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Nowhere differentiable intrinsic Lipschitz graphs

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Abstract

We construct intrinsic Lipschitz graphs in Carnot groups with the property that, at every point, there exist infinitely many different blow-up limits, none of which is a homogeneous subgroup. This provides counterexamples to a Rademacher theorem for intrinsic Lipschitz graphs.

The notion of Lipschitz submanifolds in sub-Riemannian geometry was introduced, at least in the setting of Carnot groups, by Franchi, Serapioni and Serra Cassano in a series of seminal papers [5–7] through the theory of *intrinsic Lipschitz graphs*. One of the main open questions concerns the differentiability properties for such graphs: in this paper, we provide examples of intrinsic Lipschitz graphs of codimension 2 (or higher) that are nowhere differentiable, that is, that admit no homogeneous tangent subgroup at any point.

Recall that a Carnot group $\mathbb{G}$ is a connected, simply connected and nilpotent Lie group whose Lie algebra is stratified, that is, it can be decomposed as the direct sum $\oplus_{j=1}^{s} V_j$ of subspaces such that

$$V_{j+1} = [V_1, V_j] \text{ for every } j = 1, \dots, s-1, \qquad [V_1, V_s] = \{0\}, \qquad V_s \neq \{0\}.$$

We shall identify the group $\mathbb{G}$ with its Lie algebra via the exponential map $\exp : \oplus_{j=1}^{s} V_j \to \mathbb{G}$, which is a diffeomorphism. In this way, for $\lambda > 0$, one can introduce the homogeneous dilations $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ as the group automorphisms defined by $\delta_\lambda(p) = \lambda^j p$ for every $p \in V_j$. A subgroup of $\mathbb{G}$ is said to be homogeneous if it is dilation-invariant. Assume that a splitting $\mathbb{G} = \mathbb{W}\mathbb{V}$ of $\mathbb{G}$ as the product of homogeneous and complementary (that is, such that $\mathbb{W} \cap \mathbb{V} = \{0\}$) subgroups is fixed; we say that a function $\phi : \mathbb{W} \to \mathbb{V}$ *intrinsic Lipschitz* if there is an open nonempty cone U such that $\mathbb{V} \setminus \{0\} \subset U$ and

$$pU \cap \Gamma_\phi = \emptyset \qquad \text{for all} p \in \Gamma_\phi,$$

where $\Gamma_\phi = \{w\phi(w) : w \in \mathbb{W}\}$ is the intrinsic graph of ϕ. We say that a set $\Sigma \subset \mathbb{G}$ is a *blow-up* of Γ_ϕ at $\hat{p} = \hat{w}\phi(\hat{w})$ if there exists a sequence $(\lambda_n)_n$ such that $\lambda_n \to +\infty$ and the limit

$$\lim_{n\to\infty} \delta_{\lambda_n}(\hat{p}^{-1}\Gamma_\phi) = \Sigma$$

holds with respect to the local Hausdorff convergence. It is worth recalling that, if ϕ is intrinsic Lipschitz, then every blow-up is automatically the intrinsic Lipschitz graph of a map $\mathbb{W} \to \mathbb{V}$. Eventually, we say that ϕ is *intrinsically differentiable* at $\hat{w} \in \mathbb{W}$ if the blow-up of Γ_ϕ at $\hat{p} = \hat{w}\phi(\hat{w})$ is unique and it is a homogeneous subgroup of $\mathbb{G}$. See [8] for details.

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We say that a group $\mathbb{G}$ along with a splitting $\mathbb{W}\mathbb{V}$ satisfies an *intrinsic Rademacher Theorem* if all intrinsic Lipschitz maps $\phi : \mathbb{W} \to \mathbb{V}$ are intrinsically differentiable almost everywhere (that is, for almost all points of $\mathbb{W}$ equipped with its Haar measure). It was proved in [**6**] that this is the case when $\mathbb{V} \simeq \mathbb{R}$ and $\mathbb{G}$ is of step two; other partial results for graphs with codimension 1 ($\mathbb{V} \simeq \mathbb{R}$) are contained in [**4, 9**]. If $\mathbb{V}$ is a normal subgroup, the Rademacher Theorem has been proved for general $\mathbb{G}$ by Antonelli and Merlo in [**2**]. Recently, the third-named author [**12**] proved that Heisenberg groups (with any splitting) satisfy an intrinsic Rademacher Theorem. The question has been open for a long time if $\mathbb{G}$ is the Engel group (which has step 3) and $\mathbb{V} \simeq \mathbb{R}$ (see [**1**]). In this paper, we prove a result in the negative direction: namely, we provide examples of intrinsic Lipschitz graphs that are nowhere intrinsically differentiable. Let us state our main result:

THEOREM 1. *Let $\mathbb{G}$ be a Carnot group with stratification $\bigoplus_{j=1}^{s} V_j$. Let $\mathbb{W}\mathbb{V}$ be a splitting of $\mathbb{G}$ such that $\mathbb{W} \cap V_2 \not\subset [\mathbb{W}, \mathbb{W}]$ and there exists $v_0 \in \mathbb{V} \cap V_1$ such that $v_0 \neq 0$ and $[v_0, \mathbb{W}] = 0$. Then there is an intrinsic Lipschitz function $\phi : \mathbb{W} \to \mathbb{V}$ that is nowhere intrinsically differentiable.*

Moreover, ϕ can be constructed in such a way that, for every $p \in \Gamma_\phi$, the following properties hold.

(a) *There exist infinitely many different blow-ups of Γ_ϕ at p.*
(b) *No blow-up of Γ_ϕ at p is a homogeneous subgroup.*

The proof of Theorem 1 is postponed in order to first provide some comments.

REMARK 1. The simplest example of a Carnot group where Theorem 1 applies is $\mathbb{G} = \mathbb{H} \times \mathbb{R}$, where $\mathbb{H}$ is the first Heisenberg group. As customary, we consider generators X, Y, T of the Lie algebra of $\mathbb{H}$ such that $[X, Y] = T, [X, T] = [Y, T] = 0$ and fix the exponential coordinates $(x, y, t) = \exp(xX + yY + tT)$. Using coordinates (x, y, t, r) on $\mathbb{H} \times \mathbb{R}$ with $r \in \mathbb{R}$, we can consider the splitting $\mathbb{H} \times \mathbb{R} = \mathbb{W}\mathbb{V}$ given by the vertical subgroup $\mathbb{W} = \{x = r = 0\}$ of $\mathbb{H}$ and the horizontal Abelian subgroup $\mathbb{V} = \{y = t = 0\}$. Then $V_2 \cap \mathbb{W} \not\subset [\mathbb{W}, \mathbb{W}] = \{0\}$ and $v_0 = (0, 0, 0, 1)$ commutes with $\mathbb{W}$. Hence, this splitting of $\mathbb{H} \times \mathbb{R}$ satisfies the conditions of Theorem 1 and it does not satisfy an intrinsic Rademacher Theorem.

It is worth observing that, in this setting, the map $\phi : \mathbb{W} \to \mathbb{V}$ provided in the proof of Theorem 1 takes the form $\phi(y, t) = (0, u(t))$, where u is the $\frac{1}{2}$-Hölder continuous function constructed in the Appendix. In particular, the intrinsic graph Γ_ϕ is the set $\{(0, y, t, u(t)) : y, t \in \mathbb{R}\}$ and it is contained in the Abelian subgroup $\mathbb{W} \times \mathbb{R}$. One of the properties of u is that the limit

$$\lim_{s \to t} \frac{|u(t) - u(s)|}{\sqrt{|t - s|}}$$

does not exists at any $t \in \mathbb{R}$ and this is the ultimate reason for the nondifferentiability of ϕ.

Similar counterexamples can be constructed in any codimension $k \geqslant 2$: in fact one can consider $\mathbb{H}^{k-1} \times \mathbb{R} = (\mathbb{R}_x^{k-1} \times \mathbb{R}_y^{k-1} \times \mathbb{R}_t) \times \mathbb{R}_r$ with splitting $\mathbb{W}\mathbb{V}$ defined by $\mathbb{W} = \{x = 0, r = 0\}, \mathbb{V} = \{y = 0, t = 0\}$. It can be easily checked that the map $\phi(y, t) = (0, u(t))$ defines an intrinsic Lipschitz graph of codimension k for which the properties (a) and (b) in Theorem 1 hold at every point.

REMARK 2. The measure $\mu = \mathcal{H}^d \llcorner \Gamma_\phi$, where d is the Hausdorff dimension of $\mathbb{W}$ and $\mathcal{H}^d$ is the d-dimensional Hausdorff measure, does not have a unique tangent measure at any point. Indeed, first, any tangent measure of μ is supported on a blow-up of Γ_ϕ. Second, by [**7**, Theorem 3.9], μ and all its dilations are uniformly d-Ahlfors regular, and thus any tangent measure of μ is

d-Ahlfors regular. We then conclude that if μ_1 and μ_2 are two tangent measures of μ supported on different blow-ups of Γ_ϕ, then they are two distinct measures. Since blow-ups of Γ_ϕ are not unique, so are tangent measures. Observe also that no tangent measure can be flat, that is, supported on a homogeneous subgroup. In particular, Γ_ϕ is purely C^1_H-unrectifiable, that is, $\mathcal{H}^d(\Gamma_\phi \cap \Sigma) = 0$ for every submanifold Σ of class C^1_H (see, for example, [**3**, § 2.5 and 6.1]).

Remark 3. If $\mathbb{W}$ is a homogeneous subgroup of $\mathbb{G}$ with codimension 1, then the conditions of Theorem 1 cannot be met because $\bigoplus_{j=2}^s V_j = [\mathbb{W},\mathbb{W}] + [\mathbb{W},\mathbb{V}]$. Actually, intrinsic Lipschitz graphs of codimension 1 are boundaries of sets with finite perimeter in $\mathbb{G}$ (see, for example, [**11**, Theorem 1.2]), hence at almost every point they possess at least one blow-up which is a homogeneous subgroup of codimension 1, see [**1**]. Therefore, any possible counterexample to the Rademacher Theorem in codimension 1 cannot be as striking as the one provided by Theorem 1, in the sense that property (b) cannot hold on a set with positive measure.

Remark 4. Following the same proof strategy, one can extend Theorem 1 to the case $\mathbb{W} \cap V_j \not\subset [\mathbb{W},\mathbb{W}]$ for some $j > 2$ and $v_0 \in V_k \cap \mathbb{V} \setminus \{0\}$ with $k < j$ and $[v_0, \mathbb{W}] = 0$, by taking a k/j-Hölder analogue of the function u constructed in the appendix.

Proof of Theorem 1. Let $\beta : \mathbb{W} \to \mathbb{R}$ be a nonzero linear function such that $\mathbb{W} \cap V_j \subset \ker\beta$ whenever $j \neq 2$ and $[\mathbb{W},\mathbb{W}] \subset \ker\beta$; such a β exists[†] because $\mathbb{W} \cap V_2 \not\subset [\mathbb{W},\mathbb{W}]$. Note that such a function β is in fact a group morphism $\mathbb{W} \to \mathbb{R}$.

Consider a 1/2-Hölder continuous function $u : \mathbb{R} \to \mathbb{R}$ with the following properties. First, the difference quotients

$$\Delta(s,t) = \frac{u(s) - u(t)}{\operatorname{sgn}(s-t)|s-t|^{1/2}}$$

are bounded, namely,

$$|\Delta(s,t)| \leqslant 1 \qquad \text{for every} s, t \in \mathbb{R}. \tag{1}$$

Second, there exist $c_1 > 0$ and $c_2 > 0$ such that, for every $t_0 \in \mathbb{R}$ and $\delta \in (0, 1]$, there exist $s_1, s_2 \in \mathbb{R}$ such that

$$\begin{aligned} &\operatorname{sgn}(s_1 - t_0) = \operatorname{sgn}(s_2 - t_0) \\ &c_1\delta \leqslant |s_1 - t_0| \leqslant \delta \\ &c_1\delta \leqslant |s_2 - t_0| \leqslant \delta \\ &|\Delta(s_1, t_0) - \Delta(s_2, t_0)| \geqslant c_2. \end{aligned} \tag{2}$$

Such a function exists, as we show in the Appendix.

We can then define $\phi : \mathbb{W} \to \mathbb{V}$ as

$$\phi(w) = u(\beta(w))v_0.$$

Note that the condition $[v_0, \mathbb{W}] = 0$ implies

$$vw = wv \qquad \text{for all} w \in \mathbb{W} \text{and} v \in \mathbb{R}v_0. \tag{3}$$

Therefore, by the Baker–Campbell–Hausdorff formula, the intrinsic graph of ϕ is the set of points $w\phi(w) = w + u(\beta(w))v_0$ for $w \in \mathbb{W}$.

[†]For instance, one can consider $\beta(x) = \langle x, w_0\rangle$ for some $w_0 \in (\mathbb{W} \cap V_2) \setminus [\mathbb{W},\mathbb{W}]$ and a scalar product on $\mathbb{W}$ adapted to the grading $\bigoplus_{j=1}^s \mathbb{W} \cap V_j$ of $\mathbb{W}$.

Claim 1. The map ϕ is intrinsic Lipschitz.

Fix a homogeneous norm $\|\cdot\|$ on $\mathbb{G}$. Note that, since $\beta(\delta_\lambda x) = \lambda^2\beta(x)$ for all $x \in \mathbb{W}$, there is a constant C such that $|\beta(x)| \leqslant C\|x\|^2$, for all $x \in \mathbb{W}$. We check that Γ_ϕ has the cone property for the cone (see [7, Definition 10])

$$U = \{wv : w \in \mathbb{W},\ v \in \mathbb{V}, \|v\| > 2\sqrt{C}\|v_0\|\|w\|\}.$$

Given $\hat{w}, w \in \mathbb{W}$, by (3) we have $(\hat{w}\phi(\hat{w}))^{-1}(w\phi(w)) = (\hat{w}^{-1}w)(\phi(\hat{w})^{-1}\phi(w))$ and

$$\begin{aligned}\|\phi(\hat{w})^{-1}\phi(w)\| &= |u(\beta(w)) - u(\beta(\hat{w}))|\|v_0\| \leqslant |\beta(w) - \beta(\hat{w})|^{1/2}\|v_0\| \\ &= |\beta(\hat{w}^{-1}w)|^{1/2}\|v_0\| \leqslant \sqrt{C}\|\hat{w}^{-1}w\|\|v_0\|.\end{aligned}$$

Thus, $(\hat{w}\phi(\hat{w}))^{-1}\Gamma_\phi \cap U = \emptyset$ for all $\hat{w} \in \mathbb{W}$, that is, Γ_ϕ is an intrinsic Lipschitz graph.

Claim 2. For $p \in \Gamma_\phi$, none of the blow-ups of Γ_ϕ at p is a homogeneous subgroup.

We first observe that, if $\mathbb{V}_0 \subset \mathbb{V} \cap V_1$ is the horizontal subgroup generated by v_0 and $L : \mathbb{W} \to \mathbb{V}_0$ parameterizes a homogeneous subgroup Γ_L of $\mathbb{G}$, then $L|_{\mathbb{W}\cap V_2} = 0$. Indeed, the homogeneity of Γ_L implies that for every $w \in \mathbb{W} \cap V_2$ one has $L(2w) = \sqrt{2}\, L(w)$, because

$$(2w)(\sqrt{2}\, L(w)) = \delta_{\sqrt{2}}(w)\delta_{\sqrt{2}}(L(w)) = \delta_{\sqrt{2}}(wL(w)) \in \Gamma_L,$$

while the fact that Γ_L is a subgroup (plus the fact that $\mathbb{V}_0$ and $\mathbb{W}$ commute) gives $L(2w) = 2L(w)$, because

$$wwL(w)L(w) = (wL(w))(wL(w)) \in \Gamma_L.$$

This proves that $L = 0$ on $\mathbb{W} \cap V_2$.

We now prove the claim. Assume by contradiction that there exist $\hat{p} = \hat{w}\phi(\hat{w}) \in \Gamma_\phi$, a map $L : \mathbb{W} \to \mathbb{V}$ such that the intrinsic graph Γ_L of L is a homogeneous subgroup and a sequence $(\lambda_n)_n$ with $\lambda_n \to +\infty$, and

$$\lim_{n\to\infty} \delta_{\lambda_n}(\hat{p}^{-1}\Gamma_\phi) = \Gamma_L.$$

Observe that for every $w \in \mathbb{W}$ and every n

$$\begin{aligned}\delta_{\lambda_n}((\hat{w}\phi(\hat{w}))^{-1}(w\phi(w))) &= \delta_{\lambda_n}(\hat{w}^{-1}w\phi(\hat{w})^{-1}\phi(w)) \\ &= \delta_{\lambda_n}(\hat{w}^{-1}w)\left(\frac{u(\beta(w)) - u(\beta(\hat{w}))}{1/\lambda_n}v_0\right).\end{aligned}$$

If we set $w = \hat{w}\delta_{1/\lambda_n}w'$, then $\beta(w) = \beta(\hat{w}) + \beta(w')/\lambda_n^2$. Therefore, the set $\delta_{\lambda_n}(\hat{p}^{-1}\Gamma_\phi)$ is the intrinsic graph of the function from $\mathbb{W}$ to $\mathbb{V}$ given by

$$\phi_{\hat{p},\lambda_n}(w') = \frac{u(\beta(\hat{w}) + \beta(w')/\lambda_n^2) - u(\beta(\hat{w}))}{1/\lambda_n}v_0.$$

Since the maps $\phi_{\hat{p},\lambda_n}$ take values in $\mathbb{V}_0$, L is also $\mathbb{V}_0$-valued and, as we saw above, this implies that $L|_{\mathbb{W}\cap V_2} = 0$.

Write $\hat{t} = \beta(\hat{w})$ and let $w_0 \in \mathbb{W} \cap V_2$ be such that $\beta(w_0) = 1$; then for every $h \in \mathbb{R}$

$$\phi_{\hat{p},\lambda_n}(hw_0) = (\operatorname{sgn} h)|h|^{1/2}\Delta(\hat{t} + h/\lambda_n^2, \hat{t})v_0. \tag{4}$$

By (2), there exists a sequence $(h_n)_n$ such that for every n

$$|h_n| \in [c_1, 1] \qquad \text{and} \qquad \|\phi_{\hat{p},\lambda_n}(h_n w_0)\| \geqslant \sqrt{c_1}\, c_2\|v_0\|/2.$$

Up to passing to a subsequence we can also assume that $h_n \to \bar{h}$ with $|\bar{h}| \in [c_1, 1]$; since

$$\|\phi_{\hat{p},\lambda_n}(h_n w_0) - \phi_{\hat{p},\lambda_n}(\bar{h} w_0)\| = \left|\frac{u(\hat{t} + h_n/\lambda_n^2) - u(\hat{t} + \bar{h}/\lambda_n^2)}{1/\lambda_n}\right| \|v_0\|, \\ \leqslant |h_n - \bar{h}|^{1/2}\|v_0\|$$

we obtain

$$\|L(\bar{h} w_0)\| = \lim_n \|\phi_{\hat{p},\lambda_n}(\bar{h} w_0)\| = \lim_n \|\phi_{\hat{p},\lambda_n}(h_n w_0)\| \geqslant \sqrt{c_1}\, c_2 \|v_0\|/2.$$

This contradicts the fact that $L(\bar{h} w_0) = 0$, and the claim is proved.

Claim 3. For $p \in \Gamma_\phi$, there exist infinitely many different blow-ups of Γ_ϕ at p.

Let $\hat{p} = \hat{w}\phi(\hat{w}) \in \Gamma_\phi$ be fixed and let $\hat{t} = \beta(\hat{w})$; as before, fix also $w_0 \in \mathbb{W} \cap V_2$ such that $\beta(w_0) = 1$. By (2), we can find infinitesimal sequences $(s_n^1)_n, (s_n^2)_n$ such that

$$\operatorname{sgn}(s_n^1) = \operatorname{sgn}(s_n^2) \text{ for every } n, \\ \Delta(\hat{t} + s_n^1, \hat{t}\,) \geqslant \Delta(\hat{t} + s_n^2, \hat{t}\,) + c_2.$$

Up to passing to a subsequence, we can assume that there exists $\sigma \in \{1, -1\}$ and $\Delta^1, \Delta^2 \in \mathbb{R}$ such that

$$\operatorname{sgn}(s_n^1) = \operatorname{sgn}(s_n^2) = \sigma \text{for every} n, \\ \Delta(\hat{t} + s_n^1, \hat{t}\,) \to \Delta^1 \text{ and } \Delta(\hat{t} + s_n^2, \hat{t}\,) \to \Delta^2 \text{ as } n \to \infty, \\ \Delta^1 \geqslant \Delta^2 + c_2.$$

Due to the continuity of $s \mapsto \Delta(\hat{t} + s, \hat{t}\,)$ for $s \neq 0$, given $\Delta \in (\Delta^2, \Delta^1)$ one can find an infinitesimal sequence $(s_n)_n$ such that, for every n, $\operatorname{sgn}(s_n) = \sigma$ and $\Delta(\hat{t} + s_n, \hat{t}\,) = \Delta$. Now, as in (4) the set $\delta_{|s_n|^{-1/2}}(\hat{p}^{-1}\Gamma_\phi)$ is the intrinsic graph of a map $\phi_{\hat{p},|s_n|^{-1/2}} : \mathbb{W} \to \mathbb{V}$ such that

$$\phi_{\hat{p},|s_n|^{-1/2}}(\sigma w_0) = \sigma\Delta(\hat{t} + s_n, \hat{t})v_0 = \sigma\Delta v_0.$$

Since the family $(\phi_{\hat{p},|s_n|^{-1/2}})_n$ is uniformly Hölder continuous, up to extracting a subsequence it converges locally uniformly to a map $\psi : \mathbb{W} \to \mathbb{V}$ such that $\psi(\sigma w_0) = \sigma\Delta v_0$. The arbitrariness of $\Delta \in (\Delta^2, \Delta^1)$ implies that there are infinitely many different blow-ups at $\hat{p}$, and this concludes the proof. □

Appendix

We are now going to construct the function u used in the proof of Theorem 1: this function, in a sense, provides a counter-example to a Rademacher property for Lipschitz functions from $(\mathbb{R}, |\cdot|^{1/2})$ to $(\mathbb{R}, |\cdot|)$. We will use a classical procedure producing a self-similar function: although these ideas are well-known (see, for example, [**10**] and the references therein), we prefer to include a detailed construction because we were not able to find in the literature explicit statements for the precise estimates (2) we need.

We construct a function $u : [0, 1] \to [0, 1]$ whose difference quotients

$$\Delta(s, t) = \frac{u(s) - u(t)}{\operatorname{sgn}(s - t)|s - t|^{1/2}}$$

satisfy

$$|\Delta(s, t)| \leqslant 1 \qquad \text{for every} s, t \in [0, 1]. \tag{A.1}$$

We will construct u in such a way that there exist $c_1 > 0$ and $c_2 > 0$ with the property that, for every $t \in [0, 1]$ and $\delta \in (0, 1]$, one can find $s_1, s_2 \in [0, 1]$ such that the conditions in (2)

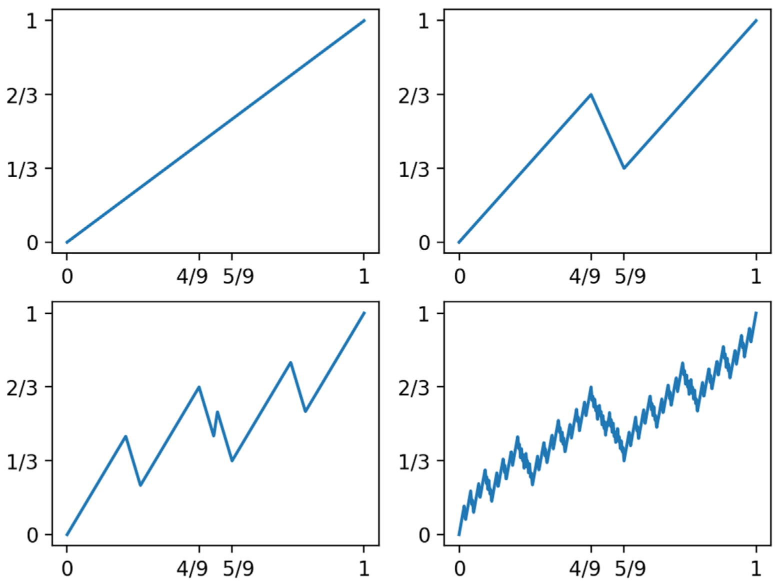

FIGURE A.1 (colour online). *Four instances of the functions* u_n *defined in* (A.2).

hold. One can then extend u to $\mathbb{R}$ by setting $u(t) = u(-t)$ for $t \in [-1, 0]$ and $u(t + 2n) = u(t)$ for all $n \in \mathbb{Z}$: this extended u does satisfy (1) and (2).

The function u is obtained as the limit of a sequence $(u_n)_{n\in\mathbb{N}}$ where $u_0(t) = t$. The function u_{n+1} is obtained from u_n on setting

$$u_{n+1}(t) = \begin{cases} \frac{2}{3}u_n\left(\frac{9}{4}t\right) & \text{if } t \in \left[0, \frac{4}{9}\right], \\ \frac{2}{3} - \frac{1}{3}u_n\left(9\left(t - \frac{4}{9}\right)\right) & \text{if } t \in \left[\frac{4}{9}, \frac{5}{9}\right], \\ \frac{1}{3} + \frac{2}{3}u_n\left(\frac{9}{4}\left(t - \frac{5}{9}\right)\right) & \text{if } t \in \left[\frac{5}{9}, 1\right]. \end{cases} \tag{A.2}$$

The first few of the functions $u_0, u_1, u_2, \ldots$ are plotted in Figure A.1. Let us note that $u_n(0) = 0$ and $u_n(1) = 1$ for every n, hence $u_n(4/9) = 2/3$ and $u_n(5/9) = 1/3$ for every $n \geqslant 1$.

Note (see Figure A.2) that the graph of u_{n+1} is the union of three affine copies of the graph of u_n, via the following maps (acting on $p \in \mathbb{R}^2$):

$$\begin{aligned} A_0(p) &= \begin{pmatrix} 4/9 & 0 \\ 0 & 2/3 \end{pmatrix} p, \\ A_{4/9}(p) &= \begin{pmatrix} 1/9 & 0 \\ 0 & -1/3 \end{pmatrix} p + \begin{pmatrix} 4/9 \\ 2/3 \end{pmatrix}, \\ A_{5/9}(p) &= \begin{pmatrix} 4/9 & 0 \\ 0 & 2/3 \end{pmatrix} p + \begin{pmatrix} 5/9 \\ 1/3 \end{pmatrix}. \end{aligned} \tag{A.3}$$

Claim 1. The functions u_n converge uniformly on $[0, 1]$ to a function u for which (A.1) holds.

The fact that u_n uniformly converge to a continuous function u is a consequence of the estimate

$$\|u_{n+1} - u_n\|_{C^0([0,1])} \leqslant \frac{2}{3}\|u_n - u_{n-1}\|_{C^0([0,1])}.$$

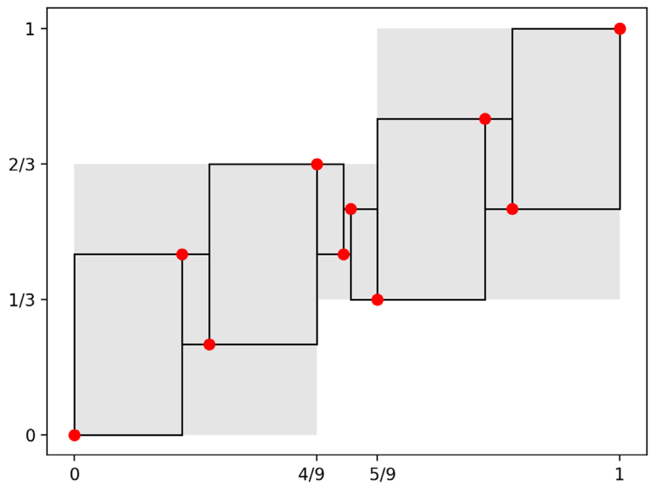

FIGURE A.2 (colour online). *Iterated images of the unit square under the affine maps in* (A.3)*; dots are the images of* (0,0) *and* (1,1)*, and they belong to the graph of the limit function* u

This estimate follows directly from the definition (A.2): for instance, for $t \in [0, 4/9]$, one has

$$|u_{n+1}(t) - u_n(t)| = \frac{2}{3}|u_n(9t/4) - u_{n-1}(9t/4)| \leqslant \frac{2}{3}\|u_n - u_{n-1}\|_{C^0([0,1])}.$$

Similarly, one can treat the other two cases $t \in [4/9, 5/9]$ and $t \in [5/9, 1]$.

The bound (A.1) on the difference quotients of u follows from the fact that the same is true for all u_n in the sequence, as we are now going to prove by induction on n. The statement is clearly true for $n = 0$. Suppose that u_n satisfies

$$|u_n(t) - u_n(s)| \leqslant |t - s|^{1/2} \text{for every} s, t \in [0, 1],$$

we will prove that also $|u_{n+1}(t) - u_{n+1}(s)| \leqslant |t - s|^{1/2}$ for every $s, t \in [0, 1]$. We distinguish several cases depending on which intervals ($[0, 4/9]$, $[4/9, 5/9]$ or $[5/9, 1]$) the points s and t belong to. We can suppose that $s < t$.

Case 1: s and t are in the same interval. We can use (A.2) and the induction hypothesis to conclude.

Case 2: $s \in [0, 4/9]$ and $t \in [4/9, 5/9]$. Since $0 \leqslant u_n \leqslant 1$, one sees from the definition of u_{n+1} that $\max(u_{n+1}(s), u_{n+1}(t)) \leqslant 2/3 = u_{n+1}(4/9)$. Thus

$$\begin{aligned}|u_{n+1}(t) - u_{n+1}(s)| &\leqslant \max(u_{n+1}(4/9) - u_{n+1}(t), u_{n+1}(4/9) - u_{n+1}(s)) \\ &\leqslant \max((t - 4/9)^{1/2}, (4/9 - s)^{1/2}) \leqslant (t - s)^{1/2},\end{aligned}$$

where the second inequality follows from Case 1.

Case 3: $s \in [4/9, 5/9]$ and $t \in [5/9, 1]$. Due to the symmetry $u_n(x) = 1 - u_n(1 - x)$, this is similar to Case 2.

Case 4: $s \in [0, 4/9]$ and $t \in [5/9, 1]$. Then either $|u_{n+1}(t) - u_{n+1}(s)| \leqslant 1/3$, and we are done because $|t - s| \geqslant 1/9$, or $|u_{n+1}(t) - u_{n+1}(s)| > 1/3$, and then necessarily $u_{n+1}(s) < u_{n+1}(t)$ (otherwise, $0 \leqslant u_{n+1}(s) - u_{n+1}(t) \leqslant u_{n+1}(4/9) - u_{n+1}(5/9) = 2/3 - 1/3 = 1/3$) and

$$\begin{aligned}|u_{n+1}(t) - u_{n+1}(s)| &= u_{n+1}(t) - u_{n+1}(s) \\ &= u_{n+1}(t) - u_{n+1}(5/9) - 1/3 + u_{n+1}(4/9) - u_{n+1}(s) \\ &\leqslant (t - 5/9)^{1/2} - 1/3 + (4/9 - s)^{1/2},\end{aligned}$$

where in the last inequality we used Case 1. By squaring the right-hand side of the last inequality, we obtain

$$\begin{aligned}&\Big((t-5/9)^{1/2}-1/3+(4/9-s)^{1/2}\Big)^2\\&\quad=(t-s)+2(t-5/9)^{1/2}(4/9-s)^{1/2}-\frac{2}{3}(t-5/9)^{1/2}-\frac{2}{3}(4/9-s)^{1/2}\\&\quad=(t-s)+(t-5/9)^{1/2}\Big((4/9-s)^{1/2}-2/3\Big)+(4/9-s)^{1/2}\Big((t-5/9)^{1/2}-2/3\Big)\\&\quad\leqslant t-s,\end{aligned}$$

where we used the fact that $4/9-s\leqslant 4/9$ and $t-5/9\leqslant 4/9$. This is enough to conclude.

Claim 2. There exist $d_1>0$ and $d_2>0$ such that, for every $t_0\in[0,1]$, one can find $s_1,s_2\in[0,1]$ such that

$$\begin{aligned}&\operatorname{sgn}(s_1-t_0)=\operatorname{sgn}(s_2-t_0)\\&d_1\leqslant|s_1-t_0|\leqslant 1\\&d_1\leqslant|s_2-t_0|\leqslant 1\\&|\Delta(s_1,t_0)-\Delta(s_2,t_0)|\geqslant d_2.\end{aligned}\tag{A.4}$$

In fact, we will prove Claim 2 for $d_1=1/18$ and

$$d_2=\min\left\{\frac{1}{3}\Big(\big(1-\tfrac{4}{81}\big)^{-1/2}-1\Big),\ \frac{7}{9}-\frac{3}{5},\ \frac{1}{\sqrt{5}}\right\}.$$

We distinguish several cases.

Case 1: $t_0\in[0,4/9]$. In this case, it suffices to consider $s_1=5/9$ and $s_2=1$, as we now show. Observe that the distances of s_1,s_2 from t_0 are both greater than $1/9>d_1$.

If $u(t_0)\geqslant 2/9$, by (A.1) and the equality $u(0)=0$, we have $t_0^{1/2}\geqslant u(t_0)$, hence $t_0\geqslant 4/81$; since $u(t_0)\leqslant 2/3$, we obtain

$$\Delta(1,t_0)\geqslant\frac{\frac{1}{3}}{\sqrt{1-\frac{4}{81}}}\qquad\text{and}\qquad\Delta(5/9,t_0)\leqslant\frac{\frac{1}{3}-\frac{2}{9}}{\sqrt{\frac{1}{9}}}=\frac{1}{3},$$

so that $\Delta(1,t_0)-\Delta(5/9,t_0)\geqslant d_2$.

If $u(t_0)\leqslant 2/9$, then $(4/9-t_0)^{1/2}\geqslant 2/3-2/9=4/9$, hence $5/9-t_0\geqslant 1/9+(4/9)^2=(5/9)^2$ and

$$\Delta(1,t_0)\geqslant\frac{7}{9}\qquad\text{and}\qquad\Delta(5/9,t_0)\leqslant\frac{\frac{1}{3}}{\frac{5}{9}}=\frac{3}{5}$$

and again $\Delta(1,t_0)-\Delta(5/9,t_0)\geqslant d_2$.

Case 2: $t_0\in[4/9,1/2]$. In this case, we take $s_1=5/9$ and $s_2=1$. The distances of s_1,s_2 from t_0 are both no less than $1/18=d_1$ and, since $1/3\leqslant u(t_0)\leqslant 2/3$, one gets

$$\Delta(1,t_0)\geqslant\frac{\frac{1}{3}}{\sqrt{\frac{5}{9}}}=\frac{1}{\sqrt{5}}\qquad\text{and}\qquad\Delta(5/9,t_0)\leqslant 0.$$

Case 3: $t_0\in[1/2,1]$. We proved that, if $t_0\in[0,1/2]$, the claim can be proved on choosing $s_1=5/9$ and $s_2=1$. Therefore, due to the symmetry $u(x)=1-u(1-x)$, when $t_0\in[1/2,1]$ it is enough to take $s_1=0$ and $s_2=4/9$.

Claim 3. There exist $c_1>0$ and $c_2>0$ such that, for every $t\in[0,1]$ and $\delta\in(0,1]$, one can find $s_1,s_2\in[0,1]$ for which the conditions in (2) hold.

By self-similarity, the graph of u over the interval $[0,1] \cap [t-\delta, t+\delta]$ contains the image of the graph of u over $[0,1]$ under an affine map $L : \mathbb{R}^2 \to \mathbb{R}^2$ which is a finite composition $L = A_{j_1} \circ \cdots \circ A_{j_N}$ of maps $(A_{j_k})_{k=1,\dots,N}$ for j_k in $\{0, 4/9, 5/9\}$. Observe that L is an affine map of the form $L(x,y) = (L_1(x), L_2(y))$ for suitable affine maps $L_1, L_2 : \mathbb{R} \to \mathbb{R}$ and it is not restrictive to assume that the length of the interval $L_1([0,1])$, which is contained in $[t-\delta, t+\delta]$, is at least $\delta/9$: this implies that there exists $\delta/9 \leqslant c \leqslant \delta$ such that $|L_1(x) - L_1(y)| = c|x-y|$ for every $x, y \in [0,1]$. Let $t_0 \in [0,1]$ be such that $L(t_0, u(t_0)) = (t, u(t))$. If $s_1, s_2 \in [0,1]$ are such that (A.4) holds, then we have

$$\begin{aligned}
&\operatorname{sgn}(L_1(s_1) - t) = \operatorname{sgn}(L_1(s_2) - t),\\
&\tfrac{d_1}{9}\delta \leqslant |L_1(s_1) - t| \leqslant \delta,\\
&\tfrac{d_1}{9}\delta \leqslant |L_1(s_2) - t| \leqslant \delta.
\end{aligned}$$

Since the maps A_{j_k} do not modify the difference quotients, also L has this property, that is,

$$|\Delta(L_1(s_1), t) - \Delta(L_1(s_2), t)| = |\Delta(s_1, t_0) - \Delta(s_2, t_0)| \geqslant d_2.$$

This concludes the proof.

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