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SIMPLE MODULES OVER THE MULTIPARAMETER QUANTUM FUNCTION ALGEBRA AT ROOTS OF 1

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Abstract. We construct essentially all the irreducible modules for the multiparameter quantum function algebra $F_\varepsilon^\varphi[G]$, where G is a simple simply connected complex algebraic group, and ε is a root of unity.

1. Introduction

In this paper we continue the study of the representation theory of the multiparameter quantum function algebra $F_\varepsilon^\varphi[G]$ which began with [CV2], [CV3]. Here ε is a primitive ℓ-th root of unity in the complex field $\mathbb{C}$, and ℓ is a $\varphi-good$ integer. In [CV2] we showed that $F_\varepsilon^\varphi[G]$ contains a central subalgebra isomorphic to the coordinate ring of G, where G is a simple simply connected complex algebraic group. Given a simple module V over $F_\varepsilon^\varphi[G]$, one can define its central character in G. It turns out that the representation theory of $F_\varepsilon^\varphi[G]$ is constant on the varieties $X_{w_1,w_2} := B^- w_1 B^- \cap B w_2 B$, where w_1, w_2 are in the Weyl group, and B, B^- are opposite Borel subgroups. This allows us to study the representation theory of $F_\varepsilon^\varphi[G]$ by means of certain localized quotients $F_\varepsilon^\varphi[G]_{w_1,w_2}$ of $F_\varepsilon^\varphi[G]$. In [CV3] we showed that these are Azumaya algebras, so that their spectrum S_{w_1,w_2}^φ coincides with the spectrum of their centers. Moreover S_{w_1,w_2}^φ is a Galois covering of X_{w_1,w_2}. Here we are interested in the construction of simple modules over $F_\varepsilon^\varphi[G]$. Such problems have been studied by Levendorski and Soibelman for compact groups in [LS], and by Hodges, Levasseur and Toro in the algebraic case at generic q in [HLT].

The main result of this paper is to show that the procedure introduced in [DP] for the construction of simple modules over $F_\varepsilon[G]$ also works in the multiparameter case (Theorem 3.5). For each pair (w_1, w_2) in $W \times W$, we

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shall therefore construct all the simple $F_\varepsilon^\varphi[G]$-modules lying over a dense subset of X_{w_1,w_2}.

We also show that the representation theory of $F_\varepsilon^\varphi[B]$ is closely related to the representation theory of $F_\varepsilon[B]$ (Theorem 4.6). We shall first recall the definition of the objects we deal with, and results from [DP], [CV2] and [CV3].

Notations.

We denote by $\mathbb{C}$ the complex numbers, by $\mathbb{R}$ the reals, and by $\mathbb{Z}$ the integers.

Let $A = (a_{ij})$ be a finite indecomposable Cartan matrix of rank n. To A there is associated a root system Φ, a simple Lie algebra $\mathfrak{g}$ and a simple simply connected algebraic group G over $\mathbb{C}$. We fix a maximal torus T of G, and a Borel subgroup B containing T. We denote by $\mathfrak{h}$ the Lie algebra of T. Then Φ is the set of roots relative to $\mathfrak{h}$, and B determines the set of positive roots Φ^+, and the simple roots $\{\alpha_1, \dots, \alpha_n\}$. Let $\mathbb{R}P$ be the real subspace of $\mathfrak{h}^*$ spanned by the roots. This is a Euclidean space, endowed with the scalar product $(\alpha_i, \alpha_j) = d_i a_{ij}$. Here $\{d_1, \dots, d_n\}$ are relatively prime positive integers such that if D is the diagonal matrix with entries $d_1, \dots, d_n$, then DA is symmetric. P is the weight lattice, Q the root lattice, and W the Weyl group; s_i is the simple reflection associated to α_i, $\{\omega_1, \dots, \omega_n\}$ are the fundamental weights and $\rho = \sum_{i=1}^n \omega_i$. We denote by w_0 is the longest element of W, $B^- := w_0 B w_0$. Let N be the number of positive roots. Let $w_0 = s_{i_1} \dots s_{i_N}$ be a reduced expression of w_0. Then one gets a total (convex) order on Φ^+ by

$$\beta_1 = \alpha_{i_1},\ \beta_2 = s_{i_1}\alpha_{i_2}, \dots, \beta_N = s_{i_1} \dots s_{i_{N-1}} \alpha_{i_N}.$$

Let q be an indeterminate over $\mathbb{C}$. We put $k = \mathbb{C}(q)$, $R = \mathbb{C}[q, q^{-1}]$. Moreover, for $i = 1, \dots, n$ we put $q_i = q^{d_i}$.

The *multiparameter* (or *twisted*) *quantum group* $U_q^\varphi(\mathfrak{g})$ is a k-Hopf algebra whose comultiplication depends on a parameter φ. The conditions we assume on φ are the following:

φ is an endomorphism of $\mathbb{R}P$, skew relative to $(\ ,\ )$,
$\varphi Q \subseteq 2Q$.

Let $\tau_i := \frac{1}{2}\varphi\alpha_i = \sum_j y_{ji}\alpha_j$, for every $i = 1, \dots, n$. The conditions on φ are then equivalent to

$$Y := (y_{ij}) \in M_n(\mathbb{Z}) \cap (DA)^{-1} A_n(\mathbb{Z}), \tag{1.1}$$

where $A_n(\mathbb{Z})$ is the group of antisymmetric integral matrices. In particular the φ's form a lattice isomorphic to $A_n(\mathbb{Z})$.

For technical reasons we shall also assume

$$\varphi P \subseteq 2P. \tag{1.2}$$

This assumption is not essential, but allows us to give a better description of the picture.

Remark. The extra condition (1.2) corresponds to the condition that also AYA^{-1} is integral. We note that (1.1) implies that AYA^{-1} is integral if G is not of type B_n, with $n \geq 3$. In fact, $DAY(DA)^{-1}$ is integral, since ${}^tY = -DAY(DA)^{-1}$. Hence, if $D = 1$ or $\det(A) = 1$, then AYA^{-1} lies in $M_n(\mathbb{Z})$. We are left with the cases B_n or C_n. In these cases $\det(A) = 2$, so that $2A^{-1}$ lies in $M_n(\mathbb{Z})$, hence $2AYA^{-1}$ is integral. If G is of type C_n, a direct evaluation of A^{-1} allows us to prove that AYA^{-1} is integral. For type B_n, with $n \geq 3$, one may have Y integral, but AYA^{-1} not integral. Note that anyway (1.1) always implies $\varphi P \subseteq P$.

We still denote by φ the endomorphisms of $\mathfrak{h}^*$ and of P induced by φ. The *multiparameter simply connected quantum group* $U_q^{sc,\varphi}(\mathfrak{g})$ is the k-algebra generated by the elements E_i, F_i, K_λ for $i = 1, \dots, n$ and λ in P satisfying suitable q-analogues of the Serre relations (cf. [R], [DKP]), and with the Hopf-algebra structure given by

$$\begin{cases} \Delta_\varphi E_i = E_i \otimes K_{-\tau_i} + K_{\alpha_i+\tau_i} \otimes E_i, \\ \Delta_\varphi F_i = F_i \otimes K_{-\alpha_i+\tau_i} + K_{-\tau_i} \otimes F_i, \\ \Delta_\varphi K_\lambda = K_\lambda \otimes K_\lambda, \end{cases} \begin{cases} \varepsilon_\varphi E_i = 0, \\ \varepsilon_\varphi F_i = 0, \\ \varepsilon_\varphi K_\lambda = 1, \end{cases} \begin{cases} S_\varphi E_i = -K_{-\alpha_i} E_i, \\ S_\varphi F_i = -F_i K_{\alpha_i}, \\ S_\varphi K_\lambda = K_{-\lambda}. \end{cases}$$

Note that the counit and the antipode are independent of φ.

$U_q^\varphi(\mathfrak{g})$ is the k-subalgebra generated by the E_i's, F_i's and K_β for β in Q. We also consider the subalgebras

$$\begin{aligned} U^0 &= k[K_\beta \mid \beta \in Q], \\ U_q^\varphi(\mathfrak{b}^+) &= k[E_i, K_\beta \mid i = 1, \dots, n, \beta \in Q], \\ U_q^\varphi(\mathfrak{b}^-) &= k[F_i, K_\beta \mid i = 1, \dots, n, \beta \in Q]. \end{aligned}$$

Let $w_0 = s_{i_1} \dots s_{i_N}$ be a reduced expression of w_0, and let $\beta_1, \dots, \beta_N$ be the convex ordering of Φ^+ associated to this reduced expression. Then (according to Lusztig) one defines root vectors $E_{\beta_1}, \dots, E_{\beta_N}$ in $U_q^\varphi(\mathfrak{b}^+)$ and $F_{\beta_1}, \dots, F_{\beta_N}$ in $U_q^\varphi(\mathfrak{b}^-)$.

Let $\mathcal{C}$ be the category of finite dimensional representations of $U_q^\varphi(\mathfrak{g})$ on which the spectrum of the K_{α_i}'s consists of powers of q. The *multiparameter quantum function algebra* $F_q^\varphi[G]$ is the k-vector space generated by the matrix coefficients from $\mathcal{C}$. We recall that given V in $\mathcal{C}$, v in V and f in V^*, the matrix coefficient $c_{f,v}$ is defined by $c_{f,v}(x) = f(xv)$ for every x in $U_q^\varphi(\mathfrak{g})$. $F_q^\varphi[G]$ is a sub-Hopf algebra of the restricted dual of $U_q^\varphi(\mathfrak{g})$. When $\varphi = 0$, one of course gets the Hopf algebras studied in [DL], [DP]. Since this case plays a major role even in the study of the general case, throughout this paper we shall refer to it as to the *classical case.* Whenever we consider algebras in the classical case, we shall drop the 0. In particular it is clear that $F_q^\varphi[G]$ is $F_q[G]$ as a coalgebra. Given a dominant weight λ, let $V(\lambda)$ be the irreducible $U_q^\varphi(\mathfrak{g})$-module with highest weight λ. Then $F_q^\varphi[G]$ is linearly spanned by the matrix coefficients of the $V(\lambda)$'s (recall that v in a $U_q^\varphi(\mathfrak{g})$-module V has weight μ if $K_\beta.v = q^{(\mu,\beta)}v$ for every β in Q). The Hopf algebras $F[T]$, $F_q^\varphi[B]$ and $F_q^\varphi[B^-]$ are defined by restricting the elements of $F_q^\varphi[G]$ respectively to U^0, $U_q^\varphi(\mathfrak{b}^+)$ and $U_q^\varphi(\mathfrak{b}^-)$.

The following formula relates the twisted case to the classical case. If V_1, V_2 are in $\mathcal{C}$, then

$$(1.3)\qquad m_\varphi(c_{f_1,v_1}\otimes c_{f_2,v_2}) = q^{\frac{1}{2}((\varphi\mu_1,\mu_2)-(\varphi\lambda_1,\lambda_2))}m_0(c_{f_1,v_1}\otimes c_{f_2,v_2}),$$

where v_i is in $(V_i)_{\mu_i}$ and f_i is in $(V_i^*)_{\lambda_i}$ for $i=1,2$ (cf. [CV2]).

From this, one can deduce the commutation rules among the $c_{\phi,v}$. If V, W are in $\mathcal{C}$, then

$$(1.4)\qquad \begin{aligned} m_\varphi(c_{\phi,v}\otimes c_{\psi,w}) = q^{((1+\varphi)\mu_1,\mu_2)-((1+\varphi)\lambda_1,\lambda_2))}&m_\varphi(c_{\psi,w}\otimes c_{\phi,v}) \\ &+\sum m_\varphi(c_{\psi_i,w_i}\otimes c_{\phi_i,v_i}), \end{aligned}$$

where v is in V_{μ_1}, ϕ is in $(V^*)_{\lambda_1}$, w is in W_{μ_2} and ψ is in $(W^*)_{\lambda_2}$. The weights of v_i, ϕ_i, w_i and ψ_i are respectively $\mu_1-\beta_i$, $\lambda_1-\gamma_i$, $\mu_2+\beta_i$ and $\lambda_2+\gamma_i$, where β_i and γ_i are in Q^+, and for each i at least one of β_i and γ_i is nonzero (this comes from (1.2.2) in [DP], and (1.3). It can also be directly deduced from the R-matrix of $U_q^\varphi(\mathfrak{g})$, as in [HLT]).

All the various actions considered in [DP] can also be used in the twisted case. So $F_q^\varphi[G]$ is naturally a left $U_q^\varphi(\mathfrak{g})\otimes U_q^\varphi(\mathfrak{g})$ module. For a,b,c in $U_q^\varphi(\mathfrak{g})$, f in $F_q^\varphi[G]$ we have

$$((a\otimes b)f)(c) = f(S(a)cb).$$

This is independent of φ, and we get the decomposition

$$F_q^\varphi[G] = \bigoplus_{\lambda\in P^+} V_{-w_0\lambda}^*\otimes V_{-w_0\lambda}$$

of left $U_q^\varphi(\mathfrak{g})\otimes U_q^\varphi(\mathfrak{g})$ modules.

We finally recall that the generalized braid group $\mathcal{B}$ associated to Φ acts on each irreducible module $V(\lambda)$ (cf. [DL]). Given w in W, let t_w be the element of $\mathcal{B}$ associated to w. For every λ in P^+, choose a lowest weight vector $v_{-\lambda}$ in $V(-w_0\lambda)$ and a highest weight vector ϕ_λ in $V(-w_0\lambda)^*$ such that $\phi_\lambda(v_{-\lambda})=1$. We put

$$z_w^\lambda = c_{t_w(\phi_\lambda),v_{-\lambda}}\ ,\ \zeta_w^\lambda = c_{\phi_\lambda,t_{w^{-1}}(v_{-\lambda})}.$$

We shall write z_w (resp. ζ_w) for z_w^ρ (resp. ζ_w^ρ).

2. The twisted case

In order to construct simple modules over $F_\varepsilon^\varphi[G]$, the results from [CV3] are crucial. For convenience of the reader, we recall them, and give the proof of one of the statements.

We put $F_i^\varphi = F_i K_{\tau_i}, E_i^\varphi = E_i K_{\tau_i}$. Then

$$\begin{aligned} \Delta_\varphi E_i^\varphi &= E_i^\varphi \otimes 1 + K_{(1+\varphi)\alpha_i} \otimes E_i^\varphi, \\ \Delta_\varphi F_i^\varphi &= F_i^\varphi \otimes K_{-(1-\varphi)\alpha_i} + 1 \otimes F_i^\varphi. \end{aligned} \tag{2.1}$$

We introduce the k-subalgebras

$$\begin{aligned} D_q^{\varphi,+} &= k[F_i^\varphi, K_{(1-\varphi)\lambda} \mid i = 1, \dots, n, \lambda \in P] \ , \\ D_q^{\varphi,-} &= k[E_i^\varphi, K_{(1+\varphi)\lambda} \mid i = 1, \dots, n, \lambda \in P]. \end{aligned}$$

$D_q^{\varphi,+}$ and $D_q^{\varphi,-}$ are sub-Hopf algebras of $U_q^{sc,\varphi}(\mathfrak{g})$, as follows from (2.1) and easy calculations with the antipode. Note that there is a natural inclusion of algebras

$$D_q^{\varphi,+} \leq D_q^+, \ D_q^{\varphi,-} \leq D_q^-.$$

We recall that one has perfect Hopf algebra pairings

$$\pi_\varphi : (D_q^{\varphi,+})_{\mathrm{op}} \otimes U_q^\varphi(\mathfrak{b}^+) \to k, \qquad \begin{cases} (K_{(1-\varphi)\lambda}, K_\alpha) \mapsto q^{-(\lambda,\alpha)}, \\ (K_{(1-\varphi)\lambda}, E_i) \mapsto 0, \\ (F_i^\varphi, E_j) \mapsto \delta_{i,j}(q_i^{-1} - q_i)^{-1}, \\ (F_i^\varphi, K_\alpha) \mapsto 0. \end{cases}$$

$$\overline{\pi}_\varphi : (D_q^{\varphi,-})_{\mathrm{op}} \otimes U_q^\varphi(\mathfrak{b}^-) \to k, \qquad \begin{cases} (K_{(1+\varphi)\lambda}, K_\alpha) \mapsto q^{(\lambda,\alpha)}, \\ (K_{(1+\varphi)\lambda}, F_i) \mapsto 0, \\ (E_i^\varphi, F_j) \mapsto \delta_{i,j}(q_i - q_i^{-1})^{-1}, \\ (E_i^\varphi, K_\alpha) \mapsto 0. \end{cases}$$

The subscript op means opposite comultiplication. They induce Hopf algebra isomorphisms (cf. [CV2])

$$F_q^\varphi[B] \cong (D_q^{\varphi,+})_{\mathrm{op}} \ , \ F_q^\varphi[B^-] \cong (D_q^{\varphi,-})_{\mathrm{op}}.$$

We are interested mainly in the multiplicative structure of $F_q^\varphi[B]$ and $F_q^\varphi[B^-]$, hence on the algebras $D_q^{\varphi,+}$ and $D_q^{\varphi,-}$.

One can define, for each of the above algebras, suitable integer forms over R, and then specialize at $q = \varepsilon$, where ε is a primitive ℓ-th root of one in $\mathbb{C}$, to obtain the $\mathbb{C}$-algebras $F_\varepsilon^\varphi[G]$, $F_\varepsilon^\varphi[B]$ and $F_\varepsilon^\varphi[B^-]$. Here ℓ is a positive odd integer, prime to 3 if G is of type G_2. The forms $R_q^\varphi[G]$, $R_q^\varphi[B^+]$ and $R_q^\varphi[B^-]$ are defined in the same way as in [DL]. We recall the forms of

$D_q^{\varphi,+}$ and $D_q^{\varphi,-}$. For every β in Φ^+, we put $e_\beta^\varphi := (q_\beta - q_\beta^{-1})E_\beta K_{\frac{1}{2}\varphi\beta}$ and $f_\beta^\varphi := (q_\beta - q_\beta^{-1})F_\beta K_{\frac{1}{2}\varphi\beta}$, where $q_\beta := q^{(\beta,\beta)/2}$. Then we define

$$R_q^\varphi[B]' = R[f_\beta^\varphi, K_{(1-\varphi)\lambda} \mid \beta \in \Phi^+, \lambda \in P],$$

$$R_q^\varphi[B^-]' = R[e_\beta^\varphi, K_{(1+\varphi)\lambda} \mid \beta \in \Phi^+, \lambda \in P].$$

One has

$$R_q^\varphi[B^+] \cong R_q^\varphi[B]' \ , \ R_q^\varphi[B^-] \cong R_q^\varphi[B^-]'. \tag{2.2}$$

as R-algebras.

Our aim is to define ideals J_w^φ of $F_q^\varphi[B]$ for every w in W, and show that as sets they are independent of φ. We consider the Hopf algebra pairing (cf. [CV1])

$$\tilde{\pi}_\varphi\colon (k[F_1, \dots, F_n, K_\lambda \mid \lambda \in P])_{\mathrm{op}} \otimes k[E_1^\varphi, \dots, E_n^\varphi, K_{(1+\varphi)\alpha} \mid \alpha \in Q] \to k$$

$$\begin{aligned}(K_\lambda, K_{(1+\varphi)\alpha}) &\mapsto q^{-(\lambda,\alpha)},\\ (K_\lambda, E_i^\varphi) &\mapsto 0,\\ (F_i, E_j^\varphi) &\mapsto \delta_{i,j}(q_i^{-1} - q_i)^{-1},\\ (F_i, K_{(1+\varphi)\alpha}) &\mapsto 0.\end{aligned}$$

For every i, let $\tilde{\alpha}_i : P \to \mathbb{Z}$ be given by $\lambda \mapsto ((1-\varphi)\lambda, \alpha_i)$, and let n_i be the positive integer such that $d_i n_i \mathbb{Z}$ is the image of $\tilde{\alpha}_i$. Consider the subalgebra $U_q^\varphi(\mathfrak{b}_i)$ of $U_q^\varphi(\mathfrak{b})$ generated by E_i^φ and $K_{(1+\varphi)\alpha_i}^{\pm 1}$. This is a sub-Hopf algebra by (2.1). Via the pairing $\tilde{\pi}_\varphi$, we can consider $S_i := k[F_i, K_{\omega_i}^{\pm 1}]$ as an algebra of functions on $U_q^\varphi(\mathfrak{b}_i)$. Restriction of maps from $U_q^\varphi(\mathfrak{b})$ to $U_q^\varphi(\mathfrak{b}_i)$ gives rise to the algebra homomorphism

$$\rho_i^\varphi : D_q^{\varphi,+} \to S_i$$

whose image is $k[F_i, K_{n_i\omega_i}^{\pm 1}]$. Here, ρ_i^φ maps F_j^φ to 0 if $j \neq i$, F_i^φ to F_i, and $K_{(1-\varphi)\lambda}$ to $K_{\omega_i}^{((1-\varphi)\lambda,\alpha_i)/d_i}$. We observed that $D_q^{\varphi,+}$ is a subalgebra of D_q^+. Then ρ_i^φ coincides with the restriction of ρ_i to $D_q^{\varphi,+}$. Similarly we consider the algebra $\mathcal{T}_i := k[K_{\omega_i}^{\pm 1}]$ as an algebra of functions on $\mathcal{T}_i^{*,\varphi} := k[K_{(1+\varphi)\alpha_i}^{\pm 1}]$. Restriction of maps from $U_q^\varphi(\mathfrak{b})$ to $\mathcal{T}_i^{*,\varphi}$ gives rise to the algebra homomorphism

$$\eta_i^\varphi : D_q^{\varphi,+} \to \mathcal{T}_i$$

Here, η_i^φ maps F_j^φ to 0 for $j = 1, \dots, n$, and $K_{(1-\varphi)\lambda}$ to $K_{\omega_i}^{((1-\varphi)\lambda,\alpha_i)/d_i}$.

Let w be in W, and let $w = s_{i_1} \dots s_{i_k}$ be a reduced expression. Choose a reduced expression $w_0 = s_{i_1} \dots s_{i_k} s_{i_{k+1}} \dots s_{i_N}$ and consider the algebra homomorphism

$$\pi_k^\varphi : F_q^\varphi[B] \xrightarrow{\Delta^N} F_q^\varphi[B] \otimes F_q^\varphi[B] \otimes \dots \otimes F_q^\varphi[B] \to \otimes_{s=1}^k S_{i_s} \bigotimes \otimes_{s=k+1}^N \mathcal{T}_{i_s}.$$

The second arrow in the definition of π_k^φ is given by the isomorphism between $F_q^\varphi[B]$ and $D_q^{\varphi,+}$, and the map

$$\rho_{i_1}\otimes\ldots\otimes\rho_{i_k}\otimes\eta_{i_{k+1}}\otimes\ldots\otimes\eta_{i_N}: \underbrace{D_q^{\varphi,+}\otimes\ldots\otimes D_q^{\varphi,+}}_{N \text{ copies}} \to \otimes_{s=1}^k S_{i_s} \bigotimes \otimes_{s=k+1}^N \mathcal{T}_{i_s}.$$

We claim that $\ker(\pi_k^\varphi) = \ker(\pi_k)$. Note that

$$S_{i_1}\otimes S_{i_2}\otimes\ldots\otimes S_{i_k}\otimes \mathcal{T}_{i_{k+1}}\otimes\ldots\otimes\mathcal{T}_{i_N}$$

is an algebra of linear functions on

$$U_q^\varphi(\mathfrak{b}_{i_1})\otimes U_q^\varphi(\mathfrak{b}_{i_2})\otimes\ldots\otimes U_q^\varphi(\mathfrak{b}_{i_k})\otimes \mathcal{T}_{i_{k+1}}^{*,\varphi}\otimes\ldots\otimes\mathcal{T}_{i_N}^{*,\varphi}$$

We consider

$$\mathcal{H}^\varphi = \bigoplus_{\lambda\in P^+} V_{-w_0\lambda}^*\otimes v_{-\lambda} \tag{2.3}$$

which, as a vector space, is independent of φ. $\mathcal{H}^\varphi$ is an algebra by [DP, Sec. 2.3] and (1.3). Let $r_+^\varphi: F_q^\varphi[G]\to F_q^\varphi[B]$ be the restriction homomorphism.

Lemma 2.1. *The restriction of r_+^φ to $\mathcal{H}^\varphi$ extends to an isomorphism between $\mathcal{H}^\varphi[z_e^{-1}]$ and $F_q^\varphi[B]$.*

Proof. We observe that normality of z_e follows from (1.4). The result then comes from Lemma 2.3 in [DP] and (1.3). □

We identify $\mathcal{H}^\varphi$ with its image in $F_q^\varphi[B]$.

Definition 2.2. We put $J_k^\varphi := \ker(\pi_k^\varphi)$, $I_k^\varphi := J_k^\varphi\cap\mathcal{H}^\varphi$.

We denote by $U^{0,\varphi}$ the subalgebra $k[K_{(1+\varphi)\alpha}|\ \alpha\in Q]$.

Proposition 2.3. *The decomposition in* (2.3) *is the decomposition of $\mathcal{H}^\varphi$ under the action of $1\otimes U^{0,\varphi}$.*

Proof. (2.3) is the decomposition of $\mathcal{H}$ into weight spaces under the action of $1\otimes U^0$. The result follows from the fact that $(1+\varphi)Q$ spans $\mathbb{R}P$. □

Proposition 2.4. $I_k^\varphi = I_k$.

Proof. $\ker(\pi_k)\cap\mathcal{H}$ is a direct sum of weight spaces under U^0 (Proposition 2.4 in [DP]). Similarly one proves that $\ker(\pi_k^\varphi)\cap\mathcal{H}$ is the direct sum of weight spaces under $U^{0,\varphi}$. Now, for λ in P^+, we have

$$(E_{i_1}^\varphi)^{r_1}\ldots(E_{i_k}^\varphi)^{r_k}K_{(1+\varphi)\alpha_{i_{k+1}}}^{r_{k+1}}\ldots K_{(1+\varphi)\alpha_{i_N}}^{r_N}v_{-\lambda} = q^M E_{i_1}^{r_1}\ldots E_{i_k}^{r_k}K_{\alpha_{i_{k+1}}}^{r_{k+1}}\ldots K_{\alpha_{i_N}}^{r_N}v_{-\lambda}$$

for some integer M. Hence

$$\ker(\pi_k^\varphi) \cap (V^*_{-w_0\lambda} \otimes v_{-\lambda}) = \ker(\pi_k) \cap (V^*_{-w_0\lambda} \otimes v_{-\lambda})$$

and we are done. □

Note that $U^{0,\varphi} \otimes U^{0,\varphi}$ acts on $\mathcal{H}^\varphi$ giving the weight space decomposition

$$\mathcal{H}^\varphi = \bigoplus_{\lambda \in P^+} \bigoplus_{\beta \in Q^+} (V^*_{-w_0\lambda})_{\lambda-\beta} \otimes v_{-\lambda}.$$

Since $\ker(\pi_k^\varphi)$ is invariant under $U^{0,\varphi} \otimes U^{0,\varphi}$, $\ker(\pi_k^\varphi) \cap \mathcal{H}^\varphi$ is the direct sum of $U^{0,\varphi} \otimes U^{0,\varphi}$ weight spaces.

Proposition 2.5. $J_k^\varphi = J_k$.

Proof. From (1.3) we get

$$m_\varphi(c_{f,v_{-\lambda}} \otimes c_{-\rho,\rho}) = q^{\frac{1}{2}(\varphi\rho,\beta)} m_0(c_{f,v_{-\lambda}} \otimes c_{-\rho,\rho}), \tag{2.4}$$

where f is in $(V^*_{-w_0\lambda})_{\lambda-\beta}$.

Now let c_λ be in $\ker(\pi_k) \cap (V^*_{-w_0\lambda} \otimes v_{-\lambda})$, and write $c_\lambda = \sum d_{\beta,\lambda}$, where $d_{\beta,\lambda} = f_\beta \otimes v_{-\lambda}$, with f_β of weight $\lambda - \beta$. By the observation preceding the proposition, $d_{\beta,\lambda}$ lies in $\ker(\pi_k)$ for each β and, by (2.4)

$$\begin{aligned} &m_\varphi(d_{\beta,\lambda} \otimes \underbrace{c_{-\rho,\rho} \otimes \ldots \otimes c_{-\rho,\rho}}_{r \text{ copies}}) \\ &= q^{\frac{1}{2}(r\varphi\rho,\beta)} m_0(d_{\beta,\lambda} \otimes \underbrace{c_{-\rho,\rho} \otimes \ldots \otimes c_{-\rho,\rho}}_{r \text{ copies}}). \end{aligned} \tag{2.5}$$

Recall that we have $J_k^\varphi = I_k^\varphi[c_{-\rho,\rho}]$, $J_k = I_k[c_{-\rho,\rho}]$ (z_e is invertible in $F_q^\varphi[B]$, with inverse $c_{-\rho,\rho}$). We prove that $I_k^\varphi[c_{-\rho,\rho}] = I_k[c_{-\rho,\rho}]$. Let d be in $(V^*_{-w_0\lambda})_{\lambda-\beta} \otimes v_{-\lambda}$. It is enough to show that if d lies in I_k $(= I_k^\varphi)$, then $m_\varphi(d \otimes \underbrace{c_{-\rho,\rho} \otimes \ldots \otimes c_{-\rho,\rho}}_{r \text{ copies}})$ lies in J_k and $m_0(d \otimes \underbrace{c_{-\rho,\rho} \otimes \ldots \otimes c_{-\rho,\rho}}_{r \text{ copies}})$ lies in J_k^φ, for each r. This follows by (2.5). □

Example. The quotient $F_q^\varphi[B]/J_e^\varphi$ is the commutative (polynomial) algebra generated by $K_{(1-\varphi)\lambda}$, λ in P. It is clearly contained in the algebra generated by K_λ, λ in P, and in this case these algebras are isomorphic.

The algebra $F_q^\varphi[B]/J_{s_i}^\varphi$ is isomorphic to the subalgebra of $D_q^{\varphi,+}$ generated by F_i^φ, $K_{(1-\varphi)\lambda}$, λ in P.

We are now in a position to deal with roots of 1. The map π_k^φ induces a map of integral forms, which specializes to $\pi_{\varepsilon,k}^\varphi$ at $q = \varepsilon$. We denote by $J_{\varepsilon,k}^\varphi$ the kernel of $\pi_{\varepsilon,k}^\varphi$. Since it does not depend on the reduced expression $s_{i_1} \ldots s_{i_k}$ of w, we shall write $J_{\varepsilon,w}^\varphi$ for $J_{\varepsilon,k}^\varphi$.

A similar construction can be done for $F_q^\varphi[B^-]$: this allows us to define the ideals $J_w^{\varphi,-}$, $J_{\varepsilon,w}^{\varphi,-}$ of $F_q^\varphi[B^-]$, $F_\varepsilon^\varphi[B^-]$ respectively. For every (w_1, w_2) in $W \times W$, we put

$$J_{w_1,w_2}^\varphi = (\gamma^\varphi)^{-1}(J_{w_1}^\varphi \otimes F_q^\varphi[B^-] + F_q^\varphi[B] \otimes J_{w_2}^{\varphi,-}),$$

$$J_{\varepsilon,w_1,w_2}^\varphi = (\gamma_\varepsilon^\varphi)^{-1}(J_{\varepsilon,w_1}^\varphi \otimes F_\varepsilon^\varphi[B^-] + F_\varepsilon^\varphi[B] \otimes J_{\varepsilon,w_2}^{\varphi,-}).$$

Here $\gamma_\varepsilon^\varphi$ is the map induced by

$$\gamma^\varphi : R_q^\varphi[G] \xrightarrow{\Delta} R_q^\varphi[G] \otimes R_q^\varphi[G] \longrightarrow R_q^\varphi[B] \otimes R_q^\varphi[B^-].$$

Let w, w_1 and w_2 be elements of W. We define

$$F_\varepsilon^\varphi[B]_w = F_\varepsilon[B]/J_{\varepsilon,w}^\varphi[z_w^{-1}]\ ,\ F_\varepsilon^\varphi[G]_{w_1,w_2} = F_\varepsilon^\varphi[G]/J_{\varepsilon,w_1,w_2}^\varphi[z_{w_1}^{-1}, \zeta_{w_2}^{-1}].$$

We now recall the definition of φ-good integer (cf. [CV2]) and the main theorem in [CV3]. Denote by ϑ the isometry $(1-\varphi)(1+\varphi)^{-1}$ of $\mathbb{R}P$. For (w_1, w_2) in $W \times W$, we put

$$e_{w_1,w_2}^\varphi = 1 - w_1 \vartheta w_2^{-1} \vartheta^{-1} : \mathbb{R}P \to \mathbb{R}Q.$$

We consider the ring $\mathbb{Z}^\varphi = \mathbb{Z}[(2d_1 \dots d_n \det(1-\varphi))^{-1}]$. It is clear that $\mathbb{Z}^\varphi P$ and $\mathbb{Z}^\varphi Q$ are invariant under ϑ. Note that the image of $\mathbb{Z}^\varphi P$ lies in $\mathbb{Z}^\varphi Q$. In fact one can introduce the group of isometries W^φ of $\mathbb{R}P$ generated by the reflections s_v for v in $\Phi^+ \cup \vartheta\Phi^+$. Then for each of these reflections, and for each λ in P, we have $s_v(\lambda) - \lambda \in \mathbb{Z}^\varphi Q$.

Define $\ell(\varphi)$ to be the least positive integer for which, for every (w_1, w_2) in $W \times W$, the image of $\mathbb{Z}^\varphi[\ell(\varphi)^{-1}]P$ under e_{w_1,w_2}^φ is a split summand of $\mathbb{Z}^\varphi[\ell(\varphi)^{-1}]Q$. An integer m is said to be a *φ-good* integer if it is prime to the $2d_i$'s, $\det(1-\varphi)$ and $\ell(\varphi)$. From now on ℓ is assumed to be φ-good. We denote by S the ring $\mathbb{Z}^\varphi[\ell(\varphi)^{-1}]$.

Definition 2.6. We denote the map $(1+\varphi)w_1(1-\varphi) - (1-\varphi)w_2(1+\varphi)$ by δ_{w_1,w_2}^φ. We put $c_{w_1,w_2}^\varphi = \ell(w_1) + \ell(w_2) + \mathrm{rk}(\delta_{w_1,w_2}^\varphi)$.

We recall that c_{w_1,w_2}^φ is the dimension of a symplectic leaf of G contained in $X_{w_1,w_2} := B^- w_1 B^- \cap B w_2 B$, with respect to the Poisson structure on G induced by φ (cf. [CV2]). It is clear that $\mathrm{rk}(\delta_{w_1,w_2}^\varphi) = \mathrm{rk}(e_{w_1,w_2}^\varphi)$ for every (w_1, w_2) in $W \times W$. In fact the map $\lambda \mapsto s_i w_1(1-\varphi)\lambda$ defines an isomorphism from $\ker(\delta_{w_1,w_2}^\varphi)$ onto $\ker(e_{w_1,w_2}^\varphi)$.

The spectrum of $F_\varepsilon^\varphi[G]_{w_1,w_2}$ is, by definition, the set of equivalence classes of irreducible representations of $F_\varepsilon^\varphi[G]_{w_1,w_2}$. If λ is in P, write $\lambda = \lambda_+ - \lambda_-$, with $\lambda_\pm$, dominant weights. We put $z_w^\lambda := z_w^{\lambda_+}(z_w^{\lambda_-})^{-1}$ and similarly for ζ_w^λ.

Definition 2.7. Let (w_1, w_2) be in $W \times W$. We put

$$P^{\varphi}_{w_1,w_2} = P \cap \ker(\delta^{\varphi}_{w_1,w_2}) \ , \quad T^{\varphi}_{w_1,w_2} = \operatorname{Spec} \mathbb{C}[P^{\varphi}_{w_1,w_2}].$$

Moreover, for λ in $P^{\varphi}_{w_1,w_2}$ we put

$$d^{\varphi}_{\lambda,w_1,w_2} = z_{w_1}^{(1-\varphi)\lambda} \zeta_{w_2}^{w_2(1+\varphi)\lambda}.$$

The main result in [CV3] is

Theorem 2.8. *Let ℓ be a φ-good integer, and let (w_1, w_2) be in $W \times W$. Then*

a) *$F^{\varphi}_{\varepsilon}[G]_{w_1,w_2}$ is an Azumaya algebra of degree $\ell^{\frac{1}{2} c^{\varphi}_{w_1,w_2}}$;*

b) *the center of $F^{\varphi}_{\varepsilon}[G]_{w_1,w_2}$ is the $\mathbb{C}[X_{w_1,w_2}]$-algebra generated by $d^{\varphi}_{\lambda,w_1,w_2}$ for λ in $P^{\varphi}_{w_1,w_2}$;*

c) *the spectrum of $F^{\varphi}_{\varepsilon}[G]_{w_1,w_2}$ is a Galois covering of X_{w_1,w_2} with Galois group $\{x \in T^{\varphi}_{w_1,w_2} \mid x^{\ell} = 1\}$.* □

We need a more precise description of the center $Z^{\varphi}_{w_1,w_2}$ of $F^{\varphi}_{\varepsilon}[G]_{w_1,w_2}$. One shows (with the same argument used in [DP]) that $Z^{\varphi}_{w_1,w_2}$ contains the coordinate ring $\mathbb{C}[X_{w_1,w_2}]$, and the algebra $Z^{\varphi}_{1,w_1,w_2} = \mathbb{C}[d^{\varphi}_{\lambda,w_1,w_2} \mid \lambda \in P^{\varphi}_{w_1,w_2}]$. Then

$$\mathbb{C}[X_{w_1,w_2}] \cap Z^{\varphi}_{1,w_1,w_2} = \mathbb{C}[d^{\varphi}_{\ell\lambda,w_1,w_2} \mid \lambda \in P^{\varphi}_{w_1,w_2}]$$

and

$$Z^{\varphi}_{w_1,w_2} = \mathbb{C}[X_{w_1,w_2}] \otimes_{\mathbb{C}[X_{w_1,w_2}] \cap Z^{\varphi}_{1,w_1,w_2}} Z^{\varphi}_{1,w_1,w_2}.$$

If we denote by $S^{\varphi}_{w_1,w_2}$ the spectrum of $F^{\varphi}_{\varepsilon}[G]_{w_1,w_2}$, which identifies with $\operatorname{Spec} Z^{\varphi}_{w_1,w_2}$, then $S^{\varphi}_{w_1,w_2}$ is the fiber product

$$\begin{array}{ccc} S^{\varphi}_{w_1,w_2} & \xrightarrow{\kappa} & T^{\varphi}_{w_1,w_2} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \ell\text{-th power}} \\ X_{w_1,w_2} & \xrightarrow{\bar{\kappa}} & T^{\varphi}_{w_1,w_2} \end{array}$$

where all the maps are induced by inclusion.

3. The main result

In this section we construct essentially all the simple modules over $F^{\varphi}_{\varepsilon}[G]$. Our purpose is to show that the procedure used in [DP] can also be used in the twisted case. We need a lemma about Euclidean spaces.

Lemma 3.1. *Let V be a Euclidean space, w an isometry of V, and r_v the reflection relative to the nonzero vector v. Then*

$$v \notin \ker(w-1)^\perp \implies \operatorname{Im}(r_v w - 1) = \operatorname{Im}(w-1) \oplus \langle v \rangle ,$$
$$v \in \ker(w-1)^\perp \implies \operatorname{Im}(w-1) = \operatorname{Im}(r_v w - 1) \oplus \langle v \rangle .$$

Proof. It is clear that $\ker(w-1)^\perp = \operatorname{Im}(w-1)$. By intersecting with $\langle v \rangle^\perp$, it follows that $\operatorname{rk}(w-1) - 1 \leq \operatorname{rk}(r_v w - 1) \leq \operatorname{rk}(w-1) + 1$. Suppose $v \notin \operatorname{Im}(r_v w - 1)$, and let γ be in $\ker(w-1)$. Then $(w\gamma, v) = 0$, so that v lies in $\ker(w-1)^\perp$, since $\ker(w-1)^\perp$ is w-invariant.

Case 1: $v \notin \ker(w-1)^\perp$. Then we must have $v \in \operatorname{Im}(r_v w - 1)$, so that $\ker(r_v w - 1) \leq \langle v \rangle^\perp$. Moreover we get $\ker(r_v w - 1) \leq \ker(w-1)$, and finally $\ker(r_v w-1) = \ker(w-1) \cap \langle v \rangle^\perp$ and $\operatorname{Im}(r_v w - 1) = \operatorname{Im}(w-1) \oplus \langle v \rangle$.

Case 2: $v \in \ker(w-1)^\perp$. We show that in this case we have $v \notin \ker(r_v w - 1)^\perp$. Then we conclude as in case 1. Let $K := \ker(w-1)^\perp$. We still call w the isometry induced by w on K. Note that $w - 1$ is invertible on K. We put $z = (w+1)(w-1)^{-1} v$. Then $-v + z$ lies in $\ker(r_v w - 1)$, and z is orthogonal to v since the endomorphism $(w+1)(w-1)^{-1}$ of K is skew. Hence $v \notin \ker(r_v w - 1)^\perp$, and we are done. □

Corollary 3.2. *Let w be an isometry of $\mathbb{R}P$, and r_v the reflection relative to the nonzero vector v. Then*

$$P \cap \ker(r_v w - 1) = P \cap \ker(w-1) \cap \langle v \rangle^\perp \quad \text{if } v \notin \ker(w-1)^\perp,$$
$$P \cap \ker(w - 1) = P \cap \ker(r_v w-1) \cap \langle v \rangle^\perp \quad \text{if } v \in \ker(w-1)^\perp.$$

The next lemma is a crucial step in our construction. We recall that S is the ring $\mathbb{Z}^\varphi[\ell(\varphi)^{-1}]$.

Lemma 3.3. *Let (w_1, w_2) be in $W \times W$, α in Φ. If $\operatorname{rk}(e^\varphi_{s_\alpha w_1, w_2}) = \operatorname{rk}(e^\varphi_{w_1, w_2}) - 1$, then there exists λ in $SP \cap \ker(e^\varphi_{s_\alpha w_1, w_2})$ such that $(\lambda, \alpha) = 1$.*

Proof. Let $w = w_1 \vartheta w_2^{-1} \vartheta^{-1}$. We have $\operatorname{Im}(w-1) = \operatorname{Im}(s_\alpha w - 1) \oplus \langle \alpha \rangle$ by Lemma 3.1. We show that the same decomposition holds over S. Since $(w-1)SP$ is a split summand of SQ, and α lies in $\operatorname{Im}(w-1) \cap Q$, it follows that α lies in $(w-1)SP$. Since for every λ in P, we have $(1-w)\lambda = (1-s_\alpha w)\lambda - \frac{2(w\lambda, \alpha)}{(\alpha, \alpha)} \alpha$, we get $(1-w)SP = (1-s_\alpha w)SP \oplus S\alpha$. Let $(x_1, \ldots, x_r)$ be an S-basis of $(1 - s_\alpha w)SP$. Then $(\alpha, x_1, \ldots, x_r)$ is an S-basis of $(1-w)SP$. But $(1-w)SP$ is a split summand of SQ, so that $(\alpha, x_1, \ldots, x_r)$ can be extended to an S-basis $(\alpha, x_1, \ldots, x_r, y_1, \ldots, y_s)$ of SQ. Since the pairing $SP \times SQ \to S$ is perfect, we can consider the dual basis $(\lambda, \lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_s)$ in SP. It is then clear that $(\lambda, \alpha) = 1$, and $(\lambda, x_i) = 0$ for every $i = 1, \ldots, r$. □

We can now start the construction of simple modules. The one-dimensional modules are exactly those lying over the points of the torus T. The Picard group is described in the same way as in the classical case (cf. Theorem 10.8 in [DL]). In fact we already showed that $F^\varphi_\varepsilon[G]_{e,e} \cong F_\varepsilon[G]_{e,e} \cong \mathbb{C}[T]$.

Next we consider the part $S^{\varphi}_{s_i,e}$ of the spectrum, where s_i is a simple reflection. We know that the algebra $F^{\varphi}_q[B]/J^{\varphi}_{s_i}$ is isomorphic to the subalgebra of $D^{\varphi,+}_q$ generated by F^{φ}_i, $K_{(1-\varphi)\lambda}$, λ in P. Moreover, since z_{s_i} corresponds to the element $q^m f^{\varphi}_{\alpha_i} K_{(1-\varphi)\rho}$ for some integer m, it follows that $F^{\varphi}_{\varepsilon}[G]_{s_i,e}$ is isomorphic to the algebra $\mathbb{C}[F^{\varphi}_i, (F^{\varphi}_i)^{-1}, K_{(1-\varphi)\lambda} \mid \lambda \in P]$. As in the classical case, this algebra is the ordinary Laurent polynomial ring in $n-1$ variables over a quantum torus $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]$, with $xy = \eta yx$, η a primitive ℓ-th root of one. To prove this, we consider the map $\tilde{\alpha}_i$ previously defined. Choose ω^{φ}_i in P such that $((1-\varphi)\omega^{\varphi}_i, \alpha_i) = d_i n_i$, and let $\lambda_1, \dots, \lambda_{n-1}$ be a basis of $\ker \tilde{\alpha}_i$. We can take $x = F^{\varphi}_i$, $y = K_{(1-\varphi)\omega^{\varphi}_i}$, $x_i = K_{(1-\varphi)\lambda^{\varphi}_i}$, for $i = 1, \dots, n-1$. Note that $xy = \varepsilon^{d_i n_i} yx$, and that $\varepsilon^{d_i n_i}$ is also a primitive ℓ-th root of one, since ℓ is prime to d_i and to $\det(1-\varphi)$. Therefore the representation theory over the strata $X_{s_i,e}$ and X_{e,s_i} is completely described.

For the induction step we consider the map introduced in [DP, Sec. 5.2]. Let (w_1, w_2) be in $W \times W$, s_i a simple reflection such that $\ell(s_i w_1) = \ell(w_1)+1$. We denote by

$$\psi^{\varphi}: F^{\varphi}_q[G] \to F^{\varphi}_q[G]/J^{\varphi}_{s_i,e} \otimes F^{\varphi}_q[G]/J^{\varphi}_{w_1,w_2}$$

the map obtained by composing Δ with projections. Note that it is independent of φ, by Proposition 2.5. It can be specialized to ε and it gives rise, after localization, to

$$\psi_{\varepsilon}: F^{\varphi}_{\varepsilon}[G]_{s_i w_1, w_2} \to F^{\varphi}_{\varepsilon}[G]_{s_i,e} \otimes F^{\varphi}_{\varepsilon}[G]_{w_1,w_2}.$$

Lemma 3.4. *Suppose $\ell(s_i w_1) = \ell(w_1)+1$, and let λ be in P. Then*

1) $\psi_{\varepsilon}(z^{\lambda}_{s_i w_1}) = z^{w_1(\lambda)}_{s_i} \otimes z^{\lambda}_{w_1}$.
2) $\psi_{\varepsilon}(\zeta^{\lambda}_{w_2}) = \zeta^{\lambda}_{e} \otimes \zeta^{\lambda}_{w_2}$.

Proof. This comes from Lemma 5.2 in [DP]. □

Take w_1, w_2 in W such that $\ell(s_i w_1) = \ell(w_1) + 1$. In the notation of the Introduction, let (p,q) be in $S^{\varphi}_{s_i,e} \times S^{\varphi}_{w_1,w_2}$, and let V_p and V_q be the corresponding irreducible representations of $F^{\varphi}_{\varepsilon}[G]$. Then $\dim(V_p) = \ell$, $\dim(V_q) = \ell^{\frac{1}{2}c^{\varphi}_{w_1,w_2}}$. In light of the discussion after Theorem 2.8, we may suppose that $p = (g_1, s)$, $q = (g_2, t)$, with g_1 in $X_{s_i,e}$, s in $T^{\varphi}_{s_i,e}$, g_2 in X_{w_1,w_2} and t in $T^{\varphi}_{w_1,w_2}$. Then $V_p \otimes V_q$ lies over the point $g_1 g_2$ of $X_{s_i w_1, w_2}$. The simple modules lying over $X_{s_i w_1, w_2}$ all have dimension $\ell^{\frac{1}{2}c^{\varphi}_{s_i w_1, w_2}}$. By Lemma 3.1, we have two cases.

1) $c^{\varphi}_{s_i w_1, w_2} = c^{\varphi}_{w_1,w_2} + 2$. We get $\dim(V_p \otimes V_q) = \ell^{\frac{1}{2}c^{\varphi}_{s_i w_1, w_2}}$, so that $V_p \otimes V_q$ is irreducible. We can describe explicitly the map

$$j^{\varphi}: \; S^{\varphi}_{s_i,e} \times S^{\varphi}_{w_1,w_2} \to S^{\varphi}_{s_i w_1, w_2}$$

we obtained.

Let λ be in $P^\varphi_{s_iw_1,w_2}$. Our assumption implies that λ is in $P^\varphi_{w_1,w_2}$ and $s_iw_1(1-\varphi)\lambda = w_1(1-\varphi)\lambda$. Then $\psi_\varepsilon(d^\varphi_{s_iw_1,w_2,\lambda}) = z_{s_i}^{w_1(1-\varphi)\lambda} z_e^{w_2(1+\varphi)\lambda} \otimes d^\varphi_{w_1,w_2,\lambda}$; $z_{s_i}^{w_1(1-\varphi)\lambda} z_e^{w_2(1+\varphi)\lambda}$ lies in $S^\varphi_{s_i,e}$, since given α, β in P, one has that $z^\alpha_{s_i} z^\beta_e$ lies in $S_{s_i,e}$ if and only if $(1+\varphi)\alpha = (1-\varphi)\beta$ and $s_i\alpha = \alpha$. Let $\delta := w_1(1-\varphi)\lambda + w_2(1+\varphi)\lambda$. We have $(1-\varphi)\delta = 2w_1(1-\varphi)\lambda$, so that $\delta = 2(1-\varphi)^{-1}w_1(1-\varphi)\lambda$. If we prove that δ lies in $2P$, say $\delta = 2\mu$, then we have $z_{s_i}^{w_1(1-\varphi)\lambda} z_e^{w_2(1+\varphi)\lambda} = d^\varphi_{s_i,e,\mu}$.

From $(1-\varphi)\delta = 2w_1(1-\varphi)\lambda$ we get $\delta = 2w_1(1-\varphi)\lambda + \varphi\delta$. Since by assumption $\varphi P \subseteq 2P$, it follows that δ lies in $2P$. Hence $\psi_\varepsilon(d^\varphi_{s_iw_1,w_2,\lambda}) = d^\varphi_{s_i,e,(1-\varphi)^{-1}w_1(1-\varphi)\lambda} \otimes d^\varphi_{w_1,w_2,\lambda}$. If we define

$$u^\varphi\colon\ T^\varphi_{s_i,e} \times T^\varphi_{w_1,w_2} \to T^\varphi_{s_iw_1,w_2}, \qquad (s,t) \mapsto h,$$

by $h(\lambda) = s((1-\varphi)^{-1}w_1(1-\varphi)\lambda)t(\lambda)$ for every λ in $P^\varphi_{s_iw_1,w_2}$, then j^φ is given by $j^\varphi((g_1,s),(g_2,t)) = (g_1g_2, u^\varphi(s,t))$.

One may define the product of p and q by $p\cdot_\varphi q = j^\varphi(p,q)$, so that $V_p \otimes V_q$ has central character $p\cdot_\varphi q$.

Suppose g in $X_{s_iw_1,w_2}$ is such that $g = g_1g_2$ for some g_1 in $X_{s_i,e}$, g_2 in X_{w_1,w_2}. We show that for every (g,h) in $S^\varphi_{s_iw_1,w_2}$, there exists s in $T^\varphi_{s_i,e}$, t in $T^\varphi_{w_1,w_2}$ such that $(g,h) = j^\varphi((g_1,s)),(g_2,t))$. Let $(\lambda_1,\ldots,\lambda_{r+1})$ be a basis of $P^\varphi_{w_1,w_2}$ such that $(\lambda_1,\ldots,\lambda_r)$ is a basis of $P^\varphi_{s_iw_1,w_2}$, and put $\mu_i := (1-\varphi)^{-1}w_1(1-\varphi)\lambda_i$ for $i = 1,\ldots,r$. Choose s in $T^\varphi_{s_i,e}$ such that (g_1,s) lies in $S^\varphi_{s_i,e}$. We then define t on $P^\varphi_{w_1,w_2}$ by $t(\lambda_i) := h(\lambda_i)s(\mu_i)^{-1}$ for $i = 1,\ldots,r$, and $t(\lambda_{r+1}) = c$, with $c^\ell = g_2(\ell\lambda_{r+1})$. By the compatibility conditions between g and h, it follows that (g_2,t) lies in $S^\varphi_{w_1,w_2}$. Then it is clear that $j^\varphi((g_1,s),(g_2,t)) = (g,h)$.

2) $c^\varphi_{s_iw_1,w_2} = c^\varphi_{w_1,w_2}$. By Corollary 3.2, we have

$$P^\varphi_{w_1,w_2} = P^\varphi_{s_iw_1,w_2} \cap <(1+\varphi)w_1^{-1}\alpha_i>^\perp .$$

We prove that, as in the classical case, $V_p \otimes V_q$ is the direct sum of ℓ simple modules. In this case the center $Z^\varphi_{s_iw_1,w_2}$ of $S^\varphi_{s_iw_1,w_2}$ does not act on $V_p \otimes V_q$ by a central character. Consider the map

$$\gamma:\ P^\varphi_{s_iw_1,w_2} \to \mathbb{Z}/\ell\mathbb{Z}, \quad \lambda \mapsto (w_1(1-\varphi)\lambda, \alpha_i) + \ell\mathbb{Z}.$$

We claim that γ is surjective. By Lemma 3.3, there exists μ in $SP \cap \ker(\varepsilon^\varphi_{s_iw_1,w_2})$ such that $(\mu,\alpha_i) = 1$. Let $\lambda = (s_iw_1(1-\varphi))^{-1}\mu$, and let r be an integer, invertible in S, such that $r\lambda$ lies in P. Then $(s_iw_1(1-\varphi)r\lambda, \alpha_i) = r$, which is prime to ℓ, since ℓ is φ-good. Therefore γ is surjective. It follows that $\ker(\gamma) = P^\varphi_{w_1,w_2} + \ell P^\varphi_{s_iw_1,w_2}$. Let Z_{0,s_iw_1,w_2} be the subalgebra of $Z^\varphi_{s_iw_1,w_2}$ corresponding to the coordinate ring of $X_{s_iw_1,w_2}$. We put

$$Z^\varphi_{2,s_iw_1,w_2} = Z_{0,s_iw_1,w_2}[d^\varphi_{\lambda,s_iw_1,w_2} \mid \lambda \in P^\varphi_{w_1,w_2}].$$

Since $d^{\varphi}_{\ell\lambda,s_iw_1,w_2}$ lies in Z_{0,s_iw_1,w_2} for every λ in $P^{\varphi}_{s_iw_1,w_2}$, and $Z^{\varphi}_{s_iw_1,w_2} = Z_{0,s_iw_1,w_2} \otimes_{Z_{0,s_iw_1,w_2}\cap Z^{\varphi}_{1,s_iw_1,w_2}} Z^{\varphi}_{1,s_iw_1,w_2}$, we get

$$Z^{\varphi}_{2,s_iw_1,w_2} = Z_{0,s_iw_1,w_2} \otimes_{Z_{0,s_iw_1,w_2}\cap Z^{\varphi}_{1,s_iw_1,w_2}} \mathbb{C}[d^{\varphi}_{\lambda,s_iw_1,w_2} \mid \lambda \in \ker(\gamma)].$$

The inclusion of $Z^{\varphi}_{2,s_iw_1,w_2}$ into $Z^{\varphi}_{s_iw_1,w_2}$ gives rise to the Galois covering

$$\nu^{\varphi} : S^{\varphi}_{s_iw_1,w_2} \to \tilde{S}^{\varphi}_{s_iw_1,w_2} := \operatorname{Spec} Z^{\varphi}_{2,s_iw_1,w_2}$$

with Galois group cyclic of order ℓ. Moreover $\operatorname{Spec}\mathbb{C}[d^{\varphi}_{\lambda,s_iw_1,w_2} \mid \lambda \in \ker(\gamma)]$ is isomorphic to $T^{\varphi}_{s_iw_1,w_2}/\ker(\gamma)^{\perp}$, where $\ker(\gamma)^{\perp} := \{t \in T^{\varphi}_{s_iw_1,w_2} \mid x(t) = 1$ for every $x \in \ker(\gamma)\}$ is cyclic of order ℓ.

Suppose λ is in $P^{\varphi}_{w_1,w_2}$. Then, as in case 1, $(1-\varphi)^{-1}w_1(1-\varphi)\lambda$ lies in $P^{\varphi}_{s_i,e}$ and $\psi(d^{\varphi}_{s_iw_1,w_2,\lambda}) = d^{\varphi}_{s_i,e,(1-\varphi)^{-1}w_1(1-\varphi)\lambda} \otimes d^{\varphi}_{w_1,w_2,\lambda}$. Therefore we get a morphism

$$j^{\varphi} \colon S^{\varphi}_{s_i,e} \times S^{\varphi}_{w_1,w_2} \to \tilde{S}^{\varphi}_{s_iw_1,w_2}$$

and $Z^{\varphi}_{2,s_iw_1,w_2}$ acts by the central character $p \cdot_{\varphi} q := j^{\varphi}(p,q)$ on $V_p \otimes V_q$. More explicitly, if $(\lambda_0, \ldots, \lambda_r)$ is a basis of $P^{\varphi}_{s_iw_1,w_2}$ such that $(\lambda_1, \ldots, \lambda_r)$ is a basis of $P^{\varphi}_{w_1,w_2}$, then $\ker(\gamma) = <\ell\lambda_0, \lambda_1 \ldots, \lambda_r>$, and $j^{\varphi}((g_1,s),(g_2,t)) = (g_1g_2, h)$, where $h(\lambda) = s((1-\varphi)^{-1}w_1(1-\varphi)\lambda)t(\lambda)$ for every λ in $P^{\varphi}_{w_1,w_2}$, and $h(\ell\lambda_0) = d^{\varphi}_{s_iw_1,w_2,\ell\lambda_0}(g_1g_2)$.

The fibers of ν^{φ} consist of ℓ simple modules of dimension $\dim(V_p \otimes V_q)/\ell$. Therefore, to prove that $V_p \otimes V_q$ is the direct sum of the representations in $(\nu^{\varphi})^{-1}p \cdot_{\varphi} q$, it is enough to show that there exists λ in $P^{\varphi}_{s_iw_1,w_2}$ such that $d^{\varphi}_{s_iw_1,w_2,\lambda}$ has at least ℓ distinct eigenvalues on $V_p \otimes V_q$ (each eigenspace is then an $F^{\varphi}_{\varepsilon}[G]$-submodule, and each of these has a composition series with factors of the same dimension). Let λ be in $P^{\varphi}_{s_iw_1,w_2}$, with $(w_1(1-\varphi)\lambda, \alpha_i) = md_i \equiv d_i \bmod \ell$. Then

$$\psi_{\varepsilon}(d^{\varphi}_{s_iw_1,w_2,\lambda}) = z_{s_i}^{w_1(1-\varphi)\lambda} z_e^{w_2(1+\varphi)\lambda} \otimes z_{w_1}^{(1-\varphi)\lambda}\zeta_{w_2}^{w_2(1+\varphi)\lambda}.$$

To show that $d^{\varphi}_{s_iw_1,w_2,\lambda}$ has ℓ distinct eigenvalues, it is enough to show that $z_{s_i}^{w_1(1-\varphi)\lambda} z_e^{w_2(1+\varphi)\lambda}$ has ℓ distinct eigenvalues on V_p.

Under the isomorphism between $F^{\varphi}_q[B]$ and $D^{\varphi,+}_q$, for every μ in P^+, $z^{\mu}_{s_i}$, z^{μ}_e are mapped, up to nonzero multiplicative constants, to $(F^{\varphi}_i)^k K_{(1-\varphi)\mu}$, $K_{(1-\varphi)\mu}$, respectively, where $k = (\mu, \alpha_i)/d_i$. Hence $z_{s_i}^{w_1(1-\varphi)\lambda} z_e^{w_2(1+\varphi)\lambda}$ is mapped to $(F^{\varphi}_i)^m K_{(1-\varphi)\mu}$ for some μ in P. From the representation theory of the quantum torus, and the fact that $m \equiv 1$ *mod* ℓ, it follows that $(F^{\varphi}_i)^m K_{(1-\varphi)\mu}$ has ℓ distinct eigenvalues on V_p, and we are done.

As in case 1, one can show that if g in $X_{s_iw_1,w_2}$ is such that $g = g_1g_2$ for some g_1 in $X_{s_i,e}$, g_2 in X_{w_1,w_2}, then for every (g,h) in $\tilde{S}^{\varphi}_{s_iw_1,w_2}$, there exists s in $T^{\varphi}_{s_i,e}$, t in $T^{\varphi}_{w_1,w_2}$ such that $(g,h) = j^{\varphi}((g_1,s)),(g_2,t))$.

We have therefore proved the announced generalization of the results obtained by De Concini and Procesi in the classical case.

Theorem 3.5. *Let ℓ be a φ-good integer. Suppose s_i, w_1 and w_2 are elements of W, such that s_i is a simple reflection and $\ell(s_iw_1) = \ell(w_1)+1$. Let (p,q) be in $S^\varphi_{s_i,e} \times S^\varphi_{w_1,w_2}$, and let V_p, V_q be the corresponding simple $F_\varepsilon^\varphi[G]$-modules. Then*

1) if $\mathrm{rk}(\delta^\varphi_{s_iw_1,w_2}) = \mathrm{rk}(\delta^\varphi_{w_1,w_2})+1$, there is a morphism

$$j^\varphi\colon\ S^\varphi_{s_i,e} \times S^\varphi_{w_1,w_2} \to S^\varphi_{s_iw_1,w_2}$$

such that $V_p \otimes V_q$ is the simple module corresponding to $j^\varphi(p,q)$;

2) if $\mathrm{rk}(\delta^\varphi_{s_iw_1,w_2}) = \mathrm{rk}(\delta^\varphi_{w_1,w_2})-1$, there are the morphisms

$$j^\varphi\colon\ S^\varphi_{s_i,e} \times S^\varphi_{w_1,w_2} \to \tilde{S}^\varphi_{s_iw_1,w_2} \ \text{and}\ \ \nu^\varphi:\ S^\varphi_{s_iw_1,w_2} \to \tilde{S}^\varphi_{s_iw_1,w_2}$$

such that $V_p \otimes V_q$ is the direct sum of the simple modules in the fiber $(\nu^\varphi)^{-1}j^\varphi(p,q)$.

Similarly one constructs simple modules over points of $X^\varphi_{w_1',w_2's_j}$ when $\ell(w_2's_j) = \ell(w_2')+1$. □

For every (w_1,w_2) in $W\times W$, let $X^0_{w_1,w_2}$ be the dense subset of X_{w_1,w_2} of elements in the image of the iterated multiplication map

$$\prod_{t=1}^{h} X_{s_{i_t},e} \times \prod_{r=1}^{k} X_{e,s_{j_r}} \to X_{w_1,w_2}$$

for some reduced expressions $w_1 = s_{i_1}\dots s_{i_h}$, $w_2 = s_{j_1}\dots s_{j_k}$. We have therefore constructed all the simple $F_\varepsilon^\varphi[G]$-modules lying over points of

$$\bigcup_{(w_1,w_2)\in W\times W} X^0_{w_1,w_2}$$

Moreover, as in the classical case, we get

Corollary 3.6. *Let (w_1,w_2), (w_1',w_2') in $W\times W$ be such that $\ell(w_1w_1') = \ell(w_1)+\ell(w_1')$ and $\ell(w_2w_2') = \ell(w_2)+\ell(w_2')$. Suppose U, V are simple $F_\varepsilon^\varphi[G]$-modules lying respectively over a point of $X^0_{w_1,w_2}$ and of $X^0_{w_1',w_2'}$. Then $U\otimes V$ is semisimple.* □

4. The solvable case

In this section we consider the solvable case. Our aim is to determine a relation between the representation theory of $F_\varepsilon^\varphi[B]$ and $F_\varepsilon[B]$. So we assume that ℓ besides being φ-good, is also prime to the bad primes of G.

The inclusion of R-algebras $R_q^\varphi[B]' \le R_q[B]'$ gives rise to the inclusion of $\mathbb{C}$-algebras $C_\varepsilon^\varphi \le C_\varepsilon$, where C_ε^φ is the specialization of $R_q^\varphi[B]'$

at $q = \varepsilon$ (cf. [DL] §6). If we consider the central subalgebra $C^{\varphi}_{\varepsilon,0} = \mathbb{C}[(f^{\varphi}_{\alpha})^{\ell}, K_{\ell(1-\varphi)\lambda} \mid \alpha \in \Phi^+, \lambda \in P]$, then we get

$$C^{\varphi}_{\varepsilon,0} \leq C_{\varepsilon,0}. \tag{4.1}$$

Since Spec $C^{\varphi}_{\varepsilon,0} = B =$ Spec $C_{\varepsilon,0}$, and $C_{\varepsilon,0}$ is finite over $C^{\varphi}_{\varepsilon,0}$, we get a surjective morphism $f_{\varphi}: B \longrightarrow B$ (since the inclusion (4.1) is not of coalgebras, f_{φ} is not a homomorphism).

Using the isomorphism (2.2) between $R^{\varphi}_{q}[B^+]$ and $R^{\varphi}_{q}[B]'$, we get an isomorphism $\iota^{\varphi}_{\varepsilon}: F^{\varphi}_{\varepsilon}[B] \to C^{\varphi}_{\varepsilon}$, and therefore the injection i^{φ} of $\mathbb{C}$-algebras from $F^{\varphi}_{\varepsilon}[B]$ into $F_{\varepsilon}[B]$ by composing

$$F^{\varphi}_{\varepsilon}[B] \xrightarrow{\iota^{\varphi}_{\varepsilon}} C^{\varphi}_{\varepsilon} \hookrightarrow C_{\varepsilon} \xrightarrow{\iota^{-1}_{\varepsilon}} F_{\varepsilon}[B].$$

Consider the image Z_0 of the Frobenius morphism (Theorem 1.6 in [DP], §3 in [CV2]) $\mathcal{F}^{\varphi}_{+}: \mathbb{C}[B] \longrightarrow F^{\varphi}_{\varepsilon}[B]$, which is independent of φ. Under $\iota^{\varphi}_{\varepsilon}$, Z_0 is mapped onto $C^{\varphi}_{\varepsilon,0}$. It follows that i^{φ} induces an injection j^{φ} of Z_0 into Z_0. The corresponding map between spectra is the map f_{φ} defined above.

To describe f_{φ} we may assume $\varepsilon = 1$. We observe that the vanishing ideals of T and U (the unipotent radical of B) in $\mathbb{C}[B]$ correspond to the ideals $\mathcal{I}_{\varphi} := (f^{\varphi}_{\alpha} \mid \alpha \in \Phi^+)$ and $\mathcal{J}_{\varphi} = (K_{(1-\varphi)\lambda} - 1 \mid \lambda \in P)$ in C^{φ}_{1}. Since $\mathcal{I}_{\varphi} \subseteq \mathcal{I}$ and $\mathcal{J}_{\varphi} \subseteq \mathcal{J}$, f_{φ} induces the restrictions

$$f_{\varphi|T}: T \longrightarrow T, \quad f_{\varphi|U}: U \longrightarrow U.$$

It is clear that $f_{\varphi|T}$ is a homomorphism with kernel isomorphic to $P/(1-\varphi)P$, while $f_{\varphi|U}$ is the identity.

The Poisson structure $\{\_,\_\}_{\varphi}$ on B is defined via the natural isomorphism between $R^{\varphi}_{q}[B^+]/(q-1)$ and $\mathbb{C}[B]$ in the following way (cf. §1.5 in [DP], §3.1 in [CV2]). Given f, g in $\mathbb{C}[B]$ we have

$$\{f,g\}_{\varphi} = \frac{[\tilde{f},\tilde{g}]_{\varphi}}{q-1} \mod (q-1),$$

where $\tilde{f}$, $\tilde{g}$ are representatives of f, g in $R^{\varphi}_{q}[B^+]$. Since $\{j^{\varphi}(f), j^{\varphi}(g)\} = j^{\varphi}(\{f,g\}_{\varphi})$ for every f, g in $\mathbb{C}[B]$,

$$f_{\varphi}: (B, \{,\}) \longrightarrow (B, \{,\}_{\varphi})$$

is a Poisson morphism. Our aim is to prove that $f_{\varphi}(X_w) = X_w$ for every w in W (here $X_w = B^- w B^- \cap B$).

For this purpose we consider the algebra automorphisms of $F_{\varepsilon}[B]$ introduced in [DP], §4.5. For t in T, we have $t(K_{\lambda}) = t^{\lambda}(K_{\lambda})$, $t(f_{\beta}) = f_{\beta}$ for every λ in P, β in Φ^+. Moreover $J_{\varepsilon,w}$ is stable under the action of T. We introduce another group of automorphisms of $F_{\varepsilon}[B]$. Let $Q^{\vee} = \mathrm{Hom}(Q, \mathbb{C}^*)$, and ν in $Q^\vee$. Let x be an element of weight β in C_ε. We put $\nu(x) = \beta^\nu x$. Extending ν by linearity, we get an algebra automorphism of $F_\varepsilon[B]$, such that $\nu(K_\lambda) = K_\lambda$, $\nu(f_\beta) = \beta^\nu f_\beta$, leaving $J_{\varepsilon,w}$ stable. For every t in T, we define the character ν_t by $\beta^{\nu_t} = t^{-\frac{1}{2}\varphi\beta}$, and we define the algebra automorphism ψ_t by $\psi_t(g) = \nu_t t(g)$. Note that

$$\psi_t(K_\lambda) = t^\lambda(K_\lambda), \qquad \psi_t(f_\alpha^\varphi) = f_\alpha^\varphi. \tag{4.2}$$

We identify T with the group $\{\psi_t \mid t \in T\}$. It follows from (4.2), that T fixes C_ε^φ.

Definition 4.1. We put $T_\varphi := \{t \in T \mid t^{(1-\varphi)P} = 1\}$.

Proposition 4.2. *The morphism f_φ is an unramified Galois covering with Galois group $P/(1-\varphi)P$.*

Proof. The group T_φ acts trivially on C_1^φ. Since $\psi_t(K_\lambda) = t^\lambda(K_\lambda)$, and T_φ has character group $P/(1-\varphi)P$, it follows that T_φ acts transitively and faithfully on the fibers of f_φ. □

We now prove that $f_\varphi(\overline{X}_w) = \overline{X}_w$. We have to show that $i^\varphi(J_{1,w}) \subseteq J_{1,w}$. To check this inclusion, it is convenient to work with the pairings. The integral form $\Gamma_q^\varphi(\mathfrak{b}_i)$ of $U_q^\varphi(\mathfrak{b}_i)$ is the R-algebra generated by $E_i^{\varphi(p)}$, $\binom{K_{(1+\varphi)\alpha_i};0}{t}$, $p \geq 0$, $t \geq 1$, and $K^{\pm 1}_{(1+\varphi)\alpha_i}$. $E_i^{\varphi(p)}$ is the φ-analogue of Lusztig's divided power, and

$$\binom{K_{(1+\varphi)\alpha_i};0}{t} = \prod_{s=1}^{t} \frac{K_{(1+\varphi)\alpha_i} q_i^{1-s} - 1}{q_i^s - 1}.$$

The integral form of $\mathcal{T}_i^{*,\varphi}$ is the R-algebra $\Gamma_q^{*,\varphi}(i)$ generated by $\binom{K_{(1+\varphi)\alpha_i};0}{t}$, $t \geq 1$, and $K^{\pm 1}_{(1+\varphi)\alpha_i}$. We also denote by $\Gamma_q^\varphi(\mathfrak{t})$ the R-algebra generated by $\mathcal{T}_1^{*,\varphi}, \ldots, \mathcal{T}_n^{*,\varphi}$.

We fix w in W, and a reduced expression $w = s_{i_1} \ldots s_{i_k}$.

Lemma 4.3. $f_\varphi(\overline{X}_w) = \overline{X}_w$.

Proof. It is enough to show that $f_\varphi(\overline{X}_w) \subseteq \overline{X}_w$, since f_φ is finite, and X_w is irreducible ([J] §A.4.5). Let a be in C_1^φ. We have to show that if $\pi_\varphi(a, x^\varphi) = 0$ for every monomial x^φ of type

$$x^\varphi = u_{i_1}^\varphi \ldots u_{i_k}^\varphi m_{i_{k+1}}^\varphi \ldots m_{i_N}^\varphi,$$

with $u_{i_j}^\varphi$, $m_{i_s}^\varphi$ in the corresponding $\mathbb{C}$-algebras $\Gamma_1^\varphi(\mathfrak{b}_{i_j})$, $\Gamma_1^{*,\varphi}(i_s)$, then $\pi(a, x) = 0$ for every monomial x of type

$$x = u_{i_1} \ldots u_{i_k} m_{i_{k+1}} \ldots m_{i_N},$$

with u_{i_j}, m_{i_s} in $\Gamma_1(\mathfrak{b}_{i_j})$, $\Gamma_1^*(i_s)$ respectively. We may assume a to be homogeneous of weight β. Then

$$a = \sum_{(r,\lambda)} c_{(r,\lambda)} (f^{\varphi}_{\beta_N})^{r_N} \dots (f^{\varphi}_{\beta_1})^{r_1} K_{(1-\varphi)\lambda}, \tag{4.3}$$

with $r = (r_N, \dots, r_1)$, $\sum r_i \beta_i = \beta$. There is a natural way to associate a monomial of type $u^{\varphi}_{i_1} \dots u^{\varphi}_{i_k} m^{\varphi}_{i_{k+1}} \dots m^{\varphi}_{i_N}$ to a monomial of type $u_{i_1} \dots u_{i_k} m_{i_{k+1}} \dots m_{i_N}$, so that E^{φ}_i corresponds to E_i, and $K_{(1+\varphi)\alpha_i}$ to K_{α_i}.

For short we write f^{φ} for $(f^{\varphi}_{\beta_N})^{r_N} \dots (f^{\varphi}_{\beta_1})^{r_1}$. It is enough to show that $\pi(a, EM) = 0$ with

$$E = E^{(p_1)}_{i_1} \dots E^{(p_k)}_{i_k}, \quad M \in \Gamma_1(\mathfrak{t}).$$

Now $\pi(f^{\varphi} K_{(1-\varphi)\lambda}, EM) = \pi(\Delta(f^{\varphi} K_{(1-\varphi)\lambda}), M \otimes E) = \pi(K_{\frac{1}{2}\varphi\beta + (1-\varphi)\lambda}, M) \pi(f^{\varphi}, E)$, since $q = 1$. Besides, $\pi_{\varphi}(f^{\varphi} K_{(1-\varphi)\lambda}, E^{\varphi} M^{\varphi}) = \pi_{\varphi}(\Delta_{\varphi}(f^{\varphi} K_{(1-\varphi)\lambda}), M^{\varphi} \otimes E^{\varphi}) = \pi_{\varphi}(K_{(1-\varphi)\lambda}, M^{\varphi}) \pi(f^{\varphi}, E^{\varphi})$. Note that

$$\pi_{\varphi}(K_{(1-\varphi)\mu}, K_{(1+\varphi)\alpha}) = q^{-(\mu,(1+\varphi)\alpha)} = q^{-((1-\varphi)\mu,\alpha)} = \pi(K_{(1-\varphi)\mu}, K_{\alpha}),$$

so that $\pi_{\varphi}(K_{(1-\varphi)\mu}, M^{\varphi}) = \pi(K_{(1-\varphi)\mu}, M)$, and $\pi(f^{\varphi}, E^{\varphi}) = \pi(f^{\varphi}, E)$, by Lemma 1.5 in [CV2], since $q = 1$. From (4.3), it follows that $aK_{-\frac{1}{2}\varphi\beta}$ lies in $\iota_1(J_{1,w})$. Then a lies in $\iota_1(J_{1,w})$, and we are done. □

Corollary 4.4. *Let y be in $\overline{X}_w$. Then the fiber of y is contained in $\overline{X}_w$.*

Proof. By Lemma 4.3, there exists x in $\overline{X}_w$ such that $f_{\varphi}(x) = y$ and, by Proposition 4.2, the fiber of y is the orbit of x under T_{φ}. Since the automorphisms ψ_t leaves $J_{\varepsilon,w}$ stable, the claim follows. □

Proposition 4.5. *$f_{\varphi}(X_w) = X_w$ for every w in W.*

Proof. Let x be in X_w. By Lemma 4.3, $f_{\varphi}(x)$ lies in $\overline{X}_w$. Hence there exists y in W, with $y \leq w$ in the Bruhat order such that $f_{\varphi}(x)$ lies in X_y. By Corollary 4.4, x lies in $\overline{X}_y$. But x was in X_w, so that $w \leq y$. Hence $y = w$, and $f_{\varphi}(x)$ lies in X_w. □

It also follows that $f_{\varphi}: X_w \to X_w$ is an unramified Galois covering with Galois group $P/(1-\varphi)P$.

Remark. Since φ is a Poisson morphism, if $\mathcal{O}$ is a symplectic leaf of $(B, \{,\})$, then $f_{\varphi}(\mathcal{O})$ is contained in a symplectic leaf $\mathcal{O}^{\varphi}$ of $(B, \{,\}_{\varphi})$. If $\mathcal{O}$ is contained in X_w, it follows from Proposition 4.5, that $\mathcal{O}^{\varphi}$ is also contained in X_w. We have $\dim\mathcal{O} = \ell(w) + \mathrm{rk}(1-w) = \ell(w) + \mathrm{rk}((1+\varphi)w(1-\varphi) - (1-\varphi)(1+\varphi)) = \dim\mathcal{O}^{\varphi}$.

We can now determine a relation between the representation theory of $F_\varepsilon[B]^\varphi$ and $F_\varepsilon[B]$. Suppose V is a simple $F_\varepsilon[B]$-module, with central character b in X_w. Then via i^φ, V becomes an $F_\varepsilon^\varphi[B]$-module V^φ, with central character $f_\varphi(b)$ in X_w by Proposition 4.5. But then, by the previous remark, every factor in a composition series of V^φ has dimension equal to the dimension of V. Hence V^φ is simple, and we get a map

$$F_\varphi\colon \operatorname{Spec} F_\varepsilon[B] \longrightarrow \operatorname{Spec} F_\varepsilon^\varphi[B].$$

On the other hand, for every b in B, the $\mathbb{C}$-algebras $F_\varepsilon^\varphi[B]/m_b F_\varepsilon^\varphi[B]$, where m_b is the vanishing ideal of b in Z_0, have all dimension $\ell^{\dim B}$. Moreover, for every b in B, the composite map

$$F_\varepsilon^\varphi[B] \xrightarrow{i^\varphi} F_\varepsilon[B] \to F_\varepsilon[B]/m_b F_\varepsilon[B]$$

is surjective since $(\ell, \det(1-\varphi)) = 1$, and its kernel contains $m_{f_\varphi(b)}$. Since $\dim F_\varepsilon[B]/m_b F_\varepsilon[B] = \dim F_\varepsilon^\varphi[B]/m_{f_\varphi(b)} F_\varepsilon^\varphi[B]$, i^φ induces the isomorphism

$$i_b^\varphi\colon F_\varepsilon^\varphi[B]/m_{f_\varphi(b)} F_\varepsilon^\varphi[B] \longrightarrow F_\varepsilon[B]/m_b F_\varepsilon[B].$$

It follows that, for every b in B and every M in Spec $F_\varepsilon^\varphi[B]$ lying over $f_\varphi(b)$, there exists a unique V in Spec $F_\varepsilon[B]$ lying over b, such that $M = V^\varphi$.

Theorem 4.6. *Let b be in B, and let $\{b_1, \ldots, b_m\}$ be the fiber $f_\varphi^{-1}(b)$. Then $m = \det(1-\varphi)$, and for every M in* Spec $F_\varepsilon^\varphi[B]$ *lying over b, the fiber $F_\varphi^{-1}(M)$ consists of m simple $F_\varepsilon[B]$-modules $V_1, \ldots, V_m$, one over each b_i, and it consists of a single orbit under the action of $P/(1-\varphi)P$.*

Proof. We only need to prove the last statement. The group T acts in a natural way on Spec $F_\varepsilon[B]$ via the automorphisms ψ_t previously introduced. Let V be in $F_\varphi^{-1}(M)$, with central character g. Then it is clear that $F_\varphi(t(V)) = M$ for every t in T_φ. Since $t(V)$ lies over $\psi_t(g)$, it follows that $F_\varphi^{-1}(M) = \{t(V) \mid t \in T_\varphi\}$. □

We summarize the previous results in the following diagrams. Recall we denote by χ^φ the central character map from $\operatorname{Spec} F_\varepsilon[B]^\varphi$ onto B.

$$\begin{array}{ccc} \operatorname{Spec} F_\varepsilon[B] & \xrightarrow{F_\varphi} & \operatorname{Spec} F_\varepsilon[B]^\varphi \\ {\scriptstyle\chi}\downarrow & & \downarrow{\scriptstyle\chi^\varphi} \\ B & \xrightarrow{f_\varphi} & B \end{array} \qquad \begin{array}{ccc} \operatorname{Spec} F_\varepsilon[B]_w & \xrightarrow{F_\varphi} & \operatorname{Spec} F_\varepsilon[B]_w^\varphi \\ {\scriptstyle\chi}\downarrow & & \downarrow{\scriptstyle\chi^\varphi} \\ X_w & \xrightarrow{f_\varphi} & X_w \end{array}$$

for every w in W.

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