Well-posedness for thermo-electro-viscoelasticity of Green-Naghdi type

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Abstract. We study the linear theory of thermo-electro-viscoelasticity of Green-Naghdi type for the case of a one-dimensional body. For the corresponding mathematical model, we prove a uniqueness theorem of the solution to the mixed boundary-initial-value problem by means of the Laplace transform after rewriting the constitutive equations in an appropriate form. Moreover, we derive a result of continuous dependence upon the supply terms.

1 Introduction

Green & Naghdi [1, 2] in the early 90's developed in a general context a thermo-mechanical theory of deformable continua that is based on an entropy balance law rather than an entropy inequality. A theory of thermoelastic bodies based on such new entropy balance law has been derived. The linearized form of this theory leads to three different models of heat conduction: type I, which adopts Fourier's law, type II and type III, respectively. They involve a new scalar variable α, which is called *thermal displacement* and represents a kind of time primitive of the empirical temperature T. In [1, p.180] we read "The temperature T (on the macroscopic scale) is generally regarded as representing (on the molecular scale) some 'mean' velocity magnitude or 'mean' (kinetic energy). With this in mind, we introduce a scalar $\alpha = \alpha(\mathbf{X}, t)$ through an integral of the form"

$$\alpha = \alpha(\mathbf{X}, t) = \int_0^t T(\mathbf{X}, \tau)\, d\tau + \alpha_0(\mathbf{X}), \qquad t > 0. \tag{1}$$

... "In view of the above interpretation associated with T and the physical dimension of the quantity defined by (1), the variable α may justifiably be called thermal displacement magnitude or simply thermal displacement. Alternatively, we may regard the scalar α (on the macroscopic scale) as representing a 'mean' displacement magnitude on the molecular scale and then"

$$T(\boldsymbol{X}, t) = \dot{\alpha}(\boldsymbol{X}, t).$$

The Green-Naghdi linear model of type III permits the transmission of heat as thermal waves at finite speed [3, 4]. Moreover, the heat flux vector is determined by the same potential function that determines the stress [5]. The Green-Naghdi theories meet great research interest and have been studied in a lot of papers (see, for instance, [5, 6, 7], [8], [9], [10] and references therein).

Incidentally, in paper [11], on moving from the classic papers by Einstein and Langevin on Brownian motion from the beginning of 900 (see quotations therein), two consistent statistical interpretations are given for the thermal displacement in a fluid, in terms of the mean value of the squared diffusive displacement of a system of particles in Brownian motion suspended in that fluid.

Later, in [12] the procedure designed by Green and Naghdi for thermoelasticity is extended to simple thermo-electro-elastic bodies, both isotropic and transversely isotropic, that are finitely deformable, heat conducting, electrically polarizable, interacting with the electric field; again, the restrictions on the constitutive relations are obtained using an energy equation that is suitable for the considered type of material. Then paper [13] extends [12] to thermo-electro-mechanical simple materials (finitely deformable, heat conducting, electrically polarizable, interacting with the electric field) that have a fading memory.

In [14] Wilkes uses the local form of the Clausius-Duhem inequality to obtain restrictions on the relaxation functions of thermoviscoelastic materials that have fading memory in the sense of Coleman and Noll [15]. For the purpose, Wilkes uses the restrictions of the constitutive equations, the dissipation inequality, and the minimality of the free energy in equilibrium that are found and used by Coleman [16].

In [17] the nonlinear theory [13] for thermo-electro-mechanical simple materials with fading memory is used to set up a Green-Naghdi thermo-electro viscoelastic theory by linearization using the Riesz representation theorem. Of course, the presences of the electrical vector and of the thermal displacement derivatives in the independent variables, imply the existence of more relaxation functions than in [14]. Hence several restrictions on them are found that extend the ones in [14] for thermoviscoelastic materials within a theory that uses the Clausius-Duhem inequality. The restrictions are obtained from the internal dissipation inequality, which is a consequence of the dissipation inequality adopted here. Following [18], the last one is to assume that the internal rate of supply of entropy per unit mass is non-negative in every process. The theoretical frame is then completed with a proposal of constitutive equations for the internal rate of entropy supply and heat flux. The linearized (infinitesimal) theory of thermo-electro-viscoelasticity is deduced as first-order approximation of the finite theory and the field equations are explicitly deduced in the simplest case of a one-dimensional body.

One-dimensional such bodies are considered in literature. In biomechanics the study of a tissue constituent typically is evaluated from the uniaxial behavior. For instance, Zeng [19] considers the Cauchy problem of a one-dimensional purely mechanical nonlinear viscoelastic model with fading memory; Babaei et al. [20], to characterize the viscoelastic behavior of biological tissues, consider a one-dimensional purely mechanical viscoelastic model.

The goal of this article is to study the mathematical model proposed in [17] for thermo-electro-viscoelasticity of Green-Naghdi type in the linear case. The approach of Ciarletta and Scalia [21] has proved useful here.

To be more specific, we prove the uniqueness of the solution by means of the Laplace transform after rewriting the constitutive equations in an appropriate form. Moreover, we derive a result of continuous dependence upon the supply terms.

Uniqueness and continuous dependence results for thermoelasticity of type III were for example proven in [22, 23]. Similar topics were analysed in [24], [25], [26], [27].

The article is structured in four sections. After the introduction. we give some preliminaries on the mathematical model that we study. Finally, there are two sections that present separately the uniquenss and the continuous dependence results.

2 Preliminaries

Let $\mathcal{B}$ be a one-dimensional body composed of a thermo-electro-viscoelastic material. We consider the natural homogeneous reference configuration, i.e. the straight line segment $B = [0, L]$ on an axis X with zero stress, at a uniform temperature T_0, and with a uniform electric potential ϕ_0.

We consider a material which is viscoelastic with a fading memory. The constitutive equations of the mathematical model are of integral type with a genuine memory of the past history.

The infinitesimal kinetic process is described by the three scalar functions

$$U = U(X,t), \alpha = \alpha(X,t), \phi = \phi(X,t), 0 \leq X \leq L, t \in \mathbb{R}, \tag{2}$$

where U is the displacement that is defined as

$$U(X,t) = \chi(X,t) - X, \tag{3}$$

with $\chi(X,t)$ being the motion, α is the thermal displacement [1, 2], and ϕ is the electric potential. In the following we will always assume that these functions are of class C^2 in $B \times \mathbb{R}$.

Following [17], for

$$\gamma = U, U_{,X}, \dot{U}, \alpha, \alpha_{,X}, \dot{\alpha}, \phi, \phi_{,X}, \dot{\phi}, \tag{4}$$

we will use the difference history up to time t,

$$\gamma_d^t(u) = \gamma(t-u) - \gamma^\dagger(t-u) = (\gamma - \gamma^\dagger)(t-u), u \in [0,\infty), \tag{5}$$

where $\gamma^\dagger(\cdot)$ is the constant history up to time t, that is defined as

$$\gamma^\dagger(t-u) = \gamma(t), u \in [0,+\infty). \tag{6}$$

Hence $\gamma_d^t(0) = 0$ and for the past difference history of the derivative $\dot{\gamma}_d^t(\cdot)$ we have

$$\dot{\gamma}_d^t(u) = \dot{\gamma}(t-u) - \dot{\gamma}^\dagger(t-u) = \left(\dot{\gamma} - \dot{\gamma}^\dagger\right)(t-u), \tag{7}$$

where $\dot{\gamma}^\dagger(\cdot)$ is the derivative of the constant history

$$\dot{\gamma}^\dagger(t-u) = \dot{\gamma}(t), \quad u \in [0,\infty). \tag{8}$$

Unlike [17], here we will use the notation $(\gamma - \gamma^\dagger)(\cdot)$ rather than $\gamma_d^t(\cdot)$.

We present the local balance laws in the linear case for a thermo-electro-viscoelastic process (2) associated with the body force f and heat supply s

$$\begin{cases} \rho\ddot{U} = \frac{d}{dX}\tilde{\sigma} + \rho f \\ \rho\dot{\tilde{\eta}} = \rho s - \frac{d}{dX}\tilde{p} \\ \frac{d}{dX}\tilde{D} = 0 \end{cases}, \tag{9}$$

where ρ is the mass density, $\tilde{\sigma}$ is the uniaxial stress, $\tilde{\eta}$ is the specific entropy per unit mass, $\tilde{p}$ is the entropy flux, and

$$\tilde{D} = \varepsilon_0 W + \tilde{P} \tag{10}$$

is the electric displacement, where $W = -\phi_{,X}$ is the electric field, $\tilde{P}$ is the electric polarization and ε_0 is the (constant) vacuum electric permittivity. Note that $s = r/\theta = s(X,t)$ and $f = f(X,t)$, where r is the external heat supply.

The constitutive equations for the stress, electric polarization, and entropy respectively write as [17, Eqs. (149)-(154)]

$$\begin{aligned} \rho^{-1}\tilde{\sigma} &= \Sigma_1 U_{,X} + \int_0^\infty \dot{m}_3(u)(U - U^\dagger)_{,X}(t-u)du + \\ &+ \int_0^\infty (\dot{\alpha} - \dot{\alpha}^\dagger)(t-s)\dot{B}_2(s)ds + \int_0^\infty (\alpha - \alpha^\dagger)_{,X}(t-s)\dot{M}_1(s)ds - \\ &- \int_0^\infty (\phi - \phi^\dagger)_{,X}(t-s)\dot{M}_3(s)ds, \end{aligned} \tag{11}$$

$$\begin{aligned} \rho^{-1}\tilde{P} &= \Sigma_2 \phi_{,X} + \int_0^\infty \dot{m}_4(u)(\phi - \phi^\dagger)_{,X}(t-u)du - \\ &- \int_0^\infty (\dot{\alpha} - \dot{\alpha}^\dagger)(t-s)\dot{B}_3(s)ds - \int_0^\infty (\alpha - \alpha^\dagger)_{,X}(t-s)\dot{M}_2(s)ds - \\ &- \int_0^\infty \dot{M}_3(u)(U - U^\dagger)_{,X}(t-u)du, \end{aligned} \tag{12}$$

$$\begin{aligned} \zeta_0^{-1}\tilde{\eta} &= -\Sigma_3(T - T_0) - \int_0^\infty \dot{m}_1(u)(\dot{\alpha} - \dot{\alpha}^\dagger)(t-u)du - \\ &- \int_0^\infty \dot{B}_1(u)(\alpha - \alpha^\dagger)_{,X}(t-u)du - \int_0^\infty \dot{B}_2(u)(U - U^\dagger)_{,X}(t-u)du + \\ &+ \int_0^\infty \dot{B}_3(u)(\phi - \phi^\dagger)_{,X}(t-u)du. \end{aligned} \tag{13}$$

We denote by θ_0 the absolute temperature, $\zeta = T'(\theta)$, $\zeta_0 = T'(\theta_0)$, $\kappa_0 = \kappa(\theta_0)$, with $T = \dot{\alpha}$ being the empirical temperature ("thermal displacement rate"). We consider that a mass density $\rho = \rho(X)$ is given in the reference configuration. Moreover, for the internal rate of entropy supply and heat flux, we have

$$\rho\tilde{\xi} = \frac{\kappa_0}{\theta_0}\left(\dot{\alpha}_{,X}\right)^2, \tag{14}$$

$$\tilde{q} = -\zeta_0\theta_0\rho\left(\frac{\partial\tilde{\psi}}{\partial\beta} + \rho^{-1}\kappa_0\dot{\alpha}_{,X}\right), \tag{15}$$

$$\tilde{p} = \frac{1}{\theta_0}\tilde{q}, \tag{16}$$

where

$$\begin{aligned}\frac{\partial\tilde{\psi}}{\partial\beta} = \frac{\partial\tilde{\Sigma}}{\partial\beta} + \int_0^\infty \dot{m}_2(u)(\alpha-\alpha^\dagger)_{,X}(t-u)du + \int_0^\infty(\dot{\alpha}-\dot{\alpha}^\dagger)(t-s)\dot{B}_1(s)ds+\\ + \int_0^\infty \dot{M}_1(u)(U-U^\dagger)_{,X}(t-u)du - \int_0^\infty \dot{M}_2(u)(\phi-\phi^\dagger)_{,X}(t-u)du.\end{aligned} \tag{17}$$

From (10) we have

$$\begin{aligned}&\rho^{-1}\tilde{D} = \left(\Sigma_2 - \rho^{-1}\varepsilon_0\right)\phi_{,X} + \int_0^\infty \dot{m}_4(u)(\phi-\phi^\dagger)_{,X}(t-u)du-\\ &-\int_0^\infty(\dot{\alpha}-\dot{\alpha}^\dagger)(t-s)\dot{B}_3(s)ds - \int_0^\infty(\alpha-\alpha^\dagger)_{,X}(t-s)\dot{M}_2(s)ds-\\ &-\int_0^\infty \dot{M}_3(u)(U-U^\dagger)_{,X}(t-u)du.\end{aligned} \tag{18}$$

In Eqs. (11)-(18) the classical notations are adopted for the relaxation functions $\dot{m}_3$, $\dot{B}_2$, etc. To be precise, we have

$$\dot{m}_3(u) = \frac{\partial m_3(0,u)}{\partial u}, \dot{B}_2(s) = \frac{\partial B_2(s,0)}{\partial s}, etc. \tag{19}$$

Throughout in the following we will assume that the relaxation functions m_i, M_j, B_j $(i = 1,2,3, j = 1,2,3)$ are continuously differentiable as many times as will be required in their manipulations of the proofs. Moreover, they satisfy the inequalities [17, p. 25]

$$m_i(s,u) > 0, \dot{m}_i(s) < 0, \forall s,u, i = 1,\ldots,4. \tag{20}$$

A simple choice of the relaxation functions that satisfy all the conditions above is made in Sect. 15.2 of [17].

3 Uniqueness

Note that $\Sigma_1 U_{,X}$ represents the "instantaneous elastic response" of the material that is due to the deformation $U_{,X}$. It is added to the response due to the past history of the kinetic process (U, α, ϕ). In the sequel, we assume that the response functionals are purely viscoelastic, i.e. they do not have instantaneous elastic responses. Hence we put $\Sigma_1 = 0$, $\Sigma_2 = 0$, $\Sigma_3 = 0$, $\partial\tilde{\Sigma}/\partial\beta = 0$.

Proposition 1 *We assume that*

i) the relaxation functions have zero limit for $s \to +\infty$, *i.e.* $m_3(0, \infty) = 0$, $B_2(\infty, 0) = 0$, $M_1(\infty, 0) = 0$, $M_3(\infty, 0) = 0$, $m_4(0, \infty) = 0$, $B_3(\infty, 0) = 0$, $M_2(\infty, 0) = 0$, $m_1(0, \infty) = 0$, $B_1(0, \infty) = 0$, $m_2(0, \infty) = 0$,

ii)

$$\Sigma_1 = 0, \Sigma_2 = 0, \Sigma_3 = 0, \frac{\partial\tilde{\Sigma}}{\partial\beta} = 0. \tag{21}$$

Then the constitutive equations (11)-(17) and (18) are equivalent to

$$\begin{aligned} \rho^{-1}\tilde{\sigma} = & \int_{-\infty}^{t} m_3(0, t-u')\dot{U}_{,X}(u')du' + \int_{-\infty}^{t} \ddot{\alpha}(s')B_2(t-s', 0)ds' + \\ & + \int_{-\infty}^{t} \dot{\alpha}_{,X}(s')M_1(t-s', 0)ds' - \int_{-\infty}^{t} \dot{\phi}_{,X}(s')M_3(t-s', 0)ds', \end{aligned} \tag{22}$$

$$\begin{aligned} \rho^{-1}\tilde{P} = & \int_{-\infty}^{t} m_4(0, t-u')\dot{\phi}_{,X}(u')du' - \int_{-\infty}^{t} \ddot{\alpha}(s')B_3(t-s', 0)ds' - \\ & - \int_{-\infty}^{t} \dot{\alpha}_{,X}(s')M_2(t-s', 0)ds' - \int_{-\infty}^{t} M_3(0, t-u')\dot{U}_{,X}(u')du', \end{aligned} \tag{23}$$

$$\begin{aligned} \zeta_0^{-1}\tilde{\eta} = & -\int_{-\infty}^{t} m_1(0, t-u')\ddot{\alpha}(u')du' - \int_{-\infty}^{t} B_1(0, t-u')\dot{\alpha}_{,X}(u')du' - \\ & - \int_{-\infty}^{t} B_2(0, t-u')\dot{U}_{,X}(u')du' + \int_{-\infty}^{t} B_3(0, t-u')\dot{\phi}_{,X}(u')du', \end{aligned} \tag{24}$$

$$\begin{aligned} \frac{\partial\tilde{\psi}}{\partial\beta} = & \int_{-\infty}^{t} m_2(0, t-u')\dot{\alpha}_{,X}(u')du' + \int_{-\infty}^{t} \ddot{\alpha}(s')B_1(t-s', 0)ds' + \\ & + \int_{-\infty}^{t} M_1(0, t-u')\dot{U}_{,X}(u')du' - \int_{-\infty}^{t} M_2(0, t-u')\dot{\phi}_{,X}(u')du'. \end{aligned} \tag{25}$$

Proof. Step 1. With the specific notations (5) to (8) and (19), by the constitutive equation (11) for $\tilde{\sigma}$ we obtain

$$\rho^{-1}\tilde{\sigma} = \int_0^\infty \frac{\partial m_3(0,u)}{\partial u}(U-U^\dagger)_{,X}(t-u)du+ \\ +\int_0^\infty (\dot{\alpha}-\dot{\alpha}^\dagger)(t-s)\frac{\partial B_2(s,0)}{\partial s}ds + \int_0^\infty (\alpha-\alpha^\dagger)_{,X}(t-s)\frac{\partial M_1(s,0)}{\partial s}ds- \\ -\int_0^\infty (\phi-\phi^\dagger)_{,X}(t-s)\frac{\partial M_3(s,0)}{\partial s}ds. \tag{26}$$

We apply integration by parts and we obtain

$$\begin{aligned} &\rho^{-1}\tilde{\sigma} = m_3(0,\infty)(U-U^\dagger)_{,X}(t-\infty)+ \\ &+\int_0^\infty m_3(0,u)\dot{U}_{,X}(t-u)du + (\dot{\alpha}-\dot{\alpha}^\dagger)(t-\infty)B_2(\infty,0)+ \\ &+\int_0^\infty \ddot{\alpha}(t-s)B_2(s,0)ds + (\alpha-\alpha^\dagger)_{,X}(t-\infty)M_1(\infty,0)+ \\ &+\int_0^\infty \dot{\alpha}_{,X}(t-s)M_1(s,0)ds - (\phi-\phi^\dagger)_{,X}(t-\infty)M_3(\infty,0)- \\ &-\int_0^\infty \dot{\phi}_{,X}(t-s)M_3(s,0)ds. \end{aligned} \tag{27}$$

To be more precise, we have the following computations

$$\begin{aligned} &\int_0^\infty \frac{\partial m_3(0,u)}{\partial u}\left[U_{,X}(t-u)-U_{,X}(t)\right]du = \\ &= m_3(0,u)\left[U_{,X}(t-u)-U_{,X}(t)\right]\Big|_{u=0}^{u=\infty} - \\ &-\int_0^\infty m_3(0,u)\frac{\partial}{\partial u}\left[U_{,X}(t-u)-U_{,X}(t)\right]du. \end{aligned} \tag{28}$$

Then we consider that $\frac{\partial}{\partial u}U_{,X}(t)=0$ and $\frac{\partial}{\partial u}U_{,X}(t-u) = -\frac{\partial}{\partial t}U_{,X}(t-u)$ and we denote $\frac{\partial}{\partial t}U_{,X}(t-u) = \dot{U}_{,X}(t-u)$. Hence, the expression above is further equal to

$$m_3(0,\infty)\left[U_{,X}(t-\infty)-U_{,X}(t)\right] + \int_0^\infty m_3(0,u)\dot{U}_{,X}(t-u)du. \tag{29}$$

After we make the change of variable $t-u=u'$ and $t-s=s'$ in (27), we obtain

$$\rho^{-1}\tilde{\sigma} = R + \int_{-\infty}^{t} m_3(0,t-u')\dot{U}_{,X}(u')du' + \int_{-\infty}^{t} \ddot{\alpha}(s')B_2(t-s',0)ds' + \\ + \int_{-\infty}^{t} \dot{\alpha}_{,X}(s')M_1(t-s',0)ds' - \int_{-\infty}^{t} \dot{\phi}_{,X}(s')M_3(t-s',0)ds', \tag{30}$$

where

$$\begin{aligned} R &= m_3(0,\infty)(U-U^{\dagger})_{,X}(t-\infty) + (\dot{\alpha}-\dot{\alpha}^{\dagger})(t-\infty)B_2(\infty,0) + \\ &+ (\alpha-\alpha^{\dagger})_{,X}(t-\infty)M_1(\infty,0) - (\phi-\phi^{\dagger})_{,X}(t-\infty)M_3(\infty,0). \end{aligned} \tag{31}$$

Hypotheses (i)-(v) imply $R=0$, hence Eq. (22) holds true. Conversely, hypotheses (i)-(v) yield $R=0$ in (31), and thus the steps from (30) to (26), that is (11), can be traversed backwards.

Step 2. With the specific notations, we obtain by the constitutive equations for $\tilde{P}$ (12)

$$\begin{aligned} \rho^{-1}\tilde{P} &= \int_0^{\infty} \frac{\partial m_4(0,u)}{\partial u}[\phi_{,X}(t-u)-\phi_{,X}(t)]du - \\ &- \int_0^{\infty} [\dot{\alpha}(t-s)-\dot{\alpha}(t)]\frac{\partial B_3(s,0)}{\partial s}ds - \\ &- \int_0^{\infty} [\alpha_{,X}(t-s)-\alpha_{,X}(t)]\frac{\partial M_2(s,0)}{\partial s}ds - \\ &- \int_0^{\infty} \frac{\partial M_3(0,u)}{\partial u}[U_{,X}(t-u)-U_{,X}(t)]du. \end{aligned} \tag{32}$$

We apply integration by parts and we obtain

$$\begin{aligned} &\rho^{-1}\tilde{P} = m_4(0,\infty)[\phi_{,X}(t-\infty)-\phi_{,X}(t)] + \\ &+ \int_0^{\infty} m_4(0,u)\dot{\phi}_{,X}(t-u)du - [\dot{\alpha}(t-\infty)-\dot{\alpha}(t)]B_3(\infty,0) - \\ &- \int_0^{\infty} \ddot{\alpha}(t-s)B_3(s,0)ds - [\alpha_{,X}(t-\infty)-\alpha_{,X}(t)]M_2(\infty,0) - \\ &- \int_0^{\infty} \dot{\alpha}_{,X}(t-s)M_2(s,0)ds - M_3(0,\infty)[U_{,X}(t-\infty)-U_{,X}(t)] - \\ &- \int_0^{\infty} M_3(0,u)\dot{U}_{,X}(t-u)du. \end{aligned} \tag{33}$$

We do a change of variable and by the assumptions $\Sigma_2=0$, $m_4(0,\infty)=0$, $B_3(\infty,0)=0$, $M_2(\infty,0)=0$ and $M_3(0,\infty)=0$, we obtain (23).

Step 3. With the specific notations, we obtain by the constitutive equations for $\tilde{\eta}$ (13)

$$\begin{aligned}\zeta_0^{-1}\tilde{\eta} = & -\int_0^\infty \frac{\partial m_1(0,u)}{\partial u}[\dot{\alpha}(t-u)-\dot{\alpha}(t)]du- \\ & -\int_0^\infty \frac{\partial B_1(0,u)}{\partial u}[\alpha_{,X}(t-u)-\alpha_{,X}(t)]du- \\ & -\int_0^\infty \frac{\partial B_2(0,u)}{\partial u}[U_{,X}(t-u)-U_{,X}(t)]du+ \\ & +\int_0^\infty \frac{\partial B_3(0,u)}{\partial u}[\phi_{,X}(t-u)-\phi_{,X}(t)]du.\end{aligned} \tag{34}$$

We apply integration by parts and we obtain

$$\begin{aligned}&\zeta_0^{-1}\tilde{\eta} = -m_1(0,\infty)[\dot{\alpha}(t-\infty)-\dot{\alpha}(t)]- \\ &-\int_0^\infty m_1(0,u)\ddot{\alpha}(t-u)du - B_1(0,\infty)[\alpha_{,X}(t-\infty)-\alpha_{,X}(t)]- \\ &-\int_0^\infty B_1(0,u)\dot{\alpha}_{,X}(t-u)du - B_2(0,\infty)[U_{,X}(t-\infty)-U_{,X}(t)]- \\ &-\int_0^\infty B_2(0,u)\dot{U}_{,X}(t-u)du + B_3(0,\infty)[\phi_{,X}(t-\infty)-\phi_{,X}(t)]+ \\ &+\int_0^\infty B_3(0,u)\dot{\phi}_{,X}(t-u)du.\end{aligned} \tag{35}$$

We do a change of variable and by the assumptions $m_1(0,\infty) = 0$, $B_1(0,\infty) = 0$, $B_2(0,\infty) = 0$ and $B_3(0,\infty) = 0$, we obtain (24).

Step 4. We have the constitutive equations for $\tilde{\psi}$.

With the specific notations, we obtain

$$\begin{aligned}\frac{\partial\tilde{\psi}}{\partial\beta} = & \int_0^\infty \frac{\partial m_2(0,u)}{\partial u}\left[\alpha_{,X}(t-u)-\alpha_{,X}(t)\right]du+ \\ & +\int_0^\infty \left[\dot{\alpha}(t-s)-\dot{\alpha}(t)\right]\frac{\partial B_1(s,0)}{\partial s}ds+ \\ & +\int_0^\infty \frac{\partial M_1(0,u)}{\partial u}\left[U_{,X}(t-u)-U_{,X}(t)\right]du- \\ & -\int_0^\infty \frac{\partial M_2(0,u)}{\partial u}\left[\phi_{,X}(t-u)-\phi_{,X}(t)\right]du.\end{aligned} \tag{36}$$

We do an integration by parts and we obtain

$$
\begin{aligned}
\frac{\partial\tilde{\psi}}{\partial\beta} &= m_2(0,\infty)\left[\alpha_{,X}(t-\infty)-\alpha_{,X}(t)\right]+ \\
&+\int_0^\infty m_2(0,u)\dot{\alpha}_{,X}(t-u)du+\left[\dot{\alpha}(t-\infty)-\dot{\alpha}(t)\right]B_1(\infty,0)+ \\
&+\int_0^\infty \ddot{\alpha}(t-s)B_1(s,0)ds+M_1(0,\infty)\left[U_{,X}(t-\infty)-U_{,X}(t)\right]+ \\
&+\int_0^\infty M_1(0,u)\dot{U}_{,X}(t-u)du-M_2(0,\infty)\left[\phi_{,X}(t-\infty)-\phi_{,X}(t)\right]- \\
&-\int_0^\infty M_2(0,u)\dot{\phi}_{,X}(t-u)du.
\end{aligned} \tag{37}
$$

We do a change of variable and by the assumptions $m_2(0,\infty)=0$, $B_1(\infty,0)=0$, $M_1(0,\infty)=0$ and $M_2(0,\infty)=0$, we obtain (25). ■

In the sequel, we define a convolution product of the form

$$(f*g)(X,t)=\int_0^t f(X,t-\tau)g(X,\tau)d\tau. \tag{38}$$

Lemma 1 *Under the assumptions of Proposition 1, the constitutive equations can be written in the form*

$$\rho^{-1}\tilde{\sigma}=\rho^{-1}\underline{\tilde{\sigma}}+\frac{d}{dt}\left(m_3*U_{,X}+B_2*\dot{\alpha}+M_1*\alpha_{,X}+M_3*\phi_{,X}\right), \tag{39}$$

$$\rho^{-1}\tilde{P}=\rho^{-1}\underline{\tilde{P}}+\frac{d}{dt}\left(m_4*\phi_{,X}-B_3*\dot{\alpha}-M_2*\alpha_{,X}-M_3*U_{,X}\right), \tag{40}$$

$$\zeta_0^{-1}\tilde{\eta}=\zeta_0^{-1}\underline{\tilde{\eta}}+\frac{d}{dt}\left(-m_1*\dot{\alpha}-B_1*\alpha_{,X}-B_2*U_{,X}+B_3*\phi_{,X}\right), \tag{41}$$

$$\frac{\partial\tilde{\psi}}{\partial\beta}=\underline{\tilde{\psi}}_{,\beta}+\frac{d}{dt}\left(m_2*\alpha_{,X}+B_1*\dot{\alpha}+M_1*U_{,X}-M_2*\phi_{,X}\right), \tag{42}$$

where

$$
\begin{aligned}
\rho^{-1}\underline{\tilde{\sigma}} &= \int_{-\infty}^0 [\dot{m}_3(t-s)U_{,X}(s)+\dot{B}_2(t-s)\dot{\alpha}(s)+ \\
&+\dot{M}_1(t-s)\alpha_{,X}(s)-\dot{M}_3(t-s)\phi_{,X}(s)]ds,
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
\rho^{-1}\underline{\tilde{P}} &= \int_{-\infty}^0 [\dot{m}_4(t-s)\phi_{,X}(s)-\dot{B}_3(t-s)\dot{\alpha}(s)- \\
&-\dot{M}_2(t-s)\alpha_{,X}(s)-\dot{M}_3(t-s)U_{,X}(s)]ds,
\end{aligned} \tag{44}
$$

$$\zeta_0^{-1}\underline{\tilde{\eta}} = \int_{-\infty}^{0} [-\dot{m}_1(t-s)\dot{\alpha}(s) - \dot{B}_1(t-s)\alpha_{,X}(s) - \\ - \dot{B}_2(t-s)U_{,X}(s) + \dot{B}_3(t-s)\phi_{,X}(s)]ds, \tag{45}$$

$$\underline{\tilde{\psi}}_{,\beta} = \int_{-\infty}^{0} [\dot{m}_2(t-s)\alpha_{,X}(s) + \dot{B}_1(t-s)\dot{\alpha}(s) + \\ + \dot{M}_1(t-s)U_{,X}(s) - \dot{M}_2(t-s)\phi_{,X}(s)]ds. \tag{46}$$

Proof. We only show this equivalence for the first term in (22). The other terms follow similarly.

We have

$$\int_{-\infty}^{t} m_3(0,t-s)\dot{U}_{,X}(s)ds = \int_{-\infty}^{t} m_3(0,t-s)\frac{d}{ds}U_{,X}(s)ds = \\ = \int_{-\infty}^{t} \left[\frac{d}{ds}\Big(m_3(0,t-s)U_{,X}(s)\Big) - \frac{d}{ds}m_3(0,t-s)U_{,X}(s)\right] ds = \\ = m_3(0,t-s)U_{,X}(s)|_{s=-\infty}^{s=t} + \int_{-\infty}^{t} \frac{d}{dt}m_3(0,t-s)U_{,X}(s)ds = \\ = m_3(0,0)U_{,X}(t) - m_3(0,\infty)U_{,X}(-\infty) + \int_{-\infty}^{0} \frac{d}{dt}m_3(0,t-s)U_{,X}(s)ds + \\ + \int_{0}^{t} \frac{d}{dt}m_3(0,t-s)U_{,X}(s)ds. \tag{47}$$

We already assumed that

$$m_3(0,\infty) = 0. \tag{48}$$

Moreover, we have

$$\frac{d}{dt}(m_3 * U_{,X}) = \frac{d}{dt}\int_{0}^{t} m_3(t-s)U_{,X}(s)ds = \\ = m_3(0)U_{,X}(t) + \int_{0}^{t} \frac{d}{dt}m_3(t-s)U_{,X}(s)ds. \tag{49}$$

This proves that

$$\int_{-\infty}^{t} m_3(t-s)\dot{U}_{,X}(s)ds = \int_{-\infty}^{0} \frac{d}{dt}m_3(t-s)U_{,X}(s)ds + \frac{d}{dt}(m_3 * U_{,X}). \tag{50}$$

For convenience, we suppressed the argument X in the relations above. ∎

In the following, we consider two solutions of the problem (9), namely $\left(U^{(1)},\phi^{(1)},\alpha^{(1)}\right)$, $\left(U^{(2)},\phi^{(2)},\alpha^{(2)}\right)$. Then we consider their difference $U =$

$U^{(1)} - U^{(2)}$, $\phi = \phi^{(1)} - \phi^{(2)}$, $\alpha = \alpha^{(1)} - \alpha^{(2)}$. This new solution corresponds to null initial and boundary conditions and to null external body forces and heat supply. Then the linear local balance laws write as

$$\begin{cases} \rho \ddot{U} = \frac{d}{dX}\tilde{\sigma} \\ \rho \dot{\tilde{\eta}} = -\frac{d}{dX}\tilde{p} \; . \\ \frac{d}{dX}\tilde{D} = 0 \end{cases} \tag{51}$$

In the sequel, we impose some initial and boundary conditions. We consider that S_i, with $i = 1, 2, \cdots, 6$ are subsets of ∂B such that $S_1 \cap S_2 = S_3 \cap S_4 = S_5 \cap S_6 = \emptyset$ and $S_1 \cup S_2 = S_3 \cup S_4 = S_5 \cup S_6 = \partial B$. Then we assume

$$\begin{aligned} &U = \bar{U} \text{ on } S_1 \times I, \tilde{\sigma} n = \bar{\sigma} \text{ on } S_2 \times I, \\ &\phi = \bar{\phi} \text{ on } S_3 \times I, \tilde{D} n = \bar{D} \text{ on } S_4 \times I, \\ &\alpha = \bar{\alpha} \text{ on } S_5 \times I, \tilde{p} n = \bar{p} \text{ on } S_6 \times I, \end{aligned} \tag{52}$$

where $\bar{U}, \bar{\sigma}, \bar{\phi}, \bar{D}, \bar{\alpha}, \bar{p}$ are prescribed functions, which are null in the case of the difference solution and $I = [0, \infty)$.

As far as the initial conditions are concerned, we consider that

$$\begin{aligned} &U(X, 0) = a_1(X), \dot{U}(X, 0) = a_2(X), \\ &\alpha(X, 0) = a_3(X), \dot{\alpha}(X, 0) = a_4(X), \\ &\phi(X, 0) = a_5(X), X \in B, \end{aligned} \tag{53}$$

where $a_i = 0$, $i = 1, 2, \cdots, 5$. Unlike U and α, in $(53)_3$ there is no expression for $\dot{\phi}$ because here we use the quasi-static approximation [29], in which Maxwell equations write as $W = -\phi_{,X}$, $d\tilde{D}/dX = 0$ and do not involve time derivatives. Moreover, we impose a null history initial condition. To be more precise, we assume that

$$U(X, t) = 0, \alpha(X, t) = 0, \dot{\alpha}(X, t) = 0, \phi(X, t) = 0 \tag{54}$$

on $\bar{B} \times (-\infty, 0)$.

We present a uniqueness theorem for the solution to the mixed boundary-initial-value problem that adopts the approach of M. Ciarletta and A. Scalia on the linear theory of viscoelasticity for materials with voids [21].

We say that a function $f(X, t)$ has a Laplace transform $\hat{f}(X, t)$ with respect to t if there exists a real number s_0 such that for all real s greater than s_0 the integrals

$$\hat{f}(X, s) = \int_0^\infty e^{-st} f(X, t) dt \tag{55}$$

converge uniformly on $\bar{B}$.

In the sequel, unless otherwise specified, we suppress the argument X.

We present an inequality that will be useful in proving the uniqueness result.

Lemma 2 *Let $A > 0$, $C > 0$. The inequality*

$$Cx^2 + Ay^2 \geq -Bxy, \tag{56}$$

holds true for each $(x, y) \in \mathbb{R} \times \mathbb{R}$ if and only if

$$B^2 \leq 4AC.$$

Proof. Putting $y = mx$ in (56), we obtain the inequality

$$Cx^2 + Am^2x^2 \geq -Bmx^2, \tag{57}$$

that is equivalent to

$$C + Am^2 \geq -Bm. \tag{58}$$

Now in the Cartesian plane (m, n) the parabola $n = Am^2 + C$ with $A, C > 0$ is not below the straight line $n = -Bm$ for each $m \in \mathbb{R}$ when they do not intersect or are tangents to each other. And this is true if and only if the equation

$$Am^2 + Bm + C = 0$$

has at most one solution, that is, if and only if

$$\Delta = B^2 - 4AC \leq 0.$$

In such a case, the inequality (57) holds for each $m \in \mathbb{R}$ and each $x \in \mathbb{R}$, hence for each $(x, y) \in \mathbb{R} \times \mathbb{R}$ since $y = mx$. ■

Now we can state the uniqueness result.

Theorem 1 (Uniqueness) *We make the following assumptions*

(H1) the body is homogeneous;

(H2) the density ρ is strictly positive;

(H3) the constitutive coefficients possess Laplace transforms;

(H4) $\varepsilon_0 > 0$, $k_0 > 0$, $\theta_0 > 0$, $\zeta_0 = T'(\theta_0) > 0$;

(H5) the relaxation functions satisfy

$$\left(\hat{B}_3\right)^2 \leq \hat{\dot{m}}_1 \hat{\dot{m}}_4, \quad (\hat{m}_3)^2 \leq 8\hat{m}_2\hat{M}_1.$$

Then there is at most one solution to the mixed boundary-initial-value problem which has a Laplace transform with respect to time.

Proof. We assume that the process $s = (U, \alpha, \phi, \tilde{\sigma}, \tilde{\eta}, \tilde{p}, \tilde{D})$ is associated to zero data. By applying the Laplace transform to the linear local balance laws (51), we obtain

$$\begin{cases} \rho s^2 \hat{U} = \widehat{\tilde{\sigma}}_{,X} \\ \rho s \hat{\tilde{\eta}} = -\hat{\tilde{p}}_{,X} \\ \hat{\tilde{D}}_{,X} = 0 \end{cases} \tag{59}$$

by using the null initial conditions (53). By applying the Laplace transform (55) to the constitutive equations (39)-(42) and by (54), it follows that

$$\rho^{-1} \hat{\tilde{\sigma}} = s \left(\hat{m}_3 \hat{U}_{,X} + \hat{B}_2 \hat{\dot{\alpha}} + \hat{M}_1 \hat{\alpha}_{,X} - \hat{M}_3 \hat{\phi}_{,X} \right), \tag{60}$$

$$\rho^{-1} \hat{\tilde{P}} = -s \left(\hat{M}_3 \hat{U}_{,X} + \hat{B}_3 \hat{\dot{\alpha}} + \hat{M}_2 \hat{\alpha}_{,X} - \hat{m}_4 \hat{\phi}_{,X} \right), \tag{61}$$

$$\zeta_0^{-1} \hat{\tilde{\eta}} = -s \left(\hat{B}_2 \hat{U}_{,X} + \hat{m}_1 \hat{\dot{\alpha}} + \hat{B}_1 \hat{\alpha}_{,X} - \hat{B}_3 \hat{\phi}_{,X} \right), \tag{62}$$

$$\widehat{\frac{\partial \tilde{\psi}}{\partial \beta}} = s \left(\hat{M}_1 \hat{U}_{,X} + \hat{B}_1 \hat{\dot{\alpha}} + \hat{m}_2 \hat{\alpha}_{,X} - \hat{M}_2 \hat{\phi}_{,X} \right) \tag{63}$$

and

$$\hat{\tilde{D}} = \varepsilon_0 \hat{W} + \hat{\tilde{P}} = -\varepsilon_0 \hat{\phi}_{,X} - \rho s \left(\hat{M}_3 \hat{U}_{,X} + \hat{B}_3 \hat{\dot{\alpha}} + \hat{M}_2 \hat{\alpha}_{,X} - \hat{m}_4 \hat{\phi}_{,X} \right), \tag{64}$$

$$\hat{\tilde{p}} = \theta_0^{-1} \hat{\tilde{q}} = -\zeta_0 \rho \left(\widehat{\frac{\partial \tilde{\psi}}{\partial \beta}} + \rho^{-1} \kappa_0 \hat{\dot{\alpha}}_{,X} \right). \tag{65}$$

The zero boundary conditions (52) lead to

$$\hat{\tilde{\sigma}} \hat{U} n = 0, \quad \hat{\tilde{D}} \hat{\phi} n = 0, \quad \hat{\tilde{p}} \hat{\alpha} n = 0 \text{ on } \partial \mathcal{B} \times I. \tag{66}$$

Let us consider the function

$$E = E_1 - E_3 - E_2 \frac{1}{\zeta_0}, \tag{67}$$

where

$$E_1 = \hat{\tilde{\sigma}} \hat{U}_{,X}, \tag{68}$$

$$E_2 = \hat{\tilde{p}} \hat{\alpha}_{,X}, \tag{69}$$

$$E_3 = \hat{\tilde{D}} \hat{\phi}_{,X} \tag{70}$$

and $\hat{\tilde{p}} = \frac{1}{\theta_0}\hat{\tilde{q}}$ in the linear case. By the constitutive equations (60)-(65), we obtain

$$\begin{aligned}
E &= \rho s \left(\hat{m}_3 \hat{\dot{U}}_{,X} \hat{U}_{,X} + \hat{B}_2 \hat{\dot{\alpha}} \hat{U}_{,X} + \hat{M}_1 \hat{\dot{\alpha}}_{,X} \hat{U}_{,X} - \hat{M}_3 \hat{\dot{\phi}}_{,X} \hat{U}_{,X} \right) \\
&+ \varepsilon_0 \hat{\dot{\phi}}_{,X} \hat{\phi}_{,X} + \rho s \left(-\hat{m}_4 \hat{\dot{\phi}}_{,X} \hat{\phi}_{,X} + \hat{B}_3 \hat{\dot{\alpha}} \hat{\phi}_{,X} + \hat{M}_2 \hat{\dot{\alpha}}_{,X} \hat{\phi}_{,X} + \hat{M}_3 \hat{\dot{U}}_{,X} \hat{\phi}_{,X} \right) \\
&+ \rho s \left(\hat{m}_2 \hat{\dot{\alpha}}_{,X} \hat{\alpha}_{,X} + \hat{B}_1 \hat{\dot{\alpha}} \hat{\alpha}_{,X} + \hat{M}_1 \hat{\dot{U}}_{,X} \hat{\alpha}_{,X} - \hat{M}_2 \hat{\dot{\phi}}_{,X} \hat{\alpha}_{,X} \right) \\
&+ \kappa_0 \hat{\dot{\alpha}}_{,X} \hat{\alpha}_{,X}.
\end{aligned} \tag{71}$$

Moreover, from (67)-(70) in the linear local balance laws (59), we obtain

$$\begin{aligned}
E &= \left(\hat{\tilde{\sigma}} \hat{\dot{U}} \right)_{,X} - \hat{\tilde{\sigma}}_{,X} \hat{\dot{U}} - \left(\hat{\tilde{D}} \hat{\dot{\phi}} \right)_{,X} + \hat{\tilde{D}}_{,X} \hat{\dot{\phi}} - \left(\hat{\tilde{p}} \hat{\dot{\alpha}} \right)_{,X} \frac{1}{\zeta_0} + \hat{\tilde{p}}_{,X} \hat{\dot{\alpha}} \frac{1}{\zeta_0} = \\
&= \left(\hat{\tilde{\sigma}} \hat{\dot{U}} \right)_{,X} - \left(\hat{\tilde{D}} \hat{\dot{\phi}} \right)_{,X} - \left(\hat{\tilde{p}} \hat{\dot{\alpha}} \right)_{,X} \frac{1}{\zeta_0} - \rho s^2 \hat{U} \hat{\dot{U}} - \rho s \hat{\tilde{\eta}} \hat{\dot{\alpha}} \frac{1}{\zeta_0}.
\end{aligned} \tag{72}$$

We equate the two expressions (71) and $(72)_2$ of E and we replace $-\rho s \hat{\tilde{\eta}} \hat{\dot{\alpha}} \frac{1}{\zeta_0}$ by $\rho s^2 \left(\hat{m}_1 \hat{\dot{\alpha}} \hat{\alpha} + \hat{B}_1 \hat{\dot{\alpha}} \hat{\alpha}_{,X} + \hat{B}_2 \hat{U}_{,X} \hat{\dot{\alpha}} - \hat{B}_3 \hat{\dot{\alpha}} \hat{\phi}_{,X} \right)$ from (62). Note that $s\hat{\alpha} = \hat{\dot{\alpha}}$. By integration on B and using the divergence theorem and the boundary conditions (66), we obtain

$$\begin{aligned}
&\int_B \Big[\rho s \left(\hat{m}_3 \hat{\dot{U}}_{,X} \hat{U}_{,X} + 2\hat{M}_1 \hat{\dot{\alpha}}_{,X} \hat{U}_{,X} + \hat{m}_2 \hat{\dot{\alpha}}_{,X} \hat{\alpha}_{,X} - \hat{m}_4 \hat{\dot{\phi}}_{,X} \hat{\phi}_{,X} + \right. \\
&\left. +2\hat{B}_3 \hat{\dot{\alpha}} \hat{\phi}_{,X} - \hat{m}_1 \hat{\dot{\alpha}} \hat{\dot{\alpha}} \right) + \rho s^2 \hat{U} \hat{\dot{U}} + \varepsilon_0 \hat{\dot{\phi}}_{,X} \hat{\phi}_{,X} + \kappa_0 s \hat{\alpha}_{,X} \hat{\alpha}_{,X} \Big] dv = 0.
\end{aligned} \tag{73}$$

We rewrite the integral as

$$\begin{aligned}
&\int_B \rho s \left(\hat{m}_3 \hat{\dot{U}}_{,X} \hat{U}_{,X} + 2\hat{M}_1 \hat{\dot{\alpha}}_{,X} \hat{U}_{,X} + \hat{m}_2 \hat{\dot{\alpha}}_{,X} \hat{\alpha}_{,X} \right) dv + \\
&+ \int_B \rho \left(s^2 2\hat{B}_3 \hat{\alpha} \hat{\phi}_{,X} - \hat{\dot{m}}_4 \hat{\phi}_{,X} \hat{\phi}_{,X} - s^2 \hat{\dot{m}}_1 \hat{\alpha} \hat{\alpha} \right) dv + \\
&+ \int_B \left(\rho s^2 \hat{U} \hat{U} + \varepsilon_0 \hat{\dot{\phi}}_{,X} \hat{\phi}_{,X} + \kappa_0 s \hat{\alpha}_{,X} \hat{\alpha}_{,X} \right) dv = 0.
\end{aligned} \tag{74}$$

Note that $\hat{\dot{m}}_1 < 0$, $\hat{\dot{m}}_4 < 0$. Hence, the last two terms in the second integrand are positive and the latter writes as

$$2\rho s^2 \hat{B}_3 \hat{\alpha} \hat{\phi}_{,X} - \rho \hat{\dot{m}}_4 \hat{\phi}_{,X} \hat{\phi}_{,X} - \rho s^2 \hat{\dot{m}}_1 \hat{\alpha} \hat{\alpha} \geq 0, \tag{75}$$

that is,

$$(-s^2 \hat{\dot{m}}_1) |\hat{\alpha}|^2 + (-\hat{\dot{m}}_4) |\hat{\phi}_{,X}|^2 \geq -2s^2 \hat{B}_3 \hat{\alpha} \hat{\phi}_{,X}. \tag{76}$$

For $C := -s^2\hat{m}_1 \geq 0$, $A := -\hat{\dot{m}}_4 \geq 0$, $B := 2s^2\hat{B}_3$, $x := \hat{\alpha}$, $y := \hat{\dot{\phi}}_{,X}$, it writes as

$$Cx^2 + Ay^2 \geq -Bxy, \tag{77}$$

and by (H5) in Lemma 2, the inequality (75) holds true in every process since $C^2 \leq 4AB \Leftrightarrow \left(\hat{B}_3\right)^2 \leq \hat{\dot{m}}_4\hat{m}_1$.

The same reasoning can be used to show that also the integrand of the first integral in (74) is nonnegative, putting $C := \hat{m}_3$, $A := \hat{m}_2$, $B := 2\hat{M}_1$, $x := \hat{U}_{,X}$, $y := \hat{\alpha}_{,X}$ since, now, $C^2 \leq 4AB \Leftrightarrow (\hat{m}_3)^2 \leq 8\hat{m}_2\hat{M}_1$. Of course, the integrand of the third integral in (74) is nonnegative.

Since the sum of the three nonnegative integrals in (74) is zero, each of them is zero. In particular, the third integral jointly with its integrand are zero.

Then we can conclude that there exists a number $s_1 > 0$ such that if $s > s_1$, we have $\hat{U} = 0$, $\hat{\phi}_{,X} = 0$ and $\hat{\alpha}_{,X} = 0$ on $\bar{B} \times [s_1, \infty)$. By the smoothness of U, ϕ, α, this implies (cf. [21, p. 154]) that $U = 0$, $\phi_{,X} = 0$, $\alpha_{,X} = 0$ on $\bar{B} \times [0, \infty)$. Since we impose null boundary conditions, we have $\phi = 0$ and $\alpha = 0$ on $\bar{B} \times [0, \infty)$. ■

4 Continuous dependence

In this section, we present a continuous dependence result based on the approach of M. Ciarletta and A. Scalia from [21].

Now, we present two lemmas that will be useful in proving the continuous dependence result.

Lemma 3 *Under the assumptions in Section 2, along any kinetic process (2), the following relations hold true:*

$$\begin{aligned}
&\dot{\phi}_{,X}(t)\int_0^\infty \dot{m}_4(u)\phi_{,X}{}_d^t(u)du = -\frac{1}{2}\frac{\partial}{\partial u}\int_0^\infty \dot{m}_4(u)\phi_{,X}{}_d^t(u)\phi_{,X}{}_d^t(u)du \\
&+\frac{1}{2}\int_0^\infty \ddot{m}_4(u)\phi_{,X}{}_d^t(u)\phi_{,X}{}_d^t(u)du - \frac{1}{2}\frac{\partial}{\partial t}\int_0^\infty \dot{m}_4(u)\phi_{,X}{}_d^t(u)\phi_{,X}{}_d^t(u)du,
\end{aligned} \tag{78}$$

$$\begin{aligned}
&\dot{U}_{,X}(t)\int_0^\infty \dot{m}_3(u)U_{,X}{}_d^t(u)du = -\frac{1}{2}\frac{\partial}{\partial u}\int_0^\infty \dot{m}_3(u)U_{,X}{}_d^t(u)U_{,X}{}_d^t(u)du \\
&+\frac{1}{2}\int_0^\infty \ddot{m}_3(u)U_{,X}{}_d^t(u)U_{,X}{}_d^t(u)du - \frac{1}{2}\frac{\partial}{\partial t}\int_0^\infty \dot{m}_3(u)U_{,X}{}_d^t(u)U_{,X}{}_d^t(u)du,
\end{aligned} \tag{79}$$

$$\dot{\alpha}_{,X}(t)\int_0^\infty \dot{m}_2(u)\alpha_{,X}{}^t_d(u)du = -\frac{1}{2}\frac{\partial}{\partial u}\int_0^\infty \dot{m}_2(u)\alpha_{,X}{}^t_d(u)\alpha_{,X}{}^t_d(u)du$$
$$+\frac{1}{2}\int_0^\infty \ddot{m}_2(u)\alpha_{,X}{}^t_d(u)\alpha_{,X}{}^t_d(u)du - \frac{1}{2}\frac{\partial}{\partial t}\int_0^\infty \dot{m}_2(u)\alpha_{,X}{}^t_d(u)\alpha_{,X}{}^t_d(u)du, \tag{80}$$

$$\int_0^\infty \dot{M}_1(s)\left[\dot{U}_{,X}(t)\alpha_{,X}{}^t_d(s) + \dot{\alpha}_{,X}(t)U_{,X}{}^t_d(s)\right]ds =$$
$$= -\frac{\partial}{\partial s}\int_0^\infty \dot{M}_1(s)\alpha_{,X}{}^t_d(s)U_{,X}{}^t_d(s)ds + \int_0^\infty \ddot{M}_1(s)\alpha_{,X}{}^t_d(s)U_{,X}{}^t_d(s)ds \tag{81}$$
$$-\frac{\partial}{\partial t}\int_0^\infty \dot{M}_1(s)\alpha_{,X}{}^t_d(s)U_{,X}{}^t_d(s)ds.$$

Proof. In each step, we prove one of the relations from the statement of the lemma.

Step 1. We prove (78) by considering the term

$$\dot{\phi}_{,X}(t)\int_0^\infty \dot{m}_4(u)\phi_{,X}{}^t_d(u)du. \tag{82}$$

First, we compute

$$\frac{\partial}{\partial t}\int_0^\infty \dot{m}_4(u)\phi_{,X}{}^t_d(u)\phi_{,X}{}^t_d(u)du = 2\int_0^\infty \dot{m}_4(u)\phi_{,X}{}^t_d(u)\frac{\partial}{\partial t}\phi_{,X}{}^t_d(u)du. \tag{83}$$

On the other hand, we have

$$\frac{\partial}{\partial u}\int_0^\infty \dot{m}_4(u)\phi_{,X}{}^t_d(u)\phi_{,X}{}^t_d(u)du$$
$$= \int_0^\infty \left[\ddot{m}_4(u)\phi_{,X}{}^t_d(u)\phi_{,X}{}^t_d(u) + 2\dot{m}_4(u)\phi_{,X}{}^t_d(u)\frac{\partial}{\partial u}\phi_{,X}{}^t_d(u)\right]du. \tag{84}$$

Since

$$\phi_{,X}{}^t_d(u) = \phi_{,X}(t-u) - \phi_{,X}(t), \tag{85}$$

we obtain

$$\frac{\partial}{\partial t}\phi_{,X}{}^t_d(u) + \frac{\partial}{\partial u}\phi_{,X}{}^t_d(u) = -\frac{\partial}{\partial t}\phi_{,X}(t). \tag{86}$$

By using the relation (86), we replace $\frac{\partial}{\partial u}\phi_{,X}{}^t_d(u)$ in the relation (84). Hence, we obtain

$$\frac{\partial}{\partial u}\int_0^\infty \dot{m}_4(u)\phi_{,X}{}^t_d(u)\phi_{,X}{}^t_d(u)du = \int_0^\infty \left[\ddot{m}_4(u)\phi_{,X}{}^t_d(u)\phi_{,X}{}^t_d(u) -\right.$$
$$\left. -2\dot{m}_4(u)\phi_{,X}{}^t_d(u)\frac{\partial}{\partial t}\phi_{,X}(t) - 2\dot{m}_4(u)\phi_{,X}{}^t_d(u)\frac{\partial}{\partial t}\phi_{,X}{}^t_d(u)\right]du. \tag{87}$$

Then we replace the last term with the expression from (83). We obtain

$$\frac{\partial}{\partial u}\int_0^\infty \dot m_4(u)\phi_{,X_d^t}(u)\phi_{,X_d^t}(u)du = \int_0^\infty \ddot m_4(u)\phi_{,X_d^t}(u)\phi_{,X_d^t}(u)du- \\ -2\int_0^\infty \dot m_4(u)\phi_{,X_d^t}(u)\frac{\partial}{\partial t}\phi_{,X}(t)du - \frac{\partial}{\partial t}\int_0^\infty \dot m_4(u)\phi_{,X_d^t}(u)\phi_{,X_d^t}(u)du. \tag{88}$$

Hence, from (83) and (88) we have

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial u}\right)\int_0^\infty \dot m_4(u)\phi_{,X_d^t}(u)\phi_{,X_d^t}(u)du = -2\dot\phi_{,X}(t)\int_0^\infty \dot m_4(u)\phi_{,X_d^t}(u)du+ \\ +\int_0^\infty \ddot m_4(u)\phi_{,X_d^t}(u)\phi_{,X_d^t}(u)du,$$

that is, (78) holds true.

Steps 2, 3. The proofs for (79), (80) are similar to Step 1.

Step 4. We prove (81). First we compute

$$\begin{aligned}&\frac{\partial}{\partial t}\int_0^\infty \dot M_1(s)\alpha_{,X_d^t}(s)U_{,X_d^t}(s)ds \\ &=\int_0^\infty \dot M_1(s)\left[\frac{\partial}{\partial t}\alpha_{,X_d^t}(s)U_{,X_d^t}(s)+\alpha_{,X_d^t}(s)\frac{\partial}{\partial t}U_{,X_d^t}(s)\right]ds.\end{aligned} \tag{89}$$

On the other hand, we obtain

$$\begin{aligned}&\frac{\partial}{\partial s}\int_0^\infty \dot M_1(s)\alpha_{,X_d^t}(s)U_{,X_d^t}(s)ds \\ &=\int_0^\infty \left[\ddot M_1(s)\alpha_{,X_d^t}(s)U_{,X_d^t}(s)+\dot M_1(s)\frac{\partial}{\partial s}\alpha_{,X_d^t}(s)U_{,X_d^t}(s)+\right. \\ &\left.+\dot M_1(s)\alpha_{,X_d^t}(s)\frac{\partial}{\partial s}U_{,X_d^t}(s)\right]ds.\end{aligned} \tag{90}$$

Replacing in (90) the expression

$$\frac{\partial}{\partial s}\psi_{,X_d^t}(s) = -\frac{\partial}{\partial t}\psi_{,X}(t)-\frac{\partial}{\partial t}\psi_{,X_d^t}(s) \tag{91}$$

for $\psi = U,\alpha$, we obtain

$$\frac{\partial}{\partial s}\int_0^\infty \dot M_1(s)\alpha_{,X_d^t}(s)U_{,X_d^t}(s)ds = \int_0^\infty \ddot M_1(s)\alpha_{,X_d^t}(s)U_{,X_d^t}(s)ds- \\ -\int_0^\infty \dot M_1(s)\frac{\partial}{\partial t}\alpha_{,X}(t)U_{,X_d^t}(s)ds - \int_0^\infty \dot M_1(s)\frac{\partial}{\partial t}\alpha_{,X_d^t}(s)U_{,X_d^t}(s)ds- \\ -\int_0^\infty \dot M_1(s)\alpha_{,X_d^t}(s)\frac{\partial}{\partial t}U_{,X}(t)ds - \int_0^\infty \dot M_1(s)\alpha_{,X_d^t}(s)\frac{\partial}{\partial t}U_{,X_d^t}(s)ds. \tag{92}$$

Hence, by (89) and (92) we have

$$\left(\frac{\partial}{\partial s}+\frac{\partial}{\partial t}\right)\int_0^\infty \dot{M}_1(s)\alpha_{,X}{}_d^t(s)U_{,X}{}_d^t(s)ds=\int_0^\infty \ddot{M}_1(s)\alpha_{,X}{}_d^t(s)U_{,X}{}_d^t(s)ds-$$
$$-\int_0^\infty \dot{M}_1(s)\left[\dot{\alpha}_{,X}(t)U_{,X}{}_d^t(s)+\alpha_{,X}{}_d^t(t)\dot{U}_{,X}(t)\right]ds,$$

that is, (81). ■

Lemma 4 *Under the assumptions in Section 2, along any kinetic process, the following relations hold true:*

$$\begin{aligned}
&-\rho\dot{U}_{,X}\int_0^\infty \phi_{,X}{}_d^t(s)\dot{M}_3(s)ds+\rho\dot{\phi}_{,X}\int_0^\infty \dot{M}_3(u)U_{,X}{}_d^t(u)du=\\
&=\frac{\partial}{\partial s}\rho\int_0^\infty \dot{M}_3(s)\phi_{,X}{}_d^t(s)U_{,X}{}_d^t(s)ds-\rho\int_0^\infty \ddot{M}_3(s)\phi_{,X}{}_d^t(s)U_{,X}{}_d^t(s)ds+\\
&+\frac{\partial}{\partial t}\rho\int_0^\infty \dot{M}_3(s)\phi_{,X}{}_d^t(s)U_{,X}{}_d^t(s)ds-\\
&-2\rho\int_0^\infty \dot{M}_3(s)\left(\frac{\partial}{\partial s}+\frac{\partial}{\partial t}\right)\phi_{,X}{}_d^t(s)U_{,X}{}_d^t(s)ds,
\end{aligned}\tag{93}$$

$$\begin{aligned}
&-\rho\dot{\alpha}_{,X}\int_0^\infty \phi_{,X}{}_d^t(s)\dot{M}_2(s)ds+\rho\dot{\phi}_{,X}\int_0^\infty \dot{M}_2(u)\alpha_{,X}{}_d^t(u)du=\\
&=\frac{\partial}{\partial s}\rho\int_0^\infty \dot{M}_2(s)\phi_{,X}{}_d^t(s)\alpha_{,X}{}_d^t(s)ds-\rho\int_0^\infty \ddot{M}_2(s)\phi_{,X}{}_d^t(s)\alpha_{,X}{}_d^t(s)ds+\\
&+\frac{\partial}{\partial t}\rho\int_0^\infty \dot{M}_2(s)\phi_{,X}{}_d^t(s)\alpha_{,X}{}_d^t(s)ds-\\
&-2\rho\int_0^\infty \dot{M}_2(s)\left(\frac{\partial}{\partial s}+\frac{\partial}{\partial t}\right)\phi_{,X}{}_d^t(s)\alpha_{,X}{}_d^t(s)ds,
\end{aligned}\tag{94}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\int_0^\infty \dot{B}_1(u)\alpha_{,X}{}_d^t(u)du\right)\rho\dot{\alpha}(t)-\rho\dot{\alpha}_{,X}(t)\int_0^\infty \dot{\alpha}_d^t(u)\dot{B}_1(u)du=\\
&=\rho\int_0^\infty \dot{B}_1(u)\left[\dot{\alpha}(t)\right]^2\frac{\partial}{\partial X}\left[\frac{\frac{\partial}{\partial t}\alpha_d^t(u)}{\dot{\alpha}(t)}\right]du,
\end{aligned}\tag{95}$$

$$\frac{\partial}{\partial t}\left(\int_0^\infty \dot{B}_2(u)U_{,X}{}_d^t(u)du\right)\rho\dot{\alpha}(t) - \rho\dot{U}_{,X}\int_0^\infty \dot{\alpha}_d^t(u)\dot{B}_2(u)du =$$
$$= \rho\frac{\partial}{\partial u}\int_0^\infty \dot{B}_2(u)U_{,X}{}_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u)du-$$
$$-\rho\int_0^\infty \ddot{B}_2(u)U_{,X}{}_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u)du - \rho\frac{\partial}{\partial t}\int_0^\infty \dot{B}_2(u)U_{,X}{}_d^t(u)\frac{\partial}{\partial u}\alpha_d^t(u)du, \tag{96}$$

$$\frac{\partial}{\partial t}\left(\int_0^\infty \dot{m}_1(u)\dot{\alpha}_d^t(u)du\right)\rho\dot{\alpha}(t) = -\frac{1}{2}\rho\frac{\partial}{\partial t}\int_0^\infty \dot{m}_1(u)\dot{\alpha}(t)\dot{\alpha}(t)du+$$
$$+\frac{1}{2}\rho\frac{\partial}{\partial t}\int_0^\infty \dot{m}_1(u)\frac{\partial}{\partial u}\alpha_d^t(u)\frac{\partial}{\partial u}\alpha_d^t(u)du + \frac{1}{2}\rho\frac{\partial}{\partial u}\int_0^\infty \dot{m}_1(u)\frac{\partial}{\partial t}\alpha_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u)du-$$
$$-\frac{1}{2}\rho\int_0^\infty \ddot{m}_1(u)\frac{\partial}{\partial t}\alpha_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u)du, \tag{97}$$

$$-\frac{\partial}{\partial t}\left(\int_0^\infty \dot{B}_3(u)\phi_{,X}{}_d^t(u)du\right)\rho\dot{\alpha}(t) - \rho\dot{\phi}_{,X}(t)\int_0^\infty \dot{\alpha}_d^t(u)\dot{B}_3(u)du =$$
$$= \rho\frac{\partial}{\partial u}\int_0^\infty \dot{B}_3(u)\phi_{,X}{}_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u)du - \rho\int_0^\infty \ddot{B}_3(u)\phi_{,X}{}_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u)du+$$
$$+\rho\frac{\partial}{\partial t}\int_0^\infty \dot{B}_3(u)\phi_{,X}{}_d^t(u)\frac{\partial}{\partial u}\alpha_d^t(u)du + 2\rho\frac{\partial}{\partial t}\int_0^\infty \dot{B}_3(u)\phi_{,X}{}_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u)du-$$
$$-2\rho\int_0^\infty \dot{B}_3(u)\phi_{,X}{}_d^t(u)\frac{\partial}{\partial t}\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial t}\right)\alpha_d^t(u)du. \tag{98}$$

Proof. Step 1. Proof of (93). Recall that

$$\frac{\partial}{\partial t}\phi_{,X}(t) = -\frac{\partial}{\partial t}\phi_{,X}{}_d^t(s) - \frac{\partial}{\partial s}\phi_{,X}{}_d^t(s), \tag{99}$$

$$-\frac{\partial}{\partial t}U_{,X}(t) = \frac{\partial}{\partial s}U_{,X}{}_d^t(s) + \frac{\partial}{\partial t}U_{,X}{}_d^t(s). \tag{100}$$

Therefore, we obtain

$$
\begin{aligned}
&-\rho\dot{U}_{,X}\int_0^\infty \phi_{,X_d^t}(s)\dot{M}_3(s)ds+\rho\dot{\phi}_{,X}\int_0^\infty \dot{M}_3(u)U_{,X_d^t}(u)du=\\
&=\rho\int_0^\infty \dot{M}_3(s)\left\{\left[-\frac{\partial}{\partial t}\phi_{,X_d^t}(s)-\frac{\partial}{\partial s}\phi_{,X_d^t}(s)\right]U_{,X_d^t}(s)+\right.\\
&+\left.\left[\frac{\partial}{\partial s}U_{,X_d^t}(s)+\frac{\partial}{\partial t}U_{,X_d^t}(s)\right]\phi_{,X_d^t}(s)\right\}ds=\\
&=\rho\int_0^\infty \dot{M}_3(s)\left[\frac{\partial}{\partial s}U_{,X_d^t}(s)\phi_{,X_d^t}(s)-\frac{\partial}{\partial s}\phi_{,X_d^t}(s)U_{,X_d^t}(s)+\right.\\
&\left.+\frac{\partial}{\partial t}U_{,X_d^t}(s)\phi_{,X_d^t}(s)-\frac{\partial}{\partial t}\phi_{,X_d^t}(s)U_{,X_d^t}(s)\right]ds=\\
&=\rho\int_0^\infty \dot{M}_3(s)\left[\phi_{,X_d^t}(s)\right]^2\left\{\frac{\partial}{\partial s}\left[\frac{U_{,X_d^t}(s)}{\phi_{,X_d^t}(s)}\right]+\frac{\partial}{\partial t}\left[\frac{U_{,X_d^t}(s)}{\phi_{,X_d^t}(s)}\right]\right\}ds.
\end{aligned}
\tag{101}
$$

Note that

$$
\begin{aligned}
&\rho\int_0^\infty \dot{M}_3(s)\left[\phi_{,X_d^t}(s)\right]^2\frac{\partial}{\partial s}\left[\frac{U_{,X_d^t}(s)}{\phi_{,X_d^t}(s)}\right]ds=\\
&=\frac{\partial}{\partial s}\rho\int_0^\infty \dot{M}_3(s)\phi_{,X_d^t}(s)U_{,X_d^t}(s)ds-\rho\int_0^\infty \ddot{M}_3(s)\phi_{,X_d^t}(s)U_{,X_d^t}(s)ds-\\
&-\rho\int_0^\infty \dot{M}_3(s)2\frac{\partial}{\partial s}\phi_{,X_d^t}(s)U_{,X_d^t}(s)ds
\end{aligned}
\tag{102}
$$

and

$$
\begin{aligned}
&\rho\int_0^\infty \dot{M}_3(s)\left[\phi_{,X_d^t}(s)\right]^2\frac{\partial}{\partial t}\left[\frac{U_{,X_d^t}(s)}{\phi_{,X_d^t}(s)}\right]ds=\\
&=\frac{\partial}{\partial t}\rho\int_0^\infty \dot{M}_3(s)\phi_{,X_d^t}(s)U_{,X_d^t}(s)ds-\rho\int_0^\infty \dot{M}_3(s)2\frac{\partial}{\partial t}\phi_{,X_d^t}(s)U_{,X_d^t}(s)ds.
\end{aligned}
\tag{103}
$$

By plugging relations (102) and (103) into (101), we obtain (93).

Step 2. Eq. (94) follows from the previous one with u replaced by α and M_3 replaced by M_2.

Step 3. Proof of (95). Let us write the left-hand side in the form

$$
\rho\int_0^\infty \dot{B}_1(u)\left\{\frac{\partial}{\partial X}\left[\frac{\partial}{\partial t}\alpha_d^t(u)\right]\dot{\alpha}(t)-\frac{\partial}{\partial X}\dot{\alpha}(t)\frac{\partial}{\partial t}\alpha_d^t(u)\right\}du \tag{104}
$$

and recall the differentiation rule of a quotient of functions.

Step 4. Proof of (96). Recall that

$$\frac{\partial}{\partial t}\alpha(t) = -\frac{\partial}{\partial u}\alpha_d^t(u) - \frac{\partial}{\partial t}\alpha_d^t(u), \tag{105}$$

$$\frac{\partial}{\partial t}U_{,X}(t) = -\frac{\partial}{\partial u}U_{,X}{}_d^t(u) - \frac{\partial}{\partial t}U_{,X}{}_d^t(u). \tag{106}$$

Then for the left-hand side of (96) we have

$$\begin{aligned} S &:= \frac{\partial}{\partial t}\left(\int_0^\infty \dot{B}_2(u)U_{,X}{}_d^t(u)du\right)\rho\dot{\alpha}(t) - \rho\dot{U}_{,X}\int_0^\infty \dot{\alpha}_d^t(u)\dot{B}_2(u)du \\ &= \rho\int_0^\infty \dot{B}_2(u)\left[-\frac{\partial}{\partial u}\alpha_d^t(u) - \frac{\partial}{\partial t}\alpha_d^t(u)\right]\frac{\partial}{\partial t}U_{,X}{}_d^t(u)du \\ &- \rho\int_0^\infty \dot{B}_2(u)\left[-\frac{\partial}{\partial u}U_{,X}{}_d^t(u) - \frac{\partial}{\partial t}U_{,X}{}_d^t(u)\right]\dot{\alpha}_d^t(u)du. \end{aligned} \tag{107}$$

By replacing $\dot{\alpha}_d^t(u)$ in the last integral by

$$\dot{\alpha}_d^t(u) = \frac{\partial\alpha}{\partial t}(t-u) - \frac{\partial\alpha}{\partial t}(t) \tag{108}$$

we obtain

$$\begin{aligned} S &= \rho\int_0^\infty \dot{B}_2(u)\left[-\frac{\partial}{\partial u}\alpha_d^t(u)\frac{\partial}{\partial t}U_{,X}{}_d^t(u) + \frac{\partial}{\partial u}U_{,X}{}_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u)\right]du \\ &= \rho\int_0^\infty \dot{B}_2(u)\frac{\partial}{\partial u}\left(U_{,X}{}_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u)\right)du \\ &- \rho\int_0^\infty \dot{B}_2(u)\frac{\partial}{\partial t}\left(U_{,X}{}_d^t(u)\frac{\partial}{\partial u}\alpha_d^t(u)\right)du \\ &= \rho\int_0^\infty \frac{\partial}{\partial u}\left(\dot{B}_2(u)U_{,X}{}_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u)\right)du \\ &- \rho\int_0^\infty \ddot{B}_2(u)U_{,X}{}_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u)du - \rho\frac{\partial}{\partial t}\int_0^\infty \dot{B}_2(u)U_{,X}{}_d^t(u)\frac{\partial}{\partial u}\alpha_d^t(u)du. \end{aligned} \tag{109}$$

Step 5. Proof of (97). Note that by (105) we have

$$\frac{\partial}{\partial t}\dot{\alpha}_d^t(u) = -\frac{\partial}{\partial t}\dot{\alpha}(t) - \frac{\partial}{\partial u}\dot{\alpha}_d^t(u) \tag{110}$$

and the left-hand side of (97) writes as

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\int_0^\infty \dot{m}_1(u)\dot{\alpha}_d^t(u)du\right)\rho\dot{\alpha}(t) = \rho\int_0^\infty \dot{m}_1(u)\frac{\partial}{\partial t}\left(\dot{\alpha}_d^t(u)\right)\dot{\alpha}(t)du = \\ &= -\rho\int_0^\infty \dot{m}_1(u)\dot{\alpha}(t)\ddot{\alpha}(t)du - \rho\int_0^\infty \dot{m}_1(u)\dot{\alpha}(t)\frac{\partial}{\partial u}\dot{\alpha}_d^t(u)du. \end{aligned} \tag{111}$$

Now, recalling (105), we compute the last integral

$$
\begin{aligned}
&-\rho\int_0^\infty \dot m_1(u)\dot\alpha(t)\frac{\partial}{\partial u}\dot\alpha_d^t(u)du=\\
&=\rho\int_0^\infty \dot m_1(u)\frac{\partial}{\partial u}\alpha_d^t(u)\frac{\partial}{\partial u}\dot\alpha_d^t(u)du+\rho\int_0^\infty \dot m_1(u)\frac{\partial}{\partial t}\alpha_d^t(u)\frac{\partial}{\partial u}\dot\alpha_d^t(u)du\\
&=\frac{1}{2}\rho\frac{\partial}{\partial t}\int_0^\infty \dot m_1(u)\frac{\partial}{\partial u}\alpha_d^t(u)\frac{\partial}{\partial u}\alpha_d^t(u)du+\frac{1}{2}\rho\frac{\partial}{\partial u}\int_0^\infty \dot m_1(u)\frac{\partial}{\partial t}\alpha_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u)du\\
&-\frac{1}{2}\rho\int_0^\infty \ddot m_1(u)\frac{\partial}{\partial t}\alpha_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u)du.
\end{aligned}
\tag{112}
$$

Step 6. Proof of (98). Employing (105) several times in the left-hand side of (98), we have

$$
\begin{aligned}
&-\frac{\partial}{\partial t}\left(\int_0^\infty \dot B_3(u)\phi_{,X}{}_d^t(u)du\right)\rho\dot\alpha(t)-\rho\dot\phi_{,X}(t)\int_0^\infty \dot\alpha_d^t(u)\dot B_3(u)du\\
&=-\rho\int_0^\infty \dot B_3(u)\left\{\frac{\partial}{\partial t}\phi_{,X}{}_d^t(u)\left[-\frac{\partial}{\partial u}\alpha_d^t(u)-\frac{\partial}{\partial t}\alpha_d^t(u)\right]\right.\\
&\left.+\left[-\frac{\partial}{\partial u}\phi_{,X}{}_d^t(u)-\frac{\partial}{\partial t}\phi_{,X}{}_d^t(u)\right]\dot\alpha_d^t(u)\right\}du
\end{aligned}
\tag{113}
$$

$$
\begin{aligned}
&=\rho\int_0^\infty \dot B_3(u)\left[\frac{\partial}{\partial t}\phi_{,X}{}_d^t(u)\frac{\partial}{\partial u}\alpha_d^t(u)+\frac{\partial}{\partial u}\phi_{,X}{}_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u)\right.\\
&\left.+2\frac{\partial}{\partial t}\phi_{,X}{}_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u)\right]du=\rho\int_0^\infty \dot B_3(u)\left[\frac{\partial}{\partial u}\left(\phi_{,X}{}_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u)\right)\right.\\
&+\frac{\partial}{\partial t}\left(\phi_{,X}{}_d^t(u)\frac{\partial}{\partial u}\alpha_d^t(u)\right)+2\frac{\partial}{\partial t}\left(\phi_{,X}{}_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u)\right)\\
&\left.-2\phi_{,X}{}_d^t(u)\left(\frac{\partial^2}{\partial t\partial u}\alpha_d^t(u)+\frac{\partial^2}{\partial t^2}\alpha_d^t(u)\right)\right]du.
\end{aligned}
\tag{114}
$$

After writing the derivatives in a suitable way, we derive the final result. This finishes the proof. ■

In the sequel, we use the notations below that will be useful in proving the continuous dependence result

$$\begin{aligned}
W(t,u) &= -\dot{m}_4(u)\phi_{,X}{}_d^t(u)\phi_{,X}{}_d^t(u) + \dot{m}_3(u)U_{,X}{}_d^t(u)U_{,X}{}_d^t(u) + \\
&+ \dot{m}_2(u)\alpha_{,X}{}_d^t(u)\alpha_{,X}{}_d^t(u) + 2\dot{M}_1(u)\alpha_{,X}{}_d^t(u)U_{,X}{}_d^t(u) - \\
&- 2\dot{M}_3(u)\phi_{,X}{}_d^t(u)U_{,X}{}_d^t(u) - 2\dot{M}_2(u)\phi_{,X}{}_d^t(u)\alpha_{,X}{}_d^t(u) - \\
&- 2\dot{B}_2(u)U_{,X}{}_d^t(u)\frac{\partial}{\partial u}\alpha_d^t(u) + \dot{m}_1(u)\frac{\partial}{\partial u}\alpha_d^t(u)\frac{\partial}{\partial u}\alpha_d^t(u) - \\
&- \dot{m}_1(u)\dot{\alpha}(t)\dot{\alpha}(t) + 2\dot{B}_3(u)\phi_{,X}{}_d^t(u)\frac{\partial}{\partial u}\alpha_d^t(u) + \\
&+ 4\dot{B}_3(u)\phi_{,X}{}_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u),
\end{aligned} \tag{115}$$

$$\begin{aligned}
M(t,u) &= -\ddot{m}_4(u)\phi_{,X}{}_d^t(u)\phi_{,X}{}_d^t(u) + \ddot{m}_3(u)U_{,X}{}_d^t(u)U_{,X}{}_d^t(u) + \\
&+ \ddot{m}_2(u)\alpha_{,X}{}_d^t(u)\alpha_{,X}{}_d^t(u) + 2\ddot{M}_1(u)\alpha_{,X}{}_d^t(u)U_{,X}{}_d^t(u) - \\
&- 2\ddot{M}_3(u)\phi_{,X}{}_d^t(u)U_{,X}{}_d^t(u) - 2\ddot{M}_2(u)\phi_{,X}{}_d^t(u)\alpha_{,X}{}_d^t(u) + \\
&+ 2\ddot{B}_2(u)U_{,X}{}_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u) + \ddot{m}_1(u)\frac{\partial}{\partial t}\alpha_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u) + \\
&+ 2\ddot{B}_3(u)\phi_{,X}{}_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u),
\end{aligned} \tag{116}$$

$$\begin{aligned}
Q(t,u) &= 2\dot{M}_3(u)\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial t}\right)\phi_{,X}{}_d^t(u)U_{,X}{}_d^t(u) + \\
&+ 2\dot{M}_2(u)\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial t}\right)\phi_{,X}{}_d^t(u)\alpha_{,X}{}_d^t(u) - \\
&- 2\dot{B}_3(u)\phi_{,X}{}_d^t(u)\frac{\partial}{\partial t}\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial t}\right)\alpha_d^t(u) + \\
&+ \dot{B}_1(u)\left[\dot{\alpha}(t)\right]^2\frac{\partial}{\partial X}\left[\frac{\frac{\partial}{\partial t}\alpha_d^t(u)}{\dot{\alpha}(t)}\right],
\end{aligned} \tag{117}$$

$$\mathcal{U}(t) = \int_B \left[\rho\dot{U}(t)\dot{U}(t) + \varepsilon_0\phi_{,X}(t)\phi_{,X}(t) + \kappa_0\int_0^t (\dot{\alpha}_{,X}(\tau))^2 d\tau\right] dv. \tag{118}$$

Using the notations introduced above, we derive the following result.

Theorem 2 *Under the assumptions in Section 2 and in Theorem 1, along any kinetic process (2) associated with the body force f and heat supply s, we have*

$$\begin{aligned}
\dot{\mathcal{U}} &= \int_{\partial B}\left(\tilde{\sigma}\dot{U}n - \tilde{D}\dot{\phi}n - \tilde{p}\dot{\alpha}n\frac{1}{\zeta_0}\right) + \int_B \rho\left(f\dot{U} + s\dot{\alpha}\frac{1}{\zeta_0}\right) - \\
&- \frac{1}{2}\rho\int_0^\infty\int_B M dv ds + \frac{1}{2}\rho\frac{d}{dt}\left(\int_0^\infty\int_B W dv ds\right) + \rho\int_0^\infty\int_B Q dv ds.
\end{aligned} \tag{119}$$

Proof. Along a thermo-electro-viscoelastic process associated with the body force f and heat supply s, we have

$$\begin{aligned}
&\tilde{\sigma}\dot{U}_{,X} - \tilde{D}\dot{\phi}_{,X} - \tilde{p}\dot{\alpha}_{,X}\frac{1}{\zeta_0} = \\
&= \left(\tilde{\sigma}\dot{U}\right)_{,X} - \tilde{\sigma}_{,X}\dot{U} - \left(\tilde{D}\dot{\phi}\right)_{,X} + \tilde{D}_{,X}\dot{\phi} - \left(\tilde{p}\dot{\alpha}\right)_{,X}\frac{1}{\zeta_0} + \left(\tilde{p}_{,X}\dot{\alpha}\right)\frac{1}{\zeta_0} = \\
&= \left(\tilde{\sigma}\dot{U}\right)_{,X} - \left(\tilde{D}\dot{\phi}\right)_{,X} - \left(\tilde{p}\dot{\alpha}\right)_{,X}\frac{1}{\zeta_0} - \rho\ddot{U}\dot{U} + \rho f\dot{U} - \rho\dot{\tilde{\eta}}\frac{1}{\zeta_0}\dot{\alpha} + \rho s\dot{\alpha}\frac{1}{\zeta_0}.
\end{aligned} \tag{120}$$

The latter equality follows from the local balance laws (9). Note that by the constitutive equation (13) for $\tilde{\eta}$ we deduce that we can rewrite the following term on the right-hand side

$$\begin{aligned}
\rho\dot{\tilde{\eta}}\frac{1}{\zeta_0}\dot{\alpha} = \Big\{ &-\frac{\partial}{\partial t}\left[\Sigma_3(T-T_0)\right] - \frac{\partial}{\partial t}\int_0^\infty \dot{m}_1(u)\dot{\alpha}_d^t(u)du - \\
&- \frac{\partial}{\partial t}\int_0^\infty \dot{B}_1(u)\alpha_{,X}{}_d^t(u)du - \frac{\partial}{\partial t}\int_0^\infty \dot{B}_2(u)U_{,X}{}_d^t(u)du + \\
&+ \frac{\partial}{\partial t}\int_0^\infty \dot{B}_3(u)\phi_{,X}{}_d^t(u)du \Big\}\rho\dot{\alpha}.
\end{aligned} \tag{121}$$

By replacing the constitutive equations (11) for $\tilde{\sigma}$, (12) for $\tilde{P}$ and (17) for $\frac{\partial\tilde{\psi}}{\partial\beta}$, we obtain

$$\begin{aligned}
&\tilde{\sigma}\dot{U}_{,X} - \tilde{D}\dot{\phi}_{,X} - \tilde{p}\dot{\alpha}_{,X}\frac{1}{\zeta_0} = \\
&= \rho\dot{U}_{,X}\Big[\Sigma_1 U_{,X} + \int_0^\infty \dot{m}_3(u)U_{,X}{}_d^t(u)du + \int_0^\infty \dot{\alpha}_d^t(s)\dot{B}_2(s)ds + \\
&+ \int_0^\infty \alpha_{,X}{}_d^t(s)\dot{M}_1(s)ds - \int_0^\infty \phi_{,X}{}_d^t(s)\dot{M}_3(s)ds\Big] - \rho\dot{\phi}_{,X}\Big[(\Sigma_2 - \\
&- \rho^{-1}\varepsilon_0)\phi_{,X} + \int_0^\infty \dot{m}_4(u)\phi_{,X}{}_d^t(u)du - \int_0^\infty \dot{\alpha}_d^t(s)\dot{B}_3(s)ds - \\
&- \int_0^\infty \alpha_{,X}{}_d^t(s)\dot{M}_2(s)ds - \int_0^\infty \dot{M}_3(u)U_{,X}{}_d^t(u)du\Big] + \rho\dot{\alpha}_{,X}\Big[\frac{\partial\tilde{\Sigma}}{\partial\beta} + \\
&+ \int_0^\infty \dot{m}_2(u)\alpha_{,X}{}_d^t(u)du + \int_0^\infty \dot{\alpha}_d^t(s)\dot{B}_1(s)ds + \int_0^\infty \dot{M}_1(u)U_{,X}{}_d^t(u)du - \\
&- \int_0^\infty \dot{M}_2(u)\phi_{,X}{}_d^t(u)du\Big] + \kappa_0(\dot{\alpha}_{,X})^2.
\end{aligned} \tag{122}$$

By equating the two expressions (120) and (122) and by considering the relation (121), we obtain

$$\begin{aligned}
&\rho\dot{U}_{,X}\left[\Sigma_1 U_{,X}+\int_0^\infty \dot{m}_3(u)U_{,X}{}_d^t(u)du+\int_0^\infty \dot{\alpha}_d^t(s)\dot{B}_2(s)ds+\right.\\
&\left.+\int_0^\infty \alpha_{,X}{}_d^t(s)\dot{M}_1(s)ds-\int_0^\infty \phi_{,X}{}_d^t(s)\dot{M}_3(s)ds\right]-\rho\dot{\phi}_{,X}\left[(\Sigma_2-\right.\\
&-\rho^{-1}\varepsilon_0)\phi_{,X}+\int_0^\infty \dot{m}_4(u)\phi_{,X}{}_d^t(u)du-\int_0^\infty \dot{\alpha}_d^t(s)\dot{B}_3(s)ds-\\
&\left.-\int_0^\infty \alpha_{,X}{}_d^t(s)\dot{M}_2(s)ds-\int_0^\infty \dot{M}_3(u)U_{,X}{}_d^t(u)du\right]+\rho\dot{\alpha}_{,X}\left[\frac{\partial\tilde{\Sigma}}{\partial\beta}+\right.\\
&+\int_0^\infty \dot{m}_2(u)\alpha_{,X}{}_d^t(u)du+\int_0^\infty \dot{\alpha}_d^t(s)\dot{B}_1(s)ds+\int_0^\infty \dot{M}_1(u)U_{,X}{}_d^t(u)du-\\
&\left.-\int_0^\infty \dot{M}_2(u)\phi_{,X}{}_d^t(u)du\right]+\kappa_0(\dot{\alpha}_{,X})^2=\\
&=\left(\tilde{\sigma}\dot{U}\right)_{,X}-\left(\tilde{D}\dot{\phi}\right)_{,X}-(\tilde{p}\dot{\alpha})_{,X}\frac{1}{\zeta_0}-\rho\ddot{U}\dot{U}+\rho f\dot{U}+\rho s\dot{\alpha}\frac{1}{\zeta_0}-\\
&-\left\{-\frac{\partial}{\partial t}\left[\Sigma_3(T-T_0)\right]-\frac{\partial}{\partial t}\int_0^\infty \dot{m}_1(u)\dot{\alpha}_d^t(u)du-\right.\\
&-\frac{\partial}{\partial t}\int_0^\infty \dot{B}_1(u)\alpha_{,X}{}_d^t(u)du-\frac{\partial}{\partial t}\int_0^\infty \dot{B}_2(u)U_{,X}{}_d^t(u)du+\\
&\left.+\frac{\partial}{\partial t}\int_0^\infty \dot{B}_3(u)\phi_{,X}{}_d^t(u)du\right\}\rho\dot{\alpha}.
\end{aligned}\tag{123}$$

Note that the expression

$$-\rho\dot{\alpha}_{,X}\frac{\partial\tilde{\Sigma}}{\partial\beta}+\rho\dot{\phi}_{,X}\Sigma_2-\rho\dot{U}_{,X}\Sigma_1 U_{,X}+\frac{\partial}{\partial t}\Sigma_3(T-T_0)\rho\dot{\alpha}\tag{124}$$

is zero since by (21) we do not have instantaneous elastic responses.

Finally, by using the results from Lemma 3 and Lemma 4, we obtain the following result

$$\begin{aligned}
&\int_B \left[\rho \ddot{U}\dot{U} + \varepsilon_0 \dot{\phi}_{,X}\phi_{,X} + \kappa_0 \left(\dot{\alpha}_{,X}\right)^2\right] dv = \\
&= \int_{\partial B} \left(\tilde{\sigma}\dot{U}n - \tilde{D}\dot{\phi}n - \tilde{p}\dot{\alpha}n\frac{1}{\zeta_0}\right) da + \int_B \left(\rho f \dot{U} + \rho s \dot{\alpha}\frac{1}{\zeta_0}\right) dv + \\
&+ \int_B -\rho \left[-\frac{1}{2}\frac{\partial}{\partial u}\int_0^\infty \dot{m}_3(u) U_{,X}{}^t_d(u) U_{,X}{}^t_d(u) du + \right. \\
&+ \left. \frac{1}{2}\int_0^\infty \ddot{m}_3(u) U_{,X}{}^t_d(u) U_{,X}{}^t_d(u) du - \frac{1}{2}\frac{\partial}{\partial t}\int_0^\infty \dot{m}_3(u) U_{,X}{}^t_d(u) U_{,X}{}^t_d(u) du\right] - \\
&- \rho \left[-\frac{\partial}{\partial s}\int_0^\infty \dot{M}_1(s)\alpha_{,X}{}^t_d(s) U_{,X}{}^t_d(s) ds + \int_0^\infty \ddot{M}_1(s)\alpha_{,X}{}^t_d(s) U_{,X}{}^t_d(s) ds - \right. \\
&- \left. \frac{\partial}{\partial t}\int_0^\infty \dot{M}_1(s)\alpha_{,X}{}^t_d(s) U_{,X}{}^t_d(s) ds\right] + \rho \left[-\frac{1}{2}\frac{\partial}{\partial u}\int_0^\infty \dot{m}_4(u)\phi_{,X}{}^t_d(u)\phi_{,X}{}^t_d(u) du + \right. \\
&+ \left. \frac{1}{2}\int_0^\infty \ddot{m}_4(u)\phi_{,X}{}^t_d(u)\phi_{,X}{}^t_d(u) du - \frac{1}{2}\frac{\partial}{\partial t}\int_0^\infty \dot{m}_4(u)\phi_{,X}{}^t_d(u)\phi_{,X}{}^t_d(u) du\right] + \\
&+ \rho \left[-\frac{\partial}{\partial s}\int_0^\infty \dot{M}_3(s)\phi_{,X}{}^t_d(s) U_{,X}{}^t_d(s) ds + \int_0^\infty \ddot{M}_3(s)\phi_{,X}{}^t_d(s) U_{,X}{}^t_d(s) ds - \right. \\
&- \left. \frac{\partial}{\partial t}\int_0^\infty \dot{M}_3(s)\phi_{,X}{}^t_d(s) U_{,X}{}^t_d(s) ds + 2\int_0^\infty \dot{M}_3(s)\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial t}\right)\phi_{,X}{}^t_d(s) U_{,X}{}^t_d(s) ds\right] + \\
&+ \rho \left[-\frac{\partial}{\partial s}\int_0^\infty \dot{M}_2(s)\phi_{,X}{}^t_d(s)\alpha_{,X}{}^t_d(s) ds + \int_0^\infty \ddot{M}_2(s)\phi_{,X}{}^t_d(s)\alpha_{,X}{}^t_d(s) ds - \right. \\
&- \left. \frac{\partial}{\partial t}\int_0^\infty \dot{M}_2(s)\phi_{,X}{}^t_d(s)\alpha_{,X}{}^t_d(s) ds + 2\int_0^\infty \dot{M}_2(s)\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial t}\right)\phi_{,X}{}^t_d(s)\alpha_{,X}{}^t_d(s) ds\right] - \\
&- \rho \left[-\frac{1}{2}\frac{\partial}{\partial u}\int_0^\infty \dot{m}_2(u)\alpha_{,X}{}^t_d(u)\alpha_{,X}{}^t_d(u) du + \frac{1}{2}\int_0^\infty \ddot{m}_2(u)\alpha_{,X}{}^t_d(u)\alpha_{,X}{}^t_d(u) du - \right. \\
&- \left. \frac{1}{2}\frac{\partial}{\partial t}\int_0^\infty \dot{m}_2(u)\alpha_{,X}{}^t_d(u)\alpha_{,X}{}^t_d(u) du\right] + \rho \int_0^\infty \dot{B}_1(u)\left[\dot{\alpha}(t)\right]^2 \frac{\partial}{\partial X}\left[\frac{\frac{\partial}{\partial t}\alpha^t_d(u)}{\dot{\alpha}(t)}\right] du +
\end{aligned} \tag{125}$$

$$
\begin{aligned}
&+\rho\left[\frac{\partial}{\partial u}\int_0^\infty\left(\dot{B}_2(u)U,_X{}_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u)\right)du-\int_0^\infty\ddot{B}_2(u)U,_X{}_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u)du-\right.\\
&\left.-\frac{\partial}{\partial t}\int_0^\infty\dot{B}_2(u)U,_X{}_d^t(u)\frac{\partial}{\partial u}\alpha_d^t(u)du\right]-\frac{1}{2}\rho\frac{\partial}{\partial t}\int_0^\infty\dot{m}_1(u)\dot{\alpha}(t)\dot{\alpha}(t)du+\\
&+\frac{1}{2}\rho\frac{\partial}{\partial t}\int_0^\infty\dot{m}_1(u)\frac{\partial}{\partial u}\alpha_d^t(u)\frac{\partial}{\partial u}\alpha_d^t(u)du+\frac{1}{2}\rho\frac{\partial}{\partial u}\int_0^\infty\dot{m}_1(u)\frac{\partial}{\partial t}\alpha_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u)du-\\
&-\frac{1}{2}\rho\int_0^\infty\ddot{m}_1(u)\frac{\partial}{\partial t}\alpha_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u)du+\rho\frac{\partial}{\partial u}\int_0^\infty\dot{B}_3(u)\phi,_X{}_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u)du-\\
&-\rho\int_0^\infty\ddot{B}_3(u)\phi,_X{}_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u)du+\rho\frac{\partial}{\partial t}\int_0^\infty\dot{B}_3(u)\phi,_X{}_d^t(u)\frac{\partial}{\partial u}\alpha_d^t(u)du+\\
&+2\rho\frac{\partial}{\partial t}\int_0^\infty\dot{B}_3(u)\phi,_X{}_d^t(u)\frac{\partial}{\partial t}\alpha_d^t(u)du-2\rho\int_0^\infty\dot{B}_3(u)\phi,_X{}_d^t(u)\frac{\partial}{\partial t}\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial t}\right)\alpha_d^t(u)du.
\end{aligned}
\tag{126}
$$

We group the terms with the first order derivative of the relaxation functions to form W. Then we group the terms with the second order derivative of the relaxation functions to form M. All the other terms are included in Q. ■

Let $r^{(1)}$ and $r^{(2)}$ be two solutions which are associated to the same initial and boundary conditions and to the supply terms $(f^{(1)}, f^{(2)})$ and $(s^{(1)}, s^{(2)})$, respectively. We consider the differences $r = r^{(1)} - r^{(2)}$, $f = f^{(1)} - f^{(2)}$, $s = s^{(1)} - s^{(2)}$. Hence, r is a solution that can be associated to the supply terms (f, s) and to zero initial and boundary conditions. Moreover, we make the following assumptions

C1) there exist positive constants $\tilde{M}_1, \tilde{M}_2, \tilde{M}_3, \tilde{M}_4, \tilde{M}_5$ and $t_1 > 0$ such that

$$
\begin{aligned}
&\int_0^{t_1}\int_B f^2 dvdt \le \tilde{M}_1^2, \quad \int_0^{t_1}\int_B s^2 dvdt \le \tilde{M}_2^2,\\
&\int_0^{t_1}\int_B \dot{U}^2 dvdt \le \tilde{M}_3^2, \quad \int_0^{t_1}\int_B \dot{\phi}^2 dvdt \le \tilde{M}_4^2, \quad \int_0^{t_1}\int_B \dot{\alpha}^2 dvdt \le \tilde{M}_5^2;
\end{aligned}
\tag{127}
$$

C2) the inequality

$$
\int_0^t\int_0^\infty\int_B M dvdsdt - \int_0^\infty\int_B W dvds - \int_0^t\int_0^\infty\int_B Q dvdsdt \ge 0 \tag{128}
$$

holds for all $t \in [0, t_1)$ along any kinetic process (2).

In the sequel, we prove a result of continuous dependence upon the supply terms.

Theorem 3 *Under the assumptions of Theorem 2, let the hypotheses C1) and C2) hold true. Moreover, we consider that $\tilde{r}$ is a solution of the initial boundary value problem associated to zero initial and boundary conditions and to the supply terms (f, s). Therefore, we have*

$$\mathcal{U}(t) \leq \tilde{M}_3 \left(\int_0^{t_1} \int_B f^2 dvdt \right)^{\frac{1}{2}} + \tilde{M}_5 \left(\int_0^{t_1} \int_B s^2 dvdt \right)^{\frac{1}{2}}, \tag{129}$$

for $t \in [0, t_1)$.

Proof. First, we consider the null boundary conditions in the relation (119) from Theorem 2. It follows that $\int_{\partial B} \left(\tilde{\sigma} \dot{U} n - \tilde{D} \dot{\phi} n - \tilde{p} \dot{\alpha} n \frac{1}{\zeta_0} \right) = 0$. Then we integrate the relation (119) from 0 to t and consider the null initial conditions. This means that $\mathcal{U}(0) = 0$. Therefore, we obtain

$$\begin{aligned} \mathcal{U}(t) = & \int_0^t \int_B \rho \left(f \dot{U} + s \dot{\alpha} \frac{1}{\zeta_0} \right) dvdt - \frac{1}{2} \rho \int_0^t \int_0^\infty \int_B M dvdsdt + \\ & + \frac{1}{2} \rho \frac{d}{dt} \left(\int_0^t \int_0^\infty \int_B W dvdsdt \right) + \rho \int_0^t \int_0^\infty \int_B Q dvdsdt. \end{aligned} \tag{130}$$

Finally, we use the Schwarz inequality for the two terms in the first integral and the hypothesis C2) for the last three integrals in order to obtain the desired result. ■

5 Conclusions

We have shown theorems on uniqueness and continuous data dependence that are appropriate to the boundary value problems for a linear thermo-electro-viscoelastic body $\mathcal{B}$ of dimension one.

Many authors assume that to perform an experiment corresponds to posing an initial-boundary-value problem and that the outcome of an experiment is given by the solution of it (cf. e.g. [30]).

Experiments are a tool to test the material parameters of a theoretical model. This topic is studied for hyperelasticity, e.g., in [31], whose line of thought is as follows: to validate a model for a continuous media many experiments have to be numerically simulated; then the informations about the material parameters implemented in the model can be adjusted in order to recover the experimental data.

Here the material parameters are relaxation functions. In [17] some simple solutions of the field equations for $\mathcal{B}$ are shown. A future development of the present work may be to find some class of less simple solutions, interpreted as experiments, to choose and test physically reliable relaxation functions.

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