Spherical birational sheets in reductive groups

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Abstract

We classify the spherical birational sheets in a complex simple simply-connected algebraic group. We use the classification to show that, when G is a connected reductive complex algebraic group with simply-connected derived subgroup, two conjugacy classes $\mathcal{O}_1$, $\mathcal{O}_2$ of G lie in the same birational sheet, up to a shift by a central element of G, if and only if the coordinate rings of $\mathcal{O}_1$ and $\mathcal{O}_2$ are isomorphic as G-modules. As a consequence, we prove a conjecture of Losev for the spherical subvariety of the Lie algebra of G.

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1 Introduction

Let G be a complex connected reductive algebraic group acting on a variety X. A sheet of X is an irreducible component of the locally closed subset $\{x \in X \mid \dim(G \cdot x) = d\}$ for some fixed d: then X is the finite union of its sheets. Let B be a Borel subgroup of G, the complexity of X is the codimension of a generic B-orbit in X. The variety X is *spherical* if has complexity zero. By [2, Proposition 1], the complexity of orbits as homogeneous spaces of G is constant along the sheets. In particular it follows that the property of being spherical is preserved along sheets. We say that the sheet S is *spherical* if the orbits in S are spherical. Now assume $X = G$, and the action is given by conjugation. Let T be a maximal torus of B, with Weyl group W. From the Bruhat decomposition $G = \bigcup_{w \in W} BwB$, it follows that for every conjugacy class $\mathcal{O}$ of G there exists a unique $w_{\mathcal{O}} \in W$ such that $\mathcal{O} \cap Bw_{\mathcal{O}}B$ is dense in $\mathcal{O}$. Similarly, for S a sheet of conjugacy classes, there is a unique $w_S \in W$ such that $S \cap Bw_SB$ is dense in S. By [10, Proposition 5.3] if S is a spherical sheet, then for every conjugacy class $\mathcal{O}$ lying in S we have $w_{\mathcal{O}} = w_S$.

A natural question is to consider the ring of regular functions $\mathbb{C}[\mathcal{O}]$ as $\mathcal{O}$ varies in a sheet S and ask whether the G-modules $\mathbb{C}[\mathcal{O}]$ are isomorphic. When G acts via the adjoint action on its Lie algebra $\mathfrak{g}$, some answers were obtained in [4]: for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ the G-module structure of $\mathbb{C}[\mathcal{O}]$ is preserved along sheets, but in this fails in general. In [20], Losev refined the notion of sheets of adjoint orbits by introducing the definition of birational sheets. In [20, Theorem 4.4], it is proven that birational sheets are locally closed subvarieties partitioning $\mathfrak{g}$. A remarkable result (see [20, Remark 4.11]) states that if $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are adjoint orbits of $\mathfrak{g}$ lying in the same birational sheet, then their G-module structures are isomorphic. In the same Remark, Losev conjectured that the viceversa is also true, aiming for an intrinsic characterization of birational sheets of the Lie algebra.

In this paper we deal with this problem with respect to spherical orbits both in the setting of conjugacy classes in G with simply-connected derived subgroup and in the setting of adjoint orbits in $\mathfrak{g}$. We recall the definition of birational sheet in $\mathfrak{g}$ from [20] and in G from [1]. A birational sheet is a certain union of G-orbits and is contained in a sheet, hence the property of being spherical is preserved along birational sheets. We shall call *spherical birational sheet* any birational sheet consisting of spherical orbits. For G simple simply-connected, we classify the spherical birational sheets and observe that the union G_{sph} of all spherical conjugacy classes in G is the disjoint union of spherical birational sheets. If $\mathcal{O}$ is a spherical conjugacy class, then $\mathbb{C}[\mathcal{O}]$ is multiplicity-free, i.e. a simple G-module occurs in $\mathbb{C}[\mathcal{O}]$ with multiplicity at most 1. Therefore, $\mathbb{C}[\mathcal{O}]$ is completely determined as a G-module by its weight monoid, i.e. by the highest dominant weights λ for which the simple G-module with highest weight λ occurs in the decomposition of $\mathbb{C}[\mathcal{O}]$.

In [14] the weight monoids are explicitely described for every spherical conjugacy class of G simple simply-connected. Using these results and the classification of spherical birational sheets, we shall prove the main result of this paper: let G be a complex connected reductive algebraic group with simply-connected derived subgroup and let $\mathcal{O}_1$ and $\mathcal{O}_2$ be spherical conjugacy classes in G. Let S_1^{bir} (resp. S_2^{bir}) be the birational sheet containing $\mathcal{O}_1$ (resp. $\mathcal{O}_2$). Then $\mathbb{C}[\mathcal{O}_1]$ is isomorphic to $\mathbb{C}[\mathcal{O}_2]$ as a G-module if and only if $S_2^{bir} = zS_1^{bir}$ for some $z \in Z(G)$ (the assumption on the derived subgroup of G cannot be relaxed).

>From this we also deduce the validity of Losev's conjecture in the case of spherical adjoint orbits in $\mathfrak{g}$. We also show that Losev's conjecture (resp. the corresponding group anologue) is true in the case $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ (resp. $G = \mathrm{SL}_n(\mathbb{C})$).

2 Definitions and notations

Let G be a connected reductive algebraic group over $\mathbb{C}$ and let $\mathfrak{g}$ be its Lie algebra. If K is a closed subgroup of G, we denote by K° its identity component, by K' its derived subgroup and by $Z(K)$ its centre. Similarly, if $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$, we denote by $\mathfrak{z}(\mathfrak{k})$ its centre.

If X is a K-set, we denote by X/K the set of K-orbits of elements in X. When K acts regularly on a variety X and $x \in X$, the K-orbit of x is denoted by $K \cdot x$. For $Y_1, Y_2 \subset X$, we write $Y_1 \sim_K Y_2$ if $\{K \cdot y_1 \mid y_1 \in Y_1\} = \{K \cdot y_1 \mid y_1 \in Y_1\}$; if $Y_i = \{y_i\}$ for $i = 1, 2$, we write $y_1 \sim_K y_2$. For any $n \in \mathbb{N}$, we define the locally closed subsets $X_{(n)} := \{x \in X \mid \dim(K \cdot x) = n\}$ of X. A *sheet* of X for the action of K is an irreducible component of $X_{(n)}$ for some $n \in \mathbb{N}$ such that $X_{(n)} \neq \varnothing$. For $Y \subseteq X$, the regular locus of Y is $Y^{reg} = Y \cap X_{(\bar{n})}$, where $\bar{n} = \max\{n \in \mathbb{N} \mid Y \cap X_{(n)} \neq \varnothing\}$, an open subset of Y, and the normalizer of Y in K is $N_K(Y) := \{k \in K \mid k \cdot Y = Y\}$. For $x \in X$, its stabilizer is $K_x := \{k \in K \mid k \cdot x = x\}$. When we consider the conjugacy (resp. the adjoint) action of G on itself (resp. on $\mathfrak{g}$) we adopt the following notation for orbits and stabilizers. Dealing with K-conjugacy classes or K-adjoint orbits, we shall use the notation $\mathcal{O}_g^K := K \cdot g$, $\mathfrak{O}_\xi^K := \mathrm{Ad}(K)(\xi)$. We shall omit superscripts whenever $K = G$. For $x \in G$ and $\eta \in \mathfrak{g}$, we write:

$$C_G(x) := G_x = \{g \in G \mid gxg^{-1} = x\}; \qquad C_G(\eta) := G_\eta = \{g \in G \mid (\mathrm{Ad}\, g)(\eta) = \eta\};$$
$$\mathfrak{c}_{\mathfrak{g}}(x) := \{\xi \in \mathfrak{g} \mid (\mathrm{Ad}\, x)(\xi) = \xi\}; \qquad \mathfrak{c}_{\mathfrak{g}}(\eta) := \mathfrak{g}_\eta = \{\xi \in \mathfrak{g} \mid [\eta, \xi] = 0\}.$$

For a subset $Y \subseteq G$, we set $C_G(Y) := \bigcap_{y \in Y} C_G(y)$.

We write $\mathcal{U}_K$ for the unipotent variety of K and $\mathcal{N}_{\mathfrak{k}}$ for the nilpotent cone of $\mathfrak{k} := \mathrm{Lie}(K)$; we also set $\mathcal{U} := \mathcal{U}_G$ and $\mathcal{N} := \mathcal{N}_{\mathfrak{g}}$. The set of all K-conjugacy classes of K is denoted K/K.

When we write $g = su \in G$ we implicitly assume that su is the Jordan decomposition of g, with s semisimple and u unipotent. Similarly for $\xi = \xi_s + \xi_n \in \mathfrak{g}$.

Let B be a Borel subgroup of G and T a maximal torus of B. We denote by Φ the root system of G with respect to T, by Δ the base of Φ individuated by B and by Φ^+ the corresponding subset of positive roots. The one-parameter subgroup of G corresponding to the root $\alpha \in \Phi$ will be denoted by U_α. We call *Levi subgroup* of G every Levi factor of a parabolic subgroup of G.

A standard parabolic subgroup is a subgroup containing B: it is of the form $P_\Theta = \langle B, U_{-\alpha} \mid \alpha \in \Theta\rangle$ for $\Theta \subseteq \Delta$. We have $P_\Theta = L_\Theta U_\Theta$, where the Levi factor $L_\Theta := \langle T, U_\alpha, U_{-\alpha} \mid \alpha \in \Theta\rangle$ is called a *standard Levi subgroup* and U_Θ is the unipotent radical of P_Θ. We also set $\mathrm{Lie}(T) = \mathfrak{h}$, $\mathrm{Lie}(B) = \mathfrak{b}$, $\mathrm{Lie}(U_\alpha) = \mathfrak{g}_\alpha$ for all $\alpha \in \Phi$.

A *pseudo-Levi subgroup* is the connected centralizer of a semisimple elements of G.

Finite-dimensional irreducible G-modules are parametrized by $X(T)^+$, the set of dominant weights of T (with respect to Φ^+), and we write $V(\lambda)$ for the irreducible G-module of highest weight λ.

Let X be a conjugacy class in G/G or an adjoint orbit in $\mathfrak{g}/G$. We have a decomposition into simple G-modules of the ring of regular functions $\mathbb{C}[X]$:

$$\mathbb{C}[X] \simeq_G \bigoplus_{\lambda \in X(T)^+} n_\lambda V(\lambda),$$

where n_λ is the *multiplicity* with which $V(\lambda)$ occurs in $\mathbb{C}[X]$, denoted by $[\mathbb{C}[X] : V(\lambda)]$. We denote by $\Lambda(X)$ the monoid of dominant weights occurring in $\mathbb{C}[X]$. If a Borel subgroup of G has a dense orbit on X, we call X *spherical*. Since X is quasi-affine, this is equivalent to the fact that $\mathbb{C}[X]$ is *multiplicity-free*, i.e. $n_\lambda \in \{0, 1\}$ for every $\lambda \in X(T)^+$: hence $\mathbb{C}[X] \simeq_G \bigoplus_{\lambda\in\Lambda(X)} V(\lambda)$.

A closed subgroup $H \leq G$ is said to be *spherical* if the homogeneous space G/H is a spherical variety.

We denote by G_{sph} (resp. $\mathfrak{g}_{sph}$) the union of all spherical conjugacy classes in G (resp. spherical adjoint orbits in $\mathfrak{g}$): these are closed subsets by [2, Corollary 2].

When G is simple, we denote the simple roots by $\alpha_1, \ldots, \alpha_n$: we shall use the numbering and the description of the simple roots in terms of the canonical basis $(e_1, \ldots, e_k)$ of an appropriate $\mathbb{R}^k$ as in [5, Planches I–IX]. We denote by $\mathtt{P}$ the weight lattice, by $\mathtt{P}^+$ the monoid of dominant weights. Also, $\alpha_1^\vee, \ldots, \alpha_n^\vee$ are the co-roots, $\omega_1, \ldots \omega_n$ are the fundamental weights and $\breve{\omega}_1, \ldots \breve{\omega}_n$ are the fundamental co-weights: these are the elements $\breve{\omega}_j$ of $\mathfrak{h}$ defined by $\alpha_i(\breve{\omega}_j) = \delta_{ij}$ for $1 \leq i, j \leq n$. The Weyl group of G is denoted by W, for $w \in W$ we use the notation $\dot{w}$ for an element of $N_G(T)$ representing $w \in W \simeq N_G(T)/T$. We write s_i for the simple reflection with respect to the simple root α_i, for $i = 1, \ldots, n$. Let $\beta = \sum_{j=1}^n c_j\alpha_j$ be the highest root in Φ: we define $\widetilde{\Delta} = \Delta \cup \{-\beta\}$. For the exceptional groups, we shall write $\beta = (c_1, \ldots, c_n)$ For $\Theta \subsetneq \widetilde{\Delta}$, set $L_\Theta := \langle T, U_\alpha, U_{-\alpha} \mid \alpha \in \Theta\rangle$. Following the terminology introduced in [27], we say that L_Θ is a *standard pseudo-Levi subgroup* of G. By [27, Proposition 2], pseudo-Levi subgroups are conjugates of standard pseudo-Levi subgroups.

An element $su \in G$ is *isolated* if $C_G(Z(C_G(s)^\circ)^\circ) = G$, as in [21, Definition 2.6]; in this case we say that $\mathcal{O}_{su}$ is an *isolated class*.

A partition of $n \in \mathbb{N} \setminus \{0\}$ is a sequence of non-increasing positive integers $\mathbf{d} = [d_1, \ldots, d_r]$ such that $\sum_{i=1}^r d_i = n$: we write $\mathbf{d} = [d_1, \ldots, d_r] \vdash n$. If $\mathbf{d} \vdash n$, the dual partition is $\mathbf{d}^t = \mathbf{f}$, where $f_i = |\{j \mid d_j \geq i\}|$ for all i. We will also use the compact notation $\mathbf{d} = [e_1^{m_1}, \ldots, e_s^{m_s}]$ where $e_1 > \cdots > e_s > 0$ by grouping equal d_i's. Partitions will be used to denote nilpotent orbits in classical Lie algebras, whereas for exceptional Lie algebras we will use the Bala-Carter labeling, as in [13].

We use the symbol $\sqcup$ to denote a disjoint union.

3 Jordan classes, sheets and birational sheets

3.1 Lie algebra case

Let $\mathfrak{l} \subseteq \mathfrak{g}$ be a Levi subalgebra and embed it in a parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{p}$. Let $P \leq G$ such that $\mathrm{Lie}(P) = \mathfrak{p}$, and let $P = LU_P$ be its Levi decomposition with $\mathrm{Lie}(L) = \mathfrak{l}$ and $\mathrm{Lie}(U_P) = \mathfrak{n}$. Let $\mathfrak{O}^L \in \mathcal{N}_{\mathfrak{l}}/L$. Then P acts on the closed subvariety $\overline{\mathfrak{O}^L} + \mathfrak{n} \subseteq \mathfrak{g}$ via the adjoint action. The *generalized Springer map* is:

$$\gamma\colon G \times^P (\overline{\mathfrak{O}^L} + \mathfrak{n}) \to \mathrm{Ad}(G)(\overline{\mathfrak{O}^L} + \mathfrak{n}), \quad g * \xi \mapsto (\mathrm{Ad}\, g)(\xi). \tag{1}$$

The image of γ is the closure of a single orbit $\mathfrak{O} \in \mathcal{N}/G$, and $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathfrak{O}^L := \mathfrak{O}$ is the orbit *induced* from $\mathfrak{O}^L$. It only depends on the pair $(\mathfrak{l}, \mathfrak{O}^L)$, not on the parabolic subgroup P chosen to define (1). If $\mathfrak{O} \in \mathcal{N}/G$ cannot be induced from a nilpotent orbit $\mathfrak{O}^L$ in a proper Levi subalgebra $\mathfrak{l} \subsetneq \mathfrak{g}$, then $\mathfrak{O}$ is said to be *rigid*. For a complete exposition on induction, refer to [13, §7].

A *decomposition datum* of $\mathfrak{g}$ consists of a pair of a Levi subalgebra $\mathfrak{l} \subseteq \mathfrak{g}$ and an orbit $\mathfrak{O}^L \in \mathcal{N}_{\mathfrak{l}}/L$, see [3, §1.6]. To any element $\xi = \xi_s + \xi_n \in \mathfrak{g}$ we can associate its decomposition datum $(\mathfrak{c}_{\mathfrak{g}}(\xi_s), \mathfrak{O}_{\xi_n}^{C_G(\xi_s)})$.

We denote by $\mathscr{D}(\mathfrak{g})$ the set of decomposition data of $\mathfrak{g}$. G acts by simultaneous conjugacy on the elements of $\mathscr{D}(\mathfrak{g})$. We say that two elements of $\mathfrak{g}$ are *Jordan equivalent* if their decomposition data are conjugate in G. The *Jordan class* of $\xi \in \mathfrak{g}$ is the set $\mathfrak{J}(\xi)$ consisting of all elements which are Jordan equivalent to ξ. If $\xi \in \mathfrak{g}$ has decomposition datum $(\mathfrak{l}, \mathfrak{O}^L)$, then $\mathfrak{J}(\xi) = \mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L) = (\mathrm{Ad}\, G)(\mathfrak{z}(\mathfrak{l})^{reg} + \mathfrak{O}^L)$. Jordan classes form a partition of $\mathfrak{g}$ into finitely many irreducible subvarieties parametrized by the (finite) set $\mathscr{D}(\mathfrak{g})/G$. They consist of unions of equidimensional adjoint orbits and their closure $\overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}$ (resp. *regular closure* $\overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}^{reg}$) is a union of Jordan classes.

Sheets for the adjoint action of G on the Lie algebra $\mathfrak{g}$ have been studied in [4, 3]. They are parametrized by the G-equivalence classes of decomposition data $(\mathfrak{l}, \mathfrak{O}^L) \in \mathscr{D}(\mathfrak{g})$ such that $\mathfrak{O}^L \in \mathcal{N}_{\mathfrak{l}}/L$ is rigid. The sheet $\mathfrak{S}(\mathfrak{l}, \mathfrak{O}^L)$ corresponding to the (class of) decomposition datum $(\mathfrak{l}, \mathfrak{O}^L)$ with $\mathfrak{O}^L$ is rigid is:

$$\mathfrak{S}(\mathfrak{l}, \mathfrak{O}^L) = \overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}^{reg} = \bigcup_{\xi \in \mathfrak{z}(\mathfrak{l})} (\mathrm{Ad}\, G)(\xi + \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{c}_{\mathfrak{g}}(\xi)} \mathfrak{O}^L).$$

Every sheet $\mathfrak{S}(\mathfrak{l}, \mathfrak{O}^L)$ contains a unique nilpotent orbit, i.e. $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathfrak{O}^L$. The dimension of a sheet has been determined explicitly in [24], [25].

If $\mathfrak{g}$ is simple of type A, its sheets are disjoint and the G-module structure of the rings of functions $\mathbb{C}[\mathfrak{O}]$ is preserved along sheets, see [4]. In general, these properties do not hold and sheets intersect non-trivially.

In [20], Losev introduced birational sheets of $\mathfrak{g}$ by restricting conditions on induction. Let $(\mathfrak{l}, \mathfrak{O}^L) \in \mathscr{D}(\mathfrak{g})$. As in [20, §4], we say that $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathfrak{O}^L$ is *birationally induced* from $(\mathfrak{l}, \mathfrak{O}^L)$ if, for a (hence any) parabolic subalgebra $\mathfrak{p}$ with Levi factor $\mathfrak{l}$, the generalized Springer map as in (1) is birational. If $\mathfrak{O} \in \mathcal{N}/G$ cannot be induced birationally from a proper Levi subalgebra, we say that $\mathfrak{O}$ is *birationally rigid*; all rigid orbits are birationally rigid. For any $(\mathfrak{l}, \mathfrak{O}^L) \in \mathscr{D}(\mathfrak{g})$, one can define, as in [20, §4], the set

$$\mathrm{Bir}(\mathfrak{z}(\mathfrak{l}), \mathfrak{O}^L) = \{\xi \in \mathfrak{z}(\mathfrak{l}) \mid \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{c}_{\mathfrak{g}}(\xi)} \mathfrak{O}^L \text{ is birationally induced}\}.$$

Since $\mathfrak{O}^L = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{l}} \mathfrak{O}^L$ is birationally induced from $(\mathfrak{l}, \mathfrak{O}^L)$, the inclusion $\mathfrak{z}(\mathfrak{l})^{reg} \subset \mathrm{Bir}(\mathfrak{z}(\mathfrak{l}), \mathfrak{O}^L)$ holds. By [20, Proposition 4.2], the set $\mathrm{Bir}(\mathfrak{z}(\mathfrak{l}), \mathfrak{O}^L)$ is open in $\mathfrak{z}(\mathfrak{l})$ and it is independent of the parabolic group chosen for induction. For $(\mathfrak{l}, \mathfrak{O}^L) \in \mathscr{D}(\mathfrak{g})$, the *birational closure* of $\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)$ is defined by as follows:

$$\overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}^{bir} = \bigcup_{\xi \in \mathrm{Bir}(\mathfrak{z}(\mathfrak{l}), \mathfrak{O}^L)} (\mathrm{Ad}\, G)(\xi + \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{c}_{\mathfrak{g}}(\xi)} \mathfrak{O}^L).$$

In particular $\overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}^{bir}$ is open in $\overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}^{reg}$ and in $\overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}$, hence it is irreducible and contained in a sheet.

Definition 3.1. For $(\mathfrak{l}, \mathfrak{O}^L) \in \mathscr{D}(\mathfrak{g})$ with $\mathfrak{O}^L$ birationally rigid, the *birational sheet* corresponding to $(\mathfrak{l}, \mathfrak{O}^L)$ is defined as $\overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}^{bir}$.

In [20, Theorem 4.4], it is proven that birational sheets are locally closed subvarieties partitioning the Lie algebra $\mathfrak{g}$; they are paramatrized by G-equivalence classes of pairs $(\mathfrak{l}, \mathfrak{O}^L) \in \mathscr{D}(\mathfrak{g})$ where $\mathfrak{O}^L \in \mathcal{N}_{\mathfrak{l}}/L$ is birationally rigid.

We state a remarkable result on birational sheets obtained by Losev, see [20, Remark 4.11].

Proposition 3.2. *If $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are two orbits of $\mathfrak{g}$ lying in the same birational sheet, then their G-module structure is isomorphic.*

In addition, Losev conjectured that the viceversa is also true, giving hope for an intrinsic characterization of birational sheets of the Lie algebra.

Conjecture 3.3. *If $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are two orbits of $\mathfrak{g}$ with isomorphic G-module structure, then they lie in the same birational sheet.*

3.2 Group case

Before its introduction in the case of the adjoint action on the Lie algebra, induction was defined by Lusztig and Spaltenstein for unipotent conjugacy classes in a connected reductive algebraic group, see [22]. Consider a parabolic subgroup $P \leq G$ with Levi decomposition $P = LU_P$ and $\mathcal{O}^L \in \mathcal{U}_L/L$. Then P acts on $\overline{\mathcal{O}^L}U_P$ via conjugacy and one can define the generalized Springer map:

$$\gamma \colon G \times^P \overline{\mathcal{O}^L}U_P \to G \cdot (\overline{\mathcal{O}^L}U_P), \quad g * x \mapsto gxg^{-1}. \tag{2}$$

The image of γ is the closure of a single conjugacy class $\mathcal{O} \in \mathcal{U}/G$, and $\mathrm{Ind}_L^G(\mathcal{O}^L) := \mathcal{O}$ is the *conjugacy class induced* from $(L, \mathcal{O}^L)$. When γ is birational, we say that $\mathrm{Ind}_L^G(\mathcal{O}^L)$ is *birationally induced* from $(L, \mathcal{O}^L)$. If $\mathcal{O}$ is a unipotent class in G which cannot be induced (resp. birationally induced) from $(L, \mathcal{O}^L)$ from any proper Levi subgroup L of G and $\mathcal{O}^L \in \mathcal{U}_L/L$, we say that it is *rigid* (resp. *birationally rigid*). All these notions are independent of the chosen parabolic subgroup P, see [1, Lemma 3.5].

Remark 3.4. Thanks to the bijective correspondences between parabolic subgroups, Levi subgroups, unipotent classes in G and parabolic subalgebras, Levi subalgebras, nilpotent orbits in $\mathfrak{g}$, we have that γ in (1) is birational if and only if γ in (2)is so, see [1, Remark 3.4].

Definition 3.5. Consider a pseudo-Levi subgroup $M \leq G$, let $Z := Z(M)$ and $z \in Z$. We say that the connected component $Z^\circ z$ *satisfies the regular property* (RP) for M if

$$C_G(Z^\circ z)^\circ = M. \tag{RP}$$

Observe that, for a pseudo-Levi subgroup $M \leq G$ and $z \in Z := Z(M)$, we have that $Z^\circ z$ satisfies (RP) for M if and only if $Z^{reg} \cap Z^\circ z \neq \varnothing$ if and only if $Z(M) = \langle Z^\circ, Z(G), z \rangle$ (see [11, Remark 3.6]) if and only if M is a Levi subgroup of $C_G(z)^\circ$ (see [1, Lemma 3.3]).

Remark 3.6. Assume G simple, let $M = L_\Theta$ for $\Theta \subset \widetilde{\Delta}$, let s be such that $M = C(s)^\circ$ and set $Z := Z(M)$. Observe that $Z^\circ s$ satisfies (RP) for M. Let $z \in Z$ such that $Z^\circ z$ satisfies (RP) for M, then, by [10, Theorem 4.1] (see also [27, Theorem 7]), there is $w \in W$ such that $w(\Theta) = \Theta$ and $\dot{w}(Z^\circ z)\dot{w}^{-1} = Z^\circ \hat{z} s$ for a certain $\hat{z} \in Z(G)$. Let $W_1 = \{w \in W \mid wsw^{-1}s^{-1} \in Z^\circ Z(G)\}$, $W_2 = \{w \in W \mid wsw^{-1}s^{-1} \in Z^\circ\}$. The assignment $w \mapsto wsw^{-1}s^{-1}Z^\circ$ defines a group homomorphism $W_1 \to \frac{Z^\circ Z(G)}{Z^\circ}$ with kernel W_2. Then the number of different G-classes of pairs $(M, Z^\circ z)$ for a fixed M with $Z^\circ z$ satisfying (RP) for M is

$$d_M := \left[\frac{Z(G)}{Z(G) \cap Z^\circ} : W_1/W_2 \right]. \tag{3}$$

Remark 3.7. Let $L \leq G$ be a pseudo-Levi subgroup and let $Z := Z(L)$, then L is a Levi subgroup if and only if $Z(L) = Z(G)Z(L)^\circ$ if and only if $Z(L)^\circ z$ satisfies (RP) for all $z \in Z(L)$.

Lemma 3.8. *Let $L \leq G$ be a Levi subgroup. Then two connected components of $Z := Z(L)$ are conjugate in G if and only if they are equal.*

Proof. This is clear from Remark 3.7. □

A *decomposition datum* of G consists of a triple $(M, Z(M)^\circ z, \mathcal{O}^M)$ such that:
(a) M is a pseudo-Levi subgroup of G;
(b) $Z(M)^\circ z$ is a connected component of $Z(M)$ satisfying (RP) for M;
(c) $\mathcal{O}^M$ is a unipotent conjugacy class of M.

To any element $su \in G$ we can associate its decomposition datum $(C_G(s)^\circ, Z(C_G(s)^\circ)^\circ s, \mathcal{O}_u^{C_G(s)^\circ})$: any decomposition datum is of this form.

The set of all decomposition data of G is denoted by $\mathscr{D}(G)$ and G acts on this set by simultaneous conjugacy on the triples.

Two elements $g_1, g_2 \in G$ are said to be *Jordan equivalent* if their decomposition data are conjugate in G. The *Jordan class* of su is the set of all elements which are Jordan equivalent to su: it is denoted $J(su)$.

If $\tau = (C_G(s)^\circ, Z(C_G(s)^\circ)^\circ s, \mathcal{O}_u^{C_G(s)^\circ})$ is the decomposition datum of su, then

$$J(su) = J(\tau) = G \cdot ((Z(C_G(s)^\circ)^\circ s)^{reg} \mathcal{O}_u^{C_G(s)^\circ}).$$

The group G is partitioned into its Jordan classes, which are finitely many locally closed irreducible subvarieties parametrized by the finite set $\mathscr{D}(G)/G$. Jordan classes are unions of equidimensional conjugacy classes. The closure of a Jordan class is a union of Jordan classes.

Sheets for the conjugacy action of G on itself were studied in [11]. They are parametrized by the G-equivalence classes of decomposition data $\tau = (M, Z(M)^\circ s, \mathcal{O}^M) \in \mathscr{D}(G)$ with $\mathcal{O}^M \in \mathcal{U}_M/M$ rigid: the sheet corresponding to τ is

$$S(\tau) := \overline{J(\tau)}^{reg} = \bigcup_{z \in Z(M)^\circ s} G \cdot \left(s \operatorname{Ind}_M^{C_G(z)^\circ} \mathcal{O}^M \right).$$

In the remainder of the paper, unless differently specified, we work under the assumption G' *simply-connected*: as a consequence, centralizers of semisimple elements are connected.

As in [1, §5.1], for $(M, Z(M)^\circ s, \mathcal{O}^M) \in \mathscr{D}(G)$ we define the set:

$$\operatorname{Bir}(Z(M)^\circ s, \mathcal{O}^M) = \{z \in Z(M)^\circ s \mid \operatorname{Ind}_M^{C_G(z)} \mathcal{O}^M \text{ is birationally induced}\}.$$

This is an open subset of $Z(M)^\circ s$, independent of the parabolic group chosen for induction ([1, Remark 5.2, Proposition 5.1]): it contains $(Z(M)^\circ s)^{reg}$, since $\mathcal{O}^M = \operatorname{Ind}_M^M \mathcal{O}^M$ is birationally induced from $(M, \mathcal{O}^M)$. For $\tau = (M, Z(M)^\circ s, \mathcal{O}^M) \in \mathscr{D}(G)$, the *birational closure* of $J(\tau)$ is

$$\overline{J(\tau)}^{bir} := \bigcup_{z \in \operatorname{Bir}(Z(M)^\circ s, \mathcal{O}^M)} G \cdot \left(z \operatorname{Ind}_M^{C_G(z)} \mathcal{O}^M\right).$$

Then $J(\tau) \subseteq \overline{J(\tau)}^{bir} \subseteq \overline{J(\tau)}^{reg}$: in particular, being $\overline{J(\tau)}^{reg}$ irreducible, it is contained in a sheet, hence so is $\overline{J(\tau)}^{bir}$. In fact $\overline{J(\tau)}^{bir}$ is an irreducible locally closed subvariety of G and a union of Jordan classes ([1, Proposition 5.2, Corollary 5.3]).

Definition 3.9. We define the set

$$\mathscr{B}\mathscr{B}(G) := \{(M, Z(M)^\circ s, \mathcal{O}^M) \in \mathscr{D}(G) \mid \mathcal{O}^M \in \mathcal{U}_M/M \text{ birationally rigid}\}.$$

For $\tau \in \mathscr{B}\mathscr{B}(G)$, we define the *birational sheet* of G corresponding to (the class of) τ as $\overline{J(\tau)}^{bir}$.

It follows from [1, Theorem 5.1] that the birational sheets of G form a partition of G.

Remark 3.10. For G semisimple, a birational sheet coincides with a single conjugacy class if and only if it is $\mathcal{O}_{su}$ with s isolated and $\mathcal{O}_u^{C_G(s)}$ a birationally rigid unipotent class of $C_G(s)$.

3.3 Criteria for birational induction

We recollect some results from [1, Lemmas 3.2, 3.6]: they will be used to classify birational sheets containing spherical conjugacy classes.

Lemma 3.11. *Let $P \leq G$ be a parabolic subgroup with Levi decomposition $P = LU$, let $\mathcal{O}^L \in \mathcal{U}_L/L$, let $\mathcal{O} = \operatorname{Ind}_L^G \mathcal{O}^L$ and let γ be as in (2). The following are equivalent:*
(i) γ is birational;
(ii) for all $x \in \mathcal{O} \cap \overline{\mathcal{O}^L}U$, we have $C_G(x) = C_P(x)$;
(iii) there exists $x \in \mathcal{O} \cap \overline{\mathcal{O}^L}U$ such that $C_G(x) = C_P(x)$. □

Lemma 3.12. *Let $\phi\colon \mathcal{N} \to \mathcal{U}$ denote a Springer's isomorphism and let $\overline{G}$ be the adjoint group in the same isogeny class of G. Let $\nu \in \mathcal{N}$. Suppose that $C_{\overline{G}}(\nu)$ is connected. If $\mathcal{O}_{\phi(\nu)} = \operatorname{Ind}_L^G \mathcal{O}^L$ for a Levi subgroup $L \leq G$ and $\mathcal{O}^L \in \mathcal{U}_L/L$, then $\mathcal{O}_{\phi(\nu)}$ is birationally induced from $(L, \mathcal{O}^L)$.* □

Remark 3.13. Let $G = \mathrm{SL}_n(\mathbb{C})$, then the condition in Lemma 3.12 is always fulfilled, hence a unipotent class in G (resp. a nilpotent orbit in $\mathfrak{g}$) is rigid if and only if it is birationally rigid if and only if it is $\{1\}$ (resp. $\{0\}$), see [1, Example 3.4]. Moreover, sheets coincide with sheets in $\mathfrak{g}$ and in G, see [1, Corollary 5.4].

3.4 Birationally rigid unipotent classes

In this section we assume G simple and we recollect the complete list of birationally rigid conjugacy classes in $\mathcal{U}$ (equivalently of birationally rigid adjoint orbits in $\mathcal{N}$).

Namikawa gave in [26] a criterion to test when a nilpotent orbit is birationally rigid for simple classical Lie algebras. If $\mathfrak{g}$ is of type A, then the only birationally rigid orbit is the only rigid orbit, i.e. the null orbit. Now let $\mathfrak{g}$ be of type $\mathsf{B}, \mathsf{C}, \mathsf{D}$. Let $\mathbf{d} = [d_1, \dots, d_r]$ denote the partition corresponding to a nilpotent orbit $\mathfrak{O}$. Then $\mathfrak{O}$ is birationally rigid in $\mathfrak{g}$ if and only if $\mathbf{d}$ has *full members*, i.e. $1 = d_r$ and $d_i - d_{i+1} \leq 1$ for all $i = 1, \dots, r-1$, with the exception of the case $\mathbf{d} = [2^{n-1}, 1^2]$ in D_n for $n = 2m+1, m \geq 1$, which is birationally induced as a Richardson orbit.

Fu worked out the exceptional types in [16]: birationally rigid orbits coincide with rigid ones, except in type E_7, where also $A_2 + A_1$ and $A_4 + A_1$ are birationally rigid, and in type E_8, where also $A_4 + A_1$ and $A_4 + 2A_1$ are birationally rigid.

For a complete list of rigid nilpotent orbits in the exceptional cases, see [23, Appendix 5.7]. It follows that every spherical nilpotent orbit is (birationally) rigid, apart from $2A_1$ in type E_6 and $(3A_1)''$ in type E_7.

Remark 3.14. Recall from [3, Lemma 3.9] that all nilpotent orbits $\mathfrak{O}$ in $\mathfrak{g}$ simple are characteristic, except for:

(1) $\mathfrak{g}$ of type D_4: $\mathrm{Aut}(\mathfrak{g})$ acts transitively on $\{\mathfrak{O}_{[4^2]}, \mathfrak{O}'_{[4^2]}, \mathfrak{O}_{[5,1^3]}\}$ and on $\{\mathfrak{O}_{[2^4]}, \mathfrak{O}'_{[2^4]}, \mathfrak{O}_{[3,1^5]}\}$.

(2) $\mathfrak{g}$ of type $\mathsf{D}_{2m}, m \geq 3$: the graph automorphism permutes $\mathfrak{O}_{\mathbf{d}}$ and $\mathfrak{O}'_{\mathbf{d}}$ for every very even partition $\mathbf{d} \vdash 4m$.

It follows that all birationally rigid nilpotent orbits in simple Lie algebras are characteristic, analogously for all birationally rigid unipotent classes in simple algebraic groups.

3.5 Birational sheets and translation by central elements

Let $\tau := (M, Z(M)^\circ s, \mathcal{O}^M) \in \mathscr{D}(G)$. For each $z \in Z(G)$, let $\tau_z := (M, Z(M)^\circ zs, \mathcal{O}^M)$. Then we have $\overline{J(\tau_z)}^{bir} = z\overline{J(\tau)}^{bir}$, so that the union of all $\overline{J(\tau_z)}^{bir}$ as z varies in $Z(G)$ is

$$Z(G)\overline{J(\tau)}^{bir} := \bigcup_{z \in Z(G)} z\overline{J(\tau)}^{bir}. \tag{4}$$

We shall be interested in $Z(G)\overline{J(\tau)}^{bir}$ for $\tau \in \mathscr{BB}(G)$: to describe it, it is enough to describe $\overline{J(\tau)}^{bir}$ and to count the number of birational sheets in $Z(G)\overline{J(\tau)}^{bir}$.

Remark 3.15. For G simple, let $\tau := (M, Z(M)^\circ z, \mathcal{O}^M) \in \mathscr{D}(G)$ and set $Z := Z(M)$. We have seen that the number of different G-classes of pairs $(M, Z^\circ z)$ for a fixed M, with $Z^\circ z$ satisfying (RP) for M equals the index $d_M = \left[\frac{Z(G)}{Z(G) \cap Z^\circ} : W_1/W_2\right]$, defined in Remark 3.6. If $\mathcal{O}^M$ is characteristic in M, the number of different G-classes of triples $(M, Z(M)^\circ z, \mathcal{O}^M)$ for fixed M and $\mathcal{O}^M$, with $Z^\circ z$ satisfying (RP) for M is again the index d_M.

4 The ring of regular functions as an invariant

We open this section with an analysis of the behaviour of the ring of regular functions on conjugacy classes belonging to the same Jordan class. After that, we focus on G simple of type A: in this case the relation between (birational) sheets and the decomposition into simple G-modules of rings of regular functions on orbits is completely understood; we give an overview of the problem in the Lie algebra and we conclude similar results in the simply-connected group.

It is proven in [8, §3.7] that $\xi_1, \xi_2 \in \mathfrak{g}$ belong to the same Jordan class if and only if their centralizers $C_G(\xi_1)$ and $C_G(\xi_2)$ are G-conjugate: in this case $\mathfrak{O}_{\xi_1}$ and $\mathfrak{O}_{\xi_2}$ are isomorphic as G-homogeneous spaces and their rings of regular functions are isomorphic as G-modules.

We address the similar problem in the group case, where the presence of a non trivial centre $Z(G)$ yields $C_G(x) = C_G(zx)$ for all $x \in G, z \in Z(G)$. The Springer–Steinberg Theorem on connectedness of centralizers of semisimple elements allows to prove

Proposition 4.1. *Suppose G' is simply-connected and let $x, y \in G$. If $Z(G)J(x) = Z(G)J(y)$, then $C_G(x)$ and $C_G(y)$ are G-conjugate. In particular, for any Jordan class J in G and any pair of classes $\mathcal{O}_1, \mathcal{O}_2 \subset Z(G)J$ we have $\mathbb{C}[\mathcal{O}_1] \simeq_G \mathbb{C}[\mathcal{O}_2]$.*

As far as the other implication is concerned, we have

Proposition 4.2. *Let G be simple and simply-connected, let $x, y \in G$. If $C_G(x) \sim_G C_G(y)$, then $Z(G)J(x) = Z(G)J(y)$ except for the following cases:*

(i) G of type E_6, and $\{\mathcal{O}_x, \mathcal{O}_y\} = \{\mathcal{O}_{\varphi u_1 u_2 u_3}, \mathcal{O}_{\varphi^{-1} u_1 u_2 u_3}\}$, where $\varphi = \exp(2\pi i \check{\omega}_4/3)$ and $u_1, u_2, u_3 \in \mathrm{SL}_3(\mathbb{C})$ unipotent, with $u_i \not\sim_{\mathrm{SL}_3(\mathbb{C})} u_j$ for $i \neq j$;

(ii) G of type E_8, and $\{\mathcal{O}_x, \mathcal{O}_y\} = \{\mathcal{O}_{\varphi u_1 u_2}, \mathcal{O}_{\varphi^2 u_1 u_2}\}$, where $\varphi = \exp(2\pi i \check{\omega}_5/5)$ and $u_1, u_2 \in \mathrm{SL}_5(\mathbb{C})$ unipotent such that $u_1 \not\sim_{\mathrm{SL}_5(\mathbb{C})} u_2$.

Proof. We may assume $C_G(x) = C_G(y)$, with $x = su$ and $y = s'u'$. Note that $C_G(s)$ is the unique minimal pseudo-Levi subgroup containing $C_G(x)$: if $M \geq C_G(x)$ is a pseudo-Levi in G, then $Z(M) \leq Z(C_G(x))$. The structure of $Z(C_G(x))$ described in [19, Theorem 2.1] implies $Z(M) \leq Z(C_G(s))$, which yields $C_G(s) \leq M$. Similarly, $C_G(s')$ is the unique minimal pseudo-Levi subgroup containing $C_G(y)$. This implies $C_G(s) = C_G(s') =: H$ and $C_H(u) = C_H(u')$, equivalently $u \sim_H u'$, by [19, Theorem 2.1].

Hence we may assume $x = su$ and $y = s'u$, where $C_G(s) = C_G(s') =: H \sim_G M_\Theta$, for $\Theta \subset \widetilde{\Delta}$. By [27, Proposition 7], the group $Z(H)/Z(G)Z(H)^\circ$ is cyclic of order $d_\Theta := \gcd\{c_i \mid \alpha_i \in \widetilde{\Delta} \setminus \Theta\}$ and it is generated by the cosets of s and s'. Note that when $s \equiv s' \mod Z(G)Z(H)^\circ$, then $Z(G)J(x) = Z(G)J(y)$: we will therefore concentrate on the other cases. Recall that $1 \leq d_\Theta \leq 6$.

If $d_\Theta \in \{1, 2\}$, then we always have $s \equiv s' \mod Z(G)Z(H)^\circ$ and we conclude.

If $d_\Theta \in \{3, 4, 6\}$ and $(G, d_\Theta) \neq (\mathsf{E}_6, 3)$, then the cosets of s, s^{-1} are the only two generators of $Z(H)/Z(G)Z(H)^\circ$. If $s' \equiv s^{-1} \mod Z(G)Z(H)^\circ$, then as in the proof of [27, Proposition 7], there exists $w \in W$ such that $s' \equiv wsw^{-1} \mod Z(G)Z(H)^\circ$; moreover w fixes the irreducible components of Θ, which are of type A_n, D_5 or E_6, so that any lift $\dot{w} \in N_G(T)$ preserves the class $\mathcal{O}_u^H$: this allows to conclude that $Z(G)J(x) = Z(G)J(y)$.

In the remaining cases, $(G, d_\Theta) = (\mathsf{E}_6, 3)$ or $(\mathsf{E}_8, 5)$, H is semisimple hence $Z(G)J(x) = Z(G)J(y)$ if and only if $x \sim_G zy$ for some $z \in Z(G)$.

We discuss the case $(G, d_\Theta) = (\mathsf{E}_6, 3)$: we may assume $s = \varphi := \exp(2\pi i \check{\omega}_3/3)$ and $s' = z^i \varphi^{-1}$, where $z := \exp(-2\pi i \check{\omega}_1) \in Z(G)$ and $i \in \{0, 1, 2\}$. We have $H = H_1 H_2 H_3$ with $H_1 = \langle X_{\pm\alpha_1}, X_{\pm\alpha_3}\rangle$, $H_2 = \langle X_{\pm\alpha_5}, X_{\pm\alpha_6}\rangle$, $H_3 = \langle X_{\pm\alpha_2}, X_{\pm\beta}\rangle$ and $H_i \simeq \mathrm{SL}_3(\mathbb{C})$ for $i = 1, 2, 3$. We write $u = u_1 u_2 u_3$, with $u_i \in H_i$. There exists w of order 2 in W as in the proof of [27, Proposition 7] such that $w\varphi w^{-1} = \varphi^{-1}$. We may choose a lift $\dot{w} \in N_G(T)$ inducing the automorphism of $\mathrm{SL}_3(\mathbb{C}) \times \mathrm{SL}_3(\mathbb{C}) \times \mathrm{SL}_3(\mathbb{C})$, $(g_1, g_2, g_3) \mapsto (g_2, g_1, \gamma(g_3))$, where γ is the graph automorphism of $\mathrm{SL}_3(\mathbb{C})$ ([17, Table 4.7.1]). Moreover, there exists ρ of order 3 in W such that $\rho\varphi\rho^{-1} = z\varphi$. We may choose a lift $\dot{\rho} \in N_G(T)$ inducing the automorphism of $\mathrm{SL}_3(\mathbb{C}) \times \mathrm{SL}_3(\mathbb{C}) \times \mathrm{SL}_3(\mathbb{C})$, $(g_1, g_2, g_3) \mapsto (g_3, g_1, g_2)$. Since $N_G(H)/H \simeq \langle w, \rho\rangle \simeq S_3$, we have $Z(G)\mathcal{O}_{\varphi u} = Z(G)\mathcal{O}_{\varphi^{-1}u}$ if and only if $u_i \sim_{\mathrm{SL}_3(\mathbb{C})} u_j$ for some $i \neq j$.

We are left with the case $(G, d_\Theta) = (\mathsf{E}_8, 5)$: we may assume $s = \varphi := \exp(2\pi i \check{\omega}_5/5)$ and $s' \in \{\varphi^2, \varphi^3, \varphi^4\}$. We have $H = H_1 H_2$ with $H_i \simeq \mathrm{SL}_5(\mathbb{C})$ and write $u = u_1 u_2$ with $u_i \in H_i$, for $i = 1, 2$. There exists w of order 4 in W as in the proof of [27, Proposition 7] such that $w\varphi w^{-1} = \varphi^2$. We may choose a lift $\dot{w} \in N_G(T)$ inducing the automorphism of $\mathrm{SL}_5(\mathbb{C}) \times \mathrm{SL}_5(\mathbb{C})$, $(g_1, g_2) \mapsto (\gamma(g_2), g_1)$, where γ is the graph automorphism of $\mathrm{SL}_5(\mathbb{C})$ ([17, Table 4.7.1]). It follows that $\varphi u \sim_G \varphi^4 u$ and $\varphi^2 u \sim_G \varphi^3 u$. Since $N_G(H)/H \simeq \langle w\rangle$, we have $\mathcal{O}_{\varphi u} = \mathcal{O}_{\varphi^2 u}$ if and only if $u_1 \sim_{\mathrm{SL}_5(\mathbb{C})} u_2$. $\square$

We note that the assumption on G' in Proposition 4.1 cannot be removed.

Example 4.3. Consider $G = \mathrm{SL}_2(\mathbb{C})$, let $\overline{G} = \mathrm{PSL}_2(\mathbb{C})$ and let $\pi\colon G \to \overline{G}$, $\pi(g) = \bar{g}$ be the isogeny. Let us consider the torus $T \leq G$ given by diagonal elements: $T = \{t_k = \mathrm{diag}[k, k^{-1}] \mid$

$k \in \mathbb{C}^\times\}$. We have the following situation for the centralizer of a regular element $\bar{t}_k, k \neq \pm 1$:

$$C_{\overline{G}}(\bar{t}_k) = \begin{cases} \overline{H} := N_{\overline{G}}(\overline{T}) & \text{if } k = \pm i; \\ \overline{T} = \overline{H}^\circ & \text{if } k \notin \{\pm i, \pm 1\} \end{cases}$$

Observe that $\overline{G} \cdot (\overline{T}^{reg})$ is the Jordan class in $\overline{G}$ consisting of regular semisimple elements. Along $\overline{G} \cdot (\overline{T}^{reg})$ neither the $\overline{G}$-module structure nor the $\overline{G}$-homogeneous space structure of conjugacy classes is preserved:

k	$\mathbb{C}[\mathcal{O}_{\bar{t}_k}^{\overline{G}}]$	$\Lambda(\mathcal{O}_{\bar{t}_k}^{\overline{G}})$
$k = \pm i$	$\mathbb{C}[\overline{G}/\overline{H}]$	$4n\omega$
$k \in \mathbb{C}^\times \setminus \{\pm 1, \pm i\}$	$\mathbb{C}[\overline{G}/\overline{T}]$	$2n\omega$

Table 1: Regular semisimple spherical classes in $\mathrm{PSL}_2(\mathbb{C})$.

4.1 Type A

The study of sheets of $\mathfrak{g}$ for the adjoint action of G in [4] was a first attempt to classify sets of orbits $\mathfrak{O}$ such that the decomposition of $\mathbb{C}[\mathfrak{O}]$ into simple G-modules is constant: the main result in this direction is summed up in the following statement.

Theorem 4.4 ([4, Theorems 3.8 and 6.3]). *Let G be simple and adjoint. Let $\mathfrak{S}$ be a sheet of $\mathfrak{g}$ and let $\mathfrak{O}$ be the unique nilpotent orbit in $\mathfrak{S}$, let $\nu \in \mathfrak{O}$. Suppose that $\overline{\mathfrak{O}}$ is normal and that $C_G(\nu)$ is connected. Then $\mathbb{C}[\overline{\mathfrak{O}}] \simeq_G \mathbb{C}[\mathfrak{O}] \simeq_G \mathbb{C}[\mathfrak{O}'] \simeq_G \mathbb{C}[\overline{\mathfrak{O}'}]$ for all orbits $\mathfrak{O}'$ in $\mathfrak{S}$.*

Suppose G is simple and adjoint of type A, then the sheets of $\mathfrak{g}$ are disjoint and parametrized by the unique nilpotent orbit contained in them [15]. Let $\mathfrak{O}$ be a nilpotent oribt of $\mathfrak{g}$, then the hypothess of Theorem 4.4 are fulfilled: normality of $\overline{\mathfrak{O}}$ follows from [18] and all centralizers in G of elements in $\mathfrak{g}$ are connected. Therefore, Borho and Kraft could conclude in [4, *Nachtrag bei der Korrektur*] that the multiplicities of simple G-modules in the decomposition of the algebras of regular functions on adjoint orbits are preserved along sheets of $\mathfrak{g}$.

The natural question is: does the invariant given by the multiplicities separate distinct sheets of $\mathfrak{sl}_n(\mathbb{C})$? The answer is affirmative, as recorded in the following statement: we are indebted to Eric Sommers for suggesting the use of small modules in the proof.

Proposition 4.5. *Let G be simple of type A. If $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are two distinct nilpotent orbits in $\mathfrak{g}$, then $\Lambda(\mathfrak{O}_1) \neq \Lambda(\mathfrak{O}_2)$. In particular $\mathbb{C}[\mathfrak{O}_1] \not\simeq_G \mathbb{C}[\mathfrak{O}_2]$.*

Proof. Let $\mathbb{C}$ denote the trivial representation of a group. Let $\lambda \in \mathsf{P}^+$ and let $V(\lambda)_0 = V(\lambda)^T$ be the zero weight subspace in $V(\lambda)$. Then $V(\lambda)_0$ is a W-module, in general reducible. Let L be a Levi subgroup of G with Weyl group W_L. By [7, Proof of Corollary 1], if $V(\lambda)$ is small (i.e. if twice a root never occurs as a weight of λ), we have $V(\lambda)^L = (V(\lambda)_0)^{W_L}$ and, by Frobenius reciprocity, $\dim(V(\lambda)_0)^{W_L} = [\mathrm{Ind}_{W_L}^W(\mathbb{C}) : V(\lambda)_0]$. In the case of $\mathfrak{sl}_n(\mathbb{C})$, for every irreducible S_n-module M there exists λ in $\mathsf{P}^+ \cap \mathbb{Z}\Phi$ such that $V(\lambda)$ is small and $V(\lambda)_0 \simeq_{S_n} M$, see [7, example p. 389]. Conjugacy classes of Levi subgroups of $\mathrm{SL}_n(\mathbb{C})$ are indexed by partitions $\mathbf{d} = [d_1, \ldots, d_k]$ of n with $d_k > 0$: the induced Richardson nilpotent class is $\mathfrak{O}_{\mathbf{d}^t}$. Let $L_{\mathbf{d}}$ be the standard Levi subgroup, with Weyl group $S_{\mathbf{d}} = S_{d_1} \times \cdots \times S_{d_k}$, corresponding to the partition $\mathbf{d} = [d_1, \ldots, d_k]$. We know that $[\mathbb{C}[\mathfrak{O}_{\mathbf{d}^t}] : V] = \dim V^{L_{\mathbf{d}}}$ for every simple $\mathrm{SL}_n(\mathbb{C})$-module V. We put $U_{\mathbf{d}} = \mathrm{Ind}_{S_{\mathbf{d}}}^{S_n}(\mathbb{C})$ and denote by $V_{\mathbf{d}}$ the simple S_n-module (Specht module) corresponding to $\mathbf{d}$. Then

$$U_{\mathbf{d}} = V_{\mathbf{d}} \oplus \bigoplus_{\mathbf{f}>\mathbf{d}} K_{\mathbf{f}\,\mathbf{d}} V_{\mathbf{f}} \tag{5}$$

where the coefficients $K_{\mathbf{f}\,\mathbf{d}}$ are the Kostka numbers and $<$ is the lexicographic total order on partitions of n. Let $\mathbf{d}$, $\mathbf{f}$ be different partitions of n: we may assume $\mathbf{d} > \mathbf{f}$. By the previous discussion, there exists a small simple $\mathrm{SL}_n(\mathbb{C})$-module $V(\lambda)$ such that $V_{\mathbf{f}} \simeq_{S_n} V(\lambda)_0$. Then $[\mathbb{C}[\mathfrak{O}_{\mathbf{f}}] : V(\lambda)] = 1 \neq 0 = [\mathbb{C}[\mathfrak{O}_{\mathbf{d}}] : V(\lambda)]$ and this allows to conclude. $\square$

We gather the arguments above in the following result.

Theorem 4.6. *Let G be simple of type A. Then two adjoint orbits $\mathfrak{O}_1, \mathfrak{O}_2$ in $\mathfrak{g}$ belong to the same sheet if and only if $\mathbb{C}[\mathfrak{O}_1] \simeq_G \mathbb{C}[\mathfrak{O}_2]$ if and only if $\Lambda(\mathfrak{O}_1) = \Lambda(\mathfrak{O}_2)$.*

Remark 4.7. One direction of Theorem 4.6 is a particular case of Proposition 3.2 while the other direction proves Conjecture 3.3 for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, as in this case birational sheets coincide with sheets.

Since pseudo-Levi subgroups of $\mathrm{SL}_n(\mathbb{C})$ are Levi subgroups, we deduce a group analogue of Theorem 4.6.

Theorem 4.8. *Let $G = \mathrm{SL}_n(\mathbb{C})$, $\mathcal{O}_1$ and $\mathcal{O}_2$ be conjugacy classes of G and let S_1 (resp. S_2) be the (birational) sheet containing $\mathcal{O}_1$ (resp. $\mathcal{O}_2$). Then $\mathbb{C}[\mathcal{O}_1] \simeq_G \mathbb{C}[\mathcal{O}_2]$ if and only if $\Lambda(\mathcal{O}_1) = \Lambda(\mathcal{O}_2)$ if and only if $S_2 = zS_1$ for some $z \in Z(G)$.*

Proof. Recall that (birational) sheets in G are disjoint and are parameterized by G-classes of pairs $(L, Z(L)^\circ z)$, with L a Levi sugbroup of G and a certain z in $Z(G)$, see [1, Corollary 5.4]. For every $x \in G$ there exists $\xi \in \mathfrak{g}$ such that $C_G(\xi) = C_G(x)$: if the sheet of G containing $\mathcal{O}_x$ corresponds to the G-class of $(L, Z(L)^\circ z)$, then the sheet of $\mathfrak{g}$ containing ξ corresponds to the G-class of $\mathrm{Lie}(L)$. Let $x_i \in \mathcal{O}_i$ and $\xi_i \in \mathfrak{g}$ such that $C_G(\xi_i) = C_G(x_i)$, and let $(L_i, Z(L_i)^\circ z_i)$ correspond to S_i for $i = 1, 2$. Then $\mathbb{C}[\mathcal{O}_i] = \mathbb{C}[\mathfrak{O}_{\xi_i}]$ for $i = 1, 2$. Therefore $\mathbb{C}[\mathcal{O}_1] \simeq_G \mathbb{C}[\mathcal{O}_2]$ if and only if $L_1 \sim_G L_2$ by Theorem 4.6, hence if and only if $S_2 = zS_1$ for some $z \in Z(G)$. The equivalence $\mathbb{C}[\mathcal{O}_1] \simeq_G \mathbb{C}[\mathcal{O}_2]$ if and only if $\Lambda(\mathfrak{O}_1) = \Lambda(\mathfrak{O}_2)$ follows from Theorem 4.6. $\square$

5 Spherical birational sheets

This section is dedicated to the main result of the paper: the classification of spherical conjugacy classes grouped in birational sheets and the verification of the analogues of Proposition 3.2 and Conjecture 3.3 in the case of G connected reductive with G' simply-connected.

The property of being spherical is preserved along sheets, as proven in [2, Proposition 1]. A *spherical sheet* is a sheet consisting of spherical orbits, as in [10]. Since every birational sheet is irreducible, it is contained in a sheet, and the following definition is well-posed.

Definition 5.1. Let $\tau \in \mathscr{BB}(G)$. We say that the birational sheet $\overline{J(\tau)}^{bir}$ is *spherical* if one of the following equivalent properties is satisfied:

(i) all conjugacy classes $\mathcal{O} \subset \overline{J(\tau)}^{bir}$ are spherical;

(ii) there exists a spherical conjugacy class $\mathcal{O} \subset \overline{J(\tau)}^{bir}$;

(iii) $\overline{J(\tau)}^{bir}$ is contained in a spherical sheet.

As recalled in the Introduction, if $\mathcal{O}$ (resp. S) is a conjugacy class (resp. a sheet) of G and we denote by $w_{\mathcal{O}}$ (resp. w_S) the unique element of W such that $\mathcal{O} \cap Bw_{\mathcal{O}}B$ is dense in $\mathcal{O}$ (resp. $S \cap Bw_SB$ is dense in S), then if S is spherical, for every conjugacy class $\mathcal{O}$ lying in S we have $w_{\mathcal{O}} = w_S$. For a birational sheet $\overline{J(\tau)}^{bir}$ we may define w_τ as the unique element of W such that $\overline{J(\tau)}^{bir} \cap Bw_\tau B$ is dense in $\overline{J(\tau)}^{bir}$. It follows that for a spherical birational sheet $\overline{J(\tau)}^{bir}$, we have $w_\tau = w_{\mathcal{O}}$ for every conjugacy class $\mathcal{O} \subset \overline{J(\tau)}^{bir}$ and $w_\tau = w_S$ for every sheet S containing $\overline{J(\tau)}^{bir}$.

We state our main result.

Theorem 5.2. *Let G be a complex connected reductive algebraic group with G' simply-connected. Then the spherical birational sheets form a partition of G_{sph}. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be spherical conjugacy classes in G. Let $\overline{J(\tau_1)}^{bir}$ (resp. $\overline{J(\tau_2)}^{bir}$) be the birational sheet containing $\mathcal{O}_1$ (resp. $\mathcal{O}_2$). Then $\mathbb{C}[\mathcal{O}_1]$ is isomorphic to $\mathbb{C}[\mathcal{O}_2]$ as a G-module if and only if $\overline{J(\tau_2)}^{bir} = z\overline{J(\tau_1)}^{bir}$ for some $z \in Z(G)$.*

Since the birational sheets form a partition of G, the spherical birational sheets form a partition of G_{sph}. The remainder of this section is devoted to the proof of Theorem 5.2: it is enough to assume G simple.

>From the list of spherical conjugacy classes in G simple simply-connected [9, 14], we compute the list of spherical birational sheets in G, proceeding as follows.

If $z \in Z(G)$, then $\mathcal{O}_z = \{z\}$, $w_z = 1$ and $\mathbb{C}[\mathcal{O}_z] = \mathbb{C}$: then $\{z\}$ is the unique sheet and the unique birational sheet containing z. Therefore, we shall deal only with non-central spherical conjugacy classes.

First, we compute all spherical birational sheets containing semisimple elements. These are exactly those obtained as $\overline{J(\tau)}^{bir}$ with $\tau = (M, Z(M)^\circ s, \{1\}) \in \mathscr{BB}(G)$ and M a spherical pseudo-Levi subgroup of G. All such possible subgroups M can be deduced from [9, 14]: by inspection, we have two possibilities.

(i) If M is a spherical Levi subgroup, the birational sheet $\overline{J(\tau)}^{bir}$ is dense in the spherical sheet $\overline{J(\tau)}^{reg}$. Moreover, it turns out that $\overline{J(\tau)}^{reg} \setminus J(\tau)$ is a union of isolated classes: by checking whether each of these classes is birationally induced by means of Lemmas 3.11 and 3.12, we produce $\overline{J(\tau)}^{bir}$.

(ii) If M is a spherical pseudo-Levi subgroup which is not Levi, then M is semisimple and $\overline{J(\tau)}^{bir} = \overline{J(\tau)}^{reg} = J(\tau)$ is an isolated class.

At this point we are left with considering all non-semisimple spherical conjugacy classes which are not birationally induced as in (i). By inspecting the lists in [9, 14], these are spherical classes $\mathcal{O}_{su}$ with s semisimple isolated and $\mathcal{O}_u^{C_G(s)}$ birationally rigid: we conclude that these classes are spherical birational sheets.

As recalled in the introduction, for a spherical conjugacy class $\mathcal{O}$ the G-module structure of $\mathbb{C}[\mathcal{O}]$ is completely determined by the weight lattice $\Lambda(\mathcal{O})$. We collect the list of spherical birational sheets in a table. In the first column there is a certain $\tau = (M, Z(M)^\circ s, \mathcal{O}_u^M) \in \mathscr{BB}(G)$ with $M = C_G(s)$ and $\mathcal{O}_{su}$ spherical in G. In the second column we describe $\overline{J(\tau)}^{bir}$. From the tables in [14] we verify that the weight monoid is constant on the orbits in $\overline{J(\tau)}^{bir}$ and we describe its elements in the third column. In the cases when $Z(G)$ is non-trivial, we list also a fourth column indicating the number of (disjoint) birational sheets in $Z(G)\overline{J(\tau)}^{bir}$. This is produced by applying Remark 3.15 in all cases, except for one case where G is of type C_{2p} and M is of type $\mathsf{C}_p\mathsf{C}_p$, see Remark 5.8).

The fact that $\Lambda(\mathcal{O})$ is independent of the orbit $\mathcal{O}$ in $\overline{J(\tau)}^{bir}$ (and hence in $Z(G)\overline{J(\tau)}^{bir}$) proves the group analogue of Proposition 3.2 (for spherical conjugacy classes in G simple simply-connected). To prove the validity of the group analogue of Conjecture 3.3 one has to check that the entries in the third column are pairwise distinct.

For $k = 1, \dots, n$, we put

$$\sigma_k := \exp\left(\frac{2\pi i}{c_k}\check{\omega}_k\right); \tag{6}$$

$$\Theta_k := \Delta \setminus \{\alpha_k\} \qquad\qquad L_k := L_{\Theta_k}; \tag{7}$$

$$\widetilde{\Theta}_k := \widetilde{\Delta} \setminus \{\alpha_k\} \qquad\qquad M_k := L_{\widetilde{\Theta}_k} = C_G(\sigma_k). \tag{8}$$

We shall freely use the notation from [14]. For $K \leq G$, K simple, we will consider the isogeny $\pi_K \colon K \to \overline{K} := K_{ad}$ to the adjoint group, omitting subscripts when $K = G$.

5.1 Type A_n, $n \geq 1$

Here $G = \mathrm{SL}_{n+1}(\mathbb{C})$, for $n \geq 1$. Theorem 5.2 holds for G, as a consequence of Theorem 4.8. For the sake of completeness, we list the spherical birational sheets of G and the weight monoids of classes contained in them. Set $m = \lfloor \frac{n+1}{2} \rfloor$.

Lemma 5.3. *Let $\mathbf{d} = [d_1, \dots, d_r] \vdash n+1$ and let $L_{\mathbf{d}}$ be the standard Levi subgroup of G indexed by $\mathbf{d}$. Then $Z(L_{\mathbf{d}})$ has exactly $\gcd\{d_i \mid d_i \in \mathbf{d}\}$ connected components, pairwise not conjugate in G.*

Proof. We have $Z(L) \simeq S := \{(z_1, \dots, z_r) \in (\mathbb{C}^\times)^r \mid z_1^{d_1} \cdots z_r^{d_r} = 1\}$. If $d = \gcd\{d_i \mid i = 1, \dots, r\}$, we have $Z(L)/Z(L)^\circ \simeq S/S^\circ \simeq \mathbb{Z}/d\mathbb{Z}$. The last assertion follows from Lemma 3.8. □

For $n = 1$, every conjugacy class of G is spherical and there are three (birational) sheets: $\{-1\}, \{1\}$ and G^{reg}.

Let $n \geq 2$. Consider the Levi subgroups L_i, for all $i = 1, \dots, m$. Then $L_i' \simeq \mathrm{SL}_{n+1-i} \times \mathrm{SL}_i$, the centre $Z(L_i)$ is one-dimensional and consists of $d = \gcd(n+1-i, i) = \gcd(i, n+1)$ distinct connected components which are not conjugate in G. Let $\mathbf{d}_i = [n+1-i, i]$ and let $\tau_i := (L_i, Z(L_i)^\circ, \{1\})$, then $Z(L_i)^\circ = \exp(\mathbb{C}\check{\omega}_i)$, $Z(G) \cap Z(L_i)^\circ$ has order $\frac{n+1}{d}$ and

$$\overline{J(\tau_i)}^{bir} = \overline{J(\tau_i)}^{reg} = \bigcup_{z \in Z(L_i)^\circ} G \cdot (z \operatorname{Ind}_{L_i}^{C_G(z)}\{1\}) = \bigcup_{\zeta \in \mathbb{C}\setminus 2\pi i\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_i)} \sqcup \bigsqcup_{z \in Z(G) \cap Z(L_i)^\circ} z\mathcal{O}_{\mathbf{d}_i^t},$$

by [13, Theorem 7.2.3]. Moreover the unipotent class $\mathcal{O}_{\mathbf{d}_i^t}$ is the class denoted by X_i in [14, §4.1].

τ	$\overline{J(\tau)}^{bir}$	$\Lambda(\mathcal{O})$	d
$(L_\ell, Z(L_\ell)^\circ, \{1\})$ $\ell = 1 \dots, m-1$	$\bigcup_{\zeta \in \mathbb{C}\setminus 2\pi i\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_\ell)} \sqcup$ $\sqcup (Z(G) \cap Z(L_i)^\circ) X_\ell$	$\sum_{k=1}^{\ell} n_k(\omega_k + \omega_{n-k+1})$	$\gcd(\ell, n+1)$
$(L_m, Z(L_m), \{1\})$ $n = 2m$	$\bigcup_{\zeta \in \mathbb{C}\setminus 2\pi i\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_m)} \sqcup Z(G)X_m$	$\sum_{k=1}^{m} n_k(\omega_k + \omega_{n-k+1})$	1
$(L_m, Z(L_m)^\circ, \{1\})$ $n+1 = 2m$	$\bigcup_{\zeta \in \mathbb{C}\setminus 2\pi i\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_m)} \sqcup \pm X_m$	$\sum_{k=1}^{m-1} n_k(\omega_k + \omega_{n-k+1}) + 2n_m\omega_m$	m

Table 2: Type $\mathsf{A}_n, n \geq 1, m = \lfloor \frac{n+1}{2} \rfloor$.

5.2 Type $\mathsf{C}_n, n \geq 2$

We have $Z(G) = \langle \hat{z} \rangle$ with $\hat{z} = \prod_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \alpha_{2i-1}^{\vee}(-1)$. We set $p := \lfloor \frac{n}{2} \rfloor$.

5.2.1 Type C_2

Lemma 5.4. *Let $G = \mathrm{Sp}_4(\mathbb{C})$. Let $S_i := \overline{J_G(\tau_i)}^{reg}$ with $\tau_i := (L_i, Z(L_i)^\circ, \{1\})$ for $i = 1, 2$. Then $S_2 = \overline{J(\tau_2)}^{bir}$ is a birational sheet and $S_1 = \overline{J(\tau_1)}^{bir} \sqcup \mathcal{O}_{[2^2]}$.*

Proof. Observe that L_2 is maximal and $Z(L_2)$ is connected. Then

$$S_2 = \bigcup_{z \in Z(L_2)} G \cdot (z \operatorname{Ind}_{L_2}^{C_G(z)}\{1\}) = G \cdot ((Z(L_2))^{reg}) \sqcup \operatorname{Ind}_{L_2}^{G}\{1\} \sqcup \hat{z} \operatorname{Ind}_{L_2}^{G}\{1\}.$$

We have $\operatorname{Ind}_{L_2}^{G}\{1\} = \mathcal{O}_{[2^2]}$, and $u = x_{\beta_1}(1)x_{\beta_2}(1) \in \mathcal{O}_{[2^2]} = X_2$ satisfies $C_G(u) \leq P_{\Theta_2}$, so that $\mathcal{O}_{[2^2]}$ is birationally induced from $(L_2, \{1\})$ by Lemma 3.11 and S_2 is a birational sheet.

For S_1, observe that $L_1 < M_1 < G$, where $Z(L_1) = Z(L_1)^\circ \sqcup \hat{z}Z(L_1)^\circ$. We have

$$S_1 = \bigcup_{z \in Z(L_1)^\circ} G \cdot (z \operatorname{Ind}_{L_1}^{C_G(z)}\{1\}) = G \cdot ((Z(L_1)^\circ)^{reg}) \sqcup G \cdot (\sigma_1 \operatorname{Ind}_{L_1}^{C_1}\{1\}) \sqcup \operatorname{Ind}_{L_1}^{G}\{1\}.$$

Observe that M_1 is of type $\mathsf{A}_1\mathsf{A}_1$, so the class $\operatorname{Ind}_{L_1}^{M_1}\{1\} = \mathcal{O}_{x_\beta(1)}^{M_1}$ is birationally induced by Remark 3.13. The subregular unipotent class $\mathcal{O}_{[2^2]} = \operatorname{Ind}_{L_1}^{G}\{1\}$ is not birationally induced from $(L_1, \{1\})$, so that $\overline{J(\tau_1)}^{bir} = \bigcup_{\zeta \in \mathbb{C} \setminus \pi i \mathbb{Z}} \mathcal{O}_{\exp(\zeta \check{\omega}_1)} \sqcup \mathcal{O}_{\sigma_1 x_\beta(1)}$ and $Z(G)\overline{J(\tau_1)}^{bir} = \overline{J(\tau_1)}^{bir} \sqcup \hat{z}\overline{J(\tau_1)}^{bir}$. ☐

There is only one more spherical pseudo-Levi subgroup M_1 giving rise to the (birational) sheet $\mathcal{O}_{\sigma_1}$. Note that σ_1 and $\hat{z}\sigma_1$ are conjugate, hence $Z(G)\mathcal{O}_{\sigma_1} = \mathcal{O}_{\sigma_1}$.

Up to central elements, there is only one more spherical conjugacy classes X_1 corresponding to the partition $[2, 1^2]$: this is a birationally rigid unipotent conjugacy class in G.

τ	$\overline{J(\tau)}^{bir}$	$\Lambda(\mathcal{O})$	d
$(L_2, Z(L_2), \{1\})$	$\bigcup_{\zeta \in \mathbb{C} \setminus 2\pi i \mathbb{Z}} \mathcal{O}_{\exp(\zeta \check{\omega}_2)} \sqcup X_2 \sqcup \hat{z}X_2$	$2n_1\omega_1 + 2n_2\omega_2$	1
$(L_1, Z(L_1)^\circ, \{1\})$	$\bigcup_{\zeta \in \mathbb{C} \setminus \pi i \mathbb{Z}} \mathcal{O}_{\exp(\zeta \check{\omega}_1)} \sqcup \mathcal{O}_{\sigma_1 x_\beta(1)}$	$2n_1\omega_1 + n_2\omega_2$	2
$(M_1, \{\sigma_1\}, \{1\})$	$\mathcal{O}_{\sigma_1}$	$n_2\omega_2$	1
$(G, \{1\}, \mathcal{O}_{[2,1^2]})$	X_1	$2n_1\omega_1$	2

Table 3: Type C_2.

Remark 5.5. The subregular unipotent class $\mathcal{O}_{[2^2]}$ lies in both the sheets S_1 and S_2. This agrees with what is stated in [4, §6(c)]: $\mathcal{O}_{[2^2]}$ can be deformed in semisimple classes of both types $\mathcal{O}_{\exp(\zeta\check{\omega}_1)}$ and $\mathcal{O}_{\exp(\zeta\check{\omega}_2)}$, but in general the multiplicities of the weights can decrease. Indeed, for $\zeta \in \mathbb{C} \setminus \pi i \mathbb{Z}$, we have $\Lambda(\mathcal{O}_{\exp(\zeta\check{\omega}_1)}) = \langle 2\omega_1, \omega_2 \rangle > \langle 2\omega_1, 2\omega_2 \rangle = \Lambda(\mathcal{O}_{[2^2]}) = \Lambda(\mathcal{O}_{\exp(\zeta\check{\omega}_2)})$.

Remark 5.6. The sheet S_1 is not a union of birational sheets.

5.2.2 Type C_n, $n \geq 3$

Lemma 5.7. *Let $n \geq 3$. Then:*

(i) Let $\tau_1 = (L_1, Z(L_1)^\circ, \{1\})$; then the spherical sheet $S_1 := \overline{J(\tau_1)}^{reg}$ decomposes as the union of $\overline{J(\tau_1)}^{bir}$ and the unipotent birationally rigid class $\mathcal{O}_{\mathbf{d}}$ with $\mathbf{d} = [2^2, 1^{2(n-1)}]$; similarly for the birational sheet $\hat{z}S_1$.

(ii) Let $\tau_n = (L_n, Z(L_n), \{1\})$; then the spherical sheet $S_n := \overline{J(\tau_n)}^{reg}$ is a birational sheet containing the unipotent class $\mathcal{O}_{\mathbf{f}}$, with $\mathbf{f} = [2^n]$.

Proof. (i) L_1 is of type $\mathsf{T}_1\mathsf{C}_{n-1}$ and $L_1 < M_1 < G$ and $Z(L_1) = Z(L_1)^\circ \sqcup \hat{z}Z(L_1)^\circ$. Then

$$S_1 = \bigcup_{z \in Z(L_1)^\circ} G \cdot (z \operatorname{Ind}_{L_1}^{C_G(z)}\{1\}) = G \cdot ((Z(L_1)^\circ)^{reg}) \sqcup G \cdot (\sigma_1 \operatorname{Ind}_{L_1}^{M_1}\{1\}) \sqcup \operatorname{Ind}_{L_1}^{G}\{1\}.$$

The class $\operatorname{Ind}_{L_1}^{M_1}\{1\}$ is birationally induced, by Remark 3.13. The unipotent class $\mathcal{O}_{\mathbf{d}} = \operatorname{Ind}_{L_1}^{G}\{1\}$ is not birationally induced from $(L_1, \{1\})$, indeed it is birationally rigid by §3.4 and it coincides with a whole birational sheet. Hence

$$\overline{J(\tau_1)}^{bir} = \bigcup_{\zeta \in \mathbb{C} \setminus \pi i \mathbb{Z}} \mathcal{O}_{\exp(\zeta \check{\omega}_1)} \sqcup \mathcal{O}_{\sigma_1 x_{\beta_1}(1)}$$

and $Z(G)\overline{J(\tau_1)}^{bir} = \overline{J(\tau_1)}^{bir} \sqcup \hat{z}\overline{J(\tau_1)}^{bir}$.

(ii) L_n is maximal of type $\mathsf{T}_1\tilde{\mathsf{A}}_{\mathsf{n-1}}$ and $Z(L_n)$ is connected, as $L_n = C_G(\exp \check{\omega}_n)$ and $\exp(2\pi i \check{\omega}_n) = \hat{z}$.

$$S_n = \bigcup_{z \in Z(L_n)} G \cdot (z \operatorname{Ind}_{L_n}^{C_G(z)}\{1\}) = G \cdot (Z(L_n)^{reg}) \sqcup \operatorname{Ind}_{L_n}^{G}\{1\} \sqcup \hat{z} \operatorname{Ind}_{L_n}^{G}\{1\}.$$

We have $\operatorname{Ind}_{L_n}^{G}\{1\} = \mathcal{O}_{\mathbf{f}}$, with $u = x_{\beta_1}(1) \cdots x_{\beta_n}(1) \in \mathcal{O}_{\mathbf{f}} = X_n$ satisfies $C_G(u) \leq P_{\Theta_n}$, so that X_n is birationally induced and S_n is a birational sheet

$$\overline{J(\tau_n)}^{bir} = \overline{J(\tau_n)}^{reg} = S_n = \bigcup_{\zeta \in \mathbb{C} \setminus 2\pi i \mathbb{Z}} \mathcal{O}_{\exp(\zeta \check{\omega}_n)} \sqcup X_n \sqcup \hat{z}X_n$$

and $Z(G)\overline{J(\tau_n)}^{bir} = \overline{J(\tau_n)}^{bir}$. □

We consider the remaining spherical pseudo-Levi subgroups.

(i) For $\ell = 1, \ldots, p$, M_ℓ is maximal of type $\mathsf{C}_\ell\mathsf{C}_{n-\ell}$ and $Z(M_\ell) = \langle \sigma_\ell \rangle \times Z(G)$. Then, for $(M_\ell, \{\sigma_\ell\}, \{1\})$ we get $\mathcal{O}_{\sigma_\ell}$, a (birational) sheet consisting of an isolated class. We have $Z(G)\mathcal{O}_{\sigma_\ell} = \mathcal{O}_{\sigma_\ell} \sqcup \hat{z}\mathcal{O}_{\sigma_\ell}$ except when $n = 2p$, $\ell = p$, in which case σ_ℓ and $\hat{z}\sigma_\ell$ are G-conjugate and $Z(G)\mathcal{O}_{\sigma_p} = \mathcal{O}_{\sigma_p}$.

(ii) For $\ell = 2, \ldots, p$, the pseudo-Levi M_ℓ of type $\mathsf{C}_\ell\mathsf{C}_{n-\ell}$ admits the birationally rigid unipotent class $\mathcal{O}_{x_{\beta_1}(1)}^{M_\ell}$ of the form $[2, 1^{2\ell-2}] \times \{1\}$. Then $\mathcal{O}_{\sigma_\ell x_{\beta_1}(1)}$ is a (birational) sheet consisting of an isolated class.

(iii) For $\ell = 1, \ldots, p$, the pseudo-Levi M_ℓ of type $\mathsf{C}_\ell\mathsf{C}_{n-\ell}$ has the birationally rigid unipotent class $\mathcal{O}_{x_{\alpha_n}(1)}^{C_G(\sigma_\ell)}$ of the form $\{1\} \times [2, 1^{2(n-\ell)-2}]$. Then $\mathcal{O}_{\sigma_\ell x_{\alpha_n}(1)}$ is a (birational) sheet consisting of an isolated class.

In cases (ii) and (iii), we have $Z(G)\mathcal{O}_{\sigma_\ell x_{\beta_1}(1)} = \mathcal{O}_{\sigma_\ell x_{\beta_1}(1)} \sqcup \hat{z}\mathcal{O}_{\sigma_\ell x_{\beta_1}(1)}$ and $Z(G)\mathcal{O}_{\sigma_\ell x_{\alpha_n}(1)} = \mathcal{O}_{\sigma_\ell x_{\alpha_n}(1)} \sqcup \hat{z}\mathcal{O}_{\sigma_\ell x_{\alpha_n}(1)}$. The only case which needs further explanation is when $n = 2p$, $\ell = p$: then σ_p and $\hat{z}\sigma_p$ are G-conjugate, but $\sigma_p x_{\beta_1}$ and $\hat{z}\sigma_p x_{\beta_1}$ are not G-conjugate.

Remark 5.8. This is an example of $(M, Z^\circ s_1, \mathcal{O}^M)$, $(M, Z^\circ s_2, \mathcal{O}^M)$ in $\mathscr{BB}(G)$ with $(M, Z^\circ s_1) \sim_G (M, Z^\circ s_2)$, but $(M, Z^\circ s_1, \mathcal{O}^M) \not\sim_G (M, Z^\circ s_2, \mathcal{O}^M)$: in this case the rigid orbit $\mathcal{O}^M$ of M is not characteristic.

Up to central elements, the remaining spherical conjugacy classes in G are X_ℓ corresponding to the partition $[2^\ell, 1^{2n-2\ell}]$, for $\ell = 1, \dots, n-1$: these are all birationally rigid unipotent conjugacy classes in G, see §3.4.

τ	$\overline{J(\tau)}^{bir}$	$\Lambda(\mathcal{O})$	d
$(L_n, Z(L_n), \{1\})$	$\bigcup_{\zeta\in\mathbb{C}\setminus\pi i\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_n)} \sqcup X_n \sqcup \hat{z}X_n$	$\sum_{i=1}^{n} 2n_i\omega_i$	1
$(L_1, Z(L_1)^\circ, \{1\})$	$\bigcup_{\zeta\in\mathbb{C}\setminus\pi i\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_1)} \sqcup \mathcal{O}_{\sigma_1 x_{\beta_1}(1)}$	$2n_1\omega_1 + n_2\omega_2$	2
$(M_\ell, \{\sigma_\ell\}, \{1\})$ $\ell = 1, \dots, p-1$	$\mathcal{O}_{\sigma_\ell}$	$\sum_{i=1}^{\ell} n_{2i}\omega_{2i}$	2
$(M_p, \{\sigma_p\}, \{1\})$ if $n = 2p+1$	$\mathcal{O}_{\sigma_p}$	$\sum_{i=1}^{p} n_{2i}\omega_{2i}$	2
$(M_p, \{\sigma_p\}, \{1\})$ if $n = 2p$			1
$(M_p, \{\sigma_p\}, \mathcal{O}^{M_p}_{\{1\}\times[2,1^{2(n-p)-2}]})$	$\mathcal{O}_{\sigma_p x_{\alpha_n}(1)}$	$\sum_{i=1}^{n} n_i\omega_i \mid \sum_{i=1}^{\lfloor\frac{n+1}{2}\rfloor} n_{2i-1} \in 2\mathbb{N}$	2
$(M_\ell, \{\sigma_\ell\}, \mathcal{O}^{M_\ell}_{\{1\}\times[2,1^{2(n-\ell)-2}]})$ $\ell = 1, \dots, p-1$	$\mathcal{O}_{\sigma_\ell x_{\alpha_n}(1)}$	$\sum_{i=1}^{2\ell+1} n_i\omega_i \mid \sum_{i=1}^{\ell+1} n_{2i-1} \in 2\mathbb{N}$	2
$(M_\ell, \{\sigma_\ell\}, \mathcal{O}^{M_\ell}_{[2,1^{2\ell-2}]\times\{1\}})$ $\ell = 2, \dots, p$	$\mathcal{O}_{\sigma_\ell x_{\beta_1}(1)}$	$\sum_{i=1}^{2\ell} n_i\omega_i \mid \sum_{i=1}^{\ell} n_{2i-1} \in 2\mathbb{N}$	2
$(G, \{1\}, \mathcal{O}_{[2^\ell, 1^{2n-2\ell}]})$ $\ell = 1, \dots, n-1$	X_ℓ	$\sum_{i=1}^{\ell} 2n_i\omega_i$	2

Table 4: Type $\mathsf{C}_n, n \geq 3, p = \lfloor \frac{n}{2} \rfloor$.

5.3 Type $\mathsf{B}_n, n \geq 3$

We have $Z(G) = \langle \hat{z} \rangle$ with $\hat{z} = \alpha_n^\vee(-1)$.

The following result holds indepedently of the parity of n.

Lemma 5.9. *Let $\tau_1 = (L_1, Z(L_1), \{1\})$. Then $S_1 := \overline{J(\tau_1)}^{reg} = \overline{J(\tau_1)}^{bir}$ is a spherical birational sheet of G containing the unipotent class $\mathcal{O}_{\mathbf{d}}$, with $\mathbf{d} = [3, 1^{2n-2}]$.*

Proof. L_1 is maximal of type $\mathsf{T}_1\mathsf{B}_{n-1}$ and $Z(L_1)$ is connected since $L_1 = C_G(\exp \check{\omega}_1)$ and $\exp(2\pi i \check{\omega}_1) = \hat{z}$.

$$S_1 = \bigcup_{z \in Z(L_1)} G \cdot (z \operatorname{Ind}_{L_1}^{C_G(z)} \{1\}) = G \cdot (Z(L_1)^{reg}) \sqcup \mathcal{O}_{\mathbf{d}} \sqcup \hat{z}\mathcal{O}_{\mathbf{d}}.$$

We have $\operatorname{Ind}_{L_1}^G \{1\} = \mathcal{O}_{\mathbf{d}}$, with $\mathbf{d} = [3, 1^{2n-2}]$ and $u = x_{\gamma_1}(1) \in \mathcal{O}_{\mathbf{d}} = Z_1$, where $\gamma_1 = e_1$ is the highest short root of G, satisfies $C_G(u) \leq P_{\Theta_1}$, so that Z_1 is birationally induced and S_1 is a birational sheet. Therefore

$$S_1 = \bigcup_{\zeta \in \mathbb{C}\setminus 2\pi i\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_1)} \sqcup Z_1 \sqcup \hat{z}Z_1$$

and $Z(G)\overline{J(\tau_1)}^{bir} = \overline{J(\tau_1)}^{bir}$. □

5.3.1 Type $\mathsf{B}_{2m+1}, m \geq 1$

In this section we deal with cases $n = 2m+1$, $m \geq 1$. We set $u_k := \prod_{i=1}^{k} x_{\beta_i}(1)$ for $k = 1, \ldots, m$.

Lemma 5.10. *Let $\tau_n = (L_n, Z(L_n), \{1\})$ and let $S_n := \overline{J(\tau_n)}^{reg}$. Then $S_n = \overline{J(\tau_n)}^{bir}$ is a spherical birational sheet in G.*

Proof. L_n is of type $\mathsf{T}_1\mathsf{A}_{n-1}$ and $L_n < M_n < G$, where $Z(L_n)$ is connected since $L_n = C_G(\exp \check{\omega}_n)$ and $\sigma_n^2 = \hat{z}$. Then

$$\begin{aligned} S_n = & \bigcup_{z \in Z(L_n)} G \cdot (z \operatorname{Ind}_{L_n}^{C_G(z)}\{1\}) = \\ & = G \cdot (Z(L_n)^{reg}) \cup G \cdot (\sigma_n \operatorname{Ind}_{L_n}^{M_n}\{1\}) \cup G \cdot (\sigma_n^{-1} \operatorname{Ind}_{L_n}^{M_n}\{1\}) \cup \operatorname{Ind}_{L_n}^{G}\{1\} \cup \hat{z} \operatorname{Ind}_{L_n}^{G}\{1\}. \end{aligned}$$

We show that $\operatorname{Ind}_{L_n}^{M_n}\{1\}$ is birationally induced from $(L_n, \{1\})$. Observe that M_n is of type D_n and $\operatorname{Ind}_{L_n}^{M_n}\{1\}$ is the unipotent class corresponding to $[2^{2n-1}, 1^2]$ in $\mathrm{SO}_{2n}(\mathbb{C})$. Let $K := M_n$, for $u \in \operatorname{Ind}_{L_n}^{K}$, we have $C_{\overline{K}}(\pi_K(u))$ is connected by [12, p. 399], so the claim follows. Also $\operatorname{Ind}_{L_n}^{G}\{1\}$, the unipotent class corresponding to the partition $[3, 2^{n-1}]$, denoted by Z_{m+1} in [14], is birationally induced from $(L_n, \{1\})$. Indeed, for $u \in \operatorname{Ind}_{L_n}^{G}\{1\}$, the centralizer $C_{\overline{G}}(\pi(u))$ is connected by [12, p. 399], and we conclude. Thus S_n is a birational sheet in G. We observe moreover that $G \cdot (\sigma_n \operatorname{Ind}_{L_n}^{M_n}\{1\}) = G \cdot (\sigma_n^{-1} \operatorname{Ind}_{L_n}^{M_n}\{1\}) = \mathcal{O}_{\sigma_n u_m}$, as w_0 conjugates σ_n to its inverse and $\mathcal{O}_{u_m}^{M_n}$ is characteristic in M_n. Therefore

$$S_n = \overline{J(\tau_n)}^{bir} = \bigcup_{\zeta \in \mathbb{C} \setminus \pi i \mathbb{Z}} \mathcal{O}^{G}_{\exp(\zeta \check{\omega}_n)} \sqcup \mathcal{O}_{\sigma_n u_m} \sqcup Z_{m+1} \sqcup \hat{z} Z_{m+1}$$

and $Z(G) S_n = S_n$. □

We consider the remaining spherical pseudo-Levi subgroups:

(i) For $\ell = 2, \ldots, n$, the pseudo-Levi M_ℓ is maximal of type $\mathsf{D}_\ell\mathsf{B}_{n-\ell}$. If ℓ is even we have $\sigma_\ell^2 = 1$ and $Z(M_\ell) = \langle \sigma_\ell \rangle \times Z(G)$; if ℓ is odd we have $\sigma_\ell^2 = \hat{z}$ and $Z(M_\ell) = \langle \sigma_\ell \rangle$. In any case σ_ℓ and $\hat{z}\sigma_\ell$ are G-conjugate (via the reflection s_{e_1}). Then $\mathcal{O}_{\sigma_\ell}$ is a (birational) sheet consisting of an isolated class, and $Z(G)\mathcal{O}_{\sigma_\ell} = \mathcal{O}_{\sigma_\ell}$.

(ii) For $\ell = n$, the subgroup M_n of type D_n admits the birationally rigid unipotent class $\mathcal{O}_{u_k}^{M_n}$, corresponding in $\mathrm{SO}_{2n}(\mathbb{C})$ to the partition $[2^{2k}, 1^{2(n-2k)}]$, for $k = 1, \ldots, m-1$. Since $\sigma_n \sim_W \sigma_n^{-1}$ by s_n and $\mathcal{O}_{u_k}^{M_n}$ is characteristic, $Z(G)\mathcal{O}_{\sigma_n u_k} = \mathcal{O}_{\sigma_n u_k}$ is a (birational) sheet consisting of an isolated class.

Up to central elements, the remaining spherical conjugacy classes in G are X_ℓ corresponding to the partition $[2^{2\ell}, 1^{2n+1-4\ell}]$, for $\ell = 1, \ldots, m$ and Z_ℓ corresponding to the partition $[3, 2^{2(\ell-1)}, 1^{2n+2-4\ell}]$, for $\ell = 2, \ldots, m$: these are all birationally rigid unipotent conjugacy classes in G.

τ	$\overline{J(\tau)}^{bir}$	$\Lambda(\mathcal{O})$	d
$(L_1, Z(L_1), \{1\})$	$\bigcup_{\zeta\in\mathbb{C}\setminus 2\pi i\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_1)} \sqcup Z_1 \sqcup \hat{z}Z_1$	$2n_1\omega_1 + n_2\omega_2$	1
$(L_n, Z(L_n), \{1\})$	$\bigcup_{\zeta\in\mathbb{C}\setminus \pi i\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_n)} \sqcup$ $\sqcup \mathcal{O}_{\sigma_n u_m} \sqcup Z(G)Z_{m+1}$	$\sum_{i=1}^{n-1} n_i\omega_i + 2n_n\omega_n$	1
$(M_\ell, \{\sigma_\ell\}, \{1\})$ $\ell = 2, \dots, m$	$\mathcal{O}_{\sigma_\ell}$	$\sum_{i=1}^{2\ell-1} 2n_i\omega_i + n_{2\ell}\omega_{2\ell}$	1
$(M_\ell, \{\sigma_\ell\}, \{1\})$ $\ell = m+2, \dots, n$	$\mathcal{O}_{\sigma_\ell}$	$\sum_{i=1}^{2(n-\ell)} 2n_i\omega_i + n_{2(n-\ell)+1}\omega_{2(n-\ell)+1}$	1
$(M_{m+1}, \{\sigma_{m+1}\}, \{1\})$	$\mathcal{O}_{\sigma_{m+1}}$	$\sum_{i=1}^{n} 2n_i\omega_i$	1
$(M_n, \{\sigma_n\}, \mathcal{O}_{u_\ell}^{M_n})$ $\ell = 1, \dots, m-1$	$\mathcal{O}_{\sigma_n u_\ell}$	$\sum_{i=1}^{2\ell+1} n_i\omega_i$	1
$(G, \{1\}, \mathcal{O}_{[2^{2\ell}, 1^{2n+1-4\ell}]})$ $\ell = 1, \dots, m$	X_ℓ	$\sum_{i=1}^{\ell} n_{2i}\omega_{2i}$	2
$(G, \{1\}, \mathcal{O}_{[3, 2^{2(\ell-1)}, 1^{2n+2-4\ell}]})$ $\ell = 2, \dots, m$	Z_ℓ	$\sum_{i=1}^{2\ell} n_i\omega_i \mid \sum_{i=1}^{\ell} n_{2i-1} \in 2\mathbb{N}$	2

Table 5: Type B_n, $n = 2m+1$, $m \geq 1$.

5.3.2 Type $\mathsf{B}_{2m}, m \geq 2$

In this section we assume $n = 2m$, $m \geq 2$. We set $u_k := \prod_{i=1}^{k} x_{\beta_i}(1)$ for $k = 1, \dots, m$.

Lemma 5.11. *Let $\tau_n = (L_n, Z(L_n)^\circ, \{1\})$ and let $S_n := \overline{J(\tau_n)}^{reg}$. Then $S_n = \overline{J(\tau_n)}^{bir} \sqcup \mathcal{O}_{[3,2^{2(m-1)},1^2]}$, where $\mathcal{O}_{[3,2^{2(m-1)},1^2]}$ is a birationally rigid unipotent class in G.*

Proof. We have L_n of type $\mathsf{T}_1\mathsf{A}_{n-1}$ and $L_n < M_n < G$, where $\sigma_n^2 = 1$ and $Z(L_n) = Z(L_n)^\circ \sqcup \hat{z}Z(L_n)^\circ$ with $Z(L_n)^\circ = \exp(\mathbb{C}\check{\omega}_n)$. Then

$$S_n = \bigcup_{z\in Z(L_n)^\circ} G\cdot(z\,\mathrm{Ind}_{L_n}^{C_G(z)}\{1\}) =$$
$$= G\cdot((Z(L_n)^\circ)^{reg}) \cup G\cdot(\sigma_n\,\mathrm{Ind}_{L_n}^{M_n}\{1\}) \cup \mathrm{Ind}_{L_n}^{G}\{1\}$$

where the last two members in the union are the isolated classes in S_n. We show that $\mathrm{Ind}_{L_n}^{M_n}\{1\}$ is birationally induced. We have M_n of type D_n, and $u_m = x_{\beta_1}(1)\cdots x_{\beta_m}(1)$ is an element of $\mathrm{Ind}_{L_n}^{M_n}\{1\}$. Then if $K := M_n$, the centralizer $C_{\overline{K}}(\pi_K(u_m))$ is connected by [12, p. 399], and the claim follows.

We have $\mathrm{Ind}_{L_n}^{G}\{1\} = \mathcal{O}_{\mathbf{d}}$ with $\mathbf{d} = [3, 2^{2(m-1)}, 1^2]$ a full-member partition (see §3.4), hence $\mathcal{O}_{\mathbf{d}}$ is birationally rigid, hence not birationally induced from $(L_n, \{1\})$, and it forms a single birational sheet. Therefore

$$\overline{J(\tau_n)}^{bir} = \bigcup_{\zeta\in\mathbb{C}\setminus\pi i\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_n)} \sqcup \mathcal{O}_{\sigma_n u_m}$$

and $Z(G)\overline{J(\tau_n)}^{bir} = \overline{J(\tau_n)}^{bir} \sqcup \hat{z}\overline{J(\tau_n)}^{bir}$. Also $S_n = \overline{J(\tau_n)}^{bir} \sqcup Z_m$. □

We consider the remaining spherical pseudo-Levi subgroups:

(i) For $\ell = 2, \dots, n$, the subgroup M_ℓ is maximal of type $\mathsf{D}_\ell\mathsf{B}_{n-\ell}$. If ℓ is even we have $\sigma_\ell^2 = 1$ and $Z(L) = \langle\sigma_\ell\rangle \times Z(G)$; if ℓ is odd we have $\sigma_\ell^2 = \hat{z}$ and $Z(L) = \langle\sigma_\ell\rangle$. In any case σ_ℓ and $\hat{z}\sigma_\ell$ are G-conjugate (via the reflection s_{e_1}). Then $\mathcal{O}_{\sigma_\ell}$ is a (birational) sheet consisting of an isolated class, and $Z(G)\mathcal{O}_{\sigma_\ell} = \mathcal{O}_{\sigma_\ell}$.

(ii) For $\ell = n$, we get M_n maximal of type D_n. Then M_n admits the birationally rigid unipotent class $\mathcal{O}_{u_k}^{M_n}$, corresponding to the partition $[2^{2k}, 1^{2(n-2k)}]$ in $\mathrm{SO}_{2n}(\mathbb{C})$, for $k = 1, \dots, m-1$. Since $\sigma_n \sim_W \hat{z}\sigma_n$ via s_n and $\mathcal{O}_{u_k}^{M_n}$ is characteristic in M_n, we have that $Z(G)\mathcal{O}_{\sigma_\ell u_k} = \mathcal{O}_{\sigma_\ell u_k}$ is a (birational) sheet consisting of an isolated class, for $k = 1, \dots, m-1$.

Up to central elements, the remaining spherical conjugacy classes in G are X_ℓ corresponding to the partition $[2^{2\ell}, 1^{2n+1-4\ell}]$, for $\ell = 1, \dots, m$ and Z_ℓ corresponding to the partition $[3, 2^{2(\ell-1)}, 1^{2n+2-4\ell}]$, for $\ell = 2, \dots, m$: these are all birationally rigid unipotent conjugacy classes in G.

τ	$\overline{J(\tau)}^{bir}$	$\Lambda(\mathcal{O})$	d
$(L_1, Z(L_1), \{1\})$	$\bigcup_{\zeta\in\mathbb{C}\setminus 2\pi i\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_1)} \sqcup Z_1 \sqcup \hat{z}Z_1$	$2n_1\omega_1 + n_2\omega_2$	1
$(L_n, Z(L_n)^\circ, \{1\})$	$\bigcup_{\zeta\in\mathbb{C}\setminus \pi i\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_n)} \sqcup \mathcal{O}_{\sigma_n u_m}$	$\sum_{i=1}^{n-1} n_i\omega_i + 2n_n\omega_n$	2
$(M_\ell, \{\sigma_\ell\}, \{1\})$ $\ell = 2, \dots, m-1$	$\mathcal{O}_{\sigma_\ell}$	$\sum_{i=1}^{2\ell-1} 2n_i\omega_i + n_{2\ell}\omega_{2\ell}$	1
$(M_\ell, \{\sigma_\ell\}, \{1\})$ $\ell = m+1, \dots, n$	$\mathcal{O}_{\sigma_\ell}$	$\sum_{i=1}^{2(n-\ell)} 2n_i\omega_i + n_{2(n-\ell)+1}\omega_{2(n-\ell)+1}$	1
$(M_m, \{\sigma_m\}, \{1\})$	$\mathcal{O}_{\sigma_m}$	$\sum_{i=1}^{n} 2n_i\omega_i$	1
$(M_n, \{\sigma_n\}, \mathcal{O}_{u_\ell}^{M_n})$ $\ell = 1, \dots, m-1$	$\mathcal{O}_{\sigma_n u_\ell}$	$\sum_{i=1}^{2\ell+1} n_i\omega_i$	1
$(G, \{1\}, \mathcal{O}_{[2^{2\ell}, 1^{2n+1-4\ell}]})$ $\ell = 1, \dots, m-1$	X_ℓ	$\sum_{i=1}^{\ell} n_{2i}\omega_{2i}$	2
$(G, \{1\}, \mathcal{O}_{[2^{2m}, 1]})$	X_m	$\sum_{i=1}^{m-1} n_{2i}\omega_{2i} + 2n_n\omega_n$	2
$(G, \{1\}, \mathcal{O}_{[3, 2^{2(\ell-1)}, 1^{2n+2-4\ell}]})$ $\ell = 2, \dots, m-1$	Z_ℓ	$\sum_{i=1}^{2\ell} n_i\omega_i \mid \sum_{i=1}^{\ell} n_{2i-1} \in 2\mathbb{N}$	2
$(G, \{1\}, \mathcal{O}_{[3, 2^{2(m-1)}, 1^2]})$	Z_m	$\sum_{i=1}^{n} n_i\omega_i \mid \sum_{i=1}^{m} n_{2i-1}, n_n \in 2\mathbb{N}$	2

Table 6: Type B_n, $n = 2m$, $m \geq 2$.

5.4 Type $\mathsf{D}_n, n \geq 4$

We fix the following notation:

$$\hat{z}_1 = \sigma_1 = \alpha_{n-1}^\vee(-1)\alpha_n^\vee(-1), \quad \hat{z}_{n-1} = \sigma_{n-1}, \quad \hat{z}_n = \sigma_n.$$

Then:

- if $n = 2m$ is even, $\hat{z}_{n-1}$ and $\hat{z}_n$ are involutions and $\hat{z}_n\hat{z}_{n-1} = \hat{z}_1$; in particular,

$$\prod_{j=0}^{m-1} \alpha_{2j+1}^\vee(-1) = \begin{cases} \hat{z}_n & m \text{ even} \\ \hat{z}_{n-1} & m \text{ odd} \end{cases},$$

hence $Z(G) = \langle\hat{z}_1, \hat{z}_n\rangle = \langle\hat{z}_1, \hat{z}_{n-1}\rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

- If $n=2m+1$ is odd, $\hat{z}_n=\hat{z}_{n-1}^{-1}$ has order 4 and $\hat{z}_n^2=\hat{z}_1$, hence $Z(G)=\langle\hat{z}_n\rangle=\langle\hat{z}_{n-1}\rangle\simeq\mathbb{Z}_4$.

The following result holds for any $n\geq 4$:

Lemma 5.12. *Let $\tau_1:=(L_1,Z(L_1)^\circ,\{1\})\in\mathscr{D}(G)$. Then the sheet $S_1:=\overline{J(\tau_1)}^{reg}=\overline{J(\tau_1)}^{bir}$ is a spherical birational sheet containing the unipotent class $\mathcal{O}_{\mathbf{d}}$, with $\mathbf{d}=[3,1^{2n-3}]$.*

Proof. L_1 is maximal of type $\mathsf{T}_1\mathsf{D}_{n-1}$ and $Z(L_1)^\circ=\exp(\mathbb{C}\check{\omega}_1)$, $Z(L_1)=Z(L_1)^\circ\sqcup Z(L_1)^\circ\hat{z}_n$. We have

$$S_1=\bigcup_{z\in Z(L_1)^\circ}G\cdot(z\,\mathrm{Ind}_{L_1}^{C_G(z)}\{1\})=G\cdot((Z(L_1)^\circ)^{reg})\cup\mathrm{Ind}_{L_1}^G\{1\}\cup\hat{z}_1\,\mathrm{Ind}_{L_1}^G\{1\}.$$

Then the only unipotent isolated class in S_1 is $\mathrm{Ind}_{L_1}^G\{1\}=\mathcal{O}_{\mathbf{d}}$, where $\mathbf{d}=[3,1^{2n-3}]$ in $\mathrm{SO}_{2n}(\mathbb{C})$. We show that $\mathrm{Ind}_{L_1}^G\{1\}$ is birationally induced from $(L_1,\{1\})$. Let $u\in\mathrm{Ind}_{L_1}^G\{1\}$, the centralizer $C_{\overline{G}}(\pi(u))$ is connected by [12, p. 399], and the claim follows. Moreover the class $\mathcal{O}_{[3,1^{2n-3}]}$ is the class denoted by Z_1 in [14]. Therefore

$$S_1=\bigcup_{\zeta\in\mathbb{C}\setminus 2\pi i\mathbb{Z}}\mathcal{O}_{\exp(\zeta\check{\omega}_1)}\sqcup Z_1\sqcup\hat{z}_1Z_1$$

and $Z(G)S_1=S_1\sqcup\hat{z}_nS_1$ □

5.4.1 Type D_{2m}, $m\geq 2$

Let ϑ denote the graph automorphism of G swapping α_{n-1} and α_n.

Lemma 5.13. *The following spherical sheets of G are spherical birational sheets.*

(i) $S_n:=\overline{J(\tau_n)}^{reg}=\overline{J(\tau_n)}^{bir}$, where $\tau_n:=(L_n,Z(L_n)^\circ,\{1\})$;

(ii) $S_{n-1}:=\vartheta(S_n)=\overline{J(\tau_{n-1})}^{reg}=\overline{J(\tau_{n-1})}^{bir}$, where $\tau_{n-1}:=(L_{n-1},Z(L_{n-1})^\circ,\{1\})$,

Proof. Consider L_n of type $\mathsf{T}_1\mathsf{A}_{n-1}$ and maximal; moreover, $Z(L_n)^\circ=\exp(\mathbb{C}\check{\omega}_n)$ and $Z(L_n)=Z(L_n)^\circ\sqcup Z(L_n)^\circ\hat{z}_1$. We have

$$S_n=\bigcup_{z\in Z(L_n)^\circ}G\cdot(z\,\mathrm{Ind}_{L_n}^{C_G(z)}\{1\})=G\cdot((Z(L_n)^\circ)^{reg})\cup\mathrm{Ind}_{L_n}^G\{1\}\cup\hat{z}_n\,\mathrm{Ind}_{L_n}^G\{1\}.$$

Let $\mathcal{O}=\mathrm{Ind}_{L_n}^G\{1\}$, then $\mathcal{O}$ is one of the two unipotent classes corresponding to the very even partition $[2^n]$, the one denoted by X_m in [14]. We show that $\mathrm{Ind}_{L_n}^G\{1\}$ is birationally induced from $(L_n,\{1\})$. Let $u\in\mathrm{Ind}_{L_n}^G\{1\}$, then $C_{\overline{G}}(\pi(u))$ is connected by [12, p. 399], and the claim follows.

Therefore

$$S_n=\bigcup_{\zeta\in\mathbb{C}\setminus 2\pi i\mathbb{Z}}\mathcal{O}_{\exp(\zeta\check{\omega}_n)}\sqcup X_m\sqcup\hat{z}_nX_m$$

is a spherical birational sheet and $Z(G)S_n=S_n\sqcup\hat{z}_1S_n$. By applying the automorphism ϑ we get

$$S_{n-1}=\bigcup_{\zeta\in\mathbb{C}\setminus 2\pi i\mathbb{Z}}\mathcal{O}_{\exp(\zeta\check{\omega}_{n-1})}\sqcup X_m'\sqcup\hat{z}_{n-1}X_m'$$

and $Z(G)S_{n-1}=S_{n-1}\sqcup\hat{z}_1S_{n-1}$. □

We consider the remaining spherical pseudo-Levi subgroups. For $\ell = 2, \ldots, m$, the subgroup M_ℓ is maximal of type $\mathsf{D}_\ell\mathsf{D}_{n-\ell}$. If ℓ is even we have $\sigma_\ell^2 = 1$ and $Z(M_\ell) = \langle \sigma_\ell \rangle \times Z(G)$; if ℓ is odd we have $\sigma_\ell^2 = \hat{z}_1$ and $Z(M_\ell) = \langle \sigma_\ell \rangle \times \langle z_n \rangle$. For $\ell = 2, \ldots, m-1$, σ_ℓ is not G-conjugate to $\hat{z}_n\sigma_\ell$ and $\hat{z}_{n-1}\sigma_\ell$, as one can see by passing to $\mathrm{SO}_{2n}(\mathbb{C})$. On the other hand, for $\ell = 2, \ldots, m$, ω_ℓ is W-conjugate to $\omega_\ell - 2\omega_1$, therefore σ_ℓ is G-conjugate to $\hat{z}_1\sigma_\ell$. Moreover, ω_m is W-conjugate to $\omega_m - 2\omega_1$, $\omega_m - 2\omega_{n-1}$, $\omega_m - 2\omega_n$, hence σ_m is G-conjugate to $\hat{z}_1\sigma_m$, $\hat{z}_{n-1}\sigma_m$ and $\hat{z}_n\sigma_m$. Then $\mathcal{O}_{\sigma_\ell}$ is a (birational) sheet consisting of an isolated class, and $Z(G)\mathcal{O}_{\sigma_\ell} = \mathcal{O}_{\sigma_\ell} \sqcup \hat{z}_n\mathcal{O}_{\sigma_\ell}$ for $\ell = 2, \ldots, m-1$, whereas $Z(G)\mathcal{O}_{\sigma_m} = \mathcal{O}_{\sigma_m}$.

Up to central elements, the remaining spherical conjugacy classes in G are X_ℓ corresponding to the partition $[2^{2\ell}, 1^{2n-4\ell}]$, for $\ell = 1, \ldots, m-1$ and Z_ℓ corresponding to the partition $[3, 2^{2(\ell-1)}, 1^{2n-4\ell+1}]$, for $\ell = 2, \ldots, m$: these are all birationally rigid unipotent conjugacy classes in G.

τ	$\overline{J(\tau)}^{bir}$	$\Lambda(\mathcal{O})$	d
$(L_1, Z(L_1)^\circ, \{1\})$	$\bigcup_{\zeta \in \mathbb{C}\setminus 2\pi i\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_1)} \sqcup Z_1 \sqcup \hat{z}_1 Z_1$	$2n_1\omega_1 + n_2\omega_2$	2
$(L_n, Z(L_n)^\circ, \{1\})$	$\bigcup_{\zeta \in \mathbb{C}\setminus 2\pi i\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_n)} \sqcup X_m \sqcup \hat{z}_n X_m$	$\sum_{i=1}^{m-1} n_{2i}\omega_{2i} + 2n_n\omega_n$	2
$(L_{n-1}, Z(L_{n-1})^\circ, \{1\})$	$\bigcup_{\zeta \in \mathbb{C}\setminus 2\pi i\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_{n-1})} \sqcup X'_m \sqcup \hat{z}_{n-1} X'_m$	$\sum_{i=1}^{m-1} n_{2i}\omega_{2i} + 2n_{n-1}\omega_{n-1}$	2
$(M_\ell, \{\sigma_\ell\}, \{1\})$ $\ell = 2, \ldots, m-1$	$\mathcal{O}_{\sigma_\ell}$	$\sum_{i=1}^{2\ell-1} 2n_i\omega_i + n_{2\ell}\omega_{2\ell}$	2
$(M_m, \{\sigma_m\}, \{1\})$	$\mathcal{O}_{\sigma_m}$	$\sum_{i=1}^{n} 2n_i\omega_i$	1
$(G, \{1\}, \mathcal{O}_{[2^{2\ell}, 1^{2n-4\ell}]})$ $\ell = 1, \ldots, m-1$	X_ℓ	$\sum_{i=1}^{\ell} n_{2i}\omega_{2i}$	4
$(G, \{1\}, \mathcal{O}_{[3, 2^{2(\ell-1)}, 1^{2n-4\ell+1}]})$ $\ell = 2, \ldots, m-1$	Z_ℓ	$\sum_{i=1}^{2\ell} n_i\omega_i \mid \sum_{i=1}^{\ell} n_{2i-1} \in 2\mathbb{N}$	4
$(G, \{1\}, \mathcal{O}_{[3, 2^{2(m-1)}, 1]})$	Z_m	$\sum_{i=1}^{n} n_i\omega_i \mid$ $\sum_{i=1}^{m} n_{2i-1}, n_{n-1} + n_n \in 2\mathbb{N}$	4

Table 7: Type $\mathsf{D}_n, n = 2m, m \geq 2$.

5.4.2 Type D_{2m+1}, $m \geq 2$

Lemma 5.14. *Let $\tau_n = (L_n, Z(L_n), \{1\}) \in \mathscr{D}(G)$. The spherical sheet $S_n := \overline{J(\tau_n)}^{reg}$ is a birational spherical sheet, containing the unipotent class $\mathcal{O}_{\mathbf{d}}$, with $\mathbf{d} = [2^{n-1}, 1^2]$.*

Proof. Consider L_n: it is maximal of type $\mathsf{T}_1\mathsf{A}_{n-1}$ and $Z(L_n) = \exp(\mathbb{C}\check{\omega}_n)$ is connected. We have:

$$\begin{aligned} S_n = &\bigcup_{z \in Z(L_n)} G \cdot (z \operatorname{Ind}_{L_n}^{C_G(z)}\{1\}) = \\ &= G \cdot (Z(L_n)^{reg}) \cup \operatorname{Ind}_{L_n}^G\{1\} \cup \hat{z}_1 \operatorname{Ind}_{L_n}^G\{1\} \cup \hat{z}_{n-1} \operatorname{Ind}_{L_n}^G\{1\} \cup \hat{z}_n \operatorname{Ind}_{L_n}^G\{1\}. \end{aligned}$$

Let $\mathcal{O} = \operatorname{Ind}_{L_n}^G\{1\}$, then $\mathcal{O}$ is the unipotent class corresponding to the partition $[2^{n-1}, 1^2]$ in $\mathrm{SO}_{2n}(\mathbb{C})$, the unipotent class denoted by X_m in [14]. We show that $\operatorname{Ind}_{L_n}^G\{1\}$ is birationally induced from $(L_n, \{1\})$. Let $u \in \operatorname{Ind}_{L_n}^G\{1\}$, then $C_{\overline{G}}(\pi(u))$ is connected by [12, p. 399], and the claim follows. Therefore

$$S_n = \bigcup_{\zeta \in \mathbb{C}\setminus 2\pi i\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_n)} \sqcup X_m \sqcup \hat{z}_1 X_m \sqcup \hat{z}_{n-1} X_m \sqcup \hat{z}_n X_m$$

is a spherical birational sheet and $Z(G)S_n = S_n$ □

We consider the remaining spherical pseudo-Levi subgroups. For $\ell = 2, \ldots, m$, M_ℓ is maximal of type $\mathsf{D}_\ell \mathsf{D}_{n-\ell}$. If ℓ is even we have $\sigma_\ell^2 = 1$ and $Z(M_\ell) = \langle \sigma_\ell \rangle \times Z(G) \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$; if ℓ is odd we have $\sigma_\ell^2 = \hat{z}_1$ and $Z(M_\ell) = \langle \sigma_\ell, z_n \rangle \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$. For $\ell = 2, \ldots, m$, σ_ℓ is not G-conjugate to $\hat{z}_n \sigma_\ell$ (and $\hat{z}_{n-1}\sigma_\ell$), as one can see by passing to $\mathrm{SO}_{2n}(\mathbb{C})$. On the other hand, for $\ell = 2, \ldots, m$, ω_ℓ is W-conjugate to $\omega_\ell - 2\omega_1$ and therefore σ_ℓ is G-conjugate to $\hat{z}_1\sigma_\ell$. Then $\mathcal{O}_{\sigma_\ell}$ is a (birational) sheet consisting of an isolated class, and $Z(G)\mathcal{O}_{\sigma_\ell} = \mathcal{O}_{\sigma_\ell} \cup \hat{z}_n \mathcal{O}_{\sigma_\ell}$ for $\ell = 2, \ldots, m$.

Up to central elements, the remaining spherical conjugacy classes in G are X_ℓ corresponding to the partition $[2^{2\ell}, 1^{2n-4\ell}]$, for $\ell = 1, \ldots, m-1$ and Z_ℓ corresponding to the partition $[3, 2^{2(\ell-1)}, 1^{2n+1-4\ell}]$, for $\ell = 2, \ldots, m$: these are all birationally rigid unipotent conjugacy classes in G.

τ	$\overline{J(\tau)}^{bir}$	$\Lambda(\mathcal{O})$	d
$(L_1, Z(L_1)^\circ, \{1\})$	$\bigcup_{\zeta\in\mathbb{C}\setminus 2\pi i\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_1)} \sqcup Z_1 \sqcup \hat{z}_1 Z_1$	$2n_1\omega_1 + n_2\omega_2$	2
$(L_n, Z(L_n), \{1\})$	$\bigcup_{\zeta\in\mathbb{C}\setminus 2\pi i\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_n)} \sqcup X_m \sqcup$ $\sqcup \hat{z}_1 X_m \sqcup \hat{z}_{n-1} X_m \sqcup \hat{z}_n X_m$	$\sum_{i=1}^{m-1} n_{2i}\omega_{2i} + n_{n-1}(\omega_{n-1}+\omega_n)$	1
$(M_\ell, \{\sigma_\ell\}, \{1\})$ $\ell = 2, \ldots, m-1$	$\mathcal{O}_{\sigma_\ell}$	$\sum_{i=1}^{2\ell-1} 2n_i\omega_i + n_{2\ell}\omega_{2\ell}$	2
$(M_m, \{\sigma_m\}, \{1\}$	$\mathcal{O}_{\sigma_m}$	$\sum_{i=1}^{n-2} 2n_i\omega_i + n_{n-1}(\omega_{n-1}+\omega_n)$	2
$(G, \{1\}, \mathcal{O}_{[2^{2\ell}, 1^{2n-4\ell}]})$ $\ell = 1, \ldots, m-1$	X_ℓ	$\sum_{i=1}^{\ell} n_{2i}\omega_{2i}$	4
$(G, \{1\}, \mathcal{O}_{[3, 2^{2(\ell-1)}, 1^{2n-4\ell+1}]})$ $\ell = 2, \ldots, m-1$	Z_ℓ	$\sum_{i=1}^{2\ell} n_i\omega_i \mid \sum_{i=1}^{\ell} n_{2i-1} \in 2\mathbb{N}$	4
$(G, \{1\}, \mathcal{O}_{[3, 2^{2(m-1)}, 1^3]})$	Z_m	$\sum_{i=1}^{n-2} n_i\omega_i + n_{n-1}(\omega_{n-1}+\omega_n) \mid$ $\sum_{i=1}^{m} n_{2i-1} \in 2\mathbb{N}$	4

Table 8: Type $\mathsf{D}_n, n = 2m+1, m \geq 2$.

5.5 Type E_6

We have $Z(G) = \langle \hat{z} \rangle$, $\hat{z} = \alpha_1^\vee(\xi)\alpha_6^\vee(\xi^{-1})\alpha_3^\vee(\xi^{-1})\alpha_5^\vee(\xi)$ where ξ is a primitive third root of 1.

Lemma 5.15. *Let $\tau_1 = (L_1, Z(L_1), \{1\})$. Then the spherical sheet $S_1 := \overline{J(\tau_1)}^{reg}$ is a spherical birational sheet containing the unipotent class $2A_1$.*

Proof. L_1 is maximal of type $\mathsf{D}_5\mathsf{T}_1$ and $Z(L_1)$ is connected since $L_1 = C_G(\exp\check{\omega}_1)$ and $\exp(2\pi i\check{\omega}_1) = \hat{z}$. Then

$$S_1 = \bigcup_{z\in Z(L_1)} G\cdot(z\,\mathrm{Ind}_{L_1}^G\{1\}) = \bigcup_{\zeta\in\mathbb{C}\setminus 2\pi i\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_1)} \cup Z(G)\,\mathrm{Ind}_{L_1}^G\{1\}.$$

The class $\mathrm{Ind}_{L_1}^{G}\{1\} = 2A_1$ is birationally induced from $(L_1, \{1\})$ by Lemma 3.12. Indeed, for $u \in \mathrm{Ind}_{L_1}^{G}\{1\}$, the subgroup $C_{\overline{G}}(\bar{u})$ is connected, by [12, p. 402]. Hence

$$S_1 = \overline{J(\tau_1)}^{bir} = \bigcup_{\zeta \in \mathbb{C}\setminus 2\pi i\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_1)} \sqcup Z(G)2A_1$$

and $Z(G)\overline{J(\tau_1)}^{bir} = \overline{J(\tau_1)}^{bir}$. $\square$

There is only one more spherical pseudo-Levi subgroup, M_2 of type $\mathsf{A}_1\mathsf{A}_5$. Observe that $\sigma_2^2 = 1$ and $Z(M_2) = \langle\sigma_2\rangle \times Z(G)$. M_2 gives rise to the (birational) sheet $\mathcal{O}_{\sigma_2}$ which coincides with an isolated class. We have $Z(G)\mathcal{O}_{\sigma_2} = \mathcal{O}_{\sigma_2} \sqcup \hat{z}\mathcal{O}_{\sigma_2} \sqcup \hat{z}^2\mathcal{O}_{\sigma_2}$. Up to central elements, the remaining spherical conjugacy classes in G are A_1 and $3A_1$: these are birationally rigid unipotent conjugacy classes in G.

τ	$\overline{J(\tau)}^{bir}$	$\Lambda(\mathcal{O})$	d
$(L_1, Z(L_1), \{1\})$	$\bigcup_{\zeta\in\mathbb{C}\setminus 2\pi i\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_1)} \sqcup Z(G)2A_1$	$n_1(\omega_1+\omega_6)+n_2\omega_2$	1
$(M_2, \{\sigma_2\}, \{1\})$	$\mathcal{O}_{\sigma_2}$	$n_1(\omega_1+\omega_6)+n_3(\omega_3+\omega_5)+2n_2\omega_2+2n_4\omega_4$	3
$(G, \{1\}, A_1)$	A_1	$n_2\omega_2$	3
$(G, \{1\}, 3A_1)$	$3A_1$	$n_1(\omega_1+\omega_6)+n_3(\omega_3+\omega_5)+n_2\omega_2+n_4\omega_4$	3

Table 9: Type E_6.

5.6 Type E_7

We have $Z(G) = \langle\hat{z}\rangle$, $\hat{z} = \alpha_2^\vee(-1)\alpha_5^\vee(-1)\alpha_7^\vee(-1)$.

Lemma 5.16. *Let $\tau_7 = (L_7, Z(L_7), \{1\})$. Then the spherical sheet $S_7 := \overline{J(\tau_7)}^{reg}$ is a spherical birational sheet containing the unipotent class $(3A_1)''$.*

Proof. L_7 is maximal of type $\mathsf{E}_6\mathsf{T}_1$ and $Z(L_7)$ is connected since $L_7 = C_G(\exp\check{\omega}_7)$ and $\exp(2\pi i\check{\omega}_7) = \hat{z}$. Then

$$S_7 = \bigcup_{z\in Z(L_7)} G\cdot(z\,\mathrm{Ind}_{L_7}^{G}\{1\}) = \bigcup_{\zeta\in\mathbb{C}\setminus 2\pi i\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_7)} \cup Z(G)\,\mathrm{Ind}_{L_7}^{G}\{1\}.$$

The isolated class $\mathrm{Ind}_{L_7}^{G}\{1\} = (3A_1)''$ is birationally induced: for $u \in \mathrm{Ind}_{L_7}^{G}\{1\}$, the group $C_{\overline{G}}(\pi(u))$ is connected, by [12, p. 403]. Hence

$$S_7 = \overline{J(\tau_7)}^{bir} = \bigcup_{\zeta\in\mathbb{C}\setminus 2\pi i\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_7)} \sqcup Z(G)(3A_1)''$$

and $Z(G)\overline{J(\tau_7)}^{bir} = \overline{J(\tau_7)}^{bir}$. $\square$

We consider the remaining spherical pseudo-Levi subgroups:

(i) Consider M_2, maximal of type A_7. We have $\sigma_2^2 = \hat{z}$ and $Z(M_2) = \langle\sigma_2\rangle$, σ_2 and $\hat{z}\sigma_2 = \sigma_2^{-1}$ are G-conjugate via w_0. Then $\mathcal{O}_{\sigma_2}$ is a (birational) sheet consisting of an isolated class, and $Z(G)S = S$.

(ii) Consider M_1, maximal of type $\mathsf{D}_6\mathsf{A}_1$. We have $\sigma_1^2 = 1$ and $Z(M_1) = \langle\hat{z}, \sigma_1\rangle$, σ_1 and $\hat{z}\sigma_1$ are not G-conjugate (in fact G has 2 classes of non-central involutions: $\mathcal{O}_{\sigma_1}$ and $\mathcal{O}_{\hat{z}\sigma_1}$). Then $\mathcal{O}_{\sigma_1}$ is a (birational) sheet consisting of an isolated class, and $Z(G)\mathcal{O}_{\sigma_1} = \mathcal{O}_{\sigma_1} \sqcup \hat{z}\mathcal{O}_{\sigma_1}$.

Up to central elements, the remaining spherical conjugacy classes in G are A_1, $2A_1$, $(3A_1)'$ and $4A_1$: these are birationally rigid unipotent conjugacy classes in G.

τ	$\overline{J(\tau)}^{bir}$	$\Lambda(\mathcal{O})$	d
$(L_7, Z(L_7), \{1\})$	$\bigcup_{\zeta\in\mathbb{C}\setminus 2\pi i\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_7)} \sqcup Z(G)(3A_1)''$	$n_1\omega_1 + n_6\omega_6 + 2n_7\omega_7$	1
$(M_1, \{\sigma_1\}, \{1\})$	$\mathcal{O}_{\sigma_1}$	$2n_1\omega_1 + 2n_3\omega_3 + n_4\omega_4 + n_6\omega_6$	2
$(M_2, \{\sigma_2\}, \{1\})$	$\mathcal{O}_{\sigma_2}$	$\sum_{i=1}^{7} 2n_i\omega_i$	1
$(G, \{1\}, A_1)$	A_1	$n_1\omega_1$	2
$(G, \{1\}, 2A_1)$	$2A_1$	$n_1\omega_1 + n_6\omega_6$	2
$(G, \{1\}, (3A_1)')$	$(3A_1)'$	$n_1\omega_1 + n_3\omega_3 + n_4\omega_4 + n_6\omega_6$	2
$(G, \{1\}, 4A_1)$	$4A_1$	$\sum_{i=1}^{7} n_i\omega_i \mid n_2 + n_5 + n_7$ even	2

Table 10: Type E_7.

5.7 Type E_8

There are no spherical proper Levi subgroups. We list the spherical pseudo-Levi subgroups.

(i) Consider M_8, maximal of type $\mathsf{A}_1\mathsf{E}_7$. We have $\sigma_8^2 = 1$ and $Z(M_8) = \langle\sigma_8\rangle$. Then $\mathcal{O}_{\sigma_8}$ is a (birational) sheet consisting of an isolated class.

(ii) Consider M_1, maximal of type D_8. We have $\sigma_1^2 = 1$ and $Z(M_1) = \langle\sigma_1\rangle$. Then $\mathcal{O}_{\sigma_1}$ is a (birational) sheet consisting of an isolated class.

The remaining spherical conjugacy classes in G are A_1, $2A_1$, $3A_1$ and $4A_1$: these are birationally rigid unipotent conjugacy classes in G.

τ	$\overline{J(\tau)}^{bir}$	$\Lambda(\mathcal{O})$
$(M_8, \{\sigma_8\}, \{1\})$	$\mathcal{O}_{\sigma_8}$	$n_1\omega_1 + n_6\omega_6 + 2n_7\omega_7 + 2n_8\omega_8$
$(M_1, \{\sigma_1\}, \{1\})$	$\mathcal{O}_{\sigma_1}$	$\sum_{i=1}^{8} 2n_i\omega_i$
$(G, \{1\}, A_1)$	A_1	$n_8\omega_8$
$(G, \{1\}, 2A_1)$	$2A_1$	$n_1\omega_1 + n_8\omega_8$
$(G, \{1\}, 3A_1)$	$3A_1$	$n_1\omega_1 + n_6\omega_6 + n_7\omega_7 + n_8\omega_8$
$(G, \{1\}, 4A_1)$	$4A_1$	$\sum_{i=1}^{8} n_i\omega_i$

Table 11: Type E_8.

5.8 Type F_4

There are no spherical proper Levi subgroups. We list the spherical pseudo-Levi subgroups.

(i) Consider M_1, maximal of type $\mathsf{A}_1\mathsf{C}_3$. We have $\sigma_1^2 = 1$ and $Z(M_1) = \langle\sigma_1\rangle$. Then $\mathcal{O}_{\sigma_1}$ is a (birational) sheet consisting of an isolated class.

(ii) Consider M_4, maximal of type B_4. We have $\sigma_4^2 = 1$ and $Z(M_4) = \langle\sigma_4\rangle$. Then $\mathcal{O}_{\sigma_4}$ is a (birational) sheet consisting of an isolated class.

(iii) M_4 admits the birationally rigid unipotent class $\mathcal{O}^{M_4}_{x_{\beta_1}(1)}$, corresponding to the partition $[2^2, 1^5]$ in $\mathrm{SO}_9(\mathbb{C})$. Then $\mathcal{O}_{\sigma_4 x_{\beta_1}(1)}$ is a (birational) sheet consisting of an isolated class.

The remaining spherical conjugacy classes in G are A_1, $\widetilde{A}_1$ and $A_1 + \widetilde{A}_1$: these are birationally rigid unipotent conjugacy classes in G.

τ	$\overline{J(\tau)}^{bir}$	$\Lambda(\mathcal{O})$
$(M_4, \{\sigma_4\}, \{1\})$	$\mathcal{O}_{\sigma_4}$	$n_4\omega_4$
$(M_1, \{\sigma_1\}, \{1\})$	$\mathcal{O}_{\sigma_1}$	$\sum_{i=1}^{4} 2n_i\omega_i$
$(M_4, \{\sigma_4\}, \mathcal{O}^{M_4}_{[2^2,1^5]})$	$\mathcal{O}_{\sigma_4 x_{\beta_1}(1)}$	$\sum_{i=1}^{4} n_i\omega_i$
$(G, \{1\}, A_1)$	A_1	$n_1\omega_1$
$(G, \{1\}, \widetilde{A}_1)$	$\widetilde{A}_1$	$n_1\omega_1 + 2n_4\omega_4$
$(G, \{1\}, A_1 + \widetilde{A}_1)$	$A_1 + \widetilde{A}_1$	$n_1\omega_1 + n_2\omega_2 + 2n_3\omega_3 + 2n_4\omega_4$

Table 12: Type F_4.

5.9 Type G_2

There are no spherical proper Levi subgroups. We list the spherical pseudo-Levi subgroups.

(i) Consider M_2, maximal of type $\mathsf{A}_1\tilde{\mathsf{A}}_1$. We have $\sigma_2^2 = 1$ and $Z(M_2) = \langle\sigma_2\rangle$. Then $\mathcal{O}_{\sigma_2}$ is a (birational) sheet consisting of an isolated class.

(ii) Consider M_1, maximal of type A_2. We have $\sigma_1^3 = 1$ and $Z(M_1) = \langle\sigma_1\rangle$; moreover, σ_1 and σ_1^{-1} are G-conjugate. Then $\mathcal{O}_{\sigma_1}$ is a (birational) sheet consisting of an isolated class.

The remaining spherical conjugacy classes in G are A_1, $\widetilde{A}_1$: these are birationally rigid unipotent conjugacy classes in G.

τ	$\overline{J(\tau)}^{bir}$	$\Lambda(\mathcal{O})$
$(M_2, \{\sigma_2\}, \{1\})$	$\mathcal{O}_{\sigma_2}$	$2n_1\omega_1 + 2n_2\omega_2$
$(M_1, \{\sigma_1\}, \{1\})$	$\mathcal{O}_{\sigma_1}$	$n_1\omega_1$
$(G, \{1\}, A_1)$	A_1	$n_2\omega_2$
$(G, \{1\}, \widetilde{A}_1)$	$\widetilde{A}_1$	$n_1\omega_1 + 2n_2\omega_2$

Table 13: Type G_2.

Proof of Theorem 5.2. From the tables in [14], for each group G the weight monoid is preserved along the classes in each $Z(G)\overline{J(\tau)}^{bir}$. On the other hand, the entries in the third column of the tables in §5 are pairwise distinct, except for one case in Table 4, with $n = 2p$ for $p \in \mathbb{N}, p \geq 2$:

$$\tau_1 = (M_p, \{\sigma_p\}, \mathcal{O}^{M_p}_{\{1\}\times[2,1^{n-2}]}) = (M_p, \{\sigma_p\}, \mathcal{O}^{M_p}_{x_{\alpha_n}(1)})$$

$$\tau_2 = (M_p, \{\sigma_p\}, \mathcal{O}^{M_p}_{[2,1^{n-2}]\times\{1\}}) = (M_p, \{\sigma_p\}, \mathcal{O}^{M_p}_{x_{\beta_1}(1)}).$$

In this case σ_p and $\hat{z}\sigma_p$ are G-conjugate, and so are $\sigma_p x_{\alpha_n}(1)$ and $\hat{z}\sigma_p x_{\beta_1}(1)$. Therefore τ_1 and $(M_p, \{\hat{z}\sigma_p\}, \mathcal{O}^{M_p}_{x_{\beta_1}(1)})$ are G-conjugate, i.e. $\overline{J(\tau_1)}^{bir} = \hat{z}\overline{J(\tau_2)}^{bir}$. ☐

We conclude this Section with another characterization of spherical birational sheets up to central elements. If H is a spherical subgroup of G, by [6, Theorem 1], there exists a flat deformation of G/H to a homogeneous spherical space G/H_0, where H_0 contains a maximal unipotent subgroup of G: such an homogeneous space is called *horospherical*, and H_0 a *horospherical contraction* of H, see also [28]. Moreover, if G/H is (isomorphic to) a conjugacy class, then $\mathbb{C}[G/H] \simeq_G \mathbb{C}[G/H_0]$, see [14, Theorem 3.15].

Proposition 5.17. *Let G be a complex connected reductive algebraic group with G' simply-connected. Let $x_1, x_2 \in G_{sph}$. Then $\mathcal{O}_{x_1}$ and $\mathcal{O}_{x_2}$ are contained in the same birational sheet up to a central element if and only if $C_G(x_1)$ and $C_G(x_2)$ have the same horospherical contraction.*

Proof. Let $x \in G_{sph}$ and $H = C_G(x)$. We recall the description of the horospherical contraction H_0 of H containing U from [14, Corollary 3.8]. Let w be the unique element in W such that $\mathcal{O}_x \cap BwB$ is dense in $\mathcal{O}_x$. By choosing $x \in wB$, the dense B-orbit in $\mathcal{O}_x$ is $\mathcal{O}_x^B$. Then $P := \{g \in G \mid g \cdot \mathcal{O}_x^B = \mathcal{O}_x^B\}$ is a parabolic subgroup containing B. Let $\Theta \subseteq \Delta$ be such that $P = P_\Theta$. One has $H_0 = \langle U^-, U_{w_\Theta}, T_x \rangle$, where, $w := w_0 w_\Theta$, $U_{w_\Theta} := U \cap L_\Theta$, $T_x := T \cap C_G(x)$.

We may assume that x_i lies in the dense B-orbit $\mathcal{O}_{x_i}^B$ $(\subseteq Bw_iB)$, for $i = 1, 2$. We have seen that $\mathcal{O}_{x_1}$ and $\mathcal{O}_{x_1}$ are contained in the same birational sheet up to a central element if and only if $\Lambda(\mathcal{O}_{x_1}) = \Lambda(\mathcal{O}_{x_2})$. The last equality is equivalent to $w_1 = w_2$ and $T_{x_1} = T_{x_2}$ by [14, Lemma 3.9, Theorem 3.23]. □

Remark 5.18. From the classification it follows that the birationally rigid unipotent conjugacy class $\mathcal{O}^M$ appearing in the decomposition datum $\tau = (M, Z(M)^\circ z, \mathcal{O}^M)$ is in fact rigid, except in the cases

(i) $(G, \{1\}, X_2)$ in type C_n, $n \geq 3$;

(ii) $(G, \{1\}, Z_m)$ in type B_{2m}, $m \geq 2$.

In the first (resp. second) case $\overline{J(\tau)}^{bir}$ is contained only in the (spherical) sheet corresponding to $(L_1, Z(L_1)^\circ, \{1\})$ (resp. $(L_n, Z(L_n)^\circ, \{1\})$).

In the other cases $\overline{J(\tau)}^{bir}$ is contained only in the sheet $\overline{J(\tau)}^{reg}$: in particular every spherical birational sheet is contained in a unique sheet.

6 Remarks for Lie algebras

By [2, Proposition 1], the subset $\mathfrak{g}_{sph}$ consisting of spherical ajoint orbits is a union of sheets. Since every birational sheet is contained in a sheet, $\mathfrak{g}_{sph}$ is a union of spherical birational sheets. Being birational sheets disjoint, $\mathfrak{g}_{sph}$ is a disjoint union of spherical birational sheets.

Having described the spherical birational sheets in G, from the tables in §5 one can easily deduce the corresponding classification of spherical birational sheets in $\mathfrak{g}$. In each table we have a spherical birational sheet $\overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}^{bir}$ for each $\tau = (L, Z(L)^\circ, \mathcal{O}^L)$ with L a Levi subgroup of G: here $\mathfrak{l} := \mathrm{Lie}(L)$ and $\mathfrak{O}^L$ is the nilpotent orbit in $\mathfrak{l}$ corresponding to $\mathcal{O}^L$. Moreover, if $L = G$, then $\overline{\mathfrak{J}(\mathfrak{g}, \mathfrak{O})}^{bir} = \mathfrak{O}$; if $L \lneq G$, then L is a maximal Levi subgroup L_i of G and $\mathcal{O}^{L_i} = \{1\}$ so that $\mathfrak{O}^{L_i} = \{0\}$. There are two possibilities: either $\overline{J(\tau)}^{bir}$ does not contain unipotent conjugacy classes, or it contains a unique unipotent conjugacy class $\mathcal{O}_u$. In this case, let $\mathfrak{O}_\nu$ be the corresponding nilpotent orbit in $\mathfrak{g}$. Then we have $\overline{\mathfrak{J}(\mathfrak{l}_i, \{0\})}^{bir} = \bigcup_{\zeta \neq 0} \mathfrak{O}_{\zeta \check{\omega}_i}$ in the first case and $\overline{\mathfrak{J}(\mathfrak{l}_i, \{0\})}^{bir} = \bigcup_{\zeta \neq 0} \mathfrak{O}_{\zeta \check{\omega}_i} \cup \mathfrak{O}_\nu$ in the second case. In particular this proves Losev's Conjecture 3.3 for $\mathfrak{g}_{sph}$:

Theorem 6.1. *Let $\mathfrak{g}$ be reductive and let $\mathfrak{O}_1$ and $\mathfrak{O}_2$ be spherical adjoint orbits of $\mathfrak{g}$. Then $\Lambda(\mathfrak{O}_1) = \Lambda(\mathfrak{O}_2)$ if and only if $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are contained in the same birational sheet of $\mathfrak{g}$.* □

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