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Can Offshoring Increase Quality? A Dynamic Model

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Abstract

We propose a dynamic model to study the problem of relocating production to a foreign country, where production costs are lower. The idea is to investigate the impact of offshoring on the quality of the manufactured goods in a simple dynamic model, which is analytically treatable. What we will show is that, under the assumptions of our model, if the trading costs are sufficiently convenient, the quality increases with offshoring.

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1 Introduction

In recent decades, because of a decline in trade costs, many firms have outsourced the production of their goods. If the production cost of a good in a foreign country is less than in the home country, then a national firm can seek the maximum profit by purchasing the good from the foreign country.

Many papers address this topic using macroeconomic models. The most interesting ones are the models that describe the trade and endogenous growth in two countries. An example of this kind of models is introduced by Saito in the paper [3] to describe the possibility of outsourcing the production of some intermediate goods.

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On the other hand, the problems of production inventory planning have received a considerable amount of attention in Operations Research and Management Science. Applications to optimal production inventory planning are also widely reported in the literature; we suggest the paper [1] and the references therein for an interesting introduction to such problems.

In this paper, we want to use an extension of the simple model described in [2] to deal with both aspects of the same model together.

By the way, it is difficult to introduce both offshoring and production inventory planning aspects in the same model. The main risk in this type of models is to make them too complicated and that it becomes difficult to obtain results that are not just numerical simulations. Even if the model described in this paper is just a toy model, it allows us to analyze an interesting issue. How offshoring can affect the quality of a good?

We will see that the answer is not straightforward. Moreover, when the production costs in the foreign country are low, then it is optimal for the producer to increase the quality of the good.

The rest of the paper is organized as follows: In Section 2, we introduce offshoring in the model of [2]. In Section 3, we characterize the optimal solution using the Pontryagin Maximum Principle conditions and we prove that, if the treading costs are sufficiently convenient, the quality increases with offshoring. In Section 4, we suggest further analysis that can improve the interest of the model.

2 The model

The model presented in this section is an extension of the original model [2]. We assume that a firm can produce the same good in two different countries. In line with the literature, we denote as N (North) the home country and as S (South) the foreign country. The company has already amortized the costs of the plants in the two countries and is able to produce a good of the same quality in the two production plants.

We assume that the firm has to produce the quantity $B > 0$ of good and it has to deliver it at the time $T > 0$. The firm can decide where to produce the good. It can decide the production flows in two plans, one is located in the home country (North), the other is located in the foreign country (South).

We denote by $u_N(t)$ and $u_S(t)$ the production flows of the North and the South plant, respectively. The inventory levels in the two plants are described by the (state) variables $x_N(t)$ and $x_S(t)$ which satisfy the following ODEs:

$$\dot{x}_i(t) = u_i(t), \qquad i \in \{N, S\}, \qquad t \in [0, T]. \tag{1}$$

The plant obsolescence effect is neglected in this model. We perfectly know that this is a strong assumption, however we prefer to manage a model which can give us closed-form solutions.

We assume that there are production costs c_i^{prod}, depending on production intensity u_i and on quality q.

The produced goods have to be delivered in the home country, hence at the end of the programming interval T the firm has to pay trade costs c^{trade} for the quantity of goods produced in the foreign country.
The market price of the good (delivered at time T), depends on the quality level q. Hence, the revenue of the firm when the production quality is q is given by the product $p(q)B$.
Summarising all assumptions, we obtain the following optimal control problem: the firm wants to choose $u_i \in L^1([0,T];[0,+\infty))$ with $i \in \{N,S\}$ and $q \in [1,+\infty)$ to maximize

$$-\int_0^T \sum_{i\in\{N,S\}} c_i^{prod}(u_i(t),q)\,dt - c_S^{trade}(x_S(T)) + p(q)B \tag{2}$$

subject to constraints (1) and to the boundary conditions

$$x_i(0)=0, \qquad i\in\{N,S\}, \qquad x_N(T)+x_S(T)=B \tag{3}$$

For the sake of clarity, we have written the model in a general form, however in the following discussion we use some specific assumptions about the data of the model.

We assume that the production costs are quadratic in the production intensity in both plants and linear in the production quality q (which is a decision variable of the firm):

$$c_i^{prod}(u_i,q) := \kappa_i q u_i^2/2, \qquad i\in\{N,S\} \tag{4}$$

with $\kappa_N > \kappa_S > 0$ and $q \geq 1$.

We assume that the trade costs are linear in the quantity of goods produced in the foreign country:

$$c_S^{trade}(x_S) := \chi x_S, \qquad \chi > 0 \tag{5}$$

The market price for the good at a quality level q is given by

$$p(q) := \frac{\pi}{2}\left(1-\frac{1}{q}\right), \qquad \pi > 0 \tag{6}$$

where $q \geq 1$.

3 Optimality conditions

We analyze the optimal control problem described in the previous section in two steps. First of all, we consider the quality parameter q as a given quantity and we maximize with respect to the two production flows. Then we face the problem of maximizing with respect to the parameter q.

Theorem 3.1. *Let q be given and let us consider the optimal control problem*

$$\max_{u_N(t),u_S(t)} -\int_0^T \sum_{i\in\{N,S\}} \kappa_i q u_i^2(t)/2dt - \chi x_S(T) \tag{7}$$

subject to the motion equations (1) and the initial and terminal conditions (3).

Then there are two different kinds of optimal paths.

1. *If $\chi \geq q\kappa_N B/T$, then the optimal production flows are*

$$u_N^*(t) \equiv B/T, \qquad u_S^*(t) \equiv 0. \tag{8}$$

2. *If $\chi < q\kappa_N B/T$, then the optimal production flows are*

$$u_N^*(t) \equiv \frac{\kappa_S}{\kappa_N+\kappa_S}\left(\frac{B}{T}+\frac{\chi}{q\kappa_S}\right), \quad u_S^*(t) \equiv \frac{\kappa_N}{\kappa_N+\kappa_S}\left(\frac{B}{T}-\frac{\chi}{q\kappa_N}\right). \tag{9}$$

Proof. Let us assume that $(u_N^*(t), u_S^*(t), x_N^*(t), x_S^*(t))$ is an optimal path. Then there exist two adjoint functions $p_N(t)$, $p_S(t)$, and two constants p_0, v_0 such that (see [4] pages 85, 178, 179, 182):

I) $(p_0, v_0) \neq (0,0)$

II) $u_i^*(t)$ maximizes the function $u_i \mapsto -p_0\kappa_i q u_i^2/2 + p_i(t)u_i$, $u_i \geq 0$, $i \in \{N,S\}$

III) $\dot{p}_i(t) = 0$, for all $i \in \{N,S\}$

IV) $p_0 \in \{0,1\}$

V) $p_N(T) = v_0$, while $p_S(T) = -p_0\chi + v_0$

Let us assume that $p_0 = 0$, then by I), II), III), and V) it must be that $p_N(t) \equiv p_S(t) \equiv v_0 < 0$. However, this implies that $u_N^*(t) \equiv u_S^*(t) \equiv 0$, therefore $x_N^*(t) \equiv x_S^*(t) \equiv 0$ and this contradicts $x_N^*(T) + x_S^*(T) = B$.

Then it must be $p_0 = 1$. The adjoint functions are constant by III); hence from II) we obtain that $u_N^* \equiv [v_0/q\kappa_N]^+$, while $u_S^* \equiv [(-\chi+v_0)/q\kappa_S]^+$. If $v_0 \leq 0$ then $u_N^*(t) \equiv u_S^*(t) \equiv 0$ and we obtain again the previous contradiction. Hence, it must be $v_0 > 0$. Now, we can have the two following cases.

If $v_0 \le \chi$ then $u_N^*(t) \equiv v_0/q\kappa_N$, and $u_S^*(t) \equiv 0$. Therefore, using the motion equation, we obtain $x_N^*(T) = Tv_0/q\kappa_N$, which is feasible if and only if $x_N^*(T) = B$. This allows us to find the value $v_0 = q\kappa_N B/T$ and to characterize the two optimal production flows.

If $v_0 > \chi$ then $u_N^*(t) \equiv v_0/q\kappa_N$, and $u_S^*(t) \equiv (v_0 - \chi)/q\kappa_N$. Using the motion equation and the final constraint $x_N^*(T) + x_S^*(T) = B$, we can find that

$$v_0 = \frac{\kappa_N \kappa_S}{\kappa_N + \kappa_S}\left(\frac{qB}{T} + \frac{\chi}{\kappa_S}\right) \tag{10}$$

And this quantity allows us to determine the optimal production flows.

Finally, let us introduce the Hamiltonian function

$$H(u_N, u_S, p_N, p_S) = \sum_{i \in \{N,S\}} -p_0 q\kappa_i u_i^2/2 + p_i u_i \tag{11}$$

We notice that it does not depend on the state variables and it is concave in the control, moreover the scrap-value function is concave, hence the necessary conditions are also sufficient (see [4] page 105, 182).

□

The idea behind this result is straightforward: if the trade costs are high, then it is convenient to produce in the home country, otherwise it is better to offshore part of the production in the foreign country.

However, what about the quality? First of all, we have to redefine the model using the results described in the previous theorem.

3.1 High trade costs

When the trade costs are high (i.e., if $\chi \ge q\kappa_N B/T$), then the profit function becomes

$$\Pi(q) = -\frac{\kappa_N q B^2}{2T} + \frac{\pi}{2}\left(1 - \frac{1}{q}\right)B \tag{12}$$

This function is strictly concave and it has its maximum at

$$q^* = \sqrt{\frac{\pi T}{B\kappa_N}} \tag{13}$$

This solution is feasible if and only if $\chi \ge q^* \kappa_N B/T$, therefore

$$\chi \ge \sqrt{\frac{\pi \kappa_N B}{T}} \tag{14}$$

In the following, we denote as q_h^* the optimal quality reached when the trade costs are high, i.e.

$$q_h^* \doteq \sqrt{\frac{\pi T}{B\kappa_N}} \tag{15}$$

3.2 Low trade costs

When the trade costs are low (i.e., if $\chi < q\kappa_N B/T$), then the profit function is more complex because we have to consider both production flows. After some computations, we can write the profit function:

$$\Pi(q) = \frac{\kappa_N\kappa_S}{\kappa_N+\kappa_S}\left(\frac{\chi^2 T}{2q\kappa_N\kappa_S} - \frac{qB^2}{2T} - \frac{\chi B}{\kappa_S}\right) + \frac{\pi}{2}\left(1-\frac{1}{q}\right)B \tag{16}$$

Its first derivative is

$$\Pi'(q) = -\frac{\kappa_N\kappa_S}{\kappa_N+\kappa_S}\left(\frac{\chi^2 T}{2q^2\kappa_N\kappa_S} + \frac{B^2}{2T}\right) + \frac{\pi}{2}\frac{B}{q^2} \tag{17}$$

whereas the second derivative is

$$\Pi''(q) = \left(\frac{\chi^2 T}{\kappa_N+\kappa_S} - \pi B\right)\frac{1}{q^3} \tag{18}$$

In the following, we assume that the quantity of goods is large enough to assure that this function is concave (i.e., we assume that $B > \chi^2 T/[\pi(\kappa_N+\kappa_S)]$ is satisfied). Under these hypotheses, we obtain that the optimal production quality is

$$q^* = \sqrt{\frac{(B\pi(\kappa_N+\kappa_S) - \chi^2 T)\,T}{\kappa_N\kappa_S B^2}} \tag{19}$$

This solution is feasible if and only if $\chi < q^*\kappa_N B/T$, therefore

$$\chi < \sqrt{\frac{\pi\kappa_N B}{T}} \tag{20}$$

In the following, we denote as q_ℓ^* the optimal quality reached when the trade costs are low, i.e.

$$q_\ell^* \doteq \sqrt{\frac{(B\pi(\kappa_N+\kappa_S) - \chi^2 T)\,T}{\kappa_N\kappa_S B^2}} \tag{21}$$

In the following remark, we check the continuity of the solution when we move from high trade cost to low trade costs.

Remark 3.2. *If $\chi = q^*\kappa_N B/T$, the optimal production quality in the case when the trade costs are high, i.e., q_h^*, converges to the one in the case when the trade costs are low, i.e., q_ℓ^*. It is enough to substitute the value $q^*\kappa_N B/T$ in place of χ in equation (19), to find q^* as in (13).*

3.3 Comparison between solutions

We close this section by comparing the optimal quality obtained in the previous two subsections. We are wondering how the optimal quality changes when we move from high trade costs to low trade costs. The following theorem answers this question.

Theorem 3.3. *The optimal production quality with low trade costs q_ℓ^*, when it exists, is always higher than the one with high trade costs q_h^*. It means that, if $\chi < q^* k_N B/T$, then*

$$q_\ell^* > q_h^*. \tag{22}$$

Proof. Using the definition of q_ℓ^* in (21) and the definition of q_h^* in (15) we note that (22) holds true if and only if

$$\frac{B\pi(k_N + k_S) - \chi^2 T}{k_N k_S B^2} > \frac{\pi T}{B k_N} \quad \Leftrightarrow \quad B\pi(k_N + k_S) - \chi^2 T > B\pi k_S$$
$$\Leftrightarrow \quad B\pi k_N > \chi^2 T \quad \Leftrightarrow \quad \chi < \sqrt{\frac{B\pi k_N}{T}}.$$

□

4 Conclusions

We have proposed a simple model to treat the relationship between the relocation of production and the quality of the product. We know that the assumptions made are strong, but our purpose is to find closed-form solutions. This is to be considered as a starting point for future research questions in which to generalize the proposed model, to find less uniform results.
A first extension concerns the plant obsolescence effect, neglected here, with the introduction of inventory costs, different for each of the two countries involved, as proposed in [1].
Another interesting generalization relates to the case, in which the quantity of the good to be delivered at a fixed time depends both on the price and on the quality of the manufactured good, as studied in [2].

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