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Hamilton–Jacobi Homogenization and the Isospectral Problem

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Abstract: We consider the homogenization theory for Hamilton–Jacobi equations on the one-dimensional flat torus in connection to the isospectrality problem of Schrödinger operators. In particular, we link the equivalence of effective Hamiltonians provided by the weak KAM theory with the class of the corresponding operators exhibiting the same spectrum.

1. Introduction

Let us consider the one-dimensional flat torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ and two functions $V, A \in C^\infty(\mathbb{T}; \mathbb{R})$ and define $H(x, p) := \frac{1}{2}|p + A(x)|^2 + V(x)$. Consider the Hamilton–Jacobi equation

$$\frac{1}{2}|P + \nabla_x S(P, x) + A(x)|^2 + V(x) = \overline{H}(P), \quad P \in \mathbb{R}, \tag{1}$$

where the convex map $P \mapsto \overline{H}(P)$ is the effective Hamiltonian (see, e.g., [1–8]), whereas S is a viscosity solution, in this case unique (see [5,9–11] and references therein). We recall the following inf-sup formula:

$$\overline{H}(P) = \inf_{v \in C^{1,1}(\mathbb{T})} \sup_{x \in \mathbb{T}} \frac{1}{2}|P + \nabla_x v(x) + A(x)|^2 + V(x). \tag{2}$$

The target of this paper is to study the functions H with the same effective Hamiltonian

$$\overline{H}_1 = \overline{H}_2 \tag{3}$$

in connection to the class of Schrödinger operators

$$\widehat{H} := \frac{1}{2}(-i\hbar\nabla_x + A(x))^2 + V(x) \tag{4}$$

and related isospectral problem

$$\mathrm{Spec}(\widehat{H}_1) = \mathrm{Spec}(\widehat{H}_2) \qquad \forall\, 0 < \hbar \leq 1, \tag{5}$$

In order to do so, we make use of the inf-sup Formula (2) together with the well known Bohr–Sommerfeld rules on the (discrete) spectrum of (4), which we here recall in Section 2.2, to prove the main result of the paper.

More precisely, the objective is to show that the (semiclassical) isospectrality condition implies a constraint on the related two effective Hamiltonians.

The content of our main result is the following:

Theorem 1. *Let $H_\alpha(x,p) := \frac{1}{2}|p + A_\alpha(x)|^2 + V_\alpha(x)$ with $\alpha = 1, 2$ such that $\max V_1 = \max V_2$ and $\int_\mathbb{T} A_1(x)dx = \int_\mathbb{T} A_2(x)dx$. If*

$$\mathrm{Spec}(\widehat{H}_1) = \mathrm{Spec}(\widehat{H}_2) \qquad \forall\, 0 < \hbar \le 1, \tag{6}$$

then

$$\overline{H}_1(P) = \overline{H}_2(P) \qquad \forall P \in \mathbb{R}. \tag{7}$$

Conversely, if (7) holds true, then $\mathrm{Spec}(\widehat{H}_1) = \mathrm{Spec}(\widehat{H}_2) \bmod \mathcal{O}(\hbar^2)$ for $E > \min \overline{H}$, i.e., the two spectrums are close up to a remainder of order $\mathcal{O}(\hbar^2)$.

The assumptions on V and A are not restrictive, since these are in fact necessary conditions to have the equality (7), as shown in Remark 1.

We recall that in Section 4 of [7] a result is shown on the link between the homogenization theory of the Hamilton–Jacobi equation and the spectrum of the Hill operator $-\frac{1}{2}\frac{d^2}{dx^2} + V(x)$, namely when $\hbar = 1$. In this setting, the authors proved that the isospectrality implies the same 'viscous' effective Hamiltonian $\overline{H}_{\mathrm{visc}}$, namely the function such that

$$-\Delta_x w(P,x) + \frac{1}{2}|P + \nabla_x w(P,x)|^2 + V(x) = \overline{H}_{\mathrm{visc}}(P), \quad P \in \mathbb{R}, \tag{8}$$

for a unique C^2 - function $w : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$. Moreover, in [7] and [12], various results on the inverse problem in the theory of periodic homogenization of Hamilton–Jacobi equations are shown.

Our Theorem 1 provides a further link between the homogenization theory and spectral problem, for a larger class of Hamiltonians and by using other arguments with respect to the paper [7]. Moreover, the second statement in Theorem 1 shows that the effective Hamiltonian can also be associated to the equivalent class of operators with the same spectrum mod $\mathcal{O}(\hbar^2)$ above the energy value $\min \overline{H}$.

We now recall Theorem 5.2 in [13], which works in one dimension and under some nondegeneracy conditions on the Hamiltonian dynamics, showing that for such isospectral operators there exists (locally in a neighborhood of energies) a symplectic map $\psi : \mathbb{R} \times \mathbb{T} \to \mathbb{R} \times \mathbb{T}$ such that

$$H_2 = H_1 \circ \psi. \tag{9}$$

With respect to this observation, we also underline a symplectic invariance property of $\overline{H}$ in arbitrary dimension n. As shown in [14], for the whole time one Hamiltonian flows $\varphi \equiv \varphi^1 : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{R}^n$ with C^1-regularity, we have the invariance

$$\overline{H \circ \varphi} = \overline{H}. \tag{10}$$

Unluckily, the equality (9) cannot be applied to recover the equivalence of effective Hamiltonians (10) since it is well known that a symplectic map could not be (in general) a Hamiltonian flow.

Our paper is the first attempt (in the simple one dimensional case) to provide a direct connection between the Hamilton–Jacobi homogenization and the inverse spectral problem for operators of kind (4) without passing through equation (8) but taking into account only (1). We hope that the full generalization of Theorem 1 towards the n-dim case and arbitrary smooth potentials can be given by following the same ideas of the current paper, by using more general tools of spectral theory in place of Bohr–Sommerfeld rules that work only in a integrable setting such as our one dimensional case.

The fact that the isospectrality in Theorem 1 is for all the values $0 < \hbar \le 1$ is not a restrictive assumption. Indeed, this can be found for example also in the work [13], as well as in [15] where such a semiclassical isospectrality is considered for quantum integrable Ψdo and related to the convex hull of the image of the classical momentum map.

We remind that the original idea to apply techniques of Weak KAM theory into the framework of quantum mechanics goes back to L.C. Evans [1,2] in order to study certain semiclassical approximation problems of Schrödinger eigenfunctions as $\hbar \to 0$.

The work [16] deals with some inverse spectral results for operators of kind $\Delta_x + Q(x)$, namely the Laplacian for an Hermitian line bundle L plus a potential function Q on a manifold M. In particular, it is shown that the spectra $\mathrm{Spec}(Q; \nabla, L)$ when ∇ ranges over all the translation invariant connections uniquely determines the potential Q. Notice that in our paper we consider operators such as $-\hbar^2 \Delta_x/2 - i\hbar A(x) \cdot \nabla_x + Q(x)$ for $0 < \hbar \leq 1$, hence we deal with the family of connections given by $\hbar \nabla_x$ on $\mathbb{T}$.

On the link between the Schrödinger spectral problem and KAM tori into the phase space, we recall that, in [17] for Schrödinger operators on $\mathbb{T}^n$ and under the assumptions of the KAM Theorem, the authors provided semiclassical expansions for the eigenfunctions and eigenvalues.

We underline that a complete study of the link between the effective Hamiltonian, viscosity solutions of Hamilton–Jacobi equation and Schrödinger eigenvalue problem should also involve the phase space analysis of eigenfunctions (and energy quasimodes). In this direction, some preliminary results for the n-dim case have been obtained [18–20]. For the time evolution of WKB-type wave functions, we address the reader to the works [21–23] (and references therein).

To conclude, we underline that, for the class of one-dimensional Hamiltonians in Theorem 1, we have

$$\overline{H}(P) = \overline{\mathcal{H}}(P + \bar{A}), \quad \forall P \in \mathbb{R}, \quad \bar{A} := \frac{1}{2\pi} \int_{\mathbb{T}} A(y)\,dy, \tag{11}$$

(see Lemma 1). Moreover,

$$\overline{\mathcal{H}}(P) = \overline{\mathcal{H}}(-P) \qquad \forall P \in \mathbb{R}, \tag{12}$$

namely for $\overline{\mathcal{H}}$ there is a symmetry property with respect to the map $P \mapsto -P$. This can be easily seen thanks to (19). In higher dimensions, we stress the meaningful open problem to write the effective Hamiltonian $\overline{H}$ as the composition of a more general $\overline{\mathcal{H}}$ with other kinds of symmetries realized by translations or more general volume preserving maps. This use of symmetry properties should clarify the general study of the homogenization for Hamilton–Jacobi equations and the isospectrality of the corresponding Schrödinger operators.

2. Preliminaries and Settings

2.1. Hamilton–Jacobi Equation

In this section, we recall standard results about Hamilton–Jacobi equation and effective Hamiltonian on $\mathbb{T}^n \times \mathbb{R}^n$.

Let H be a Tonelli Hamiltonian, namely $H \in C^2(\mathbb{T}^n \times \mathbb{R}^n; \mathbb{R})$ is such that the map $p \mapsto H(x, p)$ is convex with positive definite Hessian and in addition $H(x, p)/\|p\| \to +\infty$ as $\|p\| \to +\infty$.

For any $P \in \mathbb{R}^n$, it is known that there exists a unique real number $c = \bar{H}(P)$ such that the problem on $\mathbb{T}^n$:

$$H(x, P + \nabla_x S) = c, \tag{13}$$

has a solution $S = S(P, x)$ in the viscosity sense (see, e.g., [5] and references therein). Moreover, as shown in [5], any viscosity solution is also a weak KAM solution of negative type and belongs to $C^{0,1}(\mathbb{T}^n; \mathbb{R})$. Furthermore, as shown in [24], any viscosity solution S exhibits $C^{1,1}_{\mathrm{loc}}$-regularity outside the closure of its singular set $\Sigma(S)$ and $\mathbb{T}^n \backslash \overline{\Sigma(S)}$ is an open and dense subset of $\mathbb{T}^n$.

The function $\overline{H}$ is called the effective Hamiltonian. It is a convex function and can be represented or approximated in various ways (see, e.g., [1–7]).

In particular (see [14,25] and references therein), we have a useful inf-sup formula. Let $v \in C^{1,1}(\mathbb{T}^n)$ and $\Gamma := \{(x, \nabla_x v(x)) \in \mathbb{T}^n \times \mathbb{R}^n \mid x \in \mathbb{T}^n\}$, denote by $\mathcal{G}$ the set of all Γ. The effective Hamiltonian can be represented by the formula

$$\overline{H}(P) = \inf_{\Gamma \in \mathcal{G}} \sup_{(x,p) \in \Gamma} H(x, p + P). \tag{14}$$

Moreover, such a value equals the Mather $\alpha(H)$ function (see [5,8]).

As shown in Proposition 1 of [14], for any fixed time one Hamiltonian flows $\varphi \equiv \varphi^1 : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{R}^n$ with C^1-regularity, we have the following invariance property:

$$\overline{H} \circ \varphi = \overline{H}. \tag{15}$$

In the mechanical case $\mathcal{H}(x, p) := \frac{1}{2}|p|^2 + V(x)$, we have

$$\overline{\mathcal{H}}(P) = \inf_{v \in C^{1,1}(\mathbb{T}^n)} \sup_{x \in \mathbb{T}^n} \frac{1}{2}|P + \nabla_x v(x)|^2 + V(x) \tag{16}$$

and it is easily seen that $\overline{H}(0) = \max V$. The so-called viscous version of the Hamilton–Jacobi equation

$$-\Delta_x w(P, x) + \frac{1}{2}|P + \nabla_x w(P, x)|^2 + V(x) = \overline{H}_{\text{visc}}(P), \quad P \in \mathbb{R}^n, \tag{17}$$

has a unique C^2-solution $w : \mathbb{R}^n \times \mathbb{T}^n \longrightarrow \mathbb{R}$ (see Theorem 5 in [26]).

As shown in [6], in the one-dimensional setting, the effective Hamiltonian can be given by the inversion of the map

$$E \mapsto \mathcal{J}(E) := \frac{1}{2\pi} \int_0^{2\pi} \sqrt{2(E - V(x))} \, dx, \quad E \geq \max V, \tag{18}$$

namely

$$\overline{\mathcal{H}}(P) = \begin{cases} \max V & \text{if } |P| \leq \mathcal{J}(\max V), \\ \mathcal{J}^{-1}(P) & \text{otherwise.} \end{cases} \tag{19}$$

In Lemma 1, we show the effective Hamiltonian for $H(x, p) := \frac{1}{2}|p + A(x)|^2 + V(x) = \mathcal{H}(x, p + A(x))$, which turns out to be directly related to formula (19).

2.2. Bohr–Sommerfeld Rules

The leading term of the Weyl Law (see [27]) for the number of the eigenvalues smaller than E (which is here supposed to be greater than $\max V$) is given by

$$\mathcal{J}(E) := \text{Vol}\{(x, p) \in \mathbb{T} \times \mathbb{R} : \mathcal{H}(x, p) \leq E\}. \tag{20}$$

The so-called Bohr–Sommerfeld rules (see [28–30]) in our one-dimensional and periodic setting take the form

$$\mathcal{S}_\hbar(E_{\hbar,\ell}) = 2\pi\hbar\,\ell \qquad \text{for } \ell = 1, 2, \ldots \tag{21}$$

where $E_{\hbar,\ell}$ are the eigenvalues of $-\frac{1}{2}\hbar^2 \Delta_x + V(x)$, and

$$\mathcal{S}_\hbar(E) \sim \sum_{j=0}^\infty \hbar^j \, \mathrm{S}_j(E) = 2\pi\mathcal{J}_0(E) + \frac{1}{2}\hbar\pi\mu(E) + \mathcal{O}(\hbar^2), \tag{22}$$

where

$$\mathcal{J}_0(E) = \int_{\gamma_E} p \, dx \tag{23}$$

is the Action integral for the classical curve γ_E at energy E and positive momentum, i.e., $p > 0$. When $E > \max V$, it is easily seen that

$$\mathcal{J}_0(E) = \mathcal{J}(E) \tag{24}$$

as given above. In particular, the value $\mu(E)$ is the Maslov index of γ_E seen as a Lagrangian submanifold of $\mathbb{T} \times \mathbb{R}$. Moreover, for any fixed constant $a \in \mathbb{R}$ and the translated Hamiltonian $\mathcal{H}(x, p + a)$, we have that the related action is modified as

$$\mathcal{J}_0^{(a)}(E) = \mathcal{J}_0(E) - a. \tag{25}$$

As shown in Proposition 5.2 of [28], the semiclassical series in (21) is locally uniform in E, which implies that the remainder vanishes as an $\mathcal{O}(\hbar^2)$ term when E is in a fixed, bounded interval. Thus, the above equalities (21) and (22) imply that two systems with the same Bohr–Sommerfeld rules necessarily have the same effective Hamiltonian. This fact is one of the main ingredients used in the proof of Theorem 1.

3. Results

In this section, we provide two preliminary results and then we show the proof of Theorem 1.

Lemma 1. *Let $V, A \in C^\infty(\mathbb{T}; \mathbb{R})$, $\bar{A} := \frac{1}{2\pi} \int_{\mathbb{T}} A(y)\, dy$. The effective Hamiltonian of $H(x, p) := \frac{1}{2}(p + A(x))^2 + V(x)$ is linked to the effective Hamiltonian of $\mathcal{H}(x, p) := \frac{1}{2}p^2 + V(x)$ in the following way*

$$\overline{H}(P) = \overline{\mathcal{H}}(P + \bar{A}) \quad \forall P \in \mathbb{R}. \tag{26}$$

Furthermore,

$$\min_{P \in \mathbb{R}} \overline{H}(P) = \max_{x \in \mathbb{T}} V(x). \tag{27}$$

Proof. We begin by recalling that

$$\overline{H}(P) = \inf_{v \in C^{1,1}(\mathbb{T})} \sup_{x \in \mathbb{T}} \frac{1}{2}|P + \nabla_x v(x) + A(x)|^2 + V(x). \tag{28}$$

On the one-dim flat torus $\mathbb{T}$, we can write $A(x) = \bar{A} + \nabla_x \phi(x)$ and define $u(x) := v(x) + \phi(x)$ so that

$$\overline{H}(P) = \inf_{u \in C^{1,1}(\mathbb{T})} \sup_{x \in \mathbb{T}} \frac{1}{2}|P + \bar{A} + \nabla_x u(x)|^2 + V(x) \tag{29}$$

namely we have

$$\overline{H}(P) = \overline{\mathcal{H}}(P + \bar{A}). \tag{30}$$

where $\overline{\mathcal{H}}$ is explicitly shown in (19). To conclude,

$$\min_{P \in \mathbb{R}} \overline{H}(P) = \min_{P \in \mathbb{R}} \overline{\mathcal{H}}(P + \bar{A}) = \min_{Q \in \mathbb{R}} \overline{\mathcal{H}}(Q) = \max_{x \in \mathbb{T}} V(x). \tag{31}$$

$\square$

Remark 1. *In view of the above lemma, we have that $\max V_1 = \max V_2$ and $\int_{\mathbb{T}} A_1(x)dx = \int_{\mathbb{T}} A_2(x)dx$ are necessary conditions for the equality*

$$\overline{H}_1(P) = \overline{H}_2(P), \quad \forall P \in \mathbb{R}. \tag{32}$$

Indeed, $\min_{P \in \mathbb{R}} \overline{H}(P) = \max_{x \in \mathbb{T}} V(x)$ implies that the maximum of V must be the same same. Moreover, $\overline{H}(P) = \overline{\mathcal{H}}(P + \bar{A})$ and, thus, recalling (19), $\overline{H}$ is a constant function on the interval $I := [-\bar{A} - \mathcal{J}(\max V), -\bar{A} + \mathcal{J}(\max V)]$. As a consequence, (32) could be fulfilled if $I_1 = I_2$. On the other hand, the two functions must be the same also outside I and this gives the equality of functions $\mathcal{J}_1 = \mathcal{J}_2$. We conclude that it must be $\bar{A}_1 = \bar{A}_2$.

Lemma 2. *Let us define* $\widehat{H} := \frac{1}{2}(-i\hbar\nabla_x + A(x))^2 + V(x)$ *and* $\widehat{G} := \frac{1}{2}(-i\hbar\nabla_x + \bar{A})^2 + V(x)$ *where* $\bar{A} := \frac{1}{2\pi}\int_{\mathbb{T}} A(y)\,dy$. *Then,*

$$\text{Spec}(\widehat{H}) = \text{Spec}(\widehat{G}) \qquad \forall\, 0 < \hbar \leq 1, \tag{33}$$

Proof. Define the unitary operator on $L^2(\mathbb{T})$

$$(U\varphi)(x) := e^{-\frac{i}{\hbar}\phi(x)}\varphi(x) \tag{34}$$

where $\phi \in C^\infty(\mathbb{T};\mathbb{R})$ is such that $A(x) - \bar{A} = \nabla\phi(x)$. This provides the unitary conjugation

$$U^\dagger \circ \widehat{H} \circ U = \widehat{G}. \tag{35}$$

The proof of this fact can be done by taking the orthogonal set $e_k(x) := e^{ik\cdot x}$ with $k \in \mathbb{Z}$ and then computing the action of the operators on the righthand side and also on the lefthand side of (35) on e_k, to check the equality.

Now, easily observe that, since the functions $H(x,p) := \frac{1}{2}|p + A(x)|^2 + V(x)$ and $G(x,p) := \frac{1}{2}|p + \bar{A}|^2 + V(x)$, have compact sub-level sets in $\mathbb{T} \times \mathbb{R}$, then both the spectrums of $\widehat{H}$ and $\widehat{G}$ are discrete (see, e.g., [27] with a more general class of Ψdo on manifolds). Whence, the unitary equivalence (35) directly implies the same spectrum. □

Remark 2. *The unitary conjugation (35) still holds true for operators of this kind on Sobolev space* $W^{2,2}(\mathbb{R};\mathbb{C})$ *and suitable assumptions on smooth* V *and* A *(see, e.g., Section 1 of [31]). However, we stress that on* $\mathbb{R}$ *one can always write* $A(x) = \nabla_x\phi(x)$, *whereas on* $\mathbb{T}$ *we must take into account a possible nonzero* $\bar{A}$. *This makes not trivial our study on the equivalence of effective Hamiltonians, as well as the use of these operators.*

Proof of Theorem 1. As a consequence of Lemmas 1 and 2, in what follows, we can consider only the case $A(x) = \bar{A}$.

In view of (21) and (22), we can write

$$2\pi\mathcal{J}(E_{\hbar,\ell}) - 2\pi\bar{A} + \frac{1}{2}\hbar\pi\mu(E_{\hbar,\ell}) + \mathcal{O}(\hbar^2) = 2\pi\hbar\,\ell. \tag{36}$$

Recalling (18) and (19), we have

$$\overline{\mathcal{H}}(\mathcal{J}(E)) = E \quad \forall E \geq \max V. \tag{37}$$

and thus the equality

$$\overline{\mathcal{H}}(\mathcal{J}(E) - \bar{A} + \bar{A}) = E. \tag{38}$$

reads, thanks to (26),

$$\overline{H}(\mathcal{J}(E) - \bar{A}) = E. \tag{39}$$

As a consequence, for any $E_{\hbar,\ell} > \max V$ we have $\mu(E_{\hbar,\ell}) = 0$ and

$$\overline{H}(\hbar\,\ell - r_{\hbar,\ell}) = E_{\hbar,\ell}, \quad r_{\hbar,\ell} = \mathcal{O}(\hbar^2). \tag{40}$$

Notice in particular that the remainder $r_{\hbar,\ell}$ depends from $\bar{A}$.

Any vector $P \in \mathbb{R}$ in a bounded interval can be approximated by a sequence of kind $\ell_j\hbar_j$ where $j \to +\infty$, $\ell_j \to +\infty$ and $\hbar_j \to 0^+$.

We now remind the continuity of the map $P \longmapsto \overline{H}(P)$ for any $P \in \mathbb{R}$ and the continuity of $P \longmapsto \nabla\overline{H}(P)$ for those P such that $\overline{H}(P) \geq \min\overline{H} + \varepsilon$ for some fixed $\varepsilon > 0$. In particular, notice that $\min\overline{H} = \max V$.

In what follows, we consider

$$\Omega := \{P \in \mathbb{R} \mid \overline{H}(P) > \min\overline{H} + \varepsilon; \quad |\nabla\overline{H}(P)| < \lambda\} \tag{41}$$

for some fixed $\lambda > 0$ large enough (recall that $\overline{H}$ is a convex map).

Thus, we can use the values $E_{\hbar,\ell}$ of the spectrum to recover the value $\overline{H}(P)$. Namely,

$$
\begin{align}
\overline{H}(P) &= \lim_{j \to +\infty} \overline{H}(\ell_j \hbar_j) \tag{42} \\
&= \lim_{j \to +\infty} \overline{H}(\ell_j \hbar_j - r_{\hbar_j} + r_{\hbar_j}) \tag{43} \\
&= \lim_{j \to +\infty} E_{\hbar_j, \ell_j} + R_{\hbar_j, \ell_j}. \tag{44}
\end{align}
$$

The above remainder is defined as

$$
R_{\hbar,\ell} := \overline{H}(\ell\hbar) - \overline{H}(\ell\hbar - r_{\hbar,\ell}). \tag{45}
$$

We recall that $r_{\hbar,\ell} = \mathcal{O}(\hbar^2)$ when $E_{\hbar,\ell}$ lies in a bounded interval. Moreover, the sequence $\ell_j \hbar_j$ also lies in a bounded interval since we assume that $P \in \mathbb{R}$ belongs to a fixed bounded interval. The continuity of $\nabla \overline{H}$ with its uniformly boundedness on the prescribed set of P and the previous observations ensure that

$$
\begin{align}
R_{\hbar,\ell} &\leq \sup_{0 \leq \alpha \leq 1} |\nabla \overline{H}(\alpha[\ell\hbar] + (1-\alpha)[\ell\hbar - r_\hbar])| \, |r_\hbar| \\
&\leq \|\nabla \overline{H}\|_{C^0(\Omega')} |r_\hbar|. \tag{46}
\end{align}
$$

The above convex combination belongs, for any j, to a suitable large bounded set $\Omega' \supseteq \Omega$ since $\forall P \in \Omega$ we have $\ell_j \hbar_j \to P$ and $r_{\hbar_j, \ell_j} \to 0$ as $j \to +\infty$. The limit (44) together with (46) allow recovering

$$
\overline{H}(P) = \lim_{j \to +\infty} E_{\hbar_j, \ell_j}. \tag{47}
$$

Since we are assuming that for two potentials with the same maximum we have Schrödinger operators with the same spectrum, we necessarily have the same equality (47). For any fixed P as prescribed above we have

$$
\overline{H}_1(P) = \overline{H}_2(P). \tag{48}
$$

We recall that $\overline{H}(P) = \overline{\mathcal{H}}(P - \bar{A})$ and $\overline{\mathcal{H}}(P) = \max V$ when $|P| \leq \mathcal{J}(\max V)$ (see Section 2.1). Thus, we also have the equivalence of the minimum points of the two effective Hamiltonians.

To conclude, in order to prove that

$$
\mathrm{Spec}(\widehat{H}_1) = \mathrm{Spec}(\widehat{H}_2) \bmod \mathcal{O}(\hbar^2) \tag{49}
$$

for $E > \min \overline{H}$, we simply use the equivalence (48) and recall (40). Applying again the estimate (46), we have the remainder $\mathcal{O}(\hbar^2)$. $\square$

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