J. Algebra Appl. (2023) 2350046.
https://arxiv.org/abs/2011.13639, https://doi.org/10.1142/S0219498823500469

Metrizability of spaces of valuation domains associated to pseudo-convergent sequences

G. Peruginelli*
D. Spirito†

November 23, 2021

Abstract

Let V be a valuation domain of rank one with quotient field K. We study the set of extensions of V to the field of rational functions $K(X)$ induced by pseudo-convergent sequences of K from a topological point of view, endowing this set either with the Zariski or with the constructible topology. In particular, we consider the two subspaces induced by sequences with a prescribed breadth or with a prescribed pseudo-limit. We give some necessary conditions for the Zariski space to be metrizable (under the constructible topology) in terms of the value group and the residue field of V.

Keywords: pseudo-convergent sequence, pseudo-limit, metrizable space, Zariski-Riemann space, constructible topology.

MSC Primary 12J20, 13A18, 13F30.

1 Introduction

Let D be an integral domain with quotient field K, and let L be a field extension of K. The *Zariski space* $\mathrm{Zar}(L|D)$ of L over D is the set of all valuation domains containing D and having L as quotient field. This set was originally studied by Zariski during its study of the problem of resolution of singularities [19, 20]; to this end, he introduced a topology (later called the *Zariski topology*) that makes $\mathrm{Zar}(L|D)$ into a compact space that is not Hausdorff [21, Chapter VI, Theorem 40].

*Dipartimento di Matematica "Tullio Levi-Civita", University of Padova, Via Trieste 63, 35121 Padova, Italy. E-mail: gperugin@math.unipd.it

†Dipartimento di Matematica "Tullio Levi-Civita", University of Padova, Via Trieste 63, 35121 Padova, Italy. E-mail: spirito@math.unipd.it

A second topology that can be considered on the Zariski space is the *constructible topology* (or *patch topology*), that can be constructed from the Zariski topology in the same way as it is constructed on the spectrum of a ring. The Zariski space $\mathrm{Zar}(L|D)$ endowed with the constructible topology, which we denote by $\mathrm{Zar}(L|D)^{\mathrm{cons}}$, is more well-behaved than the starting space $\mathrm{Zar}(L|D)$ with the Zariski topology, since beyond being compact it is also Hausdorff; furthermore, it keeps its link with the spectra of rings, in the sense that there is a ring A such that $\mathrm{Spec}(A)$ is homeomorphic to $\mathrm{Zar}(L|D)^{\mathrm{cons}}$ [7].

Suppose now that $D = V$ is a valuation domain. In this case, the study of $\mathrm{Zar}(L|V)$ often concentrates on the subset of the *extensions* of V to L, i.e., to the valuation domains $W \in \mathrm{Zar}(L|V)$ such that $W \cap K = V$. When $L = K(X)$ is the field of rational functions over K, there are several ways to construct extensions of V to $K(X)$, among which we can cite key polynomials [9, 17], monomial valuations, and minimal pairs [1, 2]. Another approach is by means of *pseudo-monotone sequences* and, in particular, *pseudo-convergent sequences*: the latter are a generalization of the concept of Cauchy sequences that were introduced by Ostrowski [10] and later used by Kaplansky to study immediate extensions and maximal valued fields [8]. Pseudo-monotone sequences were introduced by Chabert in [4] to describe the polynomial closure of subsets of rank one valuation domains. In particular, Ostrowski introduced pseudo-convergent sequences in order to describe all rank one extensions of a rank one valuation domain when the quotient field K of V is algebraically closed (*Ostrowski's Fundamentalsatz*, see [10, §11, IX, p. 378]); recently, the authors used pseudo-monotone sequences to extend Ostrowski's result to arbitrary rank when the completion $\widehat{K}$ of K with respect to the v-adic topology is algebraically closed [14, Theorem 6.2].

Motivated by these results, in this paper we are interested in the subspace $\mathcal{V}$ of $\mathrm{Zar}(K(X)|V)$ containing the extensions of V defined by pseudo-convergent sequences, under the hypothesis that V has rank 1 (see §2 for the definition of this kind of extensions). The study of $\mathcal{V}$ was started in [13], where it was shown that $\mathcal{V}$ is always a regular space (even under the Zariski topology) [13, Theorem 6.15] and that the Zariski and the constructible topology agree on $\mathcal{V}$ if and only if the residue field of V is finite [13, Proposition 6.11]. We continue the study of this space by concentrating on the problem of metrizability: more precisely, we are interested on conditions under which $\mathcal{V}$ and some distinguished subsets of $\mathcal{V}$ are metrizable. More generally, we look for conditions under which the whole Zariski space (endowed with the constructible topology) is metrizable. To do so, we consider two partitions of $\mathcal{V}$.

In Section 3, we study the spaces $\mathcal{V}(\bullet, \delta) \subset \mathcal{V}$ consisting of those extensions of V induced by pseudo-convergent sequences having the same (fixed) breadth $\delta \in \mathbb{R} \cup \{\infty\}$ (see Section 2 for the definition); this can be seen as a generalization of the study of valuation domains associated to elements of the completion of K tackled in [12], which in our notation reduces to the special case $\delta = \infty$. In particular, we show that $\mathcal{V}(\bullet, \delta)$ can be seen as a complete ultrametric space under a very natural distance function (Theorem 3.5) which induces both the Zariski and the constructible topology (that in particular coincide, see Proposition 3.4); however, these distances (as δ ranges in $\mathbb{R} \cup \{\infty\}$), cannot be unified into a metric encompassing all of $\mathcal{V}$ (Proposition 3.8).

In Section 4, we study the spaces $\mathcal{V}(\beta, \bullet) \subset \mathcal{V}$ consisting of those extensions of V induced by pseudo-convergent sequences having a (fixed) pseudo-limit $\beta \in \overline{K}$ (with respect to some prescribed extension of V to $\overline{K}$). We show that these spaces are closed, with respect to the Zariski topology (Proposition 4.2), and that the constructible and the Zariski topology agree on each $\mathcal{V}(\beta, \bullet)$ (Proposition 4.6); furthermore, we represent $\mathcal{V}(\beta, \bullet)$ through a variant of the upper limit topology (Theorem 4.4), and we show that it is metrizable if and only if the value group of V is countable (Proposition 4.7). As a consequence, we get that, when the value group of V is not countable, the space $\mathrm{Zar}(K(X)|V)^{\mathrm{cons}}$ is not metrizable (Corollary 4.8).

In Section 5, we look at the same partitions, but on the sets $\mathcal{V}_{\mathrm{div}}$ and $\mathcal{V}_{\mathrm{stat}}$ of extensions induced, respectively, by pseudo-divergent and pseudo-stationary sequences (the other type of pseudo-monotone sequences beyond the pseudo-convergent ones, see [4, 11, 14]). Using a quotient onto the space $\mathrm{Zar}(k(t)|k)$ (where k is the residue field of V) we first show that $\mathrm{Zar}(K(X)|V)^{\mathrm{cons}}$ is not metrizable if k is uncountable (Proposition 5.3); then, with a similar method, we show that $\mathcal{V}_{\mathrm{div}}(\bullet, \delta)$ is not Hausdorff (with respect to the Zariski topology) when δ belongs to the value group of V (Proposition 5.4). On the other hand, we show that fixing a pseudo-limit (i.e., considering $\mathcal{V}_{\mathrm{div}}(\beta, \bullet)$) we get a space homeomorphic to $\mathcal{V}(\beta, \bullet)$ (Proposition 5.5). For pseudo-stationary sequences, we show that both partitions $\mathcal{V}_{\mathrm{stat}}(\bullet, \delta)$ and $\mathcal{V}_{\mathrm{stat}}(\beta, \bullet)$ give rise to discrete spaces (Proposition 5.6).

2 Background and notation

Let D be an integral domain and L be a field containing D (not necessarily the quotient field of D). The *Zariski space of D in L*, denoted by $\mathrm{Zar}(L|D)$, is the set of valuation domains of L containing D endowed with the so-called *Zariski topology*, i.e., with the topology generated by the subbasic open sets

$$B(\phi) = \{W \in \mathrm{Zar}(L|D) \mid \phi \in W\},$$

where $\phi \in L$. Under this topology, $\mathrm{Zar}(L|D)$ is a compact space [21, Chapter VI, Theorem 40], but it is usually not Hausdorff nor T_1 (indeed, $\mathrm{Zar}(L|D)$ is a T_1 space if and only if D is a field and L is an algebraic extension of D). The *constructible topology* on $\mathrm{Zar}(L|D)$ is the coarsest topology such that the subsets $B(\phi_1, \ldots, \phi_k) = B(\phi_1) \cap \cdots \cap B(\phi_n)$ are both open and closed. The constructible topology is finer than the Zariski topology, but $\mathrm{Zar}(L|D)^{\mathrm{cons}}$ (i.e., $\mathrm{Zar}(L|D)$ endowed with the constructible topology) is always compact and Hausdorff [7, Theorem 1].

From now on, and throughout the article, we assume that V is a valuation domain of rank one; we denote by K its quotient field, by M its maximal ideal and by v the valuation associated to V. Its value group is denoted by Γ_v.

If L is a field extension of K, a valuation domain W of L *lies over* V if $W \cap K = V$; we also say that W is an *extension* of V to L. In this case, the residue field of W is naturally an extension of the residue field of V and similarly the value group of W is an extension of the value group of V.

We denote by $\widehat{K}$ and $\widehat{V}$ the completion of K and V, respectively, with respect to the topology induced by the valuation v. We still denote by v the unique extension of v to $\widehat{K}$ (whose valuation domain is precisely $\widehat{V}$). We denote by $\overline{K}$ a fixed algebraic closure of K.

Since V has rank one, we can consider Γ_v as a subgroup of $\mathbb{R}$. If u is an extension of v to $\overline{K}$, then the value group of u is $\mathbb{Q}\Gamma_v = \{q\gamma \mid q \in \mathbb{Q}, \gamma \in \Gamma_v\}$.

The valuation v induces an ultrametric distance d on K, defined by

$$d(x, y) = e^{-v(x-y)}.$$

In this metric, V is the closed ball of center 0 and radius 1. Given $s \in K$ and $\gamma \in \Gamma_v$, the closed ball of center s and radius $r = e^{-\gamma}$ is:

$$\{x \in K \mid d(x, s) \leq r\} = \{x \in K \mid v(x - s) \geq \gamma\}.$$

The basic objects of study of this paper are pseudo-convergent sequences, introduced by Ostrowski in [10] and used by Kaplansky in [8] to describe immediate extensions of valued fields. Related concepts are *pseudo-stationary* and *pseudo-divergent* sequences introduced in [4], which we will define and use in Section 5.

Definition 2.1. Let $E = \{s_n\}_{n\in\mathbb{N}}$ be a sequence in K. We say that E is a *pseudo-convergent* sequence if $v(s_{n+1} - s_n) < v(s_{n+2} - s_{n+1})$ for all $n \in \mathbb{N}$.

In particular, if $E = \{s_n\}_{n\in\mathbb{N}}$ is a pseudo-convergent sequence and $n \geq 1$, then $v(s_{n+k} - s_n) = v(s_{n+1} - s_n)$ for all $k \geq 1$. We shall usually denote this quantity by δ_n; following [18, p. 327] we call the sequence $\{\delta_n\}_{n\in\mathbb{N}}$ the *gauge* of E. We call the quantity

$$\delta_E = \lim_{n\to\infty} v(s_{n+1} - s_n) = \lim_{n\to\infty} \delta_n$$

the *breadth* of E. The breadth δ_E is an element of $\mathbb{R} \cup \{\infty\}$, and it may not lie in Γ_v.

Definition 2.2. The *breadth ideal* of E is

$$\mathrm{Br}(E) = \{b \in K \mid v(b) > v(s_{n+1} - s_n), \forall n \in \mathbb{N}\} = \{b \in K \mid v(b) \geq \delta_E\}.$$

In general, $\mathrm{Br}(E)$ is a fractional ideal of V and may not be contained in V. If $\delta = +\infty$, then $\mathrm{Br}(E)$ is just the zero ideal and E is a Cauchy sequence in K. If V is a discrete valuation ring, then every pseudo-convergent sequence is actually a Cauchy sequence.

The following definition has been introduced in [8], even though an equivalent concept already appears in [10, p. 375] (see [10, X, p. 381] for the equivalence).

Definition 2.3. An element $\alpha \in K$ is a *pseudo-limit* of E if $v(\alpha - s_n) < v(\alpha - s_{n+1})$ for all $n \in \mathbb{N}$, or, equivalently, if $v(\alpha - s_n) = \delta_n$ for all $n \in \mathbb{N}$. We denote the set of pseudo-limits of E by $\mathcal{L}_E$, or $\mathcal{L}_E^v$ if we need to emphasize the valuation.

If $\mathrm{Br}(E)$ is the zero ideal then E is a Cauchy sequence in K and converges to an element of $\widehat{K}$, which is the unique pseudo-limit of E. Kaplansky proved the following more general result.

Lemma 2.4. *[8, Lemma 3] Let $E \subset K$ be a pseudo-convergent sequence. If $\alpha \in K$ is a pseudo-limit of E, then the set of pseudo-limits of E in K is equal to $\alpha + \mathrm{Br}(E)$.*

Lemma 2.4 can also be phrased in a geometric way: if $\alpha \in \mathcal{L}_E$, then $\mathcal{L}_E$ is the closed ball of center α and radius $e^{-\delta_E}$.

The following concepts have been given by Kaplansky in [8] in order to study the different kinds of immediate extensions of a valued field K, i.e., extensions $V \subseteq W$ of valuation rings where neither the residue field nor the value group change.

Definition 2.5. Let E be a pseudo-convergent sequence. We say that E is of *transcendental type* if, for every $f \in K[X]$, the value $v(f(s_n))$ eventually stabilizes; on the other hand, if $v(f(s_n))$ is eventually strictly increasing for some $f \in K[X]$, we say that E is of *algebraic type.*

The main difference between these two kinds of sequences is the nature of the pseudo-limits: if E is of algebraic type, then it has pseudo-limits in the algebraic closure $\overline{K}$ (for some extension u of v), while if E is of transcendental type then it admits a pseudo-limit only in a transcendental extension [8, Theorems 2 and 3].

The central point of [13] is the following: if $E = \{s_n\}_{n \in \mathbb{N}} \subset K$ is a pseudo-convergent sequence, then the set

$$V_E = \{\phi \in K(X) \mid \phi(s_n) \in V, \text{ for all but finitely many } n \in \mathbb{N}\} \quad (1)$$

is a valuation domain of $K(X)$ extending V [13, Theorem 3.8]. If E, F are pseudo-convergent sequences of algebraic type, then $V_E = V_F$ if and only if $\mathcal{L}_E^u = \mathcal{L}_F^u$ for some extension u of v to $\overline{K}$ [13, Theorem 5.4]. In general, we say that two pseudo-convergent sequences E, F are *equivalent* if $V_E = V_F$; this condition can also be expressed by means of a notion analogous to the one defined classically for Cauchy sequences (see [13, Definition 5.1]).

We are interested in the study of the following subspace of $\mathrm{Zar}(K(X)|V)$:

$$\mathcal{V} = \{V_E \mid E \subset K \text{ is a pseudo-convergent sequence}\}.$$

The space $\mathcal{V}$ is always regular under both the Zariski and the constructible topologies [13, Theorem 6.15]; however, these two topologies coincide if and only if the residue field of V is finite [13, Proposition 6.11].

3 Fixed breadth

In this section, we study the subsets of $\mathcal{V}$ obtained by fixing the breadth of the pseudo-convergent sequences.

Definition 3.1. Let $\delta \in \mathbb{R} \cup \{+\infty\}$. We denote by $\mathcal{V}(\bullet, \delta)$ the set of valuation domains V_E such that the breadth of E is δ.

If $\delta = \infty$, then the elements of $\mathcal{V}(\bullet, \delta)$ are the rings defined through pseudo-convergent sequences with $\mathrm{Br}(E) = (0)$, i.e., from pseudo-convergent sequences that are also Cauchy sequences. In this case, E has a unique limit $\alpha \in \widehat{K}$, and by [13, Remark 3.10] we have

$$V_E = W_\alpha = \{\phi \in K(X) \mid v(\phi(\alpha)) \geq 0\}.$$

Therefore, there is a natural bijection between $\widehat{K}$ and $\mathcal{V}(\bullet, \infty)$, given by $\alpha \mapsto W_\alpha$; by [12, Theorem 3.4], such a bijection is also a homeomoprhism, when $\widehat{K}$ is endowed with the v-adic topology and $\mathcal{V}(\bullet, \infty)$ with the Zariski topology. In particular, it follows that the latter is an ultrametric space. Note that when V is a discrete valuation ring, $\mathcal{V} = \mathcal{V}(\bullet, \infty)$.

Proposition 3.2. *Let V be a discrete valuation ring. Then, $\mathcal{V} \simeq \widehat{K}$ is an ultrametric space.*

Proof. The claim follows from the previous discussion and the fact that if V is discrete then every pseudo-convergent sequence has infinite breadth. □

The purpose of this section is to see how the homeomorphism $\mathcal{V}(\bullet, \infty) \simeq \widehat{K}$ generalizes when we consider pseudo-convergent sequences with fixed breadth $\delta \in \mathbb{R}$.

Fix $\delta \in \mathbb{R} \cup \{\infty\}$, and set $r = e^{-\delta}$. Given two pseudo-convergent sequences $E = \{s_n\}_{n \in \mathbb{N}}$ and $F = \{t_n\}_{n \in \mathbb{N}}$, with $V_E, V_F \in \mathcal{V}(\bullet, \delta)$, we set

$$d_\delta(V_E, V_F) = \lim_{n \to \infty} \max\{d(s_n, t_n) - r, 0\}.$$

It is clear that if $r = 0$ (or, equivalently, $\delta = +\infty$) then $d_\delta(V_E, V_F) = d(\alpha, \beta)$, where α and β are the (unique) limits of E and F, respectively; so in this case we get the same distance as in [12]. We shall interpret d_δ in a similar way in Proposition 3.6; we first show that it is actually a distance.

Proposition 3.3. *Preserve the notation above.*

(a) d_δ is well-defined.

(b) d_δ is an ultrametric distance on $\mathcal{V}(\bullet, \delta)$.

Proof. (a) Let $E = \{s_n\}_{n\in\mathbb{N}}$ and $F = \{t_n\}_{n\in\mathbb{N}}$ be two pseudo-convergent sequences. We start by showing that the limit of $a_n = \max\{d(s_n, t_n) - r, 0\}$ exists. If all subsequences of $\{a_n\}_{n\in\mathbb{N}}$ go to zero, we are done. Otherwise, there is a subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ with a positive (possibly infinite) limit; in particular, there is a $\overline{\delta} < \delta$ and $k_0 \in \mathbb{N}$ such that $v(s_{n_k} - t_{n_k}) < \overline{\delta}$ for all $k \geq k_0$. Choose $k_1 \in \mathbb{N}$ such that $\overline{\delta} < \min\{\delta_{k_1}, \delta'_{k_1}\}$ (where $\{\delta_n\}_{n\in\mathbb{N}}$ and $\{\delta'_n\}_{n\in\mathbb{N}}$ are the gauges of E and F, respectively). Fix an $m = n_l$ such that $m > k_1$ and $l > k_0$. Then, for all $n > m$, we have

$$v(s_n - t_n) = v(s_n - s_m + s_m - t_m + t_m - t_n) = v(s_m - t_m)$$

since $v(s_n - s_m) = \delta_m > \delta_{k_1} > \overline{\delta} > v(s_{n_l} - t_{n_l}) = v(s_m - t_m)$, and likewise for $v(t_n - t_m)$. Hence, a_n is eventually constant (more precisely, equal to $e^{-v(s_m - t_m)} - e^{-\delta}$); in particular, $\{a_n\}_{n\in\mathbb{N}}$ has a limit.

In order to show that d_δ is well-defined, we need to show that, if $V_E = V_{E'}$, where $E = \{s_n\}_{n\in\mathbb{N}}$ and $E' = \{s'_n\}_{n\in\mathbb{N}}$, then

$$\lim_{n\to\infty} \max\{d(s_n, t_n) - r, 0\} = \lim_{n\to\infty} \max\{d(s'_n, t_n) - r, 0\}.$$

Let l be the limit on the left hand side and l' the limit on the right hand side.

If F is equivalent to E and E', by [13, Definition 5.1 and Theorem 5.4] for every k there are i_0, j_0, i'_0, j'_0 such that $v(s_i - t_j) > \delta_k$, $v(s'_{i'} - t'_{j'}) > \delta'_k$ for $i \geq i_0$, $j \geq j_0$, $i' \geq i'_0$, $j' \geq j'_0$. Hence, both l and l' are equal to 0, and in particular they are equal.

Suppose that F is not equivalent to E and E'. If l is positive, and $\eta = -\log(l + r)$, then $v(s_n - t_n) = \eta$ for large n, and $\eta < \delta_k$ for some k; since E and E' are equivalent there is a i_0 such that $v(s_i - s'_i) > \delta_k$ for all $i \geq i_0$. Hence, for all large n,

$$v(s'_n - t_n) = v(s'_n - s_n + s_n - t_n) = v(s_n - t_n) = \eta,$$

as claimed. The same reasoning applies if $l' > 0$; furthermore, if $l = 0 = l'$ then clearly $l = l'$. Hence, $l = l'$ always, as claimed.

(b) d_δ is obviously symmetric. Clearly $d_\delta(V_E, V_E) = 0$; if $d_\delta(V_E, V_F) = 0$, for every $r_k = e^{-\delta'_k} < r$ (where $\delta'_k = v(t_{k+1} - t_k)$) there is i_0 such that $d(s_i, t_i) < r_k$ for all $i \geq i_0$. Thus, if $i, j \geq i_0$, then

$$d(s_i, t_j) = \max\{d(s_i, t_i), d(t_i, t_j)\} = r_k.$$

Hence, E and F are equivalent and $V_E = V_F$. The strong triangle inequality follows from the fact that $d(s_n, t_n) \leq \max\{d(s_n, s'_n), d(s'_n, t_n)\}$ for all $s_n, s'_n, t_n \in K$. Therefore, d_δ is an ultrametric distance. □

Let $\mathcal{V}_K(\bullet, \delta)$ be the subset of $\mathcal{V}(\bullet, \delta)$ corresponding to pseudo-convergent sequences with a pseudo-limit in K. We recall that by [13, Theorem 5.4] the map $V_E \mapsto \mathcal{L}_E$, from $\mathcal{V}_K(\bullet, \delta)$ to the set of closed balls in K of radius $e^{-\delta}$, is a one-to-one correspondence. When $\delta = \infty$, $\mathcal{V}_K(\bullet, \infty)$ corresponds to K under the homeomorphism between $\mathcal{V}(\bullet, \infty)$ and $\widehat{K}$; in particular, $\mathcal{V}(\bullet, \infty)$ is the completion of $\mathcal{V}_K(\bullet, \infty)$ under d_∞. An analogous result holds for $\delta \in \mathbb{R}$.

Proposition 3.4. *Let $\delta \in \mathbb{R}$. Then $\mathcal{V}(\bullet, \delta)$ is the completion of $\mathcal{V}_K(\bullet, \delta)$ under the metric d_δ. In particular, $\mathcal{V}(\bullet, \delta)$, under d_δ, is a complete metric space.*

Proof. Let $\{\zeta_k\}_{k \in \mathbb{N}} \subset \Gamma$ be an increasing sequence of real numbers with limit δ and, for every k, let z_k be an element of K of valuation ζ_k; let $Z = \{z_k\}_{k \in \mathbb{N}}$. It is clear that Z is a pseudo-convergent sequence with 0 as a pseudo-limit and having breadth δ. Then, for every $s \in K$, $s + Z = \{s + z_k\}_{k \in \mathbb{N}}$ is a pseudo-convergent sequence with pseudo-limit s and breadth δ.

Let $E = \{s_n\}_{n \in \mathbb{N}}$ be a pseudo-convergent sequence with breadth δ, and let $F_n = s_n + Z$. By above, $V_{F_n} \in \mathcal{V}_K(\bullet, \delta)$, for each $n \in \mathbb{N}$. We claim that $\{V_{F_n}\}_{n \in \mathbb{N}}$ converges to V_E in $\mathcal{V}(\bullet, \delta)$. Indeed, fix $t \in \mathbb{N}$, and take $k > t$ such that $\zeta_k > \delta_t$. Then,

$$u(s_t + z_k - s_k) = u(s_t - s_k + z_k) = \delta_t;$$

hence, $d(V_E, V_{F_n}) = e^{-\delta_n} - e^{-\delta}$. In particular, the distance goes to 0 as $n \to \infty$, and thus V_E is the limit of V_{F_n}.

Conversely, let $\{V_{F_n}\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathcal{V}_K(\bullet, \delta)$, and let $s_n \in K$ be a pseudo-limit of F_n. Then, $s_n + Z$ is another pseudo-convergent sequence with limit s_n and breadth δ; by [13, Theorem 5.4] it follows that $V_{F_n} = V_{s_n + Z}$. There is a subsequence of $E = \{s_n\}_{n \in \mathbb{N}}$ which is pseudo-convergent; indeed, it is enough to take $\{s_{n_k}\}_{k \in \mathbb{N}}$ such that $d(s_{n_k}, s_{n_{k+1}}) < d(s_{n_{k-1}}, s_{n_k})$. Hence, without loss of generality E itself is pseudo-convergent; we claim that V_E is a limit of $\{V_{F_n}\}_{n \in \mathbb{N}}$. Indeed, as above, $u(s_t + z_k - s_k) = \delta_t$ for large k, and thus $d_\delta(V_E, V_{s_n + Z}) = e^{-\delta_t} - e^{-\delta}$. Thus, $\{V_{F_n}\}_{n \in \mathbb{N}}$ has a limit, namely V_E. Therefore, $\mathcal{V}(\bullet, \delta)$ is the completion of $\mathcal{V}_K(\bullet, \delta)$. □

We now prove that the topology induced by d_δ is actually the Zariski topology.

Theorem 3.5. *Let $\delta \in \mathbb{R} \cup \{\infty\}$. On $\mathcal{V}(\bullet, \delta)$, the Zariski topology, the constructible topology and the topology induced by d_δ coincide.*

Proof. If $\delta = \infty$, then the Zariski topology and the topology induced by d_δ coincide by [12, Theorem 3.4].

Suppose now that V is nondiscrete and fix $\delta \in \mathbb{R}$. Let $V_E \in \mathcal{V}(\bullet, \delta)$ and $\rho \in \mathbb{R}$, $\rho > 0$: we show that the open ball $\mathcal{B}(V_E, \rho) = \{V_F \in \mathcal{V}(\bullet, \delta) \mid d_\delta(V_E, V_F) < \rho\}$ of the ultrametric topology induced by d_δ is open in the Zariski topology. Since by Proposition 3.4 $\mathcal{V}_K(\bullet, \delta)$ is dense in $\mathcal{V}(\bullet, \delta)$ under the metric d_δ, without loss of generality we may assume that $V_E \in \mathcal{V}_K(\bullet, \delta)$, i.e., E has a pseudo-limit b in K. To ease the notation, we denote by $B(\phi)$ the intersection $B(\phi) \cap \mathcal{V}(\bullet, \delta)$.

Let $\gamma < \delta$ be such that $\rho = e^{-\gamma} - e^{-\delta}$. We claim that

$$\mathcal{B}(V_E, \rho) = \bigcup_{\delta > v(c) > \gamma} B\left(\frac{X - b}{c}\right).$$

Indeed, suppose $V_F \in \mathcal{B}(V_E, \rho)$, where $F = \{t_n\}_{n \in \mathbb{N}}$. If F is equivalent to E then $V_E = V_F$ and $v\left(\frac{t_n - b}{c}\right) = \delta_n - v(c)$; since $\gamma < \delta$ and Γ is dense in $\mathbb{R}$, there is a $c \in K$ such that $\gamma < v(c) < \delta$, and for such a c the limit of $\delta_n - v(c)$ is positive; hence, V_E belongs to the union. If F is not equivalent to E, then $0 < d_\delta(V_E, V_F) < \rho$, that is, $e^{-\delta} < \lim_n d(s_n, t_n) < e^{-\delta} + \rho$. By the proof of Proposition 3.3(a), $v(s_n - t_n)$ is eventually constant, and thus there is an $\epsilon > 0$ such that $\delta > v(s_n - t_n) \geq \gamma + \epsilon$ for all large n. Let $c \in K$ be of value comprised between γ and $\gamma + \epsilon$ (such a c exists because Γ is dense in $\mathbb{R}$); then,

$$v\left(\frac{t_n - b}{c}\right) = v(t_n - b) - v(c) = v(t_n - s_n + s_n - b) - v(c) \geq \min\{\gamma + \epsilon, \delta_n\} - v(c) > 0$$

since δ_n becomes bigger than $\gamma + \epsilon$. Hence, $\frac{X-b}{c} \in V_F$, or equivalently $V_F \in B\left(\frac{X-b}{c}\right)$.

Conversely, suppose $V_F \neq V_E$ belongs to $B\left(\frac{X-b}{c}\right)$ for some $c \in K$ such that $\gamma < v(c) < \delta$. Since $\mathcal{L}_E \cap \mathcal{L}_F = \emptyset$ by [13, Theorem 5.4], b is not a pseudo-limit of F; therefore, $v(t_n - s_n) = v(t_n - b + b - s_n) = v(b - t_n) \geq v(c) > \gamma$ for sufficiently large n. Thus,

$$d_\delta(V_E, V_F) = \lim_n d(s_n, t_n) - e^{-\delta} = \lim_n d(b, t_n) - e^{-\delta} < e^{-\gamma} - e^{-\delta} = \rho,$$

i.e., $V_F \in \mathcal{B}(V_E, \rho)$. Thus, being the union of sets that are open in the Zariski topology, $\mathcal{B}(V_E, \rho)$ is itself open in the Zariski topology. Therefore, the ultrametric topology is finer than the Zariski topology.

Let now δ be arbitrary, $\phi \in K(X)$ be a rational function, and suppose $V_E \in B(\phi)$ for some $V_E \in \mathcal{V}(\bullet, \delta)$. We want to show that for some $\rho > 0$

there is a ball $\mathcal{B}(V_E,\rho)\subseteq B(\phi)$, and thus that $B(\phi)$ is open in the ultrametric topology induced by d_δ. We distinguish two cases.

Suppose that E is of algebraic type, and let $\beta\in\mathcal{L}_E^u$ for some extension u of v to $\overline{K}$. By [13, Lemma 6.6], there is an annulus $C=\mathcal{C}(\beta,\tau,\delta)=\{s\in\overline{K}\mid \tau<u(s-\beta)<\delta\}$ such that $\phi(s)\in V$ for every $s\in C$. Let $\epsilon=e^{-\tau}-e^{-\delta}$. Let $F=\{t_n\}_{n\in\mathbb{N}}$ be a pseudo-convergent sequence with $d_\delta(V_E,V_F)<\epsilon$. Then, for every n such that $e^{-\delta_n}-e^{-\delta}>d_\delta(V_E,V_F)$, we have

$$d(t_n,\beta)=\max\{d(t_n,s_n),d(s_n,\beta)\}=e^{-\delta_n},$$

and in particular $v(t_n-\beta)$ becomes larger than τ. Hence, t_n is eventually in C and $\phi(t_n)\in V$ for all large n, and thus $\phi\in V_F$; therefore, $\mathcal{B}(V_E,\epsilon)\subseteq B(\phi)$.

Suppose that E is of transcendental type. Let $\phi(X)=c\prod_{i=1}^{A}(X-\alpha_i)^{\epsilon_i}$ over $\overline{K}$, where each ϵ_i is either 1 or -1. Then, there is an N such that $u(s_n-\alpha_i)$ is constant for every i and every $n\geq N$. Let δ' be the maximum among such constants; then, $\delta'<\delta$ (otherwise the α_i where such maximum is attained would be a pseudo-limit of E, in contrast to the fact that E is of transcendental type). Let ϵ be such that $e^{-\delta}+\epsilon<e^{-\delta'}$ and let $V_F\in\mathcal{B}(V_E,\epsilon)$, with $F=\{t_n\}_{n\in\mathbb{N}}$. For all i, and all large n,

$$d(t_n,\alpha_i)=\max\{d(t_n,s_n),d(s_n,\alpha_i)\}=d(s_n,\alpha_i),$$

and thus $u(t_n-\alpha_i)=u(s_n-\alpha_i)$. It follows that $v(\phi(t_n))=v(\phi(s_n))$ for large n; in particular, $v(\phi(t_n))$ is positive, and $\phi\in V_F$. Hence, $\mathcal{B}(V_E,\epsilon)\subseteq B(\phi)$.

Hence, $B(\phi)$ is open under the topology induced by d_δ and therefore the Zariski topology and the topology induced by d_δ on $\mathcal{V}(\bullet,\delta)$ are the same.

In order to prove that these topologies coincide also with the constructible topology, we need only to show that every $B(\phi)$, $\phi\in K(X)$, is closed in the Zariski topology. Let then $V_E\notin B(\phi)$. If E is of transcendental type, exactly as above there exists $\epsilon>0$ such that for each $V_F\in\mathcal{B}(V_E,\epsilon)$, where $F=\{t_n\}_{n\in\mathbb{N}}$, $v(\phi(t_n))=v(\phi(s_n))$ for large n; in particular, $v(\phi(t_n))$ is negative, and $\phi\notin V_F$; thus $\mathcal{B}(V_E,\epsilon)$ is disjoint from $B(\phi)$. If E is of algebraic type, then by [13, Remark 6.7], there exists an annulus $\mathcal{C}=\mathcal{C}(\beta,\tau,\delta)$ such that $\phi(s)\notin V$ for every $s\in\mathcal{C}$. As above, for every pseudo-convergent sequence $F=\{t_n\}_{n\in\mathbb{N}}$ with $d_\delta(V_E,V_F)<\epsilon$, with $\epsilon=e^{-\tau}-e^{-\delta}$, we have $t_n\in\mathcal{C}$ for all but finitely many $n\in\mathbb{N}$, so that $\phi(t_n)\notin V$. Again, this shows that $\mathcal{B}(V_E,\epsilon)$ is disjoint from $B(\phi)$. □

Joining Proposition 3.4 with Theorem 3.5, we obtain that the set $\mathcal{V}_K=\{V_E\in\mathcal{V}\mid\mathcal{L}_E\cap K\neq\emptyset\}=\bigcup_\delta\mathcal{V}_K(\bullet,\delta)$ of all the extensions arising from pseudo-convergent sequences with pseudo-limits in K is dense in $\mathcal{V}$, with respect to both the Zariski and the constructible topology. This result can also be obtained as a corollary of [13, Proposition 6.9].

If we restrict to pseudo-convergent sequences of algebraic type, the distance d_δ can be interpreted in a different way.

Proposition 3.6. *Let $E, F \subset K$ be pseudo-convergent sequences of algebraic type with breadth δ, and let u be an extension of v to $\overline{K}$. If $\beta \in \mathcal{L}_E^u$ and $\beta' \in \mathcal{L}_F^u$, then*

$$d_\delta(V_E, V_F) = \max\{d_u(\beta, \beta') - e^{-\delta}, 0\}.$$

Proof. If $d_u(\beta, \beta') \leq e^{-\delta}$, then the pseudo-limits of E and F coincide, and thus $V_E = V_F$ by [13, Theorem 5.4]; hence, $d_\delta(V_E, V_F) = 0$. On the other hand, if $d_u(\beta, \beta') > e^{-\delta}$ then $u(\beta - \beta') < \delta$ and thus, for large n,

$$v(s_n - t_n) = u(s_n - \beta + \beta - \beta' + \beta' - t_n) = u(\beta - \beta');$$

hence, $d_\delta(V_E, V_F) = d_u(\beta, \beta') - e^{-\delta}$, as claimed. $\square$

If V is a DVR, then $\mathcal{V} = \mathcal{V}(\bullet, \infty)$, so, in this case, the distance d_∞ is an ultrametric distance on the whole $\mathcal{V}$. On the other hand, if V is not discrete, it is not possible to unify the metrics d_δ in a single metric defined on the whole $\mathcal{V}$. To this end, we need the following lemma.

Lemma 3.7. *Let $\delta \in \mathbb{R} \cup \{\infty\}$. Then the closure of $\mathcal{V}(\bullet, \delta)$ in $\mathcal{V}$ is equal to* $\bigcup_{\delta' \leq \delta} \mathcal{V}(\bullet, \delta')$.

Proof. If V is discrete, then the statement is a tautology (see Proposition 3.2). We assume henceforth that V is not discrete.

Let $E = \{s_n\}_{n \in \mathbb{N}}$ be a pseudo-convergent sequence with breadth $\delta' < \delta$; we want to show that V_E is in the closure of $\mathcal{V}(\bullet, \delta)$. By Proposition 3.4, $\mathcal{V}(\bullet, \delta')$ is contained in the closure of $\mathcal{V}_K(\bullet, \delta')$; hence, we can suppose that E has a pseudo-limit in K.

For each $n \in \mathbb{N}$, let E_n be a pseudo-convergent sequence with pseudo-limit s_n and breadth δ: since $\delta' < \delta$, by [13, Proposition 6.9] V_E is the limit of V_{E_n} in the Zariski topology, and thus it belongs to the closure of $\mathcal{V}_K(\bullet, \delta')$, as claimed. If $\delta = \infty$ we are done; suppose for the rest of the proof that $\delta < \infty$.

Suppose $\delta' > \delta$; we claim that if $E = \{s_n\}_{n \in \mathbb{N}}$ is pseudo-convergent sequence with breadth δ' then there is an open set containing V_E and disjoint from $\mathcal{V}(\bullet, \delta)$. Let $\gamma \in \Gamma_v$ be such that $\delta' > \gamma > \delta$; then, there is an N such that $v(s_n - s_{n+1}) > \gamma$ for all $n \geq N$. Take $s = s_N$, and consider the open set $B\left(\frac{X-s}{c}\right)$, where $c \in K$ has value γ. Then, $V_E \in B\left(\frac{X-s}{c}\right)$ since $v(s_n - s_N) = \delta'_N > \gamma$ for all $n \geq N$. On the other hand, if $F = \{t_n\}_{n \in \mathbb{N}} \subset K$ is a pseudo-convergent sequence of breadth δ and $V_F \in B\left(\frac{X-s}{c}\right)$, then F would be eventually contained in the ball of center s and radius γ, and in particular $v(t_n - t_{n+1}) \geq \gamma$ for all large n. However, $v(t_n - t_{n+1}) < \delta < \gamma$, a contradiction. Therefore, $V_F \notin B\left(\frac{X-s}{c}\right)$ and so V_E is not in the closure of $\mathcal{V}(\bullet, \delta)$. $\square$

Proposition 3.8. *Let V be a rank one non-discrete valuation domain. Suppose $\mathcal{V}$ is metrizable with a metric d. Then, for any $\delta \in \mathbb{R} \cup \{\infty\}$, the restriction of d to $\mathcal{V}(\bullet, \delta)$ is* not *equal to d_δ.*

Proof. If the restriction of d is equal to d_δ, then by Proposition 3.4 $\mathcal{V}(\bullet, \delta)$ would be complete with respect to d. However, this would imply that $\mathcal{V}(\bullet, \delta)$ is closed, in contrast to Lemma 3.7. □

To conclude this section, we analyze the relationship among the sets $\mathcal{V}(\bullet, \delta)$, as δ ranges in $\mathbb{R} \cup \{\infty\}$. Recall that two metric spaces (X, d) and (X', d') are *similar* if there is a map $\psi : X \longrightarrow X'$ and a constant $r > 0$ such that $d'(\psi(x), \psi(y)) = rd(x, y)$ for every $x, y \in X$. We call such a map ψ a *similitude*.

Proposition 3.9. *If $\delta_1 - \delta_2 \in \Gamma_v$, then the metric spaces $(\mathcal{V}(\bullet, \delta_1), d_{\delta_1})$ and $(\mathcal{V}(\bullet, \delta_2), d_{\delta_2})$ are similar; in particular, they are homeomorphic when endowed with the Zariski topology.*

Proof. Given a pseudo-convergent sequence $E = \{s_n\}_{n\in\mathbb{N}}$ and $c \in K$, $c \neq 0$, we denote by cE the sequence $\{cs_n\}_{n\in\mathbb{N}}$. Clearly, cE is again pseudo-convergent, it has breadth $\delta_E + v(c)$, and two sequences E and F are equivalent if and only if cE and cF are equivalent.

Let $c \in K$ be such that $v(c) = \delta_1 - \delta_2$. Then, the map

$$\begin{aligned} \Psi_c \colon \mathcal{V}(\bullet, \delta_2) &\longrightarrow \mathcal{V}(\bullet, \delta_1) \\ V_E &\longmapsto V_{cE} \end{aligned}$$

is well-defined and bijective (its inverse is $\Psi_{c^{-1}} : \mathcal{V}(\bullet, \delta_1) \longrightarrow \mathcal{V}(\bullet, \delta_2)$). We claim that Ψ_c is a similitude. Indeed, let $E = \{s_n\}_{n\in\mathbb{N}}$ and $F = \{t_n\}_{n\in\mathbb{N}}$ be pseudo-convergent sequences of breadth δ_2, and suppose $V_E \neq V_F$. By the proof of Proposition 3.3, there is an N such that $v(s_n - t_n) = v(s_N - t_N)$ for all $n \geq N$. Hence, for these n's,

$$e^{-v(cs_n - ct_n)} - e^{-\delta_1} = e^{-v(c)}e^{-v(s_n - t_n)} - e^{-\delta_1} = e^{-v(c)}[e^{-v(s_n - t_n)} - e^{-\delta_2}]$$

so that, passing to the limit, $d_{\delta_1}(V_{cE}, V_{cF}) = e^{-v(c)}d_{\delta_2}(V_E, V_F)$. Hence, Ψ_c is an similitude, and in particular a homeomorphism when $\mathcal{V}(\bullet, \delta_1)$ and $\mathcal{V}(\bullet, \delta_2)$ are endowed with the metric topology. Since this topology coincides with the Zariski topology (Theorem 3.5), they are homeomorphic also under the Zariski topology. □

4 Fixed pseudo-limit

In the previous section, we considered valuation domains induced by pseudo-convergent sequences having the same breadth; in this section, we reverse the situation by considering pseudo-convergent sequences having a prescribed pseudo-limit. Note that, in particular, these pseudo-convergent sequences are of algebraic type.

Throughout this section, let u be a fixed extension of v to $\overline{K}$.

Definition 4.1. Let $\beta \in \overline{K}$. We set

$$\mathcal{V}^u(\beta, \bullet) = \{V_E \in \mathcal{V} \mid \beta \in \mathcal{L}_E^u\}$$

To ease the notation, we set $\mathcal{V}^u(\beta, \bullet) = \mathcal{V}(\beta, \bullet)$.

Equivalently, a valuation domain V_E is in $\mathcal{V}(\beta, \bullet)$ if β is a center of $\mathcal{L}_E^u$, i.e., if $\mathcal{L}_E^u = \{x \in \overline{K} \mid u(x - \beta) \geq \delta_E\}$. Note that if $V_E \in \mathcal{V}^u(\beta, \bullet)$ then E must be of algebraic type, since it must have a pseudo-limit in $\overline{K}$.

If V is a DVR, then $\mathcal{V}(\beta, \bullet)$ reduces to the single element $W_\beta = \{\phi \in K(X) \mid \phi(\beta) \in V\}$ (see [13, Remark 3.10]), which corresponds to any Cauchy sequence $E \subset K$ converging to β.

We start by showing that each $\mathcal{V}(\beta, \bullet)$ is closed in $\mathcal{V}$.

Proposition 4.2. *Let $\beta \in \overline{K}$, and let u be an extension of v to $\overline{K}$. Then, $\mathcal{V}(\beta, \bullet) = \mathcal{V}^u(\beta, \bullet)$ is closed in $\mathcal{V}$ with respect to the Zariski topology.*

Proof. If V is discrete, then $\mathcal{V}(\beta, \bullet)$ has just one element (see the comments above). By [12, Theorem 3.4] each point of $\mathcal{V}$ is closed, so the statement is true in this case. Henceforth, for the rest of the proof we assume that V is non discrete.

Let $V_E \notin \mathcal{V}(\beta, \bullet)$. We distinguish two cases.

Suppose first that $E = \{s_n\}_{n \in \mathbb{N}}$ is of algebraic type, and let $\alpha \in \overline{K}$ be a pseudo-limit of E with respect to u. Since $\beta \notin \mathcal{L}_E \Leftrightarrow u(\alpha - \beta) < \delta_E$ (Lemma 2.4) it follows that there is $m \in \mathbb{N}$ such that $u(\alpha - \beta) < u(\alpha - s_m)$. Let $s = s_m$. Choose a $d \in K$ such that

$$u(\beta - \alpha) = u(\beta - s) < v(d) < u(\alpha - s) < \delta_E,$$

and let $\phi(X) = \frac{X - s}{d}$; we claim that $V_E \in B(\phi)$ but $B(\phi) \cap \mathcal{V}(\beta, \bullet) = \emptyset$.

Indeed,

$$v(\phi(s_n)) = v\left(\frac{s_n - s}{d}\right) = v(s_n - s) - v(d) > 0$$

since $v(s_n - s) = u(s_n - \alpha + \alpha - s) = u(\alpha - s)$ for large n; hence $V_E \in B(\phi)$. On the other hand, if $F = \{t_n\}_{n \in \mathbb{N}}$ has pseudo-limit β, then $v(t_n - s) = u(t_n - \beta + \beta - s) = u(\beta - s)$ for large n and so

$$v(\phi(t_n)) = u(\beta - s) - v(d) < 0,$$

i.e., $V_F \notin B(\phi)$. The claim is proved.

Suppose now that $E = \{s_n\}_{n \in \mathbb{N}}$ is of transcendental type: then, $u(s_n - \beta)$ is eventually constant, say equal to λ. Then, $\lambda < \delta$, for otherwise β would be a pseudo-limit of E; hence, we can take a $d \in K$ such that $\lambda < v(d) < \delta$. Choose an N such that $u(s_N - \beta) = \lambda$ and such that $v(d) < \delta_N$, and define $\phi(X) = \frac{X - s_N}{d}$. Then, $v(\phi(s_n)) = \delta_N - v(d) > 0$ for $n > N$, and thus $V_E \in B(\phi)$. Suppose now $v(\phi(t)) \geq 0$. Then, $v(t - s_N) \geq v(d) > \lambda$; however, $v(t - s_N) = u(t - \beta + \beta - s_N)$, and since $u(\beta - s_N) = \lambda$ we must have $u(t - \beta) = \lambda$. In particular, there is no annulus C of center β such that $\phi(t) \in V$ for all $t \in C$; hence, by [13, Lemma 6.6], $V_F \notin B(\phi)$ for every $V_F \in \mathcal{V}(\beta, \bullet)$, i.e., $\mathcal{V}(\beta, \bullet) \cap B(\phi) = \emptyset$. The claim is proved. $\square$

We now want to characterize the Zariski topology of $\mathcal{V}(\beta, \bullet)$. By [13, Theorem 5.4], there is a natural injective map

$$\begin{aligned} \Sigma_\beta \colon \mathcal{V}(\beta, \bullet) &\longrightarrow (-\infty, +\infty] \\ V_E &\longmapsto \delta_E. \end{aligned} \tag{2}$$

In general this map is not surjective: for example, there might be some $\beta \in \overline{K}$ which is not the limit of any Cauchy sequence in K (with respect to u) and thus $\delta_E \neq +\infty$ for every $V_E \in \mathcal{V}(\beta, \bullet)$. By [13, Proposition 5.5] the image of Σ_β is $(-\infty, \delta(\beta, K)]$, where $\delta(\beta, K)$ is defined as

$$\delta(\beta, K) = \sup\{u(\beta - x) \mid x \in K\}.$$

In order to study the Zariski topology on $\mathcal{V}(\beta, \bullet)$, we introduce a topology on the interval $(-\infty, \delta(\beta, K)]$.

Definition 4.3. Let $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$, with $a < b$, and let $\Lambda \subseteq \mathbb{R}$. The Λ-*upper limit topology* on $(a, b]$ is the topology generated by the sets $(\alpha, \lambda]$, for $\lambda \in \Lambda \cup \{\infty\}$ and $\alpha \in (a, b]$. We denote this space by $(a, b]_\Lambda$.

The Λ-upper limit topology is a variant of the upper limit topology (see e.g. [16, Counterexample 51]), and in fact the two topologies coincide when $\Lambda = \mathbb{R}$.

For the next theorem we need to recall a definition and a result from [13]. Let $E = \{s_n\}_{n \in \mathbb{N}}$ be a pseudo-convergent sequence; we can associate to E the map

$$\begin{aligned} w_E \colon K(X) &\longrightarrow \mathbb{R} \cup \{\infty\} \\ \phi &\longmapsto \lim_{n \to \infty} v(\phi(s_n)); \end{aligned}$$

this map is always well-defined, and it is possible to characterize when it is a valuation on $K(X)$ [13, Propositions 4.3 and 4.4]. Given $s \in K$ and $\gamma \in \mathbb{R}$, we set

$$\Omega(s, \gamma) = \{V_F \in \mathcal{V} \mid w_F(X - s) \leq \gamma\};$$

this set is always open and closed in $\mathcal{V}$ (with respect to the Zariski topology) [13, Lemma 6.14].

Theorem 4.4. *Suppose V is not discrete, and let $\beta \in \overline{K}$ be a fixed element. The map Σ_β defined in* (2) *is a homeomorphism between $\mathcal{V}(\beta, \bullet)$ (endowed with the Zariski topology) and $(-\infty, \delta(\beta, K)]_{\mathbb{Q}\Gamma_v}$.*

Proof. To shorten the notation, let $\mathcal{X} = (-\infty, \delta(\beta, K)]_{\mathbb{Q}\Gamma_v}$.

We start by showing that Σ_β is continuous. Clearly, $\Sigma_\beta^{-1}(\mathcal{X}) = \mathcal{V}(\beta, \bullet)$ is open.

Suppose $\gamma \in \mathbb{Q}\Gamma_v$ satisfies $\gamma < \delta(\beta, K)$. Then, there is a $t \in K$ such that $u(t - \beta) > \gamma$; we claim that

$$\Sigma_\beta^{-1}((-\infty, \gamma]) = \Omega(t, \gamma) \cap \mathcal{V}(\beta, \bullet).$$

Indeed, let $E = \{s_n\}_{n \in \mathbb{N}}$ be a pseudo-convergent sequence having β as a pseudo-limit. If $\delta_E \leq \gamma$, then (since $u(\beta - t) > \gamma$)

$$w_E(X - t) = \lim_{n \to \infty} v(s_n - t) = \lim_{n \to \infty} u(s_n - \beta + \beta - t) = \delta_E$$

and so $V_E \in \Omega(t, \gamma)$. Conversely, if $V_E \in \Omega(t, \gamma) \cap \mathcal{V}(\beta, \bullet)$ then $w_E(X - t) \leq \gamma$, and thus (using again $u(\beta - t) > \gamma$)

$$\delta_E = \lim_{n \to \infty} u(s_n - \beta) = \lim_{n \to \infty} u(s_n - t + t - \beta) = \lim_{n \to \infty} u(s_n - t) = w_E(X - t) \leq \gamma,$$

i.e., $\Sigma_\beta(V_E) \leq \gamma$.

By [13, Lemma 6.14], $\Omega(t, \gamma)$ is open and closed in $\mathcal{V}$; hence, $\Sigma_\beta^{-1}((-\infty, \gamma])$ and $\Sigma_\beta^{-1}((\gamma, \delta(\beta, K)])$ are both open. If now $(a, b]$ is an arbitrary basic open set of $\mathcal{X}$, with $b \in \mathbb{Q}\Gamma$, then

$$\Sigma_\beta^{-1}((a, b]) = \Sigma_\beta^{-1}((-\infty, b]) \cap \left(\bigcup_{\substack{c \in \mathbb{Q}\Gamma_v \\ c > a}} \Sigma_\beta^{-1}((c, \delta(\beta, K)]) \right)$$

is open. Hence, Σ_β is continuous.

Let now ϕ be an arbitrary nonzero rational function over K, and for ease of notation let $B(\phi)$ denote the intersection $B(\phi) \cap \mathcal{V}(\beta, \bullet)$. Suppose $\delta \in \Sigma_\beta(B(\phi))$, and let $E = \{s_n\}_{n \in \mathbb{N}}$ be a pseudo-convergent sequence of breadth δ having β as a pseudo-limit. By [13, Lemma 6.6] there are $\theta_1, \theta_2 \in \mathbb{Q}\Gamma_v$ such that $\theta_2 < \delta \leq \theta_1$ and such that $v(\phi(t)) \geq 0$ for all $t \in \mathcal{C}(\beta, \theta_1, \theta_2)$. In particular, if $V_F \in \mathcal{V}(\beta, \bullet)$, $F = \{t_n\}_{n \in \mathbb{N}}$, is such that $\Sigma_\beta(V_F) \in (\theta_1, \theta_2]$ we have that $t_n \in \mathcal{C}(\beta, \theta_1, \theta_2)$ for each $n \geq N$, for some $N \in \mathbb{N}$, so that $v(\phi(t_n)) \geq 0$ for each $n \geq N$, thus $\phi \in V_F$. Hence, $(\theta_1, \theta_2] \subseteq \Sigma_\beta(B(\phi))$, and thus $(\theta_1, \theta_2]$ is an open neighbourhood of δ in $\Sigma_\beta(B(\phi))$, which thus is open.

Hence, Σ_β is open, and thus Σ_β is a homeomorphism. □

Let $\mathcal{W}$ be the set of valuation domains of $K(X)$ associated to the valuations w_E defined above, as E ranges through the set of pseudo-convergent sequences of K such that w_E is a valuation. When V is not discrete, we obtain a new proof of the non-compactness of $\mathcal{W}$, independent from [13, Proposition 6.4].

Corollary 4.5. *The spaces $\mathcal{V}$ and $\mathcal{W}$ are not compact.*

Proof. If V is a DVR, then $\mathcal{V}$ is homeomorphic to $\widehat{K}$ ([12, Theorem 3.4]). In particular, it is not compact. The space $\mathcal{W}$ is not compact by [13, Proposition 6.4].

Suppose that V is not discrete, and let $\beta \in \overline{K}$ be a fixed element. By Proposition 4.2, $\mathcal{V}(\beta, \bullet)$ is closed in $\mathcal{V}$; hence if $\mathcal{V}$ were compact so would be $\mathcal{V}(\beta, \bullet)$. By Theorem 4.4, it would follow that $\mathcal{X} = (-\infty, \delta(\beta, K)]_{\mathbb{Q}\Gamma_v}$ is compact. However, let $\gamma_1 > \gamma_2 > \cdots$ be a decreasing sequence of elements in $\mathbb{Q}\Gamma_v$, with $\delta(\beta, K) > \gamma_1$. Then, the family $(\gamma_1, \delta(\beta, K)], (\gamma_2, \gamma_1], \ldots, (\gamma_{n+1}, \gamma_n], \ldots$ is an open cover of $\mathcal{X}$ without finite subcovers: hence, $\mathcal{X}$ is not compact, and so neither is $\mathcal{V}$.

Let $\Psi : \mathcal{W} \longrightarrow \mathcal{V}$ be the map $W_E \mapsto V_E$ (see [13, Proposition 6.13]). Since Ψ is continuous, if $\mathcal{W}$ were compact then so would be its image $\mathcal{V}_0$. Hence, as in the previous part of the proof, also $\mathcal{V}_0 \cap \mathcal{V}(\beta, \bullet)$ would be compact; however, since $\Sigma_\beta(\mathcal{V}_0 \cap \mathcal{V}(\beta, \bullet)) = (-\infty, \delta(\beta, K)] \setminus \{+\infty\}$, we can use the same method as above (eventually substituting $(\gamma_1, +\infty]$ with $(\gamma_1, +\infty))$ to show that this set can't be compact. Hence, $\mathcal{W}$ is not compact, as claimed. $\square$

We note that, when V is a DVR, $\widehat{K}$ (and thus $\mathcal{V}$) is locally compact if and only if the residue field of V is finite [3, Chapt. VI, §5, 1., Proposition 2]. We conjecture that $\mathcal{V}$ is locally compact also when V is not discrete.

Proposition 4.6. *Let $\beta \in \overline{K}$, and let u be an extension of v to $\overline{K}$. Then, the Zariski and the constructible topologies agree on $\mathcal{V}(\beta, \bullet) = \mathcal{V}^u(\beta, \bullet)$.*

Proof. It is enough to show that $B(\phi) \cap \mathcal{V}(\beta, \bullet)$ is closed for every $\phi \in K(X)$. Suppose $\delta \in C = \Sigma_\beta(\mathcal{V}(\beta, \bullet) \setminus B(\phi))$ and let $V_E \in \mathcal{V}(\beta, \bullet) \setminus B(\phi)$: by [13, Lemma 6.6 and Remark 6.7], there is an annulus $\mathcal{C} = \mathcal{C}(\beta, \theta_1, \theta_2)$ with $\theta_1, \theta_2 \in \mathbb{Q}\Gamma_v$, $\theta_1 < \delta \leq \theta_2$ and such that $\phi(t) \notin V$ for all $t \in \mathcal{C}$. Hence, $(\theta_1, \theta_2]$ is an open neighborhood of δ in $(-\infty, \delta(\beta, K)]_{\mathbb{Q}\Gamma_v}$ contained in C; thus, C is open and $B(\phi) \cap \mathcal{V}(\beta, \bullet)$ is closed, being the complement of the image of C under the homeomorphism Σ_β^{-1} (see Theorem 4.4). $\square$

To conclude, we study the metrizability of $\mathcal{V}(\beta, \bullet)$ and $\mathcal{V}$. It is well-known [16, Counterexample 51(4)] that the upper limit topology is not metrizable, since it is separable but not second countable. Something similar happens for $(a, b]_\Lambda$.

Proposition 4.7. *Let Λ be a subset of $(a, b]$ that is dense in the Euclidean topology. The following are equivalent:*

(i) *Λ is countable;*
(ii) *$(a, b]_\Lambda$ is second-countable;*
(iii) *$(a, b]_\Lambda$ is metrizable;*

(iv) $(a,b]_\Lambda$ *is an ultrametric space.*

Proof. (iii) $\Longrightarrow$(ii) follows from the fact that $(a,b]_\Lambda$ is separable (since, for example, $\mathbb{Q}\cap(a,b]$ is dense in $(a,b]_\Lambda$); (iv) $\Longrightarrow$ (iii) is obvious.

(ii) $\Longrightarrow$ (i) Any basis of $(a,b]_\Lambda$ must contain an open set of the form $(\alpha,\lambda]$, for each $\lambda\in\Lambda$ (and some $\alpha\in(-\infty,\lambda)$). Hence, if $(a,b]_\Lambda$ is second-countable then Λ must be countable.

(i) $\Longrightarrow$ (iv) Suppose that Λ is countable, and fix an enumeration $\{\lambda_1,\lambda_2,\ldots\}$ of Λ. Let $r:\Lambda\longrightarrow\mathbb{R}$ be the map sending λ_i to $1/i$; then, for each $x,y\in(a,b]$ we set

$$d(x,y)=\begin{cases}\max\{r(\lambda)\mid\lambda\in[\min(x,y),\max(x,y))\cap\Lambda\}, & \text{if } x\neq y\\ 0, & \text{if } x=y.\end{cases}$$

We claim that d is a metric on $(a,b]$ whose topology is exactly $(a,b]_\Lambda$.

Note first that d is well-defined and nonnegative; it is also clear from the definition (and the fact that Λ is dense in $\mathbb{R}$) that $d(x,y)=0$ if and only if $x=y$ and that $d(x,y)=d(y,x)$. Let now $x,y,z\in(a,b]$, and suppose without loss of generality that $x\leq y$. If $z\leq x$, then $[z,y)\supseteq[x,y)$, and thus $d(x,y)\leq d(y,z)$; in the same way, if $y\leq z$ then $[x,z)\supseteq[x,y)$ and $d(x,y)\leq d(x,z)$. If $x\leq z\leq y$, then $[x,y)=[x,z)\cup[z,y)$; hence, $d(x,y)=\max\{d(x,z),d(y,z)\}$. In all cases, we have $d(x,y)\leq\max\{d(x,z),d(y,z)\}$, and thus d induces an ultrametric space.

Let now $x\in\Lambda\subseteq(a,b]$ and $\rho\in\mathbb{R}$ be positive; we claim that the open ball $B=B_d(x,\rho)=\{t\in(a,b]\mid d(x,t)<\rho\}$ is equal to $(y,z]$, where

$$y=\max\{\lambda\in\Lambda\cap(-\infty,x)\mid r(\lambda)\geq\rho\},$$
$$z=\min\{\lambda\in\Lambda\cap(x,+\infty)\mid r(\lambda)\geq\rho\}$$

(with the convention $\max\emptyset=a$ and $\min\emptyset=b$). Note that since $\rho>0$, there are only a finite number of λ with $r(\lambda)\geq\rho$; in particular, $y,z\in\Lambda$ and by definition, $y<x<z$.

Let $t\in(a,b]$. If $t<y$, then $r(\lambda)\geq\rho$ for some $\lambda\in(t,x)\cap\Lambda$, and thus $d(t,x)\geq\rho$, and so $t\notin B$; in the same way, if $y<t<x$, then $r(\lambda)<\rho$ for every $\lambda\in(t,x)\cap\Lambda$, and thus $t\in B$. Symmetrically, if $x<t<z$ then $t\in B$, while if $z<t$ then $t\notin B$. We thus need to analyze the cases $t=y$ and $t=z$.

By definition,

$$d(x,z)=\max\{r(\lambda)\mid\lambda\in[x,z)\cap\Lambda\};$$

since by definition $r(\lambda)<\rho$ for every $\lambda\in[x,z)\cap\Lambda$, we have $d(x,z)<\rho$ and $z\in B_d(x,\rho)$.

Since $y\in\Lambda$, we have $r(y)\geq\rho$. Thus,

$$d(x,y)=\max\{r(\lambda)\mid\lambda\in[y,x)\cap\Lambda\}\geq r(y)\geq\rho$$

and $y \notin B_d(x, \rho)$. Thus, $B_d(x, \rho) = (y, z]$ as claimed; therefore, $B_d(x, \rho)$ is open in $(a, b]_\Lambda$.

The family of the intervals $(y, z]$, as z ranges in Λ and y in $(a, b]$, is a basis of $(a, b]_\Lambda$; therefore, the topology induced by d on $(a, b]$ is exactly the Λ-upper limit topology. Hence, $(a, b]_\Lambda$ is an ultrametric space, as claimed. □

As a consequence, we obtain a necessary condition for metrizability, while in [13, Corollary 6.16] we obtained a sufficient condition, namely, if V is countable, then $\mathcal{V}$ is metrizable.

Corollary 4.8. *Let V be a valuation ring with uncountable value group. Then, $\mathcal{V}$ and $\mathrm{Zar}(K(X)|V)^{\mathrm{cons}}$ are not metrizable.*

Proof. If $\mathcal{V}$ were metrizable, so would be $\mathcal{V}(\beta, \bullet)$, in contrast to Theorem 4.4 and Proposition 4.7 (note that, if the value group of V is uncountable, in particular V is not discrete). Similarly, if $\mathrm{Zar}(K(X)|V)^{\mathrm{cons}}$ were metrizable, so would be $\mathcal{V}(\beta, \bullet)$, endowed with the constructible topology. Since the Zariski and the constructible topologies agree on $\mathcal{V}(\beta, \bullet)$ (Proposition 4.6), this is again impossible. □

5 Beyond pseudo-convergent sequences

Corollary 4.8 gives a condition for the non-metrizability of $\mathrm{Zar}(K(X)|V)^{\mathrm{cons}}$ that depends on the value group of V. In this section we prove a similar criterion, but based on the residue field of V.

Lemma 5.1. *Let V be a valuation ring with quotient field K, let L be an extension field of K and let W be an extension of V to L. Let $\pi : W \longrightarrow W/M$ be the quotient map. Then, the map*

$$\begin{aligned} \{Z \in \mathrm{Zar}(L|V) \mid Z \subseteq W\} &\longrightarrow \mathrm{Zar}(W/M_W|V/M_V), \\ Z &\longmapsto \pi(Z) \end{aligned}$$

is a homeomorphism, when both sets are endowed with either the Zariski or the constructible topology.

Proof. Apply [15, Lemma 4.2] with $D = V$. □

Lemma 5.2. *Let X be an uncountable compact topological space with at most one limit point. Then, X is not metrizable.*

Proof. Since X is infinite and compact it has a limit point, say x_0, which is also unique by assumption. Suppose that X is metrizable, and let d be a metric inducing the topology. For each integer $n > 0$, let $C_n = \{y \in X \mid 1/n \leq d(y, x_0)\}$. By construction, $x_0 \notin C_n$, and thus all points of C_n are isolated. Furthermore, C_n is closed (since it is the complement of an open ball), and thus it is compact; therefore, C_n must be finite. Hence, the countable union $\bigcup_{n>0} C_n$ is a countable set, against the fact that the union is equal to the uncountable set $X \setminus \{x_0\}$. Therefore, X is not metrizable. $\square$

Proposition 5.3. *Let V be a valuation ring with uncountable residue field. Then,* $\mathrm{Zar}(K(X)|V)^{\mathrm{cons}}$ *is not metrizable.*

Proof. Let W be the Gaussian extension of V (see e.g. [6]); then, W is an extension of V to $K(X)$ having the same value group of V and whose residue field is $k(t)$, where $k = V/M$ is the quotient field of V and t is an indeterminate. Consider $\Delta = \{Z \in \mathrm{Zar}(K(X)|V) \mid Z \subseteq W\}$; by Lemma 5.1, Δ is homeomorphic to $\mathrm{Zar}(k(t)|k)$, when both sets are endowed with the constructible topology. Hence, it is enough to prove that $\mathrm{Zar}(k(t)|k)^{\mathrm{cons}}$ is not metrizable.

The points of $\mathrm{Zar}(k(t)|k)$ are $k(t)$, $k[t^{-1}]_{(t^{-1})}$ and the valuation rings of the form $k[t]_{(f(t))}$, where $f \in k[t]$ is an irreducible polynomial. The points different from $k(t)$ are isolated: indeed, $k[t^{-1}]_{(t^{-1})}$ is the only point in the open set $\mathrm{Zar}(k(t)|k) \setminus B(t)$, while $k[t]_{(f(t))}$ is the only point in the open set $\mathrm{Zar}(k(t)|k) \setminus B(f(t)^{-1})$. Since $\mathrm{Zar}(k(t)|k)^{\mathrm{cons}}$ is compact the claim follows from Lemma 5.2. $\square$

In the following, we study more deeply spaces like $\{Z \in \mathrm{Zar}(L|V) \mid Z \subseteq W\}$ by using two classes of sequences that are similar to pseudo-convergent sequences. Let $E = \{s_n\}_{n\in\mathbb{N}}$ be a sequence in K; then, we say that:

- E is a *pseudo-divergent sequence* if $v(s_n - s_{n+1}) > v(s_{n+1} - s_{n+2})$ for every $n \in \mathbb{N}$;
- E is a *pseudo-stationary sequence* if $v(s_n - s_m) = v(s_{n'} - s_{m'})$ for every $n \neq m$, $n' \neq m'$.

These two kinds of sequences have been introduced in [4] and together with the class of pseudo-convergent sequences introduced by Ostrowski form the class of *pseudo-monotone sequences* [11]. Most of the notions introduced for pseudo-convergent sequences, like the breadth and the valuation domain V_E, can be generalized to pseudo-monotone sequences, see [14]. In particular, the notion of pseudo-limit generalizes as well; however, there are fewer subsets that can be the set $\mathcal{L}_E$ of pseudo-limits of E. More precisely:

- if E is a pseudo-divergent sequence, then there is an $\alpha \in K$ such that $\mathcal{L}_E = \{x \in K \mid v(x - \alpha) > \delta_E\}$, where δ_E is the breadth of E; if $\delta_E = v(c) \in \Gamma_v$, in particular, $\mathcal{L}_E = \alpha + cM$;
- if E is pseudo-stationary sequence, then there is an $\alpha \in K$ such that $\mathcal{L}_E = \alpha + cV$, where $v(c) = \delta_E$ is the breadth of E.

Furthermore, for every set $\mathcal{L}$ of this kind (with the additional hypothesis that the residue field of V is infinite for pseudo-stationary sequences) there is a sequence E of the right type with $\mathcal{L} = \mathcal{L}_E$. In particular, both pseudo-divergent sequences and pseudo-stationary sequences always have a pseudo-limit in E, and the elements of E themselves are pseudo-limits of E ([14, Lemma 2.5]). For both pseudo-divergent and pseudo-stationary sequences the ring V_E is uniquely determined by the pseudo-limits: i.e., if E, F are pseudo-divergent (respectively, pseudo-stationary) then $V_E = V_F$ if and only if $\mathcal{L}_E = \mathcal{L}_F$. For details, see [14, Section 2.4].

Suppose that the residue field k of V is infinite, and let $Z = \{z_t\}_{t\in k}$ be a complete set of residues of k. Fix two elements $\alpha, c \in K$, and let $\delta = v(c)$. Let $\mathcal{L} = \{x \in K \mid v(x-\alpha) \geq \delta\} = \alpha + cV$ be the closed ball of center α and radius δ. Then, there are a pseudo-convergent sequence E and a pseudo-stationary sequence F such that $\mathcal{L}_E = \mathcal{L} = \mathcal{L}_F$; by [14, Proposition 7.1], $V_E \subsetneq V_F$.

For every $z \in Z$, there is also a pseudo-divergent sequence D_z such that $\mathcal{L}_{D_z} = \alpha - cz + cM$. Then, $V_{D_z} \neq V_E$ and $V_{D_z} \subsetneq V_F$ for every $z \in Z$; furthermore, $V_{D_z} \neq V_{D_{z'}}$ if $z \neq z'$. Let

$$\mathcal{X}_{\alpha,\delta} = \{V_E, V_F, V_{D_z} \mid z \in Z\}$$

be the set of the rings in this form. By [14, Proposition 7.2], the map $\widetilde{\pi}$ of Lemma 5.1 restricts to

$$\begin{aligned} \widetilde{\pi}\colon \mathcal{X}_{\alpha,\delta} &\longrightarrow \mathrm{Zar}(k(t)|k) \\ V_F &\longmapsto k(t), \\ V_E &\longmapsto k[1/t]_{(1/t)}, \\ V_{D_z} &\longmapsto k[t]_{(t-\pi(z))}, \end{aligned}$$

and the lemma guarantees that $\widetilde{\pi}$ is also a homeomorphism between $\mathcal{X}_{\alpha,\delta}$ and its image.

In particular, we get the following; we denote by $\mathcal{V}_{\mathrm{div}}$ the set of valuation rings V_E, as E ranges among the pseudo-divergent sequences.

Proposition 5.4. *Let $\mathcal{V}_{\mathrm{div}}(\bullet, \delta) = \{V_E \mid E$ is a pseudo-divergent sequence with $\delta_E = \delta\}$. Then:*

(a) if $\delta \notin \Gamma_v$, then $\mathcal{V}_{\mathrm{div}}(\bullet, \delta) = \mathcal{V}_K(\bullet, \delta)$;

(b) if $\delta \in \Gamma_v$ and the residue field of V is finite, then $\mathcal{V}_{\mathrm{div}}(\bullet, \delta)$ is discrete (with respect to the Zariski and the constructible topology);

(c) if $\delta \in \Gamma_v$ and the residue field of V is infinite, then $\mathcal{V}_{\mathrm{div}}(\bullet, \delta)$ is not Hausdorff (with respect to the Zariski topology).

In particular, if the residue field of V is infinite then $\mathcal{V}_{\mathrm{div}}$ is not Hausdorff, with respect to the Zariski topology.

Proof. (a) If $\delta \notin \Gamma_v$, then for every $\beta \in K$ we have $\{x \in K \mid v(x-\beta) \geq \delta\} = \{x \in K \mid v(x-\beta) > \delta\}$. Hence, if E, F are, respectively, a pseudo-convergent and a pseudo-divergent sequence having β as a pseudo-limit and having breadth δ then $\mathcal{L}_E = \mathcal{L}_F$, and thus by [14, Proposition 5.1] $V_E = V_F$. Since every pseudo-divergent sequence has pseudo-limits in K, it follows that $\mathcal{V}_{\mathrm{div}}(\bullet, \delta) = \mathcal{V}_K(\bullet, \delta)$.

(b) Suppose that $\delta \in \Gamma_v$, and let $c \in K$ be such that $v(c) = \delta$. Let E be a pseudo-divergent sequence with breadth δ, and let $\alpha \in \mathcal{L}_E$; since the residue field is finite we can find $\beta_1, \ldots, \beta_k \in K$ such that $0, \frac{\alpha-\beta_1}{c}, \ldots, \frac{\alpha-\beta_k}{c}$ is a complete set of residues of the residue field of V.

We claim that

$$\{V_E\} = B\left(\frac{X-\alpha}{c}\right) \cap B\left(\frac{c}{X-\beta_k}\right) \cap \cdots \cap B\left(\frac{c}{X-\beta_1}\right) \cap \mathcal{V}_{\mathrm{div}}(\bullet, \delta).$$

Let Ω be the intersection on the right hand side. Since $\alpha \in \mathcal{L}_E$ the value $v(s_n - \alpha)$ decreases to δ, and thus $v\left(\frac{s_n-\alpha}{c}\right)$ is always positive; in particular, $V_E \in B\left(\frac{X-\alpha}{c}\right)$. On the other hand, $v(s_n - \beta_i) = v(s_n - \alpha + \alpha - \beta_i) = v(\alpha - \beta_i) = \delta$ for every $i \in \{1, \ldots, k\}$ and every n, and thus $v\left(\frac{c}{s_n-\beta_i}\right) = 0$, i.e., $V_E \in B\left(\frac{c}{X-\beta_1}\right)$. Hence, $V_E \in \Omega$.

Suppose now that $F = \{t_n\}_{n\in\mathbb{N}}$ is a pseudo-divergent sequence such that $V_F \in \Omega$. Then, $V_F \in B\left(\frac{X-\alpha}{c}\right)$, i.e., $v(t_n - \alpha) \geq \delta$ for all large n, and thus F must be eventually contained in the closed ball $\{x \in K \mid v(x-\alpha) \geq \delta\} = \alpha + cV = \beta_i + cV$ (for every i). Since F has breadth δ, by the discussion after Proposition 5.3 its set $\mathcal{L}_F$ of pseudo-limits is in the form $z + cM_V$, where z is any element of $\mathcal{L}_F$; therefore, $\mathcal{L}_F$ is either $\alpha + cM_V$ or $\beta_i + cM_V$ for some i (by the assumption on the β_i's). However, if $\mathcal{L}_F = \beta_i + cM_V$ then $v(t_n - \beta_i) > \delta$ for all n, which implies that $V_F \notin B\left(\frac{c}{X-\beta_i}\right)$, against $V_F \in \Omega$; therefore, $\mathcal{L}_F = \alpha + cM_V = \mathcal{L}_E$ and thus $V_F = V_E$ by by [14, Proposition 5.1]. Therefore, $\Omega = \{V_E\}$ and V_E is isolated. Since V_E was arbitrary, $\mathcal{V}_{\mathrm{div}}(\bullet, \delta)$ is discrete.

(c) Suppose that $\delta \in \Gamma_v$ and that the residue field is infinite. With the notation as before the statement, consider the set $\mathcal{X}_d(\alpha, \delta) = \mathcal{X}_{\alpha,\delta} \setminus \{V_F, V_E\}$: then, $\mathcal{X}_d(\alpha, \delta)$ is a subset of $\mathcal{V}_{\mathrm{div}}(\bullet, \delta)$, and by Lemma 5.1 it is homeomorphic to $\Lambda = \{k[t]_{(t-z)} \mid z \in k\} \subseteq \mathrm{Zar}(k(t)|k)$. The map $\mathrm{Spec}(k[t]) \longrightarrow \mathrm{Zar}(k[t])$, $P \mapsto K[t]_P$, is a homeomorphism (when both sets are endowed with the respective Zariski topologies) [5, Lemma 2.4], and thus Λ is homeomorphic to $\Lambda_s = \{(t-z) \mid z \in k\} \subseteq \mathrm{Max}(k[t])$. The Zariski topology on $\mathrm{Max}(k[t])$ coincides with the cofinite topology; since Λ_s is infinite, it follows that Λ_s is not Hausdorff; thus, neither Λ nor $\mathcal{X}_d(\alpha, \delta)$ nor $\mathcal{V}_{\mathrm{div}}(\bullet, \delta)$ are Hausdorff. $\square$

On the other hand, if we fix a pseudo-limit, we obtain a situation very similar to the pseudo-convergent case.

Proposition 5.5. *Let $\beta \in K$, and let $\mathcal{V}_{\mathrm{div}}(\beta, \bullet) = \{V_E \mid E$ is a pseudo-divergent sequence with $\beta \in \mathcal{L}_E\}$. Then,*

$$\mathcal{V}_{\mathrm{div}}(\beta, \bullet) \simeq \mathcal{V}(\beta, \bullet) \simeq (-\infty, +\infty]_{\mathbb{Q}\Gamma_v}.$$

Proof. For every $\beta, \beta' \in K$, we have $\mathcal{V}_{\mathrm{div}}(\beta, \bullet) \simeq \mathcal{V}_{\mathrm{div}}(\beta', \bullet)$ and $\mathcal{V}(\beta, \bullet) \simeq \mathcal{V}(\beta', \bullet)$, so we can suppose $\beta = 0$.

Consider the map

$$\begin{aligned} \psi\colon K(X) &\longrightarrow K(X) \\ \phi(X) &\longmapsto \phi(1/X). \end{aligned}$$

Then, ψ is a K-automorphism of $K(X)$ that coincide with its own inverse, and thus it induces a self-homeomorphism

$$\begin{aligned} \overline{\psi}\colon \mathrm{Zar}(K(X)|V) &\longrightarrow \mathrm{Zar}(K(X)|V) \\ V_E &\longmapsto \psi(V_E). \end{aligned}$$

We claim that $\overline{\psi}$ sends $\mathcal{V}_{\mathrm{div}}(0, \bullet)$ to $\mathcal{V}(0, \bullet)$, and conversely.

Note first that, for every $\phi \in K(X)$ and every $t \in K$, we have $\phi(t) = (\psi(\phi))(t^{-1})$.

Suppose $E = \{s_n\}_{n\in\mathbb{N}}$ is a pseudo-divergent sequence having 0 as a pseudo-limit; without loss of generality, $0 \neq s_n$ for every n. Then, $\delta_n = v(s_n)$ is decreasing, and thus $F = \{s_n^{-1}\}_{n\in\mathbb{N}}$ is a pseudo-convergent sequence having 0 as a pseudo-limit. Then, $\phi(s_n) = (\psi(\phi))(s_n^{-1})$, and thus $\phi \in V_E$ if and only if $\psi(\phi) \in V_F$, i.e., $\psi(V_E) = V_F$, so that $\overline{\psi}(\mathcal{V}_{\mathrm{div}}(0, \bullet)) \subseteq \mathcal{V}(0, \bullet)$. Conversely, if $F = \{t_n\}_{n\in\mathbb{N}}$ is a pseudo-convergent sequence having 0 as a pseudo-limit, then $E = \{t_n^{-1}\}_{n\in\mathbb{N}}$ is a pseudo-divergent sequence with $0 \in \mathcal{L}_E$, and as above $\phi \in V_F$ if and only if $\psi(\phi) \in V_E$, i.e., $\overline{\psi}(\mathcal{V}(0, \bullet)) \subseteq \mathcal{V}_{\mathrm{div}}(0, \bullet)$.

Since $\overline{\psi}$ is idempotent, it follows that $\overline{\psi}(\mathcal{V}(0, \bullet)) = \mathcal{V}_{\mathrm{div}}(0, \bullet)$, and so $\mathcal{V}_{\mathrm{div}}(0, \bullet)$ and $\mathcal{V}(0, \bullet)$ are homeomorphic. The homeomorphism $\mathcal{V}(0, \bullet) \simeq (-\infty, +\infty]_{\mathbb{Q}\Gamma_v}$ follows from Theorem 4.4. □

Note that, while the homeomorphism between $\mathcal{V}(\beta, \bullet)$ and $(-\infty, +\infty]_{\mathbb{Q}\Gamma_v}$ is constructed by sending V_E to δ_E (Theorem 4.4), the one between $\mathcal{V}_{\mathrm{div}}(\beta, \bullet)$ and $(-\infty, +\infty]_{\mathbb{Q}\Gamma_v}$ sends V_E to $-\delta_E$.

We conclude with analyzing the pseudo-stationary case, showing that the two partitions give rise to especially uninteresting spaces.

Proposition 5.6. *The following hold.*

(a) For every $\delta \in \Gamma_v$, the set

$$\mathcal{V}_{\mathrm{stat}}(\bullet, \delta) = \{V_E \mid E \text{ is a pseudo-stationary sequence with } \delta_E = \delta\}$$

is discrete, with respect to the Zariski and the constructible topology.

(b) For every $\beta \in K$, the set

$$\mathcal{V}_{\text{stat}}(\beta, \bullet) = \{V_E \mid E \text{ is a pseudo-stationary sequence with } \beta \in \mathcal{L}_E\}$$

is discrete, with respect to the Zariski and the constructible topology.

Proof. Since the constructible topology is finer than the Zariski topology, it is enough to prove the claim for the latter.

(a) Take a pseudo-stationary sequence $E = \{s_n\}_{n \in \mathbb{N}}$ of breadth δ, and let $\beta \in \mathcal{L}_E$; let also $c \in K$ be such that $v(c) = \delta$. Consider the function $\phi(X) = \frac{X-\beta}{c}$; we claim that $B(\phi) \cap \mathcal{V}_{\text{stat}}(\bullet, \delta) = \{V_E\}$.

Indeed, for large n we have $v(s_n - \beta) = \delta$, and thus $v(\phi(s_n)) = v(s_n - \beta) - v(c) = 0$, so that $\phi \in V_E$, i.e., $V_E \in B(\phi)$. Conversely, suppose $V_F \in B(\phi)$, where $F = \{t_n\}_{n \in \mathbb{N}}$ is pseudo-stationary with breadth δ. Then, for large n, we must have $v(t_n - \beta) \geq \delta$. Since $v(t_n - t_m) = \delta$ for $n \neq m$, we must have $v(t_n - \beta) = \delta$, i.e., β is a pseudo-limit of F. Thus, $\mathcal{L}_E = \beta + cV = \mathcal{L}_F$ and $V_E = V_F$ by [14, Proposition 5.1],

Therefore, $B(\phi) \cap \mathcal{V}_{\text{stat}}(\bullet, \delta) = \{V_E\}$ and V_E is an isolated point of $\mathcal{V}_{\text{stat}}(\bullet, \delta)$. Since V_E was arbitrary, $\mathcal{V}_{\text{stat}}(\bullet, \delta)$ is discrete, as claimed.

(b) Let $E = \{s_n\}_{n \in \mathbb{N}}$ be a pseudo-stationary sequence having β as a pseudo-limit, and let $c \in K$ be such that $v(c) = \delta_E$. Let $\phi(X) = \frac{X-\beta}{c}$; we claim that $B(\phi, \phi^{-1}) \cap \mathcal{V}_{\text{stat}}(\beta, \bullet) = \{V_E\}$.

The proof that $V_E \in B(\phi, \phi^{-1})$ follows as in the previous case. Suppose now that $F = \{t_n\}_{n \in \mathbb{N}}$ is in the intersection. Then, we must have $v(\phi(t_n)) \geq 0$ and $v(\phi^{-1}(t_n)) = -v(\phi(t_n)) \geq 0$; thus, $v(t_n - \beta) = \delta_E$ for large n. However, since β is a pseudo-limit of F, we also have $v(t_n - \beta) = \delta_F$; hence, $\delta_E = \delta_F$ and $V_E = V_F$. Therefore, as above, V_E is an isolated point of $\mathcal{V}_{\text{stat}}(\beta, \bullet)$, which thus is discrete. □

References

[1] V. Alexandru, N. Popescu, Al. Zaharescu, *A theorem of characterization of residual transcendental extensions of a valuation.* J. Math. Kyoto Univ. 28 (1988), no. 4, 579-592.

[2] V. Alexandru, N. Popescu, Al. Zaharescu, *Minimal pairs of definition of a residual transcendental extension of a valuation*, J. Math. Kyoto Univ. 30 (1990), no. 2, 207-225.

[3] N. Bourbaki, *Algèbre commutative*, Hermann, Paris, 1961.

[4] J.-L. Chabert, *On the polynomial closure in a valued field*, J. Number Theory 130 (2010), 458-468.

[5] D. E. Dobbs, R. Fedder and M. Fontana, *Abstract Riemann surfaces of integral domains and spectral spaces*, Ann. Mat. Pura Appl. (4) **148** (1987) 101–115.

[6] R. Gilmer, *Multiplicative ideal theory*, Marcel Dekker Inc., New York, 1972, Pure and Applied Mathematics, No. 12.

[7] M. Hochster, *Prime ideal structure in commutative rings*, Trans. Amer. Math. Soc., 142 (1969), 43–60.

[8] I. Kaplansky, *Maximal Fields with Valuation*, Duke Math. J. 9 (1942), 303-321.

[9] S. MacLane, *A construction for absolute values in polynomial rings*, Trans. Amer. Math. Soc. 40 (1936), no. 3, 363–395.

[10] A. Ostrowski, *Untersuchungen zur arithmetischen Theorie der Körper*, Math. Z. 39 (1935), 269-404.

[11] G. Peruginelli, *Prüfer intersection of valuation domains of a field of rational functions*, J. Algebra 509 (2018), 240-262.

[12] G. Peruginelli, *Transcendental extensions of a valuation domain of rank one*, Proc. Amer. Math. Soc. 145 (2017), no. 10, 4211-4226.

[13] G. Peruginelli, D. Spirito, *The Zariski-Riemann space of valuation domains associated to pseudo-convergent sequences*, Trans. Amer. Math. Soc. 373 (2020), no. 11, 7959–7990.

[14] G. Peruginelli, D. Spirito, *Extending valuations to the field of rational functions using pseudo-monotone sequences*, J. Algebra 586 (2021), 756-786.

[15] D. Spirito, *Isolated points of the Zariski space*, preprint, arXiv: https://arxiv.org/abs/2009.11141.

[16] L. A. Steen and J. A. Seebach, *Counterexamples in Topology.* Springer-Verlag, New York-Heidelberg, second edition, 1978.

[17] M. Vaquié, Michel, *Extension d'une valuation*, Trans. Amer. Math. Soc. 359 (2007), no. 7, 3439–3481.

[18] S. Warner, *Topological Fields.* North-Holland Mathematics Studies, 157.

[19] O. Zariski, *The reduction of the singularities of an algebraic surface*, Ann. of Math. (2), 40:639–689, 1939.

[20] O. Zariski, *The compactness of the Riemann manifold of an abstract field of algebraic functions*, Bull. Amer. Math. Soc., 50:683–691, 1944.

[21] O. Zariski, P. Samuel, *Commutative Algebra, vol. II*, Springer-Verlag, New York-Heidelberg-Berlin, 1975.