



Self-similar aftershock rates

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In many important systems exhibiting crackling noise—an intermittent avalanchelike relaxation response with power-law and, thus, self-similar distributed event sizes—the “laws” for the rate of activity after large events are not consistent with the overall self-similar behavior expected on theoretical grounds. This is particularly true for the case of seismicity, and a satisfying solution to this paradox has remained outstanding. Here, we propose a generalized description of the aftershock rates which is both self-similar and consistent with all other known self-similar features. Comparing our theoretical predictions with high-resolution earthquake data from Southern California we find excellent agreement, providing particularly clear evidence for a unified description of aftershocks and foreshocks. This may offer an improved framework for time-dependent seismic hazard assessment and earthquake forecasting.

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I. INTRODUCTION

Many natural and man-made systems exhibit an intermittent avalanchelike response to changing external conditions [1,2]. Sequences of such sudden responses or events often constitute the most crucial features of the evolutionary dynamics of complex systems, in terms of their description, characterization, and understanding. Prominent examples include earthquakes on fault systems [3], frictional sliding [4], irreversible plastic deformation in solids [5–7], fracture [8–13], materials failure [14,15], magnetization processes [16,17], solar flare emissions [18,19], financial markets [20–23], Internet traffic [24], and media coverage [25]. The avalanchelike response—often called crackling noise—is characterized by discrete, impulsive events spanning a broad range of energies E , with a power-law frequency distribution $P(E) \propto E^{-\epsilon}$ [26–29]. The empirical Gutenberg-Richter (GR) relation for earthquakes is one specific example [30]: Energies released by earthquakes follow a power-law distribution and are conveniently handled in the logarithmic scale of the magnitude m with $E \propto 10^{\frac{3}{2}m}$ such that $P(m) \propto 10^{-bm}$, where $b = \frac{3}{2}(\epsilon - 1)$. There is also a good degree of universality, as ϵ is close to 1.5 for many systems exhibiting crackling noise [29]. The associated absence of characteristic scales indicates scale-free or self-similar behavior.

Significant progress has been made in understanding the self-similar distribution of event sizes and the associated universality of crackling noise by using mean-field and renormalization approaches [17,26–29,31–34]. These theoretical approaches elevate self-similar behavior to a general principle as in the case of critical phenomena in equilibrium systems. Thus, it is important to establish whether other properties often associated with crackling noise also exhibit self-similar behavior. This includes those spatiotemporal correlations between events that reflect the intrinsic or endogenous dynamics of a

given system and are a consequence of event-event triggering as for aftershocks [6,35–39]. This is most clearly reflected in the time-varying (local) event rates following large events, which are empirically found to follow—across a wide range of scales and systems from friction and fracture to socioeconomic systems [10,12,13,19–25,37,40–43]—the Omori-Utsu (OU) relation,

$$r(t) = \frac{K}{(t+c)^p} \equiv \frac{1}{\tau(t/c+1)^p}, \quad (1)$$

first proposed for earthquakes [44]. Here, t measures the time after the large event or trigger, p is typically close to 1 ($p \gtrsim 1$ if one only considers directly triggered events [42]), and $\tau \equiv c^p/K$. K is found to increase with the energy of the trigger, according to the *productivity* relation $K = K_0 E^{2\alpha/3}$. Its equivalent formulation in terms of the magnitude of the trigger, M , is $K = K_0 10^{\alpha M}$. The exponent α is less than b across many systems exhibiting crackling noise [6,10,12,13,22,37,45]. The parameter K_0 naturally depends on the observational threshold m_{th} [31,46]; by lowering it one counts more triggered events and it was reported $K_0 \sim 10^{-\beta m_{\text{th}}}$ [42,47–49]. In principle the exponents α and β and the b value for triggered events or aftershocks [37,50], b_{as} , appearing in the mentioned scalings may be different and indeed this is often observed [37,42,45,51–53].

A consequence of the difference between α and b_{as} across many systems exhibiting crackling noise is the breakdown of self-similarity in the triggering process in those cases [54]. Specifically, the number of triggered events of a given energy will explicitly depend on the energy of the trigger and not just their energy ratio: Since the magnitudes of the triggered events or aftershocks are distributed according to $P(m_{\text{as}}) \propto 10^{-b_{\text{as}} m_{\text{as}}}$, the number of events with magnitude m_{as} triggered by an event of magnitude M scales as $10^{\alpha M - b_{\text{as}} m_{\text{as}}}$ for constant c in Eq. (1). This translates into a scaling with $10^{(\alpha - b_{\text{as}})M + b_{\text{as}} \Delta m}$ with $\Delta m = M - m_{\text{as}}$ that includes an explicit dependence on M for $\alpha \neq b_{\text{as}}$, in disagreement with the hypothesis of a self-similar triggering process that only depends on Δm [31,46,54–59]. An explicit dependence on M would imply that the triggering

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process inducing a number of events of, say, magnitude 6 due to an event of magnitude 8 is fundamentally different from the triggering process inducing a number of events of magnitude 3 due to an event of magnitude 5.

To reconcile different values of α and b_{as} with self-similar triggering, we build on the well-established behavior of equilibrium critical phenomena and propose a natural generalization of the OU relation that is consistent with a self-similar triggering process. Specifically, we build on the fact that critical phenomena can be characterized not only by critical exponents but also by universal scaling functions that describe the behavior near equilibrium critical points [60]. A general way to cast the event-event triggering rates into such a scaling form under the condition that the rates should only depend on the energy ratios between trigger and triggered event or, equivalently, their magnitude difference $\Delta m = M - m_{\text{as}}$ is the following ansatz:

$$r(m_{\text{as}}, t | M, 0) = \frac{1}{\tau_{\Delta m}} f\left(\frac{t}{c_{\Delta m}}\right), \quad (2)$$

where $\tau_{\Delta m}$ and $c_{\Delta m}$ are two time scales varying only with Δm . In fact, an approach based on a limited scaling form with a constant $\tau_{\Delta m} = \tau$ was previously introduced by Lippiello *et al.* [58,59]. In this paper, by analyzing high-resolution earthquake data from Southern California, we show that only the full self-similar generalization of the OU relation, in a form following Eq. (2) with both $c_{\Delta m}$ and $\tau_{\Delta m}$ scaling with Δm and with a specific functional form of f , is consistent with all empirical relations. In this generalized form of the OU relation, self-similarity is present even in the case of a nonconstant c and $\alpha \neq b_{\text{as}}$.

II. GENERALIZED OU RELATION

Self-similarity in the OU relation (1) is violated if the number of the directly triggered events of magnitude m_{as} does explicitly depend on the main shock magnitude M , and not only on the magnitude difference. To ensure self-similarity and consistency with empirical observations, we propose the following generalization of the OU relation for *conditional* event-event triggering rates:

$$r(m_{\text{as}}, t | M, 0) = \frac{1}{\tau_{\Delta m}} \left(\frac{t}{c_{\Delta m}} + 1\right)^{-p}, \quad (3)$$

with time scales

$$c_{\Delta m} = c_0 10^g \Delta m \quad \text{and} \quad \tau_{\Delta m} = \tau_0 10^{-z} \Delta m \quad (4)$$

scaling with Δm ($g \geq 0$ and $z \geq 0$ are supposedly universal scaling exponents while c_0 and τ_0 are constant prefactors). Specifically, $r(m_{\text{as}}, t | M, 0)$ is the rate of events or aftershocks of magnitude m_{as} at time t triggered directly by an event of magnitude M at time zero. This generalized OU relation corresponds to Eq. (2) with $f(x) = (1+x)^{-p}$; hence, it is a natural generalization in the sense that f is consistent with the standard OU form and all parameters now depend explicitly and exclusively on Δm . Note also that Eq. (3) does not require $M > m_{\text{as}}$ but it is applicable to all magnitude or energy combinations such that it encompasses what is often considered main shock-aftershock pairs as well as

foreshock-main shock pairs. Here, a foreshock is an event that triggers another event with a larger magnitude.

With a simple mathematical derivation (see Ref. [55] for a somewhat similar derivation in a context in which the magnitude of the largest aftershock is assumed to play a significant role), we may show that Eq. (3) ensures self-similarity, as the total number of triggered events of magnitude m_{as} ,

$$N(m_{\text{as}} | M) \equiv \int_0^\infty r(m_{\text{as}}, t | M, 0) dt = \frac{c_{\Delta m}}{\tau_{\Delta m}(p-1)}, \quad (5)$$

depends only on Δm . Such self-similar generalization of the OU relation is consistent with the GR relation since

$$\begin{aligned} N_{>}(m_{\text{th}} | M) &\equiv \int_{m_{\text{th}}}^\infty N(m_{\text{as}} | M) dm_{\text{as}} \\ &= \frac{c_0}{\tau_0(p-1) \ln 10(g+z)} 10^{(g+z)(M-m_{\text{th}})}. \end{aligned} \quad (6)$$

Thus, we have the scaling relation

$$b_{\text{as}} = g + z, \quad (7)$$

indicating that only two out of these three scaling exponents are independent. Hence, the generalized OU relation does not introduce an additional independent parameter compared to the standard OU relation with its associated productivity relation. In particular, for $K_{\Delta m} \equiv c_{\Delta m}^p / \tau_{\Delta m}$ we also have a generalized productivity relation

$$K_{\Delta m} = K_0 10^{\alpha \Delta m} \quad (8)$$

with $K_0 = c_0^p / \tau_0$ and

$$\alpha = z + pg = b_{\text{as}} + g(p-1). \quad (9)$$

Note that this implies that $K_{\Delta m}$ and $N(m_{\text{as}} | M)$ do *not* scale the same way with Δm if $g \neq 0$, which explicitly allows $b_{\text{as}} \neq \alpha$ in our self-similar framework.

A related consequence of the generalized OU relation given by Eqs. (3) and (4) is that the GR relation for triggered events needs to be modified if only triggered events over short time intervals are considered. For example, the number of triggered events or aftershocks of magnitude m_{as} up to time t^* ,

$$N(m_{\text{as}}, t^* | M) \equiv \int_0^{t^*} r(m_{\text{as}}, t | M, 0) dt, \quad (10)$$

only follows the GR relation (5) for $t^* \rightarrow \infty$. For finite t^* , $N(m_{\text{as}}, t^* | M)$ has two power-law regimes: For small m_{as} , it decays with exponent z while it decays with exponent $b_{\text{as}} = g + z$ for large m_{as} . The transition point is around a magnitude m^* for which $c_{M-m^*} \approx t^*$. Thus, the transition point moves to lower magnitudes for increasing t^* , recovering the full GR relation for $t^* \rightarrow \infty$. Note that only for $g = 0$ —corresponding to a constant $c_{\Delta m}$ —the GR relation holds for all t^* .

In contrast to the conditional rates in Eqs. (2) and (3), the classic OU relation (1) considers the integrated rates $r_{>}(m_{\text{th}}, t | M, 0) \equiv \int_{m_{\text{th}}}^\infty r(m_{\text{as}}, t | M, 0) dm_{\text{as}}$. For the proposed self-similar OU conditional rates in Eq. (3), one easily verifies that such integrated rates inherit the scale invariance with respect to $M - m_{\text{th}}$. Moreover, these integrated rates have a functional form very similar to the classic OU relation for realistic situations. See the Appendix for more details.

To summarize, the proposed self-similar form of the conditional event-event triggering rates can indeed be considered a realistic generalization of the classic OU relation as it is consistent with the GR relation, the productivity relation, as well as the classic form of the OU relation for integrated rates.

III. COMPARISON WITH DATA

To test the validity of the self-similar OU relation, we analyze the event-event triggering for earthquakes in Southern California. Specifically, we analyze the relocated high-resolution Southern California catalog [61]. We consider all local earthquakes with magnitude $m \geq 2$ from 1982 to 2011 (101 991 events).

The triggering relations between earthquakes are identified using the established methodology described in Refs. [37,41,53,62,63]. The method quantifies the correlation between an event i and a following event j via the expression

$$n(i, j) = C |t_i - t_j| |\vec{r}_i - \vec{r}_j|^{D_f} 10^{-bm_i}, \quad (11)$$

which estimates the expected number of events in the space-time window spanned by i and j with magnitude larger than or equal to m_i . Here t_i denotes the time of occurrence of event i , and m_i its magnitude. $b = 1.05$ is the estimated b value for the full catalog. As in Ref. [37], we use hypocenters \vec{r}_i and the parameter $D_f = 2.3$ for the fractal dimension and set $C = 1$ without loss of generality. This leads to a threshold value $\log_{10} n^* = 10.0$ for the identification of triggered events; i.e., only events with $n(i, j) < n^*$ are considered as plausible main shock-aftershock pairs. Among all events i preceding j , the identification of the (most likely) trigger of j results from selecting that with the lowest $n(i, j)$. Further details on the methodology can be found in Ref. [37].

For our analysis of triggering rates, we focus only on directly triggered events; namely, we do not consider aftershocks of aftershocks. To obtain sufficient statistics, rates for all events in a given small magnitude range are stacked and averaged. In particular, triggered events with magnitude $M - m_{as} \in [\Delta m, \Delta m + 0.5)$ are selected for each trigger having a magnitude M in the range $[M, M + 0.5)$.

A. Self-similarity of conditional rates

In Fig. 1(a) we show the conditional triggering rates for different combinations of magnitudes. The striking feature is that all rates with a given Δm are quite indistinguishable from each other, regardless of the magnitude of the trigger. This strongly suggests that Δm is the relevant quantity determining the triggering rates and, thus, supports the hypothesis of self-similarity as formulated in Eq. (2). Since this behavior is independent on whether Δm is positive or negative [see Fig. 1(a)], it also provides a unified description of aftershocks and foreshocks.

To further establish that the dependence on Δm is correctly captured by Eq. (4), we recall that Eq. (3) is an example of the general scaling form (2). This implies that all curves should collapse onto a single master curve, the scaling function $f(x)$, under appropriate rescaling with $c_{\Delta m}$ and $\tau_{\Delta m}$. This is indeed what we observe in Fig. 1(b) for $g \approx 0.66$ and

$z \approx 0.24$ providing direct support for the scaling proposed in Eq. (4).

B. Scaling function

Further support for the proposed self-similar OU relation comes from fitting the conditional triggering rates in Fig. 1 to Eq. (3) using a standard maximum likelihood estimator (MLE) for p , c , and K [42], with $\tau \equiv c^p/K$ as in Eq. (1). Specifically, this allows us to estimate $c_{\Delta m}$ and $K_{\Delta m}$ (and consequently $\tau_{\Delta m}$) and their behavior in a more quantitative way as well as to establish whether the form of the scaling function (3) with a constant p is appropriate. The corresponding results for p are shown in Fig. 2(b). There are no significant differences in the estimates for fixed Δm , though there is an increasing trend in p with Δm for large Δm . This can be attributed to the fact that the MLE slightly overestimates p for large Δm due to a detection issue of triggered events at late times [64]. Indeed, an inspection of the rescaled rates [Fig. 1(d)] confirms that for large Δm the decay t^{-p} is well described by $p \approx 1.15$. This is also compatible with the values of p reported in Fig. 2(b) for smaller values of Δm , which suffer much less from the detection issue due to the more extended range of their power-law decay and follow the proposed functional form (3) very well [see Fig. 1(a) for an example].

Moreover, the direct estimates for c and K show a clear scaling with Δm ; see Figs. 2(c) and 2(d). Not only are the estimates statistically indistinguishable in almost all cases for fixed Δm but the scaling exponent g is also consistent with the value obtained from the rescaling analysis above: Best fits give $g = 0.66 \pm 0.04$ and $\alpha = 1.10 \pm 0.03$, respectively. Here, the error bars correspond to 95% confidence intervals (however, the systematic uncertainties mentioned above lead to higher error bars). The behavior of K shows in particular that the generalized productivity relation (8) holds. This is further supported by Fig. 1(c): The triggering rates collapse for sufficiently large values of t under appropriate rescaling with $\alpha \approx 1$.

C. Scaling relations

The self-similar OU relation provides a consistent picture where several direct MLE estimates of scaling exponents (g , z , p , b_{as} , and α) match estimates from scaling relations (7) and (9) between the different exponents. Having estimated the values of g , z , and α fully determines p and b_{as} . This gives $b_{as} \approx 0.90$ and $p \approx 1.15$. Thus, the value of p is consistent with the directly observed one. This is also true for b_{as} . Figure 2(a) shows MLE estimates of b_{as} [65] as a function of the lower magnitude threshold m_{th} for different main shock ranges: They yield $b_{as} \approx 0.90$, which is clearly consistent with the value of b_{as} obtained from the scaling arguments. This value emerges independently of $m_{th} \geq 2.5$.

Deviations from $b_{as} \approx 0.90$, for main shocks with $M < 3.5$, are consistent with the established presence of earthquake swarms in Southern California [66]. Swarms are typically associated with very specific geological settings and triggering mechanisms and are phenomena dominated by small magnitude events and characterized by larger b_{as} values. Importantly, these higher b_{as} for small main shocks do not significantly

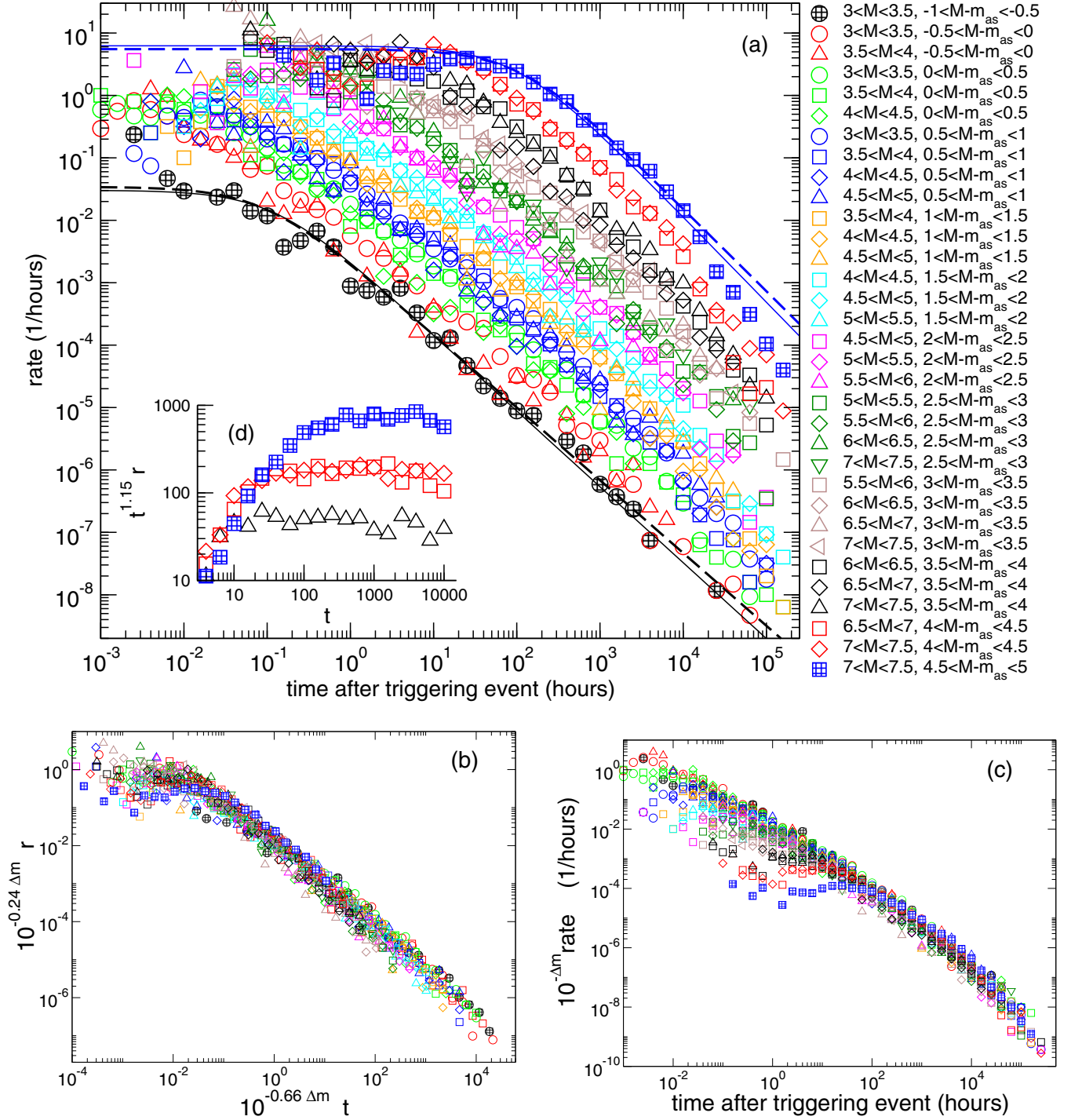


FIG. 1. (a) Averaged conditional aftershock rates for different ranges of main shock and aftershock magnitudes. Note that all curves with the same magnitude difference collapse. The solid lines correspond to OU fits over the full range of t , while the dashed lines correspond to fits up to $t = 10^4$ hours only. See Fig. 2 for all the estimated OU parameters. (b) All curves collapse onto a unique scaling function under appropriately chosen rescaling with the magnitude difference as predicted by the scaling ansatz (3). Data with $t > 10^4$ h are not considered here due to the natural detection problem discussed in the main text. (c) Conditional rates rescaled by $10^{-\alpha \Delta m}$ with $\alpha = 1$. (d) Some conditional rates rescaled by t^p with $p = 1.15$.

affect the quality of the scaling collapse in Fig. 1(b), where only four curves have $M < 3.5$. Yet, we expect that swarm activity will lead to deviations from the self-similar OU relation for smaller magnitudes if not excluded from the triggering analysis.

IV. PHYSICAL ORIGINS VERSUS OBSERVATIONAL LIMITATIONS

Short-term aftershock incompleteness (STAI) is intrinsic to many earthquake catalogs. STAI arises from overlapping

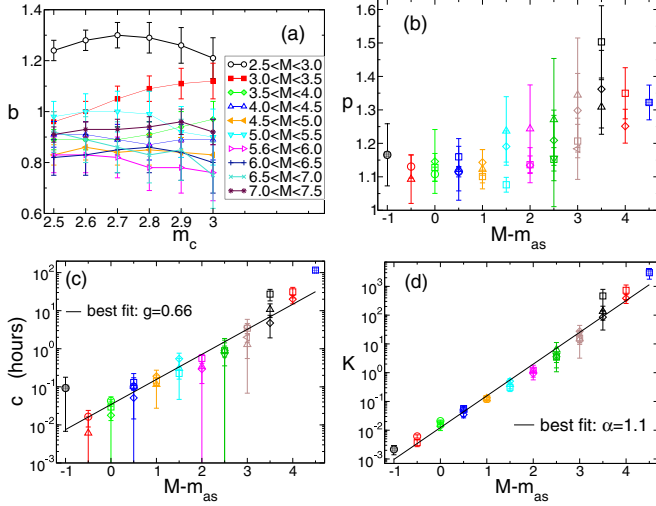


FIG. 2. (a) Estimates of b_{as} as a function of the lower magnitude threshold m_{th} for different main shock ranges. (b)–(d) Estimated parameters of the Omori-Utsu relation (1) for the conditional rates shown in Fig. 1(a).

wave forms and/or detector saturation, in particular after large shocks, such that events are missed in the coda of preceding ones [67–69]. This detection problem is not specific to earthquakes but a general problem related to crackling noise and the identification of “events” from recordings. One important consequence of STAI is an increase in the local magnitude of completeness [70] and, hence, an overestimation of the c value in Eq. (1) for large events [69,71]. Thus, variations in c with Δm for large Δm are typically expected due to STAI [67,68].

Let us discuss several points that exclude STAI as a source of the self-similarity in the underlying triggering process observed for the data from Southern California. First, STAI is not relevant for foreshocks ($\Delta m < 0$) and it has only minor effects for large aftershocks. Yet, the same scaling emerges over the whole range, namely for $-1 \leq \Delta m \leq 5$; see Fig. 1. Moreover, the scaling collapse shown in Fig. 1(b) does even improve if we exclude rates with large Δm : The rates at small times for large Δm are systematically smaller than what the scaling collapse of the other rates suggests. Similarly, the estimated c values are also systematically higher for the largest Δm 's [see Fig. 2(d)]. Both effects are consistent with STAI. Thus, STAI is present, and it leads to minor deviations from the proposed self-similar OU relation, but it cannot explain the observed self-similar behavior itself.

Second, the direct estimates of α , p , and b_{as} are not significantly affected by STAI, since they either reflect the behavior at later times (α , p) or do not vary with magnitude threshold m_{th} [$b_{as} \approx 0.9$, see Fig. 2(a)]. With these values and the scaling relations (7) and (9), the other two exponents g and z are fully determined. Hence, the scaling of $c_{\Delta m}$ with Δm is needed for consistency of the empirical picture where the complete set of five exponents is redundantly determined by direct estimates and scaling relations. One should thus understand the physical mechanisms generating nontrivial $c_{\Delta m}$ [48,72,73].

Third, the limited effect of STAI is also evident from the number of triggered events over finite time intervals, defined

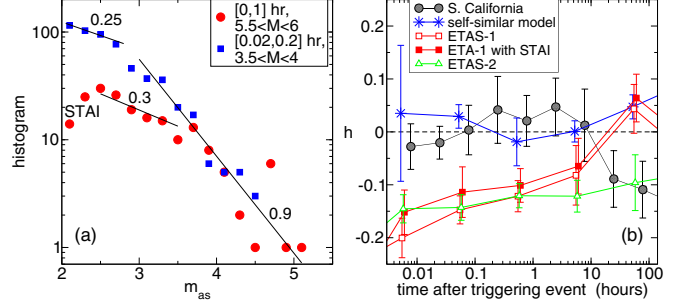


FIG. 3. (a) Stacked time-limited frequency-magnitude distributions for aftershocks. Different regimes are visible consistent with STAI and the proposed self-similar generalization of the OU relation. (b) Indicator of departure from self-similarity as a function of time, for the observed catalog and for four synthetic model catalogs. Only the self-similar model is compatible with the null indicator observed within 1 day from main shocks for the observed data.

in Eq. (10). Figure 3(a) shows two examples. For main shocks with $5.5 \leq M \leq 6.0$ and considering only aftershocks up to time $t^* = 1$ h, we observe the two regimes predicted by the proposed self-similar generalization of the OU relation: one at low m consistent with the value of $z \simeq 0.24$ determined above, and a second regime at larger m with an exponent consistent with $b_{as} \simeq 0.9$. The typical effect of STAI of a temporary increase in the local magnitude of completeness is visible at the lowest magnitudes, where it leads to missing events and strong deviations from the proposed scaling behavior for aftershocks with $m_{as} \leq 2.5$. This effect is not visible for smaller main shocks as the second example in Fig. 3(a) shows: For $3.5 \leq M \leq 4.0$, no deviations from the behavior predicted by the self-similar OU relation are visible on time scales longer than about 70 s.

The prediction by the self-similar OU relation of two power laws in the time-limited frequency-magnitude distribution plus a third regime due to STAI as shown in Fig. 3(a) also allows us to revisit previously published work from a new perspective. In fact, it might be possible to partially connect documented temporal variations of the b value [70] with (previously unnoticed) behavior in the time-limited frequency-magnitude distribution. Reference [74] provides a specific example using a high-resolution catalog that has very carefully addressed the issue of STAI: The time-limited frequency-magnitude distribution of aftershocks in Japan shown in their Fig. 8(b) provides evidence for two different regimes above the magnitude of completeness. This supports the proposed self-similar OU relation beyond the catalog studied here, while fully taking STAI into account.

V. MODEL SIMULATION OF THE SELF-SIMILAR OU RELATION

As a final step to confirm that our analysis is able to distinguish the proposed self-similar OU relation from non-self-similar versions for the conditional rates, we repeat it on synthetic catalogs generated by (a) the standard epidemic-type aftershock sequence (ETAS) model (which is not self-similar if the unconditional rates follow the classic OU relation with constant c and $\alpha \neq b_{as}$) [41,54,75], and (b) our own variant built on the self-similar structure

of Eq. (3) (see the Supplemental Material [76] for more details). If the conditional rates are not self-similar, there remains a dependence on the main shock magnitude M that should be detectable in plots like that shown in Fig. 1. By looking at groups of ETAS conditional rates with the same Δm and increasing M 's, we observe systematic decreases (see the Supplemental Material [76]). The following estimator quantifies the trend with M : for each time t and each Δm we have performed a linear fit of the log-log trend of $r(M - \Delta m, t | M, 0)$ vs M . This yields a slope $h_{\Delta m}(t) \equiv \text{Cov}(\log_{10} r, \log_{10} M) / \text{Var}(\log_{10} M)$, which is supposed to be the same for all Δm 's and all t 's. Hence the average slope $h(t) \equiv \sum_{\Delta m} h_{\Delta m}(t) / \sum_{\Delta m} 1$ is an indicator of the departure from self-similarity in the data. In Fig. 3(b) we can see that, both for the data from Southern California and for the self-similar model, the indicator $h \approx 0$ in the range $t \lesssim 1$ day, while at later times the statistics is not sufficient to perform its reliable estimate. For the ETAS models, the expected value $h = \alpha - b \approx -0.2$ is fairly well detected as well. We also find that self-similar behavior in the ETAS model cannot typically be induced by STAI: If STAI is introduced via one of the typical relations [68], our procedure continues to detect the absence of self-similarity $h \neq 0$ [Fig. 3(b)]. See also the additional analysis in the Supplemental Material [76]. Thus, the comparison with synthetic data shows that our analysis scheme is sensible enough to detect self-similar behavior in the conditional OU rates, corroborating that the conditional triggering rates in Southern California are indeed self-similar.

VI. CONCLUSIONS

In summary, we have described a natural generalization of the OU relation that embodies self-similarity for event-event triggering in crackling noise. From a conceptual point of view, this provides some closure and an important piece in our understanding of event-event triggering where energy, distance, and time appear in several scale-free empirical relations, either singularly (e.g., the GR relation) or combined together. It has also profound consequences for probabilistic forecasting of aftershocks as well as modeling, as it implies that synthetic catalogs of relevant examples of crackling noise, including earthquakes, should be generated with algorithms reproducing the observed self-similarity. An important challenge for the future is to understand its physical origin, possibly with the help of laboratory experiments [10,12,77].

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APPENDIX: CONDITIONAL VERSUS INTEGRATED SELF-SIMILAR OMORI-UTSU RATES

If the conditional aftershock decay rate $r(m, t | M, 0)$ given by Eq. (3) is assumed, the integrated rate $r_{>}(m_{th}, t | M, 0) \equiv \int_{m_{th}}^{\infty} r(m, t | M, 0) dm$ that arises,

$$r_{>}(m_{th}, t | M, 0) = \frac{1}{\tau_0 \ln 10} \left[\frac{10^{(M-m_{th})z}}{z} {}_2F_1\left(p, -\frac{z}{g}, 1 - \frac{z}{g}, -\frac{t}{c_0} 10^{-(M-m_{th})g}\right) - \frac{1}{z} {}_2F_1\left(p, -\frac{z}{g}, 1 - \frac{z}{g}, -\frac{t}{c_0}\right) + \frac{(t/c_0)^{-p}}{z + pg} {}_2F_1\left(p, p + \frac{z}{g}, 1 + p + \frac{z}{g}, -\frac{c_0}{t}\right) \right], \tag{A1}$$

includes hypergeometric functions

$${}_2F_1(a, b, c, x) \equiv \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!}, \tag{A2}$$

where $(a)_k \equiv a(a + 1)(a + 2) \cdots (a + k)$.

Despite the complicated form, the integrated rates described by Eq. (A1) have a functional form very similar to the classical Omori-Utsu relation for realistic parameters as shown in Fig. 4, the only difference being that the transition region between the constant regime at short times and the power-law decay at longer times is a little bit broader than in the classic Omori-Utsu relation. This difference vanishes for $g = 0$. Most importantly, the integrated rates inherit the self-similarity with respect to $M - m_{th}$: Magnitudes enter in Eq. (A1) only through the combination $M - m_{th}$. A consequence is that, for example, the rate of aftershocks with magnitude ≥ 4 of a main shock with $M = 7$ is the same as that of aftershocks with magnitude ≥ 2 of a main shock with $M = 5$, because $M - m_{th} = 3$ in both cases.

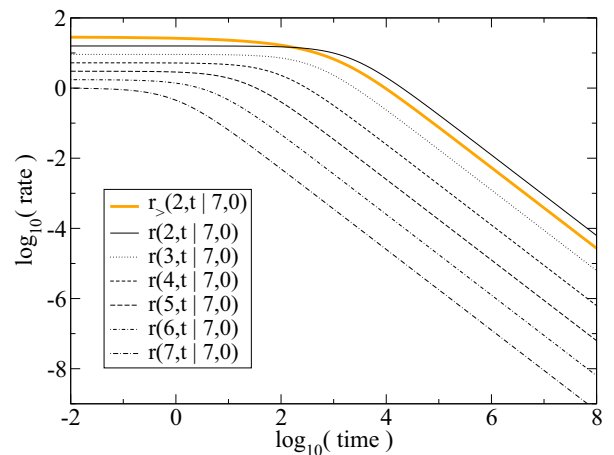


FIG. 4. Comparison between the conditional and the integrated self-similar Omori-Utsu relation for $M = 7$, $m_{th} = 2$, and several values of the aftershock magnitudes. In this example we set $c_0 = \tau_0 = 1$, $g = 0.66$, $z = 0.24$, and $p = 1.15$.

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