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Rational reductions of the 2D-Toda hierarchy and mirror symmetry

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Abstract. We introduce and study a two-parameter family of symmetry reductions of the two-dimensional Toda lattice hierarchy, which are characterized by a rational factorization of the Lax operator into a product of an upper diagonal and the inverse of a lower diagonal formal difference operator. They subsume and generalize several classical 1+1 integrable hierarchies, such as the bigraded Toda hierarchy, the Ablowitz–Ladik hierarchy and E. Frenkel's q-deformed Gelfand–Dickey hierarchy. We establish their characterization in terms of block Toeplitz matrices for the associated factorization problem, and study their Hamiltonian structure. At the dispersionless level, we show how the Takasaki–Takebe classical limit gives rise to a family of non-conformal Frobenius manifolds with flat identity. We use this to generalize the relation of the Ablowitz–Ladik hierarchy to Gromov–Witten theory by proving an analogous mirror theorem for the general rational reduction: in particular, we show that the dual-type Frobenius manifolds we obtain are isomorphic to the equivariant quantum cohomology of a family of toric Calabi–Yau threefolds obtained from minimal resolutions of the local orbifold line.

Keywords. Rational reductions, Gromov–Witten, integrable hierarchies, mirror symmetry, 2D-Toda, Ablowitz–Ladik

1. Introduction

The two-dimensional Toda equation

$$(\partial_x^2 - \partial_t^2)x_n = \mathrm{e}^{x_{n+1}} - 2\mathrm{e}^{x_n} + \mathrm{e}^{x_{n-1}}, \qquad n \in \mathbb{Z}, \tag{1.1}$$

is among the archetypical examples in classical field theory of integrable non-linear dynamical systems in two space dimensions. Besides its intrinsic interest in the theory of

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integrable systems [18, 50, 66, 68], the hierarchy of commuting flows of (1.1)—the so-called *2D-Toda hierarchy*—has provided a unifying framework for a variety of problems in various branches of mathematics and mathematical physics, ranging from the combinatorics of matrix integrals [3, 36] to enumerative geometry [45, 56] and applications to classical and quantum physics [33, 54, 55].

The purpose of this paper is to construct and study an infinite family of symmetry reductions of the two-dimensional Toda hierarchy, which we dub the *rational reductions of 2D-Toda* (henceforth, RR2T). Their defining feature is the following factorization property of the 2D-Toda Lax operators:

$$L_1^a = AB^{-1}, \qquad L_2^b = BA^{-1}, \tag{1.2}$$

where A and B are respectively a degree $a \geq 1$ upper diagonal and a degree $b \geq 1$ lower diagonal difference operator; this property is preserved by the Toda flows. It turns out that the resulting hierarchies enjoy remarkable properties both from the point of view of the theory of integrable systems, as well as from the vantage of their applications to the topology of moduli spaces of stable maps.

1.1. Main results

The RR2T, which are the natural counterpart in the 2D-Toda world of the "constrained reductions" of the KP hierarchy of [4, 6], are distinguished in a number of ways. First off, the embedding into the Toda hierarchy recovers and ties together a host of known classical integrable hierarchies in 1+1 dimensions: notable examples include the Ablowitz–Ladik system [1, 9], the bi-graded Toda hierarchy [17], and the q-deformed version of the Gelfand–Dickey hierarchy [35]. Moreover, rational reductions have a natural characterization in the associated factorization problem, where they correspond to the block Toeplitz condition on the moment matrix; in the semi-infinite case this naturally generalizes the ordinary Toeplitz condition arising in the theory of unitary matrix models. Thirdly, the analysis of the relation of the Hamiltonian structure on the reduced system to the (second) Poisson structure of the parent 2+1 hierarchy reveals that the reduction itself is remarkable in that it is a *purely kinematical phenomenon*, whose ultimate cause is completely independent of the particular form of the Hamiltonians: the submanifold in field space where the Lax operator factorizes comes along with an infinite-dimensional degeneration of the Poisson tensor, whose pointwise kernel contains the conormal fibers to the factorization locus. Fourthly, the semi-classical Lax–Sato formalism for the dispersionless limit of the hierarchy gives rise to a host of (old and new) solutions of WDVV in the form of a family of semi-simple, non-conformal Frobenius dual-type structures on a genus zero double Hurwitz space[1] having covariantly constant identity. For $b \leq 1$, they are *bona fide* dual in the sense of Dubrovin [29] of conformal Frobenius manifolds of charge $d = 1$, with possibly non-flat unit. The double Hurwitz space picture entails, on the one hand, the existence of a bi-Hamiltonian structure of Dubrovin–Novikov type at the dispersionless level for several subcases, as well as a tri-Hamiltonian structure as in

[1] We borrow the terminology from [60].

[58,60] for $a = b$; on the other, it furnishes for all (a, b) a one-dimensional B-model-type Landau–Ginzburg description for the dual-type Frobenius structure. Generalizing a result of [9], we show that the resulting non-conformal Frobenius manifolds are isomorphic to the $(\mathbb{C}^*)^2$-equivariant orbifold cohomology of the local $\mathbb{P}^1$-orbifolds with two stacky points of order a and b [46], or equivalently [22] of the $(\mathbb{C}^*)^2$-equivariant cohomology of one of their toric minimal resolutions (the *toric trees*). This establishes a (novel) version of equivariant mirror symmetry for these targets via one-dimensional logarithmic Landau–Ginzburg models, which has various applications to the study of wall-crossings in Gromov–Witten theory as anticipated in [10], and it leads us to conjecture that the full descendent Gromov–Witten potential for these targets is a τ-function of the RR2T, a statement that we verify in genus less than or equal to one.

The paper is organized as follows. In Section 2, after reviewing the Lax formalism for the 2D-Toda hierarchy, we first construct the RR2T in the bi-infinite case, study the reduction of the 2D-Toda flows, and discuss various examples. We then illustrate their relation to biorthogonal ensembles on the unit circle and the factorization problem of block Toeplitz matrices, and discuss the Hamiltonian structure of the hierarchy. Section 2.6 is devoted to the study of the dispersionless limit of the flows. We analyze the Takasaki–Takebe limit of the equations in the framework of Frobenius structures on double Hurwitz spaces and determine explicitly the dual-type structures that arise, as well as the extra flat structures that occur in special cases. Finally, Section 3 is devoted to the relation to Gromov–Witten theory. We prove an equivariant mirror theorem for toric trees, and outline the range of its implications. First of all, we verify up to genus one that the full descendent Gromov–Witten potential is a τ-function of the RR2T, upon establishing a Miura equivalence between the dispersive expansion of the RR2T to quadratic order and the analogue of the Dubrovin–Zhang quasi-Miura formalism applied to the local theory of the orbifold line. Moreover, we discuss in detail the properties of the A-model Dubrovin connection in the light of its connection with RR2T, prove that its flat sections are multivariate hypergeometric functions of type F_D, and discuss its implications for the Crepant Resolution Conjecture at higher genus.

1.2. Relation to other work

Several instances of RR2T have made a more or less covert appearance in the literature. In a prescient work [38], Gibbons and Kupershmidt[2] constructed a Lax formalism for a relativistic generalization of the one-dimensional Toda hierarchy which would correspond in our language to the RR2T of bidegree $(a, 1)$, where the dependent variable in the denominator has been frozen to a parameter equal to the speed of light. More recent examples include the Ablowitz–Ladik hierarchy treated by the authors [9], corresponding to the case $(a, b) = (1, 1)$, and the somewhat degenerate example of the lattice analogue of KdV [1], to which RR2T boils down for $b = 0$. Dual-type structures for the dispersionless limit of the RR2T have been computed in the special case of the bigraded Toda hierarchy [59] and the RR2T of bidegree (a, a) (see also [63,72]). Closer to the discus-

[2] Building on earlier work of Bruschi–Ragnisco [11]; see also [48,61].

sion of Section 3 is a recent paper of Takasaki [65], where the (full-dispersive) RR2T of bidegree (b, b) with suitable initial data is considered in connection with the partition function of the melting crystal model [57] for the so-called "generalized conifolds" deformed by shift symmetries [64]. As the generalized conifolds correspond precisely to the toric Calabi–Yau threefolds of Section 3 for $a = b$, it would be intriguing to bridge Takasaki's approach with our own, and in particular to intepret the 2D-Toda evolution in the crystal model as suitable gravitational deformations of our prepotentials. We will leave this open for future work.

2. Rational reductions of 2D-Toda

2.1. The 2D-Toda hierarchy

Denote by $\mathcal{A} = \{(a_{ij} \in \mathbb{C})_{i,j\in\mathbb{Z}}\}$ the vector space of doubly-infinite matrices with complex coefficients. Equivalently, this is the space of formal difference operators $\sum_{r\in\mathbb{Z}} a_r \Lambda^r$ where a_r for every r is an element of the space $\mathscr{F}$ of $\mathbb{C}$-valued functions on $\mathbb{Z}$, and the shift operator Λ acts on $f \in \mathscr{F}$ by $\Lambda^k f(n) = f(n+k)$. For $\Delta = \sum_{r\in\mathbb{Z}} a_r \Lambda^r \in \mathcal{A}$, the $\mathbb{C}$-linear projections

$$\Delta_+ = \sum_{r\in\mathbb{Z}^+} a_r \Lambda^r, \tag{2.1}$$

$$\Delta_- = \sum_{r\in\mathbb{Z}_0^-} a_r \Lambda^r \tag{2.2}$$

define a canonical decomposition $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$, corresponding to the projections of Δ to its upper/strictly lower triangular part. We will denote by Δ^T its transpose

$$\Delta^T = \sum_{r\in\mathbb{Z}} \Lambda^{-r} a_r \tag{2.3}$$

and, whenever defined, we denote its positive/negative order $\mathrm{ord}_\pm \Delta$ as the degree of its projections to $\mathcal{A}_\pm$ as formal difference operators,

$$\mathrm{ord}_\pm \Delta = \deg_{\Lambda^{\pm 1}} (\Delta)_\pm. \tag{2.4}$$

Armed with these definitions, we construct an infinite-dimensional dynamical system over an affine subspace of $\mathcal{A} \oplus \mathcal{A}$, as follows. The 2-*dimensional Toda lattice* [68] is the system $(\partial_{s_r^{(1)}}, \partial_{s_r^{(2)}}, r > 0)$ of commuting flows given by the Lax equations

$$\partial_{s_r^{(1)}} L_i = [(L_1^r)_+, L_i], \qquad \partial_{s_r^{(2)}} L_i = [-(L_2^r)_-, L_i], \qquad i = 1, 2, \tag{2.5}$$

where the *2D-Toda Lax operators* are the formal difference operators

$$L_1 = \Lambda + \sum_{j\geq 0} u_j^{(1)} \Lambda^{-j}, \qquad L_2 = \sum_{j\geq -1} u_j^{(2)} \Lambda^j \tag{2.6}$$

with $u_j^{(k)} \in \mathscr{F}$ for all $j \in \mathbb{N} \cup \{-1\}$, $k = 1, 2$. Commutativity of these flows follows from the (simplified form of) the zero-curvature equations

$$\partial_{s_q^{(j)}} L_i^r - \partial_{s_r^{(i)}} L_j^q + [(L_i^r)_+, (L_j^q)_+] - [(L_i^r)_-, (L_j^q)_-] = 0, \tag{2.7}$$

which in turn is equivalent to a compatibility condition for the Zakharov–Shabat spectral problem

$$\begin{aligned} &L_1\Psi_1 = w\Psi_1, && L_2^T\Psi_2 = w\Psi_2, && \partial_{s_q^{(1)}}\Psi_1 = (L_1^q)_+\Psi_1, \\ &\partial_{s_q^{(1)}}\Psi_2 = -(L_1^q)_+^T\Psi_2^*, && \partial_{s_q^{(2)}}\Psi_1 = (L_2^q)_-^T\Psi_1, && \partial_{s_q^{(2)}}\Psi_2 = -(L_2^q)_-^T\Psi_2^*. \end{aligned} \tag{2.8}$$

for wave vectors $\Psi_i \in \mathbb{C}((w)) \otimes \mathscr{F}$, $i = 1, 2$ [68].

An equivalent formulation of the 2D-Toda hierarchy can be given in terms of the *Sato equations*

$$\partial_{s_r^{(i)}} S_1 = -(L_i^r)_- S_1, \qquad \partial_{s_r^{(i)}} S_2 = -(L_i^r)_- S_2 \tag{2.9}$$

for the *dressing operators*

$$S_1 = 1 + p_1^{(1)}\Lambda^{-1} + \cdots, \qquad S_2 = p_0^{(2)} + p_1^{(2)}\Lambda + \cdots. \tag{2.10}$$

The Lax operators are expressed in terms of the dressing operators by

$$L_1 = S_1 \Lambda S_1^{-1}, \qquad L_2 = S_2 \Lambda^{-1} S_2^{-1}, \tag{2.11}$$

and the commutativity of the flows $\partial_r^{(i)}$ on S_i again follows from (2.7).

Under suitable assumptions, the initial value problem for the 2D-Toda equation can be solved in terms of a factorization problem [62]. Let $\mu \in \mathcal{A}$ be a matrix depending on the times $s_r^{(i)}$ according to

$$\frac{\partial \mu}{\partial s_r^{(1)}} = \Lambda^r \mu, \tag{2.12}$$

$$\frac{\partial \mu}{\partial s_r^{(2)}} = \mu \Lambda^{-r}, \tag{2.13}$$

or equivalently

$$\mu = \exp\Big(\sum_{r\geq 1} s_r^{(1)} \Lambda^r\Big) \mu_0 \exp\Big(\sum_{r\geq 1} s_r^{(2)} \Lambda^{-r}\Big). \tag{2.14}$$

Assume the factorization

$$\mu = S_1^{-1} S_2 \tag{2.15}$$

exists and uniquely determines S_1 and S_2 as in (2.10). Differentiating this expression with respect to $s_r^{(i)}$ and projecting it onto $\mathcal{A}_\pm$ we find that S_1, S_2 satisfy the Sato equations (2.9), hence the associated Lax operators of (2.11) solve (2.5). In the semi-infinite case the factorization problem can be directly solved using biorthogonal polynomials, as we will show in Section 2.4.

2.2. The rational reductions

Consider now the difference operators

$$A = \Lambda^a + \alpha_{a-1}\Lambda^{a-1} + \cdots + \alpha_0 \in \mathcal{A}_+, \tag{2.16}$$

$$B = 1 + \beta_1\Lambda^{-1} + \cdots + \beta_b\Lambda^{-b} \in 1 + \mathcal{A}_- \tag{2.17}$$

for $a, b > 0$. We define two factorization maps $L_i : \mathcal{A}_+ \oplus \mathcal{A}_- \to \mathcal{A}$ by

$$L_1^a = AB^{-1}, \quad L_2^b = BA^{-1}; \tag{2.18}$$

notice that they give Lax operators in the form of (2.6). It is convenient to define also the dual operators $\widehat{L}_1, \widehat{L}_2$ by

$$\widehat{L}_1^a = B^{-1}A, \quad \widehat{L}_2^b = A^{-1}B. \tag{2.19}$$

Theorem 2.1. *For $i = 1, 2$, $r > 0$, the equations*

$$\partial_{s_r^{(i)}} A = (L_i^r)_+ A - A(\widehat{L}_i^r)_+, \tag{2.20}$$

$$\partial_{s_r^{(i)}} B = (L_i^r)_+ B - B(\widehat{L}_i^r)_+ \tag{2.21}$$

define commutative flows on A, B that induce the 2D-Toda Lax equations (2.5).

Proof. We first check that these flows are well-defined. From

$$A^{-1}L_1A = ((A^{-1}L_1A)^a)^{1/a} = (A^{-1}L_1^aA)^{1/a} = (B^{-1}A)^{1/a} = \widehat{L}_1 \tag{2.22}$$

we obtain

$$L_i^r A = A\widehat{L}_i^r, \tag{2.23}$$

and similarly

$$L_i^r B = B\widehat{L}_i^r. \tag{2.24}$$

With the aid of these identities we can rewrite (2.20) and (2.21) as

$$\partial_{s_r^{(i)}} A = -(L_i^r)_- A + A(\widehat{L}_i^r)_-, \tag{2.25}$$

$$\partial_{s_r^{(i)}} B = -(L_i^r)_- B + B(\widehat{L}_i^r)_-. \tag{2.26}$$

The r.h.s. in both (2.20) and (2.25) is a difference operator in $\mathcal{A}_+$ of order $\mathrm{ord}_+ = a - 1$, hence the flow given by (2.20) is well-defined on operators of the form (2.16). Similarly (2.21) gives a well-defined flow on operators of the form (2.17). In general, if $\partial_t A = WA - A\widehat{W}$ and $\partial_t B = WB - B\widehat{W}$ for some difference operators W, $\widehat{W}$, then

$$\partial_t L_i = [W, L_i], \quad \partial_t \widehat{L}_i = [\widehat{W}, \widehat{L}_i]. \tag{2.27}$$

Hence from (2.20), (2.21), (2.25) and (2.26) it follows that the operators L_i satisfy the 2D-Toda Lax equations (2.5). To prove commutativity, observe that if $\partial_{t_i} A = W^i A - A\widehat{W}^i$ for some difference operators W^i, $\widehat{W}^i$, $i = 1, 2$, then

$$\partial_{t_1}\partial_{t_2} A - \partial_{t_2}\partial_{t_1} A = (W^1_{t_2} - W^2_{t_1} + [W^1, W^2])A - A(\widehat{W}^1_{t_2} - \widehat{W}^2_{t_1} + [\widehat{W}^1, \widehat{W}^2]). \quad (2.28)$$

Applying this formula to the flows defined by (2.20) and (2.21) we see that the right-hand side vanishes because of (2.7), hence the flows commute. □

Remark 2.2. Notice that the dual Lax operators also satisfy the Lax equations (2.5) with $\widehat{L}_i$ instead of L_i,

$$\partial_{s_r^{(1)}}\widehat{L}_i = [(\widehat{L}_1^r)_+, \widehat{L}_i], \qquad \partial_{s_r^{(2)}}\widehat{L}_i = [-(\widehat{L}_2^r)_-, \widehat{L}_i], \qquad i = 1, 2. \quad (2.29)$$

Remark 2.3. The inverses of A and B appearing in (2.18) and (2.19) are defined as the following upper (resp. lower) diagonal matrices:

$$A^{-1} = \sum_{k\geq 0}(1 - \alpha_0^{-1}A)^k\alpha_0^{-1}, \qquad B^{-1} = \sum_{k\geq 0}(1 - B)^k. \quad (2.30)$$

The pairs of matrices of the rational form given by (2.18) form a submanifold of the 2D-Toda phase space of pairs of Lax operators (2.6). The previous theorem shows that, on such a submanifold, the 2D-Toda flows coincide with the push-forward under the factorization map (2.18) of the vector fields defined by (2.20) and (2.21) on the space of pairs $\{(A, B) \in \mathcal{A}_+ \oplus \mathcal{A}_-\}$, where A and B are of the form given by (2.16) and (2.21). This clearly implies that the submanifold of rational 2D-Toda Lax operators given by (2.18) is invariant under the 2D-Toda flows.

Definition 2.4. A *rational reduction of the 2D-Toda hierarchy* (RR2T) of bidegree (a, b) is the hierarchy of flows induced by the 2D-Toda flows on the invariant subset of matrices of the form (2.18).

We may more generally consider Lax operators of the form

$$L_1 = (\Lambda^m A B^{-1})^{1/(a+m)}, \qquad L_2 = (BA^{-1}\Lambda^{-m})^{1/(b+m)}. \quad (2.31)$$

The same analysis of Theorem 2.1 carries through to this case as well. Notice that in this case the flows in (2.25) should be defined in terms of the operator $\hat{A} := \Lambda^m A$, rather than A.

Definition 2.5. Let (L_1, L_2) be as in (2.31). The associated reduction of the 2D-Toda lattice hierarchy will be called the *m-generalized RR2T* of bidegree (a, b).

Remark 2.6. We can partially lift the condition that $a, b > 0$ by considering the case when $a = 0$ (resp. $b = 0$) as the degenerate situation in which only one half of the flows given by $\partial_{s_r^{(2)}}$ (resp. $\partial_{s_r^{(1)}}$) is defined by (2.5) and (2.18). All of the above then carries through to this setting.

As it turns out, Theorem 2.1 gives rise to a variety of new reductions of the 2D-Toda hierarchy, incorporating at the same time several known infinite-dimensional lattice integrable systems.

Example 2.7 (The Ablowitz–Ladik hierarchy). The Ablowitz–Ladik system [1] is a discretization of the complexified non-linear Schrödinger equation given by the second order system

$$\mathrm{i}\dot{x}_n = -\tfrac{1}{2}(1 - x_n y_n)(x_{n+1} + x_{n-1}) + x_n, \tag{2.32}$$

$$\mathrm{i}\dot{y}_n = \tfrac{1}{2}(1 - x_n y_n)(y_{n+1} + y_{n-1}) - y_n, \tag{2.33}$$

for $n \in \mathbb{Z}$. This system is Hamiltonian, and it possesses an infinite number of local conserved currents in involution [1]. As shown in [9], after work of Adler–van Moerbeke [3] and Cafasso [15] in the semi-infinite case, its integrability is bequeathed from a rational embedding into the 2D-Toda hierarchy. Explicitly, introduce lattice variables $\alpha, \beta \in \mathscr{F}$ through

$$\alpha_n = -\frac{y_n}{y_{n+1}}, \tag{2.34}$$

$$\beta_n = \frac{(1 - x_n y_n) y_{n-1}}{y_n}. \tag{2.35}$$

Then [9] the Ablowitz–Ladik hierarchy is the pull-back under (2.34) and (2.35) of the rational reduction of the 2D-Toda flows of bidegree $(a, b) = (1, 1)$.

Example 2.8 (The q-deformed Gelfand–Dickey hierarchy). Denote by D_q the scaling (q-difference) operator on the real line, $D_q f(x) = f(xq)$, and write $Q_\pm$ for the projection of a q-difference operator Q onto its q-differential/strictly q-pseudodifferential part. Lax equations in the form

$$\partial_{t_m} \mathfrak{L} = [\mathfrak{L}, (\mathfrak{L}^m)_+] \tag{2.36}$$

for the q-pseudodifference operator

$$\mathfrak{L} \triangleq D_q + \sum_{j \geq 0} u_j(x) D_q^{-j} \tag{2.37}$$

were proposed by E. Frenkel [35] as a q-analogue of the KP hierarchy. In particular, the natural q-analogue of the Gelfand–Dickey (n-KdV) hierarchy,

$$\mathfrak{L}^{n+1} = D_q^{n+1} + \sum_{j \geq 1}^{n} \tau_j(x) D_q^j, \tag{2.38}$$

give rise to a completely integrable bi-Hamiltonian system. By rewriting the q-difference Lax equations (2.36) and (2.38) as ordinary Lax equations for a discrete operator L [2], the system (2.36) can be recast in the form of a reduction of the 2D-Toda flows under the constraint

$$(L^{n+1})_- = 0. \tag{2.39}$$

This corresponds to the RR2T of bidigree $(a, b) = (n + 1, 0)$.

Example 2.9 (The bigraded Toda hierarchy). The bigraded Toda lattice hierarchy of [17] also enjoys a representation as a (generalized) RR2T. By (2.18) and (2.31), the Lax operator for (N, M) bigraded Toda

$$L = \Lambda^N + u_{N-1}\Lambda^{N-1} + \cdots + u_{-M}\Lambda^{-M} \tag{2.40}$$

indeed corresponds to the Lax operator L_1^{N+M} for the $-M$-generalized RR2T of bidegree $(N+M, 0)$. Notice that in this formulation we can only recover as reductions of the 2D-Toda flows the standard flows and not the extended or logarithmic ones.

2.3. Rational reductions and the factorization problem

It is illuminating to consider the form of the constraint leading to the RR2T at the level of dressing operators. By Remark 2.2, the dual Lax operators $\widehat{L}_i$ satisfy the 2D-Toda Lax equations, (2.29). By introducing the corresponding 2D-Toda dressing operators $\widehat{S}_i$ as in (2.9) and (2.10), which satisfy the Sato equations

$$\partial_{s_r^{(i)}}\widehat{S}_1 = -(\widehat{L}_i^r)_-\widehat{S}_1, \qquad \partial_{s_r^{(i)}}\widehat{S}_2 = -(\widehat{L}_i^r)_-\widehat{S}_2, \tag{2.41}$$

the RR2T of bidegree (a, b) can be translated into the pair of constraints

$$S_1\Lambda^a\widehat{S}_1^{-1} = S_2\widehat{S}_2^{-1} \triangleq A, \tag{2.42a}$$

$$S_1\widehat{S}_1^{-1} = S_2\Lambda^{-b}\widehat{S}_2^{-1} \triangleq B. \tag{2.42b}$$

Proposition 2.10. *The constraints given by* (2.42) *are preserved by the Sato equations for S_i, $\hat{S}_i$, hence define a reduction of 2D-Toda at the level of dressing operators that corresponds to the rational reduction of bidegree* (a, b).

Proof. Notice that in this case the operators A, B arise naturally as a combination of the dressing operators of two copies of the 2D-Toda hierarchy. Clearly (2.42) implies that the operators A, B are of the form (2.16), (2.17). The corresponding Lax operators L_i, $\hat{L}_i$, defined through (2.11), factorize as in (2.18), (2.19), i.e.

$$L_1^a = S_1\Lambda^a S_1^{-1} = S_1\Lambda^a\hat{S}_1^{-1}\cdot\hat{S}_1S_1^{-1} = AB^{-1}, \quad \text{etc.} \tag{2.43}$$

and the Sato equations induce the flows (2.20), (2.21). It follows that the constraints (2.42) are preserved by the Sato equations. □

As the simplest non-trivial rational reduction of the 2D-Toda hierarchy gives rise to the Ablowitz–Ladik hierarchy [9], which is in turn related to a factorization problem for a Toeplitz moment matrix, it is natural to ask whether the generic rational reduction may be interpreted in the same way.

Definition 2.11. We say that $\mu \in \mathcal{A}$ is a *block Toeplitz operator* of bidegree (a, b) if

$$\Lambda^a\mu\Lambda^{-b} = \mu. \tag{2.44}$$

Equivalently, its matrix entries satisfy $\mu_{i+a,j+b} = \mu_{ij}$, which reduces to the usual Toeplitz condition when $a = b = 1$. Clearly the property of being block Toeplitz of bidegree (a, b) is preserved by the time evolution as in (2.14).

Let now $(\mu_{ij})_{i,j\in\mathbb{Z}}$ be a block Toeplitz matrix of bidegree (a, b) depending on the times $s_r^{(i)}$ as in (2.14) and such that the factorization problems

$$\mu = S_1^{-1} S_2, \tag{2.45a}$$

$$\mu\Lambda^{-b} = \widehat{S}_1^{-1}\widehat{S}_2 \tag{2.45b}$$

admit solutions for S_i, $\widehat{S}_i$ of the form (2.10). We have the following

Proposition 2.12. *The dressing matrices S_i, $\widehat{S}_i$ satisfy the Sato equations with the constraints in* (2.42)*. The corresponding Lax operators* (2.11) *give a solution of the RR2T of bidegree* (a, b).

Proof. By substituting (2.45a) into (2.45b) we get

$$S_1^{-1} S_2 \Lambda^{-b} = \widehat{S}_1^{-1}\widehat{S}_2. \tag{2.46}$$

Left-multiplication by S_1 and right-multiplication by $\widehat{S}_2^{-1}$ give (2.42b). By the block Toeplitz property, (2.44), we can rewrite (2.45b) as

$$\Lambda^{-a}\mu = \widehat{S}_1^{-1}\widehat{S}_2. \tag{2.47}$$

Performing the same substitution as before and rearranging terms we obtain (2.42a). □

2.4. Semi-infinite block Toeplitz matrices and biorthogonal polynomials on the unit circle

All statements of the previous sections can be transferred almost verbatim to the so-called semi-infinite case, given by the algebra $\mathcal{A}^{\frac{\infty}{2}} = \{(a_{ij} \in \mathbb{C})_{i,j\in\mathbb{Z}_{\geq 0}}\}$ of complex semi-infinite matrices. In this case Λ and Λ^{-1} denote the semi-infinite matrices

$$(\Lambda)_{ij} := \delta_{i+1,j}, \qquad (\Lambda^{-1})_{ij} := (\Lambda^T)_{ij} = \delta_{i,j+1}. \tag{2.48}$$

Here, with an abuse of notation, we denote by Λ^{-1} the transpose of Λ, which is in fact only a right inverse of Λ. We have

$$\Lambda^{-1}\Lambda = 1 - E_{11} \tag{2.49}$$

where $(E_{11})_{ij} = \delta_{i,0}\delta_{j,0}$.

2.4.1. The factorization problem for 2D-Toda and biorthogonal polynomials. In the semi-infinite case and for generic initial data for the 2D-Toda flows, a sufficient condition for the existence of the factorization (2.15) is given by Gauss' elimination: if all the leading principal minors of $\mu \in \mathcal{A}^{\frac{\infty}{2}}$ are non-zero, this leads to the LU decomposition of (2.15). The factorization problem can then be interpreted as the construction of biorthogonal polynomials with respect to the bilinear form $\langle\,,\rangle_\mu$ associated to μ. More

precisely, let $\mu \in \mathcal{A}^{\frac{\infty}{2}}$ and let $\langle\ ,\ \rangle_\mu$ be the $\mathbb{C}$-bilinear form on $\mathbb{C}[z]$ defined by

$$\langle z^i, z^j \rangle_\mu = \mu_{ij}. \tag{2.50}$$

Let $p_j^{(i)}(z)$, $i = 1, 2$, $j \geq 0$, be monic polynomials in $\mathbb{C}[z]$ of degree j. The factorization problem for μ is equivalent to the requirement that $p_j^{(i)}(z)$ form a biorthogonal basis in $\mathbb{C}[z]$ with respect to $\langle\ ,\ \rangle_\mu$, i.e.

$$\langle p_i^{(1)}, p_j^{(2)} \rangle_\mu = \delta_{ij} h_i. \tag{2.51}$$

Indeed, the coefficients of the biorthogonal polynomials are related to the matrices S_1, S_2 by

$$p_i^{(1)}(z) = \sum_{k=0}^{i} (S_1)_{ik} z^k, \tag{2.52a}$$

$$p_i^{(2)}(z) = h_i \sum_{k=0}^{i} (S_2^{-1})_{ki} z^k. \tag{2.52b}$$

The biorthogonality property (2.51) turns into

$$S_1 \mu S_2^{-1} h = h, \tag{2.53}$$

i.e. the factorization (2.15) of the moment matrix. Denote now by $p^{(i)}$ (resp. $\widehat{p}^{(i)}$) the semi-infinite vector having $p_j^{(i)}$ (resp. $\widehat{p}_j^{(i)}$) as its j^{th} entry. By (2.11) and (2.15), the Lax operators L_i act on biorthogonal polynomials as

$$L_1 p^{(1)}(z) = z p^{(1)}(z), \tag{2.54}$$

$$h L_2^T h^{-1} p^{(2)}(z) = z p^{(2)}(z). \tag{2.55}$$

2.4.2. Semi-infinite block Toeplitz matrices. Let us now turn to the study of the (a, b) RR2T in the semi-infinite case, or equivalently to the factorization problem for semi-infinite block Toeplitz matrices. We start by defining two sets of biorthogonal polynomials associated with the $\mathbb{C}$-bilinear forms

$$\langle z^i, z^j \rangle_\mu = \mu_{ij}, \tag{2.56}$$

$$\langle z^i, z^j \rangle_{\widehat{\mu}} = \widehat{\mu}_{ij} = \mu_{i,j+b}, \tag{2.57}$$

where $\widehat{\mu} = \mu \Lambda^{-b}$. Both μ and $\widehat{\mu}$ satisfy the Toeplitz property, which translates, at the level of bilinear forms, into

$$\langle z^a f(z), z^b g(b) \rangle_\bullet = \langle f(z), g(z) \rangle_\bullet \tag{2.58}$$

for any $f, g \in \mathbb{C}[z]$. The monic polynomials $p_j^{(i)}$ and $\widehat{p}_j^{(i)}$ satisfy the biorthogonality conditions

$$\langle p_i^{(1)}, p_j^{(2)} \rangle_\mu = \delta_{ij} h_i, \tag{2.59a}$$

$$\langle \widehat{p}_i^{(1)}, \widehat{p}_j^{(2)} \rangle_{\widehat{\mu}} = \delta_{ij} \widehat{h}_i. \tag{2.59b}$$

The corresponding dressing matrices S_i, $\widehat{S}_i$ are defined through (2.52); such matrices solve the factorization problems (2.45a) and (2.45b). If we assume that the moment matrix μ depends on the times $s_r^{(i)}$ as in (2.14), then, according to Proposition 2.12, $(S_i, \widehat{S}_i)$ give a solution of the (a, b)-graded RR2T.

Proposition 2.13. *The biorthogonal polynomials $p_j^{(i)}$ and the dual biorthogonal polynomials $\widehat{p}_j^{(i)}$ are related by the identities*

$$A\widehat{p}^{(1)} = z^a p^{(1)}, \tag{2.60a}$$
$$\widehat{h}A^T h^{-1} p^{(2)} = \widehat{p}^{(2)}, \tag{2.60b}$$
$$B\widehat{p}^{(1)} = p^{(1)}, \tag{2.60c}$$
$$\widehat{h}B^T h^{-1} p^{(2)} = z^b \widehat{p}^{(2)}. \tag{2.60d}$$

Proof. Let us prove the first relation. Applying A to (2.52a) we get

$$(A\widehat{p}^{(1)})_i = \sum_{k\geq 0} (A\widehat{S}_1)_{ik} z^k, \tag{2.61}$$

where we have used the fact that the sum in (2.52a) can be extended to ∞ due to the triangular structure of S_1. The first part of (2.42a) gives

$$A\widehat{S}_1 = S_1 \Lambda^a, \tag{2.62}$$

which substituted above gives (2.60a). The remaining relations are proved in a similar way. □

As a straightforward consequence, we obtain recursion relations for the biorthogonal polynomials $p_j^{(2)}$ and $\widehat{p}_j^{(1)}$.

Corollary 2.14. *The biorthogonal polynomials $p_j^{(2)}$, $\widehat{p}_j^{(1)}$ satisfy the relations*

$$A\widehat{p}^{(1)} = z^a B\widehat{p}^{(1)}, \tag{2.63a}$$
$$B^T h^{-1} p^{(2)} = z^b A^T h^{-1} p^{(2)}. \tag{2.63b}$$

Remark 2.15. For $a = b = 1$ we get from (2.63a)

$$\widehat{p}_{i+1}^{(1)} + \alpha_0(i)\widehat{p}_i^{(1)} = z(\widehat{p}_i^{(1)} + \beta_1(i)\widehat{p}_{i-1}^{(1)}), \tag{2.64}$$

and from (2.63b)

$$p_i^{(2)} h_i^{-1} + p_{i+1}^{(2)} h_{i+1}^{-1} \beta_1(i+1) = z(p_{i-1}^{(2)} h_{i-1}^{-1} + p_i^{(2)} h_i^{-1} \alpha_0(i)) \tag{2.65}$$

for $i \geq 0$, and assuming $p_j^{(i)} = \widehat{p}_j^{(i)} = 0$ when $j < 0$. Notice that in the general (a, b) case the recursions in (2.63) involve $a + b + 2$ terms.

Remark 2.16. For the Ablowitz–Ladik lattice, $(a, b) = (1, 1)$, the moment matrix can be seen to arise from the scalar product on functions on the unit circle,

$$\langle f, g\rangle_\mu = \frac{1}{2\pi \mathrm{i}} \int_{S^1} f(z) g(z^{-1}) \mathrm{e}^{\sum_{i>0}(s_i^{(1)} z^i - s_i^{(2)} z^{-1})} \frac{\mathrm{d}z}{z}. \tag{2.66}$$

Correspondingly, the associated 2D-Toda τ-function is the partition of the unitary matrix model,

$$Z_{U(N)} = \prod_{i=0}^{n-1} h_n, \tag{2.67}$$

and the recursion relations of (2.64) and (2.65) imply the three-term recursion relations of [43, 48] for the unitary ensemble. The general (a, b) case corresponds to complex integrals of the form

$$\langle f, g\rangle_\mu = \frac{1}{2\pi \mathrm{i}} \int_{S^1} f(z^b) g(z^{-a}) \mathrm{e}^{\sum_{i>0}(s_i^{(1)} z^i - s_i^{(2)} z^{-1})} \frac{\mathrm{d}z}{z}. \tag{2.68}$$

Notice that the bilinear form on $\mathbb{C}[z]$ thus defined is not symmetric anymore as soon as $a \neq b$, and the unitary matrix model interpretation is correspondingly less obvious.

2.5. Hamiltonian structure

Since the 2D-Toda hierarchy admits a triplet of compatible Poisson structures [16], a natural question arises whether the RR2T flows admit a Hamiltonian formulation. Unlike the case of the extended bigraded Toda hierarchy, the generic RR2T is not given by an affine submanifold in field space, and correspondingly the Dirac reduction of the parent Poisson structures is not straightforward. Remarkably, however, at least one Poisson structure can always be reduced to the locus defined by the factorization of the Lax operator as in (2.18). The key to this is a degeneration property of the corresponding Poisson tensor, as we now illustrate.

It is well-known that the 2D-Toda hierarchy can be formulated in terms of two Lax operators of the form

$$\bar{L}_1 = \Lambda^a + \sum_{j \geq -a+1} \bar{u}_j^{(1)} \Lambda^{-j}, \qquad \bar{L}_2 = \sum_{j \geq -b} \bar{u}_j^{(2)} \Lambda^j, \tag{2.69}$$

for two fixed integers $a, b \geq 1$. They are related to the Lax operators defined in (2.6) by $\bar{L}_1 = L_1^a$ and $\bar{L}_2 = L_2^b$. In the rest of this subsection we will always use the formulation in terms of the Lax operators (2.69), and to keep the notation simple, we will drop the bars and denote them by L_1 and L_2.

Denote by $(\dot{L}_1, \dot{L}_2)$ an element in the tangent space $T\mathcal{A}^{\mathrm{2DT}} = \{(\dot{L}_1 = \sum_{j \leq a-1} \dot{u}_j^{(1)} \Lambda^j, \dot{L}_2 = \sum_{j \geq -b} \dot{u}_j^{(2)} \Lambda^j)\}$ of the 2D-Toda phase space and introduce the bilinear pairing

$$\langle (\dot{L}_1, \dot{L}_2), (X, Y)\rangle = \mathrm{Tr}(\dot{L}_1 X + \dot{L}_2 Y) \tag{2.70}$$

to induce differential forms in $T^*\mathcal{A}^{\mathrm{2DT}}$ from operators (X, Y) of the form $X=\sum_{k>n} x_k\Lambda^k$ and $Y = \sum_{k<m} y_k\Lambda^k$ for some $n, m \in \mathbb{Z}$. Similarly, we denote by $(\dot{A}, \dot{B})$ an element of the tangent space $T\mathcal{A}^{\mathrm{RR}} = \{(\dot{A} = \dot{\alpha}_{a-1}\Lambda^{a-1} + \cdots + \dot{\alpha}_0, \dot{B} = \dot{\beta}_1\Lambda^{-1} + \cdots + \dot{\beta}_b\Lambda^{-b})\}$ to the phase space $\mathcal{A}^{\mathrm{RR}}$ of rational reductions. The same bilinear pairing described above produces a differential form on $\mathcal{A}^{\mathrm{RR}}$ starting this time from an operator (X, Y) of the more general form $X = \sum_{k\in\mathbb{Z}} x_k\Lambda^k$ and $Y = \sum_{k\in\mathbb{Z}} y_k\Lambda^k$.

It was shown in [16] that, for $a = b = 1$, $\mathcal{A}^{\mathrm{2DT}}$ can be endowed with three compatible Poisson structures with respect to which the 2D-Toda flows are Hamiltonian. The construction of [16] can be easily extended to the general $a, b \geq 1$ case. In particular, what was referred to in [16] as the "second" Poisson tensor reads as follows. When applied to a differential form corresponding via the pairing to the operator (X_1, X_2), it gives the vector

$$\begin{aligned} P(\langle\cdot, (X_1, X_2)\rangle) = \big(&\tfrac{1}{2}[L_1, (L_1X_1 + X_1L_1)_- - (L_2X_2 + X_2L_2)_-] \\ &+ \tfrac{1}{2}[L_1, (\Lambda^a + 1)(\Lambda^a - 1)^{-1}\,\mathrm{Res}([L_1, X_1] + [L_2, X_2])] \\ &- \tfrac{1}{2}L_1([L_1, X_1] + [L_2, X_2])_{\leq 0} - \tfrac{1}{2}([L_1, X_1] + [L_2, X_2])_{\leq 0}L_1, \\ &\tfrac{1}{2}[L_2, (L_2X_2 + X_2L_2)_+ - (L_1X_1 + X_1L_1)_+] \\ &+ \tfrac{1}{2}[L_2, (\Lambda^a + 1)(\Lambda^a - 1)^{-1}\,\mathrm{Res}([L_1, X_1] + [L_2, X_2])] \\ &- \tfrac{1}{2}L_2([L_1, X_1] + [L_2, X_2])_{>0} - \tfrac{1}{2}([L_1, X_1] + [L_2, X_2])_{>0}L_2\big). \end{aligned} \tag{2.71}$$

This Poisson structure degenerates on the submanifold of $\mathcal{A}^{\mathrm{2DT}}$ given by the image of $\mathcal{A}^{\mathrm{RR}}$, as shown by the following lemma, hence it yields, simply by restriction, a well-defined Poisson structure on that submanifold.

Lemma 2.17. *For $L_1 = AB^{-1}$, $L_2 = BA^{-1}$, we have*

$$P(\langle\cdot, (X_1, X_2)\rangle) = \big((\dot{A} - AB^{-1}\dot{B})B^{-1}, (\dot{B} - BA^{-1}\dot{A})A^{-1}\big)$$

where $(\dot{A}, \dot{B}) \in T\mathcal{A}^{\mathrm{RR}}$ is given by

$$\begin{aligned} \dot{A} &= \big((X_2BA^{-1} - AB^{-1}X_1)_- + (\Lambda^{-a} - 1)^{-1}\zeta\big)A \\ &\quad - A\big((A^{-1}X_2B - B^{-1}X_1A)_- + (1 - \Lambda^a)^{-1}\zeta\big), \\ \dot{B} &= ((BA^{-1}X_2 - X_1AB^{-1})_- + (1 - \Lambda^a)^{-1}\zeta)B \\ &\quad - B\big((A^{-1}X_2B - B^{-1}X_1A)_- + (1 - \Lambda^a)^{-1}\zeta\big), \end{aligned}$$

and

$$\zeta = \mathrm{Res}([L_1, X_1] + [L_2, X_2]).$$

In other words, the vector given by the image by the Poisson tensor of the differential form $\langle\cdot, (X_1, X_2)\rangle$ is the push-forward of a vector in $T\mathcal{A}^{\mathrm{RR}}$, i.e., it is tangent to $\mathcal{A}^{\mathrm{RR}}$.

For any functional $f = f(L_1, L_2)$ on $\mathcal{A}^{\mathrm{2DT}}$ we denote by $\left(\frac{\delta f}{\delta L_1}, \frac{\delta f}{\delta L_2}\right)$ a pair of operators such that we can express the derivative of f along $(\dot{L}_1, \dot{L}_2)$ as

$$\dot{f} = \left\langle \left(\frac{\delta f}{\delta L_1}, \frac{\delta f}{\delta L_2}\right), (\dot{L}_1, \dot{L}_2)\right\rangle. \tag{2.72}$$

In other words, the vector $\left(\frac{\delta f}{\delta L_1}, \frac{\delta f}{\delta L_2}\right)$ is a preimage of the differential $\mathrm{d}f$ with respect to the bilinear pairing above. The Poisson bracket of two functionals f, g on $\mathcal{A}^{\mathrm{2DT}}$ is

$$\{f, g\} = \left\langle \left(\frac{\delta f}{\delta L_1}, \frac{\delta f}{\delta L_2}\right), P\left(\left\langle \cdot, \left(\frac{\delta g}{\delta L_1}, \frac{\delta g}{\delta L_2}\right)\right\rangle\right)\right\rangle. \tag{2.73}$$

From the lemma and skew-symmetry, it follows that $\{f, g\}$, when restricted on $\mathcal{A}^{\mathrm{RR}}$, does not depend on the choice of the functional f (resp. g) on $\mathcal{A}^{\mathrm{2DT}}$ as long as it restricts to the same $f_{|\mathcal{A}^{\mathrm{RR}}}$ (resp. $g_{|\mathcal{A}^{\mathrm{RR}}}$). In other words, $\mathcal{A}^{\mathrm{RR}}$ is a Poisson submanifold of $\mathcal{A}^{\mathrm{2DT}}$.

The explicit form of RR2T Poisson brackets for the coefficients $\alpha_0, \dots, \alpha_{a-1}$, $\beta_1, \dots, \beta_b$ of A and B can be computed starting from the 2D-Toda (second) Poisson bracket for the first a and b coefficients $u_0^{(1)}, \dots, u_{a-1}^{(1)}, u_{-1}^{(2)}, \dots, u_b^{(2)}$ of L_1 and L_2 respectively[3] and applying the change of coordinates induced by the equations $L_1^a = AB^{-1}$, $L_2^b = BA^{-1}$.

In case $(a, b) = (1, 1)$, where $A = \Lambda + \alpha$ and $B = 1 + \beta\Lambda^{-1}$, one readily computes

$$\begin{aligned} \{\alpha(n), \alpha(m)\} &= 0, \\ \{\log\alpha(n), \log\beta(m)\} &= \delta(n-m+1) - \delta(n-m), \\ \{\log\beta(n), \log\beta(m)\} &= \delta(n-m+1) - \delta(n-m-1), \end{aligned} \tag{2.74}$$

which coincides with the Poisson structure introduced by Adler–van Moerbeke [3] for the Ablowitz–Ladik hierarchy.

Since the 2D-Toda flows are Hamiltonian with respect to (2.71), with Hamiltonian functions given by

$$H_i^{(j)} = -\frac{1}{i}\operatorname{Tr} L_j^i, \qquad j = 1, 2, \tag{2.75}$$

the Ablowitz–Ladik flows are Hamiltonian with respect to (2.74), with the same Hamiltonian functions.

2.6. *Long-wave limit and semi-classical Lax formalism*

Starting from the 2D-Toda lattice hierarchy of Section 2.1, a continuous integrable system of $2 + 1$ evolutionary PDEs can be constructed by interpolation. For a fixed real parameter $\epsilon > 0$—the "lattice spacing"—introduce dependent variables $U_j^{(i)}(x)$ such that $U_j^{(i)}(\epsilon n) = (u_j^{(i)})_n$, and accordingly define a shift operator $\Lambda_\epsilon = \mathrm{e}^{\epsilon\partial_x}$ by one unit of lattice spacing. Replacing the unit shift Λ_1 by the ϵ-shift $\mathrm{e}^{\epsilon\partial_x}$ and rescaling the time variables by $t_r^{(i)} \triangleq \epsilon s_r^{(i)}$ gives a system of evolutionary partial differential equations in the time variables $t_r^{(i)}$ in the form

$$\begin{aligned} \partial_{t_r^{(p)}} U_j^{(i)}(x) &= \sum_{g\geq 0} \epsilon^{2g} \mathcal{P}_{i,j}^{[g],p,r}(U, U_x, \dots, U^{(2g)}) \\ &= \sum_{k,l} \mathcal{P}_{k,l,i,j}^{[0],p,r}(U)\partial_x U_k^{(l)} + O(\epsilon^2), \end{aligned} \tag{2.76}$$

[3] See [16] for explicit formulas.

where $\mathcal{P}_{i,j}^{[g],p,r}(U, U_x, \dots, U^{(2g)})$ is an element of the vector space $\mathcal{I}_g$ of differential polynomials in $U(x)$ homogeneous of degree $2g+1$ with respect to the independent variable x. Following [19], we will call this the *interpolated 2D-Toda lattice*.

We will be particularly interested in the quasi-linear limit of the interpolated 2D-Toda lattice, where the dispersion parameter ϵ is set to zero. As noticed in [66], the dispersionless limit $\epsilon \to 0$ of (2.76) can be formulated as the quasi-classical (Ehrenfest) limit of the Lax equations (2.5), as follows. Write $\lambda_i(z) \triangleq \sigma_\Lambda(L_i) \in \mathbb{C}((z))$ for the total symbol in the variable $z \in \mathbb{C}$ of the difference operators L_i in (2.6),

$$\lambda_1(z) = z + \sum_{j\geq 0} U_j^{(1)} z^{-j}, \qquad \lambda_2(z) = \sum_{j\geq -1} U_j^{(2)} z^j. \tag{2.77}$$

Furthermore, define the *Orlov functions*

$$\mathcal{B}_n^{(1)}(z) \triangleq [(\lambda_1)^n]_+, \qquad \mathcal{B}_n^{(2)}(z) \triangleq [(\lambda_2)^n]_-, \tag{2.78}$$

where $[f]_\pm$ denotes the projection to the analytic/purely principal part of $f \in \mathbb{C}((z))$, and for $f, g \in \mathbb{C}((x, z))$ define the Poisson bracket

$$\{f, g\}_{\text{Lax}} = z\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial z} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial z}\right). \tag{2.79}$$

Then the *semi-classical Lax equations*

$$\frac{\partial \lambda_i}{\partial t_r^{(j)}} \triangleq \{\mathcal{B}_r^{(j)}, \lambda_i\}_{\text{Lax}}, \tag{2.80}$$

where the time-derivatives are understood to be taken at fixed z, induce the dispersionless limit of the interpolated 2D-Toda flows of (2.76) on the coefficients $U_k^{(l)}$ of λ_l,

$$\partial_{t_r^{(p)}} U_j^{(i)}(x) = \sum_{k,l} \mathcal{P}_{k,l,i,j}^{[0],p,r}(U)\partial_x U_k^{(l)}. \tag{2.81}$$

Consistency of the dispersionless Lax equations (2.80) requires the existence of a potential function $\mathcal{F}$ of the long-wave time variables $t_r^{(j)}$ such that

$$\mathcal{B}_n^{(i)}(z(\lambda_j)) = \delta_{ij}\lambda_j^{s_j n} + \delta_{j2}\frac{\partial^2 \mathcal{F}}{\partial t_0^{(1)} \partial t_n^{(i)}} - \sum_{m>0} \frac{\partial^2 \mathcal{F}}{\partial t_n^{(i)} \partial t_m^{(j)}} \frac{1}{m\lambda_j^{s_i m}}, \tag{2.82}$$

where $s_i = (-1)^{i+1}$. By the general dToda theory [66], the potential $\mathcal{F}$ yields the eikonal limit of the logarithm of the long-wave limit of the 2D-Toda τ-function,

$$\mathcal{F} = \log \tau_{\text{dToda}}. \tag{2.83}$$

2.7. Rational reductions and Frobenius manifolds

The integration of the consistency conditions for $\mathcal{F}$ has a natural formulation in the language of Frobenius manifolds [18]. An even more poignant picture emerges in the case of RR2T: by [27, 60] the dispersionless limit (henceforth denoted as dRR2T) coincides

with the Principal Hierarchy of the Frobenius manifold defined on a genus zero double Hurwitz space, as we now show.

2.7.1. Flat structures and the Principal Hierarchy. We start by giving the following

Definition 2.18. Let M be a complex manifold, $\dim_{\mathbb{C}} M = n$. A holomorphic *Frobenius structure* $\mathcal{M} = (M, \eta, \cdot)$ on M is the datum of a holomorphic symmetric $(0, 2)$-tensor η, which is non-degenerate and with flat Levi-Civita connection ∇, and a commutative, associative fiberwise product law $X \cdot Y$ with unit on vector fields $X, Y \in \mathcal{X}(M)$, which is tensorial and satisfies

Compatibility:

$$\eta(X \cdot Y, Z) = \eta(X, Y \cdot Z) \quad \text{for all vector fields } X, Y, Z; \tag{2.84}$$

Flatness: the pencil of affine connections

$$\nabla^{(\zeta)}_X Y \triangleq \nabla_X Y + \zeta X \cdot Y, \quad \zeta \in \mathbb{C}, \tag{2.85}$$

is identically flat for all $\zeta \in \mathbb{C}$.

Following the terminology introduced in [60], extra flat structures on $\mathcal{M}$ will be characterized according to the following

Definition 2.19. Let $\mathcal{M} = (M, \eta, \cdot)$ be a holomorphic Frobenius manifold structure on M, and let $e \in \mathcal{X}(M)$ be the unit of the $\cdot$-product. We will say that $\mathcal{M}$ is

- *semi-simple* if the product structure $\cdot|_p$ on T_pM has no nilpotent elements for a generic $p \in M$;
- of *dual type* if there exists $d \in \mathbb{Z}$ such that for all $f \in \mathcal{O}_M$,

$$\nabla \mathrm{d} f = 0 \;\Rightarrow\; \left(\partial_e + \frac{d-1}{2}\right) f = c_f \tag{2.86}$$

for some constant $c_f \in \mathbb{C}$;
- *conformal* if $\nabla e = 0$ and there exists $E \in \mathcal{X}(M)$ such that $\nabla E \in \Gamma(\mathrm{End}(TM))$ is diagonalizable and horizontal with respect to ∇, and the pencil of affine connections (2.85) extends to a flat meromorphic connection $\nabla^{(\zeta)}$ on $M \times \mathbb{P}^1_\zeta$ via

$$\nabla^{(\zeta)} \frac{\partial}{\partial \zeta} = 0, \tag{2.87}$$

$$\nabla^{(\zeta)}_{\partial/\partial\zeta} X = \frac{\partial}{\partial \zeta} X + E \cdot X - \frac{1}{\zeta} \widehat{\mu} X, \tag{2.88}$$

where $\widehat{\mu}$ is the traceless part of $-\nabla E$;
- *tri-Hamiltonian* if it is conformal, n is even and $\widehat{\mu}$ has only two eigenvalues $\pm d/2$ with multiplicity $n/2$, where $d = 2(1 - \mathrm{Tr}(\nabla E))$.

A Frobenius manifold structure $\mathcal{M}$ on M embodies the existence of a Hamiltonian hierarchy of quasi-linear commuting flows on its loop space [27]. Let $\mathrm{t} = \{\tau^\alpha_{(\zeta)} \in \mathcal{O}_M\}_{\alpha=1}^n$ be the datum of a marked system of flat coordinates for $\nabla^{(\zeta)}$ depending holomorphically on ζ around $\zeta = 0$; this is determined up to a $\mathbb{C}[[\zeta]]$-valued affine transformation in general, a freedom which reduces to a complex affine transformation when $\mathcal{M}$ is conformal by virtue of (2.88). Write $h_{\alpha,p} \triangleq \eta_{\alpha\beta}([\zeta^p]\tau^\alpha_{(\zeta)}) \in \mathcal{O}_M$ for the p^{th} Taylor coefficient of $\eta_{\alpha,\beta}\tau^\alpha_{(\zeta)}$ at $\zeta = 0$. In terms of the flat metric η, we define [27] a hydrodynamic Poisson structure $\{\,,\,\}_\eta$ on the loop space $\mathcal{L}_\mathcal{M} = \mathrm{Maps}(S^1, M)$ as

$$\{\tau^\alpha_{(0)}(X), \tau^\beta_{(0)}(Y)\}_\eta = \eta^{\alpha\beta}\delta'(X - Y), \tag{2.89}$$

where $X, Y \in S^1$ are coordinates on the base of the loop space, as well as an infinite set of quasi-linear Hamiltonian flows via

$$\frac{\partial \tau^\beta}{\partial t^{\alpha,p}} \triangleq \{\tau^\beta, H_{\alpha,p}\}_\eta = \partial_X \partial^\beta h_{\alpha,p}. \tag{2.90}$$

These flows generate a commuting family of Hamiltonian conservation laws [27], which is complete as long as $\mathcal{M}$ is semi-simple [67].

Definition 2.20. The hierarchy of hydrodynamic type equations (2.90) will be called the *Principal Hierarchy* associated to $(\mathcal{M}, \mathrm{t})$.

2.7.2. Frobenius dual-type structures for the RR2T. Let $a, b \in \mathbb{Z}_+^2$ and $m \in \mathbb{Z}$. In this section we will construct a Frobenius dual-type structure [60] on the space of symbols of the Lax operator $L_1^{a+m} = L_2^{-b-m}$ of the generalized RR2T of Definition 2.5.

Definition 2.21. Let $v, q_{-a+1}, \ldots, q_{b-1} \in \mathbb{C}$, $a, b \in \mathbb{Z}^+$ and $\nu \in \mathbb{C}^*$. We define $\mathcal{H}_{a,b,\nu}$ to be the space of multivalued functions on $\mathbb{P}^1$ of the form

$$\lambda(z) = \mathrm{e}^v z^{\nu+b} \frac{\prod_{k=0}^{a-1}(z - \mathrm{e}^{q_{-k}})}{\prod_{l=0}^{b-1}(z - \mathrm{e}^{-q_l})}. \tag{2.91}$$

Remark 2.22. Write

$$z_k = \begin{cases} 0 & \text{for } k = 1, \\ \mathrm{e}^{q_{2-k}} & \text{for } k = 2, \ldots, a+1, \\ \mathrm{e}^{-q_{k+2-a}} & \text{for } k = a+2, \ldots, a+b+1, \\ \infty & \text{for } k = a+b+2. \end{cases} \tag{2.92}$$

Then the meromorphic function $z^{-\nu}\lambda(z)$ has, for generic values of the parameters, a zero of order b at z_1, simple zeroes at z_{k+2}, $k = 0, \ldots, a-1$, a pole of order a at $z_{a+b+2} \triangleq \infty$, and simple poles at $z_{a+2+k} \triangleq e^{-q_k}$, $k = 0, \ldots, b-1$. When $\nu = m \in \mathbb{Z}_0$, this function is the total symbol of the $(a+m)^{\text{th}}$ power of the Lax operator L_1 (or equivalently

the $(b+m)^{\text{th}}$ inverse power of L_2) of the m-generalized RR2T of bidegree (a,b), up to a trivial rescaling of the argument z. The space $\mathcal{H}_{a,b,m}$ is then a *genus zero double Hurwitz space*, a moduli space of rational curves with a marked meromorphic function $\lambda : \mathbb{P}^1 \to \mathbb{P}^1$ having specified ramification profile $\kappa \in \mathbb{Z}^{a+b+2}$ at zero and infinity. In our case, the latter reads

$$\kappa = (b+m, \underbrace{1, \dots, 1}_{a}, \underbrace{-1, \dots, -1}_{b}, -a-m). \tag{2.93}$$

For $\nu = m \in \mathbb{Z}$, we define on the $(a+b)$-dimensional complex manifold $\mathcal{H}_{a,b,\nu}$ a triplet $(\eta^{(1)}, \eta^{(2)}, \eta^{(3)})$, where $\eta^{(i)} \in \Gamma(\mathrm{Sym}^2 T^*\mathcal{H}_{a,b,m})$, $\det \eta^{(i)} \neq 0$, by the Landau–Ginzburg formulas

$$\eta^{(1)}(X,Y) = \sum_{i=1}^{a+b+2} \mathrm{Res}_{z_i} \frac{X(\lambda)Y(\lambda)}{\mathrm{d}\lambda} \left(\frac{\mathrm{d}z}{z}\right)^2, \tag{2.94}$$

$$\eta^{(2)}(X,Y) = \sum_{i=1}^{a+b+2} \mathrm{Res}_{z_i} \frac{X(\log\lambda)Y(\log\lambda)}{\mathrm{d}\log\lambda} \left(\frac{\mathrm{d}z}{z}\right)^2, \tag{2.95}$$

$$\eta^{(3)}(X,Y) = \sum_{i=1}^{a+b+2} \mathrm{Res}_{z_i} \frac{X(\lambda^{-1})Y(\lambda^{-1})}{\mathrm{d}\lambda^{-1}} \left(\frac{\mathrm{d}z}{z}\right)^2, \tag{2.96}$$

for $X, Y \in \mathcal{X}(\mathcal{H}_{a,b,\nu})$. We further equip $T_\lambda \mathcal{H}_{a,b,m}$ with a triplet $(\bullet, \star, *)$ of commutative, associative products defined by

$$\eta^{(1)}(X \bullet Y, Z) = \sum_{i=1}^{a+b+2} \mathrm{Res}_{z_i} \frac{X(\lambda)Y(\lambda)Z(\lambda)}{\mathrm{d}\lambda} \left(\frac{\mathrm{d}z}{z}\right)^2, \tag{2.97}$$

$$\eta^{(2)}(X \star Y, Z) = \sum_{i=1}^{a+b+2} \mathrm{Res}_{z_i} \frac{X(\log\lambda)Y(\log\lambda)Z(\log\lambda)}{\mathrm{d}\log\lambda} \left(\frac{\mathrm{d}z}{z}\right)^2, \tag{2.98}$$

$$\eta^{(3)}(X * Y, Z) = \sum_{i=1}^{a+b+2} \mathrm{Res}_{z_i} \frac{X(\lambda^{-1})Y(\lambda^{-1})Z(\lambda^{-1})}{\mathrm{d}\lambda^{-1}} \left(\frac{\mathrm{d}z}{z}\right)^2, \tag{2.99}$$

depending holomorphically on the base-point $\lambda \in \mathcal{H}_{a,b,m}$. When $\nu \notin \mathbb{Z}$, equations (2.94), (2.96), (2.97) and (2.99) are ill-defined, but the definitions (2.95) and (2.98) of the metric and product $(\eta^{(2)}, \star)$ carry through unscathed. The main result of this section is the following

Theorem 2.23. *Let $a, b \in \mathbb{Z}^+$, $\nu \in \mathbb{C}$. Then*

(i) *Equations (2.95) and (2.98) define on $\mathcal{H}_{a,b,\nu}$ a semi-simple Frobenius structure $\mathcal{M}^{(2)}_{a,b,\nu} = (\mathcal{H}_{a,b,\nu}, \eta^{(2)}, \star)$ of dual type of charge one.*

(ii) *Let $\nu = m \in \mathbb{Z}$ and suppose that both $b+m, -a-m$ are either equal to one or negative. Then (2.94) and (2.97) define a conformal Frobenius structure $\mathcal{M}^{(1)}_{a,b,m} =$*

$(\mathcal{H}_{a,b,m}, \eta^{(1)}, \bullet)$ of charge one on $\mathcal{H}_{a,b,m}$. The unit of this structure is flat iff $m \neq 1-b$ and $m \neq -a-1$, and we have

$$\mathcal{M}^{(2)}_{a,b,m} = \mathcal{D}(\mathcal{M}^{(1)}_{a,b,m}), \tag{2.100}$$

where $\mathcal{D}$ is Dubrovin's duality morphism of Frobenius structures [29].

(iii) *Let $b = a$ and $\nu = 1 - a$. Then* (2.94)–(2.99) *define a tri-Hamiltonian Frobenius structure on $\mathcal{H}_{a,a,1-a}$.*

Proof. Theorem 2.23 is essentially a verbatim translation of [60, Theorem 2] to the setting of RR2T. We sketch the main points of the proof below. For (i), flatness of the residue pairing $\eta^{(2)}$ follows from checking, through a direct computation of (2.95), that the coordinates $v, q_{-a+1}, \dots, q_{b-1}$ form in fact a flat coordinate frame for $\eta^{(2)}$. Further, by the explicit form of (2.95) and (2.98), the $\star$-product structure is clearly compatible with the metric $\eta^{(2)}$ in the sense that the two form a Frobenius algebra on the tangent spaces of $\mathcal{H}_{a,b,\nu}$; it is immediate to check that the algebra is unital, the identity being the flat vector field $e = \partial_v$. Moreover the $a + b$ critical values of $\log \lambda$,

$$u_i \triangleq \log \lambda(y_i), \qquad y_i \in \mathbb{P}^1 \text{ with } \lambda'(y_i) = 0, i = 1, \dots, a+b, \tag{2.101}$$

are a set of local coordinates on $\mathcal{H}_{a,b,\nu} \setminus \Delta_{a,b,\nu}$, where the discriminant $\Delta_{a,b,\nu}$ is $\{\lambda \in \mathcal{H}_{a,b,\nu} \mid u_i \neq u_j \ \forall i \neq j\}$. In these coordinates, the product and the metric take the form

$$\begin{aligned} \partial_{u_i} \star \partial_{u_j} &= \delta_{ij} \partial_{u_i}, \\ \eta^{(2)}(\partial_{u_i}, \partial_{u_j}) &= \eta^{(2)}_{ii}(u) \delta_{ij} \end{aligned} \tag{2.102}$$

for functions $\eta^{(2)}_{ii}(u) \in \mathcal{O}(\mathcal{H}_{a,b,\nu} \setminus \Delta_{a,b,\nu})$, possibly singular on $\Delta_{a,b,\nu}$. Moreover, thanks to the flatness of $\eta^{(2)}$ and its compatibility with the product, we can write

$$\eta^{(2)}_{ii}(u) = \eta^{(2)}(\partial_{u_i}, \partial_{u_i}) = \eta^{(2)}(e, \partial_{u_i}),$$

and by the flatness of e we get $\eta^{(2)}_{ii} = \partial_{u_i} t_1(u)$, where $\mathrm{d}t_1(u) = \eta^{(2)}(e, \cdot)$. This means that $\eta^{(2)}$ is a Egoroff metric, which implies (see for instance [44]) that $\nabla_X \eta^{(2)}(Y \star Z, K)$ is symmetric in all four vector fields X, Y, Z, K.

The above proves that (2.95) and (2.98) endow $\mathcal{H}_{a,b,\nu}$ with a semi-simple Frobenius dual-type structure, which has charge one by the flatness of the unit vector field.

As for (ii), notice that if $\nu = m \in \mathbb{Z}$, then λ is single-valued and $\mathcal{H}_{a,b,\nu}$ is a genus zero double Hurwitz space. Under the further condition that the zeroes of λ be simple, $\mathcal{H}_{a,b,m}$ becomes a Hurwitz space in a standard sense, with the only proviso that the divisor where λ has multiple zeroes is removed. Then under the conditions of (ii) the existence of a conformal Frobenius manifold structure is a direct corollary of [27, Theorem 5.1] for $m \neq 1-a, 1-b$; when $m = 1-a$ or $1-b$, the proof of the above theorem goes through almost unscathed except for the covariant constancy of the unit vector field, which fails to be satisfied in these cases. Furthermore, (2.100) follows from a standard argument (see [30, Proposition 5.1]), which together with (i) above proves semi-simplicity and the charge one condition.

Finally, (iii) is an immediate consequence of (ii) together with [60, Theorem 2]. □

Under the conditions of (ii), the statement of Theorem 2.23 implies that the metrics $\eta^{(1)}$ and $\eta^{(2)}$ form a flat pencil, which is exact if and only if $m \neq 1-a, 1-b$: $\eta^{(2)}$ is the (inverse) of the intersection form on $\mathcal{M}^{(1)}_{a,b,m}$. Moreover, when λ has only simple zeroes and poles, this is enhanced to a triple of compatible flat metrics $\eta^{(1)}, \eta^{(2)}, \eta^{(3)}$. And finally, if the unit of the first structure is flat, the resulting Frobenius structure is tri-Hamiltonian.

By comparing the formulas for the flat coordinates for $\eta^{(2)}$ and $\eta^{(1)}$ one easily sees when the pencil $(\eta^{(2)})^{-1} - \epsilon(\eta^{(1)})^{-1}$ is resonant, namely, when $\eta^{(1)}$ and $\eta^{(2)}$ have common flat coordinates. This happens if and only if λ has more than one pole; there is one common flat coordinate for each pole after the first.

As an immediate consequence of Theorem 2.23, the semi-classical limit of the RR2T with equations (2.80) and (2.81) has a neat description in terms of the Principal Hierarchy of $\mathcal{M}^{(i)}_{a,b,v}$, $i = 1, 2$.

Corollary 2.24. (1) *For any* $(a, b) \in \mathbb{Z}^2_+$, $m \in \mathbb{Z}$ *and* $\mathtt{t} \in \mathrm{Aff}_{a+b}(\mathbb{C}[[z]])$, *the Principal Hierarchy of* $(\mathcal{M}^{(2)}_{a,b,m}, \mathtt{t})$ *is a complete system of commuting Hamiltonian conservation laws of the m-generalized dRR2T of bidegree* (a, b).

(2) *Let* $-a - m < 0$, $b + m < 0$ *as in Theorem* 2.23(ii), *and fix* $\mathtt{t} \in \mathrm{Aff}_{a+b}(\mathbb{C})$ *such that*

$$h_{\alpha,p} = -\operatorname{Res}_{z=\infty} \frac{\lambda^{\frac{\alpha}{m+a}+p}}{\left(\frac{\alpha}{m+a}\right)_{1+p}} \frac{\mathrm{d}z}{z}, \quad \alpha = 1, \dots, m+a, \tag{2.103}$$

$$h_{\alpha+m+a,p} = -\operatorname{Res}_{z=0} \frac{\lambda^{\frac{\alpha}{-b-m}+p}}{\left(\frac{\alpha}{-b-m}\right)_{1+p}} \frac{\mathrm{d}z}{z}, \quad \alpha = 1, \dots, -m-b-1, \tag{2.104}$$

where $(x)_n \triangleq \Gamma(x+n)/\Gamma(x)$. *Then the Hamiltonian flows of the Principal Hierarchy* (2.90) *of* $\mathcal{M}^{(1)}_{a,b,m}$ *associated to* $h_{\alpha,p}$, $\alpha = 1, \dots, a-b-1$, *coincide with the semi-classical Lax flows* (2.80) *for the m-generalized dRR2T of bidegree* (a, b) *upon identifying*

$$t_{\alpha,p} \mapsto \frac{\left(\frac{\alpha}{m+a}\right)_{1+p}}{\alpha + p(m+a)} t^{(1)}_{\alpha+p(m+a)}, \quad \alpha = 1, \dots, m+a, \tag{2.105}$$

$$t_{\alpha+m+a,p} \mapsto \frac{\left(\frac{\alpha}{-b-m}\right)_{1+p}}{\alpha - p(m+b)} t^{(2)}_{\alpha-p(b+m)}, \quad \alpha = 1, \dots, -m-b-1. \tag{2.106}$$

Proof. Point (2) is an immediate application of Proposition 6.3 and Theorem 6.5 in [27]. Notice in particular that Proposition 6.3 warrants the existence of a flat coordinate system $\mathtt{t}$ for the deformed connection on $\mathcal{H}_{a,b,m} \times \mathbb{C}$ ((2.87) and (2.88)) which is compatible with (2.103) and (2.104); the scaling factors in (2.105) and (2.106) are required for consistency with the definition of the semiclassical Lax flows.[4]

To see why (1) holds, consider the Taylor expansion in the variable ζ of the deformed flatness equations of $\mathcal{M}^{(1)}_{a,b,m}$ in the tangent directions to $\mathcal{H}_{a,b,m}$ (see (2.85)). Then, from

[4] See e.g. [20, Section 1].

(2.94), (2.95), (2.97) and (2.98), writing the p^{th} Taylor coefficient in flat coordinates for the intersection form $\eta^{(2)}$ yields the deformed flatness equations of the dual Frobenius structure $\mathcal{M}^{(2)}_{a,b,m}$ with $\zeta = p$ [9,29]. The first statement then follows immediately. □

Remark 2.25. The dispersionless limit of the Poisson structure for RR2T obtained as a reduction of the second Poisson bracket of the 2D-Toda hierarchy [16] corresponds to the Poisson structure associated to the metric $\eta^{(2)}$ via (2.89), as one can promptly check by computing the Poisson brackets for the coefficients of A and B, as described in Section 2.5, and taking their quasi-classical limit.

Remark 2.26. Under the conditions of Corollary 2.24(2), it should be stressed that the dToda Hamiltonian flows, (2.80), are generated by a strict subset of the flat coordinates of the deformed connection, (2.87), (2.88). The remaining flows, which by semi-simplicity of $\mathcal{M}^{(1)}_{a,b,m}$ make the Principal Hierarchy a complete family of conservation laws [67], are a genuine extension of the dRR2T, analogous to the extension of the ordinary 1D-Toda hierarchy [19]. On the other hand, as soon as the conditions of (2) are not matched, it can readily be checked in examples that the metric in (2.94) is typically curved if either of $b+m$ or $-a-m$ is greater than one. The conditions on the range of m leave only two possibilities for $m \geq 0$: $b = 0,\ m = 1$ or $b = 1,\ m = 0$. The case $m < 0$ displays instead a wealth of flat structures: as long as $-a-1 \leq m \leq 1-b$ and $m \neq -a, -b$, the metric $\eta^{(1)}$ in (2.94) is flat. Equivalently, for any fixed bidegree (a, b) there exist $a+b+1$ generalized RR2T (see Definition 2.5) such that their semi-classical limit has a dispersionless bi-Hamiltonian structure of Dubrovin–Novikov type. This structure is exact when both $b+m$ and $-a-m$ are negative, and tri-Hamiltonian if $a = b, m = -a+1$.

Remark 2.27 (Flat coordinates of $\eta^{(1)}$). Flat coordinates for the first Frobenius structure on $\mathcal{H}_{a,b,m}$ can be constructed using standard methods from [27, 60]. For definiteness, consider the case when $b = 1$ and $m = 0$. By applying the change of variables $z \mapsto \mathrm{e}^{-q_0}(z+1)$ we obtain

$$\lambda = \mathrm{e}^{v-aq_0}(z+1)^a\left(1-\frac{\mathrm{e}^{2q_0}-1}{z}\right)\prod_{k=1}^{a-1}\left(1-\frac{\mathrm{e}^{q-k+q_0}}{z+1}\right), \qquad \phi = \frac{\mathrm{d}z}{z+1}. \tag{2.107}$$

We denote by $z = z(\lambda, q)$ a local inverse of the function $\lambda(z, q)$, and from the equation $\partial_q(\lambda(z(\lambda, q), q)) = 0$ we obtain the "thermodynamic identity" $\partial_q\lambda = -(\partial_z\lambda)(\partial_q z)$, from which we can rewrite the residue formula (2.94) as

$$\eta(X, Y) = \sum_{i=1}^{a+3} \mathrm{Res}_{z_i}\, X(\log(z+1))Y(\log(z+1))\mathrm{d}\lambda. \tag{2.108}$$

Now notice that we can expand the local solutions $\log(z(\lambda, q)+1)$ in the following way as a series of λ:

$$\begin{aligned}
&\log(z+1) = \tfrac{1}{a}[\log\lambda - (v-aq_0) - \textstyle\sum_{k=1}^{a}\tau_k/\lambda^{k/a}] + O(1/\lambda^{1+1/a}), && z \to \infty,\\
&\log(z+1) = O(1/\lambda), && z \to 0,\\
&\log(z+1) = \log\lambda + c_0 + O(\lambda), && z \to -1,\\
&\log(z+1) = c_j + O(\lambda), && z \to e^{q-j+q_0} - 1.
\end{aligned} \tag{2.109}$$

This shows that the only contribution to the sum in (2.94) comes from $z = \infty$ and that the coefficients

$$\tau_0 = v - aq_0,$$
$$\tau_k = \frac{a}{k}\,\mathrm{Res}_{\lambda^{1/a}=\infty}\left[\lambda^{k/a}\frac{\partial}{\partial\lambda^{1/a}}\log(z+1)\mathrm{d}\lambda^{1/a}\right] = \frac{a}{k}\,\mathrm{Res}_{z=\infty}\frac{\lambda^{k/a}}{z+1}\mathrm{d}z, \qquad k = 1,\dots,a,$$

are flat coordinates for $\eta^{(1)}$.

Example 2.28 (Bi-Hamiltonian structure of q-deformed dispersionless 2-KdV). Let us consider the dispersionless limit of the q-deformed Gelfand–Dickey hierarchy of Example 2.8 for $n = 2$. The symbol of the Lax operator reads

$$\lambda(z) = z^3 + az^2 + bz + c \tag{2.110}$$

and a quick inspection of the semi-classical Lax equations reveals that c is invariant under the flows of (2.80). When $c = 0$, the hierarchy manifestly reduces to the generalized dRR2T of bidegree $(a, b) = (2, 0)$ with $\nu = m = 1$, $v = 0$.

By Theorem 2.23, the space of coefficients $\mathcal{H}_{2,0,1}$ is endowed with a conformal Frobenius manifold structure $\mathcal{M}^{(1)}_{2,0,1} = (\mathcal{H}_{2,0,1}, \eta^{(1)}, \bullet)$ of charge one. The discussion of Remark 2.27 shows that flat coordinates for the metric $\eta^{(1)}$ are given by

$$t_1 = -a/3, \qquad t_2 = b - a^2/6. \tag{2.111}$$

In this chart, $\eta^{(1)}$ takes the off-diagonal form $\eta^{(1)}_{ij} = \delta_{i+j,2}$, and the algebra structure on $\mathcal{M}^{(1)}_{2,0,1}$ is induced by the polynomial prepotential

$$F^{(1)}(t_1, t_2) = \tfrac{12}{5}t_1^6 - t_2t_1^4 + \tfrac{1}{4}t_2^2t_1^2 - t_2^3/144. \tag{2.112}$$

As far as the dual-type Frobenius structure $\mathcal{M}^{(2)}_{2,0,1} = (\mathcal{H}_{2,0,1}, \eta^{(2)}, \star)$ is concerned, from the proof of Theorem 2.23(i) we know that the zeroes $(\mathrm{e}^{q_0}, \mathrm{e}^{q_{-1}})$ of λ are exponentiated flat coordinates of $\eta^{(2)}$. Then the Miura transformation

$$t_1 = \tfrac{1}{3}(\mathrm{e}^{q_0} + \mathrm{e}^{q_{-1}}), \tag{2.113}$$
$$t_2 = \tfrac{1}{6}(4\mathrm{e}^{q_0+q_{-1}} - \mathrm{e}^{2q_0} - \mathrm{e}^{2q_{-1}}) \tag{2.114}$$

and (2.95) yield $\eta^{(2)}_{ij} = (3 + (-1)^{i+j})/2$ in the chart (q_0, q_{-1}). Finally, the $\star$-product is given by (2.98) by the dual prepotential

$$F^{(2)}(q_0, q_{-1}) = \tfrac{5}{6}q_0^3 + \tfrac{1}{2}q_{-1}q_0^2 + q_{-1}^2q_0 + \tfrac{2}{3}q_{-1}^3 - \mathrm{Li}_3(\mathrm{e}^{q_{-1}-q_0}), \tag{2.115}$$

where $\mathrm{Li}_3(x) = \sum_{n>0} x^n/n^3$ is the polylogarithm function of order 3.

Example 2.29 (Tri-Hamiltonian structure of dispersionless Ablowitz–Ladik). Let now $a = b = 1$, $m = 0$. This case corresponds to the dispersionless limit of the Ablowitz–Ladik hierarchy of Example 2.7. For this case, the Frobenius manifold structures $\mathcal{M}^{(1)}_{1,1,0}$ and $\mathcal{M}^{(2)}_{1,1,0}$ on $\mathcal{H}_{1,1,0}$ were constructed in [9]; we will review and expand on that in light of the general result of Theorem 2.23. In this case, the symbol of the Lax operator reads

$$\lambda(z) = \mathrm{e}^{v} z \frac{z - \mathrm{e}^{q_0}}{z - \mathrm{e}^{-q_0}}. \tag{2.116}$$

By the proof of Theorem 2.23(i) we know that (v, q_0) are flat coordinates for the metric $\eta^{(2)}$ defined by (2.95). Furthermore, Theorem 2.23(ii) implies that the metric $\eta^{(1)}$ is flat in this case. By the discussion of Remark 2.27, flat coordinates for $\eta^{(1)}$ are given by

$$v = \tfrac{1}{2}(\log(t_1 + \mathrm{e}^{t_2}) + t_2), \tag{2.117}$$

$$q_0 = \tfrac{1}{2}(\log(t_1 + \mathrm{e}^{t_2}) - t_2). \tag{2.118}$$

Notice that $t_2 v - q_0$ is a flat coordinate for both $\eta^{(1)}$ and $\eta^{(2)}$, and the flat pencil is resonant in this case. The Frobenius potentials in the respective flat frames are

$$F^{(1)}(t_1, t_2) = \tfrac{1}{2} t_2 t_1^2 + \mathrm{e}^{t_2} s_1 + \tfrac{1}{2} s_1^2 \log(s_1), \tag{2.119}$$

$$F^{(2)}(v, q_0) = v^2 q_0 + 2 v q_0^2 + \tfrac{7}{3} q_0^3 + \mathrm{Li}_3(\mathrm{e}^{2q_0}). \tag{2.120}$$

A further consequence of Theorem 2.23 is the existence of a third compatible flat metric $\eta^{(3)}$, along with the corresponding Frobenius manifold structure $\mathcal{M}^{(3)}_{1,1,0}$. Introducing a local chart (s_1, s_2) via

$$v = -\tfrac{1}{2}(s_2 + 3\log(\mathrm{e}^{-s_2} - s_1)), \tag{2.121}$$

$$q_0 = \tfrac{1}{2}(s_2 + \log(\mathrm{e}^{-s_2} - s_1)) \tag{2.122}$$

gives a flat coordinate system for $\eta^{(3)}$ as defined in (2.96), as indeed $\eta^{(3)}_{ij} = \delta_{i+j,3}$; the pencil $(\eta^{(3)})^{-1} - \epsilon(\eta^{(2)})^{-1}$ is again resonant, since $s_2 = v + 3q_0$. It follows from (2.99) that the third product structure is induced by the prepotential

$$F^{(3)}(s_1, s_2) = \tfrac{1}{2} s_2 s_1^2 - \mathrm{e}^{-s_2} s_1 - \tfrac{1}{2} s_1^2 \log(s_1), \tag{2.123}$$

which shows that the first and third Frobenius structures are isomorphic,

$$\mathcal{M}^{(1)}_{1,1,0} \simeq \mathcal{M}^{(3)}_{1,1,0}. \tag{2.124}$$

The isomorphism is non-trivial, in that $\eta^{(1)}$ and $\eta^{(3)}$ do not share a common flat system and the associated Frobenius structures are not related by an affine change of flat coordinates.

3. Equivariant mirror symmetry of toric trees

Let X be a smooth quasi-projective variety over $\mathbb{C}$ with vanishing odd cohomology, and T an algebraic torus action on X with projective fixed loci $i_j : X_j^T \hookrightarrow X$, $j = 1, \dots, r \in \mathbb{N}$. If X is projective, the equivariant Gromov–Witten invariants of (X, T) [39] are defined as

$$\langle \phi_{\alpha_1}, \dots, \phi_{\alpha_n} \rangle_{g,n,\beta}^{X,T} \triangleq \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]_T^{\mathrm{vir}}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\phi_{\alpha_i}) \in H_T(\mathrm{pt}), \tag{3.1}$$

where $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is the stable compactification [49] of the moduli space of degree $\beta \in H_2(X, \mathbb{Z})$ morphisms from n-pointed, genus g curves to X, $[\overline{\mathcal{M}}_{g,n}(X, \beta)]_T^{\mathrm{vir}}$ is the T-equivariant virtual fundamental class of $\overline{\mathcal{M}}_{g,n}(X, \beta)$, $\phi_{\alpha_i} \in H_T(X)$ are arbitrary equivariant cohomology classes of X, and $\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(X, \beta) \to X$ is the evaluation map at the i^{th} marked point. Equation (3.1) still makes sense if X is non-compact as long as X_i^T is for all i; in that case, we define invariants by their localization to the fixed locus by the Graber–Pandharipande virtual localization formula [42, 49]. For T-equivariant cohomology classes $\phi_1, \phi_2 \in H_T(X)$, write η for the non-degenerate inner product

$$\eta(\phi_1, \phi_2) \triangleq \sum_{j=1}^{r} \int_{X_j^T} \frac{i_j^*(\phi_1 \cup \phi_2)}{e(N_{X_j/X^T})}, \tag{3.2}$$

where $e(N_{X_j/X^T})$ is the T-equivariant Euler class of the normal bundle to the fixed locus X^T. We will denote by the same symbol the flat non-degenerate pairing on $T(H_T(X))$ obtained from (3.2) by identifying $T_\tau H_T(X) \simeq H_T(X)$ for all $\tau \in H_T(X)$. For vector yields $\varphi_i \in \mathcal{X}(H_T(X))$, $i = 1, 2$, the genus zero equivariant Gromov–Witten invariants (3.1) define further a product structure $\varphi_1 \circ \varphi_2$ on the tangent fiber at τ through

$$\eta(\varphi_1, \varphi_2 \circ \varphi_3) \triangleq \sum_{n \geq 0} \sum_{\beta \in H_2(X,\mathbb{Z})} \langle \phi_1, \phi_2, \phi_3, \tau^{\otimes n} \rangle_{0,n+3,\beta}^{X,T}, \tag{3.3}$$

which is commutative, associative, and compatible with η [39]. The corresponding Frobenius manifold structure $QH_T(X) \triangleq (H_T(X), \eta, \circ)$ on $H_T(X)$ is the *T-equivariant quantum cohomology of X*.

Let $\mu_i = c_1(\mathcal{O}_{BT_i}(1))$ be the hyperplane class on the classifying space BT_i of the i^{th} factor of $T = (\mathbb{C}^*)^l$, and write $\mathbb{K} \triangleq \mathbb{C}(\mu_1, \dots, \mu_n)$ for the field of fractions of $H^\bullet(BT)$. Then $QH_T(X)$ is a finite-dimensional dual-type Frobenius manifold over $\mathbb{K}$ of charge one: it has a flat identity by the Fundamental Class Axiom of Gromov–Witten theory, and it is generally non-conformal as a consequence of the non-trivial grading of the ground field $\mathbb{K}$. The purpose of this section is to exhibit an isomorphism of such Frobenius dual-type structures with the second Frobenius structure on $\mathcal{H}_{a,b,\mu}$ of Theorem 2.23 for a suitable family of targets. When X is the total space of the bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ and $T \simeq \mathbb{C}^*$ is the one-torus action that covers the trivial action on the base and scales the fibers with opposite weights, it was already shown in [8, 9] that $QH_T(X) \simeq \mathcal{M}_{1,1,0}^{(2)}$. Moreover, it was proved in [10] that $\mathcal{M}_{a,0,\nu}^{(2)}$ is isomorphic to the T-equivariant orbifold cohomology of the A_{a-1}-surface singularity, where $T \simeq \mathbb{C}^*$ acts with generic weights specified by ν. We will see how this correspondence with Gromov–Witten theory generalizes to arbitrary (a, b, ν).

3.1. Toric data

Let $\mathcal{S}_{a,b} = \{v_i \in \mathbb{Z}^3\}_{i=1}^{a+b+2}$ be the set of three-dimensional integer vectors

$$v_i = \begin{cases} (0, a+1-i, 1), & i = 1, \dots, a+1, \\ (1, a+2-i, 1), & i = a+2, \dots, a+b+2. \end{cases} \tag{3.4}$$

$\mathcal{S}_{a,b}$ is the skeleton of the fan of a toric variety, given by the cone over a triangulation of the rays v_i (Figures 1 and 2). We can construct it as a GIT quotient $\mathbb{C}^{a+b+2} /\!\!/ (\mathbb{C}^*)^{a+b-1}$ [23] by considering the exact sequence

$$0 \to \mathbb{Z}^{a+b-1} \xrightarrow{M} \mathbb{Z}^{a+b+2} \xrightarrow{N} \mathbb{Z}^3 \to 0, \tag{3.5}$$

where

$$M^T = \begin{pmatrix} 1 & -2 & 1 & 0 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & \dots & 0 & \dots & 0 \\ & \vdots & \ddots & & & \vdots & & \vdots & & \\ 0 & \dots & 1 & -2 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & -1 & -1 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 & -2 & 1 & \dots & 0 \\ & \vdots & & & & \ddots & \vdots & & & \\ 0 & \dots & 0 & \dots & 0 & \dots & 0 & 1 & -2 & 1 \end{pmatrix}, \tag{3.6}$$

$$N = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ 0 & 1 & 2 & \dots & a & 0 & -1 & \dots & -b \\ 1 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \end{pmatrix}. \tag{3.7}$$

A triangulation of the fan corresponds to a choice of chamber in the GIT problem, as in Figures 1 and 2. The picture in Figure 1 corresponds to the orbifold chamber in the secondary fan of (3.6) and (3.7); we will denote by $X_{a,b}$ the resulting singular variety. It is obtained by deleting the unstable locus

$$X_{a,b}^{\mathrm{us}} \triangleq V\Big(\prod_{i=2}^{a-1} x_i \prod_{j=2}^{b-1} x_{a+j}\Big) \tag{3.8}$$

in $\mathbb{C}^{a+b+2}$ and quotienting by the $(\mathbb{C}^*)^{a+b-1}$-action with weights specified by M in (3.6). The picture in Figure 2 corresponds instead to the smooth (large volume) chamber: we remove the Zariski-closed set $Y_{a,b}^{\mathrm{us}}$ defined by

$$Y_{a,b}^{\mathrm{us}} \triangleq V\Big(\prod_{j>i+1,\, j\neq a+1,a+2} \langle x_i, x_j\rangle \prod_{j=1}^{a-1} \langle x_{a+1}, x_j\rangle \prod_{j=a+4}^{a+b+2} \langle x_{a+2}, x_j\rangle\Big) \tag{3.9}$$

and then quotient by the $(\mathbb{C}^*)^{a+b-1}$-action with weights specified by M in (3.6). The resulting variety, which we will denote by $Y_{a,b}$, is a smooth quasi-projective Calabi–Yau threefold, and the variation of GIT given by moving from Figure 1 to Figure 2 is a crepant resolution of the singularities of $X_{a,b}$.

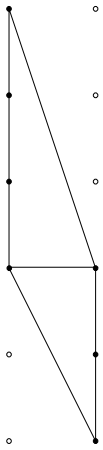

Fig. 1. The toric diagram of the orbifold $X_{a,b}$ for $a = 3, b = 2$.

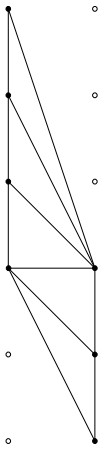

Fig. 2. The toric diagram of the minimal resolution $Y_{a,b}$ for $a = 3, b = 2$.

3.1.1. T-equivariant cohomology. The resolution $Y_{a,b}$ can be visualized as a tree of two chains $\{L_i\}_{i=1}^{a-1}$ and $\{L_i\}_{i=a+1}^{a+b-1}$ of $\mathbb{P}^1$ with normal bundle $\mathcal{O} + \mathcal{O}(-2)$, which are then connected along a $(-1, -1)$ curve L_a. We will refer to the resulting geometry as a *toric tree*, to reflect the shape of the corresponding web diagram (Figure 3). Explicitly, we have

$$L_i \triangleq \begin{cases} V(x_{i+1}, x_{a+2}), & i < a, \\ V(x_{a+1}, x_{a+2}), & i = a, \\ V(x_{a+1}, x_{i+2}), & i > a. \end{cases} \tag{3.10}$$

The fundamental cycles $[L_j] \in H_2(Y_{a,b}, \mathbb{Z})$ of the links of the chain are a system of generators for $H_2(Y_{a,b}, \mathbb{Z}) \simeq \mathbb{Z}^{a+b-1}$. Define $\omega_j \in H^2(Y_{a,b}, \mathbb{Z})$ to be their cohomology duals, and $\mathcal{O}(\omega_j)$ the corresponding line bundles; by definition, they restrict to $\mathcal{O}(1)$ on L_j, and to the trivial bundle on L_i, $i \neq j$. Consider now the following $T \simeq (\mathbb{C}^*)^2$-action on $\mathbb{C}^{a+b+2}$:

$$(x_i; \sigma_1, \sigma_2) \to \begin{cases} \sigma_1^{-1} x_a, & i = a, \\ \sigma_2 \sigma_1 x_{a+1}, & i = a + 1, \\ \sigma_2^{-1} x_{a+2}, & i = a + 2, \\ x_i, & \text{else}. \end{cases} \tag{3.11}$$

This descends to an effective torus action on $X_{a,b}$, which preserves $K_{Y_{a,b}} \simeq \mathcal{O}_{Y_{a,b}}$. Let $\{p_i\}_{i=1}^{a+b}$ denote the fixed points of the torus action, so that p_i and p_{i+1} correspond to the poles of each $\mathbb{P}^1$ in the chain. Turning on a torus action as in (3.11) we obtain an action on the bundles over the links of the chain, linearized as in (3.11); their equivariant first Chern classes provide lifts of ω_j to T-equivariant cohomology, which we will denote by the same symbol $\omega_j \in H_T(Y_{a,b})$.

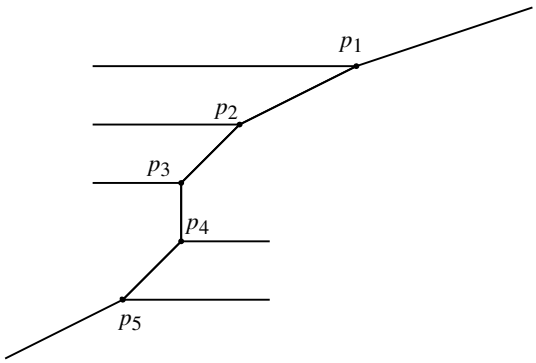

Fig. 3. The toric web diagram of $Y_{a,b}$ for $a = 3, b = 2$.

3.2. Mirror symmetry

Denote $\mu_i \triangleq c_1(\mathcal{O}_{BT_i}(1))$ where $\mathbb{C}^* \simeq T_i \hookrightarrow T$ are the two cartesian projections of the two-torus T acting on $Y_{a,b}$. We have the following

Theorem 3.1. *Let* (a, b, ν) *be as in Definition* 2.21. *Then*

$$QH_T(Y_{a,b}) \simeq \mathcal{M}^{(2)}_{a,b,\nu} \tag{3.12}$$

upon identifying $\nu = \mu_1/\mu_2$.

Proof. The proof is given by explicit calculation of both sides of (3.12). For the r.h.s., we will use the fact that in positive degree all genus zero Gromov–Witten invariants can be computed by a combined use of the deformation invariance of GW invariants and the Aspinwall–Morrison formula [5, 13, 70]. The result is [13, 47]

$$\begin{aligned}
&\langle \omega_{i_1}, \dots, \omega_{i_n} \rangle^{Y_{a,b},T}_{0,n,\beta} \\
&= \begin{cases}
1/d^3 & \text{if } i_j = a \text{ for some } j,\ \beta = d([L_a] + \sum_{i=k_a}^{a-1}[L_i] + \sum_{j=a+k_b}^{a+b+1}[L_j]), \\
& \quad k_\bullet = \min(\{i_j\}, \bullet), \\
-1/d^3 & \text{if } k_+ = \max(\{i_j\}) < a \text{ or } k_- = \min(\{i_j\}) > a, \\
& \quad \beta = d(L_{k_-} + \cdots + L_{k_+}), \\
0 & \text{else.}
\end{cases}
\end{aligned} \tag{3.13}$$

When $\beta = 0$ and $n = 3$, Gromov–Witten invariants are defined as the equivariant triple intersection numbers of $Y_{a,b}$, which can be computed explicitly by localization to the T-fixed points from (3.11). Explicitly, the restrictions of the Kähler classes to the fixed loci read

$$\omega_i|_{p_j} = \begin{cases}
(a-i)\mu_2 + \mu_1 & \text{for } j \le i \le a-1, \\
0 & \text{for } i \le a-1,\ j > i, \\
0 & \text{for } i \ge a,\ j \le i, \\
(a-i)\mu_2 - \mu_1 & \text{for } j > i \ge a,
\end{cases} \tag{3.14}$$

and the moving part contribution to the Euler class is computed as

$$e_T(TM)|_{p_i} = \begin{cases} -\mu_2((a-i)\mu_2+\mu_1)(\mu_1+\mu_2(a-i+1)) & \text{for } i \le a, \\ \mu_2(\mu_1+(i-a-1)\mu_2)((i-a)\mu_2+\mu_1) & \text{for } i \ge a+1. \end{cases} \tag{3.15}$$

Then, denoting $s_{i,c} \triangleq \sum_{k=i}^{c}(e_T(TM))^{-1}|_{p_k}$, we get

$$s_{i,c} = \begin{cases} \dfrac{i-c-1}{\mu_2((a-c)\mu_2+\mu_1)((a+1-i)\mu_2+\mu_1)} & \text{for } i < c \le a, \\ \dfrac{c-i+1}{\mu_2((a-c)\mu_2-\mu_1)((a+1-i)\mu_2-\mu_1)} & \text{for } a < i \le c, \end{cases} \tag{3.16}$$

and therefore

$$\langle \mathbf{1}^3 \rangle_{0,3,0}^{Y_{a,b},T} = s_{1,a+b} = s_{1,a-b} = \frac{b-a}{\mu_2(b\mu_2+\mu_1)(a\mu_2+\mu_1)}. \tag{3.17}$$

Furthermore, for $i \le j \le k < a$,

$$\langle \mathbf{1}^2, \omega_i \rangle_{0,3,0}^{Y_{a,b},T} = ((a-i)\mu_2+\mu_1)s_{1,i} = -\frac{i}{\mu_2(a\mu_2+\mu_1)}, \tag{3.18}$$

$$\langle \mathbf{1}, \omega_i, \omega_j \rangle_{0,3,0}^{Y_{a,b},T} = \frac{-i((a-j)\mu_2+\mu_1)}{(a\mu_2+\mu_1)}, \tag{3.19}$$

$$\langle \omega_i, \omega_j, \omega_k \rangle_{0,3,0}^{Y_{a,b},T} = \frac{i((a-j)\mu_2+\mu_1)((a-k)\mu_2+\mu_1)}{\mu_2(a\mu_2+\mu_1)}, \tag{3.20}$$

and for $i \ge j \ge k \ge a$,

$$\langle \mathbf{1}^2, \omega_i \rangle_{0,3,0}^{Y_{a,b},T} = (-(i-a)\mu_2-\mu_1)s_{i+1,a+b} = \frac{i-a-b}{\mu_2(b\mu_2+\mu_1)}, \tag{3.21}$$

$$\langle \mathbf{1}, \omega_i, \omega_j \rangle_{0,3,0}^{Y_{a,b},T} = \frac{(i-a-b)((a-j)\mu_2-\mu_1)}{\mu_2(b\mu_2+\mu_1)}, \tag{3.22}$$

$$\langle \omega_i, \omega_j, \omega_k \rangle_{0,3,0}^{Y_{a,b},T} = \frac{(i-a-b)((a-j)\mu_2-\mu_1)((a-k)\mu_2-\mu_1)}{\mu_2(b\mu_2+\mu_1)}. \tag{3.23}$$

Write $\tau = \tau_0\mathbf{1} + \sum_{i=1}^{a+b-1}\tau_i\omega_i$ for $\tau \in H_T(Y_{a,b})$, where $\omega_0 \triangleq \mathbf{1}_Y$. Then (3.13)–(3.23) imply that the generating function $F_{\mathrm{GW}}^{Y_{a,b},T}$ of the genus zero Gromov–Witten invariants of $Y_{a,b}$ takes the form

$$\begin{aligned} F_{\mathrm{GW}}^{Y_{a,b},T}(\tau) &\triangleq \sum_{n,\beta}\left\langle \frac{\tau^{\otimes n}}{n!} \right\rangle_{0,n,\beta}^{Y_{a,b},T} \\ &= \sum_{i,j,k}\langle \omega_i, \omega_j, \omega_k \rangle_{0,3,0}^{Y_{a,b},T}\tau_i\tau_j\tau_k + \sum_{l=0}^{a-1}\sum_{k=0}^{b-1}\mathrm{Li}_3(\mathrm{e}^{\tau_a+\tau_{a-1}+\cdots+\tau_{a-l}+\tau_{a+b-1}+\cdots+\tau_{a+b-k}}) \\ &\quad - \sum_{k\le l=1}^{a-1}\mathrm{Li}_3(\mathrm{e}^{\tau_k+\cdots+\tau_l}) - \sum_{k\le l=a+1}^{a+b-1}\mathrm{Li}_3(\mathrm{e}^{\tau_k+\cdots+\tau_l}). \end{aligned} \tag{3.24}$$

As far as the r.h.s. of (3.12) is concerned, the prepotential of $\mathcal{M}^{(2)}_{a,b,\nu}$ can be computed analytically in closed form from (2.95) and (2.98). A rather tedious, but completely straightforward residue calculation shows that the prepotentials coincide,

$$F^{\mathcal{M}^{(2)}_{a,b,\nu}}(v, q_{-i}, q_j) = F^{Y_{a,b},T}_{\mathrm{GW}}(\tau), \tag{3.25}$$

upon identifying flat coordinates as

$$v = \tau_0/\mu_2 + (a-\nu)\tau_a/2 + \sum_{j=1}^{a-1} j\tau_j, \tag{3.26}$$

$$q_{-k} = -(\tau_a/2 + \tau_{a-1} + \cdots + \tau_{a-k}), \quad k = 0, \ldots, a-1, \tag{3.27}$$

$$q_l = -(\tau_a/2 + \tau_{a+1} + \cdots + \tau_{a+l}), \quad l = 1, \ldots, b-1. \tag{3.28}$$

□

Theorem 3.1 prompts the following immediate generalization of the conjectural correspondence of [8] for the Ablowitz–Ladik hierarchy.

Conjecture 3.2. *The full descendent all-genus Gromov–Witten potential of* $(Y_{a,b}, T)$ *for* $\mu_1 = m\mu_2$ *is the logarithm of a* τ*-function of the* m*-generalized RR2T of bidegree* (a, b).

In other words, the parameter m in Definition 2.5 corresponds to a choice of weights of a resonant subtorus $\mathbb{C}^* \simeq T' \subset T$. Its proof up to genus one will be the subject of Section 3.4.

Remark 3.3. When $b = 0$, the GIT quotient in (3.5) yields $Y^{a+1,0} \simeq \mathbb{C} \times \mathcal{A}_a$, where $\mathcal{A}_a$ is the canonical resolution of the A_a surface singularity. Conjecture 3.2 then suggests that a suitable τ-function of the q-deformed a-KdV hierarchy should yield the total GW potential of $\mathbb{C} \times \mathcal{A}_a$. This has interesting implications already for the case $a = 1$ and $\mathcal{A}_0 = \mathbb{C}^2$, where it would imply that the τ-function of the scalar hierarchy highlighted in [7] to be underlying the generating functions of triple Hodge integrals on $\mathcal{M}_{g,n}$ should be a τ-function of the q-deformed KdV hierarchy of [35].

3.3. Twisted periods and the Dubrovin connection

Information on the genus zero gravitational invariants of $Y_{a,b}$ is encoded into the pencil of affine connections of (2.85), or the *Dubrovin connection* on $QH_T(Y_{a,b})$. An immediate spin-off of Theorem 3.1 is an explicit characterization of its space of solutions.

Let $\nu = m \in \mathbb{Z}$ and $\pi : \mathcal{U}_{a,b,m} \to \mathcal{H}_{a,b,m}$ be the universal curve over the genus zero double Hurwitz space $\mathcal{H}_{a,b,m}$. For $\lambda \in \mathcal{H}_{a,b,m}$ we write C_λ for the fiber of π at λ, and $C_{[\lambda]} \triangleq C_\lambda \setminus \{\mathrm{e}^{q_0}, \mathrm{e}^{-q_0}, \{\mathrm{e}^{\mathrm{sgn}(k)q_k}\}_{k\neq 0=1-a}^{b-1}\}$. Let now $p : \tilde{C}_{[\lambda]} \to C_{[\lambda]}$ be the universal covering map and, for $\zeta \in \mathbb{C}$, fix a choice of principal branch for $\lambda^\zeta = \exp(\zeta \log \lambda)$ as

$$\lambda^\zeta(z) = z^{\zeta(m+b)} \prod_{i=1-a}^{0} |z - q_i|^\zeta \mathrm{e}^{\mathrm{i}\zeta \arg_{i,+}(z)} \prod_{j=0}^{b-1} |z - q_j^{-1}|^{-\zeta} \mathrm{e}^{-\mathrm{i}\zeta \arg_{j,-}(z)} \tag{3.29}$$

where $\arg_{i,\pm}(z) \in [0, 2\pi)$ is the angle formed by $z - \mathrm{e}^{\pm q_i}$ with $\Im\mathrm{m}(z) = 0$. On the complex line L_λ parametrized by λ^ζ, we have a monodromy representation $\rho_\lambda : \pi_1(C_{[\lambda]}) \to L_\lambda \simeq \mathbb{C}$ defined by local coefficients l_{q_i} around e^{q_i} resulting in multiplication by $\mathfrak{q}_i := \rho_\lambda(l_{q_i}) = \mathrm{e}^{2\pi\mathrm{i}\zeta\sigma_i}$, where $\sigma_i = i+m+b+1$ or $i+m+b+a-1$ for $i > 0$ or $i < 0$ respectively, and we set $\mathfrak{q}_\pm = \mathfrak{q}_{0^\pm}$. Then the sheaf of sections of $\tilde{C}_{[\lambda]} \times_{\pi_1(C_{[\lambda]})} L_\lambda \to C_{[\lambda]}$ defines a locally constant sheaf L_λ on $C_{[\lambda]}$, and we denote by $H_\bullet(C_{[\lambda]}, L_\lambda)$ (resp. $H^\bullet(C_{[\lambda]}, L_\lambda)$) the homology (resp. cohomology) groups of $C_{[\lambda]}$ twisted by the set of local coefficients determined by $\mathfrak{q}_i$. Integrating $\lambda^\zeta\phi \in H^1(C_{[\lambda]}, L_\lambda)$ over $\gamma \in H_1(C_{[\lambda]}, L_\lambda)$ defines the *twisted period mapping*

$$\Pi_\lambda : H_1(C_{[\lambda]}, L_\lambda) \to \mathcal{O}(\mathcal{H}_{a,b,m}), \qquad \gamma \mapsto \int_\gamma \lambda^\zeta \, \mathrm{d}z/z. \tag{3.30}$$

Let now $\mathrm{Sol}_{a,b,\nu,\zeta}$ be the $(a+b)$-dimensional $\mathbb{C}(\zeta)$-vector space of horizontal sections for the Dubrovin connection,

$$\mathrm{Sol}_{a,b,\nu,\zeta} = \{s \in \mathcal{X}(\mathcal{H}_{a,b,m}) \mid \nabla^{(\eta^{(2)},\zeta)} s = 0\}. \tag{3.31}$$

As for the ordinary periods of $\mathcal{M}^{(1)}_{a,b,m}$ [28], twisted periods are an affine basis for the space of flat coordinates of the deformed flat connection on $\mathcal{M}^{(2)}_{a,b,m}$.

Proposition 3.4 ([29]). *The gradients with respect to $\eta^{(2)}$ of the twisted periods of* (3.30) *generate over $\mathbb{C}(\zeta)$ the solution space of the horizontality condition* (2.85) *for the Dubrovin connection on $QH_T(Y_{a,b})$,*

$$\mathrm{Sol}_{a,b,m,\zeta} = \mathrm{span}_{\mathbb{C}(\zeta)}\{\nabla^{\eta^{(2)}} \Pi_\lambda(\gamma)\}_{\gamma \in H_1(C_{[\lambda]}, L_\lambda)}. \tag{3.32}$$

Remark 3.5. Except for the double Hurwitz space interpretation, all of the above generalizes trivially to the case when $\nu \in \mathbb{C}$.

3.3.1. The twisted period mapping for $\mathcal{M}^{(2)}_{a,b,\nu}$. For generic monodromy weights, the homology with local coefficients L_λ coincides with the integral homology of the Riemannian covering [69] of $C_{[\lambda]}$,

$$H^\bullet(C_{[\lambda]}, L_\lambda) \simeq H^\bullet(\tilde{C}_{[\lambda]}/[\pi_1(C_{[\lambda]}), \pi_1(C_{[\lambda]})], \mathbb{Z}). \tag{3.33}$$

A basis of $H_1(C_{[\lambda]}, L_\lambda)$ can then be presented in the form of compact loops $\gamma_k = [l_0, l_{\mathrm{e}^{q_k \mathrm{sgn}(k)}}]$, $\gamma_\pm = [l_0, l_{\mathrm{e}^{\pm q_0}}]$ given by the commutator of simple oriented loops around zero and each of the punctures of $C_{[\lambda]}$. Then the twisted periods

$$\Omega_\pm \triangleq \frac{\Pi_\lambda(\gamma_k)}{(1 - \mathrm{e}^{2\pi\mathrm{i}\zeta\nu})(1 - \mathrm{e}^{\mp 2\pi\mathrm{i}\zeta})}, \tag{3.34}$$

$$\Omega_k \triangleq \frac{\Pi_\lambda(\gamma_k)}{(1 - \mathrm{e}^{2\pi\mathrm{i}\zeta\nu})(1 - \mathrm{e}^{-\mathrm{sgn}(k)2\pi\mathrm{i}\zeta})}, \qquad k \neq 0, \tag{3.35}$$

give a $\mathbb{C}(\zeta, \nu)$-basis of $\mathrm{Sol}_{a,b,\nu,\zeta}$ [10, 69, 71]. In turn, the period integrals of (3.34) and (3.35) are hypergeometric functions in exponentiated flat variables for $\eta^{(2)}$.

Proposition 3.6. *The twisted periods of* $\mathcal{M}^{(2)}_{a,b,v,\zeta}$ *are given by*

$$\Omega_{\pm} = \frac{\Gamma(\xi)\Gamma(1\pm\zeta)}{\Gamma(1+\xi\pm\zeta))}\mathrm{e}^{\zeta(v+2q_0)}\mathrm{e}^{\pm\xi q_0}\prod_{j=1-a,\ j\neq 0}^{b-1}\mathrm{e}^{\zeta q_j} \times \Phi^{[a-\theta(\pm 1),b-\theta(\pm 1)]}(\xi,\zeta,-\zeta,1+\xi\pm\zeta;\{\mathrm{e}^{\pm q_0-q_i}\}_{i=1-a}^{-1},\mathrm{e}^{\pm 2q_0},\{\mathrm{e}^{\pm q_0+q_i}\}_{i=1}^{b-1}), \tag{3.36}$$

$$\Omega_k = \frac{\Gamma(\xi)\Gamma(1+\mathrm{sgn}(k)\zeta)}{\Gamma(1+\xi+\mathrm{sgn}(k)\zeta)}\mathrm{e}^{\zeta(v+2q_0)}\mathrm{e}^{\xi\,\mathrm{sgn}(k)q_k}\prod_{j=1-a,\ j\neq 0}^{b-1}\mathrm{e}^{\zeta q_j} \times \Phi^{[a-\theta(k),b-\theta(k)]}(\xi,\zeta,-\zeta,1+\xi+\mathrm{sgn}(k)\zeta;\{\mathrm{e}^{\mathrm{sgn}(k)q_k-q_i}\}_{i\neq k=1-a}^{0},\{\mathrm{e}^{\mathrm{sgn}(k)q_k+q_i}\}_{i\neq k=0}^{b-1}), \tag{3.37}$$

where $\theta(x)$ *is Heaviside's step function and we define*

$$\Phi^{[M,N]}(a,b_1,b_2,c,w_1,\dots,w_{M+N}) \triangleq F_D^{(M+N)}(a;\overbrace{b_1,\dots,b_1}^{M\text{ times}},\overbrace{b_2,\dots,b_2}^{N\text{ times}};c;w_1,\dots,w_{M+N}), \tag{3.38}$$

and $\xi \triangleq \zeta(v+b)$.

In (3.38), $F_D^{(M)}(a;b_1,\dots,b_M;c;w_1,\dots,w_M)$ is the Lauricella function of type D [34]:

$$F_D^{(M)}(a;b_1,\dots,b_M;c;w_1,\dots,w_M) \triangleq \sum_{i_1,\dots,i_M}\frac{(a)_{\sum_j i_j}}{(c)_{\sum_j i_j}}\prod_{j=1}^{M}\frac{(b_j)_{i_j}w_j^{i_j}}{i_j!}, \tag{3.39}$$

where we use the Pochhammer symbol $(x)_m \triangleq \Gamma(x+m)/\Gamma(x)$. The proof is an immediate consequence of (3.34) and (3.35) and the Euler integral representation of the Lauricella function,

$$F_D^{(M)}(a;b_1,\dots,b_M;c;w_1,\dots,w_M) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 z^{a-1}(1-z)^{c-a-1}\prod_{i=1}^{M}(1-w_i z)^{-b_i}\,\mathrm{d}z. \tag{3.40}$$

3.4. Dispersive deformation and elliptic Gromov–Witten invariants

In this section we study the dispersive deformation of the m-generalized RR2T at order $O(\epsilon^2)$, and describe in detail the workflow of the proof of Conjecture 3.2 at the genus one approximation. In order to do so, we first offer a reformulation of Conjecture 3.2 in the language of the theory of formal loop spaces.

3.4.1. Conjecture 3.2 as a Miura equivalence of dispersive hierarchies. Recall that the Principal Hierarchy of Definition 2.20 can be thought of as a triplet $(\mathcal{M}, \{\,,\,\}^{[0]}, \mathcal{H}^{[0]})$ where $\mathcal{M}$ is an n-dimensional complex Frobenius manifold, $\{\,,\,\}^{[0]} \triangleq \{\,,\,\}_\eta$ in (2.89) is a local Poisson structure on the loop space $\mathcal{L}_\mathcal{M}$, and $\mathcal{H}^{[0]} = (H_{\alpha,p})_{\alpha,p}$ is a family of local functionals $H^{[0]}_{\alpha,p} = \int_{S^1} h^{[0]}_{\alpha,p}\,\mathrm{d}x$, $h^{[0]}_{\alpha,p} \in \mathcal{O}_\mathcal{M}$ for $\alpha = 1,\dots,n$ and $p \in \mathbb{Z}^+$, giving rise to commuting Hamiltonian vector fields on $\mathcal{L}_\mathcal{M}$ as in (2.90). When $\mathcal{M} = QH^\bullet_T(Y_{a,b}) \simeq \mathcal{M}^{(2)}_{a,b,v}$, the isomorphism of Theorem 3.1 induces a Poisson morphism $\mathcal{L}_{QH^\bullet_T(Y_{a,b})} \simeq \mathcal{L}_{\mathcal{M}^{(2)}_{a,b,v}}$ such that the dispersionless Toda densities $h_{\alpha,p}$ pull back to the expansion of the Hamiltonian densities of the Principal Hierarchy of $QH^\bullet_T(Y_{a,b})$, proving Conjecture 3.2 at the genus zero approximation.

For the higher genus theory, we have two, a priori inequivalent deformations of $\{\,,\,\}^{[0]}$ and $\mathcal{H}^{[0]}$, depending on a formal parameter ϵ. The first one is the spatial interpolation of the Toda lattice of Section 2.6 applied to the 2D-Toda Hamiltonians (2.75) and to the second 2D-Toda Poisson structure reduced on the factorization locus $\mathcal{A}^{\mathrm{RR}}$ (Section 2.5): we call this the *RR2T deformation*. The second is the Buryak–Posthuma–Shadrin deformation of the Poisson structure and Hamiltonians induced by Givental's formula for the higher genus Gromov–Witten potential [14,41]; we will refer to this as the *GW deformation*. In either case, $(\{\,,\,\}^{[0]}, (H^{[0]}_{\alpha,p})_{\alpha,p})$ deforms as

$$\begin{aligned}\{\tau^\alpha(X),\tau^\beta(Y)\}^{[0]} &\mapsto \{\tau^\alpha(X),\tau^\beta(Y)\}^{[\epsilon]}\\ &= \{\tau^\alpha(X),\tau^\beta(Y)\}^{[0]} + \sum_{g=1}^\infty \epsilon^g \sum_{s=0}^{g+1} \mathscr{P}^{\alpha,\beta}_{g,s}(\tau,\tau_X,\dots,\tau^{(s)})\delta^{(s)}(X-Y),\\ h^{[0]}_{\alpha,p} &\mapsto h^{[\epsilon]}_{\alpha,p} = h^{[0]}_{\alpha,p}(\tau) + \sum_{g=1}^\infty h^{[g]}_{\alpha,p}(\tau,\tau_X,\dots,\tau^{(g)}).\end{aligned} \tag{3.41}$$

where $h^{[g]}_{\alpha,p}$, $\mathscr{P}^{\alpha,\beta}_{g,s}$ are polynomials in the jet variables $\tau^{(i)} = \partial^i_X\tau$ $(i>0)$, graded homogeneous of degrees g and $g-s+1$ respectively; these vanish for the GW deformation when g is odd.

Both deformations come with a canonical system of coordinates for the jet space of $\mathcal{M}$: the tau-symmetric coordinates τ^α for the Gromov–Witten deformation, and the coefficients (α,β) of the Lax operators (2.16)–(2.18) in the deformation by lattice interpolation. Conjecture 3.2 can then be stated as the existence of an ϵ-dependent Poisson morphism which matches the Poisson structures and the Hamiltonian densities, up to total derivatives, of the two deformations. Such a morphism, if it exists, should take the form of an element of the *polynomial Miura group* of transformations of the form [32]

$$(\alpha,\beta) \mapsto \tau(\alpha,\beta) + \sum_{g>0} \epsilon^g \mathscr{F}_{[g]}(\alpha,\beta,\alpha_x,\beta_x,\dots,\alpha^{(g)},\beta^{(g)}). \tag{3.42}$$

The leading order in ϵ of the sought-for Miura transformation is just the change of variables to flat coordinates given by (2.16), (2.17), (2.91) and (3.26). We can then rephrase Conjecture 3.2 as follows:

Conjecture 3.2 (reloaded). *There exists a polynomial Miura transformation* (3.42) *matching the GW and the* (a, b) *RR2T deformations to all orders of the dispersive expansion.*

3.4.2. The genus one case—strategy of the proof. On the RR2T side, we have all the ingredients that are needed to compute the dispersive deformation of the Principal Hierarchy: all we have to do is to take the spatial interpolation of (2.71) and (2.75). On the other hand, closed-form expressions for the Gromov–Witten dispersive deformation of the Poisson bracket and the Hamiltonians from Givental's formula require control to all orders of the steepest-descent asymptotics of the oscillating integrals of $\mathcal{M}$, which is typically out of computational reach.[5] However, a workaround to this problem exists in genus one, corresponding to the $O(\epsilon^2)$ approximation. In this case, the rational Miura transformation [31]

$$\tau_\alpha(x) \mapsto \tau_\alpha(x) + \frac{\epsilon^2}{24}\frac{\partial^2}{\partial_x \partial_{t^{\alpha,0}}}(\log\det M + G(\tau)), \tag{3.43}$$

where

$$M_{\alpha,\beta} = c_{\alpha\beta\gamma}\tau_x^\gamma, \qquad c_{\alpha\beta\gamma} = \partial^3_{\alpha\beta\gamma}F(\tau), \tag{3.44}$$

deforms the Principal Hierarchy associated to quantum cohomology to the $O(\epsilon^2)$ truncation of the full higher genus hierarchy; here F and G denote respectively the genus 0 and 1 primary Gromov–Witten potentials. Dubrovin–Zhang show [31,32] that the associated τ-function satisfies the genus one topological recursion relations, and it restricts (by construction) on the small phase space to the primary Gromov–Witten potential.

As all the ingredients in (3.43) are explicitly known by localization in our case, the proof of Conjecture 3.2 at order $O(\epsilon^2)$ becomes practically feasible. Our strategy to prove it can be structured in the following four steps.

Step 1: Compute the deformation of the Poisson structure and the Hamiltonian densities on the phase space of the Principal Hierarchy from the quasi-Miura transformation (3.43).

Step 2: Compute the reduction of the second Poisson structure for the 2D-Toda lattice on the phase space of the (a, b) RR2T, from (2.71), and the associated Toda Hamiltonian densities, from (2.75). Interpolate and expand in the lattice spacing to $O(\epsilon^2)$.

Step 3: Find a family of Miura transformations matching the deformed Poisson tensors of Steps 1 and 2.

Step 4: Find a Miura group element such that the Hamiltonian densities agree after pullback, up to total derivatives.

[5] An alternative approach, which would lead to a proof of Conjecture 3.2 sidestepping the issue of the Hamiltonian structure, would be to derive the Hirota bilinear equations for the RR2T directly from Givental's formula—an approach successfully pioneered by Milanov and Tseng [51,52] for the extended bigraded Toda hierarchy. Unfortunately, the fact that we are dealing with the dual Frobenius structure hampers a straightforward generalization to the case at hand.

This method of proof can be automated for given (a, b) and verified symbolically; a parametric statement in (a, b) hinges on performing Step 1 (in particular the computation of (2.71) on the factorization locus) parametrically in these two variables. The relevant computer code is available upon request.

Remark 3.7. A priori there is no guarantee that a Miura group element satisfying Steps 3–4 exists. However, solutions to Step 3 are guaranteed to exist by the vanishing of the loop space Poisson cohomology in degree 1 and 2, as soon as $\mathcal{M}$ has trivial topology [24, 32, 37]: in this case there are Miura group elements $(\mathscr{F}_{\mathrm{RR2T}}, \mathscr{F}_{\mathrm{GW}})$ such that the deformed Poisson brackets are trivialized to their $\epsilon = 0$ limit,

$$\mathscr{F}^*_{\mathrm{RR2T}}\{\,,\,\}^{[\epsilon]}_{\mathrm{RR2T}} = \{\,,\,\}^{[0]} = \mathscr{F}^*_{\mathrm{GW}}\{\,,\,\}^{[\epsilon]}_{\mathrm{GW}}, \tag{3.45}$$

to all orders in ϵ. Furthermore, such Miura group elements are far from unique: for any formal ϵ-series K with values in graded-homogeneous differential polynomials,

$$K = \sum_{g\geq 0} \epsilon^g K_{[g]}(\tau, \dots, \tau^{(g)}), \qquad \deg K_{[g]} = g, \tag{3.46}$$

composing $\mathscr{F}^*_{\mathrm{RR2T}}$, $\mathscr{F}^*_{\mathrm{GW}}$ on the left with the time-ϵ canonical transformation,

$$\tau^\alpha \mapsto \tau^\alpha + \sum_{g>0} \frac{\epsilon^g}{g!} \overbrace{\Big\{K, \big\{K, \dots, \{K, \tau^\alpha\}^{[0]}\big\}^{[0]}\Big\}^{[0]}}^{g \text{ times}} \tag{3.47}$$

leaves $\{\,,\,\}^{[0]}$ invariant to all orders in ϵ. Proving Step 4 amounts then to showing that there exists (at least) one such K up to $O(\epsilon^2)$ such that the Toda-deformed Hamiltonians pull back to the GW-deformed ones under composition.

Remark 3.8. In fact, when it comes to Step 4 it is sufficient to show that the two deformations agree on a *single* Hamiltonian $\bar{H}^{[\epsilon]}$. Once this is done, the involutivity condition with the perturbed Hamiltonian,

$$\{\bar{H}^{[\epsilon]}, H^{[\epsilon]}_{\alpha,p}\} = 0, \tag{3.48}$$

admits, order by order in ϵ, a unique solution for the dispersive deformation of the Hamiltonian densities in Section 3.4.1 [30]. The simplest choice is to pick $\bar{H}^{[0]}$ to be the dispersionless limit of the Toda Hamiltonian given by $\mathrm{Tr}\, L_1$, i.e.,

$$\bar{H}^{[0]} = \int_{S^1} \mathrm{Res}_{z=0}\, \lambda(z)\, \frac{\mathrm{d}z}{z}, \tag{3.49}$$

with the RR2T and GW perturbations computed from (2.16), (2.17) and (3.43) respectively.

Remark 3.9. A further simplification in the computations comes from the fact that it is sufficient to prove Conjecture 3.2 for the genus one deformation of the Principal Hierarchy with $G = 0$; switching G—the elliptic GW potential—to an arbitrary function on the small phase space amounts to composing the result with an explicit, polynomial Miura group element. This simplifies the proof of Conjecture 3.2 considerably.

3.4.3. Step 1. We start with the following technical

Lemma 3.10 ([31]). *The genus* 1 *topological deformation of the Principal Hierarchy associated to a semi-simple Frobenius manifold with potential F, flat coordinates $\tau^1, \dots, \tau^N$ and flat metric η is Miura equivalent, up to higher genera, to the following deformation of the Poisson structure:*

$$\{\tau^\alpha(x), \tau^\beta(y)\}^{[\epsilon]}_{\mathrm{GW}} = \eta^{\alpha\beta}\delta'(x-y) + \frac{\epsilon^2}{24}\big(c^{\alpha\beta\mu}_\mu(\tau(x)) + c^{\alpha\beta\mu}_\mu(\tau(y))\big)\delta'''(x-y) - \frac{\epsilon^2}{24}\big(\partial_x(c^{\alpha\beta\mu}_\mu(\tau(x))) + \partial_y(c^{\alpha\beta\mu}_\mu(\tau(y)))\big)\delta'(x-y) + O(\epsilon^4) \tag{3.50}$$

and to Hamiltonian densities

$$h^{\mathrm{GW}}_{\beta,p} = h^{[0]}_{\beta,p} + \frac{\epsilon^2}{24}\left(\frac{\partial h^{[0]}_{\beta,p-1}}{\partial u^\zeta}(c^\zeta_{\nu\gamma}c^{\mu\nu}_{\alpha\mu} - c^\zeta_{\mu\nu\alpha}c^{\mu\nu}_\gamma) - \frac{\partial h^{[0]}_{\beta,p-2}}{\partial u^\zeta}c^{\zeta}_{\delta\sigma}c^{\sigma\mu}_\mu c^\delta_{\alpha\gamma}\right)\tau^\alpha_x\tau^\gamma_x + O(\epsilon^4), \tag{3.51}$$

where $c_{\alpha\beta\gamma}$ and $c_{\alpha\beta\gamma\delta}$ denote the third and fourth derivatives of F, respectively, and the indices are raised and lowered by η.

As per Remark 3.9, the Miura equivalence appearing in the above theorem is a change of coordinates of the form

$$\tilde\tau^\alpha = \tau^\alpha + \epsilon^2\big(A^\alpha_{\mu\nu}(\tau)\tau^\mu_x\tau^\nu_x + B^\alpha_\mu(\tau)\tau^\mu_{xx}\big) + O(\epsilon^4), \tag{3.52}$$

which can be explicitly computed in terms of the G-function of the Frobenius manifold.

Remark 3.11. Equation (3.51) expresses the dispersive deformation of the p^{th} Taylor coefficient of the canonically normalized flat sections of the Dubrovin connection for $Y_{a,b}$. However, by Remark 3.8, we will be mainly interested in deforming the dToda flow generated by the residue of the Lax symbol at infinity: since we are dealing with the second structure $\mathcal{M}_{a,b,v}$ on $\mathcal{M}$, this is equivalent to the twisted period around a Pochhammer loop encircling 1 and ∞ (see (3.30)), with the parameter ζ in (3.29) set equal to one. This little twist in the story amounts to resumming $\zeta^p h^{\mathrm{GW}}_{\beta,p}$ with respect to p in (3.51), and then evaluating the result at $\zeta = 1$, which gives

$$\overline{h}^{\mathrm{GW}} = \overline{h}^{[0]} + \frac{\epsilon^2}{24}\frac{\partial \overline{h}^{[0]}}{\partial \tau^\rho}(c^\rho_{\nu\gamma}c^{\mu\nu}_{\alpha\mu} - c^\rho_{\mu\nu\alpha}c^{\mu\nu}_\gamma - c^\rho_{\delta\sigma}c^{\sigma\mu}_\mu c^\delta_{\alpha\gamma})\tau^\alpha_x\tau^\gamma_x + O(\epsilon^4). \tag{3.53}$$

Example 3.12 ($(a,b,m) = (1,1,0)$). This is the case of the Ablowitz–Ladik hierarchy. Here, (3.24) and (3.50) together imply that the deformation of the Poisson bracket is trivial at $O(\epsilon^2)$,

$$\{\tau^\alpha(x), \tau^\beta(y)\}^{[\epsilon]}_{\mathrm{GW}} = -\mu_2^{-2}\delta^{\alpha+\beta,1}\delta'(x-y) + O(\epsilon^4), \tag{3.54}$$

whereas the first Hamiltonian density gets corrected as

$$\bar h^{\mathrm{GW}} = \mathrm{e}^{-\tau_0/\mu_2}(1-\mathrm{e}^{\tau_1}) + \frac{\epsilon^2\mathrm{e}^{\tau_1-\tau_0/\mu_2}}{24\mu_2(\mathrm{e}^{\tau_1}-1)}\big[2\mu_2\big((\mathrm{e}^{\tau_1}-1)(\tau_0'(x))^2 + \mathrm{e}^{\tau_1}(\tau_1'(x))^2\big) - \big(4(\mathrm{e}^{\tau_1}-1)\tau_0'(x) - \tau_1'(x)\big)\tau_1'(x)\big] + O(\epsilon^4). \tag{3.55}$$

Example 3.13 ($(a, b) = (1, 2, 0)$). In this case the dispersionless Poisson bracket does get corrected from (3.50). Setting $\mu_2 = 1$ for notational simplicity, we find

$$\{\tau^\alpha(x), \tau^\beta(y)\}^{[\epsilon]}_{\mathrm{GW}} = \{\tau^\alpha(x), \tau^\beta(y)\}^{[0]} + \epsilon^2 \mathcal{T}(\tau, \tau_x, \tau_{xx}) \begin{cases} 0, & \alpha = 0 \text{ or } \beta = 0, \\ 1, & \alpha = \beta = 1, \\ -2, & (\alpha, \beta) = (2, 1), (1, 2), \\ 4, & \alpha = \beta = 2, \end{cases} \tag{3.56}$$

with

$$\begin{aligned} \mathcal{T}(\tau, \tau_x, \tau_{xx}) = \frac{\mathrm{e}^{\tau_2(x)}}{12(\mathrm{e}^{\tau_2(x)} - 1)^4} \big[&(\mathrm{e}^{\tau_2(x)} - 1)\big(2\delta^{(3)}(x - y) + 3(\mathrm{e}^{\tau_2(x)} + 1)\tau_2'(x)\delta''(x - y)\big) \\ &+ \big((4\mathrm{e}^{\tau_2(x)} + \mathrm{e}^{2\tau_2(x)} + 1)\tau_2'(x)^2 - (\mathrm{e}^{2\tau_2(x)} - 1)\tau_2''(x)\big)\delta'(x - y)\big]. \end{aligned} \tag{3.57}$$

The first dToda density reads here

$$\bar{h}^{[0]} = \mathrm{e}^{\tau_0(x)}(\mathrm{e}^{\tau_1(x)} + \mathrm{e}^{\tau_1(x)+\tau_2(x)} - 1), \tag{3.58}$$

and its full $O(\epsilon^2)$ GW-deformation can be read off from (3.53).

3.4.4. Step 2. This step is a straightforward application of ϵ-interpolation to (2.71) and (2.75). For the sake of readability, we exemplify it in the two instances considered above.

Example 3.14 ($(a, b, m) = (1, 1, 0)$). As opposed to the GW-deformation, the RR2T-deformed Poisson bracket receives in this case corrections to all (even and odd) orders in ϵ, as is apparent from (2.74). The continuous interpolation leads to

$$\{\alpha(x), \alpha(y)\}^{[\epsilon]}_{\mathrm{RR2T}} = 0, \tag{3.59}$$

$$\begin{aligned} \{\log\alpha(x), \log\beta(y)\}^{[\epsilon]}_{\mathrm{RR2T}} &= \epsilon^{-1}\big(\delta(x - y + \epsilon) - \delta(x - y)\big) \\ &= \delta'(x - y) + \tfrac{1}{2}\epsilon\delta''(x - y) + \tfrac{1}{6}\epsilon^2 + O(\epsilon^3), \end{aligned} \tag{3.60}$$

$$\begin{aligned} \{\log\beta(x), \log\beta(y)\}^{[\epsilon]}_{\mathrm{RR2T}} &= \epsilon^{-1}\big(\delta(x - y + \epsilon) - \delta(x - y - \epsilon)\big) \\ &= 2\delta'(x - y) + \tfrac{1}{3}\epsilon^2\delta'''(x - y) + O(\epsilon^4). \end{aligned} \tag{3.61}$$

In the same vein, (2.75) for $i = 1$ gives

$$\begin{aligned} \bar{h}^{\mathrm{RR2T}} = \alpha(x) - \beta(x + \epsilon) &= \alpha(x) - \beta(x) + \epsilon\beta'(x) - \tfrac{1}{2}\epsilon^2\beta''(x) + O(\epsilon^3), \\ &= \alpha(x) - \beta(x) + (\text{total derivative}). \end{aligned} \tag{3.62}$$

Example 3.15 ($(a,b,m) = (1,2,0)$). Equation (2.71) computes the full-dispersive Poisson bracket on the factorization locus as

$$\begin{aligned}
\{\alpha_1(x),\alpha_1(y)\}^{[\epsilon]}_{\mathrm{RR2T}} &= 0,\\
\epsilon\{\alpha_1(x),\beta_1(y)\}^{[\epsilon]}_{\mathrm{RR2T}} &= \beta_1(y)\big(\alpha_1(y-\epsilon)\delta(x-y+\epsilon)-\alpha_1(y)\delta(x-y)\big),\\
\epsilon\{\alpha_1(x),\beta_2(y)\}^{[\epsilon]}_{\mathrm{RR2T}} &= \beta_2(y)\big(\alpha_1(y-2\epsilon)\delta(x-y+2\epsilon)-\alpha_1(y)\delta(x-y)\big),\\
\epsilon\{\beta_1(x),\beta_1(y)\}^{[\epsilon]}_{\mathrm{RR2T}} &= \big(\beta_2(y+\epsilon)-\beta_1(y)\beta_1(y+\epsilon)\big)\delta(x-y-\epsilon)\\
&\quad+\big(\beta_1(y)\beta_1(y-\epsilon)-\beta_2(y)\big)\delta(x-y+\epsilon),\\
\epsilon\{\beta_1(x),\beta_2(y)\}^{[\epsilon]}_{\mathrm{RR2T}} &= \beta_2(y)\big(\beta_1(y-2\epsilon)\delta(x-y+2\epsilon)-\alpha_2(y+\epsilon)\delta(x-y-\epsilon)\big),\\
\epsilon\{\beta_2(x),\beta_2(y)\}^{[\epsilon]}_{\mathrm{RR2T}} &= \beta_2(y)\big[(-\beta_2(y+2\epsilon)\delta(x-y-2\epsilon)-\beta_2(y+\epsilon)\delta(x-y-\epsilon)\\
&\quad+\alpha_3(y-\epsilon)\delta(x-y+\epsilon)+\beta_2(y-2\epsilon)\delta(x-y+2\epsilon)\big].
\end{aligned}\tag{3.63}$$

It should be noticed that the Poisson bracket is not logarithmically constant in these coordinates. The full-dispersive deformation is given by Taylor-expanding the r.h.s. in ϵ. As before, the full-dispersive first Hamiltonian is here given as

$$\bar{h}^{\mathrm{RR2T}} = \alpha_1(x) - \beta_1(x+\epsilon) = \alpha_1(x) - \beta_1(x) + (\text{total derivative}). \tag{3.64}$$

3.4.5. Step 3. The next step is to match the Poisson structures $\{\,,\,\}^{[\epsilon]}_{\mathrm{GW}}$ and $\{\,,\,\}^{[\epsilon]}_{\mathrm{RR2T}}$ computed in Steps 1–2. We will do this by explicitly computing the trivializing polynomial Miura transformation that transforms them back to their undeformed expression. We start from the GW-deformation.

Lemma 3.16. *The Miura transformation*

$$\tau^\alpha \mapsto \tau^\alpha - \tfrac{1}{24}\epsilon^2(\partial_x^2 c^{\alpha\mu}_{\mu}) + O(\epsilon^4) \tag{3.65}$$

transforms the Poisson bracket (3.50) *to*

$$\{\tau^\alpha(x),\tau^\beta(y)\} = \eta^{\alpha\beta}\delta'(x-y) + O(\epsilon^4) \tag{3.66}$$

and the Hamiltonian densities (3.51) *to*

$$h_{\beta,p} = h^{[0]}_{\beta,p} - \tfrac{1}{24}\epsilon^2\left(\frac{\partial h^{[0]}_{\beta,p-1}}{\partial\tau^\zeta}c^\zeta_{\mu\nu\alpha}c^{\mu\nu}_\gamma + \frac{\partial h^{[0]}_{\beta,p-2}}{\partial\tau^\zeta}c^\zeta_{\delta\sigma}c^{\sigma\mu}_{\mu}c^\delta_{\alpha\gamma}\right)\tau^\alpha_x\tau^\gamma_x + O(\epsilon^4). \tag{3.67}$$

Proof. The proof is an immediate consequence of the formula $\tilde{P}^{\alpha\beta} = (L^*)^\alpha_\mu \circ P^{\mu\nu}\circ L^\beta_\nu$ for the transformation of the differential operator $P^{\alpha\beta}$ associated to the Poisson bracket, where $(L^*)^\alpha_\mu = \sum_{s\geq 0}\frac{\partial\tilde{\tau}^\alpha}{\partial\tau^\mu_s}\partial_x^s$ and $L^\beta_\nu = \sum_{s\geq 0}(-\partial_x)^s\circ\frac{\partial\tilde{\tau}^\beta}{\partial\tau^\nu_s}$, with $\tau^\alpha_s = \partial_x^s\tau^\alpha$. For the Hamiltonians one simply evaluates the functions at the shifted values and performs Taylor's expansion. □

One by-product of the lemma is that the expression for the deformed Hamiltonian densities simplifies as well in this Miura deformed coordinates. On the RR2T side, we act in the same way—by plugging an arbitrary Miura transformation that trivializes $\{\,,\,\}^{[\epsilon]}_{\mathrm{RR2T}}$ to $O(\epsilon^2)$ and solving the ensuing overconstrained differential system.

Example 3.17 ($(a, b, m) = (1, 2, 0)$). In this case, a trivialization of the RR2T-deformed Poisson bracket reads, at $O(\epsilon^2)$,

$$\alpha_1(x) \mapsto \alpha_1(x) - \frac{1}{2}\epsilon\alpha_1'(x) + \epsilon^2\left(\frac{5}{24}\alpha_1''(x) - \frac{\alpha_1'(x)^2}{12\alpha_1(x)}\right) + O(\epsilon^3), \tag{3.68}$$

$$\begin{aligned}\beta_1(x) \mapsto\ & \beta_1(x) + \frac{\epsilon^2}{24(\beta_1(x)^2 - 4\beta_2(x))^2\beta_2(x)^2}\big[2\beta_1(x)^5\beta_2'(x)^2 - \beta_2(x)^2\beta_1(x)^3\beta_1'(x)^2 \\ & -14\beta_2(x)\beta_1(x)^3\beta_2'(x)^2 + 16\beta_2(x)^2\beta_1(x)^2\beta_1'(x)\beta_2'(x) - 20\beta_2(x)^3\beta_1(x)\beta_1'(x)^2 \\ & + 32\beta_2(x)^3\beta_1'(x)\beta_2'(x) - 2\beta_2(x)\beta_1(x)^5\beta_2''(x) + \beta_2(x)^2\beta_1(x)^4\beta_1''(x) \\ & [2pt] + 10\beta_2(x)^2\beta_1(x)^3\beta_2''(x) + 4\beta_2(x)^3\beta_1(x)^2\beta_1''(x) - 8\beta_2(x)^3\beta_1(x)\beta_2''(x) \\ & - 32\beta_2(x)^4\beta_1''(x)\big] + O(\epsilon^3),\end{aligned} \tag{3.69}$$

$$\beta_2(x) \mapsto \beta_2(x) + \frac{1}{2}\epsilon\beta_2'(x) + \epsilon^2\left(\frac{\beta_2'(x)^2}{4\beta_2(x)} - \frac{1}{8}\beta_2''(x)\right) + O(\epsilon^3), \tag{3.70}$$

so that in the new variables we have

$$\begin{aligned}\{\alpha_1(x), \alpha_1(y)\}^{[0]} &= 0,\\ \{\alpha_1(x), \beta_1(y)\}^{[0]} &= \alpha_1(x)\delta(x-y)\beta_1'(x) + \alpha_1(x)\beta_1(x)\delta'(x-y),\\ \{\alpha_1(x), \beta_2(y)\}^{[0]} &= 2\alpha_1(x)\delta(x-y)\beta_2'(x) + 2\alpha_1(x)\beta_2(x)\delta'(x-y),\\ \{\beta_1(x), \beta_1(y)\}^{[0]} &= (2\beta_1(x)\beta_1'(x) - \beta_2'(x))\delta(x-y)\\ &\quad + 2(\beta_1(x)^2 - \beta_2(x))\delta'(x-y),\\ \{\beta_1(x), \beta_2(y)\}^{[0]} &= 3\beta_1(x)\delta(x-y)\beta_2'(x) + 3\beta_1(x)\beta_2(x)\delta'(x-y),\\ \{\beta_2(x), \beta_2(y)\}^{[0]} &= 6\beta_2(x)\delta(x-y)\beta_2'(x) + 6\beta_2(x)^2\delta'(x-y).\end{aligned} \tag{3.71}$$

Relating now (α, β) to q as in (2.91) and composing with (3.26) to go to τ-variables returns $\{\,,\,\}^{[0]} = \{\,,\,\}_\eta$, the dispersionless Poisson bracket in flat coordinates for the metric $\eta^{(2)}$ of $\mathcal{M}^{(2)}_{1,2,0}$. The general Miura group element trivializing $\{\,,\,\}^{\mathrm{RR2T}}$ is obtained by composing (3.71) with (3.47), for an arbitrary K.

3.4.6. Step 4. All that is left to do at this stage is to find a canonical generator K such that $\bar{h}^{\mathrm{GW}}$ matches $\bar{h}^{\mathrm{RR2T}}$ in the resulting trivializing coordinate system, up to total derivatives. The quickest way to do this is as follows: choose K such that the transformed $\bar{h}^{\mathrm{RR2T}}$ has 1) no linear term in ϵ and 2) no linear terms in τ^α_{xx} at $O(\epsilon^2)$: this amounts to the solution of two inhomogeneous linear systems of rank $a + b$. Then compose $\bar{h}^{\mathrm{RR2T}}$ in the resulting coordinate system with a further canonical transformation generated by a differential polynomial $\tilde{K}$, with vanishing linear term in ϵ. Now imposing that the difference of the transformed $\bar{h}^{\mathrm{RR2T}}$ with h^{GW} is a total derivative is equivalent to a rank $\binom{a+b+1}{2}$ linear system in the derivatives of the components of $\tilde{K}$; checking compatibility of the solution then concludes the proof.

Example 3.18 ($(a,b,m)=(1,2,0)$). Let us see this explicitly at work in the case when $(a,b,m)=(1,2,0)$. The GW- and RR2T-deformed Hamiltonian density in the coordinates for which the Poisson is in standard form (3.67) read here

$$\begin{aligned}
\bar{h}^{\mathrm{GW}} &= \mathrm{e}^{\tau_0}(\mathrm{e}^{\tau_1}+\mathrm{e}^{\tau_1+\tau_2}-1) - \frac{\epsilon^2}{24(\mathrm{e}^{\tau_1}-1)(\mathrm{e}^{\tau_2}-1)^2(\mathrm{e}^{\tau_1+\tau_2}-1)} \\
&\quad\times\big[\mathrm{e}^{\tau_0+\tau_1+\tau_2}(\mathrm{e}^{\tau_1}-1)(3\mathrm{e}^{\tau_2}-\mathrm{e}^{2\tau_2}+5\mathrm{e}^{\tau_1+\tau_2}-5\mathrm{e}^{\tau_1+2\tau_2}+2\mathrm{e}^{\tau_1+3\tau_2}-4)(\tau_2')^2 \\
&\quad-2(\mathrm{e}^{\tau_2}-1)(\mathrm{e}^{\tau_1+\tau_2}-1)\tau_2'') + 2(\mathrm{e}^{\tau_1}-1)(\mathrm{e}^{\tau_2}-1)^2(\mathrm{e}^{\tau_2}+1)(\mathrm{e}^{\tau_1+\tau_2}-1)(\tau_0')^2 \\
&\quad+4(\mathrm{e}^{\tau_1}-1)(\mathrm{e}^{\tau_2}-1)(\mathrm{e}^{\tau_1+\tau_2}-1)((\mathrm{e}^{2\tau_2}-1)\tau_1'+\mathrm{e}^{\tau_2}(\mathrm{e}^{\tau_2}-2)\tau_2')\tau_0' \\
&\quad+(\mathrm{e}^{\tau_2}-1)^2(-2\mathrm{e}^{\tau_1}+\mathrm{e}^{\tau_2}-2\mathrm{e}^{\tau_1+\tau_2}+2\mathrm{e}^{2(\tau_1+\tau_2)}+2\mathrm{e}^{2\tau_1+\tau_2}-2\mathrm{e}^{\tau_1+2\tau_2}+1)(\tau_1')^2 \\
&\quad+2\mathrm{e}^{\tau_2}(\mathrm{e}^{\tau_1}-1)(\mathrm{e}^{\tau_2}-1)(-\mathrm{e}^{\tau_2}-4\mathrm{e}^{\tau_1+\tau_2}+2\mathrm{e}^{\tau_1+2\tau_2}+3)\tau_1'\tau_2'\big]+O(\epsilon^4), \qquad (3.72)\\
\bar{h}^{\mathrm{Toda}} &= \mathrm{e}^{\tau_0}(\mathrm{e}^{\tau_1}+\mathrm{e}^{\tau_1+\tau_2}-1)+\tfrac{\epsilon}{2}\mathrm{e}^{\tau_0}\big((3\mathrm{e}^{\tau_1}+3\mathrm{e}^{\tau_1+\tau_2}-2)\tau_0'+3\mathrm{e}^{\tau_1}((\mathrm{e}^{\tau_2}+1)\tau_1'+\mathrm{e}^{\tau_2}\tau_2')\big) \\
&\quad+\frac{\epsilon^2\mathrm{e}^{\tau_0}}{24(\mathrm{e}^{\tau_2}-1)^2}\big[(\mathrm{e}^{\tau_2}-1)^2\big(3(9\mathrm{e}^{\tau_1}+9\mathrm{e}^{\tau_1+\tau_2}-4)(\tau_0')^2+27\mathrm{e}^{\tau_1}(\mathrm{e}^{\tau_2}+1)(\tau_1')^2\big) \\
&\quad+54\mathrm{e}^{\tau_1+\tau_2}(\mathrm{e}^{\tau_2}-1)^2\tau_1'\tau_2'+54\mathrm{e}^{\tau_1}(\mathrm{e}^{\tau_2}-1)^2\tau_0'((\mathrm{e}^{\tau_2}+1)\tau_1'+\mathrm{e}^{\tau_2}\tau_2')+30\mathrm{e}^{\tau_1+\tau_2}(\tau_2')^2 \\
&\quad-51\mathrm{e}^{\tau_1+2\tau_2}(\tau_2')^2+27\mathrm{e}^{\tau_1+3\tau_2}(\tau_2')^2+10(3\mathrm{e}^{\tau_1}+2\mathrm{e}^{\tau_2}-\mathrm{e}^{2\tau_2}-3\mathrm{e}^{\tau_1+\tau_2}-3\mathrm{e}^{\tau_1+2\tau_2} \\
&\quad+3\mathrm{e}^{\tau_1+3\tau_2}-1)\tau_0''+10(3\mathrm{e}^{\tau_1}\tau_1''-3\mathrm{e}^{\tau_1+\tau_2}\tau_1''-3\mathrm{e}^{\tau_1+2\tau_2}\tau_1''+3\mathrm{e}^{\tau_1+3\tau_2})\tau_1'' \\
&\quad+(2\mathrm{e}^{\tau_1}+30\mathrm{e}^{\tau_1+\tau_2}-60\mathrm{e}^{\tau_1+2\tau_2}+28\mathrm{e}^{\tau_1+3\tau_2})\tau_2''\big]+O(\epsilon^3). \qquad (3.73)
\end{aligned}$$

Let us first get rid of the linear term in ϵ in $\bar{h}^{\mathrm{RR2T}}$, as well as of the terms linear in the second derivatives. This is accomplished by an $O(\epsilon^2)$ transformation generated by

$$\begin{aligned}
K_1 &= (-\tau_0-\tau_1-\tau_2/2)-\frac{\epsilon\mathrm{e}^{-\tau_2}}{24(\mathrm{e}^{\tau_2}-1)}\big[(-26\mathrm{e}^{\tau_2}+26\mathrm{e}^{2\tau_2}+\mathrm{e}^{3\tau_2}-1)\tau_0' \\
&\quad-10(\mathrm{e}^{\tau_2}-\mathrm{e}^{2\tau_2})\tau_1'+(-8\mathrm{e}^{\tau_2}+2\mathrm{e}^{2\tau_2}-\mathrm{e}^{3\tau_2}-1)\tau_2'\big]+O(\epsilon^2). \qquad (3.74)
\end{aligned}$$

Composing this with a canonical transformation generated by

$$K_2=\epsilon\big(K_2^{(0)}(\tau)\tau_x^0+K_2^{(1)}(\tau)\tau_x^1+K_2^{(2)}(\tau)\tau_x^2\big) \qquad (3.75)$$

such that $h^{\mathrm{RR2T}}-h^{\mathrm{GW}}=\partial_x f$ gives a system of six linear equations in the τ-derivatives of $K_2^{(i)}$, which is solved by

$$\begin{aligned}
\partial_{\tau^0}K_2^{(1)} &= \partial_{\tau^1}K_2^{(0)}-\frac{\mathrm{e}^{\tau_1}+\mathrm{e}^{\tau_1+\tau_2}-2}{24(\mathrm{e}^{\tau_1}-1)(\mathrm{e}^{\tau_1+\tau_2}-1)}, \\
\partial_{\tau^0}K_2^{(2)} &= \partial_{\tau^2}K_2^{(0)}-\frac{\mathrm{e}^{-\tau_2}(-\mathrm{e}^{\tau_2}+\mathrm{e}^{2\tau_2}-\mathrm{e}^{3\tau_2}+\mathrm{e}^{\tau_1+\tau_2}+\mathrm{e}^{\tau_1+4\tau_2}-1)}{24(\mathrm{e}^{\tau_2}-1)(\mathrm{e}^{\tau_1+\tau_2}-1)}, \qquad (3.76)\\
\partial_{\tau^1}K_2^{(2)} &= \partial_{\tau^2}K_2^{(1)}-\frac{\mathrm{e}^{\tau_1+\tau_2}(\mathrm{e}^{\tau_1}+\mathrm{e}^{\tau_1+\tau_2}-2)}{24(\mathrm{e}^{\tau_1}-1)(\mathrm{e}^{\tau_2}-1)(\mathrm{e}^{\tau_1+\tau_2}-1)},
\end{aligned}$$

which is immediately shown to be compatible.

3.5. *Further applications*

Proposition 3.6 has a number of applications to the study of the gravitational quantum cohomology of $Y_{a,b}$ as well as of its higher genus Gromov–Witten theory. When $b = 0$, these were explored in detail in [10]; we highlight below the main features of their generalization to arbitrary (a, b).

3.5.1. Twisted periods and the J-function. A distinguished basis of flat coordinates for the Dubrovin connection is given by the generating function of genus zero one-point descendent invariants of $Y_{a,b}$, or *J-function* [40],

$$J^{\alpha}_{Y_{a,b}}(\tau,\zeta) \triangleq \frac{\delta^{\alpha,0}}{\zeta} + \tau^{\alpha} + \zeta \sum_{n\geq 0}\sum_{\beta\in\mathbb{Z}} \frac{1}{n!}\left\langle \frac{\omega_{\alpha}}{1-\zeta\psi}, \tau^{\otimes n}\right\rangle^{Y_{a,b}}_{0,n+1,\beta}, \tag{3.77}$$

where ψ is a cotangent line class and the denominator is a formal geometric series expansion in $\zeta\psi$. Write $i_j : p_j \hookrightarrow Y_{a,b}$ for the embedding of the j^{th} fixed point into $Y_{a,b}$, and define $u^{\text{cl}}_j \mathbf{1}_j \triangleq i_j^*(\tau^{\alpha}\omega_{\alpha}) \in H_T(\{p_j\})$. The coefficients u^{cl}_j are linear functions $u^{\text{cl}}_j(\tau) = \sum c_{j\alpha}\tau^{\alpha}$, and they are canonical coordinates for the classical equivariant cohomology algebra of $Y_{a,b}$. By the Divisor Axiom of Gromov–Witten theory, the coefficients $c_{j\alpha} = \omega_{\alpha}|_{p_j}$ are local exponents of (2.85) at the Fuchsian point $\text{LR} = \{\tau_{\alpha} = -\infty\}$, and the vector of the localized components $J^{j}_{Y_{a,b}}\mathbf{1}_j = i_j^*(J^{\alpha}_{Y_{a,b}}\omega_{\alpha})$ of the J-function diagonalizes the monodromy around LR with weights $c_{j\alpha} \in \mathbb{C}$,

$$J^{j}_{Y_{a,b}} \sim \zeta \mathrm{e}^{\zeta u^{\text{cl}}_j}(1 + O(\mathrm{e}^{\tau})). \tag{3.78}$$

The asymptotic behavior (3.78) at LR characterizes uniquely the localized components of the J-function as a flat coordinate system for (2.85). Knowledge of the monodromy properties of the twisted periods of (3.36) and (3.37) at LR is then sufficient, by Proposition 3.6, to give a closed form expression for the J-functions as a hypergeometric function in exponentiated flat variables. This can be achieved via an iterated use of the connection formula at infinity for the Gauss function, as explained in [10, Appendix C]. In vector notation, the final result in our case is

$$J_{Y_{a,b}} = (\mathcal{A}^{(a)} \oplus \mathcal{A}^{(b)})\Pi, \tag{3.79}$$

where

$$\mathcal{A}^{(a)}_{ij} = \begin{cases} \mathrm{e}^{\pi \mathrm{i} i\zeta}\dfrac{\zeta\Gamma(1+\zeta\nu-(i+1)\zeta)}{\Gamma(1-\zeta)\Gamma(\zeta\nu-i\zeta)}, & i = -j, \\ \mathrm{e}^{-\mathrm{i}\pi(\zeta\nu-\zeta(2j+1))}\dfrac{\zeta\sin(\pi\zeta)\Gamma(1-\zeta\nu+\zeta i)\Gamma(1+\zeta\nu-\zeta(i+1))}{\pi\Gamma(1-\zeta)}, & i > -j, \\ 0, & i < -j. \end{cases} \tag{3.80}$$

$$\mathcal{A}_{ij}^{(b)} = \begin{cases} \mathrm{e}^{-\pi \mathrm{i} j \zeta} \dfrac{\zeta \Gamma(1+\xi+(a+j+1)\zeta)}{\Gamma(1+\zeta)\Gamma(\xi+(j+a)\zeta)}, & i = j+a, \\ \mathrm{e}^{-\mathrm{i}\pi(\xi+\zeta(2j+1+a))} \dfrac{\zeta \sin(\pi\zeta)\Gamma(1-\xi-\zeta(j+a))\Gamma(1+\xi+\zeta(a+j+1))}{\pi\Gamma(1+\zeta)}, & j > i+a, \\ 0, & j < i+a. \end{cases} \tag{3.81}$$

In the same vein, let $H_{\mathrm{orb},T}(X_{a,b})$ denote the T-equivariant Chen–Ruan cohomology of $[X_{a,b}]$. This has two torus fixed orbi-points $[p^{(a)}]$ and $[p^{(b)}]$—the North and South pole of the base weighted projective line—with stackiness $\mathbb{Z}_a$ and $\mathbb{Z}_b$ respectively. By the Atiyah–Bott isomorphism, $H_{\mathrm{orb},T}(X_{a,b})$ is then generated by the (Thom push-forwards) of $\mathbf{1}_{i/a}, \mathbf{1}_{j/b}, \alpha = 0, \dots, a-1, \beta = 0, \dots, b-1$. Write $x \triangleq \sum_{c\in\{a,b\},\, \alpha\in\mathbb{Z}_c} x^{\alpha,c}\mathbf{1}_{\alpha/c}$ for a point $x \in H_{\mathrm{orb},T}(X_{a,b})$. Then the orbifold J-function

$$J_{X_{a,b}}^{\gamma,c}(x,\zeta) \triangleq \frac{\delta^{\alpha,0}}{\zeta} + x^{\alpha} + \zeta \sum_{n\geq 0}\sum_{\beta\in\mathbb{Z}} \frac{1}{n!}\left\langle \frac{\mathbf{1}_{\gamma/c}}{1-\zeta\psi}, x^{\otimes n}\right\rangle_{0,n+1,\beta}^{X_{a,b}} \tag{3.82}$$

gives a system of flat coordinates for the Dubrovin connection on $T(H_{\mathrm{orb},T}(X_{a,b}))$. As Frobenius manifolds, the quantum cohomologies of $X_{a,b}$ and $Y_{a,b}$ are isomorphic [21], with the undeformed flat coordinates related as [12]

$$x^{\alpha,a} = \sum_{\gamma=0}^{a-1} \varepsilon_a^{\alpha\gamma}\tau^{a-1-\gamma} - \varepsilon_a^{\alpha}, \qquad x^{\beta,b} = \sum_{\beta=0}^{b-1} \varepsilon_b^{\beta\gamma}\tau^{\gamma+a+1} - \varepsilon_b^{\beta}, \tag{3.83}$$

where $\varepsilon_c \triangleq \mathrm{e}^{2\pi\mathrm{i}/c}$, and by the Divisor Axiom the localized components of $J^{X_{a,b}}$ are the unique set of flat coordinates of the Dubrovin connection such that

$$J_{X_{a,b}}^{\gamma,c}(x,\zeta) \simeq \zeta \mathrm{e}^{\zeta x^{0,c}}(1+O(x)). \tag{3.84}$$

This can be compared with the behavior of the twisted periods at $x = 0$, where the integrals appearing in (3.34)–(3.37) can be explicitly evaluated in terms of the Euler Beta function. The result is

$$\Pi = (\mathcal{B}^{(a)} \oplus \mathcal{B}^{(b)}) J_{\mathcal{X}_{a,b}}, \tag{3.85}$$

where

$$\mathcal{B}_{jk}^{(a)} = \varepsilon_a^{(j-n/2)\zeta\nu} \frac{\varepsilon_a^{-jk}}{a} \begin{cases} -\varepsilon_a^{k/2} \dfrac{\Gamma((\zeta\nu+k)/a)\Gamma(1-\zeta)}{\Gamma((\zeta\nu+k)/a-\zeta)} & \text{for } 1 \leq k \leq a-1, \\ \dfrac{\Gamma(\zeta\nu/a)\Gamma(1-\zeta)}{\zeta\Gamma(1+\zeta(1+\nu/a))} & \text{for } k = 0, \end{cases} \tag{3.86}$$

$$\mathcal{B}_{jk}^{(b)} = \varepsilon_b^{(j-n/2)(\xi+a)} \frac{\varepsilon_b^{-jk}}{b} \begin{cases} -\varepsilon_b^{k/2} \dfrac{\Gamma((\xi+a-k)/b+1)\Gamma(1+\zeta)}{\Gamma((\xi+a-k)/b-\zeta+1)} & \text{for } 1 \leq k \leq b-1, \\ \dfrac{\Gamma((a+\xi)/b)\Gamma(1+\zeta)}{\zeta\Gamma(1+(\xi+a)/b-\zeta)} & \text{for } k = 0. \end{cases} \tag{3.87}$$

The composition $\mathcal{U} \triangleq \mathcal{A}^{(a)}\mathcal{B}^{(a)} \oplus \mathcal{A}^{(b)}\mathcal{B}^{(b)}$ gives the transition matrix from the vector form of the orbifold J-function to the one of the resolution. Closed-form knowledge of $\mathcal{U}$

has important applications to the Crepant Resolution Conjecture [22], as well as to its generalization to open Gromov–Witten theory [10]; in particular, by the block diagonal form of (3.80), (3.81), (3.86) and (3.87), the genus zero results in [10] generalize immediately to the case at hand. Similarly, Proposition 3.6 makes it an exercise in book-keeping to generalize to arbitrary (a, b) the quantized Crepant Resolution Conjecture proven in [10] for $b = 0$.

3.5.2. Pure braid group actions in quantum cohomology. A further application of Theorem 3.1 and Proposition 3.6 is a complete characterization of the monodromy group of the Dubrovin connection. By Theorems 2.23 and 3.1, the open set $\mathcal{M}^{(2),\mathrm{reg}}_{a,b,\nu} \triangleq \mathcal{M}^{(2)}_{a,b,\nu} \setminus \Delta_{a,b,\nu}$ of regular points for the pencil of flat connections of (2.85) on $\mathcal{M}^{(2)}_{a,b,\nu}$ is the complement of the arrangement of hyperplanes $\{q_i = q_j\}_{i\neq j}$. Equivalently, it is isomorphic to the configuration space of $a + b$ distinct points in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$,

$$\mathcal{M}^{(2),\mathrm{reg}}_{a,b,\nu} \simeq M_{0,a+b+3}. \tag{3.88}$$

Any simple loop σ in $\mathcal{M}^{(2),\mathrm{reg}}_{a,b,\nu}$ then gives a monodromy action on Sol_λ,

$$M_\sigma : \pi_1(\mathcal{M}^{(2),\mathrm{reg}}_{a,b,\nu}) \to \mathrm{Aut}(\mathrm{Sol}_\lambda), \tag{3.89}$$

which is a representation of the colored braid group in $a+b+2$ strands, as $\pi_1(M_{0,a+b+3}) \simeq \mathrm{PB}_{a+b+2}$. Monodromy matrices in the twisted period basis can be computed explicitly [53]; the resulting representation is the Gassner representation [25], with weights specified as in (2.91).

Remark 3.19. By the T-equivariant version of Iritani's integral structures in quantum cohomology, this pure braid group action carries through to an action on the T-equivariant K-groups of $X_{a,b}$ and $Y_{a,b}$. Very recently, pure braid group actions on the derived category of coherent sheaves were constructed in [26] for a family of toric Calabi–Yau obtained from deformations of resolutions of type A surface singularities; when the variety is a threefold, their examples coincide precisely with $Y_{a,b}$. It would be interesting to establish a clear link between our D-module construction and theirs.

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