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Corrigendum

Relative regular Riemann–Hilbert correspondence (*Proc. London Math. Soc.* (3) 122 (2021) 434–457)

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Abstract

We correct a wrong argument in Proposition 3.3.

The numbers for the statements or the bibliography refer to the main text [1]. We have noticed a wrong argument in the proof of Proposition 3.3. On page 454, line 1, we claimed that “Since $i_s^*(\beta_{\mathcal{M},\mathcal{N}})$ is an isomorphism according to the regularity assumption and the absolute case…”. The regularity assumption together with the Riemann–Hilbert correspondence in the absolute case imply an isomorphism

$$\begin{aligned} Li_s^* R\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M},\mathcal{N}) &\overset{(1)}{\simeq} R\mathcal{H}om_{\mathcal{D}_X}(Li_s^*\mathcal{M}, Li_s^*\mathcal{N}) \\ &\overset{(2)}{\simeq} R\mathcal{H}om_{\mathbb{C}_X}({}^{\mathrm{p}}\mathrm{Sol} Li_s^*\mathcal{N}, {}^{\mathrm{p}}\mathrm{Sol} Li_s^*\mathcal{M}) \\ &\overset{(3)}{\simeq} R\mathcal{H}om_{\mathbb{C}_X}(Li_s^{*}{}^{\mathrm{p}}\mathrm{Sol}_X\mathcal{N}, Li_s^{*}{}^{\mathrm{p}}\mathrm{Sol}_X\mathcal{M}) \\ &\overset{(4)}{\simeq} Li_s^* R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}({}^{\mathrm{p}}\mathrm{Sol}_X\mathcal{N}, {}^{\mathrm{p}}\mathrm{Sol}_X\mathcal{M}), \end{aligned}$$

where (1) and (3) have been observed in [15, Proposition 3.1], (2) is explained in (i) of the proof of [11, Corollary 4.3.5], and (4) is [15, Proposition 2.10]. We have wrongly inferred that $\mathcal{H}^0 \circ Li_s^*$ of the natural morphism

$$\beta_{\mathcal{M},\mathcal{N}} : \mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M},\mathcal{N}) \longrightarrow \mathcal{H}^0 R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}({}^{\mathrm{p}}\mathrm{Sol}_X\mathcal{N}, {}^{\mathrm{p}}\mathrm{Sol}_X\mathcal{M}),$$

which is our $i_s^*(\beta_{\mathcal{M},\mathcal{N}})$, is an isomorphism (note that $\mathcal{H}^0 R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}({}^{\mathrm{p}}\mathrm{Sol}_X\mathcal{N}, {}^{\mathrm{p}}\mathrm{Sol}_X\mathcal{M})$ is denoted $\mathcal{H}om_{\mathrm{perv}(p_X^{-1}\mathcal{O}_S)}({}^{\mathrm{p}}\mathrm{Sol}_X\mathcal{N}, {}^{\mathrm{p}}\mathrm{Sol}_X\mathcal{M})$ in the main text, cf. Lemma 2.10). Our assumption of strictness of $\mathcal{N}$ only implies an isomorphism

$$\mathcal{H}^0 R\mathcal{H}om_{\mathcal{D}_X}(Li_s^*\mathcal{M}, Li_s^*\mathcal{N}) \simeq \mathcal{H}om_{\mathcal{D}_X}(i_s^*\mathcal{M}, i_s^*\mathcal{N}) \xrightarrow{\sim} \mathcal{H}^0 Li_s^* R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}({}^{\mathrm{p}}\mathrm{Sol}_X\mathcal{N}, {}^{\mathrm{p}}\mathrm{Sol}_X\mathcal{M}),$$

where both left-hand and right-hand sides can be different from the source and target of $i_s^*(\beta_{\mathcal{M},\mathcal{N}})$ (we have an example for the left-hand side taken from [15, Example 3.12]). However, once Theorems 1 and 3 are proved as explained below, the morphism (14) on page 452 is an isomorphism and by taking its $\mathcal{H}^0$ with $F = {}^{\mathrm{p}}\mathrm{Sol}_X(\mathcal{N})$ (so that $\mathrm{RH}_X^S(F) \simeq \mathcal{N}$), we

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recover that $\beta_{\mathcal{M},\mathcal{N}}$ is an isomorphism, from which we deduce the statement wrongly inferred *a priori*.

We show how to get around this error. For that purpose, Sections 3.2– 3.4 of the main text have to be replaced with Sections E.3.2–E.3.4 below (Sections 3.1 and 3.5 remain unchanged). The main change is the new statement of Proposition E.3.3 (which essentially corresponds to Corollary 3.4 of the published text) and its new proof. The end of the proof of Theorem 3 has been adapted correspondingly. As a consequence of the new proof of Theorem 3, the statement of Proposition 3.3 in the main text turns out to be true, as well as all other statements of the article. We only modify the way to obtain them.

E.3.2. *Proof of Theorem 3: first step*

For a regular holonomic $\mathcal{D}_{X\times S/S}$-module $\mathcal{M}$, let us set (see Proposition 1.5)

$$\begin{aligned}\mathrm{Char}(\mathcal{M}) := \bigcup_j \mathrm{Char}(\mathcal{H}^j\mathcal{M}) = \bigcup_{i\in I}\Lambda_i\times T_i\\ \mathrm{Supp}(\mathcal{M}) = \bigcup_{i\in I} Y_i\times T_i,\quad \mathrm{Supp}_X(\mathcal{M}) = Z = Z_{\mathcal{M}} := \bigcup_{i\in I} Y_i.\end{aligned}\tag{E.1}$$

PROPOSITION E.3.3. *Let $\mathcal{M}$ be a strict regular holonomic $\mathcal{D}_{X\times S/S}$-module with X-support Z. Let $Y\subset X$ be a hypersurface containing the singular locus $\mathrm{Sing}(Z)$ and all subsets Y_i with $\dim Y_i<\dim Z$. Then the localized $\mathcal{D}_{X\times S/S}$-module $\mathcal{M}(*(Y\times S))$ is regular holonomic and locally isomorphic to the projective pushforward of a relative $\mathcal{D}$-module of D-type.*

Proof. The question is local. The assumption on Y implies that $Z^o := Z\setminus(Y\cap Z)$ is smooth of pure dimension $\dim Z$ and the characteristic variety of $\mathcal{M}_{|(X\setminus Y)\times S}$ is contained in $(T^*_{Z^o}X)\times S$. By Kashiwara's equivalence, $\mathcal{M}_{|(X\setminus Y)\times S}$ is the pushforward by the inclusion map of a coherent $\mathcal{O}_{Z^o\times S}$-module with flat relative connection. The strictness assumption entails that this flat relative connection is of the form $(\mathcal{O}_{Z^o\times S}\otimes_{p^{-1}\mathcal{O}_S}F, \mathrm{d}_{Z^o\times S/S})$ for some locally constant $p_{Z^o}^{-1}\mathcal{O}_S$-module F which is *locally free of finite rank*.

One can find a complex manifold X' together with a divisor with normal crossings $Y'\subset X'$ and a projective morphism $\pi: X'\to X$ which induces a biholomorphism $X'\setminus Y'\xrightarrow{\sim} Z^o$. We set $\delta=\dim Z-\dim X=\dim X'-\dim X\leqslant 0$. For each ℓ, we consider the $\mathcal{D}_{X'\times S/S}$-module $\mathcal{M}'^\ell := \mathcal{H}^\ell{}_{\mathrm{D}}\pi^*\mathcal{M}$. Although it is not yet known to be coherent, it is locally an inductive limit (union) of coherent $\mathcal{D}_{X'\times S/S}$-submodules, hence also of $\mathcal{O}_{X'\times S}$-coherent submodules (cf. [**2**, Proposition 2.1]). We simply say that $\mathcal{M}'^\ell$ is quasi-coherent (over $\mathcal{D}_{X'\times S/S}$ or over $\mathcal{O}_{X'\times S}$). We will use the following property, that is deduced from the similar one for coherent $\mathcal{O}_{X'\times S}$-modules:

(∗) A quasi-coherent $\mathcal{O}_{X'\times S}$-module which is zero on $(X'\setminus Y')\times S$ becomes zero after being tensored with $\mathcal{O}_{X'\times S}(*(Y'\times S))$.

If $\ell\neq\delta$, the sheaf-theoretic restriction of $\mathcal{M}'^\ell$ to $(X'\setminus Y')\times S$ is zero, so $\mathcal{M}'^\ell(*(Y'\times S))=0$ owing to quasi-coherence, according to (∗). Since $\mathcal{O}_{X'\times S}(*(Y'\times S))$ is flat over $\mathcal{O}_{X'\times S}$, we conclude that

$${}_{\mathrm{D}}\pi^*\big(\mathcal{M}(*(Y\times S))\big)[\delta]\simeq\big({}_{\mathrm{D}}\pi^*\mathcal{M}\big)(*(Y'\times S))[\delta]\simeq\mathcal{M}'^\delta(*(Y'\times S)).\tag{E.2}$$

We will first check that $\mathcal{M}'^\delta(*(Y'\times S))$ is strict (i.e., $Li_s^*\mathcal{M}'^\delta(*(Y'\times S))$ has cohomology in degree zero only, cf. [18, Lemma 1.13]). Strictness of $\mathcal{M}(*(Y\times S))$ follows from flatness of $\mathcal{O}_{X\times S}(*(Y\times S))$ over $\mathcal{O}_{X\times S}$. Furthermore, as a complex of $\mathcal{O}_{X'\times S}$-modules, ${}_{\mathrm{D}}\pi^*\big(\mathcal{M}(*(Y\times S))\big)$ is nothing but $L\pi^*\big(\mathcal{M}(*(Y\times S))\big)$. We then have, for each $s\in S$,

$$\begin{aligned} Li_s^*\mathcal{M}'^\delta(*(Y'\times S)) &\simeq Li_s^*L\pi^*\bigl(\mathcal{M}(*(Y\times S))\bigr)[\delta] \quad \text{(according to (E.2))}\\ &\simeq L\pi^*(i_s^*\mathcal{M})(*Y)[\delta] \qquad \text{(strictness of } \mathcal{M}(*(Y\times S)))\\ &\simeq L^\delta\pi^*(i_s^*\mathcal{M})(*Y) \qquad \text{(same argument as (E.2))} \end{aligned}$$

has cohomology in degree zero only, as wanted. The same argument shows that, while $\mathcal{M}'^\delta(*(Y'\times S))$ may *a priori* be non-$\mathcal{D}_{X'\times S/S}$-coherent, its restriction by i_s^* is regular holonomic (hence $\mathcal{D}_{X'}$-coherent) for each $s\in S$.

We now take up the argument of [18, Proof of Proposition 2.11] and show that $\mathcal{M}'^\delta(*(Y'\times S))$ is regular holonomic and of D-type with respect to Y'. As noticed at the beginning of the proof, $F := \mathcal{H}om_{\mathcal{D}_{X'\times S/S}}(\mathcal{O}_{X'\times S}, \mathcal{M}'^\delta)|_{(X'\setminus Y')\times S}$ is locally free of finite rank. Let $j' : X'\setminus Y' \hookrightarrow X'$ denote the inclusion. The isomorphism

$$j'^{-1}\mathcal{M}'^\delta \xrightarrow{\sim} (\mathcal{O}_{(X'\setminus Y')\times S}\otimes_{p^{-1}\mathcal{O}_S} F, \mathrm{d}_{X'\times S/S}) =: (V,\nabla)$$

extends as a morphism of $\mathcal{D}_{X'\times S/S}(*(Y'\times S))$-modules

$$\psi : \mathcal{M}'^\delta(*(Y'\times S)) \longrightarrow j'_*(V,\nabla).$$

Let m be a local section of $\mathcal{M}'^\delta(*(Y'\times S))$. Since for each $s\in S$, $i_s^*\bigl(\mathcal{M}'^\delta(*(Y'\times S))\bigr) = (i_s^*\mathcal{M}'^\delta)(*Y')$ is regular holonomic, the image $m(\cdot,s)$ of m in the latter module has moderate growth in the sense of [6, p. 862] when restricted to $X'\setminus Y'$. According to [18, Lemma 2.12], $\psi(m)$ is a local section of the Deligne extension $\widetilde{V}$ of (V,∇), which is $\mathcal{D}_{X'\times S/S}$-coherent by Theorem 1.13(a). Then $\operatorname{im}\psi$, being quasi-coherent, is a coherent $\mathcal{D}_{X'\times S/S}$-submodule of $\widetilde{V}$. By applying $(*)$ to the kernel and cokernel of ψ, we obtain that ψ is an isomorphism.

According to Proposition 1.12, ${}_{\mathrm{D}}\pi_*\widetilde{V}$ has regular holonomic cohomology. Furthermore, since $\mathcal{H}^j{}_{\mathrm{D}}\pi_*\widetilde{V}$ is supported on $Y\times S$ for $j\neq 0$, and since $\widetilde{V} = \widetilde{V}(*(Y'\times S))$, so that ${}_{\mathrm{D}}\pi_*\widetilde{V} \simeq {}_{\mathrm{D}}\pi_*\widetilde{V}(*(Y\times S))$, we have

$${}_{\mathrm{D}}\pi_*\widetilde{V} \simeq \mathcal{H}^0{}_{\mathrm{D}}\pi_*\widetilde{V} \simeq \mathcal{H}^0{}_{\mathrm{D}}\pi_*\widetilde{V}(*(Y\times S)).$$

On the other hand, there is a natural adjunction morphism (cf. [5, Lemma 4.28 and Proposition 4.34])

$${}_{\mathrm{D}}\pi_*{}_{\mathrm{D}}\pi^*\mathcal{M}[\delta] \longrightarrow \mathcal{M},$$

which induces a morphism of coherent $\mathcal{D}_{X\times S/S}(*(Y\times S))$-modules

$$\mathcal{H}^0{}_{\mathrm{D}}\pi_*\widetilde{V} \simeq (\mathcal{H}^0{}_{\mathrm{D}}\pi_*\mathcal{M}'^\delta)(*(Y\times S)) \longrightarrow \mathcal{M}(*(Y\times S)),$$

where the left-hand side is $\mathcal{D}_{X\times S/S}$-coherent and regular holonomic. Its cokernel is zero on $(X\setminus Y)\times S$ and $\mathcal{D}_{X\times S/S}(*(Y\times S))$-coherent, hence it is zero according to $(*)$, so that this morphism is an isomorphism. In conclusion, $\mathcal{M}(*(Y\times S))$ is regular holonomic. ☐

After the proof of Theorem 1, Proposition E.3.3 can be improved:

Corollary E.3.4 (of Theorem 1). *For any $\mathcal{M}$ in $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$ and any hypersurface $Y\subset X$, the complexes $R\Gamma_{[Y\times S]}(\mathcal{M})$ and $\mathcal{M}(*(Y\times S))$ belong to $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$.*

Proof. In view of the equivalence of Theorem 1, this reduces to Proposition 2.6. ☐

E.3.3. *End of the proof of Theorem* 2

We can argue by induction on the length of $\mathcal{M}$ and then reduce to the cases of a projection and of a closed embedding. The first case was proved in Section 1.4.1. The case of a closed embedding $i : Y \hookrightarrow X$ is a consequence of Corollary E.3.4.

E.3.4. *End of the proof of Theorem* 3

We refer to [11, Lemma 4.1.4] which contains the guidelines for the proof of Theorem 3. In what follows, for a complex manifold X and $\mathcal{M}\in \mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$ we consider the statement

$$P_X(\mathcal{M}):\quad R\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M},\mathrm{RH}^S_X(F))\in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}\mathrm{c}}(p_X^{-1}\mathcal{O}_S)\quad \forall F\in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}\mathrm{c}}(p_X^{-1}\mathcal{O}_S),$$

in other words, $\mathcal{M}$ satisfies Theorem 3.

LEMMA E.3.6. *The statement P satisfies the following properties.*

(a) *For any manifold X and any open covering $(U_i)_{i\in I}$ of X,*

$$P_X(\mathcal{M}) \iff P_{U_i}(\mathcal{M}_{|U_i})\qquad \forall\, i\in I.$$

(b) $P_X(\mathcal{M})\Rightarrow P_X(\mathcal{M}[n])\ \forall\, n\in\mathbb{Z}$.

(c) *For any distinguished triangle $\mathcal{M}'\to\mathcal{M}\to\mathcal{M}''\xrightarrow{+1}$ in $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$,*

$$P_X(\mathcal{M}')\wedge P_X(\mathcal{M}'')\Rightarrow P_X(\mathcal{M}).$$

(d) *For any regular relative holonomic $\mathcal{D}_{X\times S/S}$-modules $\mathcal{M}$ and $\mathcal{M}'$,*

$$P_X(\mathcal{M}\oplus\mathcal{M}')\Rightarrow P_X(\mathcal{M}).$$

(e) *For any projective morphism $f:X\to Y$ and any regular holonomic $\mathcal{D}_{X\times S/S}$-module $\mathcal{M}$ which is f-good,*

$$P_X(\mathcal{M})\Rightarrow P_Y({}_{\mathrm{D}}f_*\mathcal{M}).$$

(f) *If $\mathcal{M}=\mathcal{H}^0(\mathcal{M})$ is torsion, then $P_X(\mathcal{M})$ is true.*

Proof. It is clear that $P_X(\bullet)$ satisfies Properties E.3.6(a), (b), (c), (d). Then Property (e) follows, by adjunction, Proposition 2.9 and by the stability of S-$\mathbb{C}$-constructibility under proper direct image. Last, Property (f) has been seen in Section 3.1. ☐

End of the proof of Theorem 3 (*and hence that of Theorem* 1). We wish to prove that $P_X(\mathcal{M})$ is true for any X and $\mathcal{M}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$.

We proceed by induction on the dimension of $Z_{\mathcal{M}}$ (cf. (E.1)). If $\dim Z_{\mathcal{M}}=0$, then $P_X(\mathcal{M})$ holds true by Kashiwara's equivalence and E.3.6(e), since $P_X(\mathcal{M})$ obviously holds if X has dimension zero.

Let us suppose $P_X(\mathcal{N})$ true for any $\mathcal{N}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$ such that $\dim Z_{\mathcal{N}}<k$ (with $k\geqslant 1$) and let us prove the truth of $P_X(\mathcal{M})$ for $\mathcal{M}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$ with $\dim Z_{\mathcal{M}}=k$.

By E.3.6(b) and (c), we are reduced to proving $P_X(\mathcal{M})$ in the case where $\mathcal{M}$ is a regular holonomic $\mathcal{D}_{X\times S/S}$-module with $\dim Z_{\mathcal{M}}=k$.

Following the notation of Section 1.1, let $t(\mathcal{M})$ (respectively $f(\mathcal{M})$) be the torsion part (respectively the strict quotient) of $\mathcal{M}$. According to E.3.6(c) (applied to the distinguished triangle $t(\mathcal{M})\to\mathcal{M}\to f(\mathcal{M})\xrightarrow{+1}$) and to E.3.6(f), we are reduced to proving $P_X(f(\mathcal{M}))$. Note that $\dim Z_{f(\mathcal{M})}\leqslant k$ since $Z_{f(\mathcal{M})}\subseteq Z_{\mathcal{M}}$. If $\dim Z_{f(\mathcal{M})}<k$, $P_X(f(\mathcal{M}))$ holds true by induction. Hence we are reduced to proving $P_X(\mathcal{M})$ in the case where $\mathcal{M}$ is a strict regular holonomic $\mathcal{D}_{X\times S/S}$-module such that $\dim Z_{\mathcal{M}}=k$, a property that we now assume to hold. Locally (recall that $P_X(\mathcal{M})$ is a local statement by E.3.6(a)), there exists a hypersurface Y in X satisfying the assumptions of Proposition E.3.3.

On the one hand, it is enough to check the property $P_X(\mathcal{M})$ for those $F\in\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}\mathrm{c}}(p_X^{-1}\mathcal{O}_S)$ such that $F=F\otimes\mathbb{C}_{(X\setminus Y)\times S}$. Indeed, let us check that it holds for those F such that $F=F\otimes$

$\mathbb{C}_{Y\times S}$. For any $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}\mathrm{c}}(p_X^{-1}\mathcal{O}_S)$, the complex $\mathcal{N} := \mathrm{RH}^S_X(F \otimes \mathbb{C}_{Y\times S}) \simeq R\Gamma_{[Y\times S]}(\mathrm{RH}^S_X(F))$ belongs to $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$ according to Proposition 2.6(b), and we have, by [16, (3)],

$$R\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \mathcal{N}) \simeq R\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\boldsymbol{D}\mathcal{N}, \boldsymbol{D}\mathcal{M}).$$

The duality functor preserves $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ by [20, Proposition 2.5] and also $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$ since it does so in the absolute case and $Li_s^*(\boldsymbol{D}\mathcal{M}) \simeq \boldsymbol{D}(Li_s^*\mathcal{M})$. Let us also notice that $\boldsymbol{D}\mathcal{M} = \mathcal{H}^0\boldsymbol{D}\mathcal{M}$ is strict holonomic (cf. [18, Proposition 2]). Since $\mathcal{N}$ has $\mathcal{D}_{X\times S/S}$-coherent cohomology and is supported on $Y \times S$, we have

$$R\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\boldsymbol{D}\mathcal{N}, (\boldsymbol{D}\mathcal{M})(*(Y\times S))) = 0.$$

Furthermore, $\boldsymbol{D}\mathcal{M}$ being regular holonomic and strict, so is $(\boldsymbol{D}\mathcal{M})(*(Y\times S))$ by Proposition E.3.3, hence $R\Gamma_{[Y\times S]}(\boldsymbol{D}\mathcal{M})$ is also regular holonomic, as well as $\mathcal{M}' := \boldsymbol{D}R\Gamma_{[Y\times S]}(\boldsymbol{D}\mathcal{M})$. Finally, applying once more [16, (3) and (1)], we obtain

$$R\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \mathcal{N}) \simeq R\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M}', \mathcal{N}),$$

with $\dim Z_{\mathcal{H}^j\mathcal{M}'} < k$ for any j, so the latter complex is S-$\mathbb{C}$-constructible by the induction hypothesis.

On the other hand, $\mathcal{M}(*(Y\times S))$ is regular holonomic, according to Proposition E.3.3. We can now apply E.3.6(c) to the triangle $R\Gamma_{[Y\times S]}(\mathcal{M}) \to \mathcal{M} \to \mathcal{M}(*(Y\times S)) \xrightarrow{+1}$ (which is a distinguished triangle in $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$). By the induction hypothesis, $P_X(R\Gamma_{[Y\times S]}(\mathcal{M}))$ holds true.

We thus assume that $\mathcal{M} = \mathcal{M}(*(Y\times S))$ is strict, and $F = F \otimes \mathbb{C}_{(X\setminus Y)\times S}$. Let $\pi : X' \to X$ be as in Proposition E.3.3 and set $\delta = \dim X' - \dim X$. Note that the assumption on F entails

$$\pi^{-1}F = \pi^{-1}F \otimes \mathbb{C}_{(X'\setminus Y')\times S},$$

while ${}_{\mathrm{D}}\pi^*\mathcal{M}[\delta]$ is concentrated in degree zero and is of D-type along Y'. According to Lemma 2.14, $R\mathcal{H}om_{\mathcal{D}_{X'\times S/S}}({}_{\mathrm{D}}\pi^*\mathcal{M}[\delta], \mathrm{RH}^S_{X'}(\pi^{-1}F))$ is an object of $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}\mathrm{c}}(p_{X'}^{-1}\mathcal{O}_S)$, isomorphic to $R\mathcal{H}om_{\mathcal{D}_{X'\times S/S}}({}_{\mathrm{D}}\pi^*\mathcal{M}, {}_{\mathrm{D}}\pi^*\,\mathrm{RH}^S_X(F))$ by Proposition 2.9, and thus $R\pi_*$ of the latter is an object of $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}\mathrm{c}}(p_X^{-1}\mathcal{O}_S)$. By adjunction we have (cf. [5, Theorem 4.33])

$$\begin{aligned} R\pi_* R\mathcal{H}om_{\mathcal{D}_{X'\times S/S}}({}_{\mathrm{D}}\pi^*\mathcal{M}, {}_{\mathrm{D}}\pi^*\,\mathrm{RH}^S_X(F)) &\simeq R\mathcal{H}om_{\mathcal{D}_{X\times S/S}}({}_{\mathrm{D}}\pi_{*\mathrm{D}}\pi^*\mathcal{M}[\delta], \mathrm{RH}^S_X(F)) \\ &\simeq R\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \mathrm{RH}^S_X(F)), \end{aligned}$$

since the adjunction ${}_{\mathrm{D}}\pi_{*\mathrm{D}}\pi^*[\delta] \to \mathrm{Id}$ is an isomorphism when applied to $\mathcal{D}_{X\times S/S}(*(Y\times S))$-modules. This ends the proof of Theorem 3. □

References

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