Converse Lyapunov theorems for control systems with unbounded controls ☆

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Abstract

In this paper we extend well-known relationships between global asymptotic controllability, sample stabilizability, and the existence of a control Lyapunov function to a wide class of control systems with unbounded controls, including control-polynomial systems. In particular, we consider open loop controls and discontinuous stabilizing feedbacks, which may be unbounded approaching the target, so that the corresponding trajectories may present a chattering behaviour. A key point of our results is to prove that global asymptotic controllability, sample stabilizability, and existence of a control Lyapunov function for these systems or for an *impulsive extension* of them are equivalent.

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1. Introduction

In this paper we extend classic equivalence results between global asymptotic controllability to a set $\mathcal{C}$, sample stabilizability to $\mathcal{C}$, and the existence of a control Lyapunov function to control systems *with an impulsive character.* In particular, we consider a control system of the form

$$\dot{x}(t) = f(x(t), u(t)) \text{ a.e.}, \qquad u(t) \in U \text{ a.e.}, \tag{1}$$

where the (unbounded) control set $U \subseteq \mathbb{R}^m$ is a closed cone, the target set $\mathcal{C} \subset \mathbb{R}^n$ is closed with compact boundary, and the function $f : \mathbb{R}^n \times U \to \mathbb{R}^n$

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satisfies suitable growth assumptions in the control variable, which include control-polynomial dependence (see hypothesis **(Hg)** below). The extension lies in the fact that, following [1, 2, 3], we consider notions of global asymptotic controllability and sample stabilizability which involve open loop controls $u \in L^\infty_{loc}$ and locally bounded feedback laws $K : \mathbb{R}^n \setminus \mathcal{C} \to U$ with possibly $\limsup_{x\to\bar{x}\in\partial\mathcal{C}} |K(x)| = +\infty$, respectively.

The problems of asymptotic controllability and feedback stabilization to a point or to a set of control systems that are nonlinear (and for which linearization fails), and their relationships with the existence of control Lyapunov functions have been central topics in control theory since the 1980s. It is now well-known that a smooth control Lyapunov function, which guarantees the asymptotic controllability of the system, may not exist and a continuous stabilizing feedback fails in general to exist either, even if the dynamics function f is smooth and bounded and the control set U is compact (see [4, 5, 6, 7, 8, 9, 10]). These problems were solved in [11] by the introduction of nonsmooth control Lyapunov functions, discontinuous feedback laws $K = K(x)$, and a "sample and hold" solution concept, similar to that used in differential games in [12].

In this context, converse Lyapunov theorems have been established. Such results consist in showing how global asymptotic controllability (GAC) (equivalent to sample stabilizability by [11]) implies the existence of continuous (see [7]), and actually locally Lipschitz and even semiconcave control Lyapunov functions. The latter property plays a fundamental role in the the explicit construction of stabilizing feedback strategies (see [13, 14, 15, 16, 17, 18]). We have limited ourselves to mentioning only a few key articles and those most related to the present work. For a broader overview we refer e.g. to [19, 20, 21, 22, 23] and references therein. There are two key hypotheses in the above literature:

(i) f is continuous in (x, u) and Lipschitzean in x on compact subsets of $\mathbb{R}^n$ (or 'uniform in distance to the set $\mathcal{C}$', as in [17]), uniformly with respect to U;

(ii) the vector field $f(x, u)$ associated to $u = u(t)$ or $u = K(x)$ and steering trajectories of (1) to $\mathcal{C}$ in a uniform way, is bounded in any neighborhood of the target. Actually, it is usually assumed that these open-loop and feedback controls are themselves bounded for states close to $\mathcal{C}$.

Our aim is to extend the above results to a wide class of control systems with impulsive character, where conditions (i), (ii) do not hold. Before introducing these systems precisely, let us point out that we do not achieve the extension by refining the techniques used in the case of classic assumptions on f. Rather, using a reparameterization technique commonly adopted in optimal impulsive control, as generalized in [24, 25, 26], our main result consists in proving that GAC, sample stabilizability, and existence of a Lyapunov function for the original control system (1) are equivalent to GAC, sample stabilizability, and existence of a Lyapunov function, respectively, for a related extended control system with bounded controls (see the Converse Lyapunov Theorem 4.3). This equivalence makes it possible to reduce the study of the stabilizability of systems with an impulsive character to the stabilizability of systems satisfying the classical hypotheses (i), (ii). In particular, given a semiconcave Lyapunov function for the extended system we are able to construct a locally bounded stabilizing feedback strategy for system (1) (see Subsection 4.2 below).

More in detail, the key hypothesis on the dynamics function f that we will assume in our main results is:

(Hg) *there exists a strictly increasing, bijective function* $\nu : [0, +\infty) \to [0, +\infty)$, *to which we refer to as* growth rate, *such that*

(i) *the function* $\bar{f} : (\mathbb{R}^n \setminus \mathcal{C}) \times U \to \mathbb{R}^n$, *defined by* $\bar{f}(x, u) := \dfrac{f(x, u)}{1 + \nu(|u|)}$, *is uniformly continuous on* $\mathcal{K} \times U$ *for any compact set* $\mathcal{K} \subset \mathbb{R}^n \setminus \mathcal{C}$ *and bounded on* $(B_R(\mathcal{C}) \setminus \mathcal{C}) \times U$ *for any* $R > 0$;

(ii) *the function* $F : \overline{(\mathbb{R}^n \setminus \mathcal{C})} \times [0, +\infty) \times U \to \mathbb{R}^n$, *defined as*

$$F(x, w_0, w) := \lim_{r \to w_0^+} \bar{f}\left(x, \frac{w}{|w|}\,\nu^{-1}\left(\frac{|w|}{r}\right)\right) = \lim_{r \to w_0^+} f\left(x, \frac{w}{|w|}\,\nu^{-1}\left(\frac{|w|}{r}\right)\right) r \text{ [1]}$$

is not identically zero, it is continuous, and locally bounded on $\overline{(\mathbb{R}^n \setminus \mathcal{C})} \times \overline{\mathbb{U}}$, *where* $\mathbb{U} := \{(w_0, w) \in (0, +\infty) \times U : \quad w_0 + |w| = 1\}$.

The function ν represents the maximal growth of f in the control u. We will refer to $\bar{f}$ and F as *rescaled dynamics* and *extended dynamics* function, respectively. This extension consists essentially in a control-compactification, obtained by adding the scalar control w_0, so that the pairs $(w_0, w) = (0, w)$ with $|w| = 1$ represent the points of U 'at infinity'. An important class of control systems satisfying hypothesis **(Hg)** is given by control-polynomial systems:

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{\bar{d}} \left(\sum_{\substack{\alpha \in \mathbb{N}^m \\ \alpha_1 + \cdots + \alpha_m = i}} f_\alpha(x(t))\, u_1^{\alpha_1}(t) \cdots u_m^{\alpha_m}(t) \right), \quad u(t) \in U, \tag{2}$$

with continuous coefficients f_0, f_α. In this case, a growth rate is $\nu(r) := r^{\bar{d}}$ and the related extended dynamics function F reads

$$F(x, w_0, w) = f_0(x) w_0 + \sum_{i=1}^{\bar{d}} \sum_{\substack{\alpha \in \mathbb{N}^m \\ \alpha_1 + \cdots + \alpha_m = i}} \frac{w_0^{1 - i/\bar{d}}}{|w|^{i - i/\bar{d}}} w_1^{\alpha_1} \cdots w_m^{\alpha_m} f_\alpha(x)$$

for $x \in \mathbb{R}^n \setminus \mathcal{C}$, $(w_0, w) \in \overline{\mathbb{U}} = \{(w_0, w) \in [0, +\infty) \times \mathbb{R}^m \mid w_0 + |w| = 1\}$. Notice that, because of the unboundedness of the cone U, control polynomial systems violate assumptions (i), (ii) even if all vector fields f_0, f_α are smooth. System (2) plays a relevant role, for instance, in applications to Lagrangian mechanics where part of the coordinates act as controls. The evolution of the

[1] For any $w \in \mathbb{R}^m$, when $w = 0$ we mean that $\frac{w}{|w|} = 0$.

remaining coordinates is then described by an "impulsive" control system, where the dynamics function is linearly or quadratically dependent on the derivatives of the controlled coordinates, derivatives which are identifiable with unbounded controls (see [27, 28], and [29] with references therein). For this kind of systems it is quite common that GAC and stabilizability cannot be achieved by means of bounded controls, but it is possible (and completely reasonable) to stabilize the system using unbounded controls. Let us mention e.g. [29], where the authors exhibit examples of mechanical systems controlled by moving holonomic constraints, for which stabilization can only be achieved by "vibrating controls", namely allowing unbounded inputs.

We establish our main result, the Converse Lyapunov Theorem 4.3, under hypothesis **(Hg)** and assumptions allowing to apply a classic version of the Converse Lyapunov Theorem to the extended dynamics F (see hypothesis **(Hg)*** in Section 4). The statement of Theorem 4.3 can be roughly summarized in the equivalence between global asymptotic controllability, sample stabilizability, and existence of a Control Lyapunov Function, for either the original control system $\dot{x} = f(x, u)$ or the related extended control system $\dot{x} = F(x, w_0, w)$. The proof of this result relies on three key results: Theorem 2.5, dealing with two equivalent notions of GAC for unbounded dynamics, which is crucial in order to prove that GAC of $\dot{x} = f(x, u)$ and GAC of the rescaled control system $\dot{x} = \bar{f}(x, u)$ are equivalent (see Theorems 3.1, 3.2); the interplay between the rescaled and the extended control system (see Propositions 3.1, 4.1); the fact that sample stabilizability implies GAC also for dynamics which are merely continuous on $\mathbb{R}^n \setminus \mathcal{C}$ (see Theorem 2.9).

The paper is organized as follows. In Section 2 we introduce the notions of GAC and GAC with $U \cap \sigma$ controls, prove that they are equivalent, and show that sample stabilizability implies GAC. Section 3 is devoted to establish some relationships between the rescaled and the extended system. In Section 4 we prove the converse Lyapunov theorem and describe how related explicit feedback constructions for the original and for the extended control system can be implemented. In Section 5, an example illustrates the results of the paper.

1.1. Notations

For $a, b \in \mathbb{R}$, we set $a \vee b := \max\{a, b\}$, $a \wedge b := \min\{a, b\}$. Let $\Omega \subseteq \mathbb{R}^N$ for some integer $N \geq 1$ be a nonempty set. For every $r \geq 0$, we set $B_r(\Omega) := \{x \in \mathbb{R}^n : \ d(x, \Omega) \leq r\}$, where d is the usual Euclidean distance. We use $\overline{\Omega}$, $\partial\Omega$, and $\mathring{\Omega}$ to denote the closure, the boundary, and the interior of Ω, respectively. For any interval $I \subseteq \mathbb{R}$, $L^\infty(I, \Omega)$, $AC(I, \Omega)$ are the sets of functions $x : I \to \Omega$, which are essentially bounded or absolutely continuous, respectively, on I. We use $L^\infty_{loc}(I, \Omega)$, $AC_{loc}(I, \Omega)$ to denote the sets of functions $x : I \to \Omega$, which are essentially bounded or absolutely continuous on any compact subset $J \subset I$. When no confusion may arise, we simply write $L^\infty(I)$, $AC(I)$, $L^\infty_{loc}(I)$, $AC_{loc}(I)$.

2. Global asymptotic controllability and sample stabilizability

We introduce two concepts of global asymptotic controllability and prove that they are equivalent. Furthermore, we show that sample stabilizability implies global asymptotic controllability, as in the case of bounded controls.

Unless otherwise specified, we assume $f : (\mathbb{R}^n \setminus \mathcal{C}) \times U \to \mathbb{R}^n$ continuous. Let us set $\mathbf{d}(x) := d(x, \mathcal{C})$.

2.1. Equivalent concepts of global asymptotic controllability

Definition 2.1 (Admissible trajectory-control pair)**.** *A couple (x, u) is called an* admissible trajectory-control pair *for (f, U) if there exists $T_x \leq +\infty$ such that: the control u belongs to $L^\infty_{loc}([0, T_x), U)$; the trajectory $x \in AC_{loc}([0, T_x), \mathbb{R}^n \setminus \mathcal{C})$ verifies*

$$\dot{x}(t) = f(x(t), u(t)) \qquad a.e.\ t \in [0, T_x); \tag{3}$$

and, if $T_x < +\infty$, one has $\lim_{t \to T_x^-} \mathbf{d}(x(t)) = 0$. If (x, u) is an admissible trajectory-control pair for (f, U) and $T_x < +\infty$, we extend x to $[0, +\infty)$ by setting $x(t) := \bar{z}$ for any $t \geq T_x$, where $\bar{z}$ is an arbitrary point of the set

$$\mathcal{C}_x := \left\{ \zeta \in \partial\mathcal{C} : \exists \tau_i \uparrow T_x \ \text{ as } i \to +\infty \text{ and } \lim_{i \to +\infty} x(\tau_i) = \xi \right\}.^{2}$$

When no confusion may arise, we will simply call *admissible trajectory-control pair* any admissible trajectory-control pair for (f, U).

As customary, we use $\mathcal{KL}$ to denote the set of all continuous functions $\beta : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ such that: (1) $\beta(0, t) = 0$ and $\beta(\cdot, t)$ is strictly increasing and unbounded for each $t \geq 0$; (2) $\beta(r, \cdot)$ is strictly decreasing for each $r \geq 0$; (3) $\beta(r, t) \to 0$ as $t \to +\infty$ for each $r \geq 0$. We refer to any function $\beta \in \mathcal{KL}$ as a *descent rate.*

Definition 2.2 (GAC)**.** *The system* (3) *is called* Globally Asymptotically Controllable (GAC) to $\mathcal{C}$ *if there exists a function $\beta \in \mathcal{KL}$ such that for any initial point $z \in \mathbb{R}^n \setminus \mathcal{C}$ there is an admissible trajectory-control pair (x, u) for (f, U) with $x(0) = z$, such that*

$$\mathbf{d}(x(t)) \leq \beta(\mathbf{d}(z), t) \quad \forall t \geq 0. \tag{4}$$

Definition 2.3 (GAC with $U \cap \sigma$ controls)**.** *Let $\sigma : (0, +\infty) \to (0, +\infty)$ be a continuous function. We say that the system* (3) *is* Globally Asymptotically Controllable (GAC) to $\mathcal{C}$ with $U \cap \sigma$ controls *if there exists a descent rate $\beta \in \mathcal{KL}$ such that for any initial point $z \in \mathbb{R}^n \setminus \mathcal{C}$ there is an admissible trajectory-control pair (x, u) for (f, U) with $x(0) = z$, such that*

$$\mathbf{d}(x(t)) \leq \beta(\mathbf{d}(z), t) \quad \forall t \geq 0$$

[2] By the very definition of T_x, we have that $\mathcal{C}_x \neq \emptyset$, as $\partial\mathcal{C}$ is assumed to be compact. Hence, for each admissible trajectory-control pair (x, u) for (f, U), the trajectory x, possibly extended as above, is always defined on the whole interval $[0, +\infty)$.

and

$$|u(t)| \le \sigma(\mathbf{d}(x(t)) \quad \textit{for a.e. } t \in [0, T_x). \tag{5}$$

Remark 2.4. The concept of GAC with $U \cap \sigma$ controls introduced above only apparently coincides with the definition considered, e.g., in the survey papers [30, 17] from which the notation $U \cap \sigma$ is mutuated. In fact, in the previous literature the function σ was supposed to be increasing, in order to prevent unbounded inputs around the target. On the contrary, in Definition 2.3 it may happen that $\lim_{r \to 0^+} \sigma(r) = +\infty$, thus allowing for controls with L^∞ norm diverging to $+\infty$ as the trajectory approaches $\mathcal{C}$.

The concept of GAC with $U \cap \sigma$ controls, although apparently stronger than GAC, when f is locally Lipschitz continuous in x, is equivalent to GAC.

Precisley, let us consider the following hypothesis:

(Hl) *the function $f : (\mathbb{R}^n \setminus \mathcal{C}) \times U \to \mathbb{R}^n$ is continuous and, for every pair of compact sets $\mathcal{K} \subset \mathbb{R}^n \setminus \mathcal{C}$, $U_1 \subset U$, there is some constant $L > 0$ such that*

$$|f(x, u) - f(y, u)| \le L\,|x - y| \qquad \forall x, y \in \mathcal{K},\ \forall u \in U_1.$$

Theorem 2.5. *Let f satisfy* **(Hl)**. *Then, system* (3) *is GAC to $\mathcal{C}$ if and only if it is GAC to $\mathcal{C}$ with $U \cap \sigma$ controls.*

Proof. If system (3) is GAC to $\mathcal{C}$ with $U \cap \sigma$, it is trivially GAC to $\mathcal{C}$. So, let us assume that (3) is GAC to $\mathcal{C}$ and prove that it is GAC with $U \cap \sigma$ controls by building a continuous positive function σ and a descent rate $\bar{\beta}$.

Step 1. (β-admissible trajectory-control pairs) Without loss of generality, in the definition of GAC we can assume the inequality in (4) strict, namely, that there exists a descent rate $\beta \in \mathcal{KL}$ such that for all $z \in \mathbb{R}^n \setminus \mathcal{C}$ there is some admissible trajectory-control pair (x, u) with $x(0) = z$, such that

$$\mathbf{d}(x(t)) < \beta(\mathbf{d}(z), t) \qquad \forall t \ge 0. \tag{6}$$

We refer to such (x, u) as a β-*admissible trajectory-control pair from* z. Let us define the set $\mathcal{A}_\beta$, given by the triplets (x, u, z), where $z \in \mathbb{R}^n \setminus \mathcal{C}$ and, for each fixed z, we select exactly one (x, u) among the admissible trajectory-control pairs from z. Note that, by (6), one has

$$\beta(R, 0) > R \quad \forall R > 0. \tag{7}$$

Step 2. (β-strips) Let $r_0 := 1$ and recursively define $(r_i)_{i \in \mathbb{Z}}$ by

$$r_{i-1} = \beta(r_i, 0) \qquad i \in \mathbb{Z},$$

so that, for instance, r_1 is the solution of $\beta(r_1, 0) = 1 = r_0$ and $r_{-1} = \beta(r_0, 0)$. By (7) and by the definition of $\mathcal{KL}$ functions, we have that the sequence $(r_i)_{i \in \mathbb{Z}}$ is strictly decreasing, positive and

$$\lim_{i \to +\infty} r_i = 0 \quad \text{and} \quad \lim_{i \to -\infty} r_i = +\infty.$$

For every $i \in \mathbb{Z}$, set $\mathcal{B}_i := \{z \in \mathbb{R}^n \setminus \mathcal{C} : \ \mathbf{d}(z) \in [r_i, r_{i-1}]\}$. We define the *i-th β-strip* as the set $\mathcal{A}_\beta^i := \{(x, u, z) \in \mathcal{A}_\beta : \quad z \in \mathcal{B}_i\}$. Note that for all $i \in \mathbb{Z}$, $r_{i-2} = \beta(r_{i-1}, 0)$, therefore, for every $(x, u, z) \in \mathcal{A}_\beta^i$ one has $\mathbf{d}(x(t)) < r_{i-2}$ for all $t \geq 0$.

Fix $i \in \mathbb{Z}$ and consider a triplet $(x, u, z) \in \mathcal{A}_\beta^i$. Define

$$T_{i,z} := \inf \left\{ t \geq 0 : \ \mathbf{d}(x(t)) = \frac{r_i + r_{i+1}}{2} \right\}. \tag{8}$$

Clearly, $0 < T_{i,z} < T_x$ and the fact that $u \in L^\infty_{loc}([0, T_x))$ implies

$$\|u\|_{L^\infty([0,T_{i,z}])} < +\infty. \tag{9}$$

Set

$$\tilde{\varepsilon}_{i,z} := \inf \left\{ \frac{1}{2}(\beta(\mathbf{d}(z), t) - \mathbf{d}(x(t)) : \ t \in [0, T_{i,z}] \right\}.$$

Note that, by the continuity of β and x, $\tilde{\varepsilon}_{i,z}$ is actually a minimum. Furthermore, $\tilde{\varepsilon}$ is positive in view of (6). Define

$$\bar{\varepsilon}_i := \frac{r_i - r_{i+1}}{4}, \qquad \varepsilon_{i,z} := \min\{\tilde{\varepsilon}_{i,z}, \bar{\varepsilon}_i\}.$$

In view of the Lipschitzianity hypothesis **(Hl)**, there exists $\delta_{i,z} > 0$ such that, for all $\bar{z} \in \mathbb{R}^n \setminus \mathcal{C}$ verifying $|z - \bar{z}| < \delta_{i,z}$, the Cauchy problem $\dot{x} = f(x, u)$, $x(0) = \bar{z}$, admits a unique solution, denoted in the following by $x(\cdot\,; u, \bar{z})$, which is defined on the whole interval $[0, T_{i,z}]$ and verifies

$$|x(t) - x(t; u, \bar{z})| < \varepsilon_{i,z} \quad \forall t \in [0, T_{i,z}]. \tag{10}$$

From the definition of $\varepsilon_{i,z}$, it follows that

$$\begin{aligned} \mathbf{d}(x(t; u, \bar{z})) &< \mathbf{d}(x(t)) + \frac{1}{2}(\beta(\mathbf{d}(z), t) - \mathbf{d}(x(t)) < \beta(\mathbf{d}(z), t) \\ &\leq \beta(r_{i-1}, 0) = r_{i-2} \quad \forall t \in [0, T_{i,z}], \end{aligned}$$

while the definition of $T_{i,z}$ yields

$$\mathbf{d}(x(t; u, \bar{z})) \geq \mathbf{d}(x(T_{i,z})) - \tilde{\varepsilon}_i > r_{i+1} \quad \forall t \in [0, T_{i,z}].$$

In conclusion, for any $(x, u, z) \in \mathcal{A}_\beta^i$ there exists some $\delta_{i,z} > 0$ such that, for all $\bar{z} \in \mathbb{R}^n \setminus \mathcal{C}$ verifying $|z - \bar{z}| < \delta_{i,z}$, one obtains

$$\begin{cases} \mathbf{d}(x(t; u, \bar{z})) \in (r_{i+1}, r_{i-2}) & \forall t \in [0, T_{i,z}], \\ \bar{x}(T_{i,z}) \in \mathcal{B}_{i+1}. \end{cases} \tag{11}$$

Step 3. (Construction of σ and $\bar{\beta}$) Fix $i \in \mathbb{Z}$ and consider the cover of $\mathcal{B}_i$ given by the collection of open balls $\mathring{B}_{\delta_{i,z}}(\{z\})$, with $z \in \mathcal{B}_i$. Since $\partial\mathcal{C}$ is compact, then the i-th strip $\mathcal{B}_i$ is compact, as well, and consequently it admits a finite subcover $\{\mathring{B}_{\delta_{i,z}}(\{z\})\}_{z\in\mathcal{Z}_i}$, where $\mathcal{Z}_i$ is a finite subset of $\mathcal{B}_i$. Now, define

$$\bar{\sigma}(i) := \max\{\|u\|_{L^\infty([0,T_{i,z}])} : \quad (x,u,z) \in \mathcal{A}^i_\beta, \quad z \in \mathcal{Z}_i\}, \tag{12}$$

where $T_{i,z}$ is as in (8), and $\sigma : (0,+\infty) \to (0,+\infty)$, given by

$$\sigma(r) := \max\{\bar{\sigma}(i-i), \bar{\sigma}(i), \bar{\sigma}(i+1)\} \qquad \forall r \in [r_i, r_{i-1}).$$

Note that, for every $i \in \mathbb{Z}$, one has

$$\sigma(r) \geq \bar{\sigma}(i) \qquad \forall r \in [r_{i+1}, r_{i-2}). \tag{13}$$

To build a new descent rate function, (to be associated to controls u such that $|u(t)| \leq \sigma(\mathbf{d}(x(t)))$ for every $t \geq 0$), set

$$T_i := \max\{T_{i,z} : \ z \in \mathcal{Z}_i\}. \tag{14}$$

Replacing any time T_i with a larger value if necessary, we can assume that, for every $i \in \mathbb{Z}$, one has $\sum_{j=0}^{+\infty} T_{i+j} = +\infty$. Then, for every $i \in \mathbb{Z}$ and $N \in \mathbb{N}$, define

$$\bar{T}_{i,-1} := 0, \qquad \bar{T}_{i,N} := \sum_{j=0}^{N} T_{i+j}. \tag{15}$$

Note that $\bar{T}_{i,N} \to +\infty$ as $N \to \infty$ for every fixed $i \in \mathbb{Z}$. Consider the piecewise constant function $b : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$, given by

$$\begin{cases} b(R,t) := r_{i+N-2} & \text{if } R \in [r_i, r_{i-1}) \text{ and } t \in [\bar{T}_{i,N-1}, \bar{T}_{i,N}), \\ b(0,t) = 0 & \forall t \geq 0, \end{cases}$$

for all $i \in \mathbb{Z}$ and $N \in \mathbb{N}$. To make notation more compact, we introduce the decreasing, integer valued function $i(R) := i \in \mathbb{Z}$, such that $R \in [r_i, r_{i-1})$. Note that $i(R) \to +\infty$ as $R \to 0$ and $i(R) \to -\infty$ as $R \to +\infty$. We then rewrite the definition of b as follows

$$b(R,t) := r_{i(R)+N-2} \qquad \text{if } t \in [\bar{T}_{i(R),N-1}, \bar{T}_{i(R),N}),$$

for $N \in \mathbb{N}$. Since (r_i) vanishes as $i \to +\infty$, for all $R \in (0,+\infty)$, $b(R,t) \to 0$ as $t \to +\infty$. Moreover, since $r_{i(R)} \to +\infty$ as $R \to +\infty$, for every $t \geq 0$, we have that $b(R,t) \to +\infty$ as $R \to +\infty$. Then b can be dominated by some $\mathcal{KL}$ function, (say, a larger, continuous linear interpolation) that we call $\bar{\beta}$.

Step 4. (GAC with $U \cap \sigma$ controls) To conclude, for every initial datum $\bar{z} \in \mathbb{R}^n \setminus \mathcal{C}$ we need to provide a $\bar{\beta}$-admissible trajectory control pair $(\bar{x}, \bar{u})$ from $\bar{z}$, satisfying $|\bar{u}(t)| \leq \sigma(\mathbf{d}(\bar{x}(t))$ for all $t \geq 0$.

To this aim, fix $\bar{z} \in \mathbb{R}^n \setminus \mathcal{C}$ and, for brevity, set $i := i(\mathbf{d}(\bar{z}))$, so that $\bar{z} \in \mathcal{B}_i$. By Step 3 it follows that $\bar{z} \in \mathring{B}_{\delta_{i,z_0}}(\{z_0\})$ for some $z_0 \in \mathcal{Z}_i$ and, if $(x_0, u_0, z_0) \in \mathcal{A}^i_\beta$

(taking into account also the inequality (13)), there exists $\hat{t}_0 := T_{i,z_0} \leq T_i$ such that the trajectory $\bar{x} := x(t; u_0, \bar{z})$, satisfies

$$\mathbf{d}(\bar{x}(t)) \in (r_{i+1}, r_{i-2}) \quad \forall t \in [0, \hat{t}_0], \tag{16}$$

$$\bar{x}(\hat{t}_0)) \in \mathcal{B}_{i+1}, \tag{17}$$

$$|u_0(t)| \leq \bar{\sigma}(i) \leq \sigma(\mathbf{d}(\bar{x}(t))) \quad \forall t \in [0, \hat{t}_0]. \tag{18}$$

By repeating the same argument starting from the point $\bar{z}_1 := \bar{x}(\hat{t}_0) \in \mathcal{B}_{i+1}$, one obtains a time $\hat{t}_1 \leq T_{i+1}$ and a control $u_1 \in L^\infty([0, \hat{t}_1], U)$ which satisfy an updated version of (16)-(18), with i replaced by $i+1$. Iterating this procedure, one gets a sequence of times $(\hat{t}_N)_N$ and a sequence of controls (u_N), such that $\hat{t}_N \leq T_{i+N}$ and $u_N \in L^\infty([0, \hat{t}_N], U)$ for every N. Therefore, setting for every $N \in \mathbb{N}$,

$$\hat{T}_{-1} := 0, \quad \hat{T}_N := \textstyle\sum_{j=0}^N \hat{t}_j, \quad \hat{T}_\infty := \sum_{j=0}^{+\infty} \hat{t}_j,$$
$$\hat{u}(t) := u_N(t - \hat{T}_{N-1}) \quad \forall t \in (\hat{T}_{N-1}, \hat{T}_N],$$

one obtains a control $\hat{u} \in L^\infty_{loc}([0, \hat{T}_\infty), U)$ such that the corresponding trajectory $\hat{x} := x(t; \hat{u}, \bar{z})$ is defined on the whole interval $[0, \hat{T}_\infty)$ and enjoys the following properties:

$$\mathbf{d}(\hat{x}(t)) \in (r_{i+N+1}, r_{i+N-2}) \quad \forall t \in [\hat{T}_{N-1}, \hat{T}_N], \tag{19}$$

$$\hat{x}(\hat{T}_N) \in \mathcal{B}_{i+N+1}, \tag{20}$$

$$|\hat{u}(t)| \leq \sigma(\mathbf{d}(\hat{x}(t))) \quad \forall t \in [0, \hat{T}_N]. \tag{21}$$

In particular, $\mathbf{d}(\hat{x}(t)) \to 0$ as $t \to \hat{T}_\infty^-$.

At this point, a simple inductive argument shows that

$$\mathbf{d}(\hat{x}(t)) < r_{i+N-1} \quad \forall t \geq \hat{T}_N.$$

Furthermore, for every fixed N, $\hat{T}_N$, which depends on $\bar{z}$, is bounded above by a constant which depends only on $\mathbf{d}(\bar{z})$. Indeed, by construction, $\hat{T}_N \leq \bar{T}_{i,N}$, for all $N \geq 0$. Hence, one finally obtains that

$$\mathbf{d}(\hat{x}(t)) < r_{i(\mathbf{d}(z))+N-2} = b(\mathbf{d}(\bar{z}), t) \leq \bar{\beta}(\mathbf{d}(\bar{z}), t), \ t \in [\bar{T}_{i(\mathbf{d}(z)),N-1}, \bar{T}_{i(\mathbf{d}(z)),N}).$$

Since $\bar{T}_{i(\mathbf{d}(\bar{z})),N} \to +\infty$ as $N \to \infty$, this concludes the proof. □

2.2. Sample stabilizability

The feedback counterpart of a notion of GAC involving admissible trajectory-control pairs (x, u) *with controls u in* $L^\infty_{loc}([0, T_x), U)$, requires necessarily to consider *locally* bounded feedback functions $K : \mathbb{R}^n \setminus \mathcal{C} \to U$, which may have $\limsup_{x \to \bar{x} \in \partial\mathcal{C}} |K(x)| = +\infty$.

In correspondence of such feedbacks, following [2] we adopt the notion of *sample stabilizability* below.

A *partition* (of $[0, +\infty)$) is a sequence $\pi = (t_k)$ such that $t_0 = 0$, $t_{k-1} < t_k$ $\forall k \geq 1$, and $\lim_{k \to +\infty} t_k = +\infty$. The value $\mathrm{diam}(\pi) := \sup_{k \geq 1}(t_k - t_{k-1})$ will be called the *diameter* or the *sampling time* of the partition π.

Definition 2.6 (Sampling trajectory-control pair). *Given a locally bounded feedback $K : \mathbb{R}^n \setminus \mathcal{C} \to U$, a partition $\pi = (t_k)$, and a point $z \in \mathbb{R}^n \setminus \mathcal{C}$, we call* π-sampling trajectory-control pair *for* $\dot{x} = f(x, u)$ from z, *a pair (x, u), where the sampling trajectory x is a continuous function defined by recursively solving*

$$\dot{x} = f(x(t), K(x(t_{k-1}))) \qquad a.e.\ t \in [t_{k-1}, t_k],\ (x(t) \in \mathbb{R}^n \setminus \mathcal{C})$$

from the initial time t_{k-1} up to time

$$\tau_k := t_{k-1} \vee \sup\{\tau \in [t_{k-1}, t_k] :\ x \text{ is defined on } [t_{k-1}, \tau)\},$$

such that $x(t_0) = x(0) = z$. In this case, the trajectory x is defined on the right-open interval from time zero up to time $T^- := \inf\{\tau_k :\ \tau_k < t_k\}$. Accordingly, for every $k \geq 1$ and for all $t \in [t_{k-1}, t_k) \cap [0, T^-)$, the sampling control *is defined as*

$$u(t) := K(x(t_{k-1})) \quad \forall t \in [t_{k-1}, t_k) \cap [0, T^-), \quad k \geq 1. \tag{22}$$

If $T^- = T_x < +\infty$ such that $\lim_{t \to T_x^-} \mathbf{d}(x(t)) \to 0$, we extend x to $[0, +\infty)$ as described in Definition 2.1.

Definition 2.7 (Sample stabilizability). *A locally bounded feedback $K : \mathbb{R}^n \setminus \mathcal{C} \to U$ is said to* sample stabilize the control system $\dot{x} = f(x, u)$ to $\mathcal{C}$ *if there is a descent rate $\beta \in \mathcal{KL}$ satisfying the following: for each pair $0 < r < R$ there exists $\delta = \delta(R, r) > 0$, such that, for every partition π with $diam(\pi) \leq \delta$ and for any $z \in \mathbb{R}^n \setminus \mathcal{C}$ such that $\mathbf{d}(z) \leq R$, any π-sampling trajectory-control pair (x, u) with $x(0) = z$ is admissible and verifies:*

$$\mathbf{d}(x(t)) \leq \max\{\beta(\mathbf{d}(z), t), r\} \qquad \forall t \in [0, +\infty). \tag{23}$$

We call $\dot{x} = f(x, u)$ sample stabilizable to $\mathcal{C}$ *if there is a feedback K as above.*

Remark 2.8. Given a discontinuous feedback K, in this paper we only consider sampling trajectories, which are classical solutions corresponding to piecewise constant controls. We just point out that, because of the mere continuity of f and the unboundedness of K, sampling trajectories can have a finite blow-up time and chattering phenomena may occur. As a consequence, classical Euler solutions –defined in [13] as uniform limits of sampling solutions– may not exist. For this reason, in [2] (see also [3]) we proposed a notion of *weak* Euler solution, given by the pointwise limit of a sequence of suitably truncated sampling trajectories. In particular, in [2] it has been shown that sample stabilizability implies weak Euler stabilizability.

The main result of this subsection is:

Theorem 2.9. *Let $f : (\mathbb{R}^n \setminus \mathcal{C}) \times U \to \mathbb{R}^n$ be continuous. Then, if the control system $\dot{x} = f(x, u)$ is sample stabilizable to $\mathcal{C}$, it is GAC to $\mathcal{C}$.*

Proof. Assume that $\dot{x} = f(x, u)$ is sample stabilizable to $\mathcal{C}$. Let $K : \mathbb{R}^n \setminus \mathcal{C} \to U$ be a locally bounded, sample stabilizing feedback, let $\beta \in \mathcal{KL}$ be an associated descent rate, and, for any $r, R \in (0, +\infty)$ with $r < R$, let $\delta(R, r) > 0$ be as in Definition 2.7.

Consider the sequence $(r_i)_{i\in\mathbb{Z}}$ introduced in Step 2 of the proof of Theorem 2.5, defined in a recursive way by setting

$$r_0 := 1, \qquad r_{i-1} = \beta(r_i, 0) \quad \forall i \in \mathbb{Z}.$$

As already observed, this sequence is positive, strictly decreasing, and satisfies $\lim_{i\to-\infty} r_i = +\infty$, $\lim_{i\to+\infty} r_i = 0$. For every $i \in \mathbb{Z}$, let $\mathcal{B}_i := \{z \in \mathbb{R}^n \setminus \mathcal{C} : \ \mathbf{d}(z) \in (r_i, r_{i-1}]\}$ and choose a positive sequence $(\hat{t}_i)_{i\in\mathbb{Z}}$ such that $\beta(r_{i-1}, \hat{t}_i) \leq r_i$ for all $i \in \mathbb{Z}$, and also satisfying

$$\sum_{n=0}^{+\infty} \hat{t}_{i+n} = +\infty.$$

For each $i \in \mathbb{Z}$, set $\hat{T}_{i,-1} := 0$ and $\hat{T}_{i,N} := \sum_{n=0}^{N} \hat{t}_{i+n}$ for any $N \in \mathbb{N}$. Hence, define the piecewise constant function $b : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$, given by

$$b(R, t) := \begin{cases} r_{i-2+N} & \text{if } R \in [r_i, r_{i-1}), t \in [\hat{T}_{i,N-1}, \hat{T}_{i,N}) \\ 0 & \text{if } R = 0, t \geq 0, \end{cases}$$

for all $i \in \mathbb{Z}$ and $N \in \mathbb{N}$. Note that if $t \in [0, \hat{t}_i) \subseteq [\hat{T}_{i,-1}, \hat{T}_{i,0})$ then

$$b(r_i, t) = r_{i-2} = \beta(r_{i-1}, 0) \geq \beta(r_{i-1}, t), \tag{24}$$

for all $i \in \mathbb{Z}$. As observed in the proof of Theorem 2.5, we can approximate this function b with a $\mathcal{KL}$ function $\bar{\beta}$ such that $\bar{\beta}(R, t) \geq b(R, t)$ for all $(R, t) \in [0, +\infty) \times [0, +\infty)$.

Now, fixed $i \in \mathbb{Z}$, define $\delta_i := \delta(r_i, r_{i-1})$ and consider the partition $\pi_i := (t_{i,k})_{k\geq 0}$ where $t_{i,k} := k\delta_i$. For any $z \in \mathcal{B}_i$, select a π_i sampling-trajectory $x_i(t; z)$ from z associated to the sample stabilizing feedback K. Then, by the above definitions,

$$\mathbf{d}(x_i(t; z)) \leq \min\{\beta(\mathbf{d}(z), t), r_i\} \quad \text{for } t \geq 0.$$

In particular, also in view of (24), we have

$$\mathbf{d}(x_i(t; z)) \leq \beta(r_{i-1}, t) \leq b(r_i, t) \leq \bar{\beta}(r_i, t) \leq \bar{\beta}(\mathbf{d}(z), t) \text{ for } t \in [0, \hat{t}_i]. \tag{25}$$

and, since $r_i \geq \beta(r_{i-1}, t)$ for all $t \geq \hat{t}_i$,

$$\mathbf{d}(x_i(t; z)) \leq r_i \quad \text{for } t \geq \hat{t}_i. \tag{26}$$

Let $u_i(t)$ be the sampling control associated to $x_i(t; z)$.

Consider the map $i : \mathbb{R}^n \setminus \mathcal{C} \to \mathbb{Z}$, defined as $i(z) := i$ whenever $z \in \mathcal{B}_i$. Fix $z \in \mathbb{R}^n \setminus \mathcal{C}$. Let us build in a recursive way an increasing sequence of times

$\{T_N\}_{N\geq 0}$ such that $T_N \leq \hat{T}_{i(z),N}$ for all $N \geq 0$, and a trajectory-control pair (x, u) defined in $[0, T_N]$ such that

$$\mathbf{d}(x(t)) \leq \bar{\beta}(\mathbf{d}(z), t) \qquad \forall t \in [0, T_N], \qquad \mathbf{d}(x(T_N)) = r_{i(z)+N}.$$

Precisely, for $N = 0$, we define $T_0 := \inf\{t > 0 : \; x_{i(z)}(t; z) \in \mathcal{B}_{i(z)+1}\}$. Note that, in view of (26), $T_0 \leq \hat{t}_{i(z)} = \hat{T}_{i(z),0}$. Set

$$x(t) := x_{i(z)}(t; z), \qquad u(t) := u_{i(z)}(t) \qquad t \in [0, T_0].$$

From (25) and from the definition of T_0 we derive that

$$\mathbf{d}(x(t)) \leq \bar{\beta}(\mathbf{d}(z), t) \qquad \forall t \in [0, T_0]; \qquad \mathbf{d}(x(T_0)) = r_{i(z)}.$$

Let now $N > 0$ and let be defined $T_0, \dots, T_{N-1}$ (satisfying $T_n \leq \hat{T}_{i(z),n}$ for all $n = 0, \dots, N-1$) and a trajectory-control pair (x, u) on $[0, T_{N-1}]$ satisfying

$$\mathbf{d}(x(t)) \leq \bar{\beta}(\mathbf{d}(z), t), \qquad \forall t \in [0, T_{N-1}], \qquad \mathbf{d}(x(T_{N-1})) = r_{i(z)+N-1}.$$

Set $z_N := x(T_{N-1})$ and observe that $z_N \in \mathcal{B}_{i(z)+N}$. Define $t_N := \inf\{t > 0 : \; x_{i(z)+N}(t; z_N) \in \mathcal{B}_{i_0+N+1}\}$. Then, in view of (26), $t_N \leq \hat{t}_{i(z)+N}$. Set $T_N := T_{N-1} + t_N$ and note that $T_N \leq \hat{T}_{i(z),N}$. We extend the definition of (x, u) to $(T_{N-1}, T_N]$ as follows

$$x(t) = x_{i(z)+N}(t - T_{N-1}; z_N), \quad u(t) = u_{i(z)+N}(t - T_{N-1}), \quad t \in (T_{N-1}, T_N].$$

From (25) and (24), we deduce that

$$\mathbf{d}(x(t)) = \mathbf{d}(x_{i(z)+N}(t - T_{N-1}; z_N)) \leq r_{i(z)+N-1}, \qquad t \in (T_{N-1}, T_N]. \tag{27}$$

On the other hand, one has

$$r_{i(z)+N-1} = b(\mathbf{d}(z), t) \leq \bar{\beta}(\mathbf{d}(z), t), \qquad t \in [\hat{T}_{i(z),N-1}, \hat{T}_{i(z),N}].$$

Since $\bar{\beta}(\mathbf{d}(z), \cdot)$ is decreasing, then $r_{i(z)+N-1} \leq \bar{\beta}(\mathbf{d}(z), t)$ for all $t \in [0, \hat{T}_{i(z),N}]$. In particular, $r_{i(z)+N-1} \leq \bar{\beta}(\mathbf{d}(z), t)$ for all $t \in [0, T_N]$, because $T_N \leq \hat{T}_{i(z),N}$. This, together with (27), implies that

$$\mathbf{d}(x(t)) \leq \bar{\beta}(\mathbf{d}(z), t) \qquad \forall t \in (T_{N-1}, T_N]. \tag{28}$$

Moreover, we have by construction $\mathbf{d}(x(T_N)) = r_{i(z)+N}$.

So far, we iteratively constructed an admissible trajectory-control pair (x, u) from z, which is defined in $[0, \tilde{T})$, where $\tilde{T} := \lim_{N\to\infty} T_N$ $(\leq +\infty)$, $\mathbf{d}(x(T_N)) = r_{i(z)+N}$ for all $N \geq 0$, and such that

$$\mathbf{d}(x(t)) \leq \bar{\beta}(\mathbf{d}(z), t) \qquad \forall t \in [0, \tilde{T}).$$

Therefore, $\mathbf{d}(x(t)) \to 0$ as $t \to \tilde{T}^-$ and, extending x to $[0, +\infty)$ as in Definition 2.1 when $\tilde{T} < +\infty$, this yields

$$\mathbf{d}(x(t)) \leq \bar{\beta}(\mathbf{d}(z), t) \qquad \forall t \geq 0,$$

so concluding the proof.

□

It is worth noting that the results of Theorems 2.5, 2.9 are constructive, in the sense that, given a decrease rate β associated with the GAC with $U \cap \sigma$ controls or the sample stabilizability, respectively, we explicitly indicate how to obtain a decrease rate for the GAC. In addition, thanks to Theorem 2.9, sufficient conditions for GAC for the case of unbounded control systems, obtained in [31, 1], follow now as corollaries by the sample stabilizability results in [32, 2].

3. The rescaled system and the impulsive extension

In this section we establish some relationships between the GAC to $\mathcal{C}$ of a rescaled control system and the associated impulsive extension. These results will be crucial to obtain the Converse Lyapunov Theorem of Section 4.

3.1. GAC of the rescaled problem

Throughout this subsection, the function $f : (\mathbb{R}^n \setminus \mathcal{C}) \times U \to \mathbb{R}^n$ is continuous and satisfies the growth assumption **(Hg), (i)**, for some growth rate ν. Let $\bar{f}$ denote the associated rescaled dynamics.

For the purpose of distinguishing the admissible trajectory-control pairs of (f, U) from those of $(\bar{f}, U)$, we will denote by (x, u) the former and by (y, v) the latter. Precisely, we simply say that (x, u) is an *admissible trajectory-control pair* when $u \in L^\infty_{loc}([0, T_x), U)$, $x \in AC_{loc}([0, T_x), \mathbb{R}^n \setminus \mathcal{C})$, and x solves the *original control system*

$$\dot{x}(t) = f(x(t), u(t)) \quad \text{a.e. } t \in (0, T_x), \tag{29}$$

where $\lim_{t \to T_x^-} \mathbf{d}(x(t)) = 0$ whenever $T_x < +\infty$. We call (y, v) an *admissible rescaled trajectory-control pair* when $v \in L^\infty_{loc}([0, S_y), U)$, $y \in AC_{loc}([0, S_y), \mathbb{R}^n \setminus \mathcal{C})$, and y solves the *rescaled control system*

$$y'(s) = \bar{f}(y(s), v(s)) \quad \text{a.e. } s \in (0, S_y), \ ^3 \tag{30}$$

where $\lim_{t \to S_y^-} \mathbf{d}(y(s)) = 0$ whenever $S_x < +\infty$. When T_x $[S_y]$ is finite, we mean that x $[y]$ is extended to $[0, +\infty)$ as described in Definition 2.1.

As an easy consequence of the chain rule, admissible rescaled trajectory-control pairs (y, v) are in one-to-one correspondence with admissible trajectory-control pairs (x, u) through a time-change.

Lemma 3.1. *Assume f continuous and satisfying* **(Hg), (i)**. *Fix $z \in \mathbb{R}^n \setminus \mathcal{C}$.*

(i) *Given an admissible process (x, u) from z, set*

$$s(t) := \int_0^t (1 + \nu(|u(\tau)|))\, d\tau \ \ \forall t \in [0, T_x), \quad S_y := \lim_{t \to T_x^-} s(t), \quad t(\cdot) := s^{-1}(\cdot).$$

[3] In (30) we use the apex "$'$" to denote differentiation with respect to the new parameter s, in order to stress that it does not coincide, in general, with the time variable t, of (29), as clarified by Lemma 3.1.

Then $(y,v)(s) := (x,u) \circ t(s)$*,* $s \in [0, S_y)$*, is an admissible rescaled trajectory-control pair from* z*.*

(ii) *Vice-versa, let* (y,v) *be an admissible rescaled trajectory-control pair from* z *and set*

$$t(s) := \int_0^s (1 + \nu(|v(\sigma)|)^{-1} d\sigma \quad \forall s \in [0, S_y), \quad T_x := \lim_{s \to S_y^-} t(s), \qquad s(\cdot) := t^{-1}(\cdot).$$

Then, $(x,u)(t) := (y,v) \circ s(t)$*,* $t \in [0, T_x)$*, is an admissible trajectory-control pair from* z*.*

The following theorem establishes the equivalence between the GAC to $\mathcal{C}$ with $U \cap \sigma$ controls of the original and the rescaled control system.

Theorem 3.1. *Assume* f *continuous and satisfying* **(Hg), (i)**. *Then, the original control system* (29) *is GAC to* $\mathcal{C}$ *with* $U \cap \sigma$ *controls if and only if the rescaled control system* (30) *is GAC to* $\mathcal{C}$ *with* $U \cap \sigma$ *controls (for the same* σ*).*

Proof. Suppose first that the rescaled control system (30) is GAC to $\mathcal{C}$ with $U \cap \sigma$ controls, for some continuous function $\sigma : (0, +\infty) \to (0, +\infty)$ and some descent rate $\beta \in \mathcal{KL}$. Hence, for every $z \in \mathbb{R}^n \setminus \mathcal{C}$ there is an admissible rescaled trajectory-control pair (y, v) such that $y(0) = z$ and

$$\mathbf{d}(y(s)) \le \beta(\mathbf{d}(z), s), \quad |v(s)| \le \sigma(\mathbf{d}(y(s)) \quad \forall s \ge 0.$$

Consider now $(x,u)(t) := (y,v) \circ s(t)$ where $s(t)$ is the time-change introduced in Lemma (3.1),(ii). Since $s(t) \ge t$ for all $t \ge 0$, from the monotonicity properties of $\beta(r, \cdot)$ it follows that

$$\mathbf{d}(x(t)) = \mathbf{d}(y(s(t))) \le \beta(\mathbf{d}(z), s(t)) \le \beta(\mathbf{d}(z), t) \quad \forall t \ge 0.$$

Thus, β is a descent rate also for the original control system (29). Moreover,

$$|u(t)| = |v(s(t))| \le \sigma(\mathbf{d}(y(s(t))) = \sigma(\mathbf{d}(x(t)) \quad \forall t \in [0, T_x),$$

where $T_x = \lim_{s \to S_y^-} t(s)$, as in Lemma (3.1),(ii).

To prove the converse implication, let us assume that the original control system (29) is GAC to $\mathcal{C}$ with $U \cap \sigma$ controls, for some continuous function $\sigma : (0, +\infty) \to (0, +\infty)$ and some descent rate $\beta \in \mathcal{KL}$.

For any couple (R, r) of real numbers such that $0 < r < R$, let us set

$$N(R, r) := \nu\big(\max \sigma([r, \beta(R, 0)]\big).$$

By the continuity of ν, β, and σ and by the monotonicity properties of ν and β, one deduces immediately that the function $N : \{(R, r) : \; 0 < r < R\} \to (0, +\infty)$ is continuous, for any R, $r \mapsto N(R, r)$ is decreasing, and, for any r, $R \mapsto N(R, r)$ is increasing. By eventually enlarging N, we can assume without loss of generality that N is strictly monotone with respect to both r and R.

Now, let $S(R,r)>0$ be the value of s implicitly defined by the equation

$$\beta\left(R,\frac{s}{1+N(R,r)}\right)=r.$$

From the monotonicity and continuity properties of β and N, it follows that S is a continuous function on $\{(R,r):\ 0<r<R\}$, such that $r\mapsto S(R,r)$ is strictly decreasing and $R\mapsto S(R,r)$ is strictly increasing. As a consequence, if, for any R, we denote by $\rho=\rho(R,s)$ the inverse of the map $\rho\mapsto S(R,\rho)$, one easily obtains that ρ is a $\mathcal{KL}$ function. Moreover, we have the identity

$$\beta\left(R,\frac{s}{1+N(R,\rho(R,s))}\right)=\rho(R,s). \tag{31}$$

Let us show that ρ is a descent rate for the rescaled control system (30). To this aim, fix $z\in\mathbb{R}^n\setminus\mathcal{C}$ and let (x,u) be an admissible trajectory-control pair of (29) with $x(0)=z$ and satisfying

$$\mathbf{d}(x(t))\le\beta(\mathbf{d}(z),t),\qquad |u(t)|\le\sigma(\mathbf{d}(x(t))\quad\forall t\ge 0. \tag{32}$$

Now, define $(y,v)(s):=(x,u)\circ t(s)$ for all $s\in[0,S_y)$, where

$$s(t):=\int_0^t(1+\nu(|u(\tau)|))\,d\tau\ \ \forall t\in[0,T_x),\quad S_y:=\lim_{t\to T_x^-}s(t),\quad t(\cdot):=s^{-1}(\cdot).$$

By Lemma 3.1 (y,v) is an admissible trajectory-control pair for $(\bar f,U)$. Furthermore, the rescaled control v verifies

$$|v(s)|=|u(t(s))|\le\sigma(\mathbf{d}(x(t(s)))=\sigma(\mathbf{d}(y(s))\quad\forall s\in[0,S_y).$$

Let us show that $\mathbf{d}(y(s))\le\rho(\mathbf{d}(z),s)$ for every $s\in[0,S_y)$. By contradiction, suppose that there exists some $s\in[0,S_y)$ such that

$$\mathbf{d}(y(s))>\rho(\mathbf{d}(z),s). \tag{33}$$

Since $y(s)=x(t(s))$, we deduce that

$$\rho(\mathbf{d}(z),s)<\mathbf{d}(x(t(s))\le\beta(\mathbf{d}(z),0).$$

By the definitions of N, $t(s)$, and the monotonicity properties of β, we get

$$t(s)\ge\frac{s}{1+N(\mathbf{d}(z),\rho(\mathbf{d}(z),s))}. \tag{34}$$

Using (32) and (31) we obtain the required contradiction with (33). Indeed, (34) and the monotonicity of β, imply that

$$\begin{aligned}\mathbf{d}(y(s))=\mathbf{d}(x(t(s)))\le\beta(\mathbf{d}(z),t(s))\le\beta\left(\mathbf{d}(z),\tfrac{s}{1+N(\mathbf{d}(z),\rho(\mathbf{d}(z),s))}\right)\\=\rho(\mathbf{d}(z),s).\end{aligned}$$

At this point, if $S_y=+\infty$ the proof is concluded. If instead $S_y<+\infty$, by definition we extend the trajectory y to $[0,+\infty)$ as described in Definition 2.1, so $\mathbf{d}(y(s))=0<\rho(\mathbf{d}(z),s)$ for every $s\ge S_y$ trivially. □

From Theorems 2.5, 3.1 one derives that, under the Lipschitz continuity assumption **(Hl)**, the original system is GAC to $\mathcal{C}$ if and only if the rescaled system is GAC to $\mathcal{C}$. Precisely, one has:

Theorem 3.2. *Assume that f satisfies* **(Hl)** *and* **(Hg), (i)**. *Then, the original system* (29) *is GAC to $\mathcal{C}$ if and only if the rescaled system* (30) *is GAC to $\mathcal{C}$. In addition, for both systems being GAC to $\mathcal{C}$ is equivalent to being GAC to $\mathcal{C}$ with $U \cap \sigma$ controls.*

Proof. Theorem 3.1 establishes that the original system (29) is GAC to $\mathcal{C}$ with $U \cap \sigma$ controls if and only if the rescaled system (30) is GAC to $\mathcal{C}$ with $U \cap \sigma$ controls. To conclude the proof, it is enough to observe that, when f verifies **(Hl)** and **(Hg), (i)**, the rescaled dynamics $\bar{f}$ verifies **(Hl)** too. Hence, Theorem 2.5 applied both to f and $\bar{f}$ implies the equivalence of the notions of GAC to $\mathcal{C}$ and GAC to $\mathcal{C}$ with $U \cap \sigma$ controls both for the original system and the rescaled system. □

3.2. GAC of the impulsive extension

Let f be a continuous function satisfying **(Hg)**, and let ν, $\bar{f}$, F, $\mathbb{U}$ be as in **(Hg)**. We now embed the original system into an extended control system and show that GAC to $\mathcal{C}$ of the rescaled control system implies GAC to $\mathcal{C}$ of the extended system.

Definition 3.3 (Admissible extended trajectory-control pairs). *A triple (y, w_0, w) is called an* admissible extended trajectory-control pair *if there exists $S_y \leq +\infty$ such that: the control $(w_0, w) \in L^\infty([0, S_y), \overline{\mathbb{U}})$; the trajectory $y \in AC([0, S_y), \mathbb{R}^n \setminus \mathcal{C})$ is a solution of the* extended control system

$$y'(s) = F(y(s), w_0(s), w(s)) \qquad \text{a.e. } s \in [0, S_y); \tag{35}$$

and, if $S_y < +\infty$, one has $\lim_{s \to S_y^-} \mathbf{d}(y(s)) = 0$. If (y, w_0, w) is an admissible extended trajectory-control pair and $S_y < +\infty$, we extend y to $[0, +\infty[$ by setting $y(s) := \lim_{\sigma \to S_y^-} y(\sigma)$ for any $s \geq S_y$.[4]

The original and the rescaled system can be embedded in the extended system as follows.

Lemma 3.2. *Assume that f is a continuous function satisfying* **(Hg)** *for some growth rate ν. Let (x, u) be an admissible trajectory-control pair for (f, U). Set*

$$s(t) := \int_0^t (1 + \nu(|u(\tau)|))\, d\tau \ \ \forall t \in [0, T_x), \quad S_y := \lim_{t \to T_x^-} s(t), \quad t := s^{-1},$$

$$y(s) := x \circ t(s) \ \ \forall s \in [0, S_y),$$

$$(w_0, w)(s) := \left(1, \frac{u\,\nu(|u|)}{|u|} \circ t(s)\right) t'(s) = \left(\frac{1}{1 + \nu(|u|)}, \frac{u\,\nu(|u|)}{|u|(1 + \nu(|u|))}\right) \circ t(s)$$

[4]In this case, the limit at S_y always exists, since F is bounded on any neighborhood of the target, which has compact boundary.

for a.e. $s \in [0, S_y)$*. Then* (y, v)*, where* $v(s) := u \circ t(s)$ *for a.e.* $s \in [0, S_y)$*, is an admissible rescaled trajectory-control pair, while* (y, w_0, w) *is an admissible extended trajectory-control pair, with* $w_0 > 0$ *a.e. on* $[0, S_y)$*.*

Proof. The rescaled trajectory-control pair (y, v) is admissible by Lemma 3.1, (i). In view of the definition of (w_0, w) and of F, straightforward calculations yield that

$$y'(s) = \bar{f}(y(s), v(s)) = F(y(s), w_0(s), w(s)), \quad \text{a.e. } s \in [0, S_y). \tag{36}$$

This shows that (y, w_0, w) is an admissible extended trajectory-control pair with $w^0 > 0$ a.e.. □

Remark 3.4. When $\nu(r) = r^{\bar{d}}$ for some integer $\bar{d} \geq 1$, the extended dynamics F is equivalent to the extended dynamics introduced in [24], whose definition is based on the notion of *recession function*. Indeed, if for any $(x, w_0, w) \in \overline{(\mathbb{R}^n \setminus \mathcal{C})} \times \overline{\mathbb{U}}$ we set $(\tilde{w}_0, \tilde{w}) := \left(w_0^{1/\bar{d}}, \frac{w}{|w|}\,|w|^{1/\bar{d}}\right)$, then $\tilde{w}_0^{\bar{d}} + |\tilde{w}|^{\bar{d}} = 1$ and we obtain that

$$F(x, \tilde{w}_0, \tilde{w}) = \lim_{r \to \tilde{w}_0^+} \bar{f}\left(x, \frac{\tilde{w}}{r}\right) = \lim_{r \to \tilde{w}_0^+} f\left(x, \frac{\tilde{w}}{r}\right) r^{\bar{d}}.$$

In particular, in the case of control-affine f one has $\bar{d} = 1$ and $(\tilde{w}_0, \tilde{w}) \equiv (w_0, w)$, so that the classical impulsive extension of the graph-completion approach considered in [3], coincides with the present one.

The extension consists in considering (y, w_0, w), where w_0 may be zero on nondegenerate subintervals of $[0, S_y)$. On these intervals, the time variable $t = \int_0^s w_0(\sigma)\, d\sigma$ is constant –i.e., the time stops–, while the state variable y evolves, according to the equation $y' = F(t, y, 0, w)$, sometimes called the *fast dynamics*. For this reason, system (35) is often referred to as the *impulsive extension* of the original control system (29), despite the fact that it is an ordinary control system, as the extended controls (w_0, w) take values in the compact set $\overline{\mathbb{U}}$ and the trajectory y is absolutely continuous. A detailed discussion of this topic goes beyond the purposes of the paper. We just mention that an equivalent, t-based description of this extension, where u is no more a function and the trajectory x is a discontinuous map whose total variation is bounded on $[0, T]$ for every $T < T_x$, but possibly unbounded on $[0, T_x)$, in short $x \in BV_{loc}[0, T_x)$, could be given (see [26] and also [33, 34, 35, 36]).

Proposition 3.1. *Assume* f *continuous and satisfying* **(Hg)***. If the rescaled system* (30) *is GAC to* $\mathcal{C}$*, then the extended system* (35) *is GAC to* $\mathcal{C}$*.*

Proof. Since the rescaled control system (30) is GAC to $\mathcal{C}$, there exists a descent rate $\beta \in \mathcal{KL}$ such that for all $z \in \mathbb{R}^n \setminus \mathcal{C}$ there is an admissible rescaled trajectory-control pair (y, v), such that $y(0) = x$ and

$$\mathbf{d}(y(s)) \leq \beta(\mathbf{d}(z), s) \quad \forall s \geq 0. \tag{37}$$

Define for a.e. $s \in [0, S_y)$ the extended control

$$(w_0, w)(s) = \left(\frac{1}{1+\nu(|v(s)|)}\,,\; \frac{v(s)\,\nu(|v(s)|)}{|v(s)|(1+\nu(|v(s)|))} \right).$$

Since $(w_0, w)(s) \in \mathbb{U}$ for a.e. $s \in [0, S_y)$, from Lemma 3.2 it follows that (y, w_0, w) is an admissible extended trajectory-control pair for (35) and this, together with (37) and the arbitrariness of z, implies that the extended system (35) is GAC to $\mathcal{C}$, with the same descent rate β as the rescaled system (30). □

4. A Converse Lyapunov Theorem

In this section we state our main result, which extends to control systems with unbounded controls and their impulsive extensions well-known relationships between GAC, sample stabilizability, and existence of control Lyapunov functions. Furthermore, we relate explicit stabilizing feedback constructions for the extended and for the original system, which are based on the existence of a *semiconcave* control Lyapunov function.

4.1. Main result

To begin with, let us introduce the notion of (nonsmooth) control Lyapunov function. In the following, given an open set $\Omega \subseteq \mathbb{R}^N$, a continuous function $W : \overline{\Omega} \to [0, +\infty)$ is said *positive definite on* Ω if $W(x) > 0$ $\forall x \in \Omega$ and $W(x) = 0$ $\forall x \in \partial\Omega$. The function W is called *proper on* Ω if the pre-image $W^{-1}(K)$ of any compact set $K \subset [0, +\infty)$ is compact. As customary, $\partial_P W(x)$ refers to the *proximal subdifferential of* W *at* x (which may very well be empty). We recall that p belongs to $\partial_P W(x)$ if and only if there exist σ and $\eta > 0$ such that

$$W(y) - W(x) + \sigma|y-x|^2 \geq \langle p\,,\, y-x\rangle \qquad \forall y \in B_\eta(\{x\}).\mathcal{C}$$

The *limiting subdifferential* $\partial_L W(x)$ *of* W *at* $x \in \Omega$, is defined as

$$\partial_L W(x) := \Big\{ \lim_{i\to+\infty} p_i : \; p_i \in \partial_P W(x_i), \; \lim_{i\to+\infty} x_i = x \Big\}.$$

When the function W is locally Lipschitz continuous on Ω, the limiting subdifferential $\partial_L W(x)$ is nonempty at every point, the set-valued map $x \rightsquigarrow \partial_L W(x)$ is upper semicontinuous, and the Clarke generalized gradient at x can be derived as $\mathrm{co}\,\partial_L W(x)$. As sources for nonsmooth analysis we refer e.g. to [37, 38, 39].

For any nonempty closed set $\mathbf{U} \subseteq \mathbb{R}^M$ for some integer $M > 0$ and any continuous function $\mathbf{f} : (\mathbb{R}^n \setminus \mathcal{C}) \times \mathbf{U} \to \mathbb{R}^n$, let consider the control system

$$\dot{x} = \mathbf{f}(x, u), \qquad u \in \mathbf{U}, \tag{38}$$

and the *Hamiltonian* $H_{\mathbf{f},\mathbf{U}} : (\mathbb{R}^n \setminus \mathcal{C}) \times U \to [-\infty, +\infty)$, given by

$$H_{\mathbf{f},\mathbf{U}}(x, p) := \inf_{u\in\mathbf{U}} \{\langle p\,,\, \mathbf{f}(x,u)\rangle\}\,. \tag{39}$$

Notice that $H_{\mathbf{f},\mathbf{U}}$ may be discontinuous and equal to $-\infty$ at some points.

Definition 4.1 (Control Lyapunov Function)**.** *Let* $W : \overline{\mathbb{R}^n \setminus \mathcal{C}} \to [0, +\infty)$ *be a locally Lipschitz continuous function on* $\overline{\mathbb{R}^n \setminus \mathcal{C}}$*, which is positive definite and proper on* $\mathbb{R}^n \setminus \mathcal{C}$*. We say that* W *is a* Control Lyapunov Function, *(CLF), for the system* (38) *if there exists some continuous, strictly increasing function* $\gamma : (0, +\infty) \to (0, +\infty)$*, that we call a* decrease rate*, such that the following* (infinitesimal) decrease condition *is satisfied:*

$$H_{\mathbf{f},\mathbf{U}}(x, \partial_L W(x)) < -\gamma(W(x)) \quad \forall x \in \mathbb{R}^n \setminus \mathcal{C}.^{5} \tag{40}$$

Remark 4.2. If the continuous function $\mathbf{f} : (\mathbb{R}^n \setminus \mathcal{C}) \times \mathbf{U} \to \mathbb{R}^n$ is bounded in $(B_R(\mathcal{C}) \setminus \mathcal{C}) \times \mathbf{U}$ for some $R > 0$ and continuous in x uniformly with respect to $\mathbf{U}$ –as it is for the rescaled dynamics $\bar{f}$ and for the extended dynamics F–, then the Hamiltonian $H_{\mathbf{f},\mathbf{U}}$ is continuous and the decrease condition (40) is equivalent to the usual condition

$$H_{\mathbf{f},\mathbf{U}}(x, \partial_P W(x)) < -V(x) \quad \forall x \in \mathbb{R}^n \setminus \mathcal{C}, \tag{41}$$

expressed in terms of the proximal subdifferential, for some continuous function $V : \overline{\mathbb{R}^n \setminus \mathcal{C}} \to [0, +\infty)$, which is positive definite and proper on $\mathbb{R}^n \setminus \mathcal{C}$, used e.g. in [11, 13, 18] (see [32, Prop. 4.2]). Incidentally, in this case (40) has also an equivalent formulation, which involves the Dini derivative.

The rescaled system and the extended system share the same CLFs.

Proposition 4.1. *Assume* $f : (\mathbb{R}^n \setminus \mathcal{C}) \times U \to \mathbb{R}^n$ *continuous and satisfying assumption* **(Hg)**. *A map* $W : \mathbb{R}^n \setminus \mathcal{C} \to \mathbb{R}$ *is a CLF for the rescaled problem* (30) *if and only if it is a CLF for the extended problem* (35).

Proof. Consider the maps

$$\begin{aligned} u \mapsto (w_0, w)(u) &:= \left(\frac{1}{1 + \nu(|u|)}, \frac{u\,\nu(|u|)}{|u|(1 + \nu(|u|))} \right) \quad \forall u \in U, \\ (w_0, w) \mapsto u(w_0, w) &:= \frac{w}{|w|}\, \nu^{-1}\left(\frac{|w|}{w_0} \right) \quad \forall (w_0, w) \in \mathbb{U}, \end{aligned} \tag{42}$$

where $\mathbb{U}$ and ν are as in **(Hg)**. The definitions of $\bar{f}$ and F imply that

$$\begin{aligned} F(x, (w_0, w)(u)) &= \bar{f}(x, u) & \forall (x, u) \in (\mathbb{R}^n \setminus \mathcal{C}) \times U, \\ \bar{f}(x, u(w_0, w)) &= F(x, w_0, w) & \forall (x, w_0, w) \in (\mathbb{R}^n \setminus \mathcal{C}) \times \mathbb{U}. \end{aligned}$$

Thus, for any $x \in \mathbb{R}^n \setminus \mathcal{C}$ and $p \in \mathbb{R}^n$, one has

$$\begin{aligned} \inf_{u \in U} \left\{ \langle p\,,\, \bar{f}(x, u) \rangle \right\} &= \inf_{u \in U} \{ \langle p\,,\, F(x, (w_0, w)(u)) \rangle \} \\ &\geq \inf_{(w_0, w) \in \overline{\mathbb{U}}} \{ \langle p\,,\, F(x, w_0, w) \rangle \} \\ &= \inf_{(w_0, w) \in \mathbb{U}} \{ \langle p\,,\, F(x, w_0, w) \rangle \} \\ &= \inf_{(w_0, w) \in \mathbb{U}} \left\{ \langle p\,,\, \bar{f}(x, u(w_0, w)) \rangle \right\} \\ &\geq \inf_{u \in U} \left\{ \langle p\,,\, \bar{f}(x, u) \rangle \right\}. \end{aligned}$$

[5] This means that $H_{\mathbf{f},\mathbf{U}}(x, p) < -\gamma(W(x))$ for every $p \in \partial_L W(x)$.

Therefore $H_{\bar{f},U} \equiv H_{F,\overline{\mathbb{U}}}$. As a consequence, a map W is a CLF for (29) if and only if it is a CLF for (35). $\square$

We are ready to state the main result of the paper. To this aim, we introduce the following stronger assumptions.

(Hg)* *The function $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ is continuous and a stronger version of hypothesis* **(Hg)** *is valid, where $\mathbb{R}^n$ replaces $\mathbb{R}^n \setminus \mathcal{C}$. Furthermore, for any compact set $\mathcal{K} \subset \mathbb{R}^n$ there is some constant $\bar{L} > 0$ such that*

$$|F(x, w_0, w) - F(y, w_0, w)| \le L|x - y| \qquad \forall x, y \in \mathcal{K},\ \forall (w_0, w) \in \bar{\mathbb{U}}.$$

Theorem 4.3 (Converse Lyapunov Theorem)**.** *Assume hypothesis* **(Hg)****. Then the following properties are equivalent:*

(i) *the original control system $\dot{x} = f(x, u)$ is GAC to $\mathcal{C}$;*

(ii) *the extended control system $y' = F(y, w_0, w)$ is GAC to $\mathcal{C}$;*

(iii) *there exists a CLF for the extended control system $y' = F(y, w_0, w)$;*

(iv) *there exists a CLF for the original control system $\dot{x} = f(x, u)$;*

(v) *the system $\dot{x} = f(x, u)$ is sample stabilizable to $\mathcal{C}$;*

(vi) *the system $y' = F(y, w_0, w)$ is sample stabilizable to $\mathcal{C}$.*

Proof. Let us preliminarily observe that the Lipschitz continuity hypothesis on F in **(Hg)*** implies both that f satisfies hypothesis **(Hl)** and that $\bar{f}$ is locally Lipschitz continuous in x (on $\overline{\mathbb{R}^n \setminus \mathcal{C}}$), uniformly w.r.t. $u \in U$.

(i) $\Longrightarrow$ (ii). From Theorem 3.2 it follows that the original control system is GAC to $\mathcal{C}$ if and only if the rescaled control system $y' = \bar{f}(y, v)$ is GAC to $\mathcal{C}$. This implies that the extended control system is GAC to $\mathcal{C}$, in view of Proposition 3.1.

(ii) $\Longleftrightarrow$ (iii). The fact that the extended control system is GAC to $\mathcal{C}$ if and only if there exists a CLF for it follows from [17, Theorem 1] (see also [18, Thm 3.2, Rmk. 4]). Observe that this result is applicable to the impulsive extension essentially because the set $\overline{\mathbb{U}}$ of extended control values is bounded.

(iii) $\Longrightarrow$ (iv). Let W be a CLF for the extended control system $y' = F(y, w_0, w)$, for some decrease rate γ. Hence, by Proposition 4.1 W is also a CLF for the rescaled system $y' = \bar{f}(y, v)$, with the same γ. Since the definition of $\bar{f}$ implies that

$$H_{f,U}(x, p) \le H_{\bar{f},U}(x, p) < -\gamma(W(x)) \quad \forall x \in \mathbb{R}^n \setminus \mathcal{C},\ \ \forall p \in \partial_L W(x), \qquad (43)$$

we can finally conclude that W is a CLF also for the original control system.

(iv) $\Longrightarrow$ (v). Let W be a CLF for (29). Then, [2, Theorem 4.6] implies that $\dot{x} = f(x, u)$ is sample stabilizable to $\mathcal{C}$.

(v) $\Longrightarrow$ (i). The sample stabilizability of $\dot{x} = f(x,u)$ to $\mathcal{C}$ implies that it is GAC to $\mathcal{C}$, by Theorem 2.9.

With this, we have shown that (i),(ii),(iii),(iv), and (v) are equivalent. To conclude the proof it suffices to observe that, by the same arguments as above, (iii) $\Longrightarrow$ (vi) and (vi) $\Longrightarrow$ (ii), namely, the existence of a CLF for $y' = F(y, w_0, w)$ implies sample stabilizability of the extended system, which in turn implies that $y' = F(y, w_0, w)$ is GAC to $\mathcal{C}$. □

4.2. Semiconcave CLFs and stabilizing feedback construction

Given an open set $\Omega \subseteq \mathbb{R}^N$, a function $W : \Omega \to \mathbb{R}$ is called *locally semiconcave* if for every compact subset $\mathcal{K} \subset \Omega$ there exists $\rho > 0$ such that, for all x, $\hat{x} \in \mathcal{K}$ with $[x, \hat{x}] \subset \mathcal{K}$, one has

$$W(x) + W(\hat{x}) - 2W\left(\frac{x+\hat{x}}{2}\right) \le \rho|x-\hat{x}|^2.$$

Locally semiconcave functions are locally Lipschitz continuous and twice differentiable almost everywhere (see e.g. [37]).

Since the works by Rifford [14, 15], semiconcave control Lyapunov functions have proven to play a crucial role for the explicit construction of sample stabilizing feedback strategies. It is therefore worth noting that in Theorem 4.3 we can assume without loss of generality that CLFs are locally semiconcave on $\mathbb{R}^n \setminus \mathcal{C}$. Precisely, one has:

Proposition 4.2. *Under the assumptions of Theorem 4.3,* [6] *if there exists a CLF either for the extended control system* (35) *or for the original control system* (29)*, then there exists a CLF for* (35) *or* (29)*, respectively, which is locally semiconcave on* $\mathbb{R}^n \setminus \mathcal{C}$*.*

Proof. If there exists a locally Lipschitz continuous CLF for the extended system (35), the dynamics of which meet classical Lipschitz continuity and boundedness assumptions, then from [32, Theorem 4.3] there is also a locally semiconcave CLF for (35). On the other hand, when there exists a locally Lipschitz continuous CLF for the original system (29), [2, Theorem 4.3] guarantees the existence of a locally Lipschitz continuous CLF for the rescaled system (30). At this point, [32, Theorem 4.3] again implies the existence of a locally semiconcave CLF for (30), which, as one deduces by (43), is also a locally semiconcave CLF for (29). The proof of the claim is thus complete. □

Thanks to Proposition 4.2, we can explicitly build a stabilizing feedback for the original control system from a stabilizing feedback for the impulsive extension. To this aim, we need the following preliminary result.

[6] Actually, from the results in [32, 2] this statement is valid even if f, $\bar{f}$, and F satisfy the assumptions in $\mathbf{(Hg)}^*$ only for $x \in \overline{\mathbb{R}^n \setminus \mathcal{C}}$.

Proposition 4.3. *Assume that $F : (\mathbb{R}^n \setminus \mathcal{C}) \times \overline{\mathbb{U}} \to \mathbb{R}$ is a continuous function. Let W be a CLF for $y' = F(y, w_0, w)$, with decrease rate γ. Then, there exists a continuous function $N : (0, +\infty) \to (0, 1]$ such that*

$$H_{F,\mathbb{U}_{N(W(x))}}(x, \partial_L W(x)) < -\gamma(W(x)) \quad \forall x \in \mathbb{R}^n \setminus \mathcal{C}, \tag{44}$$

where, for every $\rho \in (0, 1]$,

$$\mathbb{U}_\rho := \{(w_0, w) \in \mathbb{U} : \ w_0 \geq \rho\}.$$

Proof. To prove the statement, we show that there is some continuous function $N : (0, +\infty) \to (0, 1]$, which is increasing in $(0, 1]$, decreasing in $[1, +\infty)$, and such that, for any $r > 0$, one has

$$H_{F,\mathbb{U}_{N(r)}}(x, \partial_L W(x)) < -\gamma(W(x)) \quad \forall x \in W^{-1}([r \wedge 1, r \vee 1]). \tag{45}$$

In fact, proven this, from the monotonicity properties of N it immediately follows that in (45) we can replace r with $W(x)$, and that implies (44) by the arbitrariness of $r > 0$.

To prove (45), fix $r > 0$ and set

$$\Gamma_r := \{(x, p) : \ \ x \in W^{-1}([r \wedge 1, r \vee 1]), \ p \in \partial_L W(x)\}.$$

Notice that the properties of W –in particular, the properness of W and the upper semicontinuity of the set-valued map $x \rightsquigarrow \partial_L W(x)$– imply that Γ_r is a compact set. For every $(x, p) \in \Gamma_r$, define

$$w_0(x, p) := \sup\{w_0 : \ (w_0, w) \in \mathbb{U} \text{ and } \langle p, F(x, w_0, w)\rangle < -\gamma(W(x))\}.$$

This set is nonempty because W is a CLF, and $w_0(x, p) \in [0, 1]$. At this point, set

$$\hat{N}(r) := \inf\{w_0(x, p) : \ (x, p) \in \Gamma_r\}.$$

By construction, $\hat{N}$ is nonnegative, increasing in $(0, 1]$ and decreasing in $[1, +\infty)$, and $\hat{N}([0, +\infty)) \subseteq [0, 1]$. If $\hat{N}(r) > 0$ for all $r > 0$, then the required N is given by any continuous, positive approximation from below of $\hat{N}$, which is increasing in $(0, 1]$ and decreasing in $[1, +\infty)$. To conclude, it only remains to prove that $\hat{N}(r) > 0$ for all $r > 0$. To this end, assume by contradiction that $\hat{N}(r) = 0$ for some $r > 0$. Then, there is some sequence $((x_k, p_k))_{k \geq 1} \subset \Gamma_r$, such that $w_0(x_k, p_k) < 1/k$ for all k. Hence, the definition of $w_0(x_k, p_k)$ yields

$$\langle p_k, F(x_k, w_0, w)\rangle < -\gamma(W(x_k)) \ \text{ for } \ (w_0, w) \in \overline{\mathbb{U}} \Longrightarrow w_0 < \frac{1}{k}. \tag{46}$$

Since Γ_r is compact, there exists a subsequence, that we still denote $((x_k, p_k))_k$, converging to some $(\bar{x}, \bar{p}) \in \Gamma_r$. Since W is a CLF with decrease rate γ, there exists some $(\bar{w}_0, \bar{w}) \in \mathbb{U}$ (with $\bar{w}_0 > 0$) such that

$$\langle \bar{p}, F(\bar{x}, \bar{w}_0, \bar{w})\rangle < -\gamma(W(\bar{x})).$$

Therefore, by the continuity of F, W and γ, for a sufficiently large k one has $1/k < \bar{w}_0$ and

$$\langle p_k, F(x_k, \bar{w}_0, \bar{w})\rangle < -\gamma(W(x_k)),$$

in contradiction with (46), so that the proof is complete. □

From Proposition 4.3 it follows that, given a semiconcave control Lyapunov function for the extended control system, we can always select a stabilizing feedback $\hat{K}(x) = (\hat{w}_0(x), \hat{w}(x))$ *which is not impulsive,* namely such that $\hat{w}_0(x) > 0$ for every $x \in \mathbb{R}^n \setminus \mathcal{C}$.

Proposition 4.4. *Consider the same assumptions as in Theorem 4.3. Let the extended system* (35) *be sample stabilizable to* $\mathcal{C}$*. Then, there exist a continuous function* $N : (0, +\infty) \to (0, 1)$ *and a stabilizing feedback* $\hat{K} : \mathbb{R}^n \setminus \mathcal{C} \to \mathbb{U}$, $\hat{K}(x) = (\hat{w}_0(x), \hat{w}(x))$ *for* (35)*, satisfying*

$$w_0(x) \geq N(W(x)) \qquad \forall x \in \mathbb{R}^n \setminus \mathcal{C}, \tag{47}$$

so that the locally bounded feedback $K : \mathbb{R}^n \setminus \mathcal{C} \to U$ *given by*

$$K(x) := \frac{\hat{w}(x)}{|\hat{w}(x)|}\, \nu^{-1}\left(\frac{|\hat{w}(x)|}{\hat{w}_0(x)}\right) \qquad \forall x \in \mathbb{R}^n \setminus \mathcal{C} \tag{48}$$

is sample stabilizing for the original system (29) *to* $\mathcal{C}$*.*

Proof. From Theorem 4.3 and Proposition 4.2 it follows that, if the extended control system $y' = F(y, w_0, w)$ is sample stabilizable to $\mathcal{C}$, then it admits a locally semiconcave CLF W. Hence, Proposition 4.3 with reference to W implies the existence of a continuous function $N : (0, +\infty) \to (0, 1]$ such that, fixed a selection $p(x) \in \partial_L W(x)$ for any $x \in \mathbb{R}^n \setminus \mathcal{C}$, any map $\hat{K} : \mathbb{R}^n \setminus \mathcal{C} \to \mathbb{U}$ such that

$$\hat{K}(x) = (\hat{w}_0(x), \hat{w}(x)) \in \underset{(w_0,w)\in \mathbb{U}_{N(W(x))}}{\arg\min} \Big\{\langle p(x), F(x, w_0, w)\rangle\Big\},$$

satisfies (47) and the inequality

$$\langle p(x), F(x, w_0, w)\rangle < -\gamma(W(x)) \qquad \forall x \in \mathbb{R}^n \setminus \mathcal{C}.$$

As shown in [2, Sect. 3], this implies that $\hat{K}$ is a sample stabilizing feedback for the extended system (35). Consider now the feedback $K : \mathbb{R}^n \setminus \mathcal{C} \to U$ associated to such $\hat{K}$, defined as in (48). It is locally bounded, since

$$|K(x)| \leq \nu^{-1}\left(\frac{1}{N(W(x))}\right) \qquad \forall x \in \mathbb{R}^n \setminus \mathcal{C}.$$

Furthermore, K is sample stabilizing for the rescaled control system $y' = \bar{f}(y, u)$, in view of the identity $\bar{f}(x, K(x)) = F(x, \hat{K}(x))$ for all $x \in \mathbb{R}^n \setminus \mathcal{C}$, from which it follows that sampling trajectories associated to $\hat{K}$ for the extended system coincide with the sampling trajectories associated to K for the rescaled system. Since by [2, Theorem 2.5] the rescaled and the original system share the same stabilizing feedbacks, K is sample stabilizing also for $\dot{x} = f(x, u)$ and the proof is concluded. □

Remark 4.4. Note that the converse relation, namely the fact that, given a stabilizing feedback K for the original system (29), it is possible to derive a feedback $\hat{K}$ for the impulsive extension, is quite obvious. Indeed, it is easy to see that the map

$$\hat{K} : x \mapsto \left(\frac{1}{1+\nu(|K(x)|)} \, , \, \frac{K(x)\,\nu(|K(x)|)}{|K(x)|(1+\nu(|K(x)|))} \right) \quad \forall x \in \mathbb{R}^n \setminus \mathcal{C}$$

is a stabilizing feedback for the extended system (35), again by the identity $F(x, \hat{K}(x)) = \bar{f}(x, K(x))$ together with [2, Theorem 2.5].

5. An example

In this section, we introduce a simple control system which is neither globally asymptotically controllable nor sample stabilizable to the origin by means of bounded controls, whereas it is by means of unbounded strategies. Furthermore, we show how to construct a stabilizing feedback for the original system, given a control Lyapunov function (and an associated stabilizing feedback) of the extend system.

Consider the target $\mathcal{C} := \{0\}$ and the one-dimensional control system

$$\dot{x}(t) = x(t) - x^3(t)u(t), \quad u(t) \in U := [0, +\infty) \quad \text{a.e.} \tag{49}$$

Using only controls taking values in a bounded subset $[0, M]$ of U for some $M > 0$, the best strategy to approach the origin is clearly to implement the constant control $u \equiv M$ and solve the differential equation

$$\dot{x}(t) = x(t) - Mx^3(t).$$

But this way, for every initial point $z \neq 0$ we get the trajectory

$$x(t) = \frac{ze^t}{\sqrt{z^2 M(e^{2t} - 1) + 1}}, \tag{50}$$

so we have $\lim_{t \to +\infty} x(t) = \dfrac{\text{sign}(z)}{\sqrt{M}} \neq 0$. Therefore, the control system (49) with controls in any given bounded subset of U is not GAC to $\{0\}$.

In view of Theorem 4.3, the global asymptotic controllability and the sample stabilizability of system (49) to $\{0\}$ when admissible pairs (x, u) with controls $u \in L^\infty_{loc}([0, T_x), [0, +\infty))$ are allowed, is equivalent to the global asymptotic controllability to $\{0\}$ of the impulsive extension

$$y'(s) = F(y(s), w_0(s), w(s)) = y(s)\, w_0(s) - y^3(s)w(s), \quad (w_0, w)(s) \in \overline{\mathbb{U}} \quad \text{a.e.,} \tag{51}$$

where $\mathbb{U} := \{(w_0, w) \in (0, +\infty) \times [0, +\infty), \quad w_0 + w = 1\}$. Here, by choosing the constant control $(w_0, w)(s) \equiv (0, 1)$ for every $s \geq 0$, for each starting point

$z \neq 0$ we get an extended trajectory (describing in the original time variable an instantaneous jump from z to the target) that satisfies

$$\mathbf{d}(y(s)) = |y(s)| = \frac{1}{\sqrt{2s + \frac{1}{z^2}}} =: \beta(|z|, s) \qquad \forall s \geq 0,$$

where $\beta \in \mathcal{KL}$. So, the original system is GAC and sample stabilizable to the origin, because the extended system is GAC to $\{0\}$.

Again Theorem 4.3 together with Propositions 4.2, 4.4 guarantees that there is a locally semiconcave control Lyapunov function for the extended system, which makes it possible to build both a sample stabilizing feedback $\hat{K}$ for (51) and a locally bounded sample stabilizing feedback K for (49). In particular, a locally semiconcave CLF for (51) is given by the function $W(x) := |x|$ for all $x \in \mathbb{R}$. Indeed, for every $x \neq 0$, one has

$$H_{F,\mathbb{U}}(x, \partial_L W(x)) = \inf_{(w_0,w)\in\mathbb{U}} \left\{ \frac{x}{|x|} \left(x w_0 - x^3 w\right) \right\} = -|x|^3 < -\gamma(W(x)),$$

if we choose the decrease rate $\gamma(r) := \frac{r^3}{2(2+r^2)}$, $r > 0$. At this point, setting $N(r) := \frac{r^2}{2+r^2}$, we can define the feedback $\hat{K} : \mathbb{R} \setminus \{0\} \to \mathbb{U}$, given by

$$\hat{K}(x) = (\hat{w}_0(x), \hat{w}(x)) := (N(|x|), 1 - N(|x|)) = \left(\frac{x^2}{2+x^2}, \frac{2}{2+x^2} \right),$$

which is sample stabilizing for the extend system, since

$$\frac{x}{|x|} \left(x\hat{w}_0(x) - x^3 \hat{w}(x)\right) = \frac{x}{|x|} \left(x\frac{x^2}{2+x^2} - x^3 \frac{2}{2+x^2} \right) = -\frac{|x|^3}{2+x^2} < -\gamma(W(x)),$$

for any $x \neq 0$. At this point, from Proposition 4.4 it follows that the locally bounded feedback $K : \mathbb{R} \setminus \{0\} \to U$, defined by

$$K(x) = \frac{\hat{w}(x)}{\hat{w}_0(x)} = \frac{2}{x^2} \qquad \forall x \neq 0,$$

is sample stabilizing for the original control system (49). In particular, an associated descent rate is $\beta(R, t) := Re^{-t/2}$ for all $(R, t) \in [0, +\infty)^2$. Indeed, let (R, r) be a pair with $0 < r < R$, and choose any sampling time $\delta(R, r) > 0$, which is continuous, r-increasing and R-decreasing, such that $\delta(R, r) \leq \ln(\varphi)$, where $\varphi := (1 + \sqrt{5})/2$ is the Golden Mean. Then, for any partition $\pi = (t_i)_i$ of $[0, +\infty)$ with sampling time $\delta(R, r)$, the π-sampling trajectory from each $z \neq 0$ with $\mathbf{d}(z) \leq R$ associated to K, satisfies the recursive relation

$$\dot{x}(t) = x(t) - \frac{1}{(x(t_n))^2} x^3(t) \quad t \in [t_n, t_{n+1}], \quad x(0) = z.$$

In view of the definition of $\delta(R, r)$, one has $e^{t-t_n} \in [1, \varphi]$ for all $t \in [t_n, t_{n+1}]$ and, in particular, this implies $e^{3(t-t_{n+1})} - 2e^{2(t-t_n)} + 1 \leq 0$ for all $t \in [t_n, t_{n+1}]$.

Using (50) (with $M = 2(x(t_n))^{-2}$) we then obtain, after few computations, the estimate

$$|x(t)| = |x(t_n)| \frac{e^{t-t_n}}{\sqrt{2e^{2(t-t_n)} - 1}} \leq |x(t_n)| e^{-(t-t_n)/2} \quad \forall t \in [t_n, t_{n+1}].$$

In particular $|x(t_{n+1})| \leq |x(t_n)| e^{-(t_{n+1}-t_n)/2}$, for all $n \geq 0$, therefore

$$|x(t)| \leq |z| e^{-t/2} = \beta(|z|, t) \qquad \forall t \geq 0.$$

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