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High order Lyapunov-like functions for optimal control ⋆

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Abstract: We consider an optimal control problem where the state has to approach asymptotically a closed target, while paying an integral cost with a non-negative Lagrangian ℓ. We generalize the dissipative relation that usually defines a Control Lyapunov Function by introducing a weaker differential inequality, which involves both the Lagrangian ℓ and higher order dynamics' directions expressed in form of iterated Lie brackets up to a certain degree k. The existence of a solution U of the resulting extended relation turns out to be sufficient for a twofold goal: on the one hand, it ensures that the system is globally asymptotically controllable to the target, and, on the other hand, it implies that the value function associated to the minimization problem is bounded above by a U-dependent function. We call such a solution U a *degree-k Minimum Restraint Function* ($k \geq 1$). An example is provided where a smooth degree-1 Minimum Restraint Function fails to exist, while the distance from the target happens to be a C^∞ degree-2 Minimum Restraint Function.

Keywords: Lyapunov functions, Lie brackets, asymptotic controllability, optimal control, exit-time problems.

1. INTRODUCTION

Let $\mathcal{T} \subseteq \mathbb{R}^n$ be a closed subset with compact boundary, which will be called the *target*. For a given cost ℓ and a controlled dynamics f, the *value function* is defined as

$$V(x) := \inf_{(y,\alpha)} \int_0^{T_y} \ell(y(t), \alpha(t))\, dt, \qquad \ell \geq 0, \tag{1}$$

where the infimum is computed over trajectory-control pairs $(y, \alpha) : [0, T_y[\to (\mathbb{R}^n \setminus \mathcal{T}) \times A$, $A \subseteq \mathbb{R}^m$, verifying

$$\begin{cases} \dot{y} = f(y, \alpha), \qquad y(0) = x \\ \lim\limits_{t \to {T_y}^-} \mathbf{d}(y(t)) = 0, \end{cases} \tag{2}$$

$\mathbf{d}(\cdot)$ denoting the distance from the target $\mathcal{T}$.

In Motta and Rampazzo (2013) we introduced a particular Control Lyapunov Function, called *Minimum Restraint Function* (MRF), which is defined as being, for some $p_0 \geq 0$, a positive definite, proper and locally semiconcave solution of the dissipative inequality [1]

$$\inf_{a \in A} \{\langle D^*U(x), f(x, a)\rangle + p_0\, \ell(x, a)\} < 0 \quad \forall x \in \mathbb{R}^n \backslash \mathcal{T}. \tag{3}$$

Specifically, for quite general f, ℓ, and A, we proved the following result:

Theorem 1.1. (Motta and Rampazzo (2013)). Let $U : \overline{\mathbb{R}^n \backslash \mathcal{T}} \to \mathbb{R}$ be a MRF for some $p_0 \geq 0$. Then

i) the system is *globally asymptotically controllable* (GAC), to $\mathcal{T}$ (see Def. 1.2);
ii) if $p_0 > 0$, one has

$$V(x) \leq \frac{U(x)}{p_0} \qquad \forall x \in \mathbb{R}^n \backslash \mathcal{T}. \tag{4}$$

Let us observe that the thesis ii) above would be almost obvious if we assumed the hypothesis $\inf_{(x,a)} \ell > 0$. Indeed, in this case we would be allowed to make use of standard comparison arguments for viscosity sub- and super-solutions (see e.g. Bardi and Capuzzo Dolcetta (1997)). Instead, the weaker condition $\inf_{(x,a)} \ell \geq 0$ is far from being a trivial drawback (see e.g. Soravia (1999); Motta and Sartori (2014, 2015)). Nevertheless, in Motta and Rampazzo (2013) one proves that, given a MRF U with some $p_0 > 0$, an explicit construction of a trajectory-control pair (y, α) can be implemented in such a way that i) y approaches the target in the uniform and stable manner prescribed for GAC systems, and ii) the cost corresponding

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[1] Here $D^*U(x)$ is the *set of limiting gradients* of U at x. Since U is locally semiconcave, it coincides with the *limiting subdifferential* $\partial_L U$. See e.g. Cannarsa and Sinestrari (2004).

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to (y,α) is dominated by the function $\frac{U(x)}{p_0}$. (See also Lai, Motta and Rampazzo (2016) for further extensions.)

By well known results on Control Lyapunov Functions, in general no smooth solutions of (3) exist. Such non-existence phenomenon, rather than being due to some non-smoothness of the data, is often the consequence of topological obstructions. Roughly speaking, it may originate from a shortage of dynamics' directions that are transversal to level sets of any alleged smooth Control Lyapunov Function (see e.g. Clarke, Ledyaev, Sontag and Subbotin (1997), Malisoff, Rifford and Sontag (2004), the book Bacciotti and Rosier (2005) and references therein). Nevertheless, the regularity issue for Lyapunov-like functions is of obvious interest, for instance, for numerical purposes or for robustness with respect to the data.

In the present paper we propose to replace inequality (3) with a new, less demanding, inequality, which involves Lie brackets. To keep the geometric picture transparent and to avoid too many technicalities, we will consider a control-affine, driftless control system

$$\dot y = f(y,\alpha) := \sum_{j=1}^{m} \alpha_j\, f_j(y), \tag{5}$$

with the controls ranging in $A := \{\pm\mathbf{e}_1,\dots,\pm\mathbf{e}_m\}$.[2] We will also make the hypothesis that the Lagrangian ℓ is independent of the control, namely $\ell(y,a)\equiv\ell(y)$. We assume the function $\ell : \mathbb{R}^n \to [0,+\infty[$ to be locally Lipschitz continuous and the vector fields $f_1,\dots,f_m : \mathbb{R}^n \to \mathbb{R}^n$ to be of class C^∞. (About the possibility of weakening these requirements see Rem. 2.4.)

Besides the usual Hamiltonian

$$H(x,p,p_0) := \inf_{a\in A}\langle p, \sum_{j=1}^{m} a_j\, f_j(x)\rangle + p_0\,\ell(x),$$

we introduce the *degree-k Hamiltonians* $H^{(k)}$ given, for every integer $k\ge 1$, by

$$H^{(k)}(x,p,p_0) := \inf_{v\in\mathcal{F}^{(k)}(x)} \langle p, v\rangle + p_0\,\ell(x). \tag{6}$$

Here $\mathcal{F}^{(k)}$ denotes the family of iterated Lie brackets of degree not greater than k of the vector fields $f_1,\dots,f_m$, and $\mathcal{F}^{(k)}(x) := \{B(x),\quad B\in\mathcal{F}^{(k)}\}$ for any $x\in\mathbb{R}^n$.

In order to anticipate the main result, let us give the notion of *degree-k Minimum Restraint Function.*

Definition 1.1. Given an integer $k\ge 1$, we say that a continuous function $U : \overline{\mathbb{R}^n\setminus\mathcal{T}} \to \mathbb{R}$ with restriction to $\mathbb{R}^n\setminus\mathcal{T}$ locally semiconcave, positive definite, and proper, is a *degree-k Minimum Restraint Function* (degree-k MRF) if for some $p_0\ge 0$ it is a solution of

$$H^{(k)}\big(x, D^*U(x), p_0\big) < 0 \quad \forall x\in\mathbb{R}^n\setminus\mathcal{T}. \tag{7}$$

Observe that, since

$$H^{(k)} \le H^{(k-1)} \cdots \le H^{(1)} \equiv H, \tag{8}$$

relation (7) is weaker than (3). As a consequence, if $k_1\ge k_2$, a degree-k_2 MRF is a degree-k_1 MRF, while the converse relation is not valid, in general. In particular, a degree-k MRF may happen to be *more regular* than a MRF (see the example in Section 3).

[2] As customary, $\{\mathbf{e}_1,\dots,\mathbf{e}_m\}$ denotes the canonical basis of $\mathbb{R}^m$.

In Motta and Rampazzo (2018) we dealt with the case when $\ell\equiv 0$. In that case, a degree-k MRF is called a *degree-k Control Lyapunov Function.* We proved that the existence of such a function implies that the system is GAC to $\mathcal{T}$. More generally, omitting details on the regularity hypotheses, our main result (Thm. 2.1) reads as follows:

Main result. Let $U : \overline{\mathbb{R}^n\setminus\mathcal{T}} \to \mathbb{R}$ be a degree-k MRF for some $p_0\ge 0$. Then

i) the system is GAC to $\mathcal{T}$;
ii) if $p_0>0$ and U, $\mathcal{T}$ verify suitable regularity assumptions , for some $N>0$ there exists a continuous, increasing function $\tilde\Psi$, $\tilde\Psi(0)=0$, such that
$$V(x) \le \tilde\Psi(U(x)) \quad \forall x\in U^{\leftarrow}(]0,N]).\,^{3} \tag{9}$$

On the one hand, this extends the quoted result on degree-k Control Lyapunov Functions, in that Lie brackets are involved. On the other hand, it also generalizes Theorem 1.1, because the thesis comprises controllability *and an optimal value estimate.*

The paper is organized as follows. In Section 2 we give the precise assumptions and the statement of Theorem 2.1. In Section 3 we illustrate the main result through an example. In Section 4 we sketch the proof of Theorem 2.1. Section 5 is devoted to some concluding remarks.

1.1 Preliminaries and notation

For $a,b\in\mathbb{R}$, let $a\vee b := \max\{a,b\}$, $a\wedge b := \min\{a,b\}$. Given a subset $\mathcal{K}\subset\mathbb{R}^n$, we set $\mathbf{d}(\mathcal{K}) := \inf_{z\in\mathcal{K}}\mathbf{d}(z)$, where $\mathbf{d}(z) := \mathbf{d}(z,\mathcal{T})$ for every $z\in\mathbb{R}^n$. For any $\nu>0$, $B(\mathcal{T},\nu) := \{z\in\mathbb{R}^n :\ \mathbf{d}(z)\le\nu\}$. We set $\mathbb{R}^+ := [0,+\infty[$ and $\mathbb{R}^+_* :=]0,+\infty[$.

For any initial condition $x\in\mathbb{R}^n\setminus\mathcal{T}$ and any measurable control $\alpha : \mathbb{R}^+\to A$, we will use $y_x(\cdot,\alpha)$ (or y, when no confusion may arise) to denote the unique (possibly local) forward solution to (5) verifying $y(0)=x$. The pair (y,α) will be called *admissible* if there exists $T_y\le+\infty$ such that y is defined on $[0,T_y[$, and

$$\mathbf{d}(y(t))>0 \ \ \forall t\in[0,T_y[, \qquad \lim_{t\to T_y^-}\mathbf{d}(y(t))=0.$$

If $T_y<+\infty$, y will be extended to $\mathbb{R}^+$ by setting $y(t) := \lim_{t\to T_y^-} y(t)$ [4] for all $t\ge T_y$.

To give the notion of global asymptotic controllability, we recall that $\mathcal{KL}$ is used to denote the set of continuous functions $\beta : \mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ such that: (1) for each $s\ge 0$, $\beta(0,s)=0$ and $\beta(\cdot,s)$ is strictly increasing and unbounded; (2) for each $\delta\ge 0$, $\beta(\delta,\cdot)$ is decreasing and $\lim_{s\to+\infty}\beta(\delta,s)=0$.

Definition 1.2. We call the control system (5) *globally asymptotically controllable to* $\mathcal{T}$ (GAC to $\mathcal{T}$) provided for any $\sigma>0$ there is a function $\beta\in\mathcal{KL}$ such that, for each initial state $x\in\mathbb{R}^n\setminus\mathcal{T}$ with $\mathbf{d}(x)\le\sigma$ there exists an admissible trajectory-control pair $(y,\alpha)(\cdot)$ verifying

$$\mathbf{d}(y(t)) \le \beta\big(\mathbf{d}(x),t\big) \qquad \forall t\in\mathbb{R}^+. \tag{10}$$

[3] We set $U^{\leftarrow}(]0,N]) := \{x\in\mathbb{R}^n\setminus\mathcal{T} :\ U(x)\le N\}$.
[4] This limit exists, since A and $\partial\mathcal{T}$ are compact and $f\in C^\infty$.

The Lie bracket of two vector fields F_1, F_2 is the vector field $[F_1, F_2]$ defined by $[F_1, F_2] := DF_2 \cdot F_1 - DF_1 \cdot F_2$, where D denotes differentiation. By repeating the bracketing procedure we obtain the so-called iterated brackets. An iterated Lie bracket of some vector fields $F_1, \dots, F_N$ is said *of degree* $k \geq 1$ if its formal expression contains k vector fields: for instance, F_3, $[F_2, F_3]$ and $[[F_2, F_3], [F_1, F_3]]$ are of degree $1, 2, 4$, respectively.

Let us summarize some basic notions and results from nonsmooth analysis (see e.g. Cannarsa and Sinestrari (2004), Clarke, Ledyaev, Stern and Wolenski (1998)).

Let Ω be an open subset of $\mathbb{R}^N$ for some $N > 0$ and let $\overline{\Omega}$ denote its closure.

Definition 1.3. Let the boundary of Ω be compact. A continuous function $F : \overline{\Omega} \to \mathbb{R}$ is said *positive definite on* Ω if $F(x) > 0 \ \forall x \in \Omega$ and $F(x) = 0 \ \forall x \in \partial\Omega$. F is called *proper on* Ω if the pre-image $F^{-1}(K)$ of any compact set $K \subset \mathbb{R}^+$ is compact.

Definition 1.4. A continuous map $F : \Omega \to \mathbb{R}$ is said to be *semiconcave on* Ω if there exists $\rho > 0$ such that

$$F(x) + F(\hat{x}) - 2F\left(\frac{x+\hat{x}}{2}\right) \leq \rho|x - \hat{x}|^2,$$

for all $x, \hat{x} \in \Omega$ such that $[x, \hat{x}] \subset \Omega$. The constant ρ above is called a semiconcavity constant for F in Ω. F is said to be *locally semiconcave on* Ω if it semiconcave on every compact subset of Ω.

Locally semiconcave functions are locally Lipschitz. Actually, they are twice differentiable almost everywhere.

Definition 1.5. Let $F : \Omega \to \mathbb{R}$ be a locally Lipschitz function. For every $x \in \Omega$, the *set of limiting gradients* of F at x is defined as

$$D^*F(x) := \left\{ \lim_k DF(x_k) : \ x_k \in \mathrm{df}_F \setminus \{x\}, \ \lim_k x_k = x \right\},$$

where df_F denotes the set of differentiability points of F.

The set-valued map $x \rightsquigarrow D^*F(x)$ is upper semicontinuous on Ω, with nonempty, possibly not convex, compact values.

Finally, one has:

Lemma 1.1. Let $F : \Omega \to \mathbb{R}$ be a locally semiconcave function. Then, for any compact set $\mathcal{K} \subset \Omega$, there exist some positive constants L and ρ such that, for any $x \in \mathcal{K}$,

$$\begin{aligned} &F(\hat{x}) - F(x) \leq \langle p, \hat{x} - x \rangle + \rho|\hat{x} - x|^2, \\ &|p| \leq L \quad \forall p \in D^*F(x), \end{aligned} \tag{11}$$

for any point $\hat{x} \in \mathcal{K}$ such that $[x, \hat{x}] \subset \mathcal{K}$.

2. MAIN RESULT

Let U be a a degree-k MRF for some $p_0 \geq 0$, $k \geq 1$.

We preliminarily notice (see (Motta and Rampazzo , 2018, Prop. 3.1)) that the dissipative inequality (7) implies that, for any $N > 0$, there exists a continuous, increasing, function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$, $\gamma(0) \geq 0$, that we will call a (*degree-k*) *N-dissipativity rate* for U, verifying

$$H^{(k)}(x, D^*U(x)) + p_0(U(x))\,\ell(x) \leq -\gamma(U(x)) \tag{12}$$

for all $x \in U^{\leftarrow}(]0, 2N])$.

Let us now relate the semiconcavity of U, the distance from the target, and an integrability assumption on γ.

Definition 2.1. Let $\mathcal{K} \subseteq \mathbb{R}^n$ be any subset. For every $\alpha \in [0, 1]$ we say that a continuous function $U : \overline{\mathbb{R}^n \setminus \mathcal{T}} \to \mathbb{R}$ *verifies the α-property* on $\mathcal{K} \setminus \mathcal{T}$, if

i) U is locally semiconcave, positive definite, and proper on $\mathbb{R}^n \setminus \mathcal{T}$, and
ii) there are some positive constants L, R, and C, such that for all $x, \hat{x} \in \mathcal{K} \setminus \mathcal{T}$ verifying $[x, \hat{x}] \subset \mathcal{K} \setminus \mathcal{T}$, $|\hat{x} - x| \leq R$, one has

$$U(\hat{x}) - U(x) \leq \langle p, \hat{x} - x \rangle + \frac{C}{\mathbf{d}^{\alpha}([x, \hat{x}])} |\hat{x} - x|^2, \tag{13}$$

$$|p| \leq L, \tag{14}$$

for every $p \in D^*U(x)$.

Remark 2.1. In view of Lemma 1.1, when $\mathcal{K}$ is a compact subset of $\mathbb{R}^n \setminus \mathcal{T}$, any function U which is locally semiconcave on $\mathbb{R}^n \setminus \mathcal{T}$ has the α-property with $\alpha = 0$ in $\mathcal{K} \setminus \mathcal{T}$. More generally, the α-property is a generalization (to any function) of a condition concerning the distance function $\mathbf{d}$ (from a closed set $\mathcal{T}$) (see e.g Marigonda (2006), Marigonda and Rigo (2015)). In particular,

1) if $\mathcal{T}$ has *boundary of class $C^{1,1}$*, then $U := \mathbf{d}$ is semiconcave in $\mathbb{R}^n \setminus \mathrm{int}(\mathcal{T})$, where $\mathrm{int}(\mathcal{T})$ denotes the interior of $\mathcal{T}$. Hence (13), (14) hold true for $\alpha = 0$, $L = 1$, and some C independent of $R > 0$.

2) If $\mathcal{T}$ satisfies the *internal sphere condition of radius* $r > 0$, namely, for all $x \in \mathcal{T}$ there exists $\bar{x} \in \mathcal{T}$ such that $x \in B(\bar{x}, r) \subset \mathcal{T}$, then $U := \mathbf{d}$ has the α-property with $\alpha = 0$ in $\mathbb{R}^n \setminus \mathcal{T}$ and satisfies (13), (14) for $L = 1$ and $C = 1/r$, for every $R > 0$.

3) If $\mathcal{T}$ *is a singleton*, $U := \mathbf{d}$ has the α-property with $\alpha = 1$ in $\mathbb{R}^n \setminus \mathcal{T}$ and (13), (14) hold for $L = C = 1$ and for every $R > 0$.

(See e.g. Cannarsa and Sinestrari (2004).)

Definition 2.2. Consider $\alpha \in [0, 1]$ and an integer $k \geq 1$. A function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ will be called *(α, k)-increasing* if it is increasing and continuous and, furthermore, the maps

$$r \mapsto \frac{1}{[\gamma(r)]^{k-1}}, \ \frac{1}{[r^{\alpha}\,\gamma(r)]^{1-1/k}} \tag{15}$$

are integrable on $[0, u]$, for any $u > 0$.

In this case, we define the continuous, increasing map $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$,

$$\Psi(u) := u \vee u^{\frac{1}{k}} \vee \int_0^u \frac{dr}{[\gamma(r)]^{k-1}} \vee \int_0^u \frac{dr}{[r^{\alpha}\,\gamma(r)]^{1-\frac{1}{k}}}. \tag{16}$$

Let us finally state our main result.

Theorem 2.1. Let $U : \overline{\mathbb{R}^n \setminus \mathcal{T}} \to \mathbb{R}$ be a degree-k MRF for some $p_0 \geq 0$. Then:

i) the system is GAC to $\mathcal{T}$.
ii) Moreover, if $p_0 > 0$, for some $(\alpha, \nu, N) \in [0, 1] \times \mathbb{R}^+_* \times \mathbb{R}^+_*$, U enjoys the α-property on $B(\mathcal{T}, \nu) \setminus \mathcal{T}$, and γ is an N-dissipativity rate for U which is (α, k)-increasing, then there exists some $\bar{C} > 0$ such that

$$V(x) \leq \bar{C}\Psi(U(x)) \quad \forall x \in U^{\leftarrow}(]0, N]), \tag{17}$$

where the map Ψ is defined as in (16).

Remark 2.2. The regularity and integrability assumptions considered in Theorem 2.1 are essential for the validity of a bound like (3). They involve a certain interplay between the dissipativity rate γ and the regularity properties of the pair $(U, \mathcal{T})$. In particular, quantities relating curvature parameters of $\partial\mathcal{T}$ and the semiconcavity of U play a key role, as observed in Marigonda (2006) and Marigonda and Rigo (2015), where some regularity issues for the minimum time function are carefully investigated.

Remark 2.3. When i) U is a degree-k MRF for some $p_0 > 0$ with the α-property on $B(\mathcal{T}, \nu) \setminus \mathcal{T}$ for some α, ν, and ii) for some $N > 0$, the N-dissipativity rate γ can be chosen equal to a positive constant (as it can be done as soon as $\gamma(0) > 0$), then γ is trivially (α, k)-increasing for every $\alpha \in [0, 1]$ and $k \geq 1$. In this case, (17) reads

$$V(x) \leq \bar{C}\, U^{1/k}(x) \quad \forall x \in U^{\leftarrow}(]0, N \wedge 1]),$$

for some $\bar{C} > 0$.

Remark 2.4. The results of Theorem 2.1 could be extended in several directions. In particular, similarly to Motta and Rampazzo (2018), we could weaken considerably the smoothness assumption on the vector fields $f_1, \ldots, f_m$ –in the case of degree-2 MRFs we could include also Lipschitz continuous vector fields (see Rampazzo and Sussmann (2007))– and give some partial results for systems with drift, as well as generalize the present results to unbounded closed targets.

3. AN EXAMPLE

As is well known, there is no need for local controllability for the existence of a Control Lyapunov Function (CLF). Similarly, the fact that the Lie algebra rank condition is verified almost everywhere is far from being necessary for a MRF of whatever degree to exist. Moreover, a system that fails to be locally controllable on large areas of its domain might not have any C^1 degree-1 MRF (or even a CLF function), while admitting a smooth degree-k MRF, for some $k > 1$, as illustrated in the following example.

Example 3.1. Let φ, $\psi : [0, +\infty[\to [0, 1]$ be C^∞ maps such that, for any $q \in \mathbb{N}$,

$$\begin{aligned} &\varphi(r) = 1 \ \text{ if } r \in [2q, 2q+1], \\ &\varphi(r) = 0 \ \text{ if } r \in [2q + (5/4), 2q + (7/4)]; \end{aligned}$$

$$\begin{aligned} &\psi(r) = 1 \ \text{ if } r \in [2q + (7/8), 2q + (17/8)], \\ &\psi(r) = 0 \ \text{ if } r \in [2q + (1/4), 2q + (3/4)] \cup [0, (1/4)]. \end{aligned}$$

Let us consider the optimal control problem in $\mathbb{R}^3$

$$V(x) := \inf_{(y,a)} \int_0^{T_y} \ell(y(t))\, dt, \qquad \ell \geq 0, \tag{18}$$

$$\begin{cases} \dot{y} = a_1\, f_1(y) + a_2\, f_2(y) + a_3\, f_3(y)\,, \\ y(0) = x, \\ \lim_{t \to T_y{}^-} \mathbf{d}(y(t)) = 0, \end{cases}$$

where $\mathcal{T} := \{0\}$,

$$l(y) = (|y|^3 \wedge 1)\ \left|\sin\left|(|y|^{-1})\right|\right| \ \ \forall y \neq 0, \quad \ell(0) = 0,$$

and

$$f_1 = \varphi(|x|)\left(\frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_3}\right), \quad f_2 = \varphi(|x|)\left(\frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}\right),$$

$$f_3 = \psi(|x|)\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}\right).$$

It is worth noticing notice that the system is not small time locally controllable at every point x such that

$$2q + (5/4) \leq |x| \leq 2q + (7/4). \tag{19}$$

Furthermore, the cost ℓ vanishes at every x such that $|x| = \frac{1}{k\pi}$, $\ k \in N$, $\ k \geq 1$.

Let us choose $U(x) := \mathbf{d}(x) = |x|$. For every $p_0 \geq 0$ and $q \in \mathbb{N}$, for all x verifying (19) we have

$$H^{(1)}(x, D^*U(x), p_0) = -|x| + p_0\, \ell(x),$$

but

$$H^{(1)}(x, D^*U(x), p_0) = +p_0\, \ell(x) \quad (\geq 0)$$

for every x such that $x_1 = x_2 = 0$ and $|x_3| \leq 1/4$ or $2q + (1/4) \leq |x_3| \leq 2q + (3/4)$, $q \geq 1$. Actually, there are no C^1 degree-1 MRFs, as it can be proved by observing that the system coincides with the *nonholonomic integrator* (see e.g. Bacciotti and Rosier (2005)) in a whole neighborhood of the target.

However, it is easy to see that, for all $x \in \mathbb{R}^3 \setminus \mathcal{T}$,

$$\begin{aligned} &H^{(2)}(x, D^*U(x), p_0) \\ &= -\frac{(|x_1 - x_3 x_2| \vee |x_2 + x_3 x_1| \vee 2|x_3|)}{|x|} + p_0\, \ell(x) \leq -C \end{aligned}$$

for $C := \frac{2}{3} - p_0$, where $C > 0$ as soon as $p_0 \in \left]0, \frac{2}{3}\right[$. Therefore, U is a (C^∞) degree-2 MRF function.

4. SKETCH OF THE PROOF OF THEOREM 2.1

Since the proof in the case without cost is already contained in Motta and Rampazzo (2018), we provide here accuracy only when the cost comes into play, and assume that:

1) U is a degree-k MRF with $p_0 > 0$, $k \geq 1$, which enjoys the α-property on $B(\mathcal{T}, \nu) \setminus \mathcal{T}$ for some $\alpha \in [0, 1]$ and $\nu > 0$;
2) an $N > 0$ is fixed and γ is an N-dissipativity rate of U, which is also (α, k)-increasing.

Step 1. (*A degree-k "feedback"*) For every $x \in U^{\leftarrow}(]0, 2N])$ let us fix a selection $p(x) \in D^*U(x)$, so that

$$H^{(k)}(x, p(x), p_0) \leq -\gamma(U(x)).$$

A selection $\mathbf{v}(x) \in \mathcal{F}^{(k)}(x)$ is called a *degree-k feedback* if there exists a positive integer $h \leq k$ such that

$$\begin{cases} \mathbf{v}(x) \in \mathcal{F}^{(h)}(x), \\ \Big\langle p(x),\, \mathbf{v}(x) \Big\rangle + p_0\, \ell(x) \leq -\gamma(U(x)), \\ \text{and, if } h > 1, \quad H^{(h-1)}(x, p(x), p_0) > -\gamma(U(x)). \end{cases} \tag{20}$$

We will call h the *degree of the feedback* $\mathbf{v}$ *at* x.

Since we are going to build trajectories that remain for all times in $U^{\leftarrow}([0, 2N])$, it is not restrictive to assume that, for some $M > 0$, any iterated bracket X in $\mathcal{F}^{(k)}$ verifies

$$\sup_{x \in \mathbb{R}^n} |X(x)| \leq M. \tag{21}$$

Under this assumption (see e.g. Feleqi and Rampazzo (2017)), we have:

Lemma 4.1. There exists some $c > 0$ such that for any $z \in U^{\leftarrow}(]0, 2N])$, any feedback $\mathbf{v}$ of degree h at z, and any $t > 0$, one can find a control $\alpha_t : [0, t] \to A$ such that

(i) α_t is constant on $\left[\frac{jt}{r}, \frac{(j+1)t}{r}\right[$ for $j = 0, \ldots, r-1$;

(ii) with the position $y_z^t(\cdot) := y_z(\cdot, \alpha_t)$, one has

$$\left| y_z^t(t) - z - \frac{\mathbf{v}(z)}{r^h}\, t^h \right| \leq \frac{c}{r^h}\, t^{h+1}, \tag{22}$$

where r is a positive integer depending on the formal bracket corresponding to $\mathbf{v}(z)$. [5]

Step 2. (*A crucial estimate*)

Lemma 4.2. Fix p, $\mathbf{v}$ as in Step 1. There exists a time-valued function $\tau :]0,N] \times \{1,\dots,k\} \to]0,1]$, such that

i) $j \mapsto \tau(u,j)^j$ and $j \mapsto \tau(u,j)^{j-1}$ are decreasing for every $u \in]0,N]$,
ii) $u \mapsto \tau(u,j)$ is increasing for every $j \in \{1,\dots,k\}$, and
iii) for all $z \in U^{\leftarrow}(]0,N])$ with the feedback $\mathbf{v}$ of degree h at z, for any $t \in [0,\, \tau(U(z),h)]$ one has

$$U(y_z^t(t)) - U(z) + p_0 \max_{s\in[0,t]} \ell(y_z^t(t)) \left(\frac{t}{r}\right)^h \leq -\frac{\gamma(U(z))}{2}\left(\frac{t}{r}\right)^h, \tag{23}$$

where r and y_z^t are as in Lemma 4.1.

Proof. The α-property on $B(\mathcal{T},\nu)$ together with the local semiconcavity of U imply that there exist positive parameters R, L and C such that, for any $z, \hat z \in U^{\leftarrow}(]0,2N])$ with $[z,\hat z] \subset U^{\leftarrow}(]0,2N]) \setminus \mathcal{T}$ and $|\hat z - z| \leq R$, one has

$$U(\hat z) - U(z) \leq \langle p, \hat z - z\rangle + C\left(1 \vee \frac{1}{\mathbf{d}([z,\hat z])^\alpha}\right)|\hat z - z|^2,$$
$$|p| \leq L \qquad \forall p \in D^*U(z). \tag{24}$$

For every $(u,j) \in \mathbb{R}_*^+ \times \{1,\dots,k\}$, let us set

$$\tau_0 := \frac{N}{ML} \wedge \frac{R}{M}, \quad \tau_1(u,j) := 1 \wedge \left(\frac{u}{2L(M+c)}\right)^{1/j}, \tag{25}$$

where M verifies (21) and c is as in Lemma 4.1. Given $z \in U^{\leftarrow}(]0,N])$, we get $U(z) \leq L\,\mathbf{d}(z)$ and

$$y(t) \in U^{\leftarrow}([0,2N]), \quad |y(t) - z| \leq R \qquad \forall t \in [0,\tau_0],$$

for any trajectory y issuing from z. Furthermore, if $\mathbf{v}$ is a feedback of degree $h \leq k$ at z, by (22), for every $t \in [0,\tau_1(U(z),h)]$ one has ($t \leq 1$ and)

$$|y_z^t(t) - z| \leq (M+ct)\left(\frac{t}{r}\right)^h \leq \frac{U(z)}{2L}. \tag{26}$$

For all $t \in [0,\tau_0 \wedge \tau_1(U(z),h)]$, by estimate (26) one has $|y_z^t(t) - z| \leq \frac{\mathbf{d}(z)}{2}$, so that

$$\mathbf{d}([z, y_z^t(t)]) \geq \frac{\mathbf{d}(z)}{2} \geq \frac{U(z)}{2L}.$$

Let Λ denote the Lipschitz constant of ℓ on $U^{\leftarrow}([0,2N])$ and, for every $(u,j) \in]0,N] \times \{1,\dots,k\}$, let $\check\tau = \check\tau(u,j) > 0$ be the (unique) solution of

$$(Lc + p_0\Lambda M)\,\check\tau + \frac{C(M+c)^2(2L)^\alpha}{[(2L)\wedge u]^\alpha}\,\check\tau^j = \frac{\gamma(u)}{2}. \tag{27}$$

We set

$$\tau(u,j) := \tau_0 \wedge \tau_1(u,j) \wedge \check\tau(u,j). \tag{28}$$

By (24), for any $t \in [0,\tau(U(z),h)]$ we have

$$\begin{aligned}
&U(y_z^t(t)) - U(z) + p_0 \max_{s\in[0,t]} \ell(y_z^t(s))\left(\frac{t}{r}\right)^h \\
&\leq \left\langle p(z), y_z^t(t) - z\right\rangle + C\left(1 \vee \frac{(2L)^\alpha}{U^\alpha(z)}\right)\left|y_z^t(t) - z\right|^2 \\
&\qquad + p_0\left[\ell(z) + \Lambda \max_{s\in[0,t]} |y_z^t(s) - z|\right]\left(\frac{t}{r}\right)^h \\
&\leq \left[\left\langle p(z), \mathbf{v}(z)\right\rangle + p_0\,\ell(z)\right]\left(\frac{t}{r}\right)^h \\
&\qquad + [Lct + p_0\Lambda M t]\left(\frac{t}{r}\right)^h \\
&\qquad + C(M+ct)^2\left(1 \vee \frac{2L}{U(z)}\right)^\alpha \left(\frac{t}{r}\right)^{2h} \\
&\leq -\frac{\gamma(U(z))}{2}\left(\frac{t}{r}\right)^h
\end{aligned} \tag{29}$$

where the last inequality follows from the definition (27) of $\check\tau$, since $t \leq 1$ and $r \geq 1$. Notice that the monotonicity properties of τ are easy consequences of its definition.

Step 3. (*A trajectory approaching the target*) Fixed x in $U^{\leftarrow}(]0,N])$, we define recursively times t_j ($j \geq 0$), trajectory-control pairs $(y_j,\alpha_j) : [s_{j-1},s_j] \to \mathbb{R}^n \times A$, $s_0 := 0$, $s_j := s_{j-1} + t_j$, and points x_j ($j \geq 1$), as follows:

1) $t_0 := s_0 = 0$, $x_1 := x$, for every $j \geq 1$, $t_j := \tau(u_j,h_j)$, where $u_j := U(x_j)$, h_j is the degree of the feedback $\mathbf{v}$ at x_j and τ is as in Lemma 4.2;
2) $(y_1,\alpha_1) := (y_{x_1}^{t_1}, \alpha_{t_1})$, and for every $j > 1$ we set $y_j(s_{j-1}) := y_{j-1}(s_{j-1}) := x_j$, and $(y_j,\alpha_j)(s) := (y_{x_j}^{t_j}, \alpha_{t_j})(s - s_{j-1})$ for all $s \in [s_{j-1}, s_j]$.

At this point, the proof that the trajectory-control pair $(y,\alpha) := (y_j,\alpha_j)$ on any interval $[s_{j-1},s_j]$, $j \geq 1$, is admissible and that there is a $\mathcal{KL}$ function β such that $\mathbf{d}(y(t)) \leq \beta(\mathbf{d}(x),t)$ for all $t \geq 0$ –namely, that the system is GAC to $\mathcal{T}$– is akin to the proof of (Motta and Rampazzo , 2018, Thm. 2.1), hence we omit it.

Step 4. (*A bound on the cost*) Given $x \in U^{\leftarrow}(]0,N])$, let (y,α) be a pair determined as in Step 3. By (23), we have

$$\max_{s\in[s_{j-1},s_j]} \ell(y(s))\left(\frac{t_j}{r_j}\right)^{h_j} \leq \frac{u_j - u_{j+1}}{p_0} \quad \forall j \geq 1,$$

if r_j denotes the integer r in (23) when $z = x_j$. Hence,

$$\int_{s_{j-1}}^{s_j} \ell(y(s))\,ds \leq \max_{s\in[s_{j-1},s_j]} \ell(y(s))\,t_j \leq \frac{r_j^{h_j}(u_j - u_{j+1})}{p_0\, t_j^{h_j-1}}. \tag{30}$$

In order to estimate the quantity $\frac{1}{t_j^{h_j-1}}$, let us notice that, by (27), either $(Lc + p_0\Lambda M)\,\check\tau \geq \gamma(u)/4$ or $\frac{C(M+c)^2(2L)^\alpha}{[(2L)\wedge u_j]^\alpha}\check\tau^h \geq \gamma(u)/4$. Being $t_j = \tau(u_j,h_j)$, one gets

$$\frac{1}{t_j^{h_j-1}} \leq \frac{1}{\tau_0^{h_j-1}} \vee 1 \vee \frac{[2L(M+c)]^{1-1/h_j}}{u_j^{1-1/h_j}} \vee \frac{[4\,(Lc + p_0\Lambda M)]^{h_j-1}}{\gamma^{h_j-1}(u_j)} \vee \frac{[4\,C\,(2L)^\alpha(M+c)^2]^{1-1/h_j}}{\{\gamma(u_j)[(2L)\wedge u_j]^\alpha\}^{1-1/h_j}}.$$

Since $\lim_j u_j = 0$ by Step 3 (and $h_j \leq k$ for all $j \geq 1$), in case $\gamma(0) = 0$ there exist some integer $\bar{\jmath}$ and some constants

[5] We recall that r is increasing with the degree and one can have different r's for feedbacks of the same degree (see e.g. Motta and Rampazzo (2018)).

$\tilde{C}_1 > 0$, $\tilde{C}_2 > 0$, such that

$$\frac{1}{t_j^{h_j-1}} \leq \tilde{C}_1 \qquad \forall j \leq \bar{\jmath},$$

$$\frac{1}{t_j^{h_j-1}} \leq \frac{\tilde{C}_2}{u_j^{1-1/k}} \vee \frac{\tilde{C}_2}{\gamma^{k-1}(u_j)} \vee \frac{\tilde{C}_2}{[u_j^{\alpha}\,\gamma(u_j)]^{1-1/k}} \quad \forall j > \bar{\jmath},$$

Since γ is increasing, when $\gamma(0) > 0$ we can simply replace $\gamma(u_j)$ in the last estimate above with $\gamma(0)$. Because of the definition of Ψ (see (16)), by (30) we finally obtain

$$\int_0^{T_y} \ell(y(s))\,ds \leq \sum_{j=1}^{+\infty} \int_{s_{j-1}}^{s_j} \ell(y(s))\,ds \leq \bar{C}\Psi(U(z))$$

with $\bar{C} := 2\max\{r_j^{h_j},\ j \in \mathbb{N}\} \cdot \frac{\tilde{C}_1 \vee \tilde{C}_2}{p_0}$.

5. CONCLUSIONS

For a class of nonlinear control systems, we have introduced a degree-k dissipativaty inequality involving the Lagrangian and some iterated Lie brackets (of the dynamics' vector fields). We have called *degree-k Minimum Restraint Function* (MRF) a solution to this inequality. (In the case when the Lagrangian is equal to zero –that is, there is no minimization involved– U reduces to a *degree-k Control Lyapunov Function.*)

The existence of a degree-k MRF U enables one to build, from any initial state, an open-loop strategy producing both the asymptotic controllability to a target and an explicit bound on the corresponding cost. It is worth noticing that the set of solutions of this degree-k inequality becomes larger and larger as k increases. In particular, the weakened condition allows for the existence of a smooth degree-k MRF ($k > 1$) in several cases when topological obstructions prevent a C^1 degree-1 MRF to exist.

Recently, in Lai and Motta (2021, 2020, 2019) it has been shown that the existence of a degree-1 MRF implies the existence of a (discontinuous) stabilizing feedback *and* provides a bound to the cost. We are led to believe that, starting from the result of the present work, one could introduce a concept of 'feedback strategy' in relation with a degree-k MRF U for $k > 1$. Let us observe that the augmented complexity due to the use of Lie brackets might be compensated by the enhanced regularity of U.

Furthermore, as a likely development of our results, one could also try to define degree-k MRFs in the case of a control system with a *drift* and with unbounded controls. Indeed, the directions along iterated brackets (possibly including the drift as well) can be run by the system also in this case (see e.g. Aronna and Motta and Rampazzo (2020) and Chittaro and Stefani (2016)).

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