

ON JORDAN CLASSES FOR VINBERG'S θ -GROUPS

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ABSTRACT. Popov has recently introduced an analogue of Jordan classes (packets, or decomposition classes) for the action of a θ -group (G_0, V) , showing that they are finitely-many, locally-closed, irreducible unions of G_0 -orbits of constant dimension partitioning V . We carry out a local study of their closures showing that Jordan classes are smooth and that their closure is a union of Jordan classes. We parametrize Jordan classes and G_0 -orbits in a given class in terms of the action of subgroups of Vinberg's little Weyl group, and include several examples and counterexamples underlying the differences with the symmetric case and the critical issues arising in the θ -situation.

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1. INTRODUCTION

Theta groups (or, equivalently, periodically graded reductive Lie algebras) were deeply studied in [31, 32] as a natural generalisation of symmetric spaces, [12, 13]. In all situations \mathfrak{g} is a \mathbb{Z}_m -graded complex reductive Lie algebra, its degree 0 part \mathfrak{g}_0 is again reductive and the focus is on the action of the corresponding connected algebraic group G_0 on the other homogeneous components \mathfrak{g}_i of \mathfrak{g} . As observed by Vinberg, there is no loss of generality in studying the action on the degree 1 component $V = \mathfrak{g}_1$ only. Key results in [31] concern n theory and include: the introduction of a little Weyl group and the analogue of the Steinberg map and Chevalley's restriction theorem and the proof that the little Weyl groups are complex reflection groups. These results were confirmed also in positive characteristic, [14], where an alternative description of the little Weyl group in terms of the usual Weyl group is proposed. Many interesting examples in representation theory can be interpreted in terms of graded Lie algebras: for instance, if \mathfrak{g} is the Lie algebra of a classical group G , a grading on the defining representation of G induces a grading on \mathfrak{g} and the G_0 -action on V can be seen as a representation of a cyclic quiver with additional structure, [18, §0.3, §9.5].

A structural feature of theta groups is that they are visible groups, that is, connected reductive algebraic groups with a representation for which each fiber of the Steinberg map consists of finitely many orbits. This property almost characterizes the theta groups: more precisely, a connected simple irreducible visible linear group is either a (commutant of a) theta group or it is isomorphic to $\mathrm{Spin}(11)$ or $\mathrm{Spin}(13)$ [11]. Various explicit descriptions of the orbits and invariants for theta groups of order $m = 2$ are known (see [35, Summary Table]) but a number of cases with $m \geq 3$ have also been considered in the literature [9, 10, 20, 23, 34].

An important application of theta group theory is in the representation theory of reductive groups over a p -adic field F . Indeed, the classification of positive rank gradings [14, 15, 25] over the residue field k of a maximal unramified extension L of F leads to the classification of non-degenerate K -types, and stable G_0 -orbits in V^* are strictly related to supercuspidal representations of the rational points of G over F attached to elliptic Z -regular elements of the Weyl group, [26]. Also, in the context of a graded version of the generalized Springer correspondence, the block decomposition of the G_0 -equivariant derived category supported on the nilpotent part of each \mathfrak{g}_i leads to the construction of representations of various graded double affine Hecke algebras with possibly unequal parameters, one for each block, [18, 19]. It emerges from these constructions that parabolic induction is no longer the right instrument in the graded setting, leading to the introduction of spirals. This shows that even though many results in the classical symmetric case have an analogue in the graded setting, generalisations to the case of $m > 2$ are not always straightforward.

This phenomenon is also visible in the study of related G_0 -stable stratifications in V . In the setting of the ungraded generalized Springer correspondence, one of the relevant stratifications is given by the decomposition into Jordan classes (packets, or decomposition classes) in a reductive group G , or Lie algebra \mathfrak{g} . In the Lie algebra setting Jordan classes were introduced in [5] and were crucial in the construction of sheets for the adjoint action of a semisimple group G on its Lie algebra. These classes are G -stable, disjoint, finitely-many, locally-closed, smooth and irreducible. The decomposition into Jordan classes in a Lie algebra turns out to coincide with the decomposition into orbit-types, i.e., into the subsets of elements with same stabilizer up to conjugation. Borho and Kraft proved that sheets are easily described as regular closures of those Jordan classes which satisfy some maximality property with respect to closure inclusion, and it was shown in [4] that the closure and regular closure of Jordan classes can be described in terms of Lusztig-Spaltenstein's parabolic induction of adjoint orbits. The symmetric analogue of Jordan classes and sheets has been studied by Tauvel and Yu, (see [29] and references in there) and their closures were studied in [7, 8]. In the latter it is again observed that parabolic induction is no longer efficient, and slice induction is proposed: one of the difficulties in working with parabolic induction is the fact that many homogenous Levi subalgebras do not necessarily lie in a homogeneous parabolic subalgebra, see the Appendix A for an example of this phenomenon. An analogue of Jordan classes for theta groups when \mathfrak{g} is semisimple has been recently introduced by Popov in [24], generalizing the classical and symmetric ones. As in these cases, Jordan classes form a partition of V into finitely-many, locally-closed, irreducible unions of G_0 -orbits of constant dimension, and so sheets for the G_0 -action on V are regular closures of some Jordan class. In this paper we introduce a local study of such Jordan classes and their closures leading us to prove that any Jordan class is smooth and that its closure is a union of Jordan classes. In order to characterize the closure relation, we provide an analogue of the results in [8] on slice induction. For our inductive arguments, we needed to extend slightly the notion of Jordan classes to the case of reductive Lie algebras. Our local approach differs from [8] because we rely on Luna's fundamental Lemma and use the Slodowy slice only after reduction to neighbourhoods of nilpotent points; Luna's slice theorem is also used for the proof of smoothness.

It is also worthwhile to notice that a different, coarser, notion of Jordan equivalence relation could have been introduced, by using regularity for the G_0 -action rather than for the action of the full group G . In the symmetric setting these two notions coincide by virtue of [13, Proposition 5], but they might differ for $m > 2$. Popov's choice of Jordan classes in V ensures that each of them is contained in a usual Jordan class in \mathfrak{g} . We devote §2.4 and §3.3 to comparisons of different notions of regularity and refer to [21, 9] for an analysis of various results on regularity in theta groups.

The paper is structured as follows. In §2, we recall the basics on periodically graded complex reductive Lie algebras, introduce the relevant notions of regularity and extend to the reductive case the general treatment in [24] of Jordan classes and sheets in V . We then focus in §3 on the local study of the closures of Jordan classes in V , the main results here are Theorem 3.11, Theorem 3.12 and Proposition 3.13. We conclude §3 with some regularity questions, including Proposition 3.15. The last section is devoted to slice induction, leading to Theorem 4.3, and to the parametrization of the Jordan classes in V and the G_0 -orbits contained in a class. The paper finishes with Example 4.11 on trivectors in 9-dimensional space and with Appendix A, dealing with obstructions to the existence of homogeneous parabolic subalgebras in \mathfrak{g} .

The results of this paper are valid over any algebraically closed field of characteristic zero, but we expect that they extend to suitable positive characteristics. Indeed the results from [16] needed in §3-§4 hold also in positive characteristic under mild restrictions, cf. [3].

During completion of this paper we were informed that Professor È. B. Vinberg had passed away. Without his work in [31] this manuscript would never have been written, so we would like to dedicate it to his memory.

2. PRELIMINARIES ON VINBERG'S θ -GROUPS AND JORDAN CLASSES

2.1. Graded Lie algebras. Let \mathfrak{g} be a complex reductive Lie algebra which is \mathbb{Z}_m -graded, that is, it admits a direct sum decomposition of vector spaces

$$\mathfrak{g} = \bigoplus_{l \in \mathbb{Z}_m} \mathfrak{g}_l \tag{2.1}$$

with $[\mathfrak{g}_i, \mathfrak{g}_l] \subset \mathfrak{g}_{i+l}$ for all $i, l \in \mathbb{Z}_m$. We note that the subspaces of (2.1) can be recovered as the eigenspaces of the automorphism θ of \mathfrak{g} defined by $\theta(x) = \omega^l x$ for $x \in \mathfrak{g}_l$, where $\omega = e^{\frac{2\pi i}{m}}$. Conversely, any automorphism θ of \mathfrak{g} of period m defines a \mathbb{Z}_m -grading. Due to this, we will denote a Lie algebra \mathfrak{g} with a \mathbb{Z}_m -grading by the triple $\{\mathfrak{g}, \theta, m\}$, or often simply by $\{\mathfrak{g}, \theta\}$. Whenever a subspace $A \subset \mathfrak{g}$ is homogeneous, i.e., it satisfies $A = \bigoplus_l (A \cap \mathfrak{g}_l)$, we will write $A_l = A \cap \mathfrak{g}_l$ and $A = \bigoplus_l A_l$.

The Lie algebra \mathfrak{g} has a decomposition into homogeneous ideals

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{s}, \quad \text{where } \mathfrak{s} := [\mathfrak{g}, \mathfrak{g}]. \tag{2.2}$$

We denote by κ a bilinear form on \mathfrak{g} that is non-degenerate, \mathfrak{g} -invariant, θ -invariant and such that $\mathfrak{z}(\mathfrak{g})$ and \mathfrak{s} are orthogonal. We call any such bilinear form *adapted*.

Lemma 2.1. *There exists an adapted bilinear form κ on \mathfrak{g} if and only if $\mathfrak{z}(\mathfrak{g})$ is symmetrically graded, i.e., $\dim \mathfrak{z}(\mathfrak{g})_l = \dim \mathfrak{z}(\mathfrak{g})_{-l}$ for all $l \in \mathbb{Z}_m$. In this case $\dim \mathfrak{g}_l = \dim \mathfrak{g}_{-l}$ for all $l \in \mathbb{Z}_m$.*

Proof. If κ is adapted, then $\kappa(\mathfrak{g}_l, \mathfrak{g}_i) = 0$ whenever $i + l \neq 0$, hence \mathfrak{g}_{-l} and \mathfrak{g}_l are dual spaces, and so are \mathfrak{s}_{-l} and \mathfrak{s}_l and also $\mathfrak{z}(\mathfrak{g})_{-l}$ and $\mathfrak{z}(\mathfrak{g})_l$. In particular $\mathfrak{z}(\mathfrak{g})$ is symmetrically graded.

Conversely, it is enough to consider an appropriate extension of the Killing form of \mathfrak{s} . \square

With the term *reductive \mathbb{Z}_m -graded Lie algebra $\{\mathfrak{g}, \theta\}$* , we will always mean a complex reductive Lie algebra $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{s}$ together with a \mathbb{Z}_m -grading such that the center $\mathfrak{z}(\mathfrak{g})$ is symmetrically graded. This is also the class of graded Lie algebras considered in [31], since they allow for adapted bilinear forms. By Lemma 2.1 we may assume κ to be an extension of the Killing form of \mathfrak{s} .

Let G be any connected algebraic group with Lie algebra $\mathfrak{Lie}(G) = \mathfrak{g}$, let S be the connected subgroup of G with $\mathfrak{Lie}(S) = \mathfrak{s}$, and let $^\circ$ denote the identity component of a closed subgroup, so $G = Z(G)^\circ S$. Let G_0 be the connected subgroup of G with $\mathfrak{Lie}(G_0) = \mathfrak{g}_0$. Unless otherwise stated, for Lie subalgebras of \mathfrak{g} we will use a gothic letter, the corresponding Roman capital letter will indicate the connected subgroup of G with that Lie algebra, a lower index 0 its intersection with G_0 . So, the decomposition $\mathfrak{g}_0 = \mathfrak{z}(\mathfrak{g})_0 \oplus \mathfrak{s}_0$, gives an almost direct product $G_0 = Z(G)_0^\circ S_0^\circ$, where S_0° is the reductive, connected subgroup of S with $\mathfrak{Lie}(S_0^\circ) = \mathfrak{s}_0$. By restricting the adjoint representation, G_0 and S_0° act on \mathfrak{g}_l , for any $l \in \mathbb{Z}_m$, with trivial action of $Z(G)_0^\circ$. The reduction process in [31, pag. 467] shows that it is enough to focus on the case of $l = 1$; we set $V := \mathfrak{g}_1$. The linear group of transformations of V associated to G_0 is called the θ -group of the graded Lie algebra $\{\mathfrak{g}, \theta\}$ and it does not depend on the choice of G in the class of locally isomorphic groups. However, by abuse of notation, we will directly refer to G_0 as the θ -group of $\{\mathfrak{g}, \theta\}$. The decomposition (2.2) in degree 1 gives a decomposition of V into G_0 -stable subspaces $V = \mathfrak{z}(\mathfrak{g})_1 \oplus \mathfrak{s}_1$ with trivial G_0 -action on $\mathfrak{z}(\mathfrak{g})_1$. Observe that $\mathfrak{z}(\mathfrak{g})_1 \neq 0$ may occur only if θ is not inner.

Let $x \in \mathfrak{g}$ and m be a Lie subalgebra of \mathfrak{g} with associated subgroup $M \subset G$. The orbit of x for the action of M is denoted by \mathcal{O}_x^M , and the stabilizer of x in M by M^x . The centralizer of x in m is denoted by m^x , with center $\mathfrak{z}(m^x)$. If $x \in V$, then \mathfrak{g}^x , $\mathfrak{z}(\mathfrak{g}^x)$ and $[\mathfrak{g}^x, \mathfrak{g}^x]$ are θ -stable, in other words homogeneous. We recall that if $x \in \mathfrak{g}$ is semisimple, then G^x is a connected subgroup of G , the Levi subgroup of a parabolic subgroup of G [28, 7.3.5]. In this case, the restriction of κ to $\mathfrak{g}^x = \mathfrak{z}(\mathfrak{g}^x) \oplus [\mathfrak{g}^x, \mathfrak{g}^x]$ is an adapted bilinear form, so $\mathfrak{z}(\mathfrak{g}^x) = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{z}(\mathfrak{s}^x)$ is symmetrically graded. We stress that $G_0^x = G^x \cap G_0$ is not connected in general.

We recall the following general results on centralizers, that we will later apply when $x \in V$.

Lemma 2.2. [29, Proposition 35.3.1, Corollary 35.3.2] *Let $x \in \mathfrak{g}$. Then*

$$[\mathfrak{g}, x]^\perp = [\mathfrak{g}, \mathfrak{z}(\mathfrak{g}^x)]^\perp = \mathfrak{g}^x, \text{ and } [\mathfrak{g}, \mathfrak{g}^x]^\perp = \mathfrak{z}(\mathfrak{g}^x). \quad (2.3)$$

The following conditions are equivalent for any $x, y \in \mathfrak{g}$:

- (i) $y \in \mathfrak{z}(\mathfrak{g}^x)$;
- (ii) $\mathfrak{g}^x \subset \mathfrak{g}^y$;
- (iii) $[\mathfrak{g}, y] \subset [\mathfrak{g}, x]$;
- (iv) $\mathfrak{z}(\mathfrak{g}^y) \subset \mathfrak{z}(\mathfrak{g}^x)$;

Corollary 2.3. *Let $g \in G_0$ and $x, y \in V$. Then the following conditions are equivalent:*

- (i) $g \cdot \mathfrak{g}^x = \mathfrak{g}^y$;
- (ii) $g \cdot \mathfrak{z}(\mathfrak{g}^x) = \mathfrak{z}(\mathfrak{g}^y)$;
- (iii) $g \cdot \mathfrak{z}(\mathfrak{g}^x)_1 = \mathfrak{z}(\mathfrak{g}^y)_1$.

Proof. Clearly (i) \Leftrightarrow (ii) by Lemma 2.2 since $g \cdot \mathfrak{g}^x = \mathfrak{g}^{g \cdot x}$ and $g \cdot \mathfrak{z}(\mathfrak{g}^x) = \mathfrak{z}(\mathfrak{g}^{g \cdot x})$, and (ii) \Rightarrow (iii). If (iii) holds, then $y \in \mathfrak{z}(\mathfrak{g}^y)_1 = \mathfrak{z}(\mathfrak{g}^{g \cdot x})_1$ and $x \in \mathfrak{z}(\mathfrak{g}^{g^{-1} \cdot y})_1$, hence $g \cdot \mathfrak{g}^x = \mathfrak{g}^y$ by Lemma 2.2. \square

2.2. The Jordan decomposition. Let $\{\mathfrak{g}, \theta\}$ be a reductive \mathbb{Z}_m -graded Lie algebra. For elements x, y, z in \mathfrak{g} , lower indices s and n will always indicate semisimple and nilpotent parts in the Jordan decomposition, i.e., they stand for $x = x_s + x_n$ with $x_s \in \mathfrak{g}$ semisimple, $x_n \in \mathfrak{g}$ nilpotent, and $[x_s, x_n] = 0$. Elements of $\mathfrak{z}(\mathfrak{g})$ are always intended to be semisimple.

Let \mathcal{S} (resp. \mathcal{N}) be the set of semisimple (resp. nilpotent) elements of \mathfrak{g} . We note that θ preserves both \mathcal{S} and \mathcal{N} , so semisimple and nilpotent parts of any $x \in \mathfrak{g}_l$ also belong to \mathfrak{g}_l . We set $\mathcal{S}_V = \mathcal{S} \cap V$, $\mathcal{N}_V = \mathcal{N} \cap V$, and stress that the number of G_0 -orbits in \mathcal{N}_V is finite [31].

Lemma 2.4. *The action of the reductive Lie algebra \mathfrak{g}_0 on \mathfrak{g} is completely reducible.*

Proof. Since $\mathfrak{g}_0 = \mathfrak{z}(\mathfrak{g})_0 \oplus \mathfrak{s}_0$, it is sufficient to prove the claim for \mathfrak{s}_0 . Now κ restricted to \mathfrak{s}_0 is non-degenerate and \mathfrak{s}_0 contains the semisimple and nilpotent parts of any of its elements. The claim then follows from, e.g., [29, Proposition 20.5.12]. \square

We emphasize that \mathfrak{g}_0 is not a subalgebra of maximal rank of \mathfrak{g} in general, that is, it might not contain any Cartan subalgebra of \mathfrak{g} . Let $x \in V$. A direct consequence of Lemma 2.2 is:

Lemma 2.5. *The tangent space $T_x \mathcal{O}_x^{G_0}$ to $\mathcal{O}_x^{G_0}$ at x is given by the subspace $[\mathfrak{g}_0, x]$ of V . Its orthogonal complement in \mathfrak{g}_{-1} coincides with \mathfrak{g}_{-1}^x .*

2.3. The Cartan subspace. A Cartan subspace of $\{\mathfrak{g}, \theta\}$ is an abelian subspace \mathfrak{c} of V which consists of semisimple elements and it is maximal in the class of such subspaces.

Theorem 2.6. [31, pag. 472] *Any two Cartan subspaces of $\{\mathfrak{g}, \theta\}$ are conjugate by the action of an element in G_0 . As a consequence, if $x \in \mathcal{S}_V$, then $\mathcal{O}_x^{G_0}$ meets any Cartan subspace of $\{\mathfrak{g}, \theta\}$.*

The dimension of a Cartan subspace of a graded Lie algebra $\{\mathfrak{g}, \theta\}$ is called the *rank* of $\{\mathfrak{g}, \theta\}$. It is clear that $\{\mathfrak{g}, \theta\}$ has zero rank if and only if $V \subset \mathcal{N}_V$. For any set R of commuting elements of \mathcal{S}_V , the centralizer $\mathfrak{c}_{\mathfrak{g}}(R) = \cap_{x \in R} \mathfrak{g}^x$ of R in \mathfrak{g} is a homogeneous Levi subalgebra of \mathfrak{g} , so

$$\mathfrak{c}_{\mathfrak{g}}(R) = \mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(R)) \oplus [\mathfrak{c}_{\mathfrak{g}}(R), \mathfrak{c}_{\mathfrak{g}}(R)] \quad (2.4)$$

and these summands are also homogeneous. We recall a useful characterization of a Cartan subspace in terms of its centralizer [31, pag. 471].

Proposition 2.7. *A subspace $\mathfrak{c} \subset V$ consisting of commuting semisimple elements is a Cartan subspace if and only if $\mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c}))_1 = \mathfrak{c}$ and the graded Lie algebra $\{[\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c}), \mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})], \theta\}$ has zero rank.*

Let \mathfrak{c} be a Cartan subspace. By the previous result and equation (2.4) for $R = \mathfrak{c}$, we have a decomposition $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})_1 = \mathfrak{c} \oplus [\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c}), \mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})]_1$, with $\mathfrak{c} \subset \mathcal{S}_V$ and $[\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c}), \mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})]_1 \subset \mathcal{N}_V$. In other words, this decomposition gives the Jordan components of any element of $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})_1$.

Corollary 2.8. *For any $x \in \mathfrak{c}$, we have $\mathfrak{z}(\mathfrak{g}^x)_1 \subset \mathfrak{c}$.*

Proof. Since $\mathfrak{z}(\mathfrak{g}^x)$ consists of semisimple elements, it follows that $\mathfrak{z}(\mathfrak{g}^x)_1 \subset \mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})_1 \cap \mathcal{S}_V = \mathfrak{c}$. \square

Before turning to the next subsection, we recall that the Weyl group in the sense of Vinberg is the group $W_{\text{Vin}} = W_{\text{Vin}}(\mathfrak{g}, \theta)$ of linear transformations of \mathfrak{c} given by $W_{\text{Vin}} \cong N_{G_0}(\mathfrak{c})/Z_{G_0}(\mathfrak{c})$, where $N_{G_0}(\mathfrak{c})$ (resp. $Z_{G_0}(\mathfrak{c})$) is the normalizer (resp. centralizer) of \mathfrak{c} in G_0 .

Theorem 2.9. [31, pag. 473] *The group W_{Vin} is finite and for $x, y \in \mathfrak{c}$ we have $y \in \mathcal{O}_x^{G_0}$ if and only if $y \in W_{\text{Vin}} \cdot x$.*

There is a geometric counterpart to this result [31, §4]. The restriction $\mathbb{C}[V] \rightarrow \mathbb{C}[\mathfrak{c}]$ of polynomial functions from V to \mathfrak{c} induces a ‘‘Chevalley-type’’ isomorphism $\mathbb{C}[V]^{G_0} \cong \mathbb{C}[\mathfrak{c}]^{W_{\text{Vin}}}$ and each fiber of the ‘‘Steinberg quotient map’’ $\varphi: V \rightarrow V//G_0 \cong \mathfrak{c}/W_{\text{Vin}}$ consists of finitely many G_0 -orbits. Here $V//G_0$ is the GIT quotient of V , and two elements of V fail to be separated by the invariants if and only if their semisimple parts lie in the same G_0 -orbit. Recall that semisimple (resp. nilpotent) orbits can also be characterized as the closed orbits (resp. orbits whose closure contains 0). Hence, each fiber of φ contains exactly one closed orbit.

2.4. Dimensions of centralizers and regularity conditions. This subsection deals with some general observations, which encompass a classical result of Kostant and Rallis (see [13] and also [21]), and motivates the introduction of two distinct notions of regularity.

Proposition 2.10. *Let $\{\mathfrak{g}, \theta\}$ be a reductive \mathbb{Z}_m -graded Lie algebra (with symmetrically graded center, as usual). Then $\dim \mathfrak{g}_l - \dim \mathfrak{g}_l^x = \dim \mathfrak{g}_{l-1} - \dim \mathfrak{g}_{l-1}^x$ for all $x \in V$ and $l \in \mathbb{Z}_m$.*

Proof. Let κ be an adapted bilinear form on \mathfrak{g} . The bilinear form given by $\kappa_x(y, z) := \kappa(x, [y, z])$ is skew-symmetric for all $y, z \in \mathfrak{g}$ and its radical is the centralizer \mathfrak{g}^x , which is homogeneous. It induces a non-degenerate bilinear form on the quotient $\mathfrak{g}/\mathfrak{g}^x = \bigoplus_{l \in \mathbb{Z}_m} \mathfrak{g}_l/\mathfrak{g}_l^x$ with the property that $\mathfrak{g}_i/\mathfrak{g}_i^x \perp \mathfrak{g}_l/\mathfrak{g}_l^x$ if $i + l + 1 \neq 0$, in particular $\mathfrak{g}_l/\mathfrak{g}_l^x \cong (\mathfrak{g}_{-l-1}/\mathfrak{g}_{-l-1}^x)^*$. \square

Corollary 2.11.

- (i) *For all $x \in V$ we have $\dim \mathcal{O}_x^G = 2 \dim \mathcal{O}_x^{G_0} + \sum_{l \neq -1, 0} (\dim \mathfrak{g}_l - \dim \mathfrak{g}_l^x)$;*

- (ii) If $x \in \mathcal{S}_V$, then $\dim \mathfrak{g}_l - \dim \mathfrak{g}_l^x = \dim \mathfrak{g}_{l+1} - \dim \mathfrak{g}_{l+1}^x$ is independent of $l \in \mathbb{Z}_m$ and we have $\dim \mathcal{O}_x^{\mathfrak{G}_0} = m \dim \mathcal{O}_x^{\mathfrak{G}_0}$;
- (iii) Let $x \in V$, then $\mathfrak{g}_0^x = \mathfrak{g}_0$ if and only if $x \in \mathfrak{z}(\mathfrak{g})_1$.

Proof. Claim (i) is immediate from Proposition 2.10. If $x \in \mathcal{S}_V$, then the restriction of κ to \mathfrak{g}^x is non-degenerate and $\dim \mathfrak{g}_l^x = \dim \mathfrak{g}_{-l}^x$ for all $l \in \mathbb{Z}_m$, so (ii) follows from Proposition 2.10 and (i). If $x \in \mathfrak{z}(\mathfrak{g})_1$, then clearly $\mathfrak{g}_0^x = \mathfrak{g}_0$. Conversely, if $\mathfrak{g}_0^x = \mathfrak{g}_0$ then $x \in \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h}_0)$, where \mathfrak{h}_0 is a Cartan subalgebra of \mathfrak{g}_0 , and x is semisimple by a classical result, see e.g. [33, pag. 116]. Then $\mathfrak{g}^x = \mathfrak{g}$ by (ii) and $x \in \mathfrak{z}(\mathfrak{g})_1$. \square

If $x, y \in V$ are two elements with $\dim \mathcal{O}_x^{\mathfrak{G}_0} = \dim \mathcal{O}_y^{\mathfrak{G}_0}$, then $\dim \mathfrak{g}_l^x = \dim \mathfrak{g}_l^y$ for $l = 0, -1$. The following simple example shows that the hypothesis $x, y \in \mathcal{S}_V$ is indeed necessary for $\dim \mathfrak{g}_l^x = \dim \mathfrak{g}_l^y$ to hold also for $l \neq 0, -1$.

Example 2.12. Let \mathfrak{g} be of type E_8 and θ be the automorphism of \mathfrak{g} of order 3 extensively studied in [34]. Here $\mathfrak{g}_1 \cong \Lambda^3 \mathbb{C}^9$, $\mathfrak{g}_0 \cong \mathfrak{sl}(9)$ and $\mathfrak{g}_{-1} \cong \Lambda^3(\mathbb{C}^9)^*$. The orbits of $SL(9)$ on $V = \Lambda^3 \mathbb{C}^9$ have been classified in *loc. cit.* Let e_i , for $1 \leq i \leq 9$, be the canonical basis vectors of \mathbb{C}^9 and let $e_{ijl} := e_i \wedge e_j \wedge e_l$. The trivector $x_s = e_{123} + e_{456} + e_{789}$ is semisimple, with centralizer \mathfrak{g}^{x_s} a reductive Lie algebra with semisimple part \mathfrak{r} of type E_6 . More precisely $\mathfrak{r} = \mathfrak{r}_{-1} \oplus \mathfrak{r}_0 \oplus \mathfrak{r}_1$ with

$$\mathfrak{r}_1 = X \otimes Y \otimes Z, \quad \mathfrak{r}_0 = \mathfrak{sl}(X) \oplus \mathfrak{sl}(Y) \oplus \mathfrak{sl}(Z), \quad \mathfrak{r}_{-1} = X^* \otimes Y^* \otimes Z^*,$$

where $X = \text{span}\{e_1, e_2, e_3\}$, $Y = \text{span}\{e_4, e_5, e_6\}$, $Z = \text{span}\{e_7, e_8, e_9\}$ and where we identified tensor products with subspaces of $\mathfrak{g}_{\pm 1}$ by mapping pure tensors to the corresponding antisymmetrizations. Since \mathfrak{g}^{x_s} has maximal rank, its center is two-dimensional and it is not difficult to see that it consists of $x_s \in \mathfrak{g}_1$ and $x_s^* \in \mathfrak{g}_{-1}$.

Now x_s is the semisimple part of trivectors $x = x_s + x_n$ in the VI family, cf. [34, Table 5]. We consider those trivectors for which $\dim \mathcal{O}_x^{\mathfrak{G}_0} = 76$, i.e., $x = x_s + x_n$ with nilpotent part:

Class 7: $x_n = e_{149} + e_{158} + e_{167} + e_{248} + e_{357}$;

Class 8: $x_n = e_{149} + e_{167} + e_{258} + e_{347}$;

Class 9: $x_n = e_{147} + e_{158} + e_{258} + e_{269}$.

In all the three cases $\dim \mathfrak{g}_0^x = 4$ and $\dim \mathfrak{g}_{-1}^x = 8$ by Proposition 2.10. However a direct computation tells us that $\mathfrak{g}_1^x = \{y \in \mathfrak{g}_1^{x_s} \mid y \wedge x_n = 0\}$ has dimension 6, 8 and 10, respectively.

Corollary 2.11 and Example 2.12 motivate the following.

Definition 2.13. For any subset $A \subset V$, we set

$$(i) \quad A^{\text{reg}} = \left\{ x \in A \mid \dim \mathfrak{g}^x \leq \dim \mathfrak{g}^y \text{ for all } y \in A \right\};$$

$$(ii) \quad A^\bullet = \left\{ x \in A \mid \dim \mathfrak{g}_0^x \leq \dim \mathfrak{g}_0^y \text{ for all } y \in A \right\}.$$

The subset A^{reg} (resp., A^\bullet) is called the regular part (resp., the G_0 -regular part) of A .

Note that $A^\bullet = \left\{ x \in A \mid \dim \mathfrak{g}_{-1}^x \leq \dim \mathfrak{g}_{-1}^y \text{ for all } y \in A \right\}$ due to Lemma 2.5. A simple relation between the two notions is given by the following.

Lemma 2.14. Let A be an irreducible subset of V (w.r.t. the induced topology). Then

$$A^{\text{reg}} = \bigcap_{l \in \mathbb{Z}_m} \left\{ x \in A \mid \dim \mathfrak{g}_l^x \leq \dim \mathfrak{g}_l^y \text{ for all } y \in A \right\} \quad (2.5)$$

and so $A^{\text{reg}} \subset A^\bullet$ as a Zariski open subset.

Proof. Clearly each subset on the R.H.S of (2.5) is non-empty and Zariski open in A . Since A is irreducible, the (finite) intersection of all such subsets is non-empty, so equal to A^{reg} . \square

Let Σ be the set of restricted roots, that is, the non-zero linear functions on \mathfrak{c} occurring in the weight space decomposition of the semisimple action of the abelian subalgebra \mathfrak{c} on \mathfrak{g} . We write $\mathfrak{g} = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{c}) \oplus \bigoplus_{\sigma \in \Sigma} \mathfrak{g}_\sigma$.

Example 2.15. A semisimple element $y \in \mathcal{S}_V$ belongs to $\mathcal{S}_V^{\text{reg}}$ if and only if it lies in \mathcal{S}_V^\bullet if and only if $\dim \mathfrak{g}^y = \dim \mathfrak{c}_\mathfrak{g}(c)$. The first equivalence follows from Corollary 2.11 (ii). For the second one, observe that y is G_0 -conjugated to some $x \in \mathfrak{c}$, whose centralizer is stabilized by \mathfrak{c} and decomposes as

$$\mathfrak{g}^x = \mathfrak{c}_\mathfrak{g}(c) \oplus \bigoplus_{\sigma \in \Sigma(x)} \mathfrak{g}_\sigma \quad (2.6)$$

with $\Sigma(x) = \{\sigma \in \Sigma \mid \sigma(x) = 0\}$. Hence, $x \in \mathcal{S}_V^{\text{reg}}$ if and only if $\sigma(x) \neq 0$ for all $\sigma \in \Sigma$, i.e., $\mathfrak{g}^x = \mathfrak{c}_\mathfrak{g}(c)$.

Example 2.16. Contrarily to the ungraded case and the $(m = 2)$ -case, an element $x_s \in \mathfrak{c} \cap \mathcal{S}_V^{\text{reg}}$ is not necessarily in V^\bullet (let alone V^{reg} or $\mathfrak{g}^{\text{reg}}$, since $\mathfrak{g}^{\text{reg}} \cap V \subset V^{\text{reg}} \subset V^\bullet$). In general, x_s extends to an element $x = x_s + x_n \in V^\bullet$ where x_n is an element in general position in $[\mathfrak{c}_\mathfrak{g}(c), \mathfrak{c}_\mathfrak{g}(c)]_1$ (recall that $[\mathfrak{c}_\mathfrak{g}(c), \mathfrak{c}_\mathfrak{g}(c)]_1$ consists of nilpotent elements). Then, if $[\mathfrak{c}_\mathfrak{g}(c), \mathfrak{c}_\mathfrak{g}(c)]_1 \neq 0$, also $x_n \neq 0$ and so $\mathfrak{g}_0^x \subsetneq \mathfrak{g}_0^{x_s}$ due to (iii) of Corollary 2.11 applied to the reductive Lie algebra \mathfrak{g}^{x_s} . The G_0 -orbits in V^\bullet have codimension in V equal to the rank of $\{\mathfrak{g}, \theta\}$, hence $\dim \mathfrak{g}_0^x = \dim \mathfrak{g}_0 - \dim V + \dim \mathfrak{c}$, see [31, Theorem 5].

2.5. Jordan classes and sheets for θ -groups. V. L. Popov has recently generalized the notion of a Jordan class to the case of semisimple \mathbb{Z}_m -graded Lie algebras $\{\mathfrak{g}, \theta\}$ and studied its main geometric properties in [24]. For $m = 1, 2$, the notion coincides with that studied in [8, 29]. We here briefly extend his general treatment to the reductive case, which is more suitable for our inductive and local arguments of §3-§4, and directly refer to [24, §3] for more details. (We warn the reader that the symbol “reg” in [24] is replaced by “•” in the present paper.)

Let $\{\mathfrak{g}, \theta\}$ be a reductive \mathbb{Z}_m -graded Lie algebra. Two elements $x = x_s + x_n$ and $y = y_s + y_n$ of V are G_0 -Jordan equivalent if there exists $g \in G_0$ such that

$$\mathfrak{g}^{y_s} = g \cdot \mathfrak{g}^{x_s}, \quad y_n = g \cdot x_n. \quad (2.7)$$

This is an equivalence relation $x \stackrel{G_0}{\sim} y$ on V , the equivalence class $J_{G_0}(x)$ of $x \in V$ is called the G_0 -Jordan class of x in V . Evidently the union of all G_0 -Jordan classes in V is a partition of V .

Remark 2.17. (1) By construction any G_0 -Jordan class is a G_0 -stable set consisting of G_0 -orbits of the same dimension. For example $\mathcal{S}_V^{\text{reg}}$ constitutes a G_0 -Jordan class, as it can be easily seen from Theorem 2.6 and Example 2.15.

- (2) The equality $\mathfrak{g}^{x_s} = \mathfrak{g}^{z+x_s}$ for any $z \in \mathfrak{z}(\mathfrak{g})_1$ and $x \in \mathfrak{g}_1$ implies that $z + x \stackrel{G_0}{\sim} x$, so the additive group underlying $\mathfrak{z}(\mathfrak{g})_1$ acts on each G_0 -Jordan class $J_{G_0}(x)$ by translations.
- (3) Since $G_0 = Z(G)_0^\circ S_0^\circ$, the element g from (2.7) can always be chosen in S_0° . Then, for $x = z + x' \in \mathfrak{z}(\mathfrak{g})_1 \oplus \mathfrak{s}_1$ and $y = w + y' \in \mathfrak{z}(\mathfrak{g})_1 \oplus \mathfrak{s}_1$, the statement $x \stackrel{G_0}{\sim} y$ holds if and only if $x' \stackrel{S_0^\circ}{\sim} y'$ holds and the decomposition of $V = \mathfrak{z}(\mathfrak{g})_1 \oplus \mathfrak{s}_1$ induces a decomposition

$$J_{G_0}(x) = J_{G_0}(x') = \mathfrak{z}(\mathfrak{g})_1 \times J_{S_0^\circ}(x') \quad (2.8)$$

where $J_{S_0^\circ}(x')$ is the S_0° -Jordan class of $x' \in \mathfrak{s}_1$ as introduced in [24].

- (4) Equality (2.8) applied to $x' \in \mathcal{N}_V \subset \mathfrak{s}_1$ gives $J_{G_0}(x') = \mathfrak{z}(\mathfrak{g})_1 \times \mathcal{O}_{x'}^{G_0} = \mathfrak{z}(\mathfrak{g})_1 \times \mathcal{O}_{x'}^{S_0^\circ}$. For $z \in \mathfrak{z}(\mathfrak{g})_1$ we then get $J_{G_0}(z) = J_{G_0}(0) = \mathfrak{z}(\mathfrak{g})_1$.

Observe that if $x = x_s + x_n \in V$, then x_n lies in the degree 1 component of the homogeneous semisimple subalgebra $[\mathfrak{g}^{x_s}, \mathfrak{g}^{x_s}]$.

Lemma 2.18. We have $\mathfrak{z}(\mathfrak{g}^x) = \mathfrak{z}(\mathfrak{g}^{x_s}) \oplus \mathfrak{z}(\mathfrak{g}^{x_n} \cap [\mathfrak{g}^{x_s}, \mathfrak{g}^{x_s}])$ and the components of an element in $\mathfrak{z}(\mathfrak{g}^x)$ with respect to this decomposition coincide with its semisimple and nilpotent parts, respectively. Thus, $\mathfrak{z}(\mathfrak{g}^x)_1 = \mathfrak{z}(\mathfrak{g}^{x_s})_1 \oplus \mathfrak{z}(\mathfrak{g}^{x_n} \cap [\mathfrak{g}^{x_s}, \mathfrak{g}^{x_s}])_1$.

Proof. The first claim is [29, Proposition 39.1.1], the second follows since $\mathfrak{z}(\mathfrak{g}^x)$ and its summands are homogeneous. \square

Lemma 2.2 tells us that

$$\begin{aligned} (\mathfrak{z}(\mathfrak{g}^x)_1)^{\text{reg}} &= \{y \in \mathfrak{z}(\mathfrak{g}^x)_1 \mid \mathfrak{g}^y = \mathfrak{g}^x\} \\ &= \{y \in \mathfrak{z}(\mathfrak{g}^x)_1 \mid \mathfrak{z}(\mathfrak{g}^y) = \mathfrak{z}(\mathfrak{g}^x)\} \\ &= \{y \in \mathfrak{z}(\mathfrak{g}^x)_1 \mid \text{rk}(\text{ad}_{\mathfrak{g}}(y)) = \text{rk}(\text{ad}_{\mathfrak{g}}(x))\}, \end{aligned} \quad (2.9)$$

which is a Zariski open subset of $\mathfrak{z}(\mathfrak{g}^x)_1$, hence irreducible. We note that this is also the set of all $y \in V$ such that $\mathfrak{g}^y = \mathfrak{g}^x$ and that $x \in (\mathfrak{z}(\mathfrak{g}^x)^{\text{reg}})_1$, so $(\mathfrak{z}(\mathfrak{g}^x)^{\text{reg}})_1 = (\mathfrak{z}(\mathfrak{g}^x)_1)^{\text{reg}}$ and we will omit the parentheses in the sequel.

The proof of the following result is as in [29, Lemma 39.1.2 & Proposition 39.1.5], once the last claim of Lemma 2.18 is taken into account. See also [24, Proposition 3.10].

Proposition 2.19. *Let $x = x_s + x_n \in V$. Then:*

- (i) *the decomposition in Lemma 2.18 induces a decomposition $\mathfrak{z}(\mathfrak{g}^x)_1^{\text{reg}} = \mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}} \times \mathfrak{z}(\mathfrak{g}^{x_n} \cap [\mathfrak{g}^{x_s}, \mathfrak{g}^{x_s}]_1^{\text{reg}, G^{x_s}})$, where "reg, G^{x_s} " is the regular part for the action of G^{x_s} ;*
- (ii) *the G_0 -Jordan class of x is the irreducible subset of V given by $J_{G_0}(x) = G_0 \cdot (\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}} + x_n)$.*

We will need the following results from [24] which readily generalize to the reductive case by virtue of (2.8).

Proposition 2.20. ([24, Proposition 3.9 and Proposition 3.17]). *Let $\{\mathfrak{g}, \theta\}$ be a reductive \mathbb{Z}_m -graded Lie algebra and $x, y \in V$. Then the following conditions are equivalent:*

- (i) $x \stackrel{G_0}{\sim} y$;
- (ii) *there exists $g \in G_0$ such that $\mathfrak{g}^y = g \cdot \mathfrak{g}^x$;*
- (iii) *there exists $g \in G_0$ such that $\mathfrak{z}(\mathfrak{g}^y) = g \cdot (\mathfrak{z}(\mathfrak{g}^x))$.*

Moreover the number of G_0 -Jordan classes in V is finite.

Corollary 2.21. *The G_0 -Jordan class of $x \in V$ coincides also with $J_{G_0}(x) = G_0 \cdot \mathfrak{z}(\mathfrak{g}^x)_1^{\text{reg}}$, it is locally closed in V (hence a subvariety of V) and $\dim J_{G_0}(x) = \dim \mathfrak{g}_0 - \dim \mathfrak{g}_0^x + \dim \mathfrak{z}(\mathfrak{g}^{x_s})_1$.*

Proof. The first two statements can be proved as in [29, Corollary 39.1.7], for the last one see [24, Proposition 3.13]. \square

It follows from Corollary 2.21 that any G_0 -Jordan class $J_{G_0}(x) = G_0 \cdot \mathfrak{z}(\mathfrak{g}^x)_1^{\text{reg}}$ is contained in the G -Jordan class $J_G(x) = G \cdot \mathfrak{z}(\mathfrak{g}^x)^{\text{reg}}$. However it is well-known that two elements $x, y \in V$ in the same G -Jordan class are not G_0 -Jordan equivalent in general (see [29, 38.7.18] for an example of nilpotent elements in V that are G -conjugate but not G_0 -conjugate). We conclude this subsection recalling the relationship between the sheets for the G_0 -action on V and the G_0 -Jordan classes.

Let H be a connected algebraic group acting on a variety X and let $d \in \mathbb{N}$. We set $X_{(d)} = \{x \in X \mid \dim \mathcal{O}_x^H = d\}$ and for any subset $A \subset X$ we set $A_{(d)} = A \cap X_{(d)}$. Each $X_{(d)}$ is locally closed and its irreducible components are called *sheets* for the H -action on X . We observe that $X_{(\leq d)} := \bigcup_{j \leq d} X_{(j)}$ is closed so $\overline{X_{(d)}} \subset X_{(\leq d)}$ [29, Proposition 21.4.4].

If $A \subset V$, and p is the largest integer with $A_{(p)} \neq \emptyset$ then, according to Definition 2.13, we have $A_{(p)} = A^\bullet$, which is a Zariski open subset of A . In particular, the set V^\bullet is a Zariski open subset of V , hence it is irreducible, and it is called the G_0 -regular sheet of V .

Proposition 2.22. ([24, Proposition 3.19]) *For any sheet S in V there exists a unique G_0 -Jordan class $J \subset S$ such that $S = \overline{J}^\bullet$. Moreover we have $\overline{S} = \overline{J}$.*

3. CLOSURE OF A G_0 -JORDAN CLASS

3.1. Closure of G_0 -Jordan classes: the semisimple parts. By virtue of Proposition 2.22, it is important to understand the closure and G_0 -regular closure of a G_0 -Jordan class and to see which classes are dense in a sheet. We start with a preliminary result and then describe which semisimple parts occur in the closure of a G_0 -Jordan class.

We first describe the image of the closure of a Jordan class through the Steinberg map $\varphi: V \rightarrow V//G_0 \cong \mathfrak{c}/W_{\text{Vin}}$.

Lemma 3.1. *Let $J = J_{G_0}(x)$ be a G_0 -Jordan class in V , with $x_s \in \mathfrak{c}$. Then, $\varphi(\bar{J}) = \frac{W_{\text{Vin}} \cdot \mathfrak{z}(\mathfrak{g}^{x_s})_1}{W_{\text{Vin}}}$. In particular, it is a closed subset of $\mathfrak{c}/W_{\text{Vin}}$, i.e., an affine variety.*

Proof. Observe that $\mathfrak{z}(\mathfrak{g}^{x_s})_1 \subset \mathfrak{c}$, cf. Corollary 2.8, so the expression for the image makes sense. Now, $\varphi(J) = \varphi(\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}})$ by Proposition 2.19, G_0 -equivariance and [31, Theorem 3], hence

$$\varphi(\bar{J}) \subset \overline{\varphi(J)} = \overline{\varphi(\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}})} = \frac{\overline{W_{\text{Vin}} \cdot \mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}}}}{W_{\text{Vin}}} = \frac{W_{\text{Vin}} \cdot \mathfrak{z}(\mathfrak{g}^{x_s})_1}{W_{\text{Vin}}}. \quad (3.1)$$

On the other hand, if $y_s \in \mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}}$, then $y = y_s + x_n \in \mathcal{O}_y^{G_0} \subset J$ and so $y_s \in \overline{\mathcal{O}_y^{G_0}} \subset \bar{J}$, [31, Proposition 4], giving $\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}} \subset \bar{J}$. It follows that

$$\mathfrak{z}(\mathfrak{g}^{x_s})_1 = \overline{\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}}} \subset \bar{J}, \quad (3.2)$$

hence $W_{\text{Vin}} \cdot \mathfrak{z}(\mathfrak{g}^{x_s})_1 \subset \bar{J}$ and

$$\frac{W_{\text{Vin}} \cdot \mathfrak{z}(\mathfrak{g}^{x_s})_1}{W_{\text{Vin}}} = \varphi(W_{\text{Vin}} \cdot \mathfrak{z}(\mathfrak{g}^{x_s})_1) \subset \varphi(\bar{J}).$$

□

Let $J = J_{G_0}(x) \subset V_{(d)}$ be a G_0 -Jordan class in V . Then its closure \bar{J} is a union of G_0 -orbits and if $\mathcal{O}_y^{G_0} \subset \bar{J}$, then $\overline{\mathcal{O}_y^{G_0}} \subset \bar{J}$. Let $\mathcal{M}_{\bar{J}}$ be the set of G_0 -orbits contained in \bar{J} which are maximal with respect to the partial order given by inclusion of orbit closures. By construction $\bar{J} = \bigcup_{\mathcal{O} \in \mathcal{M}_{\bar{J}}} \overline{\mathcal{O}}$.

The following proposition can be seen as a partial analogue of [4, §3.1].

Proposition 3.2. *Let $J = J_{G_0}(x)$ be a G_0 -Jordan class in V . Then $\bar{J}^\bullet = \bigcup_{\mathcal{O} \in \mathcal{M}_{\bar{J}}} \mathcal{O}$.*

Proof. We may assume without loss of generality that $x = x_s + x_n$ with $x_s \in \mathfrak{c}$. First of all $\bar{J} \subset \overline{V_{(d)}} \subset V_{(\leq d)}$, so $\dim \mathcal{O} \leq d$ for any $\mathcal{O} \in \mathcal{M}_{\bar{J}}$. We then consider the restriction $\psi = \varphi|_{\bar{J}}: \bar{J} \rightarrow \varphi(\bar{J})$ to \bar{J} of the Steinberg map φ .

Let $z \in \mathcal{O}$ for $\mathcal{O} \in \mathcal{M}_{\bar{J}}$. By [31, Theorem 4] the irreducible component of the fiber $\psi^{-1}\psi(z)$ containing z is the closure of a G_0 -orbit in \bar{J} , i.e., it is $\overline{\mathcal{O}}$. Since ψ is a dominant morphism of irreducible affine varieties, we may argue as in [31, Corollary 2] and the fibers of ψ are all of the same dimension, which is the maximum dimension of an orbit in \bar{J} , namely d . Hence $\dim \mathcal{O} = d$, $\mathcal{O} \subset \bar{J}^\bullet$ and

$$\bigcup_{\mathcal{O} \in \mathcal{M}_{\bar{J}}} \mathcal{O} \subset \bar{J}^\bullet.$$

The other inclusion follows because $\overline{\mathcal{O}} \setminus \mathcal{O}$ is always a union of G_0 -orbits of dimension $< d$. □

Lemma 3.3. *Let $J = J_{G_0}(x)$ be a G_0 -Jordan class and $y = y_s + y_n \in \bar{J}$. Then:*

- (i) $y_s \in \bar{J}$;
- (ii) y_s is G_0 -conjugate to an element of $\mathfrak{z}(\mathfrak{g}^{x_s})_1$;
- (iii) For any $y'_s \in \mathfrak{z}(\mathfrak{g}^{x_s})_1$ there exists a $y'_n \in \mathfrak{g}^{y'_s} \cap \mathcal{N}_V$ such that $y'_s + y'_n \in \bar{J}^\bullet$.
- (iv) If $z \in \mathfrak{z}(\mathfrak{g})_1$, then $z + y \in \bar{J}$, and in that case $z + y \in \bar{J}^\bullet$ if and only if $y \in \bar{J}^\bullet$.

Proof. Since \bar{J} is G_0 -stable, claim (i) follows from [31, Proposition 4] because $y_s \in \overline{\mathcal{O}_y^{G_0}} \subset \bar{J}$. We now turn to (ii). We may assume $y_s \in \mathfrak{c}$ by Theorem 2.6. Claim (ii) is then an immediate consequence of the following identity

$$\bar{J} \cap \mathfrak{c} = W_{\text{Vin}} \cdot \mathfrak{z}(\mathfrak{g}^{x_s})_1, \quad (3.3)$$

which we now establish.

First of all $W_{\text{Vin}} \cdot \mathfrak{z}(\mathfrak{g}^{x_s})_1 \subset \bar{J}$ by (3.2) and $W_{\text{Vin}} \cdot \mathfrak{z}(\mathfrak{g}^{x_s})_1 \subset \mathfrak{c}$ by Corollary 2.8, so one inclusion is clear. Conversely $\varphi(\bar{J} \cap \mathfrak{c}) \subset \varphi(\bar{J}) = \varphi(W_{\text{Vin}} \cdot \mathfrak{z}(\mathfrak{g}^{x_s})_1)$, by Lemma 3.1. It follows

that $\bar{J} \cap \mathfrak{c} \subset W_{\text{Vin}} \cdot \mathfrak{z}(\mathfrak{g}^{x_s})_1$, since the restriction of φ to \mathfrak{c} is just the natural projection to $\mathfrak{c}/W_{\text{Vin}}$ and both sets are W_{Vin} -stable.

We prove (iii). By Proposition 2.19 we have that $\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}} + x_n \subset J$, so $\mathfrak{z}(\mathfrak{g}^{x_s})_1 + x_n \subset \bar{J}$ and $y'_s + x_n \in \bar{J}$. Therefore the orbit $\mathcal{O}_{y'_s + x_n}^{G_0}$ is contained in the closure of an orbit \mathcal{O} in $\mathcal{M}_{\bar{J}}$. Since the fibers of the Steinberg map are closed and [31, Theorem 3] is in force, \mathcal{O} is represented by an element of the form $y'_s + y'_n$ for some $y'_n \in \mathfrak{g}^{y'_s} \cap \mathcal{N}_V$. Clearly $\mathcal{O} \subset \bar{J}^\bullet$ by Proposition 3.2.

Finally, (iv) follows from the action of $\mathfrak{z}(\mathfrak{g})_1$ on J , cf. Remark 2.17 (2). \square

Corollary 3.4. *The G_0 -regular closure \bar{J}^\bullet of a G_0 -Jordan class J contains at least a nilpotent G_0 -orbit.*

3.2. A local study of the closure of a G_0 -Jordan class. We start with a local characterization of the closure of a G_0 -Jordan class.

Lemma 3.5. *The following statements are equivalent for a G_0 -Jordan class J :*

- (i) \bar{J} is a union of G_0 -Jordan classes;
- (ii) For every $y \in \bar{J}$ there exists a Zariski open neighbourhood U_y of y in $J_{G_0}(y)$ such that $U_y \subset \bar{J}$.

Proof. The implication (i) \Rightarrow (ii) is immediate, since G_0 -Jordan classes are disjoint and we may take $U_y = J_{G_0}(y)$. Assume now that (ii) holds. Let $y \in \bar{J}$ and set $J' = J_{G_0}(y)$. Then $J' \cap \bar{J}$ is a non-empty closed subset of J' . On the other hand, condition (ii) implies that any point of $J' \cap \bar{J}$ has an open neighbourhood of J' therein, therefore $J' \cap \bar{J}$ is also open in J' . Since J' is a Zariski irreducible variety, it is connected in the Zariski topology, thus $J' \subset \bar{J}$ and (i) holds. \square

By virtue of Lemma 3.5 we shall apply a local approach and look at the closure of a G_0 -Jordan class in the neighbourhood of a point of V . For the rest of this subsection for any $y_s \in \mathcal{S}_V$ we will use the following notation: $\mathfrak{m} := \mathfrak{g}^{y_s}$; $M := G^{y_s} \leq G$; and $M_0 := M \cap G_0$ with identity component M_0° . For any subset $X \subset \mathfrak{m}_1$, we will write $X^{\text{reg}, M}$ to indicate the regular part of X for the action of M . We also recall that for any GIT quotient $\pi: X \rightarrow X//H$ of a reductive algebraic group H acting on a variety X , a subset U of X is called π -saturated or H -saturated if $U = \pi^{-1}\pi(U)$. Saturated implies H -stable, the converse is not necessarily true.

For \mathfrak{m} as above, we consider the M_0 -stable subset of \mathfrak{m}_1 defined as follows:

$$U_{\mathfrak{m}} = \{z \in \mathfrak{m}_1 \mid \mathfrak{g}^z \subset \mathfrak{m}\}.$$

Lemma 3.6. *With notations as above:*

- (i) $U_{\mathfrak{m}}$ is M_0 -saturated;
- (ii) $U_{\mathfrak{m}}$ is open in \mathfrak{m}_1 ;
- (iii) For all $z = z_s + z_n \in U_{\mathfrak{m}}$ we have $\mathfrak{z}(\mathfrak{g}^{z_s})_1^{\text{reg}} + z_n = (\mathfrak{z}(\mathfrak{m}^{z_s})_1^{\text{reg}, M} + z_n) \cap U_{\mathfrak{m}}$;
- (iv) For any G_0 -Jordan class J such that $J \cap U_{\mathfrak{m}} \neq \emptyset$, we have

$$J \cap U_{\mathfrak{m}} = \bigcup_{i \in I_J} J_{M,i} \cap U_{\mathfrak{m}}, \quad (3.4)$$

where $\{J_{M,i} \mid i \in I_J\}$ is the (finite) set of M_0° -Jordan classes in \mathfrak{m}_1 such that $J_{M,i} \cap U_{\mathfrak{m}} \cap J \neq \emptyset$. In addition, $\dim J_{M,i} = \dim J_{M,j}$ for any $i, j \in I_J$, so the closures of the $J_{M,i} \cap U_{\mathfrak{m}}$'s in (3.4) are the irreducible components of $J \cap U_{\mathfrak{m}}$;

- (v) Let $y_s \in \mathcal{S}_V$ and $y = y_s + y_n$ for $y_n \in \mathcal{N}_V \cap \mathfrak{m}$. Then

$$J_{G_0}(y) \cap U_{\mathfrak{m}} = \mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + \bigcup_{n_i \in \mathcal{N}_{G_0}(\mathfrak{m})/M_0^\circ} n_i \cdot \mathcal{O}_{y_n}^{M_0^\circ} \quad (3.5)$$

and the locally closed sets $\mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + n_i \cdot \mathcal{O}_{y_n}^{M_0^\circ}$ are the irreducible components of $J_{G_0}(y) \cap U_{\mathfrak{m}}$.

Proof. For $\mathfrak{m} = 2$, parts (i)-(ii) are [8, Lemma 2.1]. We propose a slightly different proof for (i). Saturation is equivalent to say that $\mathfrak{g}^z \subset \mathfrak{m}$ if and only if $\mathfrak{g}^{z_s} \subset \mathfrak{m}$, for any $z = z_s + z_n \in \mathfrak{m}_1$. As $\mathfrak{g}^z = \mathfrak{g}^{z_s} \cap \mathfrak{g}^{z_n}$, one implication is immediate. We will now show that $\mathfrak{g}^{z_s} \not\subset \mathfrak{g}^s$ implies $\mathfrak{g}^z \not\subset \mathfrak{g}^s$ for any semisimple element $s \in \mathfrak{g}$ and any $z \in \mathfrak{g}^s$, independently of the $\mathbb{Z}_{\mathfrak{m}}$ -grading.

Since z_s and s commute, we can always find a Cartan subalgebra \mathfrak{h} of \mathfrak{g} containing both. Then

$$\mathfrak{g}^s = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi(s)} \mathfrak{g}_\alpha \right), \quad \mathfrak{g}^{z_s} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi(z_s)} \mathfrak{g}_\alpha \right),$$

where $\Phi(\mathfrak{h})$ is the set of roots vanishing on an element $\mathfrak{h} \in \mathfrak{h}$. Since $(\Phi(s) + (\Phi \setminus \Phi(s))) \cap \Phi \subset \Phi \setminus \Phi(s)$, the reductive subalgebra $\mathfrak{g}^s \cap \mathfrak{g}^{z_s} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi(s) \cap \Phi(z_s)} \mathfrak{g}_\alpha \right)$ stabilizes the subspace $X = \bigoplus_{\alpha \in \Phi(z_s) \setminus \Phi(s)} \mathfrak{g}_\alpha$. As $z_n \in \mathfrak{g}^s \cap \mathfrak{g}^{z_s}$ acts nilpotently on X , there is a non-zero ξ in there such that $[z_n, \xi] = 0$. In other words $\xi \in \mathfrak{g}^z \setminus \mathfrak{g}^s$.

To prove (ii) we use the argument in [6, Lemma 2.1]. We may assume $y_s \in \mathfrak{c}$ and that \mathfrak{h} is a Cartan subalgebra of \mathfrak{g} , hence of \mathfrak{m} , containing \mathfrak{c} . The product $f = \prod_{\alpha \in \Phi \setminus \Phi(y_s)} \alpha$ is a homogeneous polynomial on \mathfrak{h} that is invariant for the Weyl group of \mathfrak{m} . By Chevalley's restriction theorem f extends to an M -invariant polynomial F on \mathfrak{m} . By (i), $U_m = \{z \in \mathfrak{m}_1 \mid \mathfrak{g}^{z_s} \subset \mathfrak{m}\}$, and it is not hard to verify that this is equal to $\{z \in \mathfrak{m}_1 \mid F(z) \neq 0\}$, hence it is open in \mathfrak{m}_1 .

Since U_m is M_0 -saturated, it is enough to prove (iii) for $z = z_s \in U_m$. We have $\mathfrak{g}^{z_s} = \mathfrak{m}^{z_s}$, so $\mathfrak{z}(\mathfrak{g}^{z_s})_1 = \mathfrak{z}(\mathfrak{m}^{z_s})_1$. If $x \in \mathfrak{z}(\mathfrak{g}^{z_s})_1^{\text{reg}}$ then $\mathfrak{g}^x = \mathfrak{g}^{z_s} = \mathfrak{m}^{z_s} \subset \mathfrak{m}$, so $x \in \mathfrak{z}(\mathfrak{m}^{z_s})_1^{\text{reg}, M} \cap U_m$. Conversely, if $x \in \mathfrak{z}(\mathfrak{m}^{z_s})_1^{\text{reg}, M} \cap U_m$, then $\mathfrak{g}^x \subset \mathfrak{m}$, so $\mathfrak{g}^x = \mathfrak{m}^x = \mathfrak{g}^{z_s}$ and $x \in \mathfrak{z}(\mathfrak{g}^{z_s})_1^{\text{reg}}$.

We prove (iv). Clearly $J \cap U_m \subset \bigcup_{i \in I_J} J_{M,i} \cap U_m$, and we now show the other inclusion. Let $z = z_s + z_n \in J \cap J_{M,i} \cap U_m$ for some $i \in I_J$, so $J = J_{G_0}(z)$ and $J_{M,i} = J_{M_0^o}(z)$. Combining the fact that U_m is M_0^o -stable with (iii) gives

$$\begin{aligned} J_{M,i} \cap U_m &= (M_0^o \cdot (\mathfrak{z}(\mathfrak{m}^{z_s})_1^{\text{reg}, M} + z_n)) \cap U_m = M_0^o \cdot ((\mathfrak{z}(\mathfrak{m}^{z_s})_1^{\text{reg}, M} + z_n) \cap U_m) \\ &= M_0^o \cdot (\mathfrak{z}(\mathfrak{g}^{z_s})_1^{\text{reg}} + z_n) \subset G_0 \cdot (\mathfrak{z}(\mathfrak{g}^{z_s})_1^{\text{reg}} + z_n) = J, \end{aligned}$$

establishing (3.4). Corollary 2.21 then gives

$$\dim J_{M,i} = \dim M_0^o - \dim \mathfrak{m}_0^z + \dim \mathfrak{z}(\mathfrak{m}^{z_s})_1 = \dim M_0^o - \dim \mathfrak{g}_0^z + \dim \mathfrak{z}(\mathfrak{g}^{z_s})_1,$$

which is independent of $i \in I_J$. Equation (3.4) expresses $J \cap U_m$ as the finite disjoint union of irreducible locally closed subsets of the same dimension, and the claim on the irreducible components follows directly.

Finally, we prove (v). By construction,

$$\begin{aligned} \mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + \bigcup_{n_i \in N_{G_0}(\mathfrak{m})/M_0^o} n_i \cdot \mathcal{O}_{y_n}^{M_0^o} &= U_m \cap (\mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + N_{G_0}(\mathfrak{m}) \cdot y_n) \\ &= U_m \cap (N_{G_0}(\mathfrak{m}) \cdot (\mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + y_n)) \subset U_m \cap J_{G_0}(y). \end{aligned}$$

Conversely, let $z \in J_{G_0}(y) \cap U_m$. Then, there is $g \in G_0$ such that $\mathfrak{g}^{z_s} = g \cdot \mathfrak{m}$ and $z_n = g \cdot y_n$. Saturation of U_m gives $\mathfrak{g}^{z_s} \subset \mathfrak{m}$, hence $g \cdot \mathfrak{m} \subset \mathfrak{m}$ and

$$z \in N_{G_0}(\mathfrak{m}) \cdot (\mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + y_n) = \mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + N_{G_0}(\mathfrak{m}) \cdot y_n = \mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + \bigcup_{n_i \in N_{G_0}(\mathfrak{m})/M_0^o} n_i \cdot \mathcal{O}_{y_n}^{M_0^o},$$

establishing (3.5). Last claim follows from (iv) once we prove that the sets $\mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + n_i \cdot \mathcal{O}_{y_n}^{M_0^o}$ are closed in $J_{G_0}(y) \cap U_m$. On the one hand

$$\overline{\mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + n_i \cdot \mathcal{O}_{y_n}^{M_0^o}}^{\overline{J_{G_0}(y) \cap U_m}} \subset \mathfrak{z}(\mathfrak{m})_1 + n_i \cdot \mathcal{O}_{y_n}^{M_0^o m_1} = \mathfrak{z}(\mathfrak{m})_1 + n_i \cdot \mathcal{O}_{y_n}^{M_0^o m_1}.$$

On the other hand, if $z \in \mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + n_i \cdot \mathcal{O}_{y_n}^{M_0^o}$, then (3.5) gives $z \in \mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + n_j \cdot \mathcal{O}_{y_n}^{M_0^o}$ for some j . Hence, $z \in (\mathfrak{z}(\mathfrak{m})_1 + n_j \cdot \mathcal{O}_{y_n}^{M_0^o m_1}) \cap (\mathfrak{z}(\mathfrak{m})_1 + n_i \cdot \mathcal{O}_{y_n}^{M_0^o m_1})$. Since the two nilpotent orbits have the same dimension, we necessarily have $i = j$, completing the proof. \square

Let H and L be reductive algebraic groups acting on an affine variety X , with $H \subset L$. Then H acts with trivial stabilizers on the product $L \times X$ via $h \cdot (l, x) = (lh^{-1}, h \cdot x)$: we set $L \times^H X := (L \times X)/H \cong (L \times X)//H$ and note that L acts on $L \times^H X$ by multiplication from the

left. The class of $(l, x) \in L \times X$ will be denoted by the symbol $l * x \in L \times^H X$. Since H acts on $L \times X$ with trivial stabilizer, the projection $L \times X \rightarrow L \times^H X$ is a principal H -bundle [16, III.1, Corollaire 1]. In other words, there is a surjective étale map $Y \rightarrow L \times^H X$ such that the base change of $L \times X \rightarrow L \times^H X$ through $Y \rightarrow L \times^H X$ is isomorphic to the projection $X \times Y \rightarrow Y$. Hence, A is an open subset of $L \times^H X$ if and only if its pre-image is an open H -stable subset of $L \times X$. In particular, $L \times^H A'$ is open in $L \times^H X$ for any open H -stable subset A' of X .

We consider the natural action maps

$$\tilde{\mu}: G \times \mathfrak{m} \rightarrow \mathfrak{g}, \quad \tilde{\mu}_0: G_0 \times \mathfrak{m}_1 \rightarrow V, \quad (3.6)$$

and the induced maps $\mu: G \times^{M_0} \mathfrak{m} \rightarrow \mathfrak{g}$ and $\mu_0: G_0 \times^{M_0} \mathfrak{m}_1 \rightarrow V$.

Lemma 3.7. *The maps $\tilde{\mu}, \tilde{\mu}_0$ are smooth at $(1, y_s)$ and the induced maps μ, μ_0 are étale at $1 * y_s$.*

Proof. The differential of the map $\tilde{\mu}$ at $(1, y_s)$ maps any element $(x', y') \in \mathfrak{g} \oplus \mathfrak{m}$ to $[x', y_s] + y'$, therefore it is surjective by (2.3) and since $[\mathfrak{g}, y_s] \cap \mathfrak{m} = 0$ due to the semisimplicity of y_s . The differential of the induced map μ at $1 * y_s$ is also surjective, hence bijective by dimensional reasons. The restriction of the differential of the map $\tilde{\mu}$ at $(1, y_s)$ to the degree 1 terms readily implies the surjectivity of the differential of $\tilde{\mu}_0$ at $(1, y_s)$, whence the differential of μ_0 at $1 * y_s$ is bijective. \square

We will also consider the GIT quotient maps

$$\pi_0: G_0 \times^{M_0} \mathfrak{m}_1 \rightarrow (G_0 \times^{M_0} \mathfrak{m}_1) // G_0, \quad \pi_{\mathfrak{m}_1}: \mathfrak{m}_1 \rightarrow \mathfrak{m}_1 // M_0$$

associated with multiplication from the left by G_0 , and the adjoint action of M_0 , respectively. Recall that we have natural identifications $(G_0 \times^{M_0} \mathfrak{m}_1) // G_0 \cong \mathfrak{m}_1 // M_0$. We will invoke a variant of Luna's étale slice Theorem [16] and its consequences to deduce properties of the closure of G_0 -Jordan classes.

Corollary 3.8. *There exists an affine open neighbourhood \mathcal{U} of y_s in \mathfrak{m}_1 , which is M_0 -saturated and such that the restriction of μ_0 to $G_0 \times^{M_0} \mathcal{U}$ is étale with Zariski open image $G_0 \cdot \mathcal{U}$ in V .*

Proof. By Lemma 3.7, the map μ_0 is étale at $1 * y_s$. The orbit $\mathcal{O}_{1 * y_s}^{G_0}$ is closed (since it is an orbit of minimal dimension $\dim G_0 - \dim M_0$) and so is the semisimple orbit $\mathcal{O}_{y_s}^{G_0}$, cf. [31, Prop. 3]. It is also not hard to verify that the restriction of μ_0 to $\mathcal{O}_{1 * y_s}^{G_0}$ is injective. By [16, Lemme Fondamental, §II.2] applied to $X = G_0 \times^{M_0} \mathfrak{m}_1$ and $Y = V$ there exists an affine π_0 -saturated open neighbourhood \mathcal{U}_X of $1 * y_s$ in $G_0 \times^{M_0} \mathfrak{m}_1$ such that the restriction of μ_0 to it is étale and the image is an affine open subset of V , saturated for the Steinberg map $\varphi: V \rightarrow V // G_0$. In fact \mathcal{U}_X is $G_0 \times^{M_0} \mathcal{U}$ where $\mathcal{U} = \pi_{\mathfrak{m}_1}^{-1} \pi_0(\mathcal{U}_X)$ is the desired M_0 -saturated affine open neighbourhood of y_s in \mathfrak{m}_1 . \square

Proposition 3.9. *Let \bar{J} be a G_0 -Jordan class in V and $y = y_s + y_n$, where $y_s \in \mathcal{S}_V$ and $y_n \in \mathcal{N}_V \cap \mathfrak{m}$. Then $y \in \bar{J}$ if and only if $y \in \bar{J}_{M, \mathfrak{l}}$ for some $\mathfrak{l} \in I_J$, where $\{\bar{J}_{M, \mathfrak{l}} \mid \mathfrak{l} \in I_J\}$ is as in Lemma 3.6 (iv).*

Proof. Consider the restriction of μ_0 from $G_0 \times^{M_0} \mathcal{U}$ to V determined in Corollary 3.8. Assume $y \in \bar{J}$, so $y_s \in \bar{J}$ by Lemma 3.3. We consider the commutative diagram

$$\begin{array}{ccc} G_0 \times^{M_0} (\bar{J} \cap \mathcal{U}) & \longrightarrow & G_0 \cdot \mathcal{U} \cap \bar{J} \\ j \downarrow & & \downarrow \\ G_0 \times^{M_0} \mathcal{U} & \xrightarrow{\mu_0} & G_0 \cdot \mathcal{U} \end{array}$$

We claim that it is a pull-back diagram. Indeed, the pull-back of μ_0 through the closed inclusion $\bar{J} \cap G_0 \cdot \mathcal{U} \rightarrow G_0 \cdot \mathcal{U}$ is the restriction $X := \mu_0^{-1}(\bar{J}) \rightarrow \bar{J} \cap G_0 \cdot \mathcal{U}$. Moreover, \bar{J} is a reduced subscheme and μ_0 is étale, so X is a G_0 -stable closed subscheme of $G_0 \times^{M_0} \mathcal{U}$. By [16, Lemma 3], there exists a closed M_0 -stable subscheme Y' of \mathcal{U} such that $X = G_0 \times^{M_0} Y'$ and the natural morphism $X \rightarrow G_0 \times^{M_0} \mathcal{U}$ is induced by the inclusion $Y' \subset \mathcal{U}$. Observe that $Y' = \mu_0(1 * Y') \subset \bar{J} \cap \mathcal{U}$, so the morphism $X \rightarrow G_0 \times^{M_0} \mathcal{U}$ factors through the morphism

$f: X \rightarrow G_0 \times^{M_0} (\bar{J} \cap \mathcal{U})$. Let $f': G_0 \times^{M_0} (\bar{J} \cap \mathcal{U}) \rightarrow X$ be the unique morphism coming from the universal property of the pull-back. Now $j \circ f \circ f' = j$, hence $f \circ f' = \text{id}$ whilst $f' \circ f = \text{id}$ follows from the universal property. Hence, the restriction of μ_0 to $G_0 \times^{M_0} (\bar{J} \cap \mathcal{U}) \rightarrow \bar{J}$ is again étale, whence open. As U_m is open in \mathfrak{m} , $U_m \cap \mathcal{U} \cap \bar{J}$ is open in $\mathcal{U} \cap \bar{J}$. Thus, $G_0 \times^{M_0} (U_m \cap \mathcal{U} \cap \bar{J}) \rightarrow \bar{J}$ is an open morphism, too.

If X is an irreducible component of $U_m \cap \mathcal{U} \cap \bar{J}$ containing y , then $G_0 \times^{M_0} X$ dominates the irreducible set \bar{J} , so $G_0 \cdot X \cap J \neq \emptyset$ because J is locally closed and thus $X \cap J \neq \emptyset$ by G_0 -stability. Note also that $X \cap J$ is open in X , hence it is irreducible and dense in X . Now

$$X \cap J \subset J \cap U_m = \bigcup_{i \in I_J} J_{M,i} \cap U_m$$

by Lemma 3.6 (iv), so $X \cap J \subset \overline{J_{M,l} \cap U_m}^{J \cap U_m} \subset \overline{J_{M,l}}$ for some $l \in I_J$ by irreducibility. Finally $\bar{X} = \overline{X \cap J} \subset \overline{J_{M,l}}$ by density and $y \in \overline{J_{M,l}}$ follows since $y \in X$.

Assume now $y \in \overline{J_{M,l}}$ for some $l \in I_J$. Then $\overline{J_{M,l} \cap U_m} = \overline{J_{M,l}}$ since $J_{M,l}$ is irreducible and U_m is open in \mathfrak{m}_1 , so $y \in \overline{J_{M,l}} = \overline{J_{M,l} \cap U_m} \subset \overline{J \cap U_m} \subset \bar{J}$. \square

Corollary 3.10. *Let $y_s \in \mathcal{S}_V$ and let $y = y_s + y_n$ for $y_n \in \mathcal{N}_V \cap \mathfrak{m}$. Then there exists an M_0° -stable open neighbourhood U_1 of y in \mathfrak{m}_1 such that $M_0 \cdot U_1 \subset U_m$ and*

$$J_{G_0}(y) \cap U_1 = J_{M_0^\circ}(y) \cap U_1. \quad (3.7)$$

Proof. Lemma 3.6 (v) gives

$$J_{G_0}(y) \cap U_m = (\mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + \bigcup_{n_i \in \mathcal{N}_{G_0}(\mathfrak{m})/M_0^\circ} n_i \cdot \mathcal{O}_{y_n}^{M_0^\circ}) \cap U_m = (\mathfrak{z}(\mathfrak{m})_1 + \bigcup_{n_i \in \mathcal{N}_{G_0}(\mathfrak{m})/M_0^\circ} n_i \cdot \mathcal{O}_{y_n}^{M_0^\circ}) \cap U_m,$$

since $\mathfrak{z}(\mathfrak{m})_1^{\text{reg}, M} = \mathfrak{z}(\mathfrak{m})_1$. The subsets $(\mathfrak{z}(\mathfrak{m})_1 + n_i \cdot \mathcal{O}_{y_n}^{M_0^\circ}) \cap U_m$ are the irreducible components of $J_{G_0}(y) \cap U_m$ by Lemma 3.6 (v). They are disjoint, whence open, and M_0° -stable. Hence, there exists an M_0° -stable open subset \mathcal{U}_y in \mathfrak{m}_1 such that $J_{G_0}(y) \cap U_m \cap \mathcal{U}_y = (\mathfrak{z}(\mathfrak{m})_1 + \mathcal{O}_{y_n}^{M_0^\circ}) \cap U_m$. Then $U_1 := \mathcal{U}_y \cap U_m$ is an M_0° -stable Zariski open neighbourhood of y in \mathfrak{m}_1 satisfying

$$J_{G_0}(y) \cap U_1 = (\mathfrak{z}(\mathfrak{m})_1 + \mathcal{O}_{y_n}^{M_0^\circ}) \cap U_1 = J_{M_0^\circ}(y) \cap U_1.$$

Finally $M_0 \cdot U_1 \subset M_0 \cdot U_m = U_m$ since U_m is M_0 -stable. \square

Theorem 3.11. *Let J be a G_0 -Jordan class in V . Then \bar{J} is a union of G_0 -Jordan classes and it is decomposable, i.e., it contains the semisimple and nilpotent components of all its elements.*

Proof. We will show that condition (ii) in Lemma 3.5 is satisfied for any $y = y_s + y_n \in \bar{J}$.

Let U_1 be as in Corollary 3.10 and \mathcal{U} as in Corollary 3.8. By saturation of \mathcal{U} we have $y \in U_1 \cap \mathcal{U}$. We consider the M_0 -stable open subset $U'_1 = M_0 \cdot U_1 \subset U_m$ of \mathfrak{m}_1 and apply M_0 to both sides of (3.7) to get

$$J_{G_0}(y) \cap U'_1 = M_0 \cdot (J_{G_0}(y) \cap U_1) = M_0 \cdot (J_{M_0^\circ}(y) \cap U_1) \subset (M_0 \cdot J_{M_0^\circ}(y)) \cap U'_1. \quad (3.8)$$

We then set $\mathcal{V} = U'_1 \cap \mathcal{U}$ and obtain an M_0 -stable open neighbourhood of y in \mathfrak{m}_1 . Corollary 3.8 then guarantees that $G_0 \cdot \mathcal{V}$ is an open neighbourhood of y in V . We will show that $\mathcal{U}_y := J_{G_0}(y) \cap G_0 \cdot \mathcal{V}$ is the sought neighbourhood of y in $J_{G_0}(y)$ contained in \bar{J} .

Proposition 3.9 ensures that $y \in \overline{J_{M,l}} \cap \mathcal{V}$ for some $l \in I_J$. Now $y_s \in \mathfrak{z}(\mathfrak{m})_1$, so combining (3.8), Remark 2.17 (2) and Lemma 3.3 (iv) we get

$$\begin{aligned} J_{G_0}(y) \cap \mathcal{V} &\subset (M_0 \cdot J_{M_0^\circ}(y)) \cap \mathcal{V} = \left(M_0 \cdot (\mathfrak{z}(\mathfrak{m})_1 + \mathcal{O}_{y_n}^{M_0^\circ}) \right) \cap \mathcal{V} \\ &\subset \left(M_0 \cdot \overline{J_{M,l}} \right) \cap \mathcal{V} = M_0 \cdot \left(\overline{J_{M,l}} \cap \mathcal{V} \right) \subset \bar{J} \cap \mathcal{V}, \end{aligned}$$

where for the last inclusion we invoke Proposition 3.9 once more. We finally arrive at

$$J_{G_0}(y) \cap G_0 \cdot \mathcal{V} = G_0 \cdot (J_{G_0}(y) \cap \mathcal{V}) \subset G_0 \cdot (\bar{J} \cap \mathcal{V}) \subset \bar{J},$$

This proves that \bar{J} is the union of G_0 -Jordan classes.

We finally prove that \bar{J} is decomposable. Let $y = y_s + y_n \in \bar{J}$ and $J_{G_0}(y)$ the corresponding G_0 -Jordan class. Then $y_s \in \bar{J}$ by Lemma 3.3 (i) and

$$y_n \in \mathfrak{z}(\mathfrak{m})_1 + y_n = \overline{\mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + y_n} \subset \overline{J_{G_0}(y)} \subset \bar{J},$$

where we used our previous result $J_{G_0}(y) \subset \bar{J}$. \square

Theorem 3.12. *Let J be a G_0 -Jordan class and let S be a sheet in V . Then \bar{J}^\bullet , \bar{J}^{reg} and S are unions of G_0 -Jordan classes.*

Proof. By Theorem 3.11, the closure \bar{J} is a union of G_0 -Jordan classes. Since all such classes are of constant G - and G_0 -orbit dimension, it follows that also \bar{J}^\bullet and \bar{J}^{reg} are unions of G_0 -Jordan classes. The statement for S is a direct consequence of Proposition 2.22. \square

We conclude this subsection with the following important consequence of the local study of the closure of a G_0 -Jordan class.

Proposition 3.13. *G_0 -Jordan classes are smooth.*

Proof. Let $J = J_{G_0}(y)$ be a G_0 -Jordan class in V and $\mathfrak{m} = \mathfrak{g}^{y_s}$. We will show that y has a smooth Zariski open neighbourhood in J . Let $U_{\mathfrak{m}}$ and \mathcal{U} be the open neighbourhoods of y_s and y in \mathfrak{m}_1 as in Lemma 3.6 and Corollary 3.8, respectively.

By Lemma 3.6 (v), the intersection $J \cap U_{\mathfrak{m}}$ is smooth, therefore $J \cap U_{\mathfrak{m}} \cap \mathcal{U}$ is non-empty and smooth as well. Recall that $p: G_0 \times \bar{J} \rightarrow G_0 \times^{M_0} \bar{J}$ is a principal M_0 -bundle, so there is a surjective étale map $f: Y \rightarrow G_0 \times^{M_0} \bar{J}$ such that the base change $X \rightarrow Y$ of $G_0 \times \bar{J} \rightarrow G_0 \times^{M_0} \bar{J}$ is isomorphic to the projection $\tilde{p}: M_0 \times Y \rightarrow Y$. Being the base change of an étale map, the induced morphism $\tilde{f}: M_0 \times Y \rightarrow G_0 \times \bar{J}$ is again so. By [1, Éxp 1, Corollaire 9.2], $G_0 \times (J \cap U_{\mathfrak{m}} \cap \mathcal{U})$ is smooth if and only if $\tilde{f}\tilde{p}^{-1}f^{-1}(G_0 \times^{M_0} (J \cap U_{\mathfrak{m}} \cap \mathcal{U})) = p^{-1}(G_0 \times^{M_0} (J \cap U_{\mathfrak{m}} \cap \mathcal{U}))$ is so. One may verify that the scheme-theoretic fiber of $G_0 \times^{M_0} (J \cap U_{\mathfrak{m}} \cap \mathcal{U})$ through p is $G_0 \times (J \cap U_{\mathfrak{m}} \cap \mathcal{U})$ hence $G_0 \times^{M_0} (J \cap U_{\mathfrak{m}} \cap \mathcal{U})$ is smooth. Invoking again [1, Éxp 1, Corollaire 9.2] we conclude that $\mu_0(G_0 \times^{M_0} (J \cap U_{\mathfrak{m}} \cap \mathcal{U}))$ is smooth and it is a smooth open neighbourhood of y in J . \square

3.3. Regularity questions. Let $J = J_{G_0}(x_s + x_n)$ be a G_0 -Jordan class. Then $\bar{J}^{\text{reg}} \subset \bar{J}^\bullet$ since J is irreducible, hence \bar{J} too, and Lemma 2.14 is in force. Note that $\bar{J}^\bullet = \bar{J}^{\text{reg}}$ whenever $x_s = 0$, because $J = \mathfrak{z}(\mathfrak{g})_1 \times \mathcal{O}_{x_n}^{G_0}$ and orbits are locally closed, so $J = \bar{J}^\bullet = \bar{J}^{\text{reg}}$. The equality $\bar{J}^\bullet = \bar{J}^{\text{reg}}$ is always satisfied in the symmetric case $m = 2$ due to Corollary 2.11 and one may wonder if $\bar{J}^\bullet = \bar{J}^{\text{reg}}$ also for $m \geq 3$, by combining Theorem 3.12 and the fact that G_0 -Jordan classes are defined in terms of regular parts for the action of G , cf. Corollary 2.21.

However, this is not the case. A reason is that open G_0 -orbits \mathcal{O}^{G_0} in irreducible components of the fibers of the Steinberg map $\varphi: V \rightarrow V//G_0 \cong \mathfrak{c}/W_{V_{\text{in}}}$ do *not* give rise in general to open G -orbits $G \cdot \mathcal{O}^{G_0}$ in the irreducible components of the Steinberg map $p: \mathfrak{g} \rightarrow \mathfrak{g}/G \cong \mathfrak{h}/W$. To make this more precise, we need some notions and results from [21, 22] and, for simplicity of exposition, we restrict to the case where \mathfrak{g} is semisimple.

Definition 3.14. A complex semisimple \mathbb{Z}_m -graded Lie algebra $\{\mathfrak{g}, \theta, m\}$, the corresponding grading, and the automorphism θ are called:

- (i) *S-regular* if $\mathcal{S}_V \cap \mathfrak{g}^{\text{reg}} \neq \emptyset$;
- (ii) *N-regular* if $\mathcal{N}_V \cap \mathfrak{g}^{\text{reg}} \neq \emptyset$;
- (iii) *very N-regular* if each irreducible component of \mathcal{N}_V intersects $\mathfrak{g}^{\text{reg}}$ non-trivially.

Clearly (iii) implies (ii). It is an important result of L. V. Antonyan and D. I. Panyushev in [21] that if a connected component of $\text{Aut}(\mathfrak{g})$ contains automorphisms of order m , then it contains a unique *N-regular* automorphism of that order (up to conjugation by the group of inner automorphisms of \mathfrak{g}). Moreover, as mentioned in the introduction of [21], the condition of *S-regularity* is equivalent to *N-regularity* in the symmetric case $m = 2$, but for $m \geq 3$ neither of these properties implies the other. An example of *S-regular* grading that is not

\mathcal{N} -regular is given in [21, Example 4.5]. Here \mathfrak{g} is of type E_6 with the inner automorphism of order $m = 4$ described by the Kac diagram



This is the affine Dynkin diagram of \mathfrak{g} of type E_6 , where the white and black nodes correspond to roots subspaces of degree 0 and 1, respectively. The semisimple part of \mathfrak{g}_0 is given by the subdiagram consisting of white nodes and the dimension of the centre of \mathfrak{g}_0 is the number of black nodes minus 1. We have $G_0 \cong SL(4) \times SL(2) \times (\mathbb{C}^\times)^2$ up to local isomorphism, acting on $V = \mathfrak{g}_1 \cong \mathbb{C}^4 \oplus (\mathbb{C}^4)^* \oplus (\wedge^2 \mathbb{C}^4 \boxtimes \mathbb{C}^2)$. The reader is referred to e.g. [33, Chapter 3, §3] for a detailed treatment of periodic automorphisms and their associated Kac diagrams.

Now G_0 -Jordan classes form a finite partition of V , which is irreducible, so there is one class J that is open in V . We call it the G_0 -regular Jordan class of V and note that it is the unique G_0 -Jordan class that is dense in the G_0 -regular sheet $S = V^\bullet$ of V . (See Example 2.16 for an explicit description of representatives of the G_0 -orbits in the G_0 -regular Jordan class.) Since the grading (3.9) is \mathcal{S} -regular, we have $\bar{J}^{reg} = V^{reg} = \mathfrak{g}^{reg} \cap V$ in this case. Let \mathcal{O}^{G_0} be the nilpotent G_0 -orbit that is open in one of the irreducible components of \mathcal{N}_V . We have $\mathcal{O}^{G_0} \subset \bar{J}^\bullet = V^\bullet$ by [31, Corollaries 1 and 2], but $\mathcal{O}^{G_0} \not\subset \bar{J}^{reg}$ since the grading is not \mathcal{N} -regular.

The cone \mathcal{N}_V is often reducible and a larger class of examples for which $\bar{J}^{reg} \neq \bar{J}^\bullet$ comes from \mathcal{N} -regular gradings that are not very \mathcal{N} -regular: the G_0 -regular Jordan class J satisfies $\bar{J}^{reg} = \mathfrak{g}^{reg} \cap V$ and, by an argument as above, there is a nilpotent G_0 -orbit contained in \bar{J}^\bullet but not in \bar{J}^{reg} . Exceptional \mathcal{N} -regular gradings whose nodes are not all black are classified in [9], and very \mathcal{N} -regular gradings appear to occur very rarely. Inner exceptional gradings with all nodes black are \mathcal{N} -regular but not very \mathcal{N} -regular [21, Example 4.4] and the same is true for the outer grading of E_6 with all nodes black (W. A. de Graaf, 05-05-2020, personal communication). The following result is a consequence of these observations, and the tables are a specialization of Tables 2-7 of [9].

Proposition 3.15. *Let $\{\mathfrak{g}, \theta, m\}$ be an exceptional complex simple \mathbb{Z}_m -graded Lie algebra, $m \geq 3$. Then $\{\mathfrak{g}, \theta, m\}$ is \mathcal{N} -regular but not very \mathcal{N} -regular if and only if the associated Kac diagram has all the nodes black or is one in the following tables. In all these cases we have that $\bar{J}^{reg} \subset \bar{J}^\bullet$ properly, where J is the G_0 -regular Jordan class of V .*

TABLE 1. \mathcal{N} -regular but not very \mathcal{N} -regular automorphisms of G_2 .

m	Kac diagram	# orbits in \mathcal{N}_V	# components of \mathcal{N}_V	$\dim \mathcal{N}_V$	$\dim \mathfrak{c}$
3		6	2	4	1

TABLE 2. \mathcal{N} -regular but not very \mathcal{N} -regular automorphisms of F_4 .

m	Kac diagram	# orbits in \mathcal{N}_V	# components of \mathcal{N}_V	$\dim \mathcal{N}_V$	$\dim \mathfrak{c}$
4		29	3	12	2
6		35	6	8	2
8		30	4	6	1

TABLE 3. \mathcal{N} -regular but not very \mathcal{N} -regular inner automorphisms of E_6 .

m	Kac diagram	# orbits in \mathcal{N}_V	# components of \mathcal{N}_V	$\dim \mathcal{N}_V$	$\dim \mathfrak{c}$
-----	-------------	-----------------------------	---------------------------------	----------------------	---------------------

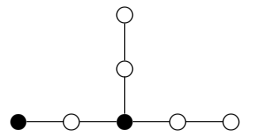
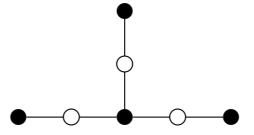
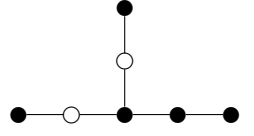
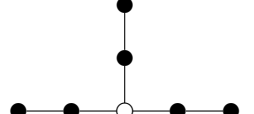
<i>N</i> -regular but not very <i>N</i> -regular inner automorphisms of E_6 .					
4		43	3	18	2
6		133	9	12	2
8		70	4	9	1
9		118	6	8	1

TABLE 4. *N*-regular but not very *N*-regular outer automorphisms of E_6 .

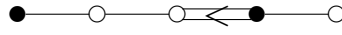
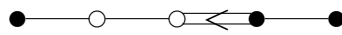
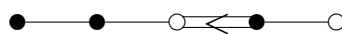
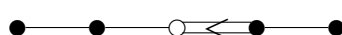
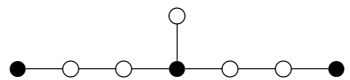
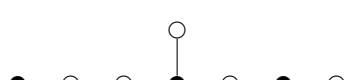
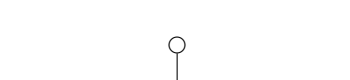
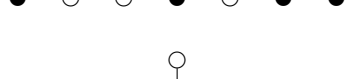
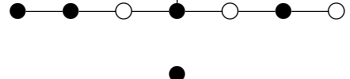
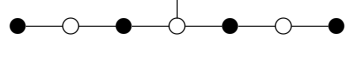
<i>m</i>	<i>Kac diagram</i>	# orbits in \mathcal{N}_V	# components of \mathcal{N}_V	$\dim \mathcal{N}_V$	$\dim \mathfrak{c}$
6		34	5	12	3
8		22	3	9	1
10		25	2	8	1
12		30	4	6	1

TABLE 5. *N*-regular but not very *N*-regular automorphisms of E_7 .

<i>m</i>	<i>Kac diagram</i>	# orbits in \mathcal{N}_V	# components of \mathcal{N}_V	$\dim \mathcal{N}_V$	$\dim \mathfrak{c}$
6		233	10	21	3
7		112	3	18	1
8		163	2	17	1
9		132	4	14	1
10		199	4	13	1
12		217	5	11	1

\mathcal{N} -regular but not very \mathcal{N} -regular automorphisms of E_7 .					
14		238	7	9	1

TABLE 6. \mathcal{N} -regular but not very \mathcal{N} -regular automorphisms of E_8 .

m	<i>Kac diagram</i>	# orbits in \mathcal{N}_V	# components of \mathcal{N}_V	$\dim \mathcal{N}_V$	$\dim \mathfrak{c}$
4		144	2	60	4
6		270	7	40	4
8		219	2	30	2
9		206	2	28	1
10		300	7	24	2
12		398	10	20	2
14		333	4	18	1
15		354	5	16	1
18		397	5	14	1
20		438	7	12	1
24		478	8	10	1

4. SLICE-INDUCTION AND PARAMETRIZATION OF ORBITS AND CLASSES

4.1. **Slice-induction.** Theorem 3.11 shows that the closure of a G_0 -Jordan class in V is a union of G_0 -Jordan classes, generalising results of [5, 8]. We aim at detecting which G_0 -Jordan classes lie in the closure of a given one. In the classical case, this can be described in terms of Lusztig-Spaltenstein’s parabolic induction of adjoint orbits, [17, 4]. Slice induction is introduced in [8] to deal with the $m = 2$ case, since orbit induction no longer works. We

will briefly show how to extend to the case of general m the construction in [8], by combining some of its general arguments with our local approach.

Let \mathfrak{m} be now a θ -stable reductive subalgebra of \mathfrak{g} and M the connected subgroup of G with $\text{Lie}(M) = \mathfrak{m}$. For a nilpotent element $e \in \mathfrak{m}_1$ we consider a graded $\mathfrak{sl}(2)$ -triple $\{e, h, f\}$ in \mathfrak{m} , so that $h \in \mathfrak{m}_0$ and $f \in \mathfrak{m}_{-1}$, and the corresponding Slodowy slice $S_{\mathfrak{m},e} = e + \mathfrak{m}^f \subset \mathfrak{m}$. Since \mathfrak{m}^f is homogeneous, we can consider its intersection with V , obtaining $S_{\mathfrak{m}_1,e} = e + \mathfrak{m}_1^f \subset \mathfrak{m}_1$. If $e = 0$, we consider the trivial triple as an $\mathfrak{sl}(2)$ -triple, so $S_{\mathfrak{m}_1,0} = \mathfrak{m}_1$. We start with two preliminary results in the case $\mathfrak{m} = \mathfrak{g}$.

Lemma 4.1. *Let $\{e, h, f\}$ be a graded $\mathfrak{sl}(2)$ -triple in \mathfrak{g} and let $X \subset V$ be an irreducible locally closed G_0 -stable subset such that $X \cap S_{\mathfrak{g}_1,e} \neq \emptyset$. Then the action morphism $\psi: G_0 \times S_{\mathfrak{g}_1,e} \rightarrow V$ is smooth, its restriction $\psi_X: G_0 \times (S_{\mathfrak{g}_1,e} \cap X) \rightarrow X$ is smooth and dominant, more precisely $\psi_X(G_0 \times C)$ is dense in X for any irreducible component C of $S_{\mathfrak{g}_1,e} \cap X$.*

Proof. For $m = 2$, this is part of [8, Proposition 2.4 (i)], we record the proof for completeness. The action morphism ψ is G_0 -equivariant with smooth domain and codomain, hence it suffices to verify that the differential is surjective at any point of the form $(1, y) \in G_0 \times S_{\mathfrak{g}_1,e}$. We note that

$$\begin{aligned} d\psi|_{(1,y)} : \mathfrak{g}_0 \times \mathfrak{g}_1^f &\rightarrow V \\ (x, z) &\rightarrow [x, y] + z \end{aligned}$$

and by $\mathfrak{sl}(2)$ -representation theory $\mathfrak{g} = [\mathfrak{g}, e] \oplus \mathfrak{g}^f$, which in degree 1 becomes $V = [\mathfrak{g}_0, e] \oplus \mathfrak{g}_1^f$, so the differential at $(1, e)$ is surjective. The contracting \mathbb{C}^* -action argument in [27, 7.4, Corollary 1] carries over to the \mathbb{Z}_m -graded case because $\{e, h, f\}$ is a graded $\mathfrak{sl}(2)$ -triple and $h \in \mathfrak{m}_0$, so ψ is smooth at any point $(1, y)$, hence everywhere.

By [27, III.5, Lemma 2] applied to the G_0 -equivariant morphism given by the inclusion of X in V , the restriction ψ_X is again smooth, whence open. Since X is irreducible, the image of any irreducible component of $G_0 \times (S_{\mathfrak{g}_1,e} \cap X)$ is dense in X . \square

Lemma 4.2. *Let J be a G_0 -Jordan class in V and $e \in \mathcal{N}_V$. Then $e \in \bar{J}$ if and only if $J \cap S_{\mathfrak{g}_1,e} \neq \emptyset$ and only if $\bar{J} \cap S_{\mathfrak{g}_1,e} \neq \emptyset$.*

Proof. We note that J is a locally closed G_0 -stable cone by Proposition 2.20 and Corollary 2.21, so when $m = 2$ these are the equivalences (i) = (iv) = (v) in [8, Theorem 2.6]. The proof of [8, Lemma 2.3] shows the existence of a contracting \mathbb{C}^* -action on $S_{\mathfrak{g}_1,e}$ and it carries over to the $m > 2$ case. If $J \cap S_{\mathfrak{g}_1,e} \neq \emptyset$ then each irreducible component of $\bar{J} \cap S_{\mathfrak{g}_1,e}$ is stable under the \mathbb{C}^* -action, so e lies in each of them. As a consequence, $e \in \bar{J}$.

Clearly $e \in \bar{J}$ gives $\bar{J} \cap S_{\mathfrak{g}_1,e} \neq \emptyset$, so it remains to show that $\bar{J} \cap S_{\mathfrak{g}_1,e} \neq \emptyset$ implies $J \cap S_{\mathfrak{g}_1,e} \neq \emptyset$. We follow the proof of [8, Proposition 2.5], establishing that $J \cap S_{\mathfrak{g}_1,e}$ is dense in $\bar{J} \cap S_{\mathfrak{g}_1,e}$.

Since J is open in \bar{J} , the subset $J \cap S_{\mathfrak{g}_1,e}$ is open in $\bar{J} \cap S_{\mathfrak{g}_1,e}$ and therefore it is enough to prove that it meets every irreducible component C of $\bar{J} \cap S_{\mathfrak{g}_1,e}$. The latter follows then from the density of $G_0 \cdot C$ in \bar{J} , guaranteed by Lemma 4.1 applied to $X = \bar{J}$. \square

For the rest of the section we resume the convention from Subsection 3.2 that whenever we have $y_s \in \mathcal{S}_V$, then $\mathfrak{m} := \mathfrak{g}^{y_s}$, and M_0° is the identity component of $M_0 := G_0^{y_s} = G_0 \cap G^{y_s}$.

Theorem 4.3. *Let J_1, J_2 be G_0 -Jordan classes in V . Then the following conditions are equivalent:*

- (i) $J_2 \subset \bar{J}_1$;
- (ii) $J_2 \cap \bar{J}_1 \neq \emptyset$;
- (iii) There exist $x \in J_1, y \in J_2$ such that $\mathfrak{g}^{x_s} \subset \mathfrak{m}$ and $J_{M_0^\circ}(x) \cap S_{\mathfrak{m}_1, y_s} \neq \emptyset$;
- (iv) There exist $x \in J_1, y \in J_2$ such that $\mathfrak{g}^{x_s} \subset \mathfrak{m}$ and $y \in \overline{J_{M_0^\circ}(x)}$.

Proof. This is the generalization of [8, Theorem 3.5] to the $m > 2$ case, but our proof is slightly different and it combines Lemma 4.2 and Lemma 3.6 with our local approach.

The equivalence (i) \Leftrightarrow (ii) is immediate from Theorem 3.11. We prove the other ones.

Claim (iii) \Leftrightarrow (iv). Lemma 4.2 applied to \mathfrak{m} , y_n and $J_{M_0^\circ}(x)$ says that $J_{M_0^\circ}(x) \cap S_{m_1, y_n} \neq \emptyset$ if and only if $y_n \in \overline{J_{M_0^\circ}(x)}$. Since $y_s \in \mathfrak{z}(\mathfrak{m})_1$, the latter condition is equivalent to $y \in \overline{J_{M_0^\circ}(x)}$ by Lemma 3.3 (iv).

Claim (iv) \Rightarrow (ii). Let x, y be as in (iv).

Since $\mathfrak{g}^x \subset \mathfrak{g}^{x_s}$, we have $x \in \mathfrak{U}_m$ and hence $J_{G_0}(x) \cap \mathfrak{U}_m \cap J_{M_0^\circ}(x) \neq \emptyset$. Lemma 3.6 (iv) gives

$$J_{G_0}(x) \cap \mathfrak{U}_m = \bigcup_{i \in I_{J_1}} J_{M, i} \cap \mathfrak{U}_m \quad (4.1)$$

and $J_{M_0^\circ}(x)$ is, by construction, one of the M_0° -Jordan classes occurring in the R.H.S. Applying Proposition 3.9 to $J = J_{G_0}(x)$ and $y \in \overline{J_{M_0^\circ}(x)}$, we see that $y \in \overline{J_{G_0}(x)}$. This shows $y \in J_2 \cap \overline{J_1} \neq \emptyset$, proving (ii).

Claim (ii) \Rightarrow (iv). Assume now $y \in J_2 \cap \overline{J_1}$. By Proposition 3.9 the element y lies in $\overline{J_{M, l}}$ where $J_{M, l}$ is an M_0° -Jordan class such that $\emptyset \neq J_{M, l} \cap \mathfrak{U}_m \subset J_1 \cap \mathfrak{U}_m$. Then, for any $x \in J_{M, l} \cap \mathfrak{U}_m$ the pair (x, y) satisfies (iv). \square

Comparing dimensions of orbits in J_1 and J_2 we readily get:

Corollary 4.4. *Let J_1, J_2 be G_0 -Jordan classes in V . Then $J_2 \subset \overline{J_1}^\bullet$ if and only if there exist $x \in J_1, y \in J_2$ such that $\mathfrak{g}^{x_s} \subset \mathfrak{m}, J_{M_0^\circ}(x) \cap S_{m_1, y_n} \neq \emptyset$ and $\dim \mathcal{O}_x^{M_0^\circ} = \dim \mathcal{O}_{y_n}^{M_0^\circ}$.*

Remark 4.5. Condition (iii) from Theorem 4.3 is called weak slice-induction in [8]. If J_2 is weakly slice-induced from J_1 and satisfies the dimension condition in Corollary 4.4, then it is called slice-induced from J_1 . Slice-induction is shown to coincide with parabolic induction in the ungraded case $m = 1$ in [8, Corollary 3.7].

Corollary 4.6. *A G_0 -Jordan class $J = J_{G_0}(y)$ contained in $V_{(d)}$ is dense in a sheet if and only if $J_{M_0^\circ}(x) \cap S_{m_1, y_n} = \emptyset$ for any $x \in V_{(d)} \setminus J$ such that $\mathfrak{g}^{x_s} \subset \mathfrak{m}$.*

Proof. First of all, the irreducible subset J is contained in some sheet S in $V_{(d)}$ and there is a unique G_0 -Jordan class $J' \subset V_{(d)}$ such that $S = \overline{J'}^\bullet$ by Proposition 2.22. The condition $J_{M_0^\circ}(x) \cap S_{m_1, y_n} = \emptyset$ for any $x \in V_{(d)} \setminus J$ such that $\mathfrak{g}^{x_s} \subset \mathfrak{m}$ is equivalent to say that there are no G_0 -Jordan classes $J \neq J'$ such that $J \subset \overline{J'}^\bullet$, in other words, that $J = J'$. \square

4.2. Parametrization of orbits and classes. We aim at a parametrization of the G_0 -orbits contained in a G_0 -Jordan class $J_{G_0}(x) = G_0 \cdot (\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}} + x_n)$. By Theorem 2.6, we may assume that $x = x_s + x_n \in V$ with $x_s \in \mathfrak{c}$, so Corollary 2.8 ensures that $\mathfrak{z}(\mathfrak{g}^{x_s})_1 \subset \mathfrak{c}$. Let

$$\Gamma := N_{W_{V_{\text{in}}}}(\mathfrak{z}(\mathfrak{g}^{x_s})_1),$$

the stabilizer of $\mathfrak{z}(\mathfrak{g}^{x_s})_1$ in $W_{V_{\text{in}}}$.

Remark 4.7. (1) Observe that $x_s \in \mathfrak{c}$ implies $Z_{G_0}(\mathfrak{c}) \subset G_0^{x_s} \subset N_{G_0}(\mathfrak{g}^{x_s})$. Corollary 2.3 gives also $N_{G_0}(\mathfrak{g}^{x_s}) = N_{G_0}(\mathfrak{z}(\mathfrak{g}^{x_s})_1) = N_{G_0}(\mathfrak{z}(\mathfrak{g}^{x_s}))$, so $\Gamma \cong (N_{G_0}(\mathfrak{c}) \cap N_{G_0}(\mathfrak{g}^{x_s})) / Z_{G_0}(\mathfrak{c})$. In other words, if $w \in \Gamma$, then any of its representatives $\dot{w} \in N_{G_0}(\mathfrak{c})$ lies in $N_{G_0}(\mathfrak{g}^{x_s})$.

(2) The group $N_{G_0}(\mathfrak{c}) \cap N_{G_0}(\mathfrak{g}^{x_s})$ normalizes $G_0^{x_s}$ and $\mathfrak{g}_1^{x_s}$ and thus acts on the set of $G_0^{x_s}$ -orbits in $\mathfrak{g}_1^{x_s}$. Since $Z_{G_0}(\mathfrak{c}) \subset G_0^{x_s}$, this action factors through an action of Γ on the set of $G_0^{x_s}$ -orbits in $\mathfrak{g}_1^{x_s}$ which preserves the set of nilpotent ones. Observe that $G_0^{x_s}$ and its orbits need not be connected.

We shall need the stabilizer Γ_n in Γ of $\mathcal{O}_{x_n}^{G_0^{x_s}}$ with respect to the action defined above:

$$\Gamma_n = \text{Stab}_\Gamma(\mathcal{O}_{x_n}^{G_0^{x_s}}).$$

Proposition 4.8. *Let $x = x_s + x_n \in V$ with $x_s \in \mathfrak{c}$. The assignment φ from $\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}}$ to the orbit set $J_{G_0}(x)/G_0$ given by $y_s \mapsto \mathcal{O}_{(y_s + x_n)}^{G_0}$ induces a homeomorphism $\overline{\varphi}: \mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}}/\Gamma_n \rightarrow J_{G_0}(x)/G_0$, where the orbit set is endowed with the quotient topology.*

Proof. The map φ is well-defined and surjective by Proposition 2.19. We prove injectivity. Let $y_s, z_s \in \mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}}$ be such that $g \cdot (y_s + x_n) = z_s + x_n$ for some $g \in G_0$, i.e.,

$$g \cdot y_s = z_s, \quad (4.2)$$

$$g \cdot x_n = x_n, \quad (4.3)$$

and consider $w \in W_{V_{\text{in}}}$ such that $w \cdot y_s = z_s$, cf. Theorem 2.9. Any representative $\dot{w} \in N_{G_0}(\mathfrak{c})$ of w satisfies $\dot{w} \cdot \mathfrak{g}^{x_s} = \dot{w} \cdot \mathfrak{g}^{y_s} = \mathfrak{g}^{z_s} = \mathfrak{g}^{x_s}$, so $w \in \Gamma$ by Remark 4.7. Moreover, $\dot{w}g^{-1} \in G^{z_s} \cap G_0 = G_0^{x_s}$ by (4.2). It follows from (4.3) that

$$\dot{w} \cdot x_n \in \mathcal{O}_{x_n}^{G_0^{x_s}} \quad \text{so} \quad \dot{w} \cdot \mathcal{O}_{x_n}^{G_0^{x_s}} = \mathcal{O}_{x_n}^{G_0^{x_s}},$$

in other words $w \in \Gamma_n$ and $\bar{\varphi}$ is injective.

Let $p: J_{G_0}(x) \rightarrow J_{G_0}(x)/G_0$ be the quotient map and U an open subset in $J_{G_0}(x)/G_0$. Then $p^{-1}(U)$ is a G_0 -stable open subset in $J_{G_0}(x)$ and its intersection

$$p^{-1}(U) \cap (\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}} \times \mathcal{O}_{x_n}^{G_0^{x_s}})$$

is an open Γ_n -stable subset of $\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}} \times \mathcal{O}_{x_n}^{G_0^{x_s}}$. Its projection onto $\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}}$ is again an open Γ_n -stable subset, and so is its image through the quotient map by the finite group Γ_n . We have therefore proved that $\bar{\varphi}$ is a continuous bijection, and it remains to show that is open.

By Corollary 2.21 and Proposition 3.13, the action morphism $G_0 \times (\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}} + x_n) \rightarrow J_{G_0}(x)$ is a morphism of smooth varieties whose induced map on the tangent spaces is surjective. Hence it is smooth, and an open morphism in the Zariski topology (see [2, VII, Remark 1.2] and [2, V, Theorem 5.1 and VII, Theorem 1.8]). From this, it is straightforward to see that $\bar{\varphi}$ is open. \square

We briefly turn to the parametrization of G_0 -Jordan classes. Thanks to Theorem 2.9 and Example 2.15 describing the centralizer of an element of \mathfrak{c} , we easily establish the following.

Lemma 4.9. *Let x_s and y_s be two elements in \mathfrak{c} . Then the centralizers \mathfrak{g}^{x_s} and \mathfrak{g}^{y_s} are G_0 -conjugate if and only if there exists $w \in W_{V_{\text{in}}}$ such that $w \cdot \Sigma(x_s) = \Sigma(y_s)$.*

The hyperplane arrangement on \mathfrak{c} determined by the restricted roots $\sigma \in \Sigma$ admits an action of $W_{V_{\text{in}}}$ and it induces a stratification on \mathfrak{c} , where two elements lie in the same stratum $\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}} = \{y_s \in \mathfrak{c} \mid \Sigma(y_s) = \Sigma(x_s)\}$ if and only if their centralizers coincide. Equivalently, the stratum associated to a closed and symmetric subset $\tilde{\Sigma} \subset \Sigma$ (as in of [33, pag. 182]) is

$$S_{\tilde{\Sigma}} = \{x \in \mathfrak{c} \mid \Sigma(x) = \tilde{\Sigma}\}$$

and the collection of $S_{\tilde{\Sigma}}$'s is a finite partition of \mathfrak{c} . Already in the ungraded case, where the class of centralizers of semisimple elements coincides with the class of Levi subalgebras, not all closed and symmetric subsets $\tilde{\Sigma}$ of Σ give rise to a non-empty stratum. In the graded case, some information on stabilizers of generic elements in S_V is to be found in [31] under the assumption that V is a simple G_0 -module. We refer to [24, Proposition 3.4] for an alternative general description of centralizers of semisimple elements. In view of Lemma 4.9, two strata $S_{\tilde{\Sigma}}$ and $S_{\tilde{\Sigma}'}$ are said to be equivalent if $w \cdot \tilde{\Sigma} = \tilde{\Sigma}'$ for some $w \in W_{V_{\text{in}}}$. Given $\tilde{\Sigma} \subset \Sigma$, we set $\mathfrak{m}(\tilde{\Sigma})$ to be the θ -stable Levi subalgebra of \mathfrak{g} constructed as in (2.6) and $M(\tilde{\Sigma}) \subset G$, $M(\tilde{\Sigma})_0 = M(\tilde{\Sigma}) \cap G_0 \subset G_0$ as usual.

Proposition 4.10. *Jordan classes in V are in one-to-one correspondence with $W_{V_{\text{in}}}$ -classes of pairs $(\tilde{\Sigma}, \mathcal{O})$ where $\tilde{\Sigma} \subset \Sigma$ satisfies $S_{\tilde{\Sigma}} \neq \emptyset$ and \mathcal{O} is a nilpotent orbit in $\mathfrak{m}(\tilde{\Sigma})_1$ for the action of $M(\tilde{\Sigma})_0$.*

Proof. Observe that $N_{G_0}(\mathfrak{c})$ acts on the set of pairs $(\tilde{\Sigma}, \mathcal{O})$ as above and that if $\mathfrak{m}(\tilde{\Sigma})$ is the centralizer of some $x_s \in \mathfrak{c}$, then $Z_{G_0}(\mathfrak{c}) \subset M(\tilde{\Sigma})_0$, hence it acts trivially on $(\tilde{\Sigma}, \mathcal{O})$. Thus the action of $N_{G_0}(\mathfrak{c})$ induces an action of $W_{V_{\text{in}}}$.

Now recall that, for $x \in V$ the assignment $J_{G_0}(x) \mapsto (\mathfrak{g}^{x_s}, \mathcal{O}_{x_n}^{G_0^{x_s}})$ establishes a one-to-one correspondence between G_0 -Jordan classes in V and G_0 -classes of pairs (l, \mathcal{O}) where l is the

stabilizer of a semisimple element in V and \mathcal{O} a nilpotent orbit in \mathfrak{l}_1 for the action of L_0 . Theorem 2.6 guarantees that we can always find a pair in the G_0 -orbit where $\mathfrak{l} = \mathfrak{m}(\tilde{\Sigma})$ for some $\tilde{\Sigma} \subset \Sigma$. Assume that for two pairs $(\mathfrak{m}(\tilde{\Sigma}), \mathcal{O})$ and $(\mathfrak{m}(\tilde{\Sigma}'), \mathcal{O}')$ of this form there is $g \in G_0$ such that $(g \cdot \mathfrak{m}(\tilde{\Sigma}), g \cdot \mathcal{O}) = (\mathfrak{m}(\tilde{\Sigma}'), \mathcal{O}')$. By Lemma 4.9 we can decompose $g = g' \dot{w}$, where $g' \in N_{G_0}(\mathfrak{m}(\tilde{\Sigma}'))$ and $\dot{w} \in N_{G_0}(\mathfrak{c})$. In addition, $g' = \mathfrak{l} \dot{\sigma}$ with $\mathfrak{l} \in M(\tilde{\Sigma}')_0$ and $\dot{\sigma} \in N_{G_0}(\mathfrak{m}(\tilde{\Sigma}')) \cap N_{G_0}(\mathfrak{c})$. In other words, we may replace g by an element in $N_{G_0}(\mathfrak{c})$, so $(\mathfrak{m}(\tilde{\Sigma}), \mathcal{O})$ and $(\mathfrak{m}(\tilde{\Sigma}'), \mathcal{O}')$ lie in the same W_{Vin} -orbit. \square

The results of [34] encompass a parametrization of the G_0 -Jordan classes, where θ is the automorphism of order $m = 3$ of $\mathfrak{g} = E_8$ for which $\mathfrak{g}_1 \cong \Lambda^3 \mathbb{C}^9$, $\mathfrak{g}_0 \cong \mathfrak{sl}(9)$ and $\mathfrak{g}_{-1} \cong \Lambda^3(\mathbb{C}^9)^*$ as in Example 2.12. This is shown in the following:

Example 4.11. By the discussion in [34, §3.4], the seven “families” described in [34, §1] parametrize the Levi subalgebras $\mathfrak{l} = \mathfrak{g}^{x_s}$ that arise from elements $x_s \in \mathcal{S}_V$ up to G_0 -conjugation, and the “classes” in Tables 1-6 of [34, §1] parametrize the nilpotent orbits in \mathfrak{l}_1 for the action of $G_0^{x_s}$. (If x_s is in family I then $\mathfrak{g}^{x_s} = \mathfrak{h}$, there is no non-trivial nilpotent orbit and only one class.) By Proposition 4.10, our G_0 -Jordan classes almost coincide with the classes of [34]: the finite group $N_{G_0}(\mathfrak{l})/G_0^{x_s}$ acts on the set of nilpotent $G_0^{x_s}$ -orbits in \mathfrak{l}_1 , possibly glueing some of them.

Hence, some of the 164 classes of [34] may correspond to the same G_0 -Jordan class. A look at Tables 1-6 tells us that this may happen only in a few cases, since centralizers of elements of a G_0 -Jordan class are G_0 -conjugate by Proposition 2.20 and $N_{G_0}(\mathfrak{l})/G_0^{x_s} = \mathbb{1}$ in the VII family:

- III family:** Classes 2-3, 4-6, and 7-8;
- V family:** Classes 7-8, and 10-11;
- VI family:** Classes 5-6, 8-9, 11-12, and 17-18.

Recall that the support of a trivector $\varphi \in \Lambda^3 \mathbb{C}^9$ is the unique minimal subspace $E \subset \mathbb{C}^9$ such that $\varphi \in \Lambda^3 E$. Its dimension is the *rank* of φ , one of the simplest discrete G_0 -invariants of a trivector. The nilpotent $G_0^{x_s}$ -orbits associated to the classes 7-8 in V family have different rank, so they are not G_0 -related. Thus, they correspond to different G_0 -Jordan classes. A similar observation works in all the remaining cases, except those of the III family and the classes 5-6 of VI family, but it is not difficult to see that the nilpotent $G_0^{x_s}$ -orbits of these last two classes are not G_0 -related. It remains therefore to deal with the III family.

First of all, the rank of the nilpotent orbit in class 4 is strictly smaller than the rank of those in classes 5 and 6. However, the permutation matrix

$$g = - \left(\begin{array}{c|c|c} \text{Id}_{3 \times 3} & 0 & 0 \\ \hline 0 & 0 & \text{Id}_{3 \times 3} \\ \hline 0 & \text{Id}_{3 \times 3} & 0 \end{array} \right) \quad (4.4)$$

is an element of $N_{G_0}(\mathfrak{l})$ and it *does* relate the nilpotent $G_0^{x_s}$ -orbits associated to classes 5-6, which then correspond to a single G_0 -Jordan class. The same is true for classes 2-3 and 7-8. In summary, the space $\Lambda^3 \mathbb{C}^9$ is partitioned into 161 G_0 -Jordan classes.

The quotient Γ/W_{x_s} of Γ with the stabilizer W_{x_s} of $x_s \in \mathfrak{c}$ in W_{Vin} was found in [34, §3.4] for all families (see also the fourth and fifth columns of [34, Table 7]). In the case of III family, it is a group of order 72 generated by complex reflections. Consider, for example, the G_0 -Jordan class III.5, represented by $\chi = x_s + x_n$. A simple check shows that g as in (4.4) normalizes also \mathfrak{c} , so $g \in N_{G_0}(\mathfrak{z}(\mathfrak{g}^{x_s})) \cap N_{G_0}(\mathfrak{c})$ and, by our previous discussion, it is not in Γ_n . The G_0 -orbits in the G_0 -Jordan class III.5 are then parametrized by the quotient $\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}}/\Gamma_n$ of $\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}}$ by a group Γ_n of order 36.

We conclude with an application of Theorem 4.3. Let $J_2 = J_{G_0}(\mathfrak{y})$ be the G_0 -Jordan class numbered III.7, i.e., the one with the representative $\mathfrak{y} = \mathfrak{y}_s + \mathfrak{y}_n$ given by

$$\mathfrak{y}_s = (\mathfrak{e}_{123} + \mathfrak{e}_{456} + \mathfrak{e}_{789}) + i(\mathfrak{e}_{147} + \mathfrak{e}_{258} + \mathfrak{e}_{369}), \quad \mathfrak{y}_n = \mathfrak{e}_{159}. \quad (4.5)$$

The centralizer $\mathfrak{m} = \mathfrak{g}^{y_s}$ is a reductive Lie algebra with semisimple part \mathfrak{r} of type $A_2 \oplus A_2$. More precisely, the center of \mathfrak{m} is 4-dimensional and sits in degrees ± 1 : it consists of the two components in brackets that defines y_s in (4.5) and of their duals. The semisimple part $\mathfrak{r} = \mathfrak{r}_{-1} \oplus \mathfrak{r}_0 \oplus \mathfrak{r}_1$ is graded as follows [34, §2.4]:

$$\begin{aligned} \mathfrak{r}_1 &= \text{span}\{\mathbf{e}_{159}, \mathbf{e}_{267}, \mathbf{e}_{348}\} \oplus \text{span}\{\mathbf{e}_{168}, \mathbf{e}_{249}, \mathbf{e}_{357}\}, \\ \mathfrak{r}_0 &= \text{span}\{\mathbf{d}_{159}, \mathbf{d}_{267}, \mathbf{d}_{348}\} \oplus \text{span}\{\mathbf{d}_{168}, \mathbf{d}_{249}, \mathbf{d}_{348}\}, \\ \mathfrak{r}_{-1} &= \text{span}\{\mathbf{e}^{159}, \mathbf{e}^{267}, \mathbf{e}^{348}\} \oplus \text{span}\{\mathbf{e}^{168}, \mathbf{e}^{249}, \mathbf{e}^{357}\}, \end{aligned} \quad (4.6)$$

where \mathbf{e}^i , for $1 \leq i \leq 9$, is the dual basis of $(\mathbb{C}^9)^*$, $\mathbf{e}^{ijl} := \mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{e}^l$ and the elements $\mathbf{d}_{ijk} = [\mathbf{e}_{ijk}, \mathbf{e}^{ijk}]$ satisfy $\mathbf{d}_{159} + \mathbf{d}_{267} + \mathbf{d}_{348} = \mathbf{d}_{168} + \mathbf{d}_{249} + \mathbf{d}_{348} = 0$. The direct sums of vector spaces in (4.6) correspond to the Lie algebra decomposition of \mathfrak{r} .

Let $J_1 = J_{G_0}(x)$ be any of the G_0 -Jordan classes in the II family, i.e., one of II.1, II.2 or II.3. The choice of representative $x = x_s + x_n$ given by

$$x_s = y_s + (\mathbf{e}_{159} + \mathbf{e}_{267} + \mathbf{e}_{348}), \quad x_n = \begin{cases} \mathbf{e}_{168} + \mathbf{e}_{249} & \text{for II.1,} \\ \mathbf{e}_{168} & \text{for II.2,} \\ 0 & \text{for II.3,} \end{cases} \quad (4.7)$$

easily allows to check that $J_2 \subset \bar{J}_1$. First of all $\mathfrak{z}(\mathfrak{g}^{x_s})_1$ is generated by the 3 vectors in brackets in (4.5) and (4.7), hence $y_s \in \mathfrak{z}(\mathfrak{g}^{x_s})_1$ and $\mathfrak{g}^{x_s} \subset \mathfrak{m}$. A graded $\mathfrak{sl}(2)$ -triple $\{e, h, f\}$ in \mathfrak{m} with $e = y_n$ is provided by $f = \mathbf{e}^{159}$ and $h = \mathbf{d}_{159}$, and the required Slodowy slice $S_{m_1, e} = e + m_1^f$ is the affine subspace in m_1 modeled on $m_1^f = \text{span}\{\mathbf{e}_{267}, \mathbf{e}_{348}\} \oplus \text{span}\{\mathbf{e}_{168}, \mathbf{e}_{249}, \mathbf{e}_{357}\} \oplus \mathfrak{z}(\mathfrak{m})_1$. It is evident that $x \in S_{m_1, e}$, so $J_2 \subset \bar{J}_1$ thanks to Theorem 4.3 (iii).

APPENDIX A. CARTAN, LEVI AND PARABOLIC SUBALGEBRAS IN \mathbb{Z}_m -GRADED LIE ALGEBRAS

Let $\{\mathfrak{g}, \theta\}$ be a reductive \mathbb{Z}_m -graded Lie algebra and $\mathfrak{c} \subset V$ a fixed Cartan subspace. The existence of a homogeneous Cartan subalgebra \mathfrak{h} of \mathfrak{g} containing \mathfrak{c} is a result probably known to experts by a long time; the proof in [25, §4.1] is stated for \mathfrak{g} simple, but its proof carries over for any reductive \mathfrak{g} .

Proposition A.1. *There exists a homogeneous Cartan subalgebra $\mathfrak{h} = \bigoplus_{l \in \mathbb{Z}_m} \mathfrak{h}_l$ of \mathfrak{g} that satisfies $\mathfrak{h} \supset \mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c}))$ and $\mathfrak{h}_1 = \mathfrak{c}$.*

Remark A.2. By [31, §3.1], the Cartan subspace \mathfrak{c} is *not* an algebraic subalgebra in general, unless $m \leq 2$. On the other hand \mathfrak{h} and $\mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c}))$ are algebraic, hence $\mathfrak{h} \supset \mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})) \supset \bar{\mathfrak{c}}$, where $\bar{\mathfrak{c}}$ is the algebraic closure of \mathfrak{c} . It is clear that $\mathfrak{h} = \mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c}))$ if and only if $[\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c}), \mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})] = 0$ but we are not aware of any general condition under which $\mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})) = \bar{\mathfrak{c}}$.

We will call *adapted* any Cartan subalgebra \mathfrak{h} of \mathfrak{g} as in Proposition A.1. For such an \mathfrak{h} , let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha} \quad (A.1)$$

be the root space decomposition of \mathfrak{g} with respect to \mathfrak{h} , with associated set of roots $\Phi \subset \mathfrak{h}^*$. The automorphism $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ permutes the root spaces in (A.1):

Lemma A.3. *For any $\alpha \in \Phi$, we have $\alpha \circ \theta \in \Phi$ and $\theta^{-1}(\mathfrak{g}_{\alpha}) = \mathfrak{g}_{\alpha \circ \theta}$.*

We note that any root $\alpha \in \Phi$ can be decomposed as $\alpha = \alpha_0 + \alpha_1 + \dots + \alpha_{m-2} + \alpha_{m-1}$, where $\alpha_l|_{\mathfrak{h}_l} = \alpha|_{\mathfrak{h}_l}$ and $\text{Ker}(\alpha_l) = \bigoplus_{k \neq l} \mathfrak{h}_k$ for any $l \in \mathbb{Z}_m$. Repeatedly applying Lemma A.3, we see that

$$\alpha \circ \theta^l = \alpha_0 + \omega^l \alpha_1 + \dots + (\omega^l)^{m-2} \alpha_{m-2} + (\omega^l)^{m-1} \alpha_{m-1}$$

is a root too, for any $l \in \mathbb{Z}_m$. In other words, we may consider the equivalence class of roots given by $[\alpha] = \{\alpha \circ \theta^l \mid l \in \mathbb{Z}_m\}$ for any $\alpha \in \Phi$. We let $[\Phi] = \{[\alpha] \mid \alpha \in \Phi\}$ be the collection of

such equivalence classes and note that the direct sum of root spaces

$$\mathfrak{g}_{[\alpha]} = \bigoplus_{\iota \in \mathbb{Z}_m} \mathfrak{g}_{\alpha \circ \theta^\iota}$$

is a homogeneous subspace of \mathfrak{g} , whence $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{[\alpha] \in [\Phi]} \mathfrak{g}_{[\alpha]}$ is a decomposition of \mathfrak{g} into homogeneous subspaces.

Now, the centralizer \mathfrak{g}^x of any $x \in \mathfrak{c}$ is a homogeneous Levi subalgebra containing $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})$. A natural question is whether there exists a parabolic subalgebra of \mathfrak{g} with Levi factor \mathfrak{g}^x that is also homogeneous: we will now see that this is rarely the case. For simplicity of exposition, we restrict to the case where \mathfrak{g} is semisimple.

Let \mathfrak{p} be a proper parabolic subalgebra of \mathfrak{g} with a homogeneous Levi factor \mathfrak{l} that contains $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})$. Then, there exists a \mathbb{Z} -grading

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}(j) \quad (\text{A.2})$$

of \mathfrak{g} such that $\mathfrak{p} = \mathfrak{g}(\geq 0) = \bigoplus_{j \geq 0} \mathfrak{g}(j)$ and $\mathfrak{l} = \mathfrak{g}(0)$. We let $Z \in \mathfrak{g}$ be the grading element of (A.2), the unique element in \mathfrak{g} that satisfies $[Z, X] = jX$ for all $X \in \mathfrak{g}(j)$, $j \in \mathbb{Z}$, see, e.g., [30].

Now $Z \in \mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c}))$, so it belongs to the adapted Cartan subalgebra $\mathfrak{h} = \bigoplus_{\iota \in \mathbb{Z}_m} \mathfrak{h}_\iota$ of \mathfrak{g} of Proposition A.1. We will write $Z = Z_0 + \dots + Z_{m-1}$, where $Z_\iota \in \mathfrak{h}_\iota$ for all $\iota \in \mathbb{Z}_m$.

Definition A.4. Let $\alpha = \alpha_0 + \dots + \alpha_{m-1} \in \Phi$ be a root with respect to \mathfrak{h} and $\iota \in \mathbb{Z}_m$. The ι^{th} mode of α is the complex number $\lambda_\iota = \alpha_\iota(Z_\iota)$.

We remark that $\alpha(Z) = \sum_{\iota \in \mathbb{Z}_m} \lambda_\iota$. Since the adjoint action of Z has integer eigenvalues, we may apply Lemma A.3 repeatedly to the roots $\alpha \circ \theta^\iota \in \Phi$ and get:

Proposition A.5. *The modes of α satisfy a system of linear equations of the form*

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{m-1} \\ 1 & \omega^2 & (\omega^2)^2 & \dots & (\omega^{m-1})^2 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \omega^{m-1} & (\omega^2)^{m-1} & \dots & (\omega^{m-1})^{m-1} \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{m-1} \end{pmatrix} = \begin{pmatrix} n_0 \\ n_1 \\ n_2 \\ \vdots \\ n_{m-1} \end{pmatrix}, \quad (\text{A.3})$$

where $n_\iota = \alpha(\theta^\iota(Z)) \in \mathbb{Z}$ for any $\iota \in \{0, \dots, m-1\}$.

The $m \times m$ matrix on the L.H.S. of (A.3) is a symmetric matrix of Vandermonde type with coefficients in the cyclotomic field $\mathbb{Q}(\omega)$. We denote it by $M(\omega)$ and compactly rewrite (A.3) as $M(\omega)\vec{\lambda} = \vec{n}$, where $\vec{\lambda} \in \mathbb{C}^m$ is the vector of modes and $\vec{n} \in \mathbb{Z}^m$. Clearly all modes are elements of $\mathbb{Q}(\omega)$, but we have the following stronger result for λ_0 .

Proposition A.6. *The identity $m\lambda_0 = \sum_{\iota \in \mathbb{Z}_m} n_\iota$ is satisfied for any α , therefore $\lambda_0 \in \frac{1}{m}\mathbb{Z}$. If $\mathfrak{h}_0 = 0$, then \mathfrak{p} is not θ -stable.*

Proof. Let $W = \{\vec{y} \in \mathbb{C}^m \mid \sum_{\iota \in \mathbb{Z}_m} y_\iota = 0\}$. All columns of $M(\omega)$ but the first one lie in W , so

$$\vec{n} - \lambda_0 \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \in W. \quad (\text{A.4})$$

Adding all entries of the above vector gives $m\lambda_0 = \sum_{\iota \in \mathbb{Z}_m} n_\iota \in \mathbb{Z}$. If $\mathfrak{h}_0 = 0$, then $Z_0 = 0$, so $\lambda_0 = 0$ and $\vec{n} \in W \cap \mathbb{Z}^m$.

Now, $\mathfrak{h} \subset \mathfrak{g}(0)$ and $\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi, \alpha(Z) \geq 0} \mathfrak{g}_\alpha$. If $\mathfrak{g}_\alpha \subset \bigoplus_{j > 0} \mathfrak{g}(j)$, then $\alpha(Z) = n_0 > 0$ and, if $\mathfrak{h}_0 = 0$, there exists $\iota \in \mathbb{Z}_m$ such that $n_\iota < 0$, i.e., $\theta^{-1}\mathfrak{g}_\alpha = \mathfrak{g}_{\alpha \circ \theta^\iota} \notin \mathfrak{p}$. \square

Example A.7. The \mathbb{Z}_m -graded Lie algebra $\{\mathfrak{g}, \theta, m\} = \{E_8, \theta, 3\}$ as in Examples 2.12 and 4.11 satisfies $\mathfrak{h}_0 = 0$. By Theorem 2.6 and Proposition A.6, *all* centralizers \mathfrak{g}^x of non-zero $x \in \mathcal{S}_V$ do *not* extend to θ -stable parabolic subalgebras.

ACKNOWLEDGMENTS

We thank Michael Bulois for useful email exchanges on sheets in symmetric Lie algebras and slice induction, Willem de Graaf for a helpful discussion on \mathcal{N} -regular gradings, Dmitri Panyushev for information on generic stabilizers of semisimple elements in V , and Vladimir Popov for pointing us reference [24]. We thank the anonymous referees for pointing out an error in the earlier version of Corollary 3.10 and for several comments, leading to an improved version of the manuscript. This research was partially supported by DOR1898721/18 and BIRD179758/17 funded by the University of Padova and FFABR2017 funded by MIUR. This project started when the third named author was holding a Type B Post doc Fellowship at the University of Padova. The third named author acknowledges that the research leading to these results has received funding from the Norwegian Financial Mechanism 2014-2021 (project registration number 2019/34/H/ST1/00636).

REFERENCES

- [1] *Revêtements étales et groupe fondamental (SGA 1)*, volume 3 of *Documents Mathématiques (Paris)* [*Mathematical Documents (Paris)*]. Société Mathématique de France, Paris, 2003. Séminaire de géométrie algébrique du Bois Marie 1960–61. [Algebraic Geometry Seminar of Bois Marie 1960-61], Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin; MR0354651 (50 #7129)].
- [2] A. Altman and S. Kleiman. *Introduction to Grothendieck duality theory*. Lecture Notes in Mathematics, Vol. 146. Springer-Verlag, Berlin-New York, 1970.
- [3] P. Bardsley and R. W. Richardson. étale slices for algebraic transformation groups in characteristic p . *Proceedings of the London Mathematical Society*, s3-51:295–317, 1985.
- [4] W. Borho. Über Schichten halbeinfacher Lie-Algebren. *Invent. Math.*, 65(2):283–317, 1981/82.
- [5] W. Borho and H. Kraft. Über Bahnen und deren Deformationen bei linearen Aktionen reduktiver Gruppen. *Comment. Math. Helv.*, 54(1):61–104, 1979.
- [6] A. Broer. Decomposition varieties in semisimple Lie algebras. *Canad. J. Math.*, 50(5):929–971, 1998.
- [7] M. Bulois. Sheets of symmetric Lie algebras and Slodowy slices. *J. Lie Theory*, 21(1):1–54, 2011.
- [8] M. Bulois and P. Hivert. Sheets in symmetric Lie algebras and slice induction. *Transform. Groups*, 21(2):355–375, 2016.
- [9] W. A. de Graaf. Computing representatives of nilpotent orbits of θ -groups. *J. Symbolic Comput.*, 46(4):438–458, 2011.
- [10] L. Yu. Galitski and D. A. Timashev. On classification of metabelian Lie algebras. *J. Lie Theory*, 9(1):125–156, 1999.
- [11] V. G. Kac. Some remarks on nilpotent orbits. *J. Algebra*, 64(1):190–213, 1980.
- [12] B. Kostant. Lie group representations on polynomial rings. *Amer. J. Math.*, 85:327–404, 1963.
- [13] B. Kostant and S. Rallis. Orbits and representations associated with symmetric spaces. *Amer. J. Math.*, 93:753–809, 1971.
- [14] P. Levy. Vinberg’s θ -groups in positive characteristic and Kostant-Weierstrass slices. *Transform. Groups*, 14(2):417–461, 2009.
- [15] P. Levy. KW-sections for Vinberg’s θ -groups of exceptional type. *J. Algebra*, 389:78–97, 2013.
- [16] D. Luna. Slices étales. *Mémoires de la S.M.F.*, 33:81–105, 1973.
- [17] G. Lusztig and N. Spaltenstein. Induced unipotent classes. *J. London Math. Soc. (2)*, 19(1):41–52, 1979.
- [18] G. Lusztig and Z. Yun. \mathbb{Z}/m -graded Lie algebras and perverse sheaves, I. *Represent. Theory*, 21:277–321, 2017.
- [19] G. Lusztig and Z. Yun. \mathbb{Z}/m -graded Lie algebras and perverse sheaves, III: Graded double affine Hecke algebra. *Represent. Theory*, 22:87–118, 2018.
- [20] A. G. Nurmiev. Orbits and invariants of third-order matrices. *Mat. Sb.*, 191(5):101–108, 2000.
- [21] D. I. Panyushev. On invariant theory of θ -groups. *J. Algebra*, 283(2):655–670, 2005.
- [22] D. I. Panyushev. Periodic automorphisms of Takiff algebras, contractions, and θ -groups. *Transform. Groups*, 14(2):463–482, 2009.
- [23] D. D. Pervushin. Invariants and orbits of the standard $(\mathrm{SL}_4(\mathbb{C}) \times \mathrm{SL}_4(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}))$ -module. *Izv. Ross. Akad. Nauk Ser. Mat.*, 64(5):133–146, 2000.

- [24] V. L. Popov. Modality of representations, and packets for θ -groups. In *Lie groups, geometry, and representation theory*, volume 326 of *Progr. Math.*, pages 459–479. Birkhäuser/Springer, Cham, 2018.
- [25] M. Reeder, P. Levy, J.-K. Yu, and B. H. Gross. Gradings of positive rank on simple Lie algebras. *Transform. Groups*, 17(4):1123–1190, 2012.
- [26] M. Reeder and J.-K. Yu. Epipelagic representations and invariant theory. *J. Amer. Math. Soc.*, 27(2):437–477, 2014.
- [27] P. Slodowy. *Simple singularities and simple algebraic groups*, volume 815 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.
- [28] T. A. Springer. *Linear algebraic groups*, volume 9 of *Progress in Mathematics*. Birkhäuser, Boston, Mass., 1981.
- [29] P. Tauvel and R. W. T. Yu. *Lie algebras and algebraic groups*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005.
- [30] A. Čap and J. Slovák. *Parabolic geometries. I*, volume 154 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2009. Background and general theory.
- [31] È. B. Vinberg. The Weyl group of a graded Lie algebra. *Math. USSR-Izv.*, 10:463–495, 1977.
- [32] È. B. Vinberg. Classification of homogeneous nilpotent elements of a semisimple graded Lie algebra. *Trudy Sem. Vektor. Tenzor. Anal.*, (19):155–177, 1979.
- [33] È. B. Vinberg, editor. *Lie groups and Lie algebras, III*, volume 41 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 1994.
- [34] È. B. Vinberg and A. G. Èlašvili. A classification of the three-vectors of nine-dimensional space. *Trudy Sem. Vektor. Tenzor. Anal.*, 18:197–233, 1978.
- [35] È. B. Vinberg and V. L. Popov. Invariant theory. In *Algebraic geometry, 4 (Russian)*, Itogi Nauki i Tekhniki, pages 137–314, 315. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1989.

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