SIAM J. MATH. ANAL.
Vol. 47, No. 1, pp. 407–426

SHAPE SENSITIVITY ANALYSIS OF THE EIGENVALUES OF THE REISSNER–MINDLIN SYSTEM*

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Abstract. We consider the eigenvalue problem for the Reissner–Mindlin system arising in the study of the free vibration modes of an elastic clamped plate. We provide quantitative estimates for the variation of the eigenvalues upon variation of the shape of the plate. We also prove analyticity results and establish Hadamard-type formulas. Finally, we address the problem of minimization of the eigenvalues in the case of isovolumetric domain perturbations. In the spirit of the Rayleigh conjecture for the biharmonic operator, we prove that balls are critical points with volume constraint for all simple eigenvalues and the elementary symmetric functions of multiple eigenvalues.

Key words. Reissner–Mindlin, plates, eigenvalues, domain perturbation

AMS subject classifications. 35J47, 35B20, 35P15, 74K20

DOI. 10.1137/140969968

1. Introduction. Let Ω be a bounded open set in $\mathbb{R}^N$ with $N \geq 2$, and $t, \lambda, \mu, k > 0$ be fixed parameters. We consider the following eigenvalue problem

$$\begin{cases} -\frac{\mu}{12}\Delta\beta - \frac{\mu+\lambda}{12}\nabla\mathrm{div}\beta - \frac{\mu k}{t^2}(\nabla w - \beta) = \frac{\gamma t^2}{12}\beta & \text{in } \Omega, \\ -\frac{\mu k}{t^2}(\Delta w - \mathrm{div}\beta) = \gamma w & \text{in } \Omega, \\ \beta = 0, \quad w = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

in the unknowns $(\beta, w) = (\beta_1, \dots, \beta_N, w)$ (the eigenvector) and γ (the eigenvalue). According to the Reissner–Mindlin model for moderately thin plates, for $N = 2$, system (1.1) describes the free vibration modes of an elastic clamped plate $\Omega \times (-t/2, t/2)$ with midplane Ω and thickness t. In that case λ and μ are the Lamé constants, k is the correction factor, w the transverse displacement of the midplane, $\beta = (\beta_1, \beta_2)$ the fiber rotation, and γt^2 the vibration frequency. We refer to Durán et al. [13] for more information and references; see also Hervella-Nieto [15]. Although $N = 2$ seems to be the case of main interest in applications, our methods allow us to treat the general case without any restriction on the space dimension.

It is well known that the spectrum of the Reissner–Mindlin system is discrete, hence problem (1.1) has a divergent sequence of positive eigenvalues of finite multiplicity

$$0 < \gamma_{1,t}[\Omega] \leq \gamma_{2,t}[\Omega] \leq \cdots \leq \gamma_{n,t}[\Omega] \leq \cdots$$

depending on t and Ω. Here each eigenvalue is repeated according to its multiplicity.

*Received by the editors May 27, 2013; accepted for publication October 20, 2014; published electronically January 13, 2015. This work was supported by the research project "Singular perturbation problems for differential operators," Progetto di Ateneo of the University of Padova. The authors are also members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

http://www.siam.org/journals/sima/47-1/96996.html

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The behavior of the solutions to Reissner–Mindlin systems as $t \to 0$ is well known. We refer to Brezzi and Fortin [2, 3] for a deep analysis of related computational problems and references; see also Lovadina, Mora, and Rodríguez [18]. In particular, it is proved in [13] for $N = 2$ that $\gamma_{n,t}[\Omega] \to \gamma_{n,0}[\Omega]$ as $t \to 0$, where $\gamma_{n,0}[\Omega]$ are the eigenvalues of the problem

$$(1.2) \qquad \begin{cases} \frac{2\mu+\lambda}{12}\Delta^2 w = \gamma w & \text{in } \Omega, \\ w = \nabla w = 0 & \text{on } \partial\Omega. \end{cases}$$

In this paper we are interested in the dependence of $\gamma_{n,t}[\Omega]$ on Ω. In section 3, we provide stability estimates in the spirit of [6, 7, 8, 9, 10]. These estimates allow us to control the variation of $\gamma_{n,t}[\Omega]$ upon variation of Ω.

First, we consider the case of domain deformations of the form $\phi(\Omega)$, where ϕ is a diffeomorphism of class $C^{1,1}$ and in Theorem 3.3 we prove the existence of a constant $c > 0$ independent of n and t such that

$$(1.3) \qquad |\gamma_{n,t}[\phi(\Omega)] - \gamma_{n,t}[\Omega]| \leq c\gamma_{n,t}[\Omega]\delta(\phi),$$

provided $\delta(\phi) < c^{-1}$, where $\delta(\phi)$ is defined by

$$(1.4) \qquad \delta(\phi) = \max_{1\leq|\alpha|\leq 2} \sup_{x\in\Omega} |D^\alpha(\phi(x) - x)|.$$

Second, we prove estimates in terms of explicit geometric quantities which measure the vicinity of two open sets Ω_1 and Ω_2. To do so, we assume that Ω_1 and Ω_2 belong to the same uniform class $C(\mathcal{A})$, where $\mathcal{A}$ is a fixed atlas by the help of which the open sets are described locally as the subgraphs of suitable continuous functions; see Definition 2.2. In this case, it is possible to prove the existence of a constant $c > 0$ independent of n and t such that

$$(1.5) \qquad |\gamma_{n,t}[\Omega_1] - \gamma_{n,t}[\Omega_2]| \leq c \max\{\gamma_{n,t}[\Omega_1], \gamma_{n,t}[\Omega_2]\} d_{\mathcal{A}}(\Omega_1, \Omega_2),$$

provided $d_{\mathcal{A}}(\Omega_1, \Omega_2) < c^{-1}$, where $d_{\mathcal{A}}(\Omega_1, \Omega_2)$ is the so-called atlas distance of Ω_1 and Ω_2; see Theorem 3.6 and Definition 2.3. We note that the atlas distance $d_{\mathcal{A}}(\Omega_1, \Omega_2)$ is an easily computable one-dimensional distance which measures the gap between the graphs describing the boundaries of Ω_1, Ω_2 and that it is possible to control it via the more familiar Hausdorff distance between $\partial\Omega_1$ and $\partial\Omega_2$; see Theorem 2.4. Importantly, the atlas class $C(\mathcal{A})$ includes open sets with strong boundary degenerations such as cusps of exponential type. In fact, if the modulus of continuity ω of the functions describing the boundaries of Ω_1 and Ω_2 is fixed and the boundary of one of the two domains is contained in an ϵ-neighborhood of the boundary of the other one, then it is possible to prove an estimate via $\omega(\epsilon)$; see Corollary 3.7.

Note that error estimates independent of t for a finite element discretization of the eigenvalue problem (1.1) on a polygon in the plane have been obtained in Durán et al. [13]. Considering that polygons are typically used in order to approximate sufficiently regular planar domains, we believe that our estimates complement those in [13].

In section 4, we consider families of open sets $\phi(\Omega)$ parametrized by Lipschitz homeomorphisms ϕ, and we prove analyticity results for the dependence of $\gamma_{n,t}[\phi(\Omega)]$ on ϕ. Following the analysis of [16], we prove that simple eigenvalues and the elementary symmetric functions of multiple eigenvalues depend real analytically on ϕ, and we establish Hadamard-type formulas for the Fréchet differentials; see Theorem 4.1.

In particular, if Ω is sufficiently smooth and $\gamma_{n,t}[\Omega]$ is simple then for perturbations of the identity I of the type $\phi_\epsilon = I + \epsilon\psi$, $\epsilon \in \mathbb{R}$, we have

$$\frac{d\gamma_{n,t}[\phi_\epsilon(\Omega)]}{d\epsilon}_{|\epsilon=0} = -\int_{\partial\Omega}\left(\frac{\mu}{12}\left|\frac{\partial\beta}{\partial n}\right|^2 + \frac{\mu+\lambda}{12}\left(\frac{\partial\beta}{\partial n}\cdot n\right)^2 + \frac{\mu k}{t^2}\left(\frac{\partial w}{\partial n}\right)^2\right)\psi\cdot n d\sigma, \tag{1.6}$$

where n is the unit outer normal to $\partial\Omega$ and (β, w) is an eigenvector associated with $\gamma_{n,t}[\Omega]$ normalized by the condition $\int_\Omega w^2 + \frac{t^2}{12}|\beta|^2 dx = 1$. The bifurcation phenomenon which occurs in the case of multiple eigenvalues is more involved and is described by the Rellich–Nagy-type Theorem 4.3.

Finally, in section 5 we address the problem of the optimization of the eigenvalues in the case of isovolumetric perturbations. Recall that the celebrated Rayleigh conjecture states that, among all bounded domains with fixed measure, the first eigenvalue of problem (1.2) is minimized by the ball. Such a conjecture has been proved for $N = 2$ by N.S. Nadirashvili and for $N = 2, 3$ by M.S. Ashbaugh and R.D. Benguria. We refer to Henrot [14] for a survey on this topic. Taking into account the limiting behavior of the Reissner–Mindlin eigenvalues as $t \to 0$, it would be natural to state the same conjecture also for the Reissner–Mindlin system. Here we give support to it by proving that the Reissner–Mindlin system exhibits the same symmetry property of biharmonic and polyharmonic operators; see [4, 5, 17]. Namely, we prove that balls are critical points with volume constraint for all simple eigenvalues and all symmetric functions of multiple eigenvalues of system (1.1); see Theorem 5.5. To do so, we characterize critical open sets as those open sets for which a suitable overdetermined system has nontrivial solutions and we prove that such overdetermined conditions are satisfied when the open set is a ball.

2. Preliminaries and notation.

In this section we introduce the eigenvalue problem under consideration and the classes of open sets which allow us to prove the quantitative estimates of section 3.

2.1. The Reissner–Mindlin eigenvalue problem.

Let Ω be an open set in $\mathbb{R}^N$ with finite measure. By $H^1_0(\Omega)$ we denote the closure in the standard Sobolev space $H^1(\Omega)$ of the space of C^∞-functions with compact support in Ω. We set $\mathcal{V}(\Omega) = (H^1_0(\Omega))^N \times H^1_0(\Omega)$ and we denote by (β, w) the generic element of $\mathcal{V}(\Omega)$, where $\beta = (\beta_1, \ldots, \beta_N) \in (H^1_0(\Omega))^N$ and $w \in H^1_0(\Omega)$.

For any fixed $t, \lambda, \mu, k > 0$, we consider the weak formulation of problem (1.1). Namely, we say that $\gamma \in \mathbb{R}$ is an eigenvalue of the Reissner–Mindlin system if and only if there exists $(\beta, w) \in \mathcal{V}(\Omega)$ with $(\beta, w) \neq 0$ such that

$$\begin{aligned}\frac{\mu}{12}\int_\Omega \nabla\beta : \nabla\eta dx + \frac{\mu+\lambda}{12}\int_\Omega \mathrm{div}\beta\,\mathrm{div}\eta dx + \frac{\mu k}{t^2}\int_\Omega (\nabla w - \beta)\cdot(\nabla v - \eta)dx \\ = \gamma\int_\Omega\left(wv + \frac{t^2}{12}\beta\cdot\eta\right)dx\end{aligned} \tag{2.1}$$

for all test functions $(\eta, v) \in \mathcal{V}(\Omega)$, in which case (β, w) is called an eigenvector associated with γ. Here by $A : B$ we denote the Frobenius product of two matrices A, B, defined by $A : B = \sum_{i,j=1}^N a_{ij}b_{ij}$. Note that β is thought of as a row vector.

As is customary in spectral theory we interpret problem (2.1) as an eigenvalue problem for a nonnegative self-adjoint operator in Hilbert space as follows. For any fixed $t > 0$, we denote by $\mathcal{L}^2_t(\Omega)$ the space $(L^2(\Omega))^N \times L^2(\Omega)$ endowed with the scalar product $< (\beta, w), (\eta, v) >_{\Omega,t}$ defined by the right-hand side of (2.1) (without γ) for any

$(\beta, w), (\eta, v) \in \mathcal{L}^2_t(\Omega)$. Clearly, for each $t > 0$ the norm induced by such a scalar product is equivalent to the standard L^2-norm. Moreover, we consider the bilinear form $Q_{\Omega,t}$ defined on $\mathcal{V}(\Omega) \times \mathcal{V}(\Omega)$ by the left-hand side of equality (2.1). We also denote by $Q_{\Omega,t}(\beta, w) = Q_{\Omega,t}((\beta, w), (\beta, w))$ the quadratic form associated with the bilinear form $Q_{\Omega,t}$ and we observe that such a quadratic form is coercive in $\mathcal{V}(\Omega)$. In particular, the corresponding norm $Q^{1/2}_{\Omega,t}(\cdot)$ is equivalent to the standard Sobolev norm in $\mathcal{V}(\Omega)$. This implies that the quadratic form $Q_{\Omega,t}(\cdot)$ is closed in $\mathcal{L}^2_t(\Omega)$, hence (see, e.g., Davies [11, Chap. 4]) there exists a nonnegative self-adjoint operator $R_{\Omega,t}$ densely defined on $\mathcal{L}^2_t(\Omega)$ such that the domain $\mathrm{Dom}(R^{1/2}_{\Omega,t})$ of the square root $R^{1/2}_{\Omega,t}$ of $R_{\Omega,t}$ is $\mathcal{V}(\Omega)$ and such that $Q_{\Omega,t}((\beta, w), (\eta, v)) =< R^{1/2}_{\Omega,t}(\beta, w), R^{1/2}_{\Omega,t}(\eta, v) >_{\Omega,t}$ for all $(\beta, w), (\eta, v) \in \mathcal{V}(\Omega)$. In particular, $(\beta, w) \in \mathrm{Dom}(R_{\Omega,t})$ if and only if $(\beta, w) \in \mathrm{Dom}(R^{1/2}_{\Omega,t})$ and there exists $(\theta, f) \in L^2(\Omega) \times (L^2(\Omega))^N$ such that $Q_{\Omega,t}((\beta, w), (\eta, v)) =< (\theta, f), (\eta, v) >_{\Omega,t}$ for all $(\eta, v) \in \mathcal{V}(\Omega)$, in which case $R_{\Omega,t}(\beta, w) = (\theta, f)$.

It follows that the eigenvalues and the eigenvectors of problem (2.1) coincide with the eigenvalues and the eigenvectors of the operator $R_{\Omega,t}$. Moreover, since $|\Omega|$ is finite, $\mathcal{V}(\Omega)$ is compactly embedded into $\mathcal{L}^2_t(\Omega)$, hence the spectrum of $R_{\Omega,t}$ is discrete and consists of a divergent sequence of positive eigenvalues of finite multiplicity, which we denote by $\gamma_{n,t}[\Omega]$, $n \in \mathbb{N}$. We note that by the Courant min-max principle, we have

$$\gamma_{n,t}[\Omega] = \min_{\substack{E \subset \mathcal{V}(\Omega) \\ \dim E = n}} \max_{(\beta,w) \in E \setminus \{0\}} \frac{Q_{\Omega,t}(\beta, w)}{\|(\beta, w)\|^2_{\mathcal{L}^2_t(\Omega)}} \tag{2.2}$$

for all $n \in \mathbb{N}$.

Remark 2.1. Assume that Ω is a bounded open set in $\mathbb{R}^2$ representing the midplane of an elastic clamped plate $\Omega \times] - t/2, t/2[$ of thickness t. In the literature (cf., e.g., [13]), the weak formulation of the eigenvalue problem for the Reissner–Mindlin system describing the free vibration modes of such a plate can be found in the form

$$t^3 a(\beta, \eta) + \mathcal{K} t \int_\Omega (\nabla w - \beta) \cdot (\nabla v - \eta) dx = \omega^2 \left(t \int_\Omega wv dx + \frac{t^3}{12} \int_\Omega \beta \cdot \eta dx \right), \tag{2.3}$$

where $a(\beta, \eta) = \frac{E}{12(1-\nu^2)} \int_\Omega [(1 - \nu)\epsilon(\beta) : \epsilon(\eta) + \nu \mathrm{div}\beta \mathrm{div}\eta] dx$ and $\mathcal{K} = \frac{Ek}{2(1+\nu)}$. Here ω is the angular vibration frequency, $\epsilon(\beta) = (\nabla\beta + \nabla^t\beta)/2$ is the linear strain tensor, ν the Poisson ratio, E the Young modulus, $\mathcal{K}$ the shear modulus, and k the correction factor (usually $k = 5/6$). By recalling Korn's indentity

$$2 \int_\Omega \epsilon(\beta) : \epsilon(\eta) dx = \int_\Omega \nabla\beta : \nabla\eta dx + \int_\Omega \mathrm{div}\beta \mathrm{div}\eta dx,$$

which holds for any $\beta, \eta \in \mathcal{V}(\Omega)$, problem (2.3) can be easily rewritten in the form (2.1) by setting $\gamma = \omega^2/t^2$ and choosing

$$\lambda = \frac{\nu E}{1 - \nu^2}, \quad \text{and} \quad \mu = \frac{E}{2(1 + \nu)}. \tag{2.4}$$

The formulation in (2.1) is somewhat more general since it allows any choice of the constants $\lambda, \mu > 0$ including the standard Lamé constants $\lambda = \nu E/[(1 + \nu)(1 - 2\nu)]$, $\mu = \frac{E}{2(1+\nu)}$.

We refer also to Bathe [1] for further details.

2.2. The atlas class and the atlas distance. For any set V in $\mathbb{R}^N$ and $\delta > 0$ we denote by V_δ the set $\{x \in V : d(x, \partial V) > \delta\}$. We shall also denote by V^δ the set $\{x \in \mathbb{R}^N : d(x, V) < \delta\}$. Here $d(x, A)$ denotes the Euclidean distance from x to a set A. We recall the following definition from [9], where by cuboid we mean a set which is the isometric image of a set of the form $\Pi_{i=1}^N]a_i, b_i[$.

Definition 2.2 (atlas class). *Let $\rho > 0$, $s, s' \in \mathbb{N}$, $s' \le s$, and $\{V_j\}_{j=1}^s$ be a family of bounded open cuboids and $\{r_j\}_{j=1}^s$ be a family of isometries in $\mathbb{R}^N$. We say that $\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ is an atlas in $\mathbb{R}^N$ with the parameters $\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s$, briefly an atlas in $\mathbb{R}^N$.*

We denote by $C(\mathcal{A})$ the family of all open sets Ω in $\mathbb{R}^N$ satisfying the following properties:

(i) $\Omega \subset \bigcup_{j=1}^{s} (V_j)_\rho$ *and* $(V_j)_\rho \cap \Omega \neq \emptyset$;

(ii) $V_j \cap \partial\Omega \neq \emptyset$ *for* $j = 1, \dots s'$, $V_j \cap \partial\Omega = \emptyset$ *for* $s' < j \le s$;

(iii) *for* $j = 1, \dots, s$

$$r_j(V_j) = \{\, x \in \mathbb{R}^N : \ a_{ij} < x_i < b_{ij},\ i = 1, \dots, N\}$$

and

$$r_j(\Omega \cap V_j) = \{x \in \mathbb{R}^N : \ a_{Nj} < x_N < g_j(\bar{x}),\ \bar{x} \in W_j\},$$

where $\bar{x} = (x_1, \dots, x_{N-1})$, $W_j = \{\bar{x} \in \mathbb{R}^{N-1} : \ a_{ij} < x_i < b_{ij},\ i = 1, \dots, N-1\}$ *and* g_j *is a continuous function defined on* $\overline{W}_j$ *(it is meant that if* $s' < j \le s$ *then* $g_j(\bar{x}) = b_{Nj}$ *for all* $\bar{x} \in \overline{W}_j$*); moreover for* $j = 1, \dots, s'$

$$a_{Nj} + \rho \le g_j(\bar{x}) \le b_{Nj} - \rho$$

for all $\bar{x} \in \overline{W}_j$.

We say that an open set Ω in $\mathbb{R}^N$ is an open set with a continuous boundary if Ω is of class $C(\mathcal{A})$ for some atlas $\mathcal{A}$.

Let $\omega : [0, \infty[\to [0, \infty[$ be a modulus of continuity, i.e., a continuous nondecreasing function such that $\omega(0) = 0$ and, for some $k > 0$, $\omega(t) \ge kt$ for all $0 \le t \le 1$. Let $M > 0$. We denote by $C_M^{\omega(\cdot)}(\mathcal{A})$ the family of all open sets Ω in $\mathbb{R}^N$ belonging to $C(\mathcal{A})$ and such that all the functions g_j in Definition 2.2(iii) *satisfy the condition*

$$|g_j(\bar{x}) - g_j(\bar{y})| \le M\omega(|\bar{x} - \bar{y}|) \tag{2.5}$$

for all $\bar{x}, \bar{y} \in \overline{W}_j$.

We also say that an open set is of class $C^{\omega(\cdot)}$ if there exists an atlas $\mathcal{A}$ and $M > 0$ such that $\Omega \in C_M^{\omega(\cdot)}(\mathcal{A})$.

The family of open sets of class $C(\mathcal{A})$ can be thought of as a metric space endowed with so-called atlas distance. We recall the definition introduced in [9].

Definition 2.3 (atlas distance). *Let $\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ be an atlas in $\mathbb{R}^N$. For all $\Omega_1, \Omega_2 \in C(\mathcal{A})$ we define the "atlas distance" $d_{\mathcal{A}}$ by*

$$d_{\mathcal{A}}(\Omega_1, \Omega_2) = \max_{j=1,\dots,s} \ \sup_{(\bar{x}, x_N) \in r_j(V_j)} |g_{1j}(\bar{x}) - g_{2j}(\bar{x})|\,, \tag{2.6}$$

where g_{1j}, g_{2j}, respectively, are the functions describing the boundaries of Ω_1, Ω_2, respectively, as in Definition 2.2 (iii).

The atlas distance depends on the chosen atlas but has the advantage of being easily computable. Moreover, we observe that it can be controlled via the Hausdorf distance. Indeed, we have the following theorem where, for the sake of completeness, we collect also other relevant properties of the atlas distance proved in [9].

Given two sets A, B in $\mathbb{R}^N$, the lower Hausdorff–Pompeiu deviation of A from B is defined in [9] by $d_{\mathcal{HP}}(A,B) = \min\{\sup_{x\in A} d(x,B), \sup_{x\in B} d(x,A)\}$. Note that the standard Hausdorff–Pompeiu distance of A and B is defined by $d^{\mathcal{HP}}(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{x\in B} d(x,A)\}$.

THEOREM 2.4. *Let $\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ be an atlas, ω a modulus of continuity as in Definition* 2.2*, and $M > 0$. Let $\tilde{\mathcal{A}} = (\rho/2, s, s', \{(V_j)_{\rho/2}\}_{j=1}^s, \{r_j\}_{j=1}^s)$. Then the following statements hold:*

(i) *$(C(\mathcal{A}), d_{\mathcal{A}})$ is a complete metric space;*
(ii) *$C_M^{\omega(\cdot)}(\mathcal{A})$ is a compact subset of $C(\mathcal{A})$;*
(iii) *there exists $c > 0$ depending only on $N, \mathcal{A}, \omega, M$ such that*

$$d^{\mathcal{HP}}(\partial\Omega_1, \partial\Omega_2) \le d_{\tilde{\mathcal{A}}}(\Omega_1, \Omega_2) \le c\,\omega(d_{\mathcal{HP}}(\partial\Omega_1, \partial\Omega_2)) \tag{2.7}$$

for all $\Omega_1, \Omega_2 \in C_M^{\omega(\cdot)}(\mathcal{A})$.

3. Quantitative estimates.

3.1. Estimates via diffeomorphisms. Given an open set Ω in $\mathbb{R}^N$ with finite measure, we consider a diffeomorphism from Ω onto another open set $\phi(\Omega)$ in $\mathbb{R}^N$ and we prove a quantitative stability estimate for $|\gamma_{n,t}[\phi(\Omega)] - \gamma_{n,t}[\Omega]|$ in terms of the measure of vicinity $\delta(\phi)$ defined by (1.4). In order to obtain an estimate independent of t, we use the special transformation C_ϕ from the space $\mathcal{V}(\Omega)$ onto $\mathcal{V}(\phi(\Omega))$ defined by

$$C_\phi(\beta, w) = (\beta\nabla\phi^{-1}, w) \circ \phi^{(-1)} \tag{3.1}$$

for all $(\beta, w) \in \mathcal{V}(\Omega)$. Here and in what follows we denote by A^{-1} the inverse of a matrix A, as opposed to the inverse of a function f which is denoted by $f^{(-1)}$; we shall also denote by A^T the transpose of A.

It is clear that in order to guarantee that C_ϕ is well-defined, it suffices to assume that ϕ is a diffeomorphism of class $C^{1,1}$, i.e., ϕ and its inverse have Lipschitz continuous gradients. In fact, it is easy to prove the following lemma that will be used in what follows.

LEMMA 3.1. *Let Ω be an open set in $\mathbb{R}^N$ and let $\phi : \Omega \to \phi(\Omega)$ be a diffeomorphism of class $C^{1,1}$ from Ω onto an open set $\phi(\Omega)$ in $\mathbb{R}^N$. Assume that*

$$\max_{1\le|\alpha|\le 2} \sup_{x\in\Omega} |D^\alpha \phi(x)| < \infty, \qquad \inf_{x\in\Omega} |\det\nabla\phi(x)| > 0.$$

Then C_ϕ is a linear homeomorphism from $\mathcal{V}(\Omega)$ onto $\mathcal{V}(\phi(\Omega))$.

Then we can prove the following.

LEMMA 3.2. *Let Ω be an open set in $\mathbb{R}^N$ with finite measure and let $\phi : \Omega \to \phi(\Omega)$ be a diffeomorphism of class $C^{1,1}$ from Ω onto an open set $\phi(\Omega)$ in $\mathbb{R}^N$. Assume that there exist $M_1, M_2 > 0$ such that*

$$\max_{1\le|\alpha|\le 2} \sup_{x\in\Omega} |D^\alpha \phi(x)| < M_1, \qquad \inf_{x\in\Omega} |\det\nabla\phi(x)| > M_2 \tag{3.2}$$

for all $x \in \Omega$. Then there exists $c > 0$ depending only on $N, M_1, M_2, \lambda, \mu,$ and $|\Omega|$ such that

$$|Q_{\phi(\Omega),t}(C_\phi(\beta, w)) - Q_{\Omega,t}(\beta, w)| \le cQ_{\Omega,t}(\beta, w)\delta(\phi) \tag{3.3}$$

for all $t > 0$ and $(\beta, w) \in \mathcal{V}(\Omega)$.

Proof. Let $(\beta, w) \in \mathcal{V}(\Omega)$. To shorten our notation, we denote by $C_\phi^{(1)}(\beta)$ the first entry of $C_\phi(\beta, w)$, i.e., $C_\phi^{(1)}(\beta) = (\beta\nabla\phi^{-1}) \circ \phi^{(-1)}$. We begin by estimating $\int_{\phi(\Omega)} |\nabla C_\phi^{(1)}(\beta)|^2 dy - \int_\Omega |\nabla\beta|^2 dx$. By means of a change of variables, we get

$$\int_{\phi(\Omega)} |\nabla C_\phi^{(1)}(\beta)|^2 dy = \int_\Omega |(\nabla(\beta\nabla\phi^{-1}))\nabla\phi^{-1}|^2 |\det\nabla\phi| dx. \tag{3.4}$$

It is easy to see that in order to estimate $\int_{\phi(\Omega)} |\nabla C_\phi^{(1)}(\beta)|^2 dy - \int_\Omega |\nabla\beta|^2 dx$ it suffices to estimate $\int_\Omega (|(\nabla(\beta\nabla\phi^{-1}))\nabla\phi^{-1}|^2 - |\nabla\beta|^2)|\det\nabla\phi| dx$. We clearly have that

$$\begin{aligned} &\left| \int_\Omega (|(\nabla(\beta\nabla\phi^{-1}))\nabla\phi^{-1}|^2 - |\nabla\beta|^2)|\det\nabla\phi| dx \right| \\ &\quad \le \|\det\nabla\phi\|_{L^\infty(\Omega)} \|(\nabla(\beta\nabla\phi^{-1}))\nabla\phi^{-1} - \nabla\beta\|_{L^2(\Omega)} \\ &\qquad \cdot (\|(\nabla(\beta\nabla\phi^{-1}))\nabla\phi^{-1}\|_{L^2(\Omega)} + \|\nabla\beta\|_{L^2(\Omega)}). \end{aligned} \tag{3.5}$$

By the triangle inequality we get

$$\begin{aligned} &\|(\nabla(\beta\nabla\phi^{-1}))\nabla\phi^{-1} - \nabla\beta\|_{L^2(\Omega)} \\ &\quad \le \|\nabla\phi^{-1}\|_{L^\infty(\Omega)} \|\nabla(\beta\nabla\phi^{-1}) - \nabla\beta\|_{L^2(\Omega)} \\ &\qquad + \|\nabla\phi^{-1} - I\|_{L^\infty(\Omega)} \|\nabla\beta\|_{L^2(\Omega)} \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \|\nabla(\beta\nabla\phi^{-1}) - \nabla\beta\|_{L^2(\Omega)} \le \|\nabla\phi^{-1} - I\|_{L^\infty(\Omega)} \|\nabla\beta\|_{L^2(\Omega)} \\ + \|\nabla(\nabla\phi^{-1})\|_{L^\infty(\Omega)} \|\beta\|_{L^2(\Omega)}. \end{aligned} \tag{3.7}$$

Moreover

$$\begin{aligned} \|\nabla(\beta\nabla\phi^{-1})\|_{L^2(\Omega)} \le \|\nabla\phi^{-1}\|_{L^\infty(\Omega)} \|\nabla\beta\|_{L^2(\Omega)} \\ + \|\nabla(\nabla\phi^{-1})\|_{L^\infty(\Omega)} \|\beta\|_{L^2(\Omega)}. \end{aligned} \tag{3.8}$$

By standard calculus it follows that there exists a constant $c > 0$ depending only on N, M_1, M_2 such that

$$\|\nabla\phi^{-1}\|_{L^\infty(\Omega)} \le c \tag{3.9}$$

and

$$\|\nabla\phi^{-1} - I\|_{L^\infty(\Omega)},\ \|\nabla(\nabla\phi^{-1}))\|_{L^\infty(\Omega)} \le c\delta(\phi). \tag{3.10}$$

By using the Poincaré inequality $\|\beta\|_{L^2(\Omega)} \le c\|\nabla\beta\|_{L^2(\Omega)}$ with c depending only on N and $|\Omega|$, and combining inequalities (3.4)–(3.10) we conclude that

$$\left| \int_{\phi(\Omega)} |\nabla C_\phi^{(1)}(\beta)|^2 dy - \int_\Omega |\nabla\beta|^2 dx \right| \le c_1\delta(\phi) \int_\Omega |\nabla\beta|^2 dx, \tag{3.11}$$

where the constant c_1 depends only on N, M_1, M_2, and $|\Omega|$.

Similarly, one can also prove the existence of a constant $c_2 > 0$ depending only on N, M_1, M_2, and $|\Omega|$ such that

$$\left| \int_{\phi(\Omega)} (\operatorname{div} C_\phi^{(1)}(\beta))^2 dy - \int_\Omega (\operatorname{div}\beta)^2 dx \right| \le c_2\delta(\phi) \int_\Omega |\nabla\beta|^2 dx. \tag{3.12}$$

Finally, we estimate $\int_{\phi(\Omega)} |\nabla(w \circ \phi^{(-1)}) - C_\phi^{(1)}(\beta)|^2 dy - \int_\Omega |\nabla w - \beta|^2 dx$. We note that

$$\int_{\phi(\Omega)} |\nabla(w \circ \phi^{(-1)}) - C_\phi^{(1)}(\beta)|^2 dy = \int_\Omega |(\nabla w - \beta) \cdot \nabla \phi^{-1}|^2 |\det \nabla \phi| dx \tag{3.13}$$

and that

$$\begin{aligned} &\int_\Omega \big| |(\nabla w - \beta) \cdot \nabla \phi^{-1}|^2 - |\nabla w - \beta|^2 \big| dx \\ &\qquad \leq \|\nabla \phi^{-1} (\nabla \phi^{-1})^T - I\|_{L^\infty(\Omega)} \int_\Omega |\nabla w - \beta|^2 dx. \end{aligned} \tag{3.14}$$

It follows that there exists $c_3 > 0$ depending only on N, M_1, M_2 such that

$$\left| \int_{\phi(\Omega)} |\nabla(w \circ \phi^{(-1)}) - C_\phi^{(1)}(\beta)|^2 dy - \int_\Omega |\nabla w - \beta|^2 dx \right| \leq c_3 \delta(\phi) \int_\Omega |\nabla w - \beta|^2 dx. \tag{3.15}$$

By combining inequalities (3.11), (3.12), (3.15), we deduce the validity of (3.3). ∎

As in the case of elliptic partial differential equations discussed in [9], we can prove the following.

THEOREM 3.3. *Let Ω be an open set in $\mathbb{R}^N$ with finite measure and $M_1, M_2 > 0$. Then there exists $c > 0$ depending only on λ, μ, M_1, M_2, and $|\Omega|$ such that estimate* (1.3) *holds for all $t > 0$ and for all diffeomorphisms ϕ of class $C^{1,1}$ from Ω onto an open set $\phi(\Omega)$ in $\mathbb{R}^N$ such that inequalities* (3.2) *are satisfied and $\delta(\phi) < c^{-1}$.*

Proof. Let ϕ be a diffeomorphism of class $C^{1,1}$ from Ω onto an open set $\phi(\Omega)$ in $\mathbb{R}^N$, satisfying inequalities (3.2). Obviously we have

$$\begin{aligned} &\left| \frac{Q_{\phi(\Omega)}(C_\phi(\beta, w))}{\|C_\phi(\beta, w))\|^2_{\mathcal{L}^2_t(\phi(\Omega))}} - \frac{Q_\Omega(\beta, w)}{\|(\beta, w))\|^2_{\mathcal{L}^2_t(\Omega)}} \right| \\ &\quad \leq \frac{|Q_{\phi(\Omega)}(C_\phi(\beta, w)) - Q_\Omega(\beta, w)|}{\|C_\phi(\beta, w))\|^2_{\mathcal{L}^2_t(\phi(\Omega))}} \\ &\qquad + \frac{Q_\Omega(\beta, w) \left| \|C_\phi(\beta, w)\|^2_{\mathcal{L}^2_t(\phi(\Omega))} - \|(\beta, w)\|^2_{\mathcal{L}^2_t(\Omega)} \right|}{\|C_\phi(\beta, w))\|^2_{\mathcal{L}^2_t(\phi(\Omega))} \|(\beta, w))\|^2_{\mathcal{L}^2_t(\Omega}}. \end{aligned} \tag{3.16}$$

As in the proof of Lemma 3.2, one can prove the existence of a constant $c > 0$ depending only on N, M_1, M_2 such that

$$\|C_\phi(\beta, w)\|^2_{\mathcal{L}^2_t(\phi(\Omega))} \geq c \|(\beta, w)\|^2_{\mathcal{L}^2_t(\Omega)} \tag{3.17}$$

and

$$\left| \|C_\phi(\beta, w)\|^2_{\mathcal{L}^2_t(\phi(\Omega))} - \|(\beta, w)\|^2_{\mathcal{L}^2_t(\Omega)} \right| \leq c \delta(\phi) \|(\beta, w)\|^2_{\mathcal{L}^2_t(\Omega)}; \tag{3.18}$$

see also Lemma 3.1. By combining inequalities (3.3) and (3.16)–(3.18) we deduce that

$$(1 - c\delta(\phi)) \frac{Q_\Omega(\beta, w)}{\|(\beta, w))\|^2_{\mathcal{L}^2_t(\Omega)}} \leq \frac{Q_{\phi(\Omega)}(C_\phi(\beta, w))}{\|C_\phi(\beta, w))\|^2_{\mathcal{L}^2_t(\phi(\Omega))}} \leq (1 + c\delta(\phi)) \frac{Q_\Omega(\beta, w)}{\|(\beta, w))\|^2_{\mathcal{L}^2_t(\Omega)}}. \tag{3.19}$$

If $1 - c\delta(\phi) > 0$, it is possible to apply the min-max principle to deduce (1.3) from (3.19) combined with Lemma 3.1. ∎

Remark 3.4. Since the weak formulation (2.1) involves only weak derivatives of the first order, one may try to obtain stability estimates also under weaker assumptions on ϕ. For example, one may think of using bi-Lipschitz domain transformations, i.e., maps ϕ of class $C^{0,1}$ together with their inverses. In this case, one would replace the measure of vicinity $\delta(\phi)$ by the natural weaker measure of vicinity

$$\tilde{\delta}(\phi) = \|\nabla\phi - I\|_{L^\infty(\Omega)}.$$

In order to prove the corresponding estimate, in the proof of Theorem 3.3 one should replace the operator C_ϕ defined in (3.1) by the operator $\tilde{C}_\phi$ defined by

$$\tilde{C}_\phi(\beta, w) = (\beta \circ \phi^{(-1)}, w \circ \phi^{(-1)})$$

for all $(\beta, w) \in \mathcal{V}(\Omega)$. The definition of the operator $\tilde{C}_\phi$ does not involve $\nabla\phi$ and establishes a linear homeomorphism between $\mathcal{V}(\Omega)$ and $\mathcal{V}(\phi(\Omega))$. Unfortunately, the summand $\int_\Omega (\nabla w - \beta) \cdot (\nabla v - \eta) dx$ in the quadratic form (2.1) does not behave well under the transformation $\tilde{C}_\phi$ and this would lead to an estimate depending on t. Namely, one would obtain the estimate

$$|\gamma_{n,t}[\phi(\Omega)] - \gamma_{n,t}[\Omega]| \leq \frac{c}{t^2}\gamma_{n,t}[\Omega]\tilde{\delta}(\phi), \tag{3.20}$$

where the presence of a better measure of vicinity $\tilde{\delta}(\phi)$ is compensated for by the presence of the factor t^2 which spoils the estimate for t close to zero.

In any case, using domain transformations ϕ of class $C^{1,1}$ and the corresponding strong measure of vicinity $\delta(\phi)$ is enough for our purpose of obtaining estimates via Hausdorff distance.

3.2. Estimates via atlas and Hausdorff distance. In general, even if two open sets Ω_1 and Ω_2 are known to be diffeomorphic, it is not easy to construct a diffeomorphism ϕ such that $\phi(\Omega_1) = \Omega_2$ and provide information on $\delta(\phi)$ in terms of explicit geometric quantities. However, if Ω_1, Ω_2 belong to the same class $C(\mathcal{A})$ then it is possible to construct a suitable diffeomorphism ϕ such that $\phi(\Omega_1) \subset \Omega_2$ and estimate $\delta(\phi)$ via the atlas distance (2.6). Such a construction was first used in Burenkov and Davies [6] and then implemented in [9]. We briefly recall it.

Let $\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ be an atlas in $\mathbb{R}^N$ and let $\{\psi_j\}_{j=1}^s$ be a partition of unity such that $\psi_j \in C_c^\infty(\mathbb{R}^N)$, $\operatorname{supp}\psi_j \subset (V_j)_{\frac{3}{4}\rho}$, $0 \leq \psi_j \leq 1$, and $\sum_{j=1}^s \psi_j(x) = 1$ for all $x \in \cup_{j=1}^s (V_j)_\rho$. For $\epsilon \geq 0$ we consider the following transformation

$$\phi_\epsilon(x) = x - \epsilon \sum_{j=1}^s \xi_j \psi_j(x), \quad x \in \mathbb{R}^N, \tag{3.21}$$

where $\xi_j = r_j^{(-1)}((0, \ldots, 1))$.

Then we recall the following technical lemma from [9].

LEMMA 3.5. *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$. Then there exist $M, M_1, M_2, E > 0$ depending only on N and $\mathcal{A}$ such that ϕ_ϵ satisfies* (3.2) *and such that $\delta(\phi_\epsilon) \leq M\epsilon$ for all $\epsilon \in [0, E[$. Moreover, $\phi_\epsilon(\Omega_1) \subset \Omega_2$ for all $\epsilon \in [0, E[$ and for all $\Omega_1, \Omega_2 \in C(\mathcal{A})$ such that $\Omega_2 \subset \Omega_1$ and $d_{\mathcal{A}}(\Omega_1, \Omega_2) < \epsilon/s$.*

Proceeding as in [9] we can prove the following.

THEOREM 3.6. *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$. Then there exists $c > 0$ depending only on $\mathcal{A}, \lambda, \mu$ such that estimate* (1.5) *holds for all $n \in \mathbb{N}$, $t > 0$, and for all $\Omega_1, \Omega_2 \in C(\mathcal{A})$ satisfying $d_{\mathcal{A}}(\Omega_1, \Omega_2) < c^{-1}$.*

Proof. Let $E > 0$ be as in Lemma 3.5 and let $\Omega_1, \Omega_2 \in C(\mathcal{A})$ be such that $d_{\mathcal{A}}(\Omega_1, \Omega_2) < \epsilon/s$. Clearly $\Omega_1 \cap \Omega_2 \in C(\mathcal{A})$ and $d_{\mathcal{A}}(\Omega_1 \cap \Omega_2, \Omega_1) d_{\mathcal{A}}(\Omega_1 \cap \Omega_2, \Omega_2) < \epsilon/s$. Thus by Lemma 3.5 we have that $\phi_\epsilon(\Omega_1), \phi_\epsilon(\Omega_2) \subset \Omega_1 \cap \Omega_2$. By the monotonicity of the eigenvalues with respect to inclusion, we immediately get

$$\gamma_{n,t}[\Omega_i] \le \gamma_{n,t}[\Omega_1 \cap \Omega_2] \le \gamma_{n,t}[\phi_\epsilon(\Omega_i)] \tag{3.22}$$

for all $i = 1, 2$. Moreover, by combining Theorem 3.3 and Lemma 3.5, we deduce that there exists c as in the statement such that

$$|\gamma_{n,t}[\Omega_i] - \gamma_{n,t}[\Omega_1 \cap \Omega_2]| \le |\gamma_{n,t}[\phi_\epsilon(\Omega_i)] - \gamma_{n,t}[\Omega_i]| \le c\gamma_{n,t}[\Omega_i]\epsilon \tag{3.23}$$

for all $i = 1, 2$, provided $\epsilon \le c^{-1}$. Inequality (1.5) easily follows by choosing $\epsilon = 2sd_{\mathcal{A}}(\Omega_1, \Omega_2)$ in (3.23). ∎

We note that by Theorem 2.4 and estimate (1.5), it immediately follows that if ω is a modulus of continuity as in Definition 2.2 then there exists $c > 0$ depending only on $\mathcal{A}, \omega, \lambda, \mu$ such that

$$|\gamma_{n,t}[\Omega_1] - \gamma_{n,t}[\Omega_2]| \le c \max\{\gamma_{n,t}[\Omega_1], \gamma_{n,t}[\Omega_2]\} \omega(d_{\mathcal{HP}}(\partial\Omega_1, \partial\Omega_2)) \tag{3.24}$$

for all $n \in \mathbb{N}$, $t > 0$, and for all $\Omega_1, \Omega_2 \in C_M^{\omega(\cdot)}(\mathcal{A})$ satisfying the condition $d_{\mathcal{HP}}(\Omega_1, \Omega_2) < c^{-1}$.

In several papers devoted to stability estimates for domain perturbation problems, the vicinity of two domains is described by means of ϵ-neighborhoods of the boundaries defined by the Euclidean distance; see, e.g., [6] and Davies [12]. This can be done also in the case of the Reissner–Mindlin system. Indeed, one can prove the following.

COROLLARY 3.7. *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$, ω a modulus of continuity as in Definition* 2.2*, and $M > 0$. Then there exists $c > 0$ depending only on $\mathcal{A}, \omega, \lambda, \mu, M$ such that*

$$|\gamma_{n,t}[\Omega_1] - \gamma_{n,t}[\Omega_2]| \le c \max\{\gamma_{n,t}[\Omega_1], \gamma_{n,t}[\Omega_2]\} \omega(\epsilon) \tag{3.25}$$

for all $n \in \mathbb{N}$, $t > 0$, $\epsilon \in]0, c^{-1}[$, and for all $\Omega_1, \Omega_2 \in C_M^{\omega(\cdot)}(\mathcal{A})$ such that

$$(\Omega_1)_\epsilon \subset \Omega_2 \subset (\Omega_1)^\epsilon \quad \text{or} \quad (\Omega_2)_\epsilon \subset \Omega_1 \subset (\Omega_2)^\epsilon. \tag{3.26}$$

Proof. Note that if Ω_1 and Ω_2 satisfy one of the inclusions in (3.26) then $d_{\mathcal{HP}}(\partial\Omega_1, \partial\Omega_2) \le \epsilon$, which combined with inequality (3.24) allows us to deduce (3.25). ∎

4. Shape differentiability. Given a bounded open set in $\mathbb{R}^N$, we denote by $C^{0,1}(\Omega; \mathbb{R}^N)$ the set of Lipschitz continuous maps from Ω to $\mathbb{R}^N$. By $\mathrm{BLip}(\Omega)$ we denote the set of functions $\phi \in C^{0,1}(\Omega; \mathbb{R}^N)$ such that ϕ is injective and the inverse $\phi^{(-1)} : \phi(\Omega) \to \Omega$ is Lipschitz continuous. We shall think of $C^{0,1}(\Omega; \mathbb{R}^N)$ as a Banach space endowed with the standard norm defined by

$$\|\phi\|_{C^{0,1}(\Omega;\mathbb{R}^N)} = \|\phi\|_{L^\infty(\Omega)} + \mathrm{Lip}(\phi)$$

for all $\phi \in C^{0,1}(\Omega; \mathbb{R}^N)$, where $\mathrm{Lip}(\phi)$ is the Lipschitz constant of ϕ. We recall that $\mathrm{BLip}(\Omega)$ is an open set in $C^{0,1}(\Omega; \mathbb{R}^N)$; see, e.g., [16, Lemma 3.11].

In this section, we prove analyticity results for the maps $\phi \mapsto \gamma_{n,t}[\phi(\Omega)]$, defined for $\phi \in \mathrm{BLip}(\Omega)$. To shorten our notation, in what follows we shall write $\gamma_{n,t}[\phi]$ instead of $\gamma_{n,t}[\phi(\Omega)]$.

As is known, when dealing with differentiability properties of the eigenvalues, it is necessary to pay attention to bifurcation phenomena associated with multiple eigenvalues. Following [16, 17], given a finite nonempty subset of $\mathbb{N}$, we set

$$\mathcal{A}_{F,t}(\Omega) = \{\phi \in \mathrm{BLip}(\Omega) : \gamma_{l,t}[\phi] \notin \{\gamma_{j,t}[\phi] : j \in F\} \ \forall l \in \mathbb{N} \setminus F\}$$

and

$$\Theta_{F,t}(\Omega) = \{\phi \in \mathcal{A}_{F,t}(\Omega) : \gamma_{j,t}[\phi] \text{ have a common value } \gamma_{F,t}[\phi] \ \forall j \in F\}.$$

Then we can prove the following real-analyticity result in the spirit of the results in [4, 5, 16].

THEOREM 4.1. *Let Ω be a bounded open set in $\mathbb{R}^N$, $t > 0$, and F a finite nonempty subset of $\mathbb{N}$. Then the following statements hold.*

(i) *The set $\mathcal{A}_{F,t}(\Omega)$ is open in $C^{0,1}(\Omega;\mathbb{R}^N)$. Moreover, for every $s \in \{1, \ldots, |F|\}$ the real valued function $\Gamma^{(s)}_{F,t}$ defined on $\mathcal{A}_{F,t}(\Omega)$ by*

$$\Gamma^{(s)}_{F,t}[\phi] = \sum_{\substack{j_1 < \cdots < j_s \\ j_1,\ldots,j_s \in F}} \gamma_{j_1,t}[\phi] \cdots \gamma_{j_s,t}[\phi]$$

for all $\phi \in \mathcal{A}_{F,t}(\Omega)$, is real analytic.

(ii) *Let $\tilde{\phi} \in \Theta_{F,t}(\Omega)$ be such that $\tilde{\phi}(\Omega)$ is of class $C^{1,1}$. Then for every $s \in \{1, \ldots, |F|\}$ the Fréchet differential of the function $\Gamma^{(s)}_{F,t}$ at the point $\tilde{\phi}$ is provided by the formula*

$$\begin{aligned} d_{|_{\phi=\tilde{\phi}}}\Gamma^{(s)}_{F,t}[\psi] = -\gamma^{s-1}_{F,t}[\tilde{\phi}]\binom{|F|-1}{s-1}\sum_{l=1}^{|F|}\int_{\partial\tilde{\phi}(\Omega)}\Bigg(\frac{\mu}{12}\left|\frac{\partial\beta^{(l)}}{\partial n}\right|^2 \\ + \frac{\mu+\lambda}{12}\left(\frac{\partial\beta^{(l)}}{\partial n}\cdot n\right)^2 + \frac{\mu k}{t^2}\left(\frac{\partial w^{(l)}}{\partial n}\right)^2\Bigg)\zeta\cdot n\,d\sigma \end{aligned} \tag{4.1}$$

for all $\psi \in C^{0,1}(\Omega;\mathbb{R}^N)$, where $\zeta = \psi \circ \tilde{\phi}^{(-1)}$ and $(\beta^{(1)}, w^{(1)}), \ldots, (\beta^{(|F|)}, w^{(|F|)})$ is an orthonormal basis in $\mathcal{L}^2_t(\tilde{\phi}(\Omega))$ for the eigenspace associated with $\gamma_{F,t}[\tilde{\phi}]$.

Proof. The proof can be deduced by the abstract results in [16] as follows. We consider the operator $R_{\phi(\Omega),t}$ as an operator acting from the space $\mathcal{V}(\phi(\Omega))$ to its dual and we pull it back to Ω by changing variables via ϕ. Namely, the pull back $\mathcal{R}_{\phi,t}$ of $R_{\phi(\Omega),t}$ is the operator defined from $\mathcal{V}(\Omega)$ to its dual which takes any $(\theta, u) \in \mathcal{V}(\Omega)$ to the functional $\mathcal{R}_{\phi,t}(\theta, u)$ defined by

$$\begin{aligned} \mathcal{R}_{\phi,t}(\theta,u)(\dot{\theta},\dot{u}) = \frac{\mu}{12}\int_\Omega \Big(\nabla(\theta\circ\phi^{(-1)}) : \nabla(\dot{\theta}\circ\phi^{(-1)})\Big)\circ\phi|\det D\phi|dx \\ + \frac{\mu+\lambda}{12}\int_\Omega\Big(\mathrm{div}(\theta\circ\phi^{(-1)})\mathrm{div}(\dot{\theta}\circ\phi^{(-1)})\Big)\circ\phi|\det D\phi|dx \\ + \frac{\mu k}{t^2}\int_\Omega(\nabla(u\circ\phi^{(-1)})\circ\phi-\theta)\cdot(\nabla(\dot{u}\circ\phi^{(-1)})\circ\phi-\dot{\theta})|\det D\phi|dx \end{aligned} \tag{4.2}$$

for all $(\dot\theta, \dot u) \in \mathcal{V}(\Omega)$. Similarly, we consider the map $\mathcal{J}_{\phi,t}$ from $\mathcal{V}(\Omega)$ to its dual defined by

$$\mathcal{J}_{\phi,t}(\theta,u)(\dot\theta,\dot u) = \int_\Omega \left(u\dot u + \frac{t^2}{12}\theta\cdot\dot\theta\right)|\det D\phi|dx \tag{4.3}$$

for all $(\theta,u),(\dot\theta,\dot u) \in \mathcal{V}(\Omega)$. Note that $\mathcal{R}_{\phi,t}(\theta,u)(\dot\theta,\dot u)$ can be considered as a scalar product in $\mathcal{V}(\Omega)$ and the corresponding norm is equivalent to the standard Sobolev norm. Accordingly, we can think of $\mathcal{V}(\Omega)$ as a Hilbert space endowed with such a scalar product. Thus, by the Riesz representation theorem applied to $\mathcal{V}(\Omega)$, it follows that the operator $\mathcal{R}_{\phi,t}$ is invertible.

It is easy to see that $(\beta,w) \in \mathcal{V}(\phi(\Omega))$ is an eigenvector associated with an eigenvalue γ of the operator $R_{\phi(\Omega),t}$ if and only if $\mathcal{R}_{\phi,t}(\beta\circ\phi, w\circ\phi) = \gamma\mathcal{J}_{\phi,t}(\beta\circ\phi, w\circ\phi)$. This implies that the eigenvalues of the operator $R_{\phi(\Omega),t}$ are the reciprocal of the eigenvalues of the operator $T_{\phi,t}$ defined from $\mathcal{V}(\Omega)$ to itself by

$$T_{\phi,t} = \mathcal{R}^{(-1)}_{\phi,t} \circ \mathcal{J}_{\phi,t}. \tag{4.4}$$

It turns out that $T_{\phi,t}$ is a compact self-adjoint operator on the Hilbert space $\mathcal{V}(\Omega)$. Note that the operators $\mathcal{R}_{\phi,t}$, $\mathcal{J}_{\phi,t}$, hence $T_{\phi,t}$, depend real analytically on ϕ, since they are obtained as composition of real analytic maps. Thus, it is possible to apply the general results in [16] and conclude that the elementary symmetric functions $\sum_{j_1<\cdots<j_s\in F}\gamma^{-1}_{j_1,t}[\phi]\cdots\gamma^{-1}_{j_s,t}[\phi]$ of the eigenvalues of $T_{\phi,t}$ depend real analytically on ϕ. Then, by arguing as in [16], one can easily deduce the validity of statement (i).

As for statement (ii), we set $\theta^{(l)} = \beta^{(l)}\circ\tilde\phi$ and $u^{(l)} = w^{(l)}\circ\tilde\phi$ for $l = 1,\dots,|F|$. By arguing as in [16] we obtain

$$d|_{\phi=\tilde\phi}(\Gamma^{(s)}_{F,t})[\psi] = -\gamma^s_{F,t}[\tilde\phi]\binom{|F|-1}{s-1}\sum_{l=1}^{|F|}\mathcal{R}_{\tilde\phi,t}\left(d|_{\phi=\tilde\phi}T_{t,\phi}[\psi](\theta^{(l)},u^{(l)})\right)\left((\theta^{(l)},u^{(l)})\right).$$

Then one can easily prove formula (4.1) using Lemma 4.2 below. ∎

LEMMA 4.2. *Let Ω be a bounded open set in $\mathbb{R}^N$ and $\tilde\phi \in \mathrm{BLip}(\Omega)$ be such that $\tilde\phi(\Omega)$ is of class $C^{1,1}$. Let $t>0$ and $(\beta^{(i)},w^{(i)}) \in \mathcal{V}(\tilde\phi(\Omega))$, $i=1,2$, be eigenvectors associated with an eigenvalue $\tilde\gamma$ of the operator $R_{\tilde\phi(\Omega),t}$. Let $\theta^{(i)} = \beta^{(i)}\circ\tilde\phi$, $u^{(i)} = w^{(i)}\circ\tilde\phi$, $i=1,2$. Then we have*

$$\begin{aligned}\mathcal{R}_{\tilde\phi,t}\left(d|_{\phi=\tilde\phi}T_{\phi,t}[\psi](\theta^{(1)},u^{(1)})\right)(\theta^{(2)},u^{(2)}) &= \tilde\gamma^{-1}\frac{\mu}{12}\int_{\partial\tilde\phi(\Omega)}\frac{\partial\beta^{(1)}}{\partial n}\cdot\frac{\partial\beta^{(2)}}{\partial n}\zeta\cdot n d\sigma \\ &+ \tilde\gamma^{-1}\frac{\mu+\lambda}{12}\int_{\partial\tilde\phi(\Omega)}\frac{\partial\beta^{(1)}}{\partial n}\cdot n\frac{\partial\beta^{(2)}}{\partial n}\cdot n\zeta\cdot n d\sigma + \tilde\gamma^{-1}\frac{\mu k}{t^2}\int_{\partial\tilde\phi(\Omega)}\frac{\partial w^{(1)}}{\partial n}\frac{\partial w^{(2)}}{\partial n}\zeta\cdot n d\sigma\end{aligned} \tag{4.5}$$

for all $\psi \in C^{0,1}(\Omega;\mathbb{R}^N)$, where $\zeta = \psi\circ\tilde\phi^{(-1)}$ and $\mathcal{R}_{\tilde\phi,t}$, $T_{\phi,t}$ are defined by (4.2), (4.4), *respectively.*

Proof. First of all, we note that by classical regularity theory, the eigenvectors $(\beta^{(i)},w^{(i)})$, $i=1,2$, belong to $(H^2(\tilde\phi(\Omega)))^N \times H^2(\tilde\phi(\Omega))$. This will be used in most of the following computations.

By standard calculus in normed spaces we have

$$\begin{aligned}(4.6)\quad & \mathcal{R}_{\tilde\phi,t}\left[\mathrm{d}|_{\phi=\tilde\phi}\left(\mathcal{R}^{(-1)}_{\phi,t}\circ\mathcal{J}_{\phi,t}[\psi](\theta^{(1)},u^{(1)}),(\theta^{(2)},u^{(2)})\right)\right]\\ &= \mathcal{R}_{\tilde\phi,t}\left[\mathcal{R}^{(-1)}_{\tilde\phi,t}\circ \mathrm{d}|_{\phi=\tilde\phi}\mathcal{J}_{\phi,t}[\psi](\theta^{(1)},u^{(1)}),(\theta^{(2)},u^{(2)})\right]\\ &\quad + \mathcal{R}_{\tilde\phi,t}\left[\mathrm{d}|_{\phi=\tilde\phi}\mathcal{R}^{(-1)}_{\phi,t}[\psi]\circ\mathcal{J}_{\tilde\phi,t}(\theta^{(1)},u^{(1)}),(\theta^{(2)},u^{(2)})\right].\end{aligned}$$

Now we note that

$$\begin{aligned}(4.7)\quad & \mathcal{R}_{\tilde\phi,t}\left[\mathrm{d}|_{\phi=\tilde\phi}\mathcal{R}^{(-1)}_{\phi,t}[\psi]\circ\mathcal{J}_{\tilde\phi,t}(\theta^{(1)},u^{(1)}),(\theta^{(2)},u^{(2)})\right]\\ &= -\mathcal{R}_{\tilde\phi,t}\left[\mathcal{R}^{(-1)}_{\tilde\phi,t}\circ \mathrm{d}|_{\phi=\tilde\phi}\mathcal{R}_{\phi,t}[\psi]\circ\mathcal{R}^{(-1)}_{\tilde\phi,t}\circ\mathcal{J}_{\tilde\phi,t}(\theta^{(1)},u^{(1)}),(\theta^{(2)},u^{(2)})\right]\\ &= -\tilde\gamma^{-1}\left(d|_{\phi=\tilde\phi}\mathcal{R}_{\phi,t}[\psi](\theta^{(1)},u^{(1)})\right)(\theta^{(2)},u^{(2)}).\end{aligned}$$

By standard calculus we have

$$(4.8)\qquad \left[\left(d|_{\phi=\tilde\phi}(\det\nabla\phi)[\psi]\right)\circ\tilde\phi^{(-1)}\right]\det\nabla\tilde\phi^{(-1)} = \mathrm{div}\zeta,$$

hence
(4.9)

$$(d|_{\phi=\tilde\phi}\mathcal{J}_{\phi,t}[\psi][(\theta^{(1)},u^{(1)})])[(\theta^{(2)},u^{(2)})] = \int_{\tilde\phi(\Omega)}\left(w^{(1)}w^{(2)}+\frac{t^2}{12}\beta^{(1)}\beta^{(2)}\right)\mathrm{div}\zeta dy.$$

Note that, in order to shorten our notation, in what follows summation symbols will be omitted. By standard calculus in normed space and changing variables we get

$$\begin{aligned}(4.10)\qquad & \left(d|_{\phi=\tilde\phi}\mathcal{R}_{t,\phi}[\psi](\theta^{(1)},u^{(1)})\right)(\theta^{(2)},u^{(2)})\\ &= -\frac{\mu}{12}\int_{\tilde\phi(\Omega)}\left(\frac{\partial\beta_i^{(1)}}{\partial y_r}\frac{\partial\beta_i^{(2)}}{\partial y_j}+\frac{\partial\beta_i^{(2)}}{\partial y_r}\frac{\partial\beta_i^{(1)}}{\partial y_j}\right)\frac{\partial\zeta_r}{\partial y_j}dy\\ &\quad+\frac{\mu}{12}\int_{\tilde\phi(\Omega)}\frac{\partial\beta_i^{(1)}}{\partial y_j}\frac{\partial\beta_i^{(2)}}{\partial y_j}\mathrm{div}\zeta dy\\ &\quad-\frac{\mu+\lambda}{12}\int_{\tilde\phi(\Omega)}\left(\frac{\partial\beta_i^{(1)}}{\partial y_r}\mathrm{div}\beta^{(2)}+\frac{\partial\beta_i^{(2)}}{\partial y_r}\mathrm{div}\beta^{(1)}\right)\frac{\partial\zeta_r}{\partial y_i}dy\\ &\quad+\frac{\mu+\lambda}{12}\int_{\tilde\phi(\Omega)}\mathrm{div}\beta^{(1)}\mathrm{div}\beta^{(2)}\mathrm{div}\zeta dy\\ &\quad-\frac{\mu k}{t^2}\int_{\tilde\phi(\Omega)}\frac{\partial w^{(1)}}{\partial y_r}\frac{\partial\zeta_r}{\partial y_i}\left(\frac{\partial w^{(2)}}{\partial y_i}-\beta_i^{(2)}\right)dy\\ &\quad-\frac{\mu k}{t^2}\int_{\tilde\phi(\Omega)}\left(\frac{\partial w^{(1)}}{\partial y_i}-\beta_i^{(1)}\right)\frac{\partial w^{(2)}}{\partial y_r}\frac{\partial\zeta_r}{\partial y_i}dy\\ &\quad+\frac{\mu k}{t^2}\int_{\tilde\phi(\Omega)}\left(\frac{\partial w^{(1)}}{\partial y_i}-\beta_i^{(1)}\right)\left(\frac{\partial w^{(2)}}{\partial y_i}-\beta_i^{(2)}\right)\mathrm{div}\zeta dy.\end{aligned}$$

Now note that

$$
\begin{aligned}
(4.11)\qquad & \int_{\tilde\phi(\Omega)} \frac{\partial \beta_i^{(1)}}{\partial y_r}\frac{\partial \beta_i^{(2)}}{\partial y_j}\frac{\partial \zeta_r}{\partial y_j}dy \\
&= \int_{\partial\tilde\phi(\Omega)} \frac{\partial \beta_i^{(1)}}{\partial n}\frac{\partial \beta_i^{(2)}}{\partial n}\zeta\cdot n d\sigma \\
&\quad - \int_{\tilde\phi(\Omega)} \Delta\beta^{(2)}\cdot(\nabla\beta^{(1)}\cdot\zeta)dy - \int_{\tilde\phi(\Omega)} \frac{\partial \beta_i^{(2)}}{\partial y_j}\frac{\partial^2 \beta_i^{(1)}}{\partial y_j\partial y_r}\zeta_r dy \\
&= - \int_{\tilde\phi(\Omega)} \Delta\beta^{(2)}\cdot(\nabla\beta^{(1)}\cdot\zeta)dy + \int_{\tilde\phi(\Omega)} \frac{\partial \beta_i^{(1)}}{\partial y_j}\frac{\partial^2 \beta_i^{(2)}}{\partial y_j\partial y_r}\zeta_r dy \\
&\quad + \int_{\tilde\phi(\Omega)} \frac{\partial \beta_i^{(1)}}{\partial y_j}\frac{\partial \beta_i^{(2)}}{\partial y_j}\mathrm{div}\zeta dy.
\end{aligned}
$$

Note that here and in what follows we also use the fact that if U is a smooth open set and $f\in H^2(U)\cap H^1_0(U)$ then $\nabla f=\frac{\partial f}{\partial n}n$ on ∂U; moreover, if $g\in (H^2(U)\cap H^1_0(U))^N$ then $\mathrm{div} g=\frac{\partial g}{\partial n}\cdot n$ on ∂U.

By (4.11) the sum of the first two integrals in the right-hand side of (4.10) equals

$$
\begin{aligned}
(4.12)\qquad & -\frac{\mu}{12}\int_{\partial\tilde\phi(\Omega)} \frac{\partial \beta^{(1)}}{\partial n}\cdot\frac{\partial \beta^{(2)}}{\partial n}\zeta\cdot n d\sigma \\
&\qquad + \frac{\mu}{12}\int_{\tilde\phi(\Omega)} \left(\Delta\beta_i^{(1)}\nabla\beta_i^{(2)}+\Delta\beta_i^{(2)}\nabla\beta_i^{(1)}\right)\cdot\zeta dy.
\end{aligned}
$$

Now we observe that

$$
\begin{aligned}
(4.13)\qquad & \int_{\tilde\phi(\Omega)} \frac{\partial \beta_i^{(1)}}{\partial y_r}\frac{\partial \zeta_r}{\partial y_i}\mathrm{div}\beta^{(2)}dy \\
&= \int_{\partial\tilde\phi(\Omega)} \frac{\partial \beta^{(1)}}{\partial n}\cdot n\,\mathrm{div}\beta^{(2)}\zeta\cdot n d\sigma \\
&\quad - \int_{\tilde\phi(\Omega)} \frac{\partial \mathrm{div}\beta^{(1)}}{\partial y_r}\zeta_r\mathrm{div}\beta^{(2)}dy - \int_{\tilde\phi(\Omega)} \frac{\partial \mathrm{div}\beta^{(2)}}{\partial y_i}\frac{\partial \beta_i^{(1)}}{\partial y_r}\zeta_r dy \\
&= - \int_{\tilde\phi(\Omega)} \frac{\partial \mathrm{div}\beta^{(2)}}{\partial y_i}\frac{\partial \beta_i^{(1)}}{\partial y_r}\zeta_r dy + \int_{\tilde\phi(\Omega)} \mathrm{div}\beta^{(1)}\mathrm{div}\beta^{(2)}\mathrm{div}\zeta dy \\
&\quad + \int_{\tilde\phi(\Omega)} \mathrm{div}\beta^{(1)}\frac{\partial \mathrm{div}\beta^{(2)}}{\partial y_r}\zeta_r dy.
\end{aligned}
$$

Thus, the sum of the third and fourth integrals in the right-hand side of (4.10) is equal to

$$
\begin{aligned}
(4.14)\qquad & \frac{\mu+\lambda}{12}\int_{\tilde\phi(\Omega)} \left(\frac{\partial \mathrm{div}\beta^{(1)}}{\partial y_i}\frac{\partial \beta_i^{(2)}}{\partial y_r}+\frac{\partial \mathrm{div}\beta^{(2)}}{\partial y_i}\frac{\partial \beta_i^{(1)}}{\partial y_r}\right)\zeta_r dy \\
&\qquad - \frac{\mu+\lambda}{12}\int_{\partial\tilde\phi(\Omega)} \frac{\partial \beta^{(1)}}{\partial n}\cdot n\frac{\partial \beta^{(2)}}{\partial n}\cdot n\zeta\cdot n d\sigma.
\end{aligned}
$$

Now note that

$$
\begin{aligned}
(4.15)\qquad & \int_{\tilde\phi(\Omega)} \frac{\partial w^{(1)}}{\partial y_r}\frac{\partial \zeta_r}{\partial y_i}\left(\frac{\partial w^{(2)}}{\partial y_i} - \beta_i^{(2)}\right) dy \\
&= \int_{\partial\tilde\phi(\Omega)} \frac{\partial w^{(1)}}{\partial n}\frac{\partial w^{(2)}}{\partial n}\zeta\cdot n d\sigma \\
&\quad - \int_{\tilde\phi(\Omega)} \frac{\partial w^{(1)}}{\partial y_r}\zeta_r\left(\Delta w^{(2)} - \mathrm{div}\beta^{(2)}\right) dy \\
&\quad - \int_{\tilde\phi(\Omega)} \frac{\partial^2 w^{(1)}}{\partial y_i\partial y_r}\zeta_r\left(\frac{\partial w^{(2)}}{\partial y_i} - \beta_i^{(2)}\right) dy \\
&= - \int_{\tilde\phi(\Omega)} \frac{\partial w^{(1)}}{\partial y_r}\zeta_r\left(\Delta w^{(2)} - \mathrm{div}\beta^{(2)}\right) dy \\
&\quad + \int_{\tilde\phi(\Omega)} \nabla w^{(1)}(\nabla w^{(2)} - \beta^{(2)})\mathrm{div}\zeta dy \\
&\quad + \int_{\tilde\phi(\Omega)} \frac{\partial w^{(1)}}{\partial y_i}\left(\frac{\partial^2 w^{(2)}}{\partial y_i\partial y_r} - \frac{\partial\beta_i^{(2)}}{\partial y_r}\right)\zeta_r dy.
\end{aligned}
$$

By using the second equality in (4.15), and the first equality in (4.15) with $(\beta^{(1)}, w^{(1)})$ replaced by $(\beta^{(2)}, w^{(2)})$, we get that the sum of the last three integrals in (4.10) is equal to

$$
\begin{aligned}
(4.16)\qquad & -\frac{\mu k}{t^2}\int_{\partial\tilde\phi(\Omega)} \frac{\partial w^{(1)}}{\partial n}\frac{\partial w^{(2)}}{\partial n}\zeta\cdot n d\sigma \\
&+ \frac{\mu k}{t^2}\int_{\tilde\phi(\Omega)} (\Delta w^{(1)} - \mathrm{div}\beta^{(1)})\nabla w^{(2)}\cdot\zeta dy \\
&+ \frac{\mu k}{t^2}\int_{\tilde\phi(\Omega)} (\Delta w^{(2)} - \mathrm{div}\beta^{(2)})\nabla w^{(1)}\cdot\zeta dy \\
&- \frac{\mu k}{t^2}\int_{\tilde\phi(\Omega)} \beta^{(1)}(\nabla w^{(2)} - \beta^{(2)})\mathrm{div}\zeta dy + \frac{\mu k}{t^2}\int_{\tilde\phi(\Omega)} \frac{\partial w^{(1)}}{\partial y_i}\frac{\partial \beta_i^{(2)}}{\partial y_r}\zeta_r dy \\
&- \frac{\mu k}{t^2}\int_{\tilde\phi(\Omega)} \beta_i^{(1)}\frac{\partial^2 w^{(2)}}{\partial y_i\partial y_r}\zeta_r dy = -\frac{\mu k}{t^2}\int_{\partial\tilde\phi(\Omega)} \frac{\partial w^{(1)}}{\partial n}\frac{\partial w^{(2)}}{\partial n}\zeta\cdot n d\sigma \\
&+ \frac{\mu k}{t^2}\int_{\tilde\phi(\Omega)} (\Delta w^{(1)} - \mathrm{div}\beta^{(1)})\nabla w^{(2)}\cdot\zeta dy \\
&+ \frac{\mu k}{t^2}\int_{\tilde\phi(\Omega)} (\Delta w^{(2)} - \mathrm{div}\beta^{(2)})\nabla w^{(1)}\cdot\zeta dy \\
&+ \frac{\mu k}{t^2}\int_{\tilde\phi(\Omega)} \left(\frac{\partial w^{(1)}}{\partial y_i} - \beta_i^{(1)}\right)\frac{\partial\beta_i^{(2)}}{\partial y_r}\zeta_r dy \\
&+ \frac{\mu k}{t^2}\int_{\tilde\phi(\Omega)} \left(\frac{\partial w^{(2)}}{\partial y_i} - \beta_i^{(2)}\right)\frac{\partial\beta_i^{(1)}}{\partial y_r}\zeta_r dy.
\end{aligned}
$$

Using the fact that

$$
-\frac{\mu}{12}\Delta\beta^{(i)} - \frac{\mu+\lambda}{12}\nabla\mathrm{div}\beta^{(i)} - \frac{\mu k}{t^2}(\nabla w^{(i)} - \beta^{(i)}) = \frac{\tilde\gamma t^2}{12}\beta^{(i)}
$$

and

$$-\frac{\mu k}{t^2}(\Delta w^{(i)} - \mathrm{div}\beta^{(i)}) = \tilde{\gamma} w^{(i)}$$

for $i = 1, 2$, we get that

$$\begin{aligned}(4.17)\quad & \left(d|_{\phi=\tilde{\phi}}\mathcal{R}_{t,\phi}[\psi](\theta^{(1)}, u^{(1)})\right)(\theta^{(2)}, u^{(2)}) \\ &= -\frac{\mu}{12}\int_{\partial\tilde{\phi}(\Omega)} \frac{\partial\beta^{(1)}}{\partial n}\cdot\frac{\partial\beta^{(2)}}{\partial n}\zeta\cdot n d\sigma - \frac{\mu+\lambda}{12}\int_{\partial\tilde{\phi}(\Omega)} \frac{\partial\beta^{(1)}}{\partial n}\cdot n\frac{\partial\beta^{(2)}}{\partial n}\cdot n\zeta\cdot n d\sigma \\ &\quad - \frac{\mu k}{t^2}\int_{\partial\tilde{\phi}(\Omega)} \frac{\partial w^{(1)}}{\partial n}\frac{\partial w^{(2)}}{\partial n}\zeta\cdot n d\sigma + \tilde{\gamma}\int_{\tilde{\phi}(\Omega)}\left(w^{(1)}w^{(2)} + \frac{t^2}{12}\beta^{(1)}\cdot\beta^{(2)}\right)\mathrm{div}\zeta dy.\end{aligned}$$

This, combined with (4.6), (4.7), (4.9), concludes the proof. ∎

In the case of domain perturbations depending real analytically on one scalar parameter, it is possible to apply the Rellich–Nagy theorem which allows us to conclude that the eigenvalues splitting from a multiple eigenvalue of multiplicity m are described by m real-analytic functions. Namely, we have the following theorem which can be proved by applying [16, Cor. 2.28] combined with Lemma 4.2.

THEOREM 4.3. *Let Ω be a bounded open set in $\mathbb{R}^N$ and $t > 0$. Let $\tilde{\phi} \in \mathrm{BLip}(\Omega)$ and $\{\phi_\epsilon\}_{\epsilon\in\mathbb{R}} \subset \mathrm{BLip}(\Omega)$ be a family depending real analytically on ϵ such that $\phi_0 = \tilde{\phi}$. Let $\tilde{\gamma}$ be an eigenvalue of $R_{\tilde{\phi}(\Omega),t}$ of multiplcity m, with $\tilde{\gamma} = \gamma_{n,t}[\tilde{\phi}] = \cdots = \gamma_{n+m-1,t}[\tilde{\phi}]$ for some $n \in \mathbb{N}$. Then there exists an open interval $\mathcal{I}$ containing zero and m real-analytic functions $g_1, \ldots, g_m$ from $\mathcal{I}$ to $\mathbb{R}$ such that $\{\gamma_{n,t}[\phi_\epsilon], \ldots, \gamma_{n+m-1,t}[\phi_\epsilon]\} = \{g_1(\epsilon), \ldots, g_m(\epsilon)\}$ for all $\epsilon \in \mathcal{I}$. Moreover, if $\tilde{\phi}(\Omega)$ is an open set of class $C^{1,1}$ then the derivatives $g_1'(0), \ldots, g_m'(0)$ at zero of the functions $g_1, \ldots, g_m$ coincide with the eigenvalues of the matrix $(D_{ij})_{i,j\in\{1,\ldots,m\}}$ defined by*

$$\begin{aligned}(4.18) D_{ij} &= -\frac{\mu}{12}\int_{\partial\tilde{\phi}(\Omega)} \frac{\partial\beta^{(i)}}{\partial n}\cdot\frac{\partial\beta^{(j)}}{\partial n}\zeta\cdot n d\sigma - \frac{\mu+\lambda}{12}\int_{\partial\tilde{\phi}(\Omega)} \frac{\partial\beta^{(i)}}{\partial n}\cdot n\frac{\partial\beta^{(j)}}{\partial n}\cdot n\zeta\cdot n d\sigma \\ &\quad - \frac{\mu k}{t^2}\int_{\partial\tilde{\phi}(\Omega)} \frac{\partial w^{(i)}}{\partial n}\frac{\partial w^{(j)}}{\partial n}\zeta\cdot n d\sigma,\end{aligned}$$

where $(\beta^{(i)}, w^{(i)})$, $i = 1, \ldots, m$, is an orthonormal basis in $\mathcal{L}_t^2(\tilde{\phi}(\Omega))$ of the eigenspace associated with $\tilde{\gamma}$.

5. Isovolumetric perturbations. Given a bounded open set Ω in $\mathbb{R}^N$, we consider isovolumetric domain perturbations, which means that we consider transformations $\phi \in \mathrm{BLip}(\Omega)$ satisfying the volume constraint

$$(5.1)\qquad |\phi(\Omega)| = \text{constant}.$$

It is then natural to consider the real-valued functional V defined on $\mathrm{BLip}(\Omega)$ by

$$(5.2)\qquad V[\phi] = \mathrm{Vol}\,\phi(\Omega)$$

for all $\phi \in \mathrm{BLip}(\Omega)$. We recall the following.

DEFINITION 5.1. *Let Ω be a bounded open set in $\mathbb{R}^N$. Let $\mathcal{F}$ be a real-valued differentiable map defined on an open subset of $\mathrm{BLip}(\Omega)$. We say that $\tilde{\phi} \in \mathrm{BLip}(\Omega)$ is a critical point for $\mathcal{F}$ with volume constraint if*

$$(5.3)\qquad \ker d_{|_{\phi=\tilde{\phi}}} V \subset \ker d_{|_{\phi=\tilde{\phi}}} \mathcal{F}.$$

As is well known this definition is related to the problem of finding local extremal points for the problems

$$\min_{V[\phi]=\mathrm{const}} \mathcal{F}[\phi] \quad \text{or} \quad \max_{V[\phi]=\mathrm{const}} \mathcal{F}[\phi].$$

Indeed if ϕ is a local minimizer or maximizer of a function $\mathcal{F}$ under condition (5.1) then inclusion (5.3) holds.

The following theorem can be proved using formula (4.1), by observing that $d_{|_{\phi=\tilde\phi}} V[\psi] = \int_{\partial\tilde\phi(\Omega)} (\psi\circ\tilde\phi^{(-1)})\cdot n d\sigma$ and by using the Lagrange multipliers theorem.

THEOREM 5.2. *Let Ω be a bounded open set in $\mathbb{R}^N$ and $t>0$. Let F be a nonempty finite subset of $\mathbb{N}$ and $s\in\{1,\dots,|F|\}$. Let $\tilde\phi\in\Theta_\Omega[F]$ be such that $\tilde\phi(\Omega)$ is of class $C^{1,1}$. Then $\tilde\phi$ is a critical point for $\Gamma^{(s)}_{F,t}$ with volume constraint if and only if there exists an orthonormal basis $(\beta^{(1)},w^{(1)}),\dots,(\beta^{(|F|)},w^{(|F|)})$ in $\mathcal{L}^2_t(\tilde\phi(\Omega))$ of the eigenspace associated with the eigenvalue $\gamma_{F,t}[\tilde\phi]$ and there exists $c\in\mathbb{R}$ such that*

$$\sum_{l=1}^{|F|}\left(\frac{\mu}{12}\left|\frac{\partial\beta^{(l)}}{\partial n}\right|^2+\frac{\mu+\lambda}{12}\left(\frac{\partial\beta^{(l)}}{\partial n}\cdot n\right)^2+\frac{\mu k}{t^2}\left(\frac{\partial w^{(l)}}{\partial n}\right)^2\right)=c \text{ on } \partial\tilde\phi(\Omega). \tag{5.4}$$

As in the case of the Laplace operator discussed in [17] and polyharmonic operators considered in [4, 5], it turns out that if $\tilde\phi(\Omega)$ is a ball then condition (5.4) is satisfied. In order to prove it, we need the following lemma. Recall that β is thought of as a row vector.

LEMMA 5.3. *Let B be a ball in $\mathbb{R}^N$ centered at zero, $t>0$, and let (β,w) be an eigenvector of $R_{B,t}$ in B associated with an eigenvalue γ. Let A be an orthogonal linear transformation in $\mathbb{R}^N$ and M the corresponding matrix. Then also $((\beta\circ A)M, w\circ A)$ is an eigenvector of $R_{B,t}$ associated with γ.*

Proof. First of all, we note that the rotational invariance of the Laplace operator yields

$$\Delta((\beta\circ A)M)=((\Delta\beta)\circ A)M \quad\text{and}\quad \Delta(w\circ A)=(\Delta w)\circ A.$$

Moreover, by standard calculus we have

$$\mathrm{div}((\beta\circ A)M) = \mathrm{Tr}\left(M^T\nabla(\beta\circ A)\right) = \mathrm{Tr}\left(M^T((\nabla\beta)\circ A)M\right) = (\mathrm{div}\beta)\circ A,$$

where Tr denotes the trace of a matrix and

$$\nabla\mathrm{div}((\beta\circ A)M)=\nabla((\mathrm{div}\beta)\circ A)=((\nabla\mathrm{div}\beta)\circ A)M.$$

By using the previous identities and the fact that (β,w) is a solution to (1.1), we get

$$\begin{aligned}
&-\frac{\mu}{12}\Delta((\beta\circ A)M)-\frac{\mu+\lambda}{12}\nabla\mathrm{div}((\beta\circ A)M)-\frac{\mu k}{t^2}(\nabla(w\circ A)-(\beta\circ A)M)\\
&\quad=-\frac{\mu}{12}((\Delta\beta)\circ A)M-\frac{\mu+\lambda}{12}((\nabla\mathrm{div}\beta)\circ A)M-\frac{\mu k}{t^2}((\nabla w)\circ A-(\beta\circ A))M\\
&\quad=\frac{\gamma t^2}{12}(\beta\circ A)M
\end{aligned}\tag{5.5}$$

and

$$-\frac{\mu k}{t^2}(\Delta(w\circ A)-(\mathrm{div}((\beta\circ A)M))=-\frac{\mu k}{t^2}(\Delta w-\mathrm{div}\beta)\circ A=\gamma w\circ A,$$

which show that $((\beta\circ A)M, w\circ A)$ is an eigenvector of $R_{B,t}$ associated with γ. ∎

We now prove the following.

THEOREM 5.4. *Let B be a ball in $\mathbb{R}^N$ centered at zero, and let γ be an eigenvalue of $R_{B,t}$. Let F be the subset of $\mathbb{N}$ of indexes j such that $\gamma_{j,t}[B] = \gamma$. Let $(\beta^{(1)}, w^{(1)}), \dots, (\beta^{(|F|)}, w^{(|F|)})$ be an orthonormal basis in $\mathcal{L}^2_t(B)$ of the eigenspace associated with γ. Then the functions*

$$\sum_{l=1}^{|F|} |\beta^{(l)}|^2, \quad \sum_{l=1}^{|F|} \left|\frac{\partial \beta^{(l)}}{\partial n}\right|^2, \quad \sum_{l=1}^{|F|} \left|\frac{\partial \beta^{(l)}}{\partial n} \cdot n\right|^2, \quad \sum_{l=1}^{|F|} |w^{(l)}|^2, \quad \sum_{l=1}^{|F|} \left|\frac{\partial w^{(l)}}{\partial n}\right|^2, \tag{5.6}$$

where $n(x) = x/|x|$ for all $x \in \bar{B} \setminus \{0\}$, are radial. In particular, there exists $c \in \mathbb{R}$ such that condition (5.4) *holds.*

Proof. Let $O_N(\mathbb{R})$ denote the group of orthogonal linear transformations in $\mathbb{R}^N$, and let $A \in O_N(\mathbb{R})$ be a transformation with associated matrix M. By Lemma 5.3 it follows that $\{((\beta^{(l)} \circ A)M, w^{(l)} \circ A) : l = 1, \dots, |F|\}$ is another orthonormal basis of the eigenspace associated with γ. Since both $\{(\beta^{(l)}, w^{(l)}) : l = 1, \dots, |F|\}$ and $\{((\beta^{(l)} \circ A)M, w^{(l)} \circ A) : l = 1, \dots, |F|\}$ are orthonormal bases, then there exists $S[A] \in O_{|F|}(\mathbb{R})$ with matrix $(S_{ij}[A])_{i,j=1,\dots,|F|}$ such that

$$((\beta^{(j)} \circ A)M, w^{(j)} \circ A) = \sum_{l=1}^{|F|} S_{jl}[A](\beta^{(l)}, w^{(l)}). \tag{5.7}$$

By (5.7) we deduce that

$$(\beta \circ A)M = S[A]\beta \quad \text{and} \quad w \circ A = S[A]w, \tag{5.8}$$

where β denotes the $l \times N$-matrix, the rows of which are given by the row vectors $\beta^{(j)}$, and w is the column vector the entries of which are given by $w^{(j)}$.

By the first equality in (5.8) we have $(\beta\beta^T) \circ A = S[A]\beta\beta^T S[A]^T$, hence

$$\sum_{l=1}^{|F|} |\beta^{(l)} \circ A|^2 = \mathrm{Tr}\,[(\beta\beta^T) \circ A] = \mathrm{Tr}\,[S[A]\beta\beta^T S[A]^T] = \mathrm{Tr}\,[\beta\beta^T] = \sum_{l=1}^{|F|} |\beta^{(l)}|^2. \tag{5.9}$$

By the arbitrary choice of A we deduce by (5.9) that $\sum_{l=1}^{|F|} |\beta^{(l)}|^2$ is a radial function. Similarly, using the second equality in (5.8), one can prove that $\sum_{l=1}^{|F|} |w^{(l)}|^2$ is a radial function as well.

We now consider the other functions in (5.6). By differentiating in the radial direction n the first equality in (5.8), we have that for every $j = 1, \dots, l$ and $s = 1, \dots, N$,

$$\sum_{r,h,k=1}^{N} \frac{\partial \beta_r^{(j)}}{\partial x_h} \circ A M_{hk} M_{rs} n_k = \sum_{l=1}^{|F|} \sum_{k=1}^{N} S_{jl}[A] \frac{\partial \beta_s^{(l)}}{\partial x_k} n_k. \tag{5.10}$$

Taking into account that $Mn = n \circ A$ we deduce by (5.10) that

$$\left(\frac{\partial\beta}{\partial n} \circ A\right) M = S[A]\frac{\partial\beta}{\partial n}. \tag{5.11}$$

By proceeding as in (5.9) we get that $\sum_{l=1}^{|F|} |\frac{\partial\beta^{(l)}}{\partial n}|^2$ is a radial function.

By multiplying both sides of (5.11) by n we also get

$$\left(\frac{\partial\beta}{\partial n} \cdot n\right) \circ A = S[A]\frac{\partial\beta}{\partial n} \cdot n, \tag{5.12}$$

which implies that $\sum_{l=1}^{|F|} |\frac{\partial\beta^{(l)}}{\partial n} \cdot n|^2$ is a radial function. Similarly, one can prove that the last function in (5.6) is radial. ◻

Combining all the results in this section we get the following.

THEOREM 5.5. *Let Ω be a bounded open set in $\mathbb{R}^N$. Let $\tilde{\phi} \in \mathrm{BLip}(\Omega)$ be such that $\tilde{\phi}(\Omega)$ is a ball. Let $\tilde{\gamma}$ be an eigenvalue of $R_{\tilde{\phi}(\Omega),t}$ and let F be the set of indexes $j \in \mathbb{N}$ such that $\gamma_{j,t}[\tilde{\phi}(\Omega)] = \tilde{\gamma}$. Then for all $s = 1, \ldots, |F|$ the elementary symmetric function $\Gamma_{F,t}^{(s)}$ has a critical point at $\tilde{\phi}$ with volume constraint.*

Acknowledgments. The authors are very thankful to Professors Carlo Lovadina, Sergei V. Rogosin, and Luis M. Hervella-Nieto for useful discussions and references.

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