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MODELING COMPETITION BETWEEN TWO PHARMACEUTICAL DRUGS USING INNOVATION DIFFUSION MODELS[1]

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The study of competition among brands in a common category is an interesting strategic issue for involved firms. Sales monitoring and prediction of competitors' performance represent relevant tools for management. In the pharmaceutical market, the diffusion of product *knowledge* plays a special role, different from the role it plays in other competing fields. This latent feature naturally affects the evolution of drugs' performances in terms of the number of packages sold. In this paper, we propose an innovation diffusion model that takes the spread of knowledge into account. We are motivated by the need of modeling competition of two antidiabetic drugs in the Italian market.

1. Introduction. The diffusion of an innovation often has to cope with the rise of many competitors that generate huge competitive effects, expansion or contraction in the market's potential size, changes in the evolutionary dynamics of certain brands, increases or decreases in life cycle length, and anticipation of the time of entry of additional products in the market. These effects can be modeled only if they are included in a single complex system that can correctly identify competition and contextual forces.

We cannot observe the complex system in which single agents (consumers) may interact and share information regarding alternative technologies, comparable solutions, similar devices, and so on. Instead, we observe the resulting aggregate emergent dynamics (level reached by diffusion; e.g., number of packages sold), and we base our analysis upon this level of observability.

Usually, the diffusion of products in a marketplace has a limited time horizon defining particular finite life cycles with different internal dynamics. We observe poor performance at the beginning of the process after launch due to limited acceptance of a newcomer to the market that interacts with previous knowledge and consumers' lifestyles. Similarly, but for different reasons, we notice a pronounced decrease in sales at the end of the commercial life cycle, when the product is perceived as an old, inefficient solution. Previous competing processes are nonstationary and nonlinear due to chilling and saturating effects within their life spans. Following this qualitative reasoning, for modeling and predictive purposes, we may

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exclude the direct use of ARMA-like (or VARMA) processes, which are strongly based on weak stationary conditions after some differencing.

The pharmaceutical market is an important example of competition among alternative drugs. The products can differ to a great extent when they are based on different active compounds, or they can differ only at the commercial level when the same active compound is sold by competing firms. Moreover, this market differs from other markets, since in many countries the cost of essential/vital drugs is paid through a welfare system. To some extent, pricing does not directly influence physicians' prescriptions. In addition, in Italy, the Ministry of Health and pharmaceutical firms negotiate the price to be paid by the national health service.

The aim of this paper is to build and apply a competition model for pharmaceutical drugs (source: IMS Health Italy). In particular, we focus on a pair of drugs with the same active compound, based on glimepiride. This is a situation of substitute products (brands) competing for the same patients. The results will be compared with the outcomes obtained by applying alternative models. In this case, we emphasize that an "explicative" model, in addition to describing the data and providing reliable forecasts, should highlight the key features of the competition among the analyzed drugs. This is a further reason, beyond nonstationarity, not to rely on traditional time series approaches.

A specific method for studying the dynamics of these special markets is based on two steps. First, to detect the mean trajectory of the processes, we use the diffusion of innovation methodology, which is strongly related to system analysis and epidemiological modeling tools. Second, to take into account seasonal autoregressive or moving average effects, we perform an analysis of residuals, thereby improving short-term prediction.

The models due to Bass and colleagues [Bass (1969), Bass, Krishnan and Jain (1994)] represent an essential step for the development of aggregate univariate diffusion patterns, and a huge number of extensions have originated from them [see, among others, Meade and Islam, (2006), Peres, Muller and Mahajan (2010)].

Conversely, the main contributions pertaining to competition modeling are rather sparse. Krishnan, Bass and Kumar (2000), Savin and Terwiesch (2005) and, recently, Guseo and Mortarino (2012) and Guseo and Mortarino (2014b) have described competition with a differential representation admitting a closed-form solution. The differential representation is typical of the models proposed in quantitative marketing literature, where an aggregate parsimonious description of real adoption processes, based on interpretable parameters, is essential to capture relevant features, deduce related managerial implications, and predict the future evolution of the market under study. The simplicity of the model's structure is obtained by introducing plausible assumptions regarding the behavior of the agents playing a role within the market. In addition, a tractable solution for estimation and prediction makes it easy to validate the model through aggregate sales data. The relevant issue in this research topic is to build an adequately large set of models to describe the different characteristics of the diffusion process. Confirmation

or rejection of the assumptions underlying each model is then attained by fitting available observed data and comparing the models' performances.

Models available in the literature to describe diffusion of competing products in a common market assume that the asymptotic market potential—that is, the total number of adoptions a product will ultimately reach at the end of its life cycle—is invariant throughout the life cycle from the products' launch. However, this assumption is almost always unrealistic. In general, knowledge and awareness of a product are not immediately disseminated throughout eligible adopters upon the entrance of a pioneering brand into the market. Moreover, new brands are often followed by competitors, whose launch may affect awareness of the earlier products. The topic arises from the consideration that awareness of a product and adoption are themselves diffusion processes. Awareness is a latent prerequisite for adoption, and the degree of penetration of a product into the market is limited by the degree of diffusion of knowledge regarding its existence and properties. For this reason, the market potential would be better described as a dynamic process than as a fixed constant, as discussed in Guseo and Guidolin (2009) for the univariate case without competition effects.

In Section 2 we briefly illustrate the standard Bass model [Bass (1969)] with its extension [Guseo and Guidolin (2009)], that introduces a dynamic market potential. The underlying reasons motivating this extension are also presented. In Section 3 we discuss how the competition model proposed in Guseo and Mortarino (2014b) can be extended to incorporate dynamic market potential. In Section 4 we illustrate the application of the new model to the description of competition between two antidiabetic drugs. In Section 5 we discuss the improvement obtained for these data with the proposed dynamic potential model and contrast it with the more common constant hypothesis and with alternative dynamic structures. In Section 6 we present concluding remarks.

2. A possible form for dynamic market potential. The simpler form of a univariate diffusion of an innovation model is given by the Bass model [Bass (1969)]. The differential representation is defined through the following equation:

$$z'(t) = m\left[p + q\frac{z(t)}{m}\right]\left[1 - \frac{z(t)}{m}\right] = \left[p + q\frac{z(t)}{m}\right][m - z(t)], \tag{2.1}$$

where $z(t)$ and $z'(t) = \partial z(t)/\partial t$ represent the mean cumulative sales and the mean instantaneous sales at time t, respectively. Parameter m is the fixed market potential (the asymptotic level of cumulative sales or the total number of adoptions at the end of the life cycle), that is, $m = \lim_{t\to\infty} z(t)$.

Equation (2.1) makes explicit that, at each time point, the increase in instantaneous mean sales is proportional to the residual market, $m - z(t)$. The proportionality factor is affected by a fixed effect, p, and by a time-varying effect, $qz(t)/m$. The former, called the innovative coefficient, is independent of the degree of diffusion reached. The higher the value of p, the more rapid the takeoff of

the life cycle, describing a process in which exogenous factors, such as advertising or institutional communication efforts, push a product's diffusion. The latter effect, $qz(t)/m$, depends upon the degree of saturation of the market and describes, through the interaction $z(t)[m - z(t)]$, how word-of-mouth from previous sales promotes further diffusion. The coefficient q is called the imitative coefficient. The higher the value of q, the more important word-of-mouth is in increasing diffusion. As $z(t)$ approaches m, the residual market, $m - z(t)$, collapses and instantaneous mean sales, $z'(t)$, reduces to zero.

Under the initial condition $z(0) = 0$, and defining $z(t) = 0$ for $t < 0$, the explicit solution of equation (2.1) is

$$z(t) = m\frac{1 - e^{-(p+q)t}}{1 + (q/p)e^{-(p+q)t}} = mw(t; p, q), \qquad t > 0,\ p, q > 0, \tag{2.2}$$

where

$$w(t; p, q) = \frac{1 - e^{-(p+q)t}}{1 + (q/p)e^{-(p+q)t}}. \tag{2.3}$$

Continuous-time modeling is a common choice throughout the diffusion of innovation literature, even when models are fitted to weekly, monthly or quarterly data. This is partially because the involved variables are measured continuously over time, even if, for administrative reasons, data are recorded at discrete times. In addition, Putsis (1996) conducts a detailed comparison and emphasizes that using seasonally adjusted quarterly data results in better estimates than using annual data. In contrast, moving from quarterly to monthly data produces only marginal statistical improvement. Boswijk and Franses (2005) indicate that the values of p and q in their discretized version of the Bass model correspond to those of the continuous time model used here whenever equally spaced data are available.

Although model (2.1) and its successive extensions proved to be extremely valuable in describing innovation diffusion processes, all are limited by the fact that market potential, m, is a fixed constant, and hence cannot evolve over time. This assumption conflicts with the common perception that knowledge may be time dependent. Some attempts have been proposed in the literature to overcome this limitation. In some papers, the dynamic market potential is modeled as a function of exogenous observed variables [see, e.g., Kim, Bridges and Srivastava (1999), and the included references]. In other cases, it is assumed to be a function of time only [e.g., Sharif and Ramanathan (1981), Centrone, Goia and Salinelli (2007), Meyer and Ausubel (1999)].

Here, we follow the approach proposed in Guseo and Guidolin (2009). In principle, the market potential can be any function $m(t)$ that defines an upper bound for cumulative sales $z(t)$, that is, $z(t) \le m(t)$ for all t. However, a parsimonious and intuitive method for specifying the form of $m(t)$ arises when we examine the communication network spreading information about the product in question. The number of potential adopters of a product can be thought of, at each time

point, as the size of the aware agents' group. We describe awareness of the product as knowledge transmitted through a network that describes the specific contacts among agents who eventually "speak" about the product. This approach is linked to the literature on social networks, often represented with random graph models where nodes denote individual social actors (agents) and edges denote specific relationships between two actors [Handcock and Gile (2010)]. Many contributions to the literature assume observability of the edges, either complete or partial, through sample data.

In our approach, conversely, the communication network evolving over time is latent and does not have to be observed or described in detail; this is also due to the high costs of reliable relational data collection. The focus is instead on the number of informed agents (active nodes). This is a key aspect, since we want to deal with all the situations where the communication network is *product-specific* (people usually choose to talk with someone—and not with someone else—according to the topic of the conversation). In these situations, the content-driven network is totally unobservable, or it is very difficult to obtain reliable *pertinent* data.

The formalized structure of such a network is described in Guseo and Guidolin (2009), where the authors explain in detail how this interpretation may lead to the following dynamic market potential function:

$$\begin{aligned} m(t) &= K\sqrt{\frac{1-e^{-(p_c+q_c)t}}{1+(q_c/p_c)e^{-(p_c+q_c)t}}} \\ &= K\sqrt{w(t;p_c,q_c)}, \qquad K, p_c, q_c > 0, t > 0, \end{aligned} \tag{2.4}$$

where K is the upper asymptotic potential (directly related to the network's size), $K = \lim_{t\to\infty} z(t)$, and p_c and q_c are evolutionary parameters describing how fast communication spreads through the network. In particular, for large values of p_c and q_c, the dynamic market potential $m(t)$ rapidly approaches K.

The expression under the square root in equation (2.4) represents the core of the Bass model [Bass (1969)] describing the latent diffusion process of communication. This is an S-shaped curve, a distribution function, whose peakedness varies according to the product's communication features.

The model proposed by Guseo and Guidolin (2009) extends the Bass model (2.2) in the following manner:

$$z'(t) = m(t)\left[p_s + q_s\frac{z(t)}{m(t)}\right]\left[1-\frac{z(t)}{m(t)}\right] + z(t)\frac{m'(t)}{m(t)}, \qquad p_s, q_s > 0, t \geq 0, \tag{2.5}$$

where $z(t)$ represents the mean cumulative sales, as in equation (2.1), and $m(t) \geq z(t)$ may be defined as in (2.4). The new parameters p_s and q_s are evolutionary parameters that describe how fast the product is adopted (whereas, as mentioned above, p_c and q_c are related to knowledge spread).

The final term of equation (2.5) requires close examination to understand its meaning. This component enables us to take into account a self-reinforcing effect that is common within marketing behavioral studies [see, e.g., Sydow and Schreyögg (2013), a recent contribution on self-reinforcing processes]. The standard adoption process described in the first part of the equation is enhanced whenever the market is growing faster. In other words, an acceleration of the number of informed people [the network's size, $m(t)$] further induces people to adopt. More generally, excluding assumption (2.4), $m'(t)$ may be negative when $m(t)$ is nonmonotonic, thereby introducing a shrinking effect on instantaneous sales due to a decreasing market potential.

3. The proposed model. The proposed model describes the diffusion of two competing brands. They are supposed to be sufficiently similar to share a common market potential, whose size grows in time as described in Section 2. The assumption of a common market potential is suitable in situations where the products are substitutes competing for the same adopters. Whenever competition concerns products that are sufficiently different to preserve product-specific market potentials, the Lotka–Volterra models should be preferred, although these structures do not allow a closed-form solution [Abramson and Zanette (1998)].

We denote the mean cumulative sales at time t of brand i by $z_i(t)$, $i = 1, 2$, and the instantaneous mean sales by $z_i'(t) = \partial z_i(t)/\partial t$, $i = 1, 2$. We now describe the category sales, $z(t) = z_1(t) + z_2(t)$, by separately describing the two brands constituting the category. The model is given by

$$
\begin{aligned}
z_1'(t) &= m(t)\left[p_1 + (q_1 + \delta)\frac{z_1(t)}{m(t)} + q_1\frac{z_2(t)}{m(t)}\right]\left[1 - \frac{z(t)}{m(t)}\right] + z_1(t)\frac{m'(t)}{m(t)},\\
z_2'(t) &= m(t)\left[p_2 + (q_2 - \delta)\frac{z_1(t)}{m(t)} + q_2\frac{z_2(t)}{m(t)}\right]\left[1 - \frac{z(t)}{m(t)}\right] + z_2(t)\frac{m'(t)}{m(t)},
\end{aligned}
\tag{3.1}
$$

where $z(t) \le m(t)$, for all t.

To obtain an equivalent formulation of model (3.1), that may be more comparable with the univariate Bass model, we can rearrange the terms in the following manner:

$$
\begin{aligned}
z_1'(t) &= m(t)\left[p_1 + q_1\frac{z(t)}{m(t)} + \delta\frac{z_1(t)}{m(t)}\right]\left[1 - \frac{z(t)}{m(t)}\right] + z_1(t)\frac{m'(t)}{m(t)},\\
z_2'(t) &= m(t)\left[p_2 + (q_2 - \delta)\frac{z(t)}{m(t)} + \delta\frac{z_2(t)}{m(t)}\right]\left[1 - \frac{z(t)}{m(t)}\right] + z_2(t)\frac{m'(t)}{m(t)}.
\end{aligned}
$$

In equation (3.1), we may observe innovators' effects (parameters p_1 and p_2) and word-of-mouth effects (parameters q_1, q_2 and δ). These parameters may be different for the two competitors to describe products with different strengths in the market. Observe that this structure is similar to the model used in Guseo and Mortarino (2014b), allowing for within-brand word-of-mouth ($q_1 + \delta$ and q_2 for

the two brands, resp.) that may be different from cross-brand word-of-mouth (q_1 and $q_2 - \delta$). In other words, this model is able to deal with situations in which word-of-mouth functions asymmetrically for the two products. In Guseo and Mortarino (2014b), however, unlike the proposed model, $m(t)$ was supposed to be constant throughout the life cycle: $m(t) = m$ for all t.

The final additive terms in equation (3.1)—which would obviously vanish for a constant $m(t)$—represent a *self-reinforcing* component, as described in the previous section. The mean sales of both products are accelerated when $m(t)$ grows faster, that is, when awareness of the product category spreads rapidly based on the collective behavior of agents. Conversely, the mean sales are further reduced by a shrinking potential induced by unfavorable signals. In the latter case, $m(t)$ could also be a nonmonotonic function, and the self-reinforcing term could be negative when the market potential undergoes a contraction.

Notice that the sum of the equations in (3.1) is equal to model (2.5). Moreover, this model can also be used with an expression for $m(t)$ that is different from equation (2.4).

Let us define $p_s = p_1 + p_2$ and $q_s = q_1 + q_2$. Through $w(t; p_s, q_s)$, defined in (2.3), and

$$y(t) = 1 + \frac{q_s}{p_s} w(t; p_s, q_s) = \frac{1 + q_s/p_s}{1 + (q_s/p_s)e^{-(p_s+q_s)t}}, \tag{3.2}$$

it is proven in Appendix 1 [Guseo and Mortarino (2015)] that, for any $m(t)$, the closed-form solution of the system (3.1) is

$$\begin{aligned} z_1(t) &= m(t)\left\{\frac{q_1}{q_s - \delta} w(t; p_s, q_s) + \left[\frac{p_s}{\delta}\left(\frac{p_1}{p_s} - \frac{q_1}{q_s - \delta}\right)\right]\left[y(t)^{\delta/q_s} - 1\right]\right\}, \\ z_2(t) &= m(t)\left\{\left(\frac{q_2 - \delta}{q_s - \delta}\right) w(t; p_s, q_s) + \left[\frac{p_s}{\delta}\left(\frac{p_2}{p_s} - \frac{q_2 - \delta}{q_s - \delta}\right)\right]\left[y(t)^{\delta/q_s} - 1\right]\right\}, \end{aligned} \tag{3.3}$$

when $\delta \neq 0$ and $\delta \neq q_s$. When $\delta = q_s$, the solution reduces to

$$\begin{aligned} z_1(t) &= m(t)\left[\left(\frac{p_1}{p_s} - \frac{q_1}{q_s}\right) w(t; p_s, q_s) + \frac{q_1 p_s}{q_s^2} y(t) \ln y(t)\right], \\ z_2(t) &= m(t)\left[\left(1 - \frac{p_1}{p_s} + \frac{q_1}{q_s}\right) w(t; p_s, q_s) - \frac{q_1 p_s}{q_s^2} y(t) \ln y(t)\right], \end{aligned} \tag{3.4}$$

while in the special case $\delta = 0$, we obtain

$$\begin{aligned} z_1(t) &= m(t)\left[\frac{q_1}{q_s} w(t; p_s, q_s) + \frac{p_s}{q_s}\left(\frac{p_1}{p_s} - \frac{q_1}{q_s}\right) \ln y(t)\right], \\ z_2(t) &= m(t)\left[\frac{q_2}{q_s} w(t; p_s, q_s) + \frac{p_s}{q_s}\left(\frac{p_2}{p_s} - \frac{q_2}{q_s}\right) \ln y(t)\right]. \end{aligned} \tag{3.5}$$

The solutions for the mean cumulative sales enable us to use a nonlinear regression model with dependent variables given by the *observed* cumulative sales

of the two brands.[2] A reasonable and robust inferential methodology for estimating and testing the performance of this structure may be implemented through the regression model

$$v_i(t) = z_i(t) + \varepsilon_i(t), \qquad i = 1, 2, \tag{3.6}$$

where $v_i(t)$ represents the observed cumulative sales data for each of the two products and $z_i(t;\beta)$ denotes the mean cumulative functions (3.3) depending on the vector of parameters $\beta = \{K, p_c, q_c, p_1, q_1, p_2, q_2, \delta\}$ and on time t. Henceforth, we use either the notation $z_i(t)$ or $z_i(t;\beta)$ to make explicit the dependence of the functions (3.3) upon β parameters. Here, we assume that $m(t)$ is modeled as in (2.4). In the rest of the paper, we will denote model (3.6)—with $m(t)$ specified as in (2.4)—with the expression Competition Dynamic Market Potential (CDMP) model. The residual term $\varepsilon_i(t)$ is usually a white noise or a more complex stationary process if seasonality or autoregressive aspects are included as stochastic components. The joint estimate of β is obtained with a single model where $v_1(t)$ and $v_2(t)$ are stacked. This estimate could be generated using the Beauchamp and Cornell technique [Beauchamp and Cornell (1966)]. However, recent results show that it is advisable to use ordinary nonlinear least squares [Guseo and Mortarino (2014a)]. Note that estimation through nonlinear least squares does not require assumptions regarding the distribution of $\varepsilon_i(t)$. The nonlinear predicted values describe the mean trajectories of the competing processes, that is, $z_1(t)$ and $z_2(t)$.

We propose a detailed simulation study in Appendix 5 [Guseo and Mortarino (2015)] to assess the performance of the CDMP model under different values of the noise-to-signal ratio when the latent market potential is correctly specified. We also consider a further improvement in the analysis of the robustness of the CDMP model for alternative $m(t)$ structures.

A different approach, based on a stochastic version of equation (3.1) including an error term with suitable assumptions, may be extremely complex. This approach is tractable, to our knowledge, only for simpler models such as the Bass model [Boswijk and Franses (2005)]. However, as mentioned in Section 2, the Bass model is too simple a structure to describe complex markets. Jha, Chaudhary and Gutpa (2011) propose a stochastic differential equation model to describe the adoption of newer successive technologies. However, their work does not present a comparison with existing deterministic models. The comparison is essential to evaluate the effective gain of the stochastic approach, whose results are obtained through non-negligible assumptions regarding the stochastic component of the model, which may be inappropriate for real (not simulated) data sets.

[2] An alternative approach using instantaneous sales is described in Appendix 2 [Guseo and Mortarino (2015)].

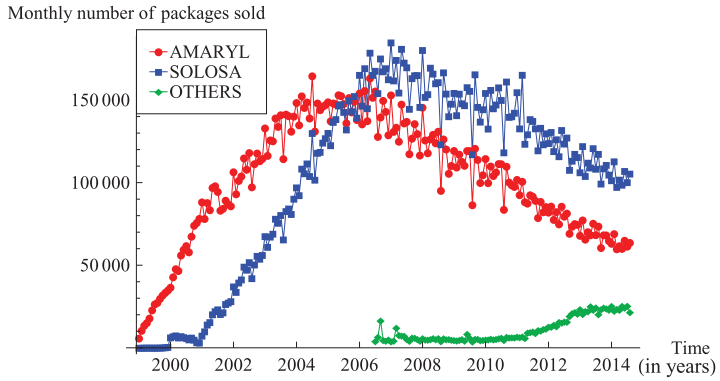

FIG. 1. *Monthly sales data for Amaryl* 2 *mg and Solosa* 2 *mg. The series of the sum of all the sales of alternative products is also presented* (*source*: *IMS Health Italy*).

4. Antidiabetic drug sales case study. Amaryl (Sanofi–Aventis) and Solosa (Lab. Guidotti) are two glimepiride-based drugs used by people with type 2 diabetes. Glimepiride belongs to the class of drugs known as sulfonylureas. It lowers hyperglycemia by causing the body to release its natural insulin. These drugs, at a dose of 2 mg, were launched in the Italian market in January 1999 and were for many years duopolists in the glimepiride market. Figure 1 shows monthly sales data (available until August 2014, for a total of 188 data points) for the two drugs separately. In addition, the figure depicts the series of the sum of all the sales of alternative products (12 generic drugs) commercialized since 2006. The more recent products have never represented an actual threat to the two oldest brands.

These two drugs are perfect substitutes from the medical viewpoint, and thus a model with a common market potential appears to be an adequate solution. Moreover, in 1999, glimepiride represented a radical novelty in the Italian market, since it was the first type of sulfonylurea available. Other dosages of the same drugs were launched much later, in 2006. These considerations suggest that awareness of the properties and efficacy of these drugs perhaps was not widespread among Italian physicians in 1999. A dynamic market potential seems conceivable for these data. The complete impossibility of observing the communication network that spread knowledge about glimepiride beginning in 1999 finally suggests that the Guseo–Guidolin model (2.4) could be an appropriate tentative solution. Of course, only good agreement between the available data and functions (3.3), which incorporate these features, could confirm or lead to rejection of these assumptions.

Joint nonlinear regression of the two main competitors' cumulative sales on functions (3.3)—that is, the CDMP model, (3.6)—gives rise to the parameter estimates shown in Table 1.

The huge value of $R^2 = 0.99996$ is unsurprising, given that we are working with cumulative data and any S-shaped fitting produces high determination indexes. A standard approach advises the use of the R^2 measure only for comparative purposes, as will be described at the beginning of Section 5. In addition, the evaluation

TABLE 1
Estimation results for the CDMP model, (3.6)

	Estimate	Standard error	95% confidence interval
K	$4.8669 * 10^7$	$2.5771 * 10^5$	$(4.81621 * 10^7, 4.9176 * 10^7)$
p_c	$2.3837 * 10^{-3}$	$6.6814 * 10^{-5}$	$(2.2523 * 10^{-3}, 2.5151 * 10^{-3})$
q_c	$4.5235 * 10^{-2}$	$4.6993 * 10^{-4}$	$(4.4311 * 10^{-2}, 4.6159 * 10^{-2})$
p_1	$3.2004 * 10^{-3}$	$6.5762 * 10^{-5}$	$(3.0711 * 10^{-3}, 3.3297 * 10^{-3})$
q_1	$1.4277 * 10^{-2}$	$3.2663 * 10^{-4}$	$(1.3635 * 10^{-2}, 1.4920 * 10^{-2})$
p_2	$-7.9208 * 10^{-4}$	$3.6160 * 10^{-5}$	$(-8.6318 * 10^{-4}, -7.2097 * 10^{-4})$
q_2	$1.2709 * 10^{-3}$	$5.5915 * 10^{-4}$	$(1.7135 * 10^{-4}, 2.3704 * 10^{-3})$
δ	$-2.2248 * 10^{-2}$	$9.6448 * 10^{-4}$	$(2.4145 * 10^{-2}, -2.0351 * 10^{-2})$

$R^2 = 0.99996$

of the squared linear correlation coefficient between observed *instantaneous* sales and fitted *instantaneous* sales yields a value of 0.9673, which is extremely high.

The agreement between the observed and fitted values can also be assessed by examining Figure 2. The two estimated profiles follow the observations very well, and discrepancies (essentially due to seasonal effects) could easily be modeled using a SARMAX approach characterizing the second step refinement for short-term prediction [see Appendix 4, Guseo and Mortarino (2015)]. The analysis of residuals is depicted in Figure 3.

Because we deal with consumables (i.e., repeatedly purchased goods), $\hat{K}$ (49 million) represents an estimate of the total number of packages of the two drugs that could be sold. Figure 4 depicts the estimated evolution of the common dynamic market potential, $m(t)$. It is very far from a fixed m pattern, since knowledge of these drugs seems to have spread slowly among physicians. This could be explained by the observation that a new active compound (as glimepiride was in

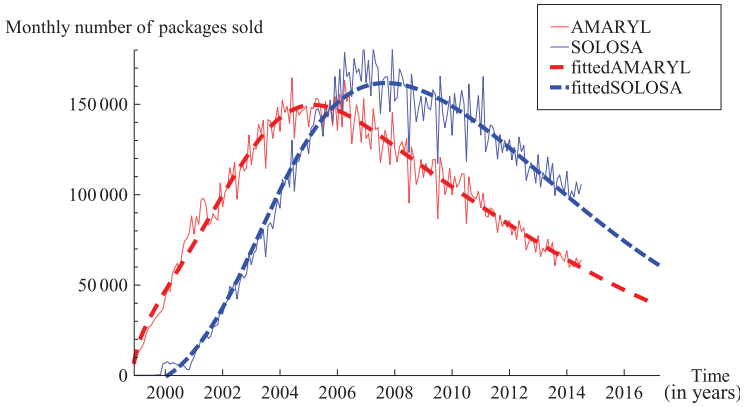

FIG. 2. *Comparison of the monthly number of packages sold and fitted values of instantaneous sales using CDMP model*, (3.6).

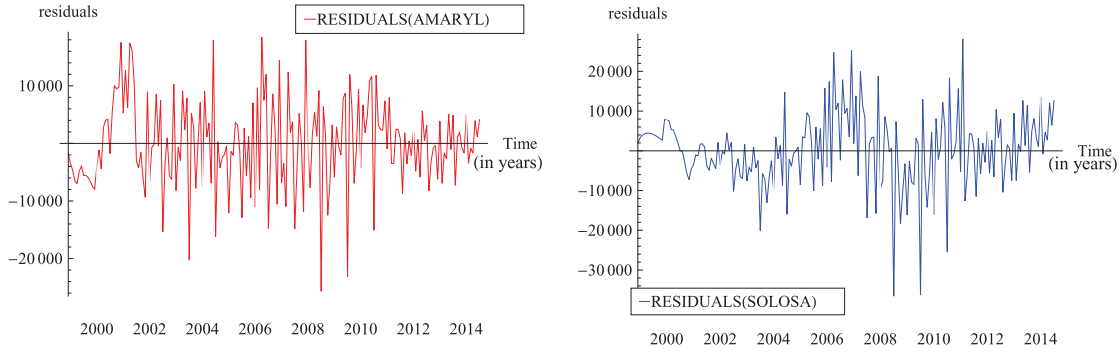

FIG. 3. *Residuals for the two products* (*instantaneous sales scale*).

the Italian market in 1999) is accepted with caution until side effects are entirely disclosed.

If we focus on innovation parameters, it is evident that this component did not play a significant role for Solosa, and this may explain its slow start. Lab. Guidotti, which launched Solosa, is a big Italian company; however, its promotional strength could not compete with the promotional efforts exerted by the international company Sanofi–Aventis, which promoted Amaryl.

Imitative parameters have to be interpreted with reference to the proposed model. If we substitute the estimates in model (3.1), we obtain the following equations:

$$z_1'(t) - z_1(t)\frac{m'(t)}{m(t)} \propto 0.0032 - 0.0080\frac{z_1(t)}{m(t)} + 0.0143\frac{z_2(t)}{m(t)},$$

$$z_2'(t) - z_2(t)\frac{m'(t)}{m(t)} \propto -0.0008 + 0.0235\frac{z_1(t)}{m(t)} + 0.0013\frac{z_2(t)}{m(t)}.$$

Amaryl was sustained by a stronger innovation effect, and its cycle began much more rapidly than its competitor's cycle (0.0032 vs. −0.0008). Sanofi–Aventis is a much larger company than Lab. Guidotti, and the former's promotional strength

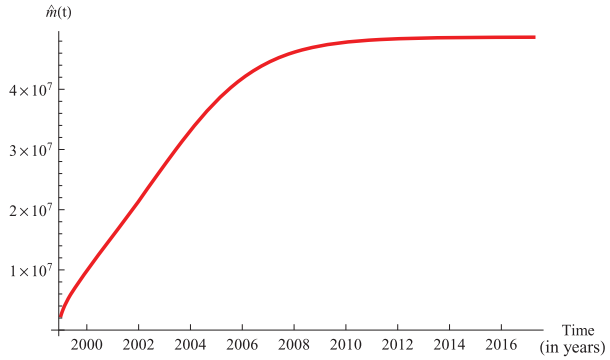

FIG. 4. *Plot of the estimated market potential function*, $\hat{m}(t)$.

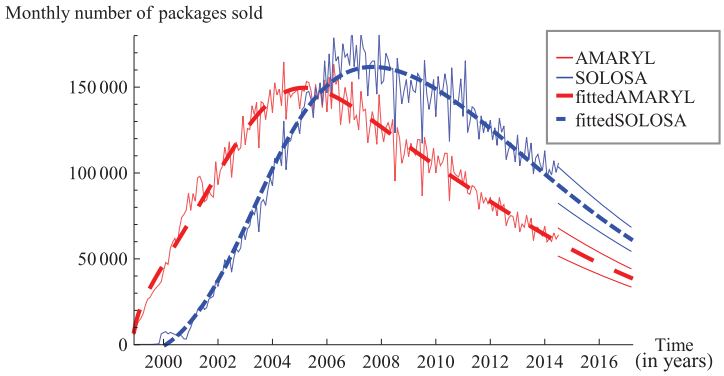

FIG. 5. *Mean sales forecast and confidence bands for estimates based on the CDMP model*, (3.6).

enabled an impressive start to Amaryl's sales. However, Amaryl experienced a negative within-brand word-of-mouth effect, in contrast with Solosa's positive effect (−0.0080 vs. 0.0013). Both products were sustained by a positive cross-brand word-of-mouth effect from the competitor, but the effect of this was to increase Solosa's sales more strongly (0.0235 vs. 0.0143). This ultimately led Solosa to outsell Amaryl. Both drugs now appear to be in a declining phase of their life cycle, due to the appearance of other active compounds in the type 2 diabetes market.

Figure 5 illustrates predictive confidence bands for the future sales of the two products. Details regarding their construction are given in Appendix 3 [Guseo and Mortarino (2015)].

5. Comparison with alternative models. The efficacy of the proposed model in this application must be proven with reference to alternative models. As mentioned in the Introduction, we will examine a set of models to identify which one performs better with available observations. The first alternative to be considered is a simpler model with constant market potential. As mentioned above, it is plausible that knowledge of the properties of the new active compound did not arise immediately at the products' launch. However, this hypothesis should be tested by examining whether a model with dynamic market potential, $m(t)$, really improves the fitting.

The model proposed in Guseo and Mortarino (2014b) fits this purpose since it can be obtained by (3.3) with the only restriction $m(t) = m$. All other features related to the evolution of the process are the same for the two models. Thus, we can claim that if model (3.6) shows a significantly better performance than Guseo and Mortarino's model (2014b), this proves that the market potential for this category evolved in a manner that differs significantly from the constant path. Note, too, that other models [e.g., those by Krishnan, Bass and Kumar (2000), Savin and Terwiesch (2005), Libai, Muller and Peres (2009), Guseo and Mortarino (2012)] are nested within the Guseo and Mortarino (2014b) model. The R^2 value for the Guseo and Mortarino (2014b) model equals 0.9988. Since this model is

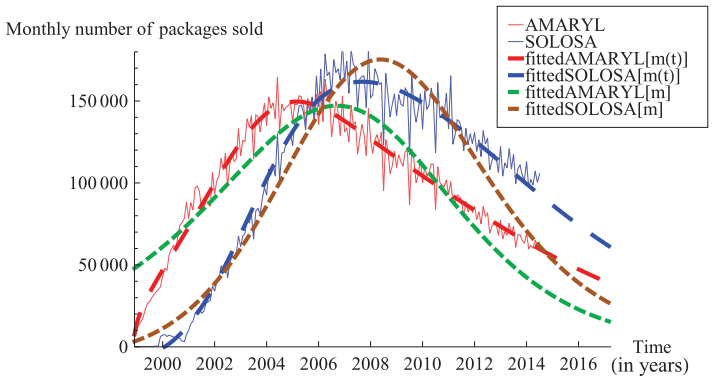

FIG. 6. *Comparison of the fitted values for the monthly number of packages sold using the CDMP model*, (3.6), *and the model used in Guseo and Mortarino* (2014b).

nested within model (3.6), an F test can be used to detect whether the gain from the simpler model to the more complex model is significant. As the first step, the squared multiple partial correlation coefficient

$$\widetilde{R}^2 = (R^2_{M1} - R^2_{M2})/(1 - R^2_{M2}) \tag{5.1}$$

is calculated (here, R^2_{M2} denotes the determination index of the reduced model that has to be compared to model $M1$). A possible test to verify the significance of the s parameters of the $M1$ model that are not included in model $M2$ may be given by

$$F = [\widetilde{R}^2(N-k)]/[(1-\widetilde{R}^2)s], \tag{5.2}$$

where N denotes the number of observations used to fit the models and k is the number of parameters included in model $M1$. Under the null hypothesis of equivalence between models $M1$ and $M2$, (5.2) is distributed as a Snedecor's F with $(s, N-k)$ degrees of freedom, if the stochastic component of the regression model is normal i.i.d. This may not be true for our case. Nevertheless, the F ratio (5.2) can be used as an approximate robust criterion for comparing model $M2$ nested in $M1$, by considering the well-known common robust threshold 4. Here, the test comparing model (3.1) with Guseo and Mortarino's (2014b) model assigns a huge value of $F = 5474.78$ ($\widetilde{R}^2 = 0.9675$), demonstrating the relevance of the extended (3.6) model.

In Figure 6, the fitted values of model (3.6) and Guseo and Mortarino's (2014b) model are compared. The rigidity of a fixed market potential makes the latter model inadequate to describe these data; even worse, for larger t values, it shows a heavy underestimation that makes forecasts totally unreliable.

Both the result of the F test and the graphical comparison prove that a constant market potential is not adequate to describe this market. Given that conclusion, it could be interesting to see whether alternative market potential functions might perform better than (2.4).

Table 2 shows the R^2 and the corresponding ρ^2 between observed and fitted values of instantaneous sales for alternative models. In detail, the formulations

TABLE 2
Comparison among alternative model specifications for market potential, $m(t)$

$m(t)$	R^2	ρ^2
(2.4)	0.999960	0.967295
Constant market potential	0.998766	0.877826
(5.3)	0.999930	0.964444
(5.4)	0.999931	0.964347

used were

$$m(t) = K\frac{1 - e^{-(p_c+q_c)t}}{1 + (q_c/p_c)e^{-(p_c+q_c)t}} \tag{5.3}$$

and

$$m(t) = K \cdot F(t) = K \cdot \int_0^t \frac{1}{\Gamma(\alpha_1)}\alpha_0^{\alpha_1} t^{\alpha_1-1} e^{-\alpha_0 t}\, dt, \tag{5.4}$$

where $\Gamma(\alpha_1) = \int_0^\infty t^{\alpha_1-1}e^{-t}\, dt$, and $\alpha_0, \alpha_1, t > 0$. The function in (5.3) represents a modification of the proposed function (2.4), while (5.4) describes the evolution of the dynamic market potential as proportional to $F(t)$, the cumulative distribution function of a Gamma random variable (with mean equal to α_1/α_0 and variance equal to α_1/α_0^2).

The values presented in Table 2 confirm that a constant market potential assumption is not adequate to describe these data. The structures (5.3) and (5.4) perform slightly worse than the proposed structure (2.4). However, the main difference is that a Gamma distribution or a structure similar to (5.3) only serves the purpose of representing a flexible monotonic function. Conversely, (2.4) has been proposed essentially because it represents the size of an informed network spreading information regarding the product category. Thus, this model structure has a substantial interpretative content. The proposed model is shown to perform best in this application. In light also of the results of the simulation study proposed in Appendix 5 [Guseo and Mortarino (2015)], our opinion is that the CDMP model represents a useful contribution in the field of competition diffusion of innovation models.

6. Concluding remarks. Diffusion of innovation methodologies have faced and are facing new challenges in parsimonious model-building in terms of incorporating the major effects that can modify the evolutionary shapes of these methodologies over time.

This paper highlights the key features of the competition between Amaryl and Solosa. These two drugs differ essentially in the persuasion effects exerted by the two companies that launched the drugs and in their acceptance through the community of physicians spreading word-of-mouth about their efficacy.

The initial novelty of the active compound of these drugs in the market suggested to us that the existing models of competition must be enriched with the introduction of dynamic market potential. This extension rests on the concept that awareness is a fundamental prerequisite for adoption. We can imagine that, at the individual level, awareness and adoption are two sequential states that subjects (here, physicians) may undergo. The first state, awareness, is latent. In addition, since individual data are unavailable in this case, the description is aggregated (as a mean profile) and leads to equation (2.4).

Similarly, although in a very different context, note that the Guseo and Guidolin (2009) paper inspired the approach followed by Furlan and Mortarino (2012) to describe and predict the death toll due to pleural mesothelioma contracted through exposure to asbestos fibers in a residential area close to a big plant. In that case, contamination (state 1)—that is, contact with lethal asbestos fibers—was the latent prerequisite for developing the disease (state 2).

Finally, we would emphasize that our proposed model is useful specifically for analyzing competition between two products. The tractability of the model, in terms of the estimation of the involved parameters, enables us to deal with a higher number of competitors only if they have entered the market simultaneously. Diachronic competition, that is, among products launched at different times, generally requires model structures with multiple regimes (a change-point in the evolution of existing products occurs whenever a new competitor appears). In this case, for more than three products, the parameter cardinality becomes too high to obtain reliable estimates, unless each regime is covered by an adequate observation period.

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SUPPLEMENTARY MATERIAL

Supplementary materials (DOI: 10.1214/15-AOAS868SUPP; .pdf). In Appendix 1 we provide details regarding the closed-form solution of the proposed model. In Appendix 2 we propose an alternative estimation method to deal with monthly sales data instead of cumulative observations. In Appendix 3 we discuss the construction of predictive confidence bands. In Appendix 4 we present a SARMAX refinement for the first-order model fitting for short-term forecasting purposes. Finally, in Appendix 5 we show the results of a simulation study to assess the reliability of inferences.

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MODELING COMPETITION BETWEEN TWO PHARMACEUTICAL DRUGS USING INNOVATION DIFFUSION MODELS

—SUPPLEMENTARY MATERIALS—

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Appendix 1. Proof. Let $z(t) = z_1(t) + z_2(t)$ denote the sum of the cumulative sales of the two products and $z'(t)$ the total instantaneous sales. If we sum the equations of system (3.1), we obtain

$$\begin{aligned} z'(t) &= z_1'(t) + z_2'(t) \\ &= m(t)\left[(p_1+p_2) + (q_1+q_2)\frac{z_1(t)}{m(t)} + (q_1+q_2)\frac{z_2(t)}{m(t)}\right]\left[1 - \frac{z(t)}{m(t)}\right] \\ &\quad + [z_1(t) + z_2(t)]\frac{m'(t)}{m(t)} \\ &= m(t)\left[(p_1+p_2) + (q_1+q_2)\frac{z(t)}{m(t)}\right]\left[1 - \frac{z(t)}{m(t)}\right] + z(t)\frac{m'(t)}{m(t)}. \end{aligned} \tag{A1.1}$$

Equation (A1.1) defines a coevolutive model (Guseo and Guidolin, 2009) with unspecified market potential $m(t)$ and adoption parameters $p_s = p_1 + p_2$ and $q_s = q_1 + q_2$. Thus, the solution of the differential equation (A1.1), with initial condition $z(0) = 0$, is given below:

$$\frac{z(t)}{m(t)} = w(t) = \frac{1 - e^{-(p_s+q_s)t}}{1 + \frac{q_s}{p_s}e^{-(p_s+q_s)t}}. \tag{A1.2}$$

In order to find a solution for $z_1(t)$, the first equation in system (3.1)

*Corresponding author

should be rearranged in the following manner:

$$
\begin{aligned}
z_1'(t) - z_1(t)\frac{m'(t)}{m(t)} &= m(t)\left[p_1 + (q_1+\delta)\frac{z_1(t)}{m(t)} + q_1\frac{z_2(t)}{m(t)}\right]\left[1 - \frac{z(t)}{m(t)}\right] \\
\frac{z_1'(t)\,m(t) - z_1(t)\,m'(t)}{m^2(t)} &= \left[p_1 + q_1\frac{z(t)}{m(t)} + \delta\frac{z_1(t)}{m(t)}\right]\left[1 - \frac{z(t)}{m(t)}\right] \\
\left[\frac{z_1(t)}{m(t)}\right]' &= \left[p_1 + q_1\frac{z(t)}{m(t)} + \delta\frac{z_1(t)}{m(t)}\right]\left[1 - \frac{z(t)}{m(t)}\right] \\
\left[\frac{z_1(t)}{m(t)}\right]' &= \left[p_1 + q_1 w(t;p_s,q_s) + \delta\frac{z_1(t)}{m(t)}\right]\left[1 - w(t;p_s,q_s)\right]. \qquad \text{(A1.3)}
\end{aligned}
$$

Equation (A1.3) perfectly matches the differential equation in Guseo and Mortarino (2014, p. 308, between (A.1) and (A.2)), where m_c is replaced by $m(t)$ and the condition $z_s = 0$ is added in the expression for $w(t;p_s,q_s)$. In our case, unlike in Guseo and Mortarino (2014), we are examining competition between two products that enter the market simultaneously. If we add these two conditions to the solution of the differential equation, we obtain exactly (3.3), (3.4), and (3.5).

Appendix 2. Regression model with instantaneous sales. Estimation of the parameters involved in diffusion models is usually performed through cumulative data, as described at the end of Section 3. The main reason is that the solution of the differential equations describing the mean evolutionary trajectory refers to $z(t)$ in the univariate case (or $z_i(t)$, $i = 1, 2$, in the competitive setup). Thus, the corresponding observed data are the cumulative sales.

As an alternative approach, instantaneous data could be used as dependent variables in a regression model, provided that the specification is modified in the following manner with respect to (3.6):

$$
s_i(t) = z_i(t+0.5;\beta) - z_i(t-0.5;\beta) + \xi_i(t), \quad i = 1, 2, \qquad \text{(A2.4)}
$$

where $s_i(t) = v_i(t+1) - v_i(t)$ represents observed instantaneous sales (here, monthly sales) and $z_i(t) = z_i(t;\beta)$ is defined as in (3.6).

The reason for using the difference $z_i(t+0.5) - z_i(t-0.5)$ in (A2.4) instead of the more intuitive $z_i(t+1) - z_i(t)$ is the known symmetric approximation of a function $F(x)$,

$$
F'(x) = \frac{F(x+h) - F(x-h)}{2h} + O(h^2),
$$

which cancels out the second-order derivative, $F''(x)$. This approximation has a simple form for $h = 0.5$.

TABLE A2.1
Estimation results for model (A2.4).

	Estimate	Standard Error	95% Confidence Interval
K	$4.9794{*}10^{7}$	$7.9471{*}10^{5}$	$(4.8231{*}10^{7}, 5.1357{*}10^{7})$
p_c	$2.2006{*}10^{-3}$	$1.9303{*}10^{-4}$	$(1.8210{*}10^{-3}, 2.5801{*}10^{-3})$
q_c	$4.4731{*}10^{-2}$	$1.5710{*}10^{-3}$	$(4.1641{*}10^{-2}, 4.7820{*}10^{-2})$
p_1	$3.4950{*}10^{-3}$	$2.4323{*}10^{-4}$	$(3.0167{*}10^{-3}, 3.9733{*}10^{-3})$
q_1	$1.4902{*}10^{-2}$	$1.3948{*}10^{-3}$	$(1.2160{*}10^{-2}, 1.7645{*}10^{-2})$
p_2	$-9.3386{*}10^{-4}$	$1.8066{*}10^{-4}$	$(-1.2891{*}10^{-3}, -5.7859{*}10^{-4})$
q_2	$-6.2316{*}10^{-4}$	$1.8463{*}10^{-3}$	$(-4.2537{*}10^{-3}, 3.0074{*}10^{-3})$
δ	$-2.5405{*}10^{-2}$	$3.5526{*}10^{-3}$	$(-3.2391{*}10^{-2}, -1.8419{*}10^{-2})$
$R^2 = 0.96777$			

TABLE A2.2
Comparison between model (A2.4) and Guseo and Mortarino's (2014) model estimated with instantaneous sales.

	R^2	ρ^2
model (A2.4)	0.9678	0.9678
Guseo and Mortarino's (2014) model	0.8920	0.8937
	$\widetilde{R}^2 = 0.7016$	$F = 432.57$

If we use model (A2.4), we obtain the parameter estimates shown in Table A2.1. A comparison with results shown in Table 1 highlights that the procedure relying on cumulative data leads uniformly to smaller standard errors for all the parameters.

From Figure A2.1, comparing the estimated mean trajectory with model (3.6) (already shown in Figure 2) and with model (A2.4), we see that the two methods give rise to almost overlapping paths, even if the squared correlation coefficient between observed and fitted values is slightly higher when instantaneous data are used (0.9678 vs. 0.9672). Further, the estimate of the dynamic market potential structure, $\hat{m}(t)$, is fully coherent with the one obtained from model (3.6) (see Figure A2.2).

In Section 5, the comparison between the CDMP model, (3.6), and the simpler model of Guseo and Mortarino (2014) was conducted through a test statistic based upon the respective R^2 values. Of course, if we use model (A2.4) for the comparison, we should estimate Guseo and Mortarino's (2014) model with the same approach (using instantaneous sales as response variables). Table A2.2 shows the comparison. The change in the estimation method does not modify the conclusions regarding the superiority of a dynamic market potential structure.

Appendix 3. Predictive confidence bands. Starting from Srinivasan and Mason (1986) and, more recently, Boswijk and Franses (2005), we

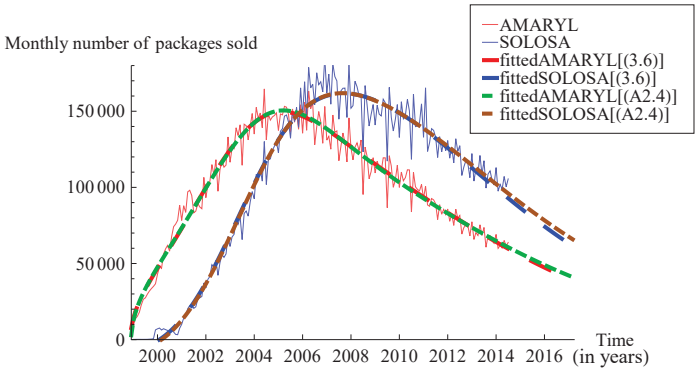


FIG A2.1. *Comparison of the observed and fitted values, instantaneous sales, model* (A2.4) *(the fitted values with the CDMP model,* (3.6)*, are also shown).*

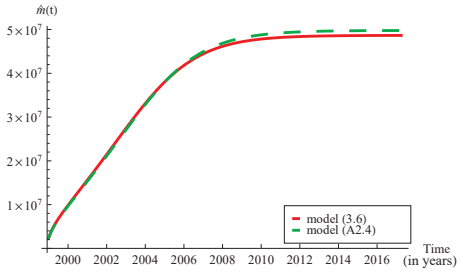


FIG A2.2. *Comparison of the estimated market potential function* $\hat{m}(t)$ *with the CDMP,* (3.6)*, and* (A2.4) *models.*

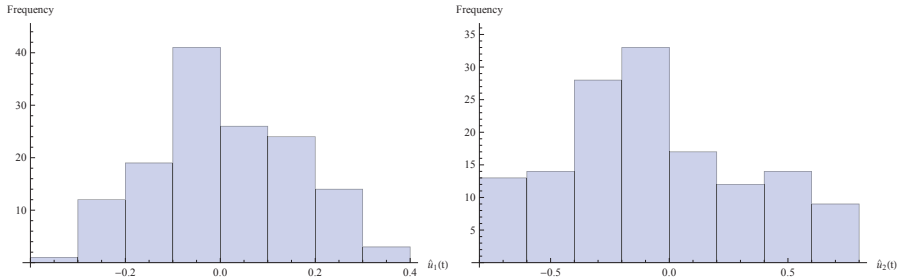


FIG A3.3. *Histograms of the estimates* $\hat{u}_i(t), i = 1, 2, t = 49, \ldots, 188$*, based on the CDMP model,* (3.6).

know that—for the models under study—estimators do not have standard asymptotic properties and the parameters cannot be estimated consistently even as the sample period increases. Intuitively, since we describe finite diffusion processes, an increase in sample size with observations for t values after the end of the process does not correspond to an increase in information.

The procedure used to estimate the parameters using cumulative data (model (3.6)) with a nonlinear least squares algorithm (NLS) does not at first require detailed assumptions about the structure of $\varepsilon_i(t)$. Following Boswijk and Franses (2005), it is, however, important to allow for heteroscedasticity in $\varepsilon_i(t)$. Alternative model specifications may be appropriate, but a sensible assumption is

$$v_i(t) = z_i(t; \beta) + \varepsilon_i(t) = z_i(t; \beta) + z_i'(t; \beta) u_i(t), \tag{A3.5}$$

where $u_i(t)$ are supposed to be normally distributed, with zero mean and constant variance, ${}_u\sigma_i^2$, for $i = 1, 2$. The structure (A3.5) takes into account the specific heterogeneity issue typical of saturating diffusion models (low variability around the mean trajectory both at the beginning and at the end of the diffusion cycle, with higher variability when the diffusion peaks).

The data described in this paper support the structure (A3.5), as is evident from Figure A3.3, representing the histograms of the estimates $\hat{u}_i(t), i = 1, 2, \ t = 49, \ldots, 188$, obtained through the residuals of model (3.6). An examination of Figure 2 reveals that for both series, the fit in the first part of the series is worse and the estimate of the stochastic component, based upon residuals inflated by a partial lack-of-fit, is biased. To avoid that effect, since the purpose is to build plausible confidence bands for the future evolution of the series, we have excluded the first 48 residuals (4 years) of both series. The normality assumption is confirmed, and the corresponding

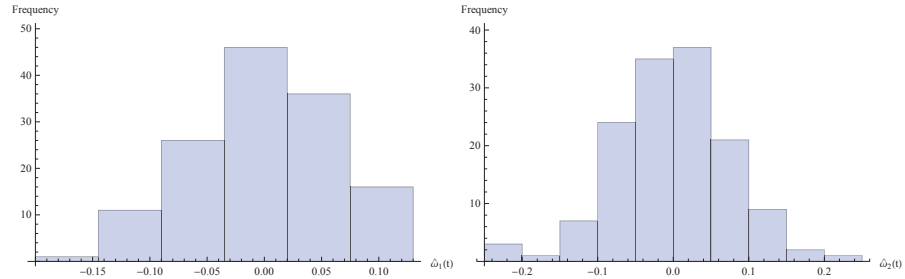


FIG A3.4. *Histograms of the estimates* $\hat{\omega}_i(t), i = 1, 2$ *based on model* (A2.4).

variance estimates are ${}_u\hat{\sigma}_1^2 = 0.0231$ and ${}_u\hat{\sigma}_2^2 = 0.1436$. Thus, we can use the following values as confidence bands for the predictions $\hat{z}_i(t) = z_i(t; \hat{\beta})$, i.e.,

$$\hat{z}_i(t) \pm 2\, {}_u\hat{\sigma}_i\, \hat{z}_i'(t), \quad t = 189, \ldots$$

When the fitted values are represented in terms of instantaneous data (as in Figure A2.1), the prediction confidence bands have to be modified in the following manner. Beginning from instantaneous observed data,

$$\begin{aligned} s_i(t) &= v_i(t+1) - v_i(t) \\ &\overset{(A3.5)}{=} z_i(t+1) + z_i'(t+1)u_i(t+1) - z_i(t) - z_i'(t)u_i(t) \\ &= z_i(t+1) - z_i(t) + z_i'(t+1)u_i(t+1) - z_i'(t)u_i(t), \end{aligned}$$

and thus the amplitude of the confidence bands around the estimated trajectory,

$$\hat{z}_i(t+1) - \hat{z}_i(t),$$

should be proportional to the standard error of

$$z_i'(t+1)u_i(t+1) - z_i'(t)u_i(t),$$

given by

$$[\hat{z}_i'(t+1)]^2 + [\hat{z}_i'(t)]^2 - 2\hat{z}_i'(t+1)\hat{z}_i'(t))\widehat{\mathrm{Cov}}[u_i(t+1), u_i(t)].$$

Note that the last term of the previous expression must not be neglected and equals 0.0207 for $i = 1$ and 0.1419 for $i = 2$. The resulting confidence bands are plotted in Figure 5.

When we turn to the regression model (A2.4), an analogous argument leads to

(A3.6)

$$s_i(t) = z_i(t+0.5) - z_i(t-0.5) + \xi_i(t) = z_i(t+0.5) - z_i(t-0.5) + z_i'(t)\omega_i(t),$$

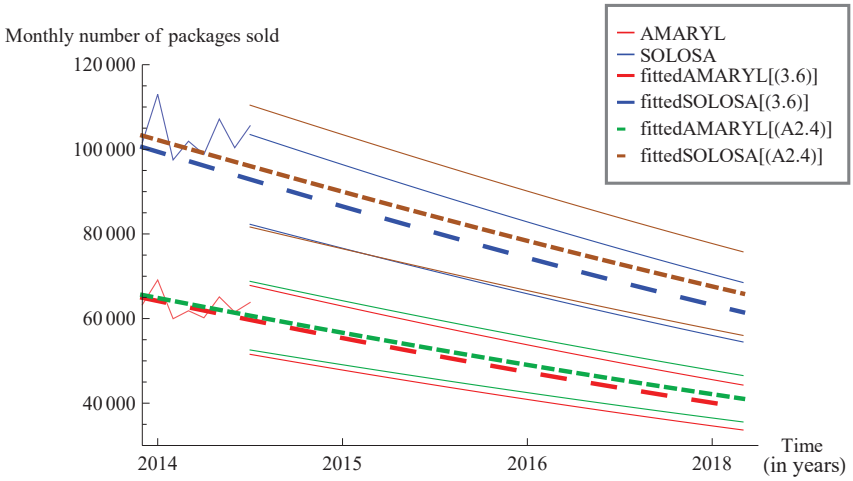


FIG A3.5. *Forecasted mean sales trajectory and confidence bands, model* (A2.4).

where $\omega_i(t)$ are supposed to be normally distributed, with zero mean and constant variance, ${}_\omega\sigma_i^2, i = 1, 2$.

The data described in this paper support the structure (A3.5), as is evident from Figure A3.4, representing the histograms of the estimates $\hat{\omega}_i(t), i = 1, 2$, obtained through the residuals of model (A2.4). The first 48 residuals of both series were excluded for the same reason described for the residuals of model (3.6). The normality assumption is confirmed and the corresponding variance estimates are ${}_\omega\hat{\sigma}_1^2 = 0.0045$ and ${}_\omega\hat{\sigma}_2^2 = 0.0056$. Thus, we can use the following confidence bands for the predictions $\hat{z}_i(t + 0.5) - \hat{z}_i(t - 0.5)$, i.e.,

$$\hat{z}_i(t + 0.5) - \hat{z}_i(t - 0.5) \pm 2\,{}_\omega\hat{\sigma}_i\,\hat{z}_i'(t), \quad t = 189, \ldots$$

The resulting confidence bands are plotted in Figure A3.5 together with the confidence bands obtained through model (3.6). Again, we observe that model (3.6) leads to greater precision: We obtained narrower bands for Solosa and very similar bands for Amaryl.

Appendix 4. SARMAX refinement. As mentioned in Section 4, for short-term prediction, we use a two-step procedure. First, we apply a robust NLS algorithm to model (3.6), which ignores the stochastic structure of $\varepsilon_i(t)$, under the well-known Levenberg-Marquardt correction of the Gauss-Newton recursive procedure (see, e.g., Seber and Wild, 2003). Second, the prediction

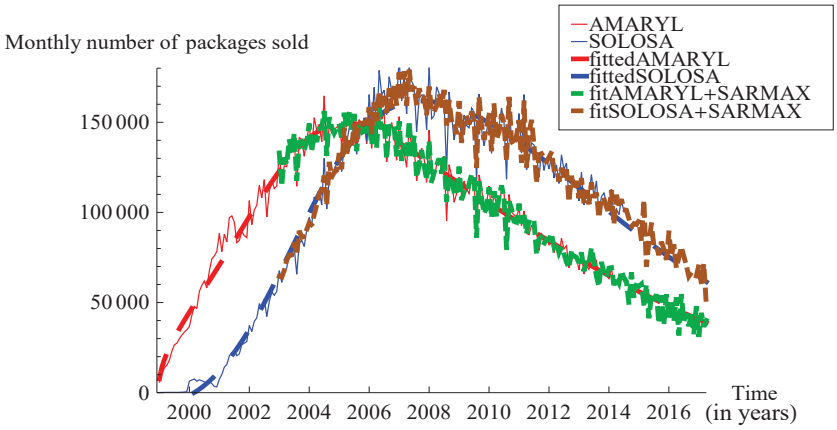


FIG A4.6. *Comparison of the fitted values using the CDMP model,* (3.6)*, and the SARMAX refinement.*

$z_i(t;\hat{\beta})$, based on an NLS solution, $\hat{\beta}$, may be used, as an input X, in a model based on a seasonal, autoregressive, moving average process (SARMAX) to improve short-term prediction, which is relevant for managerial applications. This second step is implemented if the residuals of the first stage do not follow a standard white noise pattern. The Durbin-Watson statistic may be used as an exploratory test to diagnose whether this second step is necessary. In this case, the Durbin-Watson statistic equals 0.0847, distinctly detecting a positive autocorrelation.

The SARMAX improvement for short-term predictions rests on the following equation based on a polynomial function of backward operators, namely,

(A4.7)
$$\Psi(B)\Phi(B^s)\left\{s_i(t) - c_i[z_i(t+0.5;\hat{\beta}) - z_i(t-0.5;\hat{\beta})]\right\} = \Omega(B)\Theta(B^s)a_i(t)$$

with $a_i(t)$ a White Noise process; B and B^s the standard and seasonal backward operators; and $\Psi(B)$, $\Phi(B^s)$, $\Omega(B)$, and $\Theta(B^s)$ the usual backward polynomials of order g, G, h, and H, respectively. The calibration parameters c_i allow a global assessment of the stability of the predicted regressive values stemming from the estimated models $z_i(t;\hat{\beta})$.

As above, since the first part of the series produced higher residuals and since SARMAX is meant as an improvement for predictive purposes, we

TABLE A4.3

Parameter estimates for the SARMAX refinement. [ ] t statistic. ∗: significant, 95%. *∗∗: strongly significant,* 99%. *prM denotes the fitted values with the CDMP model,* (3.6). *The subscripts of* $\tilde{R}^2$ *and* F *define the involved nested models; in particular,* $M|S$ *denotes the comparison between the CDMP model and the SARMAX model.*

Model	Parameter	**Amaryl**	**Solosa**
SARMAX + prM	AR1	-0.9096**	0.1372
	AR2	0.0446	1.0304**
	AR3	0.6661**	0.7554
	AR4	0.2504	-0.4155
	AR5	-0.4528**	-0.7863**
	AR6	–	0.2391
	MA1	-0.6233**	0.2806
	MA2	0.2330*	0.8279**
	MA3	0.4287**	0.4580
	MA4	-0.3284**	-0.5807
	MA5	-0.7675**	-0.3927*
	MA6	–	0.3771
	SAR1	0.6777**	1.0623**
	SAR2	-0.7824**	-0.0361
	SAR3	1.0223**	-0.8886**
	SAR4	–	1.8993**
	SAR5	–	-0.7673**
	SMA1	0.1788	0.6459**
	SMA2	-0.8049**	0.2042*
	SMA3	0.6395**	-0.8361**
	SMA4	–	1.5490**
	SMA5	–	-0.0185
prM	c	0.9885**	1.0067**
$[t_M]$		[32.4488]	[16.2051]
RSS		$2.3043*10^9$	$9.0195*10^9$
RSS_M		$8.3637*10^9$	$1.5361*10^{10}$
$\tilde{R}^2_{M\|S}$		0.7245	0.4128
$F_{M\|S}$		17.8228	3.3985

chose to apply it only to the second part of the series ($t > 48$), whose data are more relevant for future evolution.

The estimates of the parameters involved in Equation (A4.7) applied to instantaneous data are proposed in Table A4.3. The agreement between the observed data and the fitted values with the SARMAX refinement is almost perfect (see Figure A4.6). This confirms that the discrepancies between the observed data and the fitted values with the CDMP model were essentially due only to autoregressive/moving average components and seasonal effects.

Further, the overlapping of the fitted trajectory on the observed data strongly supports the choice of model (3.6) for the description of the mean trajectories of the sales data for both drugs.

Appendix 5. Simulation study.

A5.1. *Correct specification.* In order to assess the performance of the proposed CDMP model for different fluctuation levels around the mean trajectory, a simulation study has been performed as follows. We started from a fixed parameter configuration,

$$\beta_0 = \{10000, 0.07, 0.04, 0.02, 0.03, 0.005, 0.1, 0.05\}.$$

We simulated 1000 instantaneous datasets as

$$s_i(t) = z_i'(t; \beta_0) + z_i'(t; \beta_0)\omega_i(t), \quad i = 1, 2, \quad t = 1, \ldots, 50,$$

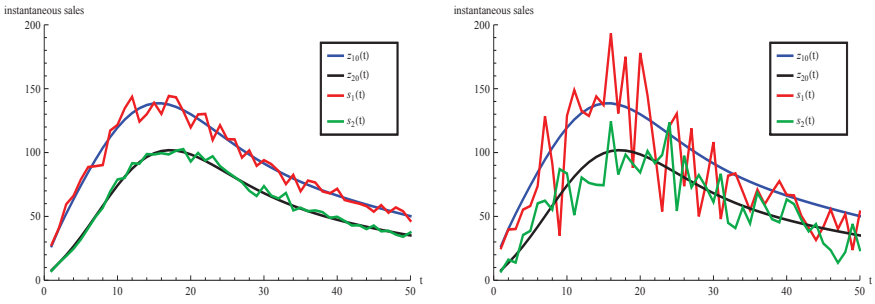
where $(\omega_1(t), \omega_2(t)) \sim \mathcal{N}_2({}_\omega\sigma_1^2, {}_\omega\sigma_2^2, -0.1{}_\omega\sigma_1{}_\omega\sigma_2)$. The correlation value has been fixed at -0.1 to represent a substitution effect between the sales of the two products.

The ${}_\omega\sigma_i$ were allowed to vary in the set $\{0.05, 0.10, 0.15, 0.20, 0.25\}$. As an example, Figure A5.7 shows the plot of the first simulated dataset in the case ${}_\omega\sigma_i = 0.05$ (a) and ${}_\omega\sigma_i = 0.25$ (b). The latter case corresponds to a situation with a very high noise–to–signal ratio. For each simulated dataset, the corresponding cumulative data have been used to fit the CDMP model, (3.6). Notice that this situation corresponds to a correct specification, because, for data simulation, we used the mean instantaneous trajectories, $z_i(t)$, with $m(t)$ specified as in (2.4).

Table A5.4 shows the mean squared error (MSE) of the parameter estimates

$$MSE(\hat{\beta}_j) = \frac{1}{1000} \sum_{k=1}^{1000} (\hat{\beta}_{jk} - \beta_{j0})^2,$$

where $\hat{\beta}_{jk}$ denotes the estimate of the j-th parameter obtained with the k-th simulated dataset.



(a) (b)

FIG A5.7. *First simulated instantaneous dataset for* ${}_{\omega}\sigma_i = 0.05$ *(a) and* ${}_{\omega}\sigma_i = 0.25$ *(b).*

TABLE A5.4
MSE of the parameter estimates

parameter (β_{j0})	K (10000)	p_c (0.007)	q_c (0.04)	p_1 (0.02)
${}_{\omega}\sigma_i = 0.05$	$7.0655*10^{5}$	$1.5130*10^{-7}$	$9.1726*10^{-5}$	$1.1897*10^{-5}$
${}_{\omega}\sigma_i = 0.10$	$7.7058*10^{5}$	$3.8166*10^{-7}$	$1.3603*10^{-4}$	$2.5986*10^{-5}$
${}_{\omega}\sigma_i = 0.15$	$1.2554*10^{6}$	$8.3944*10^{-7}$	$2.3944*10^{-4}$	$5.5103*10^{-5}$
${}_{\omega}\sigma_i = 0.20$	$4.6040*10^{7}$	$4.2749*10^{-6}$	$5.0578*10^{-4}$	$6.7359*10^{-5}$
${}_{\omega}\sigma_i = 0.25$	$5.4571*10^{8}$	$7.7279*10^{-6}$	$8.2287*10^{-4}$	$3.4129*10^{-4}$

parameter (β_{j0})	q_1 (0.03)	p_2 (0.005)	q_2 (0.1)	δ (0.05)
${}_{\omega}\sigma_i = 0.05$	$2.7558*10^{-3}$	$1.4823*10^{-6}$	$3.5251*10^{-3}$	$8.9532*10^{-3}$
${}_{\omega}\sigma_i = 0.10$	$8.3951*10^{-3}$	$6.5940*10^{-6}$	$9.1042*10^{-3}$	$2.4722*10^{-2}$
${}_{\omega}\sigma_i = 0.15$	$3.1755*10^{-2}$	$2.1710*10^{-5}$	$3.2648*10^{-2}$	$9.0742*10^{-2}$
${}_{\omega}\sigma_i = 0.20$	$6.7459*10^{-2}$	$3.1218*10^{-5}$	$6.9825*10^{-2}$	$1.8678*10^{-1}$
${}_{\omega}\sigma_i = 0.25$	$9.3869*10^{-1}$	$2.7502*10^{-4}$	$9.4350*10^{-1}$	2.6786

TABLE A5.5
MISE for the market potential function, $m(t)$

	$T = 50$	$T = 60$	$T = 70$	$T = 88$	$T = 114$
${}_{\omega}\sigma_i = 0.05$	7.9842*10^5	8.4136*10^5	9.5680*10^5	1.8808*10^6	6.7524*10^6
${}_{\omega}\sigma_i = 0.10$	1.0264*10^6	1.1166*10^6	1.3408*10^6	2.8253*10^6	9.2680*10^6
${}_{\omega}\sigma_i = 0.15$	1.6529*10^6	1.8306*10^6	2.2428*10^6	4.7968*10^6	1.5172*10^7
${}_{\omega}\sigma_i = 0.20$	3.8034*10^6	4.1396*10^6	4.8721*10^6	9.0723*10^6	2.7517*10^7
${}_{\omega}\sigma_i = 0.25$	6.3988*10^7	6.9203*10^7	8.0086*10^7	1.3918*10^7	3.8715*10^7

For ${}_{\omega}\sigma_i \leq 0.20$, we observe a gradual deterioration in the accuracy of the estimates of the evolutionary parameters (p_1, q_1, p_2, q_2, and δ). The estimates are quite unstable in the case ${}_{\omega}\sigma_i = 0.25$, corresponding to an extreme value of the noise–to–signal ratio. Notice that the MSE, as an average, may be heavily affected by some odd convergence points for a few simulated datasets. Due to the high number of datasets, the estimation procedure is unsupervised and common initial values have been used for all of them.

One interesting issue is to see whether the model allows for good estimates of the market potential component, $m(t)$. Thus the mean integrated squared error (MISE) has been evaluated as follows:

$$MISE(\hat{m}(t)) = \frac{1}{1000} \sum_{k=1}^{1000} \int_{t=0}^{T} [\hat{m}_k(t) - m_0(t)]^2 dt,$$

where $\hat{m}_k(t)$ denotes the estimate of $m(t)$ obtained with the k-th simulated dataset, $m_0(t)$ denotes the true function $m(t)$ used to generate the data, and $T = 50, 60, 70, 88, 114$. The two final values represent, respectively, the 95-th and 99-th quantiles of $z_i(t; \beta_0)$. In other words, after 114, the products' lifecycle is essentially concluded and the estimation issue is no longer interesting (firms usually stop offering a product when sales levels are negligible). Because a product's exit from the market is often anticipated due to high commercialization costs, the 95-th quantile, 88, is also an interesting endpoint to consider. Notice that the simulated data used to fit the model cover the first 50 time points. This is why the first T value has been set at 50. The values 60 and 70 represent, for forecasting purposes, a medium term and a long term. Table A5.5 shows the MISE values. Unlike the MSE values, the MISE results do not highlight a sudden jump for high ${}_{\omega}\sigma_i$ values, but they smoothly increase with ${}_{\omega}\sigma_i$ values.

The left side of Figures A5.8-A5.12 shows the plots of $m_0(t)$, $\hat{m}_k(t)$, $\overline{m}(t) = \frac{1}{1000} \sum_{k=1}^{1000} \hat{m}_k(t)$, the median trajectory, and the quantile trajectories (0.05 and 0.95). Because our main interest is in the global model function, however, the right side of Figures A5.8-A5.12 shows the plots

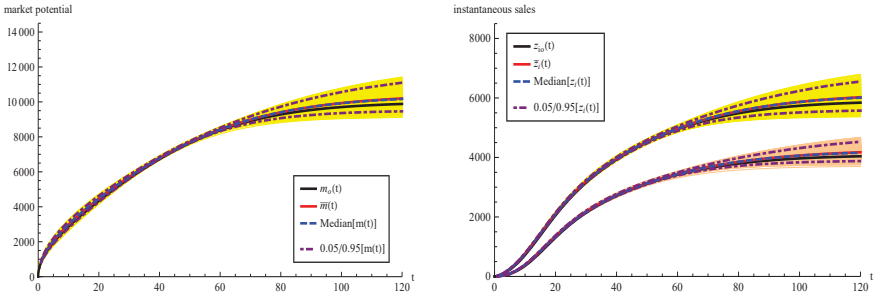


FIG A5.8. *Case* $_{\omega}\sigma_i = 0.05$. *Left side:* $\hat{m}_k(t)$ *(yellow trajectories), true* $m_0(t)$, $\overline{m}(t) = \frac{1}{1000}\sum_{k=1}^{1000}\hat{m}_k(t)$, *median trajectory, and quantile trajectories (0.05 and 0.95). Right side:* $\hat{z}_{ik}(t)$ *(yellow and orange trajectories), true* $z_{i0}(t)$, $\overline{z}_i(t) = \frac{1}{1000}\sum_{k=1}^{1000}\hat{z}_{ik}(t)$, *median trajectory, and quantile trajectories (0.05 and 0.95).*

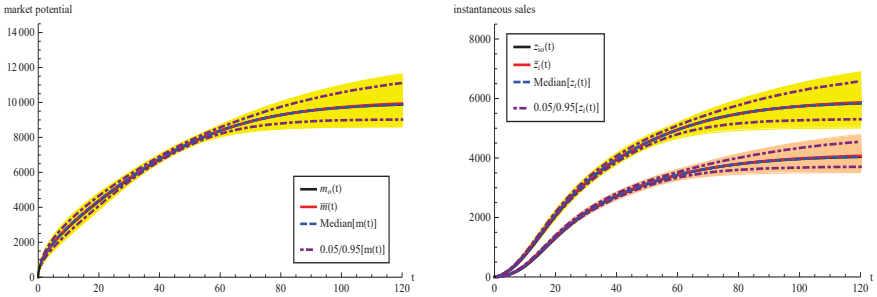


FIG A5.9. *Case* $_{\omega}\sigma_i = 0.10$. *Left side:* $\hat{m}_k(t)$ *(yellow trajectories), true* $m_0(t)$, $\overline{m}(t) = \frac{1}{1000}\sum_{k=1}^{1000}\hat{m}_k(t)$, *median trajectory, and quantile trajectories (0.05 and 0.95). Right side:* $\hat{z}_{ik}(t)$ *(yellow and orange trajectories), true* $z_{i0}(t)$, $\overline{z}_i(t) = \frac{1}{1000}\sum_{k=1}^{1000}\hat{z}_{ik}(t)$, *median trajectory, and quantile trajectories (0.05 and 0.95).*

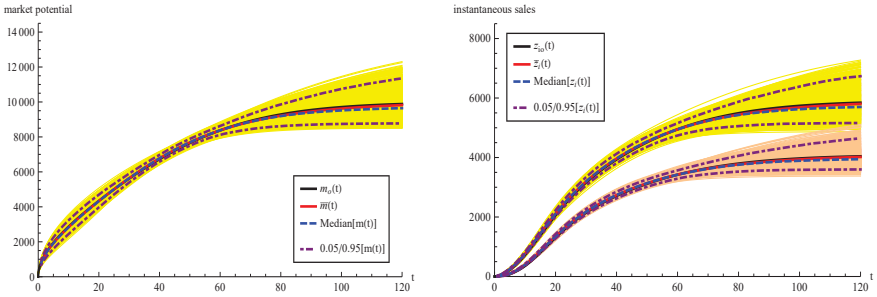


FIG A5.10. *Case* $_{\omega}\sigma_i = 0.15$. *Left side:* $\hat{m}_k(t)$ *(yellow trajectories), true* $m_0(t)$, $\overline{m}(t) = \frac{1}{1000}\sum_{k=1}^{1000}\hat{m}_k(t)$, *median trajectory, and quantile trajectories (0.05 and 0.95). Right side:* $\hat{z}_{ik}(t)$ *(yellow and orange trajectories), true* $z_{i0}(t)$, $\overline{z}_i(t) = \frac{1}{1000}\sum_{k=1}^{1000}\hat{z}_{ik}(t)$, *median trajectory, and quantile trajectories (0.05 and 0.95).*