

# $\Gamma$ -convergence and homogenisation for a class of degenerate functionals



Nicolas Dirr <sup>a,\*</sup>, Federica Dragoni <sup>a</sup>, Paola Mannucci <sup>b</sup>, Claudio Marchi <sup>c</sup>

<sup>a</sup> Cardiff School of Mathematics, Cardiff University, Cardiff, UK

<sup>b</sup> Dipartimento di Matematica "Tullio Levi-Civita", Università di Padova, Padova, Italy

<sup>c</sup> Dipartimento di Ingegneria dell'Informazione, Università di Padova, Padova, Italy

## ARTICLE INFO

### Article history:

Received 27 June 2019

Accepted 27 August 2019

Communicated by Enrico Valdinoci

### Keywords:

Gamma-convergence

Homogenisation

Carnot groups

Heisenberg group

Hörmander condition

Degenerate functionals

## ABSTRACT

This paper is on  $\Gamma$ -convergence for degenerate integral functionals related to homogenisation problems in the Heisenberg group. Here both the rescaling and the notion of invariance or periodicity are chosen in a way motivated by the geometry of the Heisenberg group. Without using special geometric features, these functionals would be neither coercive nor periodic, so classic results do not apply. All the results apply to the more general case of Carnot groups.

© 2019 The Authors. Published by Elsevier Ltd. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

## 1. Introduction

$\Gamma$ -convergence is a notion of convergence of a family of functionals  $F_\varepsilon$  to a functional  $F_\infty$  which goes back to E. de Giorgi (see [15–17]) and guarantees the convergence of minimisers of the functionals  $F_\varepsilon$  to minimisers of the limit (or effective) functional  $F_\infty$ ; for a precise definition and properties see Section 4. The convergence of minimisers implies, under suitable conditions, convergence of solutions of the Euler–Lagrange equations, and is therefore a useful tool for homogenisation problems, in particular in the random and nonlinear case, see e.g. [12]. If the functional is integral, i.e. of the form

$$u \mapsto F(u) = \int_A f(x, \nabla u(x)) dx,$$

where  $A$  is a (Borel) domain,  $\nabla u$  is the distributional gradient of the real-valued function  $u$  assumed to be in a suitable  $L^p$ -space, and  $f : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$  with some regularity and growth assumptions, then the corresponding Euler–Lagrange equation is a nonlinear divergence form equation.

\* Corresponding author.

E-mail addresses: [dirrnp@cardiff.ac.uk](mailto:dirrnp@cardiff.ac.uk) (N. Dirr), [Dragonif@cardiff.ac.uk](mailto:Dragonif@cardiff.ac.uk) (F. Dragoni), [mannucci@math.unipd.it](mailto:mannucci@math.unipd.it) (P. Mannucci), [claudio.marchi@unipd.it](mailto:claudio.marchi@unipd.it) (C. Marchi).

The above functional can be generalised to degenerate functionals in the setting of Carnot groups. In this paper we focus specifically on the  $n$ -dimensional Heisenberg group  $\mathbb{H}^n$ , which is a step 2 Carnot group defined on  $\mathbb{R}^{2n+1}$  (see Section 2 for definitions and properties). Thus the family of functionals considered here is of the form

$$u \mapsto F(u) = \int_A f(x, \nabla_{\mathcal{X}} u(x)) dx$$

where  $A$  is a domain on  $\mathbb{R}^N = \mathbb{R}^{2n+1}$  while  $\nabla_{\mathcal{X}} u$  is the horizontal gradient in the Heisenberg group which belongs to a suitable  $m$ -dimensional subspace of the “space of derivatives” (tangent space), see Definition (2.7).

Since  $m = 2n$ , we have  $m < N$ : as a consequence, such functionals are typically not coercive in the classical sense, so classical results do not apply.

Working in the setting of the Heisenberg group, the scaling needs to adapt to the underlying geometrical structure, therefore we will consider the following anisotropic scaling  $\delta_{1/\varepsilon}(\underline{x}_1, \underline{x}_2, x_3) = (\varepsilon^{-1}\underline{x}_1, \varepsilon^{-1}\underline{x}_2, \varepsilon^{-2}x_3)$ , with  $(\underline{x}_1, \underline{x}_2, x_3) \in \mathbb{R}^N \equiv \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ . Hence the scaling is anisotropic w.r.t. the last component. The anisotropy can be understood heuristically in another way: at each point, some directions are “forbidden”, i.e. paths of the associated control problem can move only on a  $2n$ -dimensional subspace of a  $(2n+1)$ -dimensional space. By varying their direction often (i.e. by the use of non-trivial commutators from the Hörmander condition) they are able to reach any given point but the cost for “zig-zagging” to get in the forbidden direction is higher, so typically they move slower in these directions, which makes a faster rescaling necessary.

The limit functional  $F_\infty$  is of the same form, i.e.

$$F_\infty(u) = \int_A f_0(\nabla_{\mathcal{X}} u(x)) dx,$$

where the integral function  $f_0$  does not depend on  $x$  anymore (however the horizontal gradient still depends on  $x$  through the vector fields). The corresponding Euler–Lagrange equation will not be elliptic but only subelliptic, we refer to [7,20] for an overview on subelliptic equations.

The study of homogenisation in subelliptic settings started with the periodic case (see e.g. [4–6,22,23,28,33]). The first result for the stochastic case in this degenerate setting is [19], where the authors studied the case Hamilton–Jacobi (first order) case for Hamiltonian depending on the horizontal gradient in the case of Carnot groups.

As  $\Gamma$ -convergence has nice compactness results, the main difficulty is in general the identification of the  $\Gamma$ -limit as again an integral functional. Here it is used that the integrand can be retrieved by considering minimisation problems over small cubes with *affine* boundary conditions, see [11]. A generalisation to the setting of the Heisenberg group requires a suitable adaption of the notion of “affine”, namely  $H$ -affine functions, see Section 3 for the definition and references. Recently some results for  $\Gamma$ -convergence of degenerate functionals in very general geometries have been proved in [27]. Here we use that the minimal normalised energy on anisotropically (Heisenberg dilations) scaled cubes is subadditive by constructing admissible functionals on large cubes and patching together translated minimisers on translated cubes. Here we need to use the specific properties of translations in the Heisenberg group. Note that cubes rescaled by an integer (i.e.  $\delta_k(Q)$ ) cannot be written as union of translations of the original cube  $Q$ , not even up to a set of measure zero. This is a crucial difference with the Euclidean case but we overcome the issue by controlling the error term.

A closely related approach can be found in [24], where the  $\Gamma$ -convergence in Cheeger–Sobolev spaces is considered. Our functional depending only on  $|\nabla_{\mathcal{X}} u|$  instead of  $\nabla_{\mathcal{X}} u$  would be a functional on a Cheeger–Sobolev space, but the natural tiling generalising periodicity in our case does not satisfy the assumptions of [24].

All the results are written in the Heisenberg group for sake of simplicity but the proofs apply to general Carnot groups.

These results can be applied to functionals related to subelliptic  $p$ -Laplace equations and generalised to stochastic functionals with short correlations (as done in Dal Maso-Modica [13]).

This paper is organised as follows.

In Section 2 we give an overview on the Heisenberg group and its geometry, in particular the scaling, the horizontal gradient and the notion of periodicity.

In Section 3 we define precisely our functionals and we recall the Sobolev spaces adapted to the structure of the Heisenberg group, in particular their embedding into  $L^p$ -spaces, through the embedding in fractional Sobolev spaces, which will be crucial for the later  $\Gamma$ -convergence results.

Section 4 is devoted to the  $\Gamma$ -convergence results. We first recall the definition and some basic properties of  $\Gamma$ -convergence. We use compactness properties of the  $\Gamma$ -convergence and we give conditions under which the  $\Gamma$ -limit is again an integral functional, thus recovering the results by Dal Maso–Modica, [11], for our degenerate functionals.

In Section 5 we prove the main result of the paper, that is the homogenisation result for Heisenberg-periodic functionals. In fact, we show a Akcoglu–Krengel type result, [1], for our anisotropic Heisenberg-periodic functionals, i.e. the convergence of normalised minimal energies over rescaled cubes. For this purpose, we exploit an underlying subadditive structure.

In Section 6 we mention some applications and further directions of research. We highlight how the results apply to more general functionals associated to Carnot group structures. We then give some connections with homogenisation for subelliptic  $p$ -Laplacian. Finally we explain how our methods can be used to generalise the results to the stochastic case with short correlations.

## 2. Preliminaries: The Heisenberg group

Carnot groups are non-commutative Lie groups: thus they are endowed both with a non-commutative algebraic structure and with a manifold structure. The lack of commutativity in the algebraic structure reflects on the manifold structure as restrictions on the admissible motions. This means that the allowed curves are constrained to have their velocities in a lower dimensional subspace of the tangent space of the manifold. Then the associated manifold structure is not Riemannian but sub-Riemannian. In this paper we give details for the Heisenberg group only but the results can be easily generalised to Carnot groups (see Section 6). We refer the reader to [7] for definitions and properties on Carnot groups and to [29] for an overview on sub-Riemannian manifolds.

To keep the paper easily readable we omit the intrinsic definition of the Heisenberg group, introducing it directly as the following non-commutative group structure on  $\mathbb{R}^N$ .

**Definition 2.1.** The  $n$ -dimensional Heisenberg group  $\mathbb{H}^n$ , with  $n \geq 1$ , is a Carnot group of step 2 isomorphic to  $\mathbb{R}^N$ , where  $N = 2n + 1$ , endowed with the following non-commutative group operation:

$$x * y := \left( \underline{x}_1 + \underline{y}_1, \underline{x}_2 + \underline{y}_2, x_3 + y_3 + \frac{\underline{x}_1 \cdot \underline{y}_2 - \underline{x}_2 \cdot \underline{y}_1}{2} \right) \tag{2.1}$$

for all  $x = (\underline{x}_1, \underline{x}_2, x_3), y = (\underline{y}_1, \underline{y}_2, y_3) \in \mathbb{R}^N \equiv \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$  and where by  $\cdot$  we indicate the standard inner product in  $\mathbb{R}^n$ .

In all Carnot groups it is possible to define a natural scaling, induced by the Lie algebra stratification, namely dilations. The dilations replace the multiplication by scalars in the standard vector space structure of the Euclidean  $\mathbb{R}^N$ .

**Definition 2.2.** The dilations in the Heisenberg group are the family of group homeomorphisms defined as, for all  $t > 0$ ,  $\delta_t : \mathbb{R}^N \rightarrow \mathbb{R}^N$  with

$$\delta_t(x) = (t \underline{x}_1, t \underline{x}_2, t^2 x_3), \quad \forall x = (\underline{x}_1, \underline{x}_2, x_3) \in \mathbb{R}^N \equiv \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}. \quad (2.2)$$

Thus the dilations in  $\mathbb{H}^n$  coincide with the standard Euclidean scaling in the first  $2n$  components while the last component scales as  $t^2$ .

The following properties of dilations are true in all Carnot groups and they can be easily checked in the Heisenberg group by using formulae (2.1) and (2.2).

**Lemma 2.1.** For all  $t, s > 0$ , the following properties hold true:

- (1)  $\delta_1 = \text{id}$ ;
- (2)  $\delta_t^{-1} = \delta_{t^{-1}}$ ;
- (3)  $\delta_t \circ \delta_s = \delta_{ts}$ ;
- (4) for every  $x, y \in \mathbb{R}^N$  one has  $\delta_t(x) * \delta_t(y) = \delta_t(x * y)$ .

We now recall the notion of *homogeneous dimension*. In a general Carnot group  $\mathbb{G}$ , the homogeneous dimension is the natural number  $\mathcal{Q} := \sum_{i=1}^r i \dim \mathfrak{g}_i$ , where  $r$  is the step of the stratified associated Lie algebra  $\mathfrak{g}$  (see e.g. [7] for more details). In  $\mathbb{H}^n$  one can easily show that

$$\mathcal{Q} = 2n + 2.$$

The homogeneous dimension is correlated to the scaling of measures since it coincides with the Hausdorff dimension w.r.t. every homogeneous metric. In the paper we always indicate simply by  $|A|$  the  $N$ -dimensional Lebesgue measure of the Borel set  $A$  of  $\mathbb{R}^N$ . Then for all  $t > 0$ ,  $|tA| = t^{\mathcal{Q}}|A|$  while one can easily show that

$$A^t := \delta_t(A) \quad \Rightarrow \quad |A^t| = t^{\mathcal{Q}}|A|. \quad (2.3)$$

Since the Heisenberg group (as all Carnot groups) is non-abelian, translations to the right or to the left determine two different families of homeomorphism on the group. As standard in this setting, we consider the *left-translations*, which are defined, for all  $y \in \mathbb{R}^N$  as  $L_y : \mathbb{R}^N \rightarrow \mathbb{R}^N$  with

$$L_y(x) := y * x,$$

where  $*$  is the group operation defined in (2.1).

Using the left-translations it is possible to define a sub-Riemannian structure on each Carnot group by introducing a suitable family of left-invariant vector fields spanning to the first layer of the Lie algebra stratification. We omit the general definition on Carnot groups (see e.g. [7]). In the specific case of the Heisenberg group, the vector fields can be found as

$$X_i(x) = dL_x(e_i), \quad (2.4)$$

where  $e_i$  are the unit vectors of the standard Euclidean basis on  $\mathbb{R}^N$  for  $i = 1, \dots, 2n$ . One can also easily show that, for all  $j = 1, \dots, n$

$$X_{2n+1}(x) = dL_x(e_{2n+1}) = [X_j, X_{n+j}](x),$$

where  $[\cdot, \cdot]$  are the standard Lie brackets (called also commutators) defined for vector fields. In the case  $n = 1$  the vector fields are

$$X_1(x) = \begin{pmatrix} 1 \\ 0 \\ -\frac{x_2}{2} \end{pmatrix} \quad \text{and} \quad X_2(x) = \begin{pmatrix} 0 \\ 1 \\ \frac{x_1}{2} \end{pmatrix}, \quad \forall x = (x_1, x_2, x_3) \in \mathbb{R}^3. \quad (2.5)$$

We recall that the previous vector fields are left-invariant by definition. For later use we introduce the following simplified notation: given any function  $u : \mathbb{R}^N \rightarrow \mathbb{R}$ , the translation  $L_z$  of the function  $u$  is simply  $u \circ L_z$ , i.e.

$$L_y(u)(x) := u(y * x).$$

Thus  $X_i$  is a left-invariant vector field if for all  $u \in C^\infty(\mathbb{R}^N)$  and for all fixed  $y \in \mathbb{R}^N$

$$X_i(L_y(u))(x) = (X_i u)(y * x), \quad \forall x \in \mathbb{R}^N, \tag{2.6}$$

(while this is in general false considering instead the right-translations). We recall that the vector fields  $X_i$  for  $i = 1, \dots, 2n$  span a bracket generating distribution with step 2 (see e.g. [29] for some details).

The previous vector fields allow us to define derivatives of any order, just considering how a vector field acts on smooth functions. Given a function  $u : \mathbb{R}^N \rightarrow \mathbb{R}$ , we denote the *horizontal gradient* of  $u$  by

$$\nabla_{\mathcal{X}} u = (X_1 u, \dots, X_{2n} u)^T. \tag{2.7}$$

In the case of  $n = 1$  the horizontal gradient can be explicitly written as

$$\nabla_{\mathcal{X}} u = \begin{pmatrix} u_{x_1} - \frac{x_2}{2} u_{x_3} \\ u_{x_2} + \frac{x_1}{2} u_{x_3} \end{pmatrix} \in \mathbb{R}^2.$$

We now recall that a differential operator  $\mathcal{L}$  on the Heisenberg group is called homogeneous of degree  $\kappa$  if for every  $u \in C^\infty(\mathbb{R}^N)$  one has

$$\mathcal{L}(\delta_t u) = t^\kappa \delta_t(\mathcal{L}u),$$

where the scaled function is defined as  $\delta_t u(x) := u(\delta_t(x))$ , for all  $x \in \mathbb{R}^N$ .

Then we have the following result.

**Lemma 2.2.** *For every  $i = 1, \dots, 2n$ , each left-invariant vector field  $X_i$ , defined in (2.4), is homogeneous of degree  $\kappa = 1$ , i.e., for any  $u \in C^\infty(\mathbb{R}^N)$  one has*

$$X_i(\delta_t u) = t \delta_t(X_i u).$$

The proof is a very simple computation in the Heisenberg group while for general Carnot groups the reader can find a proof e.g. in [21].

This in particular implies that the horizontal gradient is homogeneous of degree one with respect to the dilations  $\delta_\lambda$ , i.e., for every  $u \in C^\infty(\mathbb{R}^N)$  we have

$$\nabla_{\mathcal{X}}(\delta_t u) = t \delta_t(\nabla_{\mathcal{X}} u). \tag{2.8}$$

For later use, it is very useful to introduce the  $N \times 2n$ -matrix associated to the vector fields, that is

$$\sigma := (X_1, \dots, X_{2n}), \tag{2.9}$$

where  $X_i$  are the left-invariant vector fields defined in (2.4) and the *extended matrix of vector fields*, which is the  $N \times N$ -matrix

$$\sigma_{Ext} := (X_1, \dots, X_{2n}, X_{2n+1}), \tag{2.10}$$

where  $X_{2n+1}(x) = dL_x(e_{2n+1})$ , and  $e_{2n+1}$  is the unit vector spanning the  $2n + 1$ -direction (and associated to the second layer of the stratification for the Lie algebra).

**Example 2.1.** In the 1-dimensional Heisenberg group  $\mathbb{H}^1$ , the matrix  $\sigma$  is the  $2 \times 3$ -matrix given by

$$\sigma(x_1, x_2, x_3) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{x_2}{2} & \frac{x_1}{2} \end{pmatrix},$$

while  $\sigma_{Ext}$  is the  $3 \times 3$ -matrix given by

$$\sigma_{Ext}(x_1, x_2, x_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{x_2}{2} & \frac{x_1}{2} & 1 \end{pmatrix}.$$

A trivial computation shows the following property, which will be very useful later: given the quadratic matrix defined in (2.10), then

$$\det(\sigma_{Ext}(x)) = 1, \quad \forall x \in \mathbb{R}^N. \tag{2.11}$$

The property above means that the left-translations are an isometry for the associated  $L^p$ -spaces, i.e. informally setting for all fixed  $z \in \mathbb{R}^N$   $y := L_z(x) = z * x$ , we have  $dy = dx$ .

**Remark 2.1.** Property (2.11) can be generalised to all Carnot groups in exponential coordinates or to more general Carnot-type groups (see e.g. [2] for properties and definitions of Carnot-type groups).

**Remark 2.2.** Trivially  $\nabla_{\mathcal{X}} u = \sigma^T \nabla u$  where  $\nabla u$  denotes the standard (Euclidean) gradient of  $u$ .

*2.1. Periodicity in the Heisenberg group*

Being the Heisenberg group a Lie group, a very natural notion of periodicity can be introduced by left-translations, see [4,5,22,23]. We refer also to the Phd thesis [25] where periodicity in the Heisenberg group (but also in more general structures as Grushin spaces) is studied in detail with many properties and examples. Given any  $\Omega \subset \mathbb{R}^N$ , we say that  $\Omega$  is H-periodic with period  $T > 0$ , whenever  $L_{Tk}(\Omega) = (Tk) * \Omega = \Omega$  for all  $k \in \mathbb{Z}^N$ .

For later use in the paper we fix the period  $T = 2$ . In fact, recalling that  $L_y \circ L_z(x) = L_{y*z}(x)$ , the composition of two left-translations with period  $T = 1$  is not anymore a integer left-translation since  $k * h \notin \mathbb{Z}^N$ , because the third component becomes  $k_3 + h_3 + \frac{k_1 h_2 - k_2 h_1}{2}$ , which is in general not anymore an integer.

Instead the composition of two left-translations with period 2 is still a translation of the same type since, for all  $k, h \in \mathbb{Z}^N$ ,  $2k * 2h = 2z$  with  $z = (\underline{k}_1 + \underline{h}_1, \underline{k}_2 + \underline{h}_2, k_3 + h_3 + \underline{k}_2 \cdot \underline{h}_1 - \underline{k}_1 \cdot \underline{h}_2) \in \mathbb{Z}^N$ . (Note that  $z \neq k * h$  since the third component is different by a factor  $\frac{1}{2}$  in the mixed term.) One could very simply adapt everything to period  $T = 1$  by choosing a different representation of the Heisenberg group, where the group law is expressed by polynomials with integer coefficients; in that case the unit cell needs to be rescaled to a unit cube (e.g.  $[-\frac{1}{2}, \frac{1}{2}]^N$ ), see e.g. [22].

We introduce the following simplified notation for the left-translations with period 2, that is

$$\tau_k(x) := 2k * x, \quad \forall k \in \mathbb{R}^N, x \in \mathbb{R}^N. \tag{2.12}$$

We recall that, for all  $k, h \in \mathbb{R}^N$ , the following properties hold true:

$$\tau_k \circ \tau_h = \tau_{k*h} \quad \text{and} \quad \tau_k^{-1} = \tau_{-k}.$$

A definition of periodicity adapted to the Heisenberg group structure can be given for functions as follows.

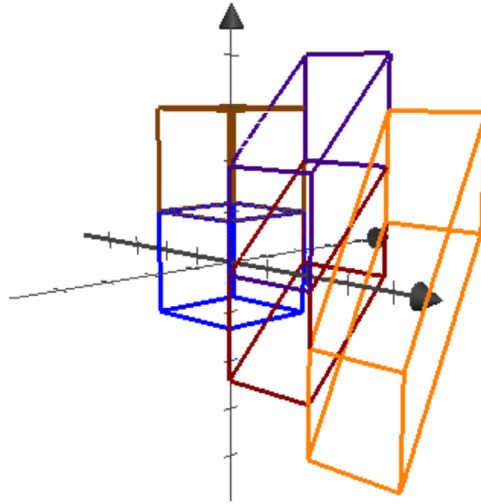


Fig. 1. Tiling in  $\mathbb{H}^1$  constructed by translating  $Q = [-1, 1]^3$ .

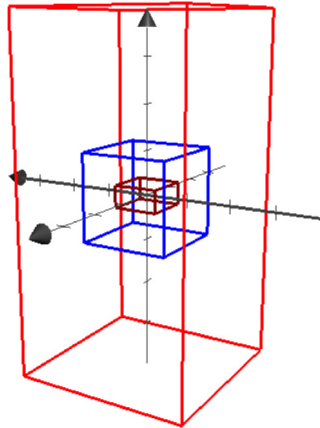


Fig. 2. Rescaling of the unit cell  $Q = [-1, 1]^3$  (which is the blue cube) w.r.t. the dilations in the 1-dimensional Heisenberg group: in particular in red one can see  $\delta_2(Q)$  while in bordeaux one can see  $\delta_{\frac{1}{2}}(Q)$ . (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

**Definition 2.3.** We say that the function  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  is  $H$ -periodic whenever

$$f(\tau_k(x)) = f(2k * x) = f(x), \quad \forall x \in \mathbb{R}^N, k \in \mathbb{Z}^N.$$

To construct a large class of periodic functions we need to introduce a  $H$ -periodic tiling of  $\mathbb{R}^N$ . Thus we consider the semiopen cube  $Q = [-1, 1)^N$ . We call  $Q$  *unit cell* and consider  $\tau_k(Q) = 2k * Q$ . Then one can easily show that the family  $\{\tau_k(Q)\}_{k \in \mathbb{Z}^N}$  fulfills

$$\bigcup_{k \in \mathbb{Z}^N} \tau_k(Q) = \mathbb{R}^N \quad \text{and} \quad \tau_k(Q) \cap \tau_h(Q) = \emptyset, \quad \forall k \neq h. \tag{2.13}$$

See Fig. 1 and [23, Lemma 2.4].

We next want to highlight a few facts about the scaling of tilings since this will be crucial later when we will study our homogenisation problem. First recall that, in the Heisenberg group, if we scale the unit cell, then we do not get anymore hypercubes but hyper-rectangles since the scaling is anisotropic. In Fig. 2 we

show how the cube  $Q$  scales for  $t > 1$  and for  $t < 1$ . Then if we want to build a tiling of  $\mathbb{R}^N$  starting by a rescaled cell, we need to be very careful and adapt the translations to the Heisenberg scaling.

**Lemma 2.3.** *Given the unit cell  $Q = [-1, 1]^N$  and a  $t > 0$ , the scaled unit cell as*

$$Q^t = \delta_t(Q) \quad \text{and} \quad Q_k^t := \tau_{\delta_t(k)}(Q^t),$$

*then the family  $\{Q_k^t\}_{k \in \mathbb{Z}^N}$  is a tiling of  $\mathbb{R}^N$  in the sense that*

$$\bigcup_{k \in \mathbb{Z}^N} Q_k^t = \mathbb{R}^N \quad \text{and} \quad Q_k^t \cap Q_h^t = \emptyset, \quad \forall k \neq h. \tag{2.14}$$

**Proof.** The result follows easily from the properties for  $t = 1$  and from the fact that  $\delta_t(k) * \delta_t(Q) = \delta_t(k * Q)$ .  $\square$

### 3. A class of degenerate functionals

Affine functions can be introduced in different ways in the Heisenberg group setting and they have been studied in [3,14]. For the purpose of the paper, we say that a function  $u : \mathbb{R}^N \rightarrow \mathbb{R}$  is  $H$ -affine (in the Heisenberg group) if

$$u(x) = q \cdot \pi_m(x) + a,$$

for  $m = 2n$  and for some  $q \in \mathbb{R}^m$  and  $a \in \mathbb{R}$ , where  $\pi_m : \mathbb{R}^N \rightarrow \mathbb{R}^m$  is the projection on the first  $m$  components and  $\cdot$  is the standard inner product on  $\mathbb{R}^m$ . The following lemma is an immediate property of  $H$ -affine functions in all Carnot-type groups and it will be key for our later results.

**Lemma 3.1.** *For all fixed  $q \in \mathbb{R}^m$ , we have*

$$\nabla_{\mathcal{X}} u = q \quad \Leftrightarrow \quad u(x) = q \cdot \pi_m(x) + a,$$

*for some  $a \in \mathbb{R}$  and for all  $x \in \mathbb{R}^N$ .*

**Proof.** One implication (from the right to the left) follows trivially from the fact that  $q \cdot \pi_m(x) + a$  does not depend on the last coordinate and the structure of the horizontal gradient.

The other implication follows from the fact that  $\nabla_{\mathcal{X}} u = q$  means  $X_i u(x) = q_i = \text{constant}$  for all  $i = 1, \dots, m$ , then

$$u_{2n+1} = [X_1, X_{n+1}]u = X_1(X_{n+1}u) - X_{n+1}(X_1u) = X_1(q_{n+1}) - X_{n+1}(q_1) = 0,$$

where we indicate by  $u_i$  the partial derivative of  $u$  w.r.t. the variable  $x_i$ , for  $i = 1, \dots, N$ . Using  $u_{2n+1}(x) = 0$ , for all  $x \in \mathbb{R}^N$ ,  $X_i u(x) = q_i$  implies  $u_i(x) = q_i$  for all  $i = 1, \dots, 2n$ , which gives  $u(x) = q \cdot \pi_m(x) + a$  for some  $a \in \mathbb{R}$ .  $\square$

We will later often use the following notation for H-linear functions:

$$l_q(x) = q \cdot \pi_m(x). \tag{3.1}$$

We next recall that the definition of Sobolev spaces in the setting of Hörmander vector fields, which applies in particular to the Heisenberg group. We refer to [31,35] for more details on these spaces.

Let  $k > 1$  be an integer,  $1 \leq \alpha \leq +\infty$  and  $A$  a domain on  $\mathbb{R}^N$ . We define the space

$$W_{\mathcal{X}}^{k,\alpha}(A) = \{u \in L^\alpha(A) \mid \mathcal{X}^J u \in L^\alpha(A), \quad \forall J \in \mathbb{N}^m, \quad |J| \leq k\},$$



where  $\mathcal{X}^J u = X_1^{J_1} \dots X_m^{J_m} u$  for  $J = (J_1, \dots, J_m)$ . Endowed with the norm

$$\|u\|_{W_{\mathcal{X}}^{k,\alpha}(A)} = \left( \sum_{|J| \leq k} \int_A |\mathcal{X}^J u|^\alpha dx \right)^{1/\alpha},$$

$W_{\mathcal{X}}^{k,\alpha}(A)$  is a Banach space, and is an Hilbert space in the case  $\alpha = 2$ .

Moreover, for any  $1 \leq \alpha < +\infty$ , the embeddings

$$W_{\mathcal{X}}^{k,\alpha}(A) \hookrightarrow W^{k/r,\alpha}(A),$$

hold true, where  $r$  is the step of the stratified associated Lie algebra, thus in Heisenberg group  $r = 2$  (see e.g. [34]). Later we will also need the following compact embedding.

**Lemma 3.2.**  *$W_{\mathcal{X}}^{k,\alpha}(A)$  is compactly embedded into  $L^\alpha(A)$ .*

**Proof.** This follows from the previous embedding and the fact that the fractional Sobolev space  $W^{k/r,\alpha}(A)$  is compactly embedded into  $L^p(A)$  (see e.g. [18]).  $\square$

**Definition 3.1.** For each domain  $A \subset \mathbb{R}^N$ , we indicate by

$$W_{\mathcal{X},0}^{k,\alpha}(A)$$

the closure of  $C_0^\infty(A)$  w.r.t. the Sobolev norm  $\|\cdot\|_{W_{\mathcal{X}}^{k,\alpha}(A)}$ .

This means that, whenever the boundary  $\partial A$  is regular enough, the trace of  $u$  vanishes on the boundary of the set.

We will use this notation to express the Dirichlet boundary conditions: more precisely

$$u - u_0 \in W_{\mathcal{X},0}^{k,\alpha}(A)$$

are all the functions  $u \in W_{\mathcal{X}}^{k,\alpha}(A)$  which coincide on  $\partial A$  (in the sense of Sobolev space) with some  $u_0 \in W_{\mathcal{X}}^{k,\alpha}(A)$ .

Next we recall the following Poincaré inequality, which is key for later results.

**Lemma 3.3.** *Given a bounded domain  $A \subset \mathbb{R}^N$ , then there exists a constant  $C > 0$  such that*

$$\int_A |u|^\alpha dx \leq C \int_A |\nabla_{\mathcal{X}} u|^\alpha dx, \quad \forall u \in W_{\mathcal{X},0}^{1,\alpha}(A).$$

**Proof.** This follows from the results in [26,34].  $\square$

Consider now a function  $f : \mathbb{R}^N \times \mathbb{R}^m \rightarrow \mathbb{R}$  with  $N = 2n + 1$  and  $m = 2n$ , we introduce the integral functional defined, for all domain  $A \subset \mathbb{R}^N$ , as

$$F(u, A) := \begin{cases} \int_A f(x, \nabla_{\mathcal{X}} u) dx, & u \in W_{\mathcal{X}}^{1,\alpha}(A), \\ +\infty, & \text{else.} \end{cases} \tag{3.2}$$

We introduce the following properties for the integrand function

$f : \mathbb{R}^N \times \mathbb{R}^m \rightarrow \mathbb{R}$  (with  $N = 2n + 1$  and  $m = 2n$ )

$$f(x, q) \text{ is measurable in } x \text{ and convex in } q; \tag{3.3}$$

$$\begin{aligned} &\exists C_1, C_2 > 0 \text{ and } \alpha > 1 \text{ such that} \\ &C_1 |q|^\alpha \leq f(x, q) \leq C_2 (|q|^\alpha + 1), \quad q \in \mathbb{R}^m, x \in \mathbb{R}^N. \end{aligned} \quad (3.4)$$

Moreover for the later homogenisation problem we will assume  $H$ -periodicity for the functional in the sense of [Definition 2.3](#); more precisely

$$f(\tau_k(x), q) = f(2k * x, q) = f(x, q), \quad \forall x \in \mathbb{R}^N, q \in \mathbb{R}^m, k \in \mathbb{Z}^N. \quad (3.5)$$

**Example 3.1.** The main example is  $f(x, q) = a(x)|q|^\alpha$ , which trivially satisfies [\(3.3\)](#) and [\(3.4\)](#) whenever  $a : \mathbb{R}^N \rightarrow \mathbb{R}$  is bounded (with a strictly positive lower bound) and measurable, while assumption [\(3.5\)](#) is equivalent to requiring that  $a(\cdot)$  is  $H$ -periodic.

We want to study the minimisation problem for  $F(u, A)$  with  $H$ -affine boundary condition, i.e.

$$m(F, u_0, A) := \min \left\{ F(u, A) \mid u - u_0 \in W_{\mathcal{X},0}^{1,\alpha}(A) \right\}, \quad (3.6)$$

with  $u_0(x) = q \cdot \pi_m(x) + a$  for some  $q \in \mathbb{R}^m$  and  $a \in \mathbb{R}$ .

**Remark 3.1.** Note that under assumptions [\(3.3\)](#) and [\(3.4\)](#), the infimum of  $F(u, A)$  on the set of functions  $u$  such that  $u - u_0 \in W_{\mathcal{X},0}^{1,\alpha}(A)$  is indeed a minimum by standard arguments, using the convexity, the embedding in [Lemma 3.2](#) and the Poincaré inequality (see [Lemma 3.3](#)).

#### 4. A $\Gamma$ -convergence result for degenerate functionals

To keep the paper self contained we next recall briefly the definition of  $\Gamma$ -convergence.

##### 4.1. Very brief introduction to $\Gamma$ -convergence

In homogenisation theory, we consider a family of solutions to equations with rapidly oscillating coefficients and investigate if they converge to a solution of a homogenised equation with slowly oscillating or constant coefficients. If these equations are the Euler–Lagrange equations of a suitable family of functionals with rapidly oscillating coefficients, and if both minimisers and solutions of the Euler–Lagrange equation are unique, then we can study convergence of the family of functionals instead.

We need a notion of convergence of functionals which guarantees that minimisers of the approximating functionals converge to minimisers of the limit functional.

A suitable mathematical setup to make this rigorous is the notion of  $\Gamma$ -convergence. Let us briefly recall the definition of  $\Gamma$ -convergence (see [\[8–10\]](#) for more details on this subject).

**Definition 4.1.** Let  $U$  be a metric space and for  $\varepsilon > 0$  let  $F_\varepsilon : U \rightarrow \overline{\mathbb{R}}$  be a family of functionals on  $U$ . We say that  $F_\varepsilon$   $\Gamma$ -converge to  $F : U \rightarrow \overline{\mathbb{R}}$  if the following conditions are verified:

1. for all  $u \in U$  and for all  $u_\varepsilon \rightarrow u$ , there holds  $\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \geq F(u)$   
( $\Gamma$ -liminf inequality);
2. for all  $u \in U$  there exist  $u_\varepsilon \rightarrow u$ , such that  $\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) = F(u)$   
( $\Gamma$ -limsup inequality).

The convergence of minimisers to minimisers is formalised in the following way, which can be easily derived from [Definition 4.1](#).

**Proposition 4.1.** *If  $F_\varepsilon$   $\Gamma$ -converge to  $F$  in  $U$ , also the corresponding minimal values (or infima) converge. Moreover, if  $u_\varepsilon$  is a minimiser of  $F_\varepsilon$  and  $u_\varepsilon \rightarrow u \in U$ , then  $u$  is a minimiser of  $F$ .*

Hence, the asymptotic behaviour of minimisers of  $F_\varepsilon$  (and therefore solutions of the Euler–Lagrange equations, see Section 6) can be partly understood by considering the  $\Gamma$ -limit of  $F_\varepsilon$ .

Moreover,  $\Gamma$ -convergence has nice compactness properties, i.e. in general it is easy to show that a  $\Gamma$ -limit along subsequences exists. The problem is then to identify this limit (see Section 5) and to show properties of this limit, in particular that it is again an integral functional.

4.2.  $\Gamma$ -Convergence limit

We say that a family  $(A_\rho)_{\rho>0}$  of open subsets of  $\mathbb{R}^N$  with Lipschitz boundary is a *substantial family* (around  $x$ ) as  $\rho \rightarrow 0^+$  if, for every positive  $\rho$ , there hold

$$A_\rho \subset B(x, \rho) := \{y \in \mathbb{R}^N \mid |y - x| < \rho\} \quad \text{and} \quad |A_\rho| \geq c |B(x, \rho)|,$$

where  $c$  is a constant independent of  $\rho$  (see the monograph [32, Ch.8] for other properties).

The following result states that the integral function  $f$  can be obtained from the minima of the Dirichlet problem for  $F$  with affine boundary data. To this purpose, for any domain  $A$  with Lipschitz boundary and for every  $u_0$   $H$ -affine data, we introduce the following regularised variational problem

$$m_{reg}(F, u_0, A) := \inf \{F(u, A) \mid u \in C^\infty(\mathbb{R}^N), u = u_0 \text{ on } \partial A\}. \tag{4.1}$$

Since the functional depends on  $u$  only through its horizontal gradient, therefore the constant  $a$  in the definition of  $H$ -affine function does not affect the results; we now consider directly the  $H$ -linear functions defined in (3.1) as boundary data. We state now a useful property, key for the later results.

**Lemma 4.1.** *Let  $A$  be a  $N$ -dimensional domain with Lipschitz boundary. For  $q \in \mathbb{R}^m$  and  $l_q(y) = q \cdot \pi_m(y)$ , and for every smooth function  $u$  such that  $u = l_q$  on  $\partial A$ , we have*

$$\int_A \nabla_{\mathcal{X}} u \, dy = q |A|. \tag{4.2}$$

**Proof.** We write  $y = (\underline{y}_1, \underline{y}_2, y_3) \in \mathbb{R}^N \equiv \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ . First we show that

$$\int_A \nabla_{\mathcal{X}} u \, dy = \int_{\partial A} (q \cdot \pi_m) \nu_0 \, d\mathcal{H}^{N-1} \in \mathbb{R}^m, \tag{4.3}$$

where  $\nu_0$  is the horizontal normal, i.e.  $\nu_0 = \sigma^T \nu$  with  $\sigma$  matrix of the vector fields defined in (2.9) and  $\nu$  the outward unit normal to  $\partial A$ , while  $d\mathcal{H}^{N-1}$  is the Hausdorff measure defined on  $\partial A$ . To prove the claim (4.3) we use that the vector fields in the Heisenberg group are divergence free and we combine a simple integration by parts with Remark 2.2, which gives

$$\int_A X_i u \, dy = \int_{\partial A} u \nu_0^i \, d\mathcal{H}^{N-1} = \int_{\partial A} (q \cdot \pi_m) \nu_0^i \, d\mathcal{H}^{N-1},$$

where  $\nu_0^i$  is the  $i$ -component of  $\nu_0$  for  $i = 1, \dots, m = 2n$ .

Then we can use the divergence theorem again (together with the fact that the vector fields in Carnot groups are divergence free) to conclude:

$$\int_{\partial A} (q \cdot \pi_m) \nu_0^i \, d\mathcal{H}^{N-1} = \int_A X_i (q \cdot \pi_m) \, dx = q_i \int_A dx = q_i |A|,$$

for all  $i = 1, \dots, m$ .  $\square$

We now use the previous lemma to show that, whenever the integrand function  $f$  does not depend on  $x$ , then  $H$ -affine functions are minimisers for problem (3.6) with  $H$ -affine boundary condition.

**Lemma 4.2.** *Given a domain with Lipschitz boundary  $A \subset \mathbb{R}^N$ , consider the problem (3.6) with  $u_0(x) = q \cdot \pi_m(x) + a$  for some  $q \in \mathbb{R}^m$  and  $a \in \mathbb{R}$ , and  $F$  defined in (3.2) with  $f(x, q) = f(q)$  convex, then*

$$m_{reg}(F, u_0, A) = \int_A f(\nabla_{\mathcal{X}} u_0) dx = f(q)|A|.$$

**Proof.** The result follows by combining Lemmas 3.1 and 4.1 with Jensen's inequality.  $\square$

Now following the arguments of Dal Maso–Modica [11] for the standard non degenerate case, we prove that the integrand function  $f$  can be retrieved in terms of  $m_{reg}$ .

**Theorem 4.1.** *Under assumptions (3.3)–(3.4), there exists a measurable subset  $\mathcal{N}$  of  $\mathbb{R}^N$  with  $|\mathcal{N}| = 0$  such that*

$$f(x, q) = \lim_{\rho \rightarrow 0^+} \frac{m_{reg}(F, l_q, A_\rho)}{|A_\rho|},$$

for every  $q \in \mathbb{R}^m$  and with  $l_q$   $H$ -linear as in (3.1),  $x \in \mathbb{R}^N \setminus \mathcal{N}$  and every substantial family  $(A_\rho)_{\rho > 0}$  around  $x$ .

**Proof.** We will use the same arguments as in [11, Theorem I] for the non-degenerate case. We just sketch the main steps.

**Step 1.** Let us at the moment assume that there exists some  $R > 0$  such that  $f$  does not depend on  $x$  for  $|q| > R$ . Then by Jensen's inequality and by Lemma 4.1, we obtain

$$\inf \left\{ \int_{A_\rho} f(x, \nabla_{\mathcal{X}} u) dy \mid u \in C^\infty(\mathbb{R}^N), u = l_q \text{ on } \partial A_\rho \right\} = \int_{A_\rho} f(x, q) dy = |A_\rho| f(x, q),$$

for every  $q \in \mathbb{R}^m$ ,  $\rho > 0$ ,  $x \in \mathbb{R}^N$  and  $l_q$  defined by (3.1). Exactly as in [11, Proposition 1.1], we can deduce

$$\left| f(x, q) - \frac{m_{reg}(F, l_q, A_\rho)}{|A_\rho|} \right| \leq \frac{1}{|A_\rho|} \int_{A_\rho} \sup_{q \in \mathbb{R}^m} |f(x, q) - f(y, q)| dy.$$

It remains to prove that there exists a measurable subset  $\mathcal{N}$  of  $\mathbb{R}^N$  with  $|\mathcal{N}| = 0$  such that

$$\lim_{\rho \rightarrow 0^+} \frac{1}{|A_\rho|} \int_{A_\rho} \phi(x, y) dy = 0,$$

for every  $x \in \mathbb{R}^N \setminus \mathcal{N}$  and every substantial family  $\{A_\rho\}$  around  $x$ , where

$$\phi(x, y) := \sup_{q \in \mathbb{R}^m} \omega(x, y, q), \quad \omega(x, y, q) := |f(x, q) - f(y, q)|, \quad \forall x, y \in \mathbb{R}^N, q \in \mathbb{R}^m.$$

We observe that  $\omega(x, y, q) = 0$  for every  $q \in \mathbb{R}^m$  with  $|q| \geq R$  and that, arguing as in [11] (recall that  $f$  is convex w.r.t.  $q$ ), there exists a positive constant  $K$  such that:  $|\omega(x, y, q_1) - \omega(x, y, q_2)| \leq K|q_1 - q_2|$  for every  $x, y \in \mathbb{R}^N$  and  $q_1, q_2 \in \mathbb{R}^m$ .

Fix a dense subset  $D$  of  $\mathbb{R}^m$ . The Lebesgue's Differentiation Theorem ensures that there exists a measurable subset  $\mathcal{N}$  of  $\mathbb{R}^N$  with  $|\mathcal{N}| = 0$  such that

$$\lim_{\rho \rightarrow 0^+} \frac{1}{|A_\rho|} \int_{A_\rho} \omega(x, y, p) dy = 0, \quad \forall x \in \mathbb{R}^N \setminus \mathcal{N}, p \in D,$$

for every substantial family  $\{A_\rho\}$  around  $x$ . Moreover, as in [11], for every  $\varepsilon > 0$ , there exists a finite set  $\{p_1, \dots, p_k\} \subset D$  such that

$$\phi(x, y) \leq \sum_{i=1}^k \omega(x, y, p_i) + K\varepsilon, \quad \forall x, y \in \mathbb{R}^N.$$

Therefore, we infer

$$\limsup_{\rho \rightarrow 0^+} \frac{1}{|A_\rho|} \int_{A_\rho} \phi(x, y) \, dy \leq K\varepsilon, \quad \forall x \in \mathbb{R}^N \setminus \mathcal{N},$$

for every substantial family  $\{A_\rho\}$  around  $x$ . By the arbitrariness of  $\varepsilon$ , we accomplish the proof.

**Step 2.** Let us now remove the additional assumption of Step 1. Taking into account the convexity and the coercivity of  $f$  w.r.t.  $q$ , by the same arguments as in [11, Theorem I], we obtain that there exists a measurable set  $\mathcal{N}' \subset \mathbb{R}^N$ , with  $|\mathcal{N}'| = 0$ , such that

$$f(x, q) \geq \limsup_{\rho \rightarrow 0^+} \frac{m_{reg}(F, l_q, A_\rho)}{|A_\rho|}, \quad \forall x \in \mathbb{R}^N \setminus \mathcal{N}', q \in \mathbb{R}^m,$$

for every family  $(A_\rho)_{\rho>0}$  as in the statement. In order to obtain the reverse inequality, we first observe that the same arguments of [11, Lemma 1.2] ensure that there exists an increasing sequence  $\{f_i\}_{i \in \mathbb{N}}$  of functions such that  $f = \sup_i f_i$  and each  $f_i$  satisfies the assumptions of step 1. For each  $i \in \mathbb{N}$ , we denote  $F_i$  and  $\mathcal{N}_i$  respectively the corresponding functional and the negligible set given by step 1. We set  $\mathcal{N}'' := \cup_{i=1}^\infty \mathcal{N}_i$ . Step 1 for  $f_i$  and the inequality  $f \geq f_i$  entail

$$f_i(x, q) = \lim_{\rho \rightarrow 0^+} \frac{m_{reg}(F_i, l_q, A_\rho)}{|A_\rho|} \leq \liminf_{\rho \rightarrow 0^+} \frac{m_{reg}(F, l_q, A_\rho)}{|A_\rho|}, \quad \forall x \in \mathbb{R}^N \setminus \mathcal{N}'', q \in \mathbb{R}^m,$$

for every family  $(A_\rho)_{\rho>0}$  as in the statement. Passing to the limit as  $i \rightarrow +\infty$ , one deduces

$$f(x, q) \leq \liminf_{\rho \rightarrow 0^+} \frac{m_{reg}(F, l_q, A_\rho)}{|A_\rho|}, \quad \forall x \in \mathbb{R}^N \setminus \mathcal{N}'', q \in \mathbb{R}^m,$$

for every family  $(A_\rho)_{\rho>0}$  as in the statement. Finally, we accomplish the proof by choosing  $\mathcal{N} = \mathcal{N}' \cup \mathcal{N}''$ .  $\square$

We denote by  $\mathcal{F} = \mathcal{F}(\alpha, C_1, C_2)$  the set of all functional  $F$  which satisfy assumptions (3.3)–(3.4) with the same constants  $\alpha, C_1$  and  $C_2$ . In the next result, we obtain a characterisation of  $\Gamma$ -convergence in terms of the convergence of the minima of problems with Dirichlet boundary conditions. We like also to mention that very recently some results in this direction have been proved in [27] in much more general geometries but with quite different techniques.

**Theorem 4.2.** *Let  $\{F_n\}_{n \in \mathbb{N}}$  be a sequence of functionals in  $\mathcal{F}$ . Let  $D$  be a dense subset of  $\mathbb{R}^m$ . Let  $\mathcal{B}$  be a family of open bounded subsets of  $\mathbb{R}^N$  which contains a substantial family around every point  $x \in \mathbb{R}^N$ . Assume that for each  $q \in D$  and for each  $B \in \mathcal{B}$  there exists  $\lim_n m(F_n, q, B)$ . Then, there exists a functional  $F_\infty \in \mathcal{F}$  such that the sequence  $\{F_n\}_{n \in \mathbb{N}}$   $\Gamma$ -converge to  $F_\infty$  and*

$$\lim_{n \rightarrow +\infty} m(F_n, l_q, A) = m(F_\infty, l_q, A),$$

for every  $q \in \mathbb{R}^m$  and for every  $A$  bounded domain of  $\mathbb{R}^N$  with Lipschitz boundary.

**Proof.** The proof follows exactly the same arguments of the proof of [11, Theorem IV] so we just sketch the main issues. We first claim that the space  $\mathcal{F}$  can be endowed with a metric  $d$  such that  $(\mathcal{F}, d)$  is a compact metric space and a sequence  $\{F_n\}_n$  of functionals in  $\mathcal{F}$  is convergent w.r.t. to  $d$  to some  $F \in \mathcal{F}$  if and only if

it  $\Gamma$ -converges to  $F$ . Indeed, this property can be obtained following the same arguments of [13, Proposition 1.21] and taking advantage of the properties of  $W_{\mathcal{X},0}^{k,\alpha}$  and of  $W_{\mathcal{X}}^{k,\alpha}$  for the Heisenberg group, in particular the Rellich compact injection and the Poincaré inequality respectively in Lemma 3.2 and in Lemma 3.3. Hence, we shall omit it.

Even if the rest of the proof follows the arguments in [11], for the sake of completeness, let us recall the role of Theorem 4.1. Let  $F_{k_1(n)}$  and  $F_{k_2(n)}$  be two subsequences of  $F_n$  which  $\Gamma$ -converge respectively to some  $F'_\infty$  and to some  $F''_\infty$ . We claim:  $F'_\infty = F''_\infty$ . Actually, we have

$$m_{reg}(F'_\infty, l_q, B) = m_{reg}(F''_\infty, l_q, B), \quad \forall q \in \mathbb{R}^m, B \in \mathcal{B}.$$

Theorem 4.1 ensures that there exists a measurable set  $\mathcal{N} \subset \mathbb{R}^N$ , with  $|\mathcal{N}| = 0$ , such that

$$f'_\infty(x, q) = f''_\infty(x, q), \quad \forall x \in \mathbb{R}^N \setminus \mathcal{N}, q \in D,$$

where  $f'_\infty$  and  $f''_\infty$  are the integrands of  $F'_\infty$  and respectively of  $F''_\infty$ . Finally, the convexity of  $f'_\infty$  and of  $f''_\infty$  permits to extend the previous equality to every  $q \in \mathbb{R}^m$ .  $\square$

### 5. Periodic homogenisation for degenerate functionals with $H$ -affine data

Given the functional  $F(u, A)$  defined in (3.2), we now introduce for all  $\varepsilon > 0$  the following rescaled functionals:

$$F_\varepsilon(u, A) = (\rho_\varepsilon^H F)(u, A) := \begin{cases} \int_A f\left(\delta_{\frac{1}{\varepsilon}}(x), \nabla_{\mathcal{X}} u(x)\right) dx, & u \in W_{\mathcal{X}}^{1,k}(A) \\ +\infty, & \text{else,} \end{cases} \tag{5.1}$$

and for all  $z \in \mathbb{R}^N$ , the following translated functionals:

$$(\tau_z^H F)(u, A) := \begin{cases} \int_A f(z * x, \nabla_{\mathcal{X}} u(x)) dx, & u \in W_{\mathcal{X}}^{k,\alpha}(A) \\ +\infty, & \text{else.} \end{cases} \tag{5.2}$$

Following the idea in [12], for all fixed  $q \in \mathbb{R}^m$ , for all bounded domain  $A \subset \mathbb{R}^N$ , and with  $N = 2n + 1$  and  $m = 2n$ , we introduce the following notation

$$\mu_q(A) := m(F, l_q, A) = \min \left\{ \int_A f(x, \nabla_{\mathcal{X}} u(x)) dx \mid u - l_q \in W_{\mathcal{X},0}^{1,\alpha}(A) \right\}, \tag{5.3}$$

where we recall that  $l_q(x) = q \cdot \pi_m(x)$  is a  $H$ -affine boundary data.

We next define

$$\tau_z^H \mu_q(A) := \mu_q(\tau_z^H(A)) = \mu_q(z * A).$$

**Lemma 5.1.** *Given a bounded domain  $A$  of  $\mathbb{R}^N$ , there holds*

$$\tau_z^H \mu_q(A) = \min \left\{ (\tau_z^H F)(w, A) \mid w - l_q \in W_{\mathcal{X},0}^{1,\alpha}(A) \right\}. \tag{5.4}$$

**Proof.** Note that since the functional  $F$  depends only on the gradient of the function,  $F(w, A) = F(w + l_q(z), A)$ . Thus to prove (5.4) is the same of proving

$$\tau_z^H \mu_q(A) = \min \left\{ (\tau_z^H F)(w + l_q(z), A) \mid w - l_q \in W_{\mathcal{X},0}^{1,\alpha}(A) \right\}. \tag{5.5}$$

In order to prove (5.5) we start looking at the right-hand side and defining  $v(x) := w(x) + l_q(z)$ . Since  $w$  and  $v$  differ only by a constant, obviously

$$\begin{aligned} & \min \left\{ (\tau_z^H F)(w + l_q(z), A) \mid w - l_q \in W_{\mathcal{X},0}^{1,\alpha}(A) \right\} \\ &= \min \left\{ \int_A f(z * x, \nabla_{\mathcal{X}} [w(x) + l_q(z)]) \, d x \mid w - l_q \in W_{\mathcal{X},0}^{1,\alpha}(A) \right\} \\ &= \min \left\{ \int_A f(z * x, \nabla_{\mathcal{X}} v(x)) \, d x \mid v - L_z(l_q) \in W_{\mathcal{X},0}^{1,\alpha}(A) \right\}, \end{aligned} \tag{5.6}$$

where we recall that  $L_z(l_q)(x) := l_q(z * x)$  by definition of translated function. Now we consider the following change of variables  $y = z * x$  (equivalently  $x = z^{-1} * y$  where  $z^{-1}$  is the inverse element w.r.t. the group law  $*$ ).

An easy computation shows that the Jacobian of the change of variables is exactly the matrix  $\sigma_{Ext}$  defined in (2.10). Then property (2.11) tells that  $|\det J| = 1$ . Since  $X_i$  are defined as left-invariant vector fields for all  $i = 1, \dots, m$  (see (2.6)) we also know that

$$\nabla_{\mathcal{X}} v(x) = \nabla_{\mathcal{X}} (L_{z^{-1}}(v))(x) = \nabla_{\mathcal{X}} v(z^{-1} * x).$$

Moreover  $x \in A$  if and only if  $y \in z * A$  and

$$v - L_z(l_q) \in W_{\mathcal{X},0}^{1,\alpha}(A) \quad \text{if and only if} \quad L_{z^{-1}}(v) - l_q \in W_{\mathcal{X},0}^{1,\alpha}(z * A),$$

where  $L_{z^{-1}}(v)(x) := v(z^{-1} * x)$ : in fact on  $\partial(z * A)$  we have  $v(z^{-1} * y) = v(x) = l_q(z * x) = l_q(z * z^{-1} * y) = l_q(y)$ . Then in the new variables  $y = z * x$  we have

$$\begin{aligned} & \min \left\{ \int_A f(z * x, \nabla_{\mathcal{X}} v(x)) \, d x \mid v - L_z(l_q) \in W_{\mathcal{X},0}^{1,\alpha}(A) \right\} \\ &= \min \left\{ \int_{z * A} f(y, \nabla_{\mathcal{X}} v(z^{-1} * y)) \, d y \mid L_{z^{-1}}(v) - l_q \in W_{\mathcal{X},0}^{1,\alpha}(z * A) \right\}. \end{aligned} \tag{5.7}$$

To conclude we now define  $u := L_{z^{-1}}(v)$ . Using again the property of left-invariant vector fields, we have  $\nabla_{\mathcal{X}} v(z^{-1} * y) = \nabla_{\mathcal{X}} u(z * z^{-1} * y) = \nabla_{\mathcal{X}} u(y)$ , then

$$\begin{aligned} & \min \left\{ \int_{z * A} f(y, \nabla_{\mathcal{X}} v(z^{-1} * y)) \, d y \mid L_{z^{-1}}(v) - l_q \in W_{\mathcal{X},0}^{1,\alpha}(z * A) \right\} \\ &= \min \left\{ \int_{z * A} f(y, \nabla_{\mathcal{X}} u(y)) \, d y \mid u - l_q \in W_{\mathcal{X},0}^{1,\alpha}(z * A) \right\} = \tau_z^H \mu_q(A). \end{aligned} \tag{5.8}$$

The chains of identities in (5.6)–(5.8) give identity (5.5) and conclude the proof.  $\square$

The following result is an immediate consequence of the previous lemma in the case of  $H$ -periodic functionals.

**Lemma 5.2.** *Assume (3.5), then, for all bounded domains  $A \subset \mathbb{R}^N$  and for all  $q \in \mathbb{R}^m$  and  $z \in \mathbb{Z}^N$*

$$\tau_z^H \mu_q(A) = \mu_q(A).$$

In the following lemma we show how the assumptions on the integrand  $f(x, q)$  are inherited by  $\mu_q(A)$ .

**Lemma 5.3.** *Let  $A$  be a bounded domain in  $\mathbb{R}^N$  with Lipschitz boundary,  $q \in \mathbb{R}^m$  and  $\mu_q(A)$  defined in (5.3) and let  $f : \mathbb{R}^N \times \mathbb{R}^m \rightarrow \mathbb{R}$  be measurable.*

(i) If  $f$  satisfies assumption (3.4), we have

$$C_1|q|^\alpha|A| \leq \mu_q(A) \leq C_2(|q|^\alpha + 1)|A|,$$

where  $C_1, C_2$  and  $\alpha$  are the same constants given in (3.4).

(ii) If  $f$  satisfies assumption (3.3), we have for all  $q_1, q_2 \in \mathbb{R}^m$  and for all  $\lambda \in (0, 1)$

$$\mu_{\lambda q_1 + (1-\lambda)q_2}(A) \leq \lambda\mu_{q_1} + (1-\lambda)\mu_{q_2}.$$

**Proof.** Using that  $l_q = q \cdot \pi_m$  is admissible for the minimum defining  $\mu_q(A)$  and that  $\nabla_{\mathcal{X}} l_q(x) = q$ , we get

$$\mu_q(A) \leq \int_A f(x, \nabla_{\mathcal{X}} l_q(x)) \, dx = \int_A f(x, q) \, dx \leq C_2(|q|^\alpha + 1)|A|.$$

Moreover

$$\begin{aligned} \mu_q(A) &= \min \left\{ \int_A f(x, \nabla_{\mathcal{X}} u(x)) \, dx \mid u - l_q \in W_{\mathcal{X},0}^{1,\alpha}(A) \right\} \\ &\geq C_1 \min \left\{ \int_A |\nabla_{\mathcal{X}} u(x)|^\alpha \, dx \mid u - l_q \in W_{\mathcal{X},0}^{1,\alpha}(A) \right\} \\ &= C_1 \int_A |\nabla_{\mathcal{X}} l_q(x)|^\alpha \, dx = C_1|q|^\alpha|A|, \end{aligned}$$

where for the last identity we use Lemma 4.2 for the convex function  $f(x, q) = |q|^\alpha$ , which tells that the minimisers are the  $H$ -affine functions, whenever  $f$  does not depend on  $x$ .

It remains to prove (ii). To this end it is enough to remark that for all functions  $u_1$  and  $u_2$  which are admissible respectively for  $\mu_{q_1}$  and  $\mu_{q_2}$ , then  $\bar{u} := \lambda u_1 + (1-\lambda)u_2$  is admissible for  $\mu_{\lambda q_1 + (1-\lambda)q_2}$ , which implies

$$\begin{aligned} \mu_{\lambda q_1 + (1-\lambda)q_2}(A) &\leq \int_A f(x, \nabla_{\mathcal{X}} \bar{u}(x)) \, dx \\ &= \int_A f(x, \lambda \nabla_{\mathcal{X}} u_1(x) + (1-\lambda) \nabla_{\mathcal{X}} u_2(x)) \, dx \\ &\leq \lambda \int_A f(x, \nabla_{\mathcal{X}} u_1(x)) \, dx + (1-\lambda) \int_A f(x, \nabla_{\mathcal{X}} u_2(x)) \, dx. \end{aligned}$$

Taking the minimum over all admissible  $u_1$  and  $u_2$ , we get property (ii).  $\square$

To prove the convergence of the functional  $F_\varepsilon(u, A)$  as  $\varepsilon \rightarrow 0^+$ , we need to show now a sort of Akcoglu–Krengel type result (see [1]) for periodic functionals, adapted to the anisotropic structure of the Heisenberg group. In [24] the authors prove a very interesting Akcoglu–Krengel type result for general metric measure spaces. We need to mention that unfortunately the result therein does not apply to our case. In fact it is quite easy to show that the Heisenberg group endowed with the Carnot–Carathéodory metric (or also with the homogeneous metric) and the Lebesgue measure is a  $(G, \{\delta_t\}_{t>0})$ -metric measure space where  $G$  is the subgroup of homeomorphisms on the Heisenberg group defined by the left-translations w.r.t. an element in  $\mathbb{Z}^N$ . Nevertheless one can also show that in general that space is not “measurable” according to the definition introduced in [24]. We give a self-contained proof which can be later adapted to the stochastic case (which will be a topic in a forthcoming paper, see Section 6).

We now recall that, defining for all  $t > 0$ ,  $Q^t = \delta_t(Q)$ , we know that  $|\delta_t(Q)| = t^{\mathcal{Q}}|Q|$  (see (2.3)), where  $\mathcal{Q}$  is the homogeneous dimension, then in  $\mathbb{H}^n$  in particular  $\mathcal{Q} = 2n + 2 = N + 1$ .

The next lemma tells that, as  $t \rightarrow +\infty$ , we can reduce to take the limits only over integer subsequences.



**Lemma 5.4.** *Assume that the limit exists for integer sequences, i.e. for  $h \in \mathbb{N}$ ,*

$$\lim_{h \rightarrow \infty} \frac{\mu_q(Q^h)}{|Q^h|} =: C.$$

*Then for all sequences  $\{t_k\} \subset \mathbb{R}$  with  $t_k \rightarrow \infty$  it holds*

$$\lim_{k \rightarrow \infty} \frac{\mu_q(Q^{t_k})}{|Q^{t_k}|} = C.$$

**Proof.** For  $t > 0$  we define

$$e_t := \frac{\mu_q(Q^t)}{|Q^t|}.$$

Fix  $\varepsilon > 0$  and choose  $N$  large enough that  $|C - e_h| < \varepsilon$  for  $h \geq N$ .

Denote by  $C^+ := \limsup_{k \rightarrow \infty} e_{t_k}$ , and  $C^- := \liminf_{k \rightarrow \infty} e_{t_k}$ , which are both finite by Lemma 5.3. We can find  $k$  such that

$$e_{t_k} \geq C^+ - \varepsilon \quad \text{and} \quad t_k > N.$$

Define  $N_k := [t_k] \geq N$ , (where by  $[\cdot]$  we indicate the integer part of a real number) and let  $u_k$  be a function with  $H$ -affine boundary conditions on  $Q^{N_k}$  such that  $F(u_k, Q^{N_k}) = \mu_q(Q^{N_k})$ . We extend  $u_k$  to  $Q^{t_k}$  by letting it equal to the boundary condition on  $Q^{t_k} \setminus Q^{N_k}$ , i.e.  $\tilde{u}_k : \mathbb{R}^N \rightarrow \mathbb{R}$  given by

$$\tilde{u}_k(x) := \begin{cases} u_k(x), & \text{if } x \in Q^{N_k}, \\ l_q(x), & \text{else,} \end{cases}$$

whose restriction to  $Q^{t_k}$  is an admissible function for  $\mu_q(Q^{t_k})$ . Note that

$$f(x, \nabla_{\mathcal{X}} \tilde{u}_k) = f(x, \nabla_{\mathcal{X}} l_q) = f(x, q) \leq C_2(|q|^\alpha + 1) \quad \text{on } Q^{t_k} \setminus Q^{N_k},$$

hence

$$F(\tilde{u}_k, Q^{t_k}) = \int_{Q^{N_k}} f(x, \nabla_{\mathcal{X}} u_k) dx + \int_{Q^{t_k} \setminus Q^{N_k}} f(x, q) dx \leq F(u_k, Q^{N_k}) + C|Q^{t_k} \setminus Q^{N_k}|,$$

where the constant depends on  $q$  and  $\alpha$ . Since  $\tilde{u}_k$  is admissible for  $Q^{t_k}$ , so  $\mu_q(Q^{t_k}) \leq F(\tilde{u}_k, Q^{t_k})$ , we estimate

$$\begin{aligned} C^+ &\leq e_{t_k} + \varepsilon \leq \frac{F(\tilde{u}_k, Q^{t_k})}{|Q^{t_k}|} + \varepsilon \leq \frac{F(u_k, Q^{N_k})}{|Q^{t_k}|} + \varepsilon + C \frac{|Q^{t_k} \setminus Q^{N_k}|}{|Q^{t_k}|} \\ &= \varepsilon + \frac{\mu_q(Q^{N_k})}{|Q^{t_k}|} + C \frac{|Q^{t_k} \setminus Q^{N_k}|}{|Q^{t_k}|} = e_{N_k} \frac{|Q^{N_k}|}{|Q^{t_k}|} + \varepsilon + C \frac{|Q^{t_k} \setminus Q^{N_k}|}{|Q^{t_k}|}. \end{aligned}$$

Note that

$$\lim_{k \rightarrow \infty} \frac{|Q^{N_k}|}{|Q^{t_k}|} = 1 \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{|Q^{t_k} \setminus Q^{N_k}|}{|Q^{t_k}|} = 0,$$

so, by choosing, if necessary,  $N$  larger, we can make the right hand side  $\leq C + 3\varepsilon$ , thus, as  $\varepsilon$  was arbitrary, we have shown  $C^+ \leq C$ .

For the opposite inequality, we use estimates similar to what we did before: we can find infinitely many  $k$  such that

$$e_{t_k} \leq C^- + \varepsilon \quad \text{and} \quad t_k > N.$$

Therefore we take  $N_k = [t_k] + 1$  and let  $u_k$  be a function with  $H$ -affine boundary condition  $l_q$  on  $Q^{t_k}$  such that  $F(u_k, Q^{t_k}) = \mu_q(Q^{t_k})$ . We extend  $u_k$  to a function  $\tilde{u}_k$  on  $Q^{N_k}$  which is admissible for  $\mu_q(Q^{N_k})$  and equals  $l_q$  on  $Q^{N_k} \setminus Q^{t_k}$ . Arguing as before we get

$$F(\tilde{u}_k, Q^{N_k}) \leq F(u_k, Q^{t_k}) + C|Q^{N_k} \setminus Q^{t_k}|.$$

Then

$$\begin{aligned} C &\leq e_{N_k} + \varepsilon \leq \frac{F(\tilde{u}_k, Q^{N_k})}{|Q^{N_k}|} + \varepsilon \leq \frac{F(u_k, Q^{t_k})}{|Q^{N_k}|} + \varepsilon + C \frac{|Q^{N_k} \setminus Q^{t_k}|}{|Q^{N_k}|} \\ &= \varepsilon + \frac{\mu_q(Q^{t_k})}{|Q^{N_k}|} + C \frac{|Q^{N_k} \setminus Q^{t_k}|}{|Q^{N_k}|} = e_{t_k} \frac{|Q^{t_k}|}{|Q^{N_k}|} + \varepsilon + C \frac{|Q^{N_k} \setminus Q^{t_k}|}{|Q^{N_k}|} \\ &\leq (C^- + \varepsilon) + (C^- + \varepsilon) \left( \frac{|Q^{t_k}|}{|Q^{N_k}|} - 1 \right) + \varepsilon + C \frac{|Q^{N_k} \setminus Q^{t_k}|}{|Q^{N_k}|}. \end{aligned}$$

By choosing, if necessary,  $N$  larger, we can make the right hand side smaller than  $C^- + (2C^- + 3)\varepsilon$ , thus we have shown  $C \leq C^-$ , but as  $C^- \leq C^+$ , we have  $C^- = C = C^+$ .  $\square$

We denote  $\mathbb{N}^*$  the set of natural numbers excluding 0. We next prove an Akcoglu–Krengel type result.

**Theorem 5.1.** *Let consider the (semiopen) unit cell  $Q = [-1, 1)^N$  and let  $q \in \mathbb{R}^m$  and  $\mu_q$  be defined in (5.3). Assume that  $f$  is measurable and satisfies (3.4) and (3.5), then*

$$\lim_{k \rightarrow +\infty} \frac{\mu_q(Q^k)}{|Q^k|} = C_q,$$

where  $C_q$  is the non-negative constant given by

$$C_q = \inf_{k \in \mathbb{N}^*} \frac{\mu_q(Q^k)}{|Q^k|}.$$

**Proof.** Note that, by Lemma 5.3(i),

$$C_1 |q|^\alpha \leq \frac{\mu_q(Q^k)}{|Q^k|} \leq C_2 (|q|^\alpha + 1), \tag{5.9}$$

so in particular  $C_q \geq 0$ .

**Step 1.** Since  $C_q$  is defined as infimum over  $\mathbb{N}^*$ , trivially  $\frac{\mu_q(Q^k)}{|Q^k|} \geq C_q$  for all  $k \in \mathbb{N}^*$ , which implies

$$\liminf_{k \rightarrow +\infty} \frac{\mu_q(Q^k)}{|Q^k|} \geq C_q.$$

**Step 2.** We next show the limsup estimate. Using the definition of infimum for  $C_q$ , for all  $\rho > 0$ , and the definition of  $\mu_q$ , then there exist  $k_\rho \in \mathbb{N}^*$  and  $u_\rho \in W_{\chi^{1,\alpha}}(Q^{k_\rho}) \cap C^\infty(Q^{k_\rho})$  and such that

$$\begin{aligned} \frac{\mu_q(Q^{k_\rho})}{|Q^{k_\rho}|} &\leq C_q + \frac{\rho}{2}, \\ \frac{F(u_\rho, Q^{k_\rho})}{|Q^{k_\rho}|} &\leq \frac{\mu_q(Q^{k_\rho})}{|Q^{k_\rho}|} + \frac{\rho}{2}, \end{aligned}$$

where  $F$  is the functional defined in (3.2). This sums up as follows

$$\frac{F(u_\rho, Q^{k_\rho})}{|Q^{k_\rho}|} \leq C_q + \rho. \tag{5.10}$$

Recall the definition of  $\tau_k$  given in (2.12); we use such translations to extend  $u_\rho$  to the whole  $\mathbb{R}^N$  by translating periodically the gradient. More precisely, let us introduce

$$j_\rho := \delta_{k_\rho}(j) \quad \text{and} \quad Q_j^\rho := \tau_{j_\rho}(Q^{k_\rho}), \quad \forall j \in \mathbb{Z}^N.$$

Using that  $\mathbb{R}^N = \bigcup_{j \in \mathbb{Z}^N} Q_j^\rho$ , we can define

$$U_\rho(x) := \sum_{j \in \mathbb{Z}^N} \left( q \cdot \pi_m(j_\rho) + u_\rho(\tau_{-j_\rho}(x)) \right) \mathbb{1}_{Q_j^\rho},$$

where by  $\mathbb{1}_A$  we indicate the characteristic function of the set  $A$ ; recall also that  $\tau_k^{-1} = \tau_{-k}$ . The function  $U_\rho$  is well-defined since  $Q_j^\rho$  are all disjoint. We can easily check that  $U_\rho$  is continuous on  $\mathbb{R}^N$ : in fact, for  $x \in Q_j^\rho$ , then  $U_\rho(x) = q \cdot \pi_m(j_\rho) + u_\rho(\tau_{-j_\rho}(x))$  and, whenever  $x \in \partial Q_j^\rho$ , we have  $\tau_{-j_\rho}(x) \in \partial Q^{k_\rho}$  which implies

$$U_\rho(x) = q \cdot \pi_m(j_\rho) + q \cdot \pi_m(\tau_{-j_\rho}(x)) = q \cdot \pi_m(j_\rho) + q \cdot (\pi_m(-j_\rho) + \pi_m(x)) = q \cdot \pi_m(x),$$

which does not anymore depend on  $j$ . The continuity of  $U_\rho$  on  $\mathbb{R}^N$ , together with the fact that  $U_\rho \in W_{\mathcal{X}}^{1,\alpha}(Q_j^\rho)$ , imply that  $U_\rho \in W_{\mathcal{X},\text{loc}}^{1,\alpha}(\mathbb{R}^N)$ .

We next introduce the following two objects:

$$\begin{aligned} S_k^\rho &:= \left\{ j \in \mathbb{Z}^N \mid Q_j^\rho \subset Q^k = \delta_k(Q) \right\}, \\ \widehat{S}_k^\rho &:= \bigcup_{j \in S_k^\rho} Q_j^\rho, \end{aligned}$$

and we construct a new function  $v_\rho$ , which is admissible for  $\mu_q(Q^k)$ , as

$$v_\rho(x) := \begin{cases} U_\rho(x), & x \in \widehat{S}_k^\rho \\ q \cdot \pi_m(x), & x \in Q^k \setminus \widehat{S}_k^\rho. \end{cases}$$

By definition  $v_\rho - l_q \in W_{\mathcal{X},0}^{1,\alpha}(Q^k)$ , and

$$\begin{aligned} F(v_\rho, Q^k) &= \int_{Q^k} f(x, \nabla_{\mathcal{X}} v_\rho(x)) \, dx \\ &= \int_{\widehat{S}_k^\rho} f(x, \nabla_{\mathcal{X}} v_\rho(x)) \, dx + \int_{Q^k \setminus \widehat{S}_k^\rho} f(x, \nabla_{\mathcal{X}} v_\rho(x)) \, dx \end{aligned}$$

First we compute

$$\int_{\widehat{S}_k^\rho} f(x, \nabla_{\mathcal{X}} v_\rho(x)) \, dx = \int_{\widehat{S}_k^\rho} f(x, \nabla_{\mathcal{X}} U_\rho(x)) \, dx = \sum_{j \in S_k^\rho} \int_{Q_j^\rho} f(x, \nabla_{\mathcal{X}} U_\rho(x)) \, dx, \tag{5.11}$$

where we have used that  $Q_j^\rho$  are disjoint. If  $x \in Q_j^\rho$ , then  $U_\rho(x) = q \cdot \pi_m(j_\rho) + u_\rho(\tau_{-j_\rho}(x))$ . By using that the vector fields are left invariant, we get

$$\nabla_{\mathcal{X}} U_\rho(x) = \nabla_{\mathcal{X}} \left( u_\rho(\tau_{-j_\rho}(x)) \right) = \nabla_{\mathcal{X}} u_\rho(\tau_{-j_\rho}(x)).$$

Thus, by using the change of variables  $y = \tau_{-j_\rho}(x)$  and recalling that the determinant of the Jacobian is 1 (see (2.11)), we get the following chain of identities:

$$\begin{aligned} \sum_{j \in S_k^\rho} \int_{Q_j^\rho} f(x, \nabla_{\mathcal{X}} U_\rho(x)) \, dx &= \sum_{j \in S_k^\rho} \int_{Q_j^\rho} f(x, \nabla_{\mathcal{X}} u_\rho(\tau_{-j_\rho}(x))) \, dx \\ &= \sum_{j \in S_k^\rho} \int_{Q^{k_\rho}} f(\tau_{j_\rho}(y), \nabla_{\mathcal{X}} u_\rho(y)) \, dy = \sum_{j \in S_k^\rho} \int_{Q^{k_\rho}} f(y, \nabla_{\mathcal{X}} u_\rho(y)) \, dy, \end{aligned} \tag{5.12}$$

where in the last identity above we have used the periodicity assumption on  $f$  (see assumption (3.5)). The integrals in the last term of (5.12) do not depend anymore on  $j$ , then

$$\begin{aligned} \sum_{j \in S_k^\rho} \int_{Q^{k\rho}} f(y, \nabla_{\mathcal{X}} u_\rho(y)) \, dy &= \text{card}(S_k^\rho) \int_{Q^{k\rho}} f(y, \nabla_{\mathcal{X}} u_\rho(y)) \, dy \\ &\leq \text{card}(S_k^\rho) (C_q + \rho) |Q^{k\rho}|, \end{aligned} \tag{5.13}$$

where the last inequality follows from (5.10).

Put together (5.11)–(5.13), we get the following estimate:

$$\int_{\widehat{S}_k^\rho} f(x, \nabla_{\mathcal{X}} v_\rho(x)) \, dx \leq \text{card}(S_k^\rho) (C_q + \rho) |Q^{k\rho}|. \tag{5.14}$$

It remains to estimate the integral on the complementary of  $\widehat{S}_k^\rho$  by using that  $v_\rho(x) = q \cdot \pi_m(x)$  for all  $x \in Q^k \setminus \widehat{S}_k^\rho$  by definition, hence

$$\int_{Q^k \setminus \widehat{S}_k^\rho} f(x, \nabla_{\mathcal{X}} v_\rho(x)) \, dx \leq \int_{Q^k \setminus \widehat{S}_k^\rho} f(x, q) \, dx \leq C_2 (|q|^\alpha + 1) |Q^k \setminus \widehat{S}_k^\rho|. \tag{5.15}$$

Estimates (5.14) and (5.15), together with the fact that  $v_\rho$  is admissible for  $\mu_q(Q^k)$ , give

$$\frac{\mu_q(Q^k)}{|Q^k|} \leq \frac{F(v_\rho, Q^k)}{|Q^k|} \leq \text{card}(S_k^\rho) (C_q + \rho) \frac{|Q^{k\rho}|}{|Q^k|} + C_2 (|q|^\alpha + 1) \frac{|Q^k \setminus \widehat{S}_k^\rho|}{|Q^k|}, \tag{5.16}$$

where in the last inequality we have used that  $\widehat{S}_k^\rho \subset Q^k$  and  $|\widehat{S}_k^\rho| = \text{card}(S_k^\rho) |Q^{k\rho}|$ , which together imply  $\frac{\text{card}(S_k^\rho) |Q^{k\rho}|}{|Q^k|} \leq 1$ .

To conclude we claim that the following limit holds true:

$$\lim_{k \rightarrow +\infty} \frac{|Q^k \setminus \widehat{S}_k^\rho|}{|Q^k|} = 0. \tag{5.17}$$

Then, by simply taking the limsup as  $k \rightarrow +\infty$  in the inequality (5.16) and using claim (5.17), we get

$$\limsup_{k \rightarrow +\infty} \frac{\mu_q(Q^k)}{|Q^k|} \leq (C_q + \rho),$$

which conclude the proof as  $\rho \rightarrow 0^+$ .

It remains only now to prove claim (5.17). By a simple rescaling we can actually show that this limit is the same as the one shown in the proof of Lemma 2.21 in [23]. In fact, set  $\varepsilon = \frac{1}{k}$ , then by using the properties of dilations (Lemma 2.1)

$$Q^\varepsilon = \delta_\varepsilon(Q) = \delta_{\frac{1}{k\rho}} \left( \delta_{k\rho}(\delta_\varepsilon(Q)) \right) = \delta_{\frac{\varepsilon}{k\rho}}(Q^{k\rho}).$$

Set  $\widetilde{Q} := \delta_\varepsilon(Q^{k\rho}) = \delta_{\frac{\varepsilon}{k\rho}}(Q^{\frac{1}{\varepsilon}})$ , by using the properties of dilations and left-translations one can easily check that

$$\tau_{\delta_\varepsilon(j)}(Q^\varepsilon) \subset \widetilde{Q} \iff \tau_{\delta_{k\rho}(j)}(Q^{k\rho}).$$

Thus by using the limit proved in [23] we conclude the proof.  $\square$

We define  $f_0 : \mathbb{R}^m \rightarrow \mathbb{R}$  as

$$f_0(q) := C_q, \tag{5.18}$$

where  $C_q$  is the limit proved in Theorem 5.1.

From Lemma 5.3, one can show that  $f_0$  keeps the properties of  $f$  simply by passing to the limit as  $k \rightarrow +\infty$ . More precisely

**Lemma 5.5.** Given  $f : \mathbb{R}^N \times \mathbb{R}^m \rightarrow \mathbb{R}$  measurable, the following properties hold:

(i) if assumption (3.4) is satisfied, then

$$C_1|q|^\alpha \leq f_0(q) \leq C_2(|q|^\alpha + 1),$$

where  $C_1, C_2$  and  $\alpha$  are the same constants given in (3.4).

(ii) if assumption (3.3) is satisfied, then for all  $q_1, q_2 \in \mathbb{R}^m$

$$f_0(\lambda q_1 + (1 - \lambda)q_2) \leq \lambda f_0(q_1) + (1 - \lambda)f_0(q_2), \quad \lambda \in (0, 1).$$

We now prove the main result of the paper.

**Theorem 5.2.** Given a bounded domain  $A \subset \mathbb{R}^N$  with Lipschitz boundary,  $u : A \rightarrow \mathbb{R}$  and the functional  $F(u, A)$  defined in (3.2). Let us assume that (3.3), (3.4) and (3.5) hold true, and  $u_0(x) = q \cdot \pi_m(x) + a$  for some  $q \in \mathbb{R}^m$  and  $a \in \mathbb{R}$ . Define the rescaled functionals  $F_\varepsilon$  introduced in (5.1) and let us consider the corresponding minimisation problems for  $u - u_0 \in W_{\mathcal{X},0}^{1,\alpha}(A)$  (see (3.6)) then

$$\lim_{\varepsilon \rightarrow 0^+} m(F_\varepsilon, u_0, A) = m(F_\infty, u_0, A),$$

where the limit functional  $F_\infty$  can be characterised as

$$F_\infty(u, A) := \begin{cases} \int_A f_0(\nabla_{\mathcal{X}} u) \, dx, & u \in W_{\mathcal{X}}^{1,\alpha}(A), \\ +\infty, & \text{else,} \end{cases}$$

and  $f_0 : \mathbb{R}^m \rightarrow \mathbb{R}$  defined as  $f_0(q) = C_q$  with  $C_q$  constant given in Theorem 5.1.

Moreover the limit function  $f_0$  is still measurable, convex and satisfies the same growth condition (3.4) satisfied by  $f$ .

**Proof.** Applying Theorem 4.2 we deduce that  $F_\varepsilon$   $\Gamma$ -converge to some functional  $F_\infty$ . Let us now prove that the limit functional  $F_\infty$  can be identified as the integral functional associated to  $f_0$  given in (5.18). Choose as substantial family  $A_\rho := [-\rho, \rho]^N$ , and fix  $t = \frac{1}{\varepsilon}$ , at the moment let us assume the following claim:

$$\mu_q(\delta_t(A_\rho)) = t^{\mathcal{Q}} m(F_\varepsilon, l_q, A_\rho), \tag{5.19}$$

with  $\mathcal{Q}$  homogeneous dimension.

By using that all the previous results in Section 5 can be obtained by replacing  $Q$  with the cube  $A_\rho$ , we have

$$\begin{aligned} \frac{m(F_\infty, l_q, A_\rho)}{|A_\rho|} &= \lim_{\varepsilon \rightarrow 0^+} \frac{m(F_\varepsilon, l_q, A_\rho)}{|A_\rho|} = \lim_{t \rightarrow +\infty} \frac{1}{|A_\rho|} \frac{\mu_q(\delta_t(A_\rho))}{t^{\mathcal{Q}}} \\ &= \lim_{t \rightarrow +\infty} \frac{\mu_q(\delta_t(A_\rho))}{|\delta_t(A_\rho)|} = C_q, \quad \forall \rho > 0. \end{aligned}$$

By Theorem 4.1, passing to the limit as  $\rho \rightarrow 0^+$ , we conclude  $f_0(q) = C_q$ .

It only remains to check claim (5.19). At this purpose, we use the change of variables  $y = \delta_{1/t}(x)$ ; hence recalling definition (5.3) and using that  $\delta_t(u)$  is the scaled function defined as  $\delta_t(u)(x) = u(\delta_t(x))$ , we have

$$\begin{aligned} \mu_q(\delta_t(A_\rho)) &= \min \left\{ \int_{\delta_t(A_\rho)} f(x, \nabla_{\mathcal{X}} u(x)) \, dx \mid u - l_q \in W_{\mathcal{X},0}^{1,\alpha}(\delta_t(A_\rho)) \right\} \\ &= t^{\mathcal{Q}} \min \left\{ \int_{A_\rho} f(\delta_t(y), \nabla_{\mathcal{X}} u(\delta_t(y))) \, dy \mid \delta_t(u) - l_{tq} \in W_{\mathcal{X},0}^{1,\alpha}(A_\rho) \right\}, \end{aligned} \tag{5.20}$$

by simply using that  $l_q(\delta_t(y)) = q \cdot \pi_m(\delta_t(y)) = q \cdot t \pi_m(y) = (tq) \cdot \pi_m(y)$  and (2.3). Defining the function  $w := \frac{1}{t} \delta_t(u)$  and using Lemma 2.2, we have that

$$\nabla_{\mathcal{X}} w(y) = \frac{1}{t} \nabla_{\mathcal{X}}(\delta_t(u))(y) = \frac{1}{t} t \nabla_{\mathcal{X}} u(\delta_t(y)) = \nabla_{\mathcal{X}} u(\delta_t(y)).$$

Moreover informally we have that, for  $y \in \partial A_\rho$ ,  $\frac{1}{t} u(\delta_t(y)) = \frac{1}{t} l_q(\delta_t(y)) = l_q(y)$ . Hence (5.20) gives

$$\begin{aligned} \mu_q(\delta_t(A_\rho)) &= t^{\mathcal{Q}} \min \left\{ \int_{A_\rho} f(\delta_t(y), \nabla_{\mathcal{X}} w(y)) \, dy \mid w - l_q \in W_{\mathcal{X},0}^{1,\alpha}(A_\rho) \right\} \\ &= t^{\mathcal{Q}} m(F_{1/t}, l_q, A_\rho), \end{aligned}$$

which proves claim (5.19).

The properties for the limit function  $f_0$  are proved in Lemma 5.5.  $\square$

### 6. Applications and generalisations

We conclude listing further directions in which we are presently working, for some of which we obtained already some partial results.

#### 6.1. Homogenisation for functionals associated to Carnot groups and the subelliptic $p$ -Laplacian.

As mentioned in the introduction all the proofs never use the specific structure of the Heisenberg group but they instead use properties true for all Carnot groups. So all the results apply without any modification to the general case of Carnot groups.

As it is well-known by Euler–Lagrange equations, we can connect minima of functionals to solutions of PDEs. Whenever uniqueness holds this correspondence is one-to-one. Then our results can be used to study homogenisation for several subelliptic PDEs and in particular for the subelliptic  $p$ -Laplacian, which is defined, for  $1 < p < +\infty$ , as

$$\operatorname{div}_{\mathcal{X}} \left( \langle \mathcal{A} \nabla_{\mathcal{X}} u, \nabla_{\mathcal{X}} u \rangle^{\frac{p-2}{2}} \mathcal{A} \nabla_{\mathcal{X}} u \right) = 0,$$

where  $\mathcal{A}(x)$  is a  $m \times m$  symmetric matrix satisfying the usual ellipticity condition. Equations of this form have been studied by many authors, see e.g. [30] and references therein. The functional associated to the  $p$ -Laplacian is

$$F_p(u, A) = \begin{cases} \int_A |\mathcal{A}(\delta_{\frac{1}{\varepsilon}}(x)) \nabla_{\mathcal{X}} u(x)|^p \, dx, & u \in W_{\mathcal{X}}^{1,p}(A) \\ +\infty, & \text{else.} \end{cases}$$

Note that  $F_p$  satisfies all our conditions for all  $1 < p < +\infty$ . Then we can apply Theorem 5.2. It remains now to show that the limit functional has still the structure of a functional associated to a subelliptic  $p$ -Laplacian equation (work in preparation).

#### 6.2. Stochastic functionals

Another generalisation is the case of random functionals, i.e. integral functionals of the form

$$u \mapsto F_\varepsilon(u, A) = \int_A f \left( \delta_{\frac{1}{\varepsilon}}(x), \omega, \nabla u(x) \right) \, dx,$$

where  $\omega$  belongs to a probability space and the integrand  $f(x, \omega, p)$  is stationary and ergodic with respect to left translations. For a precise definition of stationary ergodic in the setting of Carnot groups we refer to

[19], where the authors prove a homogenisation result for stochastic Hamilton–Jacobi equations. The general stationary ergodic case will be treated in a forthcoming paper, but we sketch here a proof for the simpler situation of short correlated random variables.

More precisely, we assume that the random integrand  $f(x, \omega, p)$  satisfies (3.3) and (3.4) uniformly in  $\omega$  and (3.5) *in law*, i.e. the random integrand and its translations are not equal, but have the same law as random variables. In addition, we require that there exists a constant  $C > 0$  such that  $f(x, \omega, p)$  and  $f(y, \omega, p)$  are independent, if  $d_h(x, y) > C$ , where by  $d_h(x, y)$  we indicate the homogeneous distance in Carnot groups, i.e. for example in 1-dimensional Heisenberg  $d_h(x, y) = |y^{-1} * x|_h$  where  $|x|_h := ((x_1^2 + x_2^2)^2 + x_3^2)^{1/4}$ . Note that this is different from being short correlated in the Euclidean distance.

Under these assumptions one can show along the lines of [13] that

$$\lim_{k \rightarrow +\infty} \frac{\mu_q(\omega, Q^k)}{|Q^k|} = C_q,$$

in probability to a constant  $C_q > 0$ , and conclude convergence of the functionals in probability to an integral functional with constant integrand  $f_0(q) = C_q$ .

As a first step, defining

$$\tilde{\mu}_q(Q) := \mathbb{E}(\mu_q(\omega, Q)),$$

one can show along the lines of Section 5 that

$$\lim_{k \rightarrow +\infty} \frac{\tilde{\mu}_q(\omega, Q^k)}{|Q^k|} = C_q,$$

for some constant  $C_q > 0$ . Note that because of the invariance in law, Lemma 5.2 holds for  $\tilde{\mu}$  but not for  $\mu(\omega, \cdot)$  with  $\omega$  fixed.

Now fix  $k_0 \gg 1$  so large that

$$\left| \frac{\tilde{\mu}_q(Q^k)}{|Q^k|} - C_q \right| < \delta/4, \quad \text{for all } k \geq k_0$$

and now fix  $k \gg k_0$ , we use the construction in step 2 of the proof of Theorem 5.1 to show that

$$\frac{\mu(\omega, Q^k)}{|Q^k|} \leq \frac{|Q^k|}{|Q^{k_0}|} \sum_{j \in S_k^{k_0}} \frac{\mu(\omega, \tau_{j k_0}(Q^{k_0}))}{|Q^{k_0}|} + o(1).$$

The r.h.s. is a normalised sum over  $(k/k_0)^{\mathcal{Q}}$  independent, identically distributed random variables with mean close to  $C_q$ . By the weak law of large numbers, we have that for  $\delta > 0$  and  $k$  sufficiently large the quantity

$$\beta(\delta) := \mathbb{P} \left( \left\{ \omega \mid \frac{\mu(\omega, Q^k)}{|Q^k|} > C_q + \delta/4 \right\} \right)$$

is small. Now define

$$\alpha(\delta) := \mathbb{P} \left( \left\{ \omega \mid \frac{\mu(\omega, Q^k)}{|Q^k|} < C_q - \sqrt{\delta} \right\} \right).$$

We have

$$\begin{aligned} C_q &\leq \frac{\mathbb{E}(\mu(\omega, Q^k))}{|Q^k|} + \delta/2 \\ &\leq \alpha(\delta)(C_q - \sqrt{\delta}) + C_2(|q|^\alpha + 1)\beta(\delta) \\ &\quad + \mathbb{P} \left( \left\{ \omega \mid C_q - \sqrt{\delta} \leq \frac{\mu(\omega, Q^k)}{|Q^k|} < C_q + \delta/4 \right\} \right) + \delta/2 \\ &\leq \alpha(\delta)(C_q - \sqrt{\delta}) + C_2(|q|^\alpha + 1)\beta(\delta) + (1 - \alpha - \beta)(C_q + \delta/4) + \delta/2 \\ &\leq C_q - \alpha(\delta)\sqrt{\delta} + (3/4)\delta + \beta(\delta)C_2(|q|^\alpha + 1). \end{aligned}$$

As we can make  $\beta(\delta)$  arbitrarily small by choosing  $k$  big, this implies that for such  $k$  also  $\alpha(\delta) \rightarrow 0$  in order to avoid the contradiction  $C_q < C_q$ .

## Acknowledgements

The first author was partially supported by EPSRC via grant EP/M028607/1. The third and the fourth author are members of GNAMPA-INdAM and have been partially supported also by the Fondazione CaRiPaRo Project “Nonlinear Partial Differential Equations: Asymptotic Problems and Mean-Field Games”. They warmly thank University of Cardiff for the kind hospitality.

## References

- [1] M.A. Akcoglu, U. Krengel, Ergodic theorems for superadditive processes, *J. Reine Angew. Math.* 323 (1981) 53–67.
- [2] M. Bardi, F. Dragoni, Convexity and semiconvexity along vector fields, *Calc. Var. Partial Differential Equations* 42 (3–4) (2011) 405–427.
- [3] M. Bardi, F. Dragoni, Subdifferential and properties of convex functions with respect to vector fields, *J. Convex Anal.* 21 (3) (2014) 785–810.
- [4] I. Birindelli, J. Wigniolle, Homogenisation of Hamilton–Jacobi equations in the Heisenberg group, *Commun. Pure Appl. Anal.* 2 (4) (2003) 461–479.
- [5] M. Biroli, U. Mosco, N.A. Tchou, Homogenisation for degenerate operators with periodical coefficients with respect to the Heisenberg group, *C. R. Acad. Sci., Paris I* 322 (5) (1996) 439–444.
- [6] M. Biroli, C. Picard, N.A. Tchou, Homogenization of the  $p$ -Laplacian associated with the Heisenberg group, *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl.* 22 (5) (1998) 23–42.
- [7] A. Bonfiglioli, E. Lanconelli, F. Uguzzoni, *Stratified Lie Groups and Potential Theory for their Sub-Laplacians*, Springer, Berlin, 2007.
- [8] A. Braides,  *$\Gamma$ -Convergence for Beginners*, Oxford University Press, Oxford, 2002.
- [9] A. Braides, A. Defranceschi, *Homogenization of Multiple Integrals*, Oxford University Press, Oxford, 1998.
- [10] G. Dal Maso, *An Introduction To  $\Gamma$ -Convergence*, Birkhäuser, Boston, 1993.
- [11] G. Dal Maso, L. Modica, Integral functionals determined by their minima, *Rend. Semin. Mat. Univ. Padova* 76 (1986) 255–267.
- [12] G. Dal Maso, L. Modica, Nonlinear stochastic homogenisation and ergodic theory, *J. Reine Angew. Math.* 368 (1986) 28–42.
- [13] G. Dal Maso, L. Modica, Nonlinear stochastic homogenization, *Ann. Mat. Pura Appl.* 144 (4) (1986) 347–389.
- [14] D. Danielli, N. Garofalo, D.M. Nhieu, Notions of convexity in Carnot groups, *Comm. Anal. Geom.* 11 (2) (2003) 263–341.
- [15] E. De Giorgi, Generalised limits in Calculus of variations, in: F. Strocchi, et al. (Eds.), *Topics in Functional Analysis 1980–81*, Quad. Scuola Norm. Sup. Pisa (1981) 117–148.
- [16] E. De Giorgi, T. Franzoni, Su un tipo di convergenza variazionale, *Atti Accad. Naz. Lincei, Rend. Cl. Sci. Mat. Fis. Natur.* (8) 58 (6) (1975) 842–850.
- [17] E. De Giorgi, S. Spagnolo, Sulla convergenza degli integrali dell’energia per operatori ellittici del secondo ordine, *Boll. Un. Mat. Ital.* (4) 8 (1973) 391–411.
- [18] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, *Bull. Sci. Math.* 136 (12) (2012) 521–573.
- [19] N. Dirr, F. Dragoni, P. Mannucci, C. Marchi, Stochastic homogenization for functionals with anisotropic rescaling and non-coercive Hamilton–Jacobi equations, *SIAM J. Math. Anal.* 50 (5) (2018) 5198–5242.
- [20] A. Domokos, J.J. Manfredi, Nonlinear subelliptic equations, *Manuscripta Math.* 130 (2) (2009) 251–271.
- [21] G. Folland, Subelliptic estimates and function spaces on nilpotent Lie groups, *Ark. Math.* 13 (2) (1975) 161–207.
- [22] B. Franchi, C.E. Gutierrez, T. van Nguyen, Homogenisation and convergence of correctors in Carnot groups, *Comm. Partial Differential Equations* 30 (10–12) (2005) 1817–1841.
- [23] B. Franchi, M.C. Tesi, Two-scale homogenisation in the heisenberg group, *J. Math. Pures Appl.* (9) 81 (6) (2002) 495–532.
- [24] O.A. Hafsá, J.-P. Mandallena,  $\Gamma$ -convergence of nonconvex integrals in Cheeger-Sobolev spaces and homogenization, *Adv. Calc. Var.* 10 (4) (2017) 381–405.
- [25] A. Jama, *Generalised Translations Periodicity and Perforated Domains with Applications To Grushin Spaces* (Ph.D. thesis), Cardiff University, 2019, <http://orca.cf.ac.uk/122191/>.
- [26] G. Lu, Weighted Poincaré and Sobolev inequalities for vector fields satisfying Hörmander’s condition and applications, *Revista Mat. Iberoamericana*, 8 (1992) 367–439.
- [27] A. Maione, A. Pinamonti, F. Serra Cassano,  $\Gamma$ -convergence for functionals depending on vector fields I. Integral representation and compactness, [arXiv:1904.06454](https://arxiv.org/abs/1904.06454).
- [28] P. Mannucci, B. Stroffolini, Periodic homogenisation under a hypoellipticity condition, *NoDEA Nonlinear Differential Equations Appl.* 22 (4) (2015) 579–600.
- [29] R. Montgomery, A tour of subriemannian geometries, their geodesics and application, in: *Mathematical Surveys and Monographs*, vol. 91, American Mathematical Society, Providence RI, 2002.
- [30] D. Riciotti,  *$P$ -Laplace Equation in the Heisenberg Group*, Springer, 2015.
- [31] L.P. Rothschild, E.M. Stein, Hypoelliptic differential operators and nilpotent groups, *Acta Math.* 137 (3–4) (1976) 247–320.
- [32] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, Singapore, 1987.



- [33] B. Stroffolini, Homogenisation of Hamilton–Jacobi equations in Carnot groups, *ESAIM Control Optim. Calc. Var.* 13 (1) (2007) 107–119.
- [34] C.-J. Xu, Subelliptic variational problems, *Bull. Soc. Math. France* 118 (2) (1990) 147–169.
- [35] C.-J. Xu, Semilinear subelliptic equations and the Sobolev inequality for vector fields satisfying Hörmander’s condition, *Chinese Ann. Math. Ser. A* 15 (2) (1994) 185–192.