On profinite groups with word values covered by nilpotent subgroups

Eloisa Detomi, Marta Morigi, and Pavel Shumyatsky

Abstract. Let $\mathcal{N}$ stand for the class of nilpotent groups or one of its well-known generalizations. For a multilinear commutator word w and a profinite group G we show that $w(G)$ is finite-by-$\mathcal{N}$ if and only if the set of w-values in G is covered by countably many finite-by-$\mathcal{N}$ subgroups. Earlier this was known only in the case where $w = x$ or $w = [x, y]$.

1. Introduction

In recent years profinite groups in which the set of word-values is covered by countably many subgroups with special properties attracted some interest (cf. [**1**]). Here we say that a set is covered by subgroups if it is contained in the set theoretical union of the subgroups. Given a group-word w in n variables and a group G, the verbal subgroup $w(G)$ of G determined by the word w is the subgroup generated by the set consisting of all values $w(g_1, \ldots, g_n)$, where $g_1, \ldots, g_n$ are elements of G. In the present paper we deal with the so called multilinear commutators (otherwise known under the name of outer commutator words). These are words which are obtained by nesting commutators, but using always different variables. For example, the word $[[x_1, x_2], [x_3, x_4, x_5], x_6]$ is a multilinear commutator while the Engel word $[x_1, x_2, x_2, x_2]$ is not.

A profinite group is a topological group that is isomorphic to an inverse limit of finite groups. In the context of profinite groups all the usual concepts of group theory are interpreted topologically. In particular, in a profinite group the verbal subgroup corresponding to the word w is the *closed* subgroup generated by all w-values. More generally, in this paper by a subgroup of a profinite group we always

2000 *Mathematics Subject Classification.* Primary 20E18; Secondary 20F12; 20F14.

Key words and phrases. Profinite groups; Coverings; Commutators.

mean a closed subgroup and by a quotient we mean a quotient over a normal closed subgroup.

In the present article we work with certain generalizations of nilpotent groups. Recall that a group G is locally nilpotent if all finitely generated subgroups of G are nilpotent. Following Shalev [**9**], we say that a group G is strongly locally nilpotent if it belongs to a locally nilpotent variety of groups. This means that, for some function f and for all positive integers d, every d-generated subgroup of G is nilpotent of class at most $f(d)$. According to Wilson and Zelmanov a profinite group is locally nilpotent if and only if it is Engel [**13**]. Such a group is strongly locally nilpotent if it is n-Engel for some positive n (see [**12**] ot [**14**]).

Throughout the present article $\mathcal{N}$ stands for one of the following classes of groups.

- The class of nilpotent groups;
- The class of pronilpotent groups;
- The class of locally nilpotent groups;
- The class of strongly locally nilpotent groups.

The class of groups G having a finite normal subgroup D such that the quotient G/D belongs to $\mathcal{N}$ is denoted by $\mathcal{FN}$.

The main result of the present article can be stated as follows.

THEOREM 1.1. *Let w be a multilinear commutator word and let G be a profinite group. The verbal subgroup $w(G)$ belongs to the class $\mathcal{FN}$ if and only if the set of w-values in G is covered by countably many $\mathcal{FN}$-subgroups.*

This generalizes the results of [**11**] and [**5**] where similar conclusions were derived in the case where w is either the word x or the word $[x, y]$. Since a profinite group G is in the class $\mathcal{FN}$ if and only if G is covered by finitely many $\mathcal{N}$-subgroups (see [**11, 5**]), we obtain the following corollary.

COROLLARY 1.2. *Let w be a multilinear commutator word and let G be a profinite group. The following statements are equivalent.*

1. *The verbal subgroup $w(G)$ belongs to the class $\mathcal{FN}$;*
2. *The set of w-values in G is covered by countably many $\mathcal{N}$-subgroups;*
3. *The set of w-values in G is covered by finitely many $\mathcal{N}$-subgroups.*

Theorem 1.1 and Corollary 1.2 are in parallel with results obtained earlier in [**3, 4**] that say that $w(G)$ is locally finite, or has finite rank, if and only if the set of w-values in G is contained in a union of countably many subgroups with the respective property. Moreover, $w(G)$ is finite if and only if the set of w-values in G is countable. In fact, the combinatorial techniques for handling multilinear commutator words developed in [**3, 4**] plays an important role in the proof of Theorem 1.1. Unsurprisingly, the proof of Theorem 1.1 is much more complicated than the proofs in the case when $w = [x, y]$ in [**11**] and [**5**].

2. Preliminary results

In any group a product of finitely many normal $\mathcal{N}$-subgroups is again a normal $\mathcal{N}$-subgroup. This is well-known when $\mathcal{N}$ is the class of nilpotent, pronilpotent or locally nilpotent groups, while for the class of strongly locally nilpotent groups it is Lemma 2.4 in [**5**]. The following lemma extends this observation to products of normal $\mathcal{FN}$-subgroups.

LEMMA 2.1. [**5**, Lemma 2.5] *In any group a product of finitely many normal $\mathcal{FN}$-subgroups is again in $\mathcal{FN}$.*

If $\mathcal{C}$ is a class of groups, a virtually-$\mathcal{C}$ group is a group with a normal $\mathcal{C}$-subgroup of finite index.

LEMMA 2.2. *In any group a product of finitely many normal virtually-$\mathcal{FN}$ subgroups is again a virtually-$\mathcal{FN}$ subgroup.*

PROOF. It is sufficient to prove the lemma for a product of two normal virtually-$\mathcal{FN}$ subgroups N_1 and N_2 of a group G. By Lemma 2.1, since N_i is virtually-$\mathcal{FN}$, there exists a unique maximal normal $\mathcal{FN}$-subgroup R_i of N_i for each $i = 1, 2$. Then R_1, R_2 are normal in G and so R_1R_2 is a normal $\mathcal{FN}$-subgroup of finite index in N_1N_2. □

If A is a subset of a group G, we write $\langle A\rangle$ for the subgroup generated by A. If B is another subset, we denote by A^B the set $\{a^b \mid a \in A \text{ and } b \in B\}$.

LEMMA 2.3. [**5**, Lemma 2.6] *Let L be a subgroup of a profinite group G such that the normalizer $N_G(L)$ is open.*

1. *If L is finite, then $\langle L^G\rangle$ is finite.*
2. *If L is in $\mathcal{FN}$ and H is a normal open subgroup of G contained in $N_G(L)$, then $\langle (L \cap H)^G\rangle$ is in $\mathcal{FN}$.*

The next two lemmas generalize [**2**, Lemma 2.1] and [**2**, Lemma 2.2] to the case of multilinear commutator words. Recall that the weight of a multilinear commutator w is just the number of different variables involved in w.

Lemma 2.4. *Let w be a multilinear commutator word of weight n. Assume that H is a normal subgroup of a group G. Let $g_1, \dots, g_n \in G$, $h \in H$ and fix $s \in \{1, \dots, n\}$. Then for every $j = 1, \dots, n$ there exist $y_j \in g_j^H$ such that*

$$w(g_1, \dots, g_{s-1}, g_s h, g_{s+1}, \dots, g_n) =$$
$$w(y_1, \dots, y_n) w(g_1, \dots, g_{s-1}, h, g_{s+1}, \dots, g_n).$$

Proof. We argue by induction on n. If $n = 1$, then the result is self-evident.

Assume that $n \geq 2$. Then there exist two multilinear commutator words w_1 and w_2 such that $w = [w_1, w_2]$. Let l be the weight of w_1 and assume that $s \leq l$. By induction, for every $j \leq l$ there exist $y_j \in g_j^H$ such that

$$w_1(g_1, \dots, g_{s-1}, g_s h, g_{s+1}, \dots, g_l) =$$
$$w_1(y_1, \dots, y_l) w_1(g_1, \dots, g_{s-1}, h, g_{s+1}, \dots, g_l).$$

Note that $\overline{h} = w_1(g_1, \dots, g_{s-1}, h, g_{s+1}, \dots, g_l) \in H$.

Using the standard commutator identities we compute

$$w(g_1, \dots, g_{s-1}, g_s h, g_{s+1}, \dots, g_n) = [w_1(y_1, \dots, y_l)\overline{h}, w_2(g_{l+1}, \dots, g_n)] =$$
$$[w_1(y_1, \dots, y_l), w_2(g_{l+1}, \dots, g_n)]^{\overline{h}} [\overline{h}, w_2(g_{l+1}, \dots, g_n)] =$$
$$[w_1(y_1^{\overline{h}}, \dots, y_l^{\overline{h}}), w_2(g_{l+1}^{\overline{h}}, \dots, g_n^{\overline{h}})][\overline{h}, w_2(g_{l+1}, \dots, g_n)],$$

and we obtain the desired result. The case $s > l$ is similar. □

Let $w = w(x_1, \dots, x_n)$ be a multilinear commutator word. If $A_1, \dots, A_n$ are subsets of a group G, we write $w(A_1, \dots, A_n)$ to denote the subgroup generated by the set of all w-values $w(a_1, \dots, a_n)$ with $a_i \in A_i$.

Let I be a subset of $\{1, \dots, n\}$. Suppose that we have a family $A_{i_1}, \dots, A_{i_s}$ of subsets of G with indices running over I and another family $B_{l_1}, \dots, B_{l_t}$ of subsets with indices running over $\{1, \dots, n\} \setminus I$. We write

$$w_I(A_i; B_l)$$

for $w(X_1, \dots, X_n)$, where $X_k = A_k$ if $k \in I$, and $X_k = B_k$ otherwise.

Lemma 2.5. *Let G be a group and let w be a multilinear commutator of weight n. Assume that $H, A_1, \dots, A_n$ are normal subgroups of G such that for some elements $a_i \in A_i$, the equality*

$$w(a_1(H \cap A_1), \dots, a_n(H \cap A_n)) = 1$$

holds. Then for any subset I of $\{1, \dots, n\}$ we have

$$w_I(H \cap A_i; a_l(H \cap A_l)) = 1.$$

PROOF. The proof is by induction on the size of I, the case $I = \emptyset$ being trivial. Let I be a non-empty subset of $\{1, \dots, n\}$ and fix $s \in I$. By induction applied to $I^* = I \setminus \{s\}$ we have

$$w_{I^*}(H \cap A_i; a_l(H \cap A_l)) = 1. \tag{1}$$

Let $\overline{w} = w(g_1, \dots, g_n)$, where $g_i \in H \cap A_i$ if $i \in I$ and $g_i \in a_i(H \cap A_i)$ otherwise. It suffices to prove that $\overline{w} = 1$.

We apply Lemma 2.4 to the element obtained by replacing in $\overline{w}$ the entry g_s with $a_s g_s$: since $g_s \in H$, there exist elements $y_i \in g_i^H \subseteq g_i(H \cap A_i)$ for every $i \neq s$ and $y_s \in a_s^H \subseteq a_s(H \cap A_s)$ such that

$$\begin{aligned} w(g_1, \dots, g_{s-1}, a_s g_s, g_{s+1}, \dots, g_n) &= w(y_1, \dots, y_n) w(g_1, \dots, g_n) \\ &= w(y_1, \dots, y_n)\overline{w}. \end{aligned}$$

Thus, $g_i, y_i \in H \cap A_i$ when $i \in I^*$ and $g_l, y_l \in a_l(H \cap A_l)$ when $l \notin I$. Since $y_s \in a_s(H \cap A_s)$, by assumption (1) we have

$$w(g_1, \dots, g_{s-1}, a_s g_s, g_{s+1}, \dots, g_n) \in w_{I^*}(H \cap A_i; a_l(H \cap A_l)) = 1,$$

and

$$w(y_1, \dots, y_n) \in w_{I^*}(H \cap A_i; a_l(H \cap A_l)) = 1.$$

Therefore $\overline{w} = 1$. □

COROLLARY 2.6. *Let G be a group and let w be a multilinear commutator of weight n. Assume that H is a normal subgroup of G such that for some elements $a_1, \dots, a_n \in G$ the equality $w(a_1 H, \dots, a_n H) = 1$ holds. Then $w(H) = 1$.*

We recall that an element of a group G is called an FC-element if it has only finitely many conjugates in G.

LEMMA 2.7. *Let $G = \langle H, a_1, \dots, a_s \rangle$ be a profinite group, where H is an open abelian normal subgroup and $a_1, \dots, a_s$ are FC-elements. Then G' is finite.*

PROOF. Note that $H \cap C_G(a_1) \cap \dots \cap C_G(a_s)$ is contained in the center of G and has finite index in G. So the result follows from Schur's theorem [**8**, 10.1.4]. □

3. The case of derived words

An important family of multilinear commutator words is formed by so-called derived words δ_k, on 2^k variables, defined recursively by

$$\delta_0 = x_1, \qquad \delta_k = [\delta_{k-1}(x_1, \dots, x_{2^{k-1}}), \delta_{k-1}(x_{2^{k-1}+1}, \dots, x_{2^k})].$$

Of course $\delta_k(G) = G^{(k)}$ is the k-th term of the derived series of G.

In the present section we deal with groups in which δ_k-values are covered by countably many $\mathcal{FN}$-subgroups. We also develop techniques that will be helpful in handling the general case of Theorem 1.1 in the subsequent section.

LEMMA 3.1. [**5**, Lemma 3.2] *Let $G = N\langle b\rangle$ be a profinite group where N is an open normal $\mathcal{FN}$-subgroup of G. Assume that there exists an open normal subgroup R of G such that $R \le N$ and $R\langle b\rangle$ is in $\mathcal{FN}$. Then G is in $\mathcal{FN}$.*

In the sequel we will often use without mentioning the fact that if w is a word, the conjugate of a w-value is again a w-value. Next lemma holds for any word w.

LEMMA 3.2. *Let w be a word and let G be a profinite group in which the set of w-values is covered by countably many $\mathcal{FN}$-subgroups G_i. Suppose that x is a w-value and N is a normal $\mathcal{FN}$-subgroup of G such that N is open in $\langle N, x\rangle$. Then the subgroup $\langle N, x\rangle$ is in $\mathcal{FN}$.*

PROOF. Let X be the set of all w-values contained in the coset xN. Of course X is non-empty. Obviously, the set X is closed and therefore compact. It is clear that X is covered by the (closed) subsets $X \cap G_i$. By the Baire category theorem (cf [**7**, p. 200]), at least one of these subsets has non-empty interior. Hence, there exist an open normal subgroup T of G, an element $b \in X$ and an index j such that all w-values contained in $X \cap bT$ belong to G_j. Let $R = T \cap N$. Notice that for every $r \in R$, the conjugate b^r is a w-value and $b^r = b[b, r] \in bR$. Since all w-values contained in bR belong to G_j, it follows that $\langle b^R\rangle \le G_j$. So $\langle b^R\rangle$ is in $\mathcal{FN}$. We observe that $\langle b, R\rangle = \langle b^R\rangle R$ is a product of two normal $\mathcal{FN}$-subgroups so it is in $\mathcal{FN}$ by Lemma 2.1. Since R is open in N, in view of Lemma 3.1 we conclude that $\langle N, b\rangle$ is in $\mathcal{FN}$. As $\langle N, b\rangle = \langle N, x\rangle$, the lemma follows. □

Throughout the rest of the article we will work under the following hypothesis.

HYPOTHESIS 3.3. *Let w be a multilinear commutator of weight n and let G be a profinite group in which the set of w-values of G is contained in a union of countably many $\mathcal{FN}$-subgroups G_i.*

LEMMA 3.4. *Assume Hypothesis 3.3. Then G contains an open normal subgroup H such that $w(H)$ is virtually-$\mathcal{FN}$.*

PROOF. For each positive integer i consider the set

$$S_i = \{(g_1, \dots, g_n) \in G \times \dots \times G \mid w(g_1, \dots, g_n) \in G_i\}.$$

Note that the sets S_i are closed in $G \times \cdots \times G$ and cover the whole group $G \times \cdots \times G$. By the Baire category theorem at least one of these sets has non-empty interior. Hence, there exist an open normal subgroup H of G, elements $a_1, \ldots a_n \in G$, and an integer j such that $w(a_1H, \ldots, a_nH) \subseteq G_j$.

Let

$$K = w(a_1H, \ldots, a_nH).$$

Note that $K \leq G_j$ and H normalizes K. Since G_j is in $\mathcal{FN}$, so is K. Let $D = K \cap H$. By Lemma 2.3, $\langle D^G \rangle$ is in $\mathcal{FN}$.

Note that $\langle D^G \rangle$ has finite index in $\langle K^G \rangle$. Indeed, suppose that $D = 1$. In this case K is a finite subgroup normalized by H and thus $\langle K^G \rangle$ is finite. Hence $\langle D^G \rangle$ has finite index in $\langle K^G \rangle$. It follows that $\langle K^G \rangle$ is virtually-$\mathcal{FN}$. The quotient $\bar{G} = G/\langle K^G \rangle$ satisfies the hypothesis of Corollary 2.6, whence $w(\bar{H}) = 1$. It follows that $w(H) \leq \langle K^G \rangle$ and hence it is virtually-$\mathcal{FN}$. □

Lemma 3.5. *Assume Hypothesis 3.3 and let $a \in G$ be a w-value. Then there exists a normal open subgroup H_a of G such that $[H_a, a]$ is in $\mathcal{FN}$.*

Proof. For each positive integer i let

$$S_i = \{x \in G \mid a^x \in G_i\}.$$

Note that the sets S_i are closed in G and cover the whole group G. By the Baire category theorem at least one of these sets has non-empty interior. Hence, there exist an open normal subgroup H of G, an element $b \in G$, and an integer j such that $bH \leq S_j$, i.e. $a^{bh} \in G_j$ for any $h \in H$. Thus $\langle a^{bH} \rangle \leq G_j$. Since $[a^b, H] \leq \langle a^{bH} \rangle$, we conclude that $[a, H^{b^{-1}}] \leq G_j^{b^{-1}}$, and the result follows. □

Proposition 3.6. *Assume Hypothesis 3.3 with $w = \delta_i$. Then $G^{(2i)}$ is virtually-$\mathcal{FN}$.*

Proof. By Lemma 3.4 there exists an open normal subgroup H such that $H^{(i)}$ is virtually-$\mathcal{FN}$. Let $K = G^{(i)}$, $L = K \cap H$. Note that L is open in K. Choose a finite set of δ_i-values $a_1, \ldots, a_s$ such that $K = \langle L, a_1, \ldots, a_s \rangle$ and let $H_{a_1}, \ldots, H_{a_s}$ be normal open subgroups of G such that $[H_{a_j}, a_j]$ is in $\mathcal{FN}$ for every j (see Lemma 3.5). Note that for each j the subgroup $[H_{a_j}, a_j]$ is a normal subgroup of H_{a_j} so, by Lemma 2.3, $\langle [H_{a_j}, a_j]^G \rangle$ is in $\mathcal{FN}$. Let $N_1 \leq G^{(i)}$ be the subgroup generated by $L^{(i)}$ and the subgroups $\langle [H_{a_j}, a_j]^G \rangle$ for $j = 1, \ldots, s$. Note that N_1 is virtually-$\mathcal{FN}$ by Lemma 2.2. The images of $a_1, \ldots, a_s$ in the quotient G/N_1 are FC-elements while the image of L in G/L' is abelian.

Therefore by Lemma 2.7 the group $KN_1/L'N_1$ has finite derived group. In other words $L'N_1$ has finite index in $K'N_1$. In particular there exist finitely many δ_i-values $b_1, \dots, b_t$ such that $K'N_1 = \langle L', b_1, \dots, b_t, N_1\rangle$.

As above, there exist normal open subgroups $H_{b_1}, \dots, H_{b_t}$ of G such that $\langle [H_{b_j}, b_j]^G\rangle$ is in $\mathcal{FN}$ for every j. Let N_2 be the subgroup generated by N_1 and the subgroups $\langle [H_{b_j}, b_j]^G\rangle$ for $j = 1, \dots, t$. Note that N_2 is virtually-$\mathcal{FN}$ by Lemma 2.2. Again, $b_1N_2, \dots, b_tN_2$ are FC-elements in G/N_2 and arguing as before we obtain that $L^{(2)}N_2$ has finite index in $K^{(2)}N_2$. By iterating this argument we get that $L^{(i)}N_i$ has finite index in $K^{(i)}N_i$ for some normal virtually-$\mathcal{FN}$ subgroup N_i, so $L^{(i)}(K^{(i)} \cap N_i)$ has finite index in $K^{(i)} = G^{(2i)}$. As $L^{(i)} \leq H^{(i)}$ is virtually-$\mathcal{FN}$ it follows that $G^{(2i)}$ is virtually-$\mathcal{FN}$, as desired. □

Proposition 3.7. *Assume Hypothesis 3.3 with $w = \delta_k$. Suppose that $G^{(k)}$ is virtually-$\mathcal{FN}$. Then $G^{(k)}$ is in $\mathcal{FN}$.*

Proof. Let N be an open characteristic $\mathcal{FN}$-subgroup of $G^{(k)}$ and let X be the set of δ_k-values in G. Lemma 3.2 tells us that for each $x \in X$ the subgroup $\langle N, x\rangle$ is in $\mathcal{FN}$. Let D_x be the (unique) minimal characteristic finite subgroup of $\langle N, x\rangle$ such that $\langle N, x\rangle/D_x$ is in $\mathcal{N}$. Since N is open in $G^{(k)}$, it follows that there are only finitely many subgroups of the form $\langle N, x\rangle$, where $x \in X$. Therefore D_x has only finitely many conjugates and so the normal closure of D_x in G is finite. It follows that $D = \langle D_x,\ x \in X\rangle$ is a finite normal subgroup of G. So we pass to the quotient G/D and we can assume that $\langle N, x\rangle$ is in $\mathcal{N}$ for every $x \in X$.

Let π be the set of primes dividing the order of $G^{(k)}/N$.

Suppose first that N is a pro-π' group. Let S be the set of all π-elements contained in procyclic subgroups generated by elements from X. Then $G^{(k)} = N\langle S\rangle$. As $\langle N, x\rangle$ is in $\mathcal{N}$ for every $x \in X$, it follows that N centralizes $\langle S\rangle$. Therefore the center of $\langle S\rangle$ has finite index in $\langle S\rangle$. So by Schur's theorem the derived group $\langle S\rangle'$ is finite. As $\langle S\rangle$ is characteristic in G, we can pass to the quotient $G/\langle S\rangle'$ and we may assume that $\langle S\rangle$ is abelian. Now $G^{(k)}$ is the product of two normal $\mathcal{N}$-subgroups, so it is in $\mathcal{N}$. This concludes the proof in the case when N is a pro-π' group.

Now suppose that N is a pro-p group for some prime $p \in \pi$. Let S_1 be the set of all p'-elements contained in procyclic subgroups generated by elements from X. Again, N centralizes S_1, so $\langle S_1\rangle'$ is finite. As above, we can assume that $\langle S_1\rangle$ is abelian. Therefore $\langle S_1\rangle$ is a p'-subgroup of $G^{(k)}$. Since $G^{(k)}$ is virtually pro-p, it follows that $\langle S_1\rangle$ is finite, so by passing to the quotient $G/\langle S_1\rangle$ we can assume that all δ_k-values of G are p-elements. Using a profinite version of Lemma 3.1

in [**10**] we obtain that $G^{(k)}$ is a pro-p group. Now we will prove that $G^{(k)}$ is in $\mathcal{N}$ by induction on $|G^{(k)} : N|$. Since $G^{(k)}/N$ is nilpotent, there is an index i such that $\gamma_i(G^{(k)})N/N$ is a nontrivial subgroup of the center of $G^{(k)}/N$ (here, as usual, $\gamma_i(G^{(k)})$ denotes the i-th term of the lower central series of $G^{(k)}$). Notice that if $x_1, \dots, x_i \in X$, then $[x_1, \dots, x_i] \in X$. As $\gamma_i(G^{(k)})$ is generated by γ_i-values whose entries are δ_k-values, there exists a δ_k-value x such that xN is a nontrivial element of the center of $G^{(k)}/N$. Therefore $\langle N, x\rangle$ is a normal $\mathcal{N}$-subgroup of $G^{(k)}/N$. The characteristic closure M of $\langle N, x\rangle$ in G is obviously again in $\mathcal{FN}$ and $|G^{(k)} : M|$ is smaller than $|G^{(k)} : N|$. Then, by induction we conclude that $G^{(k)}$ is in $\mathcal{FN}$.

Let $\pi = \{p_1, \dots, p_s\}$ and let N_π and $N_{p'}$ be the Hall π-subgroup and the Hall p'-subgroup of N, respectively. Since N is pronilpotent, the subgroups N_π, $N_{p'_i}$ are normal in G. We already know that all quotients $G^{(k)}/N_\pi$ and $G^{(k)}/N_{p'_i}$ are in $\mathcal{FN}$. Moreover, $G^{(k)}$ is isomorphic to a subgroup of the direct product

$$G^{(k)}/N_\pi \times G^{(k)}/N_{p'_1} \times \cdots \times G^{(k)}/N_{p'_s},$$

which is the product of finitely many $\mathcal{FN}$-groups. We conclude that $G^{(k)}$ is in $\mathcal{FN}$, as desired. □

Corollary 3.8. *Assume Hypothesis 3.3 with $w = \delta_i$. Then $G^{(2i)}$ is in $\mathcal{FN}$.*

Proof. By Proposition 3.6 the subgroup $G^{(2i)}$ is virtually-$\mathcal{FN}$. Then note that every δ_{2i}-value is in particular a δ_i-value, so the hypotheses of Proposition 3.7 are satisfied when $k = 2i$ and we obtain the desired result. □

4. The general case

In the present section we complete the proof of Theorem 1.1.

Recall the notation introduced in Section 2: whenever I is a subset of $\{1, \dots, n\}$ and $A_{i_1}, \dots, A_{i_s}$ and $B_{l_1}, \dots, B_{l_t}$ are families of subsets of G with indices running over I and $\{1, \dots, n\} \setminus I$, respectively, we write

$$w_I(A_i, B_l)$$

for the subgroup $w(X_1, \dots, X_n)$, where $X_k = A_k$ if $k \in I$, and $X_k = B_k$ otherwise. On the other hand, whenever $a_i \in A_i$ for $i \in I$ and $b_l \in B_l$ for $l \in \{1, \dots, n\} \setminus I$, the symbol $w_I(a_i, b_l)$ stands for the element $w(x_1, \dots, x_n)$, where $x_k = a_k$ if $k \in I$, and $x_k = b_k$ otherwise.

Lemma 4.1. *Let $A_1, \dots, A_n$ and H be normal subgroups of a group G. Let I be a subset of $\{1, \dots, n\}$. Assume that for every proper subset J of I*

$$w_J(A_i; H \cap A_l) = 1.$$

Suppose we are given elements $g_i \in A_i$ with $i \in I$ and elements $h_k \in H \cap A_k$ with $k \in \{1, \dots, n\}$. Then we have

$$w_I(g_i h_i; h_l) = w_I(g_i; h_l).$$

Proof. Let

$$\bar{w} = w_I(g_i h_i; h_l) = w(c_1, \dots, c_n)$$

where $c_i = g_i h_i$ if $i \in I$, and $c_i = h_i$ otherwise.

Fix an index $s \in I$ and let $J = I \setminus \{s\}$. We can write $g_s h_s = \bar{h} g_s$ where $\bar{h} = h_s^{g_s^{-1}} \in H \cap A_s$. Then, by Lemma 2.4,

$$\begin{aligned} \bar{w} &= w(c_1, \dots, c_{s-1}, \bar{h} g_s, c_{s+1}, \dots, c_n) \\ &= w(y_1, \dots, y_n) w(c_1, \dots, c_{s-1}, g_s, c_{s+1}, \dots, c_n) \end{aligned}$$

where $y_s \in \bar{h}^G \le H \cap A_s$ and $y_k \in c_k^G$ for every $k \neq s$. In particular $y_k \in A_k$ if $k \in J$ and $y_k \in H \cap A_k$ if $k \notin I$. Therefore

$$w(y_1, \dots, y_n) \in w_J(A_i; H \cap A_l)$$

and so $w(y_1, \dots, y_n) = 1$ by assumption. Hence

$$\bar{w} = w(c_1, \dots, c_{s-1}, g_s, c_{s+1}, \dots, c_n).$$

By repeating the argument for every $s \in I$, we get the desired conclusion. □

Lemma 4.2. *Assume Hypothesis 3.3. Let T be a normal $\mathcal{FN}$-subgroup of G and let $A_1, \dots, A_n$ be normal subgroups of G such that $w(A_1, \dots, A_n)T/T$ is abelian. Let I be a subset of $\{1, \dots, n\}$ and assume that G has an open normal subgroup H such that*

(**) $w_J(A_i; H \cap A_l) \le T$, *for every proper subset J of I.*

Then, for any given set of elements $\{g_i\}_{i \in I}$, where $g_i \in A_i$, there exist an open normal subgroup U of G, contained in H, and a normal $\mathcal{FN}$-subgroup N of G, containing T, such that

$$w_I(g_i; U \cap A_l) \le N.$$

Proof. Consider the sets

$$S_j = \{(h_1, \dots, h_n) \mid h_k \in H \cap A_k \text{ and } w_I(g_i h_i; h_l) \in G_j\}.$$

Note that the sets S_j are closed in the group $(H \cap A_1) \times \cdots \times (H \cap A_n)$ and cover the whole group. By the Baire category theorem at least one of these sets has non-empty interior. Hence, there exist an integer

r, open subgroups V_k of $H \cap A_k$, and elements $b_k \in H \cap A_k$ for every $k = 1, \ldots, n$ such that

$$w_I(g_i b_i V_i; b_l V_l) \subseteq G_r.$$

Each subgroup V_k is of the form $V_k = U_k \cap H \cap A_k$ where U_k is an open subgroup of G and we can assume that U_k is normal in G. Let $U = U_1 \cap \cdots \cap U_n \cap H$. Note that U is an open normal subgroup of G contained in H. Now let

$$K = w_I(g_i b_i(U \cap A_i); b_l(U \cap A_l)).$$

Then $K \subseteq G_r$ is in $\mathcal{FN}$ and U normalizes K. Let $D = K \cap U$. Since U has finite index in G, by Lemma 2.3, $\langle D^G \rangle$ is in $\mathcal{FN}$. So we can assume that $\langle D^G \rangle \leq T$.

Set $R = \langle K^G \rangle$. Let us examine the quotient $\tilde{G} = G/\langle D^G \rangle$. We see that $\tilde{K} = K\langle D^G \rangle/\langle D^G \rangle$ is a finite subgroup normalized by $\tilde{U}$. Thus $\tilde{R}$ is finite. Note that $T \cap R$ has finite index in R. Moreover, $R/(T \cap R)$ is isomorphic to $RT/T \leq w(A_1, \ldots, A_n)T/T$ which is abelian. As R is generated by w-values, it is the product of finitely many subgroups of the form $\langle T \cap R, x \rangle$, where x is a w-value. The subgroups $\langle T \cap R, x \rangle$ normalize each other and each of them is in $\mathcal{FN}$ by Lemma 3.2. It follows from Lemma 2.1 that R is in $\mathcal{FN}$.

Let $N = RT$. Now in the quotient group G/N the equality

$$w_I(g_i b_i(U \cap A_i); b_l(U \cap A_l)) = 1$$

holds. In view of Lemma 2.5 we deduce that

$$w_I(g_i b_i(U \cap A_i); U \cap A_l) = 1$$

in G/N. By condition (**), given that $T \leq N$, we can apply Lemma 4.1 and we obtain that

$$w_I(g_i; U \cap A_l) = w_I(g_i b_i(U \cap A_i); U \cap A_l) = 1,$$

in G/N, that is,

$$w_I(g_i; U \cap A_l) \leq N,$$

as desired. □

LEMMA 4.3. *Assume Hypothesis 3.3. Let T be a normal $\mathcal{FN}$-subgroup of G and let $A_1, \ldots, A_n$ be normal subgroups of G such that $w(A_1, \ldots, A_n)T/T$ is abelian. Let I be a subset of $\{1, \ldots, n\}$ and assume that G has an open normal subgroup H such that*

(**) $w_J(A_i; H \cap A_l) \leq T$, *for every proper subset J of I.*

Then there exist an open normal subgroup U of G, contained in H, and a normal $\mathcal{FN}$-subgroup N of G, containing T, such that

$$w_I(A_i; U \cap A_l) \leq N.$$

Proof. For each $i \in I$ choose a set R_i of coset representatives of $H \cap A_i$ in A_i. Note that all those sets are finite. Now we apply Lemma 4.2 to each choice of elements $\bar{g} = \{g_i\}_{i \in I}$, with $g_i \in R_i$: let $U_{\bar{g}}$ and $N_{\bar{g}}$ be the normal subgroups of G such that $w_I(g_i; U_{\bar{g}} \cap A_l) \leq N_{\bar{g}}$. Note that there is only a finite number of $U_{\bar{g}}$'s and $N_{\bar{g}}$'s. Then $U = \cap_{\bar{g}} U_{\bar{g}}$ is a normal open subgroup of G contained in H and $N = \prod_{\bar{g}} N_{\bar{g}}$ is a normal $\mathcal{FN}$-subgroup containing T, such that

$$w_I(g_i; U \cap A_l) \leq N$$

for every choice of $g_i \in R_i$. Note that, by condition (**) and Lemma 4.1,

$$w_I(g_i(H \cap A_i); U \cap A_l) = w_I(g_i; U \cap A_l) \leq N.$$

Since $A_i = \cup_{g_i \in R_i} g_i(H \cap A_i)$ for every $i \in I$, we conclude that

$$w_I(A_i; U \cap A_l) = \langle \cup_{\bar{g}} w_I(g_i(H \cap A_i); U \cap A_l) \rangle \leq N,$$

as desired. □

Lemma 4.4. *Assume Hypothesis 3.3. Suppose that there exists an open normal subgroup H of G such that $w(H) \leq T$, where T is a normal $\mathcal{FN}$-subgroup of G. Let $A_1, \dots, A_n$ be normal subgroups of G such that in the quotient group G/T the subgroup $w(A_1, \dots, A_n)T/T$ is abelian. Then $w(A_1, \dots, A_n)$ is in $\mathcal{FN}$.*

Proof. It is enough to prove the following statement: for every proper subgroup I of $\{1, \dots, n\}$, there exist an open normal subgroup U_I of G contained in H and a normal $\mathcal{FN}$-subgroup N_I containing T such that $w_I(A_i; U_I \cap A_l) \leq N_I$.

The proof is by induction on the size k of I. If $k = 0$, then $I = \emptyset$ and

$$w_\emptyset(A_i; H \cap A_i) = w(H \cap A_1, \dots, H \cap A_n) \leq w(H) \leq T.$$

So assume $k > 0$. Let $J_1, \dots J_s$ be all proper subsets of I. By induction, for each $t = 1, \dots, s$ there exist an open normal subgroup U_t of G contained in H and a normal $\mathcal{FN}$-subgroup N_t containing T such that $w_I(A_i; U_t \cap A_l) \leq N_t$.

Let $U = \cap_t U_t$ and $N = \langle N_t, t = 1, \dots, s \rangle$. Then $w_J(A_i; U \cap A_l) \leq N$, for every proper subgroup J of I. Now, we can apply Lemma 4.3 to I. We obtain that there exist an open normal subgroup U_I of G contained in H and a normal $\mathcal{FN}$-subgroup N_I containing T such that $w_I(A_i; U_I \cap A_l) \leq N_I$, as desired. □

We denote by $\mathbf{I}$ the set of all n-tuples $(i_1, \dots, i_n)$, where all entries i_k are non-negative integers. We will view $\mathbf{I}$ as a partially ordered set with the partial order given by the rule that

$$(i_1, \dots, i_n) \leq (j_1, \dots, j_n)$$

if and only if $i_1 \leq j_1, \dots, i_n \leq j_n$.

Given $\mathbf{i} = (i_1, \dots, i_n) \in \mathbf{I}$, we write

$$w(\mathbf{i}) = w(G^{(i_1)}, \dots, G^{(i_n)})$$

for the subgroup generated by the w-values $w(g_1, \dots, g_n)$ with $g_j \in G^{(i_j)}$. Further, let

$$w(\mathbf{i}^+) = \prod w(\mathbf{j}),$$

where the product is taken over all $\mathbf{j} \in \mathbf{I}$ such that $\mathbf{j} > \mathbf{i}$.

LEMMA 4.5. [**3**, Corollary 6] *Let $w = w(x_1, \dots, x_n)$ be a multilinear commutator word and let $\mathbf{i} \in \mathbf{I}$. If $w(\mathbf{i}^+) = 1$, then $w(\mathbf{i})$ is abelian.*

We will need the following well-known result (see for example [**10**, Lemma 4.1]).

LEMMA 4.6. *Let G be a group and let w be a multilinear commutator word on n variables. Then each δ_n-value is a w-value.*

LEMMA 4.7. *Assume Hypothesis 3.3. If $w(G)$ is virtually-$\mathcal{FN}$ then $w(G)$ is in $\mathcal{FN}$.*

PROOF. Assume that $w(G)$ is virtually-$\mathcal{FN}$. It follows from Lemma 2.1 that there exists a maximal open normal $\mathcal{FN}$-subgroup R of $w(G)$. By Lemma 4.6 every δ_n-value in G is a w-value. In view of Corollary 3.8 we deduce that $G^{(2n)}$ is in $\mathcal{FN}$. Hence $G^{(2n)} \leq R$. Since G/R is soluble, there exist only finitely many $\mathbf{i} \in \mathbf{I}$ such that $w(\mathbf{i})R/R \neq 1$.

Assume, by contradiction, that $w(G)R \neq R$ and choose $\mathbf{i} = (i_1, \dots, i_n) \in \mathbf{I}$ such that $w(\mathbf{i})R/R \neq 1$ while $w(\mathbf{j})R/R = 1$ whenever $\mathbf{i} < \mathbf{j}$. By Lemma 4.5, $w(\mathbf{i})R/R$ is abelian. As $w(\mathbf{i})$ is generated by w-values, $w(\mathbf{i})R$ is a product of finitely many normal subgroups of the form $\langle R, x\rangle$, where x is a w-value. Each subgroup $\langle R, x\rangle$ is in $\mathcal{FN}$ by Lemma 3.2. It follows that $w(\mathbf{i})R$ is in $\mathcal{FN}$. Therefore $w(\mathbf{i}) \leq R$, a contradiction. □

We are now ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1

We only need to show that if $w = w(x_1, \dots, x_n)$ is a multilinear commutator word and G is a profinite group in which the set of w-values is covered by countably many $\mathcal{FN}$-subgroups then $w(G)$ is in $\mathcal{FN}$. By Lemma 4.6 every δ_n-value in G is a w-value. In view of Corollary 3.8 we deduce that $G^{(2n)}$ is in $\mathcal{FN}$.

Let H be as in Lemma 3.4. By Lemma 4.7, $w(H)$ is in $\mathcal{FN}$. Let $T = G^{(2n)}w(H)$. Then T is in $\mathcal{FN}$ by Lemma 2.1. Since $G^{(2n)} \leq T$ it follows that G/T is soluble.

There exist only finitely many $\mathbf{i} \in \mathbf{I}$ such that $w(\mathbf{i})T/T \neq 1$. The theorem will be proved by induction on the number of such n-tuples $\mathbf{i}$.

Choose $\mathbf{i} = (i_1, \ldots, i_n) \in \mathbf{I}$ such that $w(\mathbf{i})T/T \neq 1$ while $w(\mathbf{j})T/T = 1$ whenever $\mathbf{i} < \mathbf{j}$. It follows from Lemma 4.5 that $w(\mathbf{i})T/T$ is abelian. Now we apply Lemma 4.4 and we obtain that $w(\mathbf{i})$ is in $\mathcal{FN}$. Let $N = w(\mathbf{i})T$. Then induction on the number of $\mathbf{j} \in \mathbf{I}$ such that $w(\mathbf{j}) \not\leq N$ leads us to the conclusion that $w(G)$ is in $\mathcal{FN}$. □

ACKNOWLEDGMENTS. This research was partially supported by Università di Padova (Progetto di Ricerca di Ateneo:“Invariable generation of groups”). The second author was also supported by GNSAGA (INDAM), and the third author by FAPDF and CNPq.

References

[1] C. Acciarri and P. Shumyatsky, *Coverings of commutators in profinite groups*, Rend. Sem. Mat. Univ. Padova, to appear.

[2] C. Acciarri and P. Shumyatsky, *On profinite groups in which commutators are covered by finitely many subgroups*, Math. Z. **274** (2013), 239–248.

[3] E. Detomi, M. Morigi and P. Shumyatsky, *On countable coverings of word values in profinite groups*, J. Pure Appl. Algebra **219** (2015), 1020–1030.

[4] E. Detomi, M. Morigi and P. Shumyatsky, *On conciseness of words in profinite groups*, J. Pure Appl. Algebra **220** (2016), 3010–3015.

[5] E. Detomi, M. Morigi and P. Shumyatsky, *On groups covered by locally nilpotent subgroups*, Ann. Mat. Pura Appl., doi: 10.1007/s10231-016-0627-y, to appear.

[6] D. Gorenstein, *Finite Groups*, Chelsea Publishing Company, New York, 1980.

[7] J. L. Kelley, *General Topology*, Van Nostrand, Toronto - New York - London, 1955.

[8] D. J .S. Robinson, *A course in the theory of groups*, Second edition, Graduate Texts in Mathematics, 80, Springer-Verlag, New York, 1996.

[9] A. Shalev, *Combinatorial conditions in residually finite groups II*, J. Algebra **157** (1993), 51–62.

[10] P. Shumyatsky, *Verbal subgroups in residually finite groups*, Q.J. Math. **51** (2000), 523–528.

[11] P. Shumyatsky, *On profinite groups with commutators covered by nilpotent subgroups*, Rev. Mat. Iberoam. **32** (2016), 1331–1339, doi: 10.4171/rmi/919.

[12] J. S.Wilson, *Two-generator conditions for residually finite groups*, Bull. London Math. Soc. **23** (1991), 239–248.

[13] J. S. Wilson and E. Zelmanov, *Identities for Lie algebras of pro-p groups*, J. Pure Appl. Algebra **81** (1992), 103–109.

[14] E. Zelmanov, *Some problems in the theory of groups and Lie algebras* (Russian), Mat. Sb. **180** (1989), 159–167; translation in Math. USSR-Sb. **66** (1990), 159–168.

Dipartimento di Matematica, Università di Padova, Via Trieste 63, 35121 Padova, Italy
E-mail address: detomi@math.unipd.it

Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato 5, 40126 Bologna, Italy
E-mail address: marta.morigi@unibo.it

Department of Mathematics, University of Brasilia, Brasilia-DF, 70910-900 Brazil
E-mail address: pavel@unb.br