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Combined Custom Hedging: Optimal Design, Noninsurable Exposure, and Operational Risk Management

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We develop a normative framework for the optimal design, value assessment, and risk management integration of combined custom contingent claims. A risk averse firm faces a mix of financially insurable and noninsurable risk. The firm seeks optimal positioning in a pair of custom claims, one written on the insurable term, and another written on any listed index correlated to the noninsurable term. We prove that a unique optimum always exists unless the index is redundant, and show that the optimal payoff schedules satisfy a design integral equation. We assess the firm's incremental benefit in terms of both an indifference value and an efficiency rating: this benefit increases with the correlation of the index to the noninsurable term, and it decreases with the correlation of the index to the insurable term. Our hedge proves empirically relevant for a highly risk averse firm facing a market shock (COVID-19 pandemic). In the context of a newsvendor model featuring random price and demand, we show that: (i) integrating our optimal combined custom hedge with the corresponding optimal procurement policy allows the firm to obtain a significant improvement in both risk and return; (ii) this gain may be traded off for a substantial enhancement in operational flexibility.

Key words: Integrated risk management; Noninsurable risk; Financial product design; COVID-19 pandemic.

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1. Introduction

A large body of asset pricing literature is devoted to the direct problem of financial contingent claim (or, derivative) pricing: given a payoff schedule, one looks for a rational theory to assign a corresponding fair value (Bingham and Kiesel 2004). Little attention has been paid to the inverse problem of derivative origination, whereby an economic agent seeks an affordable claim to optimize a standing business position. Yet, contingent claim design is of great relevance in the value chain of financial business operations, nested inside the functional area of product design (Xu et al. 2016). In addition, customization (*i.e.*, tailoring a payoff written on a given underlying risk term) and combination (*i.e.*, assembling payoffs written on distinct underlying risk terms) are typically performed independently of each other. This is odd given the increasing popularity of derivative combination both in portfolio allocation (Haugh and Lo (2001), Faias and Santa-Clara (2017)) and in corporate risk management (Ding et al. (2007), Chen et al. (2014)).

We hope to contribute to the existing literature by: (i) providing a comprehensive methodology for the optimal design of combined custom financial claims to hedge corporate exposure mixing financially insurable and noninsurable risk terms (Basak and Chabakauri 2012); (ii) quantitatively assessing the firm's gain over the best alternatives developed so far, and (iii) analyzing the value of integrating the optimal combined custom hedge into optimal management of physical operations, in keeping with a holistic approach to risk management (Birge 2015).

1.1. Motivation

The problem of financial contingent claim design dates back to the pioneering work of Leland (1980) and Brennan and Solanki (1981) on portfolio insurance. These authors search for a payoff function of portfolio value that maximizes expected utility. Although presented as a hedging issue, the problem is in essence a speculative gain optimization, so long as the agent's business position does not enter the utility to maximize. On the contrary, real world trades show that derivative buyers and sellers may not have a mere speculative attitude. Corporate hedgers, for instance, seek protection against cashflow variability in their business revenues; hence, their standing position must affect the hedge they select (Fraser and Simkins 2010).

Our starting point is the business position of a manufacturing firm, a commodity merchant, or a financial trader, which we refer to hereafter as a "firm". Future operating (or, business) revenues, say π, typically depend on a financially insurable risk term X (*i.e.*, a quantity that is commonly verifiable at the time of uncertainty resolution, Chod et al. (2010)) and an uninsurable risk term Y. Each term may be anything ranging from a random variable (r.v., henceforth) to a stochastic process. In addition, we allow π to depend on the firm's decision about an operational control q driving the way risk terms interact within the position in question. In general, q may range from a scalar to a more complex control policy. Revenues $\pi = \pi(X, Y; q)$ are referred to as *mixed exposure*.

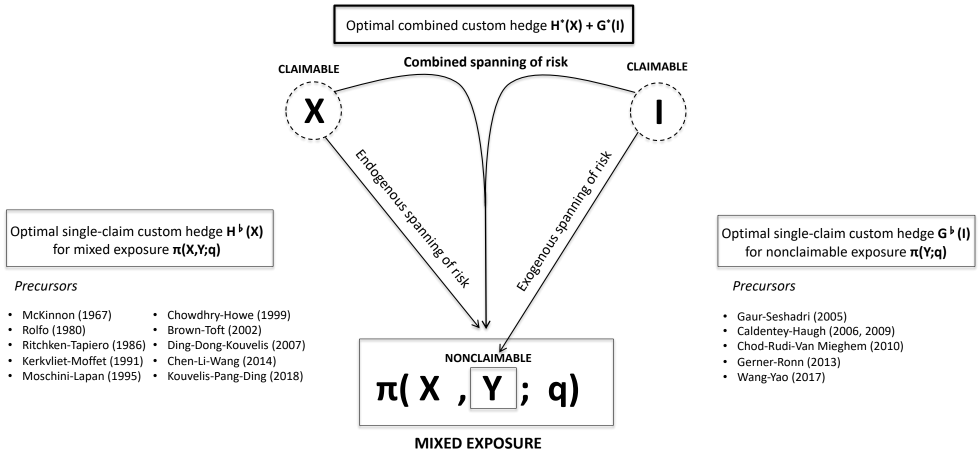
A financial intermediary acts as a claim originator in that (i) it supplies any claim H written on X with payoff function (or functional) $H(\cdot)$ in an assigned class $\mathcal{H}$; (ii) it puts a price V_H on H through a (possibly individual) state-price density (Caldentey and Haugh 2006). We say that risk term X is *claimable in* $\mathcal{H}$ provided that both of these two previous conditions are met. (Whenever the class of claims is clearly understood, we simply say the term is *claimable.*) In addition, there is a side index I that is claimable in a class $\mathcal{G}$. The firm can thus buy claims H on X and G on I, devise a static hedge $H+G$, and get to full exposure $\pi(X,Y;q)+H(X)+G(I)$ for a total hedging cost V_H+V_G. In contrast, we assume that no claim on risk term Y is available for trading whatsoever, and say that Y is *nonclaimable*. Claimability thus subsumes the attribute of being financially insurable *de facto*. Henceforth, we identify a claim, or hedge, H on X with its random payoff $H(X)$ or payoff function (or functional) $H(x)$.

Academic literature extensively examines the problem of optimizing business revenues $\pi(X,Y;q)$ through financial hedges (here, $H+G$), operational tools (here, q), or a suitable integration of the two. A first strand of research focuses on optimal positioning in a class of financial claims exclusively written on the claimable term X. The risk engendered by the nonclaimable component Y is indirectly handled through the statistical dependence on X, if any. McKinnon (1967) (resp., Rolfo (1980)) devise the minimum variance (resp., maximum mean-variance (MV)) forward hedge for a farmer exposed to a claimable soft commodity price X and a nonclaimable crop yield Y. Kerkvliet and Moffet (1991) derive futures hedge ratios in the context of an international trade involving an FX rate X and foreign revenues Y. Moschini and Lapan (1995) extend the method to expected utility preferences and enlarge the hedging opportunity space by including vanilla options and straddles. Brown and Toft (2002) adopt a value maximizing target and derive an optimal custom claim featuring a quadratic payoff function.

The aforementioned financial hedges on X may be integrated with operational management policies defined by a control q. Ritchken and Tapiero (1986) approach price-demand risk through a joint selection of physical stock procurement and financial option holding. Chowdhry and Howe (1999) put forward an integrated strategy, which addresses the price-quantity risk experienced by an international corporation featuring mean-variance preferences, and derive a genuinely custom hedge in a risk neutral setup. Ding et al. (2007) integrate real options and derivative packages to maximize MV preferences under nonzero risk premia. Chen et al. (2014) complete this research path, and derive a fully custom optimal claim in a general setup.

A second strand of research focuses on devising financial claims written on a side index I exhibiting statistical dependence on the nonclaimable term Y.Gaur and Seshadri (2005) consider fixed-price newsvendor (NV) revenues under risk averse preferences: their financial hedge is sought among alternative classes of derivative packages written on an index I that partially spans the risk engendered by demand Y. Chod et al. (2010) consider a value maximizing firm acting in a risk neutral world: they derive an optimal claim on index I and boil it down to a portfolio of vanilla instruments available for trading. Gerner and Ronn (2013) consider an airline company exposed to jet-fuel consumption Y_1 and price Y_2: as long as no futures trade on jet-fuel price, they face a

Figure 1 Single-claim *vs.* combined custom hedging of mixed claimable-nonclaimable exposure.



nonclaimable risk vector $Y := (Y_1, Y_2)$, which they manage through a suitable forward hedge on a side index I.

Extant literature thus develops two approaches to financially hedge mixed exposure: one uses financial claims written on the claimable term entering the position in question; the other adopts claims written on a claimable side index. Figure 1 illustrates this situation.

The two approaches share a common feature: financial hedging takes the form of either a single, possibly custom, contingent claim (or trading strategy), or a parametrized array of assigned derivatives. A simultaneous consideration of claim customization and combination seems to be overlooked. We contend that the effectiveness of any (possibly integrated) risk management policy may significantly be hindered by such a constraint. Our conjecture calls for a theoretical development to unify the aforementioned hedging approaches, and devise a normative method for the design, assessment, and integration of the optimal *combined custom hedge* $H^*(X) + G^*(I)$ facing mixed exposure $\pi(X, Y; q)$. Interestingly, our study owes a great deal to a research strand that extends the aforementioned static approaches to a dynamic setup (Caldentey and Haugh (2006), Caldentey and Haugh (2009), Wang and Yao (2017), and Kouvelis et al. (2018)). Appendix EC.1(D) elaborates on this connection.

1.2. Contribution

Our contribution is threefold.

1. We *design and analyze* the mean-variance *optimal combined custom hedge* $H^*(X) + G^*(I)$ of business positions featuring mixed exposure $\pi = \pi(X, Y; q)$ to financially insurable and noninsurable risk terms X and Y according to an operational control q. This allows us to unify the two aforementioned strands of research in the area of corporate risk management.

Optimal hedging may be cast as a budget-constrained functional optimization problem over a set of regular payoffs written on tradable indices only, hence under the assumption of incomplete information whereby no knowledge about nontradable risk terms may enter the hedging instrument (Caldentey and Haugh 2006). Each claim entails pure hedging and speculative terms, plus a new cross-hedge component. Both the existence and uniqueness of the optimal combined claims are guaranteed, provided the two claimable terms X and I are unrelated to one another according to a suitable measure of dependence which we identify. This measure explicitly excludes that they are *functionally dependent* on one another, meaning that $f(X) \neq h(I)$ for all functions f and h. By assuming absolutely continuous distributions for all risk variables, the optimal payoff functions satisfy a Fredholm integral equation.

2. We provide a *sharp analytical assessment of the incremental benefit* a risk averse firm ascribes to our optimal combined custom hedge $H^*(X) + G^*(I)$ compared to either optimal single-claim custom hedge $H^\flat(X)$ or $G^\flat(I)$. This benefit can be expressed in terms of indifference value (*i.e.*, a monetary amount) or relative efficiency (*i.e.*, a dimensionless rating); they allow us to quantify the power of our combined hedging proposal over the best existing alternatives.

We define combination value as the precise monetary amount which, when added to an optimal single-claim custom hedge, makes the resulting payoff as attractive as our optimal combined custom hedge's in terms of MV. We derive an explicit expression for this quantity in terms of optimal claim $G^*(I)$ and obtain two tight lower bounds for the combination value in terms of a suitable correlation measure between X and I and other model primitives. These bounds are related to a refined notion of risk aversion which allows us to differentiate between strictly and weakly risk averse firms. By assessing a combination value, we may rank any array of alternative claimable side indices to in turn decide which one to adopt. We also extend the efficiency index proposed by Brown and Toft (2002) to benchmark alternative hedges with respect to a scale ranging from no hedge to our optimal combined custom hedge. We derive a formula for efficiency in terms of optimal claim $G^*(I)$ and a tight upper bound. These results allow us to decide whether to switch from a single-claim to a combined custom hedge. A case-study shows the power of this method in the context of a global economic crisis.

3. We assess the *value enhancement from integrating* our optimal combined custom hedge $H^*(X) + G^*(I)$ within the optimal management of business and physical operations. For a generalized NV model, we experimentally measure the extent to which our integrated policy outperforms all existing primary alternatives and the effect it has on physical operations.

We consider a stylized case featuring a gas retailer exposed to random price and demand, and craft a new class of integrated risk management policies. In a realistic parametric setup, the integrated financial-operational policy consisting of our optimal combined custom hedge and the related optimal procurement provides a risk averse firm with as much as a simultaneous 38.31% reduction in risk (*i.e.*, standard deviation) and 14.22% increase in return (*i.e.*, expected value) of operating revenues. This latter comprises a financial speculative term represented by the expected payoff of the hedge, which is negative, and an operational term given by the variation in expected revenues obtained by switching from the optimally handled naked position to the naked position managed through the optimal integrated policy (+45.3% in gas procurement order), which exceeds the aforementioned figure. This underpins the hedging role of combining claims. It also shows the speculative attitude of the hedge provider who correspondingly cashes in positive expected revenues from their short hedge position. Finally, our hedge offers a key leverage to widen the profitable range of a firm's operations: gas procurement may increase anywhere from 2.2% to 54.0% while ensuring an increase in return and a reduction of risk. Ultimately, our analysis highlights the importance of combined custom hedging for integrated risk management whenever noninsurable risk is present. Achieving the three aforementioned goals in the simplest possible (*i.e.*, a one-period) time frame and the most general (*i.e.*, random-element based) state variable setup is a motive for our model.

The paper is organized as follows. Section 2 sets up the model framework. Section 3 presents our optimal combined custom hedge. Section 4 quantifies the value enhancement it yields. Section 5 analyzes a new class of integrated risk management policies. Section 6 concludes with a summary, empirical predictions, managerial insights, and avenues for future research. Complements, technical details, and all proofs are available in the Electronic Appendices.

2. Model

We develop a normative framework for the optimal design of combined custom contingent claims. Let us consider a one-period time frame $\{0, T\}$: financial hedging and an operational decision occur at time 0; uncertainty is resolved for all risk terms, and cash flows are paid out at time T. For the sake of simplicity, we assume interest rates are zero. A firm expresses their beliefs through a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Measure $\mathbb{P}$ denotes a physical probability reflecting an estimation based on a time series of observed samples, a purely subjective assessment, or any blend of the two. The sources of corporate risk are represented by a term X that is claimable in a suitable set $\mathcal{H}_X$ and a nonclaimable term Y. An additional side (*i.e.*, position external) index I is claimable in a set $\mathcal{H}_I$. (Exact hedging spaces, or classes, are defined in the next subsection.) Together, variables X, Y, and I represent the random state of the system at time T as seen from time 0. We let them be random elements defined on $(\Omega, \mathscr{F}, \mathbb{P})$. Hence, the model can handle both static and dynamic risk systems. Table EC.1 in Appendix EC.1.1 reports all terms and symbols appearing in the model.

Example 1 (Static and dynamic systems). Let 0 and T be the endpoints of a time interval $[0,T]$ and let $(X_t)_{0\leqslant t\leqslant T}$ and $(Y_t)_{0\leqslant t\leqslant T}$ be assigned stochastic processes. A standard case in finance entails figures observed exactly at time T, *i.e.*, $X := X_T$ and $Y := Y_T$. Alternatively, we may consider functionals of observed price paths. For instance, Fusai et al. (2008) consider discretely monitored Asian-style deals where payoffs depend on the arithmetic average of a commodity price monitored over a finite set $t_1,...,t_N$ of dates prior to the payoff date T. One may correspondingly set $X := N^{-1}\sum_{j=1}^{N} X(t_j)$ and $Y := N^{-1}\sum_{j=1}^{N} Y(t_j)$. More generally, risk variables can be the underlying stochastic processes, say $X := (X_t)_{0\leqslant t\leqslant T}$ and $Y := (Y_t)_{0\leqslant t\leqslant T}$.

2.1. Corporate Positions featuring Mixed Exposure

At time 0, the firm holds a naked (*i.e.*, hedge-free) position. This generates a time T cash flow $\pi = \pi(X,Y;q)$ combining claimable term X with nonclaimable term Y according to an operational policy q. Function $\pi(\cdot,\cdot;q)$ is assumed to be sufficiently regular to ensure that $\pi(X,Y;q)$ is square-integrable for any admissible control q. This quantity defines the time T state of a financially uncontrolled system, whereby no financial hedge has been entered in and an operational control policy q has been selected at time 0. In keeping with Caldentey and Haugh (2006), the operational policy domain may range from a set of scalars to a class of stochastic control policies which do not affect any system variable. Cash flow π thus accommodates a wide array of corporate exposure instances. The following examples are detailed in Appendix EC.1.2.

Example 2 (Mixed exposure). 1) Primary commodity production model (Moschini and Lapan 1995): $\pi(X,Y;q) := XYq - c(q)$. 2) Stochastic clearance price model (Caldentey and Haugh 2009): $\pi(X,Y;q) := (Y-\xi q)\,q - Xq$; 3) Generalized newsvendor model (Secomandi and Kekre 2014): $\pi(X,Y;q) := s(X)\,v(Y,q) + bid(X)\,(q-Y)_+ - ask(X)\,(Y-q)_+ - pq$; 4) Multinational production capacity allocation model (Chowdhry and Howe 1999): $\pi(\mathbf{X},\mathbf{Y};\mathbf{q}) := X_1Y_1 - c_1 o_1(\mathbf{q},\mathbf{Y}) + X_3\left[X_2Y_2 - c_2\left(Y_1+Y_2-o_1(\mathbf{q},\mathbf{Y})\right)\right]$.

2.2. The Space of Combined Custom Hedges

Hedging is determined by positioning in the custom derivative market at time 0. The firm may enter a tailored contingent claim with payoff $H(X)$ to settle at time T. However, the idiosyncratic risk portion carried over by Y remains unhedged so long as the risk term Y is nonclaimable. The firm may then identify a side index I correlated to Y and buy a custom claim with time T payoff $G(I)$. A *combined custom hedge* $H(X)+G(I)$ is defined by a pair of *combined claims* $H(X)\in\mathcal{H}_X$ and $G(I)\in\mathcal{H}_I$, where $\mathcal{H}_\#$ denotes the space of $\sigma(\#)$-measurable r.v.'s with finite variance. It also requires meeting a budget constraint $V_{H(X)} + V_{G(I)} \leqslant w$, where $V_\#$ denotes the fair value of a claim with payoff #, and w is the firm's cash available for hedging. This constraint is well-defined as long as X and I are claimable. Time T full exposure revenues can be expressed as:

$$\pi(X,Y;q)+[H(X)+G(I)], \tag{1}$$

where $(H(X),G(I))$ defines a financial control policy. They represent the time T state of a financially controlled system in our hedging problem. In general, a payoff $H(X)$ is an $\mathcal{H}_X$-measurable variable, that is a functional of random element X as a whole: later, we derive conditions whereby $H(X)$ is a genuine payoff function $x \to H(x)$ to compute on each sample value $x=X(\omega)$. Similar considerations hold for $G(I)$ and $H(X)+G(I)$.

We assume that the firm can price the two claims using arbitrage pricing theory (Bingham and Kiesel 2004). That is $V_{H(X)}=\mathbb{E}_{\mathbb{Q}}[H(X)]$ and $V_{G(I)}=\mathbb{E}_{\mathbb{Q}}[G(I)]$, where $\mathbb{Q}$ is an equivalent martingale measure (EMM) compatible with the absence of arbitrage opportunities. Following Caldentey and Haugh (2006) and (2009), we assume that $\mathbb{Q}$ is assigned through a pricing kernel $d\mathbb{Q}/d\mathbb{P}$ the firm is aware of. (Appendix EC.1.4 elaborates further on this point.) We need not assume completeness in the underlying market model. However, this property ensures the uniqueness of both the EMM $\mathbb{Q}$ and the consequent hedge. The budget constraint may be expressed as:

$$\mathbb{E}_{\mathbb{Q}}[H(X)+G(I)]\leqslant w. \tag{2}$$

Appendix EC.2.1 provides the additional regularity conditions allowing us to cast the hedge design problem within a Hilbert space setting: this framework lets us visualize the underlying issue through a neat geometrical representation (see Appendix EC.1.5) and develop a solution in full generality.

2.3. Optimal Design: Problem Statement

The quality of a combined custom hedge is assessed through a MV utility criterion $\mathscr{U}(\cdot):=\varrho\mathbb{E}[\cdot]-(a/2)\operatorname{Var}[\cdot]$, where ϱ and a are nonnegative scalars. If $\varrho>0$, then ratio a/ϱ defines the firm's *risk aversion*. If $a>0$, the reciprocal ratio ϱ/a denotes the firm's *risk propensity*. The case whereby $\varrho=0$ and $a>0$ represents a variance minimizing firm. A MV target allows us to obtain optimal hedges as the sum of pure hedging and speculative terms. This adheres to the empirical evidence about traded contingent claims in financial markets (Anderson and Danthine 1980). Correspondingly, we use the term "hedging" in the broad sense of "contingent claim positioning". Our adoption of MV preferences is supported by further considerations detailed in Appendix EC.1.3.

The firm seeks to design a pair of combined custom claims maximizing a MV target, subject to payoff regularity and a budget constraint, *i.e.*,

$$\max_{(H(X),G(I))\in\mathcal{H}_X\times\mathcal{H}_I:\ \mathbb{E}_{\mathbb{Q}}[H(X)+G(I)]\leqslant w} \mathscr{U}(\pi+[H(X)+G(I)]), \tag{3}$$

where $\pi := \pi(X, Y; q)$. Any solution (H^*, G^*) to this problem defines an optimal combined custom hedge $H^*(X) + G^*(I)$ for the firm in question. We omit explicitly indicating the dependence of π on q whenever this is immaterial to the analysis (Sections 3 and 4), but include it each time this is essential (*e.g.*, Section 5).

We can simplify the design problem (3) by showing that: 1) The maximum is attained in the geometrical variety defined by constraint $\mathbb{E}_{\mathbb{Q}}[H(X) + G(I)] = w$; 2) In a MV setup there is no loss of generality when assuming a zero cash endowment $w = 0$; 3) The maximum can be sought among value-centered claims: $\mathbb{E}_{\mathbb{Q}}[H(X)] = 0$ and $\mathbb{E}_{\mathbb{Q}}[G(I)] = 0$. All details are reported in Appendix EC.2.2. Note that property 2 also holds under the value maximizing criterion adopted in Caldentey and Haugh (2009). In the following, we exclusively focus on the value-centered problem. Appendix EC.1.4 provides a number of clarifying remarks about the extent, limitations, and possible extensions of our model, while Appendix EC.1.5 illustrates our theory within a simplified Bernoulli market model.

3. Optimal Design

3.1. Economic Characterization

We derive and analyze the economic structure of combined contingent claims that define any optimal combined custom hedge. Let $\mathscr{F}_X$ (resp., $\mathscr{F}_I$) denote the sub σ-algebra of the events space $\mathscr{F}$ generated by X (resp., I), which represents the information gleaned by the observation of X (resp., I), and $\mathbb{E}[\cdot|\mathscr{F}_\#]$ stand for the conditional expectation operator given $\mathscr{F}_\#$, which is the function of variable "#" that best approximates the argument "·" in the space of finite variance r.v.'s.

Proposition 1 **(Characterization of optimal combined custom hedge)**. *Any optimal combined custom hedge $H^*(X) + G^*(I)$ for the value-centered problem:*

$$\mathscr{U}^*_{\mathcal{H}_{XI}} := \max_{(H(X),G(I))\in\mathcal{H}_X\times\mathcal{H}_I:\ \mathbb{E}_{\mathbb{Q}}[H(X)]=\mathbb{E}_{\mathbb{Q}}[G(I)]=0} \mathscr{U}\left(\pi\left(X,Y\right) + \left[H\left(X\right) + G\left(I\right)\right]\right) \tag{4}$$

involves combined claim payoffs $H^*(X) = H_0^*(X) - \mathbb{E}_{\mathbb{Q}}[H_0^*(X)]$ and $G^*(I) = G_0^*(I) - \mathbb{E}_{\mathbb{Q}}[G_0^*(I)]$, where both $H_0^*(X)$ and $G_0^*(I)$ display a zero mean under $\mathbb{P}$ and satisfy the first-order condition:

$$\begin{cases} H_0^*(X) = -\mathbb{E}\left[(\pi - \mathbb{E}[\pi])|\,\mathscr{F}_X\right] + \mathbb{E}\left[\frac{\varrho}{a}\left(1 - \frac{d\mathbb{Q}}{d\mathbb{P}}\right)\Big|\,\mathscr{F}_X\right] - \mathbb{E}\left[G_0^*(I)|\,\mathscr{F}_X\right], \\ G_0^*(I) = -\mathbb{E}\left[(\pi - \mathbb{E}[\pi])|\,\mathscr{F}_I\right] + \mathbb{E}\left[\frac{\varrho}{a}\left(1 - \frac{d\mathbb{Q}}{d\mathbb{P}}\right)\Big|\,\mathscr{F}_I\right] - \mathbb{E}\left[H_0^*(X)|\,\mathscr{F}_I\right]. \end{cases} \tag{5}$$

Combined claims $H^*(X)$ and $G^*(I)$ are value-centered as per budget constraint. Their cash flows are driven by payoffs $H_0^*(X)$ and $G_0^*(I)$ solving system (5), each featuring as many as three financially distinct components. We analyze the structure of $H_0^*(X)$, with $G_0^*(I)$ following *mutatis mutandis.*

A first term is the best estimate one can make of the deviation of business revenues π from their average value $\mathbb{E}[\pi]$, based on the sole knowledge of the claimable term X appearing there. In a context of purely claimable exposure ($\pi = \pi(X)$) and lack of market risk premium ($\mathbb{P} = \mathbb{Q}$) this term boils down to the optimal single-claim hedge derived in Chowdhry and Howe (1999). If the market quotes a nonzero risk premium ($\mathbb{P} \neq \mathbb{Q}$), the same term arises as the variance minimizing ($\varrho = 0$) single-claim hedge (see Proposition EC.1).

A second term builds on the market risk premium $1 - d\mathbb{Q}/d\mathbb{P}$. It defines the optimal single-claim position for a pure speculator ($\pi = 0$) and represents the reward they receive for risk taking (Poitras (2013)). This cash flow acts as a rebate over the minimum variance hedging payoff. Interestingly, a speculative component enters the optimal derivative schedule proportionally to the ratio ϱ/a representing firm's propensity to risk. Notably, this term is independent of the firm's position and plays a ubiquitous role in optimal derivative positioning (Leland (1980), Brennan and Solanki (1981), Carr and Madan (2001)).

A third term seems to be new in the existing literature on the subject. It defines a cross-claim effect representing a hedge enhancement afforded by the linkage between X and I. Specifically, the greater the dependence between side index I and position's claimable X, the stronger the contribution of one claim, say $G^*(I)$, to the payoff of the other claim, say $H^*(X)$. An offsetting cross-claim effect may explain the underhedging puzzle arising in asset pricing, a phenomenon usually ascribed to utility prudence (Adam-Müller 1997), basis risk (Briys et al. 1993), defaulting OTC markets (Cummins and Mahul 2008), or liquidity needs (Mello and Parsons 2000).

Example 3 (Independent hedging index and the speculative theme). If a side index I is statistically independent of the claimable term X, the third term in system (5) vanishes, *i.e.*, $\mathbb{E}[G^*(I)|\mathscr{F}_X] = \mathbb{E}[G^*(I)] = 0$, and the (value-centered) optimal claims reduce to $H_0^*(X) = -\mathbb{E}[\mathcal{K}_0|\mathscr{F}_X]$ and $G_0^*(I) = -\mathbb{E}[\mathcal{K}_0|\mathscr{F}_I]$, where $\mathcal{K}_0 := \pi + (\varrho/a)\, d\mathbb{Q}/d\mathbb{P} - \mathbb{E}[\pi + (\varrho/a)\, d\mathbb{Q}/d\mathbb{P}]$. If index I is also statistically independent of term Y, hence $G_0^*(I) = \mathbb{E}[(\varrho/a)(1 - d\mathbb{Q}/d\mathbb{P})|\mathscr{F}_I]$, then I is useless for mitigating the firm's exposure π although it may have a speculative role. Indeed, if risk is the sole concern of the firm (*i.e.*, $\varrho = 0$), then $G_0^*(I) \equiv 0$. However, if expected revenues are part of the firm's target (*i.e.*, $\varrho > 0$), then $G_0^*(I) \neq 0$, provided that I and the market premium $d\mathbb{Q}/d\mathbb{P}$ are not statistically independent of one another.

Essential to the present study is the *optimal hedging kernel* defined as $\mathcal{K} := \pi + (\varrho/a)\, d\mathbb{Q}/d\mathbb{P}$. This quantity gathers both hedging and speculative components of the optimal combined hedge in an ideal contingent claim that simultaneously spans business and market premium risks. This can be seen by posing $G \equiv 0$ in the first equation of system (5) and allowing for positioning in claims written on a variable Z that spans both market premium risk (*i.e.*, pricing kernel $d\mathbb{Q}/d\mathbb{P}$ is $\mathcal{F}_Z$-measurable) and business risk (*i.e.*, π is $\mathcal{F}_Z$-measurable). Note that $\mathcal{K}_0$ defined in the example above is the *optimal centered hedging kernel*, *i.e.*, $\mathcal{K}_0 = \mathcal{K} - \mathbb{E}[\mathcal{K}]$.

3.2. Existence and Uniqueness

In general, the optimal combined claim pair $(H^*(X), G^*(I))$ may not be unique (see, *e.g.*, Appendix EC.1(F)). It turns out that both the existence and uniqueness of the optimal combined claims $H^*(X)$ and $G^*(I)$, and the hedge $H^*(X)+G^*(I)$ they entail, depends on the relation between X and I as measured through the *maximal correlation* index defined by:

$$r_{XI} := \sup_{\phi(X)\in\mathcal{H}_X, \psi(I)\in\mathcal{H}_I:\ \mathbb{E}[\phi(X)]=\mathbb{E}[\psi(I)]=0} \rho(\phi(X), \psi(I)) \in [0,1], \tag{6}$$

where ρ is the Pearson linear correlation (Balakrishnan and Lai 2009). Appendix EC.2.3 reports the major properties of this index.

Theorem 1 **(Existence and uniqueness of optimal combined hedge)**. *Let claimable X and index I display a maximal correlation $r_{XI} < 1$. Then, the optimal combined claim pair $(H^*(X), G^*(I))$, hence the resulting combined custom hedge $H^*(X)+G^*(I)$, exists and is unique. In addition, the value-centered optimal claim $H_0^*(X)$ solves the fixed-point functional equation:*

$$H_0^*(X) = \Pi_0^X(X) + \mathbb{E}\left[\mathbb{E}\left[H_0^*(X)\middle|\mathscr{F}_I\right]\middle|\mathscr{F}_X\right], \tag{7}$$

where term $\Pi_0^X(X) := -\mathbb{E}\left[\mathcal{K} - \mathbb{E}\left[\mathcal{K}\middle|\mathscr{F}_I\right]\middle|\mathscr{F}_X\right]$ represents the best estimate a suitable function of X can make of the portion of the optimal hedging kernel $\mathcal{K}$ that is unspanned by I.

Once we solve equation (7) for $H_0^*(X)$, then $G_0^*(I)$ stems from the second equation in the first-order system (5). This features a high level of generality, as X, Y, and I may be any kind of random element, including finite vectors and stochastic processes. We offer examples within a log-normal, a Bernoulli and an Itô dynamic market model in Appendix EC.1.6.

Example 4 (Log-normal market model). Under a standard market model featuring lognormal variables (Bingham and Kiesel 2004), things are dramatically simpler, since unitary maximal correlation is equivalent to having one variable functionally dependent on the other. If $(X,I) = (e^Z, e^W)$ and $(Z,W) \overset{\mathbb{P}}{\sim} \mathcal{N}(\mathbf{m}, C)$, condition $r_{XI} < 1$ amounts to $|\rho_{Z,W}| < 1$. In addition, $r_{XI} = 1$ entails $|\rho_{Z,W}| = 1$, *i.e.*, $W = aZ + b$: hence, the two underlying terms X and I are functionally dependent one another as long as $I = e^W = e^b(e^Z)^a = e^b X^a$ and the problem degenerates.

3.3. The Design Integral Equation

Under absolutely continuous distributions (with $\mathbb{P}$ and $\mathbb{Q}$ densities $f_\#$ and $f_\#^{\mathbb{Q}}$, respectively), combined claim payoffs $H^*(X)$ and $G^*(I)$ take the form of payoff functions $H^*(x)$ and $G^*(i)$ to compute at the values assumed by X and I, respectively; in addition, these payoff functions solve a Fredholm integral equation. Let $\mathscr{L}_{\mathbb{P}}(\#)$ be the law of a r.v. # and $L^2_\# := L^2\left(\mathbb{R}^N, \mathscr{B}_{\mathbb{R}^N}, \mathscr{L}_{\mathbb{P}}(\#)\right)$.

THEOREM 2 **(Design integral equation)**. *Let the N-dimensional r.v.'s X, Y, and I admit a joint distribution density f_{XYI} under $\mathbb{P}$. If $r_{XI} < 1$, then the optimal combined custom hedge $H^*(X) + G^*(I)$ exhibits payoff functions $H^*(x) = H_0^*(x) - c_H^*$ and $G^*(x) = G_0^*(x) - c_G^*$, with constants $c_H^* := \mathbb{E}_{\mathbb{Q}}[H_0^*(X)]$ and $c_G^* := \mathbb{E}_{\mathbb{Q}}[G_0^*(I)]$; the value-centered claim payoff function $H_0^*(x)$ in L_X^2 solves a Fredholm equation of the second kind:*

$$H_0^*(x) = \Pi_0^X(x) + \int_{\mathbb{R}^N} k(x,\xi) H_0^*(\xi) d\xi, \quad \left(x \in \mathbb{R}^N\right) \tag{8}$$

with integral kernel $k(x,\xi) := \int_{\mathbb{R}^N} \frac{f_{XI}(x,i)}{f_X(x) f_I(i)} f_{XI}(\xi,i) di$, $(x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N$, and term:

$$\begin{aligned}\Pi_0^X(x) = &- \int_{\mathbb{R}^N} \pi(x,y) \frac{f_{XY}(x,y)}{f_X(x)}\, dy + \int_{\mathbb{R}^{3N}} \pi(\xi,y) \frac{f_{XI}(\xi,i) f_{XYI}(x,y,i)}{f_X(x) f_I(i)}\, d\xi dy di \\ &- \frac{\varrho}{a}\left(\frac{f_X^{\mathbb{Q}}(x)}{f_X(x)} - \int_{\mathbb{R}^N} f_I^{\mathbb{Q}}(i) \frac{f_{XI}(x,i)}{f_X(x) f_I(i)}\, di\right), \quad \left(x \in \mathbb{R}^N\right).\end{aligned} \tag{9}$$

Expressions for the payoff function $G_0^*(i)$ in L_I^2 are derived in Appendix EC.3.3 (formulae (EC.29) and (EC.30)). Appendix EC.1.7 solves the design integral equation (8) for a large class of integral kernels, proposes a new numerical scheme, and shows a convergence result.

4. Combination Value Assessment

Firms can optimally handle mixed exposure through the optimal combined custom hedge $H^*(X) + G^*(I)$. They may nevertheless decide to hedge through a single instrument on either claimable term X or I. In this respect, two major issues present themselves:

Issue 1 *How should we select an appropriate index I within a menu of claimable terms?*

Issue 2 *When is it worth switching from a single-claim to our combined custom hedge?*

Answering these questions requires us to first assign a monetary value to our combination theme.

4.1. Financial Hedging Flexibility, Combination Value, and Risk Spanning

Access to any class $\mathcal{H} \subset L^2(\Omega)$ of contingent claims defines the *financial hedging flexibility* of a firm. Whenever it is well-posed, the corresponding hedge design problem is:

$$H_{\mathcal{H}}^* \leftarrow \max_{H \in \mathcal{H}:\ \mathbb{E}_{\mathbb{Q}}[H]=0} \mathscr{U}(\pi + H) =: \mathscr{U}_{\mathcal{H}}^*. \tag{10}$$

EXAMPLE 5. The optimal single-claim custom hedges written on either X or I are defined as $H^\flat := H_{\mathcal{H}_X}^*$ and $G^\flat := H_{\mathcal{H}_I}^*$, respectively. The optimal combined custom hedge defined in Proposition 1 satisfy $H^* + G^* = H_{\mathcal{H}_{XI}}^*$, where $\mathcal{H}_{XI} := \mathcal{H}_X + \mathcal{H}_I$. (Appendix EC.1.8 elaborates on the discrepancy between optimal single-claim and combined custom hedges in terms of their payoff functions.)

We may monetize a shift in financial hedging flexibility, say from $\mathcal{G}$ to $\mathcal{H}$, by resorting to a notion of indifference value for the MV criterion $\mathscr{U}$.

Definition 1 (Value of financial flexibility). The *value* $\mathcal{V}_{\mathcal{G}\to\mathcal{H}}$ *of a financial hedging flexibility shift from* $\mathcal{G}$ *to* $\mathcal{H}$ is the monetary amount m rendering the firm indifferent between adopting the optimal hedge in $\mathcal{G}$ plus m and the optimal hedge in $\mathcal{H}$, *i.e.*, $\mathscr{U}\left(\pi + H^*_{\mathcal{G}} + m\right) = \mathscr{U}\left(\pi + H^*_{\mathcal{H}}\right)$. The *value* $\mathcal{V}_{\mathcal{H}}$ *of financial hedging flexibility* granted by $\mathcal{H}$ is defined by $\mathcal{V}_{\{0\}\to\mathcal{H}}$.

Clearly, $\mathcal{V}_{\mathcal{G}\to\mathcal{H}}$ may be negative. It surely is nonnegative provided that $\mathcal{G}\subset\mathcal{H}$: this is the case of $\mathcal{V}_{\mathcal{H}}$ for any class $\mathcal{H}\neq\{0\}$. Since $\mathbb{E}_{\mathbb{Q}}\left[H^*_{\mathcal{G}}\right]=0$ (budget constraint), then $\mathcal{V}_{\mathcal{G}\to\mathcal{H}}$ may be interpreted as the fair price of claim $H^*_{\mathcal{G}}+m\in\mathcal{G}$ achieving the same target MV as that of the optimal claim $H^*_{\mathcal{H}}\in\mathcal{H}$. Since $\mathscr{U}\left(\#+m\right)=\mathscr{U}\left(\#\right)+\varrho m$, then $\mathscr{U}\left(\pi+H^*_{\mathcal{G}}+\mathcal{V}_{\mathcal{G}\to\mathcal{H}}\right)=\mathscr{U}^*_{\mathcal{G}}+\varrho\mathcal{V}_{\mathcal{G}\to\mathcal{H}}$, and:

$$\mathcal{V}_{\mathcal{G}\to\mathcal{H}} = \varrho^{-1}\left(\mathscr{U}^*_{\mathcal{H}} - \mathscr{U}^*_{\mathcal{G}}\right). \tag{11}$$

This formula allows us to interpret changes in optimal mean-variance in terms of monetary units. We may apply these notions to assess a monetary value to claim combination.

Definition 2 (Combination value). The *combination value* $\mathcal{CV}_{X\to X+I}$ *of* X *and* I *over* X is the *value of a financial hedging flexibility shift from single-claim to combined custom hedging, i.e.,*

$$\mathcal{CV}_{X\to X+I} := \mathcal{V}_{\mathcal{H}_X\to\mathcal{H}_{XI}}.$$

We may similarly define the combination value of X and I over I as $\mathcal{CV}_{I\to X+I} := \mathcal{V}_{\mathcal{H}_I\to\mathcal{H}_{XI}}$. Combination values are always nonnegative because $\mathcal{H}_{XI}\supset\mathcal{H}_X$. Note that these definitions focus on combination values under custom hedging: this choice averts potential valuation bias stemming from assuming any prescribed functional form for the hedge's payoff, say linear (forward hedge) or piecewise linear (combinations of vanilla instruments).

Any claim on a side index I that is functionally dependent on X (as defined in Subsection 1.2) provides the firm with no combination value over using $H^\flat(X)$. Hence, we assume that $r_{IX}<1$ so that risk variables I and X are not functionally dependent upon one another. The exact value of optimal combination under customization depends upon the ability of the claimable terms to span the risk engendered by business revenues and to leverage the market risk premium. Let I be a candidate side index. We argue that:

- *Conjecture I*: All other terms being equal, the greater the statistical linkage r_{IX} between side index I and the position's claimable X, the stronger the redundancy of I, and the smaller the value from combining a claim on X with a claim on I. Hence, on equal terms, the relevance of claim combination *vs.* single-claim positioning decreases with increasingly dependent I and X. The nonunitary assumption $r_{IX}<1$ in Theorem 1 underpins our guess.

- *Conjecture II*: All other terms being equal, the greater the statistical linkage r_{IY} between side index I and the position's nonclaimable Y, the stronger the ability of I to mitigate the risk engendered by Y, and the greater the value from combining a claim on X with a claim on I. This is actually the rationale behind the studies by Gaur and Seshadri (2005), Caldentey and Haugh (2006), Chod et al. (2010), Wang and Yao (2017) and (2019)).

These statements call for selecting a side index I which is as weakly linked to X (*i.e.*, low r_{IX}) and as strongly linked to Y (*i.e.*, high r_{IY}) as possible. However, a low value of r_{IX} does not necessarily entail a high value of r_{IY}: they may both decrease or increase, thus producing a counterbalancing effect which ought to be assessed case by case. Example EC.7 shows this may happens within the context of a Bernoulli market model. Moreover, an increase (decrease) in the speculative nature of the selected combined custom hedge may partially offset the loss (gain) in terms of net reduction of risk that results from using an index I adhering to both prescriptions above. The intricate relation between combination value and maximal correlation indices can be clarified through a number of estimates, which in turn offer a tool for the rational selection of a side index (Issue 1).

4.2. Value Estimates and Index Selection

We can compute the combination value $\mathcal{CV}_{X\to X+I}$ in terms of the additional centered claim $G_0^*(I)$ and offer a lower bound that may suffice for practical applications, *e.g.*, index selection.

THEOREM 3 **(Combination value assessment)**. *Let $r_{IX}<1$ and $\varrho>0$. Then:*

$$\mathcal{CV}_{X\to X+I} = \frac{a}{2\varrho}\mathrm{Var}\left(G_0^*(I) - \mathbb{E}\left[G_0^*(I)\middle|\mathscr{F}_X\right]\right). \tag{12}$$

$$\geqslant \frac{a}{2\varrho}(1-r_{IX}^2)\mathrm{Var}\left(\mathbb{E}\left[\mathcal{K}_0 - \mathbb{E}\left[\mathcal{K}_0\middle|\mathscr{F}_X\right]\middle|\mathscr{F}_I\right]\right), \tag{13}$$

where $\mathcal{K}_0$ is the optimal centered hedging kernel and pair $(H_0^*(X), G_0^*(I))$ solves system (5). The lower bound is tight: equality holds true provided that X and I are statistically independent.

A similar expression can be derived for $\mathcal{CV}_{I\to X+I}$. The right-hand side in these expressions is invariant upon rescaling the risk aversion parameters a and ϱ. While exact estimate (12) requires one to preliminarily compute $G_0^*(I)$, the lower bound (13) directly stems from correlation r_{IX} and other model primitives. Specifically, this bound increases proportionally to three factors: (i) The risk aversion coefficient a/ϱ: the greater its value, the stronger the firm's benefit from combined positioning on X and I; (ii) A degree of statistical independence between X and I, here measured by $1-r_{IX}^2$: this quantity attains a maximum when X and I are mutually independent (Conjecture I); (iii) The ability of a suitable payoff written on side index I to span the variance risk engendered by the portion of the optimal centered hedging kernel $\mathcal{K}_0$ that remains unspanned by using the best approximating claim written on X: this term is intimately connected to the relation between index I and the nonclaimable term Y.

We can improve our estimate (13) and seek conditions whereby a bound for $\mathcal{CV}_{X\to X+I}$ is explicitly proportional to the degree of dependence between I and Y (Conjecture II). We restrict our statement to separable mixed exposures featuring independent risk terms: a similar, yet more convoluted expression can be derived in the general case.

PROPOSITION 2 **(Combination value lower bound: strictly risk averse firms)**. *Let X and Y be statistically independent risk terms entering a separable exposure $\pi(X,Y) := \alpha(X)\beta(Y)$. If:*

$$\mathrm{Var}[\pi] \gg \left(\frac{\varrho}{a}\right)^2 \mathrm{Var}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right], \tag{14}$$

then:

$$\mathcal{CV}_{X\to X+I} \gtrapprox \frac{a}{2\varrho}(1-r_{IX}^2)\left(1-\sqrt{1-r_{IY}^2}\right)^2 \left|m_{\alpha(X)}\right| \sigma_{\beta(Y)}, \tag{15}$$

where $m_{\alpha(X)} = \mathbb{E}[\alpha(X)]$ and $\sigma^2_{\beta(Y)} = \mathrm{Var}[\beta(Y)]$. The lower bound is tight: equality holds true provided that X and I are statistically independent.

Bound (15) is approximately proportional to risk aversion coefficient a/ϱ, a degree $1-r_{IX}^2$ of statistical independence between X and I, and a degree $\left(1-\sqrt{1-r_{IY}^2}\right)^2$ of statistical dependence between I and Y: their joint effect attains a maximum value when index I is simultaneously independent of X ($r_{IX}=0$) and fully dependent on Y ($r_{IY}=1$) in terms of maximal correlation. This confirms Conjecture II. Bound (15) is also easier to compute than bound (13) and exact estimate (12) as long as it directly stems from model parameters: it thus offers the firm a practical tool to select a side hedging index within a basket of alternative claimable terms.

EXAMPLE 6 (SIDE INDEX SELECTION). Let I_1, I_2, and I_3 be claimable indices. Assume maximal correlations are $r_{I_1X}=0.1$, $r_{I_2X}=0.2$, $r_{I_3X}=0.4$, and $r_{I_1Y}=0.6$, $r_{I_2Y}=0.7$, $r_{I_3Y}=0.75$. Whereas index I_1 is the "closest" to independence of X, index I_3 is the most dependent on Y among the three indices in question. Let us consider for simplicity a variance minimizing agent ($\varrho=0$). Asymptotic estimate (15) is essentially a bound proportional to $(1-r_{IX}^2)\left(1-\sqrt{1-r_{IY}^2}\right)^2$. For a firm satisfying condition (14), it solves this puzzle and shows that I_3 is the best choice.

Condition (14) allows us to sharpen the traditional concept of risk aversion. Two firms sharing a common risk aversion a/ϱ may actually differ in terms of exposure π and beliefs $\mathbb{P}$. Hence, the pure hedging components appearing in their optimal combined claims differ as well (formula (5)). Inequality (14) states that (squared) risk aversion exceeds the market premium risk per unit of business risk. We correspondingly say the firm is *strictly risk averse*. Otherwise, the firm is said to be *weakly risk averse*. These definitions are well-posed since $1=\mathbb{E}\left[d\mathbb{Q}/d\mathbb{P}\right] \leqslant \mathbb{E}\left[(d\mathbb{Q}/d\mathbb{P})^2\right]$. For a weakly risk averse firm, we offer a lower bound in Appendix EC.1.9.

4.3. Efficiency Analysis

Estimate (12) and bounds (13), (15), and (EC.14) offer *absolute* assessments of combination values. Hence, they do not speak to the appropriateness of switching from optimal single-claim hedges to a combined custom hedge (Issue 2). According to Brown and Toft (2002), this should depend on the *relative* proximity or distance between competing hedges in terms of target performance. We can adapt their efficiency index to MV preferences and define a new one based on values of financial hedging flexibility. This index allows us to benchmark alternative hedges against a scale ranging from no hedge to our optimal combined hedge. This ranking allows a firm to assess the relative merits of alternative solutions offered by a hedge provider.

Definition 3 (Single-claim hedge efficiency). The single-claim hedge $H^{\flat}(X)$ (resp., $G^{\flat}(I)$) efficiency is the ratio of financial hedging flexibility values granted by $\mathcal{H}_X$ (resp., $\mathcal{H}_I$) and $\mathcal{H}_{XI}$:

$$\mathcal{E}_X := \frac{\mathcal{V}_{\mathcal{H}_X}}{\mathcal{V}_{\mathcal{H}_{XI}}}, \qquad (\text{resp., } \mathcal{E}_I := \frac{\mathcal{V}_{\mathcal{H}_I}}{\mathcal{V}_{\mathcal{H}_{XI}}}). \tag{16}$$

Efficiency is dimensionless, it lies in $[0,1]$, and is invariant under affine rescaling in the MV. It can also be expressed in terms of MV: $\mathcal{E}_{\#} = \left(\mathscr{U}^*_{\mathcal{H}_{\#}} - \mathscr{U}_0\right) / \left(\mathscr{U}^*_{\mathcal{H}_{XI}} - \mathscr{U}_0\right)$, with $\mathscr{U}_0 := \mathscr{U}(\pi)$. Hereafter, we focus on $\mathcal{E}_X$, the case of $\mathcal{E}_I$ being similarly dealt with. Example EC.8 ranks an array of potential hedges in a Bernoulli market model. Here follows an exact assessment for $\mathcal{E}_X$.

Proposition 3 **(Single-claim hedge efficiency assessment)**.

$$\mathcal{E}_X = \frac{\operatorname{Var}\left(\mathbb{E}\left[\mathcal{K}_0 \mid \mathscr{F}_X\right]\right)}{\operatorname{Var}\left(\mathbb{E}\left[\mathcal{K}_0 \mid \mathscr{F}_X\right]\right) + \operatorname{Var}\left(G^*(I) - \mathbb{E}\left[G^*(I) \mid \mathscr{F}_X\right]\right)}. \tag{17}$$

In particular, $\mathcal{E}_X = 1$ provided that $I = f(X)$. $\mathcal{E}_I$ can similarly be derived in terms of $H^*(X)$.

Formula (17) shows that the greater the variance of the portion of $G^*(I)$ that is unpredictable by X, the lower the capacity of single-claim hedge $H^{\flat}(X)$ to yield the MV attained by our optimal combined custom hedge $H^*(X) + G^*(I)$, and the lower the efficiency of single-claim hedge $H^{\flat}(X)$.

4.3.1. Case-study: Risk aversion and hedge combination during COVID-19 pandemic. Let $X =$ NYMEX 1m WTI oil futures price (US$/bbl), $Y =$ IEA US oil daily consumption (MMbbl), and $I =$ COMEX 1m gold futures price (US$/oz) 6-month forecasts. A representative agent of the US oil industry is exposed to operating revenues $\pi := (X - k)Y$, where $k = 22.5$ USD/bbl is an average unit production cost. When should they hedge through $H^{\flat}(X)$, $G^{\flat}(I)$, or $H^*(X) + G^*(I)$? We analyze this trilemma under normal and extreme market conditions, here represented by the first semester of 2019 and 2020 (COVID-19 pandemic), respectively. At the outset

Table 1 Volatility, correlation, and efficiency across market contexts and risk aversion levels.

Year	WTI vol	Cons vol	Gold vol	Maximal correlations			Risk averse ⟷ Risk taker				Eff
	σ_X	σ_Y	σ_I	r_{XY}	r_{XI}	r_{YI}	$100a^*$	$50a^*$	$10a^*$	a^*	
2019	0.064	0.001	0.006	0.090	0.207	0.179	0.994	0.972	0.989	0.538	$\mathcal{E}_X$
(normal)							0.048	0.036	0.037	0.400	$\mathcal{E}_I$
2020	0.962	0.026	0.003	0.849	-0.399	-0.623	0.857	0.492	0.039	0.001	$\mathcal{E}_X$
(crisis)							0.306	0.627	0.983	0.999	$\mathcal{E}_I$

of each period, we estimate a simplified version of the trivariate lognormal model (EC.18) reported in Appendix EC.2(E). 6-month estimates for volatilities, maximal correlations, and efficiencies in each time period and across four levels of risk aversion (standard: $a^* := 2\mathbb{E}[\pi]/\mathrm{Var}[\pi] \simeq 7.7 \times 10^{-4}$, med.: $10a^*$, high: $50a^*$, huge: $100a^*$) are reported in Table 1.

2019 shows a normal scenario: WTI and gold are mildly volatile, and consumption is steady. All correlations are small, with WTI-consumption smallest. Thus, revenues π almost exclusively depend on WTI and a single claim on X should hedge most of their risk, while any combination value should prove modest. Efficiency figures show this is the case ($\mathcal{E}_X \sim 1$), unless the agent is a risk taker ($a = a^*$) and claims on X and I may combine to enhance profitability regardless of risk. In contrast, 2020 exhibits an extreme scenario: WTI is highly volatile, consumption uncertainty dramatically increases, and gold prices reflect the diversification role they usually play during crises (Baur and Lucey 2010). All correlations increase in absolute value and a tradeoff emerges: on one hand, an increased r_{XY} makes consumption Y approach a function of X, hence revenues π depend on X only: hence, the best practice to hedge through a single claim on X; on the other hand, a tripled figure for $|r_{YI}|$ suggests a value in combining claims. Since $\mathcal{E}_X$ increases with risk aversion, while $\mathcal{E}_I$ increases with risk propensity, the problem can be solved as follows: for an appropriately high level of risk aversion a (resp., propensity a^{-1}), the agent should use $H^\flat(X)$ (resp., $G^\flat(I)$); on intermediate values, where both efficiency $\mathcal{E}_X$ and $\mathcal{E}_I$ are well below one, they should adopt the optimal combined hedge $H^*(X) + G^*(I)$. This applies to both the risk takers in 2019 and the highly risk averse agents in 2020. Combination strategy is thus valuable in managing mixed risk during the aforementioned economic crisis.

5. Risk Management Integration

The joint adoption of financial and operational instruments defines integrated risk management policies (Babich and Kouvelis 2018). Our theoretical development allows us to craft a new class of integrated policies to optimize business exposure.

5.1. A New Integrated Policy for the Newsvendor Position

We take the case of a commodity producer, say a gas company serving a mid-sized area in the US. They produce a stock q (operational control) to meet future random demand Y (position's non-claimable). Each unit sold yields the market price X prevailing upon delivery (position's claimable).

Gross revenues are thus: $X\min\{Y,q\}$. Residual stock, if any, entails a fixed unit cost net salvage price, say c. This leads to an overall cost equal to $c\max\{q-Y,0\}$. For the sake of simplicity, we assume that unmet demand, if any, entails no additional cost. We also disregard production costs: their consideration would increase computational effort while leaving our argument unaltered. The resulting position displays newsvendor-type operating net revenues as in Wang and Yao (2017):

$$\pi^{NV}(X,Y;q) := X\min\{Y,q\} - c\max\{q-Y,0\}. \tag{18}$$

The side index I (the position external claimable term) is a temperature record in the delivery area.

An *integrated policy* (H,q) is defined by a hedging payoff function H (financial control) in a set $\mathcal{H}$ and a procurement order q (operational control) in a set $\mathcal{Q}$. We consider seven sets $\mathcal{H}\times\mathcal{Q}$ of admissible integrated policies. They differ in terms of financial hedging space $\mathcal{H}$, while they all share a common set $\mathcal{Q}$ of feasible procurement orders. Given a set $\mathcal{H}\times\mathcal{Q}$ of admissible integrated policies, we select the one optimizing full exposure utility, namely:

$$\left(H^*_{\mathcal{H}\times\mathcal{Q}}, q^*_{\mathcal{H}\times\mathcal{Q}}\right) \leftarrow \max_{(H,q)\in\mathcal{H}\times\mathcal{Q}:\ \mathbb{E}_{\mathbb{Q}}[H]=0} \mathscr{U}\left(\pi^{NV}(X,Y;q)+H\right) =: \mathscr{U}^*_{\mathcal{H}\times\mathcal{Q}}.$$

This optimization problem can be disentangled into two nesting problems:

$$H^*_{\mathcal{H}}(q) \leftarrow \max\nolimits_{H\in\mathcal{H}:\ \mathbb{E}_{\mathbb{Q}}[H]=0} \mathscr{U}\left(\pi^{NV}(X,Y;q)+H\right) =: \mathscr{U}^*_{\mathcal{H}}(q), \tag{19}$$

$$q^*(\mathcal{H}) \leftarrow \max\nolimits_{q\in\mathcal{Q}} \mathscr{U}\left(\pi^{NV}(X,Y;q)+H^*_{\mathcal{H}}(q)\right), \tag{20}$$

so that the optimal integrated policy in $\mathcal{H}\times\mathcal{Q}$, the corresponding operating revenues, and the related utility may be expressed as $\left(H^*_{\mathcal{H}\times\mathcal{Q}}, q^*_{\mathcal{H}\times\mathcal{Q}}\right) := \left(H^*_{\mathcal{H}}(q^*(\mathcal{H})), q^*(\mathcal{H})\right)$, $\pi^*_{\mathcal{H}\times\mathcal{Q}} + H^*_{\mathcal{H}\times\mathcal{Q}} := \pi^{NV}(X,Y;q^*(\mathcal{H})) + H^*_{\mathcal{H}}(q^*(\mathcal{H}))$, and $\mathscr{U}^*_{\mathcal{H}\times\mathcal{Q}} = \mathscr{U}^*_{\mathcal{H}}(q^*(\mathcal{H}))$, respectively. We refer to maps $q\to H^*_{\mathcal{H}}(q)\in\mathcal{H}$ and $q\to\mathscr{U}^*_{\mathcal{H}}(q)\in\mathbb{R}$ as the *optimal hedge profile in* $\mathcal{H}$ and the *financially optimal utility profile over* $\mathcal{H}$, respectively. Then, the optimal hedge $H^*_{\mathcal{H}\times\mathcal{Q}}$ in $\mathcal{H}$ and utility $\mathscr{U}^*_{\mathcal{H}\times\mathcal{Q}}$ over $\mathcal{H}$ are given by the optimal hedge profile and the financially optimal utility profile over $\mathcal{H}$, each computed with the optimal procurement order $q^*(\mathcal{H})$.

The first hedging policy leaves the business position financially naked. The second to the sixth hedging policies assume positioning in a linear claim on X (Rolfo 1980), a linear claim on I (Gaur and Seshadri 2005), a custom claim on X (Chen et al. 2014), a custom claim on I (new), and a combined linear hedge on X and I (Roncoroni and Id Brik 2017), respectively. Appendix EC.2.4 offers a precise description of these six hedging classes, as well as the corresponding financially optimal utility profiles. The seventh hedging policy involves our optimal combined custom hedge, *i.e.*, $\mathcal{H}=\mathcal{H}_{XI}:=\mathcal{H}_X+\mathcal{H}_I$. The financially optimal utility profile is $\mathscr{U}^*_{\mathcal{H}_{XI}}(q) =$

$\mathscr{U}\left(\pi^{NV}(X,Y;q)+H^*(X)+G^*(I)\right)$ and $H^*_{\mathcal{H}_{XI}\times\mathcal{Q}}:=H^*(X)+G^*(I)$ is our optimal combined custom hedge. We are now ready to present our proposal.

The joint adoption of the optimal combined custom hedge with the related optimal procurement:

$$\left(H^*_{\mathcal{H}_{XI}\times\mathcal{Q}},q^*_{\mathcal{H}_{XI}\times\mathcal{Q}}\right):=\left(H^*_{\mathcal{H}_{XI}}\left(q^*\left(\mathcal{H}_{XI}\right)\right),q^*\left(\mathcal{H}_{XI}\right)\right) \tag{21}$$

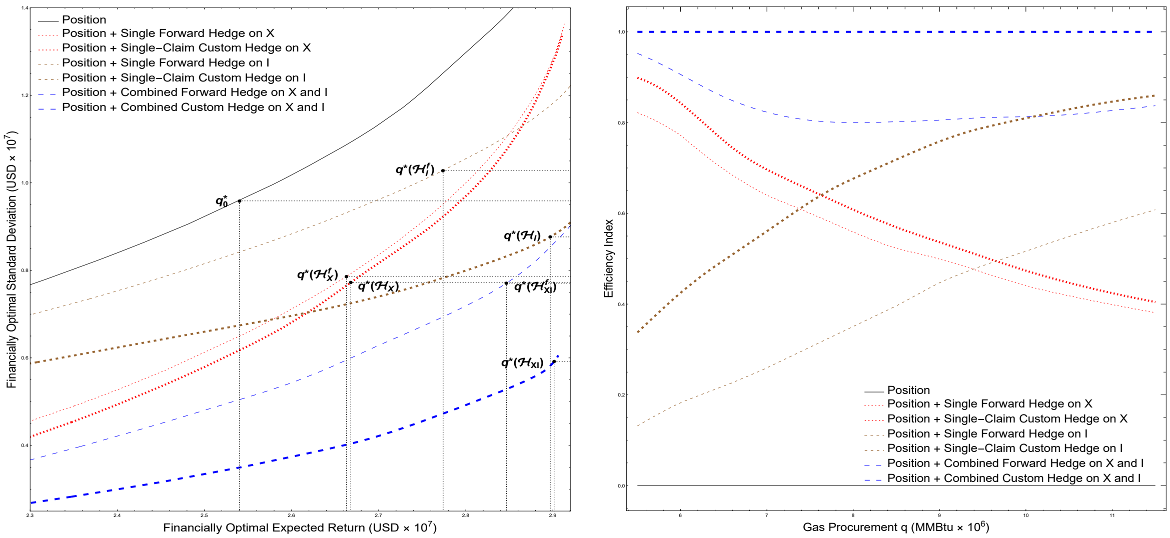
defines a new integrated risk management policy in $\mathcal{H}_{XI}\times\mathcal{Q}$ referred to as the *optimal combined custom integrated policy*. We compare this policy to the six alternatives resulting from integrating each of the aforementioned optimal hedges with the related optimal procurement order. We simplify notation and let subscripts exclusively indicate a hedging space $\mathcal{H}$.

5.2. Empirical Comparative Analysis

We assume a lognormal model for the state vector $(\log X,\log Y,\log I)$ distribution under $\mathbb{P}$ and a constant market price of risk (Cartea and Williams 2008). Appendix EC.2.5 details the model and reports a parametric setup compatible to gas prices and consumption levels recorded in the US. Hereafter, risk and return stand for standard deviation and expected value of future revenues.

5.2.1. Efficient Frontiers We first consider integrating each of the seven optimal financial hedges with a varying level of gas procurement. We assess their performance through the Wang-Yao efficient frontiers (Wang and Yao (2017) and (2019)) as depicted in Figure 2 (Left panel). Each frontier exhibits a locus of mean and standard deviation pairs $(m_{\mathcal{H}}(q),\sigma_{\mathcal{H}}(q))$ of full exposure revenues $\pi^{NV}(X,Y;q)+H^*_{\mathcal{H}}(q)$ across a range of operational levels q in $\mathcal{Q}$. Each pair thus characterizes the optimally hedged full exposure corresponding to an operational level q. It turns out that $\sigma_{\mathcal{H}}(q)$ always increases in q, while $m_{\mathcal{H}}(q)$ does so until a threshold $\bar{q}_{\mathcal{H}}$ and then decreases. Return-risk configurations $(m_{\mathcal{H}}(q),\sigma_{\mathcal{H}}(q))$ for $q>\bar{q}_{\mathcal{H}}$ are inefficient in that they entail a simultaneous increase in risk and reduction in return. Therefore, we let q range between 0 and the smallest procurement order $\bar{q}$ for which at least one of the seven efficient frontiers reverts back to inefficient return-risk configurations, namely $\bar{q}:=\min_{\mathcal{H}}\bar{q}_{\mathcal{H}}$. In the case in question, we have $\bar{q}\simeq 10.92$ MMBtu.
Descriptive statistics on both axes are reported on a \$10 MM scale. The uppermost frontier refers to a naked position, while the lowermost frontier corresponds to our optimal combined custom hedge. In between are efficient frontiers corresponding to optimal hedges in the other five classes. On each curve we indicate point $P_{\mathcal{H}}:=(m_{\mathcal{H}}(q^*_{\mathcal{H}}),\sigma_{\mathcal{H}}(q^*_{\mathcal{H}}))$ by a bold dot: it corresponds to the optimal integrated policy $(H^*_{\mathcal{H}},q^*_{\mathcal{H}})$. In Appendix EC.2(F), Table EC.4 reports optimal procurement $q^*_{\mathcal{H}}$, expected value (*i.e.*, return) and standard deviation (*i.e.*, risk) of the optimal operating revenues $\pi^*_{\mathcal{H}}+H^*_{\mathcal{H}}$ across the seven admissible integrated policy spaces, and variations of both descriptive statistics occurring upon switching between integrated policies, say from $\left(H^*_{\mathcal{G}},q^*_{\mathcal{G}}\right)$, featuring a

Figure 2 Wang-Yao return-risk efficient frontiers (left panel) and efficiency profiles (right panel) of alternative integrated financial-operational policies.



Note. Each frontier (left panel) is the locus of full exposure (*i.e.*, $\Pi_{\mathcal{H},q} := \pi^{NV}(X,Y;q) + H^*_{\mathcal{H}}(q)$) return (mean) and risk (standard deviation) pairs $(m_{\mathcal{H}}(q), \sigma_{\mathcal{H}}(q)) := (\mathbb{E}[\Pi_{\mathcal{H},q}], \mathrm{StdDev}[\Pi_{\mathcal{H},q}])$ across procurement orders q varying in the efficiency range $[0, 10.98]$ and evaluated at the corresponding optimal hedge $H^*_{\mathcal{H}}(q)$ selected in a class $\mathcal{H} = \mathcal{H}_0$ (no hedge), $\mathcal{H}^f_I$ (forward on I), $\mathcal{H}_I$ (single-claim custom on I), $\mathcal{H}^f_X$ (forward on X), $\mathcal{H}_X$ (single-claim custom on X), $\mathcal{H}^f_{XI}$ (combined forward on X and I), and $\mathcal{H}_{XI}$ (combined custom on X and I). Figures are expressed in tens of millions of dollars. A bold dot indicates the optimal procurement order $q^*(\mathcal{H}) \in \mathcal{Q}$ related to the hedging class $\mathcal{H}$ identifying the curve it lies on.

hedging class $\mathcal{G}$, to $(H^*_{\mathcal{H}}, q^*_{\mathcal{H}})$, featuring another hedging class $\mathcal{H}$. In a slight abuse of language, we refer to this modification as a change from a hedging class $\mathcal{G}$ to a hedging class $\mathcal{H}$.

We analyze these results along three dimensions: a) Nature of the underlying: X *vs.* I; b) Hedging payoff complexity: linear *vs.* custom; c) Number of underlying terms: single *vs.* combined positioning. In the background is the interplay of these terms with the naked operating revenues π^{NV}. These latter are linear in the position's claimable X, a property certainly binding for low values of q, in which case they become somewhat independent of Y. Conversely, they are nonlinear in Y for relatively low values of q, and become increasingly linear as q increases. These facts lead to a few interesting properties of the proposed hedges, which we highlight in Figure 2.

a) *Nature of underlying terms: X vs I.* Be they linear or bespoke, claims on X and claims on I show contrasting patterns. Compared to the naked position alone, while the former entail significant reductions in risk (-17.99% and -19.50% upon entering a forward on X or a custom claim on X, respectively) and mild increases in return ($+4.84\%$ and $+5.01\%$ in the aforementioned cases), the latter go the other direction ($+7.27\%$ and -8.59% in risk variation and $+9.21\%$ and $+14.05\%$ in return variation upon entering a forward on I and a custom claim on I, respectively). Hedged exposure leads to a simultaneous reduction of risk and an increase of return in all cases except one: that is upon entering the optimal forward on I, which entails a rise in both return and risk. This is because π^{NV} is linear in Y as long as $Y < q$ and then turns into a constant for $Y > q$, so that a linear hedge on index I correlated to Y is effective in the former case while it brings additional risk for low values of q. This tradeoff is resolved by the adopted MV criterion, which entails that the forward hedge on I is still preferable to a naked position. Figure 2 shows that for low procurement orders q, hedges on X push the efficient frontier downward while hedges on I display a limited effect in that sense; this feature weakens as long as q increases, until a reversion occurs: for a sufficiently high level of q, hedges on I dominate hedges on X in terms of risk reduction given a common return figure. These considerations show that, in general, mixed exposure leads to a tension in hedging effectiveness between derivatives written on the claimable term entering position's revenues and those contingent on a side index exhibiting strong correlation to the position's nonclaimable term.

b) *Hedging payoff complexity: linear vs custom.* For low values of q, hedging through claims on X leads to a remarkable improvement in the efficient frontier compared to using claims on I. This effect slows as q increases due to the increasing relevance assumed by term Y within the NV exposure. However, switching from a forward to a custom claim on X entails a steady and relatively small gain across q due to the steadily linear behavior of π^{NV} with respect to X: graphs show that the efficient frontiers related to forward and custom hedges on X are nearly matching. Conversely, hedges on I seem to be less effective than hedges on X for relatively low q, whereby π^{NV} shows a mild dependence on Y. However, switching from a forward to a custom hedge on I improves the hedge performance. This effect is exacerbated so long as the procurement order q increases, due to the ability of custom claims on I to effectively manage the increasingly relevant and nonlinear dependence of revenues π^{NV} on Y. Indeed, the efficient frontiers related to forward and custom hedges on I diverge as q increases. Also, for large levels of q, both linear and custom hedges on I perform better than any hedges on X. These facts show that optimal customization is relevant to hedging in the presence of nonlinear behavior in the underlying exposure.

c) *Number of underlyings: single-claim vs combined.* Combined hedging allows firms to resolve the aforementioned tradeoff between risk and return gain over the figure shown by naked operating revenues. In the forward hedging cases, full exposure risk (resp., return) gains resulting from combined claim positioning is comparable to risk (resp., return) gains resulting from single-claim positioning on X (resp., I), namely -19.60% *vs.* -17.99% (resp., $+12.07\%$ *vs.* 9.21%). In the custom hedging cases, optimal combination nearly doubles risk reduction when compared to the effect of a single-claim on X (-38.31% *vs.* -19.50%), and multiplies risk reduction of a single-claim on I by four (-38.31% *vs.* -8.59%). Our optimal combination approach ensures a return comparable to that of a single-claim on I ($+14.22\%$ *vs.* $+14.05\%$), which already accounts for as much as three times the return from a single-claim on X (+14.05% *vs.* +4.84%). These observations underpin our contention that optimal combination may entail a crucial improvement in hedge design.

Similar observations can be made when switching from any hedging set to another. For instance, entering the combined forward hedge yields an appreciable gain in return upon switching from either a forward (+6.89%) or a custom claim on X (+6.72%). It also entails risk savings upon switching from either a forward (−25.05%) or a custom claim on I (−12.05%). Interestingly, adopting our combined hedging strategy leads to a more efficient outcome regardless of starting position. Appendix EC.2.6 analyzes the special case of variance preferences, *i.e.*, $\varrho = 0$.

5.2.2. Hedge Efficiency across Operational Levels and Hedging Spaces We may compare optimal integrated policies across hedging classes $\mathcal{H}$ and procurement orders q by using the efficiency index defined in Section 4.3. Recall that $H^*_{\mathcal{H}}$ and $\mathscr{U}^*_{\mathcal{H}}(q)$ denote the financially optimal hedge and utility profile when hedging in $\mathcal{H}$ (formula (19)). Functions $\mathscr{U}^*_{\mathcal{H}_{XI}}(q)$ and $\mathscr{U}_0(q)$ indicate the highest and the lowest financially optimal utility profiles. Let the value $\mathcal{V}_{\mathcal{H}}(q)$ of financial hedging flexibility granted by $\mathcal{H}$ be the monetary amount m satisfying $\mathscr{U}(\pi(q)+m) = \mathscr{U}(\pi(q)+H^*_{\mathcal{H}})$. The efficiency of $\mathcal{H}$ at an operational level q can be defined as the proportion of the maximal value $\mathcal{V}_{\mathcal{H}^{cu}_{XI}}(q)$ obtained by switching from no hedging to the optimal hedging in $\mathcal{H}$, *i.e.*, $\mathcal{E}_{\mathcal{H}}(q) := \mathcal{V}_{\mathcal{H}}(q)/\mathcal{V}_{\mathcal{H}_{XI}}(q)$. Figure 2 (Right panel) shows efficiency profiles for the hedging spaces $\mathcal{H}^f_X, \mathcal{H}^f_I, \mathcal{H}_X, \mathcal{H}_I$, and $\mathcal{H}^f_{XI}$. For relatively low values of procurement order q, the largest contribution to the best hedge is carried over by claims written on price X. As q increases, this effect cedes to the increasing importance exhibited by claims written on side index I. Combining forwards is an easy way to resolve this tradeoff. In addition, custom claims perform better than linear claims, and the discrepancy between these two classes is greater for claims on I than for claims on X, a phenomenon due to the strong nonlinear dependence of NV revenues on size term Y. These observations show that, in the NV case in question, hedge nonlinearities can significantly improve hedging performance provided they enter an optimal combined positioning in the derivative market. Hence the empirical relevance of joining combination and customization themes. Appendix EC.2.6 disentangles the operational and speculative effects of hedging class switching on expected revenues, and shows the former dominates the latter upon entering any hedging class at stake.

5.2.3. The Effect of Financial Flexibility on Operational Flexibility The firm may leverage a batch of financial hedging classes in various ways. First, it can strengthen its optimal operational policy. Table EC.4 reports that optimal procurement starts at $q^*(\{0\}) = 7.087$ with no hedge and reaches $q^*(\mathcal{H}_{XI}) = 10.300$ with our optimal combined custom hedge, *i.e.*, a 45.3% increase. Second, they may use increasingly complex hedges to gain operational flexibility: by

Table 2 Operational flexibility gains generated by a change in the underlying hedging class.

Initial hedge	Initial procur.	Operational flexibility range					
	q^*	$\mathcal{Q}\left(\cdot\to\mathcal{H}_X^f\right)$	$\mathcal{Q}\left(\cdot\to\mathcal{H}_X\right)$	$\mathcal{Q}\left(\cdot\to\mathcal{H}_I^f\right)$	$\mathcal{Q}\left(\cdot\to\mathcal{H}_I\right)$	$\mathcal{Q}\left(\cdot\to\mathcal{H}_{XI}^f\right)$	$\mathcal{Q}\left(\cdot\to\mathcal{H}_{XI}\right)$
	(MMBtu)	(MMBtu)	(MMBtu)	(MMBtu)	(MMBtu)	(MMBtu)	(MMBtu)
$\mathcal{H}_0$	7.087	$[7.214, 8.651]$	$[7.214, 8.773]$	$[7.129, 7.961]$	$[7.177, 11.034]$	$[7.172, 11.034]$	$[7.241, 10.920]$
$\mathcal{H}_X^f$	7.861	-	$[7.887, 7.982]$	*	$[7.781, 8.574]$	$[7.822, 9.309]$	$[7.898, 10.920]$
$\mathcal{H}_X$	7.917	*	-	*	$[7.809, 8.400]$	$[7.854, 9.188]$	$[7.928, 10.920]$
$\mathcal{H}_I^f$	8.470	$[8.612, 8.958]$	$[8.604, 9.081]$	-	$[8.531, 11.034]$	$[8.529, 11.034]$	$[8.689, 10.920]$
$\mathcal{H}_I$	9.934	*	*	*	-	$[9.935, 10.173]$	$[10.168, 10.920]$
$\mathcal{H}_{XI}^f$	9.177	*	*	*	*	-	$[9.380, 10.920]$

switching to a larger hedging space, a firm may gain access to an array of operational procurement orders, ensuring an efficient improvement in the return-risk space.

Let us consider a starting hedging class $\mathcal{G}$. By switching to a larger class of hedges $\mathcal{H}\supset\mathcal{G}$, the firm may tune their procurement order over a range defined by a whole interval $\mathcal{Q}(\mathcal{G}\to\mathcal{H}):=[q^-,q^+]\subset\mathcal{Q}$. Each level $q\in[q^-,q^+]$ leads to a more efficient outcome compared to $P_{\mathcal{G}}:=(m_{\mathcal{G}}(q^*(\mathcal{G})),\sigma_{\mathcal{G}}(q^*(\mathcal{G})))$ in the return-risk space, that is $Q(q):=(m(q),\sigma(q))$ satisfies $m(q)\geqslant m_{\mathcal{G}}(q^*(\mathcal{G}))$ and $\sigma(q)\leqslant\sigma_{\mathcal{G}}(q^*(\mathcal{G}))$. The locus of points $Q(q)$ is clearly visible on the curves displayed in Figure 2: we consider the two-point intersection (Q_1,Q_2) between the efficient frontier corresponding to the hedging class $\mathcal{H}$ and the two semi-axes defining the lower-right orthant centered at P; they single out a portion of the efficient frontier of $\mathcal{H}$ gathering all points Q in question. The inverse images of Q_1 and Q_2 through $m(\sharp)$ are exactly q^- and q^+. Note that an operational flexibility range $\mathcal{Q}(\mathcal{G}\to\mathcal{H})$ might not exist, unless $\mathcal{G}\subseteq\mathcal{H}$, in which case it always does. Table 2 reports values for all operational flexibility sets. An asterisk indicates that no operational flexibility gain occurs. For instance, moving from $\mathcal{H}_X^f$ to $\mathcal{H}_I^f$ never leads to an optimal integrated policy $(\theta_I(i-f_I),q)$ that is more efficient than the initial policy $\left(\theta_X(x-f_X),q^*\left(\mathcal{H}_X^f\right)\right)$. In all other cases, a firm may switch between hedging classes to expand the operating range while improving its positioning in the return-risk space. How (and indeed, whether) a firm decides to increase operational flexibility is a matter of preference refinement beyond the assumed MV criterion.

6. Conclusion

6.1. Summary

We develop a new model for combined contingent claim origination, and analyze its effectiveness in a typical finance-operations context (Babich and Kouvelis 2018). From the perspective of financial economics, we solve an inverse problem of fund allocation to bespoke financial claims. More precisely, we establish a general methodology for the design of combined custom claims to optimally manage the risk engendered by any mix of insurable and noninsurable exposure faced by a firm featuring MV preferences. Under mild technical conditions, optimal custom claims exist, are unique, and their payoff schedules satisfy a design integral equation. We also provide analytical solutions and create a numerical scheme for this equation. From the standpoint of operations management, we outline a rational decision process for handling corporate risk. Specifically, we assess the monetary benefit a representative firm earns by switching from a single-claim custom hedge to our optimal combined custom hedge. A related efficiency index allows the firm to rank alternative potential hedges and make an informed choice. We also devise a new array of integrated financial-operational risk management policies which allow the firm to widen operational flexibility and improve the efficient frontier of their operating revenues. Exactly how the firm leverages these opportunities is an entirely idiosyncratic decision, outside of the scope of a claim design framework.

6.2. Empirical Predictions and Managerial Insights

Our model naturally leads to a number of testable empirical implications and managerial insights. Some proved to sharpen existing knowledge, while others have yet to be assessed.

1. *Static management of long-term risk.* Chowdhry and Howe (1999) state that firms engage either in operational hedging when facing an important demand risk or in financial hedging when demand is predictable. Hence, the former is adopted to manage long-terms exposure, while the latter is used to tackle short-term risk. We show that this need not be the case provided the hedging space allows for claim combination. Indeed, our empirical analyses demonstrate that a firm facing mixed exposure can suitably combine custom claims into an effective hedge even when demand is highly unpredictable. Static hedging may thus mitigate long-term exposure, and operational hedging can address the residual risk. This insight sharpens the Kouvelis et al. (2018) prescription whereby "*inventory decisions focus on demand risk which cannot be hedged by financial instruments*".

2. *Financial vs. operational hedging profitability.* Conventional theories state that hedging increases firm's value by alleviating market (*i.e.*, firm-exogenous) imperfections (Smith and Stulz 1985). Adam and Fernando (2006) empirically show cases where shareholder value increases through systematic gains from derivative holdings. We demonstrate that by switching between hedging classes, expected revenues vary according to a financial term leveraging market risk premia and a real component leveraging operational flexibility gains granted by the financial hedging flexibility change. Our NV exercise empirically reveals that the former may be negligible, while the latter still remains important. Hence, even in a model featuring no market imperfections, the beneficial effect of financial hedging flexibility on shareholder value may exclusively manifest through the nonfinancial (*i.e.*, firm-endogenous) channel of operational flexibility.

3. *Customization* vs. *combination.* The decision of exclusively relying on either a customization or a combination theme leads us to a new dilemma. The efficiency profiles of alternative

integrated policies in the NV case underpin the superiority of combination over customization. The ultimate choice ought to be grounded in the market context in question. We predict that the high transaction costs and price markups usually exhibited by custom claims act in favor of adopting combined forwards. To test this prediction, empirical analysts might assess the actual customization-combination mix prevailing in a batch of corporate hedging portfolios.

4. *Speculative derivative portfolios.* Poitras (2013) argues that a few financial debacles (*e.g.*, Metallgesellschaft AG) originated from a misunderstanding of the speculative side in large derivative positions. Our design system (5) exhibits two terms contributing to the overall speculative portion in a portfolio: one is idiosyncratic to the claim at stake (Chen et al. 2014); the other is a cross-claim component arising upon combination, which we first unveil. This leads to examine the relative importance of each term in financial derivative failures: we predict that the cross-claim term is relevant whenever the underlying asset correlations proved important.

5. *Decentralized risk management.* Unless maximal correlation $r_{XI} = 0$, the optimal combined hedge $H^*(X) + G^*(I)$ dominates the combination $H^\flat(X) + G^\flat(I)$ of optimal single-claim hedges. In a decentralized risk management system, should the risk engendered by a claimable X and a nonclaimable Y be handled by distinct entities, index selection might be run irrespectively of r_{XI}. This practice could increase the distance between the two combined hedges above and possibly generate an important underperformance of a hedge. When designing a decentralized framework, managers should established a centralized monitoring system to detect this kind of failure.

6. *Combination-customization complementarity.* Custom hedging on I proves increasingly effective with decreasing r_{XY} and increasing r_{IY}. Whenever customization (say, upon switching $\mathcal{H}_X^f \to \mathcal{H}_X$) is largely ineffective (say, because revenues are almost linear in X as in our NV experiment, or X and Y are poorly correlated as in Brown and Toft (2002)), optimal combination may replace ($\mathcal{H}_X \to \mathcal{H}_{XI}^f$) or add up ($\mathcal{H}_X \to \mathcal{H}_{XI}$) to it, while enhancing the firm's position. A manager can thus use combination to effectively cope with mixed revenues featuring, *e.g.*, uncorrelated risk terms X and Y. This insight settles an unresolved hedging issue in Brown and Toft (2002).

7. *Ranking alternative hedges.* A firm might wish to benchmark a hedge against an ideal reference. No hedge and our optimal combined hedge may represent the lowest and the highest points of reference, respectively. One-dimensional scales can be obtained by considering operating revenues MV, mean, or variance figures, each computed at the current level of operations. By locating and comparing hedges within a selected scale, firms may assess the relative quality of alternative hedging opportunities, rank them accordingly, and select a preferred solution.

8. *Resolving single-claim tradeoff.* Single-claim hedges may dramatically change their efficiency in response to changes of varying nature. The extreme market case-study in Subsection 4.3 shows that efficiency estimates for the optimal single-claim hedges on X and on I exhibit a tradeoff upon

varying the firm's aversion to risk. The NV case-study shows the same property upon varying the level of operations. Whatever the nature of an efficiency tradeoff is, a manager can combine claims to develop a hedge that outperforms single-claim hedges across the selected dimension.

6.3. Future Developments

Forthcoming research may target a number of theoretical enhancements and applications. A first direction is to develop concrete model instances. On the financial side, the dependence structure among state variables could be modeled using copula functions and the design integral equation could be recast in terms of the estimated copulas. This would lead to computable combined custom hedges compatible with real-world financial market contexts. On the operational side, custom claims may merge into static or dynamic operational hedging policies, such as inventory management strategies (Berling and Martínez-de-Albéniz 2011), capacity allocation options (Ding et al. 2007), or market procurement options (Secomandi and Kekre 2014), as well as combined assortment and stocking decisions (Dong et al. 2018), among others.

A second direction is to modify one or more model ingredients and examine their effect on the results we obtained. These may include: a) The class of admissible hedges: they may range from American-style to path-dependent payoff schedule; b) Naked position revenues π: they may comprise newsvendor networks (Van Mieghem 2007), decentralized supply chains(Turcic et al. 2015), and trading networks (Nadarajah and Secomandi 2018), among others; c) The target utility: risk aversion may be modeled by using risk-adjusted performance measures other than a MV criterion: they include exponential utility (Chen et al. 2007), mean-CVaR (Conditional Value-at-Risk) criterion (Zhao and Huchzermeier 2017), expected shortfall (Wang and Yao 2019), and an upper bound constraining an assigned risk measure (Park et al. 2017), among others.

A third direction involves the theoretical investigation of the integrated financial-operational risk management problem. It may prove interesting to devise a setup where the joint optimization over a functional class of custom claims and a class of operational actions could be analyzed. Operational policies might be static, dynamic, or of some intermediate kind (*e.g.*, a sequentially reoptimized rule, as in Secomandi (2015)). Analyzing the existence, uniqueness, and any characterization of a general integrated risk management problem is perhaps the most challenging issue we foresee.

A fourth direction is to use combined custom claims for purposes other than revenue hedging. Our combined custom hedge may offer a viable alternative in alleviating agency costs (Iancu et al. 2017), in increasing the cash endowment entering a budget constraint (Caldentey and Haugh 2009), or in complementing insurance contracts on side events (Dong and Tomlin 2012).

A fifth direction is to allow for the dynamic resettling of our combined custom hedge in the spirit of Chen et al. (2007), Kouvelis and Ding (2013), and Goel and Tanrisever (2017). A new setup would include a family of stochastic processes for each risk term, one process for a starting time n and state x; a dynamic balance condition whereby the fair value of a new custom hedge matches the fair value of the standing hedge; and an array of static hedge design problems, one for each time and state. Appendix EC.1.10 proposes a model setup for the optimal design of dynamic strategies based on single-claim custom hedges.

A sixth direction is to extend the role of the time T of uncertainty resolution to that of a decision variable, and examine the effect of our combined custom hedge on T. Appendix EC.1(K) gives a snapshot of conjectures put forward in Guiotto et al. (2020). An alternative path might be to explore the role of T as an entry time as in Leung and Ludkovski (2011).

Note that both issues of dynamic extension and the embedding the time T as a decision variable are still open problems even under the assumption of single-claim custom hedging. Any advance in one of the aforementioned issues would definitively constitute a step forward in the construction of a general theory of corporate risk management.

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Electronic Companion Appendices

EC.1. Complements

EC.1.1. List of Symbols

Table EC.1 A framework for the optimal design of combined custom financial contingent claims.

Symbol	Mathematical term	Financial term	Operational term
$\{0,T\}$	▶ Time set	▶ Time period	▶ Time frame
0	Initial time	Hedging setup time	Decision time
T	Final time >0	Payoff time	Realization time
$(\Omega,\mathscr{F},\mathbb{P})$	▶ Probability space	▶ Firm's beliefs	▶ Stochastic setting
X	▶ Random element	▶ Claimable (risk) term	▶ State variable 1
Y	idem	Nonclaimable (risk) term	▶ State variable 2
I	idem	Claimable (side) index	▶ State variable 3
$\mathscr{F}_{\#}$ ($\#=X$ or I)	▶ σ-algebra of events of $\#$	▶ Events gleaned by $\#$	▶ $\#$-information
$L^2(\Omega)$	▶ Finite variance r.v.	▶ Regular payoffs	▶ Regular variables
$\mathcal{H}_X$	▶ $L^2(\Omega,\mathscr{F}_X,\mathbb{P})$	▶ Payoffs $H(X)$	▶ Financial controls 1
$\mathcal{H}_I$	▶ $L^2(\Omega,\mathscr{F}_I,\mathbb{P})$	▶ Payoffs $G(I)$	▶ Financial controls 2
$\mathcal{H}_{XI}$	▶ $\mathcal{H}_X+\mathcal{H}_I$	▶ Combined p.'s $H(X)+G(I)$	▶ Financial controls 3
L^2_X	▶ $L^2(\mathbb{R}^N,\mathscr{B}_{\mathbb{R}^N},\mathscr{L}_{\mathbb{P}}(X))$	▶ Payoff functions $x\to H(x)$	▶ Financial controls 1(bis)
L^2_I	▶ $L^2(\mathbb{R}^N,\mathscr{B}_{\mathbb{R}^N},\mathscr{L}_{\mathbb{P}}(I))$	▶ Payoff functions $i\to G(i)$	▶ Financial controls 2(bis)
L^2_{XI}	▶ $L^2_X+L^2_I$	▶ Combined p. functions $H(x)+G(i)$	▶ Financial controls 3(bis)
$\mathcal{Q}$	▶ Set of scalars	▶ Operational variables q	▶ Operational controls q
$\pi=\pi(X,Y;q)$	▶ Random variable in $L^2(\Omega)$	▶ Naked exposure	▶ Fin. uncontrolled system
w	▶ Nonnegative constant	▶ Financial endowment	▶ Control constraint
$\mathbb{Q}$	▶ Martingale measure	▶ Pricing probability	▶ Control constraint
$\mathscr{U}(\cdot)$	▶ $\varrho\mathbb{E}_{\mathbb{P}}[\cdot]-(a/2)\,\mathrm{Var}_{\mathbb{P}}[\cdot]$	▶ Mean-variance utility	▶ Target reward
(no symbol)	▶ $\pi(X,Y;q)+[H(X)+G(I)]$	▶ Full exposure	▶ Fin. controlled system

EC.1.2. Examples of Mixed Exposure

EXAMPLE EC.1 (PRIMARY COMMODITY PRODUCTION MODEL). Moschini and Lapan (1995) consider a flexible producer of a primary commodity, say a farmer harvesting crops. They face risk from output price X quoted in dollars per tonne (US\$/t) and crop yield Y expressed in tonnes per hectare (t/ha). If q denotes the scale of production expressed in hectares (ha) and $c(q)$ stands for the corresponding cost, then operating net revenues may be written as:

$$\pi(X,Y;q) := XYq - c(q). \tag{EC.1}$$

This is an example of multiplicatively mixed exposure.

Example EC.2 (Stochastic clearance price model). Caldentey and Haugh (2009) develop and analyze a Stackelberg game between a producer and a retailer. They consider a background quoted price process $(X_t)_{0\leqslant t\leqslant T}$. At time 0, the producer offers the retailer a wholesale price menu X defined by a functional W evaluated at each quoted price path $X_{0T} := (X_t(\omega))_{0\leqslant t\leqslant T}$, namely: $X := W[X_{0T}]$. Then, the retailer reacts by optimizing net revenues:

$$\pi(X,Y;q) := (Y - \xi q)\,q - Xq \tag{EC.2}$$

over the class of order quantity actions q that depend on the same price path, *i.e.*, $q := Q[X_{0T}]$ for a suitable deterministic functional Q. Costs result from ordering each unit for a wholesale purchase price X. Gross revenues are the order quantity q times a retail market price, which in turn is calculated as a random market size term Y net a linear discount component ξq featuring elasticity ξ. In the original Stackelberg game model, W is a control variable determined when the producer selects a production level. Here, we focus on the retailer's viewpoint, and assume that W is assigned upfront. This is an example of additively mixed exposure.

Example EC.3 (Generalized newsvendor model). Secomandi and Kekre (2014) consider a newsvendor-like commodity merchant meeting demand Y by optimally adjusting an initially procured quantity q through suitable ex post open market trades. Let X be the quoted commodity price, $s(X)$ stand for the retailer's selling price, $bid(X)$ be the bid quote, $ask(X)$ be the ask quote, and $v(Y,q)$ denote the sales figure. In general, the merchant's net revenues can be written as sales revenues $s(X)\,v(Y,q)$, plus liquidation revenues resulting from any stock excess over demand $(q-Y)_+$, minus purchase costs of any demand excess over stock $(Y-q)_+$, net the initial procurement cost pq, *i.e.*:

$$\pi(X,Y;q) := s(X)\,v(Y,q) + bid(X)\,(q-Y)_+ - ask(X)\,(Y-q)_+ - pq.$$

Secomandi and Kekre (2014) take $s(x) = x$, $v(y,q) = y$, and p equal to the forward ask price prevailing at time 0. By setting retail price $s(x)$ equal to a constant amount, the bid price equal to a constant salvage value, the ask price equal to a constant per unit penalty cost of unsatisfied demand, and $v(Y,q) = \min\{Y,q\}$, we obtain the newsvendor position revenues considered in Caldentey and Haugh (2006).

Example EC.4 (Multinational production capacity allocation model). Chowdhry and Howe (1999) consider a multinational corporation allocating production capacity and selling their single product in two economies. Given domestic and foreign production capacities q_1 and q_2, demand figures Y_1 and Y_2, sales prices X_1 and X_2, and an exchange rate X_3, the authors show how to determine the domestic production level $o_1(\mathbf{q}, \mathbf{Y})$ depending on the demand vector $\mathbf{Y} := (Y_1, Y_2)$ and the capacity allocation vector $\mathbf{q} := (q_1, q_2)$. Foreign production level $o_2(\mathbf{q}, \mathbf{Y})$ results from a clearing condition whereby overall production $o_1 + o_2$ is required to meet global demand $Y_1 + Y_2$. Corporate net revenues may be expressed as the sum of domestic and currency converted foreign profit and losses:

$$\pi(\mathbf{X}, \mathbf{Y}; \mathbf{q}) := X_1 Y_1 - c_1 o_1(\mathbf{q}, \mathbf{Y}) + X_3 \left[X_2 Y_2 - c_2 \left(Y_1 + Y_2 - o_1(\mathbf{q}, \mathbf{Y})\right)\right],$$

where c_1 and c_2 denote domestic and foreign unit production costs, respectively.

EC.1.3. On the Adoption of Mean-Variance Preferences

There are at least three arguments in favor of adopting MV preferences in our framework. First, the MV criterion, possibly in the restricted form of a variance to minimize, is a benchmark for representing risk-aversion within risk management models in a variety of contexts. These include international trading (Ding et al. (2007), Chen et al. (2014)), operations management (Kouvelis and Gutierrez (1997), Martínez-de-Albéniz and Simchi-Levi (2006), Chen et al. (2007), Van Mieghem (2007), Kouvelis and Ding (2013)), and agricultural business optimization (Rolfo (1980), Lence (1995)), among others. Second, although the appropriateness of MV preferences has been debated for decades, there is general consensus on the dominance of this criterion over the existing alternatives in practical contexts (Markowitz 2014). Third, optimizing a MV target ensures analytical tractability of the problem. Incidentally, by replacing the variance term with a quadratic cost function, MV leads to the value maximizing criterion adopted in Stulz (1984), Mello et al. (1995), Froot and Stein (1998), Brown and Toft (2002), Chod et al. (2010), Goel and Tanrisever (2017), and Park et al. (2017), among others.

EC.1.4. Discussion on Extent, Limitations, and Extensions of our Model

We herein clarify the extent, limitations, and possible extensions of our model.

1. *Time frame and static positioning.* It is natural to cast the optimal design of a combined custom hedge as a static decision problem. This setup allows us to develop a self-contained design theory adhering to the empirical observation that most nonfinancial firms are simply insurance buyers who are either unwilling or unable to trade in financial markets (Brown and Toft 2002), hence seeking buy-and-hold hedges. This is also in line with the growing popularity of static positioning

in the hedging practice of financial intermediaries (Carr et al. 1998). A one-period setup does not require risk terms X and I to be tradable: from a buyer's perspective, they need only be claimable. Temperature records and most economic indices are examples of nontradable claimable terms.

2. *Relation to the dynamic trading approach.* In the dynamic setup of Example 1, equipped with a tradable side asset price process $(I_t)_{0\leqslant t\leqslant T}$ and risk averse preferences, Caldentey and Haugh (2006) devise a hedging payoff $V(T)$ resulting from a self-financing trading strategy (SFTS) $\theta=(\theta_t)_{0\leqslant t\leqslant T}$ that acts on the side asset price process in a complete financial market. If admissible strategies θ only depend on side asset price paths (*i.e.*, incomplete information, as opposed to complete information whereby they may also depend on the nonclaimable process $(Y_t)_{0\leqslant t\leqslant T}$), then $V(T)$ is a path-dependent payoff $G(I_{0T})$, where I_{0T} denotes a sample path of process $(I_t)_{0\leqslant t\leqslant T}$ over a time period $[0,T]$. Conversely, any path-dependent $G(I_{0T})$ can be replicated by a SFTS acting on $(I_t)_{0\leqslant t\leqslant T}$ only, *i.e.*, $\theta(t)=F[t,I_{0t}]$. Setting aside minor differences (*e.g.*, our MV target functional *vs.* their quadratic expected utility), the incomplete information solution developed by these authors corresponds to the restricted case of our combined custom hedge stemming from $H(X)\equiv 0$. Our theoretical development thus takes a step forward within the incomplete information hypothesis by jointly designing and combining a custom claim on X with a custom claim on I. In this respect, a direct design approach seems relevant to us: were we to adopt a dynamic framework, it would be hard to characterize the SFTS as achieving additively combining payoffs of form $V(T)=H(X_{0T})+G(I_{0T})$ and then define conditions ensuring their optimal coupling. In the case of path-independent claims $H(X_T)$ and $G(I_T)$, our approach also allows us to explicitly describe the corresponding payoff functions $H(x)$ and $G(i)$ through a design integral equation.

3. *Complementarity of dynamic trading and claim design approaches.* We contend that the two hedging approaches based on dynamic trading (Caldentey and Haugh (2006), Wang and Yao (2017) and (2019)) and on direct claim design (Chowdhry and Howe (1999), Chen et al. (2014), and the present work) may play a complementary role. A trading approach delivers replicating strategies, which are of a key importance for the hedge issuer. In addition, it handles trading under complete information (*i.e.*, θ depending on nontradable Y as well) whenever it makes sense, say a firm exposed to a term Y that only the firm can observe. However, claim design delivers explicit, possibly path-dependent, payoff functions or functionals incorporating desired hedging themes (*e.g.*, linearity, customization, and combination), in line with banking practice (Myint and Famery (2012), Ramirez (2011)). For instance, a firm looks for exposure to specific path dependence $F:X_{0T}\to X$, say an Asian-style average (Fusai et al. 2008) or a barrier condition (Leppard 2005). Then, the optimal claim $\hat{F}(X_{0T}):=H^*(F(X_{0T}))=H^*(X)$ is the required optimal path-dependent claim.

4. *Further linkages to dynamic financial-operational risk management.* Caldentey and Haugh (2009) use portfolio value $V_\theta(T)$ to alleviate a budget constraint in a two-stage game representing

a supply chain model with flexible contracts. Our setup covers their operating position (Example EC.2) and borrows a number of features, including the nature of the pricing kernel and the equivalence to a zero-balance constrained problem. Wang and Yao (2017) explore integrated dynamic hedging in a NV model under minimum variance preferences: by assuming demand dynamics are linked to a tradable side index, they derive MV efficient frontiers of full exposure across varying operational levels. Our model includes minimum variance as a special case (Appendix EC.2.6). We similarly leverage efficient frontiers to assess the performance of a new integrated policy we propose (Subsection 5.2.1).

5. *Market completeness.* In order to feed a budget constraint (2), the firm must be able to price any candidate contingent claim $H(X)$ and $G(I)$. This requirement is compatible with a variety of market contexts. In all cases, Y is nonclaimable, hence no contract involving Y ever needs to be priced; in particular, the distribution of Y under $\mathbb{Q}$ is not required at any step in the design process. First, the joint market for (X, I) can be complete: this is the partially complete market referred to in Smith and Nau (1995). Second, if this market is incomplete, then a pricing kernel can be identified through any criterion for resolving incompleteness (Staum 2008). Third, the firm might simply have access to the originator's pricing kernel as suggested in Caldentey and Haugh (2006). Fourth, markets for a claimable X and a claimable I can be segmented, *i.e.*, pair (X, I) is nonclaimable, whereas each market is complete on its own: the firm would then price claims using pricing kernels $d\mathbb{Q}_X/d\mathbb{P}$ or $d\mathbb{Q}_I/d\mathbb{P}$ depending on the underlying variable in question. Finally, the firm may simply specify a range of pricing kernels to in turn yield an array of optimal combined custom hedges.

EC.1.5. Combined Custom Hedging in a Bernoulli Market Model

We build a simple model drawing upon our intuition about the nature of combined custom hedging and its relevance to integrated financial-operational risk management. Within the setup proposed in Section 2, let us consider a variance minimizing firm ($\varrho = 0$) devoting no budget ($w = 0$) to hedging in a risk neutral market ($\mathbb{P} = \mathbb{Q}$). The best financial claim available so far is either the optimal custom claim written on the financially insurable price X (Chowdhry and Howe 1999):

$$H^{\flat}(X) \leftarrow \min_{H(X) \in \mathcal{H}_X \colon \mathbb{E}[H(X)]=0} \mathrm{Var}\left[\pi + H(X)\right] =: \mathscr{V}_X^*, \tag{EC.3}$$

or the optimal custom claim written on any claimable side index I exhibiting correlation to the noninsurable term Y (Gaur and Seshadri 2005):

$$G^{\flat}(I) \leftarrow \min_{G(I) \in \mathcal{H}_I \colon \mathbb{E}[G(I)]=0} \mathrm{Var}\left[\pi + G(I)\right] =: \mathscr{V}_I^*. \tag{EC.4}$$

They are referred to as optimal single-claim hedges. It is tempting to consider the naively combined custom hedge $H^\flat(X)+G^\flat(I)$, suppose it outperforms both optimal single-claim custom hedges $H^\flat(X)$ and $G^\flat(I)$, and assume it represents the optimal combined custom hedge defined as:

$$H^*(X)+G^*(I) \leftarrow \min_{(H(X),G(I))\in\mathcal{H}_X\times\mathcal{H}_I:\ \mathbb{E}[H(X)+G(I)]=0} \operatorname{Var}[\pi+H(X)+G(I)] =: \mathscr{V}^*_{XI}. \qquad \text{(EC.5)}$$

This argument would be fallacious, and it is precisely this observation which represents the springboard for our theoretical development. Specifically, it turns out that:

Claim 1. The optimal single-claim custom hedges $H^\flat(X)$ and $G^\flat(I)$, and the optimal combined custom hedge $H^*(X)+G^*(I)$ exist.

Claim 2. The optimal combined custom hedge $H^*(X)+G^*(I)$ outperforms the naively combined custom hedge $H^\flat(X)+G^\flat(I)$ unless X is independent of I; optimal single-claim custom hedges $H^\flat(X)$ and $G^\flat(I)$ can even outperform the naively combined custom hedge.

Claim 3. The optimal combined custom hedge $H^*(X)+G^*(I)$ may considerably improve the effectiveness of integrated financial-operational risk management policies encompassing any of the alternative hedges $H^\flat(X)$, $G^\flat(I)$, and $H^\flat(X)+G^\flat(I)$.

We prove the three claims in the simplest possible nontrivial setup. Let us begin with an intuitive geometric argument supporting our main idea. Under zero market premia, *i.e.*, $\mathbb{Q}=\mathbb{P}$, Chowdhry and Howe (1999) show that $H^\flat(X)=-[\mathbb{E}[\pi|X]-\mathbb{E}[\pi]]$. In general, we have the following:

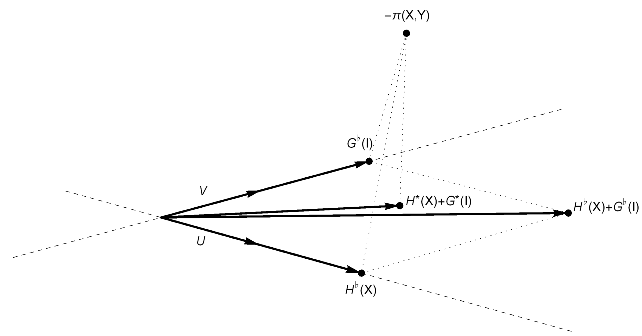
Proposition EC.1 **(Minimum variance custom hedge)**. *Consider a variance minimizing agent. Let $\mathscr{P}$ be a closed subspace of $L^2(\Omega,\mathscr{F},\mathbb{P})$ such that $1\in\mathscr{P}$ and $\mathbb{Q}\sim\mathbb{P}$. Then, the zero-valued optimal claim in $\mathscr{P}$ is given by:*

$$P^* := \arg\min_{P\in\mathscr{P}:\ \mathbb{E}_{\mathbb{Q}}[P]=0} \operatorname{Var}[\pi+P] = -\left[\operatorname{Proj}_{\mathscr{P}}\pi + c^*\right], \qquad \text{(EC.6)}$$

where constant $c^* = -\left\langle \pi, \operatorname{Proj}_{\mathscr{P}}\frac{d\mathbb{Q}}{d\mathbb{P}}\right\rangle$. (Here, $\operatorname{Proj}_{\mathscr{P}}$ indicates the orthogonal projection on $\mathscr{P}$.)

In our context, $\mathbb{Q}=\mathbb{P}$, hence $c^*=-\mathbb{E}[\pi]$ and $\mathbb{E}[H(X)]=\mathbb{E}_{\mathbb{Q}}[H(X)]=0$. The same holds for $G(I)$. Then, the optimal single-claim hedge $H^\flat$ (resp., $G^\flat$) is given by the orthogonal projection of opposite naked revenues $-\pi$ over the subspace $\mathbb{U}$ (resp., $\mathbb{V}$) of finite-variance $\mathbb{P}$-mean centered r.v.'s which can be expressed as a function of X (resp., I). Both naively combined custom hedge $H^\flat(X)+G^\flat(I)$ and optimal combined custom hedge $H^*(X)+G^*(I)$ clearly belong to the sum space $\mathbb{U}\oplus\mathbb{V}$. However, the former is the sum of projections on individual spaces $\mathbb{U}$ and $\mathbb{V}$, while the latter is the orthogonal projection over $\mathbb{U}\oplus\mathbb{V}$: clearly, the two always differ unless $\mathbb{U}$ and $\mathbb{V}$ are orthogonal (see Figure EC.1, its geometrical observation underpins the intuition behind Claim 1, provided that space orthogonality is intended as statistical independence.).

Figure EC.1 Geometry of optimal *vs.* naively combined custom hedges $H^{*}(X)+G^{*}(I)$ and $H^{\flat}(X)+G^{\flat}(I)$.



Our argument may yield closed-form expressions for these hedges if we specify a probabilistic model for all risk variables in question. The simplest nontrivial example is a *Bernoulli market model*. Specifically, let X assume values x^{+} and x^{-} with probability p_X and $1-p_X$, respectively, and I assume values i^{+} and i^{-} with probability p_I and $1-p_I$, respectively. We define $p_{XI} := \mathbb{P}(X=x^{+}, I=i^{+})$ and make no assumptions regarding the distribution of the nonclaimable risk term Y.

Theme 1: Hedge design. Optimal single-claim and combined custom hedges can be computed in closed-form.

Proposition EC.2 **(Minimum variance hedges in a risk neutral Bernoulli market model)**. *Consider a Bernoulli market model with $\mathbb{P}=\mathbb{Q}$. The optimal single-claim hedges solving problems (EC.3) and (EC.4) may be expressed as:*

$$H^{\flat}(X)=\alpha U,\ G^{\flat}(I)=\beta V, \tag{EC.7}$$

where $U := (\mathbf{1}_{X=x^{+}}-p_X)/\sqrt{p_X(1-p_X)}$ and $V := (\mathbf{1}_{I=i^{+}}-p_I)/\sqrt{p_I(1-p_I)}$ are zero mean r.v.'s with unit variance, i.e., versors in $L^2(\Omega)$, and coefficients $\alpha=-\langle\pi,U\rangle$ and $\beta=-\langle\pi,V\rangle$ are constant. In addition, the optimal combined custom hedge solving problem (EC.5) exists and is unique provided X and I are not functionally dependent on one another; it is given by:

$$H^{*}(X)+G^{*}(I)=\frac{\alpha-\rho\beta}{\sqrt{1-\rho^2}}U+\frac{\beta-\rho\alpha}{\sqrt{1-\rho^2}}V, \tag{EC.8}$$

where $|\rho| := |\langle U,V\rangle| = \left|(p_{XI}-p_Xp_I)/\sqrt{p_Xp_I(1-p_X)(1-p_I)}\right| < 1$.

If either X or I is a function of the other, any combined hedge is a claim on a single underlying. Otherwise, naively and optimal combined hedges $H^\flat(X)+G^\flat(I)$ and $H^*(X)+G^*(I)$ differ unless either $\rho=0$ or $\alpha=\beta=0$: in the first case, X and I are statistically independent variables; in the second case, any claim written on either variable is useless for hedging purposes, hence the corresponding optimal position vanishes. This proves Claim 1 stated earlier in this section.

Theme 2: Relative performance assessment. We compare our optimal combined custom hedge $H^*(X)+G^*(I)$ to the alternative hedges $H^\flat(X)$, $G^\flat(I)$, and $H^\flat(X)+G^\flat(I)$ in terms of risk mitigation performance. All the hedges in Proposition EC.2 may be written as $\xi U+\eta V$ for suitable constants ξ and η. In addition, full exposure variance is $\mathrm{Var}[\pi+(\xi U+\eta V)]=\mathrm{Var}[\pi]+\xi^2+\eta^2-2\xi\alpha-2\eta\beta+2\xi\eta\rho$. Variance figures attained by using each of the four proposed optimal hedges are:

$$\begin{aligned}\mathscr{V}^*_{\mathcal{H}_X}&=\mathrm{Var}[\pi]-\alpha^2, \quad \mathscr{V}_{XI}=\mathrm{Var}[\pi]-\left(\alpha^2+\beta^2-2\rho\alpha\beta\right),\\ \mathscr{V}^*_{\mathcal{H}_I}&=\mathrm{Var}[\pi]-\beta^2, \quad \mathscr{V}^*_{\mathcal{H}_{XI}}=\mathrm{Var}[\pi]-\left(\alpha^2+\beta^2-2\rho\alpha\beta\right)/\left(1-\rho^2\right),\end{aligned} \tag{EC.9}$$

where $\mathscr{V}_{XI}:=\mathrm{Var}\left[\pi+H^\flat(X)+G^\flat(I)\right]$. The naively combined custom hedge $H^\flat(X)+G^\flat(I)$ increases the firm's exposure compared to adopting the sole optimal single-claim custom hedge $H^\flat(X)$, i.e., $\mathscr{V}_{XI}>\mathscr{V}^*_{\mathcal{H}_X}$, provided that $\beta^2<2\alpha\beta\rho$. (If $\beta\gtrless 0$, this inequality amounts to $\beta\lessgtr 2\alpha\rho$.) Similarly, $H^\flat(X)+G^\flat(I)$ may perform worse than $G^\flat(I)$ alone. However, barring degenerate cases (e.g., $\beta=\rho\alpha$, whereby $\mathscr{V}^*_{\mathcal{H}_{XI}}=\mathscr{V}^*_{\mathcal{H}_X}$), the optimal combined custom hedge $H^*(X)+G^*(I)$ always outperforms both single-claim custom hedges $H^\flat(X)$ and $G^\flat(I)$ as well as their naive combination $H^\flat(X)+G^\flat(I)$ provided that $\rho^2<1$. These observations prove Claim 2, and show how merging the two strands of research depicted in Figure 1 is indeed a worthy pursuit.

Theme 3: Operational flexibility gain. Integrating our optimal combined custom hedge with the optimal handling of business operations may yield considerable gains in operational flexibility (Zhao and Huchzermeier 2015). Let business revenues $\pi(q)=\pi(X,Y;q)$ depend on a scalar operational control $q\in Q$, say a procurement order, a production level, or a sale price. Each financially optimal variance reported in expressions (EC.9) entails a function $q\to\mathscr{V}^*_{\#}(q)$, which we refer to as an optional variance profile.

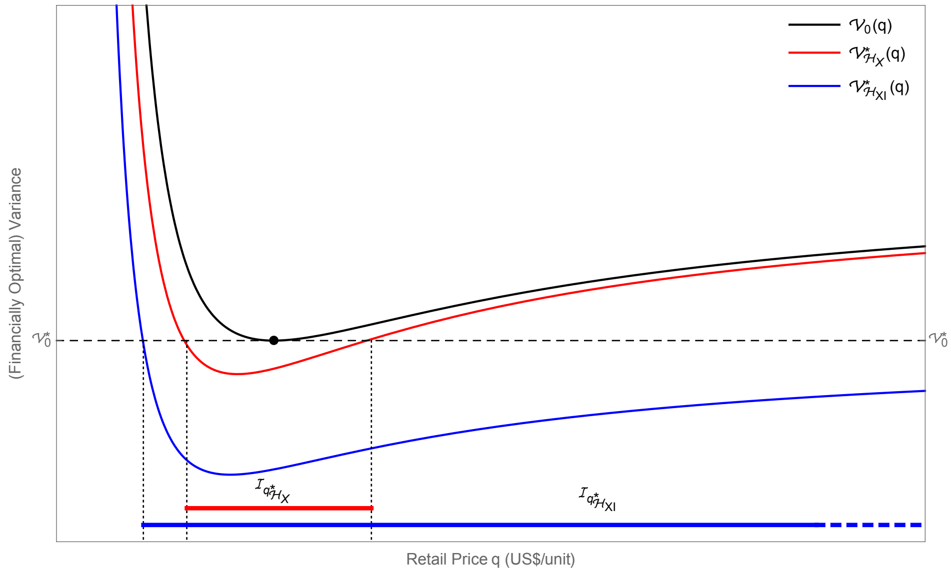
We consider three alternative optimal financial-operational hedging policies:

1. Optimal operational policy $(0,q_0^*)$, with operational level q_0^* minimizing the variance profile $\mathscr{V}_0(q):=\mathrm{Var}[\pi(q)]$ of naked position revenues;

2. Optimal single-claim integrated policy $\left(H^\flat(X),q^*_{\mathcal{H}_X}\right)$ minimizing $\mathrm{Var}[\pi(q)+H]$ over single-claim custom hedges $H(X)\in\mathcal{H}_X$ and operational levels $q\in Q$;

3. Optimal combined integrated policy $\left(H^*(X)+G^*(I),q^*_{\mathcal{H}_{XI}}\right)$ minimizing $\mathrm{Var}[\pi(q)+H+G]$ over combined custom hedges $H(X)+G(I)\in\mathcal{H}_{XI}$ and operational levels $q\in Q$.

Figure EC.2 Operational flexibility ranges and gains.



Note. Operational flexibility ranges entailed by: (i) single-claim custom hedging ($\mathcal{I}_{q^*_{\mathcal{H}_X}}$: red segment); (ii) combined custom hedging ($\mathcal{I}_{q^*_{\mathcal{H}_{XI}}}$: blue half-line). The corresponding operational flexibility gain $\mathcal{I}_{q^*_{\mathcal{H}_{XI}}} \setminus \mathcal{I}_{q^*_{\mathcal{H}_X}}$ is unbounded.

Under mild regularity conditions, the variance profiles of optimally hedged positions lie below the naked position's minimum variance $\mathscr{V}_0^* := \mathscr{V}_0(q_0^*)$ in suitable operational control ranges of values. That is, there are intervals $\mathcal{I}_{q^*_{\mathcal{H}_X}}$ and $\mathcal{I}_{q^*_{\mathcal{H}_{XI}}}$ such that $\mathscr{V}^*_{\mathcal{H}_X}(q) \leqslant \mathscr{V}_0^*$ for all $q \in \mathcal{I}_{q^*_{\mathcal{H}_X}}$ and $\mathscr{V}^*_{\mathcal{H}_{XI}}(q) \leqslant \mathscr{V}_0^*$ for all $q \in \mathcal{I}_{q^*_{\mathcal{H}_{XI}}}$. In addition, these ranges nest inside one another along with increasing hedge complexity, i.e., $\mathcal{I}_{q^*_{\mathcal{H}_X}} \subseteq \mathcal{I}_{q^*_{\mathcal{H}_{XI}}}$. We identify $\mathcal{I}_{q^*_{\mathcal{H}_X}}$ (resp., $\mathcal{I}_{q^*_{\mathcal{H}_{XI}}}$) with the operational flexibility range yielded by optimal single-claim (resp., combined) custom hedging. Thus, the range increment $\mathcal{I}_{q^*_{\mathcal{H}_{XI}}} \setminus \mathcal{I}_{q^*_{\mathcal{H}_X}}$ represents the operational flexibility gain afforded by our optimal combined hedge. We now show that this gain can be as large as an unbounded interval of values.

Example EC.5 (Electricity retailer). Let an electricity retailer act as a price taker in a competitive wholesale market (Joskow and Kahn (2001)). They buy at a competitive market price X, sell on for a selected retail price q (control variable) a load demand Y/q requested by end consumers, and earn operating revenues $\pi(q) := (q - X)Y/q$. Note that demand size is inversely proportional to retail price q. We make the natural assumption that procurement costs and demand level are positively correlated, i.e., $\rho(XY, Y) > 0$. For the sake of simplicity, we consider the selected side index I to be statistically independent of the purchase price X. (An extension to dependent variables X and I would merely increase the computational burden while leaving our argument unaffected.) Below, we derive analytical formulae for optimal variance profiles $\mathscr{V}_0^*(q)$, $\mathscr{V}^*_{\mathcal{H}_X}(q)$, $\mathscr{V}^*_{\mathcal{H}_{XI}}(q)$, their points of minimum q_0^*, $q^*_{\mathcal{H}_X}$, $q^*_{\mathcal{H}_{XI}}$, and the two operational flexibility ranges $\mathcal{I}_{q^*_{\mathcal{H}_X}}$ and $\mathcal{I}_{q^*_{\mathcal{H}_{XI}}}$. We show that each optimal variance profile assumes the form $\mathscr{V}^*_{\#}(q) = a_{\#}\left(q^{-1} - q_{\#}^{*-1}\right)^2 + b_{\#}$, where both $a_{\#} > 0$ and $b_{\#} > 0$ are decreasing with increasingly complex hedges. We also prove that for any parametric configuration entailing the two inequalities $q^{*-1}_{\mathcal{H}_X} - \sqrt{a_X^{-1}(\mathscr{V}_0^* - b_X)} > 0$ and $q^{*-1}_{\mathcal{H}_{XI}} - \sqrt{a_{XI}^{-1}(\mathscr{V}_0^* - b_{XI})} < 0$, the operational range $\mathcal{I}_{q^*_{\mathcal{H}_X}}$ is bounded while the operational range $\mathcal{I}_{q^*_{\mathcal{H}_{XI}}}$ is unbounded. In these cases, the operational flexibility gain $\mathcal{I}_{q^*_{\mathcal{H}_{XI}}} \setminus \mathcal{I}_{q^*_{\mathcal{H}_X}}$ from our combination theme is unbounded as well. Figure EC.2 illustrates this interesting phenomenon. These considerations underpin our experimental investigation in Section 5 of the extent of Claim 3 within the more general setup of Section 2.

EC.1.6. Further Examples

The following three examples compute maximal correlations, analyze their mutual dependence, and offer a case of hedge ranking in the context of a Bernoulli market model.

Example EC.6 (Maximal correlation in a Bernoulli market model). Since $\phi(X) = \alpha U$ and $\psi(I) = \beta V$ (Proposition EC.2), then: $\rho(\phi(X), \psi(I)) = |\langle U, V\rangle| = |\rho_{XI}|$, and maximal correlation is $r_{X,I} = |\rho_{XI}|$. Clearly, $\rho_{XI} = \operatorname{corr}\left(\mathbf{1}_{X=x^+} - p^+, \mathbf{1}_{I=i^+} - q^+\right)$. By the Cauchy-Schwarz inequality, $|\rho_{XI}| = 1$ whenever $V \propto U$, *i.e.*, $V = \pm U$: X and I are a function of one another.

Example EC.7 (Maximal correlation linkages in a Bernoulli market model). In the previous example, $r_{IX} = |\langle V, U\rangle| = \cos(\vartheta_{VU})$, where ϑ_{VU} denotes the angle between V and U. If Y is a Bernoulli r.v., then $r_{IY} = \cos(\vartheta_{VW})$ for some vector W. If W lies within the angle between V and U, then it satisfies $\vartheta_{VU} = \vartheta_{VW} + \vartheta_{WU}$. Since Y and X are fixed, then the angle ϑ_{WU} created between the two of them is fixed as well. By varying I, we see that $r_{IX} = \cos(\vartheta_{VU})$ decreases (or increases) if and only if $r_{IY} = \cos(\vartheta_{VW})$ decreases (or increases) as well.

Example EC.8 (Hedge ranking in a Bernoulli market model). With reference to the efficiency analysis reported in Subsection 4.3, the variance attained by using no hedge, $H^\flat(X)$, and $H^*(X) + G^*(I)$ are given by formulae (EC.9). The efficiency of single-claim hedge $H^\flat(X)$ is thus: $\mathcal{E}_X = (1-\rho^2)\alpha^2/(\alpha^2 + \beta^2 - 2\rho\alpha\beta)$, where $\alpha = -\langle \pi, U\rangle$, $\beta = -\langle \pi, V\rangle$, and $|\rho| = |\langle U, V\rangle|$. Clearly, $H^\flat(X)$ and $H^*(X) + G^*(I)$ deliver the same performance and $\mathcal{E}_X = 1$ provided that $\rho\alpha = \beta$. By using the explicit expression for $H^*(X) + G^*(I)$ in formula (EC.8), this condition amounts to having $G^*(I) = 0$.

We now show that $r_{XI} = 1$ is compatible with non-uniqueness of the optimal combined claims $H^*(X)$ and $G^*(I)$.

Example EC.9 (Non-uniqueness). Let $\mathscr{F}_I \subseteq \mathscr{F}_X$. Then, $I = f(X)$ for a measurable function f (*i.e.*, index I is redundant), system (5) entails infinitely many solution pairs. The design problem (4) reduces to $\mathscr{U}^*_{\mathcal{H}_{XI}} = \mathscr{U}^*_{\mathcal{H}_X} := \max_{\tilde{H} \in \mathcal{H}_X:\, \mathbb{E}_{\mathbb{Q}}[\tilde{H}(X)]=0} \mathscr{U}\left(\pi(X,Y) + \tilde{H}(X)\right)$, whose (unique) solution $\widetilde{H}^*(X)$ equals $H^*(X) + G^*(f(X)) = H^*(X) + G^*(I)$ for infinitely many pairs $(H^*(X), G^*(I))$ solving the original problem (4).

We conclude this appendix by exhibiting a state variable setup based on Itô diffusion processes and by computing the related maximal correlations in terms of their coefficients.

Example EC.10 (Dynamic market model). Assume Itô diffusion dynamics:

$$\begin{cases} dX_t/X_t = \alpha\ dt + \sigma_1^X dW_1 + \sigma_2^X dW_2, \\ dI_t/I_t = \beta\ dt + \sigma_1^I dW_1 + \sigma_2^I dW_2, \\ dY_t/Y_t = \gamma\ dt + \sigma_1^Y dW_1 + \sigma_2^Y\ dW_2 + \sigma_3^Y dW_3, \end{cases}$$

are assigned for given independent Brownian motions W_1, W_2, and W_3. A variety of alternative state vectors (X,Y,I) enter our setup, including 1) European-style vectors with components $X := X_T$, $Y := Y_T$, and $I := I_T$, and 2) Asian-style vectors with terms $X := (X_{t_1},\ldots,X_{t_N})$, $Y := (Y_{t_1},\ldots,Y_{t_N})$, and $I := (I_{t_1},\ldots,I_{t_N})$ (see Example 1). Appendix EC.3.9 shows that $r_{XI} < 1$ for both models provided that coefficients $\sigma^X := (\sigma_1^X, \sigma_2^X, 0)$ and $\sigma^I := (\sigma_1^I, \sigma_2^I, 0)$ are linearly independent, and $\sigma_3^Y \neq 0$.

EC.1.7. Explicit and Numerical Solution of the Design Equation

The design integral equation (8) can explicitly be solved provided that the integral kernel $k(x,\xi)$ is separable. This is the case of statistically independent variables X and I, that is $f_{XI}(x,i) = f_X(x)f_I(i)$. Then, the integral kernel is $k(x,\xi) = \int_{\mathbb{R}^N} f_{XI}(\xi,i)di = f_X(\xi)$ and the optimal hedges are easily derived (see Example 3). We may also analytically solve the design integral equation related to a class of kernels derived from a perturbation of the independence case. (A proof is reported in Appendix EC.3.11.)

Proposition EC.3 **(Separable Perturbation of the Independence Case)**. *Let variables X and I feature a joint distributions density:*

$$f_{XI}(x,i) = (1+\alpha(x)\beta(i))\, f_X(x) f_I(i), \tag{EC.10}$$

where $\alpha(x)\beta(i) \geqslant -1$ for all $(x,i) \in \mathbb{R}^N \times \mathbb{R}^N$ and $\int_{\mathbb{R}^N} \alpha(x) f_X(x)\, dx = \int_{\mathbb{R}^N} \beta(i) f_I(i)\, di = 0$. If $\|\alpha\|_{L^2_X} \|\beta\|_{L^2_{I_I}} < 1$, then the integral kernel is separable, it can be expressed as:

$$k(x,\xi) = f_X(\xi)\left(1 + \|\beta\|^2_{L^2_I} \alpha(x)\alpha(\xi)\right),$$

and the unique solution of design equation (8) is:

$$H_0^*(x) = \Pi_X(x) + \gamma \|\beta\|^2_{L^2_I} \alpha(x),\ x \in \mathbb{R}^N, \tag{EC.11}$$

where $\gamma = c^{-1}\int_{\mathbb{R}^N} \alpha(\xi)\Pi_0(\xi) f_X(\xi)\ d\xi$ and $c = 1 - \|\alpha\|^2_{L^2_X} \|\beta\|^2_{L^2_I}$.

Example EC.11. Let $N=1$. Assume that:

$$\alpha(x) := \theta(1-2F_X(x)),$$
$$\beta(i) := 1-2F_I(i),$$

where $\theta>0$ and $F_\#$ denotes the distribution function of $\#$ under $\mathbb{P}$. Both α and β satisfy the aforementioned regularity conditions and $\|\alpha\|_{L^2_{f_X}}\|\beta\|_{L^2_{f_I}}<1$. Since $\partial_x\alpha(x)=-2\theta f_X(x)$, then:

$$\mathbb{E}[\alpha(X)] = -\frac{1}{4\theta}\int_{\mathbb{R}}\partial_x(\alpha(x))^2\ dx = -\frac{\theta}{4}\left[(1-2F_X(x))^2\right]_{x=-\infty}^{x=+\infty}=0,$$

and:

$$\mathbb{E}[\alpha(X)^2] = -\frac{1}{6\theta}\int_{\mathbb{R}}\partial_x\left(\alpha(x)^3\right)\ dx = -\frac{\theta^2}{6}\left[(1-2F_X(x))^3\right]_{x=-\infty}^{x=+\infty} = \frac{\theta^2}{3}<+\infty.$$

Similarly, $\mathbb{E}[\beta(I)]=0$ and $\mathbb{E}[\beta(I)^2]<+\infty$.

The design integral equation is difficult to solve analytically. One may thus resort to a suitable numerical scheme and derive an approximated solution. It turns out that most numerical methods for integral equations rely on the compactness of the integral operator:

$$\mathscr{I}[w](x) := \int_{\mathbb{R}^N} k(x,\xi)w(\xi)\ d\xi, \qquad x\in\mathbb{R}^N, \tag{EC.12}$$

as defined in L^2_X (Hackbush 1995). In our context, $\mathscr{I}[w](X)=\mathbb{E}\left[\mathbb{E}\left[w(X)|\mathscr{F}_I\right]|\mathscr{F}_X\right]$ is a projection, hence it is compact only provided it exhibits finite dimensional rank. Unfortunately, this is not the case if (X,I) is absolutely continuous with respect to the Lebesgue measure. However, we may adapt a Galerkin finite elements method, which normally requires a compactness assumption, as a convenient means of developing our setup. Let $L^2_{X,0}:=\{H\in L^2_X:\int_{\mathbb{R}^N}H(x)f_X(x)dx=0\}$.

Theorem EC.1 **(Numerical solution of the design equation)**. *Consider a basis $(w_j)_{j\geqslant 0}$ for $L^2_{X,0}$, an orthonormal basis $(e_j)_{j=1,\dots,n}$ for $\operatorname{Span}\langle w_1,\dots,w_n\rangle$ $(n\geqslant 1)$, a matrix $T_n:=(T_{jk})$ defined by $w_j=\sum_{k=1}^n T_{kj}e_k$, operator $\mathscr{I}$ in (EC.12), and $M_n:=(\langle\mathscr{I}[e_k],e_j\rangle)_{k,j=1,\dots,n}$. If $r_{XI}<1$, then the unique solution $H_0^*\in L^2_X$ of the design equation (9) is the L^2_X-limit of payoff sequence:*

$$\left(H^*_{0,n}(x)\right)_{n\geqslant 1} := \left(\sum_{j=1}^n h_j^n w_j(x)\right)_{n\geqslant 1},$$

where $h^n\in\mathbb{R}^n$ is the unique solution of the n-dimensional linear system: $T_n^{-1}(\mathbb{I}_n-M_n)T_nh^n=\gamma^n$, matrix $\mathbb{I}_n$ is the $n\times n$ identity, and vector $\gamma^n:=(\langle\Pi_0^X,e_j\rangle)_{j=1,\dots,n}$, with Π_0^X given by (9).

In the case of Gaussian distributions, w_j may be centered Hermite polynomials; if distributions are lognormal, we may change variables according to $x=e^u$ and then center the outcome around the $\mathbb{P}$-mean. To the best of our knowledge, this result is new. We have used it for solving the design equation arising in Section 5.

EC.1.8. Combined vs. Single-Claim Hedge Payoff Analysis

Let us consider the mean-centered claims $H_0^*(X)$ and $G_0^*(I)$ solving system (5). The optimal single-claim hedge $H^\flat(X)$ results from setting $G(I) \equiv 0$ in expression (5). We see that the two claims $H_0^*(X)$ and $H_0^\flat(X)$ differ by exactly the cross-claim effect:

$$H_0^*(X) - H_0^\flat(X) = -\mathbb{E}\left[G_0^*(I)\middle|\mathscr{F}_X\right].$$

By adding $G_0^*(I)$ to both terms in the previous expression, we have:

$$\underbrace{[H_0^*(X) + G_0^*(I)]}_{\text{combined positioning}} - \underbrace{H_0^\flat(X)}_{\text{single claim}} = \underbrace{G_0^*(I) - \mathbb{E}\left[G_0^*(I)\middle|\mathscr{F}_X\right]}_{I\text{-idiosyncratic risk}}. \tag{EC.13}$$

The value-centered optimal combined custom hedge $H_0^*(X) + G_0^*(I)$ exceeds the value-centered optimal single-claim hedge $H_0^\flat(X)$ by the incremental contribution of claim $G_0^*(I)$ over its X-predictable component $\mathbb{E}\left[G_0^*(I)\middle|\mathscr{F}_X\right]$. In other words, this contribution is represented by the portion of payoff $G_0^*(I)$ that depends on whatever risk source is unspanned by X. This quantity is the I-idiosyncratic risk component of the hedge $G_0^*(I)$.

EC.1.9. Combination Value Lower Bound for Weakly Risk Averse Firms

For a weakly risk averse firm, we obtain an approximate explicit lower bound which does not depend on naked revenues π. Hence, we needn't assume anything regarding π, X, and Y.

Proposition EC.4 **(Combination value lower bound: weakly risk averse firms)**. *Assume that* $\mathrm{Var}[\pi] \ll (\varrho/a)^2\, \mathrm{Var}[d\mathbb{Q}/d\mathbb{P}]$. *Then:*

$$\mathcal{CV}_{X\to X+I} \gtrsim \frac{1}{2}(1 - r_{XI}^2)\left(1 - \sqrt{1 - r_{X^\perp I}^2}\right)^2 \frac{\varrho}{a}\mathrm{Var}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right], \tag{EC.14}$$

where $r_{X^\perp I} := \sup\left\{\rho(Z, W) : Z \in L_I^2, W \in (L_X^2)^\perp\right\}$.

If index selection is based on this bound, then it would not depend upon the firm's position revenues π or nonclaimable risk term Y. This concludes our analysis of Issue 1 stated above.

EC.1.10. A Dynamic Custom Hedging Model Setup

Let us consider our standard model setup (Section 2) and the following variations. The time frame is a finite set of points $0, \dots, N$. There is a single source of risk X, which we assume to be claimable. This is defined through a family of stochastic processes $X := (X_k^{n,x})_{0\leqslant n\leqslant k\leqslant N}$ over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, starting at $X_n^{n,x} = x$ for all times n and states x, and grouped according to the rule $X_h^{k, X_k^{n,x}} = X_h^{n,x}$ for all times $n \leqslant k \leqslant h$ and states x. We focus on one-claim hedges written on X only. For the sake of simplicity, we consider operating revenues and hedges depending on the final state $X_N^{n,x}$ only. Given a starting point in time n, a state x, and a cash endowment w, the *optimal self-financing hedging strategy* (SFHS) originating at (n,x,w) is a family $(H^*_{k,y|n,x})_{n\leqslant k\leqslant N, y=X_k^{n,x}}$ defined by the following recursion:

- At time $t=n$, optimal hedge $H^*_{n,x|n,x}$ is the payoff H determined by solving:

$$\max_{H\in L^2_X:\,\mathbb{E}_{\mathbb{Q}}[H(X_N^{n,x})]=w} \mathscr{U}\left[\pi(X_N^{n,x})+H(X_N^{n,x})\right]. \tag{EC.15}$$

This is the one-period single-claim custom hedge design problem stated in Section 4.

- At time $k+1: n\leqslant k<N$, given a standing hedge $H^*_{k,y|n,x}$ for every state $y=X_k^{n,x}$, let $z=X_{k+1}^{n,x}$ denote an admissible state. The firm seeks an optimal hedge $H:=H^*_{k+1,z|n,x}$ paying out $H^*_{k+1,z|n,x}(X_N^{k+1,z})$ at maturity N and currently worth the time $k+1$ fair value of the standing hedge $H^*_{k,X_k^{n,x}|n,x}$ in their portfolio (self-financing condition). That is:

$$\mathbb{E}_{\mathbb{Q}}[H(X_N^{k+1,z})]=\mathbb{E}_{\mathbb{Q}}\left[H^*_{k,X_k^{n,x}|n,x}(X_N^{n,x})\middle| X_{k+1}^{n,x}=z\right]. \tag{EC.16}$$

The time $k+1$ hedge $H^*_{k+1,z|n,x}$ corresponding to state z is the payoff H solving the one-period design problem:

$$\max_{H\in L^2_X:\,\mathbb{E}_{\mathbb{Q}}[H(X_N^{k+1,z})]=\mathbb{E}_{\mathbb{Q}}\left[H^*_{k,X_k^{n,x}|n,x}(X_N^{n,x})\middle| X_{k+1}^{n,x}=z\right]} \mathscr{U}\left[\pi(X_N^{k+1,z})+H(X_N^{k+1,z}))\right]. \tag{EC.17}$$

This procedure leads to the optimal SFHS:

$$H^*:=\left(H^*_{n,x|n,x},...,H^*_{k+1,\cdot|n,x},...,H^*_{N-1,\cdot|n,x}\right).$$

If $x=x_n,...,x_N$ is a sample path for the underlying risk variable, then strategy H^* entails the following sequence of payoff functions:

$$\left(H^*_{n,x_n|n,x},...,H^*_{k+1,x_{k+1}|n,x},...,H^*_{N-1,x_{N-1}|n,x}\right).$$

One might also include transaction costs and examine the tradeoff between incremental benefit and cost related to switching from a static hedge to a periodically rebalanced hedge. An issue arising upon extending any MV model to multiple periods is time inconsistency. This is of particular relevance when trying to solve the course of dimensionality entailed by the formulation above and coming up to an equivalent dynamic programming recursion. The problem may be tackled either by replacing the MV target with an expected exponential utility, as in Caldentey and Haugh (2006), or by using any of the methods developed in Basak and Chabakauri (2010) and Björk and Murgoci (2014). An extension to a dynamic setup may be found in Kouvelis et al. (2018). Further theoretical investigation is left for future research.

EC.1.11. The Term Structure of Optimal Operations

Guiotto et al. (2020) consider time T operating revenues $\pi_T(X,Y;q)$ in a dynamic setting (see, *e.g.*, Example EC.10), derive a term structure of optimal operational levels $T \to q^*(T)$, and compute the optimal lead-time defined as $T_0^* := \arg\max_{T>0} \mathscr{U}(\pi_T(X,Y;q^*(T)))$. One may include our combined custom hedge and derive the optimal operational time defined as:

$$T^*_{\mathcal{H}\oplus\mathcal{G}} := \arg\max_{T>0} \mathscr{U}\left(\pi_T(X,Y;q^*(T)) + H^*_{T,q^*(T)}(X) + G^*_{T,q^*(T)}(I)\right).$$

Our conjecture is twofold: first, $T^*_{\mathcal{H}\oplus\mathcal{G}} > T_0^*$; second, by reducing exposure to uncertainty, our optimal combined hedge allows the firm to delay any optimal operational policy change resulting from model exogenous factors (*e.g.*, a sudden increase in risk aversion due to unexpected market turmoil).

EC.2. Technical Details

EC.2.1. Regularity Conditions

Throughout the paper, functional dependence between variables is assumed to go through a Borel measurable map. We require that all r.v.'s exhibit finite variance under the physical probability $\mathbb{P}$. This assumption does not substantially impact the economics of the underlying problem. However, it ensures that any future cash flow C with finite variance has a well-defined fair price because: $\mathbb{E}_{\mathbb{Q}}[|C|]^2 \leqslant \mathbb{E}_{\mathbb{P}}\left[|C|^2\right]\mathbb{E}_{\mathbb{P}}\left[(d\mathbb{Q}/d\mathbb{P})^2\right]$. In particular, any admissible combined hedge has a finite fair price $\mathbb{E}_{\mathbb{Q}}[H(X)+G(I)]$, hence the budget constraint in problem (3) is well-defined. Let $L^2(\Omega) := L^2(\Omega,\mathscr{F},\mathbb{P})$ be the space of square integrable r.v.'s equipped with the usual inner product. We denote $\mathscr{B}_{\mathbb{R}^N}$ the Borel σ-algebra over $\mathbb{R}^N$. The assumption of variance finiteness amounts to stating that all risk variables (including the business position, pricing kernel, and hedge's payoffs) are in $L^2(\Omega)$. Should X and I be finite dimensional, then this statement is equivalent to taking payoff functions $H = H(x)$ and $G = G(i)$ in L^2_X and L^2_I, respectively. Similarly, the business position $\pi = \pi(X,Y;q)$ is assumed to be in $L^2(\Omega)$ for all $q \in \mathcal{Q}$. Clearly, the optimized business position revenues Π defined in formula (1) are in $L^2(\Omega)$ and the target MV criterion $\mathscr{U}$ is well-defined.

EC.2.2. Budget Constraint Reduction

We first show that budget constraint $\mathbb{E}_{\mathbb{Q}}[H(X)+G(I)] \leqslant w$ is binding at the optimum: any hedge $(H^*(X), G^*(I))$ that solves the optimization problem (3) meets a condition with strict equality, *i.e.*, $E_{\mathbb{Q}}[H^*(X)+G^*(I)] = w$. Otherwise, the firm may use the residual cash endowment $\delta := w - E_{\mathbb{Q}}[H^*(X)+G^*(I)] > 0$ to enhance their investment and take a derivative position $(H^*(X) + \delta, G^*(I))$. This latter clearly satisfies the budget constraint, while entailing a MV utility that strictly exceeds the one related to $(H^*(X), G^*(I))$, *i.e.*, $\mathscr{U}(\pi + (H^*(X) + \delta) + G^*(I)) > \mathscr{U}(\pi + H^*(X) + G^*(I))$. This contradicts the optimality of $(H^*(X), G^*(I))$. Next, we prove that the optimization problem (3) with an equality constraint $\mathbb{E}_{\mathbb{Q}}[H(X) + G(I)] = w$ boils down to one with equality constraints at the level of individual claims, *i.e.*, $\mathbb{E}_{\mathbb{Q}}[H(X)] = 0, \mathbb{E}_{\mathbb{Q}}[G(I)] = 0$. Let $(H^*(X), G^*(I))$ denote an optimal combined positioning that corresponds with a zero portfolio endowment. Then, any derivative positioning $(H^*(X) + w_1, G^*(I) + w_2)$, with $w_1 + w_2 = w > 0$, is optimal for the problem featuring a budget w. Indeed, they are all worth w and deliver a common MV performance. Given a worthless derivative portfolio, *i.e.*, $w = 0$, centering each payoff function by the related fair value does not alter the resulting MV target, namely $\mathscr{U}(\pi + (H(X) - V_H) + (G(I) - V_G)) = \mathscr{U}(\pi + H(X) + G(I)) - \varrho(V_H + V_G) = \mathscr{U}(\pi + H(X) + G(I))$, as long as $V_H + V_G = \mathbb{E}_{\mathbb{Q}}[H(X) + G(I)] = w = 0$, where $V_H := \mathbb{E}_{\mathbb{Q}}[H(X)]$ and $V_G := \mathbb{E}_{\mathbb{Q}}[G(I)]$. These considerations show that the firm may split their endowment w into a portion w_1 allocated to purchasing a claim $H(X)$ written on the position's claimable X and a portion w_2 devoted to buying a claim $G(I)$ written on the side index I. The optimal hedge then reads as follows:

$$(H^*(X), G^*(I)) = (\widetilde{H}^*(X) + w_1, \widetilde{G}^*(I) + w_2),$$

where claims $\widetilde{H}^*(X)$ and $\widetilde{G}^*(I)$ solve the reduced optimization problem (3) with individual constraints $\mathbb{E}_{\mathbb{Q}}[H(X)] = 0, \mathbb{E}_{\mathbb{Q}}[G(I)] = 0$.

EC.2.3. Properties of Maximal Correlation

Maximal correlation exhibits the following properties:

1. $0 \leqslant r_{XI} \leqslant 1$ (unitary range);
2. $r_{XI} = r_{IX}$ (symmetry);
3. $r_{XI} = 0 \Leftrightarrow X$ and I are statistically independent (orthogonality);
4. $r_{\Phi(X)\Psi(I)} = r_{XI}$ for all bijections Φ and Ψ (invariance under one-to-one transformation);
5. $r_{XI} = |\rho(X, I)|$ if X and I are Gaussian. (Here ρ denotes the Pearson linear correlation.)
6. For any multivariate Gaussian pair (Z_1, Z_2), $r_{Z_1,Z_2} = 1 \iff \exists a, b \neq 0 : a \cdot Z_1 = b \cdot Z_2$.

Statements 1 to 5 are proved in Schweizer and Wolff (1981). Statement 6 is proved in Guiotto (2019). Note that functional dependence between variables X and I (see Section 1.2) entails a unitary maximal correlation ($r_{XI} = 1$), while the inverse may not be true. Hence, $r_{XI} < 1$ implies that X and I are not functionally dependent on one another.

EC.2.4. Hedging Policies and Utility Profiles in the Newsvendor Model

The first hedging policy leaves the business position financially naked, *i.e.*, $\mathcal{H} = \{0\}$. We thus have a trivial hedge $H^*_{\mathcal{H}\times\mathcal{Q}} \equiv 0$, a utility profile $\mathscr{U}_0^*(q) := \mathscr{U}(\pi^{NV}(X, Y; q))$, and an optimal procurement $q_0^* := q^*(\{0\})$. The second and third hedging policies assume a positioning in linear derivatives written on either X or I. That is $\mathcal{H} = \mathcal{H}_X^f := \{x \mapsto \theta(x - f_X),\ \theta \text{ scalar}\}$ and $\mathcal{H} = \mathcal{H}_I^f := \{i \mapsto \theta(i - f_I),\ \theta \text{ scalar}\}$, where the two arbitrary constant values f_X and f_I represent the T-delivery prices agreed on at time 0 by buyer and seller. Clearly, the zero-price condition $\mathbb{E}_{\mathbb{Q}}[H] = 0$ appearing in optimization (19) entails that f_X and f_I are forward prices and the optimum is sought amongst forward contracts. The financially optimal utility profiles are $\mathscr{U}^*_{\mathcal{H}_X^f}(q) = \mathscr{U}\left(\pi^{NV}(X,Y;q) + \theta_X^\flat(X - f_X)\right)$ and $\mathscr{U}^*_{\mathcal{H}_I^f}(q) = \mathscr{U}\left(\pi^{NV}(X,Y;q) + \theta_I^\flat(I - f_I)\right)$. Here, the optimal forward sizes $\theta_X^\flat$ and $\theta_I^\flat$ define the optimal single-claim forward hedges $H^*_{\mathcal{H}_X^f \times \mathcal{Q}} : x \to \theta_X^\flat(x - f_X)$ and $H^*_{\mathcal{H}_I^f \times \mathcal{Q}} : i \to \theta_I^\flat(i - f_I)$. Size $\theta_X^\flat$ has been found by Rolfo (1980), while $\theta_I^\flat$ is computed in Gaur and Seshadri (2005). The fourth and fifth hedging policies enlarge forward hedging spaces to encompass all regular custom claims written on X and I, *i.e.*, $\mathcal{H} = \mathcal{H}_X$ and $\mathcal{H} = \mathcal{H}_I$, respectively. They yield financially optimal utility profiles $\mathscr{U}^*_{\mathcal{H}_X}(q) = \mathscr{U}\left(\pi^{NV}(X,Y;q) + H^\flat(X)\right)$ and $\mathscr{U}^*_{\mathcal{H}_I}(q) = \mathscr{U}\left(\pi^{NV}(X,Y;q) + G^\flat(I)\right)$, respectively. Here, claims $H^*_{\mathcal{H}_X \times \mathcal{Q}} := H^\flat(X)$ and $H^*_{\mathcal{H}_I \times \mathcal{Q}} := G^\flat(I)$ define the optimal single-claim custom hedges solving problem (10) for $\mathcal{H} = \mathcal{H}_X$ and $\mathcal{H} = \mathcal{H}_I$, respectively. Claim $H^\flat(X)$ has been derived in Chen et al. (2014), while $G^\flat(I)$ is, to the best of our knowledge, new. The sixth hedging policy involves an optimal combined portfolio of two forwards: one contract is written on the position's claimable X, the other on the side index I. The hedging space is $\mathcal{H} = \mathcal{H}_{XI}^f := \{(x,i) \mapsto \theta_X(x - f_X) + \theta_I(i - f_I),\ \theta_X \text{ and } \theta_I \text{ scalar}\}$. The financially optimal utility profile is $\mathscr{U}^*_{\mathcal{H}_{XI}^f}(q) = \mathscr{U}(\pi^{NV}(X,Y;q) + \theta_X^*(X - f_X) + \theta_I^*(I - f_I))$. Here, sizes θ_X^* and θ_I^* are the optimal forward gas and index units defining the optimal combined forward hedge $H^*_{\mathcal{H}_{XI}^f \times \mathcal{Q}} : (x,i) \to \theta_X^*(x - f_X) + \theta_I^*(i - f_I)$ derived in Roncoroni and Id Brik (2017).

EC.2.5. Model Implementation Algorithm

Let the state vector distribution be jointly lognormal under the physical measure $\mathbb{P}$, *i.e.*,

$$\begin{pmatrix} \log X \\ \log Y \\ \log I \end{pmatrix} \overset{\mathbb{P}}{\sim} \mathcal{N}\left(\begin{bmatrix} m_X \\ m_Y \\ m_I \end{bmatrix}, \begin{bmatrix} \sigma_X^2 & \rho_{XY}\sigma_X\sigma_Y & \rho_{XI}\sigma_X\sigma_I \\ \rho_{XY}\sigma_X\sigma_Y & \sigma_Y^2 & \rho_{YI}\sigma_Y\sigma_I \\ \rho_{XI}\sigma_X\sigma_I & \rho_{YI}\sigma_Y\sigma_I & \sigma_I^2 \end{bmatrix} \right). \tag{EC.18}$$

Risk neutral parameters $m_X^{\mathbb{Q}}$ and $m_I^{\mathbb{Q}}$ result from shifting their physical counterparts m_X and m_I by a proportion λ of the corresponding volatility terms σ_X and σ_I, where λ stands for the gas market price of risk (Kolos and Ronn 2008). A risk neutral mean $m_Y^{\mathbb{Q}}$ is not required so long as demand Y is nonclaimable and thus it does not enter arbitrage-free valuation used in the budget constraint. We assume that correlations and standard deviations are the same under $\mathbb{P}$ and $\mathbb{Q}$. In a NV model featuring operating revenues (18), we consider r.v.'s X, Y, and I representing a gas price, a consumption level, and a temperature record to be observed at time T, respectively. We assume $T = 1$ calendar year since current time 0. Estimating a trivariate lognormal model (EC.18) requires samples for X, Y, and I. In the absence of one-year market estimates for gas prices, consumption levels, and temperatures, we rely on simulated sample data. Specifically, we follow a three-step simulation procedure:

1. Identify $(\log X, \log Y, \log I)$ with the marginal $(\log X(1), \log Y(1), \log I(1))$ of a trivariate stochastic process $Z := (\log X(t), \log Y(t), \log I(t))_{t\geqslant 0}$;
2. Estimate process Z on a time series of gas prices, consumption levels, and temperature records;
3. Generate samples of (X, Y, I) by simulating paths of Z via Monte Carlo methods.

Given an array of simulated samples, we easily arrive at estimated means m_X, m_Y, m_I, standard deviations σ_X, σ_Y, σ_I, and correlations ρ_{XY}, ρ_{XI}, and ρ_{YI}. If the process has normal finite dimensional marginals, then a model instance (EC.18) is fully identified. This algorithm is fed by a time series for the three state variables in question. Whereas gas prices are open market quotes, consumption figures are usually proprietary information. In addition, temperature records should refer to the exact region where gas is actually consumed. As our goal is to assess the relevance of our integrated policy compared to major existing alternatives, we are content with deriving and adopting a realistic parametric setup. This may be obtained by crafting an estimate of the trivariate model (EC.18) based on a suitable instance of the aforementioned algorithm, which we now describe.

Gas prices are daily quoted frontline futures prices for delivery at Henry Hub in Louisiana. These quotes are posted on the Bloomberg platform (www.bloomberg.com). They are expressed in US dollars per million British thermal unit (USD/MMBtu). Our time series spans December 1, 2007 to February 28, 2009. We assume prices obey a lognormal market model featuring mean-reverting paths, *i.e.*,

$$d\log X(t) = \theta_x [\alpha_x - \log(X(t))]\, dt + \sigma_x dW_x(t), \tag{EC.19}$$

under $\mathbb{P}$. Here coefficient θ_x represents a mean reversion speed per unit of log-price discrepancy from the corresponding long-term level α_x, term σ_x denotes the instantaneous variance rate, and process W_x is a standard one-dimensional Brownian motion. This is a market model for commodity price modeling, it ensures analytical tractability, and it fits our distributional assumption (Benth et al. 2008). Next, we estimate model (EC.19) through maximum likelihood on the recorded price path. Finally, we run a Monte Carlo simulation and get to a sample mean m_X and a standard deviation σ_X for the normal variable $\log X = \log X(1)$. Throughout the implementation, we assume that time is measured in "years" according to the Actual/Actual day-count convention.

We repeat the same procedure for consumption level Y and obtain a sample mean m_Y and a standard deviation σ_Y for the normal variable $\log Y = \log Y(1)$. Here, Y is identified with the one-year estimate of the monthly average consumption in a representative region featuring 10 million inhabitants. Consumption levels for the whole US area are posted in a monthly report issued by the US Energy Information Administration (www.eia.gov). We divide each record by thirty to get an approximate figure for the region in question. Gas volumes are expressed in MMBtu. By using standard formulae linking first and second order moments to corresponding lognormal distribution parameters, *i.e.*, $m_z = 2\log\mathbb{E}[Z] - \frac{1}{2}\log\left(\mathbb{E}[Z]^2 + \mathrm{Var}[Z]\right)$ and $\sigma_z = \sqrt{\log(\mathrm{Var}[Z]/\mathbb{E}[Z]) + 1}$, with $\log(Z) \sim \mathcal{N}(m_z, \sigma_z^2)$, we obtain round values for the mean and standard deviation of the one-year forecast of gas price and volume.

Table EC.2 A realistic model setup.

	Average value	Standard deviation
Gas price X (USD/MMBtu)	4	1
Gas volume Y (MMBtu)	$10,000,000$	$5,000,000$
Temperature index I (°F)	60	30

Descriptive statistics of one-year estimate of state variables X, Y, and I. Gas price X is expressed in US dollars per million British thermal units (USD/MMBtu); Gas consumption is given in millions of British thermal units (MMBtu); temperature records are measured in degrees Fahrenheit (°F).

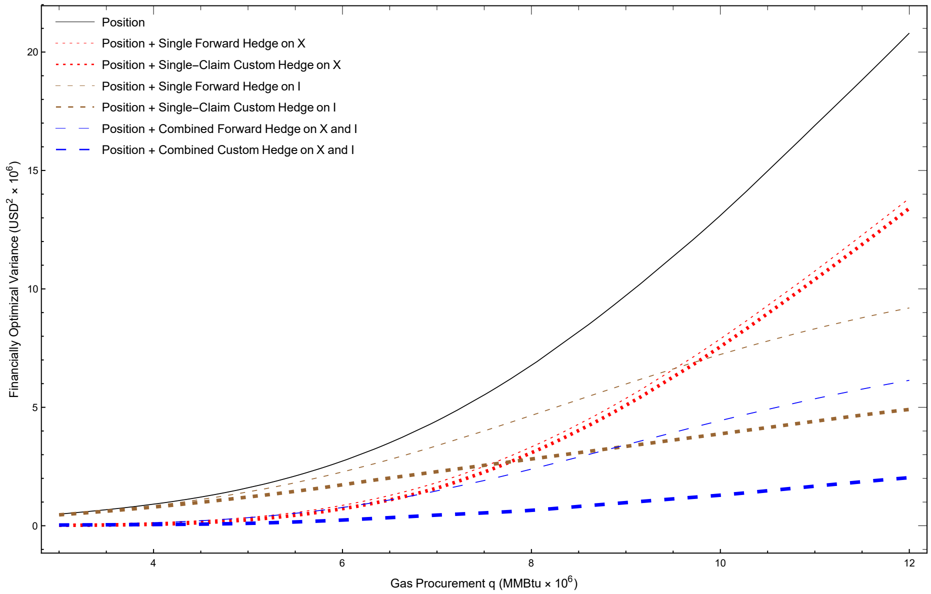
These figures are reported in Table EC.2, where temperature means and standard deviation reflect a typical continental region featuring mild climate conditions. To complete our picture of a realistic parametric setup, we assume correlations generate a mild positive dependence of gas price X on both consumption level Y and temperature record I and a strong positive dependence between I and Y. These two assumptions characterize regions where energy consumption goes in pair with warm seasons, while showing a stiff demand curve: the former exacerbates the portion of corporate exposure due to nonclaimable risk, while the latter underpins the use of temperature derivatives to cope with that type of risk. Last, initial conditions are given by recorded values $X(0) = 5.57$ USD/MMBtu, $Y(0) = 2,547,629$ MMBtu, $I(0) = 19$°F, which correspond to $\log X(0) = 1.7178$, $\log Y(0) = 14.7507$, and $\log I(0) = 2.9444$ in the natural logarithmic scale. Estimated figures are reported in Table EC.3.

Table EC.3 Parametric setup in a NV model for gas retail.

m_Y	m_X	m_I	$m_X^{\mathbb{Q}}$	$m_I^{\mathbb{Q}}$	σ_Y	σ_X	σ_I	ρ_{XY}	ρ_{XI}	ρ_{YI}	λ	ϱ	a	c
16.01	1.36	3.98	1.35	3.97	0.47	0.25	0.47	0.20	0.30	0.95	0.2	1.00	10^{-7}	2.00

This calibration procedure defines the model instance we consider in Section 5. Note that numerical computations of the hedges are run up to an approximation error lower than 10^{-5}.

Figure EC.3 Optimal variance of alternative integrated policies for a variance minimizing firm.



EC.2.6. Empirical Comparative Analysis: Further Details

Table EC.4 reports optimal procurement $q^*_{\mathcal{H}}$ as well as expected value (*i.e.*, return) and standard deviation (*i.e.*, risk) of the optimal operating revenues $\pi^*_{\mathcal{H}} + H^*_{\mathcal{H}}$ across the seven admissible integrated policy spaces. It also exhibits the variations of both mean and standard deviation that occur upon switching from an integrated policy, say, $\left(H^*_{\mathcal{G}}, q^*_{\mathcal{G}}\right)$ featuring a hedging class $\mathcal{G}$, to an integrated policy $(H^*_{\mathcal{H}}, q^*_{\mathcal{H}})$ featuring another hedging class $\mathcal{H}$.

It is interesting to study the special case of variance minimizing firms. This corresponds to setting parameter $\varrho = 0$ in a MV target. Put differently, a firm is a pure risk minimizer provided that no speculative theme affects its preferences. Figure EC.3 shows realized optimal variance against operational level q for the seven hedging strategies examined so far.

The naked position and the position hedged through our optimal combined custom claim define upper and lower bounds for the variance profile, respectively. For relatively low operational figures, hedges involving X outperform all others. However, things change as the procurement order q increases. For relatively large values of q, revenues hedged through single claims on I exhibit a lower variance than those hedged by single claims on X. Interestingly, the optimal forward on I outperforms the optimal custom claim on X for q greater than 10^6 MMBtu. As noted earlier, these facts are intimately related to the kind of functional dependence of NV revenues on price X (resp., demand Y) for relatively low (resp., high) values of procurement q. In general, by increasing hedge complexity, a firm may attain an assigned level of operations and a progressively lower variance or, equivalently, a progressively higher level of operations and a steady variance.

Table EC.4 Optimal procurement orders, full exposure return/risk figures, and related variations from switching integrated policies.

Starting class $\mathcal{G}$	$q^*_{\mathcal{G}}$	$m_{\mathcal{G}}\left(q^*_{\mathcal{G}}\right)$	$\sigma_{\mathcal{G}}\left(q^*_{\mathcal{G}}\right)$	Target class $\mathcal{H}$: Forward on X		Custom on X		Forward on I		Custom on I		Combined Forward		Combined Custom	
	MMBtu Proc.*	US$ Return*	US$ Risk*	% Return	Risk	% Return	Risk	% Return	Risk	% Return	Risk	% Return	Risk	% Return	Risk
No Hedge	7.087	2.540	0.959	4.84	-17.99	5.01	-19.50	9.21	7,27	14.05	-8.59	12.07	-19.60	14.22	-38.31
Forward on X	7.861	2.663	0.786	-		0.16	-1.84	4.17	30.80	8.78	11.47	6.89	-1.97	8.95	-24.77
Custom on X	7.917	2.668	0.772	-0.16	1.88	-		4.00	33.26	8.60	13.56	6.72	-0.12	8.77	-23.36
Forward on I	8.472	2.774	1.028	-4.00	-23.55	-3.85	24.96	-		4.42	-14.78	2.61	-25.05	4.58	-42.49
Custom on I	9.934	2.897	0.876	-8.07	-10.29	-7.92	-11.94	-4.24	17.35	-		-1.73	-12.05	0.15	-32.51
Combined Forward	9.177	2.847	0.771	-6.45	2.01	-6.29	0.12	-2.55	33.43	1.76	13.70	-		1.92	-23.27
Combined Custom	10.300	2.901	0.591	-8.21	32.93	-8.06	30.48	-4.38	-73.88	-0.15	48.18	-1.88	30.32	-	

For each hedging class $\mathcal{H} = \mathcal{H}_0$ (no hedge), $\mathcal{H}^f_X$ (forward on X), $\mathcal{H}_X$ (custom on X), $\mathcal{H}^f_I$ (forward on I), $\mathcal{H}_I$ (custom on I), $\mathcal{H}^f_{XI}$ (combined forward on X and I), and $\mathcal{H}_{XI}$ (combined custom on X and I), we report: 1) Optimal procurement order $q^*(\mathcal{H})$, 2) Full exposure return $m_{\mathcal{H}}(q^*(\mathcal{H}))$ and risk $\sigma_{\mathcal{H}}(q^*(\mathcal{H}))$, and 3) Percentage variation in revenues return and risk obtained by switching between pairs of optimal integrated policies related distinct hedging classes, say from $\mathcal{G}$ (row) to $\mathcal{H}$ (column),*i.e.*, by replacing optimal operating revenues $\pi\left(X, Y; q^*_{\mathcal{G}}\right) + H^*_{\mathcal{G}}$ with $\pi\left(X, Y; q^*_{\mathcal{H}}\right) + H^*_{\mathcal{H}}$.

Regardless of the hedging class in question, minimum variance is attained at $q=0$, a result in keeping with the pure variance minimizing firm considered in Wang and Yao (2017). By comparing this result with those reported earlier in Figure 2 for a risk averse firm, we see that the best strategy is "not to play the game" (*i.e.*, $q=0$) whenever risk is the sole concern for the firm. Should they be interested in expected revenues as well, then an array of optimal operational levels arises depending on the hedging class in question. Indeed, in the NV case under scrutiny, the greater the class of hedging opportunities, the larger the optimal procurement order (Table 2, column 2).

We conclude by disentangling the operational and speculative effects on expected revenues produced by a change in hedging classes. Suppose a firm switches from a hedging class $\mathcal{G}$ to another class $\mathcal{H}$. Then, revenues $\pi^*_{\mathcal{G}}+H^*_{\mathcal{G}}$ turn into $\pi^*_{\mathcal{H}}+H^*_{\mathcal{H}}$. Hence, expected revenue variations reported in Table EC.4 stem from two distinct channels. One is financial speculative and is represented by the variation in the expected payoff of the hedge: $\mathbb{E}\left[H^*_{\mathcal{H}}\right]-\mathbb{E}\left[H^*_{\mathcal{G}}\right]$. It stems from the ability of financial derivatives to leverage the risk premium offered by the market. The other is operational and is given by the variation in expected revenues under optimal integrated policies $\left(H^*_{\mathcal{G}},q^*_{\mathcal{G}}\right)$ and $\left(H^*_{\mathcal{H}},q^*_{\mathcal{H}}\right)$: $\mathbb{E}\left[\pi^*_{\mathcal{H}}\right]-\mathbb{E}\left[\pi^*_{\mathcal{G}}\right]$. This term originates from the effect that a change in financial hedge has on the optimal procurement order q^*, which in turn affects revenues π. For instance, let us consider entering an optimal integrated policy, starting from an optimally handled naked position $\pi^*_0:=\pi\left(q^*_0\right)$. This corresponds to switching from $\mathcal{G}:=\mathcal{H}_0$ (no hedge) to any other hedging class $\mathcal{H}$. For notational neatness, we omit indicating $\mathcal{H}_0$ as a starting hedging class. The expected relative gain $\Delta m_{\mathcal{H}}:=\left(\mathbb{E}\left[\pi^*_{\mathcal{H}}+H^*_{\mathcal{H}}\right]-\mathbb{E}\left[\pi^*_0\right]\right)/\mathbb{E}\left[\pi^*_0\right]$ can be split into a financial term $\Delta m^{fin}_{\mathcal{H}}:=\mathbb{E}\left[H^*_{\mathcal{H}}\right]/\mathbb{E}\left[\pi^*_0\right]$ and an operational term $\Delta m^{op}_{\mathcal{H}}:=\left(\mathbb{E}\left[\pi^*_{\mathcal{H}}\right]-\mathbb{E}\left[\pi^*_0\right]\right)/\mathbb{E}\left[\pi^*_0\right]$. Table EC.5 reports these quantities expressed as percentages over the no-hedge figures and for all the six non trivial hedging classes at stake.

Table EC.5 Expected gain breakdown into speculative and operational terms.

Hedging class	Procurement order	Expected gain		Speculative		Operational
$\mathcal{H}$	$q^*_{\mathcal{H}}$	$\Delta m_{\mathcal{H}}$	=	$\Delta m^{fin}_{\mathcal{H}}$	+	$\Delta m^{op}_{\mathcal{H}}$
	(MMBtu)	(%)		(%)		(%)
$\mathcal{H}^f_X$	7.861	4.84		-1.33		6.17
$\mathcal{H}_X$	7.917	5.01		-1.54		6.55
$\mathcal{H}^f_I$	8.470	9.21		-0.49		9.70
$\mathcal{H}_I$	9.934	14.05		-1.15		15.20
$\mathcal{H}^f_{XI}$	9.177	12.07		-0.84		12.91
$\mathcal{H}_{XI}$	10.300	14.22		-1.67		15.89

For each target hedging class $\mathcal{H}=\mathcal{H}^f_X$ (forward on X), $\mathcal{H}_X$ (custom on X), $\mathcal{H}^f_I$ (forward on I), $\mathcal{H}_I$ (custom on I), $\mathcal{H}^f_{XI}$ (combined forward on X and I), and $\mathcal{H}_{XI}$ (combined custom on X and I), we report optimal procurement order $q^*_{\mathcal{H}}$, full exposure expected gain $\Delta m_{\mathcal{H}_0\to\mathcal{H}}$, speculative term $\Delta m^{fin}_{\mathcal{H}_0\to\mathcal{H}}$, operational term $\Delta m^{op}_{\mathcal{H}_0\to\mathcal{H}}$. All variations are expressed as a percentage over the no-hedge figure.

Two interesting features are worth mentioning. First, expected revenues primarily increase due to an increasing optimal procurement order allowed by the adoption of a financial hedge. Put differently, operational flexibility is a major source of profitability (Zhao and Huchzermeier 2017). Second, the market risk premium conveyed by the hedge plays a relatively modest and negative role in the variation of expected revenues. The negative sign may reflect a risk averse attitude of the firm, which adheres to a "hedging pressure" (Keynes (1923), Hicks (1939)) whereby hedgers pay a premium to transfer their risk to speculators acting as hedge providers.

EC.3. Proofs

EC.3.1. Proof of Proposition 1

The statement is a direct consequence of the Lagrange multipliers theorem applied to optimal pair $(H^*(X), G^*(I))$. Let us consider the map $(H,G) \to \mathscr{U}(\pi + H + G)$. A point of maximum $(H^*(X), G^*(I))$ under constraints $\mathbb{E}_{\mathbb{Q}}[H(X)] = 0$ and $\mathbb{E}_{\mathbb{Q}}[G(I)] = 0$ satisfies the first order stationarity condition:

$$\nabla \mathscr{U}(\pi + H^*(X) + G^*(I)) = \lambda \nabla(\mathbb{E}_{\mathbb{Q}}[H^*(X)]) + \mu(\mathbb{E}_{\mathbb{Q}}[G^*(I)]), \tag{EC.20}$$

for suitable real scalars λ and μ. Here, $\nabla := (\partial_H, \partial_G)$ is the gradient operator and ∂ denotes the Fréchet differential operator. If $L^2 := L^2(\Omega, \mathscr{F}, \mathbb{P})$, straightforward calculations lead to:

$$\langle \partial_H \mathscr{U}(\pi + H + G), \delta H \rangle = \mathbb{E}\left[(\varrho - a((\pi + H + G) - \mathbb{E}[\pi + H + G]))\,\delta H\right], \ \forall \delta H(X) \in L^2, \tag{EC.21}$$

$$\langle \partial_G \mathscr{U}(\pi + H + G), \delta G \rangle = E\left[(\varrho - a((\pi + H + G) - \mathbb{E}[\pi + H + G]))\,\delta G\right], \ \forall \delta G(I) \in L^2. \tag{EC.22}$$

Clearly:

$$\langle \partial_H \mathbb{E}_{\mathbb{Q}}[H(X)], \delta H(X) \rangle = \mathbb{E}_{\mathbb{Q}}[\delta H(X)], \quad \partial_G \mathbb{E}_{\mathbb{Q}}[H(X)] = 0,$$

$$\langle \partial_G \mathbb{E}_{\mathbb{Q}}[G(I)], \delta G(I) \rangle = \mathbb{E}_{\mathbb{Q}}[\delta G(I)], \quad \partial_H \mathbb{E}_{\mathbb{Q}}[G(I)] = 0.$$

Hence, condition (EC.20) leads to a 2×2 system of equations for the optimal pair $(H^*(X), G^*(I))$:

$$\begin{cases} \mathbb{E}\left[(\varrho - a((\pi + H^*(X) + G^*(I)) - \mathbb{E}[\pi + H^*(X) + G^*(I)]))\,\delta H(X)\right] = \lambda \mathbb{E}_{\mathbb{Q}}[\delta H(X)], \\ \mathbb{E}\left[(\varrho - a((\pi + H^*(X) + G^*(I)) - \mathbb{E}[\pi + H^*(X) + G^*(I)]))\,\delta G(I)\right] = \mu \mathbb{E}_{\mathbb{Q}}[\delta G(I)], \end{cases} \tag{EC.23}$$

for any $\delta H(X), \delta G(I) \in L^2$. By taking $\delta H \equiv 1 \equiv \delta G$, we obtain Lagrange multipliers:

$$\lambda = \mathbb{E}\left[(\varrho - a((\pi + H^*(X) + G^*(I)) - \mathbb{E}[\pi + H^*(X) + G^*(I)]))\right] = \varrho,$$

and $\mu = \varrho$. By inserting these values into system (EC.23) and switching from risk neutral to physical expectation, we have:

$$\begin{cases} \mathbb{E}\left[\left(\varrho\left(1 - \frac{d\mathbb{Q}}{d\mathbb{P}}\right) - a((\pi + H^*(X) + G^*(I)) - \mathbb{E}[\pi + H^*(X) + G^*(I)])\right)\delta H(X)\right] = 0, \\ \mathbb{E}\left[\left(\varrho\left(1 - \frac{d\mathbb{Q}}{d\mathbb{P}}\right) - a((\pi + H^*(X) + G^*(I)) - \mathbb{E}[\pi + H^*(X) + G^*(I)])\right)\delta G(I)\right] = 0, \end{cases}$$

namely:

$$\begin{cases} \mathbb{E}\left[\left(\varrho\left(1-\frac{d\mathbb{Q}}{d\mathbb{P}}\right)-a\left((\pi+H^*(X)+G^*(I))-\mathbb{E}\left[\pi+H^*(X)+G^*(I)\right]\right)\right)\middle|\mathscr{F}_X\right]=0, \\ \mathbb{E}\left[\left(\varrho\left(1-\frac{d\mathbb{Q}}{d\mathbb{P}}\right)-a\left((\pi+H^*(X)+G^*(I))-\mathbb{E}\left[\pi+H^*(X)+G^*(I)\right]\right)\right)\middle|\mathscr{F}_I\right]=0. \end{cases}$$

As long as $H^*(X)$ is $\mathscr{F}_X$ measurable and $G^*(I)$ is $\mathscr{F}_I$ measurable, we may simplify the system as follows:

$$\begin{cases} H^*(X)=-\mathbb{E}\left[\pi+\frac{\varrho}{a}\frac{d\mathbb{Q}}{d\mathbb{P}}\middle|\mathscr{F}_X\right]-\mathbb{E}\left[G^*(I)\middle|\mathscr{F}_X\right]+m, \\ G^*(I)=-\mathbb{E}\left[\pi+\frac{\varrho}{a}\frac{d\mathbb{Q}}{d\mathbb{P}}\middle|\mathscr{F}_I\right]-\mathbb{E}\left[H^*(X)\middle|\mathscr{F}_I\right]+\widetilde{m}, \end{cases} \tag{EC.24}$$

for suitable constants $m,\widetilde{m}$. These latter may be computed by taking the expectation of both terms in the two equations under $\mathbb{P}$, that is:

$$\widetilde{m}=m=\mathbb{E}[H^*(X)]+\mathbb{E}\left[G^*(I)\ \right]+\mathbb{E}[\pi]+\frac{\varrho}{a}.$$

If we consider the $\mathbb{P}$-mean centered claims $H_0^*(X):=H^*(X)-\mathbb{E}[H^*(X)]$ and $G_0^*(I):=G^*(I)-\mathbb{E}[G^*(I)]$, the resulting pair $(H_0^*(X),G_0^*(I))$ satisfies the system (5). Q.E.D.

EC.3.2. Proof of Theorem 1

Our proof strategy is as follows. First, we highlight that $H_0^*(X)$ solves a fixed-point equation. Second, we prove that this equation involves a contraction in a suitable subspace of $L^2(\Omega)$. Third, we derive both existence and uniqueness of $H_0^*(X)$ by using the Banach fixed-point theorem.

To begin, we insert the second equation of system (5) into the first one and get to:

$$H_0^*(X)=\Pi_0(X)+\mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}\left[H_0^*(X)\middle|\mathscr{F}_I\right]\middle|\mathscr{F}_X\right], \tag{EC.25}$$

where:

$$\Pi_0(X):=-\mathbb{E}_{\mathbb{P}}\left[\mathcal{K}-\mathbb{E}_{\mathbb{P}}\left[\mathcal{K}\middle|\mathscr{F}_I\right]\middle|\mathscr{F}_X\right], \tag{EC.26}$$

and $\mathcal{K}:=\pi+\frac{\varrho}{a}\frac{d\mathbb{Q}}{d\mathbb{P}}$ is the optimal hedging kernel. This is a fixed-point equation. The Banach fixed-point theorem requires the right-hand side in equation (EC.25) to be a contraction in a suitable space. To show this, let us define the operator:

$$T[Z]:=\mathbb{E}\left[\mathbb{E}\left[Z\middle|\mathscr{F}_I\right]\middle|\mathscr{F}_X\right].$$

By using standard properties of conditional expectation, we easily see that T is linear and bounded on $\mathcal{H}_X$. Moreover,

$$\mathbb{E}[T[Z]]=\mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[Z\middle|\mathscr{F}_I\right]\middle|\mathscr{F}_X\right]\right]=\mathbb{E}[Z],$$

hence $\mathbb{E}[T[Z]]=0$ provided that $\mathbb{E}[Z]=0$. In other terms, T maps the Hilbert subspace $\mathbb{H}_X$ of $\mathbb{P}$-centered, $\mathscr{F}_X$-measurable, $L^2(\Omega)$ r.v.'s into itself. Also, $\Pi_0(X)\in\mathbb{H}_X$. Therefore, (EC.25) assumes the following abstract form:

$$H_0^*(X)=\Pi_0(X)+T\left[H_0^*(X)\right] \tag{EC.27}$$

in the space $\mathbb{H}_X$. Map T is a contraction over $\mathbb{H}_X$ provided that the operator norm $\|T\|$ is strictly less than 1. The following estimate proves this claim. Let $\phi(X)\in\mathbb{H}_X$ and $\psi(I)\in\mathbb{H}_I$. By using the definition of inner product in $L^2(\Omega)$ and two elementary properties of conditional expectation, we have:

$$\begin{aligned}\langle\mathbb{E}[\phi(X)\mid\mathscr{F}_I],\psi(I)\rangle &=\mathbb{E}\left[\psi(I)\mathbb{E}\left[\phi(X)\mid\mathscr{F}_I\right]\right]=\mathbb{E}\left[\mathbb{E}\left[\phi(X)\psi(I)\mid\mathscr{F}_I\right]\right]\\ &=\mathbb{E}\left[\phi(X)\psi(I)\right]=\|\phi(X)\|_2\|\psi(I)\|_2\rho(\phi(X),\psi(I)),\end{aligned}$$

hence:

$$\langle\mathbb{E}[\phi(X)\mid\mathscr{F}_I],\psi(I)\rangle\leqslant r_{XI}\|\phi(X)\|_2\|\psi(I)\|_2$$

by the very definition of maximal correlation r_{XI}. If we set $\psi(I):=\mathbb{E}[\phi(X)\mid\mathscr{F}_I]$, the following upper bound holds true:

$$\|\mathbb{E}[\phi(X)\mid\mathscr{F}_I]\|_2\leqslant r_{XI}\|\phi(X)\|_2. \tag{EC.28}$$

Consequently,

$$\|\mathbb{E}\left[\mathbb{E}[\phi(X)\mid\mathscr{F}_I]\mid\mathscr{F}_X\right]\|_2\leqslant\|\mathbb{E}[\phi(X)\mid\mathscr{F}_I]\|_2\overset{(EC.28)}{\leqslant}r_{XI}\|\phi(X)\|_2$$

for all $\phi(X)\in\mathbb{H}_X$. Hence, $\|T\|\leqslant r_{XI}$, which we assumed to be strictly smaller than 1. Therefore, T is a contraction in $\mathbb{H}_X$ and the Banach fixed-point theorem ensures that equation (EC.27) admits a unique solution $H_0^*(X)$ in $\mathbb{H}_X$. Q.E.D.

EC.3.3. Proof of Theorem 2

Since vector (X,Y,I) is absolutely continuous on $\mathbb{R}^N$, we may compute all the conditional expectations appearing in equation (7) in terms of probability densities. Indeed, conditional expectation $\mathbb{E}[\phi(X)|\mathscr{F}_I]=\psi(I)$, where:

$$\psi(i)=\mathbb{E}[\phi(X)|\,I=i]=\int_{\mathbb{R}^N}\phi(\xi)f_{X|I}(\xi|i)d\xi=\int_{\mathbb{R}^N}\phi(\xi)\frac{f_{XI}(\xi,i)}{f_I(i)}d\xi.$$

A similar formula holds for $\mathbb{E}[\psi(I)|\mathscr{F}_X]$. According to Fubini-Tonelli theorem,

$$\mathbb{E}\left[\mathbb{E}\left[H_0^*(X)|\mathscr{F}_I\right]|X=x\right]=\int_{\mathbb{R}^N}H_0^*(\xi)\left(\int_{\mathbb{R}^N}\frac{f_{XI}(x,i)}{f_X(x)}\frac{f_{XI}(\xi,i)}{f_I(i)}\,di\right)\,d\xi.$$

As for the conditional expected market premium, note that:

$$\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\psi(X)\right]=\mathbb{E}_{\mathbb{Q}}[\psi(X)]=\int_{\mathbb{R}^N}\psi(x)f_X^{\mathbb{Q}}(x)dx=\int_{\mathbb{R}^N}\psi(x)\frac{f_X^{\mathbb{Q}}(x)}{f_X(x)}\,f_X(x)dx=\mathbb{E}\left[\phi(X)\psi(X)\right],$$

where $\phi(x):=f_X^{\mathbb{Q}}(x)/f_X(x)$ and $\psi(X)\in L_X^2$. Hence, $\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\mid X=x\right]=f_X^{\mathbb{Q}}(x)/f_X(x)$. Following a similar argument, $\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\mid I=i\right]=f_I^{\mathbb{Q}}(i)/f_I(i)$. By inserting these formulae into equation (7) and the related term $\Pi_0^X(X)$, the design equation (8) follows after straightforward calculations.

A design equation for the value-centered claim payoff function $G_0^*(i)$ results from the second equation in the first-order system (5). The corresponding expression is given by:

$$G_0^*(i) = \Pi_0^I(i) + \int_{\mathbb{R}^N} H_0^*(x) \frac{f_{XI}(x,i)}{f_I(i)} dx, \quad \left(i \in \mathbb{R}^N\right) \tag{EC.29}$$

with term:

$$\begin{aligned} \Pi_0^I(i) = &-\int_{\mathbb{R}^{2N}} \pi(x,y;q) \frac{f_{XYI}(x,y,i)}{f_I(i)}\ dxdy + \int_{\mathbb{R}^{2N}} \pi(x,y;q) \frac{f_{XY}(x,y) f_{XI}(x,i)}{f_X(x) f_I(i)}\ dxdy \\ &-\frac{\varrho}{a}\left(\frac{f_I^{\mathbb{Q}}(i)}{f_I(i)} - \int_{\mathbb{R}^N} f_X^{\mathbb{Q}}(x) \frac{f_{XI}(x,i)}{f_X(x) f_I(i)}\ di\right). \quad \left(x \in \mathbb{R}^N\right) \end{aligned} \tag{EC.30}$$

Q.E.D.

EC.3.4. Proof of Theorem 3

We premise to the proof the following

LEMMA EC.1. *Let $P \in L^2(\Omega)$ be such that $\mathbb{E}_{\mathbb{Q}}[P] = 0$. Then*

$$\mathscr{U}(\pi + P) - \mathscr{U}(\pi) = \frac{a}{2}\left(\mathbb{E}[\mathcal{K}_0^2] - \mathbb{E}\left[(P_0 + \mathcal{K}_0)^2\right]\right), \tag{EC.31}$$

where $P_0 := P - \mathbb{E}[P]$.

Proof. Let $P_0 := P - \mathbb{E}[P]$ be defined such that $P = P_0 - \mathbb{E}_{\mathbb{Q}}[P_0]$. Since $\mathscr{U}(Z + k) = \mathscr{U}(Z) + \varrho k$, then:

$$\mathscr{U}(\pi + P) = \mathscr{U}(\pi + P_0) - \varrho \mathbb{E}_{\mathbb{Q}}[P_0] = \mathscr{U}(\pi + P_0) - \varrho \mathbb{E}\left[P_0 \frac{d\mathbb{Q}}{d\mathbb{P}}\right]. \tag{EC.32}$$

Because $\mathscr{U}(Z + W) = \mathscr{U}(Z) + \mathscr{U}(W) - a\ \mathrm{cov}(Z, W)$, we see that:

$$\begin{aligned} \mathscr{U}(\pi + P_0) &= \mathscr{U}(\pi) + \mathscr{U}(P_0) - a\ \mathrm{cov}(\pi, P_0) \\ &= \mathscr{U}(\pi) - \tfrac{a}{2}\mathbb{E}\left[P_0^2\right] - a\mathbb{E}[\pi P_0]. \end{aligned}$$

By inserting this expression into (EC.32), we obtain:

$$\begin{aligned} \mathscr{U}(\pi + P) &= \mathscr{U}(\pi) - \tfrac{a}{2}\left(\mathbb{E}[P_0^2] + 2\mathbb{E}\left[P_0\left(\pi + \tfrac{\varrho}{a}\tfrac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]\right) \\ &= \mathscr{U}(\pi) - \tfrac{a}{2}\left(\mathbb{E}[P_0^2] + 2\mathbb{E}\left[P_0 \mathcal{K}_0\right]\right) \\ &= \mathscr{U}(\pi) - \tfrac{a}{2}\left(\mathbb{E}\left[(P_0 + \mathcal{K}_0)^2\right] - \mathbb{E}\left[\mathcal{K}_0^2\right]\right). \end{aligned}$$

The Lemma is proved. Q.E.D.

We are now ready to prove Theorem 3. Since $\mathcal{CV}_{X \to X+I} = \varrho^{-1}\left(\mathscr{U}^*_{\mathcal{H}_{XI}} - \mathscr{U}^*_{\mathcal{H}_X}\right)$, we may focus on the utility gap $\mathscr{U}^*_{\mathcal{H}_{XI}} - \mathscr{U}^*_{\mathcal{H}_X}$. By using (EC.31) and adopting L^2 notations, we see that:

$$\mathscr{U}^*_{\mathcal{H}_{XI}} - \mathscr{U}^*_{\mathcal{H}_X} = -\frac{a}{2}\left(\left\|H_0^*(X) + G_0^*(I) + \mathcal{K}_0\right\|_2^2 - \left\|H_0^\flat(X) + \mathcal{K}_0\right\|_2^2\right). \tag{EC.33}$$

We recall that $H_0^\flat(X) = -\mathbb{E}[\mathcal{K}_0|\mathscr{F}_X]$ and:

$$H_0^* + G_0^*(I) = H_0^\flat(X) + (G_0^*(I) - \mathbb{E}[G_0^*(I)|\mathscr{F}_X]) = -\mathbb{E}[\mathcal{K}_0|\mathscr{F}_X] + (G_0^*(I) - \mathbb{E}[G_0^*(I)|\mathscr{F}_X]). \tag{EC.34}$$

The utility gap (EC.33) thus reads as follows:

$$\begin{aligned}\mathscr{U}^*_{\mathcal{H}_{XI}} - \mathscr{U}^*_{\mathcal{H}_X} &= \tfrac{a}{2}\left(\|\mathcal{K}_0 - \mathbb{E}[\mathcal{K}_0|\mathscr{F}_X]\|_2^2 - \|(\mathcal{K}_0 - \mathbb{E}[\mathcal{K}_0|\mathscr{F}_X]) + (G_0^*(I) - \mathbb{E}[G_0^*(I)|\mathscr{F}_X]\|_2^2\right)\\ &= \tfrac{a}{2}\left(-2\langle \mathcal{K}_0 - \mathbb{E}[\mathcal{K}_0|\mathscr{F}_X], G_0^*(I) - \mathbb{E}[G_0^*(I)|\mathscr{F}_X]\rangle - \|G_0^*(I) - \mathbb{E}[G_0^*(I)|\mathscr{F}_X]\|_2^2\right).\end{aligned}$$

Note that $\mathcal{K}_0 - \mathbb{E}[\mathcal{K}_0|\mathscr{F}_X]$ and $\mathbb{E}[G_0^*(I)|\mathscr{F}_X]$ are orthogonal under the standard scalar product in $L^2(\Omega,\mathbb{P})$. By using standard properties of conditional expectation, we may develop:

$$\begin{aligned}\langle \mathcal{K}_0 - \mathbb{E}[\mathcal{K}_0|\mathscr{F}_X], G_0^*(I) - \mathbb{E}[G_0^*(I)|\mathscr{F}_X]\rangle &= \langle \mathcal{K}_0 - \mathbb{E}[\mathcal{K}_0|\mathscr{F}_X], G_0^*(I)\rangle\\ &= \langle \mathbb{E}[\mathcal{K}_0 - \mathbb{E}[\mathcal{K}_0|\mathscr{F}_X]|\mathscr{F}_I], G_0^*(I)\rangle\\ &= -\langle \Pi_0^I, G_0^*(I)\rangle,\end{aligned}$$

where $\Pi_0^I := -\mathbb{E}[\mathcal{K}_0 - \mathbb{E}[\mathcal{K}_0|\mathscr{F}_X]|\mathscr{F}_I]$. A fixed-point equation for $G_0^*(I)$ can easily be derived as:

$$G_0^*(I) = \Pi_0^I + \mathbb{E}[\mathbb{E}[G_0^*(I)|\mathscr{F}_X]|\mathscr{F}_I]. \tag{EC.35}$$

By using standard properties of conditional expectation once more, we compute:

$$\begin{aligned}\langle \Pi_0^I, G_0^*(I)\rangle &= \langle G_0^*(I) - \mathbb{E}[\mathbb{E}[G_0^*(I)\ |\mathscr{F}_X]\ |\ \mathscr{F}_I], G_0^*(I)\rangle && (\text{by } (EC.35))\\ &= \langle \mathbb{E}[(G_0^*(I) - \mathbb{E}[G_0^*(I)\ |\ \mathscr{F}_X])\ |\ \mathscr{F}_I], G_0^*(I)\rangle\\ &= \langle G_0^*(I) - \mathbb{E}[G_0^*(I)\ |\ \mathscr{F}_X], G_0^*(I)\rangle && (G_0^*(I) \in \mathscr{F}_I)\\ &= \langle G_0^*(I) - \mathbb{E}[G_0^*(I)\ |\ \mathscr{F}_X], G_0^*(I) - \mathbb{E}[G_0^*(I)\ |\mathscr{F}_X]\rangle\\ &= \|G_0^*(I) - \mathbb{E}[G_0^*(I)\ |\ \mathscr{F}_X]\|_2^2,\end{aligned}$$

where the equality before the last one stems from the fact that $\mathbb{E}[G_0^*(I)\ |\ \mathscr{F}_X] \perp G_0^*(I) - \mathbb{E}[G_0^*(I)\ |\ \mathscr{F}_X]$.

The utility gain finally reads as follows:

$$\mathscr{U}^*_{\mathcal{H}_{XI}} - \mathscr{U}^*_{\mathcal{H}_X} = \frac{a}{2}\|G_0^*(I) - \mathbb{E}[G_0^*(I)\ |\ \mathscr{F}_X]\|_2^2. \tag{EC.36}$$

The adoption of a probabilistic notation whereby $\mathrm{Var}[\#] \equiv \|\#\|_2^2$ for any $\mathbb{P}$-centered argument and a conversion of utility gain to the combination value $\mathcal{CV}_{X\to X+I}$ leads to formula (12).

To prove inequality (13), we need only develop the right-hand side in expression (EC.36). By using the very definition of maximal correlation r_{XI} and the elementary inequality $2ab \leqslant a^2 + b^2$, we have

$$\begin{aligned}\|G_0^*(I) - \mathbb{E}[G_0^*(I) \mid \mathscr{F}_X]\|_2^2 &= \|G_0^*(I)\|_2^2 - 2\langle G_0^*(I), \mathbb{E}[G_0^*(I) \mid \mathscr{F}_X]\rangle + \|\mathbb{E}[G_0^*(I) \mid \mathscr{F}_X]\|_2^2 \\ &\geqslant \|G_0^*(I)\|_2^2 - 2r_{XI}\|G_0^*(I)\|_2\|\mathbb{E}[G_0^*(I) \mid \mathscr{F}_X]\|_2 + \|\mathbb{E}[G_0^*(I) \mid \mathscr{F}_X]\|_2^2 \\ &\geqslant \|G_0^*(I)\|_2^2 - r_{XI}^2\|G_0^*(I)\|_2^2 - \|\mathbb{E}[G_0^*(I) \mid \mathscr{F}_X]\|_2^2 + \|\mathbb{E}[G_0^*(I) \mid \mathscr{F}_X]\|_2^2 \\ &= (1 - r_{XI}^2)\|G_0^*(I)\|^2.\end{aligned} \tag{EC.37}$$

Formula (EC.35) leads to:

$$\|\Pi_0^I\|_2 = \|G_0^*(I) - \mathbb{E}[\mathbb{E}[G_0^*(I) \mid \mathscr{F}_X] \mid \mathscr{F}_I]\|_2 \leqslant \|G_0^*(I)\|_2. \tag{EC.38}$$

By combining (EC.37) and (EC.38), we obtain:

$$\|G_0^*(I) - \mathbb{E}[G_0^*(I) \mid \mathscr{F}_X]\|_2^2 \geqslant (1 - r_{XI}^2)\|\Pi_0^I\|_2^2, \tag{EC.39}$$

which we insert into (EC.36) and arrive at the desired lower bound:

$$\mathscr{U}_{\mathcal{H}_{XI}}^* - \mathscr{U}_{\mathcal{H}_X}^* \geqslant \frac{a}{2}(1 - r_{XI}^2)\|\Pi_0^I\|_2^2$$

after converting utility gains into the combination value $\mathcal{CV}_{X \to X+I}$. Q.E.D.

EC.3.5. Proof of Proposition 2

According to inequality (13), we must estimate:

$$\mathrm{Var}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}[(\mathcal{K}_0 - \mathbb{E}_{\mathbb{P}}[\mathcal{K}_0 \mid \mathscr{F}_X]) \mid \mathscr{F}_I] = \|\mathbb{E}_{\mathbb{P}}[(\mathcal{K}_0 - \mathbb{E}_{\mathbb{P}}[\mathcal{K}_0 \mid \mathscr{F}_X]) \mid \mathscr{F}_I]\|_2^2.$$

This is relatively straightforward: since $\mathcal{K}_0 = \pi + \frac{\varrho}{a}\frac{d\mathbb{Q}}{d\mathbb{P}}$ and $\frac{\varrho}{a}$ is small, the main contribution to the L^2-norm originates in naked position revenues π. More precisely, by using the triangular inequality,

$$\begin{aligned}&\|\mathbb{E}[(\mathcal{K}_0 - \mathbb{E}[\mathcal{K}_0 \mid \mathscr{F}_X]) \mid \mathscr{F}_I]\|_2 \\ &\quad \geqslant \|\mathbb{E}[\pi - \mathbb{E}[\pi \mid \mathscr{F}_X] \mid \mathscr{F}_I]\|_2 - \left|\tfrac{\varrho}{a}\right| \left\|\mathbb{E}\left[\tfrac{d\mathbb{Q}}{d\mathbb{P}} - \mathbb{E}\left[\tfrac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathscr{F}_X\right] \mid \mathscr{F}_I\right]\right\|_2 \\ &\quad \geqslant \|\mathbb{E}[\pi - \mathbb{E}[\pi \mid \mathscr{F}_X] \mid \mathscr{F}_I]\|_2 - \left|\tfrac{\varrho}{a}\right| \left\|\tfrac{d\mathbb{Q}}{d\mathbb{P}}\right\|_2.\end{aligned}$$

Assumption (14) leads to:

$$\|\mathbb{E}[(\mathcal{K}_0 - \mathbb{E}[\mathcal{K}_0 \mid \mathscr{F}_X]) \mid \mathscr{F}_I]\|_2 \geqslant \|\mathbb{E}[\pi - \mathbb{E}[\pi \mid \mathscr{F}_X] \mid \mathscr{F}_I]\|_2 - \varepsilon^{1/2}\mathrm{Var}[\pi]^{1/2}. \tag{EC.40}$$

By using the separable form of π, the first summand in the right-hand side of this expression can be computed as follows:

$$\pi-\mathbb{E}[\pi \mid \mathscr{F}_X]=\alpha(X)\beta(Y)-\mathbb{E}[\alpha(X)\beta(Y) \mid \mathscr{F}_X]=\alpha(X)\beta(Y)-\alpha(X)\mathbb{E}[\beta(Y)]$$
$$\equiv \alpha(X)\left(\beta(Y)-m\right)=:\alpha(X)\beta_0(X),$$

where $m:=\mathbb{E}[\beta(Y)]$. By taking the conditional expectation with respect to $\mathscr{F}_I$ and denoting the orthogonal projection on $(L^2_Y)^\perp$ by $\mathbb{E}[\cdot\ |\mathscr{F}_Y^\perp]$, we obtain:

$$\mathbb{E}\left[\pi-\mathbb{E}[\pi \mid \mathscr{F}_X] \mid \mathscr{F}_I\right]=\mathbb{E}\left[\mathbb{E}\left[\alpha(X)\beta_0(Y) \mid \mathscr{F}_Y\right] \mid \mathscr{F}_I\right]+\mathbb{E}\left[\mathbb{E}\left[\alpha(X)\beta_0(Y) \mid \mathscr{F}_Y^\perp\right] \mid \mathscr{F}_I\right]$$
$$=\mathbb{E}[\alpha(X)]\mathbb{E}[\beta_0(Y) \mid \mathscr{F}_I]+\mathbb{E}\left[\mathbb{E}\left[\alpha(X)\beta_0(Y) \mid \mathscr{F}_Y^\perp\right] \mid \mathscr{F}_I\right]$$
$$=\mathbb{E}[\alpha(X)]\beta_0(Y)-\mathbb{E}[\alpha(X)]\mathbb{E}[\beta_0(Y) \mid \mathscr{F}_I^\perp]+\mathbb{E}\left[\mathbb{E}\left[\alpha(X)\beta_0(Y) \mid \mathscr{F}_Y^\perp\right] \mid \mathscr{F}_I\right].$$

The triangular inequality leads to:

$$\begin{aligned}\left\|\mathbb{E}\left[(\pi-\mathbb{E}[\pi \mid \mathscr{F}_X]) \mid \mathscr{F}_I\right]\right\|_2 &\geqslant \|\mathbb{E}[\alpha(X)]\beta_0(Y)\|_2\\ &-\left(\|\mathbb{E}[\alpha(X)]\mathbb{E}[\beta_0(Y) \mid \mathscr{F}_I^\perp]\|_2+\left\|\mathbb{E}\left[\mathbb{E}\left[\alpha(X)\beta_0(Y) \mid \mathscr{F}_Y^\perp\right] \mid \mathscr{F}_I\right]\right\|_2\right).\end{aligned} \tag{EC.41}$$

Clearly,

$$\|\mathbb{E}[\alpha(X)]\beta_0(Y)\|_2=|\mathbb{E}[\alpha(X)]|\mathbb{E}\left[(\beta(Y)-\mathbb{E}[\beta(Y)])^2\right]^{1/2}=|m_{\alpha(X)}|\sigma_{\beta(Y)},$$

while:

$$\|\mathbb{E}[\alpha(X)]\mathbb{E}[\beta_0(Y) \mid \mathscr{F}_I^\perp]\|_2=|m_{\alpha(X)}|\|\mathbb{E}[\beta_0(Y) \mid \mathscr{F}_I^\perp]\|_2\leqslant|m_{\alpha(X)}|\sigma_{\beta(Y)}\sqrt{1-r_{YI}^2}.$$

As long as X is independent of Y, $\alpha(X)$ is $\mathscr{F}_Y^\perp$ measurable, hence:

$$\mathbb{E}\left[\alpha(X)\beta_0(Y) \mid \mathscr{F}_Y^\perp\right]=\alpha(X)\mathbb{E}\left[\beta_0(Y) \mid \mathscr{F}_Y^\perp\right]=\alpha(X)\mathbb{E}[\beta_0(Y)]=0.$$

By inserting these estimates into expression (EC.41), we arrive at:

$$\|\mathbb{E}_{\mathbb{P}}[\pi-\mathbb{E}_{\mathbb{P}}[\pi \mid \mathscr{F}_X]\ \mathscr{F}_I]\|_2\geqslant|m_{\alpha(X)}|\sigma_{\beta(Y)}\left(1-\sqrt{1-r_{YI}^2}\right).$$

Finally, by using (EC.40), letting $\varepsilon\to 0+$, and switching to combination value, we obtain the approximate estimate (15). Q.E.D.

EC.3.6. Proof of Proposition 3

Since $H_0^\flat(X)=-\mathbb{E}[\mathcal{K}_0\ |\mathscr{F}_X]$, the utility gain (EC.31) and the Pythagorean theorem yield:

$$\mathscr{U}^*_{\mathcal{H}_X}-\mathscr{U}_0=\frac{a}{2}\left(\|\mathcal{K}_0\|_2^2-\|\mathcal{K}_0-\mathbb{E}[\mathcal{K}_0 \mid \mathscr{F}_X]\|_2^2\right)=\frac{a}{2}\|\mathbb{E}[\mathcal{K}_0 \mid \mathscr{F}_X]\|_2^2. \tag{EC.42}$$

According to Theorem (3), the utility gain:

$$\begin{aligned}\mathscr{U}^*_{\mathcal{H}_{XI}}-\mathscr{U}_0&=\left(\mathscr{U}^*_{\mathcal{H}_{XI}}-\mathscr{U}^*_{\mathcal{H}_X}\right)+\left(\mathscr{U}^*_{\mathcal{H}_X}-\mathscr{U}_0\right)\\ &\overset{\text{(EC.36)}}{=}\tfrac{a}{2}\|G_0^*(I)-\mathbb{E}[G_0^*(I) \mid \mathscr{F}_X]\|_2^2+\tfrac{a}{2}\|\mathbb{E}[\mathcal{K}_0 \mid \mathscr{F}_X]\|_2^2.\end{aligned} \tag{EC.43}$$

By inserting (EC.42) and (EC.43) into the very definition of efficiency (formula (16)), closed-form expression (17) follows. Note that:

$$\mathcal{E}_X = 1, \quad \Leftrightarrow \quad \|G_0^*(I) - \mathbb{E}[G_0^*(I) \mid \mathscr{F}_X]\|_2^2 = 0, \quad \Leftrightarrow \quad G_0^*(I) = \mathbb{E}[G_0^*(I) \mid \mathscr{F}_X] \text{ a.s.-}\mathbb{P}.$$

This leads to the following alternative. If $G_0^*(I)$ is a constant, then $G_0^*(I) \equiv 0$ as long as $\mathbb{E}[G_0^*(I)] = 0$. If $G_0^*(I)$ is varying, then identity $G_0^*(I) = \mathbb{E}[G_0^*(I) \mid \mathscr{F}_X]$ implies that $G_0^*(I)$ is a non-zero r.v. that is measurable with respect to both $\mathscr{F}_I$ and $\mathscr{F}_X$. Hence, $r_{XI} = 1$. Q.E.D.

EC.3.7. Proof of Proposition EC.1

We show that the optimization problem in question may equivalently be cast as one over the space of $\mathbb{P}$-centered claims, which allows us to invoke a projection argument to derive an explicit solution. Indeed, to each claim $P \in \mathscr{P} : \mathbb{E}_{\mathbb{Q}}[P] = 0$, we associate $\mathbb{P}$-centered claim $P_0 := P - \mathbb{E}[P]$. Conversely, any $\mathbb{P}$-centered claim P_0 induces a unique $P \in \mathscr{P}$ such that $\mathbb{E}_{\mathbb{Q}}[P] = 0$, i.e.,

$$P := P_0 - \mathbb{E}_{\mathbb{Q}}[P_0]. \tag{EC.44}$$

Clearly,

$$\text{Var}[\pi + P] = \text{Var}[\pi + P_0] = \mathbb{E}\left[(\pi + P_0)^2 - \mathbb{E}[\pi]^2\right].$$

Therefore:

$$\min_{P \in \mathscr{P}:\ \mathbb{E}_{\mathbb{Q}}[P]=0} \text{Var}[\pi + P] = \left(\min_{P_0 \in \mathscr{P}\ :\ \mathbb{E}[P_0]=0} \mathbb{E}\left[(P_0 - (-\pi))^2\right]\right) - \mathbb{E}[\pi]^2 \to P_0^*.$$

That is seeking the zero $\mathbb{P}$-mean element in $\mathscr{P}$ which is the nearest to $-\pi$ in the sense of $L^2(\Omega, \mathscr{F}, \mathbb{P})$-distance. Condition $\mathbb{E}[P_0] = 0$ may be written as $\langle P_0, 1\rangle = 0$ and space $\mathscr{P}$ is closed by assumption. Then, P_0^* is the component in the orthogonal projection of $-\pi$ onto $\mathscr{P}$ that is orthogonal to 1, namely:

$$P_0^* = \text{Proj}[-\pi]_{\mathscr{P}} - \langle \text{Proj}_{\mathscr{P}}[-\pi], 1\rangle 1.$$

Orthogonal projection is self-adjoint and $1 \in \mathscr{P}$ by assumption, hence $\langle\langle \text{Proj}_{\mathscr{P}}[-\pi], 1\rangle = -\langle \pi, \text{Proj}_{\mathscr{P}} 1\rangle = -\langle \pi, 1\rangle = -\mathbb{E}[\pi]$. Therefore:

$$P_0^* = -\text{Proj}_{\mathscr{P}}\pi + \mathbb{E}[\pi]. \tag{EC.45}$$

A simple computation shows that:

$$\mathbb{E}_{\mathbb{Q}}[P_0^*] = \mathbb{E}\left[P_0^* \frac{d\mathbb{Q}}{d\mathbb{P}}\right] = -\left\langle \pi, \text{Proj}_{\mathscr{P}} \frac{d\mathbb{Q}}{d\mathbb{P}}\right\rangle + \mathbb{E}[\pi]. \tag{EC.46}$$

By replacing expressions (EC.45) and (EC.46) into (EC.44), we obtain the desired result (EC.6). Q.E.D.

EC.3.8. Proof of Proposition EC.2

Optimal hedges are orthogonal projections on the subspaces of X-measurable and I-measurable r.v.'s. As long as X admits only two values, any contingent claim written on X only exhibits two values as well. In particular, the optimal custom hedge on X may be written as $H^\flat(X) = h^+ \mathbf{1}_{X=x^+} + h^- \mathbf{1}_{X=x^-}$ for suitable constants h^+ and h^-. Here, $\mathbf{1}$ denotes the usual indicator function. Because the space of X measurable r.v.'s considered here is finite-dimensional, we may identify an orthonormal basis to compute the required orthogonal projections. Since $\mathbb{P} = \mathbb{Q}$, the assumed balance conditions yields $\mathbb{E}[H^\flat(X)] = \mathbb{E}_{\mathbb{Q}}[H^\flat(X)] = 0$. By inserting the previous expression for $H^\flat(X)$ into this equality, we easily obtain $h^- = -\frac{p_X}{1-p_X} h^+$, hence:

$$H^\flat(X) = h^+ \left(\mathbf{1}_{X=x^+} - \frac{p_X}{1-p_X} \mathbf{1}_{X=x^-} \right) = h^+ \frac{\mathbf{1}_{X=x^+} - p_X}{1-p_X}.$$

Note that:

$$\mathbb{E}[\mathbf{1}_{X=x^+} - p_X] = 0, \quad \mathbb{E}[(\mathbf{1}_{X=x^+} - p_X)^2] = p_X(1-p_X)^2 + (1-p_X)p_X^2 = p_X(1-p_X).$$

We may thus write $H^\flat(X) = \alpha U$, where:

$$U := \frac{\mathbf{1}_{X=x^+} - p_X}{\sqrt{p_X(1-p_X)}}$$

is a unit vector. Therefore, $H^\flat(X)$ lies in the linear span $\mathscr{S}_X := \mathrm{Span}(U)$ of versor U. However, we know that $H^\flat(X) = -\mathbb{E}[\pi | X] = -\langle \pi, U \rangle U$. Hence,

$$\begin{aligned} \alpha &= -\langle \pi, U \rangle = -\frac{1}{\sqrt{p_X(1-p_X)}} \mathbb{E}[\pi(X,Y;q)(\mathbf{1}_{X=x^+} - p_X)] \\ &= -\frac{1}{\sqrt{p_X(1-p_X)}} [\pi(x^+,y^+;q) p_{XY}(1-p_X) + \pi(x^+,y^-;q)(p_X - p_{XY})(1-p_X) \\ &\quad - \pi(x^-,y^+;q) p_X(p_Y - p_{XY}) - \pi(x^-,y^-;q) p_X(1-p_X-p_Y+p_{XY})], \end{aligned} \tag{EC.47}$$

where $p_{XY} := \mathbb{P}(X = x^+,\ I = i^+)$. Similarly, we can show the optimal single-claim hedge on index I may be expressed as:

$$G^\flat(I) = \beta V,$$

where β is a constant and:

$$V = \frac{\mathbf{1}_{I=i^+} - p_I}{\sqrt{p_I(1-p_I)}}$$

is a unit vector. Therefore $G^\flat(I)$ lies in $\mathscr{S}_I := \mathrm{Span}(V)$ and:

$$\begin{aligned} \beta &= -\langle \pi, V \rangle V = -\frac{1}{\sqrt{p_I(1-p_I)}} [\pi(x^+,y^+;q) p_{XYI} + \pi(x^+,y^-;q)(p_{XI} - p_{XYI}) \\ &\quad + \pi(x^-,y^+;q)(p_{YI} - p_{XYI}) + \pi(x^-,y^-;q)(p_I - p_{XI} - p_{YI} + p_{XYI}) \\ &\quad - \pi(x^+,y^+;q) p_X(p_Y - p_{XY}) - \pi(x^-,y^-;q) p_X(1-p_X-p_Y+p_{XY})] \end{aligned} \tag{EC.48}$$

with $p_{XI} := \mathbb{P}(X = x^+, I = i^+)$ and $p_{YI} := \mathbb{P}(Y = y^+, I = i^+)$. These expressions prove statement (EC.7).

Notice that U and V are linearly dependent if and only if

$$\exists \lambda \ : \ 1_{X=x^+} - p_X = \lambda(1_{I=i^+} - p_I). \tag{EC.49}$$

It is easy to verify that either $\{X = x^+\} = \{I = i^+\}$ or $\{X = x^-\} = \{I = i^+\}$. Let us focus on the first of the two. Condition (EC.49) amounts to having:

$$\begin{cases} 1 - p_X = \lambda(1 - p_I), \\ -p_X = -\lambda p_I; \end{cases}$$

whereby $\lambda = 1$ and $p_X = p_I$. If we define $f(i^+) := x^+$ and $f(i^-) := x^-$, we have $X = f(I)$. We have shown that X is a function of I (and vice versa) provided that U and V are linearly dependent. Let us assume that $X = f(I)$ for some bijection f . (The same argument applies whenever $I = g(X)$). We see that:

$$\{X = x^+\} = \{f(I) = x^+\} = \{I = f^{-1}(x^+)\}.$$

Since X and I are Bernoulli variables, then $f^{-1}(x^+)$ must match either i^+ or i^-, whereas $f^{-1}(x^-)$ assumes the other of the two values. For instance, suppose $f^{-1}(x^+) = i^+$. Then $\{X = x^+\} = \{I = i^*\}$ and $p_X = \mathbb{P}(X = x^+) = \mathbb{P}(I = i^+) = p_I$. This shows that $U = V$. We conclude that vectors U and V are linearly independent as long as X and I are not functionally dependent upon one another.

The naively combined custom hedge is just the sum of the two optimal single-claim hedges:

$$H^\flat (X) + G^\flat (I) = - \langle \pi, U \rangle U - \langle \pi, V \rangle V = \alpha U + \beta V. \tag{EC.50}$$

This hedge lies in the sum space $\mathscr{S}_X \oplus \mathscr{S}_I$ and entails variance $\mathscr{V}_{X+I} := \text{Var}\left[\pi + H^\flat (X) + G^\flat (I)\right]$.

Our optimal combined custom hedge is defined as the element $H^* (X) + G^* (I)$ in $\mathscr{S}_X \oplus \mathscr{S}_I$ attaining the minimum variance $\mathscr{V}^*_{\mathcal{H}_{XI}} := \min_{H \in \mathscr{S}_X, G \in \mathscr{S}_I} \text{Var}\left[\pi + H(X) + G(I)\right]$. That is the projection of naked position revenues π over the sum space $\mathscr{S}_X \oplus \mathscr{S}_I$. This quantity matches the sum of projections on space $\mathscr{S}_X$ and space $\mathscr{S}_I$, namely expression (EC.50), provided that they form an orthogonal basis for $\mathscr{S}_X \oplus \mathscr{S}_I$, i.e., $\langle U, V \rangle = 0$. Simple computations show that $\rho := \langle U, V \rangle = const \times (p_{XI} - p_X p_I)$. According to the Cauchy–Schwarz inequality, $\rho^2 \leqslant 1$ and $\rho^2 = 1$ if and only if $U = \pm V$, i.e., provided that X is function of I and vice versa. Here, we have $\rho^2 < 1$. Moreover, $\langle U, V \rangle = 0$ if and only if $p_{XI} = p_X p_I$, that is X and I are independent r.v.'s. In all other cases, optimal and naively combined hedges differ. To compute our optimal combined custom hedge, we need to project revenues π on an orthonormal basis for $\mathscr{S}_X \oplus \mathscr{S}_I$. If X is not statistically independent of I, pair (U, V) cannot be an orthonormal basis. We may thus perform a standard orthogonalization procedure leading to a basis (U, V^*), where:

$$V^* := \frac{V - \langle V, U \rangle U}{\| V - \langle V, U \rangle U \|} = \frac{V - \rho U}{\sqrt{1 - \rho^2}}.$$

Hence:

$$H^*(X)+G^*(I)=-\langle\pi,U\rangle U-\langle\pi,V^*\rangle V^*=\alpha U+\beta^*V^*,$$

with:

$$\beta^*=-\langle\pi,V^*\rangle=-\frac{1}{\sqrt{1-\rho^2}}\left(\langle\pi,V\rangle-\rho\langle\pi,U\rangle\right)=\frac{\beta-\rho\alpha}{\sqrt{1-\rho^2}},$$

which proves expression (EC.8). Q.E.D.

EC.3.9. Proof of Example EC.5

Naked operating revenues variance is easily computed as follows:

$$\mathscr{V}_0^*(q)=\mathrm{Var}[XY]\left(\frac{1}{q}-\frac{C(Y,XY)}{\mathrm{Var}[XY]}\right)^2+\left(\mathrm{Var}[Y]-\frac{C(Y,XY)^2}{\mathrm{Var}[XY]}\right),$$

where C denotes the covariance operator. Note that $C(Y,XY)^2=q^2\rho_{Y,XY}\mathrm{Var}[Y]^{1/2}\mathrm{Var}[XY]^{1/2}>0$ as long as $\rho_{Y,XY}>0$ by assumption. This quantity thus attains a minimum value:

$$\mathscr{V}_0^*=\mathrm{Var}[Y]-\frac{C(Y,XY)^2}{\mathrm{Var}[XY]}$$

at point:

$$q_0^*=\frac{\mathrm{Var}[XY]}{C\left(Y,XY\right)}.$$

The optimal variance under optimal single-claim custom hedging is given by:

$$\mathscr{V}_{\mathcal{H}_X}^*(q)=\mathrm{Var}[\pi(q)]-\alpha(q)^2,$$

where $\alpha(q)=-\langle\pi,U\rangle=-\left(\langle Y,U\rangle-q^{-1}\langle XY,U\rangle\right)$. By inserting $\mathscr{V}_0^*(q)$ into this expression, we obtain:

$$\begin{aligned}\mathscr{V}_{\mathcal{H}_X}^*(q)&=\left(\mathrm{Var}[XY]-\langle XY,U\rangle^2\right)\left(\tfrac{1}{q}-\tfrac{C(Y,XY)-\langle Y,U\rangle\langle XY,U\rangle}{\mathrm{Var}[XY]-\langle XY,U\rangle^2}\right)^2\\&\quad+\left(\mathrm{Var}[XY]-\langle Y,U\rangle^2\right)-\tfrac{\left(C(Y,XY)-\langle Y,U\rangle\langle XY,U\rangle\right)^2}{\mathrm{Var}[XY]-\langle XY,U\rangle^2}\\&=:a_X\left(\tfrac{1}{q}-\tfrac{1}{q_{\mathcal{H}_X}^*}\right)^2+b_X,\end{aligned}$$

where a_X, b_X, and $q_{\mathcal{H}_X}^*$ are defined accordingly. Clearly, $q_{\mathcal{H}_X}^*$ is the corresponding point of minimum.

The operational flexibility gain yielded by optimal single-claim custom hedging on X is given by the range of operations levels q such that:

$$\left|q^{-1}-q_{\mathcal{H}_X}^{*-1}\right|\leqslant\sqrt{a_X^{-1}\left(\mathscr{V}_0^*-b_X\right)},$$

namely:

$$\mathcal{I}_{q_{\mathcal{H}_X}^*}=\begin{cases}\left[\left(\frac{1}{q_{\mathcal{H}_X}^*}+\sqrt{\frac{\mathscr{V}_0^*-b_X}{a_X}}\right)^{-1},\left(\frac{1}{q_{\mathcal{H}_X}^*}-\sqrt{\frac{\mathscr{V}_0^*-b_X}{a_X}}\right)^{-1}\right], & \text{if } \frac{1}{q_{\mathcal{H}_X}^*}-\sqrt{\frac{\mathscr{V}_0^*-b_X}{a_X}}>0,\\[2ex]\left[\left(\frac{1}{q_{\mathcal{H}_X}^*}+\sqrt{\frac{\mathscr{V}_0^*-b_X}{a_X}}\right)^{-1},+\infty\right[, & \text{if } \frac{1}{q_{\mathcal{H}_X}^*}-\frac{\mathscr{V}_0^*-b_X}{a_X}<0.\end{cases}\tag{EC.51}$$

Similarly, the optimal variance under optimal combined custom hedging is:

$$\mathscr{V}^*_{\mathcal{H}_{XI}}(q) = \mathrm{Var}[\pi(q)] - \left[\alpha(q)^2 + \beta(q)^2\right],$$

where $\beta(q) = -\langle \pi, V\rangle = -\left(\langle Y, V\rangle - q^{-1}\langle XY, V\rangle\right)$. By inserting $\mathscr{V}_0^*(q)$ into this expression, we obtain:

$$\mathscr{V}^*_{\mathcal{H}_{XI}}(q) = a_{XI}\left(q^{-1} - q^{*-1}_{\mathcal{H}_{XI}}\right)^2 + b_{XI},$$

where:

$$\begin{aligned}
a_{XI} &:= a_X - \langle Y, V\rangle^2 < a_X, \\
b_{XI} &:= \left(\mathrm{Var}[XY] - \langle Y, U\rangle^2 - \langle Y, V\rangle^2\right) \\
&\quad - \frac{\left(C(Y, XY) - \langle Y, U\rangle\langle XY, U\rangle - \langle Y, V\rangle\langle XY, V\rangle\right)^2}{\mathrm{Var}[XY] - \langle XY, U\rangle^2 - \langle XY, V\rangle^2} < b_X, \\
q^{*-1}_{\mathcal{H}_{XI}} &:= \frac{C(Y, XY) - \langle Y, U\rangle\langle XY, U\rangle - \langle Y, V\rangle\langle XY, V\rangle}{\mathrm{Var}[XY] - \langle XY, U\rangle^2 - \langle VY, V\rangle^2}.
\end{aligned}$$

The operational flexibility gain yielded by optimal combined custom hedging on X and I is given by the set of operations levels q such that:

$$\left|q^{-1} - q^{*-1}_{\mathcal{H}_{XI}}\right| \leqslant \sqrt{a_{XI}^{-1}\left(\mathscr{V}_0^* - b_{XI}\right)},$$

namely:

$$\mathcal{I}_{q^*_{\mathcal{H}_{XI}}} = \begin{cases} \left[\left(\frac{1}{q^*_{\mathcal{H}_{XI}}} + \sqrt{\frac{\mathscr{V}_0^* - b_{XI}}{a_{XI}}}\right)^{-1}, \left(\frac{1}{q^*_{\mathcal{H}_{XI}}} - \sqrt{\frac{\mathscr{V}_0^* - b_{XI}}{a_{XI}}}\right)^{-1}\right], & \text{if } \frac{1}{q^*_{\mathcal{H}_{XI}}} - \sqrt{\frac{\mathscr{V}_0^* - b_{XI}}{a_{XI}}} > 0, \\ \left[\left(\frac{1}{q^*_{\mathcal{H}_{XI}}} + \sqrt{\frac{\mathscr{V}_0^* - b_{XI}}{a_{XI}}}\right)^{-1}, +\infty\right[, & \text{if } \frac{1}{q^*_{\mathcal{H}_{XI}}} - \frac{\mathscr{V}_0^* - b_{XI}}{a_{XI}} < 0. \end{cases}$$

EC.3.10. Proof of Example EC.10

The pair (X, I) defines a complete market where claims written on Y need not be replicable.

Model 1: State vector $(X(T), I(T))$ is jointly lognormal, i.e.,

$$\log\left(\frac{X(T)}{X(0)}, \frac{I(T)}{I(0)}\right) \sim \mathscr{N}(\vec{m}T, CT),$$

with:

$$\vec{m} = \begin{pmatrix} \alpha - \frac{\|\sigma^X\|^2}{2} \\ \beta - \frac{\|\sigma^I\|^2}{2} \end{pmatrix}, \quad C = \begin{pmatrix} \|\sigma^X\|^2 & \sigma^X \cdot \sigma^I \\ \sigma^X \cdot \sigma^I & \|\sigma^I\|^2 \end{pmatrix}.$$

According to property 5 reported in Appendix EC.2(C), $r_{XI} = \left|\rho_{\log X(T), \log I(T)}\right|$. Thus:

$$r_{XI} = 1 \Leftrightarrow \sigma^X \cdot \sigma^I = \pm\|\sigma^X\|\|\sigma^I\|.$$

By using the Cauchy–Schwarz inequality, the condition on the right hand side holds true provided that $\sigma^X \propto \sigma^I$, namely σ^X and σ^I are linearly dependent. This statement violates our working assumption. Hence, $r_{XI} < 1$.

Model 2: The stochastic differential equations for $X(t)$ and $I(t)$ admit solutions:

$$\frac{X(t)}{X(0)} = \exp\left(\left(\alpha - \frac{\|\sigma^X\|^2}{2}\right)t + \sigma^X \cdot W(t)\right),$$

$$\frac{I(t)}{I(0)} = \exp\left(\left(\beta - \frac{\|\sigma^I\|^2}{2}\right)t + \sigma^I \cdot W(t)\right).$$

These expressions entail the following representation:

$$\begin{aligned}(X,Y) &:= (X(t_1),\ldots,X(t_N), I(t_1),\ldots,I(t_N)) \\ &= \left(\Phi\left(\sigma^X \cdot W(t_1),\ldots,\sigma^X \cdot W(t_N)\right), \Psi\left(\sigma^I \cdot W(t_1),\ldots,\sigma^I \cdot W(t_N)\right)\right),\end{aligned}$$

for suitable functions Φ and Ψ. Maximal correlation can easily be computed by using property 4 reported in Appendix EC.2(C):

$$r_{XI} = r_{\left(\sigma^X \cdot W(t_1),\ldots,\sigma^X \cdot W(t_N)\right),\left(\sigma^I \cdot W(t_1),\ldots,\sigma^I \cdot W(t_N)\right)}.$$

Clearly, the right-hand side in the previous expression is the maximal correlation of a multivariate Gaussian variate. By using property 6 reported in Appendix EC.2(C), we see that:

$$r_{XI} = 1 \Leftrightarrow \exists a, b \neq 0 : \sum_{j=1}^{N} a_j \sigma^X \cdot W(t_j) = \sum_{j=1}^{N} b_j \sigma^I \cdot W(t_j). \tag{EC.52}$$

We claim that the condition on the right hand side violates our model assumptions. Indeed, we note that:

$$\sum_{j=1}^{N} a_j \sigma^X \cdot W(t_j) = \sum_{j=1}^{N} a_j \sigma^X \cdot (W(t_j) - W(t_1)) + \left(\sum_{j=1}^{N} a_j\right) \sigma^X \cdot W(t_1).$$

A similar condition holds for $\sum_{j=1}^{N} b_j \sigma^I \cdot W(t_j)$. By conditioning with respect to $\mathscr{F}_{t_1}$, we get to:

$$\left(\sum_{j=1}^{N} a_j\right) \sigma^X \cdot W(t_1) = \left(\sum_{j=1}^{N} b_j\right) \sigma^I \cdot W(t_1),$$

namely:

$$\left|\rho_{\sigma^X \cdot W(t_1), \sigma^I \cdot W(t_1)}\right| = 1 \Leftrightarrow \sigma^X \propto \sigma^I,$$

which contradicts our assumption of linear independent coefficients σ^X and σ^I. Consequently,

$$\sum_{j=1}^{N} a_j = \sum_{j=1}^{N} b_j = 0$$

and

$$\sum_{j=2}^{N} a_j \sigma^X \cdot \widetilde{W}(t_j) = \sum_{j=2}^{N} b_j \sigma^I \cdot \widetilde{W}(t_j),$$

where $\widetilde{W}(t) := W(t) - W(t_1)$ is defined for all $t \geqslant t_1$. Because process $\widetilde{W}$ conditioned to $W(t_1)$ is a Brownian motion starting at $t = t_1$, condition (EC.52) holds true for index $N-1$ replacing N. By following the same argument, we see that:

$$\sum_{j=2}^{N} a_j = \sum_{j=2}^{N} b_j = 0,$$

and, by iteration, we may conclude that:

$$\sum_{j=k}^{N} a_j = \sum_{j=k}^{N} b_j = 0, \ \forall k = 1, \ldots, N.$$

These conditions lead to $a_j = b_j = 0$ for all $j = 1, \ldots, N$. This result clearly contradicts the condition on the right hand side in formula (EC.52). We thus conclude that $r_{XI} < 1$. Q.E.D.

EC.3.11. Proof of Proposition EC.3

To proceed in the proof we need an auxiliary

LEMMA EC.2 **(Maximal correlation for separable integral kernels)**. *If the integral kernel k is separable, then $r_{XI} = \min\{1, \|\alpha\|_{L^2_X} \|\beta\|_{L^2_I}\}$.*

Proof. Let ϕ, ψ be Borel measurable functions satisfying:

$$\mathbb{E}[\phi(X)] = \int_{\mathbb{R}^N} \phi(x) f_X(x)\ dx = 0, \quad \mathbb{E}[\psi(I)] = \int_{\mathbb{R}^N} \psi(i) f_I(i)\ di = 0,$$

$$\mathbb{E}[\phi(X)^2] = \int_{\mathbb{R}^N} \phi(x)^2 f_X(x)\ dx = 1, \quad \mathbb{E}[\psi(I)^2] = \int_{\mathbb{R}^N} \psi(i)^2 f_I(i)\ di = 1.$$

Then:

$$\begin{aligned}
\mathbb{E}[\phi(X)\psi(I)] &= \int_{\mathbb{R}^N \times \mathbb{R}^N} \phi(x)\psi(i) f_{XI}(x,i)\ dxdi \\
&= \int_{\mathbb{R}^N \times \mathbb{R}^N} \phi(x)\psi(i) f_X(x) f_I(i) \left(1 + \alpha(x)\beta(i)\right)\ dxdi \\
&= \left(\int_{\mathbb{R}^N} \phi(x)\alpha(x) f_X(x)\ dx\right) \left(\int_{\mathbb{R}^N} \psi(i)\beta(i) f_I(i)\ di\right).
\end{aligned}$$

The Cauchy–Schwarz inequality shows that:

$$\left|\int_{\mathbb{R}^N} \phi(x)\alpha(x) f_X(x)\ dx\right| \leqslant \left(\int_{\mathbb{R}^N} \phi(x)^2 f_X(x)\ dx\right)^{1/2} \left(\int_{\mathbb{R}^N} \alpha(x)^2 f_X(x)\ dx\right)^{1/2} = \|\alpha\|_{L^2_{f_X}}.$$

Similarly, we can show that:

$$\left|\int_{\mathbb{R}^N} \psi(i)\beta(i) f_I(i)\ di\right| \leqslant \|\beta\|_{L^2_{f_I}}.$$

A bound for the maximal correlation between X and I follows:

$$
\begin{aligned}
r_{XI} &= \sup\{\mathbb{E}[\phi(X)\psi(I)] \ : \ \mathbb{E}[\psi(X)] = \mathbb{E}[\psi(I)] = 0, \ \mathbb{E}[\phi(X)^2] = \mathbb{E}[\psi(I)^2] = 1\} \\
&\leqslant \|\alpha\|_{L^2_{f_X}} \|\beta\|_{L^2_{f_I}}.
\end{aligned}
$$

By definition, $r_{XI} \leqslant 1$. Thus:

$$
r_{XI} \leqslant \min\{1, \|\alpha\|_{L^2_{f_X}} \|\beta\|_{L^2_{f_I}}\},
$$

which is the desired statement. This bound is tight: if $\phi(x) = \alpha(x)/\|\alpha\|_{L^2_{f_X}}$ and $\psi(i) = \beta(i)/\|\beta\|_{L^2_{f_I}}$, then the previous expression clearly turns into an equality. Q.E.D.

We are now ready to prove the Proposition EC.3. The proof is simply a matter of calculation of the terms appearing in the design equation. In light of the assumed form for density $f_{XI}(x,i)$, we may compute the integral kernel as follows:

$$
\begin{aligned}
k(x,\xi) &= \int_{\mathbb{R}^N} (1+\alpha(x)\beta(i))(1+\alpha(\xi)\beta(i)) f_X(\xi) f_I(i) \ di \\
&= f_X(\xi) \int_{\mathbb{R}^N} \left(1+\alpha(\xi)\beta(i)+\alpha(x)\beta(i)+\alpha(x)\alpha(\xi)\beta(i)^2\right) f_I(i) \ di \\
&= f_X(\xi)\left(1+\|\beta\|^2_{L^2_I}\alpha(x)\alpha(\xi)\right).
\end{aligned}
$$

By inserting this expression into the design equation (8), we obtain:

$$
H_0^*(x) = \Pi_0^X(x) + \int_{\mathbb{R}^N} f_X(\xi)(1+\|\beta\|^2_{L^2_I}\alpha(x)\alpha(\xi)) H_0^*(\xi) \ d\xi = \Pi_0^X(x) + \gamma\|\beta\|^2_{L^2_I}\alpha(x),
$$

where:

$$
\gamma = \int_{\mathbb{R}^N} \alpha(\xi) H_0^*(\xi) f_X(\xi) \ d\xi
$$

as long as $\mathbb{E}[H_0^*(X)] = 0$. Thus, we may further specify the expression for γ as follows:

$$
\begin{aligned}
\gamma &= \int_{\mathbb{R}^N} \alpha(\xi)\left(\Pi_0^X(\xi) + \gamma\|\beta\|^2_{L^2_I}\alpha(\xi)\right) f_X(\xi) \ d\xi \\
&= \int_{\mathbb{R}^N} \alpha(\xi)\Pi_0^X(\xi) f_X(\xi) \ d\xi + \gamma\|\alpha\|^2_{L^2_X}\|\beta\|^2_{L^2_I}.
\end{aligned}
$$

Hence:

$$
\gamma = \frac{1}{1-\|\alpha\|^2_{L^2_X}\|\beta\|^2_{L^2_I}} \int_{\mathbb{R}^N} \alpha(\xi)\Pi_0^X(\xi) f_X(\xi) \ d\xi.
$$

The proof is finished. Q.E.D.

EC.3.12. Proof of Theorem EC.1

To proceed with the proof we need the following auxiliary

LEMMA EC.3. *Let $e_1,\dots,e_n$ be orthonormal vectors in L^2_X and M_n be the $n\times n$ matrix defined as:*

$$M_n := (\langle \mathscr{I}[e_k], e_j\rangle)_{k,j=1,\dots,n}\,, \tag{EC.53}$$

where $\mathscr{I}[w] := \int_{\mathbb{R}^N} k(x,\xi)w(\xi)\; d\xi$ defined in L^2_X. Then

$$\|M_n\| \leqslant r_{XI}, \tag{EC.54}$$

where $\|.\|$ denotes the matrix norm corresponding to the standard norm in $\mathbb{R}^n$.

Proof. Let $v\in\mathbb{R}^n$. Then

$$M_n v = \left(\sum_{k=1}^n \langle \mathscr{I}[e_k], e_j\rangle v_k\right)_{j=1,\dots,n} = \left(\left\langle \mathscr{I}\left[\sum_{k=1}^n v_k e_k\right], e_j\right\rangle\right)_{j=1,\dots,n}.$$

Estimate (EC.28) leads to:

$$\|\mathscr{I}[w](X)\|_2 \leqslant r_{XI}\|w(X)\|_2,$$

where $\|\cdot\|_2$ is the $L^2(\Omega,\mathscr{F},\mathbb{P})$ norm. In particular,

$$\|\mathscr{I}[w]\|_{L^2_X} \leqslant r_{XI}\|w\|_{L^2_X}. \tag{EC.55}$$

By using Bessel inequality and this bound, we see that:

$$\|M_n v\|^2 = \sum_{j=1}^n \left\langle \mathscr{I}\left[\sum_{k=1}^n v_k e_k\right], e_j\right\rangle^2 \leqslant \left\|\mathscr{I}\left[\sum_{k=1}^n v_k e_k\right]\right\|_2^2 \overset{(EC.55)}{\leqslant} r^2_{XI}\left\|\sum_{k=1}^n v_k e_k\right\|^2_{L^2_X} = r^2_{XI}\|v\|,$$

hence our claim. Q.E.D.

We are now ready to prove the Theorem EC.1. Let us consider an approximate solution of design equation (8) in the form:

$$H^*_{0,n}(x) := \sum_{j=1}^n h^n_j w_j(x). \tag{EC.56}$$

A simple substitution in the equation shows that:

$$\sum_{j=1}^n h^n_j w_j = \Pi^X_0 + \sum_{j=1}^n h^n_j \int_{\mathbb{R}^n} k(x,\xi)w_j(\xi)\; d\xi \equiv \Pi^X_0 + \sum_{j=1}^n h^n_j \mathscr{I}[w_j]. \tag{EC.57}$$

Whereas the left-hand side lies in $\mathrm{Span}\langle w_1,\dots,w_n\rangle$, the right-hand side may not. Let

$$\mathscr{P}_n\;:\;L^2_X \to \mathrm{Span}\langle w_1,\dots,w_n\rangle$$

denote the orthogonal projection on $\mathrm{Span}\langle w_1, \dots, w_n\rangle$. We modify (EC.57) as follows:

$$\sum_{j=1}^{n} h_j^n w_j = \mathscr{P}_n \left(\Pi_0^X + \sum_{j=1}^{n} h_j^n \mathscr{I}[w_j] \right) = \mathscr{P}_n \Pi_0^X + \sum_{j=1}^{n} h_j^n \mathscr{P}_n \mathscr{I}[w_j]. \tag{EC.58}$$

This is an equation in the unknown variable $h^n := (h_j^n)_{j=1,\dots,n} \in \mathbb{R}^n$. We first prove that it admits a unique solution provided that $r_{XI} < 1$. If $e_1, \dots, e_n$ is an orthonormal basis for $\langle w_1, \dots, w_n\rangle$ (possibly obtained through the Gram-Schmidt algorithm), then:

$$\mathscr{P}_n \Pi_0^X = \sum_{j=1}^{n} \langle \Pi_0^X, e_j\rangle e_j,$$

and

$$\mathscr{P}_n (\mathscr{I}[w_j]) = \sum_{i=1}^{n} \langle \mathscr{I}[w_j], e_i\rangle e_i.$$

If $T_n := (T_{jk})$ denotes the matrix operating the following change of basis:

$$w_j = \sum_{k=1}^{n} T_{kj} e_k, \tag{EC.59}$$

then we may write:

$$\mathscr{P}_n (\mathscr{I}[w_j]) = \mathscr{P}_n \left(\sum_{\ell=1}^{n} T_{\ell j} \mathscr{I}[e_\ell] \right) = \sum_{\ell=1}^{n} T_{\ell j} \mathscr{P}_n (\mathscr{I}[e_\ell]) = \sum_{k,\ell=1}^{n} T_{\ell j} \langle \mathscr{I}[e_\ell], e_k\rangle e_k. \tag{EC.60}$$

By inserting expressions (EC.59) and (EC.60) into (EC.58), we arrive at:

$$\sum_{k=1}^{n} \left(\sum_{j=1}^{n} T_{kj} h_j^n \right) e_k = \sum_{k=1}^{n} \left(\sum_{j=1}^{n} T_{kj} \gamma_j^n \right) e_k + \sum_{k=1}^{n} \left(\sum_{j,\ell=1}^{n} T_{\ell j} h_j^n \langle \mathscr{I}[e_\ell], e_k\rangle \right) e_k. \tag{EC.61}$$

Let $h^n := \left(h_j^n\right)_j$ and $\gamma^n := (\gamma_j^n)_j$. Then expression (EC.61) may be expressed as:

$$(\mathbb{I} - M_n)\, T_n h^n = T_n \gamma^n, \tag{EC.62}$$

where M_n is the matrix defined in (EC.53). Assumption $r_{XI} < 1$ entails bound (EC.54), which in turn implies that matrix $\mathbb{I} - M_n$ is invertible. Consequently, expression (EC.62) immediately yields a unique solution of (EC.58):

$$h^n = T_n^{-1} (\mathbb{I} - M_n)^{-1} T_n \gamma^n.$$

Let us define $H_{0,n}^*(x)$ as in expression (EC.56). We can prove the convergence relation $H_{0,n}^* \xrightarrow{L_X^2} H_0^*(X)$, where $H_0^*(X)$ solves the first-order equation system (5). Since:

$$H_0^*(X) = \Pi_0^X + \mathscr{I}[H_0^*(X)] = \Pi_0^X + \mathscr{P}_n \mathscr{I}[H_0^*(X)] + (\mathbb{I} - \mathscr{P}_n)\, \mathscr{I}[H_0^*(X)],$$

$$H_{0,n}^* = \mathscr{P}_n \Pi_0^X + \mathscr{P}_n \mathscr{I}[H_{0,n}^*],$$

we have:

$$H_0^*(X) - H_{0,n}^* = (\mathbb{I} - \mathscr{P}_n)\,\Pi_0^X + \mathscr{P}_n \mathscr{I}\left[H_0^*(X) - H_{0,n}^*\right] + (\mathbb{I} - \mathscr{P}_n)\,\mathscr{I}[H_0^*(X)].$$

By taking norms on both sides in the previous identity, we obtain the following upper bound:

$$\|H_0^* - H_{0,n}^*\|_{L_X^2} \leqslant \left\|(\mathbb{I} - \mathscr{P}_n)\,\Pi_0^X\right\|_{L_X^2} + \left\|\mathscr{P}_n \mathscr{I}[H_0^*(X) - H_{0,n}^*]\right\|_{L_X^2} + \|(\mathbb{I} - \mathscr{P}_n)\,\mathscr{I}[H_0^*(X)]\|_{L_X^2}. \tag{EC.63}$$

Let us focus on the second of the three summands on the right-hand side in this expression. We recall that $\mathscr{I}$ is the nested double conditioning:

$$\mathscr{I}[H_0^*(X) - H_{0,n}^*](X) = \mathbb{E}\left[\mathbb{E}[H_0^*(X) - H_{0,n}^*(X) \mid \mathscr{F}_I] \mid \mathscr{F}_X\right].$$

Hence, bound (EC.28) and bound $\|\mathscr{P}_n\| \leqslant 1$ imply that:

$$\left\|\mathscr{P}_n \mathscr{I}[H_0^*(X) - H_{0,n}^*]\right\|_{L_X^2} \leqslant \|\mathscr{I}[H_0^*(X) - H_{0,n}^*]\|_{L_X^2} \leqslant r_{XI}\|H_0^*(X) - H_{0,n}^*\|_{L_X^2}.$$

Bound (EC.63) may thus be written as follows:

$$\|H_0^* - H_{0,n}^*\|_{L_X^2} \leqslant \left\|(\mathbb{I} - \mathscr{P}_n)\,\Pi_0^X\right\|_{L_X^2} + r_{XI}\left\|H_0^* - H_{0,n}^*\right\|_{L_X^2} + \|(\mathbb{I} - \mathscr{P}_n)\,\mathscr{I}[H_0^*]\|_{L_X^2},$$

or, equivalently,

$$\|H_0^* - H_{0,n}^*\|_{L_X^2} \leqslant \frac{1}{1 - r_{XI}}\left(\left\|(\mathbb{I} - \mathscr{P}_n)\,\Pi_0^X\right\|_{L_X^2} + \|(\mathbb{I} - \mathscr{P}_n)\,\mathscr{I}[H_0^*]\|_{L_X^2}\right). \tag{EC.64}$$

Finally, as long as $(w_n)_{n\in\mathbb{N}}$ is a basis for $L_{X,0}^2$, we have: $\|(\mathbb{I} - \mathscr{P}_n)\,\phi\|_{L_X^2} \longrightarrow 0$ for every $\phi \in L_{X,0}^2$. The convergence relation we set out to prove thus holds true. Q.E.D.

EC.3.13. Proof of Proposition EC.4

We follow a path similar to the strictly risk averse case. According to (13), we must estimate $\mathrm{Var}\left[\mathbb{E}\left[\mathcal{K}_0 - \mathbb{E}\left[\mathcal{K}_0 | \mathscr{F}_X\right] | \mathscr{F}_I\right]\right]$, where $\mathcal{K}_0 = \frac{d\mathbb{Q}}{d\mathbb{P}} - \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right]$. By assumption, π exhibits a lower order than $\frac{\varrho}{a}\frac{d\mathbb{Q}}{d\mathbb{P}}$. Hence, the main contribution in the previous expression stems from the market price of risk. By using the triangular inequality,

$$\begin{aligned}
&\mathrm{Var}\left[\mathbb{E}\left[\mathcal{K}_0 - \mathbb{E}\left[\mathcal{K}_0 | \mathscr{F}_X\right] | \mathscr{F}_I\right]\right]^{\frac{1}{2}} \\
&= \left\|\mathbb{E}\left[\pi - \mathbb{E}[\pi \mid \mathscr{F}_X] \mid \mathscr{F}_I\right] + \left(\tfrac{\varrho}{a}\right)\mathbb{E}\left[\tfrac{d\mathbb{Q}}{d\mathbb{P}} - \mathbb{E}\left[\tfrac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathscr{F}_X\right] \mid \mathscr{F}_I\right]\right\|_2 \\
&\geqslant \tfrac{\varrho}{a}\left\|\mathbb{E}\left[\tfrac{d\mathbb{Q}}{d\mathbb{P}} - \mathbb{E}\left[\tfrac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathscr{F}_X\right] \mid \mathscr{F}_I\right]\right\|_2 - \left\|\mathbb{E}\left[\pi - \mathbb{E}[\pi \mid \mathscr{F}_X] \mid \mathscr{F}_I\right]\right\|_2.
\end{aligned}$$

Let us define a new index:

$$r_{X^\perp I} := \sup\left\{\rho(Z, W)\ :\ Z \in L_I^2,\ W \in (L_X^2)^\perp\right\}.$$

We may write:

$$\|\mathbb{E}\left[\pi-\mathbb{E}[\pi \mid \mathscr{F}_X] \mid \mathscr{F}_I\right]\|_2 \leqslant r_{X^\perp I}\mathrm{Var}[\pi]^{1/2} \leqslant \varepsilon r_{X^\perp I}\left|\frac{\varrho}{a}\right|\mathrm{Var}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right]^{1/2}.$$

Following arguments used in the proof of the strictly risk averse case, we may easily derive a bound:

$$\left\|\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}-\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathscr{F}_X\right] \mid \mathscr{F}_I\right]\right\|_2 \geqslant \left(1-\sqrt{1-r^2_{X^\perp I}}\right)\mathrm{Var}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right]^{1/2}.$$

By combining these estimates together, we get to:

$$\mathrm{Var}\left[\mathbb{E}\left[\left[(\mathcal{K}_0-\mathbb{E}[\mathcal{K}_0 \mid \mathscr{F}_X]\right) \mid \mathscr{F}_I\right]\right] \geqslant \left(\frac{\varrho}{a}\right)^2\mathrm{Var}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right]\left(\left(1-\sqrt{1-r^2_{X^\perp I}}\right)^2-\varepsilon r^2_{X^\perp I}\right).$$

The approximate bound (EC.14) follows by letting $\varepsilon \to 0+$ and switching to the combination value. Q.E.D.

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