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Measurement of multipole phases in optics and electron microscopy by means of conformal transformations

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Summary. — Conformal mappings have been recently rediscovered as a practical solution to measure physical quantities in an efficient manner by implementing a unitary transformation on structured wavefields. For instance, one of the most known is the *log-pol* mapping, enabling the measurement of orbital angular momentum of optical and electron vortices. We report our latest research on a new family of conformal mappings, the circular-sector transformations, applied to wavefunctions endowed with multipole phases, showing disruptive applications in optics and matter-waves physics, in particular electron microscopy. The results suggest an innovative and promising method to measure astigmatisms and electric/magnetic dipoles in a fast and direct way.

1. – Introduction

The advent of electron microscopy started a new era in the inspection of matter, overcoming the barriers of higher resolution that had been imposed for centuries by the limitations of visible light. The technological efforts to improve the control of electron beams soon extended its range of applicability far beyond high-resolution imaging, leading to disruptive milestones in the atomic-scale characterization of materials properties and structure [1]. More recently, the flourishing field of structured light [2] inspired the possibility to investigate the exploitation of custom wavefields also in electron optics, and the generation and analysis of twisted electron beams carrying orbital angular momentum have been demonstrated [3-5]. However, while in optics it is somehow feasible to shape the phase and intensity distribution of a light beam by means of properly engineered bulk or diffractive optics, and more recently metasurfaces (or dynamically using spatial light modulators), the same task is still arduous to be achieved in an efficient and compact manner on electron waves. In our research [6] we prove how conformal transformations can overcome this limitation, providing a practical and effective solution for beam shaping and processing of charged matter waves. As a matter of fact, the condition of harmonicity satisfied by these transformations suggests their implementation in an electrostatic way by means of proper distributions of electrodes or currents [7]. Moreover, a new family of conformal transformations has been identified which allow

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the direct and efficient measurement of multipole-phase terms, paving the way to the measurements of new observables both in optics and electron microscopy.

2. – Conformal mappings and circular-sector transformations

In the paraxial regime, the propagation in free space of a field $\psi_0(x,y)$ at a distance z from a phase element is described by the Fresnel-Kirchhoff diffraction integral [8]

$$\psi_z(u,v) = \frac{e^{ikz}}{i\lambda z}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\psi_0(x,y)\,e^{i\Omega(x,y)}e^{-ik\frac{xu+yv}{z}}\mathrm{d}x\mathrm{d}y, \tag{1}$$

where $k = 2\pi/\lambda$ is the wavevector modulus, (x,y) and (u,v) are the Cartesian reference frames located at the input and output planes respectively, and $\Omega(x,y)$ is the phase imparted by the element. The stationary phase approximation [8] provides a useful tool to solve an integral of this type, by calculating the argument at the stationary points of the total phase $\Phi = \Omega - k(xu+yv)/z$, that is by solving the equation $\nabla\Phi = 0$. That is equivalent, from a physical point of view, to a ray-light approach and eventually to a coordinate transformation dictated by the following relation:

$$\nabla\Omega(x,y) = \frac{k}{z}\boldsymbol{\rho}, \tag{2}$$

where $\boldsymbol{\rho} = (u,v)$.Under the request that the transformation is conformal, in the specific anti-holomorphic, the following Cauchy-Riemann conditions are satisfied by the derivatives of u and v: $\partial_x u = -\partial_y v$ and $\partial_y u = \partial_x v$ [8]. Then, it is straightforward to prove from eq. (2) that the phase element satisfies the Laplace equation

$$\nabla^2\Omega = 0. \tag{3}$$

This apparently simple result has at least two powerful implications. In electron microscopy, it means that arbitrary conformal transformations can be produced by suitable arrangements of electrostatic/magnetostatic sources, suggesting a more efficient and versatile implementation with respect to holographic techniques. Secondly, solving eq. (3) provides a complete subset of phase patterns to build any conformal mapping, which exhibits a straightforward implementation with well-known distributions of electrodes. In fact, the general solution under polar variables (r,ϑ) separation is given by

$$\Omega(r,\vartheta) = A\cdot r^p\cdot\cos(p\vartheta+\vartheta_0), \tag{4}$$

where p is an integer. The phase in eq. (4) resembles the projected potentials of electric/magnetic multipoles, then it can be easily implemented by properly engineered distributions of charges. Using eq. (2), it is easy to show that such a phase element induces a coordinate transformation given by: $(\rho,\varphi) = \left(f\cdot A\cdot p\cdot r^{p-1}/k, (1-p)\vartheta-\vartheta_0\right)$, *i.e.*, it performs a rescaling of the azimuthal coordinate by a factor $n = 1/(1-p)$. A transformation of this kind is called an n-fold circular-sector transformation (CST) [9] and it can be exploited to measure new quantities of interest, as described in the next section.

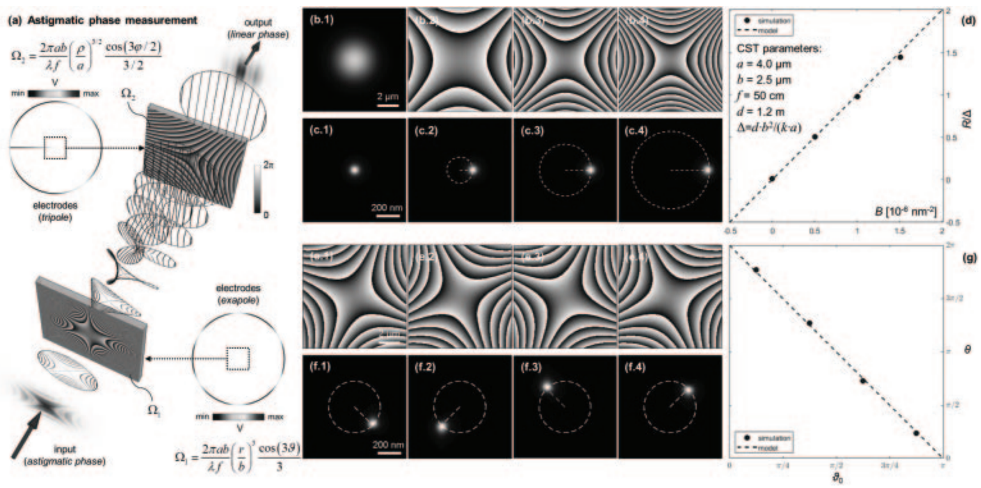

Fig. 1. – (a) Scheme of astigmatism measurement using CST with $n = -1/2$, electrodes patterns inducing the two required phase elements. (b) Simulations for increasing B, corresponding to 0 (b.1), 7 (b.2), 14 (b.3) and 21 μm (b.4) of astigmatism for a focal length of 3.33 mm, $\vartheta_0 = 0$. The far-field spot (c.1–4) shifts linearly with B (d). (e) Increasing ϑ_0: the far-field spot (f.1–4) appears at an angle $\theta = -2\vartheta_0$ (g). CST parameters in the figure, e-energy 300 KeV.

3. – Application to multipole-phase measurement

We consider a wavefunction endowed with a multipole-phase profile as follows:

$$\psi_0(r,\vartheta) = \exp(ikz) \cdot \exp(iBr^m \cos(m(\vartheta - \vartheta_0))). \tag{5}$$

Integrating the electric/magnetic potential along z shows that this is the phase of a charged particle that interacted with an in-plane electric/magnetic multipole of order $|m|$, where B is related to the multipole moment, and ϑ_0 to its orientation. If we apply to such a wavefunction a CST with a factor $n = -1/m$, we obtain

$$\psi_z(\rho,\varphi) \propto C \cdot \exp(iB'\rho\cos(\varphi + m\vartheta_0)). \tag{6}$$

The output has a linear phase with spatial frequency $B' = k/f \cdot A^{-1}(1+m)^{-1}$, A being the scaling parameter of the applied CST (eq. (4)). The far-field at a distance d will exhibit a spot forming an angle $-m\vartheta_0$ with the positive x-axis, and shifted by an amount dB'/k with respect to the origin. This finding suggests a method to measure multipole fields in an efficient and direct way by means of CST conformal mappings. As described by eq. (4), the phase element will exhibit a multipole-phase profile of order $p = 1 + m$. In order to complete the transformation on the phase, a second confocal element is required [8], having a multipole phase with different order $p = 1 + 1/m$ (see [9] for further details on CSTs). For instance, an $n = -1/2$ CST followed by a Fourier transform transforms an external quadrupole ($m = +2$) phase object into a spot, allowing one to measure the quadrupole moment and its orientation by reading them directly as a point signal on the detector (fig. 1). Even more interesting is that we can measure the strength of a dipole moment, *i.e.*, $m = -1$ by transforming the induced phase into a linear one, thanks to the trivial CST defined by $n = +1$: $\Omega = A \cdot \log(r/b)$, being b a scaling parameter (fig. 2).

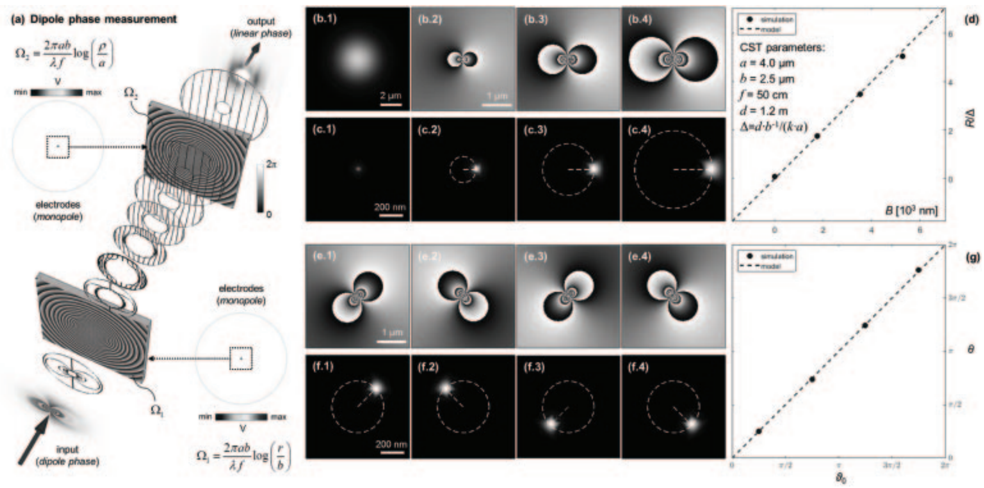

Fig. 2. – (a) Dipole measurement using CST with $n = +1$. (b) Simulations for increasing B, corresponding to a dipole of 0 (b.1), 5 (b.2) 10 (b.3) and $15 \cdot 10^6 \mu_B$ (b.4), $\vartheta_0 = 0$, and corresponding far-field (c.1–4). (d) Spot position as a function of B. (e) Increasing ϑ_0: the far-field spot (f.1–4) appears at $\theta = \vartheta_0$ (g). CST parameters in the figure, e-energy 300 KeV.

4. – Conclusions

Circular-sector transformations are able to map conformally an input multipole phase into a far-field spot whose position is dependent on the strength and orientation of the multipole momentum, thus suggesting a compact and efficient solution for the direct and real-time measurement of multipoles. Moreover, the required phase elements can be implemented in an electrostatic/magnetostatic way by proper distributions of charges/currents. These findings are expected to be especially relevant for applications in the microscopic study and characterization of magnetism, *in situ* growth, ferroelectricity, and superparamagnetism, especially on short time scales. Since the phase profile of external multipoles resembles those of astigmatisms, another useful application of general interest is the characterization of aberrations in both optics and electron optics.

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