SIAM J. CONTROL OPTIM.
Vol. 57, No. 2, pp. 1467–1492

NONDISPERSAL AND DENSITY PROPERTIES OF INFINITE PACKINGS*

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Abstract. This article is motivated by an optimization problem arising in biology. Interpreting the egg arrangements (packings) in the brood chamber as results from an optimization process, we are led to look for packings that are at the same time the most possible dense and nondispersed. We first model this issue in terms of an elementary shape optimization problem among convex bodies, involving their inradius, diameter, and area. We then solve it completely, showing that the solutions are either particular hexagons or a symmetric 2-cap body, namely the convex hull of a disk and two points lined up with the center of the disk.

Key words. shape optimization, tiling domains, density of packings

AMS subject classifications. 52A40, 52A10, 49K30, 49Q10

DOI. 10.1137/18M1181183

NOTATION.

$\mathcal{K}$	set of compact convex bodies of $\mathbb{R}^2$
$D(0,R)$	centered closed ball with radius R
$\|K\|$	area of a convex body K
$r(K)$	inradius of a convex body K
$\mathrm{Diam}(K)$	diameter of a convex body K
H_r^*	regular hexagon with inradius r

1. Introduction.

This article is devoted to investigating optimal configurations of infinite packings in the two-dimensional space $\mathbb{R}^2$. Recall that a packing associated to a convex body K with nonempty interior is an arrangement of nonoverlapping copies of K. More precisely, denoting by $\mathcal{K}$ the set of compact convex bodies of $\mathbb{R}^2$, an infinite packing $P(K)$ with pattern K is defined by

$$P(K) = \bigcup_{i\in I} \tau_i(K),$$

where I denotes a countable set of indices, and the mappings τ_i are affine isometries of $\mathbb{R}^2$ such that $\mathrm{int}(\tau_i(K)) \cap \mathrm{int}(\tau_j(K)) = \emptyset$ for all $i \neq j$.

Since we are interested in infinite packings, we will consider without loss of generality in what follows that $I = \mathbb{N}$, and we will denote by $\mathcal{P}(K)$ the set of all infinite packings of the plane with pattern K.

A close notion that will be much discussed in what follows is the one of tiling domains. Recall that, as a consequence of the definition of packings, a convex K defines a tiling domain of the plane whenever $\mathbb{R}^2 \in \mathcal{P}(K)$.

In the whole article, the notation $|\cdot|$ will denote the Lebesgue measure in $\mathbb{R}^2$.

Let us make precise the shape optimization problem we will deal with. The criterion to minimize involves two geometrical functionals denoted by d and D_∞. Let

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*Received by the editors April 16, 2018; accepted for publication (in revised form) February 11, 2019; published electronically April 23, 2019.
http://www.siam.org/journals/sicon/57-2/M118118.html
Funding: The research of the first and third authors was partially supported by the Project "Analysis and simulation of optimal shapes – application to life-sciences" of the Paris City Hall.

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us define them:

- The first one models the *density of a packing.* We choose to define it as follows; see section 2.1 for a discussion and the link with another classical quantity for the density:

$$d(K) = \frac{|K|}{|K^T|} \tag{1.1}$$

  for every convex set K, where K^T denotes the smallest convex set tiling the plane and containing K (we refer the reader to Appendix A for the proof that such a set exists). In some sense, the quantity $d(K)$ stands for a quantitative measure of the tiling ability of K. Roughly speaking, we can consider that the highest $d(K)$ is and the most tiling will be the convex set K. Note, in particular, that if K is tiling, then $d(K) = 1$.
- The second functional is defined by

$$D_\infty(K) = \frac{2\sqrt{|K|}}{\sqrt{\pi}\,\mathrm{Diam}(K)} \tag{1.2}$$

  for every convex set K, where $\mathrm{Diam}(K)$ denotes the diameter of K. As this will be highlighted in what follows, the quantity $D_\infty(K)$ is a *measure of nondispersal of any packing associated to the convex K*. Indeed, this quantity is obtained by introducing the restriction of a packing with pattern K to a disk with diameter $R > 0$, by comparing the diameter of this set with the diameter of the disk and by letting R tend to $+\infty$. Hence, trying to minimize $D_\infty(K)$ will allow us to obtain a convex K and an associated packing as "compact" as possible.

Note that modeling issues, and in particular the functionals' choices, will be discussed and commented on in section 2.1.

Finally, for a given $t \in [0,1]$, we will consider in what follows a convex combination of both previous criteria. The resulting criterion, denoted by J_t, reads

$$J_t(K) = td(K) + (1-t)\frac{1}{D_\infty(K)}.$$

Let us define the admissible set. We will deal with three kinds of constraints:

(i) The considered sets will be compact and convex subsets of $\mathbb{R}^2$.

(ii) To avoid the shapes collapsing, we impose to the considered convex sets to have a minimal inradius r_0. In what follows, we will denote by $r(K)$ the inradius of any convex set K.

(iii) Since the functionals we will deal with are invariant by homothety, it is relevant to assume the area of the pattern prescribed, equal to a positive constant A.

We now introduce the complete shape optimization problem we will solve.

Let $t \in [0,1]$, $r_0 > 0$, and $A \geqslant \pi r_0^2$ be fixed, and let $\mathcal{A}_{r_0,A}$ denote the set of compact convex sets having an inradius larger than r_0 and an area equal to A, namely

$$\mathcal{A}_{r_0,A} = \{K \in \mathcal{K} \mid r(K) \geqslant r_0 \text{ and } |K| = A\}.$$

The shape optimization problem we will consider reads

$$\boxed{\sup_{K \in \mathcal{A}_{r_0,A}} J_t(K)}. \tag{1.3}$$

It is notable that this problem is also motivated by applied considerations. Some explanations about the biological framework in which this problem naturally arises are provided at the end of this section.

Let us roughly state hereafter the main results of this article. More detailed (and technical) versions of these theorems are provided in section 2.2.

Our first result deals with generalities about tiling domains. It seems to us interesting in itself.

THEOREM A. *Among all (convex) tiling sets with a given diameter and inradius, the one of minimal area is a p-hexagon, in other words a hexagon with two parallel opposite sides with the same length. By duality, one shows that the (convex) tiling set with a given area and inradius maximizing its diameter is a p-hexagon.*

Our second result deals with the solution of problem (1.3). For the sake of clarity, we state it informally.

THEOREM B. *Under a smallness assumption on the ratio r_0^2/A, the solutions of problem* (1.3) *are either a p-hexagon or a symmetric* 2*-cap body (the convex hull of a disk and two points lined up with the center of the disk), depending on the values of the parameter t.*

Complete and extensive versions of these results are provided in Theorems 2.7 and 2.9.

We end this section by giving an interpretation of this problem in biology. The shape optimization problem (1.3) is related to the understanding of the egg shape of a class of crustaceans, subclass branchiopoda, called eulimnadia.

In a clutch, the eggs are placed in the brood chamber, which is located dorsally beneath the carapace and which is closed by the abdominal processes. To understand egg geometry, it appears relevant to interpret the observed arrangements as the result of an optimization process. This way, assuming that the resulting shapes allow the crustacean to incubate the largest number of eggs, we look for configurations guaranteeing at the same time that shapes and arrangements make the resulting packing the most "dense" (this word meaning here the "most tiling"; see the definition of $d(K)$) and the most "compact" (in the sense that the restriction of the packing to a given ball with a large radius will contain the largest number of elements). In a nutshell, denoting by K the egg shape and assuming that the clutch contains the largest possible number of eggs, it is plausible that egg arrangements look at maximizing at the same time $d(K)$ and $D_\infty(K)$. We formalize this idea by looking for patterns K maximizing a convex combination of these functionals, hence the writing of problem (1.3).

Structure of the article. This article is organized as follows. Section 2.1 is devoted to several remarks about our motivations for considering problem (1.3), as well as our functional and admissible set choices. The main results of this article are gathered in section 2.2. Section 3 is devoted to the proof of Theorem 2.7, and section 4 is devoted to the proof of Theorem 2.9.

2. Modeling and solving the optimization problem.

2.1. Modeling issues and state of the art.

Density of convex sets. Let $K \in \mathcal{K}$ and $P(K)$ be a packing with pattern K. It is standard (see [12]) to define the *density of* $\delta(P(K))$ as

$$\delta(P(K)) = \liminf_{r \to +\infty} \frac{\sharp\{i \in \mathbb{N} \mid \tau_i(K) \subset [-r/2, r/2]^2\}|K|}{r^2}. \tag{2.1}$$

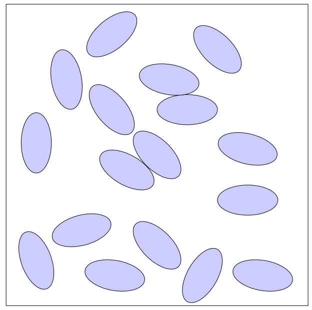

FIG. 1. *An example of a packing with ellipses; we can see the density as the ratio between the blue area (ovals) and the entire square.*

See Figures 1 and 2 for an illustration of packing density. For a fixed $r > 0$, the ratio $\sharp\{i \in \mathbb{N} \mid \tau_{i\in I}(K) \subset [-r/2, r/2]^2\}|K|/r^2$ represents the rate of the area occupied by the elements of the packing $P(K)$ contained in $[-r/2, r/2]^2$ with respect to the total area of a square with side r. Letting $r \to +\infty$ makes this definition independent of the window in which this rate is evaluated.

Having in mind to look for packings maximizing (among other criteria) the density functional, it is relevant to introduce a criterion depending only on the pattern choice, by setting $d_1(K) = \sup_{P(K)\in\mathcal{P}(K)} \delta(P(K))$, corresponding to the optimal density of a packing associated to the pattern K. This quantity is called *density of the convex* K [12].

Notice that the following elementary properties about d_1 are direct consequences of the definition.

PROPOSITION 2.1. *For every $K \in \mathcal{K}$, one has $d_1(K) \in [0, 1]$. Moreover,*

1. *if D is a disk, $d_1(D) = \frac{\pi}{2\sqrt{3}} \simeq 0.9$ [5];*
2. *if K is a tiling domain, $d_1(K) = 1$;*
3. *if $K \in \mathcal{K}$ and $T \in \mathcal{K}$ is tiling such that $K \subset T$, then $d_1(K) \geqslant \frac{|K|}{|T|}$ [10].*

The last property will be crucial in what follows since it allows one to provide a lower bound for d_1. Roughly speaking, the main ingredient consists in considering a tiling domain T such that $K \subset T$, the family of sets $\{\tau_i(T)\}_{i\in\mathbb{N}}$ defining the associated packing. We then define a packing with pattern K by placing a copy of K in each cell $\tau_i(T)$ and observing that the density of this packing will be larger than $|K|/|T|$. Moreover, it has been shown that given a convex body K, there exists a triangle T such that $K \subset T$ and $|K|/|T| \geqslant 2/3$ (see [4] by Fáry in 1950 and [2] by Courant in 1965). By considering parallelograms instead of triangles, Kuperberg obtained in 1982 in [10] the same conclusion, and this way the lower bound $d_1(K) \geqslant 3/4$ for every convex body K. This lower bound was improved in 1990 by Kuperberg and Kuperberg in [9], where it is shown that $d_1(K) \geqslant \sqrt{3}/2$, by using a particular tiling hexagon. In 1995, Doheny proved in [3] the existence of $r_0 > \sqrt{3}/2$ such that $d_1(K) \geqslant r_0$ for every convex body K. To our knowledge, the exact value of the bound $\inf\{d_1(K),\ K \in \mathcal{K}\}$ remains unknown.

Unfortunately, the precise value of $d_1(K)$ is almost never computable, even for simple choices of K. More annoying, having in mind to consider it as a criterion of an optimization problem, the quantity $d_1(K)$ appears intricate to handle. These considerations lead us to consider as an alternative and more workable definition of the density functional d defined by (1.1) involving the smallest convex tiling domain

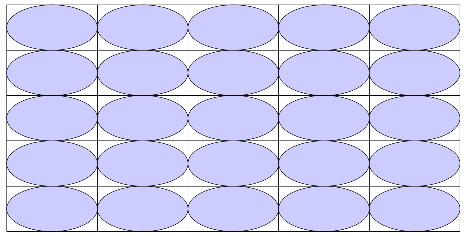

FIG. 2. *A packing with ellipses in a tiling with rectangles. It is intuitive that the density of this packing is equal to the ratio of the area of the ellipse over the area of the rectangle.*

containing K. Obviously, there holds that $d(K) \leqslant d_1(K)$ for every convex body K, and it is notable that all the properties gathered in Proposition 2.1 above remain satisfied with this new definition of density.

Nondispersal properties of convex sets. Let us first model the notion of nondispersion for packings. We start from the observation that balls are the "less dispersed" bodies, in the sense that, among all nonempty convex sets, they minimize the ratio of the diameter by the square root of their area. This leads us to define the notion of "nondispersion" of packings by comparing their diameter to that of balls. More precisely, we introduce, mimicking the definition of δ in (2.1),

$$D'_\infty(K) = \inf_{P\in\mathcal{P}(K)} \limsup_{R\to\infty} \frac{2R}{\sqrt{\sharp\{i, \tau_i(K) \subset D(0,R)\}}\,\mathrm{Diam}(K)},$$

the lim sup being used in the definition to make $D'_\infty(K)$ independent of the balls' radii. More precisely, given a packing $P \in \mathcal{P}(K)$ and $R > 0$, we consider a disk with radius R and evaluate the number of copies of K within the disk. Note also that we take the square root of this integer in the definition by observing that the maximal number of identical copies of a convex order of magnitude in a disk with radius R is[1] $\mathrm{O}(R^2)$. Finally, the diameter of K appearing in the denominator is used as a renormalization factor. This appears natural in view of defining an adimensional quantity.

First, an elementary reasoning shows that, in a disk of radius R, there cannot be more than $\pi R^2/|K|$ copies of K. As a consequence, we infer that

$$D'_\infty(K) \geqslant \frac{2\sqrt{|K|}}{\sqrt{\pi}\,\mathrm{Diam}(K)} = D_\infty(K) \tag{2.2}$$

for every $K \in \mathcal{K}$, where $D_\infty(K)$ is defined by (1.2). The following result, whose proof is postponed to Appendix B, provides fine estimates of $D'_\infty(K)$.

THEOREM 2.2. *Let $K \in \mathcal{K}$. One has*

$$\frac{2\sqrt{|K|}}{\sqrt{\pi}\,\mathrm{Diam}(K)} \leqslant D'_\infty(K) \leqslant \sqrt{\frac{2}{\sqrt{3}}}\frac{2\sqrt{|K|}}{\sqrt{\pi}\,\mathrm{Diam}(K)}. \tag{2.3}$$

Furthermore, if K is tiling, then one has $D'_\infty(K) = D_\infty(K)$.

[1]Indeed, let us provide a sketch of argument. Let us consider a rectangle tiling the plane and containing the convex body. We denote by L and ℓ its dimensions. If $R \gg L$, the number of rectangles that can be packed within a disk with radius R is $\mathrm{O}(\pi R^2)/(L\ell) = \mathrm{O}(R^2)$. Therefore, the number of copies of a convex K that can be packed within a disk is less than $\pi R^2/|K| = \mathrm{O}(R^2)$.

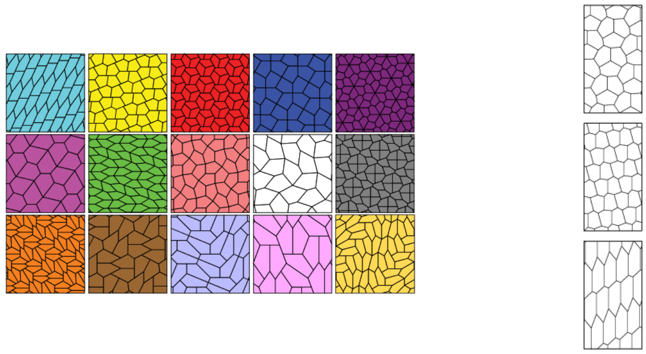

FIG. 3. *(Left) The 15 kinds of tiling pentagons (source: https://commons.wikimedia.org/wiki/File:PentagonTilings15.svg). (Right) The three kinds of tiling hexagons (source: http://mathworld.wolfram.com/HexagonTiling.html).*

According to the result above, one has $D'_\infty(K)/D_\infty(K) \in [1, 1.08)$. We infer that, in order to consider workable quantities, it will be relevant in what follows to consider D_∞ as a criterion of nondispersal.

Convex tiling domains. The previous remarks suggest that we take a short interest in convex tiling domains. Notice that a convexity argument allows us to show that a two-dimensional convex domain which is tiling in $\mathbb{R}^2$ is necessarily a polygon. More precisely, thanks to Euler's formulae, it is known that a polygon with more than six vertices cannot be tiling [1]. Moreover, any triangle or quadrilateral tiles the plane, but there exist only three kinds of tiling hexagons. The case of pentagons is more intricate. It has been recently solved in [11] by leading an exhaustive search of all families of convex pentagons tiling the plane. In particular, the authors state that there are no more than 15 kinds of pentagons tiling the plane (see Figure 3 for an illustration of tiling pentagons and hexagons).

2.2. Solving the optimization problems.

Notation. Let us define particular convex sets that will play a crucial role in what follows.

DEFINITION 2.3 (the hexagons $H_{A,r}$ and $H^{D,r}$). *Let $r > 0$ and $A \geqslant 2\sqrt{3}r^2$. Let $\mathcal{C}$ be a circle centered at the origin O with radius r and $H_{A,r}$ be the hexagon defined as follows:*

(i) *Each side of $H_{A,r}$ is tangent to $\mathcal{C}$.*

(ii) *Denoting by $\{B_i\}_{i=1,\dots,6}$ the set of tangential points ordered between $H_{A,r}$ and $\mathcal{C}$ and by θ_i the angle $\widehat{B_iOB_{i+1}}$ (with the convention that $B_7 = B_1$), one has*

$$\begin{cases} \theta_1 = \theta_4 = 4\arctan\left(\frac{2r^2+\sqrt{A^2-12r^4}}{4r^2+A}\right), \\ \theta_2 = \theta_3 = \theta_5 = \theta_6 = \frac{\pi-\theta_1}{2}. \end{cases}$$

It is notable that $H_{A,r}$ is a p-hexagon, in other words a hexagon with two parallel opposite sides with the same length (see Figure 4).

Moreover, let D and r be two positive numbers. Noting that one has[2]

$$\mathrm{Diam}(H_{A,r}) = \frac{1}{3r}\left(2A + \sqrt{A^2 - 12r^4}\right),$$

[2]We refer the reader to Appendix C for a proof of this claim.

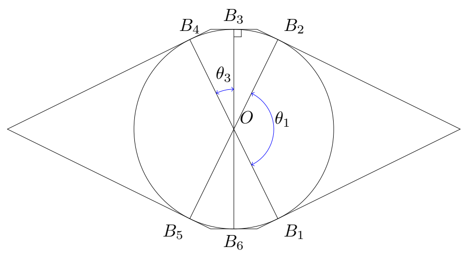

FIG. 4. *The p-hexagon $H_{A,r}$ and its inscribed circle.*

one defines the hexagon $H^{D,r}$ by $H^{D,r} = H_{A(D),r}$, where $A(D)$ is the unique solution of the equation

$$D = \alpha(A(D), r), \quad \text{with} \quad \alpha(A,r) = \frac{1}{3r}\left(2A + \sqrt{A^2 - 12r^4}\right). \tag{2.4}$$

Furthermore, one has $|H^{D,r}| = 2rD - r\sqrt{D^2 - 4r^2}$ (see Appendix C*).*

DEFINITION 2.4 (the symmetric 2-cap bodies $G^{D,r}$ and $G_{A,r}$). *Let D and r be two positive numbers such that $D \geqslant 2r$. We denote by $G^{D,r}$ the convex hull of a circle with radius r and two points at a distance of D, lined up with the circle center (see Figure* 5*). Such a convex set will be called a* symmetric 2-cap body of diameter D and inradius r.

Similarly, let A and r be two positive numbers. One defines the symmetric 2*-cap body $G_{A,r}$ by $G_{A,r} = G^{D(A),r}$, where $D(A)$ is the unique positive solution of*

$$A = r\left(\sqrt{D(A)^2 - 4r^2} + 2r \arcsin\left(\frac{2r}{D(A)}\right)\right).$$

Remark 2.5. Let A, D, r be three positive numbers. In [8], it is shown that for every convex set with area A, inradius r, and diameter D, one has

$$A \geqslant r\left(\sqrt{D^2 - 4r^2} + 2r \arcsin\left(\frac{2r}{D}\right)\right) \tag{2.5}$$

and this inequality is an equality if and only if $K = G^{D,r}$ (and thus $A = |G^{D,r}|$). This inequality can also be interpreted as follows: the convex set with diameter D and inradius r having the lowest area is $G^{D,r}$. By duality, this also means that the convex set with area A and inradius r having the maximal diameter is the convex hull of a circle with radius r and two points, lined up with the circle center.

Remark 2.6. It follows easily from geometrical observations or simple computations that the following hold:

- There exists a unique hexagon (up to rotations) fulfilling the conditions of Definition 2.3, and this construction can be led if and only if $A \geqslant 2\sqrt{3}r^2$.
- The hexagon $H_{A,r}$ is of area A and inradius r.
- The sides of $H_{A,r}$ are two by two parallels. In particular, $H_{A,r}$ is a p-hexagon (see Theorem A for the definition).
- The diameter of $H^{D,r}$ can differ from D. For instance, it is the case if $r^2 \geqslant A/2\sqrt{3}$ and $D \leqslant \min\{4/\sqrt{3}r, \mathrm{Diam}(G_{A,r})$, as noted in the proof of Lemma 4.2.

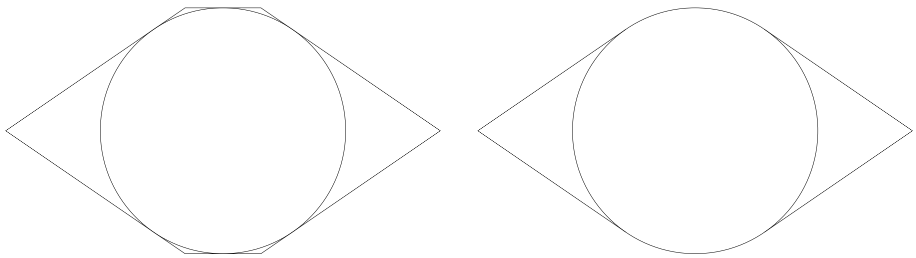

FIG. 5. *Left: the hexagon $H_{A,r}$ and its inscribed circle. Right: the symmetric 2-cap body $G^{D,r}$ and its inscribed circle.*

Statement of the main results. In the following theorem, we state several sharp inequalities for tiling domains of the plane. These results constitute key ingredients of the proof of Theorem 2.7.

THEOREM 2.7. *Let T be a compact convex tiling domain of $\mathbb{R}^2$:*

1. *There holds that*

$$|T| \geqslant 2\sqrt{3}r(T)^2 \quad \textit{and} \quad \mathrm{Diam}(T) \geqslant \frac{4}{\sqrt{3}}r(T)$$

with equality if only if T is a regular hexagon.

2. *One has*

$$\mathrm{Diam}(T) \leqslant \frac{1}{3r(T)}\left(2|T| + \sqrt{|T|^2 - 12r(T)^4}\right) \tag{2.6}$$

with equality if and only if $T = H_{A,r}$

Remark 2.8. Let $r > 0$. As a byproduct of Theorem 2.7, using, in particular, that the mapping $[2\sqrt{3}r^2, +\infty) \ni A \mapsto \alpha(A, r)$ (where α is given by (2.4)) is increasing, we get the following:

- the (convex) tiling set with diameter D and inradius r minimizing its area is the hexagon $H^{D,r}$;
- the (convex) tiling set with area A and inradius r maximizing its diameter is the hexagon $H_{A,r}$.

The first point comes from the following observation: let $A \geqslant 2\sqrt{3}r^2$ for some $r > 0$. Then, the map $F_r : A \mapsto \alpha(A, r) = \frac{1}{3r}\left(2A + \sqrt{A^2 - 12r^4}\right)$ is increasing and defines a bijection from $[2\sqrt{3}r^2, +\infty)$ to $[4/\sqrt{3}, +\infty)$. Its inverse mapping is F_r^{-1} : $[4/\sqrt{3}, +\infty) \ni D \mapsto 2rD - r\sqrt{D^2 - 4r^2}$.

Now, let T be a tiling domain of $\mathbb{R}^2$. According to the considerations above, the inequality (2.6) is equivalent to $\mathrm{Diam}(T) \leqslant F_{r(T)}(|T|)$, which rewrites $F_r^{-1}(\mathrm{Diam}(T)) \leqslant |T|$. This shows that the inequality

$$2r(T)\,\mathrm{Diam}(T) - r(T)\sqrt{\mathrm{Diam}(T)^2 - 4r(T)^2} \leqslant |T| \tag{2.7}$$

holds true for every tiling domain of $\mathbb{R}^2$. The expected conclusion follows.

The second claim is a direct consequence of (2.6) in Theorem 2.7.

THEOREM 2.9. *Let r_0 and A be two positive numbers such that $2\sqrt{3}r_0^2 < A$.*

Let us denote by X_0 ($\simeq 3.1847$) the unique zero of the function $X \mapsto \sqrt{X^2 - 4}(14 - 5X^2) + 4X(X^2 - 3)$ on $[4/\sqrt{3}, +\infty)$ and set

$$t_{A,r_0} = \frac{\sqrt{\pi}/(2\sqrt{A})}{\sqrt{\pi}/(2\sqrt{A}) + A\gamma_0/r_0^3} \in (0, 1), \tag{2.8}$$

with

$$\gamma_0 = \frac{\left(2\sqrt{X_0^2-4} - X_0\right)}{\sqrt{X_0^2-4}(2X_0 - \sqrt{X_0^2-4})^2} \simeq 0.0472. \tag{2.9}$$

1. *If* $t \in [0, t_{A,r_0}]$*, the symmetric* 2*-cap body* G_{A,r_0} *solves problem* (1.3)*.*
2. *Let us assume, moreover, that*

   $$r_0 \leqslant \gamma\sqrt{A}, \qquad \textit{where} \qquad \gamma = \frac{1}{\sqrt{2X_0 - \sqrt{X_0^2-4}}} \in [0.5069, 0.5070], \tag{2.10}$$

   and define

   $$t^*_{A,r_0} = \frac{\frac{\sqrt{\pi}}{2\sqrt{A}}\left(\mathrm{Diam}(G_{A,r_0}) - \mathrm{Diam}(H_{A,r_0})\right)}{\frac{\sqrt{\pi}}{2\sqrt{A}}\left(\mathrm{Diam}(G_{A,r_0}) - \mathrm{Diam}(H_{A,r_0})\right) + A\left(\frac{1}{|\,\mathrm{Diam}(G_{A,r_0})|} - \frac{1}{|\,\mathrm{Diam}(H_{A,r_0})|}\right)}. \tag{2.11}$$

   One has $t^*_{A,r_0} \geqslant t_{A,r_0}$*. Moreover, if* $t \in [0, t^*_{A,r_0})$*, the symmetric* 2*-cap body* G_{A,r_0} *solves problem* (1.3)*, and if* $t \in (t^*_{A,r_0}, 1]$*, the p-hexagon* H_{A,r_0} *solves problem* (1.3)*. If* $t = t^*_{A,r_0}$*, the two convex sets* H_{A,r_0} *and* G_{A,r_0} *solve problem* (1.3)*.*

Remark 2.10 (comment on the assumption (2.10)). The assumption $2\sqrt{3}r_0^2 < A$ is natural since it is a sufficient and necessary condition for ensuring the existence of the p-hexagon H_{A,r_0} (see the first item of Theorem 2.7). Note that it writes also $r_0 \leqslant \hat{\gamma}\sqrt{A}$, with $\hat{\gamma} \simeq 0.5373$.

The assumption (2.10) appears a bit technical (although relevant from an applied point of view). A refined analysis can show that if $r_0/\sqrt{A} \in (\gamma, \hat{\gamma})$, there exists $\tilde{t}_{A,r_0} \geqslant t_{A,r_0}$ such that for $t \geqslant \tilde{t}_{A,r_0}$, either the symmetric 2-cap body G_{A,r_0} or the p-hexagon H_{A,r_0} solves problem (1.3).

3. Proof of Theorem 2.7. Proving Theorem 2.7 is equivalent to determining the optimal value of the problems

$$\inf\{|K|,\ K \in \mathcal{T},\ r(K) \geqslant r_0\} \quad \text{and} \quad \inf\{\mathrm{Diam}(K),\ K \in \mathcal{T},\ r(K) \geqslant r_0\} \tag{3.1}$$

and

$$\sup\{\mathrm{Diam}(K),\ K \in \mathcal{T},\ r(K) \geqslant r_0,\ |K| = A\}, \tag{3.2}$$

where $\mathcal{T}$ denotes the set of tiling domains in $\mathbb{R}^2$. In what follows, we will solve a relaxed version of these problems, namely

$$\inf\{|K|,\ K \in \mathcal{P}_6,\ r(K) \geqslant r_0\} \quad \text{and} \quad \inf\{\mathrm{Diam}(K),\ K \in \mathcal{P}_6,\ r(K) \geqslant r_0\} \tag{3.3}$$

and

$$\sup\{\mathrm{Diam}(K),\ K \in \mathcal{P}_6,\ r(K) \geqslant r_0,\ |K| = A\}, \tag{3.4}$$

where $\mathcal{P}_6$ denotes the set of convex polygons of the plane having at most six sides, and show that the solutions are tiling domains. As a consequence, and since the new admissible set contains the previous one, the optimal values between the problems (3.1) and their relaxed version will coincide.

Before dealing with each problem separately, let us state some preliminary results allowing us to reduce the search of an optimal domain to a simpler class. The arguments used in Step 1 below hold indifferently for each problem of (3.3).

As a preliminary remark, notice that the two problems of (3.3) have a solution since $\mathcal{P}_6$ is compact for the Hausdorff topology and the functionals $K \mapsto |K|$, $K \mapsto r(K)$, $K \mapsto \text{Diam}(K)$ restricted to convex sets are continuous for this topology; see [7, Chapter 2].

Step 1. *Restricting the set of admissible domains.* The following lemmas are in order.

LEMMA 3.1. *For any problem of* (3.3) *and* (3.4)*, there exists a solution K^* that is a hexagon. Moreover, regarding the first problem of* (3.3) *and problem* (3.4)*,* every *solution of one of such problems is necessarily a hexagon.*

Proof. Let us assume by contradiction that K^* has N sides, with $N < 6$. Consider two diametral points D_1 and D_2 of K^*, and let M be any vertex of K^* different from D_1 and D_2. Then, we change K^* into $\hat{K}^*$ by removing the vertex M and creating two new vertices as follows: we cut K^* with a well-chosen hyperplane at a distance of M small enough so that the diameter and the inner radius of K^* are not modified.

- *Minimizing the area:* The area of $\hat{K}^*$ is strictly lower than the area of K^*, which contradicts the optimality of K^*. The conclusion follows.
- *Minimizing the diameter:* The diameter of $\hat{K}^*$ being equal to the one of K^*, we infer that it is possible to restrict our search to hexagons.
- *Maximizing the diameter:* Consider the set $t\hat{K}^*$, where $t > 1$ is chosen in such a way that $|t\hat{K}^*| = |K^*|$. Then, one has $r(t\hat{K}^*) = tr(\hat{K}^*) = tr_0 > r_0$ and $\text{Diam}(t\hat{K}^*) = t\,\text{Diam}(\hat{K}^*) > \text{Diam}(\hat{K}^*)$, which contradicts the optimality of K^*. The conclusion follows. ∎

Remark 3.2. It will follow from the proof that all the solutions of problems (3.3) and (3.4) are hexagons.

The proofs of the next two lemmas are exactly the same for each problem of (3.1) and (3.2). Since this last problem is more constrained and in some sense more intricate, we prove this lemma for the problem of maximizing the diameter. An easy adaptation of the proof below shows the same result for the issue of minimizing the area or the diameter.

LEMMA 3.3. *Let K^* be a solution of any problem of* (3.3) *and* (3.4)*. Then, necessarily, $r(K^*) = r_0$.*

Proof. Let K^* be a solution of problem (3.4), and let us assume by contradiction that $r(K^*) > r_0$. Since K^* is a convex polygon, there exist two vertices B and C of K^* such that $\text{Diam}(K^*) = BC$. For $t \in [0, 1]$, let ρ_t be the stretching with ratio t and direction (kept fixed) the axis (BC). Then, one has $|\rho_t(K^*)| = t|K^*|$ and $\text{Diam}(\rho_t(K^*)) = \text{Diam}(K^*)$. Noting that $[0, 1] \ni t \mapsto r(\rho_t(K^*))$ is a continuous increasing function such that $r(0) = 0$ and $r(1) = r(K^*)$, consider $r \in (r_0, r(K^*))$ and $t \in (0, 1]$ such that $r(\rho_t(K^*)) = r$. Let K_t be the range of $\rho_t(K^*)$ by the homothety centered at O, the center of the incircle, with scale factor $1/\sqrt{t} > 1$. Hence, one has $|K_t| = |K^*|$, $\text{Diam}(K_t) = \text{Diam}(K^*)/\sqrt{t}$, and $r(K_t) = r(K^*)/\sqrt{t} > r_0$. It follows that K_t is a admissible hexagon and, moreover, $\text{Diam}(K_t) > \text{Diam}(K^*)$. We have then reached a contradiction (see Figure 6 for an illustration of the proof). ∎

LEMMA 3.4. *Let K^* be a solution of any problem of* (3.3) *and* (3.4)*. Then, necessarily K^* is tangent at each side to any inscribed circle.*

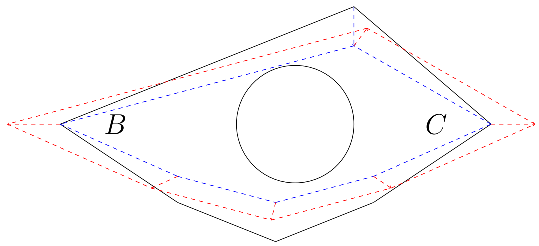

FIG. 6. *Illustration of the proof of Lemma* 3.3: *the hexagon K^* (black), the hexagon $\rho_t(K^*)$ (blue), and the hexagon K_t (red). Color is available online only.*

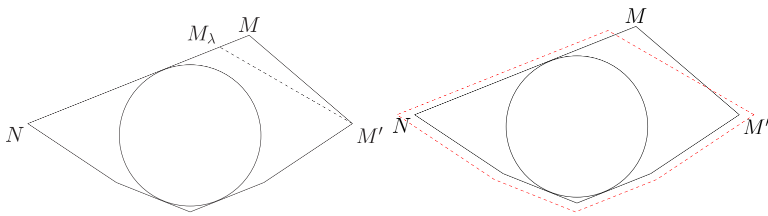

FIG. 7. *Geometrical illustration of the method: construction of $\tilde{K}$ (left) and construction of $\hat{K}$ (dotted line) from $\tilde{K}$ (right).*

Proof. We argue by contradiction by assuming that there exist an inscribed circle $\mathcal{C}$ and a side of K^* that do not meet. To reach a contradiction, we will show that one can transform K^* into a new admissible set $\hat{K}$ having a strictly larger diameter.

Consider first the case where there exists a side $[MM']$ at a positive distance of $\mathcal{C}$ such that $\mathrm{Diam}(K^*) > MM'$. Assume without loss of generality the existence of two vertices of K^* different from M and reaching its diameter. This property will allow one to construct a new set $\tilde{K}$ from K^* by slightly modifying the location of M such that $\mathrm{Diam}(\tilde{K}) = \mathrm{Diam}(K^*)$. Let N be the vertex of K^* such that M is adjacent to N and M'. Let $\lambda \in (0,1)$ and $M_\lambda = \lambda N + (1-\lambda)M$. For $\lambda > 0$ small enough, there holds that $(M'M_\lambda) \cap \mathcal{C} = \emptyset$. Hence, denoting by $\tilde{K}$ the hexagon obtained by replacing M by M_λ, one has $r(\tilde{K}) = r(K^*)$. Moreover, since $\tilde{K} \subset K^*$ and $\tilde{K} \neq K^*$, one has $|\tilde{K}| < |K^*|$. To get $\hat{K}$, we now apply a homothety to $\tilde{K}$ where the scale factor is chosen in such a way that $|\hat{K}| = |K^*|$ (see Figure 7). We then have $r(\hat{K}) > r(K^*)$ and $\mathrm{Diam}(\hat{K}) > \mathrm{Diam}(K^*)$, hence the contradiction.

Consider now the complementary case where any side which does not meet tangentially the circle $\mathcal{C}$ realizes the diameter of K^*. Hence, let us consider a side $[AB]$ of K^* realizing the diameter without meeting $\mathcal{C}$ tangentially. Notice that if such a choice of side does not exist, then we are in the previous case and we can reach a contradiction.

Denote by O the center of $\mathcal{C}$ and by M the orthogonal projection of O on (AB). Then, M belongs to the segment $[AB]$ and the distance δ of M to $\mathcal{C}$ is positive (by compactness). The new circle $\mathcal{C}'$ obtained from $\mathcal{C}$ by translation of vector $\delta\frac{\overrightarrow{OM}}{OM}$ is tangent to (AB) (see Figure 8). Let us prove that $\mathcal{C}' \subset K^*$. Let $(A, \vec{i}, \vec{j})$ be the orthonormal basis such that $\vec{i} = \overrightarrow{AB}/AB$ and K^* be contained in $\mathbb{R}^2_+$. Then, $\partial K^* \backslash (AB)$ is parametrized by a positive concave function $f : [0, AB] \mapsto \mathbb{R}_+$. For $u \in [0,1]$, let D_u be the vertical axis with equation $x = u$. Then, defining $x_1 = \min\{u \in [0,1], D_u \cap \mathcal{C} \neq \emptyset\}$ and $x_2 = \max\{u \in [0,1], D_u \cap \mathcal{C} \neq \emptyset\}$, the region

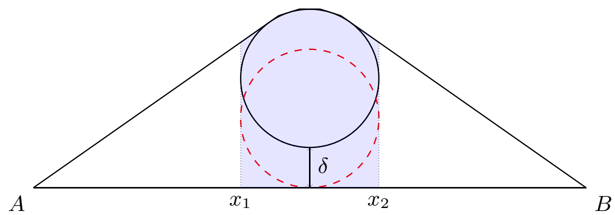

FIG. 8. *Case where the diameter is realized by the only side $[AB]$ of K^* which does not meet $\mathcal{C}$ tangentially.*

$\mathcal{R} = \{(x, y), x_1 \leqslant x \leqslant x_2, 0 \leqslant y \leqslant f(x)\}$ is contained in K^* with an easy convexity argument and, by construction, $\mathcal{C}' \subset \mathcal{R}$. Hence, $\mathcal{C}' \subset K^*$ and we are then led to the previous case. □

By combining the three lemmas we have just proved, we will recast both problems of (3.3) in a simpler way by using a convenient parametrization and some analytical arguments. For homogeneity reasons and according to Lemma 3.3, we will assume from now on that $r_0 = 1$, the solutions for the general case being easily inferred from that case.

Let K^* be a hexagon solution of a problem of (3.3). Since each problem is invariant under rotation or translation of K, we will assume without loss of generality that the center of the inscribed circle (which is uniquely located inside K^*, according to Lemma 3.4) is the origin O and that one side of K^* is included in the axis $x = 1$. Let $\{B_i\}_{i=1,\dots,6}$ be the projections of O on each side of K^* with the convention that B_1 is the projection of O on the side included in the axis $x = 1$ and the other points are located by following the trigonometric sense.

Let $\{A_i\}_{i=1,\dots,6}$ be the vertices of K^* having positive coordinates in the basis $(O; \overrightarrow{OB_i}, \overrightarrow{OB_{i+1}})$, and let $\theta_i = \widehat{B_iOB_{i+1}}$ and $\varphi_i = \widehat{B_iOA_i}$, so that

$$\sum_{i=1}^{6} \theta_i = 2\pi \quad \text{and} \quad \sum_{i=1}^{6} \varphi_i = \pi.$$

Notice that $0 \leqslant \theta_i \leqslant \pi$ and since the two triangles B_iOA_i and A_iOB_{i+1} are similar, one has $\varphi_i = \theta_i/2$ (see Figure 9).

Using this parametrization, let us rewrite each optimization problem in terms of the variables φ_i. Decomposing the hexagon K^* into the six quadrilaterals $OB_iA_iB_{i+1}$ $(i = 1, \dots, 6)$ and each quadrilateral into two similar triangles B_iOA_i and $B_{i+1}OA_i$ (whose area is equal to $\frac{OB_i.B_iA_i}{2} = \frac{\tan(\varphi_i)}{2}$), we get

$$|K^*| = \sum_{i=1}^{6} \tan \varphi_i.$$

Introduce the sets $\Theta^0 = \left\{\Phi = (\varphi_1, \dots, \varphi_6) \in [0, \pi/2]^6,\ \sum_{i=1}^{6} \varphi_i = \pi\right\}$ and $\Theta_A = \left\{\Phi \in \Theta^0,\ \sum_{i=1}^{6} \tan \varphi_i = A\right\}$. The two problems of (3.3) rewrite

$$\min_{\Phi \in \Theta^0} \sum_{i=1}^{6} \tan(\varphi_i) \qquad \text{and} \qquad \min_{\Phi \in \Theta^0} \mathrm{Diam}(H(\Phi)), \tag{3.5}$$

whereas problem (3.4) rewrites

$$\max_{\Phi \in \Theta_A} \mathrm{Diam}(H(\Phi)), \tag{3.6}$$

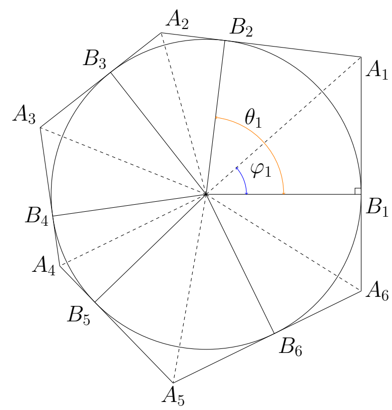

FIG. 9. *Parametrization of hexagons.*

where $H(\Phi)$ denotes the hexagon tangent at each side to the unit circle, whose semicircle center angles are the φ_i's.

Step 2. *Solving the two problems of* (3.5). Let us consider the first problem of (3.5). The proof is straightforward. Indeed, noting that the pointwise constraint $\varphi_i \leqslant \pi/2$ cannot be active, it follows from the Karush–Kuhn–Tucker theorem that there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$1 + (\tan \varphi_i)^2 = \lambda$$

for all the nonzero angles φ_i. As a consequence, all the nonzero angles are necessarily equal. Hence, investigating separately the cases where three, four, five, and six angles are nonzero yields easily the expected result.

Let us now solve the second problem of (3.5). Let K be a hexagon, and let us use the notation of Figure 9. One has

$$\min_{\Phi \in \Theta^0} \operatorname{Diam}(H(\Phi)) = \min_{K \in \mathcal{P}_6} \max_{(x,y) \in K^2} |x - y| \geqslant \min_{K \in \mathcal{P}_6} \max_{i=1,2,3} A_i A_{i+3}.$$

Let us now solve the problem $\min_{K \in \mathcal{P}_6} \max_{i=1,\dots,3} A_i A_{i+3}$. We will show that the chain of inequalities above is in fact a chain of equalities. We start with several remarks allowing us to reduce the problem. Notice that the preliminary remarks of Step 1 still hold for this problem. Consider a solution denoted by K^* associated to $\Phi^* \in \Theta^0$.

- Let us assume without loss of generality that the maximum is reached by $A_1 A_4$. Consider the hexagons $\hat{K}_i$, $i = 1, 2$, obtained by symmetrizing the quadrilaterals $A_1 A_2 A_3 A_4$ and $A_4 A_5 A_6 A_1$ with respect to the axis $(A_1 A_4)$. Assume by contradiction that A_1, A_4, and O are not aligned. Then, it is obvious that either the inradius of $\hat{K}_1$ or the one of $\hat{K}_2$ is strictly lower than 1. Assume that the inradius of $\hat{K}_1$ is strictly lower than 1. Then, applying a well-chosen homothety to $\hat{K}_1$ provides a hexagon with inradius 1 having a diameter larger than the one of K^*, which is absurd. Hence, A_1, A_4, and O are necessarily aligned and this argument can be extended to any length reaching the maximum.
- In fact, one can show that the three lengths $A_1 A_4$, $A_2 A_5$, and $A_3 A_6$ are equal. Indeed, in the converse case, assume that $A_1 A_4$ does not reach the maximum.

We replace A_1 and A_4 by $\hat{A}_1$ and $\hat{A}_4$, which are the respective images of A_1 and A_4, by a homothety centered at the middle of $[A_1A_4]$ in such a way that $\hat{A}_1\hat{A}_4 > A_1A_4$, and the maximum remains unchanged. This is a contradiction with the conclusion of Lemma 3.4.

As a result, one has necessarily $A_1A_4 = A_2A_5 = A_3A_6$ and, moreover, the points A_i, O, and A_{i+3} are aligned in this order for $i = 1, 2, 3$. According to the considerations above, and since $OA_i = 1/\cos\varphi_i^*$, $i = 1, \dots, 6$, one has

$$A_iA_{i+3} = OA_i + OA_{i+3} = \frac{1}{\cos\varphi_i^*} + \frac{1}{\cos\varphi_{i+3}^*}.$$

Therefore, we infer that

$$\min_{\Phi\in\Theta^0} \mathrm{Diam}(H(\Phi)) \geqslant \min_{K\in\mathcal{P}_6} \max_{i=1,2,3} A_iA_{i+3} = \max_{i=1,2,3} \left(\frac{1}{\cos\varphi_i^*} + \frac{1}{\cos\varphi_{i+3}^*}\right).$$

Moreover, one has

$$\max_{i=1,2,3} \frac{1}{\cos\varphi_i^*} + \frac{1}{\cos\varphi_{i+3}^*} \geqslant \frac{1}{3}\sum_{i=1}^{6} \frac{1}{\cos\varphi_i^*} \geqslant \frac{1}{3} \min_{K\in\mathcal{P}_6} \sum_{i=1}^{6} \frac{1}{\cos\varphi_i}.$$

For this last problem, let $\tilde{\Phi}$ be a solution. Notice that one has necessarily $\tilde{\varphi}_i < \pi/2$. Let us assume that $\tilde{\varphi}_i$ is positive. Hence, it follows from the Karush–Kuhn–Tucker theorem that there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$-\frac{\sin\varphi_i}{\cos^2\varphi_i} = \lambda,$$

and therefore all the nonzero angles must be equal. For $N = 3, 4, 5, 6$, assume that there are $6 - N$ zero angles and N nonzero angles (therefore equal to π/N according to the equality constraint). One shows easily that

$$\sum_{i=1}^{6} \frac{1}{\cos\tilde{\varphi}_i} = \frac{N}{\cos(\pi/N)} + (N-6) \geqslant \frac{6}{\cos(\pi/6)} = \frac{12}{\sqrt{3}}.$$

This proves that the only solution of the problem $\min_{K\in\mathcal{P}_6} \sum_{i=1}^{6} \frac{1}{\cos^2\varphi_i}$ is $\tilde{\Phi} = \frac{\pi}{6}(1, 1, 1, 1, 1, 1)$. We infer from this reasoning that

$$\min_{\Phi\in\Theta^0} \mathrm{Diam}(H(\Phi)) \geqslant \frac{4}{\sqrt{3}}.$$

We conclude by noting that this inequality is an equality as soon as $\Phi = \tilde{\Phi}$ (in other words, whenever K^* is a regular hexagon with inradius 1).

Step 3. *Solving problem* (3.6). Assume that K^* is the hexagon plotted in Figure 9. The diameter can be realized in three ways: (i) on a side, (ii) on a diagonal of the kind A_1A_4, or (iii) on a diagonal of the kind A_1A_3. In what follows, we will first consider separately each of these three cases and combine them in a second step to get the expected result. In what follows, we will denote by $\Phi^* = (\varphi_1^*, \dots, \varphi_6^*)$ a solution of (3.6) associated to a hexagon K^*.

Case (i): *the diameter is realized by a side.* Assume without loss of generality that the diameter of K^* is given by A_1A_2 (this is always possible by re-indexing the

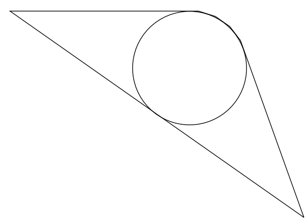

FIG. 10. *Hexagon maximizing* $D_{1,2}(\Phi)$ *for* $A = 4\sqrt{3}$.

vertices). For $\Phi \in \Theta_A$, denote by $D_{1,2}(\Phi)$ the length A_1A_2 in the hexagon $H(\Phi)$. One has $D_{1,2}(\Phi) = \tan(\varphi_1) + \tan(\varphi_2)$, and we are therefore led to solve the optimization problem

$$\max_{\Phi \in \Theta} \tan(\varphi_1) + \tan(\varphi_2).$$

It is notable that for the hexagon K^*, one has necessarily $0 < \varphi_i < \pi/2$. Indeed, the left inequality is a direct consequence of the conclusion of Lemma 3.1 for problem (3.4), and the right one comes from the area constraint. According to the Karush–Kuhn–Tucker theorem, there exists $(\lambda, \mu) \in \mathbb{R}^2$ such that $1+\tan^2(\varphi_i^*) = \lambda(1+\tan^2(\varphi_i^*))+\mu$ for $i = 1, 2$, and $0 = \lambda(1 + \tan^2(\varphi_i^*)) + \mu$ for $i = 3, 4, 5, 6$.

The two first equations yield $(\lambda, \mu) \neq (0, 0)$, and we easily infer that

$$\varphi_1^* = \varphi_2^* \quad \text{and} \quad \varphi_3^* = \varphi_4^* = \varphi_5^* = \varphi_6^*.$$

Denoting by φ the angle φ_1^* and by ψ the angle φ_3^*, it follows from the equality constraint on the φ_i's and from the area constraint that

$$\psi = \frac{\pi}{4} - \frac{\varphi}{2} \quad \text{and} \quad 2\tan(\varphi) + 4\tan(\psi) = A.$$

Let $t = \tan(\varphi/2)$. Since

$$\tan(\psi) = \tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) = \frac{1 - \tan(\varphi/2)}{1 + \tan(\varphi/2)} = \frac{1-t}{1+t},$$

the second equation rewrites $4\left(\frac{t}{1-t^2} + \frac{1-t}{1+t}\right) = A$, and hence

$$t^2\left(1 + \frac{A}{4}\right) - t + 1 - \frac{A}{4} = 0.$$

Since $A \geqslant 2\sqrt{3}$, this equation has two real roots, and the largest one is

$$t = \frac{1 + \sqrt{\frac{A^2}{4} - 3}}{2(1 + \frac{A}{4})}.$$

We then get

$$\max_{\Phi \in \Theta_A} D_{1,2}(\Phi) = D_{1,2}(\Phi^*) = \frac{4t}{1-t^2}.$$

Figure 10 illustrates the construction of the maximizing hexagon.

Case (ii): *the diameter is realized by* A_1A_4. Since $OA_1 = 1/\cos\varphi_1$, $OA_4 = 1/\cos\varphi_4$, and $\widehat{A_1OA_4} = \varphi_1 + 2\varphi_2 + 2\varphi_3 + \varphi_4$, one has $A_1A_4^2 = D_{1,4}(\Phi)$, where

$$D_{1,4}(\Phi) = \frac{1}{\cos^2\varphi_1} + \frac{1}{\cos^2\varphi_4} - \frac{2\cos(\varphi_1 + 2\varphi_2 + 2\varphi_3 + \varphi_4)}{\cos\varphi_1 \cos\varphi_4},$$

by using the Al-Kashi formula in the triangle A_1OA_4. Notice that for all $\Phi \in \Theta$, one has

(3.7)

$$D_{1,4}(\Phi) \leqslant \frac{1}{\cos^2\varphi_1} + \frac{1}{\cos^2\varphi_4} + \frac{2}{\cos\varphi_1\cos\varphi_4} = \left(\frac{1}{\cos\varphi_1} + \frac{1}{\cos\varphi_4}\right)^2 = G(\varphi_1,\varphi_4)^2,$$

where $G(x,y) = \frac{1}{\cos(x)} + \frac{1}{\cos(y)}$ for all $x, y \in [0,\pi/2]^2$.

To solve the problem of maximizing $D_{1,4}$ over Θ_A, we will maximize the mapping $\Phi \mapsto G(\varphi_1,\varphi_4)^2$ over Θ_A and use (3.7) to prove that both the optimal values and the maximizers of the aforementioned problems coincide. Hence, we investigate the optimization problem

$$\max_{\Phi\in\Theta_A} G(\varphi_1,\varphi_4).$$

With a slight abuse of notation, we denote by Φ^* a solution to this problem. Reasoning similarly to Case (i), we first notice that one has necessarily $\varphi_i^* \in (0,\frac{\pi}{2})$. Applying the Karush–Kuhn–Tucker theorem, we infer the existence of $(\lambda,\mu) \in \mathbb{R}^2$ such that

$$\frac{\sin(\varphi_i^*)}{\cos^2(\varphi_i^*)} = \lambda(1+\tan^2(\varphi_i^*)) + \mu$$

for $i = 1,4$, whereas $0 = \lambda(1+\tan^2(\varphi_i^*)) + \mu$ for $i = 2,3,5,6$. By exploiting these equalities, we get successively that $\varphi_2^* = \varphi_3^* = \varphi_5^* = \varphi_6^*$, $\mu = -\lambda(1+\tan^2(\varphi_2))$ and that φ_1^* and φ_4^* solve the equation

$$\frac{\sin\theta}{\cos^2\theta} = \lambda(\tan^2\theta - \tan^2(\varphi_2^*)). \tag{3.8}$$

Notice that $\varphi_1^* \neq \varphi_2^*$. Indeed, in the converse case, one has $\varphi_1^* = 0 = \varphi_2^* = \varphi_3^* = \varphi_5^* = \varphi_6^*$ and then $\varphi_4^* = \pi$, which is absurd. Similarly, one has $\varphi_4^* \neq \varphi_2^*$. Equation (3.8) hence rewrites

$$\frac{\sin\theta}{\cos^2\theta(\tan^2\theta - \tan^2(\varphi_2))} = \lambda.$$

We claim that the function h defined by

$$h : \theta \in [0,\varphi_2^*) \cup \left(\varphi_2^*, \frac{\pi}{2}\right) \mapsto \frac{\sin\theta}{\cos^2\theta(\tan^2\theta - \tan^2(\varphi_2^*))}$$

is one-to-one.[3] As a result, one has $\varphi_1^* = \varphi_4^*$, and we infer that

$$\varphi_1^* = \varphi_4^* = 2\arctan\left(\frac{1+\sqrt{\frac{A^2}{4}-3}}{2(1+\frac{A}{4})}\right) \quad \text{and} \quad \varphi_2^* = \varphi_3^* = \varphi_5^* = \varphi_6^* = \frac{\pi}{4} - \frac{\varphi_1^*}{2}.$$

Noticing that $\varphi_1^* + 2\varphi_2^* + 2\varphi_3^* + \varphi_4^* = \pi$ and according to the previous considerations, it follows that

$$\max_{\Phi\in\Theta_A} D_{1,4}(\Phi) = D_{1,4}(\Phi^*) = \left(\frac{1}{\cos\varphi_1^*} + \frac{1}{\cos\varphi_4^*}\right)^2 = G^2(\varphi_1^*,\varphi_4^*) = \max_{\Phi\in\Theta_A} G^2(\varphi_1,\varphi_4).$$

Moreover, the maximal value of A_1A_4 is $2/\cos\varphi_1^*$.

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[3]Indeed, since h is negative on $[0,\varphi_2)$ and positive on $(\varphi_2^*,\pi/2)$, we can deal separately with the intervals $[0,\varphi_2^*)$ and $(\varphi_2^*,\pi/2)$. On $[0,\varphi_2^*)$, one has $h(\theta) = \sin\theta\frac{1+\tan^2\theta}{\tan^2\theta-\tan^2(\varphi_2^*)}$. It follows that h is the product of the positive increasing sine function by $\theta \mapsto \frac{1+\tan^2\theta}{\tan^2\theta-\tan^2(\varphi_2^*)}$, which is negative decreasing. The conclusion follows. On $(\varphi_2^*,\pi/2)$, one has $h(\theta) = \frac{1}{\sin\theta}\big(1-\frac{\tan^2(\varphi_2^*)}{\tan^2\theta}\big)^{-1}$, and therefore h is the product of two positive decreasing functions, hence the result.

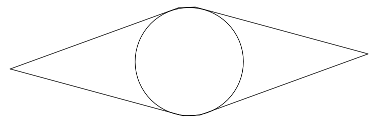

FIG. 11. *Hexagon maximizing A_1A_4 for $A = 4\sqrt{3}$.*

Case (iii): *the diameter is realized by A_1A_3.* Using computations similar to those for A_1A_4, we get

$$A_1A_3^2 = D_{1,3}(\varphi), \quad \text{with} \quad D_{1,3}(\varphi) = \frac{1}{\cos^2\varphi_1} + \frac{1}{\cos^2\varphi_3} - \frac{2\cos(\varphi_1 + 2\varphi_2 + \varphi_3)}{\cos\varphi_1\cos\varphi_3}.$$

Following along the same lines as Case (ii), and using the same notation, one shows successively that for all $\Phi \in \Theta_A$, $D_{1,3}(\Phi) \leqslant G^2(\varphi_1, \varphi_3)$ and

$$D_{1,4}(\Phi^*) = \max_{\Phi\in\Theta_A} D_{1,4}(\Phi) = \max_{\Phi\in\Theta_A} G^2(\varphi_1, \varphi_4) = \max_{\Phi\in\Theta_A} G^2(\varphi_1, \varphi_3).$$

Figure 11 illustrates the construction of the maximizing hexagon. As a consequence, there holds that

$$\max_{\Phi\in\Theta_A} D_{1,3}(\Phi) \leqslant \max_{\Phi\in\Theta_A} D_{1,4}(\Phi)$$

with equality if and only if there exists $\Phi^* \in \Theta$ such that $\pi = \varphi_1^* + 2\varphi_2^* + \varphi_3^*$. Because of the first equality constraint on the angles φ_i, it follows that $\varphi_2^* = \varphi_4^* + \varphi_5^* + \varphi_6^*$. Now, writing the optimality conditions for the problem of maximizing $D_{1,3}$ over Θ_A as for Case (ii), we infer that $\varphi_2^* = \varphi_4^* = \varphi_5^* = \varphi_6^*$. Thus, these angles are necessarily equal to 0, which contradicts Lemma 3.1. This shows that Case (iii) cannot arise.

Comparison between the three cases. According to the previous analysis, one has $A_1A_4 > A_1A_3$ for any optimal set K^*. Notice, moreover, that $\max_{\Phi\in\Theta_A} A_1A_2 = 2\tan(\Phi^*)$ and $\max_{\Phi\in\Theta_A} A_1A_4 = \frac{2}{\cos(\Phi^*)}$, with

$$\Phi^* = 2\arctan\left(\frac{1 + \sqrt{\frac{A^2}{4} - 3}}{2(1 + \frac{A}{4})}\right).$$

We then infer that the solution of problem (3.6) corresponds to Case (ii).

Therefore, the optimization problem has a unique solution (whenever $A \geqslant 2\sqrt{3}r_0^2$) given by the hexagon with inner radius r_0, which is tangent at every side to its inner circle, such that the semicircle center angles are given by

$$\varphi_1^* = \varphi_4^* = 2\arctan\left(\frac{1 + \sqrt{\frac{A^2}{4r_0^4} - 3}}{2(1 + \frac{A}{4r_0^2})}\right) \quad \text{and} \quad \varphi_2^* = \varphi_3^* = \varphi_5^* = \varphi_6^* = \frac{\pi}{4} - \frac{\varphi_1^*}{2}.$$

4. Proof of Theorem 2.9. Before solving problem (1.3), we first investigate the following auxiliary problem:

$$\max\{d(K),\ |K| = A,\ r(K) = r,\ \mathrm{Diam}(K) = D\}, \tag{4.1}$$

where (A, D, r) denote the triple of positive numbers.

To help the forthcoming analysis and since several cases must be distinguished, let us plot in Figure 12 some elements of the Blaschke–Santaló diagram for the diameter and inradius, the area being fixed.

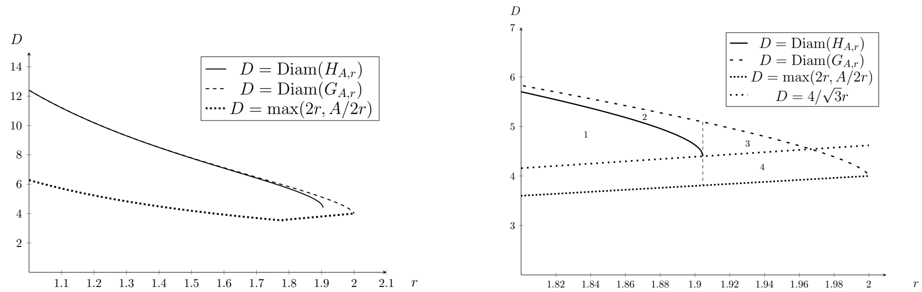

FIG. 12. *Left: Blaschke–Santaló diagram for* $(r(K), \mathrm{Diam}(K))$ *under the condition* $A = |K| = 4\pi$*. Right: zoom on the right part of the diagram.*

Remark 4.1. Let us comment on the construction of Figure 12. The lower part of the boundary consists of two pieces. The first one is obtained by using that for every convex set K, one has

$$|K| < 2\,\mathrm{Diam}(K) r(K)$$

with equality if and only if $\mathrm{int}(K) = \emptyset$ (see [6]), and the (straight) right one is obtained by using that $\mathrm{Diam}(K) \geqslant 2r(K)$ with equality if and only if K is a ball. The part of the boundary shown as a solid line is determined by using the second item of Theorem 2.7. Finally, the upper part of the boundary is obtained by using (2.5) in Remark 2.5.

First, notice that, according to the so-called isodiametric inequality, one has $r \leqslant \mathrm{Diam}(K)/2 \leqslant \sqrt{A/\pi}$ and $\mathrm{Diam}(K) \leqslant \mathrm{Diam}(G_{A,r})$ for every convex body K having as inradius r and area A, where the 2-cap body $G_{A,r}$ has been introduced in Definition 2.4.

The main ingredient of the proof of Theorem 2.9 is the following lemma about the maximization of the density functional $d(\cdot)$, whose proof is postponed to the end of this section for the sake of clarity.

LEMMA 4.2. *Let* $A > 0$ *and* $D > 0$.

1. *Let* $r \in \left(0, \sqrt{A/2\sqrt{3}}\ \right]$ *and* $D = \mathrm{Diam}(H_{A,r})$*. One has*

$$\max\{d(K),\ r(K) = r,\ \mathrm{Diam}(K) \leqslant D,\ |K| = A\} = 1.$$

2. *Let* $r \in \left(0, \sqrt{A/2\sqrt{3}}\right]$ *and* $D > \mathrm{Diam}(H_{A,r})$ *or* $(r, D) \in \{(r, D) \mid r > \sqrt{A/2\sqrt{3}}$ *and* $D \in [4/\sqrt{3}r, \mathrm{Diam}(G_{A,r})]\}$*. Then, one has*

$$\max\{d(K),\ r(K) = r,\ \mathrm{Diam}(K) = D,\ |K| = A\} = \frac{A}{|H^{D,r}|}.$$

3. *Let* $(r, D) \in \{(r, D) \mid r > \sqrt{A/2\sqrt{3}}$ *and* $D \leqslant \min\{4/\sqrt{3}r, \mathrm{Diam}(G_{A,r})\}\}$*. Then, one has*

$$\max\{d(K),\ r(K) = r,\ \mathrm{Diam}(K) = D,\ |K| = A\} = \frac{A}{|H_r^*|}.$$

Let us come back to the solution of problem (1.3).

Let us distinguish between several cases, depending on the possible values of $r(K)$ and $\mathrm{Diam}(K)$. For that purpose, let us notice that

$$\sup_{K \in \mathcal{A}_{r_0,A}} J_t(K) = \max_{1 \leqslant i \leqslant 4} \sup_{K \in \mathcal{A}^i_{r_0,A}} J_t(K)$$

with the following partition of $\mathcal{A}_{r_0,A}$:

$$\mathcal{A}^1_{r_0,A} = \{K \in \mathcal{A}_{r_0,A} \mid r(K) = r,\ r_0 \leqslant r \leqslant \sqrt{A/(2\sqrt{3})} \text{ and } \mathrm{Diam}(K) \leqslant D_{H_{A,r}})\},$$
$$\mathcal{A}^2_{r_0,A} = \{K \in \mathcal{A}_{r_0,A} \mid r(K) = r,\ r_0 \leqslant r \leqslant \sqrt{A/(2\sqrt{3})} \text{ and } \mathrm{Diam}(K) \in (D_{H_{A,r}}, D_{G_{A,r}}]\},$$
$$\mathcal{A}^3_{r_0,A} = \{K \in \mathcal{A}_{r_0,A} \mid r(K) = r,\ r > \sqrt{A/(2\sqrt{3})} \text{ and } \mathrm{Diam}(K) \in (4/\sqrt{3}r, D_{G_{A,r}})\},$$
$$\mathcal{A}^4_{r_0,A} = \{K \in \mathcal{A}_{r_0,A} \mid r(K) = r,\ r > \sqrt{A/(2\sqrt{3})} \text{ and } \mathrm{Diam}(K) \leqslant 4/\sqrt{3}r\},$$

where we introduce the notations $D_{H_{A,r}} = \mathrm{Diam}(H_{A,r})$ and $D_{G_{A,r}} = \mathrm{Diam}(G_{A,r})$ for the sake of readability. $\mathcal{A}^i_{r_0,A}$ corresponds to zone 1 in Figure 12.

Let us investigate each problem separately.

Solution of problem $\sup_{K\in\mathcal{A}^1_{r_0,A}} J_t(K)$. Let $r \in [r_0, \sqrt{A/(2\sqrt{3})}]$ and $K \in \mathcal{A}^1_{r_0,A}$ such that $r(K) = r$. According to Lemma 4.2, one has

$$J_t(K) \leqslant t + (1-t)\frac{\sqrt{\pi} D_{H_{A,r}}}{2\sqrt{A}} = t + (1-t)\frac{\sqrt{\pi}}{2\sqrt{A}}\left(\frac{1}{3r}\left(2A + \sqrt{A^2 - 12r^4}\right)\right)$$

with equality if $K = H_{A,r}$. Moreover, the mapping $r \mapsto \frac{1}{3r}\left(2A + \sqrt{A^2 - 12r^4}\right)$ is decreasing on $(0, +\infty)$. As a consequence, we infer that

$$\boxed{\max_{K\in\mathcal{A}^1_{r_0,A}} J_t(K) = J_t(H_{A,r}) = t + (1-t)\frac{\sqrt{\pi}}{2\sqrt{A}}\left(\frac{1}{3r_0}\left(2A + \sqrt{A^2 - 12r_0^4}\right)\right)},$$

and the maximum is reached by the p-hexagon H_{A,r_0}.

Solution of problem $\sup_{K\in\mathcal{A}^2_{r_0,A}\cup\mathcal{A}^3_{r_0,A}} J_t(K)$. Let $K \in \mathcal{A}^2_{r_0,A}$ and $r = r(K)$ such that $r \in [r_0, \sqrt{A/(2\sqrt{3})}]$. According to Lemma 4.2, one has

$$J_t(K) \leqslant t\frac{A}{|H^{D,r}|} + (1-t)\frac{\sqrt{\pi}D}{2\sqrt{A}}. \tag{4.2}$$

Let us first maximize the function in the right-hand side by solving the problem

$$\max_{(D,r)\in\mathcal{Z}} \psi_{t,A}(r, D), \quad \text{where} \quad \psi_{t,A}(r, D) = t\frac{A}{|H^{D,r}|} + (1-t)\frac{\sqrt{\pi}D}{2\sqrt{A}}, \tag{4.3}$$

with

$$\mathcal{Z} = \{(r, D) \mid r_0 \leqslant r \leqslant \sqrt{A/(2\sqrt{3})} \text{ and } D \in (D_{H_{A,r}}, D_{G_{A,r}}) \\ \text{or } r > \sqrt{A/(2\sqrt{3})} \text{ and } D \geqslant 4/\sqrt{3}r\}.$$

This corresponds to dealing with zones 2 and 3 of Figure 12. First, note that

$$\frac{d\psi_{t,A}}{dr}(r, D) = \frac{-tA(2D\sqrt{D^2 - 4r^2} - D^2 + 8r^2)}{\sqrt{D^2 - 4r^2}\left(2Dr - r\sqrt{D^2 - 4r^2}\right)^2}.$$

Moreover, if $D^2 \leqslant 8r^2$, we conclude directly that $2D\sqrt{D^2 - 4r^2} - D^2 + 8r^2$ is positive. In the converse case, the sign of $2D\sqrt{D^2 - 4r^2} - D^2 + 8r^2$ is also the sign of $4D^2(D^2 - 4r^2) - (D^2 - 8r^2)^2$, namely $3D^4 - 64r^4$. Notice that, in that zone, one has $D \geqslant$

$4r/\sqrt{3} \geqslant r\sqrt{8/\sqrt{3}}$, which means precisely that $3D^4 - 64r^4 > 0$. In all cases, we then have $2D\sqrt{D^2-4r^2} - D^2 + 8r^2 > 0$, and we infer that $\frac{d\psi_{t,A}}{dr}(r,D) < 0$. It follows that either $D = D_{H_{A,r}}$ or $r = r_0$. The case $D = D_{H_{A,r}}$ has been investigated when solving problem $\sup_{K\in\mathcal{A}^1_{r_0,A}} J_t(K)$ above. As a consequence, one has necessarily $r = r_0$ at the maximum.

It then remains to investigate the variations of the criterion with respect to the parameter D at $r = r_0$. One has

$$\frac{d^2\psi_{t,A}}{dD^2}(r_0,D) = -\frac{2At\left(\sqrt{D^2-4r_0^2}(14r_0^2-5D^2)+4D(D^2-3r_0^2)\right)}{r_0(D^2-4r_0^2)^{3/2}(2D-\sqrt{D^2-4r_0^2})^3}.$$

Note that $\sqrt{D^2-4r_0^2}(14r_0^2-5D^2)+4D(D^2-3r_0^2) = r_0^3\left(\sqrt{X^2-4}(14-5X^2)+4X(X^2-3)\right)$, with $X = D/r_0$. Recall that the function $X \mapsto \sqrt{X^2-4}(14-5X^2)+4X(X^2-3)$ has a unique zero X_0 on $[4/\sqrt{3},+\infty)$. Moreover, a tedious but easy analysis yields that $\sqrt{X^2-4}(14-5X^2)+4X(X^2-3) \geqslant 0$ on $[4/\sqrt{3},X_0]$ and $\sqrt{X^2-4}(14-5X^2)+4X(X^2-3) < 0$ elsewhere.

It follows that the mapping $D \mapsto \frac{d}{dD}\psi_{t,A}(r_0,\cdot)$ is decreasing on $[4\sqrt{3}r_0, X_0r_0]$ and increasing on $[X_0r_0,+\infty)$. Its minimal value is

$$\frac{d\psi_{t,A}}{dD}(r_0,X_0r_0) = \frac{-A\gamma_0}{r_0^3}t + (1-t)\frac{\sqrt{\pi}}{2\sqrt{A}},$$

where γ_0 is defined by (2.9).

The minimal value of $\frac{d\psi_{t,A}}{dD}(r_0,\cdot)$ is then nonnegative whenever $t \in [0,t_{A,r_0}]$ and negative whenever $t \in (t_{A,r_0},1]$, where t_{A,r_0} is given by (2.8). If $t \in [0,t_{A,r_0}]$, we infer from the above analysis that $D \mapsto \psi_{t,A}(r_0,D)$ is increasing on $(D_{H_{A,r}}, D_{G_{A,r}})$ and the maximum is achieved at $D = D_{G_{A,r}}$.

If $t \in (t_{A,r_0},1]$, the minimal value of $\frac{d\psi_{t,A}}{dD}\psi_{t,A}(r_0,\cdot)$ is negative. Notice, moreover, that

$$\frac{d\psi_{t,A}}{dD}(r_0,4r_0/\sqrt{3}) = \lim_{D\to+\infty}\frac{d\psi_{t,A}}{dD}(r_0,D) = \frac{(1-t)\sqrt{\pi}}{2\sqrt{A}}.$$

Combining this information about $\frac{d\psi_{t,A}}{dD}$ yields the existence[4] of $z^1_{t,r_0,A} \in [4/\sqrt{3},X_0)$ and $z^2_{t,r_0,A} \in [X_0,+\infty)$ such that the mapping $D \mapsto \psi_{t,A}(r_0,\cdot)$ is increasing on $(4/\sqrt{3}r_0, z^1_{t,r_0,A}r_0)$, decreasing on $(z^1_{t,r_0,A}, z^2_{t,r_0,A})$, and increasing on $(z^2_{t,r_0,A},+\infty)$.

Now, using that $\mathrm{Diam}(H_{A,r_0}) = \frac{1}{3r_0}\left(2A+\sqrt{A^2-12r_0^4}\right)$ and that the mapping $[2\sqrt{3},+\infty) \ni A \mapsto \frac{1}{3r_0}\left(2A+\sqrt{A^2-12r_0^4}\right)$ is increasing, we claim that

$$\frac{A}{r_0^2} \geqslant 2X_0 - \sqrt{X_0^2-4} \Longleftrightarrow D_{H_{A,r_0}} \geqslant X_0r_0.$$

Since $\psi_{t,A}(r_0,\cdot)$ decreases on $[X_0r_0, z^2_{t,r_0,A}]$ and increases on $(z^2_{t,r_0,A},+\infty)$, we infer

[4] Moreover, $z^1_{t,r_0,A}$ and $z^2_{t,r_0,A}$ are the two solutions of the equation $d\psi_{t,A}/dD(r_0,r_0z) = 0$ with unknown z on $[4/\sqrt{3},+\infty)$:

$$\frac{2\sqrt{z^2-4}-z}{\sqrt{z^2-4}(2z-\sqrt{z^2-4})^2} = \frac{(1-t)\sqrt{\pi}r_0^3}{2tA^{3/2}}.$$

that, under the smallness condition (2.10) on r_0, one has successively

$$\max_{(D,r)\in\mathcal{Z}} \psi_{t,A}(r,D) = \max_{D\in(D_{H_{A,r_0}},D_{G_{A,r_0}})} \psi_{t,A}(r_0,D)$$
$$= \max\{\psi_{t,A}(r_0,D_{H_{A,r_0}}),\psi_{t,A}(r_0,D_{G_{A,r_0}})\}.$$

To solve the problem arising in the right-hand side, let us introduce

$$\Delta_{r_0,A}(t) = \psi_{t,A}(r_0,D_{G_{A,r_0}}) - \psi_{t,A}(r_0,D_{H_{A,r_0}}).$$

One computes

$$\Delta_{r_0,A}(0) = \frac{\sqrt{\pi}}{2\sqrt{A}}(D_{G_{A,r_0}} - D_{H_{A,r_0}}), \quad \Delta_{r_0,A}(1) = A\left(\frac{1}{|D_{G_{A,r_0}}|} - \frac{1}{|D_{H_{A,r_0}}|}\right).$$

Let $M_{t,r_0,A} = \max\{\psi_{t,A}(r_0,D_{H_{A,r_0}}),\psi_{t,A}(r_0,D_{G_{A,r_0}})\}$. Hence, since $\Delta_{r_0,A}$ is affine, $\Delta_{r_0,A}(0) > 0$, and $\Delta_{r_0,A}(1) < 0$, we infer the existence of $t^*_{A,r_0} \in [0,1]$ such that on $[0,t^*_{A,r_0}]$, $M_{t,r_0,A} = \psi_{t,A}(r_0,D_{G_{A,r_0}})$ and on $(t^*_{A,r_0},1]$, $M_{t,r_0,A} = \psi_{t,A}(r_0,D_{H_{A,r_0}})$. Notice that, by construction, one has $\Delta_{r_0,A}(t^*_{A,r_0}) = 0$ leading to the expression (2.11) of t^*_{A,r_0}, and one has necessarily $t^*_{A,r_0} \geqslant t_{A,r_0}$ according to the analysis of the case where $t \in [0,t_{A,r_0}]$.

Let us come back to the solution of problem $\sup_{K\in\mathcal{A}^2_{r_0,A}\cup\mathcal{A}^3_{r_0,A}} J_t(K)$. We proved that, under the smallness assumption (2.10) on r_0, G_{A,r_0} and H_{A,r_0} are the only possible solutions of problem $\max_{(D,r)\in\mathcal{Z}}\psi_{t,A}(r,D)$. Noting that (4.2) is an equality whenever K is either equal to G_{A,r_0} or H_{A,r_0}, we infer to the end that

$$\boxed{\max_{K\in\mathcal{A}^2_{r_0,A}\cup\mathcal{A}^3_{r_0,A}} J_t(K) = \begin{cases} J_t(G_{A,r_0}) & \text{if } t\in[0,t^*_{A,r_0}],\\ J_t(H_{A,r_0}) & \text{if } t\in(t^*_{A,r_0},1].\end{cases}}$$

Estimate of $\sup_{K\in\mathcal{A}^4_{r_0,A}} J_t(K)$. According to Lemma 4.2, one has

$$J_t(K) \leqslant t\frac{A}{|H^*_r|} + (1-t)\frac{\sqrt{\pi}D}{2\sqrt{A}} = t\frac{A\sqrt{3}}{2r^2} + (1-t)\frac{\sqrt{\pi}D}{2\sqrt{A}}. \tag{4.4}$$

Since $D\mapsto t\frac{A\sqrt{3}}{2r^2}+(1-t)\frac{\sqrt{\pi}D}{2\sqrt{A}}$ is increasing, we infer that the solutions of the problem

$$\max_{(r,D)\in\hat{\mathcal{Z}}} t\frac{A\sqrt{3}}{2r^2} + (1-t)\frac{\sqrt{\pi}D}{2\sqrt{A}},$$

with $\hat{\mathcal{Z}} = \{(r,D) \mid \sqrt{A/(2\sqrt{3})} \leqslant r \leqslant \sqrt{A/\pi} \text{ and } 2r \leqslant D \leqslant 4/\sqrt{3}r\}$, satisfy necessarily $D = 4r/\sqrt{3}$. According to Lemma 4.2, we deduce successively that

$$\max_{K\in\mathcal{A}^4_{r_0,A}} J_t(K) = \max_{(r,D)\in\hat{\mathcal{Z}}} t\frac{A\sqrt{3}}{2r^2} + (1-t)\frac{\sqrt{\pi}D}{2\sqrt{A}}$$
$$\leqslant \max_{K\in\mathcal{A}^2_{r_0,A}\cup\mathcal{A}^3_{r_0,A}} J_t(K).$$

Moreover, we have proved that every solution of the last problem in the right-hand side must satisfy $r(K) = r_0$, proving that the last inequality is in fact strict.

This concludes the proof of Theorem 2.9.

Proof of Lemma 4.2. We investigate the three different cases.

Case 1. For $r \in \left(0, \sqrt{A/2\sqrt{3}}\right]$ (zone 1 of Figure 12), since $H_{A,r}$ is admissible and since $d(K) \leqslant 1$ for every convex body K, the first equality is obvious by choosing $K = H_{A,r}$.

Case 2. Let us deal with zones 2 and 3 of Figure 12. We first assume that $r \in \left(0, \sqrt{A/2\sqrt{3}}\right]$ and $D \geqslant \mathrm{Diam}(H_{A,r})$. Let K be a maximizer for the problem

$$\max\{d(K),\ r(K) = r,\ \mathrm{Diam}(K) = D,\ |K| = A\}.$$

Denoting by K^T the smallest convex set tiling the plane and containing K, one has

$$\mathrm{Diam}(K^T) \geqslant D, \qquad r(K^T) \geqslant r.$$

Then, by using Theorem 2.7 and by monotonicity of $|H^{D,r}|$ with respect to D and r, we have

$$|K^T| \geqslant |H^{\mathrm{Diam}(K^T), r(K^T)}| \geqslant |H^{D,r}|.$$

As a consequence, we infer that $d(K) = \frac{|K|}{|K^T|} \leqslant \frac{A}{|H^{D,r}|}$. Notice that the mapping $A \mapsto \mathrm{Diam}(H_{A,r})$ is increasing on its definition set. Using this remark and according to Remark 2.5, since $D \in [\mathrm{Diam}(H_{A,r}), \mathrm{Diam}(G_{A,r})]$, we have $|G^{D,r}| \leqslant A \leqslant |H^{D,r}|$. Moreover, there holds that $G^{D,r} \subset H^{D,r}$ by construction. Let us show that $(G^{D,r})^T = H^{D,r}$. Since

$$\mathrm{Diam}((G^{D,r})^T) \geqslant D \quad \text{and} \quad r((G^{D,r})^T) \geqslant r,$$

one has $|(G^{D,r})^T| \geqslant |H^{D,r}|$, showing that $(G^{D,r})^T = H^{D,r}$. Now, consider a convex set K of area A chosen such that $G^{D,r} \subset K \subset H^{D,r}$. Then, since $(G^{D,r})^T = H^{D,r}$, one has $K^T = H^{D,r}$ by continuity and $d(K) = \frac{A}{|H^{D,r}|}$. Therefore, the supremum is reached, hence the conclusion.

Now, assume that

$$(r, D) \in \{(r, D) \mid r > \sqrt{A/2\sqrt{3}} \text{ and } D \in [4/\sqrt{3}r, \mathrm{Diam}(G_{A,r})]\}.$$

Then, a convex set K with inradius r and area A cannot be tiling according to Theorem 2.7. Nevertheless, one checks easily that the diameter of the hexagon $H^{D,r}$ is equal to D if and only if $D \geqslant 4/\sqrt{3}r$. Therefore, the same argument as below allows one to conclude similarly.

Case 3. If $(r, D) \in \{(r, D) \mid r > \sqrt{A/2\sqrt{3}} \text{ and } D \leqslant \min\{4/\sqrt{3}r, \mathrm{Diam}(G_{A,r})\}\}$ (zone 4 of Figure 12), then the diameter of $H^{D,r}$ differs from D. Indeed, this is an easy consequence of the first item of Theorem 2.7.

We claim (see below for a proof), moreover, that *the regular hexagon H_r^* is the tiling convex set with inradius r and area A having the lowest diameter*, or similarly that *the regular hexagon H_r^* is the tiling convex set with inradius r and diameter D having the lowest area*.

Let K be a convex set such that $r(K) = r$ and $\mathrm{Diam}(K) = D$, with (r, D) belonging to the zone described above. One has $\mathrm{Diam}(K^T) \geqslant D$ and $r(K^T) \geqslant r$. As a consequence of the claim above, one has necessarily $\mathrm{Diam}(K^T) \geqslant D(H^*_{r(K^T)})$. Since K^T is tiling, one has

$$|K^T| \geqslant |H^{\mathrm{Diam}(K^T), r(K^T)}| \geqslant |H_r^*|$$

according to Theorem 2.7 and the claim above. It follows that for every convex K in the aforementioned zone of the Blaschke diagram, one has $d(K) \leqslant \frac{A}{|H_r^*|}$.

Let K be a convex set of area A such that $G^{\mathrm{Diam}(K^T), r(K^T)} \subset K \subset H_r^*$. We infer from the previous analysis that $K^T = H_r^*$, and $d(K) = A/|H_r^*|$, so that it maximizes the density.

To conclude, it remains to prove the claim above. For a given $r > 0$, we investigate the problem

$$\inf\{\mathrm{Diam}(T),\ T \text{ tiling, and } r(T) \geqslant r\}.$$

Notice first that, by mimicking the arguments used to prove Theorem 2.7, one shows that there exists a solution T^* to this problem, and necessarily $r(T^*) = r$.

Moreover, according to Theorem 2.7, the solution of the more constrained problem

$$\inf\{\mathrm{Diam}(T),\ T \text{ tiling},\ r(T) = r,\ \text{ and } |T| = A\},$$

with $A \geqslant 2\sqrt{3}r^2$, is the p-hexagon described in Definition 2.3. Then, by writing

$$\begin{aligned} &\inf\{\mathrm{Diam}(T),\ T \text{ tiling, and } r(T) \geqslant r\} \\ &\qquad = \inf_{A \geqslant 2\sqrt{3}r^2} \inf\{\mathrm{Diam}(T),\ T \text{ tiling},\ r(T) = r,\ \text{ and } |T| = A\}, \end{aligned}$$

and using that the area of the p-hexagon introduced in Definition 2.3 is an increasing function of the diameter (see Remark 2.8), we infer that T^* is such that $|T^*| = 2\sqrt{3}r^2$. In other words, $T^* = H_r^*$ and we are done. ∎

5. Conclusion and perspectives. In this paper, we solve several problems in convex geometry, paying attention to the class of plane tiling domains. These problems were motivated by issues in biology related to the shape of eggs of some crustaceans. Of course, the three-dimensional situation is certainly more relevant, but a complete mathematical analysis, like in this paper, seems out of range. Nevertheless, some numerical simulations will be done for this problem.

We foresee investigating a related issue in a forthcoming paper, namely the precise determination of the Blaschke–Santaló diagram; see Figure 12 for the area, diameter, and inradius (sometimes known as the A, D, r problem).

Appendix A. Existence of K^T. Since the set of convex bodies contained in a compact D is itself compact for the Hausdorff topology and since the restriction of the Lebesgue measure to this set is continuous [7], it is enough to show that the set of convex tiling domains T, with $r(T) \geqslant \varepsilon > 0$, is closed for the Hausdorff topology. To prove this claim, let $(T_n)_{n\in\mathbb{N}}$ be a sequence of convex tiling domains converging to T. Then T is necessarily convex. Since T_n is tiling for every n, there exists a sequence $(\tau_{n,i})_{i\in\mathbb{N}}$ of affine isometries such that $\mathbb{R}^2 \subset \bigcup_{i\in\mathbb{N}} \tau_{n,i}(T_n)$; in other words,

$$\forall R > 0, \qquad D(0,R) \subset \bigcup_{i\in\mathbb{N}} \tau_{n,i}(T_n).$$

Without loss of generality, we assume that every domain T_n contains the origin and the distance of $\tau_{n,i}(T_n)$ (ith copy of T_n) to the origin is nondecreasing with respect to i for a given n.

Let $D = \sup(\mathrm{Diam}(T_n))$, $R > 0$, and let $N = N(R, D, \varepsilon)$ be the minimal number of squares with circumradius ε to tessellate a disk of radius $R + D$. Then, we claim that

$$D(0,R) \subset \bigcup_{i=0}^{N} \tau_{n,i}(T_n) \subset D(0, R + 2(N+1)D)$$

for every $n \in \mathbb{N}$.

Indeed, every copy $\tau_{n,i}(T_n)$ contains such a square C_i, and any copy that contains a point of $D(0,R)$ is necessarily included in $D(0,R+D)$, and so $C_i \subset D(0,R+D)$. Let K be the smallest integer such that $D(0,R) \subset \bigcup_{i=0}^{K} \tau_{n,i}(T_n)$. Then, $\bigcup_{i=0}^{K} C_i$ is the disjoint union of sets included in $D(0,R+D)$. A volume comparison yields $K \leqslant N$ so that we have the first inclusion. The second one is straightforward since the distance of $\bigcup_{i=0}^{N} \tau_{n,i}(T_n)$ to the origin cannot be greater than $2(N+1)D$.

To show that T is tiling, let us decompose $\tau_{n,i}$ as $\tau_{n,i} = r_{n,i} + t_{n,i}$, where $r_{n,i}$ is a rotation and $t_{n,i}$ is a translation assimilated (with a slight abuse of notation) to a vector such that $\|t_{n,i}\| \leqslant R + 2(N+1)D$ for all $n \in N$ and $i \leqslant N$. Applying a compactness argument yields the existence of τ_i and $\varphi : \mathbb{N} \mapsto \mathbb{N}$ such that $\tau_{\varphi(n),i} \to \tau_i$ as $n \to +\infty$. Therefore, one has $\tau_{\varphi(n),i}(T_{\varphi(n)}) \to \tau_i(T)$ as $n \to +\infty$. Furthermore, since $\mathrm{int}(\tau_{\varphi(n),i}(T_{\varphi(n)})) \cap \mathrm{int}(\tau_{\varphi(n),j}(T_{\varphi(n)})) = \emptyset$ for $i \neq j$, we get $\mathrm{int}(\tau_i(T)) \cap \mathrm{int}(\tau_j(T)) = \emptyset$ and the sequence $\bigcup_{i=0}^{N} \tau_{\varphi(n),i}(T_{\varphi(n)})$ converges to $\bigcup_{i=0}^{N} \tau_i(T)$. Finally, by the stability of the inclusion for the Hausdorf metric, one has $D(0,R) \subset \bigcup_{i=0}^{N} \tau_i(T)$.

Using that the last inclusion holds true for every $R > 0$, we infer that T is a convex tiling domain.

Appendix B. Proof of Theorem 2.2. Let us first consider the case of tiling domains.

Case of tiling domains. Let K be a tiling domain, and set $D = \mathrm{Diam}(K)$. There exists a family $\{\tau_i\}_{i\in\mathbb{N}}$ of isometries such that $\mathbb{R}^2 = \bigcup_{i\in\mathbb{N}} \tau_i(K)$. For $R > 2D$, define $P(R) = \bigcup_{\tau_i(K)\subset D(0,R)} \tau_i(K)$.

Then, by maximality of the diameter, and since K is tiling, one has necessarily $D(0,R-D) \subset P(R)$, and therefore $\sharp\{i, \tau_i(K) \subset D(0,R)\}|K| \geqslant \pi(R-D)^2$ and

$$\frac{2R}{\sqrt{\sharp\{i,\tau_i(K)\subset D(0,R)\}}\,\mathrm{Diam}(K)} \leqslant \frac{2R\sqrt{|K|}}{\sqrt{\pi}(R-D)\,\mathrm{Diam}(K)}.$$

Letting $R \to \infty$, we obtain

$$\limsup_{R\to+\infty} \frac{2R}{\sqrt{\sharp\{i,\tau_i(K)\subset D(0,R)\}}\,\mathrm{Diam}(K)} \leqslant \frac{2\sqrt{|K|}}{\sqrt{\pi}\,\mathrm{Diam}(K)}.$$

Finally, passing to the infimum over all packings yields

$$D'_\infty(K) \leqslant \frac{2\sqrt{|K|}}{\sqrt{\pi}\,\mathrm{Diam}(K)}.$$

The conclusion follows by combining this estimate with (2.2).

We now investigate the general case.

General case. In view of proving (2.3), we will use the following result due to Kuperberg and Kuperberg [9].

PROPOSITION B.1. *Every convex set $K \in K$ is contained in a tiling hexagon K_{kup} satisfying $|K_{kup}|/|K| \leqslant 2/\sqrt{3}$. Moreover, K_{kup} is a p-hexagon, in other words a hexagon with two opposite parallel sides having the same length.*[5]

Let $K \in \mathcal{K}$, and consider the tiling K_{kup} provided by Proposition B.1. We define a packing of K by placing adequately a copy of K in each cell of K_{kup}. Denoting by

[5]Recall that every p-hexagon tiles the plane.

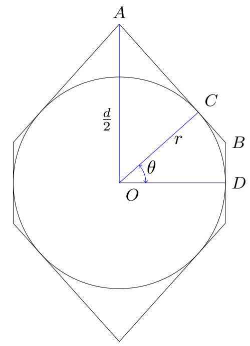

FIG. 13. *The hexagon $H^{d,r}$.*

$\{\tau_i\}_{i\in\mathbb{N}}$ the family of isometries used to define this packing, we deduce that

$$\begin{aligned} D'_\infty(K) &\leqslant \limsup_{R\to\infty} \frac{2R}{\sqrt{\sharp\{i, \tau_i(K_{kup}) \subset D(0,R)\}}\,\mathrm{Diam}(K)} \\ &= \frac{\mathrm{Diam}(K_{kup})}{\mathrm{Diam}(K)} \limsup_{R\to\infty} \frac{2R}{\sqrt{\sharp\{i, \tau_i(K_{kup}) \subset D(0,R)\}}\,\mathrm{Diam}(K_{kup})} \\ &= \frac{2\sqrt{|K_{kup}|}}{\sqrt{\pi}\,\mathrm{Diam}(K)} \leqslant \sqrt{\frac{2}{\sqrt{3}}}\frac{2\sqrt{|K|}}{\sqrt{\pi}\,\mathrm{Diam}(K)} \end{aligned}$$

by using the computation above in the case of tiling sets and Proposition B.1.

The expected conclusion follows.

Appendix C. Diameter of $H_{A,r}$ and area of $H^{D,r}$. To avoid any confusion with the notation we will use within this proof, let us denote temporarily by d the diameter of the hexagon $H^{d,r}$ we will consider and by a its area. Let us introduce the points A, B, C, D, and O, as plotted on Figure 13.

The area $|H^{d,r}|$ is equal to four times the area of the pentagon $ACBDO$, which is the sum of the area of ACO and the area of $CBDO$, which is twice the area of the triangle BDO. Hence, one has $|H^{d,r}| = 4 \times (|ACO| + 2|BDO|)$. Let $\theta = \widehat{COD}$. One has $\sin\theta = 2r/d$. Then, we compute $|ACO| = \frac{dr}{4}\cos\theta = \frac{r}{4} \times \sqrt{d^2 - 4r^2}$. In the orthonormal basis $(O; \frac{\overrightarrow{OD}}{OD}, \frac{\overrightarrow{OA}}{OA})$, the coordinates of B are $(r, r\frac{1-\cos(\theta)}{\sin(\theta)})$, and since $\theta = \arcsin(2r/d)$, we get $|BDO| = \frac{r}{4}(d - \sqrt{d^2 - 4r^2})$.

Finally, we get that $|H^{d,r}| = 2rd - r\sqrt{d^2 - 4r^2}$. By inverting the relation $a = 2rd - r\sqrt{d^2 - 4r^2}$ (whenever $a \geqslant 2\sqrt{3}r^2$ and $d \geqslant 2r$), we get that

$$d = \alpha(a, r) = \frac{1}{3r}\left(2a + \sqrt{a^2 - 12r^4}\right),$$

hence the expression of $\mathrm{Diam}(H_{a,r})$ with respect to the parameter a.

Acknowledgment. The authors warmly thank Nicolas Rabet for very stimulating discussions about modeling issues.

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