# SPECTRAL MULTIPLIERS IN A GENERAL GAUSSIAN SETTING

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Abstract. We investigate a class of spectral multipliers for an Ornstein–Uhlenbeck operator $\mathcal{L}$ in $\mathbb{R}^n$, with drift given by a real matrix $B$ whose eigenvalues have negative real parts. We prove that if $m$ is a function of Laplace transform type defined in the right half-plane, then $m(\mathcal{L})$ is of weak type $(1,1)$ with respect to the invariant measure in $\mathbb{R}^n$. The proof involves many estimates of the relevant integral kernels and also a bound for the number of zeros of the time derivative of the Mehler kernel, as well as an enhanced version of the Ornstein–Uhlenbeck maximal operator theorem.

## 1. Introduction

Given a measure space $(X,\mu)$ and a self-adjoint operator $L$ on $L^2(X,\mu)$, an important issue in harmonic analysis concerns the boundedness of the operator $m(L)$, where $m:\mathbb{R}\to\mathbb{C}$ is a Borel function. If $E$ denotes a spectral resolution of $L$ on $\mathbb{R}$, one can define $m(L)$ for many functions $m$ as

$$m(L)=\int_{\mathbb{R}} m(\nu)\,dE(\nu).$$

Great efforts have been devoted to finding minimal assumptions on the multiplier $m$ that will ensure the boundedness of $m(L)$ on the Lebesgue spaces $L^p(X,\mu)$, both in a strong and in a weak sense, when $p\neq 2$.

A few years ago, the authors started a program concerning harmonic analysis in the Ornstein–Uhlenbeck setting. In this framework, $(X,\mu)$ is the Euclidean space $\mathbb{R}^n$ equipped with a Gaussian measure $d\gamma_\infty$, known as the invariant measure and defined in Section 2. Further, $L$ is replaced by the Ornstein–Uhlenbeck operator $\mathcal{L}$, defined as

$$\mathcal{L}f=-\frac{1}{2}\,\mathrm{tr}\left(Q\nabla^2 f\right)-\langle Bx,\nabla f\rangle\,,\qquad f\in\mathcal{S}(\mathbb{R}^n), \tag{1.1}$$

where $\nabla$ and $\nabla^2$ denote the gradient and the Hessian, respectively. In this formula, $Q$ and $B$ are real $n\times n$ matrices; $Q$ is symmetric and positive definite, and the

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eigenvalues of $B$ all have negative real parts. The space $L^p(\mathbb{R}^n, d\gamma_\infty)$ will be written simply $L^p(\gamma_\infty)$.

Since in general $\mathcal{L}$ has no self-adjoint or normal extension to $L^2(\gamma_\infty)$, one cannot invoke spectral theory to define $m(\mathcal{L})$. Notice that self-adjointness and normality may fail also for the Ornstein–Uhlenbeck semigroup $(\mathcal{H}_t)_{t>0}$, generated by $\mathcal{L}$, which was first introduced in [OU]. The focus in this paper is on multipliers of Laplace transform type. This class of multipliers was introduced some fifty years ago by E. M. Stein in [St], in the context of the Littlewood–Paley theory for a sublaplacian on a connected Lie group $G$.

A function $m$ of a real variable $\lambda > 0$ is said to be of Laplace transform type if

$$m(\lambda) = \lambda \int_0^{+\infty} \varphi(t) e^{-t\lambda}\, dt = -\int_0^{+\infty} \varphi(t)\, \frac{d}{dt}\, e^{-t\lambda}\, dt, \qquad \lambda > 0, \tag{1.2}$$

for some $\varphi \in L^\infty(0, +\infty)$. Observe that such a function $m$ can be extended to an analytic function in the half-plane $\Re z > 0$. Thus we pay the price of a rather strong condition on $m$, to prove, in return, a multiplier theorem for an operator $\mathcal{L}$ which is not necessarily normal. Observe that one obtains as $m(\mathcal{L})$ the imaginary powers $\mathcal{L}^{i\gamma}$ of $\mathcal{L}$, with $\gamma \in \mathbb{R} \setminus \{0\}$, by choosing $\varphi(t) = \text{const.}\, t^{-i\gamma}$.

The exact definition of $m(\mathcal{L})$ for functions $m$ of this type will be given in Section 3. Here we present only a heuristic deduction of the kernel of $m(\mathcal{L})$. If we simply replace $\lambda$ by $\mathcal{L}$ in the last expression in (1.2), we would get

$$m(\mathcal{L}) = -\int_0^{+\infty} \varphi(t)\, \frac{d}{dt}\, e^{-t\mathcal{L}}\, dt. \tag{1.3}$$

Here $e^{-t\mathcal{L}} = \mathcal{H}_t$ is the Ornstein-Uhlenbeck semigroup, whose kernel is the Mehler kernel $K_t(x,u)$ described in Section 2. We point out that the term kernel in this paper always refers to integration with respect to $d\gamma_\infty$. Thus for each $f \in \mathcal{S}(\mathbb{R}^n)$ and all $t > 0$

$$\mathcal{H}_t f(x) = \int K_t(x,u)\, f(u)\, d\gamma_\infty(u)\,.$$

This makes it plausible that the off-diagonal kernel of $m(\mathcal{L})$ is

$$\mathcal{M}_\varphi(x,u) = -\int_0^{+\infty} \varphi(t)\, \partial_t K_t(x,u)\, dt. \tag{1.4}$$

We will verify this formula later, though after splitting the integral and under some restrictions. It will lead to an expression for the kernel in terms of $Q$ and $B$.

From now on, we assume that $m$ is of Laplace transform type.

In the standard case $Q = I$ and $B = -I$, the operator $\mathcal{L}$ is self-adjoint, and the $L^p(d\gamma_\infty)$ boundedness of $m(\mathcal{L})$ follows for all $1 < p < \infty$ from a general result due to Stein [St, Ch. 4]. Moreover, J. García-Cuerva, G. Mauceri, J. L.Torrea and the third author proved in this case the weak type $(1,1)$ of $m(\mathcal{L})$ with respect to $d\gamma_\infty$; see [GMST2, Theorem 3.8]. For more recent results in the standard case, also involving the Gaussian conical square function, we refer to [K1, K2]; see also [W1, W2], where the author investigates multiplier theorems for systems of Ornstein–Uhlenbeck operators. For an overview of this topic, we refer to Bogachev's survey [Bo] and the references therein. Very recently, several interesting results in a nonsymmetric but normal Ornstein–Uhlenbeck context have appeared in [ABQR].

In the general case, when $\mathcal{L}$ is given by (1.1), the strong $L^p(\gamma_\infty)$ boundedness of $m(\mathcal{L})$ follows for $1 < p < \infty$ from [CaD, Prop. 3.8]. In this paper, we consider the endpoint case $p = 1$, where the strong boundedness does not hold.

Our main result is the following.

**Theorem 1.1.** *If the function $m$ is of Laplace transform type, then the multiplier operator $m(\mathcal{L})$ associated to a general Ornstein–Uhlenbeck operator $\mathcal{L}$ is of weak type $(1,1)$ with respect to the invariant measure $d\gamma_\infty$.*

Thus we shall prove the inequality

$$\gamma_\infty\{x \in \mathbb{R}^n : m(\mathcal{L})\, f(x) > C\alpha\} \leq \frac{C}{\alpha}\, \|f\|_{L^1(\gamma_\infty)}, \qquad \alpha > 0, \tag{1.5}$$

for all functions $f \in L^1(\gamma_\infty)$, with $C = C(n, Q, B)$. Our theorem extends Theorem 3.8 in [GMST2] to the framework of a general, not necessarily normal, Ornstein–Uhlenbeck operator.

For a detailed study of exponential and Moser integral inequalities in the Gaussian framework we refer to [CiMP1, CiMP2, CiMP3].

What follows next is a plan of the proof of Theorem 1.1, together with a description of the structure of the paper.

In Section 2, we introduce some terminology and recall from the authors' earlier papers [CCS1, CCS2, CCS3] a few estimates which are essential in our approach. Section 3 gives a rigorous definition of the multiplier operator, and in Subsection 3.2 we split this operator by splitting the integrals in (1.2) and (1.3) into parts taken over $t < 1$ and $t > 1$. The part corresponding to $t < 1$ is further split into a local and a global part. Then in Section 4 the time derivative $\partial_t K_t$ of the Mehler kernel is computed and estimated. This leads in Section 5 to some estimates for the kernels of the different parts of the operator. There we also introduce some technical simplifications that will reduce the complexity of the proof of Theorem 1.1. This proof is given in the remaining sections, in the following way.

The operator part with $t > 1$ is dealt with in Section 6. Section 7 gives the proof for the local part mentioned above, with standard Calderón-Zygmund techniques. The remaining, global part is more delicate. For its kernel we will have a bound

$$\int_0^1 |\partial_t K_t(x,u)|\, dt \leq \sum \left| \int \partial_t K_t(x,u)\, dt \right|,$$

where the integrals in the sum are taken between consecutive zeros of $\partial_t K_t$. Therefore, we will need an estimate of the number of zeros of $\partial_t K_t(x,u)$ as $t$ runs through the interval $(0,1]$. This number turns out to be controlled by a constant depending only on $n$, as verified in Section 8. We can then complete the proof of the weak type $(1,1)$ in Section 9. There we also need an enhanced version of the Ornstein–Uhlenbeck maximal operator theorem (see [CCS2, Theorem 1.1]). Its proof is given in the Appendix (Section 10).

We will write $C < \infty$ and $c > 0$ for various constants, all of which depend only on $n$, $Q$ and $B$, unless otherwise explicitly stated. If $a$ and $b$ are positive quantities, $a \lesssim b$ or equivalently $b \gtrsim a$ means $a \le Cb$. When $a \lesssim b$ and also $b \lesssim a$, we write $a \simeq b$. By $\mathbb{N}$ we denote the set of all nonnegative integers. If $A$ is a real $n \times n$ matrix, we write $\|A\|$ for its operator norm on $\mathbb{R}^n$ with the Euclidean norm $|\cdot|$. We will adopt the dot notation for differentiation with respect to the time variable $t$, writing $\dot{K}_t = \partial_t K_t$.

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## 2. Preliminaries

In this section we collect some results from [CCS1, CCS2, CCS3] related to the Mehler kernel of a general Ornstein–Uhlenbeck semigroup.

### 2.1. **Some matrices and estimates.**

In terms of the two real $n \times n$ matrices $Q$ and $B$ introduced in Section 1, we define for $t \in (0, +\infty]$ the matrix

$$Q_t = \int_0^t e^{sB}\, Q\, e^{sB^*}\, ds. \tag{2.1}$$

Since $Q$ is real, symmetric and positive definite, and the eigenvalues of $B$ have negative real parts, this integral is convergent and the matrix $Q_t$ is symmetric and positive definite and thus invertible, for all $0 < t \le \infty$.

It will be convenient to write

$$|x|_Q = |Q_\infty^{-1/2}\, x|, \qquad x \in \mathbb{R}^n,$$

which is a norm on $\mathbb{R}^n$, and $|x|_Q \simeq |x|$. Further, we let $R(x)$ denote the (positive definite) quadratic form

$$R(x) = \frac{1}{2}\, |x|_Q^2 = \frac{1}{2}\, \langle Q_\infty^{-1}\, x, x \rangle, \qquad x \in \mathbb{R}^n.$$

The invariant measure is given by

$$d\gamma_\infty(x) = \exp(-R(x))\, dx.$$

Notice that $d\gamma_\infty$ is not normalized.

We will also use the one-parameter group of matrices

$$D_t = Q_\infty\, e^{-tB^*}\, Q_\infty^{-1}, \qquad t \in \mathbb{R}, \tag{2.2}$$

introduced in [CCS2]. We proved in [CCS2, Lemma 2.1] that

$$D_t = e^{tB} + Q_t\, e^{-tB^*}\, Q_\infty^{-1}, \qquad t > 0. \tag{2.3}$$

By means of a Jordan decomposition of $B^*$, the following estimates were proved in [CCS2, Lemma 3.1]

$$e^{ct}\, |x| \lesssim |D_t\, x| \lesssim e^{Ct}\, |x| \qquad \text{and} \qquad e^{-Ct}\, |x| \lesssim |D_{-t}\, x| \lesssim e^{-ct}\, |x|,$$

holding for $t > 0$ and all $x \in \mathbb{R}^n$. The same bounds are true with $D_t$ replaced by $e^{-tB}$ or $e^{-tB^*}$; in particular,

$$e^{ct}\,|x| \lesssim |e^{-tB}\,x| \lesssim e^{Ct}\,|x| \qquad \text{and} \qquad e^{-Ct}\,|x| \lesssim |e^{tB}\,x| \lesssim e^{-ct}\,|x| \tag{2.4}$$

for $t > 0$ and $x \in \mathbb{R}^n$.

From these inequalities one deduces (see [CCS2, Lemma 3.2])

$$\|Q_t^{-1}\| \simeq (\min(1,t))^{-1}, \tag{2.5}$$

$$\|Q_t^{-1} - Q_\infty^{-1}\| \lesssim t^{-1}\,e^{-ct}. \tag{2.6}$$

Finally, we recall the following lemma, proved in [CCS3, Lemma 2.3].

**Lemma 2.1.** *Let $x \in \mathbb{R}^n$ and $|t| \le 1$. Then*

$$|x - D_t\,x| \simeq |t|\,|x|.$$

2.2. **Spectrum and generalized eigenspaces of $\mathcal{L}$.**

Let $\lambda_1, \dots, \lambda_r$ be the eigenvalues of $B$. It is known that the spectrum of $\mathcal{L}$ is

$$\left\{ -\sum_{i=1}^{r} n_i\,\lambda_i : n_i \in \mathbb{N},\ i = 1, \dots, r \right\} \subset \{z \in \mathbb{C} :\ \Re z > 0\} \cup \{0\},$$

see [MPP, Theorem 3.1].

Each point $\lambda$ in this set is an eigenvalue of $\mathcal{L}$. The corresponding generalized eigenfunctions, i.e., the functions annihilated by $(\mathcal{L}-\lambda)^k$ for some $k \in \mathbb{N}$, are polynomials, see [LB, Theorem 9.3.20]. For each $\lambda$ they form a finite-dimensional space, and these generalized eigenspaces together span a dense subspace of $L^2(\gamma_\infty)$. In particular, 0 is an eigenvalue of $\mathcal{L}$. The corresponding eigenspace, which we denote by $\mathcal{E}_0$, is of dimension 1 and consists of the constant functions. As shown in [CCS4, Lemma 2.1], this eigenspace is orthogonal to all other generalized eigenfunctions of $\mathcal{L}$. We denote by $L^2_0(\gamma_\infty)$ the orthogonal complement of $\mathcal{E}_0$ in $L^2(\gamma_\infty)$.

2.3. **The Mehler kernel.** For $x, u \in \mathbb{R}^n$ and $t > 0$ the Mehler kernel $K_t$ is given by (see [CCS2, formula (2.6)])

$$K_t(x,u) = \Big(\frac{\det\, Q_\infty}{\det\, Q_t}\Big)^{1/2} e^{R(x)} \exp\Big[-\frac{1}{2}\big\langle (Q_t^{-1} - Q_\infty^{-1})(u - D_t\,x)\,,\, u - D_t\,x\big\rangle\Big]. \tag{2.7}$$

It is convenient to use this expression for $K_t$ when $t \le 1$. But for $t > 1$, we will use the following alternative, which can be obtained from [CCS2, first formula in the proof of Proposition 3.3],

$$K_t(x,u) = \Big(\frac{\det Q_\infty}{\det Q_t}\Big)^{1/2} e^{R(x)} \exp\Big[-\frac{1}{2}\big\langle Q_t^{-1} e^{tB}(D_{-t}\,u - x), D_t\,(D_{-t}\,u - x)\big\rangle\Big]. \tag{2.8}$$

For $0 < t \le 1$ we have the following estimates, proved in [CCS2, (2.10)]

$$\frac{e^{R(x)}}{t^{n/2}} \exp\left(-C\,\frac{|u - D_t\,x|^2}{t}\right) \lesssim K_t(x,u) \lesssim \frac{e^{R(x)}}{t^{n/2}} \exp\left(-c\,\frac{|u - D_t\,x|^2}{t}\right). \tag{2.9}$$

When $t \geq 1$ one has instead (see [CCS2, (2.11)])

$$e^{R(x)} \exp\Big[ - C\,|D_{-t}\,u - x|_Q^2 \Big] \lesssim K_t(x,u) \lesssim e^{R(x)} \exp\Big[ - \frac{1}{2}\,|D_{-t}\,u - x|_Q^2 \Big]. \quad (2.10)$$

2.4. **Polar coordinates.** We will use a variant of polar coordinates first introduced in [CCS1]. Fix $\beta > 0$ and consider the ellipsoid

$$E_\beta = \{x \in \mathbb{R}^n : \, R(x) = \beta\}\,.$$

Any $x \in \mathbb{R}^n$, $x \neq 0$, can be written uniquely as

$$x = D_s\,\tilde{x}\,,$$

for some $\tilde{x} \in E_\beta$ and $s \in \mathbb{R}$. We call $(s, \tilde{x})$ the polar coordinates of $x$.

The Lebesgue measure in $\mathbb{R}^n$ is given in terms of $(s, \tilde{x})$ by

$$dx = e^{-s\,\mathrm{tr}\,B}\,\frac{|Q^{1/2}\,Q_\infty^{-1}\tilde{x}|^2}{2\,|Q_\infty^{-1}\tilde{x}|}\,dS_\beta(\tilde{x})\,ds\,, \quad (2.11)$$

where $dS_\beta$ denotes the area measure of $E_\beta$. See [CCS2, Proposition 4.2] for a proof.

## 3. Definition and splitting of the multiplier operator

3.1. **Definition of the multiplier operator.** We use the definition described in Cowling et al. [CDMY, Section 2], which goes back to McIntosh [M]. The starting-point in this paper is an operator $T$ defined on a Hilbert (or Banach) space, which will be $L^2_0(\gamma_\infty)$ in our case. This operator is to be densely defined and one-to-one with dense range, and its spectrum must be contained in a closed sector

$$S_\omega = \{z \in \mathbb{C} : |\arg z| \leq \omega\} \cup \{0\},$$

for some $\omega \in (0, \pi/2)$. Further, the resolvent of $T$ should satisfy the estimate

$$\|(T - zI)^{-1}\| \leq C\,|z|^{-1}, \qquad z \in \mathbb{C} \setminus S_\omega, \quad (3.1)$$

for some constant $C$, where we refer to the operator norm on $L^2_0(\gamma_\infty)$.

Therefore, we define the operator $T$ as the restriction of $\mathcal{L}$ to $L^2_0(\gamma_\infty)$. We will prove Theorem 1.1 with $\mathcal{L}$ replaced by $T$. The theorem then follows, since $\mathcal{L}$ vanishes on $\mathcal{E}_0$.

From the preceding section, it is clear that $T$ has all the properties required in [CDMY] mentioned above, except possibly the inequality (3.1). We shall now verify (3.1).

According to [CFMP1, Theorem 1 and Remark 6], there exists an angle $\theta_2 \in (0, \pi/2)$ such that the semigroup $\big(e^{-tT}\big)_{t>0}$ is a contraction on $L^2_0(\gamma_\infty)$ for each $t$ in the sector $S_{\theta_2}$. Then (3.1) follows from some well-known arguments for bounded analytic semigroups (see [EN, Ch. II, Section 4.a]). Anyway, we give a concise proof.

Fix a $\theta \in (0, \theta_2)$; like $\theta_2$ this $\theta$ will only depend on $n$, $Q$ and $B$. If $z$ is on the negative real axis, the contraction property implies

$$(T - zI)^{-1} = \int_0^{+\infty} e^{-t(T - zI)}\,dt = e^{i\theta} \int_0^{+\infty} e^{-te^{i\theta}T}\,e^{te^{i\theta}zI}\,dt, \quad (3.2)$$

where we moved the path of integration to the ray $e^{i\theta}\,\mathbb{R}_+$ in $\mathbb{C}$. Here we want to let $z = re^{i\varphi}$, with $r > 0$ and $\varphi \in (\pi/2 - \theta/2, \pi]$. Then

$$0 < \theta/2 < \theta + \varphi - \pi/2 \ \leq\ \theta + \pi/2 < \pi$$

and so

$$\Re(t\,e^{i\theta} z) = t\,r\cos(\theta+\varphi) = -t\,r\sin(\theta+\varphi-\pi/2) < -c\,t\,r.$$

For such $z$ the second integral in (3.2) converges, and by analyticity it equals $e^{-i\theta}\,(T - zI)^{-1}$. Thus

$$\|(T - zI)^{-1}\| \leq \int_0^{+\infty} e^{-ctr}\,dt \leq \frac{C}{|z|},$$

which proves (3.1) for $z$ in the upper half-plane, with $\omega = \pi/2 - \theta/2$. To deal with the case when $z$ is in the lower half-plane, it is enough to take the complex conjugate of the equation (3.2) and repeat the argument, because $T$ is real. We have thus verified (3.1).

Since 0 is not in the spectrum of $T$, we have the following improvement of (3.1):

$$\|(T - zI)^{-1}\| \leq C\,(1+|z|)^{-1}, \qquad z \in \mathbb{C} \setminus S_\omega. \tag{3.3}$$

The function $m$ is of Laplace transform type and thus defined and analytic in the right half-plane. Moreover, it is bounded on any sector $S_\phi$ with $0 < \phi < \pi/2$. The definition of $m(T)$ in [CDMY] goes via a complex integral involving the resolvent of $T$. To make this integral convergent, we multiply the function $m(z)$ by $\psi(z) = 1/(1+z^2)$, following [CDMY]. With $\omega \in (0, \pi/2)$ fulfilling (3.3), we fix a $\nu \in (\omega, \pi/2)$ and let $\Gamma$ be the path

$$\Gamma(t) = |t|\,e^{i\nu\,\mathrm{sgn}\,t}, \qquad -\infty < t < \infty.$$

Now define

$$(\psi m)(T) = \frac{1}{2\pi i}\int_\Gamma \psi(z) m(z)\,(zI - T)^{-1}\,dz,$$

which is a convergent integral because of (3.3), and let

$$m(T) = \psi(T)^{-1}(\psi m)(T). \tag{3.4}$$

**Proposition 3.1.** *Let $\lambda \neq 0$ be a generalized eigenvalue of $T$ with generalized eigenspace $\mathcal{E}_\lambda$. Then the restriction to $\mathcal{E}_\lambda$ of $m(T)$ (defined above) coincides with the restriction to $\mathcal{E}_\lambda$ of the integral*

$$-\int_0^{+\infty} \varphi(t)\,\frac{d}{dt}\,e^{-tT}\,dt.$$

.

Notice that this is the integral from (1.3), and that its restriction to the finite-dimensional, $T$-invariant subspace $\mathcal{E}_\lambda$ makes perfect sense. Further, $m(T)$ is determined by these restrictions, since the $\mathcal{E}_\lambda$ together span $L^2_0(\gamma_\infty)$ and $m(T)$ is bounded on $L^2_0(\gamma_\infty)$, as proved by [CaD, Lemma 3.7].

*Proof.* Observe first that $T\big|_{\mathcal{E}_\lambda} = \lambda I + R_\lambda$, where $R_\lambda$ a nilpotent operator on $\mathcal{E}_\lambda$. For $z \in \mathbb{C} \setminus \{\lambda\}$ this leads to

$$\begin{aligned}(zI-T)^{-1}\big|_{\mathcal{E}_\lambda} &= ((z-\lambda)I - R_\lambda)^{-1} = (z-\lambda)^{-1}\left(I - \frac{R_\lambda}{z-\lambda}\right)^{-1} \\ &= \sum_0 \frac{1}{(z-\lambda)^{j+1}}\, R_\lambda^j,\end{aligned}$$

where the sum is finite. Thus

$$\begin{aligned}(\psi m)(T)\big|_{\mathcal{E}_\lambda} &= \frac{1}{2\pi i}\int_\Gamma \psi(z)\, m(z)\, (zI-T)^{-1}\big|_{\mathcal{E}_\lambda} dz \\ &= \frac{1}{2\pi i}\sum_0 \int_\Gamma \psi(z)\, m(z)\, \frac{1}{(z-\lambda)^{j+1}}\, dz\, R_\lambda^j \\ &= \sum_0 \frac{1}{j!}\, (\psi m)^{(j)}(\lambda)\, R_\lambda^j \\ &= \sum_{i,\,k} \frac{1}{i!\,k!}\, \psi^{(i)}(\lambda)\, m^{(k)}(\lambda)\, R_\lambda^{i+k} \\ &= \sum_i \frac{1}{i!}\, \psi^{(i)}(\lambda)\, R_\lambda^i \sum_k \frac{1}{k!}\, m^{(k)}(\lambda)\, R_\lambda^k \\ &= \psi(T) \sum_k \frac{1}{k!}\, m^{(k)}(\lambda)\, R_\lambda^k.\end{aligned}$$

From (3.4) and (1.2), we conclude that

$$\begin{aligned}m(T)\big|_{\mathcal{E}_\lambda} &= \sum_k \frac{1}{k!}\, m^{(k)}(\lambda)\, R_\lambda^k \\ &= -\sum_k \frac{1}{k!}\int_0^{+\infty} \varphi(t)\left(\frac{\partial}{\partial\lambda}\right)^k \frac{d}{dt}\, e^{-\lambda t}\, dt\, R_\lambda^k \\ &= -\int_0^{+\infty} \varphi(t)\, \frac{d}{dt} \sum_k \frac{1}{k!}\left(\frac{\partial^k}{\partial\lambda^k}\, e^{-\lambda t}\right) dt\, R_\lambda^k.\end{aligned}$$

Here the sum equals

$$\sum_k e^{-\lambda t}\, \frac{1}{k!}\, (-t)^k\, R_\lambda^k = e^{-\lambda t}\, e^{-tR_\lambda} = e^{-tT},$$

and the proposition follows. □

3.2. **Splitting of the multiplier operator.** Given $\varphi \in L^\infty(0,+\infty)$, we will restrict the integral in (1.2) to various intervals. For $\varepsilon > 0$ we let

$$m_\varepsilon(\lambda) = -\int_\varepsilon^{+\infty} \varphi(t)\, \frac{d}{dt}\, e^{-\lambda t}\, dt.$$

However, replacing $\epsilon$ by 0, we also define

$$m_0(\lambda) = -\int_0^1 \varphi(t)\, \frac{d}{dt}\, e^{-\lambda t}\, dt,$$

and observe that

$$m(T) = m_1(T) + m_0(T).$$

Then (1.4) hints that $m_\varepsilon(T)$ and $m_0(T)$ should have off-diagonal kernels

$$\mathcal{M}_\varepsilon(x,u) = -\int_\varepsilon^{+\infty} \varphi(t)\, \dot{K}_t(x,u)\, dt \tag{3.5}$$

and

$$\mathcal{M}_0(x,u) = -\int_0^1 \varphi(t)\, \dot{K}_t(x,u)\, dt. \tag{3.6}$$

As will be verified in Section 5, (3.5) is correct for any $\varepsilon > 0$, and we use it in Section 6 to control $m_1(T)$. But (3.6) is problematical on the diagonal $x = u$, and we need to consider separately the global and local parts, defined as follows.

First we introduce a global and a local region, having some overlap, by setting

$$G = \left\{ (x,u) \in \mathbb{R}^n \times \mathbb{R}^n \,:\, |x-u| > \frac{1}{1+|x|} \right\}$$

and

$$L = \left\{ (x,u) \in \mathbb{R}^n \times \mathbb{R}^n \,:\, |x-u| < \frac{2}{1+|x|} \right\}.$$

Let further $\eta \geq 0$ be a smooth function on $\mathbb{R}^n \times \mathbb{R}^n$, such that $\eta(x,u) = 1$ if $(x,u) \notin G$ and $\eta(x,u) = 0$ if $(x,u) \notin L$. This function shall also satisfy

$$\left|\nabla_x\, \eta(x,u)\right| + \left|\nabla_u\, \eta(x,u)\right| \lesssim |u-x|^{-1}, \qquad (x,u) \in G \cap L. \tag{3.7}$$

The global part of $m_0(T)$ is defined by

$$m_0(T)^{\mathrm{glob}} f(x) = \int \mathcal{M}_0(x,u)\, (1-\eta(x,u))\, f(u)\, d\gamma_\infty(u), \qquad f \in \mathcal{C}_0^\infty(\mathbb{R}^n). \tag{3.8}$$

Our estimate in Proposition 5.4*(i)* below will show that this integral converges. The local part of $m_0(T)$ is

$$m_0(T)^{\mathrm{loc}} = m_0(T) - m_0(T)^{\mathrm{glob}};$$

It will be seen to have off-diagonal kernel $\mathcal{M}_0(x,u)\, \eta(x,u)$, in Section 7.

## 4. The time derivative of the Mehler kernel

We compute the derivative $\dot{K}_t = \partial_t K_t(x,u)$ and estimate it for small and large $t$. As a preparation, we work out the $t$ derivatives of some of the matrices introduced in the previous section.

**Lemma 4.1.** *For all $t > 0$ one has*

$$\dot{Q}_t = e^{tB}\, Q\, e^{tB^*}; \tag{4.1}$$

$$\frac{d}{dt}\, Q_t^{-1} = -Q_t^{-1}\, \dot{Q}_t\, Q_t^{-1} = -Q_t^{-1}\, e^{tB}\, Q\, e^{tB^*}\, Q_t^{-1}; \tag{4.2}$$

$$\frac{d}{dt}\, \det Q_t = \det Q_t\, \operatorname{tr}(Q_t^{-1}\, \dot{Q}_t) = \det Q_t\, \operatorname{tr}(Q_t^{-1}\, e^{tB} Q e^{tB^*}); \tag{4.3}$$

$$\dot{D}_t = -Q_\infty\, B^*\, e^{-tB^*}\, Q_\infty^{-1} = -Q_\infty\, B^*\, Q_\infty^{-1}\, D_t. \tag{4.4}$$

*Proof.* The equality (4.1) trivially follows from (2.1). To obtain (4.2), one differentiates the equation $Q_t\, Q_t^{-1} = I$ and applies (4.1). Since $Q_t$ is nonsingular, Jacobi's formula implies (4.3) (see [B, Fact 10.11.19]). Finally, we obtain the two equalities in (4.4) from (2.2). □

It will be convenient to have two different expressions for the $t$ derivative of the Mehler kernel, as follows.

**Lemma 4.2.** *For all $(x, u) \in \mathbb{R}^n \times \mathbb{R}^n$ and $t > 0$, we have*

$$\dot{K}_t(x, u) = K_t(x, u)\, N_t(x, u),$$

*where the function $N_t$ is given by*

$$\begin{aligned} N_t(x, u) = -\frac{1}{2} \operatorname{tr}\left(Q_t^{-1}\, e^{tB}\, Q\, e^{tB^*}\right) + \frac{1}{2}\, \left|Q^{1/2}\, e^{tB^*}\, Q_t^{-1}\, (u - D_t\, x)\right|^2 \\ + \left\langle Q_\infty\, B^*\, Q_\infty^{-1}\, D_t\, x\, ,\, (Q_t^{-1} - Q_\infty^{-1})\, (u - D_t\, x)\right\rangle, \end{aligned} \tag{4.5}$$

*and also by*

$$\begin{aligned} N_t(x, u) = &- \frac{1}{2} \operatorname{tr}\left(Q_t^{-1}\, e^{tB}\, Q\, e^{tB^*}\right) + \frac{1}{2}\, \left|Q^{1/2}\, e^{tB^*}\, Q_t^{-1}\, e^{tB}\, (D_{-t}\, u - x)\right|^2 \\ &- \left\langle Q_t^{-1}\, B\, e^{tB}\, (D_{-t}\, u - x)\, ,\, e^{tB}\, (D_{-t}\, u - x)\right\rangle \\ &- \left\langle Q_t^{-1}\, e^{tB}\, Q_\infty\, B^*\, Q_\infty^{-1}\, D_{-t}\, u\, ,\, e^{tB}\, (D_{-t}\, u - x)\right\rangle \\ &- \left\langle B^*\, Q_\infty^{-1}\, D_{-t}\, u\, ,\, D_{-t}\, u - x\right\rangle \\ =&: I_t + II_t(x, u) + III_t(x, u) + IV_t(x, u) + V_t(x, u). \end{aligned} \tag{4.6}$$

*Proof.* Differentiating (2.7) with respect to $t$ and applying Lemma 4.1, one obtains

$$\begin{aligned} &\dot{K}_t(x, u) = \\ &K_t(x, u)\Big[ -\frac{1}{2} \operatorname{tr}\left(Q_t^{-1}\, e^{tB}\, Q\, e^{tB^*}\right) + \frac{1}{2} \left\langle Q_t^{-1}\, e^{tB}\, Q\, e^{tB^*}\, Q_t^{-1} (u - D_t\, x)\, ,\, u - D_t\, x\right\rangle \\ &\qquad + \left\langle (Q_t^{-1} - Q_\infty^{-1})\, Q_\infty\, B^*\, Q_\infty^{-1}\, D_t\, x\, ,\, (u - D_t\, x)\right\rangle \Big], \end{aligned}$$

from which (4.5) follows.

Next, we differentiate (2.8), applying (4.3) to the first factor, and then use (2.3) to rewrite the matrix $D_t$ in the exponent. The result will be

$$\dot{K}_t(x, u) = K_t(x, u)\Big\{ -\frac{1}{2} \operatorname{tr}\left(Q_t^{-1}\, e^{tB}\, Q\, e^{tB^*}\right)$$

$$+\frac{d}{dt}\Big[-\frac{1}{2}\big\langle Q_t^{-1}\,e^{tB}\,(D_{-t}\,u-x)\,,\,(e^{tB}+Q_t\,e^{-tB^*}\,Q_\infty^{-1})(D_{-t}\,u-x)\big\rangle\Big]\Big\}. \quad (4.7)$$

The derivative here consists of two terms, the first term being

$$\begin{aligned}
&\frac{d}{dt}\Big[-\frac{1}{2}\big\langle Q_t^{-1}\,e^{tB}\,(D_{-t}\,u-x)\,,\,e^{tB}\,(D_{-t}\,u-x)\big\rangle\Big]\\
&=\frac{1}{2}\big\langle Q_t^{-1}\,e^{tB}\,Q\,e^{tB^*}\,Q_t^{-1}\,e^{tB}\,(D_{-t}\,u-x)\,,\,e^{tB}\,(D_{-t}\,u-x)\big\rangle\\
&-\big\langle Q_t^{-1}\,B\,e^{tB}\,(D_{-t}\,u-x)\,,\,e^{tB}\,(D_{-t}\,u-x)\big\rangle\\
&-\big\langle Q_t^{-1}\,e^{tB}\,Q_\infty\,B^*\,Q_\infty^{-1}\,D_{-t}\,u\,,\,e^{tB}\,(D_{-t}\,u-x)\big\rangle,
\end{aligned}$$

where we applied (4.2) and (4.4) with $t$ replaced by $-t$. Notice that we have arrived at the terms $II_t$, $III_t$ and $IV_t$ in (4.6).

In the second term coming from the derivative in (4.7), we observe some cancellation; the term equals

$$\frac{d}{dt}\Big[-\frac{1}{2}\big\langle D_{-t}\,u-x\,,\,Q_\infty^{-1}\,(D_{-t}\,u-x)\big\rangle\Big]=-\big\langle D_{-t}\,u-x\,,\,B^*\,Q_\infty^{-1}\,D_{-t}\,u)\big\rangle=V_t(x,u),$$

where we used again (4.4). Summing up, we obtain (4.6), and the lemma is proved. □

**Lemma 4.3.** *Let $x,u\in\mathbb{R}^n$. Then for $0<t\le 1$*

$$|N_t(x,u)|\lesssim\frac{1}{t}+\frac{|u-D_t\,x|^2}{t^2}+|x|\,\frac{|u-D_t\,x|}{t} \quad (4.8)$$

*and for $t\ge 1$*

$$|N_t(x,u)|\lesssim|D_{-t}\,u-x|\,|D_{-t}\,u|+e^{-ct}\,|D_{-t}\,u-x|^2+e^{-ct}. \quad (4.9)$$

*Proof.* For $0<t\le 1$, (4.8) follows from (4.5), by means of (2.5) and (2.6).

When $t\ge 1$ we get, starting from (4.6) and using (2.4) and (2.5),

$$|I_t|=\frac{1}{2}\left|\operatorname{tr}(Q_t^{-1}\,e^{tB}\,Q\,e^{tB^*})\right|\lesssim e^{-ct}.$$

Similarly, we have

$$|II_t(x,u)|=\frac{1}{2}\left|Q^{1/2}\,e^{tB^*}\,Q_t^{-1}\,e^{tB}\,(D_{-t}\,u-x)\right|^2\lesssim e^{-ct}\,|D_{-t}\,u-x|^2,$$

and also

$$|III_t(x,u)|\lesssim e^{-ct}\,\left|D_{-t}\,u-x\right|^2.$$

Proceeding as above, we further obtain

$$|IV_t(x,u)|+|V_t(x,u)|\lesssim|D_{-t}u-x|\,|D_{-t}u|,$$

and (4.9) is proved. □

## 5. On the multiplier kernel

In this section, we estimate some parts of the multiplier kernel and verify their relevance for the corresponding parts of the operator. We also state some facts that will simplify the proofs to come.

5.1. **Estimates of kernels.** Without loss of generality, it will be assumed from now on that

$$\|\varphi\|_\infty \leq 1.$$

We first invoke a lemma from [CCS3, Lemma 5.1 and Remark 5.5].

**Lemma 5.1.** *Let $\delta > 0$. For $\sigma \in \{1, 2, 3\}$ and $x, u \in \mathbb{R}^n$, one has*

$$\int_1^{+\infty} \exp\left(-\delta\,|D_{-t}\,u - x|^2\right)\left|D_{-t}\,u\right|^\sigma dt \lesssim 1 + |x|^{\sigma-1},$$

*where the implicit constant may depend on $\delta$, in addition to $n$, $Q$ and $B$.*

**Proposition 5.2.** *(i) The integral* (3.5) *defining $\mathcal{M}_\varepsilon$ converges absolutely for any $\varepsilon > 0$ and all $x, u \in \mathbb{R}^n$. Moreover,*

$$|\mathcal{M}_1(x, u)| \lesssim e^{R(x)}, \qquad x, u \in \mathbb{R}^n, \tag{5.1}$$

*and for $0 < \varepsilon < 1$*

$$|\mathcal{M}_\varepsilon(x, u)| \lesssim \varepsilon^{-C}\, e^{R(x)}\,(1 + |x|), \qquad x, u \in \mathbb{R}^n. \tag{5.2}$$

*(ii) For any $\varepsilon > 0$, any $f \in L^2_0(\gamma_\infty)$ and a.a. $x \in \mathbb{R}^n$,*

$$m_\varepsilon(T)f(x) = \int \mathcal{M}_\varepsilon(x, u)\, f(u)\, d\gamma_\infty(u). \tag{5.3}$$

*Proof.* Aiming at *(i)*, we begin by estimating the kernel $\dot{K}_t(x, u) = K_t(x, u)N_t(x, u)$. For $1 < t < +\infty$ we use (2.10) and (4.9). Then we can neglect the factors $|D_{-t}\,u - x|$ in $N_t(x, u)$ by also reducing slightly the positive coefficient in front of the same factor in the exponent in (2.10). As a result,

$$|\dot{K}_t(x, u)| \lesssim e^{R(x)} \exp\left(-c|D_{-t}\,u - x|^2\right)(|D_{-t}\,u| + e^{-ct}), \qquad t > 1. \tag{5.4}$$

Lemma 5.1 now implies (5.1).

For $0 < t < 1$ we use instead (2.9) and (4.8), and now we can neglect all powers of $|u - D_t\,x|^2/t$ in $N_t(x, u)$. This leads to

$$|\dot{K}_t(x, u)| \lesssim e^{R(x)}\, t^{-n/2} \exp\left(-\frac{|u - D_t\,x|^2}{t}\right)\left(t^{-1} + |x|t^{-1/2}\right) \lesssim e^{R(x)}\,(1 + |x|)\, t^{-n/2-1}. \tag{5.5}$$

Integrating over $\varepsilon < t < 1$ and combining the result with (5.1), we arrive at (5.2). The claimed convergence is now clear, so *(i)* is verified.

For item *(ii)*, we need the following lemma.

**Lemma 5.3.** *Let $f \in L^2_0(\gamma_\infty)$ and $x \in \mathbb{R}^n$. Then for any $t > 0$*

$$\partial_t \int K_t(x, u)\, f(u)\, d\gamma_\infty(u) = \int \dot{K}_t(x, u)\, f(u)\, d\gamma_\infty(u).$$

*Proof.* This is easily verified by integrating $\int \dot{K}_\tau(x, u)\, f(u)\, d\gamma_\infty(u)$ from $\tau = t_0$ to $\tau = t$ for some $t_0 \in (0, t)$. Then one swaps the order of integration and differentiates with respect to $t$. To justify this swap, (5.5) is enough for $\tau < 1$. For $\tau > 1$, one estimates the quantity $|D_{-t}\,u|$ in (5.4) by $|D_{-t}\,u - x| + |x|$ and cancels $|D_{-t}\,u - x|$

against the exponential. It follows that $|\dot{K}_t(x,u)| \lesssim e^{R(x)}\,(1+|x|)$. Since the measure $d\gamma_\infty$ is finite, these estimates allow us to apply Fubini, and the lemma is proved. $\square$

To verify item *(ii)* in the proposition, we observe that because of (5.2), the right-hand side of (5.3) defines for each $x$ a functional on $L^2_0(\gamma_\infty)$, whose norm is locally uniformly bounded for $x \in \mathbb{R}^n$. Further, the operator $m_\varepsilon(T)$ is bounded on the same space (see [CaD, Lemma 3.7]). Since the generalized eigenspaces $\mathcal{E}_\lambda$ together span $L^2_0(\gamma_\infty)$, it is enough to verify (5.3) on each $\mathcal{E}_\lambda$.

So let $f \in \mathcal{E}_\lambda$ for some $\lambda$. Since $e^{-tT}\,f(x) = \int K_t(x,u)f(u)\,d\gamma_\infty(u)$, Proposition 3.1 and Lemma 5.3 imply

$$\begin{aligned} m_\varepsilon(T)f(x) &= -\int_\varepsilon^\infty \varphi(t)\,\partial_t \int K_t(x,u)f(u)\,d\gamma_\infty(u)\,dt \\ &= -\int_\varepsilon^\infty \varphi(t) \int \dot{K}_t(x,u)f(u)\,d\gamma_\infty(u)\,dt. \end{aligned}$$

Switching the order of integration, we conclude the proof of *(ii)*. $\square$

**Proposition 5.4.** *(i) The integral* (3.6) *defining* $\mathcal{M}_0$ *converges for all* $x \neq u$, *and*

$$\mathcal{M}_0(x,u) \lesssim e^{R(x)}\,(1+|x|)^C\,|x-u|^{-C}, \qquad x \neq u. \tag{5.6}$$

*(ii) For any* $f \in L^2_0(\gamma_\infty)$ *and a.a.* $x \notin \operatorname{supp} f$,

$$m_0(T)f(x) = \int \mathcal{M}_0(x,u)\,f(u)\,d\gamma_\infty(u).$$

*Proof.* Since $\mathcal{M}_0$ and $m_0(T)$ only depend on the restriction of $\varphi$ to the interval $(0,1)$, we can assume in this proof that $\varphi$ vanishes for $t \geq 1$.

To verify *(i)*, consider the first inequality in (5.5). We have $|u - D_t\,x| \geq |u-x| - |x - D_t\,x|$, and Lemma 2.1 says that $|x - D_t\,x| \lesssim t|x|$. Thus $|u - D_t\,x| \geq |u-x|/2$ if $t < c|u-x|/|x|$ for some $c > 0$, and we conclude that for $0 < t < 1 \wedge c|u-x|/|x|$

$$|\dot{K}_t(x,u)| \lesssim e^{R(x)}\,\exp\left(-c\,\frac{|u-x|^2}{t}\right)(1+|x|)\,t^{-C} \lesssim e^{R(x)}\,(1+|x|)\,|u-x|^{-2C}. \tag{5.7}$$

We use this inequality to integrate $|\dot{K}_t(x,u)|$ over $0 < t < 1 \wedge (c|u-x|/|x|)$. For the integral over $c|u-x|/|x| < t < 1$, notice that (5.5) yields $|\dot{K}_t(x,u)| \lesssim e^{R(x)}\,(1+|x|)\,t^{-C}$. In this way, *(i)* follows.

To prove *(ii)*, we will let $\varepsilon \to 0$ in (5.3), and start by considering the right-hand side.

Because of (5.7), we see from (3.5) and (3.6) that, with $\varphi$ supported in $[0,1]$, one has $\mathcal{M}_\varepsilon(x,u) \to \mathcal{M}_0(x,u)$ as $\varepsilon \to 0$, for any $x \neq u$. In the integral in the right-hand side of (5.3), we thus have pointwise convergence, and $|f(u)|\,d\gamma_\infty(u)$ is a finite measure. Further, the estimate in *(i)* holds also for $\mathcal{M}_\varepsilon$, uniformly in $\varepsilon$. By bounded convergence, we conclude

$$\int \mathcal{M}_\varepsilon(x,u)f(u)\,d\gamma_\infty(u) \to \int \mathcal{M}_0(x,u)f(u)\,d\gamma_\infty(u), \qquad \varepsilon \to 0,$$

for $x \notin \operatorname{supp} f$. Moreover, the left-hand integral here is a function of $x$ which stays locally bounded in the complement of $\operatorname{supp} f$, uniformly in $\varepsilon$. So we also have convergence in the sense of distributions in $\mathbb{R}^n \setminus \operatorname{supp} f$.

To deal with the left-hand side of (5.3), we claim that $m_\varepsilon(T)f \to m_0(T)f$ in the sense of distributions in $\mathbb{R}^n$, as $\varepsilon \to 0$. This will end the proof of *(ii)*.

With $\nu$, $\Gamma$ and $\psi(z) = 1/(1+z^2)$ as in Section 3, we have

$$m_\varepsilon(T) = (1+T^2)\, \frac{1}{2\pi i} \int_\Gamma \frac{m_\varepsilon(z)}{1+z^2}\, (zI - T)^{-1}\, dz.$$

To prove the claim, we let $f \in L^2(\gamma_\infty)$ and take $g \in C_0^\infty(\mathbb{R}^n)$. It is enough to verify that

$$\langle m_\varepsilon(T)f,\, g\rangle \to \langle m(T)f,\, g\rangle, \qquad \varepsilon \to 0,$$

with the scalar products taken in $L^2(\gamma_\infty)$. Notice that it does not matter whether we consider the convergence of the functions $m_\varepsilon(T)f$ or the measures $m_\varepsilon(T)f\, d\gamma_\infty$. We have

$$\begin{aligned} \langle m_\varepsilon(T)f,\, g\rangle &= \left\langle (1+T^2)\, \frac{1}{2\pi i} \int_\Gamma \frac{m_\varepsilon(z)}{1+z^2}\, (zI - T)^{-1} f\, dz,\, g \right\rangle \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{m_\varepsilon(z)}{1+z^2}\, \big\langle (zI - T)^{-1} f\,,\, (1 + (T^*)^2)\, g \big\rangle\, dz, \end{aligned} \tag{5.8}$$

where $T^*$ is the adjoint of $T$ in $L^2(\gamma_\infty)$, so that $(1 + (T^*)^2)\, g$ is another test function in $C_0^\infty(\mathbb{R}^n)$. Now $m_\varepsilon(z) = z \int_\varepsilon^\infty \varphi(t)\, e^{-tz}\, dt$ tends to $m(z)$ for each nonzero $z \in \Gamma$. For such $z$ we also have the bound $|m_\varepsilon(z)| \le \|\varphi\|_\infty\, |z|/\Re z \lesssim 1$. In the last integral in (5.8), the integrand thus converges pointwise, and it is also dominated by constant times

$$\frac{1}{1+|z|^2}\, \|(zI - T)^{-1} f\|_{L^2(\gamma_\infty)}\, \|(1 + (T^*)^2)\, g\|_{L^2(\gamma_\infty)},$$

which is integrable along $\Gamma$ because of (3.3). The dominated convergence theorem now implies the claim and completes the proof of Proposition 5.4. $\square$

5.2. **Simplifications.** The preceding estimates allow some observations that will simplify the proof of Theorem 1.1.

In (1.5) we take $f \ge 0$ such that $\|f\|_{L^1(\gamma_\infty)} = 1$. This allows us to assume that $\alpha$ in the same estimate is large, in particular $\alpha > 2$, since $d\gamma_\infty$ is finite.

Further, we can focus mainly on points $x$ in the ellipsoidal annulus

$$\mathcal{C}_\alpha = \left\{ x \in \mathbb{R}^n : \frac{1}{2} \log \alpha \le R(x) \le 2 \log \alpha \right\}.$$

To justify this, we will follow closely the arguments in [CCS3, Section 6]. The first observation is that the set of points $x$ for which $R(x) > 2 \log \alpha$ can be neglected, because its $d\gamma_\infty$ measure is no larger than $C/\alpha$.

The following proposition deals with the remaining part of the complement of $\mathcal{C}_\alpha$.

**Proposition 5.5.** *Let $x \in \mathbb{R}^n$ satisfy $R(x) < \frac{1}{2}\log\alpha$, where $\alpha > 2$. Then for all $u \in \mathbb{R}^n$*

$$\left|\mathcal{M}_0^{\mathrm{glob}}(x,u)\right| + \left|\mathcal{M}_1(x,u)\right| \lesssim \alpha.$$

Thus we need to take the region $\{x : R(x) < \frac{1}{2}\log\alpha\}$ into account only when considering $m_0(T)^{\mathrm{loc}}$.

*Proof.* Assume $R(x) < \frac{1}{2}\log\alpha$. The estimate for $\mathcal{M}_1$ follows immediately from (5.1). If $(x,u) \in G$, (5.6) implies

$$\mathcal{M}_0(x,u) \lesssim e^{R(x)}\,(1+|x|)^C \lesssim \alpha.$$

The proposition is verified. □

## 6. The weak type $(1,1)$ for large $t$

**Proposition 6.1.** *For any $f \in L^1(\gamma_\infty)$ such that $\|f\|_{L^1(\gamma_\infty)} = 1$ and any $\alpha > 2$,*

$$\gamma_\infty\left\{x \in \mathcal{C}_\alpha : |m_1(T)f(x)| > \alpha\right\} \lesssim \frac{1}{\alpha\sqrt{\log\alpha}}.$$

*In particular, the operator $m_1(T)$ is of weak type $(1,1)$ with respect to the invariant measure $d\gamma_\infty$.*

The estimate in this proposition means that for large $\alpha$ one has a slightly stronger estimate than the classical weak type $(1,1)$ bound. This phenomenon was already observed for the Ornstein–Uhlenbeck maximal operator in [CCS2, Section 6] and for the first-order Riesz transforms in [CCS3, Proposition 7.1]).

*Proof.* We will first use our polar coordinates to deduce a sharper version of the estimate (5.1) in Proposition 5.2*(i)*. If $x \in \mathcal{C}_\alpha$ and $u \neq 0$, we can write $x = D_s\,\tilde{x}$ and $u = D_\sigma\,\tilde{u}$ with $\tilde{x}$, $\tilde{u} \in E_{(\log\alpha)/2}$ and $s \geq 0$, $\sigma \in \mathbb{R}$.

Applying [CCS2, Lemma 4.3 *(i)*], we obtain

$$|D_{-t}\,u - x| = |D_{\sigma-t}\,\tilde{u} - D_s\,\tilde{x}| \gtrsim |\tilde{x} - \tilde{u}|.$$

Thus (2.10) implies

$$K_t(x,u) \lesssim e^{R(x)}\,\exp\Big(-c\,|\tilde{x}-\tilde{u}|^2\Big)\,\exp\Big(-c\,|D_{-t}\,u - x|^2\Big),$$

for some $c$.

Using this estimate instead of (2.10), one can follow the proof of (5.1) with an extra factor $\exp\left(-c\,|\tilde{x}-\tilde{u}|^2\right)$. The result will be

$$\left|\mathcal{M}_1(x,u)\right| \lesssim e^{R(x)}\,\exp\big(-c\,\left|\tilde{x}-\tilde{u}\right|^2\big), \qquad x \in \mathcal{C}_\alpha.$$

We can now finish the proof of Proposition 6.1 by means of the following lemma, which is the case $\sigma = 1$ of [CCS3, Lemma 7.2].

**Lemma 6.2.** *Let $f \geq 0$ be normalized in $L^1(\gamma_\infty)$. For $\alpha > 2$*

$$\gamma_\infty\left\{x = D_s\,\tilde{x} \in \mathcal{C}_\alpha :\; e^{R(x)}\int \exp\big(-c\,\left|\tilde{x}-\tilde{u}\right|^2\big)\,f(u)\,d\gamma_\infty(u) > \alpha\right\} \lesssim \frac{C}{\alpha\,\sqrt{\log\alpha}}.$$

□

## 7. The local region

To prove the weak type (1,1) of the operator $m_0(T)^{\mathrm{loc}}$, we first show that its off-diagonal kernel is $\mathcal{M}_0(x,u)\,\eta(x,u)$, as hinted at the end of Subsection 3.2, where also $\eta$ is defined. According to Proposition 5.4*(ii)*, $m_0(T)$ has off-diagonal kernel $\mathcal{M}_0(x,u)$. The definition (3.8) of $m_0(T)^{\mathrm{glob}}$ says that its kernel is $\mathcal{M}_0(x,u)\,(1-\eta(x,u))$. Thus the off-diagonal kernel of $m_0(T)^{\mathrm{loc}} = m_0(T) - m_0(T)^{\mathrm{glob}}$ is

$$\mathcal{M}_0^{\mathrm{loc}}(x,u) := \mathcal{M}_0(x,u)\,\eta(x,u) = -\int_0^1 \varphi(t)\,\dot{K}_t(x,u)\,dt\;\eta(x,u). \tag{7.1}$$

We will now verify Calderón-Zygmund estimates for this kernel, and start by recalling a result proved in [CCS3].

**Lemma 7.1.** [CCS3, Lemma 8.1] *Let $p,\, r \ge 0$ with $p + r/2 > 1$, and $(x,u) \in L$ with $x \neq u$. Then for $\delta > 0$*

$$\int_0^1 t^{-p} \exp\left(-\delta\,\frac{|u - D_t\,x|^2}{t}\right) |x|^r\,dt \lesssim C(\delta,p,r)\,|u-x|^{-2p-r+2},$$

*where $C(\delta,p,r)$ may also depend on $n$, $Q$ and $B$.*

We will use expressions for some derivatives of $K_t$ taken from [CCS3, Lemmata 4.1 and 4.2]. First,

$$\partial_{x_\ell} K_t(x,u) = K_t(x,u)\,P_\ell(t,x,u),$$

where

$$P_\ell(t,x,u) = \langle Q_\infty^{-1}\,x\,,\,e_\ell\rangle + \big\langle Q_t^{-1}\,e^{tB}\,e_\ell\,,\,u - D_t\,x\big\rangle. \tag{7.2}$$

Further,

$$\partial_{u_\ell} K_t(x,u) = -K_t(x,u)\,\big\langle Q_t^{-1}\,e^{tB}\,(D_{-t}\,u - x)\,,\,e_\ell\big\rangle. \tag{7.3}$$

The following three technical lemmata give expressions and estimates for derivatives of $\dot{K}_t$.

**Lemma 7.2.** *For $x,\, u \in \mathbb{R}^n$ and $t > 0$, one has*

$$(i)\qquad \partial_{x_\ell}\,\dot{K}_t(x,u) = K_t(x,u)\,\mathcal{S}_\ell(t,x,u);$$
$$(ii)\qquad \partial_{u_\ell}\,\dot{K}_t(x,u) = K_t(x,u)\,\mathcal{R}_\ell(t,x,u),$$

*where the factors $\mathcal{S}_\ell(t,x,u)$ and $\mathcal{R}_\ell(t,x,u)$ are given by*

$$\begin{aligned}\mathcal{S}_\ell(t,x,u) &= N_t(x,u)\,P_\ell(t,x,u) - \big\langle Q_t^{-1}\,e^{tB}\,Q\,e^{tB^*}\,Q_t^{-1}\,e^{tB}\,e_\ell\,,\,u - D_t\,x\big\rangle \\ &\quad + \big\langle Q_t^{-1}\,B\,e^{tB}\,e_\ell\,,\,u - D_t\,x\big\rangle - \big\langle Q_t^{-1}\,e^{tB}\,e_\ell\,,\,Q_\infty\,B^*\,e^{-tB^*}\,Q_\infty^{-1}\,x\big\rangle,\end{aligned} \tag{7.4}$$

*and*

$$\begin{aligned}\mathcal{R}_\ell(t,x,u) &= -N_t(x,u)\,\big\langle Q_t^{-1}\,e^{tB}\,(D_{-t}\,u - x)\,,\,e_\ell\big\rangle \\ &\quad + \big\langle Q_t^{-1}\,e^{tB}\,Q\,e^{tB^*}\,Q_t^{-1}\,e^{tB}\,(D_{-t}\,u - x)\,,\,e_\ell\big\rangle \\ &\quad - \big\langle Q_t^{-1}\,B\,e^{tB}\,(D_{-t}\,u - x)\,,\,e_\ell\big\rangle - \big\langle Q_t^{-1}\,e^{tB}\,Q_\infty\,B^*\,Q_\infty^{-1}\,D_{-t}\,u\,,\,e_\ell\big\rangle.\end{aligned} \tag{7.5}$$

*Proof.* To prove *(i)*, we start by observing that

$$\begin{aligned}\partial_{x_\ell} \dot K_t(x,u) &= \partial_t \left(K_t(x,u)\, P_\ell(t,x,u)\right)\\ &= K_t(x,u)\, N_t(x,u)\, P_\ell(t,x,u) + K_t(x,u)\, \partial_t\, P_\ell(t,x,u)\\ &= K_t(x,u)\, N_t(x,u)\, P_\ell(t,x,u) + K_t(x,u)\, \partial_t \left(\left\langle Q_t^{-1}\, e^{tB}\, e_\ell\,,\; u - D_t\, x\right\rangle\right),\end{aligned}$$

where we used (7.2). Applying (4.2) and (4.4) to the last derivative here, one arrives at (7.4), and *(i)* is verified.

To prove *(ii)*, we proceed similarly, using (7.3) to write

$$\begin{aligned}\partial_{u_\ell} \dot K_t(x,u) = -K_t(x,u)\, N_t(x,u) &\left\langle Q_t^{-1}\, e^{tB}\left(D_{-t}\, u - x\right),\, e_\ell\right\rangle\\ &- K_t(x,u)\, \partial_t \left(\left\langle Q_t^{-1}\, e^{tB}\left(D_{-t}\, u - x\right),\, e_\ell\right\rangle\right).\end{aligned}$$

As in the case of *(i)*, this leads to (7.5) and *(ii)*. □

To bound $\mathcal{S}_\ell$ and $\mathcal{R}_\ell$, we recall from [CCS3, formula (4.5)] that

$$|P_\ell(t,x,u)| \lesssim |x| + |u - D_t\, x|/t, \qquad 0 < t \le 1. \tag{7.6}$$

**Lemma 7.3.** *One has for* $0 < t \le 1$ *and all* $x,\, u \in \mathbb{R}^n$

$$|\mathcal{S}_\ell(t,x,u)| \lesssim |x|\, \frac{|u - D_t\, x|^2}{t^2} + \frac{|u - D_t\, x|^3}{t^3} + |x|^2\, \frac{|u - D_t\, x|}{t} + \frac{|u - D_t\, x|}{t^2} + \frac{|x|}{t}.$$

*Proof.* We first bound the product $N_t(x,u)\, P_\ell(t,x,u)$ appearing in (7.4). Because of (4.8) and (7.6), we have for $0 < t \le 1$

$$\begin{aligned}\left|N_t(x,u) P_\ell(t,x,u)\right| &\lesssim \left(\frac{1}{t} + \frac{|u - D_t\, x|^2}{t^2} + |x|\, \frac{|u - D_t\, x|}{t}\right)\left(|x| + \frac{|u - D_t\, x|}{t}\right)\\ &\lesssim \frac{|x|}{t} + \frac{|u - D_t\, x|}{t^2} + |x|\, \frac{|u - D_t\, x|^2}{t^2} + \frac{|u - D_t\, x|^3}{t^3} + |x|^2\, \frac{|u - D_t\, x|}{t}.\end{aligned}$$

Estimating also the other terms in (7.4), one arrives at the lemma. □

**Lemma 7.4.** *For* $t \in (0,1]$

$$|\mathcal{R}_\ell(t,x,u))| \lesssim \frac{|u - D_t\, x|}{t^2} + \frac{|u - D_t\, x|^3}{t^3} + |x|\, \frac{|u - D_t\, x|^2}{t^2} + \frac{|x|}{t}.$$

*Proof.* For $t \in (0,1]$ we have by (7.5) and (4.8)

$$\begin{aligned}\left|\mathcal{R}_\ell(t,x,u)\right| &\lesssim \left|N_t(x,u)\right| \frac{|u - D_t\, x|}{t} + \frac{|u - D_t\, x|}{t^2} + \frac{|u - D_t\, x|}{t} + \frac{|u|}{t}\\ &\lesssim \left(\frac{1}{t} + \frac{|u - D_t\, x|^2}{t^2} + |x|\, \frac{|u - D_t\, x|}{t}\right)\frac{|u - D_t\, x|}{t} + \frac{|u - D_t\, x|}{t^2} + \frac{|x|}{t}.\end{aligned}$$

Here we estimated $|u|/t$ by $|u - D_t\, x|/t^2 + |x|/t$. The lemma follows. □

**Proposition 7.5.** *For* $(x,u) \in L$, *with* $x \ne u$, *one has*

$$\int_0^1 \left|\dot K_t(x,u)\right| dt \lesssim e^{R(x)}\, |u - x|^{-n}.$$

*Proof.* From (2.9) and (4.8) we obtain

$$\begin{aligned}
&\int_0^1 \big|\dot K_t(x,u)\big|\, dt \\
&\qquad \lesssim e^{R(x)} \int_0^1 t^{-\frac{n}{2}} \exp\left(-c\,\frac{|D_t\, x-u|^2}{t}\right) \left(\frac{1}{t}+\frac{|u-D_t\, x|^2}{t^2}+|x|\,\frac{|u-D_t\, x|}{t}\right) dt \\
&\qquad \lesssim e^{R(x)} \int_0^1 t^{-\frac{n}{2}} \exp\left(-c\,\frac{|D_t\, x-u|^2}{t}\right) \left(\frac{1}{t}+\frac{|x|}{\sqrt{t}}\right) dt.
\end{aligned}$$

Because of Lemma 7.1, the last integral is controlled by $e^{R(x)}\, |u-x|^{-n}$, and the proposition is proved. □

**Proposition 7.6.** *For any $x \neq u$, the following estimates hold:*

$$\big|\mathcal{M}_0^{\mathrm{loc}}(x,u)\big| \lesssim e^{R(x)}\, |u-x|^{-n}; \tag{7.7}$$

$$\big|\nabla_x\, \mathcal{M}_0^{\mathrm{loc}}(x,u)\big| \lesssim e^{R(x)}\, |u-x|^{-n-1}; \tag{7.8}$$

$$\big|\nabla_u\, \mathcal{M}_0^{\mathrm{loc}}(x,u)\big| \lesssim e^{R(x)}\, |u-x|^{-n-1}. \tag{7.9}$$

*Proof.* Multiplying the estimate of Proposition 7.5 by $\eta(x,u)$ and using (7.1), we get (7.7).

In order to prove (7.8), we observe that (7.1) and (3.7) lead to

$$\big|\partial_{x_\ell}\, \mathcal{M}_0^{\mathrm{loc}}(x,u)\big| \lesssim \int_0^1 \big|\varphi(t)\, \partial_{x_\ell}\, \dot K_t(x,u)\big|\, dt\, \eta(x,u) + \big|\mathcal{M}_0(x,u)\big|\, |x-u|^{-1}. \tag{7.10}$$

The last term here satisfies the desired estimate because of (7.7). Using Lemma 7.2(i) and then (2.9) and Lemma 7.3, we can estimate the first term by

$$\begin{aligned}
&\int_0^1 \big|K_t(x,u)\, \mathcal{S}_\ell(t,x,u)\big|\, dt \\
&\lesssim e^{R(x)} \int_0^1 t^{-\frac{n}{2}}\, \exp\left(-c\,\frac{|u-D_t\, x|^2}{t}\right) \\
&\quad \times \left(|x|\,\frac{|u-D_t x|^2}{t^2}+\frac{|u-D_t x|^3}{t^3}+|x|^2\,\frac{|u-D_t x|}{t}+\frac{|u-D_t x|}{t^2}+\frac{|x|}{t}\right) dt \\
&\lesssim e^{R(x)} \int_0^1 t^{-\frac{n}{2}}\, \exp\left(-c\,\frac{|u-D_t\, x|^2}{t}\right) \left(\frac{|x|}{t}+\frac{1}{t\sqrt{t}}+\frac{|x|^2}{\sqrt{t}}\right) dt.
\end{aligned}$$

Lemma 7.1 says that the last expression is controlled by $e^{R(x)}\, |u-x|^{-n-1}$, and (7.8) is proved.

The verification of (7.9) is analogous. Indeed, (7.10) remains valid for derivatives with respect to $u$, and from Lemma 7.4 it follows that $R_\ell(t,x,u)$ is controlled by the right-hand side in Lemma 7.3. This implies (7.9) and ends the proof of Proposition 7.6. □

We now arrive at the goal of this section.

**Proposition 7.7.** *The operator $m_0(T)^{\mathrm{loc}}$ is of weak type $(1,1)$ with respect to the invariant measure $d\gamma_\infty$.*

*Proof.* Proposition 7.6 means that the off-diagonal kernel $\mathcal{M}_0^{\mathrm{loc}}(x,u)$ of $m_0(T)^{\mathrm{loc}}$ satisfies standard Calderón-Zygmund bounds. Thus it is enough to verify that $m_0(T)^{\mathrm{loc}}$ is bounded on $L^2(\gamma_\infty)$. For $m_0(T)$, which is of Laplace type, the $L^2$ boundedness follows from [CaD, Lemma 3.7]. We remark that this boundedness also follows from some results in [CFMP1] and [CFMP2], which can be applied here since [MPRS, Lemma 2.2] exhibits a linear change of coordinates in $\mathbb{R}^n$ reducing the setting to the case where $Q = I$ and $Q_\infty$ is a diagonal matrix.

To go from $m_0(T)$ to $m_0(T)^{\mathrm{loc}}$, we need the following lemma which gives a partition of $\mathbb{R}^n$ into cubes of local size. It is a rather standard construction; cf. [GMST1, Lemma 2.4] or [S, Lemma 4].

**Lemma 7.8.** *One can cover $\mathbb{R}^n$ by a sequence of cubes $Q_j$ which are pairwise disjoint except for boundaries and have the properties stated below. We let $c_j$ be the center of $Q_j$, and $2Q_j$ denotes the concentric cube scaled by a factor 2.*

*(1) For each $j$*

$$\frac{1}{16(1+|c_j|)} < \operatorname{diam} Q_j \le \frac{1}{4(1+|c_j|)}$$

*(2) For any $A > 0$, the balls $B(c_j, A/(1+|c_j|))$ have bounded overlap, with a bound that depends only on $A$ and $n$.*

*(3) If $x \in 2Q_j$ and $u \in Q_j$, then $|x-u| \le 1/(1+|x|)$ and thus $\eta(x,u) = 1$.*

*(4) If $u \in Q_j$ and $x \notin B(c_j, 45/(1+|c_j|))$, then $|x-u| \ge 2/(1+|x|)$ and thus $\eta(x,u) = 0$.*

Before proving this lemma, we use it to finish the proof of Proposition 7.7, by deducing the $L^2$ boundedness of $m_0(T)^{\mathrm{loc}}$ from that of $m_0(T)$.

We split a given function $f \in L^2(\gamma_\infty)$ as $f = \sum f_j$ with $f_j = f\,\chi_{Q_j}$. Item *(3)* of the lemma shows that $m_0(T)^{\mathrm{glob}} f_j = 0$ in $2Q_j$, so that $m_0(T)^{\mathrm{loc}} f_j = m_0(T) f_j$ in $2Q_j$. The weak type bound for $m_0(T)$ thus implies that

$$\|\chi_{2Q_j}\, m_0(T)^{\mathrm{loc}} f_j\|_{1,\infty} \lesssim \|f_j\|_1, \tag{7.11}$$

where we refer to the measure $d\gamma_\infty$.

Item *(4)* of the lemma shows that $m_0(T)^{\mathrm{loc}} f_j$ is supported in $B(c_j, 45/(1+|c_j|))$. But if $x \in B\left(c_j, 45/(1+|c_j|)\right) \setminus 2Q_j$ and $u \in Q_j$, then $|u-x| \simeq 1/(1+|c_j|)$, and (7.7) says that $\left|\mathcal{M}_0^{\mathrm{loc}}(x,u)\right| \lesssim e^{R(x)}\,(1+|c_j|)^n$. We conclude that $\left|m_0(T)^{\mathrm{loc}} f_j(x)\right|$ is no larger than const.$e^{R(x)}$ times the mean value of the function $\left|f_j(u)\, e^{-R(u)}\right|$ in $Q_j$. Since the density of the invariant measure is essentially constant in $B(c_j, 45/(1+|c_j|))$, this easily implies that the restriction of $m_0(T)^{\mathrm{loc}}$ is bounded from $L^1(Q_j)$ into $L^{1,\infty}(B\left(c_j, 45/(1+|c_j|)\right)) \setminus 2Q_j$, with respect to the measure $d\gamma_\infty$. Thus we can suppress the factor $\chi_{2Q_j}$ in (7.11).

The bounded overlap in Lemma 7.8*(2)* will now allow us to add the functions $m_0(T)^{\mathrm{loc}} f_j(x)$ in $L^{1,\infty}(\gamma_\infty)$ and control the quasinorm of the sum in terms of the norm of $f = \sum f_j$ in $L^1(\gamma_\infty)$. The proposition is proved, modulo Lemma 7.8. ☐

*Proof of Lemma 7.8* We start with the lattice of unit cubes with vertices in $\mathbb{Z}^n$. These cubes are repeatedly split into $2^n$ subcubes in the obvious way. This splitting is continued as long as the cubes satisfy $\operatorname{diam} Q > 1/[4(1+|c_Q|)]$, where $c_Q$ is the center of the cube $Q$. When we arrive at a cube for which $\operatorname{diam} Q \le 1/[4(1+|c_Q|)]$, this cube will be selected as one of the cubes $Q_j$ in the sequence to be constructed, and it is not split any further. It is easy to see that this leads to a sequence $(Q_j)_j$ giving a partition of $\mathbb{R}^n$, and the right-hand inequality in item *(1)* will be satisfied.

To verify the left-hand estimate of item *(1)*, assume that the selected cube $Q_j$ arose from splitting the cube $Q'$ with center $c'$. Then

$$|c_j - c'| = \frac{1}{2}\operatorname{diam} Q_j \le \frac{1}{8(1+|c_j|)} \le \frac{1}{8} \le \frac{1}{8\operatorname{diam} Q'},$$

the last step since $\operatorname{diam} Q' = 2\operatorname{diam} Q_j < 1$. It follows that

$$1+|c_j| \ge 1+|c'| - |c_j - c'| > \frac{1}{4\operatorname{diam} Q'} - \frac{1}{8\operatorname{diam} Q'} = \frac{1}{16\operatorname{diam} Q_j},$$

and item *(1)* is proved.

We move to the bounded overlap in item *(2)*. Fix $x \in \mathbb{R}^n$, and assume that the ball $B(c_j, A/(1+|c_j|))$ contains $x$. Then

$$1+|x| \le 1+|c_j| + |x-c_j| \le 1+|c_j| + \frac{A}{1+|c_j|} \le (1+A)(1+|c_j|),$$

so that $1/(1+|c_j|) \le (A+1)/(1+|x|)$. Swapping $x$ and $c_j$ in this argument, we also get $1/(1+|x|) \le (A+1)/(1+|c_j|)$. In view of item *(1)* then

$$Q_j \subset B(x, |x-c_j| + \operatorname{diam} Q_j) \subset B\left(x, \frac{A+1}{1+|c_j|}\right) \subset B\left(x, \frac{(A+1)^2}{1+|x|}\right).$$

Again because of item *(1)*, the volume of $Q_j$ satisfies

$$|Q_j| \gtrsim \frac{1}{(1+|c_j|)^n} \gtrsim \frac{1}{(A+1)^n(1+|x|)^n},$$

with implicit constants depending only on the dimension. Since the $Q_j$ are pairwise disjoint, comparison of volumes gives a bound on the number of possible $Q_j$ here, and item *(2)* follows.

Let now $x$ and $u$ be as in item *(3)*. Then $1+|x| \le 1+|c_j| + |x-c_j| \le 1+|c_j| + \operatorname{diam} Q_j \le 2(1+|c_j|)$, and so

$$|x-u| \le |x-c_j| + |u-c_j| \le \operatorname{diam} Q_j + \frac{1}{2}\operatorname{diam} Q_j \le \frac{3}{8\,(1+|c_j|)} < \frac{1}{1+|x|}.$$

This implies item *(3)*, since here $\eta(x,u) = 1$ by the definition of $\eta$.

Let finally $x$ and $u$ be as in item *(4)*. Assume first that $|x| \ge |c_j|/2$. Then $1+|c_j| \le 2(1+|x|)$, and

$$|x-u| \ge |x-c_j| - |u-c_j| \ge \frac{45}{1+|c_j|} - \frac{1}{4\,(1+|c_j|)} \ge \frac{2}{1+|x|}.$$

Assume next that $|x| < |c_j|/2$. Then $45/(1+|c_j|) < |x-c_j| \le 3|c_j|/2$, so that $|c_j|+|c_j|^2 > 30$ and $|c_j| > 5$. We get

$$|x-u| \ge |x-c_j| - |u-c_j| > \frac{|c_j|}{2} - \frac{1}{4\,(1+|c_j|)} > 2 \ge \frac{2}{1+|x|},$$

and item *(4)* is verified. □

## 8. An auxiliary bound for $0 < t \le 1$

In this section, we verify a bound on the number of zeros of the $t$ derivative of $K_t$ in the interval $(0,1]$, which will be used in the next section to control the kernel $\mathcal{M}_0^{\mathrm{glob}}$.

**Proposition 8.1.** *For $(x,u) \in \mathbb{R}^n \times \mathbb{R}^n$, the number of zeros in $I = (0,1]$ of the function $t \mapsto \dot{K}_t(x,u)$ is bounded by a positive integer depending only on $n$.*

*Proof.* Instead of $\dot{K}_t(x,u)$ we consider $\mathcal{N}_t(x,u) = 2(\det Q_t)^2\, N_t(x,u)$, since the three kernels $\dot{K}_t(x,u)$, $N_t(x,u)$ and $\mathcal{N}_t(x,u)$ have exactly the same zeros in $I$. From (4.5) we have

$$\begin{aligned}\mathcal{N}_t(x,u) = &-(\det Q_t)\,\mathrm{tr}\left((\det Q_t)Q_t^{-1}\, e^{tB}\, Q\, e^{tB^*}\right) \\ &+ \left\langle Q\, e^{tB^*}\,(\det Q_t)Q_t^{-1}\,(u - D_t\, x)\,,\, e^{tB^*}\,(\det Q_t)Q_t^{-1}\,(u-D_t\,x)\right\rangle \\ &- 2(\det Q_t)\left\langle Q_\infty\, B^*\, Q_\infty^{-1}\, D_t\, x\,,\, \left((\det Q_t)Q_t^{-1} - (\det Q_t)Q_\infty^{-1}\right)(u - D_t\,x)\right\rangle ;\end{aligned} \tag{8.1}$$

notice that here we have placed a factor $\det Q_t$ at each occurrence of $Q_t^{-1}$.

We split the argument into several claims.

Denote by $\lambda_j,\ j=1,\dots,J$, the eigenvalues of $B$. Notice that $\Re\lambda_j < 0$ for each $j$ and that the nonreal eigenvalues come in conjugate pairs.

**Claim 8.2.** *The function $t \mapsto \mathcal{N}_t(x,u)$ is a finite linear combination, with coefficients depending on $(x,u)$, of terms which are given by a product of type $\prod_{j=1}^J e^{m_j\lambda_j t}$ multiplied by a polynomial in $t$ with complex coefficients. Here $m_j \in \mathbb{Z}$. Further, the quantities $|m_j|$ and the degrees of the polynomials are all bounded by a constant depending only on $n$.*

*Proof.* Inspection shows that the last two terms in (8.1) are sums of scalar products of vectors given by multiplying $x$ or $u$ from the left by various combinations of the matrices $e^{tB}$, $e^{tB^*}$, $D_t$, $Q_t$ and $(\det Q_t)Q_t^{-1}$, the constant matrices $B^*$, $Q$, $Q_\infty$ and $Q_\infty^{-1}$, and the scalar factor $\det Q_t$. The first term in (8.1) is instead the trace of the product of some of these matrices, multiplied by $\det Q_t$. Let us examine precisely how the matrices listed here depend on $t$.

We pass from $\mathbb{R}^n$ to $\mathbb{C}^n$ and make a Jordan decomposition of $B$ via a change of coordinates in $\mathbb{C}^n$. Each Jordan block is of the form $\lambda_j(I+R)$, where $R$ is a supertriangular and thus nilpotent matrix and $I$ is the identity matrix, of some dimension. Then $\exp(t\lambda_j(I+R)) = e^{\lambda_j t}P(t)$, where $P(t)$ is a matrix with polynomial entries in $t$. To arrive at $\exp(tB)$, we put these blocks together and then change coordinates back. The result will be that in the coordinates we had before, each entry of the matrix $\exp(tB)$ is a sum over $j$ of terms of type $e^{\lambda_j t}p(t)$, where $p(t)$ is a complex polynomial that may depend on $j$ and on the entry considered. The same will be true for the entries of its adjoint $\exp(tB^*)$. From (2.2) we then see that $D_t$ is of the same form but with $e^{-\lambda_j t}$ instead of $e^{\lambda_j t}$. Considering the integral in (2.1), we see that the matrix $Q_t$ has similar entries, now with terms $e^{(\lambda_j+\lambda_{j'})t}p(t)$. Since the entries of the matrix $(\det Q_t)Q_t^{-1}$ are given by minors of $Q_t$, they will be a sum of terms which are like those described in Claim 8.2. Finally, the scalar $\det Q_t$ also has the same structure.

Claim 8.2 now follows, since the bound on the $|m_j|$ and the degrees of the polynomials is easily verified. $\square$

We observe that Claim 8.2 implies that $\mathcal{N}_t(x,u)$ can be extended to an entire function in $t$, and so the number of zeros in $(0,1]$ is finite.

**Claim 8.3.** *The function $t \mapsto \mathcal{N}_t(x,u),\ t>0$, is for each $(x,u)$ a solution of a linear ordinary differential equation in $t$, with real coefficients independent of $t$, $x$ and $u$. The order of this differential equation is bounded by a constant depending only on $n$.*

*Proof.* The preceding claim says that $\mathcal{N}_t(x,u)$ is a sum of terms given by a function of $(x,u)$ times an expression

$$\exp\left(\sum_j m_j\lambda_j t\right) P(t),$$

where the coefficients of the polynomial $P(t)$ may be complex. As a function of $t$, each such expression is annihilated by a linear differential operator with constant complex coefficients. The product of these operators will clearly annihilate $\mathcal{N}_t(x,u)$. It also follows that the order of this product operator is bounded by a constant depending only on $n$. The claim is verified, except that we found a differential operator with complex coefficients. But $\mathcal{N}_t(x,u)$ is real-valued, so one can simply delete the imaginary parts of all the coefficients of the operator. $\square$

We write the equation found in Claim 8.3 as $\mathcal{P}(D)\,\mathcal{N}_t(x,u)=0$, where $D=d/dt$ and $\mathcal{P}$ is a real polynomial with leading coefficient 1.

Proposition 8.1 is thus reduced to showing that the number of zeros of a real-valued solution of the equation $\mathcal{P}(D)\phi=0$ in $I=(0,1]$ is bounded by a constant depending only on the degree of $\mathcal{P}$.

We factorize $\mathcal{P}(D)$ into the commuting product of first-order operators of the form

$$T_\lambda = D-\lambda$$

and second-order operators of the form

$$S_{\lambda,\mu} = (D-\lambda)^2+\mu,$$

with $\lambda\in\mathbb{R}$ and $\mu>0$.

Our next claim deals with these factors of $\mathcal{P}(D)$.

**Claim 8.4.** *Let $\lambda\in\mathbb{R}$ and $\mu>0$, and let $J\subset\mathbb{R}$ be a closed interval of length less than $1/\sqrt{\mu}$. Assume $\phi\in C^2(J)$ is a real-valued function. If $S_{\lambda,\mu}\,\phi$ does not vanish in the interior $J^\circ$ of $J$, then $\phi$ has at most two zeros in $J$. Further, if $S_{\lambda,\mu}\phi$ has at* *most $k$ zeros in $J$, then $\phi$ has at most $2k+2$ zeros in the same interval. The same statements hold with $S_{\lambda,\mu}$ replaced by $T_\lambda$.*

*Proof.* To prove the first assertion about $S_{\lambda,\mu}$, we may take $\lambda = 0$ since

$$S_{0,\mu}\,\phi(t) = e^{-\lambda t}\, S_{\lambda,\mu}\big(e^{\lambda t}\,\phi(t)\big),$$

and we will write $S_\mu$ for $S_{0,\mu}$. The same trick applies to $T_\lambda$.

Assuming that $S_\mu\,\phi \neq 0$ in $J^\circ$, we may as well take $S_\mu\,\phi > 0$ there. We assume by contradiction that $t_1 < t_2 < t_3$ are three zeros of $\phi$ in $J$. Then $\phi''(t_2) = S_\mu\,\phi(t_2) > 0$. We may assume $\phi'(t_2) \geq 0$, since otherwise we consider instead the function $\phi(-t)$ in the interval $-J$. For $t > t_2$ sufficiently close to $t_2$ we then have

$$\phi(t) = \phi'(t_2)(t-t_2) + \frac{1}{2}\,\phi''(t_2)(t-t_2)^2 + o\left((t-t_2)^2\right) > 0.$$

Since $\phi(t_3) = 0$, the maximal value $M$ of $\phi$ in the interval $[t_2, t_3]$ must be assumed at some point $t_M \in (t_2, t_3)$. Clearly $M > 0$ and $\phi'(t_M) = 0$. An integration by parts yields

$$\begin{aligned} M = \int_{t_2}^{t_M} \phi'(t)\,dt &= (t-t_2)\phi'(t)|_{t_2}^{t_M} - \int_{t_2}^{t_M} (t-t_2)\phi''(t)\,dt \\ &= -\int_{t_2}^{t_M} (t-t_2)\phi''(t)\,dt. \end{aligned}$$

Since here $-\phi''(t) = \mu\phi(t) - S_\mu\,\phi(t) < \mu\phi(t) \leq \mu M$ we conclude that

$$M \leq \mu M \int_{t_2}^{t_M} (t-t_2)dt = \mu M\,\frac{(t_M-t_2)^2}{2} \leq \frac{1}{2}\,\mu M|J|^2.$$

This leads to the contradiction $|J| \geq \sqrt{2/\mu}$, which proves the first assertion of the claim. The second assertion follows from the first, applied in each of the intervals obtained by deleting from $J$ the zeros of $\phi$.

For $T_\lambda$ it is enough to apply Rolle's theorem to $T_0 = D$. □

*Conclusion of the proof of Proposition 8.1.* By Claim 8.3 we know that $\mathcal{N}_t(x,u)$ satisfies a differential equation in $t$ of the form

$$\prod_{i=1}^{K} P_i(D)\,\mathcal{N}_t(x,u) = 0,$$

where each $P_i(D)$ is some $T_{\lambda_i}$ or $S_{\lambda_i,\mu_i}$, with $\lambda_i \in \mathbb{R}$ and $\mu_i > 0$ in the case of $S_{\lambda_i,\mu_i}$. We can assume that none of the operators $P_i(D)$ can be deleted in this equation.

Choose a natural number $\kappa$ such that $\kappa^2$ is larger than all the $\mu_i$ appearing here. Then split $[0,1]$ into $\kappa$ closed subintervals of length $1/\kappa$, and let $J$ be one of these subintervals. Observe that Claim 8.4 applies to $J$, since $1/\kappa < 1/\sqrt{\mu_i}$ for each $i$.

Set for $m \in \{2, 3, \ldots, K\}$

$$\mathcal{N}_t^{(m)}(x,u) = \prod_{i=m}^{K} P_i(D)\,\mathcal{N}_t(x,u),$$

and $\mathcal{N}_t^{(K+1)}(x,u) = \mathcal{N}_t(x,u)$.

We will prove by induction that the function $t \mapsto \mathcal{N}_t^{(m)}(x,u)$ has at most $2^m - 2$ zeros in $J$, for $m \in \{2, 3, \dots, K+1\}$. Here we fix $(x,u)$. Proposition 8.1 will then follow from the case $m = K+1$.

Starting with $m = 2$, we have $P_1(D)\,\mathcal{N}_t^{(2)}(x,u) = 0$, and $\mathcal{N}_t^{(2)}(x,u)$ is not identially 0 for all $t$. By means of a conjugation with the factor $e^{\lambda_1 t}$ as in the proof of Claim 8.4, we can assume that $\lambda_1 = 0$. If $P_1(D)$ is $T_0 = D$, then $\mathcal{N}_t^{(2)}(x,u)$ is a nonzero constant, and if $P_1(D) = S_{0,\mu_1}$ we assume that $t = t_0 \in J$ is a zero of $\mathcal{N}_t^{(2)}(x,u)$. Then $\mathcal{N}_t^{(2)}(x,u)$ is proportional to $\sin\big((t-t_0)\sqrt{\mu_1}\big)$ and can have no other zero in $J$, because $|J| < 1/\sqrt{\mu}_1$. The first induction step is verified.

Assume the induction step holds for $m$. Then $P_m(D)\,\mathcal{N}_t^{(m+1)}(x,u) = \mathcal{N}_t^{(m)}(x,u)$ has at most $2^m - 2$ zeros in $J$, and Claim 8.4 implies that the number of zeros of $\mathcal{N}_t^{(m+1)}(x,u)$ in $J$ is at most $2(2^m-2)+2 = 2^{m+1}-2$. The induction is complete, and so is the proof of Proposition 8.1. $\square$

## 9. Estimates in the global region for small $t$

In this section we estimate the operator $m_0^{\mathrm{glob}}(T)$ with kernel

$$-\int_0^1 \varphi(t)\dot{K}_t(x,u)\big(1-\eta(x,u)\big)\,dt.$$

We shall need the following theorem. In order not to burden the exposition, we postpone its proof to the appendix.

**Theorem 9.1.** *The maximal operator defined by*

$$S_0^{\mathrm{glob}} f(x) = \int \sup_{0<t\le 1} K_t(x,u)\,\big(1-\eta(x,u)\big)\,|f(u)|\,d\gamma_\infty(u)$$

*is of weak type* $(1,1)$ *with respect to the invariant measure* $d\gamma_\infty$*.*

This is a sharpened version of the weak type $(1,1)$ estimate for the corresponding part of the maximal operator treated in [CCS2], since the supremum in $t$ is now placed inside the integral. As a consequence, we can prove the following result, which will complete the proof of Theorem 1.1.

**Proposition 9.2.** *The operator* $m_0^{\mathrm{glob}}(T)$ *is of weak type* $(1,1)$ *with respect to the invariant measure* $d\gamma_\infty$*.*

*Proof of Proposition 9.2.* Let $N(x,u)$ be the number of zeros in $(0,1)$ of the function $t \mapsto \dot{K}_t(x,u)$. Proposition 8.1 says that $N(x,u) \le \bar{N}$ for some constant $\bar{N} \ge 1$ that is independent of $(x,u) \in \mathbb{R}^n \times \mathbb{R}^n$. We denote these zeros by $t_1(x,u) < \dots < t_{N(x,u)}(x,u)$, and set $t_0(x,u) = 0$, $t_{N(x,u)+1}(x,u) = 1$. Since $K_t(x,u)$ vanishes at $t = 0$, it follows from the fundamental theorem of calculus that

$$\int_0^1 \left|\dot{K}_t(x,u)\right| dt = \sum_{i=0}^{N(x,u)} \left| \int_{t_i(x,u)}^{t_{i+1}(x,u)} \dot{K}_t(x,u)dt \right|$$

$$= \sum_{i=0}^{N(x,u)} \left| K_{t_{i+1}(x,u)}(x,u) - K_{t_i(x,u)}(x,u) \right|$$

$$\leq 2 \sum_{i=0}^{N(x,u)+1} K_{t_i(x,u)}(x,u) \lesssim \bar{N} \sup_{0<t\leq 1} K_t(x,u).$$

This inequality implies

$$\left| m_0^{\mathrm{glob}}(T) f(x) \right| \leq \int \int_0^1 \left| \dot{K}_t(x,u) \right| \, dt \; (1-\eta(x,u)) \, |f(u)| \, d\gamma_\infty(u)$$

$$\lesssim \bar{N} \int \sup_{0<t\leq 1} K_t(x,u) \; (1-\eta(x,u)) \, |f(u)| \, d\gamma_\infty(u),$$

and Theorem 9.1 yields

$$\gamma_\infty \left\{ x : \left| m_0^{\mathrm{glob}}(T) f(x) \right| > \alpha \right\} \lesssim \frac{1}{\alpha} \int |f(u)| \, d\gamma_\infty(u).$$

$\square$

## 10. Appendix: Proof of Theorem 9.1

In the proof of this theorem, we take $f \geq 0$ normalized in $L^1(\gamma_\infty)$. All the simplifications introduced in Subsection 5.2 will apply. In particular, we let $\alpha$ be large, and we need only consider points $x$ in $\mathcal{C}_\alpha$. We will write $x$ and $u \neq 0$ as $x = D_s \, \tilde{x}$ and $u = D_\sigma \, \tilde{u}$, respectively, where $\tilde{x}$, $\tilde{u} \in E_\beta$ with $\beta = (\log \alpha)/2$ and $s \geq 0$, $\sigma \in \mathbb{R}$.

**Lemma 10.1.** *Assume that* $(x,u) \in G$ *and* $x \in \mathcal{C}_\alpha$. *Then*

$$\sup_{0<t\leq 1} K_t(x,u) \lesssim e^{R(x)} \min \left( |\tilde{u} - \tilde{x}|^{-n}, \, |x|^n \right).$$

*Proof.* For the first bound, we use [CCS2, Lemma 4.3(i)] to get $|D_t \, x - u| \gtrsim |\tilde{x} - \tilde{u}|$, which by (2.9) yields

$$\sup_{0<t\leq 1} K_t(x,u) \lesssim e^{R(x)} \sup_{0<t\leq 1} t^{-n/2} \exp \left( -c \, \frac{|\tilde{x} - \tilde{u}|^2}{t} \right) \lesssim e^{R(x)} \, |\tilde{x} - \tilde{u}|^{-n}.$$

To get the second bound, we deduce from the definition of $G$ and Lemma 2.1 that

$$|x|^{-1} \lesssim |x - u| \leq |x - D_t \, x| + |D_t \, x - u| \lesssim t|x| + |D_t \, x - u|.$$

Thus $|x|^{-1} \lesssim t|x|$ or $|x|^{-1} \lesssim |D_t \, x - u|$. In the first case, $t^{-n/2} \lesssim |x|^n$, and the desired estimate is immediate from (2.9). In the second case,

$$K_t(x,u) \lesssim e^{R(x)} \, t^{-\frac{n}{2}} \, \exp \left( -\frac{c}{t|x|^2} \right) \lesssim e^{R(x)} \, |x|^n.$$

The lemma is proved. $\square$

To continue the proof of Theorem 9.1, we conclude from Lemma 10.1 that for $x \in \mathcal{C}_\alpha$

$$S_0^{\mathrm{glob}} f(x) \lesssim e^{R(x)} \int \min\left(|\tilde u - \tilde x|^{-n},\, |x|^n\right) f(u)\, d\gamma_\infty(u) = A(x) + B(x),$$

where

$$A(x) = |x|^n\, e^{R(x)} \int_{\{u:\, |x| \le |\tilde u - \tilde x|^{-1}\}} f(u)\, d\gamma_\infty(u)$$

and

$$B(x) = e^{R(x)} \int_{\{u:\, |x| > |\tilde u - \tilde x|^{-1}\}} |\tilde u - \tilde x|^{-n} f(u)\, d\gamma_\infty(u).$$

We will show that

$$\gamma_\infty \{x \in \mathcal{C}_\alpha : A(x) > \alpha\} \lesssim \alpha^{-1} \tag{10.1}$$

and

$$\gamma_\infty \{x \in \mathcal{C}_\alpha : B(x) > \alpha\} \lesssim \alpha^{-1}. \tag{10.2}$$

Starting with (10.1), we first observe that $A(\tilde x) < \alpha$ for $\tilde x \in E_\beta$ with $\beta = (\log \alpha)/2$, because

$$A(\tilde x) \le |\tilde x|^n\, e^{R(\tilde x)} \int_{\mathbb{R}^n} f(u)\, d\gamma_\infty(u) \lesssim (\log \alpha)^n \sqrt{\alpha} < \alpha\,,$$

and $\alpha$ is large. Further, $x = D_s\, \tilde x \in \mathcal{C}_\alpha$ implies $0 < s \lesssim 1$ in view of [CCS2, formula (4.3)]. Let

$$E_\beta^0 = \{\tilde x \in E_\beta : A(D_s\, \tilde x) > \alpha \ \text{ for some } s > 0 \text{ with } \ D_s\, \tilde x \in \mathcal{C}_\alpha\},$$

and define for $\tilde x \in E_\beta^0$

$$s_0(\tilde x) = \inf\{s : D_s\, \tilde x \in \mathcal{C}_\alpha \ \text{ and } \ A(D_s\, \tilde x) > \alpha\}.$$

Then $0 < s_0(\tilde x) \lesssim 1$ and $A(D_{s_0(\tilde x)}\, \tilde x) = \alpha$. Moreover, if $A(D_s\, \tilde x) > \alpha$ for some $D_s\, \tilde x \in \mathcal{C}_\alpha$, then $\tilde x \in E_\beta^0$ and $s > s_0(\tilde x)$. In the set $\mathcal{C}_\alpha$, the expression (2.11) for the Lebesgue measure yields $dx \simeq \sqrt{\log \alpha}\, dS_\beta\, ds$, and so

$$\gamma_\infty \{x \in \mathcal{C}_\alpha : A(x) > \alpha\} \lesssim \sqrt{\log \alpha} \int_{E_\beta^0} \int_{s_0(\tilde x)}^{C} e^{-R(D_s\, \tilde x)}\, ds\, dS_\beta(\tilde x).$$

We now write $R(D_s\, \tilde x) = R(D_{s_0(\tilde x)}\, \tilde x) + R(D_s\, \tilde x) - R(D_{s_0(\tilde x)}\, \tilde x)$ and apply the Mean Value Theorem to the difference here, observing that $\partial_s R(D_s\, \tilde x) \simeq |D_s\, \tilde x|^2 \simeq \log \alpha$ because of [CCS2, formula (4.3)]. This leads to

$$\begin{aligned}\gamma_\infty \{x \in \mathcal{C}_\alpha :\, A(x) > \alpha\} &\lesssim \sqrt{\log \alpha} \int_{E_\beta^0} e^{-R(D_{s_0(\tilde x)}\, \tilde x)} \int_{s_0(\tilde x)}^{\infty} e^{-c(s - s_0(\tilde x)) \log \alpha}\, ds\, dS_\beta(\tilde x) \\ &\lesssim \frac{1}{\sqrt{\log \alpha}} \int_{E_\beta^0} e^{-R(D_{s_0(\tilde x)}\, \tilde x)}\, dS_\beta(\tilde x).\end{aligned}$$

Now we use the equality $A(D_{s_0(\tilde{x})}\,\tilde{x}) = \alpha$, the definition of $A(x)$ and the fact that $|D_{s_0(\tilde{x})}\,\tilde{x}| \simeq \sqrt{\log\alpha}$ to see that the last expression is at most

$$\frac{1}{\alpha}\,(\log\alpha)^{\frac{n-1}{2}} \int_{E^0_\beta}\int_{\{u:\,|\tilde{u}-\tilde{x}|\lesssim(\log\alpha)^{-1/2}\}} f(u)\,d\gamma_\infty(u)\,dS_\beta(\tilde{x}) =$$

$$\frac{1}{\alpha}\,(\log\alpha)^{\frac{n-1}{2}} \int f(u)\int_{\{\tilde{x}:\,|\tilde{u}-\tilde{x}|\lesssim(\log\alpha)^{-1/2}\}} dS_\beta(\tilde{x})\,d\gamma_\infty(u) \lesssim \frac{1}{\alpha}\int f(u)\,d\gamma_\infty(u) = \frac{1}{\alpha}.$$

This proves (10.1), and we move to (10.2). Here we similarly have $B(\tilde{x}) < \alpha$ for $\tilde{x}\in E_\beta$, and we can define $E^0_\beta$ and $s_0(\tilde{x})$ as above, replacing $A(.)$ by $B(.)$. The rest of the argument is only slightly different from that for (10.1); we now have

$$\begin{aligned}\gamma_\infty\{x\in\mathcal{C}_\alpha : B(x)>\alpha\} &\lesssim \frac{1}{\sqrt{\log\alpha}}\int_{E^0_\beta}\exp\big(-R(D_{s_0(\tilde{x})}\,\tilde{x})\big)\,dS_\beta(\tilde{x})\\ &\lesssim \frac{1}{\alpha}\,\frac{1}{\sqrt{\log\alpha}}\int_{E^0_\beta}\int_{\{u:\,|\tilde{u}-\tilde{x}|\gtrsim(\log\alpha)^{-1/2}\}} |\tilde{u}-\tilde{x}|^{-n} f(u)\,d\gamma_\infty(u)\,dS_\beta(\tilde{x})\\ &= \frac{1}{\alpha}\,\frac{1}{\sqrt{\log\alpha}}\int f(u)\int_{\{\tilde{x}:\,|\tilde{u}-\tilde{x}|\gtrsim(\log\alpha)^{-1/2}\}} |\tilde{u}-\tilde{x}|^{-n}\,dS_\beta(\tilde{x})\,d\gamma_\infty(u) \lesssim \frac{1}{\alpha}.\end{aligned}$$

This is (10.2), and Theorem 9.1 is proved.

In order to prove Proposition 9.2, Theorem 9.1 is enough, as we saw in the preceding section. However, we take the opportunity to give the following related result, which strengthens Theorem 9.1 and also Theorem 1.1 in [CCS2] and may be of independent interest.

**Theorem 10.2.** *The operator $S^{\mathrm{glob}}$ defined by*

$$S^{\mathrm{glob}}f(x) = \int \sup_{0<t<\infty} K_t(x,u)\,(1-\eta(x,u))|f(u)|\,d\gamma_\infty(u), \qquad f\in L^1(\mathbb{R}^n),$$

*is of weak type $(1,1)$ for the measure $d\gamma_\infty$.*

This result is a consequence of Theorem 9.1 and the following proposition.

**Proposition 10.3.** *The operator $S_\infty$, defined by*

$$S_\infty f(x) = \int \sup_{t\ge 1} K_t(x,u)\,|f(u)|\,d\gamma_\infty(u),$$

*satisfies the inequality*

$$\gamma_\infty\{x : S_\infty f(x) > \alpha\} \lesssim \frac{1}{\alpha\sqrt{\log\alpha}} \tag{10.3}$$

*for all normalized functions $f$ in $L^1(\gamma_\infty)$ and all $\alpha > 2$.*

*Proof.* Let $t\ge 1$. The simplifications in Subsection 5.2 apply again, since $K_t(x,u)\lesssim e^{R(x)} < \alpha$ if $R(x) < (\log\alpha)/2$. For $x\in\mathcal{C}_\alpha$, a combination of (2.10) and [CCS2, Lemma 4.3(i)] implies

$$K_t(x,u) \lesssim e^{R(x)}\,\exp\big(-c\,|\tilde{u}-\tilde{x}|^2\big),$$

where we use polar coordinates with $\beta = (\log \alpha)/2$. The proposition now follows from Lemma 6.2. $\square$

**Remark 10.4.** The inequality (10.3), which is sharp as verified in [CCS2, Proposition 6.2], is slightly stronger than the weak type $(1,1)$ estimate in Theorem 10.2. The corresponding estimate for the operator $S_0^{\mathrm{glob}}$ is false, since $f$ approximating a point mass at 0 gives a counterexample.

**Remark 10.5.** In the case $Q = I$ and $B = -I$ an estimate similar to Lemma 10.1 with a kernel $\overline{M}$ controlling from above the Mehler kernel $K_t$ in the global region, has recently been proved in [Br] (see, in particular, Definition 3.2 and Proposition 3.4 therein). An earlier result of this type may be found in [MPS, Proposition 2.1]. These estimates are sharp for significant values of $(x, u)$, whereas our Theorem 10.2 is simpler, and sufficient for our needs. Moreover, Proposition 10.3 is stronger than the analogous bounds in [Br] and [MPS].

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