J. reine angew. Math. **751** (2019), 185–241
DOI 10.1515/crelle-2016-0062

Journal für die reine und angewandte Mathematik

Enhanced perversities

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Abstract. On a complex manifold, the Riemann–Hilbert correspondence embeds the triangulated category of (not necessarily regular) holonomic $\mathcal{D}$-modules into the triangulated category of $\mathbb{R}$-constructible enhanced ind-sheaves. The source category has a standard t-structure. Here, we provide the target category with a middle perversity t-structure, and prove that the embedding is exact.

In the paper, we also discuss general perversities in the framework of $\mathbb{R}$-constructible enhanced ind-sheaves on bordered subanalytic spaces.

Contents

Introduction

On a complex manifold X, the classical Riemann–Hilbert correspondence establishes an equivalence

$$\mathcal{DR}_X : \mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$$

between the derived category of $\mathcal{D}_X$-modules with regular holonomic cohomologies, and the derived category of sheaves of $\mathbb{C}$-vector spaces on X with $\mathbb{C}$-constructible cohomologies ([7]). Here,

$$\mathcal{DR}_X(\mathcal{M}) = \Omega_X \otimes^{\mathrm{L}}_{\mathcal{D}_X} \mathcal{M}$$

is the de Rham functor, and Ω_X the sheaf of top-degree holomorphic differential forms.

The first author was partially supported by grant CPDA159224, Padova University. The second author was supported by Grant-in-Aid for Scientific Research (B) 15H03608, Japan Society for the Promotion of Science.

Moreover, the functor $\mathcal{DR}_X$ interchanges the standard t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)$ with the middle perversity t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$. In particular, $\mathcal{DR}_X$ induces an equivalence between the abelian category of regular holonomic $\mathcal{D}_X$-modules and that of perverse sheaves on X.

The Riemann–Hilbert correspondence of [4] provides a fully faithful embedding

$$\mathcal{DR}^{\mathrm{E}}_X \colon \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) \rightarrowtail \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_X)$$

from the derived category of $\mathcal{D}_X$-modules with (not necessarily regular) holonomic cohomologies, into the triangulated category of $\mathbb{R}$-constructible enhanced ind-sheaves of $\mathbb{C}$-vector spaces on X. Here, $\mathcal{DR}^{\mathrm{E}}_X$ is the enhanced version of the de Rham functor. The source category $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ has a standard t-structure. In this paper, we provide the target category $\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_X)$ with a *generalized* middle perversity t-structure, and prove that $\mathcal{DR}^{\mathrm{E}}_X$ is an exact functor.

Generalized t-structures have been introduced in [10], as a reinterpretation of the notion of slicing from [3]. For example, let $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ be the derived category of sheaves of $\mathbb{C}$-vector spaces on X with $\mathbb{R}$-constructible cohomologies. Then, if X has positive dimension, $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ does not admit a middle perversity t-structure in the classical sense. That is, there is no perversity whose induced t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ is self-dual. However, it is shown in [10] that $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ has a natural middle perversity t-structure in the generalized sense. This generalized t-structure induces the middle perversity t-structure on the subcategory $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$. Moreover, it is compatible with our construction of the generalized middle perversity t-structure on $\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_X)$, since the natural embedding

$$\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X) \rightarrowtail \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_X)$$

turns out to be exact.

From now on, we shall use the term t-structure for the one in the generalized sense, and refer to the classical notion as a *classical* t-structure.

Let $\mathbf{k}$ be a field and M a real analytic manifold, or more generally a bordered subanalytic space. Let $\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_M)$ be the triangulated category of $\mathbb{R}$-constructible enhanced ind-sheaves of $\mathbf{k}$-vector spaces on M. In this paper, we also discuss the t-structures on $\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_M)$ associated with arbitrary perversities, and study their functorial properties. Let us give some details.

On the set of maps $p\colon \mathbb{Z}_{\geqslant 0} \to \mathbb{R}$, consider the involution $*$ given by $p^*(n) := -p(n) - n$. A perversity is a map $p\colon \mathbb{Z}_{\geqslant 0} \to \mathbb{R}$ such that p and p^* are decreasing.

Let $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ be the derived category of $\mathbb{R}$-constructible sheaves of $\mathbf{k}$-vector spaces on M. For a locally closed subset Z of M, let $\mathbf{k}_Z$ be the extension by zero to M of the constant sheaf on Z. For $c \in \mathbb{R}$, set

$$\begin{aligned}
{}^p\mathrm{D}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_M) := \{F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M);\ & \text{for any } k \in \mathbb{Z}_{\geqslant 0} \text{ there exists a closed} \\
& \text{subanalytic subset } Z \subset M \text{ of dimension} < k \text{ such that} \\
& H^j(\mathbf{k}_{M \setminus Z} \otimes F) \simeq 0 \text{ for } j > c + p(k)\}, \\
{}^p\mathrm{D}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_M) := \{F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M);\ & \text{for any } k \in \mathbb{Z}_{\geqslant 0} \text{ and any closed} \\
& \text{subanalytic subset } Z \subset M \text{ of dimension} \leqslant k \text{ one has} \\
& H^j \mathrm{R}\mathcal{H}om(\mathbf{k}_Z, F) \simeq 0 \text{ for } j < c + p(k)\}.
\end{aligned}$$

Then $({}^p\mathrm{D}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_M), {}^p\mathrm{D}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_M))_{c \in \mathbb{R}}$ is a t-structure in the sense of Definition 1.2.2. Moreover, the duality functor interchanges ${}^p\mathrm{D}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ and ${}^{p^*}\mathrm{D}^{\geqslant -c}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$. In particular, the t-structure $({}^{1/2}\mathrm{D}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_M), {}^{1/2}\mathrm{D}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_M))_{c \in \mathbb{R}}$ associated with the middle perversity $n \mapsto -n/2$ is self-dual.

The analogous definition for $\mathbb{R}$-constructible enhanced ind-sheaves is

$$\begin{aligned}
{}_p\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_M) := \{K \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_M);\ & \text{for any } k \in \mathbb{Z}_{\geqslant 0} \text{ there exists a closed} \\
& \text{subanalytic subset } Z \subset M \text{ of dimension} < k \text{ such that} \\
& H^j(\pi^{-1}\mathbf{k}_{M\setminus Z} \otimes K) \simeq 0 \text{ for } j > c + p(k)\}, \\
{}_p\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_M) := \{K \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_M);\ & \text{for any } k \in \mathbb{Z}_{\geqslant 0} \text{ and any closed} \\
& \text{subanalytic subset } Z \subset M \text{ of dimension} \leqslant k \text{ one has} \\
& H^j\mathrm{R}\mathcal{I}hom(\pi^{-1}\mathbf{k}_Z, K) \simeq 0 \text{ for } j < c + p(k)\}.
\end{aligned}$$

It turns out that $({}_p\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_M), {}_p\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_M))_{c\in\mathbb{R}}$ is a t-structure, but it does not behave well with respect to the duality functor $\mathrm{D}^{\mathrm{E}}_M$. Hence we set

$$\begin{aligned}
{}^p\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_M) &:= \{K \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_M);\ K \in {}_p\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_M),\ \mathrm{D}^{\mathrm{E}}_M K \in {}_{p^*}\mathrm{E}^{\geqslant -c-1/2}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_M)\}, \\
{}^p\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_M) &:= \{K \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_M);\ K \in {}_p\mathrm{E}^{\geqslant c-1/2}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_M),\ \mathrm{D}^{\mathrm{E}}_M K \in {}_{p^*}\mathrm{E}^{\leqslant -c}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_M)\}.
\end{aligned}$$

Then

$$({}^p\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_M), {}^p\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_M))_{c\in\mathbb{R}}$$

is a t-structure, and the duality functor interchanges ${}^p\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_M)$ and ${}^{p^*}\mathrm{E}^{\geqslant -c}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_M)$. In particular, the t-structure

$$({}^{1/2}\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_M), {}^{1/2}\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_M))_{c\in\mathbb{R}}$$

associated with the middle perversity $n \mapsto -n/2$ is self-dual.

Going back to the Riemann–Hilbert correspondence, the enhanced de Rham functor

$$\mathcal{DR}^{\mathrm{E}}_X \colon \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) \rightarrowtail \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_X)$$

is exact with respect to the t-structure associated with the middle perversity.

The contents of this paper are as follows. In Section 1, we recall the notion of t-structure on a triangulated category. We also recall the t-structure on the derived category of $\mathbb{R}$-constructible sheaves on a subanalytic space associated with a given perversity. In Section 2, we recall the notions of ind-sheaves and of enhanced ind-sheaves on a bordered space. In both cases we also discuss the exactness of Grothendieck operations with respect to the standard classical t-structures. In Section 3, we introduce the t-structure(s) on the derived category of $\mathbb{R}$-constructible enhanced ind-sheaves on a bordered subanalytic space associated with a given perversity. We also discuss the exactness of Grothendieck operations with respect to these t-structures. Finally, in Section 4, we prove the exactness of the embedding, provided by the Riemann–Hilbert correspondence, from the triangulated category of holonomic $\mathcal{D}$-modules on a complex manifold into that of $\mathbb{R}$-constructible enhanced ind-sheaves.

Notations

In this paper, we take a field $\mathbf{k}$ as base ring.

For a category $\mathcal{C}$, we denote by $\mathcal{C}^{\mathrm{op}}$ the opposite category of $\mathcal{C}$. One says that a full subcategory $\mathcal{S}$ of a category $\mathcal{C}$ is *strictly full* if it contains every object of $\mathcal{C}$ which is isomorphic to an object of $\mathcal{S}$.

Let $\mathcal{C}, \mathcal{C}'$ be categories and $F : \mathcal{C} \to \mathcal{C}'$ a functor. The *essential image* of $\mathcal{C}$ by F, denoted by $F(\mathcal{C})$, is the strictly full subcategory of $\mathcal{C}'$ consisting of objects which are isomorphic to $F(X)$ for some $X \in \mathcal{C}$.

For a ring A, we denote by A^{op} the opposite ring of A.

We say that a topological space is *good* if it is Hausdorff, locally compact, countable at infinity, and has finite soft dimension.

1. t-structures and perversities

The notion of t-structure on a triangulated category was introduced in [1]. As shown in [18], the derived category of a quasi-abelian category has two natural t-structures. They were presented in [9] in a unified manner, by generalizing the notion of t-structure. A further generalization is described in [10], reinterpreting the notion of slicing from [3]. In the present paper, we use the term t-structure in this more general sense, and we refer to the notion introduced in [1] as a *classical* t-structure. A basic result of [1] asserts that the *heart* of a classical t-structure is an abelian category. More generally, it is shown in [3] that small *slices* of a t-structure are quasi-abelian categories.

It is shown in [1] that, on a complex manifold, the *middle perversity* induces a self-dual classical t-structure on the triangulated category of $\mathbb{C}$-constructible sheaves. On a real analytic manifold, using results of [11], it is shown in [10] that the middle perversity induces a self-dual t-structure on the triangulated category of $\mathbb{R}$-constructible sheaves.

Here we recall these facts, considering general perversities.

1.1. Categories. References are made to [11, Chapter I], and to [18] for the notion of quasi-abelian category (see also [9, Section 2]).

Let $\mathcal{C}$ be an additive category. The *left and right orthogonal* of a subcategory $\mathcal{S}$ are the strictly full subcategories

$$
\begin{aligned}
{}^{\perp}\mathcal{S} &:= \{X \in \mathcal{C}; \mathrm{Hom}_{\mathcal{C}}(X, Y) \simeq 0 \text{ for any } Y \in \mathcal{S}\},\\
\mathcal{S}^{\perp} &:= \{X \in \mathcal{C}; \mathrm{Hom}_{\mathcal{C}}(Y, X) \simeq 0 \text{ for any } Y \in \mathcal{S}\}.
\end{aligned}
$$

Assume that $\mathcal{C}$ admits kernels and cokernels. Given $f : X \to Y$ a morphism in $\mathcal{C}$, one sets

$$\mathrm{im}\, f := \ker(Y \to \mathrm{coker}\, f), \quad \mathrm{coim}\, f := \mathrm{coker}(\ker f \to X).$$

The morphism f is called *strict* if the canonical morphism $\mathrm{coim}\, f \to \mathrm{im}\, f$ is an isomorphism.

The category $\mathcal{C}$ is called *abelian* if all morphisms are strict. It is called *quasi-abelian* if every pull-back of a strict epimorphism is a strict epimorphism, and every pushout of a strict monomorphism is a strict monomorphism.

1.2. t-structures. Let $\mathcal{T}$ be a triangulated category. Recall the notion of t-structure from [1].

Definition 1.2.1. A *classical t-structure* $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$ on $\mathcal{T}$ is a pair of strictly full subcategories of $\mathcal{T}$ such that, setting

$$\mathcal{T}^{\leqslant n} := \mathcal{T}^{\leqslant 0}[-n], \quad \mathcal{T}^{\geqslant n} := \mathcal{T}^{\geqslant 0}[-n]$$

for $n \in \mathbb{Z}$, one has:

(a) $\mathcal{T}^{\leqslant 0} \subset \mathcal{T}^{\leqslant 1}$ and $\mathcal{T}^{\geqslant 1} \subset \mathcal{T}^{\geqslant 0}$,

(b) $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}) = 0$,

(c) for any $X \in \mathcal{T}$, there exists a distinguished triangle

$$X_{\leqslant 0} \longrightarrow X \longrightarrow X_{\geqslant 1} \xrightarrow{+1}$$

in $\mathcal{T}$ with $X_{\leqslant 0} \in \mathcal{T}^{\leqslant 0}$ and $X_{\geqslant 1} \in \mathcal{T}^{\geqslant 1}$.

The following definition of [10] is a reinterpretation of the notion of slicing from [3].

Definition 1.2.2. A *t-structure* $(\mathcal{T}^{\leqslant c}, \mathcal{T}^{\geqslant c})_{c\in\mathbb{R}}$ on $\mathcal{T}$ is a pair of families of strictly full subcategories of $\mathcal{T}$ satisfying conditions (a)–(d) below, where we set

$$\mathcal{T}^{<c} := \bigcup_{c'<c} \mathcal{T}^{\leqslant c'} \quad \text{and} \quad \mathcal{T}^{>c} := \bigcup_{c'>c} \mathcal{T}^{\geqslant c'} \quad \text{for } c \in \mathbb{R}.$$

(a) $\mathcal{T}^{\leqslant c} = \bigcap_{c'>c} \mathcal{T}^{\leqslant c'}$ and $\mathcal{T}^{\geqslant c} = \bigcap_{c'<c} \mathcal{T}^{\geqslant c'}$ for any $c \in \mathbb{R}$,

(b) $\mathcal{T}^{\leqslant c+1} = \mathcal{T}^{\leqslant c}[-1]$ and $\mathcal{T}^{\geqslant c+1} = \mathcal{T}^{\geqslant c}[-1]$ for any $c \in \mathbb{R}$,

(c) $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}^{<c}, \mathcal{T}^{>c}) = 0$ for any $c \in \mathbb{R}$,

(d) for any $X \in \mathcal{T}$ and $c \in \mathbb{R}$, there are distinguished triangles in $\mathcal{T}$

$$X_{\leqslant c} \longrightarrow X \longrightarrow X_{>c} \xrightarrow{+1} \quad \text{and} \quad X_{<c} \longrightarrow X \longrightarrow X_{\geqslant c} \xrightarrow{+1}$$

with $X_L \in \mathcal{T}^L$ for L equal to $\leqslant c$, $> c$, $< c$ or $\geqslant c$.

Note that condition (c) is equivalent to either of the following:

(c)′ $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}^{\leqslant c}, \mathcal{T}^{>c}) = 0$ for any $c \in \mathbb{R}$,

(c)″ $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}^{<c}, \mathcal{T}^{\geqslant c}) = 0$ for any $c \in \mathbb{R}$.

Moreover, under condition (a), for any $c \in \mathbb{R}$ one has

$$\mathcal{T}^{\leqslant c} = \bigcap_{c'>c} \mathcal{T}^{<c'}, \quad \mathcal{T}^{\geqslant c} = \bigcap_{c'<c} \mathcal{T}^{>c'},$$

as follows from [10, Lemma 1.1].

Let $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$ be a classical t-structure. For $c \in \mathbb{R}$, set

$$\begin{aligned} \mathcal{T}^{\leqslant c} &:= \mathcal{T}^{\leqslant 0}[-n] \quad \text{for } n \in \mathbb{Z} \text{ such that } n \leqslant c < n+1, \\ \mathcal{T}^{\geqslant c} &:= \mathcal{T}^{\geqslant 0}[-n] \quad \text{for } n \in \mathbb{Z} \text{ such that } n-1 < c \leqslant n. \end{aligned}$$

Then $(\mathcal{T}^{\leqslant c}, \mathcal{T}^{\geqslant c})_{c\in\mathbb{R}}$ is a t-structure. A classical t-structure is regarded as a t-structure by this correspondence.

Conversely, if $(\mathcal{T}^{\leqslant c}, \mathcal{T}^{\geqslant c})_{c\in\mathbb{R}}$ is a t-structure, then

$$(1.2.1) \qquad (\mathcal{T}^{\leqslant c+1}, \mathcal{T}^{>c}) \quad \text{and} \quad (\mathcal{T}^{<c+1}, \mathcal{T}^{\geqslant c})$$

are classical t-structures for any $c \in \mathbb{R}$.

For $c \in \mathbb{R}$, set

$$\mathcal{T}^{c} := \mathcal{T}^{\leqslant c} \cap \mathcal{T}^{\geqslant c}.$$

Definition 1.2.3. Let $\Sigma \subset \mathbb{R}$ be a discrete subset such that $\Sigma = \Sigma + \mathbb{Z}$. A t-structure $(\mathcal{T}^{\leqslant c}, \mathcal{T}^{\geqslant c})_{c\in\mathbb{R}}$ is *indexed by* Σ if $\mathcal{T}^c = 0$ for any $c \in \mathbb{R} \setminus \Sigma$.

If Σ is non-empty, this is equivalent to the fact that for any $c \in \mathbb{R}$ one has $\mathcal{T}^{<c} = \mathcal{T}^{\leqslant s'}$, $\mathcal{T}^{\leqslant c} = \mathcal{T}^{\leqslant s''}$, $\mathcal{T}^{>c} = \mathcal{T}^{\geqslant t'}$, $\mathcal{T}^{\geqslant c} = \mathcal{T}^{\geqslant t''}$, where

$$\begin{aligned} s' &:= \max\{s \in \Sigma; s < c\}, \quad s'' := \max\{s \in \Sigma; s \leqslant c\}, \\ t' &:= \min\{s \in \Sigma; s > c\}, \quad t'' := \min\{s \in \Sigma; s \geqslant c\}. \end{aligned}$$

Classical t-structures correspond to t-structures indexed by $\mathbb{Z}$. In this paper, we will mainly consider t-structures indexed by $\frac{1}{2}\mathbb{Z}$.

The following lemma is easily proved by using [10, Lemma 1.1 (iii)].

Lemma 1.2.4. *Let $(\mathcal{T}^{\leqslant c}, \mathcal{T}^{\geqslant c})_{c\in\mathbb{R}}$ be a t-structure on $\mathcal{T}$. The following two conditions are equivalent.*

(a) *$(\mathcal{T}^{\leqslant c}, \mathcal{T}^{\geqslant c})_{c\in\mathbb{R}}$ is indexed by some discrete subset $\Sigma \subset \mathbb{R}$ such that $\Sigma = \Sigma + \mathbb{Z}$.*

(b) *For any $c \in \mathbb{R}$, there exist $a, b \in \mathbb{R}$ such that $a < c < b$, $\mathcal{T}^{<c} = \mathcal{T}^{\leqslant a}$ and $\mathcal{T}^{>c} = \mathcal{T}^{\geqslant b}$.*

1.3. Slices. Let $(\mathcal{T}^{\leqslant c}, \mathcal{T}^{\geqslant c})_{c\in\mathbb{R}}$ be a t-structure on $\mathcal{T}$. Note the following facts.

For any $c \in \mathbb{R}$, one has

$$\begin{aligned} \mathcal{T}^{>c} &= (\mathcal{T}^{\leqslant c})^{\perp}, \quad \mathcal{T}^{\leqslant c} = {}^{\perp}(\mathcal{T}^{>c}), \\ \mathcal{T}^{\geqslant c} &= (\mathcal{T}^{<c})^{\perp}, \quad \mathcal{T}^{<c} = {}^{\perp}(\mathcal{T}^{\geqslant c}). \end{aligned}$$

The embeddings $\mathcal{T}^{\leqslant c} \subset \mathcal{T}$ and $\mathcal{T}^{<c} \subset \mathcal{T}$ admit left adjoints

$$\tau^{\leqslant c}\colon \mathcal{T} \to \mathcal{T}^{\leqslant c} \quad \text{and} \quad \tau^{<c}\colon \mathcal{T} \to \mathcal{T}^{<c},$$

called the *left truncation functors*. Similarly, the embeddings $\mathcal{T}^{\geqslant c} \subset \mathcal{T}$ and $\mathcal{T}^{>c} \subset \mathcal{T}$ admit right adjoints

$$\tau^{\geqslant c}\colon \mathcal{T} \to \mathcal{T}^{\geqslant c} \quad \text{and} \quad \tau^{>c}\colon \mathcal{T} \to \mathcal{T}^{>c},$$

called the *right truncation functors*.

The distinguished triangles in Definition 1.2.2 (d) are unique up to unique isomorphism. They are, respectively, given by

$$\tau^{\leqslant c}X \longrightarrow X \longrightarrow \tau^{>c}X \xrightarrow{+1} \quad \text{and} \quad \tau^{<c}X \longrightarrow X \longrightarrow \tau^{\geqslant c}X \xrightarrow{+1}.$$

Summarizing the above notations, to a half-line L (i.e. an unbounded connected subset $L \subsetneq \mathbb{R}$) is associated a truncation functor $\tau^L\colon \mathcal{T} \to \mathcal{T}^L$. If $L' \subset \mathbb{R}$ is another half-line, there is an isomorphism of functors

$$\tau^L \circ \tau^{L'} \simeq \tau^{L'} \circ \tau^L\colon \mathcal{T} \to \mathcal{T}^L \cap \mathcal{T}^{L'}. \tag{1.3.1}$$

Let $I \subset \mathbb{R}$ be a proper interval (i.e. a bounded connected non-empty subset $I \subset \mathbb{R}$). Then there are two half-lines L, L' (unique up to ordering) such that $I = L \cap L'$. The *slice* of $\mathcal{T}$ associated with I is the additive category

$$\mathcal{T}^I := \mathcal{T}^L \cap \mathcal{T}^{L'},$$

and one denotes the functor (1.3.1) by

$$H^I : \mathcal{T} \to \mathcal{T}^I.$$

For example,

$$\mathcal{T}^{[c,c')} = \mathcal{T}^{\geqslant c} \cap \mathcal{T}^{<c'}$$

for $c < c'$, and $\mathcal{T}^{\{c\}} = \mathcal{T}^c$. One writes for short $H^c := H^{\{c\}}$.

The following proposition generalizes the fact that the heart $\mathcal{T}^0$ of a classical t-structure $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$ is abelian.

Proposition 1.3.1 (cf. [3, Lemma 4.3]). *Let* $(\mathcal{T}^{\leqslant c}, \mathcal{T}^{\geqslant c})_{c\in\mathbb{R}}$ *be a* t-structure *on* $\mathcal{T}$*, and let* $I \subset \mathbb{R}$ *be an interval.*

(i) *If* $I \to \mathbb{R}/\mathbb{Z}$ *is injective, then the slice* $\mathcal{T}^I$ *is a quasi-abelian category, and strict short exact sequences in* $\mathcal{T}^I$ *are in one-to-one correspondence with distinguished triangles in* $\mathcal{T}$ *with all vertices in* $\mathcal{T}^I$*.*

(ii) *If* $I \to \mathbb{R}/\mathbb{Z}$ *is bijective, then the slice* $\mathcal{T}^I$ *is an abelian category and the functor* $H^I : \mathcal{T} \to \mathcal{T}^I$ *is cohomological.*

Remark 1.3.2. The notion of slicing from [3] is equivalent to the datum of a t-structure $(\mathcal{T}^{\leqslant c}, \mathcal{T}^{\geqslant c})_{c\in\mathbb{R}}$ such that $\mathcal{T}$ is generated by the family of subcategories $\{\mathcal{T}^c\}_{c\in\mathbb{R}}$.

1.4. Exact functors. Let $\mathcal{S}$ and $\mathcal{T}$ be triangulated categories. Let $(\mathcal{S}^{\leqslant c}, \mathcal{S}^{\geqslant c})_{c\in\mathbb{R}}$ and $(\mathcal{T}^{\leqslant c}, \mathcal{T}^{\geqslant c})_{c\in\mathbb{R}}$ be t-structures on $\mathcal{S}$ and $\mathcal{T}$, respectively.

Definition 1.4.1. A triangulated functor $\Phi : \mathcal{S} \to \mathcal{T}$ is called

(i) *left exact* if one has $\Phi(\mathcal{S}^{\geqslant c}) \subset \mathcal{T}^{\geqslant c}$ for any $c \in \mathbb{R}$,

(ii) *right exact* if one has $\Phi(\mathcal{S}^{\leqslant c}) \subset \mathcal{T}^{\leqslant c}$ for any $c \in \mathbb{R}$,

(iii) *exact* if it is both left and right exact.

The following lemma is obvious.

Lemma 1.4.2. *Consider two triangulated functors* $\Phi : \mathcal{S} \to \mathcal{T}$ *and* $\Psi : \mathcal{T} \to \mathcal{S}$*. Assume that* (Φ, Ψ) *is an adjoint pair. (This means that* Φ *is left adjoint to* Ψ*, or equivalently that* Ψ *is right adjoint to* Φ*.) Then,* Ψ *is left exact if and only if* Φ *is right exact.*

1.5. Sheaves. Let M be a good topological space. Denote by $\mathrm{Mod}(\mathbf{k}_M)$ the abelian category of sheaves of $\mathbf{k}$-vector spaces on M, and by $\mathrm{D^b}(\mathbf{k}_M)$ its bounded derived category.

For a locally closed subset $S \subset M$, denote by $\mathbf{k}_S$ the extension to M by zero of the constant sheaf on S with stalk $\mathbf{k}$.

For $f : M \to N$ a morphism of good topological spaces, denote by $\otimes$, $\mathrm{R}\mathcal{H}om$, f^{-1}, $\mathrm{R}f_*$, $\mathrm{R}f_!$, $f^!$ the six Grothendieck operations for sheaves. The duality functor is given by $\mathrm{D}_M F = \mathrm{R}\mathcal{H}om(F, \omega_M)$ for $F \in \mathrm{D^b}(\mathbf{k}_M)$, where ω_M denotes the dualizing complex.

If M is a C^0-manifold, one has $\omega_M \simeq \mathrm{or}_M[d_M]$, where d_M denotes the dimension of M and or_M the orientation sheaf. For a map $f : M \to N$ of C^0-manifolds, the *relative orientation sheaf* is defined as $\mathrm{or}_{M/N} := f^! \mathbf{k}_N[d_N - d_M] \simeq \mathrm{or}_M \otimes f^{-1} \mathrm{or}_N$.

1.6. $\mathbb{R}$-constructible sheaves. Recall the notion of subanalytic subsets of a real analytic manifold (see [6] and [2]).

Definition 1.6.1. (i) A *subanalytic space* $M = (M, \mathcal{S}_M)$ is an $\mathbb{R}$-ringed space which is locally isomorphic to $(Z, \mathcal{S}_Z)$, where Z is a closed subanalytic subset of a real analytic manifold, and $\mathcal{S}_Z$ is the sheaf of $\mathbb{R}$-algebras of real valued subanalytic continuous functions. In this paper, we assume that subanalytic spaces are good topological spaces.

(ii) A morphism of subanalytic spaces is a morphism of $\mathbb{R}$-ringed spaces.

(iii) A subset S of M is *subanalytic* if $i(S \cap U)$ is a subanalytic subset of N for any open subset U of M, any real analytic manifold N and any subanalytic morphism $i: U \to N$ of subanalytic spaces such that i induces an isomorphism from U to a closed subanalytic subset of N.

Let M be a subanalytic space. One says that a sheaf $F \in \mathrm{Mod}(\mathbf{k}_M)$ is $\mathbb{R}$-*constructible* if there exists a locally finite family of locally closed subanalytic subsets $\{S_i\}_{i \in I}$ of M such that $M = \bigcup_{i \in I} S_i$ and F is locally constant of finite rank on each S_i. Denote by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ whose objects have $\mathbb{R}$-constructible cohomologies.

1.7. Perversities. On the set of maps $p: \mathbb{Z}_{\geqslant 0} \to \mathbb{R}$, consider the involution $*$ given by $p^*(n) := -p(n) - n$.

Definition 1.7.1. (i) A function $p: \mathbb{Z}_{\geqslant 0} \to \mathbb{R}$ is a *perversity* if both p and p^* are decreasing, i.e. if $0 \leqslant p(n) - p(m) \leqslant m - n$ for any $m, n \in \mathbb{Z}_{\geqslant 0}$ such that $n \leqslant m$.

(ii) A *classical perversity* is a $\mathbb{Z}$-valued perversity.

Let M be a subanalytic space. To a classical perversity p is associated a classical t-structure $({}^p\mathrm{D}^{\leqslant 0}_{\mathbb{R}\text{-c}}(\mathbf{k}_M), {}^p\mathrm{D}^{\geqslant 0}_{\mathbb{R}\text{-c}}(\mathbf{k}_M))$ on $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ (refer to [1] and [11, Section 10.2]). Here, slightly generalizing a construction in [10], we will associate a t-structure to a perversity.

Notation 1.7.2. Set

$$\mathrm{CS}_M := \{\text{closed subanalytic subsets of } M\}.$$

For $Z \in \mathrm{CS}_M$, denote by $i_Z: Z \to M$ the embedding. Set

$$d_Z := \dim Z \quad (\text{with } d_\varnothing = -\infty).$$

For $k \in \mathbb{Z}$, set

$$\mathrm{CS}_M^{<k} := \{Z \in \mathrm{CS}_M ; d_Z < k\}, \quad \mathrm{CS}_M^{\leqslant k} := \{Z \in \mathrm{CS}_M ; d_Z \leqslant k\}.$$

Let $(\mathrm{D}^{\leqslant 0}(\mathbf{k}_M), \mathrm{D}^{\geqslant 0}(\mathbf{k}_M))$ be the standard classical t-structure on $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$.

Definition 1.7.3. Let p be a perversity, $c \in \mathbb{R}$ and $k \in \mathbb{Z}_{\geqslant 0}$. Consider the following conditions on $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$:

$$(p_k^{\leqslant c}): \quad i^{-1}_{M \setminus Z} F \in \mathrm{D}^{\leqslant c + p(k)}(\mathbf{k}_{M \setminus Z}) \quad \text{for some } Z \in \mathrm{CS}_M^{<k},$$
$$(p_k^{\geqslant c}): \quad i^{!}_{Z} F \in \mathrm{D}^{\geqslant c + p(k)}(\mathbf{k}_Z) \quad \text{for any } Z \in \mathrm{CS}_M^{\leqslant k}.$$

We define the following strictly full subcategories of $\mathrm{D^b}(\mathbf{k}_M)$:

$$^p\mathrm{D}^{\leqslant c}(\mathbf{k}_M) := \{F \in \mathrm{D^b}(\mathbf{k}_M);\, (p_k^{\leqslant c}) \text{ holds for any } k \in \mathbb{Z}_{\geqslant 0}\},$$
$$^p\mathrm{D}^{\geqslant c}(\mathbf{k}_M) := \{F \in \mathrm{D^b}(\mathbf{k}_M);\, (p_k^{\geqslant c}) \text{ holds for any } k \in \mathbb{Z}_{\geqslant 0}\}.$$

Let us also set

$$^p\mathrm{D}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_M) := {}^p\mathrm{D}^{\leqslant c}(\mathbf{k}_M) \cap \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M),$$
$$^p\mathrm{D}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_M) := {}^p\mathrm{D}^{\geqslant c}(\mathbf{k}_M) \cap \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M).$$

Note that $({}^p\mathrm{D}^{\leqslant c}(\mathbf{k}_M), {}^p\mathrm{D}^{\geqslant c}(\mathbf{k}_M))_{c\in\mathbb{R}}$ is not a t-structure in general.

Lemma 1.7.4. *For $c \in \mathbb{R}$, $k \in \mathbb{Z}_{\geqslant 0}$ and $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$, the following conditions are equivalent.*

(i) *F satisfies $(p_k^{\leqslant c})$.*

(ii) *$\dim(\mathrm{supp}(H^jF)) < k$ for any j with $j > c + p(k)$.*

Note that $\mathrm{supp}(H^jF)$ is subanalytic, since F is $\mathbb{R}$-constructible.

Proof. It is enough to remark that $i^{-1}_{M\setminus Z}F \in \mathrm{D}^{\leqslant c+p(k)}(\mathbf{k}_{M\setminus Z})$ if and only if one has $\mathrm{supp}(H^jF) \subset Z$ for any j such that $j > c + p(k)$. □

Proposition 1.7.5. *We have the following properties.*

(i) *$({}^p\mathrm{D}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_M), {}^p\mathrm{D}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_M))_{c\in\mathbb{R}}$ is a t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$.*

(ii) *For any $c \in \mathbb{R}$, the duality functor D_M interchanges ${}^p\mathrm{D}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ and ${}^{p^*}\mathrm{D}^{\geqslant -c}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$.*

(iii) *For any interval $I \subset \mathbb{R}$ such that $I \to \mathbb{R}/\mathbb{Z}$ is injective, the prestack on M*

$$U \mapsto {}^p\mathrm{D}^{I}_{\mathbb{R}\text{-c}}(\mathbf{k}_U)$$

is a stack of quasi-abelian categories.

Proof. Note that, for (iii), it is enough to consider the case where $I \to \mathbb{R}/\mathbb{Z}$ is bijective, i.e. the case where $I = [c, c+1)$ or $I = (c, c+1]$ for some $c \in \mathbb{R}$.

(a) If p is a classical perversity, the result is due to [1]. More precisely, for the statements (i), (ii) and (iii) refer to Theorem 10.2.8, Proposition 10.2.13 and Proposition 10.2.9 of [11], respectively.

(b) Let now p be an arbitrary perversity.

For $c \in \mathbb{R}$, denote by $\lfloor c \rfloor$ the largest integer not greater than c, and by $\lceil c \rceil$ the smallest integer not smaller than c. Note that $\lceil c \rceil + \lfloor -c \rfloor = 0$.

Statements (i) and (iii) follow from (a) by noticing that for any $c \in \mathbb{R}$

$$({}^p\mathrm{D}^{<c+1}_{\mathbb{R}\text{-c}}(\mathbf{k}_M), {}^p\mathrm{D}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)) \quad \text{and} \quad ({}^p\mathrm{D}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_M), {}^p\mathrm{D}^{>c-1}_{\mathbb{R}\text{-c}}(\mathbf{k}_M))$$

are the classical t-structures associated to the classical perversities

$$p_{c,+}(n) := \lceil c + p(n) \rceil, \quad p_{c,-}(n) := \lfloor c + p(n) \rfloor,$$

respectively.

Statement (ii) follows from (a) by noticing that one has $(p_{c,\pm})^* = (p^*)_{-c,\mp}$. □

Note that $({}^p\mathrm{D}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_M), {}^p\mathrm{D}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_M))_{c\in\mathbb{R}}$ is indexed by $\bigcup_{0\leqslant k\leqslant d_M}(-p(k)+\mathbb{Z})$.

Definition 1.7.6. The *middle perversity t-structure*

$$({}^{1/2}\mathrm{D}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_M), {}^{1/2}\mathrm{D}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_M))_{c\in\mathbb{R}}$$

is the one associated with the *middle perversity* $\mathsf{m}(n) := -\frac{n}{2}$.

Note that m is the only perversity stable by $*$. In particular, the middle perversity t-structure is self-dual. It is indexed by $\frac{1}{2}\mathbb{Z}$.

2. Enhanced ind-sheaves

Let M be a good topological space. The derived category of enhanced ind-sheaves on M is defined as a quotient of the derived category of ind-sheaves on the bordered space $M\times\mathbb{R}_\infty$. We recall here these notions and some related results from [4]. We also discuss the exactness of Grothendieck operations with respect to the standard classical t-structures.

References are made to [13] for ind-sheaves, and to [4] for bordered spaces and enhanced ind-sheaves. See also [15] for enhanced ind-sheaves on bordered spaces and [16] for an exposition.

2.1. Semi-orthogonal decomposition. Let $\mathcal{T}$ be a triangulated category, and $\mathcal{N}\subset\mathcal{T}$ a strictly full triangulated subcategory. We denote by $\mathcal{T}/\mathcal{N}$ the quotient triangulated category (see e.g. [14, Section 10.2]).

Proposition 2.1.1. *Let $\mathcal{N}\subset\mathcal{T}$ be a strictly full triangulated subcategory which contains every direct summand in $\mathcal{T}$ of an object of $\mathcal{N}$. Then the following conditions are equivalent.*

(i) *The embedding $\mathcal{N}\to\mathcal{T}$ has a left adjoint.*

(ii) *The quotient functor $\mathcal{T}\to\mathcal{T}/\mathcal{N}$ has a left adjoint.*

(iii) *The composition ${}^\perp\mathcal{N}\to\mathcal{T}\to\mathcal{T}/\mathcal{N}$ is an equivalence of categories.*

(iv) *For any $X\in\mathcal{T}$ there is a distinguished triangle $X'\longrightarrow X\longrightarrow X''\xrightarrow{+1}$ with $X'\in{}^\perp\mathcal{N}$ and $X''\in\mathcal{N}$.*

(v) *The embedding ${}^\perp\mathcal{N}\to\mathcal{T}$ has a right adjoint, and $\mathcal{N}\simeq({}^\perp\mathcal{N})^\perp$.*

A similar result holds switching "left" with "right".

2.2. Ind-sheaves. Let $\mathcal{C}$ be a category and denote by $\mathcal{C}^\wedge$ the category of contravariant functors from $\mathcal{C}$ to the category of sets. Consider the Yoneda embedding $h\colon\mathcal{C}\to\mathcal{C}^\wedge$, $X\mapsto\mathrm{Hom}_{\mathcal{C}}(*,X)$. The category $\mathcal{C}^\wedge$ admits small colimits. As colimits do not commute with h, one denotes by $\varinjlim$ the colimits taken in $\mathcal{C}$, and by "$\varinjlim$" the colimits taken in $\mathcal{C}^\wedge$.

An ind-object in the category $\mathcal{C}$ is an object of $\mathcal{C}^\wedge$ isomorphic to "$\varinjlim$" φ for some functor $\varphi\colon I\to\mathcal{C}$ with I a small filtrant category. Denote by $\mathrm{Ind}(\mathcal{C})$ the strictly full subcategory of $\mathcal{C}^\wedge$ consisting of ind-objects in $\mathcal{C}$.

Let M be a good topological space. The category of ind-sheaves on M is the category

$$\mathrm{I}(\mathbf{k}_M) := \mathrm{Ind}(\mathrm{Mod}_c(\mathbf{k}_M))$$

of ind-objects in the category $\mathrm{Mod}_c(\mathbf{k}_M)$ of sheaves with compact support.

The category $\mathrm{I}(\mathbf{k}_M)$ is abelian, and the prestack on M given by $U \mapsto \mathrm{I}(\mathbf{k}_U)$ is a stack of abelian categories. There is a natural exact fully faithful functor $\iota_M \colon \mathrm{Mod}(\mathbf{k}_M) \to \mathrm{I}(\mathbf{k}_M)$ given by $F \mapsto$ “$\varinjlim$”$(\mathbf{k}_U \otimes F)$, for U running over the relatively compact open subsets of M. The functor ι_M has an exact left adjoint $\alpha_M \colon \mathrm{I}(\mathbf{k}_M) \to \mathrm{Mod}(\mathbf{k}_M)$, which sends “$\varinjlim$” φ to $\varinjlim \varphi$.

In this paper, we set for short

$$\mathrm{D}(M) := \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbf{k}_M)),$$

and denote by $(\mathrm{D}^{\leqslant 0}(M), \mathrm{D}^{\geqslant 0}(M))$ its standard classical t-structure.

For $f \colon M \to N$ a morphism of good topological spaces, denote by $\otimes$, $\mathrm{R}\mathcal{I}hom$, f^{-1}, $\mathrm{R}f_*$, $\mathrm{R}f_{!!}$, $f^!$ the six Grothendieck operations for ind-sheaves.

Since ind-sheaves form a stack, they have a sheaf-valued hom-functor $\mathcal{H}om$. One has $\mathrm{R}\mathcal{H}om \simeq \alpha_M \circ \mathrm{R}\mathcal{I}hom$.

2.3. Bordered spaces. A *bordered space* $\mathsf{M} = (M, \check{M})$ is a pair of a good topological space $\check{M}$ and an open subset M of $\check{M}$. Set $\mathring{\mathsf{M}} := M$.

Notation 2.3.1. Let $\mathsf{M} = (M, \check{M})$ and $\mathsf{N} = (N, \check{N})$ be bordered spaces. For a continuous map $f \colon M \to N$, denote by $\Gamma_f \subset M \times N$ its graph, and by $\overline{\Gamma}_f$ the closure of Γ_f in $\check{M} \times \check{N}$. Consider the projections

$$\check{M} \xleftarrow{q_1} \check{M} \times \check{N} \xrightarrow{q_2} \check{M}.$$

Bordered spaces form a category as follows: a morphism $f \colon \mathsf{M} \to \mathsf{N}$ is a continuous map $f \colon M \to N$ such that $q_1|_{\overline{\Gamma}_f} \colon \overline{\Gamma}_f \to \check{M}$ is proper; the composition of two morphisms is the composition of the underlying continuous maps.

Remark 2.3.2. (i) If $f \colon M \to N$ can be extended to a continuous map $\check{f} \colon \check{M} \to \check{N}$, then f is a morphism of bordered spaces.

(ii) The forgetful functor from the category of bordered spaces to that of good topological spaces is given by $\mathsf{M} \mapsto \mathring{\mathsf{M}}$. It has a fully faithful left adjoint $M \mapsto (M, M)$. By this functor, we consider good topological spaces as particular bordered spaces, and denote (M, M) by M. Note that $\mathsf{M} = (M, \check{M}) \mapsto \check{M}$ is not a functor.

Let $\mathsf{M} = (M, \check{M})$ be a bordered space. The continuous maps $M \xrightarrow{\mathrm{id}} M \hookrightarrow \check{M}$ induce morphisms of bordered spaces

$$M \longrightarrow \mathsf{M} \xrightarrow{j_{\mathsf{M}}} \check{M}. \tag{2.3.1}$$

Notation 2.3.3. For a locally closed subset Z of M, set $Z_\infty = (Z, \overline{Z})$, where $\overline{Z}$ is the closure of Z in $\check{M}$, and denote $i_{Z_\infty} \colon Z_\infty \to \mathsf{M}$ the morphism induced by the embedding $Z \subset M$.

Note that $M_\infty \simeq \mathsf{M}$.

Lemma 2.3.4. *Let $f\colon \mathsf{M} \to \mathsf{N}$ be a morphism of bordered spaces. Let $Z \subset \mathring{\mathsf{M}}$ and $W \subset \mathring{\mathsf{N}}$ be locally closed subsets such that $f(Z) \subset W$. Then $f|_Z\colon Z \to W$ induces a morphism $Z_\infty \to W_\infty$ of bordered spaces.*

In particular, the bordered space Z_∞ only depends on M (and not on $\check{M}$).

Definition 2.3.5. We say that a morphism $f\colon \mathsf{M} \to \mathsf{N}$ is *semi-proper* if the associated map

$$q_2|_{\overline{\Gamma}_f}\colon \overline{\Gamma}_f \to \check{N}$$

is proper. We say that f is *proper* if moreover $\mathring{f}\colon \mathring{\mathsf{M}} \to \mathring{\mathsf{N}}$ is proper.

For example, j_{M} and i_{Z_∞} are semi-proper.

Definition 2.3.6. A subset S of a bordered space $\mathsf{M} = (M, \check{M})$ is a subset of M. We say that S is open (resp. closed, locally closed) if it is so in M. We say that S is relatively compact if it is contained in a compact subset of $\check{M}$.

As seen by the following obvious lemma, the notion of relatively compact subsets only depends on M (and not on $\check{M}$).

Lemma 2.3.7. *Let $f\colon \mathsf{M} \to \mathsf{N}$ be a morphism of bordered spaces.*

(i) *If S is a relatively compact subset of M, then its image $\mathring{f}(S) \subset \mathring{\mathsf{N}}$ is a relatively compact subset of N.*

(ii) *Assume furthermore that f is semi-proper. If T is a relatively compact subset of N, then its inverse image $\mathring{f}^{-1}(T) \subset \mathring{\mathsf{M}}$ is a relatively compact subset of M.*

2.4. Ind-sheaves on bordered spaces. Let M be a bordered space. The abelian category of ind-sheaves on M is

$$\mathrm{I}(\mathbf{k}_{\mathsf{M}}) := \mathrm{Ind}(\mathrm{Mod}_c(\mathbf{k}_{\mathsf{M}})),$$

where $\mathrm{Mod}_c(\mathbf{k}_{\mathsf{M}}) \subset \mathrm{Mod}(\mathbf{k}_{\mathring{\mathsf{M}}})$ is the full subcategory of sheaves on $\mathring{\mathsf{M}}$ whose support is relatively compact in M.

There is a natural exact embedding

$$\iota_{\mathsf{M}}\colon \mathrm{Mod}(\mathbf{k}_{\mathring{\mathsf{M}}}) \to \mathrm{I}(\mathbf{k}_{\mathsf{M}}), \quad F \mapsto \text{“}\varinjlim\text{”}(\mathbf{k}_U \otimes F),$$

where U runs over the family of relatively compact open subsets of M.

We set for short

$$\mathrm{D}(\mathsf{M}) := \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbf{k}_{\mathsf{M}})),$$

and denote by $(\mathrm{D}^{\leqslant 0}(\mathsf{M}), \mathrm{D}^{\geqslant 0}(\mathsf{M}))$ its standard classical t-structure.

Let $\mathsf{M} = (M, \check{M})$, and consider the embeddings

$$\check{M} \setminus M \xrightarrow{i} \check{M} \xleftarrow{j} M.$$

The functor $\mathrm{R}i_* \simeq \mathrm{R}i_{!!}$ induces the embedding $\mathrm{D}(\check{M} \setminus M) \subset \mathrm{D}(\check{M})$, which admits a left and a right adjoint.

Proposition 2.4.1. *There is an equivalence of triangulated categories:*

$$\mathrm{D}(\mathsf{M}) \simeq \mathrm{D}(\check{M})/\mathrm{D}(\check{M} \setminus M).$$

Proof. The functor $j_!$ induces an exact functor

$$\mathrm{Mod}_c(\mathbf{k}_{\mathsf{M}}) \to \mathrm{Mod}_c(\mathbf{k}_{\check{M}}),$$

and hence an exact functor

$$\mathrm{I}(\mathbf{k}_{\mathsf{M}}) \to \mathrm{I}(\mathbf{k}_{\check{M}})$$

and a functor of triangulated categories

$$\mathrm{D}(\mathsf{M}) \to \mathrm{D}(\check{M}).$$

Composing with the quotient functor, we get the functor

$$\mathrm{D}(\mathsf{M}) \to \mathrm{D}(\check{M})/\mathrm{D}(\check{M} \setminus M).$$

On the other hand, the functor j^{-1} induces an exact functor

$$\mathrm{Mod}_c(\mathbf{k}_{\check{M}}) \to \mathrm{Mod}_c(\mathbf{k}_{\mathsf{M}}),$$

which induces an exact functor

$$\mathrm{I}(\mathbf{k}_{\check{M}}) \to \mathrm{I}(\mathbf{k}_{\mathsf{M}})$$

and a functor of triangulated categories

$$\mathrm{D}(\check{M}) \to \mathrm{D}(\mathsf{M}).$$

Since the composition $\mathrm{D}(\check{M} \setminus M) \to \mathrm{D}(\check{M}) \to \mathrm{D}(\mathsf{M})$ vanishes, we obtain a functor

$$\mathrm{D}(\check{M})/\mathrm{D}(\check{M} \setminus M) \to \mathrm{D}(\mathsf{M}).$$

It is obvious that these functors between $\mathrm{D}(\mathsf{M})$ and $\mathrm{D}(\check{M})/\mathrm{D}(\check{M} \setminus M)$ are quasi-inverse to each other. □

Thus, there are equivalences

$$\mathrm{D}(\mathsf{M}) \simeq \mathrm{D}(\check{M})/\mathrm{D}(\check{M} \setminus M) \simeq {}^{\perp}\mathrm{D}(\check{M} \setminus M) \simeq \mathrm{D}(\check{M} \setminus M)^{\perp},$$

and one has

$$\begin{aligned} {}^{\perp}\mathrm{D}(\check{M} \setminus M) &= \{F \in \mathrm{D}(\check{M}); \mathbf{k}_M \otimes F \xrightarrow{\sim} F\}, \\ \mathrm{D}(\check{M} \setminus M)^{\perp} &= \{F \in \mathrm{D}(\check{M}); \mathrm{R}\mathcal{I}hom(\mathbf{k}_M, F) \xleftarrow{\sim} F\}. \end{aligned}$$

Denote by

$$\mathrm{q}_{\mathsf{M}} \colon \mathrm{D}(\check{M}) \to \mathrm{D}(\mathsf{M}), \quad \mathrm{l}_{\mathsf{M}}, \mathrm{r}_{\mathsf{M}} \colon \mathrm{D}(\mathsf{M}) \to \mathrm{D}(\check{M})$$

the quotient functor and its left and right adjoint, respectively. For $F \in \mathrm{D}(\check{M})$, they satisfy

$$\mathrm{l}_{\mathsf{M}}\mathrm{q}_{\mathsf{M}}F \simeq \mathbf{k}_M \otimes F, \quad \mathrm{r}_{\mathsf{M}}\mathrm{q}_{\mathsf{M}}F \simeq \mathrm{R}\mathcal{I}hom(\mathbf{k}_M, F). \tag{2.4.1}$$

Remark 2.4.2. At the level of sheaves, there is a natural equivalence

$$\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M) \simeq \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\check{M}})/\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\check{M}\setminus M}).$$

There is a commutative diagram

$$\begin{array}{ccc} \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M) & \xrightarrow{\iota_{\mathsf{M}}} & \mathrm{D}(\mathsf{M}) \\ \wr\downarrow & & \wr\downarrow \\ \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\check{M}})/\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\check{M}\setminus M}) & \rightarrowtail & \mathrm{D}(\check{M})/\mathrm{D}(\check{M}\setminus M). \end{array}$$

The functor $\iota_{\mathsf{M}}\colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathring{\mathsf{M}}}) \to \mathrm{D}(\mathsf{M})$ has a left adjoint

$$\alpha_{\mathsf{M}}\colon \mathrm{D}(\mathsf{M}) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathring{\mathsf{M}}}).$$

It coincides with the composition

$$\mathrm{D}(\mathsf{M}) \longrightarrow \mathrm{D}(\mathring{\mathsf{M}}) \xrightarrow{\alpha_{\mathring{\mathsf{M}}}} \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathring{\mathsf{M}}}).$$

Let $f\colon \mathsf{M} \to \mathsf{N}$ be a morphism of bordered spaces. The six Grothendieck operations for ind-sheaves on bordered spaces

$$\begin{aligned} \otimes\colon & \mathrm{D}(\mathsf{M}) \times \mathrm{D}(\mathsf{M}) \to \mathrm{D}(\mathsf{M}), \\ \mathrm{R}\mathcal{I}hom\colon & \mathrm{D}(\mathsf{M})^{\mathrm{op}} \times \mathrm{D}(\mathsf{M}) \to \mathrm{D}(\mathsf{M}), \\ \mathrm{R}f_{!!}, \mathrm{R}f_{*}\colon & \mathrm{D}(\mathsf{M}) \to \mathrm{D}(\mathsf{N}), \\ f^{-1}, f^{!}\colon & \mathrm{D}(\mathsf{N}) \to \mathrm{D}(\mathsf{M}) \end{aligned}$$

are defined as follows. Recalling Notation 2.3.1, observe that Γ_f is locally closed in $\check{M} \times \check{N}$. For $F, F' \in \mathrm{D}(\check{M})$ and $G \in \mathrm{D}(\check{N})$, one sets

$$\begin{aligned} \mathrm{q}_{\mathsf{M}} F \otimes \mathrm{q}_{\mathsf{M}} F' &:= \mathrm{q}_{\mathsf{M}}(F \otimes F'), \\ \mathrm{R}\mathcal{I}hom(\mathrm{q}_{\mathsf{M}} F, \mathrm{q}_{\mathsf{M}} F') &:= \mathrm{q}_{\mathsf{M}} \mathrm{R}\mathcal{I}hom(F, F'), \\ \mathrm{R}f_{!!}\mathrm{q}_{\mathsf{M}} F &:= \mathrm{q}_{\mathsf{N}} \mathrm{R}q_{2!!}(\mathbf{k}_{\Gamma_f} \otimes q_1^{-1} F), \\ \mathrm{R}f_{*}\mathrm{q}_{\mathsf{M}} F &:= \mathrm{q}_{\mathsf{N}} \mathrm{R}q_{2*} \mathrm{R}\mathcal{I}hom(\mathbf{k}_{\Gamma_f}, q_1^{!} F), \\ f^{-1}\mathrm{q}_{\mathsf{N}} G &:= \mathrm{q}_{\mathsf{M}} \mathrm{R}q_{1!!}(\mathbf{k}_{\Gamma_f} \otimes q_2^{-1} G), \\ f^{!}\mathrm{q}_{\mathsf{N}} G &:= \mathrm{q}_{\mathsf{M}} \mathrm{R}q_{1*} \mathrm{R}\mathcal{I}hom(\mathbf{k}_{\Gamma_f}, q_2^{!} G). \end{aligned}$$

Remark 2.4.3. The natural embedding $\iota_{\mathsf{M}}\colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathring{\mathsf{M}}}) \to \mathrm{D}(\mathsf{M})$ commutes with the operations $\otimes$, $\mathrm{R}\mathcal{I}hom$, f^{-1}, $\mathrm{R}f_*$, $f^{!}$. If f is semi-proper, one has

$$\mathrm{R}f_{!!} \circ \iota_{\mathsf{M}} \xrightarrow{\sim} \iota_{\mathsf{N}} \circ \mathrm{R}\mathring{f}_{!}. \tag{2.4.2}$$

Remark 2.4.4. Let $\mathsf{M} = (M, \check{M})$. For the natural morphism $j_{\mathsf{M}}\colon \mathsf{M} \to \check{M}$, one has

$$\begin{aligned} \mathrm{q}_{\mathsf{M}} &\simeq j_{\mathsf{M}}^{-1} \simeq j_{\mathsf{M}}^{!}, \\ \mathrm{l}_{\mathsf{M}} &\simeq \mathrm{R}j_{M!!}, \\ \mathrm{r}_{\mathsf{M}} &\simeq \mathrm{R}j_{M*}. \end{aligned}$$

The following result generalizes (2.4.1).

Lemma 2.4.5. *Let Z be a locally closed subset of* M*, and let* $F \in \mathrm{D}(\mathsf{M})$*. Using Notation* 2.3.3*, one has*

$$\mathbf{k}_Z \otimes F \simeq \mathrm{R}i_{Z_\infty\,!!}\, i_{Z_\infty}^{-1} F,$$
$$\mathrm{R}\mathcal{I}hom(\mathbf{k}_Z, F) \simeq \mathrm{R}i_{Z_\infty\,*}\, i_{Z_\infty}^{!} F.$$

Proof. To avoid confusion, let us denote by $\mathbf{k}_{Z|\mathring{\mathsf{M}}}$ the extension by zero to $\mathring{\mathsf{M}}$ of the constant sheaf $\mathbf{k}_Z$ on Z. Since i_{Z_∞} is semi-proper, equation (2.4.2) implies $\mathbf{k}_{Z|\mathring{\mathsf{M}}} \simeq \mathrm{R}i_{Z_\infty\,!!}\mathbf{k}_Z$. Hence

$$\begin{aligned}\mathbf{k}_{Z|\mathring{\mathsf{M}}} \otimes F &\simeq (\mathrm{R}i_{Z_\infty\,!!}\mathbf{k}_Z) \otimes F \\ &\simeq \mathrm{R}i_{Z_\infty\,!!}(\mathbf{k}_Z \otimes i_{Z_\infty}^{-1} F) \\ &\simeq \mathrm{R}i_{Z_\infty\,!!}\, i_{Z_\infty}^{-1} F.\end{aligned}$$

We can prove the second isomorphism similarly. □

Let $\mathsf{M} = (M, \check{M})$ be a bordered space. By [4, Section 3.4], one has

$$\mathrm{D}^{\leqslant 0}(\mathsf{M}) = \{F \in \mathrm{D}(\mathsf{M});\, \mathrm{R}j_{\mathsf{M}\,!!}F \in \mathrm{D}^{\leqslant 0}(\check{M})\},$$
$$\mathrm{D}^{\geqslant 0}(\mathsf{M}) = \{F \in \mathrm{D}(\mathsf{M});\, \mathrm{R}j_{\mathsf{M}\,!!}F \in \mathrm{D}^{\geqslant 0}(\check{M})\}.$$

Proposition 2.4.6. *Let* M *be a bordered space. Consider the standard t-structure on* $\mathrm{D}(\mathsf{M})$*. Then:*

(i) *The bifunctor $\otimes$ is exact, i.e. for any $n, n' \in \mathbb{Z}$ one has*

$$\mathrm{D}^{\leqslant n}(\mathsf{M}) \otimes \mathrm{D}^{\leqslant n'}(\mathsf{M}) \subset \mathrm{D}^{\leqslant n+n'}(\mathsf{M}),$$
$$\mathrm{D}^{\geqslant n}(\mathsf{M}) \otimes \mathrm{D}^{\geqslant n'}(\mathsf{M}) \subset \mathrm{D}^{\geqslant n+n'}(\mathsf{M}).$$

(ii) *The bifunctor* $\mathrm{R}\mathcal{I}hom$ *is left exact, i.e. for any $n, n' \in \mathbb{Z}$ one has*

$$\mathrm{R}\mathcal{I}hom(\mathrm{D}^{\leqslant n}(\mathsf{M}), \mathrm{D}^{\geqslant n'}(\mathsf{M})) \subset \mathrm{D}^{\geqslant n'-n}(\mathsf{M}).$$

Let $f\colon \mathsf{M} \to \mathsf{N}$ be a morphism of bordered spaces. Consider the standard t-structures on $\mathrm{D}(\mathsf{M})$ *and* $\mathrm{D}(\mathsf{N})$*. Then:*

(iii) $\mathrm{R}f_{!!}$ *and* $\mathrm{R}f_*$ *are left exact.*

(iv) f^{-1} *is exact.*

Let $d \in \mathbb{Z}_{\geqslant 0}$ and assume that $f^{-1}(y) \subset \mathring{\mathsf{M}}$ has soft-dimension $\leqslant d$ for any $y \in \mathring{\mathsf{N}}$. Then:

(v) $\mathrm{R}f_{!!}(*)[d]$ *is right exact, i.e.* $\mathrm{R}f_{!!}\mathrm{D}^{\leqslant n}(\mathsf{M}) \subset \mathrm{D}^{\leqslant n+d}(\mathsf{N})$ *for any $n \in \mathbb{Z}$.*

(vi) $f^{!}(*)[-d]$ *is left exact, i.e.* $f^{!}\mathrm{D}^{\geqslant n}(\mathsf{N}) \subset \mathrm{D}^{\geqslant n-d}(\mathsf{M})$ *for any $n \in \mathbb{Z}$.*

Proof. When M and N are good topological spaces, statements (i)–(iv) follow from [13].

Let $\mathsf{M} = (M, \check{M})$ and $\mathsf{N} = (N, \check{N})$. Replacing $(M, \check{M})$ with $(M, \overline{\Gamma}_f)$, we may assume from the beginning that the morphism $f\colon M \to N$ extends to $\check{f}\colon \check{M} \to \check{N}$.

Statement (i) follows from the topological space case, using the fact that $\mathrm{R}j_{M\,!!}$ commutes with $\otimes$.

Statement (ii) follows from (i) by adjunction.

Statements (iii) and (iv) follow from the topological space case using the isomorphisms

$$\mathrm{R}f_{!!} \simeq j_{\mathsf{N}}^{-1}\mathrm{R}\check{f}_{!!}\mathrm{R}j_{\mathsf{M}!!}, \quad \mathrm{R}f_{*} \simeq j_{\mathsf{N}}^{-1}\mathrm{R}\check{f}_{*}\mathrm{R}j_{\mathsf{M}*}, \quad f^{-1} \simeq j_{\mathsf{M}}^{-1}\check{f}^{-1}\mathrm{R}j_{\mathsf{N}!!}.$$

As (vi) follows from (v) by adjunction, we are left to prove (v).

By dévissage, it is enough to show that for $F \in \mathrm{I}(\mathbf{k}_{\mathsf{M}})$ one has $H^k\mathrm{R}f_{!!}F \simeq 0$ for $k > d$. Writing $F = \text{“}\varinjlim\text{”}_i F_i$ with $F_i \in \mathrm{Mod}_c(\mathbf{k}_{\mathsf{M}})$, one has

$$H^k\mathrm{R}f_{!!}F \simeq \text{“}\varinjlim_i\text{”}\, H^k\mathrm{R}f_! F_i.$$

To conclude, note that

$$(H^k\mathrm{R}f_! F_i)_y \simeq H^k_c(f^{-1}(y); F_i|_{f^{-1}(y)}) \simeq 0$$

for any $y \in N$ and $k > d$, since $f^{-1}(y)$ has soft-dimension $\leqslant d$. □

Proposition 2.4.7. *Let $f\colon \mathsf{M} \to \mathsf{N}$ be a morphism of bordered spaces. Let $n \in \mathbb{Z}$ and $G \in \mathrm{D}(\mathsf{N})$. Assume that f is semi-proper and $\mathring{f}\colon \mathring{\mathsf{M}} \to \mathring{\mathsf{N}}$ is surjective. Then the following statements hold.*

(i) *$f^{-1}G \in \mathrm{D}^{\geqslant n}(\mathsf{M})$ implies $G \in \mathrm{D}^{\geqslant n}(\mathsf{N})$.*

(ii) *$f^{-1}G \in \mathrm{D}^{\leqslant n}(\mathsf{M})$ implies $G \in \mathrm{D}^{\leqslant n}(\mathsf{N})$.*

Proof. Let $\mathsf{M} = (M, \check{M})$ and $\mathsf{N} = (N, \check{N})$. Since f^{-1} is exact, it is enough to show that, for $G \in \mathrm{D}^0(\mathsf{N}) \simeq \mathrm{I}(\mathbf{k}_{\mathsf{N}})$ such that $f^{-1}G \simeq 0$, one has $G \simeq 0$.

Write $G = \text{“}\varinjlim\text{”}\, G_i$, for $\{G_i\}_{i\in I}$ a filtrant inductive system of objects of $\mathrm{Mod}_c(\mathbf{k}_{\mathsf{N}})$. Recall that this means that $G_i \in \mathrm{Mod}(\mathbf{k}_N)$ and $\mathrm{supp}(G_i)$ is relatively compact in $\check{N}$. Since f is semi-proper, $f^{-1}G_i \in \mathrm{Mod}_c(\mathbf{k}_{\mathsf{M}})$ by Lemma 2.3.7 (ii). Since $f^{-1}G \simeq \text{“}\varinjlim\text{”}\, f^{-1}G_i \simeq 0$, for any $i \in I$, there exists $i \to j$ in I whose induced morphism

$$f^{-1}G_i \to f^{-1}G_j$$

is the zero map. Since f is surjective, $G_i \to G_j$ is the zero map. Thus $G = 0$. □

Proposition 2.4.8. *Let $f\colon M \to N$ be a continuous map of good topological spaces, and $\{V_i\}_{i\in I}$ an open covering of N. Let $K_i \in \mathrm{D}(f^{-1}V_i)$, and let*

$$u_{ij}\colon K_j|_{f^{-1}V_i\cap f^{-1}V_j} \xrightarrow{\sim} K_i|_{f^{-1}V_i\cap f^{-1}V_j}$$

be isomorphisms. Assume that $\mathrm{R}f_*\mathrm{R}\mathcal{H}om(K_i, K_i) \in \mathrm{D}^{\geqslant 0}(\mathbf{k}_{V_i})$, and that the morphisms u_{ij} satisfy the following usual cochain condition $u_{ij} \circ u_{jk} = u_{ik}$ on $f^{-1}V_i \cap f^{-1}V_j \cap f^{-1}V_k$. Then there exist $K \in \mathrm{D}(M)$ and isomorphisms $u_i\colon K|_{f^{-1}V_i} \xrightarrow{\sim} K_i$ compatible with u_{ij}, that is, $u_{ij} \circ u_j = u_i$ on $f^{-1}V_i \cap f^{-1}V_j$. Moreover, such a K is unique up to a unique isomorphism.

Proof. The arguments we use are standard (see e.g. [10, Proposition 5.9]). Let us set $U_i := f^{-1}V_i \subset M$.

(i) Let us first discuss uniqueness. Let $K' \in \mathrm{D}(M)$ be such that there are isomorphisms $u'_i\colon K'|_{U_i} \xrightarrow{\sim} K_i$ compatible with u_{ij}. Note that for any open subset V of N, one has

$$\mathrm{Hom}_{\mathrm{D}(f^{-1}V)}(K|_{f^{-1}V}, K'|_{f^{-1}V}) \simeq H^0\mathrm{R}\Gamma(V; \mathrm{R}f_*\mathrm{R}\mathcal{H}om(K, K')).$$

From the fact that $\mathrm{R}f_*\mathrm{R}\mathcal{H}om(K,K')|_{V_i} \simeq \mathrm{R}f_*\mathrm{R}\mathcal{H}om(K_i,K_i) \in \mathrm{D}^{\geqslant 0}(\mathbf{k}_{V_i})$, we deduce

$$\mathrm{R}f_*\mathrm{R}\mathcal{H}om(K,K') \in \mathrm{D}^{\geqslant 0}(\mathbf{k}_N).$$

Hence

$$V \mapsto \mathrm{Hom}_{\mathrm{D}(f^{-1}V)}(K|_{f^{-1}V}, K'|_{f^{-1}V}) \text{ is a sheaf on } N.$$

We thus get an isomorphism $K \xrightarrow{\sim} K'$ on M by patching together the isomorphisms $u_i'^{-1} \circ u_i$ on U_i.

(ii) Let us now prove the existence of K as in the statement.

(ii-1) Assume that I is finite. In order to prove the statement, by induction we reduce to the case $I = \{1,2\}$. Set $V_0 := V_1 \cap V_2$, $K_0 := K_1|_{U_0} \simeq K_2|_{U_0}$. Let $j_i \colon U_i \to M$ $(i = 0,1,2)$ be the open inclusion. By adjunction, for $i = 1,2$ there are morphisms $\beta_i \colon \mathrm{R}j_{0\,!!}K_0 \to \mathrm{R}j_{i\,!!}K_i$. Let us complete the morphism (β_1,β_2) into a distinguished triangle

$$\mathrm{R}j_{0\,!!}K_0 \xrightarrow{(\beta_1,\beta_2)} \mathrm{R}j_{1\,!!}K_1 \oplus \mathrm{R}j_{2\,!!}K_2 \longrightarrow K \xrightarrow{+1}.$$

Then K satisfies the desired condition.

(ii-2) Assume that $I = \mathbb{Z}_{\geqslant 0}$ and that $\{V_n\}_{n\in\mathbb{Z}_{\geqslant 0}}$ is an increasing sequence of open subsets of N. Then $K_{n+1}|_{U_n} \simeq K_n$. Let $j_n \colon U_n \to M$ $(n \in \mathbb{Z}_{\geqslant 0})$ be the open inclusion. By adjunction, there are natural morphisms $\beta_n \colon \mathrm{R}j_{n\,!!}K_n \to \mathrm{R}j_{n+1\,!!}K_{n+1}$ $(n \in \mathbb{Z}_{\geqslant 0})$. Let K be the homotopy colimit of the inductive system $\{\mathrm{R}j_{n\,!!}K_n\}_{n\in\mathbb{Z}_{\geqslant 0}}$, that is, let K be the third term of the distinguished triangle

$$\bigoplus_{n\in\mathbb{Z}_{\geqslant 0}} \mathrm{R}j_{n\,!!}K_n \xrightarrow{\beta} \bigoplus_{n\in\mathbb{Z}_{\geqslant 0}} \mathrm{R}j_{n\,!!}K_n \longrightarrow K \xrightarrow{+1}.$$

Here β is the only morphism making the following diagram commute for any $m \in \mathbb{Z}_{\geqslant 0}$:

$$\begin{array}{ccc} \mathrm{R}j_{m\,!!}K_m & \xrightarrow{(\mathrm{id},-\beta_m)} & \mathrm{R}j_{m\,!!}K_m \oplus \mathrm{R}j_{m+1\,!!}K_{m+1} \\ \downarrow & & \downarrow \\ \bigoplus_{n\in\mathbb{Z}_{\geqslant 0}} \mathrm{R}j_{n\,!!}K_n & \xrightarrow{\beta} & \bigoplus_{n\in\mathbb{Z}_{\geqslant 0}} \mathrm{R}j_{n\,!!}K_n. \end{array}$$

Then K satisfies the desired condition.

(ii-3) Let I be arbitrary. Let $\{Z_n\}_{n\in\mathbb{Z}_{\geqslant 0}}$ be an increasing sequence of compact subsets of N such that $N = \bigcup_{n\in\mathbb{Z}_{\geqslant 0}} Z_n$. Let us take an increasing sequence $\{I_n\}_{n\in\mathbb{Z}_{\geqslant 0}}$ of finite subsets of I such that Z_n is covered by $\{V_i\}_{i\in I_n}$, and set $V_n' := \bigcup_{i\in I_n} V_i$, $U_n' := f^{-1}V_n'$. Applying (ii-1) with $N = V_n'$ and $I = I_n$, we can find an object $K_n \in \mathrm{D}(U_n')$ such that $K_n|_{U_i} \simeq K_i$ for any $i \in I_n$. Then we can apply (ii-2) with $V_n = V_n'$. □

2.5. Ind-sheaves with an extra variable. Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$ be the two-point compactification of the affine line. The *bordered line* is

$$\mathbb{R}_\infty := (\mathbb{R}, \overline{\mathbb{R}}).$$

Let M be a bordered space. Consider the morphisms

(2.5.1) $$\mu, q_1, q_2 \colon \mathsf{M} \times \mathbb{R}_\infty \times \mathbb{R}_\infty \to \mathsf{M} \times \mathbb{R}_\infty,$$

where $\mu(x, t_1, t_2) = (x, t_1 + t_2)$, and q_1, q_2 are the natural projections. The convolution functors

$$\overset{+}{\otimes}\colon \mathrm{D}(\mathsf{M} \times \mathbb{R}_\infty) \times \mathrm{D}(\mathsf{M} \times \mathbb{R}_\infty) \to \mathrm{D}(\mathsf{M} \times \mathbb{R}_\infty),$$
$$\mathcal{I}hom^+\colon \mathrm{D}(\mathsf{M} \times \mathbb{R}_\infty)^{\mathrm{op}} \times \mathrm{D}(\mathsf{M} \times \mathbb{R}_\infty) \to \mathrm{D}(\mathsf{M} \times \mathbb{R}_\infty)$$

are defined by

$$F_1 \overset{+}{\otimes} F_2 := \mathrm{R}\mu_{!!}(q_1^{-1} F_1 \otimes q_2^{-1} F_2),$$
$$\mathcal{I}hom^+(F_1, F_2) := \mathrm{R}q_{1*}\mathrm{R}\mathcal{I}hom(q_2^{-1} F_1, \mu^! F_2).$$

Example 2.5.1. Let $\mathsf{M} = \{\mathrm{pt}\}$ and let $a, b \in \mathbb{R}$. For $a \leqslant b$, one has

$$\mathbf{k}_{\{t \geqslant 0\}} \overset{+}{\otimes} \mathbf{k}_{\{t \geqslant a\}} \simeq \mathbf{k}_{\{t \geqslant a\}}, \qquad \mathbf{k}_{\{t \geqslant 0\}} \overset{+}{\otimes} \mathbf{k}_{\{a \leqslant t < b\}} \simeq \mathbf{k}_{\{a \leqslant t < b\}},$$
$$\mathcal{I}hom^+(\mathbf{k}_{\{t \geqslant 0\}}, \mathbf{k}_{\{t \geqslant a\}}) \simeq \mathbf{k}_{\{t < a\}}[1], \quad \mathcal{I}hom^+(\mathbf{k}_{\{t \geqslant 0\}}, \mathbf{k}_{\{a \leqslant t < b\}}) \simeq \mathbf{k}_{\{a \leqslant t < b\}}.$$

For $0 < a \leqslant b$, one has

$$\mathbf{k}_{\{0 \leqslant t < a\}} \overset{+}{\otimes} \mathbf{k}_{\{0 \leqslant t < b\}} \simeq \mathbf{k}_{\{0 \leqslant t < a\}} \oplus \mathbf{k}_{\{b \leqslant t < a+b\}}[-1].$$

Consider the standard classical t-structure on $\mathrm{D}(\mathsf{M} \times \mathbb{R}_\infty)$ discussed in Section 2.4.

Lemma 2.5.2. *Let* M *be a bordered space.*

(i) *For* $n, n' \in \mathbb{Z}$ *one has*

$$\mathrm{D}^{\leqslant n}(\mathsf{M} \times \mathbb{R}_\infty) \overset{+}{\otimes} \mathrm{D}^{\leqslant n'}(\mathsf{M} \times \mathbb{R}_\infty) \subset \mathrm{D}^{\leqslant n+n'+1}(\mathsf{M} \times \mathbb{R}_\infty),$$
$$\mathrm{D}^{\geqslant n}(\mathsf{M} \times \mathbb{R}_\infty) \overset{+}{\otimes} \mathrm{D}^{\geqslant n'}(\mathsf{M} \times \mathbb{R}_\infty) \subset \mathrm{D}^{\geqslant n+n'}(\mathsf{M} \times \mathbb{R}_\infty).$$

In particular, the bifunctor $\otimes^+$ *is left exact.*

(ii) *For* $n, n' \in \mathbb{Z}$ *one has*

$$\mathcal{I}hom^+(\mathrm{D}^{\leqslant n}(\mathsf{M} \times \mathbb{R}_\infty), \mathrm{D}^{\geqslant n'}(\mathsf{M} \times \mathbb{R}_\infty)) \subset \mathrm{D}^{\geqslant n'-n-1}(\mathsf{M} \times \mathbb{R}_\infty).$$

Proof. By the definition of the convolution functors $\otimes^+$ and $\mathcal{I}hom^+$, the statement follows from Proposition 2.4.6. □

Remark 2.5.3. There are no estimates of the form

$$\mathcal{I}hom^+(\mathbf{k}_{\{t \geqslant 0\}}, \mathrm{D}^0(\mathsf{M} \times \mathbb{R}_\infty)) \subset \mathrm{D}^{\leqslant m}(\mathsf{M} \times \mathbb{R}_\infty)$$

with $m \in \mathbb{Z}_{\geqslant 0}$ independent of M. In fact, setting, $\mathsf{M} = \mathbb{R}^n$ $(n \geqslant 1)$ and $F = \mathbf{k}_{\{x \neq 0,\ t = 1/|x|\}}$, one has

$$\mathcal{I}hom^+(\mathbf{k}_{\{t \geqslant 0\}}, F) \notin \mathrm{D}^{\leqslant n-3}(\mathsf{M} \times \mathbb{R}_\infty), \tag{2.5.2}$$

which follows from

$$\pi^{-1}\mathbf{k}_{\{x=0\}} \otimes \mathcal{I}hom^+(\mathbf{k}_{\{t \geqslant 0\}}, F) \simeq \pi^{-1}\mathbf{k}_{\{x=0\}}[1] \oplus \pi^{-1}\mathbf{k}_{\{x=0\}}[2-n].$$

Lemma 2.5.4. *For $K \in \mathrm{D}(\mathsf{M} \times \mathbb{R}_\infty)$ and $n \in \mathbb{Z}$ one has*

$$\mathbf{k}_{\{t \geqslant 0\}} \overset{+}{\otimes} \tau^{\leqslant n}(\mathbf{k}_{\{t \geqslant 0\}} \overset{+}{\otimes} K) \xrightarrow{\sim} \tau^{\leqslant n}(\mathbf{k}_{\{t \geqslant 0\}} \overset{+}{\otimes} K),$$
$$\mathbf{k}_{\{t \geqslant 0\}} \overset{+}{\otimes} \tau^{\geqslant n}(\mathbf{k}_{\{t \geqslant 0\}} \overset{+}{\otimes} K) \xrightarrow{\sim} \tau^{\geqslant n}(\mathbf{k}_{\{t \geqslant 0\}} \overset{+}{\otimes} K).$$

Let us give a proof of this result slightly different from that in [4, Proposition 4.6.2].

Proof. Consider the distinguished triangle

$$\mathbf{k}_{\{t > 0\}} \overset{+}{\otimes} \tau^{\leqslant n}(\mathbf{k}_{\{t \geqslant 0\}} \overset{+}{\otimes} K) \longrightarrow \mathbf{k}_{\{t > 0\}} \overset{+}{\otimes} (\mathbf{k}_{\{t \geqslant 0\}} \overset{+}{\otimes} K)$$
$$\longrightarrow \mathbf{k}_{\{t > 0\}} \overset{+}{\otimes} \tau^{>n}(\mathbf{k}_{\{t \geqslant 0\}} \overset{+}{\otimes} K) \xrightarrow{+1}.$$

Since the middle term vanishes, one has

$$\mathbf{k}_{\{t > 0\}} \overset{+}{\otimes} \tau^{>n}(\mathbf{k}_{\{t \geqslant 0\}} \overset{+}{\otimes} K) \simeq \mathbf{k}_{\{t > 0\}} \overset{+}{\otimes} \tau^{\leqslant n}(\mathbf{k}_{\{t \geqslant 0\}} \overset{+}{\otimes} K)[1].$$

By Lemma 2.5.2, the first term belongs to $\mathrm{D}^{>n}(\mathsf{M} \times \mathbb{R}_\infty)$ and the second term belongs to $\mathrm{D}^{\leqslant n}(\mathsf{M} \times \mathbb{R}_\infty)$. Hence they both vanish. □

2.6. Enhanced ind-sheaves. Let M be a bordered space, and consider the natural morphisms

$$\mathsf{M} \xleftarrow{\pi} \mathsf{M} \times \mathbb{R}_\infty \xrightarrow{j} \mathsf{M} \times \overline{\mathbb{R}} \xrightarrow{\overline{\pi}} \mathsf{M}.$$

Consider the full subcategories of $\mathrm{D}(\mathsf{M} \times \mathbb{R}_\infty)$

$$\mathcal{N}_\pm := \{K \in \mathrm{D}(\mathsf{M} \times \mathbb{R}_\infty); \mathbf{k}_{\{\mp t \geqslant 0\}} \overset{+}{\otimes} K \simeq 0\}$$
$$= \{K \in \mathrm{D}(\mathsf{M} \times \mathbb{R}_\infty); \mathcal{I}hom^+(\mathbf{k}_{\{\mp t \geqslant 0\}}, K) \simeq 0\},$$
$$\mathcal{N} := \mathcal{N}_+ \cap \mathcal{N}_- = \pi^{-1}\mathrm{D}(\mathsf{M}),$$

where the equalities hold by [4, Corollary 4.3.11 and Lemma 4.4.3].

The categories of enhanced ind-sheaves are defined by

$$\mathrm{E}^{\mathrm{b}}_\pm(\mathrm{I}\mathbf{k}_{\mathsf{M}}) := \mathrm{D}(\mathsf{M} \times \mathbb{R}_\infty)/\mathcal{N}_\mp, \quad \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}}) := \mathrm{D}(\mathsf{M} \times \mathbb{R}_\infty)/\mathcal{N}.$$

In this paper, we set for short

$$\mathrm{E}_\pm(\mathsf{M}) := \mathrm{E}^{\mathrm{b}}_\pm(\mathrm{I}\mathbf{k}_{\mathsf{M}}), \quad \mathrm{E}(\mathsf{M}) := \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}}).$$

By [4, Proposition 4.4.4], there are natural equivalences

$$\mathrm{E}_\pm(\mathsf{M}) \simeq \mathcal{N}_\pm/\mathcal{N} \simeq {}^\perp\mathcal{N}_\mp = \mathcal{N}_\pm \cap {}^\perp\mathcal{N},$$
$$\mathrm{E}(\mathsf{M}) \simeq {}^\perp\mathcal{N} \simeq \mathrm{E}_+(\mathsf{M}) \oplus \mathrm{E}_-(\mathsf{M}),$$

and the same equivalences hold when replacing left with right orthogonals. Moreover, one has

$${}^\perp\mathcal{N}_\mp = \{K \in \mathrm{D}(\mathsf{M} \times \mathbb{R}_\infty); \mathbf{k}_{\{\pm t \geqslant 0\}} \overset{+}{\otimes} K \xrightarrow{\sim} K\},$$
$${}^\perp\mathcal{N} = \{K \in \mathrm{D}(\mathsf{M} \times \mathbb{R}_\infty); (\mathbf{k}_{\{t \geqslant 0\}} \oplus \mathbf{k}_{\{t \leqslant 0\}}) \overset{+}{\otimes} K \xrightarrow{\sim} K\}$$
$$= \{K \in \mathrm{D}(\mathsf{M} \times \mathbb{R}_\infty); \mathrm{R}\pi_{!!} K \simeq 0\},$$

and the same equalities hold for right orthogonals, replacing $\otimes^+$ with $\mathcal{I}hom^+$ and $\mathrm{R}\pi_{!!}$ with $\mathrm{R}\pi_*$.

We use the notations

$$\mathsf{D}(\mathsf{M}\times\mathbb{R}_\infty) \underset{\mathrm{L^E},\,\mathrm{R^E}}{\overset{\mathrm{Q_M}}{\rightleftarrows}} \mathsf{E}(\mathsf{M}), \qquad \mathsf{D}(\mathsf{M}\times\mathbb{R}_\infty) \underset{\mathrm{L^E_\pm},\,\mathrm{R^E_\pm}}{\overset{\mathrm{Q^\pm_M}}{\rightleftarrows}} \mathsf{E}_\pm(\mathsf{M}),$$

for the quotient functors and their left and right adjoints, respectively. For $F \in \mathsf{D}(\mathsf{M}\times\mathbb{R}_\infty)$ one has

$$\mathrm{L^E}(\mathrm{Q_M}F) \simeq (\mathbf{k}_{\{t\geqslant 0\}} \oplus \mathbf{k}_{\{t\leqslant 0\}}) \overset{+}{\otimes} F,$$
$$\mathrm{R^E}(\mathrm{Q_M}F) \simeq \mathcal{I}hom^+(\mathbf{k}_{\{t\geqslant 0\}} \oplus \mathbf{k}_{\{t\leqslant 0\}},\ F).$$

For a locally closed subset $Z \subset M\times\mathbb{R}$, we set

(2.6.1)
$$\mathbf{k}^{\mathrm{Q}}_Z := \mathrm{Q_M}(\mathbf{k}_Z) \in \mathsf{E}(\mathsf{M}).$$

There are functors

(2.6.2)
$$\epsilon\colon \mathsf{D}(\mathsf{M}) \longrightarrow \mathsf{E}(\mathsf{M}), \qquad F \mapsto \mathbf{k}^{\mathrm{Q}}_{\{t=0\}} \otimes \pi^{-1}F,$$
$$\epsilon_\pm\colon \mathsf{D}(\mathsf{M}) \longrightarrow \mathsf{E}_\pm(\mathsf{M}), \quad F \mapsto \mathbf{k}^{\mathrm{Q}}_{\{\pm t\geqslant 0\}} \otimes \pi^{-1}F.$$

The functors $\epsilon_\pm$ are fully faithful and $\epsilon(F) \simeq \epsilon_+(F) \oplus \epsilon_-(F)$.

The bifunctors

$$\mathcal{I}hom^{\mathrm{E}}\colon \mathsf{E}(\mathsf{M})\times\mathsf{E}(\mathsf{M}) \to \mathsf{D}(\mathsf{M}),$$
$$\mathcal{H}om^{\mathrm{E}}\colon \mathsf{E}(\mathsf{M})^{\mathrm{op}}\times\mathsf{E}(\mathsf{M}) \to \mathrm{D^b}(\mathbf{k}_{\mathring{\mathsf{M}}})$$

are defined by

$$\begin{aligned}\mathcal{I}hom^{\mathrm{E}}(K,K') &:= \mathrm{R}\pi_*\mathrm{R}\mathcal{I}hom(\mathrm{L^E}K, \mathrm{L^E}K')\\ &\simeq \mathrm{R}\pi_*\mathrm{R}\mathcal{I}hom(\mathrm{L^E}K, \mathrm{R^E}K')\\ &\simeq \mathrm{R}\pi_*\mathrm{R}\mathcal{I}hom(\mathrm{R^E}K, \mathrm{R^E}K')\\ &\simeq \mathrm{R}\overline{\pi}_*\mathrm{R}\mathcal{I}hom(\mathrm{R}j_{!!}\,\mathrm{L^E}K, \mathrm{R}j_*\,\mathrm{R^E}K')\end{aligned}$$

and

$$\mathcal{H}om^{\mathrm{E}} := \alpha_{\mathsf{M}} \circ \mathcal{I}hom^{\mathrm{E}}.$$

One has

(2.6.3)
$$\mathrm{Hom}_{\mathsf{E}(\mathsf{M})}(K,K') \simeq \mathrm{Hom}_{\mathsf{D}(\mathsf{M})}(\mathbf{k}_M, \mathcal{I}hom^{\mathrm{E}}(K,K')).$$

If M is a topological space, that is, if $\mathring{\mathsf{M}} \to \mathsf{M}$ is an isomorphism, one has

$$\mathrm{Hom}_{\mathsf{E}(\mathsf{M})}(K,K') \simeq H^0\mathrm{R}\Gamma(\mathring{\mathsf{M}}; \mathcal{H}om^{\mathrm{E}}(K,K')).$$

Note, however, that $\mathrm{Hom}_{\mathsf{E}(\mathsf{M})}(K,K') \simeq H^0\mathrm{R}\Gamma(\mathring{\mathsf{M}}; \mathcal{H}om^{\mathrm{E}}(K,K'))$ *does not* hold for a general bordered space M.

Definition 2.6.1 ([4, Definition 4.6.3]). For $n \in \mathbb{Z}$, set

$$\mathsf{E}^{\leqslant n}(\mathsf{M}) := \{K \in \mathsf{E}(\mathsf{M});\ \mathrm{L^E}K \in \mathsf{D}^{\leqslant n}(\mathsf{M}\times\mathbb{R}_\infty)\},$$
$$\mathsf{E}^{\geqslant n}(\mathsf{M}) := \{K \in \mathsf{E}(\mathsf{M});\ \mathrm{L^E}K \in \mathsf{D}^{\geqslant n}(\mathsf{M}\times\mathbb{R}_\infty)\}.$$

Note that

$$\begin{aligned}\mathrm{E}^0(\mathsf{M}) &\simeq \{F \in \mathrm{I}(\mathbf{k}_{\mathsf{M}\times\mathbb{R}_\infty}); (\mathbf{k}_{\{t \geqslant 0\}} \oplus \mathbf{k}_{\{t \leqslant 0\}}) \overset{+}{\otimes} F \xrightarrow{\sim} F \text{ in } \mathrm{D}(\mathsf{M}\times\mathbb{R}_\infty)\} \\ &= \{F \in \mathrm{I}(\mathbf{k}_{\mathsf{M}\times\mathbb{R}_\infty}); \mathrm{R}\pi_{!!} F \simeq 0 \text{ in } \mathrm{D}(\mathsf{M})\}.\end{aligned}$$

Proposition 2.6.2 ([4, Proposition 4.6.2]). *We have that* $(\mathrm{E}^{\leqslant 0}(\mathsf{M}), \mathrm{E}^{\geqslant 0}(\mathsf{M}))$ *is a classical t-structure on* $\mathrm{E}(\mathsf{M})$.

Example 2.6.3. Let $a, b \in \mathbb{R}$ with $a < b$. In the category $\mathrm{E}(\{\mathrm{pt}\})$, one has

$$\begin{aligned}&\mathrm{L}^{\mathrm{E}}\,\mathbf{k}^{\mathrm{Q}}_{\{a\leqslant t\}} \simeq \mathbf{k}_{\{a \leqslant t\}}, &&\mathrm{L}^{\mathrm{E}}\,\mathbf{k}^{\mathrm{Q}}_{\{a\leqslant t<b\}} \simeq \mathbf{k}_{\{a \leqslant t < b\}},\\ &\mathrm{R}^{\mathrm{E}}\,\mathbf{k}^{\mathrm{Q}}_{\{a\leqslant t\}} \simeq \mathbf{k}_{\{t < a\}}[1], &&\mathrm{R}^{\mathrm{E}}\,\mathbf{k}^{\mathrm{Q}}_{\{a\leqslant t<b\}} \simeq \mathbf{k}_{\{a \leqslant t < b\}}.\end{aligned}$$

In particular, $\mathbf{k}^{\mathrm{Q}}_{\{a\leqslant t\}}, \mathbf{k}^{\mathrm{Q}}_{\{a\leqslant t<b\}} \in \mathrm{E}^0(\{\mathrm{pt}\})$.

Proposition 2.6.4. *Let M be a good topological space. Then the prestack on M given by $U \mapsto \mathrm{E}^0(U)$ is a stack of abelian categories.*

Proof. The statement holds since $U \mapsto \mathrm{E}^0(U)$ is a sub-prestack of the direct image by π of the stack of ind-sheaves on $M \times \mathbb{R}_\infty$. More precisely, one has

$$\mathrm{E}^0(U) \simeq \{F \in \mathrm{I}(\mathbf{k}_{U\times\mathbb{R}_\infty}); (\mathbf{k}_{\{t \geqslant 0\}} \oplus \mathbf{k}_{\{t \leqslant 0\}}) \overset{+}{\otimes} F \xrightarrow{\sim} F\}. \qquad \square$$

Lemma 2.6.5. *For any $n \in \mathbb{Z}$ one has*

$$\begin{aligned}&\mathrm{Q}_{\mathsf{M}}\mathrm{D}^{\leqslant n}(\mathsf{M}\times\mathbb{R}_\infty) \subset \mathrm{E}^{\leqslant n+1}(\mathsf{M}),\\ &\mathrm{Q}_{\mathsf{M}}\mathrm{D}^{\geqslant n}(\mathsf{M}\times\mathbb{R}_\infty) = \mathrm{E}^{\geqslant n}(\mathsf{M}).\end{aligned}$$

In particular, Q_{M} *is left exact.*

Proof. (i) For $F \in \mathrm{D}(\mathsf{M}\times\mathbb{R}_\infty)$, one has $\mathrm{L}^{\mathrm{E}}\,\mathrm{Q}_{\mathsf{M}}F \simeq (\mathbf{k}_{\{t \geqslant 0\}} \oplus \mathbf{k}_{\{t \leqslant 0\}}) \otimes^+ F$. Hence the inclusions "$\subset$" follow from Lemma 2.5.2.

(ii) It remains to show the opposite inclusion, that is, $\mathrm{Q}_{\mathsf{M}}\mathrm{D}^{\geqslant n}(\mathsf{M}\times\mathbb{R}_\infty) \supset \mathrm{E}^{\geqslant n}(\mathsf{M})$. If $K \in \mathrm{E}^{\geqslant n}(\mathsf{M})$, then $F := \mathrm{L}^{\mathrm{E}} K \in \mathrm{D}^{\geqslant n}(\mathsf{M}\times\mathbb{R}_\infty)$, and $K \simeq \mathrm{Q}_{\mathsf{M}}(F)$. $\square$

Lemma 2.6.6. *For any $n \in \mathbb{Z}$ one has*

$$\mathrm{R}^{\mathrm{E}}\,\mathrm{E}^{\geqslant n}(\mathsf{M}) \subset \mathrm{D}^{\geqslant n-1}(\mathsf{M}\times\mathbb{R}_\infty).$$

Proof. By Lemma 2.6.5, the functor $\mathrm{Q}_{\mathsf{M}}[1]$ is right exact. Hence its right adjoint $\mathrm{R}^{\mathrm{E}}[-1]$ is left exact. $\square$

Remark 2.6.7. (i) It follows from Example 2.6.3 that the estimate in Lemma 2.6.6 is optimal.

(ii) It follows from Remark 2.5.3 that there are no estimates of the form

$$\mathrm{R}^{\mathrm{E}}\,\mathrm{E}^0(\mathsf{M}) \subset \mathrm{D}^{\leqslant m}(\mathsf{M}\times\mathbb{R}_\infty)$$

with $m \in \mathbb{Z}$ independent of M.

(iii) The example in Remark 2.5.3 shows that

$$(\{K \in \mathrm{E}(\mathsf{M}); \mathrm{R}^{\mathrm{E}} K \in \mathrm{D}^{\leqslant 0}(\mathsf{M} \times \mathbb{R}_\infty)\}, \{K \in \mathrm{E}(\mathsf{M}); \mathrm{R}^{\mathrm{E}} K \in \mathrm{D}^{\geqslant 0}(\mathsf{M} \times \mathbb{R}_\infty)\})$$

is not a classical t-structure on E(M), in general.

Proposition 2.6.8. *The two functors $\mathcal{I}hom^{\mathrm{E}}$ and $\mathcal{H}om^{\mathrm{E}}$ are left exact for the standard t-structures, i.e. for $n, n' \in \mathbb{Z}$ one has:*

(i) $\mathcal{I}hom^{\mathrm{E}}(\mathrm{E}^{\leqslant n}(\mathsf{M}), \mathrm{E}^{\geqslant n'}(\mathsf{M})) \subset \mathrm{D}^{\geqslant n'-n}(\mathsf{M})$,

(ii) $\mathcal{H}om^{\mathrm{E}}(\mathrm{E}^{\leqslant n}(\mathsf{M}), \mathrm{E}^{\geqslant n'}(\mathsf{M})) \subset \mathrm{D}^{\geqslant n'-n}(\mathbf{k}_{\mathsf{M}}^{\circ})$.

Proof. By the definition of $\mathcal{I}hom^{\mathrm{E}}$, its left exactness follows from Proposition 2.4.6. This implies the left exactness of $\mathcal{H}om^{\mathrm{E}} = \alpha_{\mathsf{M}} \mathcal{I}hom^{\mathrm{E}}$, since α_{M} is exact. □

2.7. Operations. Let $f : \mathsf{M} \to \mathsf{N}$ be a morphism of bordered spaces. For enhanced ind-sheaves, the six Grothendieck operations

$$\overset{+}{\otimes} : \mathrm{E}(\mathsf{M}) \times \mathrm{E}(\mathsf{M}) \to \mathrm{E}(\mathsf{M}),$$
$$\mathcal{I}hom^{+} : \mathrm{E}(\mathsf{M})^{\mathrm{op}} \times \mathrm{E}(\mathsf{M}) \to \mathrm{E}(\mathsf{M}),$$
$$\mathrm{E}f_{!!}, \mathrm{E}f_{*} : \mathrm{E}(\mathsf{M}) \to \mathrm{E}(\mathsf{N}),$$
$$\mathrm{E}f^{-1}, \mathrm{E}f^{!} : \mathrm{E}(\mathsf{N}) \to \mathrm{E}(\mathsf{M}),$$

are defined as follows. Set $f_{\mathbb{R}_\infty} = f \times \mathrm{id}_{\mathbb{R}_\infty} : \mathsf{M} \times \mathbb{R}_\infty \to \mathsf{N} \times \mathbb{R}_\infty$. For $F, F' \in \mathrm{D}(\mathsf{M} \times \mathbb{R}_\infty)$ and $G \in \mathrm{D}(\mathsf{N} \times \mathbb{R}_\infty)$, one sets

$$\mathrm{Q}_{\mathsf{M}} F \overset{+}{\otimes} \mathrm{Q}_{\mathsf{M}} F' := \mathrm{Q}_{\mathsf{M}}(F \overset{+}{\otimes} F'),$$
$$\mathcal{I}hom^{+}(\mathrm{Q}_{\mathsf{M}} F, \mathrm{Q}_{\mathsf{M}} F') := \mathrm{Q}_{\mathsf{M}} \mathcal{I}hom^{+}(F, F'),$$
$$\mathrm{E}f_{!!} \mathrm{Q}_{\mathsf{M}} F := \mathrm{Q}_{\mathsf{N}} \mathrm{R} f_{\mathbb{R}_\infty !!} F,$$
$$\mathrm{E}f_{*} \mathrm{Q}_{\mathsf{M}} F := \mathrm{Q}_{\mathsf{N}} \mathrm{R} f_{\mathbb{R}_\infty *} F,$$
$$\mathrm{E}f^{-1} \mathrm{Q}_{\mathsf{N}} G := \mathrm{Q}_{\mathsf{M}} f_{\mathbb{R}_\infty}^{-1} G,$$
$$\mathrm{E}f^{!} \mathrm{Q}_{\mathsf{N}} G := \mathrm{Q}_{\mathsf{M}} f_{\mathbb{R}_\infty}^{!} G.$$

The duality functor is defined by

$$\mathrm{D}_{\mathsf{M}}^{\mathrm{Q}} : \mathrm{E}(\mathsf{M}) \to \mathrm{E}(\mathsf{M})^{\mathrm{op}}, \quad K \mapsto \mathcal{I}hom^{+}(K, \omega_{\mathsf{M}}^{\mathrm{Q}}),$$

where $\omega_{\mathsf{M}} := j_{\mathsf{M}}^{!} \omega_{\check{M}} \in \mathrm{D}(\mathsf{M})$ and $\omega_{\mathsf{M}}^{\mathrm{Q}} := \epsilon(\omega_{\mathsf{M}}) = \pi^{-1}\omega_{\mathsf{M}} \otimes \mathbf{k}_{\{t=0\}}^{\mathrm{Q}} \in \mathrm{E}(\mathsf{M})$.

Lemma 2.7.1 ([4, Lemma 4.3.2]). *Let $\mathsf{M} = (M, \check{M})$. For $F \in \mathrm{D}(\mathbf{k}_{M \times \mathbb{R}})$, one has*

$$\mathrm{D}_{\mathsf{M}}^{\mathrm{Q}}(\mathrm{Q}_{\mathsf{M}} F) \simeq \mathrm{Q}_{\mathsf{M}}(a^{-1} \mathrm{D}_{M \times \mathbb{R}} F),$$

where a is the involution of $M \times \mathbb{R}$ defined by $a(x, t) = (x, -t)$.

Example 2.7.2. Let $a, b \in \mathbb{R}$ with $a < b$. In the category E({pt}), one has

$$\mathrm{D}_{\{\mathrm{pt}\}}^{\mathrm{Q}} \mathbf{k}_{\{a \leqslant t\}}^{\mathrm{Q}} \simeq \mathbf{k}_{\{t < -a\}}^{\mathrm{Q}}[1] \simeq \mathbf{k}_{\{-a \leqslant t\}}^{\mathrm{Q}}, \quad \mathrm{D}_{\{\mathrm{pt}\}}^{\mathrm{Q}} \mathbf{k}_{\{a \leqslant t < b\}}^{\mathrm{Q}} \simeq \mathbf{k}_{\{-b \leqslant t < -a\}}^{\mathrm{Q}}[1].$$

In particular, $\mathrm{D}_{\{\mathrm{pt}\}}^{\mathrm{Q}} \mathbf{k}_{\{a \leqslant t\}}^{\mathrm{Q}} \in \mathrm{E}^{0}(\{\mathrm{pt}\})$ and $\mathrm{D}_{\{\mathrm{pt}\}}^{\mathrm{Q}} \mathbf{k}_{\{a \leqslant t < b\}}^{\mathrm{Q}} \in \mathrm{E}^{-1}(\{\mathrm{pt}\})$.

Proposition 2.7.3. *Let* M *be a bordered space.*

(i) *For* $n, n' \in \mathbb{Z}$ *one has*

$$\mathrm{E}^{\leqslant n}(\mathsf{M}) \overset{+}{\otimes} \mathrm{E}^{\leqslant n'}(\mathsf{M}) \subset \mathrm{E}^{\leqslant n+n'+1}(\mathsf{M}),$$
$$\mathrm{E}^{\geqslant n}(\mathsf{M}) \overset{+}{\otimes} \mathrm{E}^{\geqslant n'}(\mathsf{M}) \subset \mathrm{E}^{\geqslant n+n'}(\mathsf{M}).$$

In particular, the bifunctor $\otimes^+$ *is left exact for the standard t-structure.*

(ii) *For* $n, n' \in \mathbb{Z}$ *one has*

$$\mathcal{I}hom^+(\mathrm{E}^{\leqslant n}(\mathsf{M}), \mathrm{E}^{\geqslant n'}(\mathsf{M})) \subset \mathrm{E}^{\geqslant n'-n-1}(\mathsf{M}).$$

Let $f : \mathsf{M} \to \mathsf{N}$ *be a morphism of bordered spaces. Consider the standard t-structures on* $\mathrm{E}(\mathsf{M})$ *and* $\mathrm{E}(\mathsf{N})$. *Then:*

(iii) $\mathrm{E}f_{!!}$ *and* $\mathrm{E}f_*$ *are left exact.*

(iv) $\mathrm{E}f^{-1}$ *is exact.*

Let $d \in \mathbb{Z}_{\geqslant 0}$ *and assume that* $f^{-1}(y) \subset \mathring{\mathsf{M}}$ *has soft-dimension* $\leqslant d$ *for any* $y \in \mathring{\mathsf{N}}$. *Then:*

(v) $\mathrm{E}f_{!!}(*)[d]$ *is right exact, i.e.* $\mathrm{E}f_{!!}\mathrm{E}^{\leqslant n}(\mathsf{M}) \subset \mathrm{E}^{\leqslant n+d}(\mathsf{N})$ *for any* $n \in \mathbb{Z}$.

(vi) $\mathrm{E}f^!(*)[-d]$ *is left exact, i.e.* $\mathrm{E}f^!\mathrm{E}^{\geqslant n}(\mathsf{N}) \subset \mathrm{E}^{\geqslant n-d}(\mathsf{M})$ *for any* $n \in \mathbb{Z}$.

Proof. (i) For $K \in \mathrm{E}(\mathsf{M})$ and $K' \in \mathrm{E}(\mathsf{M})$ one has $\mathrm{L}^{\mathrm{E}}(K \otimes^+ K') \simeq \mathrm{L}^{\mathrm{E}} K \otimes^+ \mathrm{L}^{\mathrm{E}} K'$. Then the statement follows from Lemma 2.5.2.

(ii) This follows from (i) by adjunction. As we deal here with bifunctors, let us spell out the proof. Let $K \in \mathrm{E}^{\leqslant n}(\mathsf{M})$, $K' \in \mathrm{E}^{\geqslant n'}(\mathsf{M})$, and $L \in \mathrm{E}^{< n'-n-1}(\mathsf{M})$. Then one has

$$\mathrm{Hom}_{\mathrm{E}(\mathsf{M})}(L, \mathcal{I}hom^+(K, K')) \simeq \mathrm{Hom}_{\mathrm{E}(\mathsf{M})}(L \overset{+}{\otimes} K, K') \in \mathrm{Hom}_{\mathrm{E}(\mathsf{M})}(\mathrm{E}^{<n'}(\mathsf{M}), \mathrm{E}^{\geqslant n'}(\mathsf{M})) = 0.$$

Then $\mathcal{I}hom^+(K, K') \in \mathrm{E}^{<n'-n-1}(\mathsf{M})^{\perp} = \mathrm{E}^{\geqslant n'-n-1}(\mathsf{M})$.

(iii-1) Note that $\mathrm{L}^{\mathrm{E}} \circ \mathrm{E}f_{!!} \simeq \mathrm{R}f_{\mathbb{R}_\infty !!} \circ \mathrm{L}^{\mathrm{E}}$, where we recall that $f_{\mathbb{R}_\infty} := f \times \mathrm{id}_{\mathbb{R}_\infty}$. Then Proposition 2.4.6 implies that $\mathrm{E}f_{!!}$ is left exact.

(iv) This also follows from Proposition 2.4.6, since one has

$$\mathrm{L}^{\mathrm{E}} \circ \mathrm{E}f^{-1} \simeq f^{-1}_{\mathbb{R}_\infty} \circ \mathrm{L}^{\mathrm{E}}.$$

(iii-2) The fact that $\mathrm{E}f_*$ is left exact follows from (iv) by adjunction.

(v) The statement has a proof similar to (iii-1).

(vi) The statement follows from (v) by adjunction. □

Proposition 2.7.4. *Let* $f : \mathsf{M} \to \mathsf{N}$ *be a morphism of bordered spaces. Let* $n \in \mathbb{Z}$ *and* $L \in \mathrm{E}(\mathsf{N})$. *Assume that* f *is semi-proper and* $\mathring{f} : \mathring{\mathsf{M}} \to \mathring{\mathsf{N}}$ *is surjective. Then:*

(i) $f^{-1}L \in \mathrm{E}^{\geqslant n}(\mathsf{M})$ *implies* $L \in \mathrm{E}^{\geqslant n}(\mathsf{N})$.

(ii) $f^{-1}L \in \mathrm{E}^{\leqslant n}(\mathsf{M})$ *implies* $L \in \mathrm{E}^{\leqslant n}(\mathsf{N})$.

Proof. It is enough to apply Proposition 2.4.7 to the object $G = \mathrm{L}^{\mathrm{E}} L \in \mathrm{D}(\mathsf{N} \times \mathbb{R}_\infty)$ and the morphism $f_{\mathbb{R}_\infty} : \mathsf{M} \times \mathbb{R}_\infty \to \mathsf{N} \times \mathbb{R}_\infty$. □

The bifunctors

$$\pi^{-1}(*) \otimes (*)\colon \mathrm{D}(\mathsf{M}) \times \mathrm{E}(\mathsf{M}) \to \mathrm{E}(\mathsf{M}),$$
$$\mathrm{R}\mathcal{I}hom(\pi^{-1}(*), *)\colon \mathrm{D}(\mathsf{M})^{\mathrm{op}} \times \mathrm{E}(\mathsf{M}) \to \mathrm{E}(\mathsf{M})$$

are defined as follows: for $L \in \mathrm{D}(\mathsf{M})$ and $F \in \mathrm{D}(\mathsf{M} \times \mathbb{R}_\infty)$,

$$\pi^{-1}L \overset{+}{\otimes} \mathrm{Q}_{\mathsf{M}}F := \mathrm{Q}_{\mathsf{M}}(\pi^{-1}L \otimes F),$$
$$\mathrm{R}\mathcal{I}hom(\pi^{-1}L, \mathrm{Q}_{\mathsf{M}}F) := \mathrm{Q}_{\mathsf{M}}\mathrm{R}\mathcal{I}hom(\pi^{-1}L, F).$$

Lemma 2.7.5. *Let* M *be a bordered space. Consider the standard t-structures on* $\mathrm{D}(\mathsf{M})$ *and* $\mathrm{E}(\mathsf{M})$. *Then:*

(i) *The bifunctor* $\pi^{-1}(*) \otimes (*)$ *is exact.*

(ii) *The bifunctor* $\mathrm{R}\mathcal{I}hom(\pi^{-1}(*), *)$ *is left exact.*

In particular, the functor ϵ *from* (2.6.2) *is exact.*

Proof. (i) For $F \in \mathrm{D}(\mathsf{M})$ and $K \in \mathrm{E}(\mathsf{M})$ one has $\mathrm{L}^{\mathrm{E}}(\pi^{-1}F \otimes K) \simeq \pi^{-1}F \otimes \mathrm{L}^{\mathrm{E}} K$. Hence the statement follows from Proposition 2.4.6.

(ii) The statement follows by adjunction from (i). □

Let us end this section stating some facts related to Notation 2.3.3.

Lemma 2.7.6. *Let* Z *be a locally closed subset of* M, *and* $K \in \mathrm{E}(\mathsf{M})$. *One has*

$$\pi^{-1}\mathbf{k}_Z \otimes K \simeq \mathrm{E}i_{Z_\infty !!}\mathrm{E}i_{Z_\infty}^{-1} K,$$
$$\mathrm{R}\mathcal{I}hom(\pi^{-1}\mathbf{k}_Z, K) \simeq \mathrm{E}i_{Z_\infty *}\mathrm{E}i_{Z_\infty}^{!} K.$$

Proof. Note that

$$(Z \times \mathbb{R})_\infty = Z_\infty \times \mathbb{R}_\infty \quad \text{and} \quad i_{Z_\infty} \times \mathrm{id}_{\mathbb{R}_\infty} = i_{(Z\times\mathbb{R})_\infty}.$$

Hence the statement follows from Lemma 2.4.5. □

Lemma 2.7.7. *Let* Z *be a locally closed subset of* M, *and let* $Z' \subset Z$ *be a closed subset. For* $K \in \mathrm{E}(\mathsf{M})$, *there are distinguished triangles in* $\mathrm{E}(Z_\infty)$

$$\mathrm{E}i_{!!}\mathrm{E}i_{(Z\setminus Z')_\infty}^{-1} K \longrightarrow \mathrm{E}i_{Z_\infty}^{-1} K \longrightarrow \mathrm{E}i'_{!!}\mathrm{E}i_{Z'_\infty}^{-1} K \xrightarrow{+1},$$
$$\mathrm{E}i'_{*}\mathrm{E}i_{Z'_\infty}^{!} K \longrightarrow \mathrm{E}i_{Z_\infty}^{!} K \longrightarrow \mathrm{E}i_{*}\mathrm{E}i_{(Z\setminus Z')_\infty}^{!} K \xrightarrow{+1},$$

where $i\colon (Z \setminus Z')_\infty \to Z_\infty$ *and* $i'\colon Z'_\infty \to Z_\infty$ *are the natural morphisms.*

Proof. Since the proofs are similar, we shall only construct the first distinguished triangle. Consider the distinguished triangle

$$\mathbf{k}_{Z\setminus Z'} \longrightarrow \mathbf{k}_Z \longrightarrow \mathbf{k}_{Z'} \xrightarrow{+1}.$$

By Lemma 2.7.6, applying the functor $\pi^{-1}(*) \otimes K$ one gets the distinguished triangle

$$\mathrm{E}i_{(Z\setminus Z')_\infty !!}\mathrm{E}i_{(Z\setminus Z')_\infty}^{-1} K \longrightarrow \mathrm{E}i_{Z_\infty !!}\mathrm{E}i_{Z_\infty}^{-1} K \longrightarrow \mathrm{E}i_{Z'_\infty !!}\mathrm{E}i_{Z'_\infty}^{-1} K \xrightarrow{+1}.$$

Since $i_{Z'_\infty} = i_{Z_\infty} \circ i'$ and $i_{(Z\setminus Z')_\infty} = i_{Z_\infty} \circ i$, the distinguished triangle in the statement is obtained by applying the functor $\mathrm{E}i_{Z_\infty}^{-1}$ to the above distinguished triangle. □

Lemma 2.7.8. *Let $c \in \mathbb{R}$, Z a locally closed subset of* M*, and* $K \in \mathrm{E}(\mathsf{M})$.

(i) $\mathrm{E}i_{Z_\infty}^{-1} K \in \mathrm{E}^{\leqslant c}(Z_\infty)$ *if and only if* $\pi^{-1}\mathbf{k}_Z \otimes K \in \mathrm{E}^{\leqslant c}(\mathsf{M})$.

(ii) $\mathrm{E}i_{Z_\infty}^{!} K \in \mathrm{E}^{\geqslant c}(Z_\infty)$ *if and only if* $\mathrm{R}\mathcal{I}hom(\pi^{-1}\mathbf{k}_Z, K) \in \mathrm{E}^{\geqslant c}(\mathsf{M})$.

Proof. (i) By Lemma 2.7.6, one has

$$\pi^{-1}\mathbf{k}_Z \otimes K \simeq \mathrm{E}i_{Z_\infty !!}\mathrm{E}i_{Z_\infty}^{-1} K,$$
$$\mathrm{E}i_{Z_\infty}^{-1} K \simeq \mathrm{E}i_{Z_\infty}^{-1}(\pi^{-1}\mathbf{k}_Z \otimes K).$$

The statement follows, since the functors $\mathrm{E}i_{Z_\infty !!}$ and $\mathrm{E}i_{Z_\infty}^{-1}$ are exact by Proposition 2.7.3. (It follows that (i) remains true when interchanging $\leqslant c$ with $\geqslant c$.)

(ii) The statement is proved similarly. □

2.8. Stable objects. Setting

$$\mathbf{k}_{\{t \gg 0\}} := \text{“}\varinjlim_{a\to+\infty}\text{”}\ \mathbf{k}_{\{t \geqslant a\}},$$
$$\mathbf{k}_{\{t < *\}} := \text{“}\varinjlim_{a\to+\infty}\text{”}\ \mathbf{k}_{\{t < a\}},$$
$$\mathbf{k}_{\{0 \leqslant t < *\}} := \text{“}\varinjlim_{a\to+\infty}\text{”}\ \mathbf{k}_{\{0 \leqslant t < a\}},$$

there are distinguished triangles in $\mathrm{D}(\mathsf{M} \times \mathbb{R}_\infty)$

$$\mathbf{k}_{\{t \gg 0\}} \longrightarrow \mathbf{k}_{\{t < *\}}[1] \longrightarrow \mathbf{k}_{M \times \mathbb{R}}[1] \xrightarrow{+1},$$
$$\mathbf{k}_{\{0 \leqslant t < *\}} \longrightarrow \mathbf{k}_{\{t \geqslant 0\}} \longrightarrow \mathbf{k}_{\{t \gg 0\}} \xrightarrow{+1}.$$

The objects of $\mathrm{E}(\mathsf{M})$

$$\mathbf{k}_{\mathsf{M}}^{\mathrm{E}} := \mathrm{Q}_{\mathsf{M}}(\mathbf{k}_{\{t \gg 0\}}) \simeq \mathrm{Q}_{\mathsf{M}}(\mathbf{k}_{\{t < *\}}[1]),$$
$$\mathbf{k}_{\mathsf{M}}^{\mathrm{tor}} := \mathrm{Q}_{\mathsf{M}}(\mathbf{k}_{\{0 \leqslant t < *\}})$$

enter the distinguished triangle

$$\mathbf{k}_{\mathsf{M}}^{\mathrm{tor}} \longrightarrow \mathbf{k}_{\{t \geqslant 0\}} \longrightarrow \mathbf{k}_{\mathsf{M}}^{\mathrm{E}} \xrightarrow{+1}. \tag{2.8.1}$$

Note that we have

$$\mathbf{k}_{\mathsf{M}}^{\mathrm{tor}} \overset{+}{\otimes} \mathbf{k}_{\mathsf{M}}^{\mathrm{tor}} \simeq \mathbf{k}_{\mathsf{M}}^{\mathrm{tor}}, \quad \mathbf{k}_{\mathsf{M}}^{\mathrm{E}} \overset{+}{\otimes} \mathbf{k}_{\mathsf{M}}^{\mathrm{E}} \simeq \mathbf{k}_{\mathsf{M}}^{\mathrm{E}} \quad \text{and} \quad \mathbf{k}_{\mathsf{M}}^{\mathrm{tor}} \overset{+}{\otimes} \mathbf{k}_{\mathsf{M}}^{\mathrm{E}} \simeq 0.$$

Definition 2.8.1. The category $\mathrm{E}_{\mathrm{st}}(\mathsf{M})$ of stable enhanced ind-sheaves is the full subcategory of $\mathrm{E}_+(\mathsf{M})$ given by

$$\begin{aligned}\mathrm{E}_{\mathrm{st}}(\mathsf{M}) &:= \{K \in \mathrm{E}_+(\mathsf{M}); \mathbf{k}_{\mathsf{M}}^{\mathrm{tor}} \overset{+}{\otimes} K \simeq 0\}\\ &= \{K \in \mathrm{E}_+(\mathsf{M}); K \xrightarrow{\sim} \mathbf{k}_{\mathsf{M}}^{\mathrm{E}} \overset{+}{\otimes} K\}\\ &= \{K \in \mathrm{E}_+(\mathsf{M}); K \simeq \mathbf{k}_{\mathsf{M}}^{\mathrm{E}} \overset{+}{\otimes} L \text{ for some } L \in \mathrm{E}_+(\mathsf{M})\}\\ &= \{K \in \mathrm{E}_+(\mathsf{M}); K \xrightarrow{\sim} \mathbf{k}_{\{t \geqslant a\}}^{\mathrm{Q}} \overset{+}{\otimes} K \text{ for any } a \geqslant 0\},\end{aligned}$$

where the equivalences follow from (2.8.1) and [4, Proposition 4.7.5]. Similar equivalences hold by replacing $\otimes^+$ with $\mathcal{I}hom^+$.

The embedding $\mathrm{E_{st}}(\mathsf{M}) \to \mathrm{E}(\mathsf{M})$ has a left adjoint $\mathbf{k}^{\mathrm{E}}_{\mathsf{M}} \otimes^{+} *$, as well as a right adjoint $\mathcal{I}hom^{+}(\mathbf{k}^{\mathrm{E}}_{\mathsf{M}}, *)$. There is an embedding

$$e \colon \mathrm{D}(\mathsf{M}) \hookrightarrow \mathrm{E_{st}}(\mathsf{M}), \quad F \mapsto \mathbf{k}^{\mathrm{E}}_{\mathsf{M}} \otimes \pi^{-1} F. \tag{2.8.2}$$

Note that $e(F) \simeq \mathbf{k}^{\mathrm{E}}_{\mathsf{M}} \otimes^{+} \epsilon(F)$.

For a locally closed subset $Z \subset \mathring{\mathsf{M}} \times \mathbb{R}$, we set

$$\mathbf{k}^{\mathrm{E}}_{Z} := \mathbf{k}^{\mathrm{E}}_{\mathsf{M}} \overset{+}{\otimes} \mathbf{k}^{\mathrm{Q}}_{Z} \in \mathrm{E_{st}}(\mathsf{M}). \tag{2.8.3}$$

Lemma 2.8.2. *The following statements hold.*

(i) *The embedding e from* (2.8.2) *is fully faithful and exact.*

(ii) *The functor $\mathbf{k}^{\mathrm{E}}_{\mathsf{M}} \otimes^{+} (*)$ is exact.*

Proof. Statement (i) follows from [4, Proposition 4.7.15] and Lemma 2.7.5, and (ii) from [4, Lemma 4.7.4]. □

The duality functor for stable enhanced ind-sheaves is defined by

$$\mathrm{D}^{\mathrm{E}}_{\mathsf{M}} \colon \mathrm{E}(\mathsf{M}) \to \mathrm{E_{st}}(\mathsf{M})^{\mathrm{op}}, \quad K \mapsto \mathcal{I}hom^{+}(K, \omega^{\mathrm{E}}_{\mathsf{M}}),$$

where we set $\omega^{\mathrm{E}}_{\mathsf{M}} := e(\omega_{\mathsf{M}})$.

Lemma 2.8.3 ([4, Proposition 4.8.3]). *Let* $\mathsf{M} = (M, \check{M})$. *For* $F \in \mathrm{D^b}(\mathbf{k}_{M \times \mathbb{R}})$, *one has*

$$\mathrm{D}^{\mathrm{E}}_{\mathsf{M}}(\mathbf{k}^{\mathrm{E}}_{\mathsf{M}} \overset{+}{\otimes} \mathrm{Q}_{\mathsf{M}} F) \simeq \mathbf{k}^{\mathrm{E}}_{\mathsf{M}} \overset{+}{\otimes} (\mathrm{D}^{\mathrm{Q}}_{\mathsf{M}} \mathrm{Q}_{\mathsf{M}} F) \simeq \mathbf{k}^{\mathrm{E}}_{\mathsf{M}} \overset{+}{\otimes} \mathrm{Q}_{\mathsf{M}}(a^{-1} \mathrm{D}_{M \times \mathbb{R}} F),$$

where a is the involution of $M \times \mathbb{R}$ defined by $a(x,t) = (x,-t)$.

3. Perverse enhanced ind-sheaves

As we recalled in Section 1, a perversity induces a t-structure on the triangulated category of $\mathbb{R}$-constructible sheaves on a subanalytic space. Here, we extend this result to the triangulated category of $\mathbb{R}$-constructible enhanced ind-sheaves. We allow the subanalytic space to be bordered, and we also discuss exactness of the six Grothendieck operations.

3.1. Subanalytic bordered spaces.

Recall Notation 2.3.1.

Definition 3.1.1. (i) A *subanalytic bordered space* $\mathsf{M} = (M, \check{M})$ is a bordered space such that $\check{M}$ is a subanalytic space and M is an open subanalytic subset of $\check{M}$.

(ii) A morphism

$$f \colon \mathsf{M} \to \mathsf{N} = (N, \check{N})$$

of subanalytic bordered spaces is a morphism $f \colon M \to N$ of subanalytic spaces such that its graph Γ_f is a subanalytic subset of $\check{M} \times \check{N}$, and $q_1|_{\overline{\Gamma_f}}$ is proper. In particular, $f \colon \mathsf{M} \to \mathsf{N}$ is a morphism of bordered spaces.

(iii) M is *smooth* of dimension d if M is locally isomorphic to $\mathbb{R}^d$ as a subanalytic space.

(iv) A subset S of M (see Definition 2.3.6) is called subanalytic if it is subanalytic in $\check{M}$.

(v) A morphism $f\colon \mathsf{M} \to \mathsf{N}$ of subanalytic bordered spaces is said to be *submersive* if the continuous map $\mathring{f}\colon \mathring{\mathsf{M}} \to \mathring{\mathsf{N}}$ is locally (in $\mathring{\mathsf{M}}$) isomorphic to the projection $\mathring{\mathsf{N}} \times \mathbb{R}^d \to \mathring{\mathsf{N}}$ for some d.

Let $\mathsf{M} = (M, \check{M})$ be a subanalytic bordered space, and let $j_M\colon M \to \check{M}$ be the embedding.

Definition 3.1.2. Define $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_{\mathsf{M}})$ to be the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ whose objects F are such that $\mathrm{R}j_{M\,!}F$ is an $\mathbb{R}$-constructible object of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\check{M}})$. We regard $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_{\mathsf{M}})$ as a full subcategory of $\mathrm{D}(\mathsf{M})$.

Proposition 3.1.3. *Let $f\colon \mathsf{M} \to \mathsf{N}$ be a morphism of subanalytic bordered spaces.*

(i) *The functors f^{-1} and $f^{!}$ send $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_{\mathsf{N}})$ to $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_{\mathsf{M}})$.*

(ii) *If f is semi-proper, then the functors $\mathrm{R}f_{!!}$ and $\mathrm{R}f_{*}$ send $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_{\mathsf{M}})$ to $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_{\mathsf{N}})$.*

In particular, the category $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_{\mathsf{M}})$ only depends on M (and not on $\check{M}$).

Notation 3.1.4. For M a subanalytic bordered space, set

$$\mathrm{CS}_{\mathsf{M}} := \{\text{closed subanalytic subsets of } \mathsf{M}\},$$
$$\mathrm{LCS}_{\mathsf{M}} := \{\text{locally closed subanalytic subsets of } \mathsf{M}\}.$$

For $k \in \mathbb{Z}$, set

$$\mathrm{CS}^{<k}_{\mathsf{M}} := \{Z \in \mathrm{CS}_{\mathsf{M}};\, d_Z < k\}, \quad \mathrm{CS}^{\leqslant k}_{\mathsf{M}} := \{Z \in \mathrm{CS}_{\mathsf{M}};\, d_Z \leqslant k\},$$

and similarly for $\mathrm{LCS}^{<k}_{\mathsf{M}}$ and $\mathrm{LCS}^{\leqslant k}_{\mathsf{M}}$. For $Z \in \mathrm{LCS}_{\mathsf{M}}$, denote by

$$i_{Z_\infty}\colon Z_\infty \to \mathsf{M}$$

the morphism induced by the embedding $Z \subset \mathring{\mathsf{M}}$ (see Notation 2.3.3).

Definition 3.1.5. Let p be a perversity, $c \in \mathbb{R}$ and $k \in \mathbb{Z}_{\geqslant 0}$. Consider the following conditions for $F \in \mathrm{D}(\mathsf{M})$:

$$(\mathrm{I}\,p^{\leqslant c}_k)\colon \quad i^{-1}_{(M\setminus Z)_\infty} F \in \mathrm{D}^{\leqslant c+p(k)}((M \setminus Z)_\infty) \quad \text{for some } Z \in \mathrm{CS}^{<k}_{\mathsf{M}},$$
$$(\mathrm{I}\,p^{\geqslant c}_k)\colon \quad i^{!}_{Z_\infty} F \in \mathrm{D}^{\geqslant c+p(k)}(Z_\infty) \quad \text{for any } Z \in \mathrm{CS}^{\leqslant k}_{\mathsf{M}}.$$

Consider the following strictly full subcategories of $\mathrm{D}(\mathsf{M})$:

$${}^{p}\mathrm{D}^{\leqslant c}(\mathsf{M}) := \{F \in \mathrm{D}(\mathsf{M});\, (\mathrm{I}\,p^{\leqslant c}_k) \text{ holds for any } k \in \mathbb{Z}_{\geqslant 0}\},$$
$${}^{p}\mathrm{D}^{\geqslant c}(\mathsf{M}) := \{F \in \mathrm{D}(\mathsf{M});\, (\mathrm{I}\,p^{\geqslant c}_k) \text{ holds for any } k \in \mathbb{Z}_{\geqslant 0}\}.$$

Let us also set

$${}^{p}\mathrm{D}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_{\mathsf{M}}) := {}^{p}\mathrm{D}^{\leqslant c}(\mathsf{M}) \cap \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_{\mathsf{M}}),$$
$${}^{p}\mathrm{D}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_{\mathsf{M}}) := {}^{p}\mathrm{D}^{\geqslant c}(\mathsf{M}) \cap \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_{\mathsf{M}}).$$

One easily checks that

Proposition 3.1.6. *We have the following properties.*

(i) $({}^p\mathrm{D}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_\mathsf{M}), {}^p\mathrm{D}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_\mathsf{M}))_{c\in\mathbb{R}}$ *is a t-structure on* $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_\mathsf{M})$.

(ii) *For any* $c \in \mathbb{R}$, *the duality functor* D_M *interchanges* ${}^p\mathrm{D}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_\mathsf{M})$ *and* ${}^{p^*}\mathrm{D}^{\geqslant -c}_{\mathbb{R}\text{-c}}(\mathbf{k}_\mathsf{M})$.

Note that $({}^p\mathrm{D}^{\leqslant c}(\mathsf{M}), {}^p\mathrm{D}^{\geqslant c}(\mathsf{M}))_{c\in\mathbb{R}}$ is not a t-structure in general.

Lemma 3.1.7. *For any* $c \in \mathbb{R}$ *one has*

$$\alpha_\mathsf{M}({}^p\mathrm{D}^{\leqslant c}(\mathsf{M})) \subset {}^p\mathrm{D}^{\leqslant c}(\mathbf{k}_{\mathsf{M}}^{\circ}).$$

Proof. This follows from the fact that α commutes with i^{-1}. □

Remark 3.1.8. Since α does not commute with the functors $i^!$, the statement

$$\alpha_\mathsf{M}({}^p\mathrm{D}^{\geqslant c}(\mathsf{M})) \subset {}^p\mathrm{D}^{\geqslant c}(\mathbf{k}_{\overset{\circ}{\mathsf{M}}})$$

does not hold in general. For example, let $\mathsf{M} = \mathbb{R}$ and $F = \text{“}\varinjlim\text{''}_{\varepsilon\to 0+} \mathbf{k}_{[-\varepsilon,\varepsilon]}$ as in [13, Exercise 5.1]. Then

$$\alpha_\mathsf{M} F \simeq \mathbf{k}_{\{0\}} \in {}^{1/2}\mathrm{D}^0(\mathbf{k}_{\mathsf{M}}^{\circ}) \quad \text{and} \quad i^!_{\{0\}} F \simeq \mathbf{k}_{\{0\}}[-1].$$

Hence $F \in {}^{1/2}\mathrm{D}^{\geqslant 1/2}(\mathsf{M})$ but $\alpha_\mathsf{M} F \notin {}^{1/2}\mathrm{D}^{\geqslant 1/2}(\mathbf{k}_{\mathsf{M}}^{\circ})$. Here, ${}^{1/2}\mathrm{D} := {}^{\mathsf{m}}\mathrm{D}$ for the middle perversity $\mathsf{m}(n) := -\frac{n}{2}$.

3.2. Intermediate enhanced perversities. Let $\mathsf{M} = (M, \check{M})$ be a subanalytic bordered space.

Definition 3.2.1. Let p be a perversity, $c \in \mathbb{R}$ and $k \in \mathbb{Z}_{\geqslant 0}$. Consider the following conditions for $K \in \mathrm{E}(\mathsf{M})$:

$$(\mathrm{E}p_k^{\leqslant c})\colon\quad \mathrm{E}i^{-1}_{(M\setminus Z)_\infty} K \in \mathrm{E}^{\leqslant c+p(k)}((M\setminus Z)_\infty) \quad \text{for some } Z \in \mathrm{CS}^{<k}_\mathsf{M},$$
$$(\mathrm{E}p_k^{\geqslant c})\colon\quad \mathrm{E}i^{!}_{Z_\infty} K \in \mathrm{E}^{\geqslant c+p(k)}(Z_\infty) \quad \text{for any } Z \in \mathrm{CS}^{\leqslant k}_\mathsf{M}.$$

Consider the following strictly full subcategories of $\mathrm{E}(\mathsf{M})$:

$$ {}_p\mathrm{E}^{\leqslant c}(\mathsf{M}) := \{K \in \mathrm{E}(\mathsf{M});\ (\mathrm{E}p_k^{\leqslant c}) \text{ holds for any } k \in \mathbb{Z}_{\geqslant 0}\},$$
$$ {}_p\mathrm{E}^{\geqslant c}(\mathsf{M}) := \{K \in \mathrm{E}(\mathsf{M});\ (\mathrm{E}p_k^{\geqslant c}) \text{ holds for any } k \in \mathbb{Z}_{\geqslant 0}\}.$$

Note that $({}_p\mathrm{E}^{\leqslant c}(\mathsf{M}), {}_p\mathrm{E}^{\geqslant c}(\mathsf{M}))_{c\in\mathbb{R}}$ is not a t-structure if $\dim M > 0$. However, we write ${}_p\mathrm{E}^{<c}(\mathsf{M}) := \bigcup_{c'<c} {}_p\mathrm{E}^{\leqslant c'}(\mathsf{M})$, ${}_p\mathrm{E}^{c}(\mathsf{M}) := {}_p\mathrm{E}^{\leqslant c}(\mathsf{M}) \cap {}_p\mathrm{E}^{\geqslant c}(\mathsf{M})$, etc.

Remark 3.2.2. (i) Conditions $(\mathrm{E}p_k^{\leqslant c})$ and $(\mathrm{E}p_k^{\geqslant c})$ can be rewritten using the equivalences

$$\mathrm{E}i^{-1}_{(M\setminus Z)_\infty} K \in \mathrm{E}^{\leqslant c}((M\setminus Z)_\infty) \iff \pi^{-1}\mathbf{k}_{M\setminus Z} \otimes K \in \mathrm{E}^{\leqslant c}(\mathsf{M}),$$
$$\mathrm{E}i^{!}_{Z_\infty} K \in \mathrm{E}^{\geqslant c}(Z_\infty) \iff \mathrm{R}\mathcal{I}hom(\pi^{-1}\mathbf{k}_Z, K) \in \mathrm{E}^{\geqslant c}(\mathsf{M}),$$

which follow from Lemma 2.7.8.

(ii) One has

$$\mathrm{E}i^{-1}_{(M\setminus Z)_\infty} K \in \mathrm{E}^{\leqslant c}((M\setminus Z)_\infty) \implies \mathrm{E}i^{-1}_{(M\setminus Z')_\infty} K \in \mathrm{E}^{\leqslant c}((M\setminus Z')_\infty)$$

for any $Z, Z' \in \mathrm{CS}_{\mathsf{M}}$ such that $Z \subset Z'$. Similarly,

$$\mathrm{E}i^{\,!}_{Z_\infty} K \in \mathrm{E}^{\geqslant c}(Z_\infty) \implies \mathrm{E}i^{\,!}_{Z''_\infty} K \in \mathrm{E}^{\geqslant c}(Z''_\infty)$$

for any $Z \in \mathrm{CS}_{\mathsf{M}}$ and any locally closed subanalytic subset Z'' of Z. Indeed, one has

$$\mathrm{E}i^{-1}_{(M\setminus Z')_\infty} \simeq \mathrm{E}j^{-1} \circ \mathrm{E}i^{-1}_{(M\setminus Z)_\infty}$$

and

$$\mathrm{E}i^{\,!}_{Z''_\infty} \simeq \mathrm{E}j''^{\,!} \circ \mathrm{E}i^{\,!}_{Z_\infty},$$

and $\mathrm{E}j^{-1}$ is exact and $\mathrm{E}j''^{\,!}$ is left exact for the standard t-structure. Here,

$$j\colon (M\setminus Z')_\infty \to (M\setminus Z)_\infty \quad \text{and} \quad j''\colon Z''_\infty \to Z_\infty$$

are the canonical morphisms.

The following lemma is obvious.

Lemma 3.2.3. *For any $c \in \mathbb{R}$, one has*

$$\begin{aligned} \mathrm{E}^{\leqslant c+p(d_M)}(\mathsf{M}) \subset {}_p\mathrm{E}^{\leqslant c}(\mathsf{M}) \subset \mathrm{E}^{\leqslant c+p(0)}(\mathsf{M}), \\ \mathrm{E}^{\geqslant c+p(0)}(\mathsf{M}) \subset {}_p\mathrm{E}^{\geqslant c}(\mathsf{M}) \subset \mathrm{E}^{\geqslant c+p(d_M)}(\mathsf{M}). \end{aligned}$$

Note that the following lemma is a particular case of Proposition 3.3.21 below.

Lemma 3.2.4. *For any $c \in \mathbb{R}$ and any $Z \in \mathrm{LCS}_{\mathsf{M}}$, one has*

$$\begin{aligned} \mathrm{E}i^{-1}_{Z_\infty}({}_p\mathrm{E}^{\leqslant c}(\mathsf{M})) &\subset {}_p\mathrm{E}^{\leqslant c}(Z_\infty), \\ \mathrm{E}i^{\,!}_{Z_\infty}({}_p\mathrm{E}^{\geqslant c}(\mathsf{M})) &\subset {}_p\mathrm{E}^{\geqslant c}(Z_\infty), \\ \mathrm{E}i_{Z_\infty *}({}_p\mathrm{E}^{\geqslant c}(Z_\infty)) &\subset {}_p\mathrm{E}^{\geqslant c}(\mathsf{M}), \\ \mathrm{E}i_{Z_\infty !!}({}_p\mathrm{E}^{\leqslant c}(Z_\infty)) &\subset {}_p\mathrm{E}^{\leqslant c}(\mathsf{M}). \end{aligned}$$

Proof. Since the proofs are similar, let us only discuss the third inclusion.
Let $K \in {}_p\mathrm{E}^{\geqslant c}(Z_\infty)$. For $W \in \mathrm{CS}^{\leqslant k}_{\mathsf{M}}$, consider the Cartesian diagram of bordered spaces

$$\begin{array}{ccc} (Z\cap W)_\infty & \xrightarrow{\;i'\;} & W_\infty \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle i_{W_\infty}} \\ Z_\infty & \xrightarrow{\;i_{Z_\infty}\;} & \mathsf{M}. \end{array}$$

Noticing that $Z \cap W \in \mathrm{CS}^{\leqslant k}_{Z_\infty}$ and that $\mathrm{E}i'_*$ is left exact for the standard t-structures by Proposition 2.7.3, one has

$$\mathrm{E}i^{\,!}_{W_\infty}\mathrm{E}i_{Z_\infty *}K \simeq \mathrm{E}i'_*\mathrm{E}i^{\,!}K \in \mathrm{E}i'_*(\mathrm{E}^{\geqslant c+p(k)}((Z\cap W)_\infty)) \subset \mathrm{E}^{\geqslant c+p(k)}(W_\infty). \qquad \square$$

Lemma 3.2.5. *For any $c \in \mathbb{R}$ and $K \in \mathrm{E}(\mathsf{M})$, the following conditions are equivalent.*

(i) $K \in {}_p\mathrm{E}^{\geqslant c}(\mathsf{M})$.

(ii) $\mathrm{E}i_{S_\infty}^{\,!} K \in \mathrm{E}^{\geqslant c+p(k)}(S_\infty)$ *for any $k \in \mathbb{Z}_{\geqslant 0}$ and any $S \in \mathrm{LCS}_{\mathsf{M}}^{\leqslant k}$.*

(iii) $\mathrm{E}i_{S_\infty}^{\,!} K \in \mathrm{E}^{\geqslant c+p(k)}(S_\infty)$ *for any $k \in \mathbb{Z}_{\geqslant 0}$ and any smooth $S \in \mathrm{LCS}_{\mathsf{M}}^{\leqslant k}$.*

(iv) *For any $k \in \mathbb{Z}_{\geqslant 0}$ and any $Z \in \mathrm{CS}_{\mathsf{M}}^{\leqslant k}$, there exists an open subanalytic subset Z_0 of Z_∞ such that* $\dim(Z \setminus Z_0) < k$ *and* $\mathrm{E}i_{(Z_0)_\infty}^{\,!} K \in \mathrm{E}^{\geqslant c+p(k)}((Z_0)_\infty)$.

(v) *For any $k \in \mathbb{Z}_{\geqslant 0}$ and any $S \in \mathrm{LCS}_{\mathsf{M}}^{\leqslant k}$, there exists an open subanalytic subset S_0 of S_∞ such that* $\dim(S \setminus S_0) < k$ *and* $\mathrm{E}i_{(S_0)_\infty}^{\,!} K \in \mathrm{E}^{\geqslant c+p(k)}((S_0)_\infty)$.

Proof. The implications in the following diagram are clear:

$$\text{(i)} \Longrightarrow \text{(ii)} \;\begin{smallmatrix} \nearrow & \text{(iii)} & \searrow \\ \searrow & \text{(v)} & \nearrow \end{smallmatrix}\; \text{(iv)}.$$

Here the less trivial implication (i) $\Rightarrow$ (ii) follows from Remark 3.2.2 (ii).

It remains to show that (iv) $\Rightarrow$ (i). That is, we have to show that for any $Z \in \mathrm{CS}_{\mathsf{M}}^{\leqslant k}$ one has

$$\mathrm{R}\mathcal{I}hom(\pi^{-1}\mathbf{k}_Z, K) \in \mathrm{E}^{\geqslant c+p(k)}(\mathsf{M}). \tag{3.2.1}$$

We shall prove it by induction on $k \in \mathbb{Z}_{\geqslant 0}$. When $k = 0$, (3.2.1) is true, because Z_0 in (iv) coincides with Z. Assume that $k > 0$. Let $Z_0 \subset Z$ be an open subanalytic subset as in (iv), so that

$$\mathrm{R}\mathcal{I}hom(\pi^{-1}\mathbf{k}_{Z_0}, K) \in \mathrm{E}^{\geqslant c+p(k)}(\mathsf{M}).$$

Since $Z \setminus Z_0 \in \mathrm{CS}_{\mathsf{M}}^{\leqslant k-1}$, the induction hypothesis implies

$$\mathrm{R}\mathcal{I}hom(\pi^{-1}\mathbf{k}_{Z \setminus Z_0}, K) \in \mathrm{E}^{\geqslant c+p(k-1)}(\mathsf{M}) \subset \mathrm{E}^{\geqslant c+p(k)}(\mathsf{M}).$$

Then (3.2.1) follows from the distinguished triangle

$$\mathrm{R}\mathcal{I}hom(\pi^{-1}\mathbf{k}_{Z \setminus Z_0}, K) \longrightarrow \mathrm{R}\mathcal{I}hom(\pi^{-1}\mathbf{k}_Z, K) \longrightarrow \mathrm{R}\mathcal{I}hom(\pi^{-1}\mathbf{k}_{Z_0}, K) \xrightarrow{+1}. \qquad \square$$

Proposition 3.2.6. *For any $c, c' \in \mathbb{R}$, one has*

$$\mathcal{I}hom^{\mathrm{E}}({}_p\mathrm{E}^{\leqslant c}(\mathsf{M}), {}_p\mathrm{E}^{\geqslant c'}(\mathsf{M})) \subset \mathrm{D}^{\geqslant c'-c}(\mathsf{M}),$$
$$\mathcal{H}om^{\mathrm{E}}({}_p\mathrm{E}^{\leqslant c}(\mathsf{M}), {}_p\mathrm{E}^{\geqslant c'}(\mathsf{M})) \subset \mathrm{D}^{\geqslant c'-c}(\mathbf{k}_{\overset{\circ}{\mathsf{M}}}).$$

In particular, $\mathrm{Hom}_{\mathrm{E}(\mathsf{M})}({}_p\mathrm{E}^{\leqslant c}(\mathsf{M}), {}_p\mathrm{E}^{\geqslant c'}(\mathsf{M})) = 0$ *if* $c' > c$.

Proof. (i) Let $K \in {}_p\mathrm{E}^{\leqslant c}(\mathsf{M})$ and $K' \in {}_p\mathrm{E}^{\geqslant c'}(\mathsf{M})$. Reasoning by decreasing induction on $k \in \mathbb{Z}_{\geqslant -1}$, let us show the following:

(i)$_k$ there exists $Z_k \in \mathrm{CS}_{\mathsf{M}}^{\leqslant k}$ such that

$$\mathrm{R}\mathcal{I}hom(\mathbf{k}_{M \setminus Z_k}, \mathcal{I}hom^{\mathrm{E}}(K, K')) \in \mathrm{D}^{\geqslant c'-c}(\mathsf{M}).$$

The above statement is obvious for $k \geqslant d_M$. Assuming that (i)$_k$ holds true for $k \geqslant 0$, let us prove (i)$_{k-1}$. Since $K' \in {}_p\mathrm{E}^{\geqslant c'}(\mathsf{M})$, one has

$$\mathrm{R}\mathcal{I}hom(\pi^{-1}\mathbf{k}_{Z_k}, K') \in \mathrm{E}^{\geqslant c'+p(k)}(\mathsf{M}).$$

Moreover, since $K \in {}_p\mathrm{E}^{\leqslant c}(\mathsf{M})$, there exists $W_{k-1} \in \mathrm{CS}_{\mathsf{M}}^{\leqslant k-1}$ with

$$\pi^{-1}\mathbf{k}_{M \setminus W_{k-1}} \otimes K \in \mathrm{E}^{\leqslant c+p(k)}(\mathsf{M}).$$

Then

$$\begin{aligned} \mathrm{R}\mathcal{I}hom(\mathbf{k}_{Z_k \setminus W_{k-1}}, \mathcal{I}hom^{\mathrm{E}}(K, K')) &\simeq \mathrm{R}\mathcal{I}hom(\mathbf{k}_{M \setminus W_{k-1}} \otimes \mathbf{k}_{Z_k}, \mathcal{I}hom^{\mathrm{E}}(K, K')) \\ &\simeq \mathcal{I}hom^{\mathrm{E}}(\pi^{-1}\mathbf{k}_{M \setminus W_{k-1}} \otimes K, \mathrm{R}\mathcal{I}hom(\pi^{-1}\mathbf{k}_{Z_k}, K')) \\ &\in \mathcal{I}hom^{\mathrm{E}}(\mathrm{E}^{\leqslant c+p(k)}(\mathsf{M}), \mathrm{E}^{\geqslant c'+p(k)}(\mathsf{M})) \\ &\subset \mathrm{D}^{\geqslant c'-c}(\mathsf{M}), \end{aligned}$$

where the last inclusion follows from Proposition 2.6.8.

Considering the distinguished triangle

$$\begin{aligned} \mathrm{R}\mathcal{I}hom(\mathbf{k}_{Z_k \setminus W_{k-1}}, \mathcal{I}hom^{\mathrm{E}}(K, K')) &\longrightarrow \mathrm{R}\mathcal{I}hom(\mathbf{k}_{M \setminus (Z_k \cap W_{k-1})}, \mathcal{I}hom^{\mathrm{E}}(K, K')) \\ &\longrightarrow \mathrm{R}\mathcal{I}hom(\mathbf{k}_{M \setminus Z_k}, \mathcal{I}hom^{\mathrm{E}}(K, K')) \xrightarrow{+1}, \end{aligned}$$

we deduce (i)$_{k-1}$ for $Z_{k-1} = Z_k \cap W_{k-1}$.

(ii) The second inclusion follows from the first since $\mathcal{H}om^{\mathrm{E}} \simeq \alpha_{\mathsf{M}}\, \mathcal{I}hom^{\mathrm{E}}$.

(iii) The last assertion follows from (2.6.3). □

Lemma 3.2.7. *For any $c, c' \in \mathbb{R}$, one has*

$$\mathcal{I}hom^{\mathrm{E}}(\mathrm{E}^{\leqslant c}(\mathsf{M}), {}_p\mathrm{E}^{\geqslant c'}(\mathsf{M})) \subset {}^p\mathrm{D}^{\geqslant c'-c}(\mathsf{M}),$$

and in particular,

$$\mathcal{I}hom^{\mathrm{E}}(\mathbf{k}_{\mathsf{M}}^{\mathrm{Q}}, {}_p\mathrm{E}^{\geqslant c}(\mathsf{M})) \subset {}^p\mathrm{D}^{\geqslant c}(\mathsf{M}).$$

Proof. Let $k \in \mathbb{Z}_{\geqslant 0}$, $Z \in \mathrm{CS}_{\mathsf{M}}^{\leqslant k}$, $K \in \mathrm{E}^{\leqslant c}(\mathsf{M})$ and $K' \in {}_p\mathrm{E}^{\geqslant c'}(\mathsf{M})$. One has

$$\begin{aligned} \mathrm{R}\mathcal{I}hom(\mathbf{k}_Z, \mathcal{I}hom^{\mathrm{E}}(K, K')) &\simeq \mathcal{I}hom^{\mathrm{E}}(K, \mathrm{R}\mathcal{I}hom(\pi^{-1}\mathbf{k}_Z, K')) \\ &\in \mathcal{I}hom^{\mathrm{E}}(\mathrm{E}^{\leqslant c}(\mathsf{M}), \mathrm{E}^{\geqslant c'+p(k)}(\mathsf{M})) \\ &\subset \mathrm{D}^{\geqslant c'-c+p(k)}(\mathsf{M}), \end{aligned}$$

where the last inclusion follows from Proposition 2.6.8. □

Remark 3.2.8. For $c, c' \in \mathbb{R}$, the inclusion

$$\mathcal{H}om^{\mathrm{E}}(\mathrm{E}^{\leqslant c}(\mathsf{M}), {}_p\mathrm{E}^{\geqslant c'}(\mathsf{M})) \subset {}^p\mathrm{D}^{\geqslant c'-c}(\mathbf{k}_{\mathsf{M}}^{\circ})$$

does not hold in general. For example, with notations as in Remark 3.1.8, let $\mathsf{M} = \mathbb{R}$, $K = \mathbf{k}_{\mathsf{M}}^{\mathrm{E}}$ and $K' = \mathbf{k}_{\mathsf{M}}^{\mathrm{E}} \otimes \pi^{-1}F$. Then $K \in \mathrm{E}^0(\mathsf{M})$, $K' \in {}_{1/2}\mathrm{E}^{\geqslant 1/2}(\mathsf{M})$ and

$$\mathcal{H}om^{\mathrm{E}}(K, K') \simeq \alpha_{\mathsf{M}} F \notin {}^{1/2}\mathrm{D}^{\geqslant 1/2}(\mathbf{k}_{\mathsf{M}}).$$

Here, ${}_{1/2}\mathrm{E} := {}_{\mathsf{m}}\mathrm{E}$ and ${}^{1/2}\mathrm{D} := {}^{\mathsf{m}}\mathrm{D}$ for $\mathsf{m}(n) := -\frac{n}{2}$ the middle perversity.

Proposition 3.2.9. *For $c \in \mathbb{R}$ one has*

$$({}_p\mathrm{E}^{<c}(\mathsf{M}))^{\perp} = {}_p\mathrm{E}^{\geqslant c}(\mathsf{M}).$$

Proof. One has

$$ {}_p\mathrm{E}^{\geqslant c}(\mathsf{M}) \subset ({}_p\mathrm{E}^{<c}(\mathsf{M}))^{\perp} $$

by Proposition 3.2.6.

Let $K \in ({}_p\mathrm{E}^{<c}(\mathsf{M}))^{\perp}$. We have to show that one has

$$ \mathrm{E}i^{\,!}_{Z_\infty} K \in \mathrm{E}^{\geqslant c+p(k)}(Z_\infty) $$

for any $k \in \mathbb{Z}_{\geqslant 0}$ and $Z \in \mathrm{CS}^{\leqslant k}_{\mathsf{M}}$. Since $\mathrm{E}^{\geqslant c+p(k)}(Z_\infty) = (\mathrm{E}^{<c+p(k)}(Z_\infty))^{\perp}$, this is equivalent to showing that for any $L \in \mathrm{E}^{<c+p(k)}(Z_\infty)$ one has

$$ \mathrm{Hom}_{\mathrm{E}(Z_\infty)}(L, \mathrm{E}i^{\,!}_{Z_\infty} K) \simeq 0. $$

By Lemma 3.2.3, one has

$$ \mathrm{E}^{<c+p(k)}(Z_\infty) \subset {}_p\mathrm{E}^{<c}(Z_\infty). $$

Hence $\mathrm{E}i_{Z_\infty\,!!} L \in {}_p\mathrm{E}^{<c}(\mathsf{M})$ by Lemma 3.2.4. Then

$$ \mathrm{Hom}_{\mathrm{E}(Z_\infty)}(L, \mathrm{E}i^{\,!}_{Z_\infty} K) \simeq \mathrm{Hom}_{\mathrm{E}(\mathsf{M})}(\mathrm{E}i_{Z_\infty\,!!} L, K) \simeq 0. \qquad \square $$

Proposition 3.2.10. *Let M be a subanalytic space. For any interval $I \subset \mathbb{R}$ such that $I \to \mathbb{R}/\mathbb{Z}$ is injective, the prestack on M*

$$ U \mapsto {}_p\mathrm{E}^{I}(U) $$

is a stack.

Proof. (i) Let $K, L \in {}_p\mathrm{E}^{I}(M)$. By Proposition 3.2.6, one has

$$ \mathcal{H}om^{\mathrm{E}}(K, L) \in \mathrm{D}^{>-1}(M) = \mathrm{D}^{\geqslant 0}(M). $$

Hence the presheaf

$$ U \mapsto \mathrm{Hom}_{{}_p\mathrm{E}^{I}(U)}(\mathrm{E}i_U^{-1} K, \mathrm{E}i_U^{-1} L) \simeq \Gamma(U; H^0(\mathcal{H}om^{\mathrm{E}}(K, L))) $$

is a sheaf. Thus $U \mapsto {}_p\mathrm{E}^{I}(U)$ is a separated prestack on M.

(ii) Let $M = \bigcup_{a \in A} U_a$ be an open cover, $K_a \in {}_p\mathrm{E}^{I}(U_a)$, and let

$$ u_{ab} \colon K_b|_{U_a \cap U_b} \xrightarrow{\sim} K_a|_{U_a \cap U_b} $$

be isomorphisms such that $u_{ab} \circ u_{bc} = u_{ac}$ on $U_a \cap U_b \cap U_c$ $(a, b, c \in A)$. We have to show that there exist $K \in {}_p\mathrm{E}^{I}(M)$ and isomorphisms $u_a \colon K|_{U_a} \xrightarrow{\sim} K_a$ such that $u_{ab} \circ u_b = u_a$ on $U_a \cap U_b$ $(a, b \in A)$. This follows from Proposition 2.4.8 by applying it with $f = \overline{\pi}$ to $\mathrm{R}j_{a\,!!}\, \mathrm{L}^{\mathrm{E}} K_a \in \mathrm{D}(U_a \times \overline{\mathbb{R}})$, where $j_a \colon U_a \times \mathbb{R}_\infty \to U_a \times \overline{\mathbb{R}}$ is the canonical morphism. $\square$

Lemma 3.2.11. *Let M be a bordered space. Let $c \in \mathbb{R}$, $Z \in \mathrm{CS}_{\mathsf{M}}$ and $K \in \mathrm{E}(\mathsf{M})$. Set $U = \mathring{\mathsf{M}} \setminus Z$, and consider the morphisms $i \colon Z_\infty \to \mathsf{M}$ and $j \colon U_\infty \to \mathsf{M}$. Then one has:*

(i) *$K \in {}_p\mathrm{E}^{\leqslant c}(\mathsf{M})$ if and only if $\mathrm{E}i^{-1} K \in {}_p\mathrm{E}^{\leqslant c}(Z_\infty)$ and $\mathrm{E}j^{-1} K \in {}_p\mathrm{E}^{\leqslant c}(U_\infty)$;*

(ii) *$K \in {}_p\mathrm{E}^{\geqslant c}(\mathsf{M})$ if and only if $\mathrm{E}i^{\,!} K \in {}_p\mathrm{E}^{\geqslant c}(Z_\infty)$ and $\mathrm{E}j^{\,!} K \in {}_p\mathrm{E}^{\geqslant c}(U_\infty)$.*

Proof. Since the proofs are similar, let us only discuss (i).

If $K \in {}_p\mathrm{E}^{\leqslant c}(\mathsf{M})$, then $\mathrm{E}i^{-1} K$ and $\mathrm{E}j^{-1} K$ satisfy the required conditions since the functors $\mathrm{E}i^{-1}$ and $\mathrm{E}j^{-1}$ are right exact by Lemma 3.2.4.

Conversely, assume that $\mathrm{E}i^{-1}K \in {}_p\mathrm{E}^{\leqslant c}(Z_\infty)$ and $\mathrm{E}j^{-1}K \in {}_p\mathrm{E}^{\leqslant c}(U_\infty)$. For $k \in \mathbb{Z}_{\geqslant 0}$, let $S_U \in \mathrm{CS}^{<k}_{U_\infty}$ be such that

$$\pi^{-1}\mathbf{k}_{U \setminus S_U} \otimes \mathrm{E}j^{-1}K \in \mathrm{E}^{\leqslant c+p(k)}(U_\infty),$$

and $S_Z \in \mathrm{CS}^{<k}_{Z_\infty}$ be such that

$$\pi^{-1}\mathbf{k}_{Z \setminus S_Z} \otimes \mathrm{E}i^{-1}K \in \mathrm{E}^{\leqslant c+p(k)}(Z_\infty).$$

Set $S = S_Z \cup \overline{S_U} \in \mathrm{CS}^{<k}_{\mathsf{M}}$ and $S'_Z = S_Z \cup (Z \cap \overline{S_U}) \in \mathrm{CS}^{<k}_{Z_\infty}$. (Here the closure of S_U is taken in $\mathring{\mathsf{M}}$.) Then $S \cap U = S_U$ and $S \cap Z = S'_Z$. Since

$$\pi^{-1}\mathbf{k}_{U \setminus S_U} \otimes K \in \mathrm{E}^{\leqslant c+p(k)}(\mathsf{M}), \quad \pi^{-1}\mathbf{k}_{Z \setminus S'_Z} \otimes K \in \mathrm{E}^{\leqslant c+p(k)}(\mathsf{M}),$$

one concludes that $\pi^{-1}\mathbf{k}_{M \setminus S} \otimes K \in \mathrm{E}^{\leqslant c+p(k)}(\mathsf{M})$ by considering the distinguished triangle

$$\pi^{-1}\mathbf{k}_{U \setminus S_U} \otimes K \longrightarrow \pi^{-1}\mathbf{k}_{M \setminus S} \otimes K \longrightarrow \pi^{-1}\mathbf{k}_{Z \setminus S'_Z} \otimes K \xrightarrow{+1}. \qquad \square$$

A *subanalytic stratification* $\{M_\alpha\}_{\alpha \in A}$ of $\mathsf{M} := (M, \check{M})$ is a locally finite (in $\check{M}$) family of smooth $M_\alpha \in \mathrm{LCS}_{\mathsf{M}}$ such that $M = \bigsqcup_{\alpha \in A} M_\alpha$ and $\overline{M}_\alpha \cap M_\beta \neq \varnothing$ implies $\overline{M}_\alpha \supset M_\beta$.

Proposition 3.2.12. *Let $\{M_\alpha\}_{\alpha \in A}$ be a subanalytic stratification of M, and set for short $\mathsf{M}_\alpha = (M_\alpha)_\infty$. Let $K \in \mathrm{E}(\mathsf{M})$.*

(i) $K \in {}_p\mathrm{E}^{\leqslant c}(\mathsf{M})$ *if and only if* $\mathrm{E}i^{-1}_{\mathsf{M}_\alpha}K \in {}_p\mathrm{E}^{\leqslant c}(\mathsf{M}_\alpha)$ *for any* $\alpha \in A$.

(ii) $K \in {}_p\mathrm{E}^{\geqslant c}(\mathsf{M})$ *if and only if* $\mathrm{E}i^{!}_{\mathsf{M}_\alpha}K \in {}_p\mathrm{E}^{\geqslant c}(\mathsf{M}_\alpha)$ *for any* $\alpha \in A$.

Proof. The statement follows from Lemma 3.2.11. $\square$

3.3. $\mathbb{R}$-constructible enhanced ind-sheaves. In this subsection, we extend to the case of subanalytic bordered spaces the definition of $\mathbb{R}$-constructible enhanced ind-sheaves from [4, Section 4.9].

Let $\mathsf{M} = (M, \check{M})$ be a subanalytic bordered space.

Definition 3.3.1. (i) An object $K \in \mathrm{E}(\mathsf{M})$ is $\mathbb{R}$-*constructible* if for any relatively compact subanalytic open subset U of M, one has

$$\mathrm{E}i^{-1}_{U_\infty}K \simeq \mathbf{k}^{\mathrm{E}}_{U_\infty} \overset{+}{\otimes} \mathrm{Q}_{U_\infty}F \quad \text{in } \mathrm{E}(U_\infty) \text{ for some } F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_{U_\infty \times \mathbb{R}_\infty}).$$

In particular, K is stable.

(ii) $\mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M})$ is the strictly full subcategory of $\mathrm{E}(\mathsf{M})$ whose objects are $\mathbb{R}$-constructible.

Recall the morphism $j_{\mathsf{M}} \colon \mathsf{M} \to \check{M}$.

Lemma 3.3.2. *Let $K \in \mathrm{E}(\mathsf{M})$. Then $K \in \mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M})$ if and only if $\mathrm{E}j_{\mathsf{M}!!}K \in \mathrm{E}_{\mathbb{R}\text{-c}}(\check{M})$.*

Proposition 3.3.3 ([4]). *Let $f \colon \mathsf{M} \to \mathsf{N}$ be a morphism of subanalytic bordered spaces.*

(i) $\mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M})$ *is a triangulated subcategory of* $\mathrm{E}(\mathsf{M})$.

(ii) *The duality functor* $\mathrm{D}^{\mathrm{E}}_{\mathsf{M}}$ *gives an equivalence* $\mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M})^{\mathrm{op}} \xrightarrow{\sim} \mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M})$*, and there is a canonical isomorphism of functors* $\mathrm{id}_{\mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M})} \xrightarrow{\sim} \mathrm{D}^{\mathrm{E}}_{\mathsf{M}} \circ \mathrm{D}^{\mathrm{E}}_{\mathsf{M}}$.

(iii) *The functors* $\mathrm{E}f^{-1}$ *and* $\mathrm{E}f^{!}$ *send* $\mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{N})$ *to* $\mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M})$*, and*

$$\mathrm{D}^{\mathrm{E}}_{\mathsf{M}} \circ \mathrm{E}f^{-1} \simeq \mathrm{E}f^{!} \circ \mathrm{D}^{\mathrm{E}}_{\mathsf{N}} \quad \textit{and} \quad \mathrm{D}^{\mathrm{E}}_{\mathsf{M}} \circ \mathrm{E}f^{!} \simeq \mathrm{E}f^{-1} \circ \mathrm{D}^{\mathrm{E}}_{\mathsf{N}}.$$

(iv) *Assume that* f *is semi-proper. Then the functors* $\mathrm{E}f_{*}$ *and* $\mathrm{E}f_{!!}$ *send* $\mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M})$ *to* $\mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{N})$*, and*

$$\mathrm{D}^{\mathrm{E}}_{\mathsf{N}} \circ \mathrm{E}f_{*} \simeq \mathrm{E}f_{!!} \circ \mathrm{D}^{\mathrm{E}}_{\mathsf{M}} \quad \textit{and} \quad \mathrm{D}^{\mathrm{E}}_{\mathsf{N}} \circ \mathrm{E}f_{!!} \simeq \mathrm{E}f_{*} \circ \mathrm{D}^{\mathrm{E}}_{\mathsf{M}}.$$

See [4, Corollary 4.9.4, Theorem 4.9.12, Propositions 4.9.14, 4.8.2].

Definition 3.3.4. (i) An E-*type* on M is the datum

$$\mathcal{L} = (\varphi_a, m_a, \psi_b^{\pm}, n_b)_{a\in A,\, b\in B} \tag{3.3.1}$$

consisting of finite sets A, B, integers m_a and n_b for any $a \in A$ and $b \in B$, and morphisms of subanalytic bordered spaces $\varphi_a, \psi_b^{\pm} \colon \mathsf{M} \to \mathbb{R}_\infty$ for any $a \in A$ and $b \in B$, such that $\psi_b^-(x) < \psi_b^+(x)$ for any $x \in M$.

(ii) An E-type $\mathcal{L}$ as in (3.3.1) is called *stable* if for any $b \in B$

$$\overline{\{(x,t) \in M \times \mathbb{R}; t = \psi_b^+(x) - \psi_b^-(x)\}} \cap (\check{M} \times \{+\infty\}) \neq \varnothing, \tag{3.3.2}$$

where the closure is taken in $\check{M} \times \overline{\mathbb{R}}$.

Notation 3.3.5. For an E-type $\mathcal{L}$ on M as in (3.3.1), set

$$\Phi_a := \{(x,t) \in M \times \mathbb{R}; t \geqslant \varphi_a(x)\},$$
$$\Psi_b := \{(x,t) \in M \times \mathbb{R}; \psi_b^-(x) \leqslant t < \psi_b^+(x)\},$$

and

$$\mathbf{k}^{\mathrm{Q}}_{\mathcal{L}} := \Big(\bigoplus_{a\in A} \mathbf{k}^{\mathrm{Q}}_{\Phi_a}[-m_a]\Big) \oplus \Big(\bigoplus_{b\in B} \mathbf{k}^{\mathrm{Q}}_{\Psi_b}[-n_b]\Big) \in \mathrm{E}(\mathsf{M}),$$
$$\mathbf{k}^{\mathrm{E}}_{\mathcal{L}} := \Big(\bigoplus_{a\in A} \mathbf{k}^{\mathrm{E}}_{\Phi_a}[-m_a]\Big) \oplus \Big(\bigoplus_{b\in B} \mathbf{k}^{\mathrm{E}}_{\Psi_b}[-n_b]\Big) \simeq \mathbf{k}^{\mathrm{E}}_{\mathsf{M}} \overset{+}{\otimes} \mathbf{k}^{\mathrm{Q}}_{\mathcal{L}} \in \mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M}).$$

Note that $\mathbf{k}^{\mathrm{E}}_{\Psi_b} \not\simeq 0$ if and only if (3.3.2) holds true.

Definition 3.3.6. One says that $K \in \mathrm{E}(\mathsf{M})$ is *free* (resp. *stably free*) on M if, for any connected component S of $\mathring{\mathsf{M}}$, there exists an E-type $\mathcal{L}$ on S_∞ such that $\mathrm{E}i_{S_\infty}^{-1} K \simeq \mathbf{k}^{\mathrm{Q}}_{\mathcal{L}}$ (resp. $\mathrm{E}i_{S_\infty}^{-1} K \simeq \mathbf{k}^{\mathrm{E}}_{\mathcal{L}}$). (Note that $\mathrm{E}i_{S_\infty}^{-1} \simeq \mathrm{E}i_{S_\infty}^{!}$.)

If $K \in \mathrm{E}(\mathsf{M})$ is stably free, then it is $\mathbb{R}$-constructible. If K is free, then it is constructible in the sense of Remark 3.5.12 below.

A *regular filtration* $(M_k)_{k\in\mathbb{Z}}$ of M is an increasing sequence of closed subanalytic subsets M_k of M such that $M_k = \varnothing$ for $k \leqslant -1$, $M_k = \mathring{\mathsf{M}}$ for $k \geqslant d_{\mathring{\mathsf{M}}}$, and $M_k \setminus M_{k-1}$ is smooth of dimension k. In particular,

$$\varnothing = M_{-1} \subset M_0 \subset \cdots \subset M_{d_M - 1} \subset M_{d_M} = \mathring{\mathsf{M}}.$$

Lemma 3.3.7 ([4, Lemma 4.9.9]). *For any* $K \in \mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M})$ *there exists a regular filtration* $(M_k)_{k\in\mathbb{Z}}$ *of* M *such that both* $\mathrm{E}i^{-1}_{(M_k\setminus M_{k-1})_\infty} K$ *and* $\mathrm{E}i^{!}_{(M_k\setminus M_{k-1})_\infty} K$ *are stably free.*

Definition 3.3.8. Let $\mathcal{L} = (\varphi_a, m_a, \psi_b^{\pm}, n_b)_{a\in A,\, b\in B}$ be an E-*type* on M, and assume that M is smooth of dimension d. The dual of $\mathcal{L}$, denoted by

$$\mathcal{L}^* = (\varphi_a^*, m_a^*, \psi_b^{\pm *}, n_b^*)_{a\in A,\, b\in B},$$

is the E-type on M defined by $\varphi_a^* := -\varphi_a$, $m_a^* := -m_a - d$, $\psi_b^{\pm *} := -\psi_b^{\mp}$, $n_b^* := -n_b - d - 1$. Accordingly, we set

$$\Phi_a^* := \{(x,t) \in M \times \mathbb{R};\, t \geqslant -\varphi_a(x)\},$$
$$\Psi_b^* := \{(x,t) \in M \times \mathbb{R};\, -\psi_b^+(x) \leqslant t < -\psi_b^-(x)\}.$$

Lemma 3.3.9. *Let $\mathcal{L}$ be an* E-*type on* M. *Assume that M is smooth and equidimensional. Then* $\mathrm{D}^{\mathrm{Q}}_{\mathsf{M}}\mathbf{k}^{\mathrm{Q}}_{\mathcal{L}} \simeq \mathbf{k}^{\mathrm{Q}}_{\mathcal{L}^*}$ *and* $\mathrm{D}^{\mathrm{E}}_{\mathsf{M}}\mathbf{k}^{\mathrm{E}}_{\mathcal{L}} \simeq \mathbf{k}^{\mathrm{E}}_{\mathcal{L}^*}$.

Proof. This follows from Lemma 3.3.10 below. □

Lemma 3.3.10. *Recall* Notation 3.3.5 *and* Definition 3.3.8. *If* M *is smooth of dimension d, one has*

$$\mathrm{D}^{\mathrm{Q}}_{\mathsf{M}}(\mathbf{k}^{\mathrm{Q}}_{\Phi_a}) \simeq \mathbf{k}^{\mathrm{Q}}_{\Phi_a^*}[d], \quad \mathrm{D}^{\mathrm{Q}}_{\mathsf{M}}(\mathbf{k}^{\mathrm{Q}}_{\Psi_b}) \simeq \mathbf{k}^{\mathrm{Q}}_{\Psi_b^*}[d+1],$$

and

$$\mathrm{D}^{\mathrm{E}}_{\mathsf{M}}(\mathbf{k}^{\mathrm{E}}_{\Phi_a}) \simeq \mathbf{k}^{\mathrm{E}}_{\Phi_a^*}[d], \quad \mathrm{D}^{\mathrm{E}}_{\mathsf{M}}(\mathbf{k}^{\mathrm{E}}_{\Psi_b}) \simeq \mathbf{k}^{\mathrm{E}}_{\Psi_b^*}[d+1].$$

Proof. By Lemma 2.8.3, one has

$$\mathrm{D}^{\mathrm{Q}}_{\mathsf{M}}(\mathbf{k}^{\mathrm{Q}}_{\Phi_a}) \simeq \mathbf{k}^{\mathrm{Q}}_{\{t<-\varphi_a(x)\}}[d+1] \simeq \mathbf{k}^{\mathrm{Q}}_{\Phi_a^*}[d],$$
$$\mathrm{D}^{\mathrm{Q}}_{\mathsf{M}}(\mathbf{k}^{\mathrm{Q}}_{\Psi_b}) \simeq \mathbf{k}^{\mathrm{Q}}_{\{-\psi_b^+(x)\leqslant t<-\psi_b^-(x)\}}[d+1] = \mathbf{k}^{\mathrm{Q}}_{\Psi_b^*}[d+1].$$

The other statements also follow from Lemma 2.8.3. □

Definition 3.3.11. For p a perversity and $c \in \mathbb{R}$, we set

$${}_p\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}) := {}_p\mathrm{E}^{\leqslant c}(\mathsf{M}) \cap \mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M}), \quad {}_p\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}) := {}_p\mathrm{E}^{\geqslant c}(\mathsf{M}) \cap \mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M}).$$

Proposition 3.3.12. *The following properties hold.*

(i) $({}_p\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}), {}_p\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}))_{c\in\mathbb{R}}$ *is a t-structure on* $\mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M})$.

(ii) *Assume that* $\mathsf{M} = M$ *is a subanalytic space. For any interval $I \subset \mathbb{R}$ such that $I \to \mathbb{R}/\mathbb{Z}$ is injective, the prestack on M*

$$U \mapsto {}_p\mathrm{E}^{I}_{\mathbb{R}\text{-c}}(U)$$

is a stack of quasi-abelian categories.

Plan of the proof. (i) We have to prove that the conditions in Definition 1.2.2 are satisfied. Conditions (a) and (b) are clear. Condition (c) follows from Proposition 3.2.6. Condition (d) is checked in Proposition 3.3.19 below.

(ii) This follows from Proposition 3.2.10. □

Notation 3.3.13. We denote by

$$({}_{1/2}\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}),\ {}_{1/2}\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}))_{c\in\mathbb{R}}$$

the t-structure associated with the middle perversity $\mathsf{m}(n) = -\frac{n}{2}$.

Remark 3.3.14. The t-structures $({}_{p}\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}),\ {}_{p}\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}))_{c\in\mathbb{R}}$ are not well behaved with respect to duality, as one observes in Lemma 3.3.15 below. We will come back to this point in Section 3.5.

Lemma 3.3.15. *Assume that* M *is smooth of dimension* d*. Using* Notation 3.3.5*, one has:*

(i) $\mathbf{k}^{\mathrm{E}}_{\Phi_a}, \mathbf{k}^{\mathrm{E}}_{\Psi_b} \in {}_{p}\mathrm{E}^{-p(d)}_{\mathbb{R}\text{-c}}(\mathsf{M})$.

(ii) $\mathrm{D}^{\mathrm{E}}_{\mathsf{M}}\mathbf{k}^{\mathrm{E}}_{\Phi_a} \in {}_{p^*}\mathrm{E}^{p(d)}_{\mathbb{R}\text{-c}}(\mathsf{M})$ *and* $\mathrm{D}^{\mathrm{E}}_{\mathsf{M}}\mathbf{k}^{\mathrm{E}}_{\Psi_b} \in {}_{p^*}\mathrm{E}^{p(d)-1}_{\mathbb{R}\text{-c}}(\mathsf{M})$.

Proof. (i) As the proofs are similar, let us only discuss $\mathbf{k}^{\mathrm{E}}_{\Phi_a}$.

(i-1) By Lemma 3.2.3, one has

$$\mathbf{k}^{\mathrm{E}}_{\Phi_a} \in \mathrm{E}^{\leqslant 0}_{\mathbb{R}\text{-c}}(\mathsf{M}) \subset {}_{p}\mathrm{E}^{\leqslant -p(d)}_{\mathbb{R}\text{-c}}(\mathsf{M}).$$

(i-2) Let us now show that $\mathbf{k}^{\mathrm{E}}_{\Phi_a} \in {}_{p}\mathrm{E}^{\geqslant -p(d)}_{\mathbb{R}\text{-c}}(\mathsf{M})$. We have to prove that for any smooth $Z \in \mathrm{LCS}^{\leqslant k}_{\mathsf{M}}$ one has

$$\mathrm{E}\, i^{\,!}_{Z_\infty}(\mathbf{k}^{\mathrm{E}}_{\Phi_a}) \in \mathrm{E}^{\geqslant -p(d)+p(k)}(Z_\infty).$$

We may assume that $k < d$. Consider the embedding $i\colon Z \times \mathbb{R} \to M \times \mathbb{R}$. Then one has

$$\begin{aligned}\mathrm{E}\, i^{\,!}_{Z_\infty}(\mathbf{k}^{\mathrm{E}}_{\Phi_a}) &\simeq \mathbf{k}^{\mathrm{E}}_{Z_\infty} \overset{+}{\otimes} \mathrm{Q}_{Z_\infty}(i^{\,!}\mathbf{k}_{\{t\, \geqslant\, \varphi_a(x)\}}) \\ &\simeq \mathbf{k}^{\mathrm{E}}_{Z_\infty} \overset{+}{\otimes} \mathrm{Q}_{Z_\infty}(i^{-1}\mathbf{k}_{\{t\, \geqslant\, \varphi_a(x)\}} \otimes i^{\,!}\mathbf{k}_{M\times\mathbb{R}}),\end{aligned}$$

where the first isomorphism follows from [4, Proposition 4.7.14]. Locally on Z, one has $i^{\,!}\mathbf{k}_{M\times\mathbb{R}} \simeq \mathbf{k}_{Z\times\mathbb{R}}[k-d]$. Hence

$$\mathrm{E}\, i^{\,!}_{Z_\infty}(\mathbf{k}^{\mathrm{E}}_{\Phi_a}) \in \mathrm{E}^{\geqslant d-k}(Z_\infty)$$

by Lemmas 2.6.5 and 2.8.2. One concludes since $d - k \geqslant -p(d) + p(k)$ by perversity.

(ii) Using Lemma 3.3.10 and (i), one has

$$\begin{aligned}\mathrm{D}^{\mathrm{E}}_{\mathsf{M}}\mathbf{k}^{\mathrm{E}}_{\Phi_a} &\simeq \mathbf{k}^{\mathrm{E}}_{\Phi^*_a}[d] \in {}_{p^*}\mathrm{E}^{-p^*(d)-d}_{\mathbb{R}\text{-c}}(\mathsf{M}) = {}_{p^*}\mathrm{E}^{p(d)}_{\mathbb{R}\text{-c}}(\mathsf{M}), \\ \mathrm{D}^{\mathrm{E}}_{\mathsf{M}}\mathbf{k}^{\mathrm{E}}_{\Psi_b} &\simeq \mathbf{k}^{\mathrm{E}}_{\Psi^*_b}[d+1] \in {}_{p^*}\mathrm{E}^{-p^*(d)-d-1}_{\mathbb{R}\text{-c}}(\mathsf{M}) = {}_{p^*}\mathrm{E}^{p(d)-1}_{\mathbb{R}\text{-c}}(\mathsf{M}).\end{aligned}$$

□

Lemma 3.3.16. *Assume that* $\overset{\circ}{\mathsf{M}}$ *is non-empty and smooth of dimension* d*. Given a stable* E*-type* $\mathcal{L} = (\varphi_a, m_a, \psi^{\pm}_b, n_b)_{a\in A,\, b\in B}$ *on* M*, and* $c \in \mathbb{R}$*, one has:*

(i) $\mathbf{k}^{\mathrm{E}}_{\mathcal{L}} \in {}_{p}\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M})$ *if and only if* $m_a \leqslant c + p(d)$ *and* $n_b \leqslant c + p(d)$ *for any* $a \in A$ *and* $b \in B$.

(ii) $\mathbf{k}^{\mathrm{E}}_{\mathcal{L}} \in {}_{p}\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M})$ *if and only if* $m_a \geqslant c + p(d)$ *and* $n_b \geqslant c + p(d)$ *for any* $a \in A$ *and* $b \in B$.

(iii) $\mathrm{D}^{\mathrm{E}}_{\mathsf{M}}\mathbf{k}^{\mathrm{E}}_{\mathcal{L}} \in {}_{p^*}\mathrm{E}^{\geqslant -c}_{\mathbb{R}\text{-c}}(\mathsf{M})$ *if and only if* $m_a \leqslant c + p(d)$ *and* $n_b \leqslant c + p(d) - 1$ *for any* $a \in A$ *and* $b \in B$.

(iv) $\mathrm{D}^{\mathrm{E}}_{\mathsf{M}}\mathbf{k}^{\mathrm{E}}_{\mathcal{L}} \in {}_{p^*}\mathrm{E}^{\leqslant -c}_{\mathbb{R}\text{-c}}(\mathsf{M})$ *if and only if* $m_a \geqslant c + p(d)$ *and* $n_b \geqslant c + p(d) - 1$ *for any* $a \in A$ *and* $b \in B$.

Proof. Since

$$\mathbf{k}^{\mathrm{E}}_{\mathscr{L}} = \Big(\bigoplus_{a\in A}\mathbf{k}^{\mathrm{E}}_{\Phi_a}[-m_a]\Big) \oplus \Big(\bigoplus_{b\in B}\mathbf{k}^{\mathrm{E}}_{\Psi_b}[-n_b]\Big),$$

the statement follows from Lemma 3.3.15. Note that a non-zero object of ${}_p\mathrm{E}^c(\mathsf{M})$ belongs to ${}_p\mathrm{E}^{\leqslant c'}(\mathsf{M})$ (resp. ${}_p\mathrm{E}^{\geqslant c'}(\mathsf{M})$) if and only if $c \leqslant c'$ (resp. $c \geqslant c'$) by Proposition 3.2.6. □

Corollary 3.3.17. *Assume that* M *is smooth of dimension* d*. Let* $K \in \mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M})$ *be a stably free object. Then, for* $c \in \mathbb{R}$*, one has:*

(i) $K \in {}_p\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M})$ *if and only if* $K \in \mathrm{E}^{\leqslant c+p(d)}_{\mathbb{R}\text{-c}}(\mathsf{M})$.

(ii) $K \in {}_p\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M})$ *if and only if* $K \in \mathrm{E}^{\geqslant c+p(d)}_{\mathbb{R}\text{-c}}(\mathsf{M})$.

Lemma 3.3.18. *Let* $c \in \mathbb{R}$ *and* $K \in \mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M})$*. Assume that* M *is smooth and* K *is stably free on* M*. Then there are distinguished triangles in* $\mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M})$

$$K_{\leqslant c} \longrightarrow K \longrightarrow K_{>c} \xrightarrow{+1} \quad \textit{and} \quad K_{<c} \longrightarrow K \longrightarrow K_{\geqslant c} \xrightarrow{+1}$$

with $K_L \in {}_p\mathrm{E}^{L}_{\mathbb{R}\text{-c}}(\mathsf{M})$ *for* L *equal to* $\leqslant c$, $> c$, $< c$ *or* $\geqslant c$.

Proof. It is obvious since K is a direct sum of objects belonging to ${}_p\mathrm{E}^{a}_{\mathbb{R}\text{-c}}(\mathsf{M})$ for some $a \in \mathbb{R}$ by Lemma 3.3.15. □

Proposition 3.3.19. *Let* $c \in \mathbb{R}$ *and* $K \in \mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M})$*. Then there are distinguished triangles in* $\mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M})$

$$K_{\leqslant c} \longrightarrow K \longrightarrow K_{>c} \xrightarrow{+1} \quad \textit{and} \quad K_{<c} \longrightarrow K \longrightarrow K_{\geqslant c} \xrightarrow{+1}$$

with $K_L \in {}_p\mathrm{E}^{L}_{\mathbb{R}\text{-c}}(\mathsf{M})$ *for* L *equal to* $\leqslant c$, $> c$, $< c$ *or* $\geqslant c$.

Proof. Since the proof of the existence of the second distinguished triangle follows from the first one, we will construct only the first distinguished triangle. The arguments we use are standard (see e.g. [10, Lemma 5.8]).

Let $\mathsf{M} = (M, \check{M})$. Reasoning by decreasing induction on $k \in \mathbb{Z}_{\geqslant -1}$, let us show that

$(\mathrm{dt})_k$ there exists $Z_k \in \mathrm{CS}^{\leqslant k}_{\mathsf{M}}$ and a distinguished triangle

$$K'_k \longrightarrow \mathrm{E}j_k{}^{-1}K \longrightarrow K''_k \xrightarrow{+1},$$

with $K'_k \in {}_p\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}((M \setminus Z_k)_\infty)$ and $K''_k \in {}_p\mathrm{E}^{>c}_{\mathbb{R}\text{-c}}((M \setminus Z_k)_\infty)$.

Here, j_k is the morphism indicated in the diagram below, where we picture all the morphisms that will be used in the proof:

$$\begin{array}{ccccc} (Z_k \setminus Z_{k-1})_\infty & \xrightarrow{i'_k} & (M \setminus Z_{k-1})_\infty & \xleftarrow{j'_k} & (M \setminus Z_k)_\infty \\ & {\scriptstyle i_k}\searrow & \downarrow{\scriptstyle j_{k-1}} & \swarrow{\scriptstyle j_k} & \\ & & \mathsf{M}. & & \end{array}$$

The statement $(\mathrm{dt})_k$ is obvious for $k \geqslant d_M$. Assuming that $(\mathrm{dt})_k$ holds true for some $k \geqslant 0$, let us prove $(\mathrm{dt})_{k-1}$. The morphism $K'_k \to \mathrm{E}j_k{}^{-1}K \simeq \mathrm{E}j_k{}^{!}K$ induces by adjunction a morphism

$\mathrm{E}j_{k\,!!}K'_k \to K$, that we complete in a distinguished triangle in $\mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M})$

$$\mathrm{E}j_{k\,!!}K'_k \longrightarrow K \longrightarrow L \xrightarrow{+1}.$$

Let $Z_{k-1} \in \mathrm{CS}_{\mathsf{M}}^{\leqslant k-1}$ be such that $Z_k \setminus Z_{k-1}$ is smooth and $\mathrm{E}i_k^{!}L$ is stably free. Lemma 3.3.18 gives a distinguished triangle

$$L' \longrightarrow \mathrm{E}i_k^{!}L \longrightarrow L'' \xrightarrow{+1}, \tag{3.3.3}$$

with $L' \in {}_p\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}((Z_k \setminus Z_{k-1})_\infty)$ and $L'' \in {}_p\mathrm{E}^{>c}_{\mathbb{R}\text{-c}}((Z_k \setminus Z_{k-1})_\infty)$.

The first morphism above, namely

$$L' \to \mathrm{E}i_k^{!}L \simeq \mathrm{E}i_k'^{!}\mathrm{E}j_{k-1}^{!}L,$$

induces by adjunction a morphism $\mathrm{E}i'_{k\,!!}L' \to \mathrm{E}j_{k-1}^{!}L \simeq \mathrm{E}j_{k-1}^{-1}L$, that we complete in a distinguished triangle in $\mathrm{E}_{\mathbb{R}\text{-c}}((M \setminus Z_{k-1})_\infty)$

$$\mathrm{E}i'_{k\,!!}L' \longrightarrow \mathrm{E}j_{k-1}^{-1}L \longrightarrow K''_{k-1} \xrightarrow{+1}. \tag{3.3.4}$$

Consider the composite morphism $\mathrm{E}j_{k-1}^{-1}K \to \mathrm{E}j_{k-1}^{-1}L \to K''_{k-1}$, and complete it in a distinguished triangle in $\mathrm{E}_{\mathbb{R}\text{-c}}((M \setminus Z_{k-1})_\infty)$

$$K'_{k-1} \longrightarrow \mathrm{E}j_{k-1}^{-1}K \longrightarrow K''_{k-1} \xrightarrow{+1}.$$

We claim that this satisfy $(\mathrm{dt})_{k-1}$.

Note that

$$\mathrm{E}j_k'^{-1}K''_{k-1} \simeq \mathrm{E}j_k^{-1}L \simeq K''_k \in {}_p\mathrm{E}^{>c}_{\mathbb{R}\text{-c}}((M \setminus Z_k)_\infty),$$
$$\mathrm{E}j_k'^{-1}K'_{k-1} \simeq K'_k \in {}_p\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}((M \setminus Z_k)_\infty).$$

Hence, by Lemma 3.2.11, we are reduced to prove

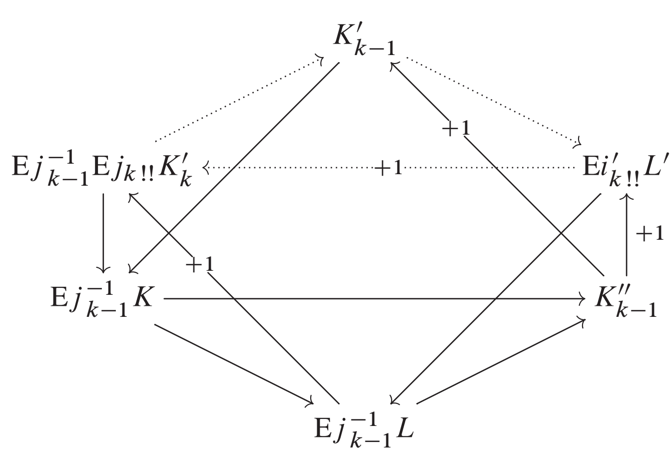
$$\mathrm{E}i_k'^{-1}K'_{k-1} \in {}_p\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}((Z_k \setminus Z_{k-1})_\infty), \tag{3.3.5}$$
$$\mathrm{E}i_k'^{!}K''_{k-1} \in {}_p\mathrm{E}^{>c}_{\mathbb{R}\text{-c}}((Z_k \setminus Z_{k-1})_\infty). \tag{3.3.6}$$

Applying the functor $\mathrm{E}i_k'^{!}$ to (3.3.4), we get a distinguished triangle

$$L' \longrightarrow \mathrm{E}i_k^{!}L \longrightarrow \mathrm{E}i_k'^{!}K''_{k-1} \xrightarrow{+1}.$$

Thus (3.3.3) gives $\mathrm{E}i_k'^{!}K''_{k-1} \simeq L'' \in {}_p\mathrm{E}^{>c}_{\mathbb{R}\text{-c}}((Z_k \setminus Z_{k-1})_\infty)$, which proves (3.3.6).

By the octahedral axiom, there is a diagram in $\mathrm{E}((M \setminus Z_{k-1})_\infty)$



and hence a distinguished triangle

$$\mathrm{E}j_{k-1}^{-1}\mathrm{E}j_{k\,!!}K'_k \longrightarrow K'_{k-1} \longrightarrow \mathrm{E}i'_{k\,!!}L' \xrightarrow{+1}.$$

Applying the functor $\mathrm{E}i_k'^{-1}$, we get

$$\mathrm{E}i_k'^{-1}K'_{k-1} \simeq L' \in {}_p\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}((Z_k \setminus Z_{k-1})_\infty),$$

which proves (3.3.5). □

Definition 3.3.20. For $p\colon \mathbb{Z}_{\geqslant 0} \to \mathbb{R}$ a perversity and $d \in \mathbb{Z}_{\geqslant 0}$, the shifted perversity $p[d]$ is given by $p[d](n) = p(d+n)$.

Note that the soft dimension of a subanalytic space is equal to its dimension.

Proposition 3.3.21. *Let $f\colon \mathsf{M} \to \mathsf{N}$ be a morphism of subanalytic bordered spaces, and $d \in \mathbb{Z}_{\geqslant 0}$. Assume that* $\dim \mathring{f}^{-1}(y) \leqslant d$ *for any $y \in \mathring{\mathsf{N}}$. Then, for any $c \in \mathbb{R}$ one has:*

(i) $\mathrm{E}f^{-1}({}_{p[d]}\mathrm{E}^{\leqslant c}(\mathsf{N})) \subset {}_p\mathrm{E}^{\leqslant c}(\mathsf{M})$.

(ii) $\mathrm{E}f^{!}({}_{p[d]}\mathrm{E}^{\geqslant c}(\mathsf{N})) \subset {}_p\mathrm{E}^{\geqslant c-d}(\mathsf{M})$.

(iii) $\mathrm{E}f_{*}({}_p\mathrm{E}^{\geqslant c}(\mathsf{M})) \subset {}_{p[d]}\mathrm{E}^{\geqslant c}(\mathsf{N})$.

(iv) $\mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{N}) \cap \mathrm{E}f_{!!}({}_p\mathrm{E}^{\leqslant c}(\mathsf{M})) \subset {}_{p[d]}\mathrm{E}^{\leqslant c+d}(\mathsf{N})$.

Proof. Let $\mathsf{M} = (M, \check{M})$ and $\mathsf{N} = (N, \check{N})$.

(i) Let $L \in {}_{p[d]}\mathrm{E}^{\leqslant c}(\mathsf{N})$. We have to prove that, for any $k \in \mathbb{Z}_{\geqslant 0}$, there exists $Z \in \mathrm{CS}^{<k}_{\mathsf{M}}$ such that one has

$$\mathrm{E}i^{-1}_{(M\setminus Z)_\infty}\mathrm{E}f^{-1}L \in \mathrm{E}^{\leqslant c+p(k)}((M \setminus Z)_\infty).$$

Let $W \in \mathrm{CS}^{<k-d}_{\mathsf{N}}$ be such that one has

$$\mathrm{E}i^{-1}_{(N\setminus W)_\infty}L \in \mathrm{E}^{\leqslant c+p(k)}((N \setminus W)_\infty).$$

Note that if $0 \leqslant k < d$, then $W = \varnothing$ will do because $L \in \mathrm{E}^{\leqslant c+p[d](0)}(\mathsf{N}) \subset \mathrm{E}^{\leqslant c+p(k)}(\mathsf{N})$ since $p(d) \leqslant p(k)$.

Then $Z := f^{-1}(W) \in \mathrm{CS}^{<k}_{\mathsf{M}}$ satisfies the desired condition. Indeed, denoting

$$f_0\colon (M \setminus Z)_\infty \to (N \setminus W)_\infty$$

the morphism induced by $f|_{M\setminus Z}$, one has

$$\begin{aligned}\mathrm{E}i^{-1}_{(M\setminus Z)_\infty}\mathrm{E}f^{-1}L &\simeq \mathrm{E}f_0^{-1}\mathrm{E}i^{-1}_{(N\setminus W)_\infty}L \\ &\in \mathrm{E}f_0^{-1}\mathrm{E}^{\leqslant c+p(k)}((N \setminus W)_\infty) \\ &\subset \mathrm{E}^{\leqslant c+p(k)}((M \setminus Z)_\infty),\end{aligned}$$

where the last inclusion follows from Proposition 2.7.3.

(ii) Let $L \in {}_{p[d]}\mathrm{E}^{\geqslant c}(\mathsf{N})$. We have to show that for any $k \in \mathbb{Z}_{\geqslant 0}$ and $Z \in \mathrm{CS}^{\leqslant k}_{\mathsf{M}}$ there exists an open subanalytic subset Z_0 of Z_∞ such that $\dim(Z \setminus Z_0) < k$ and

$$\mathrm{E}i^{!}_{(Z_0)_\infty}\mathrm{E}f^{!}L \in \mathrm{E}^{\geqslant c+p(k)-d}((Z_0)_\infty). \tag{3.3.7}$$

Recall Notation 2.3.1. Replacing $\check{M}$ with $\overline{\Gamma}_f$, we may assume that f extends to a morphism of subanalytic spaces

$$\check{f}\colon \check{M} \to \check{N}.$$

Since (3.3.7) is local on $\check{M}$, we may assume that Z is relatively compact in $\check{M}$. Then, there exists an open subanalytic subset Z_0 of Z satisfying the following properties:

(a) $\dim(Z \setminus Z_0) < k$,

(b) $Z_0 = \bigsqcup_{i\in I} S_i$, where $\{S_i\}_{i\in I}$ is a family of subanalytic smooth subsets of dimension k,

(c) $T_i := f(S_i)$ is a smooth equidimensional subset of N for any $i \in I$,

(d) f induces a submersive morphism $f_i\colon (S_i)_\infty \to (T_i)_\infty$ for any $i \in I$.

We claim that Z_0 satisfies (3.3.7). In fact, for any $i \in I$, one has

$$\mathrm{E}i^!_{(S_i)_\infty}\mathrm{E}f^!L \simeq \mathrm{E}f_i^!\mathrm{E}i^!_{(T_i)_\infty}L \in \mathrm{E}f_i^!\,\mathrm{E}^{\geqslant c+p(d_{T_i}+d)}((T_i)_\infty).$$

Since f_i is submersive, we have $\mathrm{E}f_i^! \simeq \pi^{-1}\mathrm{or}_{S_i/T_i} \otimes \mathrm{E}f_i^{-1}[d_{S_i} - d_{T_i}]$, where or_{S_i/T_i} is the relative orientation sheaf (see Section 1.5). Hence we have

$$\begin{aligned}\mathrm{E}f_i^!\,\mathrm{E}^{\geqslant c+p(d_{T_i}+d)}((T_i)_\infty) &\subset \mathrm{E}^{\geqslant c+p(d_{T_i}+d)+d_{T_i}-d_{S_i}}((S_i)_\infty)\\ &\subset \mathrm{E}^{\geqslant c+p(d_{S_i})-d}((S_i)_\infty).\end{aligned}$$

Here, the last inclusion follows from $d_{T_i} + d \geqslant d_{S_i}$ and $p(d_{T_i}+d)+d_{T_i}+d \geqslant p(d_{S_i})+d_{S_i}$ by perversity.

Thus we obtain

$$\mathrm{E}i^!_{(S_i)_\infty}\mathrm{E}f^!L \in \mathrm{E}^{\geqslant c+p(k)-d}((S_i)_\infty)$$

for any $i \in I$, which implies (3.3.7).

Statements (iii) and (iv) follow from (i) and (ii) by adjunction using, respectively, Proposition 3.2.9 and Proposition 3.3.12 (i). □

Remark 3.3.22. Concerning (iv) above, the inclusion

$$\mathrm{E}f_{!!}({}_p\mathrm{E}^{\leqslant c}(\mathsf{M})) \subset {}_{p[d]}\mathrm{E}^{\leqslant c+d}(\mathsf{N})$$

does not hold in general, since ${}_p\mathrm{E}^{\leqslant c}(\mathsf{M})$ is not stable by "$\bigoplus$". For example, let $M = \mathbb{R} \setminus \{0\}$, $N = \mathbb{R}$, and let $f\colon M \to N$ be the inclusion map. For $n \in \mathbb{Z}_{\geqslant 1}$, let $x_n = \frac{1}{n}$ and set

$$F_n = \pi^{-1}\mathbf{k}_{\{x_n\}} \otimes \mathbf{k}_{\{t\,\geqslant\,0\}},$$

an object of $\mathrm{Mod}(\mathbf{k}_{M\times\mathbb{R}_\infty})$. Let $K = \mathrm{Q}_M(\bigoplus_{n\geqslant 1} F_n) \in \mathrm{E}(M)$. Then we have $K \in {}_{1/2}\mathrm{E}^0(M)$ but $\mathrm{E}f_{!!}K \simeq \mathrm{Q}_N(\text{“}\bigoplus\text{”}_{n\geqslant 1} f_{\mathbb{R}!!}F_n) \in \mathrm{E}(N)$ does not belong to ${}_{1/2}\mathrm{E}^{\leqslant 0}(N)$. Here

$$f_{\mathbb{R}} := f \times \mathrm{id}_{\mathbb{R}}\colon M \times \mathbb{R} \to N \times \mathbb{R}.$$

Indeed, there is no $Z \in \mathrm{CS}^{<1}(N)$ such that $\mathrm{E}i^{-1}_{(N\setminus Z)_\infty}\mathrm{E}f_{!!}K \in \mathrm{E}^{\leqslant -1/2}((N \setminus Z)_\infty)$, i.e. such that $\mathrm{E}i^{-1}_{(N\setminus Z)_\infty}\mathrm{E}f_{!!}K \simeq 0$.

3.4. Dual intermediate enhanced perversity. Let p be a perversity and let M be a subanalytic bordered space. Since the t-structure $({}_p\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}), {}_p\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}))_{c\in\mathbb{R}}$ is not well behaved with respect to duality, we consider also its dual t-structure.

Notation 3.4.1. For $c \in \mathbb{R}$, set

$$\begin{aligned}{}_p'\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}) &:= \{K \in \mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M});\, \mathrm{D}^{\mathrm{E}}_{\mathsf{M}}K \in {}_{p^*}\mathrm{E}^{\geqslant -c}_{\mathbb{R}\text{-c}}(\mathsf{M})\},\\ {}_p'\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}) &:= \{K \in \mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M});\, \mathrm{D}^{\mathrm{E}}_{\mathsf{M}}K \in {}_{p^*}\mathrm{E}^{\leqslant -c}_{\mathbb{R}\text{-c}}(\mathsf{M})\}.\end{aligned}$$

The following result is a consequence of Proposition 3.3.12.

Proposition 3.4.2. *We have that* $({}_{p}'\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}),\ {}_{p}'\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}))_{c\in\mathbb{R}}$ *is a t-structure on* $\mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M})$.

Note that, by the definition, for any $c \in \mathbb{R}$ the duality functor $\mathrm{D}^{\mathrm{E}}_{\mathsf{M}}$ interchanges ${}_{p}\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M})$ and ${}_{p^*}'\mathrm{E}^{\geqslant -c}_{\mathbb{R}\text{-c}}(\mathsf{M})$, as well as ${}_{p}\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M})$ and ${}_{p^*}'\mathrm{E}^{\leqslant -c}_{\mathbb{R}\text{-c}}(\mathsf{M})$.

Lemma 3.4.3. *Let* M *be a bordered space. Let* $c \in \mathbb{R}$*,* $Z \in \mathrm{CS}_{\mathsf{M}}$*, and* $K \in \mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M})$*. Set* $U = \mathring{\mathsf{M}} \setminus Z$*. Then, considering the morphisms* $i\colon Z_\infty \to \mathsf{M}$ *and* $j\colon U_\infty \to \mathsf{M}$*, one has:*

(i) $K \in {}_{p}'\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M})$ *if and only if* $\mathrm{E}i^{-1}K \in {}_{p}'\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(Z_\infty)$ *and* $\mathrm{E}j^{-1}K \in {}_{p}'\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(U_\infty)$*;*

(ii) $K \in {}_{p}'\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M})$ *if and only if* $\mathrm{E}i^{!}K \in {}_{p}'\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(Z_\infty)$ *and* $\mathrm{E}j^{!}K \in {}_{p}'\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(U_\infty)$.

Proof. The statement follows from Lemma 3.2.11, noticing that

$$\mathrm{D}^{\mathrm{E}}_{Z_\infty}\mathrm{E}i^{-1}K \simeq \mathrm{E}i^{!}\mathrm{D}^{\mathrm{E}}_{\mathsf{M}}K, \qquad \mathrm{D}^{\mathrm{E}}_{U_\infty}\mathrm{E}j^{-1}K \simeq \mathrm{E}j^{!}\mathrm{D}^{\mathrm{E}}_{\mathsf{M}}K,$$
$$\mathrm{D}^{\mathrm{E}}_{Z_\infty}\mathrm{E}i^{!}K \simeq \mathrm{E}i^{-1}\mathrm{D}^{\mathrm{E}}_{\mathsf{M}}K, \qquad \mathrm{D}^{\mathrm{E}}_{U_\infty}\mathrm{E}j^{!}K \simeq \mathrm{E}j^{-1}\mathrm{D}^{\mathrm{E}}_{\mathsf{M}}K,$$

which is a consequence of Proposition 3.3.3. □

Lemma 3.4.4. *For any* $c \in \mathbb{R}$ *one has*

$${}_{p}'\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}) \subset {}_{p}\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}) \subset {}_{p}'\mathrm{E}^{\leqslant c+1}_{\mathbb{R}\text{-c}}(\mathsf{M}),$$
$${}_{p}\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}) \subset {}_{p}'\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}) \subset {}_{p}\mathrm{E}^{\geqslant c-1}_{\mathbb{R}\text{-c}}(\mathsf{M}).$$

Proof. Let $K \in \mathrm{E}(\mathsf{M})$. By Lemma 3.3.7, there exists a regular filtration $(M_k)_{k\in\mathbb{Z}}$ of M such that both $\mathrm{E}i^{-1}_{(M_k\setminus M_{k-1})_\infty}K$ and $\mathrm{E}i^{!}_{(M_k\setminus M_{k-1})_\infty}K$ are stably free. In order to check the inclusions in the statement, by Lemmas 3.2.11 and 3.4.3, we may assume that M is smooth equidimensional, and that K is stably free. Then one concludes using Lemma 3.3.16. □

Proposition 3.4.5. *Let* $f\colon \mathsf{M} \to \mathsf{N}$ *be a morphism of subanalytic bordered spaces, and* $d \in \mathbb{Z}_{\geqslant 0}$*. Assume that* $\dim f^{-1}(y) \leqslant d$ *for any* $y \in \mathring{\mathsf{N}}$*. Then, for any* $c \in \mathbb{R}$ *one has:*

(i) $\mathrm{E}f^{-1}({}_{p[d]}'\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{N})) \subset {}_{p}'\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M})$.

(ii) $\mathrm{E}f^{!}({}_{p[d]}'\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{N})) \subset {}_{p}'\mathrm{E}^{\geqslant c-d}_{\mathbb{R}\text{-c}}(\mathsf{M})$.

(iii) $\mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{N}) \cap \mathrm{E}f_{*}({}_{p}'\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M})) \subset {}_{p[d]}'\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{N})$.

(iv) $\mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{N}) \cap \mathrm{E}f_{!!}({}_{p}'\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M})) \subset {}_{p[d]}'\mathrm{E}^{\leqslant c+d}_{\mathbb{R}\text{-c}}(\mathsf{N})$.

Proof. (i) Let $K \in {}_{p[d]}'\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{N})$, that is, $\mathrm{D}^{\mathrm{E}}_{\mathsf{N}}K \in {}_{p[d]^*}\mathrm{E}^{\geqslant -c}_{\mathbb{R}\text{-c}}(\mathsf{M})$. Noticing that one has

$$p[d]^*(n) = p^*[d](n) + d,$$

Proposition 3.3.21 implies

$$\mathrm{D}^{\mathrm{E}}_{\mathsf{M}}\mathrm{E}f^{-1}K \simeq \mathrm{E}f^{!}\mathrm{D}^{\mathrm{E}}_{\mathsf{N}}K \in {}_{p^*}\mathrm{E}^{\geqslant -c}_{\mathbb{R}\text{-c}}(\mathsf{M}).$$

Hence $\mathrm{E}f^{-1}K \in {}_{p}'\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M})$.

Statement (ii) is proved similarly, and statement (iii) and (iv) follow from (i) and (ii) by adjunction. □

3.5. Enhanced perversity. Let p be a perversity and M a subanalytic bordered space.

Definition 3.5.1. For $c \in \mathbb{R}$, consider the strictly full subcategories of $\mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M})$ given by

$$\begin{aligned} {}^{p}\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}) &:= {}_{p}\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}) \cap {}_{p}'\mathrm{E}^{\leqslant c+1/2}_{\mathbb{R}\text{-c}}(\mathsf{M}) \\ &= \{K \in \mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M}); K \in {}_{p}\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}),\ \mathrm{D}^{\mathrm{E}}_{\mathsf{M}} K \in {}_{p^*}\mathrm{E}^{\geqslant -c-1/2}_{\mathbb{R}\text{-c}}(\mathsf{M})\}, \\ {}^{p}\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}) &:= {}_{p}\mathrm{E}^{\geqslant c-1/2}_{\mathbb{R}\text{-c}}(\mathsf{M}) \cap {}_{p}'\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}) \\ &= \{K \in \mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M}); \mathrm{D}^{\mathrm{E}}_{\mathsf{M}} K \in {}^{p}\mathrm{E}^{\leqslant -c}_{\mathbb{R}\text{-c}}(\mathsf{M})\} \\ &= \{K \in \mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M}); K \in {}_{p}\mathrm{E}^{\geqslant c-1/2}_{\mathbb{R}\text{-c}}(\mathsf{M}),\ \mathrm{D}^{\mathrm{E}}_{\mathsf{M}} K \in {}_{p^*}\mathrm{E}^{\leqslant -c}_{\mathbb{R}\text{-c}}(\mathsf{M})\}. \end{aligned}$$

By Lemma 3.4.4 one has

$$(3.5.1)\quad {}_{p}'\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}) \subset {}^{p}\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}) \subset {}_{p}\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}), \qquad {}_{p}\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}) \subset {}^{p}\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}) \subset {}_{p}'\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}).$$

In the rest of this section, we will give a proof of the following result.

Theorem 3.5.2. *Let* M *be a subanalytic bordered space.*

(i) $({}^{p}\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}), {}^{p}\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}))_{c\in\mathbb{R}}$ *is a t-structure on* $\mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M})$.

(ii) *For any* $c \in \mathbb{R}$, *the duality functor* $\mathrm{D}^{\mathrm{E}}_{M}$ *interchanges* ${}^{p}\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M})$ *and* ${}^{p^*}\mathrm{E}^{\geqslant -c}_{\mathbb{R}\text{-c}}(\mathsf{M})$.

(iii) *Assume that* $\mathsf{M} = M$ *is a subanalytic space. For any interval* $I \subset \mathbb{R}$ *such that* $I \to \mathbb{R}/\mathbb{Z}$ *is injective, the prestack on* M

$$U \mapsto {}^{p}\mathrm{E}^{I}_{\mathbb{R}\text{-c}}(U)$$

is a stack of quasi-abelian categories.

Plan of the proof. As in the proof of Proposition 3.3.12, statement (i) follows from Propositions 3.5.4 and 3.5.5 below.

Statement (ii) is clear from the definitions, and statement (iii) has a proof analogous to that of Proposition 3.2.10. □

Lemma 3.5.3. *Assume that* $\mathring{\mathsf{M}}$ *is non-empty and smooth of dimension* d. *For* $c \in \mathbb{R}$ *and* $\mathscr{L}$ *a stable* E*-type on* M *as in* (3.3.1), *one has:*

(i) $\mathbf{k}^{\mathrm{E}}_{\mathscr{L}} \in {}^{p}\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M})$ *if and only if* $m_a \leqslant c + p(d)$ *and* $n_b \leqslant c + p(d) - \frac{1}{2}$ *for any* $a \in A$ *and* $b \in B$.

(ii) $\mathbf{k}^{\mathrm{E}}_{\mathscr{L}} \in {}^{p}\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M})$ *if and only if* $m_a \geqslant c + p(d)$ *and* $n_b \geqslant c + p(d) - \frac{1}{2}$ *for any* $a \in A$ *and* $b \in B$.

Proof. The statement follows from Lemma 3.3.16. □

Proposition 3.5.4. *The bifunctors* $\mathscr{I}hom^{\mathrm{E}}$ *and* $\mathscr{H}om^{\mathrm{E}}$ *are left exact, that is, for any* $c, c' \in \mathbb{R}$ *one has*

$$\begin{aligned} \mathscr{I}hom^{\mathrm{E}}({}^{p}\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}), {}^{p}\mathrm{E}^{\geqslant c'}_{\mathbb{R}\text{-c}}(\mathsf{M})) &\subset \mathrm{D}^{\geqslant c'-c}(\mathsf{M}), \\ \mathscr{H}om^{\mathrm{E}}({}^{p}\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}), {}^{p}\mathrm{E}^{\geqslant c'}_{\mathbb{R}\text{-c}}(\mathsf{M})) &\subset \mathrm{D}^{\geqslant c'-c}(\mathbf{k}_{\mathring{\mathsf{M}}}). \end{aligned}$$

In particular, $\mathrm{Hom}_{\mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M})}({}^{p}\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}), {}^{p}\mathrm{E}^{\geqslant c'}_{\mathbb{R}\text{-c}}(\mathsf{M})) = 0$ *if* $c < c'$.

Proof. The second inclusion follows from the first one, since $\mathcal{H}om^{\mathrm{E}} \simeq \alpha_{\mathsf{M}} \mathcal{I}hom^{\mathrm{E}}$. Let us prove the first inclusion.

Let $K \in {}^p\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M})$ and $K' \in {}^p\mathrm{E}^{\geqslant c'}_{\mathbb{R}\text{-c}}(\mathsf{M})$. As in the proof of Proposition 3.2.6, reasoning by decreasing induction on $k \in \mathbb{Z}_{\geqslant -1}$, let us show the following statement:

(i)$_k$ there exists $Z_k \in \mathrm{CS}^{\leqslant k}_{\mathsf{M}}$ such that

$$\mathrm{R}\mathcal{I}hom(\mathbf{k}_{M \setminus Z_k}, \mathcal{I}hom^{\mathrm{E}}(K, K')) \in \mathrm{D}^{\geqslant c'-c}(\mathsf{M}).$$

The above statement is obvious for $k \geqslant d_M$. Assuming that (i)$_k$ holds true for some k, let us prove (i)$_{k-1}$. There exists $Z_{k-1} \in \mathrm{CS}^{\leqslant k-1}_{\mathsf{M}}$ such that $Z_{k-1} \subset Z_k$, $Z_k \setminus Z_{k-1}$ is smooth of dimension k, and $\mathrm{E}i^{-1}_{(Z_k \setminus Z_{k-1})_\infty} K$ and $\mathrm{E}i^{!}_{(Z_k \setminus Z_{k-1})_\infty} K'$ are stably free. Consider the distinguished triangle

$$\begin{aligned}\mathrm{R}\mathcal{I}hom(\mathbf{k}_{Z_k \setminus Z_{k-1}}, \mathcal{I}hom^{\mathrm{E}}(K, K')) &\longrightarrow \mathrm{R}\mathcal{I}hom(\mathbf{k}_{M \setminus Z_{k-1}}, \mathcal{I}hom^{\mathrm{E}}(K, K')) \\ &\longrightarrow \mathrm{R}\mathcal{I}hom(\mathbf{k}_{M \setminus Z_k}, \mathcal{I}hom^{\mathrm{E}}(K, K')) \xrightarrow{+1} .\end{aligned}$$

Then (i)$_{k-1}$ will follow if we show that

$$\mathrm{R}\mathcal{I}hom(\mathbf{k}_{Z_k \setminus Z_{k-1}}, \mathcal{I}hom^{\mathrm{E}}(K, K')) \in \mathrm{D}^{\geqslant c'-c}(\mathsf{M}).$$

This is equivalent to $i^!_{S_\infty} \mathcal{I}hom^{\mathrm{E}}(K, K') \in \mathrm{D}^{\geqslant c'-c}(S_\infty)$ for any connected component S of $Z_k \setminus Z_{k-1}$. One has

$$i^!_{S_\infty} \mathcal{I}hom^{\mathrm{E}}(K, K') \simeq \mathcal{I}hom^{\mathrm{E}}(\mathrm{E}i^{-1}_{S_\infty} K, \mathrm{E}i^!_{S_\infty} K').$$

By assumption,

$$\mathrm{E}i^{-1}_{S_\infty} K \simeq \mathbf{k}^{\mathrm{E}}_{\mathcal{L}} \quad \text{and} \quad \mathrm{E}i^!_{S_\infty} K' \simeq \mathbf{k}^{\mathrm{E}}_{\mathcal{L}'}$$

for some stable E-types $\mathcal{L} = (\varphi_a, m_a, \psi^{\pm}_b, n_b)_{a\in A,\, b\in B}$, $\mathcal{L}' = (\varphi_{a'}, m_{a'}, \psi^{\pm}_{b'}, n_{b'})_{a'\in A',\, b'\in B'}$. Then we are reduced to prove

$$\mathcal{I}hom^{\mathrm{E}}(\mathbf{k}^{\mathrm{E}}_{\mathcal{L}}, \mathbf{k}^{\mathrm{E}}_{\mathcal{L}'}) \in \mathrm{D}^{\geqslant c'-c}(S_\infty). \tag{3.5.2}$$

Recall that

$$\begin{aligned}\mathbf{k}^{\mathrm{E}}_{\mathcal{L}} &= \Big(\bigoplus_{a\in A} \mathbf{k}^{\mathrm{E}}_{\Phi_a}[-m_a]\Big) \oplus \Big(\bigoplus_{b\in B} \mathbf{k}^{\mathrm{E}}_{\Psi_b}[-n_b]\Big) \in {}^p\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(S_\infty), \\ \mathbf{k}^{\mathrm{E}}_{\mathcal{L}'} &= \Big(\bigoplus_{a'\in A'} \mathbf{k}^{\mathrm{E}}_{\Phi_{a'}}[-m_{a'}]\Big) \oplus \Big(\bigoplus_{b'\in B'} \mathbf{k}^{\mathrm{E}}_{\Psi_{b'}}[-n_{b'}]\Big) \in {}^p\mathrm{E}^{\geqslant c'}_{\mathbb{R}\text{-c}}(S_\infty).\end{aligned}$$

By Lemma 3.5.3 and Proposition 2.6.8, one has

$$\begin{aligned}&\mathcal{I}hom^{\mathrm{E}}(\mathbf{k}^{\mathrm{E}}_{\Psi_b}[-n_b], \mathbf{k}^{\mathrm{E}}_{\Psi_{b'}}[-n_{b'}]) \\ &\qquad \in \mathcal{I}hom^{\mathrm{E}}(\mathrm{E}^{\leqslant c+p(k)-1/2}(S_\infty), \mathrm{E}^{\geqslant c'+p(k)-1/2}(S_\infty)) \subset \mathrm{D}^{\geqslant c'-c}(S_\infty).\end{aligned}$$

Similarly, one has

$$\begin{aligned}\mathcal{I}hom^{\mathrm{E}}(\mathbf{k}^{\mathrm{E}}_{\Phi_a}[-m_a], \mathbf{k}^{\mathrm{E}}_{\Psi_{b'}}[-n_{b'}]) &\in \mathrm{D}^{\geqslant c'-c-1/2}(S_\infty), \\ \mathcal{I}hom^{\mathrm{E}}(\mathbf{k}^{\mathrm{E}}_{\Psi_b}[-n_b], \mathbf{k}^{\mathrm{E}}_{\Phi_{a'}}[-m_{a'}]) &\in \mathrm{D}^{\geqslant c'-c+1/2}(S_\infty), \\ \mathcal{I}hom^{\mathrm{E}}(\mathbf{k}^{\mathrm{E}}_{\Phi_a}[-m_a], \mathbf{k}^{\mathrm{E}}_{\Phi_{a'}}[-m_{a'}]) &\in \mathrm{D}^{\geqslant c'-c}(S_\infty).\end{aligned}$$

Hence (3.5.2) reduces to show that for any $a \in A$ and $b' \in B'$,

$$H^m \mathcal{I}hom^{\mathrm{E}}(\mathbf{k}^{\mathrm{E}}_{\Phi_a}[-m_a], \mathbf{k}^{\mathrm{E}}_{\Psi_{b'}}[-n_{b'}]) \simeq 0$$

for any $m \in \mathbb{Z}$ such that $c' - c - 1/2 \leqslant m < c' - c$. Since we have

$$H^m \mathcal{I}hom^{\mathrm{E}}(\mathbf{k}^{\mathrm{E}}_{\Phi_a}[-m_a], \mathbf{k}^{\mathrm{E}}_{\Psi_{b'}}[-n_{b'}]) \simeq H^{m+m_a-n_{b'}} \mathcal{I}hom^{\mathrm{E}}(\mathbf{k}^{\mathrm{E}}_{\Phi_a}, \mathbf{k}^{\mathrm{E}}_{\Psi_{b'}}),$$

we may assume that $m + m_a - n_{b'} \geqslant 0$. Since $m_a \leqslant c + p(k)$ and $n_{b'} \geqslant c' + p(k) - \frac{1}{2}$, one has $m + m_a - n_{b'} \leqslant m - c' + c + \frac{1}{2} < \frac{1}{2}$. Then, we have $m + m_a - n_{b'} = 0$.

Let $\pi\colon S_\infty \times \mathbb{R}_\infty \to S_\infty$ and $\overline{\pi}\colon S_\infty \times \overline{\mathbb{R}} \to S_\infty$ be the projections. Then one concludes by noticing that

$$\begin{aligned}
H^0 \mathcal{I}hom^{\mathrm{E}}(\mathbf{k}^{\mathrm{E}}_{\Phi_a}, \mathbf{k}^{\mathrm{E}}_{\Psi_{b'}}) &\simeq H^0 \mathcal{I}hom^{\mathrm{E}}(\mathbf{k}^{\mathrm{Q}}_{\Phi_a}, \mathbf{k}^{\mathrm{E}}_{\Psi_{b'}}) \\
&\simeq H^0 \mathrm{R}\pi_* \mathrm{R}\mathcal{I}hom(\mathbf{k}_{\{t \geqslant \varphi_a(x)\}}, \mathbf{k}_{\{t \gg 0\}} \overset{+}{\otimes} \mathbf{k}_{\{\psi^-_{b'}(x) \leqslant t < \psi^+_{b'}(x)\}}) \\
&\simeq H^0 \mathrm{R}\overline{\pi}_{!!} \mathrm{R}\mathcal{I}hom\Big(\mathbf{k}_{\{t \geqslant \varphi_a(x)\}}, \text{“}\varinjlim_{s \to +\infty}\text{”}\ \mathbf{k}_{\{\psi^-_{b'}(x)+s \leqslant t < \psi^+_{b'}(x)+s\}}\Big) \\
&\simeq \overline{\pi}_{!!} \mathcal{I}hom\Big(\mathbf{k}_{\{t \geqslant \varphi_a(x)\}}, \text{“}\varinjlim_{s \to +\infty}\text{”}\ \mathbf{k}_{\{\psi^-_{b'}(x)+s \leqslant t < \psi^+_{b'}(x)+s\}}\Big) \\
&\underset{(*)}{\simeq} \text{“}\varinjlim_{s \to +\infty}\text{”}\ \overline{\pi}_{!!} \mathcal{I}hom(\mathbf{k}_{\{t \geqslant \varphi_a(x)\}}, \mathbf{k}_{\{\psi^-_{b'}(x)+s \leqslant t < \psi^+_{b'}(x)+s\}}) \\
&\simeq \text{“}\varinjlim_{s \to +\infty}\text{”}\ \iota_{S_\infty} \mathring{\pi}_* \mathcal{H}om(\mathbf{k}_{\{t \geqslant \varphi_a(x)\}}, \mathbf{k}_{\{\psi^-_{b'}(x)+s \leqslant t < \psi^+_{b'}(x)+s\}}) \simeq 0,
\end{aligned}$$

where $\iota_{S_\infty}\colon \mathrm{Mod}(\mathbf{k}_S) \to \mathrm{I}(\mathbf{k}_{S_\infty})$ is the natural embedding. Note that $(*)$ holds because $\overline{\pi}_{!!}$ and $\mathcal{I}hom(\mathbf{k}_{\{t \geqslant \varphi_a(x)\}}, \bullet)$ commute with inductive limits. □

Proposition 3.5.5. *For any $c \in \mathbb{R}$ and $K \in \mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M})$ there are distinguished triangles in $\mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M})$*

$$K_{\leqslant c} \longrightarrow K \longrightarrow K_{>c} \xrightarrow{+1}, \quad K_{<c} \longrightarrow K \longrightarrow K_{\geqslant c} \xrightarrow{+1},$$

with $K_L \in {}^p\mathrm{E}^L_{\mathbb{R}\text{-c}}(\mathsf{M})$ for L equal to $\leqslant c$, $> c$, $< c$ or $\geqslant c$.

Proof. Since the proofs are similar, we will construct only the first distinguished triangle.

As in the proof of Proposition 3.3.19, one reduces to the case where M is smooth and connected, and K is stably free. Then K is a direct sum of objects in ${}^p\mathrm{E}^a_{\mathbb{R}\text{-c}}(\mathsf{M})$ for some $a \in \mathbb{R}$ by Lemma 3.5.3. □

As a corollary of Propositions 3.3.21 and 3.4.5, one has the following result.

Proposition 3.5.6. *Let $f\colon \mathsf{M} \to \mathsf{N}$ be a morphism of subanalytic bordered spaces, and $d \in \mathbb{Z}_{\geqslant 0}$. Assume that $\dim \mathring{f}^{-1}(y) \leqslant d$ for any $y \in \mathring{\mathsf{N}}$. Then, for any $c \in \mathbb{R}$ one has:*

(i) $\mathrm{E}f^{-1}({}^{p[d]}\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{N})) \subset {}^p\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M})$.

(ii) $\mathrm{E}f^{!}({}^{p[d]}\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{N})) \subset {}^p\mathrm{E}^{\geqslant c-d}_{\mathbb{R}\text{-c}}(\mathsf{M})$.

(iii) $\mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{N}) \cap \mathrm{E}f_*({}^p\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M})) \subset {}^{p[d]}\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{N})$.

(iv) $\mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{N}) \cap \mathrm{E}f_{!!}({}^p\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M})) \subset {}^{p[d]}\mathrm{E}^{\leqslant c+d}_{\mathbb{R}\text{-c}}(\mathsf{N})$.

Proof. Statements (i) and (ii) follow from Propositions 3.3.21 and 3.4.5, and (iii) and (iv) follow from them by adjunction. □

Proposition 3.5.7. *Let* M *be a subanalytic bordered space. The embedding*

$$e\colon \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_{\mathsf{M}}) \hookrightarrow \mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M})$$

induced by (2.8.2) *is exact, i.e. for any* $c \in \mathbb{R}$ *one has*

$$e({}^{p}\mathrm{D}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_{\mathsf{M}})) \subset {}^{p}\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}),$$
$$e({}^{p}\mathrm{D}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_{\mathsf{M}})) \subset {}^{p}\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}).$$

Proof. It follows from the exactness of e with respect to the standard t-structures and

$$\pi^{-1}\mathbf{k}_S \otimes e(F) \simeq e(\mathbf{k}_S \otimes F),$$
$$\mathrm{R}\mathcal{I}hom(\pi^{-1}\mathbf{k}_S, e(F)) \simeq e(\mathrm{R}\mathcal{H}om(\mathbf{k}_S, F))$$

for any $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathsf{M}})$ and $S \in \mathrm{LCS}(\mathsf{M})$, by [4, Corollary 4.7.11]. □

Definition 3.5.8. The *enhanced middle perversity t-structure*

$$({}^{1/2}\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}), {}^{1/2}\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathsf{M}))_{c\in\mathbb{R}}$$

is the one associated with the middle perversity $\mathsf{m}(n) := -\frac{n}{2}$. It is a self-dual t-structure indexed by $\frac{1}{2}\mathbb{Z}$.

Example 3.5.9. Let $\mathsf{M} = \{\mathrm{pt}\}$. Note that one has:

(i) $\mathbf{k}^{\mathrm{E}}_{\{a\leqslant t<b\}} \simeq 0$ for $a, b \in \mathbb{R}$ with $a < b$.

(ii) $\mathbf{k}^{\mathrm{E}}_{\{t\geqslant a\}} \simeq \mathbf{k}^{\mathrm{E}}_{\mathsf{M}}$ for $a \in \mathbb{R}$.

(iii) $\mathrm{D}^{\mathrm{E}}\mathbf{k}^{\mathrm{E}}_{\mathsf{M}} \simeq \mathbf{k}^{\mathrm{E}}_{\mathsf{M}}$.

Hence $\mathbf{k}^{\mathrm{E}}_{\mathsf{M}} \in {}^{1/2}\mathrm{E}^{0}_{\mathbb{R}\text{-c}}(\{\mathrm{pt}\})$, and any object of $\mathrm{E}_{\mathbb{R}\text{-c}}(\{\mathrm{pt}\})$ is a finite direct sum of shifts of copies of $\mathbf{k}^{\mathrm{E}}_{\mathsf{M}}$.

Example 3.5.10. Let $\mathsf{M} = M = \mathbb{R}$ and $K = \mathbf{k}^{\mathrm{E}}_{\{x>0,\ 0\leqslant t<1/x\}\cup\{x=0,\ t\geqslant 0\}}$, so that

$$\mathrm{D}^{\mathrm{E}}_{M} K \simeq \mathbf{k}^{\mathrm{E}}_{\{x>0,\ -1/x\leqslant t<0\}}[2].$$

Noticing that

$$\mathrm{E}i^{!}_{\{0\}} K \simeq \mathrm{D}^{\mathrm{E}}_{\{0\}}\mathrm{E}i^{-1}_{\{0\}}\mathrm{D}^{\mathrm{E}}_{M} K \simeq 0,$$
$$\mathrm{E}i^{!}_{\{0\}}\mathrm{D}^{\mathrm{E}}_{M} K \simeq \mathrm{D}^{\mathrm{E}}_{\{0\}}\mathrm{E}i^{-1}_{\{0\}} K \simeq \mathrm{D}^{\mathrm{E}}_{\{0\}}\mathbf{k}^{\mathrm{E}}_{\{0\}} \simeq \mathbf{k}^{\mathrm{E}}_{\{0\}},$$

one has $K \in {}_{1/2}\mathrm{E}^{1/2}_{\mathbb{R}\text{-c}}(\mathbb{R})$ and $\mathrm{D}^{\mathrm{E}}K \in {}_{1/2}\mathrm{E}^{-3/2}_{\mathbb{R}\text{-c}}(\mathbb{R})$, so that $K \in {}'_{1/2}\mathrm{E}^{3/2}_{\mathbb{R}\text{-c}}(\mathbb{R})$. Hence we have $K \in {}^{1/2}\mathrm{E}^{1}_{\mathbb{R}\text{-c}}(\mathbb{R})$.

Example 3.5.11. Let $\{M_\alpha\}_\alpha$ be a subanalytic stratification of M, and set $\mathsf{M}_\alpha := (M_\alpha)_\infty$. Let $K \in \mathrm{E}_{\mathbb{R}\text{-c}}(\mathsf{M})$. Assume that $\mathrm{E}i^{-1}_{\mathsf{M}_\alpha}K$ and $\mathrm{E}i^{!}_{\mathsf{M}_\alpha}K$ are stably free. Recall Notation 3.3.5. Even if only direct summands containing Φ_a appear in $\mathrm{E}i^{-1}_{\mathsf{M}_\alpha}K$, direct summands containing Ψ_b can appear in $\mathrm{E}i^{!}_{\mathsf{M}_\alpha}K$. For example, let $M = \mathbb{R}^2 \times \mathbb{R}_{>0}$ with coordinates (x, y, z), consider the bordered space $\mathsf{M} := (M, \mathbb{R}^2 \times \overline{\mathbb{R}})$, and set $K := \mathbf{k}^{\mathrm{E}}_{S} \in \mathrm{E}(\mathsf{M})$ for

$$S := \left\{(x, y, z, t) \in M \times \mathbb{R};\ x \geqslant 0,\ y > 0,\ t \geqslant \frac{zx}{x+y}\right\}.$$

Set $Z = \{x = y = 0\} \subset M$. Then one has

$$\mathrm{E}i_{Z_\infty}^{-1} K \simeq 0, \qquad \mathrm{E}i_{Z_\infty}^{!} K \simeq \mathbf{k}^{\mathrm{E}}_{\{0 \leqslant t < z\}}[-1],$$
$$\mathrm{E}i_{Z_\infty}^{!} \mathrm{D}^{\mathrm{E}}_{\mathsf{M}} K \simeq 0, \quad \mathrm{E}i_{Z_\infty}^{-1} \mathrm{D}^{\mathrm{E}}_{\mathsf{M}} K \simeq \mathrm{D}^{\mathrm{E}}_{Z_\infty} \mathrm{E}i_{Z_\infty}^{!} K \simeq \mathbf{k}^{\mathrm{E}}_{\{-z \leqslant t < 0\}}[3].$$

We deduce that $K \in {}_{1/2}\mathrm{E}^{3/2}_{\mathbb{R}\text{-c}}(\mathsf{M})$ and $\mathrm{D}^{\mathrm{E}}_{\mathsf{M}} K \in {}_{1/2}\mathrm{E}^{-3/2}_{\mathbb{R}\text{-c}}(\mathsf{M})$. Hence $K \in {}'_{1/2}\mathrm{E}^{3/2}_{\mathbb{R}\text{-c}}(\mathsf{M})$, so that $K \in {}^{1/2}\mathrm{E}^{3/2}_{\mathbb{R}\text{-c}}(\mathsf{M})$.

Remark 3.5.12. Let M be a subanalytic space. The triangulated category of enhanced sheaves on M (cf. [19] and [5]) is defined by

$$\mathrm{E}^{\mathrm{b}}(\mathbf{k}_M) := \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M \times \mathbb{R}})/\pi^{-1}\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M),$$

where $\pi\colon M \times \mathbb{R} \to M$ is the projection. One similarly defines $\mathrm{E}^{\mathrm{b}}_{\pm}(\mathbf{k}_M)$, so that

$$\mathrm{E}^{\mathrm{b}}(\mathbf{k}_M) \simeq \mathrm{E}^{\mathrm{b}}_{+}(\mathbf{k}_M) \oplus \mathrm{E}^{\mathrm{b}}_{-}(\mathbf{k}_M).$$

Note that

$$\mathrm{E}^{\mathrm{b}}_{\pm}(\mathbf{k}_M) \simeq \{K \in \mathrm{E}^{\mathrm{b}}_{\pm}(M); \mathrm{L}^{\mathrm{E}}\, K \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M \times \mathbb{R}_\infty})\}.$$

We say that an object $K \in \mathrm{E}^{\mathrm{b}}_{+}(\mathbf{k}_M)$ is $\mathbb{R}$-constructible if so is $\mathrm{L}^{\mathrm{E}}\, K \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M \times \mathbb{R}_\infty})$. Let $p\colon \mathbb{Z}_{\geqslant 0} \to \mathbb{R}$ be a perversity. Then, with obvious notations, $({}^{p}\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_M), {}^{p}\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(\mathbf{k}_M))_{c \in \mathbb{R}}$ satisfies the analogue of Theorem 3.5.2. Moreover, a description analogous to that in Lemma 3.3.7 holds, replacing "stably free" with "free".

Remark 3.5.13. Let M be a subanalytic space. It is shown in [16] that $\mathcal{H}om^{\mathrm{E}}$ induces a functor

$$\mathcal{H}om^{\mathrm{E}}(\mathbf{k}^{\mathrm{E}}_M, *)\colon \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(M) \to \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M).$$

This is neither left nor right exact with respect to the middle perversity t-structures. For example, let $M = \mathbb{R}^n$ and $K = \mathbf{k}^{\mathrm{E}}_{\{x \neq 0,\ t = -1/|x|\}}$. Then

$$K \in {}^{1/2}\mathrm{E}^{n/2}_{\mathbb{R}\text{-c}}(M).$$

Moreover, by [16, Corollary 6.6.6.], one has

$$F := \mathcal{H}om^{\mathrm{E}}(\mathbf{k}^{\mathrm{E}}_M, K) \simeq \mathbf{k}_{\{x \neq 0\}}.$$

Hence, ${}^{1/2}H^{n/2}(F) \simeq \mathbf{k}_M$ and ${}^{1/2}H^{1}(F) \simeq \mathbf{k}_{\{0\}}$ when $n \geqslant 3$. Therefore, $\mathcal{H}om^{\mathrm{E}}(\mathbf{k}^{\mathrm{E}}_{t \geqslant 0}, *)$ is not left exact. Since $\mathcal{H}om^{\mathrm{E}}(\mathbf{k}^{\mathrm{E}}_{t \geqslant 0}, *)$ commutes with duality, $\mathcal{H}om^{\mathrm{E}}(\mathbf{k}^{\mathrm{E}}_{t \geqslant 0}, *)$ is not right exact either.

4. Riemann–Hilbert correspondence

On a complex manifold, the Riemann–Hilbert correspondence embeds the triangulated category of holonomic $\mathcal{D}$-modules into that of $\mathbb{R}$-constructible enhanced ind-sheaves. We prove here the exactness of the embedding, when the target category is endowed with the middle perversity t-structure.

4.1. Subanalytic ind-sheaves. For subanalytic sheaves and ind-sheaves we refer to [13] (where subanalytic sheaves are called ind-$\mathbb{R}$-constructible sheaves).

Let M be a subanalytic space. An ind-sheaf on M is called subanalytic if it is isomorphic to a small filtrant ind-limit of $\mathbb{R}$-constructible sheaves. Then, being subanalytic is a local property.

Let us denote by $\mathrm{I}_{\mathrm{suban}}(\mathbf{k}_M)$ the category of subanalytic ind-sheaves. Note that it is a strictly full subcategory of $\mathrm{I}(\mathbf{k}_M)$ stable by kernels, cokernels and extensions.

Let $\mathrm{Op}_{M_{\mathrm{sa}}}$ be the category of relatively compact subanalytic open subsets of M, whose morphisms are inclusions.

Definition 4.1.1 (cf. [12, 13]). A *subanalytic sheaf* F is a functor $\mathrm{Op}_{M_{\mathrm{sa}}}^{\mathrm{op}} \to \mathrm{Mod}(\mathbf{k})$ which satisfies

(i) $F(\varnothing) = 0$,

(ii) for $U, V \in \mathrm{Op}_{M_{\mathrm{sa}}}$, the sequence

$$0 \longrightarrow F(U \cup V) \xrightarrow{r_1} F(U) \oplus F(V) \xrightarrow{r_2} F(U \cap V)$$

is exact. Here r_1 is given by the restriction maps and r_2 is given by the restriction $F(U) \to F(U \cap V)$ and the opposite of the restriction $F(V) \to F(U \cap V)$.

Denote by $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$ the category of subanalytic sheaves.

The following result is proved in [13].

Proposition 4.1.2. *The category* $\mathrm{I}_{\mathrm{suban}}(\mathbf{k}_M)$ *of subanalytic ind-sheaves and the category* $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$ *of subanalytic sheaves are equivalent by the functor sending* $F \in \mathrm{I}_{\mathrm{suban}}(\mathbf{k}_M)$ *to the subanalytic sheaf*

$$\mathrm{Op}_{M_{\mathrm{sa}}} \ni U \mapsto \mathrm{Hom}_{\mathrm{I}(\mathbf{k}_M)}(\mathbf{k}_U, F).$$

4.2. Enhanced tempered distributions. *Hereafter, we take the complex number field* $\mathbb{C}$ *as the base field* $\mathbf{k}$.

Let M be a real analytic manifold. Denote by $\mathcal{D}b_M$ the sheaf of Schwartz's distributions on M. The subanalytic sheaf of tempered distributions on M is defined by

$$\mathcal{D}b_M^{\mathrm{t}}(U) := \mathrm{Im}(\mathcal{D}b_M(M) \to \mathcal{D}b_M(U)) \simeq \mathcal{D}b_M(M)/\Gamma_{M \setminus U}(M; \mathcal{D}b_M)$$

for any $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. We still denote by $\mathcal{D}b_M^{\mathrm{t}}$ the corresponding subanalytic ind-sheaf.

Denote by P the real projective line, and let $t \in \mathbb{R} \subset \mathsf{P}$ be the affine coordinate. Considering the natural morphism of bordered spaces

$$j \colon M \times \mathbb{R}_\infty \to M \times \mathsf{P},$$

one sets

$$\mathcal{D}b_M^{\mathrm{T}} := j^!(\mathcal{D}b_{M \times \mathsf{P}}^{\mathrm{t}} \xrightarrow{\partial_t - 1} \mathcal{D}b_{M \times \mathsf{P}}^{\mathrm{t}}) \in \mathrm{D}(M \times \mathbb{R}_\infty),$$

where the above complex sits in degrees -1 and 0.

By the results in [4, Section 8.1] one has:

Proposition 4.2.1. *The following statements hold.*

(i) *There are isomorphisms in* $\mathrm{D}(M \times \mathbb{R}_\infty)$

$$\mathcal{D}b_M^{\mathrm{T}} \xrightarrow{\sim} \mathcal{I}hom^+(\mathbb{C}_{\{t \geqslant 0\}}, \mathcal{D}b_M^{\mathrm{T}}) \xleftarrow{\sim} \mathcal{I}hom^+(\mathbb{C}_{\{t \geqslant a\}}, \mathcal{D}b_M^{\mathrm{T}}) \quad \textit{for any } a \geqslant 0.$$

(ii) *The complex* $\mathcal{D}b_M^{\mathrm{T}}$ *is concentrated in degree* -1.

(iii) *There are natural monomorphisms in* $\mathrm{I}(\mathbb{C}_{M \times \mathbb{R}_\infty})$

$$\mathbb{C}_{\{t<*\}} \otimes \pi^{-1}\mathcal{D}b_M^{\mathrm{t}} \rightarrowtail H^{-1}\mathcal{D}b_M^{\mathrm{T}} \rightarrowtail \pi^{-1}\mathcal{D}b_M.$$

The enhanced ind-sheaf of tempered distributions is defined by

$$\mathcal{D}b_M^{\mathrm{E}} := \mathrm{Q}_M(\mathcal{D}b_M^{\mathrm{T}}) \in \mathrm{E}(M).$$

Part (iii) in the following proposition is new.

Proposition 4.2.2. *The following statements hold.*

(i) $\mathcal{D}b_M^{\mathrm{E}}$ *is stable, i.e.* $\mathbb{C}_M^{\mathrm{E}} \otimes^+ \mathcal{D}b_M^{\mathrm{E}} \simeq \mathcal{D}b_M^{\mathrm{E}}$.

(ii) $\mathrm{R}^{\mathrm{E}}\,\mathcal{D}b_M^{\mathrm{E}} \simeq \mathcal{D}b_M^{\mathrm{T}}$. *In particular, it is concentrated in degree* -1.

(iii) $\mathcal{D}b_M^{\mathrm{E}} \in \mathrm{E}^0(M)$. *In other words, the complex* $\mathrm{L}^{\mathrm{E}}\,\mathcal{D}b_M^{\mathrm{E}}$ *is concentrated in degree* 0.

Proof. (i) This follows from Proposition 4.2.1 (i).

(ii) By Proposition 4.2.1 (i), one has $\mathrm{R}^{\mathrm{E}}\,\mathcal{D}b_M^{\mathrm{E}} \simeq \mathcal{D}b_M^{\mathrm{T}}$. This is concentrated in degree -1 by Proposition 4.2.1 (ii).

(iii) By (ii), $\mathrm{R}^{\mathrm{E}}\,\mathcal{D}b_M^{\mathrm{E}} \simeq \mathcal{D}b_M^{\mathrm{T}}$ is concentrated in degree -1. Hence Lemma 2.5.2 implies

$$\mathrm{L}^{\mathrm{E}}\,\mathcal{D}b_M^{\mathrm{E}} \simeq \mathbb{C}_{\{t \geqslant 0\}} \overset{+}{\otimes} \mathcal{D}b_M^{\mathrm{T}} \in \mathrm{D}^{[-1,0]}(M \times \mathbb{R}_\infty),$$

and we are reduced to prove that $H^{-1}\,\mathrm{L}^{\mathrm{E}}\,\mathcal{D}b_M^{\mathrm{E}} \simeq 0$.

By [4, Proposition 4.3.10], there is a distinguished triangle

$$\pi_M^{-1}\mathrm{R}\pi_{M\,!!}\mathcal{D}b_M^{\mathrm{T}} \longrightarrow \mathrm{L}^{\mathrm{E}}\,\mathcal{D}b_M^{\mathrm{E}} \longrightarrow \mathcal{D}b_M^{\mathrm{T}} \xrightarrow{+1}.$$

By Proposition 4.2.1 (iii),

$$H^{-1}\mathrm{R}\pi_{M\,!!}\mathcal{D}b_M^{\mathrm{T}} \simeq \pi_{M\,!!}H^{-1}\mathcal{D}b_M^{\mathrm{T}} \subset \pi_{M\,!!}\pi^{-1}\mathcal{D}b_M = 0.$$

Thus, the above distinguished triangle induces the exact sequence

$$0 \to H^{-1}\,\mathrm{L}^{\mathrm{E}}\,\mathcal{D}b_M^{\mathrm{E}} \to H^{-1}\mathcal{D}b_M^{\mathrm{T}} \xrightarrow{\gamma} \pi^{-1}\mathrm{R}^1\pi_{M\,!!}H^{-1}\mathcal{D}b_M^{\mathrm{T}}.$$

To conclude, we have to show that γ is a monomorphism.

By Proposition 4.2.1 (iii), there is a commutative diagram

$$\begin{array}{ccc} H^{-1}\mathcal{D}b_M^{\mathrm{T}} & \xrightarrow{\gamma} & \pi_M^{-1}\mathrm{R}^1\pi_{M\,!!}H^{-1}\mathcal{D}b_M^{\mathrm{T}} \\ \downarrow & & \downarrow \\ \pi_M^{-1}\mathcal{D}b_M & \xrightarrow{\sim} & \pi_M^{-1}\mathrm{R}^1\pi_{M\,!!}\pi_M^{-1}\mathcal{D}b_M. \end{array}$$

Hence γ is a monomorphism. □

4.3. $\mathcal{D}$-modules. Let X be a complex manifold. We denote by $d_X^{\mathbb{C}}$ its complex dimension. Denote by $\mathcal{O}_X$ and $\mathcal{D}_X$ the sheaves of algebras of holomorphic functions and of differential operators, respectively. Denote by Ω_X the sheaf of differential forms of top degree.

Denote by $\mathrm{Mod}(\mathcal{D}_X)$ the category of left $\mathcal{D}_X$-modules, and by $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$ its bounded derived category. For $f\colon X \to Y$ a morphism of complex manifolds, denote by $\otimes^{\mathsf{D}}$, $\mathsf{D}f^*$, $\mathsf{D}f_*$ the operations for $\mathcal{D}$-modules.

Consider the dual of $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$ given by

$$\mathbb{D}_X \mathcal{M} = \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1})[d_X^{\mathbb{C}}].$$

A $\mathcal{D}_X$-module $\mathcal{M}$ is called *quasi-good* if, for any relatively compact open subset $U \subset X$, $\mathcal{M}|_U$ is isomorphic (as an $\mathcal{O}_X|_U$-module) to a filtrant inductive limit of coherent $\mathcal{O}_X|_U$-submodules. A $\mathcal{D}_X$-module $\mathcal{M}$ is called *good* if it is quasi-good and coherent.

To a coherent $\mathcal{D}_X$-module $\mathcal{M}$ one associates its characteristic variety $\mathrm{char}(\mathcal{M})$, a closed conic involutive complex analytic subset of the cotangent bundle T^*X. If $\mathrm{char}(\mathcal{M})$ is Lagrangian, then $\mathcal{M}$ is called holonomic. For the notion of regular holonomic $\mathcal{D}_X$-module, refer e.g. to [8, Section 5.2].

Denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ and $\mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)$ the full subcategories of $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$ whose objects have holonomic and regular holonomic cohomologies, respectively. These are triangulated categories.

Let $f\colon X \to Y$ be a morphism of complex manifolds. For $x_0 \in X$ consider

$$\mathrm{rank}^{\mathbb{C}}_{x_0}(f) := \mathrm{rank}^{\mathbb{C}}(T_{x_0}X \xrightarrow{df(x_0)} T_{f(x_0)}Y) \quad \text{and} \quad \text{flat-dim}_{\mathcal{D}_{X,x_0}}(\mathcal{D}_{X\to Y,x_0}),$$

the complex dimension of the image of $df(x_0)$, and the flat dimension of the transfer bimodule $\mathcal{D}_{X\to Y,x_0}$ as a left $\mathcal{D}_{X,x_0}$-module, respectively.

Proposition 4.3.1. *Let $f\colon X \to Y$ be a morphism of complex manifolds. For $x_0 \in X$ one has*

$$\text{flat-dim}_{\mathcal{D}_{X,x_0}}(\mathcal{D}_{X\to Y,x_0}) \leqslant d_X^{\mathbb{C}} - \mathrm{rank}^{\mathbb{C}}_{x_0}(f).$$

Proof. Set $n = d_X^{\mathbb{C}}$, $m = d_Y^{\mathbb{C}}$, $d = \mathrm{rank}^{\mathbb{C}}_{x_0}(f)$, and $y_0 = f(x_0)$.

Choose a system of local coordinates $y = (y_1, \dots, y_m)$ of Y on a neighborhood of y_0 such that $\partial_{y_1}, \dots, \partial_{y_d}$ generate $df(x_0)(T_{x_0}X) \subset T_{f(x_0)}Y$. Set $x_k = y_k \circ f$ for $k \leqslant d$, and complete them to a system of local coordinates $x = (x_1, \dots, x_n)$ of X on a neighborhood of x_0.

Consider the subring

$$\mathcal{R} := \mathcal{O}_{X,x_0}[\partial_{x_1}, \dots, \partial_{x_d}] \subset \mathcal{D} := \mathcal{D}_{X,x_0}.$$

Then $\mathcal{D}_{X\to Y,x_0} \simeq \mathcal{O}_{X,x_0} \otimes_{\mathcal{O}_{Y,y_0}} \mathcal{D}_{Y,y_0}$ is a free $\mathcal{R}$-module. In fact, one has

$$\mathcal{D}_{X\to Y,x_0} \simeq \bigoplus_{\beta \in \{0\}^d \times \mathbb{Z}^{m-d}_{\geqslant 0} \subset \mathbb{Z}^m_{\geqslant 0}} \mathcal{R}\partial_y^\beta.$$

The statement follows by Lemma 4.3.2 below. □

Lemma 4.3.2. *Use notations as in the proof above. Let $\mathcal{M}$ be a left $\mathcal{D}$-module. If $\mathcal{M}$ is flat as a left $\mathcal{R}$-module, then*

$$\text{flat-dim}_{\mathcal{D}}(\mathcal{M}) \leqslant n - d.$$

Proof. Set $\mathcal{O} := \mathcal{O}_{X,x_0}$ and $\mathcal{D}' := \mathcal{O}[\partial_{x_{d+1}}, \dots, \partial_{x_n}]$, so that $\mathcal{D} \simeq \mathcal{D}' \otimes_{\mathcal{O}} \mathcal{R}$. Set $K := \mathbb{C}\partial_{x_{d+1}} \oplus \cdots \oplus \mathbb{C}\partial_{x_n}$. Then the Spencer resolution of $\mathcal{M}$, considered as a $\mathcal{D}'$-module, is

$$0 \to (\mathcal{D}' \otimes \bigwedge^{n-d} K) \otimes_{\mathcal{O}} \mathcal{M} \to \cdots \to \mathcal{D}' \otimes_{\mathcal{O}} \mathcal{M} \to \mathcal{M} \to 0.$$

Since $\mathcal{D}' \otimes_{\mathcal{O}} \mathcal{R} \simeq \mathcal{D}$, the above resolution reads as

$$0 \to (\mathcal{D} \otimes \bigwedge^{n-d} K) \otimes_{\mathcal{R}} \mathcal{M} \to \cdots \to \mathcal{D} \otimes_{\mathcal{R}} \mathcal{M} \to \mathcal{M} \to 0.$$

Since $\mathcal{M}$ is a flat left $\mathcal{R}$-module, this is a flat resolution of $\mathcal{M}$ as a left $\mathcal{D}$-module. □

For a category $\mathcal{C}$, let $\mathrm{Pro}(\mathcal{C})$ be the category of pro-objects in $\mathcal{C}$, and let "$\varprojlim$" be the projective limit in $\mathrm{Pro}(\mathcal{C})$.

Lemma 4.3.3. *Let $\mathcal{M}$ be a quasi-good $\mathcal{D}_X$-module, flat over $\mathcal{D}_X$. Let $\{\mathcal{M}_i\}_{i \in I}$ be a filtrant inductive system of coherent $\mathcal{D}_X$-modules such that $\mathcal{M} \simeq \varinjlim_{i \in I} \mathcal{M}_i$. Then, for any $x \in X$ and any $k \neq 0$ one has*

$$\text{“}\varprojlim_{i \in I}\text{”}\, \mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}_i, \mathcal{D}_X)_x \simeq 0 \quad \textit{in } \mathrm{Pro}(\mathrm{Mod}(\mathcal{D}^{\mathrm{op}}_{X,x})).$$

Proof. There exists a filtrant inductive system $\{L_j\}_{j \in J}$ of free $\mathcal{D}_{X,x}$-modules of finite rank such that

$$\mathcal{M}_x \simeq \varinjlim_j L_j$$

(see [17]). It implies that

$$\text{“}\varinjlim_i\text{”}\, \mathcal{M}_{i,x} \simeq \text{“}\varinjlim_j\text{”}\, L_j$$

in $\mathrm{Ind}(\mathrm{Mod}(\mathcal{D}_{X,x}))$. Hence, for any $i \in I$ there exist $j \in J$, a morphism $u\colon i \to i'$ in I and a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{i,x} & \longrightarrow & \mathcal{M}_{i',x} \\ & \searrow \quad \nearrow & \\ & L_j. & \end{array}$$

It follows that the morphism induced by u,

$$\mathcal{E}xt^k_{\mathcal{D}_{X,x}}(\mathcal{M}_{i',x}, \mathcal{D}_{X,x}) \to \mathcal{E}xt^k_{\mathcal{D}_{X,x}}(\mathcal{M}_{i,x}, \mathcal{D}_{X,x}),$$

is the zero morphism. □

For a hypersurface Y of X, denote by $\mathcal{O}_X(*Y)$ the sheaf of meromorphic functions on X with poles in Y. We set

$$\mathcal{D}_X(*Y) := \mathcal{O}_X(*Y) \otimes_{\mathcal{O}_X} \mathcal{D}_X \simeq \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(*Y).$$

It is a sheaf of $\mathbb{C}$-algebras on X. For a $\mathcal{D}_X$-module $\mathcal{M}$, we set

$$\mathcal{M}(*Y) := \mathcal{D}_X(*Y) \otimes_{\mathcal{D}_X} \mathcal{M}.$$

Lemma 4.3.4. *Let $Y \subset X$ be a closed complex analytic hypersurface, and let $\mathcal{M}$ be a quasi-good $\mathcal{D}_X$-module. Assume that $\mathcal{M}|_{X \setminus Y}$ is flat over $\mathcal{D}_{X \setminus Y}$. Let $\{\mathcal{M}_i\}_{i \in I}$ be a filtrant inductive system of coherent $\mathcal{D}_X$-modules such that $\mathcal{M}(*Y) \simeq \varinjlim_{i \in I} \mathcal{M}_i$. Then, for any $V \subset\subset X$:*

(i) *For any $k \neq 0$,*

$$\text{“}\varprojlim_{i \in I}\text{”}\, \mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}_i, \mathcal{D}_X(*Y))|_V \simeq 0$$

in $\mathrm{Pro}(\mathrm{Mod}(\mathcal{D}_V^{\mathrm{op}}))$.

(ii) *One has*

$$\text{“}\varprojlim_{i \in I}\text{”}\, \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_i, \mathcal{D}_X(*Y))|_V \simeq \text{“}\varprojlim_{i \in I}\text{”}\, \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_i, \mathcal{D}_X(*Y))|_V$$

in $\mathrm{Pro}(\mathrm{D^b}(\mathcal{D}_V^{\mathrm{op}}))$.

Proof. (i) For $i \in I$, denote by I^i the category whose objects are morphism $i \to i'$ in I with source i, and whose morphisms are commutative diagrams in I

$$\begin{array}{ccc} & i & \\ \swarrow & & \searrow \\ i' & \longrightarrow & i''. \end{array}$$

It is enough to show that for any $i \in I$ there exists $(u_0 : i \to i_0) \in I^i$ such that the induced morphism

$$u_0' : \mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}_{i_0}, \mathcal{D}_X(*Y))|_V \to \mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}_i, \mathcal{D}_X(*Y))|_V$$

is the zero morphism. For $(u : i \to i') \in I^i$, set

$$\mathcal{N}_u = \mathrm{Im}(\mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}_{i'}, \mathcal{D}_X) \to \mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}_i, \mathcal{D}_X)).$$

It is a coherent $\mathcal{D}_X^{\mathrm{op}}$-module. Note that the decreasing family of closed complex analytic subsets $\{\mathrm{supp}(\mathcal{N}_u)\}_{u \in I^i}$ is locally stationary. Since I^i is filtrant by [14, Corollary 3.2.3], there exists $(u_0 : i \to i_0) \in I^i$ such that

$$\mathrm{supp}(\mathcal{N}_{u_0}|_V) = \bigcap_{u \in I^i} \mathrm{supp}(\mathcal{N}_u|_V).$$

By Lemma 4.3.3, one has $\bigcap_{u \in I^i} \mathrm{supp}(\mathcal{N}_u|_V) \subset Y$. Thus $\mathrm{supp}(\mathcal{N}_{u_0}|_V) \subset Y$, and one has $0 \simeq (\mathcal{N}_{u_0} \otimes_{\mathcal{D}_X} \mathcal{D}_X(*Y))|_V \simeq \mathrm{Im}(u_0')$. Hence we obtain (i).

(ii) This follows from (i). □

Proposition 4.3.5. *Let $Y \subset X$ be a closed complex analytic hypersurface, and let $\mathcal{M}$ be a quasi-good $\mathcal{D}_X$-module. Assume that $\mathcal{M}|_{X \setminus Y}$ is flat over $\mathcal{D}_{X \setminus Y}$. Then $\mathcal{M}(*Y)$ is a flat $\mathcal{D}_X$-module.*

Proof. The question being local, we can write $\mathcal{M}(*Y) \simeq \varinjlim_i \mathcal{M}_i$ with $\{\mathcal{M}_i\}_{i \in I}$ a filtrant inductive system of coherent $\mathcal{D}_X$-modules. Set

$$\mathcal{M}_i^* := \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_i, \mathcal{D}_X).$$

Then $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_i, \mathcal{D}_X(*Y)) \simeq \mathcal{M}_i^*(*Y)$. By Lemma 4.3.4, one has

$$\text{(4.3.1)} \qquad \text{“}\varprojlim_{i}\text{”}\, \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_i, \mathcal{D}_X(*Y)) \simeq \text{“}\varprojlim_{i}\text{”}\, \mathcal{M}_i^*(*Y) \quad \text{in } \mathrm{Pro}(\mathrm{D^b}(\mathcal{D}_X^{\mathrm{op}})),$$

by shrinking X if necessary. Let $\mathcal{P} \in \mathrm{Mod}(\mathcal{D}_X^{\mathrm{op}})$. We have to show that, for $k < 0$,

$$H^k(\mathcal{P} \otimes^{\mathrm{L}}_{\mathcal{D}_X} \mathcal{M}(*Y)) \simeq 0. \tag{4.3.2}$$

One has

$$H^k(\mathcal{P} \otimes^{\mathrm{L}}_{\mathcal{D}_X} \mathcal{M}(*Y)) \simeq H^k(\mathcal{P}(*Y) \otimes^{\mathrm{L}}_{\mathcal{D}_X} \mathcal{M}(*Y)) \simeq \varinjlim_i H^k(\mathcal{P}(*Y) \otimes^{\mathrm{L}}_{\mathcal{D}_X} \mathcal{M}_i).$$

Moreover,

$$\begin{aligned}
\text{“}\varinjlim_i\text{”}\, \mathcal{P}(*Y) \otimes^{\mathrm{L}}_{\mathcal{D}_X} \mathcal{M}_i &\simeq \text{“}\varinjlim_i\text{”}\, \mathrm{R}\mathcal{H}om_{\mathcal{D}_X^{\mathrm{op}}}(\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_i, \mathcal{D}_X),\ \mathcal{P}(*Y)) \\
&\simeq \text{“}\varinjlim_i\text{”}\, \mathrm{R}\mathcal{H}om_{\mathcal{D}_X^{\mathrm{op}}(*Y)}(\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_i, \mathcal{D}_X(*Y)), \mathcal{P}(*Y)) \\
&\underset{(*)}{\simeq} \text{“}\varinjlim_i\text{”}\, \mathrm{R}\mathcal{H}om_{\mathcal{D}_X^{\mathrm{op}}(*Y)}(\mathcal{M}_i^*(*Y), \mathcal{P}(*Y)),
\end{aligned}$$

where $(*)$ follows from (4.3.1). Hence we obtain

$$H^k(\mathcal{P} \otimes^{\mathrm{L}}_{\mathcal{D}_X} \mathcal{M}(*Y)) \simeq \varinjlim_i H^k \mathrm{R}\mathcal{H}om_{\mathcal{D}_X^{\mathrm{op}}(*Y)}(\mathcal{M}_i^*(*Y), \mathcal{P}(*Y)),$$

which vanishes for $k < 0$. □

Let us denote by $\mathrm{E}(\mathcal{D}_X^{\mathrm{op}})$ the category of enhanced ind-sheaves on X with $\mathcal{D}_X^{\mathrm{op}}$-action (see [4, Section 4.10] where $\mathrm{E}(\mathcal{D}_X^{\mathrm{op}})$ is denoted by $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\,\mathcal{D}_X^{\mathrm{op}})$).

Consider the forgetful functor

$$\mathrm{for}\colon \mathrm{E}(\mathcal{D}_X^{\mathrm{op}}) \to \mathrm{E}(X).$$

Lemma 4.3.6. *Let $c \in \mathbb{R}$, X a complex manifold, $Y \subset X$ a closed complex analytic subset, $K \in \mathrm{E}(\mathcal{D}_X^{\mathrm{op}})$, and $\mathcal{M}$ a quasi-good $\mathcal{D}_X$-module. Set $U = X \setminus Y$. Assume*

(a) $K \simeq \mathrm{R}\mathcal{I}hom(\pi^{-1}\mathbf{k}_U, K)$,

(b) $\mathrm{for}(K) \in \mathrm{E}^{\geqslant c}(X)$,

(c) *$\mathcal{M}|_U$ is flat over $\mathcal{D}_U$.*

Then

$$K \otimes^{\mathrm{L}}_{\mathcal{D}_X} \mathcal{M} \in \mathrm{E}^{\geqslant c}(X).$$

Proof. We may assume that Y is a proper subset of X, as otherwise the statement is trivial.

(i) Let $\varphi\colon X' \to X$ be a projective morphism such that $Y' := \varphi^{-1}(Y)$ is a hypersurface, and φ induces an isomorphism $U' := \varphi^{-1}(U) \xrightarrow{\sim} U$. Set

$$\begin{aligned}
K' &:= \mathrm{R}\mathcal{I}hom(\pi^{-1}\mathbb{C}_{U'}, \mathrm{E}\varphi^{-1}K \otimes^{\mathrm{L}}_{\varphi^{-1}\mathcal{D}_X} \mathcal{D}_{X\leftarrow X'}) \in \mathrm{E}(\mathcal{D}_{X'}^{\mathrm{op}}), \\
\mathcal{M}' &:= (\mathsf{D}\varphi^*\mathcal{M})(*Y').
\end{aligned}$$

Then we have $\mathrm{for}(K') \in \mathrm{E}^{\geqslant c}(X')$. Note that $\mathcal{M}'$ is concentrated in degree zero. Moreover, by Proposition 4.3.5, $\mathcal{M}'$ is a flat $\mathcal{D}_{X'}$-module. Since

$$K \otimes^{\mathrm{L}}_{\mathcal{D}_X} \mathcal{M} \simeq \mathrm{E}\varphi_*(K' \otimes^{\mathrm{L}}_{\mathcal{D}_{X'}} \mathcal{M}'),$$

and since $\mathrm{E}\varphi_*$ is left exact, we reduce to the case where $\mathcal{M}$ is flat over $\mathcal{D}_X$.

(ii) Let $\mathcal{M}$ be a quasi-good flat $\mathcal{D}_X$-module. Let $\{\mathcal{M}_i\}_{i\in I}$ be a filtrant inductive system of coherent $\mathcal{D}_X$-modules such that $\mathcal{M} \simeq \varinjlim_i \mathcal{M}_i$. Set

$$\mathcal{M}_i^* := \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_i, \mathcal{D}_X).$$

Then Lemma 4.3.4 implies that

$$\text{“}\varprojlim_i\text{”}\, \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_i, \mathcal{D}_X) \simeq \text{“}\varprojlim_i\text{”}\, \mathcal{M}_i^* \quad \text{in } \mathrm{Pro}(\mathrm{D^b}(\mathcal{D}_X^{\mathrm{op}})),$$

by shrinking X if necessary. Hence one has

$$\begin{aligned} H^k(K \otimes^{\mathrm{L}}_{\mathcal{D}_X} \mathcal{M}) &\simeq \text{“}\varinjlim_i\text{”}\, H^k(K \otimes^{\mathrm{L}}_{\mathcal{D}_X} \mathcal{M}_i) \\ &\simeq \text{“}\varinjlim_i\text{”}\, H^k \mathrm{R}\mathcal{H}om_{\mathcal{D}_X^{\mathrm{op}}}(\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_i, \mathcal{D}_X), K) \\ &\simeq \text{“}\varinjlim_i\text{”}\, H^k \mathrm{R}\mathcal{H}om_{\mathcal{D}_X^{\mathrm{op}}}(\mathcal{M}_i^*, K) \simeq 0 \end{aligned}$$

for $k < c$. □

Proposition 4.3.7. *Let $\ell \in \mathbb{Z}_{\geqslant 0}$, $c \in \mathbb{R}$, X a complex manifold, $Y \subset X$ a closed complex analytic subset, $K \in \mathrm{E}(\mathcal{D}_X^{\mathrm{op}})$, and $\mathcal{M}$ a quasi-good $\mathcal{D}_X$-module. Set $U = X \setminus Y$. Assume*

(a) $\mathrm{for}(K) \in \mathrm{E}^{\geqslant c}(X)$,

(b) $\text{flat-dim}_{\mathcal{D}_{X,x}}(\mathcal{M}_x) \leqslant \ell$ *for any* $x \in U$.

Then

$$\mathrm{R}\mathcal{I}hom(\pi^{-1}\mathbf{k}_U, K) \otimes^{\mathrm{L}}_{\mathcal{D}_X} \mathcal{M} \in \mathrm{E}^{\geqslant c-\ell}(X).$$

Proof. We may assume that Y is a proper subset of X, as otherwise the statement is trivial. Replacing K with $\mathrm{R}\mathcal{I}hom(\pi^{-1}\mathbf{k}_U, K)$, we may assume from the beginning that $K \simeq \mathrm{R}\mathcal{I}hom(\pi^{-1}\mathbf{k}_U, K)$. We proceed by induction on ℓ. The case $\ell = 0$ follows from Lemma 4.3.6. Let $\ell > 0$. Then, there is locally a short exact sequence

$$0 \to \mathcal{N} \to \mathcal{L} \to \mathcal{M} \to 0,$$

with a free $\mathcal{D}_X$-module $\mathcal{L}$. It follows that $\mathcal{N}$ is a quasi-good $\mathcal{D}_X$-module satisfying the condition $\text{flat-dim}_{\mathcal{D}_{X,x}}(\mathcal{N}_x) \leqslant \ell - 1$ for any $x \in U$. One has $K \otimes^{\mathrm{L}}_{\mathcal{D}_X} \mathcal{N} \in \mathrm{E}^{\geqslant c-\ell+1}(X)$ by the induction hypothesis. Moreover, $K \otimes^{\mathrm{L}}_{\mathcal{D}_X} \mathcal{L} \in \mathrm{E}^{\geqslant c}(X)$ since $\mathcal{L}$ is free. One concludes by considering the distinguished triangle

$$K \otimes^{\mathrm{L}}_{\mathcal{D}_X} \mathcal{L} \longrightarrow K \otimes^{\mathrm{L}}_{\mathcal{D}_X} \mathcal{M} \longrightarrow K \otimes^{\mathrm{L}}_{\mathcal{D}_X} \mathcal{N}[1] \xrightarrow{+1}. \qquad \square$$

4.4. Enhanced tempered holomorphic functions. Let X be a complex manifold, and denote by $d_X^{\mathbb{C}}$ its dimension.

Denote by $X_{\mathbb{R}}$ the real analytic manifold underlying X, and by $\overline{X}$ the conjugate complex manifold. The ind-sheaf of tempered holomorphic functions

$$\mathcal{O}_X^{\mathrm{t}} := \mathrm{R}\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \mathcal{D}b^{\mathrm{t}}_{X_{\mathbb{R}}}) \in \mathrm{D}(X)$$

is the Dolbeault complex with values in tempered distributions. It is not concentrated in degree zero if $d_X^{\mathbb{C}} > 1$. Note that $\mathcal{O}_X^{\mathrm{t}}$ inherits from $\mathcal{D}b^{\mathrm{t}}_{X_{\mathbb{R}}}$ a natural $\mathcal{D}_X$-action.

Denote by $\mathbb{P}$ the complex projective line, and let τ be its affine coordinate. The enhanced ind-sheaf of tempered holomorphic functions is defined by

$$\mathcal{O}_X^{\mathrm{E}} := i^! \mathrm{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{P}}}(\mathcal{D}_{\mathbb{P}} e^\tau, \mathcal{O}^{\mathrm{t}}_{X\times\mathbb{P}})[2] \in \mathrm{E}(X),$$

where $i\colon X\times\mathbb{R}_\infty \to X\times\mathbb{P}$ is the natural morphism, and $\mathcal{D}_{\mathbb{P}} e^\tau$ is the exponential $\mathcal{D}_{\mathbb{P}}$-module generated by e^τ. Note that $\mathcal{O}_X^{\mathrm{E}}$ inherits from $\mathcal{O}^{\mathrm{t}}_{X\times\mathbb{P}}$ a natural $\mathcal{D}_X$-action.

Proposition 4.4.1. *One has* $\mathcal{O}_X^{\mathrm{E}} \in {}_{1/2}\mathrm{E}^{\geqslant d_X^{\mathbb{C}}}(X)$.

Proof. By Lemma 3.2.5, it is enough to show that for any $k\in\mathbb{Z}_{\geqslant 0}$ and any $Z\in \mathrm{CS}^{\leqslant k}_{X_{\mathbb{R}}}$ there exists an open subanalytic subset Z_0 of Z such that $\dim(Z\setminus Z_0)<k$ and

$$\mathrm{E}i^!_{(Z_0)_\infty}\mathcal{O}_X^{\mathrm{E}} \in \mathrm{E}^{\geqslant d_X^{\mathbb{C}}-k/2}((Z_0)_\infty). \tag{4.4.1}$$

Since the question is local on X, we may assume from the beginning that Z is compact. Let Z_0, $W_0\subset N$, $\ell = d_N$ and $g\colon N\to M$ be as obtained by Lemma 4.4.3 below, for $M = X_{\mathbb{R}}$ the real analytic manifold underlying X. There exists a complexification Y of N such that $g\colon N\to X$ extends to a holomorphic map $f\colon Y\to X$. Then, $d_Y^{\mathbb{C}}=\ell$ and there is a commutative diagram

$$\begin{array}{ccccc}
(W_0)_\infty{:=}(W_0,N) & \xrightarrow{i_{(W_0)_\infty}} & N & \xrightarrow{i_N} & Y \\
\downarrow{\scriptstyle g_0} & & & & \downarrow{\scriptstyle f} \\
(Z_0)_\infty{:=}(Z_0,Z) & & \xrightarrow{\quad i_{(Z_0)_\infty}\quad} & & X.
\end{array}
\qquad (j = i_N\circ i_{(W_0)_\infty}\colon (W_0)_\infty\to Y)$$

Note that for any $w\in W_0$, setting $x=f(w)\in Z_0$, one has

$$\mathrm{rank}_w^{\mathbb{C}}(f) = \dim^{\mathbb{C}}(T_xZ_0+\sqrt{-1}T_xZ_0)\geqslant(\dim T_xZ_0)/2 = k/2. \tag{4.4.2}$$

Set

$$V := \left\{ y\in Y;\mathrm{rank}_y^{\mathbb{C}}(f)\geqslant\frac{k}{2}\right\}.$$

Then V is an open subset of Y such that $Y\setminus V$ is a closed complex analytic subset. Moreover, $W_0\subset V$. Hence Proposition 4.3.1 implies

$$\text{flat-dim}_{\mathcal{D}^{\mathrm{op}}_{Y,y}}(\mathcal{D}_{X\leftarrow Y}) \leqslant d_Y^{\mathbb{C}}-\frac{k}{2} = \ell-\frac{k}{2} \quad \text{for any } y\in V. \tag{4.4.3}$$

By Proposition 2.7.4, in order to see (4.4.1) it is enough to show

$$\mathrm{E}g_0^{-1}\mathrm{E}\,i^!_{(Z_0)_\infty}\mathcal{O}_X^{\mathrm{E}} \in \mathrm{E}^{\geqslant d_X^{\mathbb{C}}-k/2}((W_0)_\infty). \tag{4.4.4}$$

Since $W_0\to Z_0$ is smooth, one has

$$\begin{aligned}
\mathrm{E}g_0^{-1}\mathrm{E}\,i^!_{(Z_0)_\infty}\mathcal{O}_X^{\mathrm{E}} &\simeq \mathrm{or}_{W_0/Z_0}\otimes\mathrm{E}g_0^!\mathrm{E}\,i^!_{(Z_0)_\infty}\mathcal{O}_X^{\mathrm{E}}[d_{Z_0}-d_N] \\
&\simeq \mathrm{or}_{W_0/Z_0}\otimes\mathrm{E}\,i^!_{(W_0)_\infty}\mathrm{E}i_N^!\mathrm{E}f^!\mathcal{O}_X^{\mathrm{E}}[k-\ell] \\
&\simeq \mathrm{or}_{W_0/Z_0}\otimes\mathrm{E}j^!\mathrm{E}i_{N*}\mathrm{E}i_N^!\mathrm{E}f^!\mathcal{O}_X^{\mathrm{E}}[k-\ell],
\end{aligned}$$

where $\mathrm{or}_{W_0/Z_0}:=H^{k-\ell}(g_0^!\mathbb{C}_{Z_0})$ is the relative orientation sheaf.

By [4, Theorem 9.1.2], one has

$$\mathrm{E} f^{!}\mathcal{O}_X^{\mathrm{E}} \simeq \mathcal{D}_{X\leftarrow Y} \otimes^{\mathrm{L}}_{\mathcal{D}_Y} \mathcal{O}_Y^{\mathrm{E}}[\ell - d_X^{\mathbb{C}}].$$

Moreover, denoting by $\mathrm{or}_{N/Y} \simeq i_N^{!}\mathbb{C}_Y[\ell]$ the relative orientation sheaf, one has

$$\mathrm{E}i_N^{!}\mathcal{O}_Y^{\mathrm{E}} \simeq \mathrm{or}_{N/Y} \otimes \mathcal{D}b_N^{\mathrm{E}}[-\ell].$$

Thus, we obtain

$$\begin{aligned}
&\mathrm{or}_{W_0/Z_0} \otimes \mathrm{E}g_0^{-1}\mathrm{E}\, i_{(Z_0)_\infty}^{!}\mathcal{O}_X^{\mathrm{E}} \\
&\quad\simeq \mathrm{E}j^{!}\mathrm{E}i_{N*}\mathrm{E}i_N^{!}(\mathcal{D}_{X\leftarrow Y} \otimes^{\mathrm{L}}_{\mathcal{D}_Y} \mathcal{O}_Y^{\mathrm{E}})[k - d_X^{\mathbb{C}}] \\
&\quad\simeq \mathrm{E}j^{!}(\mathcal{D}_{X\leftarrow Y} \otimes^{\mathrm{L}}_{\mathcal{D}_Y} \mathrm{E}i_{N*}(\mathrm{or}_{N/Y} \otimes \mathcal{D}b_N^{\mathrm{E}}))[k - d_X^{\mathbb{C}} - \ell] \\
&\quad\simeq \mathrm{E}j^{!}(\mathcal{D}_{X\leftarrow Y} \otimes^{\mathrm{L}}_{\mathcal{D}_Y} \mathrm{R}\mathcal{I}hom(\pi^{-1}\mathbb{C}_V, \mathrm{E}i_{N*}(\mathrm{or}_{N/Y} \otimes \mathcal{D}b_N^{\mathrm{E}}))[k - d_X^{\mathbb{C}} - \ell].
\end{aligned}$$

By Proposition 4.2.2, one has

$$\mathrm{E}i_{N*}(\mathrm{or}_{N/Y} \otimes \mathcal{D}b_N^{\mathrm{E}}) \in \mathrm{E}^{\geqslant 0}(Y).$$

Hence Proposition 4.3.7 and (4.4.3) imply that

$$\mathcal{D}_{X\leftarrow Y} \otimes^{\mathrm{L}}_{\mathcal{D}_Y} \mathrm{R}\mathcal{I}hom(\pi^{-1}\mathbb{C}_V, \mathrm{E}i_{N*}(\mathrm{or}_{N/Y} \otimes \mathcal{D}b_N^{\mathrm{E}})) \in \mathrm{E}^{\geqslant k/2-\ell}(Y).$$

Finally, we obtain (4.4.4). □

Corollary 4.4.2. *One has* $\mathcal{O}_X^{\mathrm{t}} \in {}^{1/2}\mathrm{D}^{\geqslant d_X^{\mathbb{C}}}(X)$.

Proof. Since $\mathcal{O}_X^{\mathrm{t}} \simeq \mathcal{I}hom^{\mathrm{E}}(\mathbb{C}_X^{\mathrm{E}}, \mathcal{O}_X^{\mathrm{E}})$, the statement follows from Proposition 4.4.1 and Lemma 3.2.7. □

Here is the lemma which is used in the course of the proof of Proposition 4.4.1.

Lemma 4.4.3. *Let M be a real analytic manifold, and let $Z \in \mathrm{CS}_M^{\leqslant k}$ for $k \in \mathbb{Z}_{\geqslant 0}$. Assume that Z is compact. Then there exist*

(i) *an open subset Z_0 of Z which is a real analytic submanifold of dimension k,*

(ii) *a real analytic manifold N of dimension $\ell \geqslant k$,*

(iii) *a real analytic proper map $g\colon N \to M$,*

(iv) *an open subanalytic subset W_0 of N*

such that one has

(a) $\dim(Z \setminus Z_0) < k$,

(b) *$g(N) = Z$, $g(W_0) = Z_0$ and g induces a smooth morphism $W_0 \to Z_0$ of real analytic manifolds.*

Proof. It follows immediately from the existence of a real analytic manifold N and a proper real analytic map $g\colon N \to M$ such that $g(N) = Z$. Note that we may assume that N is equidimensional, by multiplying each connected component of N with a sphere if necessary. □

4.5. Riemann–Hilbert correspondence. Let X be a complex manifold. The enhanced de Rham and solution functors are defined by

$$\begin{aligned} \mathcal{DR}^{\mathrm{E}}_X &: \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X) \to \mathrm{E}(X), && \mathcal{M} \mapsto \Omega^{\mathrm{E}}_X \otimes^{\mathrm{L}}_{\mathcal{D}_X} \mathcal{M}, \\ \mathcal{S}ol^{\mathrm{E}}_X &: \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)^{\mathrm{op}} \to \mathrm{E}(X), && \mathcal{M} \mapsto \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}^{\mathrm{E}}_X), \end{aligned}$$

where $\Omega^{\mathrm{E}}_X := \Omega_X \otimes^{\mathrm{L}}_{\mathcal{O}_X} \mathcal{O}^{\mathrm{E}}_X$.

The Riemann–Hilbert correspondence of [4, Theorem 9.5.3] implies that these functors induce fully faithful functors

$$\mathcal{DR}^{\mathrm{E}}_X : \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) \to \mathrm{E}_{\mathbb{R}\text{-c}}(X), \quad \mathcal{S}ol^{\mathrm{E}}_X : \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} \to \mathrm{E}_{\mathbb{R}\text{-c}}(X). \tag{4.5.1}$$

Theorem 4.5.1. *The functors $\mathcal{DR}^{\mathrm{E}}_X$ and $\mathcal{S}ol^{\mathrm{E}}_X[d^{\mathbb{C}}_X]$ are exact. That is, for any $c \in \mathbb{R}$ one has*

$$\begin{aligned} \mathcal{DR}^{\mathrm{E}}_X(\mathrm{D}^{\leqslant c}_{\mathrm{hol}}(\mathcal{D}_X)) &\subset {}^{1/2}\mathrm{E}^{\leqslant c}_{\mathbb{R}\text{-c}}(X), & \mathcal{S}ol^{\mathrm{E}}_X(\mathrm{D}^{\leqslant c}_{\mathrm{hol}}(\mathcal{D}_X)) &\subset {}^{1/2}\mathrm{E}^{\geqslant d^{\mathbb{C}}_X - c}_{\mathbb{R}\text{-c}}(X), \\ \mathcal{DR}^{\mathrm{E}}_X(\mathrm{D}^{\geqslant c}_{\mathrm{hol}}(\mathcal{D}_X)) &\subset {}^{1/2}\mathrm{E}^{\geqslant c}_{\mathbb{R}\text{-c}}(X), & \mathcal{S}ol^{\mathrm{E}}_X(\mathrm{D}^{\geqslant c}_{\mathrm{hol}}(\mathcal{D}_X)) &\subset {}^{1/2}\mathrm{E}^{\leqslant d^{\mathbb{C}}_X - c}_{\mathbb{R}\text{-c}}(X). \end{aligned}$$

In particular, there are commutative diagrams of embeddings

$$\begin{array}{ccc} \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X) & \xrightarrow{\mathcal{DR}^{\mathrm{E}}_X} & {}^{1/2}\mathrm{E}^{0}_{\mathbb{R}\text{-c}}(X) \\ \uparrow & & \uparrow \\ \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X) & \xrightarrow{\mathcal{DR}} & {}^{1/2}\mathrm{D}^{0}_{\mathbb{R}\text{-c}}(\mathbb{C}_X) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} & \xrightarrow{\mathcal{S}ol^{\mathrm{E}}_X} & {}^{1/2}\mathrm{E}^{d^{\mathbb{C}}_X}_{\mathbb{R}\text{-c}}(X) \\ \uparrow & & \uparrow \\ \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X)^{\mathrm{op}} & \xrightarrow{\mathcal{S}ol_X} & {}^{1/2}\mathrm{D}^{d^{\mathbb{C}}_X}_{\mathbb{R}\text{-c}}(\mathbb{C}_X). \end{array}$$

Proof. It is enough to show that for any $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$ one has

$$\mathcal{DR}^{\mathrm{E}}_X(\mathcal{M}) \in {}^{1/2}\mathrm{E}^{0}_{\mathbb{R}\text{-c}}(X), \quad \mathcal{S}ol^{\mathrm{E}}_X(\mathcal{M}) \in {}^{1/2}\mathrm{E}^{d^{\mathbb{C}}_X}_{\mathbb{R}\text{-c}}(X).$$

(i) By Proposition 4.4.1, $\mathcal{O}^{\mathrm{E}}_X \in {}_{1/2}\mathrm{E}^{\geqslant d^{\mathbb{C}}_X}(X)$. Hence

$$\mathcal{S}ol^{\mathrm{E}}_X(\mathcal{M}) = \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}^{\mathrm{E}}_X) \in {}_{1/2}\mathrm{E}^{\geqslant d^{\mathbb{C}}_X}_{\mathbb{R}\text{-c}}(X) \subset {}^{1/2}\mathrm{E}^{\geqslant d^{\mathbb{C}}_X}_{\mathbb{R}\text{-c}}(X),$$

where the inclusions follow from (3.5.1). As $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$, one has

$$\mathcal{DR}^{\mathrm{E}}_X(\mathcal{M}) \simeq \mathcal{S}ol^{\mathrm{E}}_X(\mathbb{D}_X \mathcal{M})[d^{\mathbb{C}}_X] \in {}^{1/2}\mathrm{E}^{\geqslant 0}_{\mathbb{R}\text{-c}}(X).$$

(ii) By [4, Theorem 9.4.8], $\mathrm{D}^{\mathrm{E}}_X \mathcal{DR}^{\mathrm{E}}_X(\mathcal{M}) \simeq \mathcal{DR}^{\mathrm{E}}_X(\mathbb{D}_X \mathcal{M})$. We thus get from (i)

$$\mathcal{DR}^{\mathrm{E}}_X(\mathcal{M}) \in {}^{1/2}\mathrm{E}^{\leqslant 0}_{\mathbb{R}\text{-c}}(X),$$

and hence

$$\mathcal{S}ol^{\mathrm{E}}_X(\mathcal{M}) \simeq \mathcal{DR}^{\mathrm{E}}_X(\mathbb{D}_X \mathcal{M})[-d^{\mathbb{C}}_X] \in {}^{1/2}\mathrm{E}^{\leqslant d^{\mathbb{C}}_X}_{\mathbb{R}\text{-c}}(X). \qquad \square$$

Acknowledgement. The first author acknowledges the kind hospitality at RIMS, Kyoto University, during the preparation of this paper.

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Eingegangen 18. Dezember 2015, in revidierter Fassung 8. Juli 2016