - \frac{\pi}{2}$, by (23), relatively to $\mathbf{i}_3$. Therefore, $g = \pi$ corresponds to $\mathbf{P} \parallel \mathbf{j}_3$.

with

$$
\begin{aligned}
\widehat{G}^2_\pm &= \widehat{\mathcal{E}} - \frac{\delta^2}{2} \pm \sqrt{\left(\widehat{\mathcal{E}} - \frac{\delta^2}{2}\right)^2 + \delta^2 - \widehat{\mathcal{E}}^2} \\
&= \widehat{\mathcal{E}} - \frac{\delta^2}{2} \pm \delta\sqrt{1 + \frac{\delta^2}{4} - \widehat{\mathcal{E}}}
\end{aligned}
\tag{45}
$$

one sees the equality (43) is well defined for

$$
\widehat{G}_{\min} \le \widehat{G} \le \widehat{G}_{\max}
\tag{46}
$$

where

$$
\widehat{G}^2_{\min} := \max\{\widehat{G}^2_-, 0\} , \qquad \widehat{G}^2_{\max} := \min\{\widehat{G}^2_+, 1\} .
$$

Note that, when $\widehat{\mathcal{E}}$ takes its maximum value $1 + \frac{\delta^2}{4}$, one has $\widehat{G}^2_+ = \widehat{G}^2_- = 1 - \frac{\delta^2}{4}$. Therefore, by (43) and (46), the level set with $\widehat{\mathcal{E}} = 1 + \frac{\delta^2}{4}$ reduces to the maximum point $(0, \pm\sqrt{1 - \frac{\delta^2}{4}})$. Writing

$$
\widehat{G}^2_- = \frac{\widehat{\mathcal{E}}^2 - \delta^2}{\widehat{\mathcal{E}} - \frac{\delta^2}{2} + \delta\sqrt{1 + \frac{\delta^2}{4} - \widehat{\mathcal{E}}}}
$$

and noticing that

$$
1 - \widehat{G}^2_+ = 1 - \left(\widehat{\mathcal{E}} - \frac{\delta^2}{2} + \delta\sqrt{1 + \frac{\delta^2}{4} - \widehat{\mathcal{E}}}\right) = \left(\frac{\delta}{2} - \sqrt{1 + \frac{\delta^2}{4} - \widehat{\mathcal{E}}}\right)^2 \ge 0 ,
$$

one finds that

$$
\widehat{G}_{\min} = \begin{cases} 0 & \text{if } -\delta \le \widehat{\mathcal{E}} \le \delta \\ \widehat{G}_- & \text{if } \widehat{\mathcal{E}} > \delta \end{cases} , \qquad \widehat{G}_{\max} = \widehat{G}_+ .
\tag{47}
$$

Observe that

$$
\lim_{\widehat{\mathcal{E}} \to \delta} G^2_{\min} = \lim_{\widehat{\mathcal{E}} \to \delta} G^2_- = 0 , \quad \lim_{\widehat{\mathcal{E}} \to \delta} G^2_{\max} = \lim_{\widehat{\mathcal{E}} \to \delta} G^2_+ = \delta(2 - \delta)
$$

and

$$
\lim_{\widehat{\mathcal{E}} \to 1} G^2_{\max} = \lim_{\widehat{\mathcal{E}} \to 1} G^2_+ = 1 , \quad \lim_{\widehat{\mathcal{E}} \to 1} \widehat{G}^2_- = 1 - \delta^2 , \quad \lim_{\widehat{\mathcal{E}} \to 1} \widehat{G}^2_{\min} = \max\{1 - \delta^2, 0\} ,
$$

which are obtained using

$$
\lim_{\widehat{\mathcal{E}} \to \delta} \widehat{G}^2_\pm = \delta - \frac{\delta^2}{2} \pm \delta\left(1 - \frac{\delta}{2}\right) , \quad \lim_{\widehat{\mathcal{E}} \to 1} \widehat{G}^2_\pm = 1 - \frac{\delta^2}{2} \pm \frac{\delta^2}{2}
$$

in turn implied by (45).

In particular, $G_{\min}$, is continuous for $\widehat{\mathcal{E}} = \delta$. The inequality (46) defines a symmetric domain of $\widehat{G}$ with respect to the origin, consisting of the union

$$
\widehat{\mathcal{D}} = \widehat{\mathcal{D}}_- \cup \widehat{\mathcal{D}}_+
\tag{48}
$$

of two symmetric intervals

$$
\widehat{\mathcal{D}}_- = \left[-\widehat{G}_{\max} -\widehat{G}_{\min}\right] , \qquad \widehat{\mathcal{D}}_+ = \left[\widehat{G}_{\min} \widehat{G}_{\max}\right] .
$$

Observe that, for $\widehat{\mathcal{E}} > \delta$ and $\widehat{\mathcal{E}} \ne 1$, the union (48) is disjoint, since, in this case, $\widehat{G}_{\min} > 0$ (se (47)). The functions $g_\pm$ in (43) are even functions of $\widehat{G}$ on such symmetric domain.

We now study the curves in (39) in the plane $(g, \widehat{G})$, for $\widehat{\mathcal{E}}$ as in (41). By symmetry, we limit to study the behavior of g_+ for $\widehat{G} \in \widehat{\mathcal{D}}_+$. We denote as

$$\underline{g} := \cos^{-1}\left(\frac{\widehat{\mathcal{E}} - \widehat{G}_{\min}^2}{\delta\sqrt{1 - \widehat{G}_{\min}^2}}\right) \quad , \quad \overline{g} := \cos^{-1}\left(\frac{\widehat{\mathcal{E}} - \widehat{G}_{\max}^2}{\delta\sqrt{1 - \widehat{G}_{\max}^2}}\right) \tag{49}$$

the values that g_+ takes at the extrema of $\widehat{\mathcal{D}}_+$. The explicit value of $\underline{g}, \overline{g}$ is

$$\underline{g} = \begin{cases} 0 & \text{if} & \widehat{\mathcal{E}} > \delta \\ \cos^{-1}\frac{\widehat{\mathcal{E}}}{\delta} & \text{if} & -\delta \le \widehat{\mathcal{E}} \le \delta \end{cases} \quad , \quad \overline{g} = \begin{cases} \pi & \text{if} & \widehat{\mathcal{E}} < 1 \\ \frac{\pi}{2} & \text{if} & \widehat{\mathcal{E}} = 1 \\ 0 & \text{if} & \widehat{\mathcal{E}} > 1 \end{cases} \tag{50}$$

This follows from the definitions in (45) and (47). In particular, from (45) one finds, for $(\sigma, \widehat{\mathcal{E}}) \ne (+, 1)$

$$\frac{\widehat{\mathcal{E}} - \widehat{G}_\sigma^2}{\delta\sqrt{1 - \widehat{G}_\sigma^2}} = \frac{\widehat{\mathcal{E}} - \left(\widehat{\mathcal{E}} - \frac{\delta^2}{2} + \sigma\delta\sqrt{1 + \frac{\delta^2}{4} - \widehat{\mathcal{E}}}\right)}{\delta\sqrt{1 - \left(\widehat{\mathcal{E}} - \frac{\delta^2}{2} + \sigma\delta\sqrt{1 + \frac{\delta^2}{4} - \widehat{\mathcal{E}}}\right)}}$$

$$= \text{sign}\left(\frac{\delta}{2} - \sigma\sqrt{1 + \frac{\delta^2}{4} - \widehat{\mathcal{E}}}\right)$$

$$= \begin{cases} +1 & \text{for} & \sigma = - \\ -1 & \text{for} & \sigma = + \ \& \ \widehat{\mathcal{E}} < 1 \\ +1 & \text{for} & \sigma = + \ \& \ \widehat{\mathcal{E}} > 1 \end{cases}$$

while, for $(\sigma, \widehat{\mathcal{E}}) = (+, 1)$,

$$\frac{\widehat{\mathcal{E}} - \widehat{G}_+^2}{\delta\sqrt{1 - \widehat{G}_+^2}} = \frac{\sqrt{1 - \widehat{G}_+^2}}{\delta} = \frac{\sqrt{1 - \left(1 - \frac{\delta^2}{2} + \delta\sqrt{1 + \frac{\delta^2}{4} - 1}\right)}}{\delta} = 0$$

Let us study the graph of g_+ as a function of $\widehat{G}$, for $\widehat{G} \in \widehat{\mathcal{D}}_+$. From the formula

$$\partial_{\widehat{G}} g_+ = \frac{\widehat{G}}{\sqrt{(\widehat{G}^2 - \widehat{G}_-^2)(\widehat{G}_{\max}^2 - \widehat{G}^2)}} \frac{2 - \widehat{\mathcal{E}} - \widehat{G}^2}{1 - \widehat{G}^2} . \tag{51}$$

one sees that $\widehat{G} = \widehat{G}_0 := \sqrt{2 - \widehat{\mathcal{E}}} \notin \widehat{\mathcal{D}}_+$ is an extremal point, as soon as $\widehat{G}_0 \in \widehat{\mathcal{D}}_+$. Using

$$\widehat{G}_0^2 - \widehat{G}_{\max}^2 = 2 - \widehat{\mathcal{E}} - \left(\widehat{\mathcal{E}} - \frac{\delta^2}{2} + \delta\sqrt{1 + \frac{\delta^2}{4} - \widehat{\mathcal{E}}}\right)$$

$$= \sqrt{1 + \frac{\delta^2}{4} - \widehat{\mathcal{E}}}\left(2\sqrt{1 + \frac{\delta^2}{4} - \widehat{\mathcal{E}}} - \delta\right)$$

$$= 2\frac{\sqrt{1 + \frac{\delta^2}{4} - \widehat{\mathcal{E}}}}{\sqrt{1 + \frac{\delta^2}{4} - \widehat{\mathcal{E}}} + \delta}(1 - \widehat{\mathcal{E}})$$

and

$$\widehat{G}_0^2 - \widehat{G}_{\min}^2 \geq 2 - \widehat{\mathcal{E}} - \left(\widehat{\mathcal{E}} - \frac{\delta^2}{2} - \delta\sqrt{1 + \frac{\delta^2}{4} - \widehat{\mathcal{E}}}\right)$$

$$= \sqrt{1 + \frac{\delta^2}{4} - \widehat{\mathcal{E}}}\left(2\sqrt{1 + \frac{\delta^2}{4} - \widehat{\mathcal{E}}} + \delta\right) \geq 0 .$$

we see that

$$g_0 \begin{cases} \geq \widehat{G}_{\max} & \text{for} \quad \widehat{\mathcal{E}} < 1 \\ \in \widehat{\mathcal{D}}_+ & \text{for} \quad \widehat{\mathcal{E}} \geq 1 \end{cases} .$$

As a consequence,

- For $\widehat{\mathcal{E}} < 1$, $\widehat{G}_0 > \widehat{G}_{\max}$ and hence g_+ increases, in $\widehat{\mathcal{D}}_+$, from $\underline{g}$ to $\overline{g}$.
- For $\widehat{\mathcal{E}} > 1$, g_+ increases from $\underline{g}$ to g_0 for $\widehat{G}_{\min} \leq \widehat{G} \leq \widehat{G}_0$ and decreases from g_0 to $\overline{g}$, for $\widehat{G}_0 \leq \widehat{G} \leq \widehat{G}_{\max}$.

4.2. **The collisional manifold and its motions.** The E_0–level set through the saddle $(G, g) = (0, 0)$, which in the previous section, was denoted $\mathcal{S}_0$, exists only for $\delta \in (0, 2)$ and has equation, by (39) and (40),

$$\mathcal{S}_0 : \quad \frac{G^2}{\Lambda^2} + \frac{r}{a}\sqrt{1 - \frac{G^2}{\Lambda^2}} \cos g = \frac{r}{a} ,$$

as the left hand side takes the value r/a at the saddle. A first observation is that, if we solve for r, we find

$$r = a \frac{\frac{G^2}{\Lambda^2}}{1 - \sqrt{1 - \frac{G^2}{\Lambda^2}} \cos g} = \frac{a(1 - e^2)}{1 - e \cos g}$$

where e is the eccentricity. This equation tells us that, while G and g move on $\mathcal{S}_0$, $\mathbf{x}'$ belongs to the instantaneous ellipse through $\mathbf{x}$, in correspondence of the true anomaly g. For this reason, it is to be remarked that, while motions of $\mathcal{S}_0$ are perfectly meaningful for E_0, they might not exist for U. We also refer $\mathcal{S}_0$ as the "collisional manifold".

The second aspect we aim to point out is that the motions of (G, g) under E_0 along this manifold can be explicitly computed. Indeed, with

$$\sigma^2 := \delta(2 - \delta) , \quad \beta^2 := 2 - \delta , \quad \delta \in (0, 2) \tag{52}$$

the Hamilton equation for G is

$$\dot{G} = m^2 \mathcal{M} r \sqrt{1 - \frac{G^2}{\Lambda^2}} \sin g$$

Eliminating g through (39), i.e.,

$$\sin^2 g = 1 - \cos^2 g = \frac{\frac{G^2}{\Lambda^2}\left(\sigma^2 - \frac{G^2}{\Lambda^2}\right)}{\delta^2\left(1 - \frac{G^2}{\Lambda^2}\right)}$$

we immediately obtain the closed equation

$$\dot{G} = -G\sqrt{\Lambda^2\sigma^2 - G^2} .$$

It can be solved by separation. The overall solution is

$$
\begin{cases}
\mathrm{G}(t) = \dfrac{\sigma\Lambda}{\cosh\sigma\Lambda(t-t_0)} \\[2ex]
\mathrm{g}(t) = \pm\cos^{-1}\dfrac{1 - \dfrac{\beta^2}{\cosh^2\sigma\Lambda(t-t_0)}}{\sqrt{1 - \dfrac{\sigma^2}{\cosh^2\sigma\Lambda(t-t_0)}}}
\end{cases} .
$$

We remark that in the case $\delta = 1$, hence, $\sigma = \beta = 1$, it reduces to the solution of the classical pendulum.

4.3. Asymptotic action–angle coordinates.

The explicit construction of the action–angle coordinates for any value of r and Θ requires the use of elliptic integrals. This is true even in the case $\Theta = 0$, in which the phase portrait is, as discussed, explicit. One possibility to deal with this situation in practical problems is to start with "approximate coordinates"; i.e., coordinates such that E_0, even though being integrable, looks also as close–to–be–integrable. This would allow us to apply the machinery of perturbative methods. The first candidate to look at is the distance r of the two fixed centers. If r is small, E_0 is very close to G^2 and the approximate action–angle coordinates coincide with the initial $\mathcal{K}$–coordinates. The case of large r is investigated in the present section.

When r is large, the leading part of E_0 is

$$
\mathrm{E}_{0,1} = \mathrm{r}m^2\mathcal{M}\sqrt{1 - \frac{\mathrm{G}^2}{\Lambda^2}}\cos\mathrm{g} .
$$

The integration of $\mathrm{E}_{0,1}$ relies on solving equation

$$
\sqrt{1 - \frac{\mathrm{G}^2}{\Lambda^2}}\cos\mathrm{g} = \mathcal{E} \tag{53}
$$

for G. The level curves (53) exist only for[9] $|\mathcal{E}| \le 1$ and are all closed. The associated action coordinate $\mathcal{G}$ corresponds to be the area of the figure enveloped by such curves; the angle γ is the time needed to reach, on a fixed level set, a given point, starting from a fixed one. The computation of $\mathcal{G}$ and γ is explicit, as now we show. Solving (53) for G, we obtain

$$
\mathrm{G} = \pm\mathcal{L}\sqrt{1 - \frac{\mathcal{E}^2}{\cos^2\mathrm{g}}} \quad\text{with}\quad |\mathcal{E}| \le |\cos\mathrm{g}| \le 1, \quad \mathrm{sign}(\mathcal{E}) = \mathrm{sign}(\cos\mathrm{g}) \tag{54}
$$

with $\mathcal{L} = \Lambda$. We define the action coordinate $\mathcal{G}(\mathcal{L}, \mathcal{E})$ so that $\mathcal{G} = 0$ for $\mathcal{E} = 0$. Then

$$
\mathcal{G}(\mathcal{L}, \mathcal{E}) =
\begin{cases}
-\mathcal{L} + \dfrac{\mathcal{L}}{\pi}\displaystyle\int_{-\arccos|\mathcal{E}|}^{\arccos|\mathcal{E}|}\sqrt{1 - \dfrac{\mathcal{E}^2}{\cos^2\mathrm{g}}}\,d\mathrm{g} & -1 < \mathcal{E} < 0 \\[3ex]
\mathcal{L} - \dfrac{\mathcal{L}}{\pi}\displaystyle\int_{-\arccos\mathcal{E}}^{\arccos\mathcal{E}}\sqrt{1 - \dfrac{\mathcal{E}^2}{\cos^2\mathrm{g}}}\,d\mathrm{g} & 0 < \mathcal{E} < 1
\end{cases}
$$

The period of the orbit is given by

$$
\mathcal{T}(\mathcal{L}, \mathcal{E}) = 2\pi\mathcal{G}_\mathcal{E}(\mathcal{L}, \mathcal{E})
$$

---

[9]Observe that condition $|\mathcal{E}| \le 1$ corresponds to $|\widehat{\mathcal{E}}| \le \delta$ in the notations of Section 4.1. Equation (54) represents, in the plane (G, g), closed curves encircling the equilibria $(0, 0)$, $(0, \pi)$ (for $\mathcal{E} > 0$, $\mathcal{E} < 0$, respectively) corresponding to the limiting curves of the third panel in Figure 1 as $\delta \to \infty$.

With the change of variable

$$w = \frac{|\mathcal{E}|}{\sqrt{1 - \mathcal{E}^2}} \, \mathrm{tg} \, \mathrm{g} \tag{55}$$

we obtain

$$\mathcal{T}(\mathcal{L}, \mathcal{E}) = 4\mathcal{L}|\mathcal{E}| \int_0^{\arccos|\mathcal{E}|} \frac{1}{\cos^2 \mathrm{g}} \frac{d\mathrm{g}}{\sqrt{1 - \frac{\mathcal{E}^2}{\cos^2 \mathrm{g}}}} = 4\mathcal{L} \int_0^1 \frac{dw}{\sqrt{1 - w^2}} = 2\pi\mathcal{L}$$

whence (using that $\mathcal{G}$ takes the value 0 at $\mathcal{E} = 0$) the action–angle coordinates are found to be

$$\mathcal{G} = \mathcal{L}\mathcal{E} \, , \quad \gamma = \frac{\tau}{\mathcal{L}} \tag{56}$$

where τ is the time the flows employs to reach the value (G, g) on the level set $\mathcal{E}$, starting from $(\mathcal{L}\sqrt{1 - \mathcal{E}^2}, 0)$ $((\mathcal{L}\sqrt{1 - \mathcal{E}^2}, \pi)$, respectively). Looking at the generating function

$$S(\mathcal{L}, \mathcal{G}, \ell, \mathrm{g}) = \mathcal{L}\ell + \int^{\mathrm{g}} \sqrt{\mathcal{L}^2 - \frac{\mathcal{G}^2}{\cos^2 \mathrm{g}'}} \, d\mathrm{g}'$$

one obtains the transformation of coordinates

$$\begin{cases} \Lambda = \mathcal{L} \\ \\ \ell = \lambda + \arg\left(\cos\gamma, \ \frac{\mathcal{L}}{|\mathcal{G}|}\sin\gamma\right) \end{cases} \qquad \begin{cases} \mathrm{G} = \sqrt{\mathcal{L}^2 - \mathcal{G}^2}\cos\gamma \\ \\ \mathrm{tg}\,\mathrm{g} = -\frac{\mathcal{L}}{\mathcal{G}}\sqrt{1 - \frac{\mathcal{G}^2}{\mathcal{L}^2}}\sin\gamma \\ \\ \mathrm{sign}\,\cos\mathrm{g} = \ \mathrm{sign}\,\mathcal{G} \end{cases} \tag{57}$$

The expression of $\mathcal{E}$ in in action–angle coordinates is

$$\mathcal{E} = \frac{\mathcal{G}}{\mathcal{L}} \, . \tag{58}$$

Using these "approximate" coordinates, one obtains the expression of E_0 as a close–to–be–integrable system for large r:

$$\mathrm{E}_0 = \mathrm{rm}^2\mathcal{M}\frac{\mathcal{G}}{\mathcal{L}} + (\mathcal{L}^2 - \mathcal{G}^2)\cos^2\gamma$$

5. **An application to the three–body problem (sketch).** In this section we propose an application to the classical three–body problem. As said in the introduction, we choose to focus on the full problem, as opposed to the restricted one.

When the Newtonian part (i.e., the third term) in (4) is much smaller compared to the Keplerian terms, one expects, by perturbation theory, that the relevant part of the dynamics of J is played by the "secular terms"

$$\mathrm{J}_\mathrm{s}(\mathrm{r}, \Lambda, \Theta, \mathrm{G}, \mathrm{g}) = -\frac{\mathrm{m}^3\mathcal{M}^2}{2\Lambda^2} + \mathrm{U}(\mathrm{r}, \Lambda, \Theta, \mathrm{G}, \mathrm{g}) \tag{59}$$

where U is the average of the Newtonian potential, defined in (34). By the previous sections, under such perturbative assumption, the motions of J reduce substantially to the motions of E_0.

A natural question is now whether a Hamiltonian sufficiently "close" to J may generate motions that can be regarded as a "continuation" of the motions of E_0. Concretely, one might ask whether, in the specific case of the planar case, the motions represented in Figure 1 may be continued to some physical system close

to the planar two–centre problem. The first thought goes of course to the three–body problem. As we are going to describe, the question demands several technical difficulties. Therefore it is just sketchily treated here.

To fix the ideas, consider the three–body problem Hamiltonian with equal masses

$$m_0 = m_1 = m_2$$

with the translational symmetry reduced via the heliocentric method. Denoting as

$$\mathrm{m}' = \mathrm{m} = \frac{m_0}{2} \ , \quad \mathcal{M}' = \mathcal{M} = 2m_0$$

the "reduced masses", the Hamiltonian of the system is

$$
\begin{aligned}
\mathrm{H}(\underline{\mathbf{y}}, \underline{\mathbf{x}}) \ &= \ \frac{\|\mathbf{y}\|^2}{2\mathrm{m}} - \frac{\mathrm{m}\mathcal{M}}{\|\mathbf{x}\|} - \frac{\mathrm{m}\mathcal{M}}{\|\mathbf{x} - \mathbf{x}'\|} - \frac{\mathrm{m}'\mathcal{M}'}{\|\mathbf{x}'\|} \\
&+ \ \frac{\|\mathbf{y}'\|^2}{2\mathrm{m}'} + \frac{\mathbf{y}' \cdot \mathbf{y}}{m_0} \ .
\end{aligned}
\tag{60}
$$

We consider the Hamiltonian (60), in the planar case, written in $\mathcal{K}$–coordinates. Choosing the two angular momenta parallel one to the other (see Appendix A.1), its expression is the following:

$$
\begin{aligned}
\mathrm{H}(\mathrm{R}, \Lambda, \mathrm{G}, \mathrm{r}, \ell, \mathrm{g}) \ &= \ -\frac{\mathrm{m}^3\mathcal{M}^2}{2\Lambda^2} - \frac{\mathrm{m}\mathcal{M}}{\sqrt{\mathrm{r}^2 + 2\mathrm{r}ap + a^2\varrho^2}} - \frac{\mathrm{m}'\mathcal{M}'}{\mathrm{r}} \\
&+ \frac{\mathrm{R}^2}{2\mathrm{m}'} + \frac{(\mathrm{C} - \mathrm{G})^2}{2\mathrm{m}'\mathrm{r}^2} \\
&+ \frac{1}{m_0}\left(\frac{\mathrm{C} - \mathrm{G}}{\mathrm{r}}y_1(\Lambda, \mathrm{G}, \ell, \mathrm{g}) - \mathrm{R}y_2(\Lambda, \mathrm{G}, \ell, \mathrm{g})\right)
\end{aligned}
\tag{61}
$$

where $y_1(\Lambda, \mathrm{G}, \ell, \mathrm{g})$, $y_2(\Lambda, \mathrm{G}, \ell, \mathrm{g})$ are the expressions of the coordinates of the vector $\overline{\mathbf{y}}$ defined in Equation (69) below in terms of $\mathcal{K}$–coordinates, given by

$$
\begin{aligned}
y_1(\Lambda, \mathrm{G}, \ell, \mathrm{g}) \ &= \ \frac{\mathrm{m}^2\mathcal{M}}{\Lambda(1 - \mathrm{e}(\Lambda, \mathrm{G})\cos\xi(\Lambda, \mathrm{G}, \ell))}\Big(-\sin\mathrm{g}\sin\xi(\Lambda, \mathrm{G}, \ell) \\
&\quad + \frac{\mathrm{G}}{\Lambda}\cos\mathrm{g}\cos\xi(\Lambda, \mathrm{G}, \ell)\Big) \\
y_2(\Lambda, \mathrm{G}, \ell, \mathrm{g}) \ &= \ \frac{\mathrm{m}^2\mathcal{M}}{\Lambda(1 - \mathrm{e}(\Lambda, \mathrm{G})\cos\xi(\Lambda, \mathrm{G}, \ell))}\Big(\cos\mathrm{g}\sin\xi(\Lambda, \mathrm{G}, \ell) \\
&\quad + \frac{\mathrm{G}}{\Lambda}\sin\mathrm{g}\cos\xi(\Lambda, \mathrm{G}, \ell)\Big)
\end{aligned}
$$

and the remaining symbols as in (27)–(28).

As mentioned, we defer a rigorous analysis of the Hamiltonian (61) to a forthcoming paper. Here, we limit to report the results of a numerical experiment.

The experiment has been conducted on the Hamiltonian (61). The initial datum has been chosen as follows:

$$
\begin{aligned}
&\mathrm{R} = 7.071067E - 005 \ , \quad \Lambda = 2.236067E - 002 \ , \quad \mathrm{G} = 1.596860E - 002 \\
&\mathrm{r} = 100.000000 \ , \quad \ell = 0.751906 \ , \quad \mathrm{g} = \pi
\end{aligned}
\tag{62}
$$

and the total angular momentum's length $\mathrm{C} = 7.087036$. The projections of the motion in the planes (g, G), (ℓ, Λ), (r, R) are reported in Figures 2, 3 and 4. We interpret the motion of the couple (G, g) in Figure 2 as a continuation of the

motions of this couple in the last panel of Figure 1, by the following heuristic considerations. We rewrite the Hamiltonian (61) as

$$\mathrm{H}(\mathrm{R}, \Lambda, \mathrm{G}, \mathrm{r}, \ell, \mathrm{g}) = \mathrm{J}(\Lambda, \mathrm{G}, \mathrm{r}, \ell, \mathrm{g}) + \mathrm{K}(\mathrm{R}, \mathrm{r}) + f(\mathrm{R}, \Lambda, \mathrm{G}, \mathrm{r}, \ell, \mathrm{g}) \qquad (63)$$

with J as in (26) with $\Theta = 0$ and

$$
\begin{aligned}
\mathrm{K}(\mathrm{R}, \mathrm{r}) \quad &:= \quad \frac{\mathrm{R}^2}{2\mathrm{m}'} + \frac{\mathrm{C}^2}{2\mathrm{m}'\mathrm{r}^2} - \frac{\mathrm{m}'\mathcal{M}'}{\mathrm{r}} \\
f(\mathrm{R}, \Lambda, \mathrm{G}, \mathrm{r}, \ell, \mathrm{g}) \quad &:= \quad \frac{-2\mathrm{CG} + \mathrm{G}^2}{2\mathrm{mr}^2} + \frac{1}{m_0}\Big(\frac{\mathrm{C} - \mathrm{G}}{\mathrm{r}} y_1(\Lambda, \mathrm{G}, \ell, \mathrm{g}) \\
& \qquad\qquad -\mathrm{R}y_2(\Lambda, \mathrm{G}, \ell, \mathrm{g})\Big) \qquad\qquad\qquad (64)
\end{aligned}
$$

The large gap between the initial size of $a = 10^{-3}$ in (28) and $\mathrm{r} = 10^2$ makes the full Hamiltonian (63) to be well represented by its ℓ–average

$$\overline{\mathrm{H}}(\mathrm{R}, \Lambda, \mathrm{G}, \mathrm{r}, \ell, \mathrm{g}) \quad = \quad -\frac{\mathrm{m}^2\mathcal{M}^2}{2\Lambda^2} + \mathrm{U}(\Lambda, \mathrm{G}, \mathrm{r}, \mathrm{g}) + \mathrm{K}(\mathrm{R}, \mathrm{r}) + \frac{-2\mathrm{CG} + \mathrm{G}^2}{2\mathrm{mr}^2}$$

having used the last term in f in (64) has zero average. When the last term is neglected, $\overline{\mathrm{H}}$ reduces to

$$\mathrm{h}(\Lambda, \mathrm{G}, \mathrm{r}, \mathrm{g}) = -\frac{\mathrm{m}^2\mathcal{M}^2}{2\Lambda^2} + \mathrm{U}(\Lambda, \mathrm{G}, \mathrm{r}, \mathrm{g}) + \mathrm{K}(\mathrm{R}, \mathrm{r}) \ .$$

This Hamiltonian is not integrable. However, $\mathrm{K}(\mathrm{R}, \mathrm{r})$ has an equilibrium at $(\mathrm{R}, \mathrm{r}) = \left(0, \frac{\mathrm{C}^2}{\mathrm{m}'^2\mathcal{M}'}\right)$. If C is sufficiently large, such equilibrium is an "approximate" equilibrium to h, so the motions of h are approximately decoupled, and are obtained combining small oscillations (generated by K) of the couple (R, r) about the equilibrium with the motions (generated by E_0) for (G, g) depicted in the last panel of Figure 1 (since $\delta = 10^5$), with Keplerian motions for the couple (Λ, ℓ) generated by the Keplerian part in (59). Observe that the initial values of R and r in (62) are very close to the ones at the equilibrium (with the above choices of C and of the masses, $\frac{\mathrm{C}^2}{\mathrm{m}'^2\mathcal{M}'} = 100.452159$), so we expect that this picture of the motion is preserved in H for a long time, at least on a positive measure set of initial data. However, a rigorous treatment of these arguments is a bit delicate for the following reasons. A first difficulty is that the unperturbed motions of G are very close to $\mathrm{G} = 0$. This occurrence makes the Hamiltonian singular, even though the unperturbed part is not so. At this singularity we ascribe the divergences of Λ in Figure 3. Another difficulty is that, as r is large compared to a, besides the perturbative term, U is also small. Therefore, in order that the motions of (G, g) are in fact "led" by U, a careful balance between U and $\overline{f} := \frac{-2\mathrm{CG} + \mathrm{G}^2}{2\mathrm{mr}^2}$ is required. We should not miss to mention another delicate question. When $0 < \delta < 2$ (the first and the second panel in Figure 1), the previous heuristic arguments do not seem to apply. Moreover, as outlined in Section 4.2, the separatrix $\mathcal{S}_0$ appearing in such figures is a singular manifold for U. These considerations convinced the author that a rigorous treatment of a possible application of the results of Sections 2–4 to the three–body problem would require such an accurate analysis to exceed the purposes of the present note.

6. **Conclusion.** We propose an alternative approach to the analysis of the two–centre problem. We dismiss the separability property [2] that one usually gains using the ellipsoidal coordinates. Rather, we regard the two–centre Hamiltonian J in (4) as a small perturbation of the Kepler problem. This is possible when the

primaries are very far or their masses are sensitively different. We analyse the Hamiltonian of the problem using a special set of canonical coordinates, denoted as $\mathcal{K}$, and defined in Section 2, in terms of which the Hamiltonian has two degrees of freedom. This is because $\mathcal{K}$ includes, among its coordinates, all the first integrals of J but one. The lack of separation is compensated by the fact that, as proved in [15], the associated secular Hamiltonian in *renormalizably integrable*, meaning that it can be expressed as a function of a "normalising", much simpler, function. The phase portrait of such normalising function can be studied exactly, at least in the case of the planar problem, in correspondence of any value of the ratio $\delta = \mathrm{r}/a$, where $\mathrm{r} = \|\mathbf{x}'\|$ and a is the semi–major axis of the instantaneous ellipse generated by the Keplerian part of J (Section 4.1; Figure 1). Such phase portrait shows, for any value of δ, the existence of small oscillations of the perihelion of the instantaneous ellipse accompanied by a periodic change of the shape of the ellipse that have large eccentricity and, at every period, squeezes to a segment while the body changes its direction on it. We call such motions *perihelion librations*. Moreover, when $0 < \delta < 2$, the phase portrait includes a saddle equilibrium point and a separatrix $\mathcal{S}_0$ through it. It is also possible to compute, exactly, the motion on $\mathcal{S}_0$ (Section 4.2) as well as at least a first order approximation of the action–angle coordinates in the case that δ is very large (Section 4.3). In Section 5 we conjecture it is possible to use the perturbative approach proposed in the paper in order to prove, in the classical three–body problem, the existence motions including perihelion librations, at least in the case that δ is very large. We propose one numerical experiment and a heuristic argument that seem to evidence the conjecture (Figures 2, 3 and 4).

Appendix A. Explicit formulae of the $\mathcal{K}$–map. Here we provide the analytical expression of the map

$$\phi: \quad \mathcal{K} = \Big(\mathrm{Z}, \mathrm{C}, \Theta, \mathrm{G}, \Lambda, \mathrm{R}, \zeta, g, \vartheta, \mathrm{g}, \ell, \mathrm{r} \Big) \to (\underline{\mathbf{y}}, \underline{\mathbf{x}}) = (\mathbf{y}', \mathbf{y}, \mathbf{x}', \mathbf{x}) \ . \tag{65}$$

which is needed to write J and E in terms of $\mathcal{K}$.

Let

$$i = \cos^{-1}\left(\frac{\mathrm{Z}}{\mathrm{C}}\right) \ , \quad i_1 = \cos^{-1}\left(\frac{\Theta}{\mathrm{C}}\right) \ , \quad i_2 = \cos^{-1}\left(\frac{\Theta}{\mathrm{G}}\right) \tag{66}$$
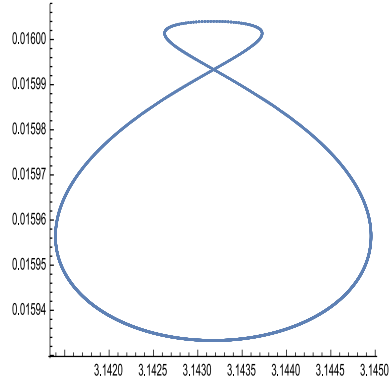
and

$$\mathcal{R}_1(\alpha) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \quad \mathcal{R}_3(\alpha) := \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{67}$$

Using the definitions in (23) and the observation that, if $\mathrm{F} \to^{(\mathrm{Y},\mathrm{X},x)} \mathrm{F}'$, the transformation of coordinates which relates the coordinates $\mathbf{x}'$ relatively to F' to the coordinates $\mathbf{x}$ relatively to F is

$$\mathbf{x} = \mathcal{R}_3(x)\mathcal{R}_1(\iota)\mathbf{x}'$$

where $\mathcal{R}_1$, $\mathcal{R}_3$ are as in (67), while $\iota := \cos^{-1}\frac{\mathrm{Y}}{\mathrm{X}}$, for the map (65) we find the following analytical expression

FIGURE 2. Projection of the motion in the plane (g, G).
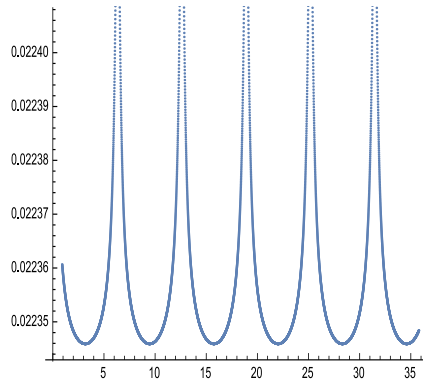
FIGURE 3. Projection of the motion in the plane (ℓ, Λ).
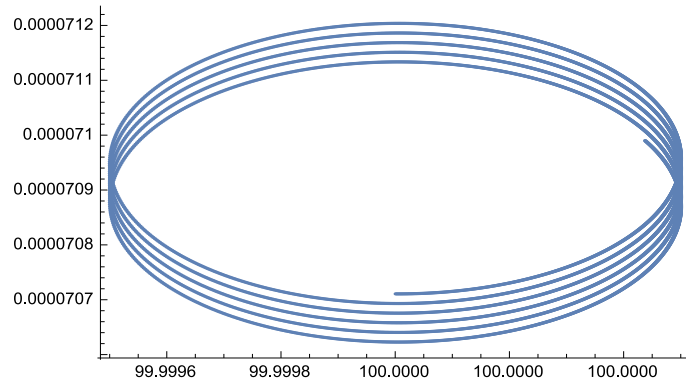
FIGURE 4. Projection of the motion in the plane (r, R).

$$\phi : \begin{cases} \mathbf{x} = \mathcal{R}_3(\zeta)\mathcal{R}_1(i)\mathcal{R}_3(g)\mathcal{R}_1(i_1)\mathcal{R}_3(\vartheta)\mathcal{R}_1(i_2)\overline{\mathbf{x}}(\Lambda, \mathrm{G}, \ell, \mathrm{g}) \\[2mm] \mathbf{y} = \mathcal{R}_3(\zeta)\mathcal{R}_1(i)\mathcal{R}_3(g)\mathcal{R}_1(i_1)\mathcal{R}_3(\vartheta)\mathcal{R}_1(i_2)\overline{\mathbf{y}}(\Lambda, \mathrm{G}, \ell, \mathrm{g}) \\[2mm] \mathbf{x}' = \mathrm{r}\mathcal{R}_3(\zeta)\mathcal{R}_1(i)\mathcal{R}_3(g)\mathcal{R}_1(i_1)\mathbf{k} \\[2mm] \mathbf{y}' = \frac{\mathrm{R}'}{\mathrm{r}'}\mathbf{x}' + \frac{1}{\mathrm{r}'^2}\mathbf{M}' \times \mathbf{x}' \end{cases} \qquad (68)$$

with

$$\overline{\mathbf{x}} = \frac{\Lambda^2}{\mathrm{m}^2\mathcal{M}}\mathcal{R}_3(\mathrm{g} - \pi/2) \begin{pmatrix} \cos\xi(\Lambda, \mathrm{G}, \ell) - \mathrm{e} \\ \frac{\mathrm{G}}{\Lambda}\sin\xi(\Lambda, \mathrm{G}, \ell) \\ 0 \end{pmatrix}$$

$$\overline{\mathbf{y}} = \frac{\mathrm{m}^2\mathcal{M}}{\Lambda(1 - \mathrm{e}(\Lambda, \mathrm{G})\cos\xi(\Lambda, \mathrm{G}, \ell))}\mathcal{R}_3(\mathrm{g} - \pi/2) \begin{pmatrix} -\sin\xi(\Lambda, \mathrm{G}, \ell) \\ \frac{\mathrm{G}}{\Lambda}\cos\xi(\Lambda, \mathrm{G}, \ell) \\ 0 \end{pmatrix}$$

$$\mathbf{C} = \mathrm{C}\mathcal{R}_3(\zeta)\mathcal{R}_1(i)\mathbf{k}, \quad \mathbf{M} = \mathrm{G}\mathcal{R}_3(\zeta)\mathcal{R}_1(i)\mathcal{R}_3(g)\mathcal{R}_1(i_1)\mathcal{R}_3(\vartheta)\mathcal{R}_1(i_2)\mathbf{k}$$

$$\mathbf{M}' = \mathbf{C} - \mathbf{M} \qquad (69)$$

where $\mathrm{e}(\Lambda, \mathrm{G})$ and $\xi(\Lambda, \mathrm{G}, \ell)$ are as in (27)–(28).

A.1. **The planar case.** The planar motions are obtained setting $\Theta = 0$ and $\vartheta = 0$, π. Indeed, the planar motions correspond to take $\mathbf{C} \parallel \sigma\mathbf{M} = \sigma(\mathbf{C} - \mathbf{M}')$, with $\sigma = \pm 1$, so

$$\Theta = \frac{\mathbf{M} \cdot \mathbf{x}'}{\|\mathbf{x}'\|} = \mathbf{0}.$$

Moreover, from the definitions in (22), we have $\mathbf{i}_2 \parallel (-\sigma\mathbf{i}_3)$, so

$$\vartheta = \alpha_{\mathbf{x}'}(\mathbf{i}_2, \mathbf{i}_3) = \begin{cases} \pi & \text{if} \quad \sigma = +1 \\[2mm] 0 & \text{if} \quad \sigma = -1 . \end{cases}$$

In such cases, the map (65) reduces to

$$\phi_{\mathrm{pl}} : \quad \mathcal{K} = \Big(\mathrm{Z}, \mathrm{C}, \mathrm{G}, \Lambda, \mathrm{R}, \zeta, g, \mathrm{g}, \ell, \mathrm{r}\Big) \to (\underline{\mathbf{y}}, \underline{\mathbf{x}}) = (\mathbf{y}', \mathbf{y}, \mathbf{x}', \mathbf{x}) . \qquad (70)$$

with

$$\begin{cases} \mathbf{x} = \mathcal{R}_3(\zeta)\mathcal{R}_1(i)\mathcal{R}_3(g)(\overline{\mathbf{x}}_1\mathbf{i} + \overline{\mathbf{x}}_2\mathbf{j}) \\[2mm] \mathbf{y} = \mathcal{R}_3(\zeta)\mathcal{R}_1(i)\mathcal{R}_3(g)(\overline{\mathbf{y}}_1\mathbf{i} + \overline{\mathbf{y}}_2\mathbf{j}) \\[2mm] \mathbf{x}' = -\mathrm{r}\mathcal{R}_3(\zeta)\mathcal{R}_1(i)\mathcal{R}_3(g)\mathbf{j} \\[2mm] \mathbf{y}' = -\mathrm{R}\mathcal{R}_3(\zeta)\mathcal{R}_1(i)\mathcal{R}_3(g)\mathbf{j} + \dfrac{\mathrm{C} - \sigma\mathrm{G}}{\mathrm{r}}\mathcal{R}_3(\zeta)\mathcal{R}_1(i)\mathcal{R}_3(g)\mathbf{i} \end{cases} \qquad (71)$$

We call *planar and prograde* the case $\sigma = +1$; while *planar and retrograde* the case $\sigma = -1$.

A.2. **Derivation of the formulae** (26)**.** Using the general formulae in (68), we find

$$\mathbf{x}' \cdot \mathbf{x} \quad = \quad \mathbf{k} \cdot \mathcal{R}_1(i_2)\overline{\mathbf{x}}(\Lambda, \mathrm{G}, \ell, \mathrm{g}) = -\mathrm{ra}\sqrt{1 - \frac{\Theta^2}{\mathrm{G}^2}}\mathrm{p} \tag{72}$$

with p as in (28), and where we have used $\mathcal{R}_3^{\mathrm{t}}(\vartheta)\mathbf{k} = \mathbf{k}$, the relation

$$\sin i_2 = \sqrt{1 - \frac{\Theta^2}{\mathrm{G}^2}}$$

(which is implied by the definition of i_2 in (66)) and the expression for $\overline{\mathbf{x}}$ in (69). Equations (72), (69) and the definition of $\mathrm{r} = \|\mathbf{x}'\|$ then imply that the Euclidean distance between $\mathbf{x}'$ and $\mathbf{x}$ has the expression

$$\|\mathbf{x}' - \mathbf{x}\|^2 = \mathrm{r}^2 + 2\mathrm{ra}\sqrt{1 - \frac{\Theta^2}{\mathrm{G}^2}}\mathrm{p} + a^2\varrho^2 \ . \tag{73}$$

Recall that the eccentricity vector $\mathbf{L}$ in (6) is related to e and $\mathbf{P}$ via $\mathbf{L} = \mathrm{m}^2\mathcal{M}\mathrm{e}\mathbf{P}$. The expression of $\mathbf{P}$ is obtained from the one for $\mathbf{x}$ in (68) taking $\nu = \ell = 0$ and normalizing. Namely,

$$\mathbf{P} = \frac{\overline{\mathbf{x}}}{a\varrho} = \mathcal{R}_3(\zeta)\mathcal{R}_1(i)\mathcal{R}_3(g)\mathcal{R}_1(i_1)\mathcal{R}_3(\vartheta)\mathcal{R}_1(i_2)\overline{\mathbf{P}}$$

with

$$\overline{\mathbf{P}} = \begin{pmatrix} \sin \mathrm{g} \\ -\cos \mathrm{g} \\ 0 \end{pmatrix}$$

Then, analogously to (72), we find, for the inner product $\mathbf{x}' \cdot \mathbf{P}$ the expression

$$\mathbf{x}' \cdot \mathbf{P} = -\mathrm{r}\sqrt{1 - \frac{\Theta^2}{\mathrm{G}^2}}\cos \mathrm{g} \tag{74}$$

Using the formulae in (73), (74), the definition of $\mathrm{G} = \|\mathbf{x} \times \mathbf{y}\|$, we find that the functions J, E, written terms of the coordinates $\mathcal{K}$, are as in (26).

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