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ASYMPTOTIC CONTROLLABILITY AND LYAPUNOV-LIKE FUNCTIONS DETERMINED BY LIE BRACKETS*

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Abstract. For a given closed target we embed the dissipative relation that defines a control Lyapunov function in a more general differential inequality involving Hamiltonians built from iterated Lie brackets. The solutions of the resulting extended relation, here called *degree-k control Lyapunov functions* ($k \geq 1$), turn out to be still sufficient for the system to be globally asymptotically controllable to the target. Furthermore, we work out some examples where no standard (i.e., degree-1) smooth control Lyapunov functions exist while a C^∞ degree-k control Lyapunov function does exist for some $k > 1$. The extension is performed under very weak regularity assumptions on the system, to the point that, for instance, (set-valued) Lie brackets of locally Lipschitz vector fields are considered as well.

Key words. Lyapunov functions, Lie brackets, global asymptotic controllability, partial differential inequalities

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1. Introduction.

A *control Lyapunov function* (CLF) for a control system,

$$\begin{cases} \dot{y} = f(y,a) \\ y(0) = x \in \mathbb{R}^n \setminus \mathcal{T}, \end{cases} \tag{1}$$

where the control parameter a ranges over a compact set of controls and the (closed) subset $\mathcal{T} \subset \mathbb{R}^n$ is regarded as a *target*, is a positive definite function $U : \overline{\mathbb{R}^n \setminus \mathcal{T}} \to \mathbb{R}$ such that, at each point $x \in \mathbb{R}^n \setminus \mathcal{T}$, the dynamics $f(x,a)$ points in a direction along which U is strictly decreasing for a suitable choice of $a \in A$. A wide literature investigates the links between the existence of a CLF and some properties of the system-target pair. Standard regularity assumptions include local semiconcavity of U in the interior of the domain of U, which, in particular, allows defining the set of limiting gradients[1] $D^*U(x)$ at each $x \in \mathbb{R}^n \setminus \mathcal{T}$. Therefore, the monotonicity of U along suitable directions of f can be expressed by means of the *dissipative* differential inequality

$$H(x, D^*U(x)) < 0 \qquad \forall x \in \mathbb{R}^n \setminus \mathcal{T}, \tag{2}$$

where

$$H(x,p) := \inf_{a \in A} \Big\langle p\,, f(x,a) \Big\rangle.$$

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[1]See Definition 2.3. Under this hypothesis, $D^*U(x)$ coincides with the *limiting subdifferential* $\partial_L U(x)$, largely used in the literature on Lyapunov functions.

Relation (2) has to be interpreted as the occurrence, at each x, of the inequality $H(x,p) < 0$ for every $p \in D^*U(x)$. Since U is assumed to be (proper and) positive definite, by choosing controls verifying (2) one is ideally looking for trajectories that run closer and closer to the target. More precisely, one has the following.

THEOREM 1.1. *If there exists a* CLF, *system* (1) *is* GAC *to* $\mathcal{T}$.

As customary, GAC to $\mathcal{T}$ is an acronym for *globally asymptotically controllable to* $\mathcal{T}$ (see Definition 1.2), which means that for any initial point x there exists a system trajectory $y(\cdot)$, $y(0) = x$, approaching the target $\mathcal{T}$ (in possibly infinite time), uniformly with respect to the distance $\mathbf{d}(x, \mathcal{T})$.

Results like Theorem 1.1—of which some "inverse" versions exist as well—lie at the basis of various constructions dealing, in particular, with stabilizability (see, e.g., [S2], [Ri], and the references therein). Nonsmoothness is crucial for control Lyapunov functions: Though relation (2) is a partial differential inequality—so admitting many more solutions than the corresponding Hamilton–Jacobi equation—in general no smooth control Lyapunov functions exist. A great deal of effective ideas have been flourishing during the last four decades to deal with this unavoidable lack of regularity (see, e.g., [CLSS], [MaRS], the books [CLSW] and [BR], and the references therein). Nevertheless, the regularity issue is of obvious interest from a numerical point of view. In addition, any feedback stabilizing strategy would likely benefit from smoothness properties of a CLF (or of some suitable CLF replacement)—in particular, in reference with sensitivity to data errors.

As an attempt to reduce the unavoidability of nonsmoothness, in the present paper we replace relation (2) with a less demanding inequality which involves Lie brackets.[2] Let us assume that the dynamics is driftless control affine, namely,

$$\begin{cases} \dot{y} = \displaystyle\sum_{i=1,\dots,m} a_i f_i(y) \\ \\ y(0) = x \in \mathbb{R}^n \backslash \mathcal{T}, \end{cases} \tag{3}$$

and let $A := \{\pm e_1, \dots, \pm e_m\}$.[3] Assume the vector fields $f_1, \dots, f_m$ are of class C^{k-1} for some integer $k \geq 1$. We will define[4] the *degree-k Hamiltonian* $H^{(k)}(x,p)$ by setting

$$H^{(k)}(x,p) := \inf_{v \in \mathcal{F}^{(k)}(x)} \langle p, v \rangle \qquad \forall (x,p) \in (\mathbb{R}^n \backslash \mathcal{T}) \times \mathbb{R}^n, \tag{4}$$

where $\mathcal{F}^{(k)}$ denotes the family of iterated Lie brackets of degree $\leq k$ of the vector fields $f_1, \dots, f_m$. (Notice, in particular, that $H^{(1)} = H$.)

A function $U : \overline{\mathbb{R}^n \backslash \mathcal{T}} \to \mathbb{R}$ will be called a *degree-k control Lyapunov function*– shortly, degree-k CLF– if (it is positive definite, proper, semiconcave on domain's interior, and) it verifies inequality

$$H^{(k)}(x, D^*U(x)) < 0 \qquad \forall x \in \mathbb{R}^n \backslash \mathcal{T}. \tag{5}$$

[2]We remind the reader that the *Lie bracket* of two C^1 vector fields X, Y is defined (on any coordinate chart) as $[X,Y] := DY \cdot X - DX \cdot Y$.

[3]More general control systems can be considered; see Remark 2.2 and subsection 5.3.

[4]See section 4 for an extension of the notion of $H^{(2)}$ when the vector fields are locally Lipschitz but not *smooth*.

Observe that, because of

$$H^{(k)} \leq H^{(k-1)} \cdots \leq H^{(1)}, \tag{6}$$

relation (5) is weaker than (2).

Still, in view of Theorem 1.2 below, the inequality (5) is sufficient for the system to be GAC to $\mathcal{T}$, as stated in the following result.

THEOREM 1.2. *Let a degree-k CLF exist for some positive integer k. Then system (3) is GAC to $\mathcal{T}$.*

The use of Lie brackets as *higher-order directions* is widespread in control theory, both within necessary conditions for optimality and within sufficient conditions for various kinds of controllability (see, e.g., [AgSa], [BP], [Co], [K], [S1], [Su], [FHT]). Furthermore, they are involved in boundary conditions ensuring uniqueness for Hamilton–Jacobi equations, e.g., in relation with continuity properties of the corresponding value function (see, e.g., [BCD], [So]). However, in the present paper Lie brackets are directly involved in the proposed differential inequalities.

Let us point out that, in relation with systems having a nonzero drift, a set of results involving Lie-bracket–based Lyapunov-like functions can be found, e.g., in [MTT], [T], [TT]. These works aim directly to the main goal of various investigations on asymptotic controllability, namely, feedback stabilizability. Actually, it will be interesting to bridge some of the general notions and results contained in these papers with the ones presented here. In particular, in section 5 we give some partial results for systems with drifts. Let us point out, however, that the hypotheses considered, e.g., in [T, Proposition 2] and in [TT, Proposition 3] cannot be met by any driftless system.

As for the regularity issue, we wish to remark that a degree-k control Lyapunov function, $k > 1$, may happen to be more regular than a standard (i.e., degree-1) control Lyapunov function. It may even occur in the case where $H^{(k)}(x, D^*U(x)) < 0$ for some C^∞ function U, while no smooth U satisfies the standard inequality $H(x, D^*U(x)) < 0$ (see Examples 2.1–2.3 below). Let us observe that the two reasons why a (degree-1) control Lyapunov function may result not differentiable are (i) the *shape* of the target's boundary $\partial\mathcal{T}$ and (ii) the *shortage* of the dynamics' directions. While there is nothing one can do to remedy (i), the introduction of Hamiltonians $H^{(k)}$ $(k > 1)$, which are minima over larger sets of directions, is a way to reduce the effects of (ii).

The regularity hypotheses in the case of degree-2 control Lyapunov functions are relaxed in section 4 in order to include Lipschitz continuous vector fields. Since the classical brackets $[f_i, f_j]$ may happen to be not even defined at possibly infinitely many points, we make use of the generalized, set-valued brackets defined in [RS1]. Accordingly, the Hamiltonian $H^{(2)}$ is computed as a min-max value. Let us remark that the degree-2 control Lyapunov function of Example 4.1 is C^∞ despite the fact the vector fields are not even C^1.

The paper is organized as follows. Section 2 comprises the definition of degree-k control Lyapunov function, the main result (namely, Theorem 2.1), and a few examples; section 3 is entirely devoted to the proof of Theorem 2.1; by making use of a set-valued notion of Lie bracket, in section 4 we prove a nonsmooth version of the main result for the case $k = 2$; in section 5, after considering feedback constructions and establishing a connection with the theory of viscosity supersolutions, we prove a partial result for the case with drift (Theorem 5.1) as well as a generalization of the main result to unbounded closed targets (Theorem 5.2).

1.1. Preliminaries and notation. For the readers convenience, some classical concepts, like global asymptotic controllability to a set $\mathcal{T}$, in short GAC to $\mathcal{T}$, and a few technical definitions are here recalled.

Given an integer $k \geq 1$ and an open subset $\Omega \subseteq \mathbb{R}^n$, we write $C^k(\Omega)$ to denote the set of vector fields of class C^k on Ω, namely, $C^k(\Omega) := C^k(\Omega, \mathbb{R}^n)$. The subset $C^k_b(\Omega) \subset C^k(\Omega)$ of functions with bounded derivatives (up to the order k) will be endowed with the norm

$$\|f\|_k := \sum_{i=0,\dots,k} \sup_{x\in\Omega} |f^{(i)}(x)| \qquad (f^{(0)} := f)$$

(which makes it a Banach space). Similarly, $C^{k-1,1}(\Omega) \subset C^{k-1}(\Omega)$ denotes the subset of vector fields whose $k-1$th derivative is locally Lipschitz continuous and $C^{k-1,1}_b(\Omega)$ is the subset of $C^{k-1}_b(\Omega)$ with (globally) Lipschitz continuous $k-1$th derivative.

Definition 1.1. *Let $k \geq 1$ be an integer, and let $f_1, \dots, f_m$ be vector fields belonging to $C^{k-1}(\mathbb{R}^n \setminus \mathcal{T})$. For any initial condition $x \in \mathbb{R}^n \setminus \mathcal{T}$ and any measurable control $\alpha : [0, +\infty) \to A$, a trajectory-control pair $(y, \alpha)(\cdot)$ will be called* admissible *if there exists $T \leq +\infty$ such that $y(\cdot)$ is a solution of* (3) *defined on* $[0, T)$ *and*

$$\lim_{t\to T} \mathbf{d}(y(t)) = 0,$$

where $\mathbf{d}(\cdot) := \mathbf{d}(\cdot, \mathcal{T})$. When $k > 1$, we will use $y_x(\cdot, \alpha)$ to denote the unique (possibly local) forward solution to the Cauchy problem (3).

Remark 1.1. The main object of the paper consists in establishing relations involving Lie brackets, so that a certain regularity is necessary when $k > 1$ (see also section 4). However, observe that as soon as $k = 1$, the vector fields $f_1, \dots, f_m$ are just continuous, so that solutions of the Cauchy problem (3) for a given control may be not unique.

To give the notion of global asymptotic controllability, we recall that $\mathcal{K}L$ is used to denote the set of continuous functions $\beta : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ such that (1) $\beta(0, s) = 0$ and $\beta(\cdot, s)$ is strictly increasing and unbounded for each $s \geq 0$, (2) $\beta(\delta, \cdot)$ is decreasing for each $\delta \geq 0$, and (3) $\beta(\delta, s) \to 0$ as $s \to +\infty$ for each $\delta \geq 0$.

Definition 1.2. *The control system in* (3) *is* globally asymptotically controllable *to $\mathcal{T}$—shortly,* (3) *is* GAC *to $\mathcal{T}$—provided there is a function $\beta \in \mathcal{K}L$ such that for each initial state $x \in \mathbb{R}^n \setminus \mathcal{T}$, there exists an admissible trajectory-control pair $(y, \alpha)(\cdot)$ such that*

$$\mathbf{d}(y(t)) \leq \beta\big(\mathbf{d}(x), t\big) \qquad \forall t \in [0, +\infty).^5 \tag{7}$$

Let us recall that if g_1, g_2 are C^1 vector fields on a differential manifold (of class C^2), their *Lie bracket* $[g_1, g_2]$ is the (continuous) vector field which is defined (on coordinate charts) by

$$[g_1, g_2] = Dg_2 \cdot g_1 - Dg_2 \cdot g_1.$$

Since $[g_1, g_2]$ turns out to be a vector field, provided sufficient regularity is assumed, one can iterate the bracketing process, so obtaining *iterated Lie brackets*. We call

[5] By convention, we fix an arbitrary $\bar{x} \in \partial\mathcal{T}$ and formally establish that, if $T < +\infty$, the trajectory $y(\cdot)$ is prolonged to $[0, +\infty)$ by setting $y(t) = \bar{x}$ for all $t \geq T$.

degree of a given iterated bracket B the number of objects appearing in B (regarded as a formal object) when commas and left and right brackets are deleted. For instance, the degrees of $[[g_2, g_3], g_2]$, $[[g_2, g_3], [g_2, g_4]]$, and $[g_4, [g_4, [g_4, [g_4, g_6]]]]$ are $3, 4$, and 5, respectively.

Let us summarize some basic notions in nonsmooth analysis (see, e.g., [CS], [CLSW] for a thorough treatment).

DEFINITION 1.3 (positive definite and proper functions). *A continuous function* $F : \overline{\mathbb{R}^n \setminus \mathcal{T}} \to \mathbb{R}$ *is said to be* positive definite on $\mathbb{R}^n \setminus \mathcal{T}$ *if* $F(x) > 0\ \forall x \in \mathbb{R}^n \setminus \mathcal{T}$ *and* $F(x) = 0\ \forall x \in \partial\mathcal{T}$. *The function* F *is called* proper on $\mathbb{R}^n \setminus \mathcal{T}$ *if the preimage* $F^{-1}(K)$ *of any compact set* $K \subset [0, +\infty[$ *is compact.*

DEFINITION 1.4. (semiconcavity). *Let* $\Omega \subset \mathbb{R}^n$. *A continuous function* $F : \Omega \to \mathbb{R}$ *is said to be* semiconcave on Ω *if there exist* $\rho > 0$ *such that*

$$F(z_1) + F(z_2) - 2F\left(\frac{z_1 + z_2}{2}\right) \le \rho|z_1 - z_2|^2$$

for all z_1, $z_2 \in \Omega$ *such that* $[z_1, z_2] \subset \Omega$. *The constant* ρ *above is called a* semiconcavity constant *for* F *in* Ω. F *is said to be* locally semiconcave on Ω *if it is semiconcave on every compact subset of* Ω.

Let us remind the reader that locally semiconcave functions are locally Lipschitz. Actually, they are twice differentiable almost everywhere.

DEFINITION 1.5. (limiting gradient). *Let* $\Omega \subset \mathbb{R}^n$ *be an open set, and let* $F : \Omega \to \mathbb{R}$ *be a locally Lipschitz function. For every* $x \in \Omega$ *let us set*

$$D^*F(x) \doteq \Big\{ w \in \mathbb{R}^n : \quad w = \lim_k \nabla F(x_k), \quad x_k \in DIFF(F) \setminus \{x\}, \quad \lim_k x_k = x \Big\},$$

where ∇ *denotes the classical gradient operator and* $DIFF(F)$ *is the set of differentiability points of* F. $D^*F(x)$ *is called the* set of limiting gradients *of* F *at* x.

The set-valued map $x \mapsto D^*F(x)$ is upper semicontinuous on Θ, with nonempty, compact values. Notice that $D^*F(x)$ is not convex. When F is a locally semiconcave function, D^*F coincides with the limiting subdifferential $\partial_L F$, namely,

$$D^*F(x) = \partial_L F(x) := \{\lim p_i : \ p_i \in \partial_P F(x_i), \ \lim x_i = x\} \quad \forall x \in \Theta,$$

where $\partial_P F$ denotes the proximal subdifferential, largely used in the literature on Lyapunov functions.

2. Degree-k control Lyapunov functions.

2.1. The main result. Let $k \ge 1$ be an integer. Throughout the whole paper we assume that the target $\mathcal{T} \subset \mathbb{R}^n$ is a closed set with compact boundary and that $f_1, \dots, f_m$ are vector fields belonging to $C_b^{k-1}(\Omega \setminus \mathcal{T})$ for any open, bounded subset $\Omega \subset \mathbb{R}^n$ (see subsection 1.1).

DEFINITION 2.1. *Let us consider the family of vector fields*

$$\mathcal{F}^{(1)} := \left\{ f = \sum_{i=1}^m a_i f_i, \quad a \in A \right\} = \big\{ \pm f_i, \quad i = 1, \dots, m \big\}.$$

Moreover, if $k > 1$ *for every positive integer* h *such that* $2 \le h \le k$, *set*

$$\mathcal{F}^{(h)} := \big\{ B, \quad B \textit{ is a iterated Lie bracket of degree} \le h \quad \textit{of } f_1, \dots, f_m \big\}.$$

Clearly, every element of $\mathcal{F}^{(h)}$ is a vector field belonging to $C_b^{k-h}(\Omega \setminus \mathcal{T})$ for any open, bounded subset $\Omega \subset \mathbb{R}^n$. Notice that

$$\mathcal{F}^{(1)} \subseteq \mathcal{F}^{(2)} \subseteq \cdots \subseteq \mathcal{F}^{(k)}. \tag{8}$$

For every $h = 1, \ldots, k$, let us introduce the set-valued map

$$\mathcal{F}^{(h)}(x) := \Big\{ X(x), \quad X \in \mathcal{F}^{(h)} \Big\} \qquad \forall x \in \mathbb{R}^n \setminus \mathcal{T}.$$

Definition 2.2. *For any integer* $1 \leq h \leq k$*, let us define the* degree-h Hamiltonian $H^{(h)}$ *corresponding to the control system* (3) *by setting*

$$H^{(h)}(x,p) := \inf_{v \in \mathcal{F}^{(h)}(x)} \langle p,\, v \rangle \qquad \forall (x,p) \in (\mathbb{R}^n \setminus \mathcal{T}) \times \mathbb{R}^n.$$

Under the above hypotheses the Hamiltonians $H^{(h)}$ are well defined and continuous. As already mentioned in the Introduction, the degree-1 Hamiltonian $H^{(1)}$ coincides with the standard Hamiltonian:

$$H^{(1)}(x,p) = H(x,p) := \inf_{a \in A} \Big\langle p,\, \sum_{i=1}^{m} a_i f_i(x) \Big\rangle.$$

Morever, by (8) one gets

$$H^{(k)} \leq H^{(k-1)} \leq \cdots \leq H^{(1)}. \tag{9}$$

Definition 2.3. *We call* degree-k control Lyapunov function*—in short,* degree-k CLF*—any continuous function* $U : \overline{\mathbb{R}^n \setminus \mathcal{T}} \to \mathbb{R}$ *such that the restriction to* $\mathbb{R}^n \setminus \mathcal{T}$ *is locally semiconcave, positive definite, and proper and verifies*

$$H^{(k)}(x, D^*U(x)) < 0 \quad \forall x \in \mathbb{R}^n \setminus \mathcal{T}, \tag{10}$$

the latter inequality meaning $H^{(k)}(x,p) < 0$ *for each* $p \in D^*U(x)$*.*

In Theorem 2.1 below we prove that the existence of a degree-k control Lyapunov function, $k > 1$, is sufficient for the system to be globally asymptotically controllable to $\mathcal{T}$ (GAC to $\mathcal{T}$; see Definition 1.2), as in the classical case $k = 1$.

Theorem 2.1. *Let us assume that for some integer* $k \geq 1$*, a degree-*k *control Lyapunov function exists. Then system* (3) *is GAC to* $\mathcal{T}$*.*

We postpone the proof of Theorem 2.1 to the next section and make some general remarks. Furthermore, we give some examples where, in particular, the distance function is a (possibly smooth) degree-k CLF for some $k > 1$ and *is not* a degree-1 CLF.

2.2. Remarks and examples.

Remark 2.1. The regularity assumptions can be slightly weakened in some cases by observing that, in order that certain degree-k brackets ($k > 3$) are defined, it is not necessary that the vector fields are $k-1$ times differentiable. For instance, the bracket $[[f_1, f_2], [f_3, f_4]]$ is well defined as soon as the vector fields f_1, f_2, f_3, f_4 are two times differentiable.

Remark 2.2. By suitably rescaling time, one can easily generalize Theorem 2.1 to the case when the control set A contains a ball of $\mathbb{R}^m$ with positive radius. By means of linear algebraic and relaxation arguments one can also try to extend the result up to the point of admitting sets A such that 0 is contained in the interior of the convex hull $co(A)$.

Remark 2.3. It is easy to adapt Theorem 2.1 to the case when the state space is an open set $\Omega \subset \mathbb{R}^n$, $\Omega \supset \mathcal{T}$. In fact, the thesis keeps unchanged as soon as one requires the degree-k CLF $U : \Omega \setminus \overset{\circ}{\mathcal{T}} \to \mathbb{R}$ to verify all the assumptions in Definition 2.3 in Ω, plus the following one:

$$\exists U_0 \in (0, +\infty] : \quad \lim_{x \to x_0,\ x \in \Omega} U(x) = U_0 \quad \forall x_0 \in \partial\Omega; \quad U(x) < U_0 \quad \forall x \in \Omega \setminus \overset{\circ}{\mathcal{T}}.$$

Remark 2.4. While the fact that U is a degree-k control Lyapunov function implies that U is also a degree-$\bar{k}$ control Lyapunov function for every $\bar{k} > k$, the converse is in general false (see Example 2.1). On the other hand, coupling Theorem 2.1 with an *inverse Lyapunov* result like in [S2], [Ri], it is easy to verify that the existence of a degree-k control Lyapunov function, $k > 1$, implies the existence of a standard (i.e., degree-1) control Lyapunov function.

Remark 2.5. As in the case of standard (i.e., degree-1) CLF, the notion of degree-k CLF is intrinsic, for vector fields, their Lie brackets, and the set of limiting gradients D^*U are chart independent. In particular the results in Theorems 2.1 and 4.1 are fit to be extended to Riemannian manifolds (where, of course, the notion of distance should coincide with the considered Riemannian metric). Incidentally, let us notice that we can define the Hamiltonians $H^{(k)}$ in terms of *Poisson brackets*.[6] Indeed, setting for every vector field X

$$H_X(x,p) := \langle p, X(x) \rangle,$$

one has

$$H^{(1)}(x,p) = \inf \Big\{ \pm \langle p, f_i \rangle,\ i = 1, \dots, m \Big\} = \inf \Big\{ -|H_{f_i}(x,p)|,\ i = 1, \dots, m \Big\}.$$

Moreover by utilizing the well-known identity

$$\{H_X, H_Y\} = H_{[X,Y]},$$

we get

$$H^{(2)}(x,p) = \inf \Big\{ H^{(1)}(x,p), -|\{H_{f_i}, H_{f_j}\}(x,p)|,\ i,j = 1, \dots, m \Big\}.$$

Similarly, we obtain

$$H^{(3)}(x,p) = \inf \Big\{ H^{(2)}(x,p), -|\{H_{f_i}, \{H_{f_j}, H_{f_\ell}\}\}(x,p)|,\ i,j,\ell = 1, \dots, m \Big\}$$

and so on for higher degrees.

Example 2.1. Consider the so-called *nonholonomic integrator*

$$\dot{y} = a_1\, f_1(y) + a_2\, f_2(y)\,,$$

where $f_1 := \dfrac{\partial}{\partial x_1} - x_2 \dfrac{\partial}{\partial x_3}$, $f_2 := \dfrac{\partial}{\partial x_2} + x_1 \dfrac{\partial}{\partial x_3}$.

[6] If $H(x,p)$ and $K(x,p)$ are differentiable functions, the *Poisson bracket* $\{H, K\}$ is defined by

$$\{H, K\}(x,p) := \sum_{i=1}^{k} \left(\frac{\partial H}{\partial x_i} \frac{\partial K}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial K}{\partial x_i} \right)(x,p).$$

By $[f_1, f_2] = 2\dfrac{\partial}{\partial x_3}$ we get

$$H^{(1)}(x,p) = -\max\{|p_1 - p_3x_2|\,,\,|p_2 + p_3x_1|\}$$

and

$$H^{(2)}(x,p) = -\max\{|p_1 - p_3x_2|\,,\,|p_2 + p_3x_1|\,,\,2|p_3|\}\,.$$

Let $\mathcal{T}$ be a compact target, and let $U(\cdot)$ coincide with the distance $\mathbf{d}(\cdot)$ from $\mathcal{T}$. If there exists a point $\bar{x} \in (\{0\}\times\{0\}\times\mathbb{R})\cap(\mathbb{R}^3\backslash\mathcal{T})$ such that $D^*(U)(x)\cap(\{0\}\times\{0\}\times\mathbb{R}) \neq \emptyset$, then $H^{(1)}(\bar{x}, D^*U(\bar{x})) = 0$. Therefore, in this case the distance function U fails to be a degree-1 CLF. For instance, this is the case when $\mathcal{T} = \{x,\,|\,|x| \le \rho\}$ for some $\rho \ge 0$. Indeed,

$$H^{(1)}((0,0,x_3), D^*U((0,0,x_3))) = 0$$

for all $|x_3| > \rho$. In fact, when $\rho = 0$, no degree-1 CLF of class C^1 exist (see [BR], [Ri]).

Yet, U *is a* degree-2 CLF, for whichever compact target $\mathcal{T}$. Indeed, for every $x \in \mathbb{R}^3\backslash\mathcal{T}$, one has $|p| = 1$ for all $p \in D^*U(x)$, which implies

$$H^{(2)}(x, D^*U(x)) \le \max_{|p|=1}\Big\{-\max\{|p_1 - p_3x_2|\,,\,|p_2 + p_3x_1|\,,\,2|p_3|\}\Big\} < 0$$

for all $x \in \mathbb{R}^3\backslash\mathcal{T}$. In the case when $\mathcal{T} = \{x,\,|\,|x| \le \rho\}$ for some $\rho \ge 0$, U is a C^1—actually, C^∞—degree-2 CLF (hence, $D^*U(x) = \{\nabla U(x)\}$). Furthermore, one has

$$H^{(2)}(x, D^*U(x)) \le -\frac{2}{3} \qquad \forall x \in \mathbb{R}^3\backslash\mathcal{T}. \tag{11}$$

Example 2.2. Now let us consider the system

$$\dot{y} = a_1\, f_1(y) + a_2\, f_2(y),$$

where $f_1 := \dfrac{\partial}{\partial x_2} + x_2^2\dfrac{\partial}{\partial x_3}$, $f_2 := \dfrac{\partial}{\partial x_2} + x_1^2\dfrac{\partial}{\partial x_3}$.
Let us compute the brackets of degree less than or equal to 3:

$$[f_1, f_2](x) = 2(x_1 - x_2)\frac{\partial}{\partial x_3}, \quad \big[f_1, [f_1, f_2]\big](x) = -\big[f_2, [f_1, f_2]\big](x) = 2\frac{\partial}{\partial x_3}.$$

Therefore,

$$H^{(1)}(x,p) = -\max\left\{|p_1 + p_3x_2^2|\,,\,|p_2 + p_3x_1^2|\right\},$$

$$H^{(2)}(x,p) = -\max\left\{|p_1 + p_3x_2^2|\,,\,|p_2 + p_3x_1^2|\,,\,2|p_3(x_1 - x_2)|\right\},$$

and

$$H^{(3)}(x,p) = -\max\left\{|p_1 + p_3x_2^2|\,,\,|p_2 + p_3x_1^2|\,,\,2|p_3(x_1 - x_2)|\,,\,2|p_3|\right\}.$$

For simplicity let us consider only the target $\mathcal{T} = \{0\}$. Once again, the distance function $U(x) := |x|$ is not a degree-1 CLF since $H^{(1)}(x, D^*U(x)) = 0$ for all $x \in \{0\}\times\{0\}\times(\mathbb{R}\setminus\{0\})$. U is not even a degree-2 CLF for $H^{(2)}(x, D^*U(x)) = 0$ for all $x \in \{0\}\times\{0\}\times(\mathbb{R}\setminus\{0\})$. However, $|\nabla U(x)| = 1$ (and $D^*U(x) = \{\nabla U(x)\}$) so that

$$H^{(3)}(x, D^*U(x)) \le \max_{|p|=1} H^{(3)}(x,p) < 0,$$

and the distance U is a (C^∞) degree-3 CLF.

Remark 2.6. The control systems in Examples 2.1 and 2.2 verify a Lie algebra rank condition at each point.[7] Hence, by the Chow–Rashevsky theorem, they are small time locally controllable at every point x; that is, the interior of the reachable set from x at any time contains x. Actually, with similar arguments it is not difficult to prove the following general fact.

Let $\mathcal{T} \subset \mathbb{R}^n$ be any target with compact boundary. If a system verifies the Lie algebra rank condition at every point by means of brackets of degree $\leq k$, the distance function $\mathbf{d}(\cdot)$ from $\mathcal{T}$ is a degree-k CLF.

Indeed, since $|p| = 1$ for every $p \in D^*\mathbf{d}(x)$ and every $x \in \mathbb{R}^n \backslash \mathcal{T}$, the Lie algebra rank condition implies that for every such x and p there must exist $w \in \mathcal{F}^k(x)$ such that $\langle p, w \rangle < 0$.

While in the previous examples the minimum time function is finite at each point, this is not the case for the following example, where no trajectories issuing from points (x_1, x_2, x_3) such that $x_3 \neq 0$ can reach the target (in finite time). Notice incidentally that the Lie algebra rank condition is violated at each point belonging to the plane $x_3 = 0$.

Example 2.3. Consider the system

$$\dot{y} = a_1\, f_1(y) + a_2\, f_2(y)\,,$$

where

$$f_1 := \frac{\partial}{\partial x_1} - x_2\phi(x_3)\frac{\partial}{\partial x_3} \quad f_2 := \frac{\partial}{\partial x_2} + x_1\phi(x_3)\frac{\partial}{\partial x_3},$$

$\phi : \mathbb{R} \to [0, +\infty[$ being a C^1 function such that $\phi(x_3) = 0$ if and only if $x_3 = 0$. By $[f_1, f_2](x) = 2\phi(x_3)\frac{\partial}{\partial x_3}$ one gets

$$H^{(1)}(x,p) = -\max\left\{|p_1 - p_3x_2\phi(x_3)|\,,\, |p_2 + p_3x_1\phi(x_3)|\right\}$$

and

$$H^{(2)}(x,p) = -\max\left\{|p_1 - p_3x_2\phi(x_3)|\,,\, |p_2 + p_3x_1\phi(x_3)|\,,\, 2\phi(x_3)|p_3|\right\}.$$

Let us consider again the target $\mathcal{T} = \{0\}$. Also in this case the distance function $U(x) := |x|$ is not a degree-1 CLF since

$$H^{(1)}(x, D^*U(x)) = 0 \quad \forall x \in \{0\} \times \{0\} \times (\mathbb{R} \setminus \{0\}),$$

and U is a still degree-2 CLF since

$$H^{(2)}(x, D^*U(x)) < 0$$

for all $x \in \{0\} \times \{0\} \times (\mathbb{R} \setminus \{0\})$.

Of course the fact that the Lie algebra rank condition is verified almost everywhere—as in the previous examples—is far from being necessary for a CLF of whatever degree to exist. In fact, a system that fails to be small time locally controllable on large areas of its domain might not have any C^1 degree-1 CLF while admitting a smooth degree-k CLF for some $k > 1$, as illustrated in the following example.

[7] A system verifies the Lie algebra rank condition at x if the iterated Lie brackets linearly span $\mathbb{R}^n$.

Example 2.4. Let $\varphi,\,\psi : [0,+\infty) \to [0,1]$ be C^∞ maps such that for any $q \in \mathbb{N}$,

$$\begin{aligned}\varphi(r) &= 1 \quad \text{if } r \in [2q, 2q+1],\\ \varphi(r) &= 0 \quad \text{if } r \in [2q+(5/4), 2q+(7/4)];\end{aligned}$$

$$\begin{aligned}\psi(r) &= 1 \quad \text{if } r \in [2q+(7/8), 2q+(17/8)],\\ \psi(r) &= 0 \quad \text{if } r \in [2q+(1/4), 2q+(3/4)] \cup [0,(1/4)].\end{aligned}$$

Let us consider the control system

$$\dot{y} = a_1\, f_1(y) + a_2\, f_2(y) + a_3\, f_3(y)\,,$$

where

$$f_1 = \varphi(|x|)\left(\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_3}\right), \quad f_2 = \varphi(|x|)\left(\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3}\right),$$

$$f_3 = \psi(|x|)\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}\right).$$

Clearly the system is not small time locally controllable at every point x such that $2q+(5/4) \le |x| \le 2q+(7/4)$. Let the target $\mathcal{T}$ coincide with the origin $\{0\}$, and, again, let us set $U(x) := \mathbf{d}(x) = |x|$. For every $q \in \mathbb{N}$ one has

$$H^{(1)}(x, D^*U(x)) = -|x| \quad \text{for all } x \text{ such that } 2q+(5/4) \le |x| \le 2q+(7/4).$$

Furthermore, $H^{(1)}(x, D^*U(x)) = 0$ for every x such that $x_1 = x_2 = 0$ and $|x_3| \le 1$ or $2q+(1/4) \le |x_3| \le 2q+(3/4)$, $q \ge 1$.[8] However, one easily checks that

$$H^{(2)}(x, D^*U(x)) = -\frac{1}{|x|}\max\{|x_1 - x_3x_2|\,,\, |x_2 + x_3x_1|\,,\, 2|x_3|\} \le -\frac{2}{3}$$

for all $x \in \mathbb{R}^3 \setminus \mathcal{T}$ so that U is a (C^∞) degree-2 Lyapunov function.

3. Proof of Theorem 2.1.

The case when $k=1$ has already been proved in [MR], where the hypotheses are even weaker than the ones assumed here (for instance, vector fields are allowed to be unbounded near the target). So we will always assume $k > 1$: In particular, there will be a unique trajectory $y_x(\cdot,\alpha)$ corresponding to an initial condition x and a control $\alpha(\cdot)$.

3.1. Preliminary facts.

To begin with, let us point out that the 0 in the dissipative relation can be replaced by a nonnegative function of U.

PROPOSITION 3.1. *Let $U : \overline{\mathbb{R}^n \setminus \mathcal{T}} \to \mathbb{R}$ be a continuous function such that U is locally semiconcave, positive definite, and proper on $\mathbb{R}^n \setminus \mathcal{T}$. Then the two conditions below are equivalent:*

(i) *U verifies*

$$H^{(k)}(x, D^*U(x)) < 0 \tag{12}$$

for all $x \in \mathbb{R}^n \setminus \mathcal{T}$;

[8] Actually, there are no C^1 degree-1 CLF, as it can be proved by noticing that the system coincides with the nonholonomic integrator in a whole neighborhood of the target.

(ii) *for every* $\sigma > 0$ *there exists a continuous, strictly increasing function* $\gamma : [0,+\infty) \to: [0,+\infty)$ *such that*

$$H^{(k)}(x, D^*U(x)) \leq -\gamma(U(x)) \tag{13}$$

for all $x \in U^{-1}((0, 2\sigma])$.

Notice that the only nontrivial implication, namely, (i) $\Longrightarrow$ (ii), is a simple consequence of the upper semicontinuity of the set-valued map $x \mapsto D^*U(x)$ on the compact sets $U^{-1}([u, 2\sigma])$ ($u \in (0, 2\sigma)$) and of the upper semicontinuity of $H^{(k)}$. For a detailed proof, we refer the reader to [MR, Proposition 3.1].

Remark 3.1. In a good deal of literature on control Lyapunov functions, one utilizes the proximal subdifferential $\partial_P U(x)$ as a nonsmooth substitute for the derivative of U (see [CLSW]). However, the use of the set of limiting gradients $D^*U(x)$ is equivalent to the use of the proximal subdifferential. Indeed, for any locally semiconcave function U, $D^*U(x)$ coincides with the limiting subdifferential $\partial_L U(x)$ defined as

$$\partial_L U(x) := \{\lim p_i : \ p_i \in \partial_P U(x_i), \ \lim x_i = x\} \quad \text{for any } x \in \mathbb{R}^n \setminus \mathcal{T} \tag{14}$$

(see subsection 1.1), so the (equivalent) assertions (i), (ii) of Proposition 3.1 hold true when $\partial_L U(x)$ replaces $D^*U(x)$. Since $\partial_P U(x) \subset \partial_L U(x)$, (ii) implies that

(ii)$'$ *for* σ *and* γ *as above,*

$$H^{(k)}(x, \partial_P U(x)) \leq -\gamma(U(x)) \quad \textit{for all } x \in U^{-1}((0, 2\sigma]). \tag{15}$$

In fact, by (14) and the continuity of $H^{(k)}(\cdot)$ we get that the equivalence

$$\text{(i)} \Longleftrightarrow \text{(ii)} \Longleftrightarrow \text{(ii)}'.$$

Second, basic properties of the semiconcave functions imply the following fact (see, e.g., [CS]).

LEMMA 3.1. *Let* $U : \overline{\mathbb{R}^n \setminus \mathcal{T}} \to \mathbb{R}$ *be a continuous function such that* U *is locally semiconcave, positive definite, and proper on* $\mathbb{R}^n \setminus \mathcal{T}$. *Then for any compact set* $\mathcal{K} \subset \mathbb{R}^n \setminus \mathcal{T}$ *there exist some positive constants* L *and* ρ *such that for any* $x \in \mathcal{K}$,[9]

$$\begin{aligned} &|p| \leq L \quad \forall p \in D^*U(x), \\ &U(\hat{x}) - U(x) \leq \langle p, \hat{x} - x\rangle + \rho|\hat{x} - x|^2 \end{aligned} \tag{16}$$

for any point $\hat{x} \in \mathcal{K}$ *such that* $[x, \hat{x}] \subset \mathcal{K}$.

Moreover, if $\mathcal{K}_1, \mathcal{K}_2 \subset \mathbb{R}^n \setminus \mathcal{T}$ *are compact subsets and* $\mathcal{K}_1 \subseteq \mathcal{K}_2$, *we can choose the corresponding constants* L_1, ρ_1 *and* L_2, ρ_2 *such that* $L_1 \leq L_2$ *and* $\rho_1 \leq \rho_2$.

3.2. A degree-k "feedback". Let U be a degree-k control Lyapunov function. Correspondingly, we are introducing a notion of degree-k feedback. For a given $\sigma > 0$, let γ be a function as in Proposition 3.1, and let $x \mapsto p(x)$ be a selection of $x \mapsto D^*U(x)$ on $U^{-1}((0, 2\sigma])$, so that

$$H^{(k)}(x, p(x)) \leq -\gamma(U(x)) \qquad \forall x \in U^{-1}((0, 2\sigma]).$$

[9] The inequality (16) is usually formulated with the proximal superdifferential $\partial^P F$. However, this does not make a difference here since $\partial^P F = \partial_C F = coD^*F$ as soon as F is locally semiconcave. Hence, (16) is true in particular for D^*U.

DEFINITION 3.1. *For a given $\sigma > 0$, let $\gamma(\cdot)$ and $p(\cdot)$ be chosen as above. A selection*

$$\mathbf{v} : U^{-1}((0, 2\sigma]) \to \mathbb{R}^n, \qquad x \mapsto \mathbf{v}(x) \in \mathcal{F}^{(k)}(x)$$

is called a degree-k feedback *(corresponding to U, σ, $\gamma(\cdot)$, and $p(\cdot)$) if for every $x \in U^{-1}((0, 2\sigma])$ there exists a positive integer $h \leq k$ such that*

$$\begin{cases} \mathbf{v}(x) \in \mathcal{F}^{(h)}(x), \\[2ex] \Big\langle p(x),\, \mathbf{v}(x) \Big\rangle \leq -\gamma(U(x)), \\[2ex] \text{and if } h > 1\text{:} \quad H^{(h-1)}(x, p(x)) > -\gamma(U(x)). \end{cases} \tag{17}$$

The number h will be called the degree of the feedback $\mathbf{v}$ at x*.*

Let us momentarily assume that there exists $\hat{M} \geq 0$ such that

$$\|f_i\|_{k-1} \leq \hat{M} \qquad \forall i = 1, \dots, m \tag{18}$$

in the whole set $\mathbb{R}^n \setminus \mathcal{T}$, which, in view of the compactness of the control set A, implies that there is $M \geq 0$ verifying

$$\|X\|_0 \leq M \tag{19}$$

for any iterated bracket X in $\mathcal{F}^{(k)}$. Under this assumption one can regard each vector $\mathbf{v}(x)$ as a *tangent* vector to a curve that is a suitable composition of flows, as stated in the following result (see, e.g., [FR1], [FR2]).

LEMMA 3.2. *Under assumption* (18) *there exists a real constant $c > 0$ such that for any $x \in \mathbb{R}^n \setminus \mathcal{T}$, any feedback $\mathbf{v}(\cdot)$ of degree h at x, and any $t > 0$, one can find a control $\alpha_t : [0, t] \to A$ such that*

(i) *$\alpha_t(\cdot)$ is constant on intervals $\left[\dfrac{jt}{r}, \dfrac{(j+1)t}{r}\right)$, $j = 0, \dots, r-1$;*

(ii) *the estimate*

$$\left| y_x(t, \alpha_t) - x - \frac{\mathbf{v}(x)}{r^h}\, t^h \right| \leq \frac{c}{r^h}\, t^{h+1} \tag{20}$$

holds true, where r is an integer depending on the formal Lie bracket corresponding to $\mathbf{v}(x)$ and is increasing with the degree.

For instance, $r = 1, 4, 10$ if $h = 1, 2, 3$, respectively. In particular, if $\mathbf{v}(x) = [[f_1, f_2], f_3](x)$ one sets

$$\alpha_t(s) := \begin{cases} e_1 & \text{if } s \in [0, t/10) \cup [6t/10, 7t/10) \\ e_2 & \text{if } s \in [t/10, 2t/10) \cup [5t/10, 6t/10) \\ e_3 & \text{if } s \in [4t/10, 5t/10) \\ -e_1 & \text{if } s \in [2t/10, 3t/10) \cup [8t/10, 9t/10) \\ -e_2 & \text{if } s \in [3t/10, 4t/10) \cup [7t/10, 8t/10) \\ -e_3 & \text{if } s \in [9t/10, t). \end{cases}$$

Let us point out that one can have different r's for feedbacks having the same degree.[10]

[10]Precisely, for each formal bracket B, the corresponding $r = r(B)$ is defined recursively: One sets $r(B) = 1$ if B has degree 1, while if $B = [B_1, B_2]$ and $r_1 = r(B_1)$, $r_2 = r(B_2)$, one sets $r(B) := 2(r_1 + r_2)$. For instance, $r([g_1, g_2]) = 4$, $r([g_1, [g_2, g_3]]) = 10$, $r([g_1, [g_2, [g_3, g_4]]]) = 22$, and $r([[g_1, g_2], [g_3, g_4]] = 16$.

3.3. A step of degree $h \leq k$. Now let us choose $z \in U^{-1}((0,\sigma])$ and a feedback $\mathbf{v}$ of degree k (surely existing by (10)). Let the feedback $\mathbf{v}$ have degree h at z. We shall rely on the following result.

Claim 3.1. *Let us consider a* degree-k CLF U*, and let* σ*,* $\gamma(\cdot)$ *and* $p(\cdot)$ *be chosen as above. Furthermore, let* $\mathbf{v}(\cdot)$ *be a degree-k feedback corresponding to these data. Then there exists a time-valued function*

$$\tau : (0,\sigma] \times \{1,\dots,k\} \to (0,1]$$

such that

(i) $j \mapsto \tau(u,j)^j$ *and* $j \mapsto \tau(u,j)^{j-1}$ *are decreasing for every* $u \in]0,\sigma]$*;*

(ii) $u \mapsto \tau(u,j)$ *is increasing for every* $j \in \{1,\dots,k\}$*;*

(iii) *for all* $z \in U^{-1}((0,\sigma])$ *with a feedback* $\mathbf{v}(\cdot)$ *of degree* h *at* z*, one has*

$$U(y_z(t,\alpha_t)) - U(z) \leq -\frac{\gamma(U(z))}{2}\left(\frac{t}{r}\right)^h \qquad \forall t \in [0,\, \tau(U(z),h)]\ ,$$

where r *and* $y_z(\cdot,\alpha_t)$ *are an integer and a trajectory associated to* $\mathbf{v}(z)$ *as in Lemma* 3.2.

Proof. Let $\nu > 0$ be such that $U^{-1}((0,2\sigma]) \subset B\left(\mathcal{T}, \frac{\nu}{2}\right)$, and fix $z \in U^{-1}((0,\sigma])$ with a feedback $\mathbf{v}(\cdot)$ of degree h at z. To begin with we wish to choose a time $\bar\tau$ such that for any $t \in [0,\bar\tau]$,

(i) $y(t) \in B(\mathcal{T},\nu)$ for any system's trajectory issuing from a point of $U^{-1}((0,\sigma])$;

(ii) $y_z^t(t) \in B\left(z, \frac{\mathbf{d}(z)}{2}\right)$ for any trajectory $y_z^t(\cdot) := y_z(\cdot,\alpha_t)$ associated to $\mathbf{v}(z)$ as in Lemma 3.2.

For this purpose, it is clearly sufficient to set

$$\bar\tau(u,j) := \min\left\{\frac{\nu}{2M}, \sqrt[j]{\frac{\mathbf{d}(U^{-1}(u))}{2M}}\right\} \qquad \forall (u,j) \in (0,\sigma] \times \{1,\dots,k\}$$

and to choose

$$\bar\tau := \bar\tau(U(z),h).$$

Because of $\mathbf{d}(U^{-1}(U(z))) \leq \mathbf{d}(z)$ and (20), the distance $\mathbf{d}([z,y_z^t(t)])$ between the segment $[z,y_z^t(t)]$ and the target $\mathcal{T}$ verifies

$$\mathbf{d}([z,y_z^t(t)]) \geq \frac{\mathbf{d}(z)}{2} \geq \frac{\mathbf{d}(U^{-1}(U(z)))}{2}$$

for every $t \in [0,\bar\tau]$. For every $u \in (0,\sigma]$, in relation with the compact set

$$\mathcal{K}(u) := \left\{x : \ \frac{\mathbf{d}(U^{-1}(u))}{2} \leq \mathbf{d}(x) \leq \nu\right\},$$

let $L(u)$ and $\rho(u)$ be a Lipschitz continuity and a semiconcavity constant, whose existence is stated in Lemma 3.1. Let us set $L := L(U(z))$ and $\rho := \rho(U(z))$. By (20), for any $t \in [0, \bar{\tau}]$, we get

$$
\begin{aligned}
&U(y_z^t(t)) - U(z) \le \Big\langle p(z), y_z^t(t) - z \Big\rangle + \rho \big| y_z^t(t) - z \big|^2 \le \\
&\Big\langle p(z), \mathbf{v}(z) \Big\rangle \left(\frac{t}{r}\right)^h + |p(z)| ct \left(\frac{t}{r}\right)^h + \rho \left(\frac{t}{r}\right)^{2h} (M+ct)^2 \le \\
&-\gamma(U(z)) \left(\frac{t}{r}\right)^h + |p(z)| ct \left(\frac{t}{r}\right)^h + \rho \left(\frac{t}{r}\right)^{2h} (M+ct)^2 \le \\
&\left(-\gamma(U(z)) + Lct + \rho \left(\frac{t}{r}\right)^h (M+ct)^2 \right) \left(\frac{t}{r}\right)^h .
\end{aligned} \tag{21}
$$

Let us observe that $\left(\frac{t}{r}\right)^h \le t \le 1$ as soon as $t \le 1$. Therefore, if we define for every u

$$
\check{\tau}(u) := \frac{\gamma(u)}{2[L(u)c + \rho(u)(M+c)^2]} \tag{22}
$$

and we set

$$
\tau(u,j) := \min\{1, \bar{\tau}(u,j), \check{\tau}(u)\} \quad \forall (u,j) \in (0,\sigma] \times \{1,\dots,k\}, \tag{23}
$$

we get

$$
Lct + \rho \left(\frac{t}{r}\right)^h (M+ct)^2 \le t[Lc + \rho(M+c)^2] \le \frac{\gamma(U(z))}{2}
$$

as soon as $t \in [0, \tau]$, $\tau := \tau(U(z), h)$.

Therefore, with this choice of τ we obtain

$$
U(y_z^t(t)) - U(z) \le -\frac{\gamma(U(z))}{2} \left(\frac{t}{r}\right)^h \quad \forall t \in [0, \tau]. \tag{24}
$$

Moreover, $u \mapsto \tau(u,j)$ is increasing for every j: indeed, by Lemma 3.1, the constants $L(u)$ and $\rho(u)$ turn out to be decreasing in u. Finally, the fact that $j \mapsto \tau(u,j)^j$ and $j \mapsto \tau(u,j)^{j-1}$ are decreasing is an easy consequence of the definition of $\tau(u,j)$ in (23). The claim is now proved. ∎

3.4. Piecewise C^1 trajectories approaching the target. Now let us define recursively a sequence of times $(t_j)_{j \ge 0}$ of trajectory-control pairs $(y_j(\cdot), \alpha_j(\cdot)) : [s_{j-1}, s_j] \to \mathbb{R}^n \times A$, $j \ge 1$, $s_0 := 0$, $s_j := s_{j-1} + t_i$, and points x_j as follows:

- $t_0 := s_0 = 0$, $x_1 := x$;
- if $j \ge 1$, $t_j := \tau(U(x_j), h_j)$, where h_j is the degree of the feedback $\mathbf{v}$ at x_j and $\tau(\cdot,\cdot)$ is as in Claim 3.1;
- $(y_1, \alpha_1) : [s_0, s_1] \to \mathbb{R}^n \times A$ is the trajectory-control pair defined as $(y_1, \alpha_1) := (y_{x_1}^{t_1}, \alpha_{t_1})$;
- for every $j > 1$, $y_j(s_{j-1}) := y_{j-1}(s_{j-1}) := x_j$, and the pair $(y_j(\cdot), \alpha_j(\cdot)) : [s_{j-1}, s_j] \to \mathbb{R}^n \times A$ is given by $(y_j, \alpha_j) := (y_{x_j}^{t_j}, \alpha_{t_j})(s - s_{j-1})$ for every $s \in [s_{j-1}, s_j]$.

Let us consider the real sequence

$$
u_j := U(x_j) \qquad j \in \mathbb{N},
$$

and let us show that

$$\lim_{j\to\infty} u_j = 0.$$

Indeed, the degree h_j of the feedback $\mathbf{v}$ at every x_j is bounded by k. Moreover, if we use r_j to denote the positive integer appearing in formula (20) in relation with the feedback $\mathbf{v}$ at x_j, we get

$$r_j^{h_j} \le (r(k))^k$$

if we set

$$r(k) := \max\{r_j,\ j \in \mathbb{N}\}^{11}.$$

Therefore, by Claim 3.1 we obtain

$$\begin{aligned} u_{j+1} - u_j = U(x_{j+1}) - U(x_j) &\le -\tfrac{\gamma(u_j)}{2} \left(\tfrac{\tau(u_j,h_j)}{r_j}\right)^{h_j} \\ &\le -\tfrac{\gamma(u_j)}{2} \left(\tfrac{\tau(u_j,k)}{r(k)}\right)^{k} < 0 \end{aligned} \tag{25}$$

for all $j \ge 1$. Hence, the sequence (u_j) is positive and decreasing, so there exists the limit

$$\lim_{j\to\infty} u_j = \eta \ge 0.$$

Let us show that $\eta = 0$. If, on the contrary, η were strictly positive, by Claim 3.1 one would have $\lim_{j\to\infty} \tau(u_j,k) \ge \tau(\eta,k) > 0$. Hence, taking the limit in (25) one would obtain

$$0 = \eta - \eta \le -\lim_{j\to\infty} \frac{\gamma(u_j)\tau^k(u_j,k)}{2(r(k))^k} \le -\frac{\gamma(\eta)\tau^k(\eta,k)}{2(r(k))^k} < 0,$$

a contradiction. Therefore,

$$\lim_{j\to\infty} U(x_j) = \lim_{j\to\infty} u_j = 0.$$

Hence, setting

$$S := \lim_{j\to\infty} s_j \ = \sum_{i=1}^{\infty} t_i$$

and

$$(y,\alpha)(s) := (y_j,\alpha_j)(s) \quad \forall j \ge 1,\ \forall s \in [s_{j-1}, s_j],$$

one finds that

$$\lim_{j\to\infty} \mathbf{d}\left(y(s_j)\right) = 0.$$

Actually, the stronger limit relation

$$\lim_{s\to S^-} \mathbf{d}\left(y(s)\right) = 0$$

holds, as it follows from the construction of the function β below.

[11] This maximum clearly exists and depends (monotonically) on k. For instance, $r(2) = 4, r(3) = 10, r(4) = \max\{22, 16\} = 22$.

3.5. Construction of a bounding $\mathcal{KL}$ function. In order to conclude the proof that the system is GAC to $\mathcal{T}$, we have to establish the existence of a $\mathcal{KL}$ function β such that $\mathbf{d}(y(s)) \leq \beta(\mathbf{d}(y(0)), s)$ for every $s \geq 0$, as in Definition 1.2.

By Claim 3.1, for any $t_j = \tau(u_j, h_j)$, one has $t_j^{h_j-1} \geq \tau^{k-1}(u_j, k)$. Moreover, as already remarked, $(r_j)^{h_j} \leq (r(k))^k$ (recall that we are using r_j to denote the positive integer appearing in formula (20) in relation with the feedback $\mathbf{v}$ at x_j) . Hence, for any $j \geq 1$, we have

$$U(y_j(s_j)) - U(y_j(s_{j-1})) = u_{j+1} - u_j$$

$$\leq -\tfrac{\gamma(u_j)}{2} \left(\tfrac{t_j}{r_j}\right)^{h_j} \leq -\tfrac{\gamma(u_j)\tau^{k-1}(u_j,k)}{2(r(k))^k}\, t_j.$$

Let us define the function $\tilde{\gamma} : (0, \sigma] \to \mathbb{R}$ by setting

$$\tilde{\gamma}(u) := \frac{\gamma(u)\,\tau^{k-1}(u,k)}{2(r(k))^k}. \tag{26}$$

Clearly, by the monotonicity of $u \mapsto \tau(u, k)$, $\tilde{\gamma}$ is (positive and) strictly increasing. Therefore, since $U(y(s_j)) \leq U(y(s_i))$ for every $i = 1, \ldots, j$, we get

$$U(y(s_j)) - U(z) = [U(y(s_j)) - U(y(s_{j-1}))] + [U(y(s_{j-1})) - U(y(s_{j-2}))] + \ldots$$

$$+[U(y(s_1)) - U(y(0))] \leq -\textstyle\sum_{i=1}^{j} \tilde{\gamma}(U(y(s_i)))\,[s_i - s_{i-1}] \leq -\tilde{\gamma}(U(y(s_j)))\,s_j\,.$$

In particular, we have

$$U(y(s_j)) + \tilde{\gamma}(U(y(s_j)))\,s_j \leq U(z)\,. \tag{27}$$

We now replace the function $\tilde{\gamma}$ with the slightly different function $\hat{\gamma} : [0, +\infty) \to [0, +\infty)$ defined by $\hat{\gamma}(u) \doteq \min\{u, \tilde{\gamma}(u)\}$ for all $u \in [0, +\infty)$. Notice that $\hat{\gamma}$ is continuous and strictly increasing and $\hat{\gamma}(u) > 0 \quad \forall u > 0$, $\hat{\gamma}(0) = 0$. Then for any $j \geq 1$,

$$\hat{\gamma}(U(y(s_j)))(1 + s_j) \leq U(z),$$

so that

$$U(y(s_j)) \leq \hat{\gamma}^{-1}\left(\frac{U(z)}{1 + s_j}\right). \tag{28}$$

Let δ_-, $\delta_+ : [0, +\infty) \to [0, +\infty)$ be the continuous, strictly increasing, unbounded functions defined by

$$\delta_-(u) \doteq \min\{\mathbf{d}(x) :\ U(x) \geq u\}, \quad \delta_+(u) \doteq \max\{\mathbf{d}(x) :\ U(x) \leq u\}, \tag{29}$$

and let us set $\hat{\delta}_-(u) := \min\{\delta_-(u), u\}$. Notice that $\hat{\delta}_-(0) = \delta^+(0) = 0$ and

$$\hat{\delta}_-(U(x)) \leq \mathbf{d}(x) \leq \delta^+(U(x))$$

$\forall x \in U^{-1}((0, \sigma])$. Therefore, setting

$$\hat{\beta}(\delta, s) := \delta_+ \circ \hat{\gamma}^{-1}\left(\frac{\hat{\delta}_-^{-1}(\delta)}{1 + s}\right) \quad \forall (\delta, s) \in [0, +\infty) \times [0, +\infty), \tag{30}$$

by (28) we get

$$\mathbf{d}(y(s_j)) \leq \delta_+ \left(U(y(s_j)) \leq \delta_+ \left(\hat{\gamma}^{-1}\left(\frac{U(z)}{1+s_j}\right)\right) \leq \hat{\beta}(\mathbf{d}(z), s_j) \tag{31}$$

for every $j \geq 1$. The estimate (31) says that the function $\hat{\beta}$ bounds the distance of the trajectory $y(\cdot)$ from the target $\mathcal{T}$ at the discrete times s_j. Hence, in order to get a bound at all times, we need to slightly modify $\hat{\beta}$. For this purpose, given any $x \in \mathbb{R}^n \setminus \mathcal{T}$, let us select a point $\pi(x) \in \mathcal{T}$ such that $\mathbf{d}(x) = |x - \pi(x)|$. Notice that for any $s \in [s_j, s_{j+1}]$, one has

$$\begin{aligned}\mathbf{d}(y(s)) &\leq |y(s) - \pi(y(s_j))| \leq |y(s) - y(s_j)| + |y(s_j) - \pi(y(s_j))| \\ &\leq M[s_{j+1} - s_j] + \mathbf{d}(y(s_j)).\end{aligned}$$

Furthermore, by the definition of τ (see Claim 3.1) it follows that

$$s_{j+1} - s_j = t_{j+1} \leq \tau(u_{j+1}, k) \leq \tau(\delta_-^{-1}(\mathbf{d}(y(s_j))), k). \tag{32}$$

Therefore,

$$\begin{aligned}\mathbf{d}(y(s)) &\leq M\tau(\delta_-^{-1}(\mathbf{d}(y(s_j))), k) + \mathbf{d}(y(s_j)) \\ &\leq M\tau(\delta_-^{-1}(\hat{\beta}(\mathbf{d}(z), s_j)), k) + \hat{\beta}(\mathbf{d}(z), s_j)\,.\end{aligned}$$

Since for all δ the function $s \mapsto \hat{\beta}(\delta, s)$ is decreasing, one obtains

$$\mathbf{d}(y(s)) \leq \beta(\mathbf{d}(z), s) \qquad \forall s \in [0, +\infty[,$$

where we have set for all $s \in \left[0, \tau(\delta_-^{-1}(\delta), k)\right]$

$$\beta(\delta, s) := M\tau(\delta_-^{-1}(\hat{\beta}(\delta, 0)), k) + \hat{\beta}(\delta, 0)$$

and, if $s > \tau(\delta_-^{-1}(\delta), k)$,

$$\beta(\delta, s) := M\tau(\delta_-^{-1}(\hat{\beta}(\delta, s - \tau(\delta_-^{-1}(\delta), k))), k) + \hat{\beta}(\delta, s - \tau(\delta_-^{-1}(\delta), k)).$$

3.6. Removal of the fictitious C^{k-1}-bound. Let us see now that, by means of a cutoff argument, we can remove the auxiliary boundedness hypothesis (19). Let $\psi : \mathbb{R}^n \to [0,1]$ be a C^∞ map such that

$$\psi = 1 \quad \text{on} \quad \overline{B(\mathcal{T}, \nu) \setminus \mathcal{T}}, \qquad \psi = 0 \quad \text{on} \ \mathbb{R}^n \backslash B(\mathcal{T}, 2\nu), \tag{33}$$

and consider the control system

$$\xi' = \sum_{i=1}^{m} a_i \ (\psi f_i(\xi))\,. \tag{34}$$

Notice that the functions $(\psi\, f_i)$ belong to $C_b^{k-1}(\mathbb{R}^n \setminus \mathcal{T})$ because of the cutoff factor ψ. Moreover, any trajectory $\xi(\cdot)$ of (34) with the initial condition $z \in U^{-1}((0, \sigma])$ exists globally and cannot exit the compact set $\overline{B(\mathcal{T}, 2\nu) \setminus \mathcal{T}}$. Owing to the previous step, there exists a trajectory ξ which approaches asymptotically the target and verifies $\mathbf{d}(\xi(s)) \leq \beta(\mathbf{d}(z), s) \quad \forall s \in [0, +\infty[$. Moreover, $\xi(s)$ belongs to $B(\mathcal{T}, \nu)$ for every $s \geq 0$. Therefore, ξ is a solution of the original system, proving that (3) has the GAC property in $U^{-1}((0, \sigma])$.

By the arbitrariness of $\sigma > 0$, it is easy to extend these constructions from $U^{-1}((0, \sigma])$ to the whole set $\mathbb{R}^n \setminus \mathcal{T}$. This concludes the proof of Theorem 2.1.

4. The case of nonsmooth dynamics. Let us begin with an example where the vector fields are not C^1.

Example 4.1. Let us consider the system

$$\dot{y} = a_1\, f_1(y) + a_2\, f_2(y)\,, \tag{35}$$

where

$$f_1 := \frac{\partial}{\partial x_1} + (|x_2| - 2x_2)\frac{\partial}{\partial x_3}\,, \quad f_2 := \frac{\partial}{\partial x_2} + (|x_1| + 2x_1)\frac{\partial}{\partial x_3},$$

and let the target $\mathcal{T}$ coincide with the origin. For the same reason as in the nonholonomic integrator, a smooth degree-1 CLF does not exist (see Example 2.1). On the other hand, the given notion of degree-2 control Lyapunov function is not even meaningful here in that the classical bracket $[f_1, f_2]$ is not defined at points x such that $x_1 = 0$ or $x_2 = 0$. Yet, in the open, dense set $\{x : \ x_1 \neq 0, \ x_2 \neq 0\}$, the bracket is well defined, and, furthermore, the Lie algebra rank condition is verified. So it is reasonable to look for a Lyapunov-like condition involving *somehow* Lie brackets.

On the one hand, in the definition of degree-k CLF, one requires the vector fields $f_1, \dots, f_m$ to be of class C^{k-1} (see also Remark 2.1), for this guarantees that the Lie brackets up to the degree k are well defined and continuous. On the other hand, as remarked in the Introduction, the nonsmoothness of a degree-1 CLF is more related to a shortage of the dynamics' directions than to the regularity of the involved vector fields.

Let us introduce the notion of set-valued bracket for locally Lipschitz vector fields.

Definition 4.1 ([RS1]). *Let $\Omega \subset \mathbb{R}^n$ be an open set, and let f, g be vector fields belonging to $C^{0,1}(\Omega)$. For every $x \in \Omega$, let us set*

$$[f, g]_{set}(x) :=$$

$$co\Big\{v \in \mathbb{R}^n, \quad v = \lim_{x_n \to x}[f, g](x_n) \quad (x_n)_{n\in\mathbb{N}} \subset DIFF(f) \cap DIFF(g)\Big\},$$

where co means "convex hull" and $DIFF(f)$, $DIFF(g)$ denote the subsets of differentiability points of f and g, respectively. Let us observe that $DIFF(f)$ and $DIFF(g)$ have full measure; hence, they are dense in Ω.

Notice that, as in the regular case, one has $[f, f]_{set}(x) = \{0\}$ and $[f, g]_{set}(x) = -[g, f]_{set}(x)$ for every $x \in \Omega$ (for any $E \subset \mathbb{R}^n$ we use the notation $-E = \{-v : \ v \in E\}$).

Let us consider the control system

$$\begin{cases} \dot{y} = \displaystyle\sum_{i=1,\dots,m} a_i\, f_i(y) \\ \\ y(0) = x \in \mathbb{R}^n \setminus \mathcal{T}, \end{cases} \tag{36}$$

and let us assume that $f_1, \dots, f_m$ belong to $C_b^{0,1}(\Omega \setminus \mathcal{T})$ for any bounded, open set $\Omega \subset \mathbb{R}^n$ (see subsection 1.1).

The families $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$ are formally defined as in the regular (i.e., C^1) case, except that their elements are set-valued vector fields:[12]

$$\mathcal{F}^{(1)} := \Big\{ \{f_\ell(\cdot)\}\,,\, \{-f_\ell(\cdot)\} : \quad \ell = 1, \dots, m \Big\}$$

and

$$\mathcal{F}^{(2)} := \mathcal{F}^{(1)} \cup \Big\{ [f_i, f_j]_{set}(\cdot) : \quad i, j = 1, \dots, m \Big\}.$$

As in the regular case, for every $x \in \mathbb{R}^n \setminus \mathcal{T}$, we set

$$\mathcal{F}^{(1)}(x) := \Big\{ \{f_\ell(x)\}\,,\, \{-f_\ell(x)\} : \quad \ell = 1, \dots, m \Big\}$$

and

$$\mathcal{F}^{(2)}(x) := \mathcal{F}^{(1)}(x) \cup \Big\{ [f_i, f_j]_{set}(x) : \quad i, j = 1, \dots, m \Big\}.$$

Accordingly, for $h = 1, 2$, we define the *degree-h Hamiltonian* $H^{(h)}$ by setting

$$H^{(h)}(x,p) := \inf_{v \in \mathcal{F}^{(h)}(x)} \sup_{w \in v} \Big\langle p,\, w \Big\rangle$$

for all $(x,p) \in (\mathbb{R}^n \setminus \mathcal{T}) \times \mathbb{R}^n$. More explicitly, one has

$$H^{(1)}(x,p) = \inf \big\{ -\big|\big\langle p,\, f_\ell(x)\big\rangle\big| : \quad \ell = 1, \dots, m \big\}$$

and

$$H^{(2)}(x,p) = \inf_{\ell,i,j} \left\{ -\big|\big\langle p,\, f_\ell(x)\big\rangle\big|\,,\, \sup_{w \in [f_i,f_j]_{set}(x)} \big\langle p,\, w\big\rangle : \quad \ell, i, j = 1, \dots, m \right\}.$$

We can now state for the case $k = 2$ a generalization of Theorem 2.1 to the nonsmooth case.

THEOREM 4.1. *Let us assume that a degree-*2 *control Lyapunov function exists. Then system* (36) *is* GAC *to* $\mathcal{T}$.

Example 4.1 (continued). Let us come back to the system in (35) and compute the bracket $[f_1, f_2]_{set}$. It turns out that

$$[f_1, f_2]_{set}(x) = I(x) \frac{\partial}{\partial x_3},$$

where

$$I(x) := \begin{cases} \{4\} & \text{if } x_1 x_2 > 0, \\ \{2\} & \text{if } x_1 < 0 \text{ and } x_2 > 0, \\ \{6\} & \text{if } x_1 > 0 \text{ and } x_2 < 0, \\ [2,4] & \text{if either } x_1 = 0 \text{ and } x_2 > 0 \text{ or } x_1 < 0 \text{ and } x_2 = 0, \\ [4,6] & \text{if either } x_1 = 0 \text{ and } x_2 < 0 \text{ or } x_1 > 0 \text{ and } x_2 = 0, \\ [2,6] & \text{if } x_1 = x_2 = 0. \end{cases}$$

[12] $\mathcal{F}^{(1)}$ practically coincides with the one in the regular case, for its elements are set-valued maps which are singletons, indeed.

The distance function $U(x) = |x|$ is not a degree-1 CLF. Indeed, by

$$H^{(1)}(x,p) = \inf\Big\{ -|p_1 + p_3(|x_2| - 2x_2)|\,,\, -|p_2 + p_3(|x_1| + 2x_1)|\Big\},$$

one obtains

$$H^{(1)}\big((0,0,x_3), DU(0,0,x_3)\big) = 0 \quad \text{for every } x_3 \neq 0.$$

Yet, the distance function U happens to be a (C^∞) degree-2 CLF. Indeed,

$$H^{(2)}(x,p) = \inf\Big\{H^{(1)}(x,p)\,,\, \sup_{w\in I(x)} w\,p_3\,,\, \sup_{w\in -I(x)} w\,p_3\Big\} =$$

$$-\sup\Big\{|p_1 + p_3(|x_2| - 2x_2)|\,,\, |p_2 + p_3(|x_1| + 2x_1)|\,,\, 2|p_3|\Big\},$$

and since $|DU(x)| = 1$ for every $x \neq 0$, one gets

$$H^{(2)}\big(x, DU(x)\big) < 0. \tag{37}$$

Notice that for the validity of the strict inequality in (37), it is crucial that $0 \notin [f_1, f_2]_{set}(x)$ for every $x \neq 0$. Furthermore, arguing as in Remark 2.4, we know that a (possibly nonsmooth) degree-1 CLF does exist. Actually, the function

$$U(x) = \max\Big\{\sqrt{x_1^2 + x_2^2}, |x_3| - \sqrt{x_1^2 + x_2^2}\Big\}$$

introduced in [Ri] as a control Lyapunov function for the nonholonomic integrator is a degree-1 CLF also for this system.

4.1. Proof of Theorem 4.1. The proof is akin to the proof of Theorem 2.1. Yet, because of the new kind of brackets and Hamiltonians here involved, some changes are needed.

As in the regular case, the 0 in the dissipative relation can be replaced by a nonnegative function of U:

PROPOSITION 4.1. *Let* $U : \overline{\mathbb{R}^n \setminus \mathcal{T}} \to \mathbb{R}$ *be a continuous function, such that* U *is locally semiconcave, positive definite, and proper on* $\mathbb{R}^n \setminus \mathcal{T}$*. Then the conditions* (i) *and* (ii) *below are equivalent:*

(i) U *verifies*

$$H^{(2)}(x, D^*U(x)) < 0 \quad \textit{for all } x \in \mathbb{R}^n \setminus \mathcal{T};$$

(ii) *for every* $\sigma > 0$ *there exists a continuous, strictly increasing function* $\gamma : [0, +\infty) \to: [0, +\infty)$ *such that*

$$H^{(2)}(x, D^*U(x)) \leq -\gamma(U(x)) \quad \textit{for all } x \in U^{-1}((0, 2\sigma]).$$

Proof. By [RS1], for every i, j, the set-valued map $x \mapsto [f_i, g_j]_{set}(x)$ is upper semicontinuous, with compact, convex values. Moreover, basic results on marginal functions imply that $(x,p) \mapsto \sup_{w\in[f_i,g_j]_{set}(x)} \langle p, w\rangle$ is upper semicontinuous (see, e.g., [AC]). As an easy consequence, the Hamiltonian

$$(x,p) \mapsto H^{(2)}(x,p)$$

turns out to be upper semicontinuous. At this point one can conclude, arguing exactly as in Proposition 3.1. ∎

We now need to adapt the notion of degree-2 feedback to the case when Lie brackets are set-valued. For a given $\sigma > 0$, let γ be a function as in Proposition 4.1, and let $x \mapsto p(x)$ be a selection of $x \mapsto D^*U(x)$ on $U^{-1}((0, 2\sigma])$, so that

$$H^{(2)}(x, p(x)) \leq -\gamma(U(x)) \qquad \forall x \in U^{-1}((0, 2\sigma]).$$

DEFINITION 4.2. *For a given $\sigma > 0$, let $\gamma(\cdot)$ and $p(\cdot)$ be chosen as above. A selection*

$$\mathbf{v} : U^{-1}((0, 2\sigma]) \to 2^{\mathbb{R}^n}, \qquad x \mapsto \mathbf{v}(x) \in \mathcal{F}^{(2)}(x)$$

is called a degree-2 feedback *(corresponding to U, σ, γ, and $p(\cdot)$) if for every $x \in U^{-1}((0, 2\sigma])$ there exists $h \in \{1, 2\}$ such that*

$$\begin{cases} \mathbf{v}(x) \in \mathcal{F}^{(h)}(x), \\ \\ \displaystyle\sup_{w \in \mathbf{v}(x)} \Big\langle p(x),\, w \Big\rangle \leq -\gamma(U(x)), \\ \\ \text{and, if } h = 2, \quad H^{(1)}(x, p(x)) > -\gamma(U(x)). \end{cases} \tag{38}$$

The number h will be called the degree of the feedback $\mathbf{v}$ at x.

More explicitly, when $h = 1$ for some $\ell = 1, \dots, m$, one has

$$\begin{cases} \mathbf{v}(x) = \{f_\ell(x)\} \text{ or } \mathbf{v}(x) = -\{f_\ell(x)\} \\ \\ -\Big|\Big\langle p(x),\, f_\ell(x) \Big\rangle\Big| \leq -\gamma(U(x)); \end{cases}$$

if $h = 2$ for some $i, j = 1, \dots, m$ $(i \neq j)$, one has

$$\begin{cases} \mathbf{v}(x) = [f_i, f_j]_{set}(x), \\ \\ \Big\langle p(x),\, w \Big\rangle \leq -\gamma(U(x)) \quad \forall w \in [f_i, f_j]_{set}(x), \\ \\ \text{and} \quad H^{(1)}(x, p(x)) > -\gamma(U(x)). \end{cases}$$

The following claim is a version of Claim 3.1 adapted to the set-valued notion of feedback.

CLAIM 4.1. *Let $U(\cdot)$, σ, $\gamma(\cdot)$, and $p(\cdot)$ as above. Furthermore, let $\mathbf{v}(\cdot)$ be a (set-valued) degree-2 feedback corresponding to these data. Then there exists a time-valued function*

$$\tau : (0, \sigma] \times \{1, 2\} \to (0, 1]$$

such that

(i) *$j \mapsto \tau(u, j)^j$ and $j \mapsto \tau(u, j)^{j-1}$ are decreasing for every $u \in (0, \sigma]$;*

(ii) *$u \mapsto \tau(u, j)$ is increasing for every $j \in \{1, 2\}$;*

(iii) *for all $z \in U^{-1}((0, \sigma])$ with a feedback $\mathbf{v}(\cdot)$ of degree h at z, one has*

$$U(y_z(t, \alpha_t)) - U(z) \leq -\frac{\gamma(U(z))}{2} \left(\frac{t}{r}\right)^h \qquad \forall t \in [0,\, \tau(U(z), h)]\,,$$

where r and $y_z(\cdot, \alpha_t)$ are an integer and a trajectory associated to $\mathbf{v}(z)$ according to Lemma 4.1.

The proof of Claim 3.1 was based on the asymptotic formulas stated in Lemma 3.2. Similarly, Claim 4.1 results proved as soon as one applies the asymptotic formula stated in Lemma 4.1 below. Precisely, once arrived to formula (21), one simply replaces estimate (20) with (44) and observes that, by the definition of $H^{(2)}$, the inequality $\Big\langle p(z),\, w\Big\rangle \le -\gamma(U(z))$ is verified *for all* $w \in [f_i, f_j]_{set}(z)$ (see Definition 4.2).

LEMMA 4.1 ([RS2], [FR2]). *If $f_1, \dots, f_m \in C_b^{0,1}(\mathbb{R}^n \setminus \mathcal{T})$, there exists a constant $c > 0$ such that for any $x \in \mathbb{R}^n \setminus \mathcal{T}$, any feedback $\mathbf{v}(x) = [f_i, f_j]_{set}(x)$, and any $t > 0$, setting*

$$\alpha_t(s) = e_i\chi_{[0,t/4[}(s) + e_j\chi_{[t/4,t/2[}(s) - e_i\chi_{[t/2,3t/4[}(s) - e_j\chi_{[3t/4,t]}(s), \quad s \in [0,t],$$

the estimate

$$d\left(y_x(t,\alpha_t) - x\,,\, [f_i, f_j]_{set}(x)\,\frac{t^2}{16}\right) \le \frac{c}{16}\,t^3 \tag{39}$$

holds true. In particular, for any $t > 0$ there exists some $w(t) \in \mathbf{v}(x)$ such that

$$\left| y_x(t,\alpha_t) - x - w(t)\frac{t^2}{16}\right| \le \frac{c}{16}\,t^3. \tag{40}$$

Let us point out that Claim 3.1 is the starting point for the construction of an admissible trajectory-control pair by means of the recursive procedure described in the proof of Theorem 2.1. Through exactly the same arguments and in view of Claim 4.1, the proof of Theorem 4.1 can now be completed.

5. Concluding remarks.

5.1. Feedback constructions. Degree-1 control Lyapunov functions are used as a primary ingredient for the construction of feedback stabilizing strategies, a classical question that is mainly concerned with the definition of an appropriate notion of solution for discontinuous ODEs (see, e.g., [CLSS], [CLRS], [MaRS], [AB]).

It might be interesting to associate some concept of feedback strategy also in relation with a degree-k control Lyapunov function, $k > 1$, all the more so as it may happen to be smoother than the degree-1 CLF. As a matter of fact, the proofs of Theorems 2.1 and 4.1 seem to suggest a notion of "feedback" such that, depending on which bracket minimizes the Hamiltonian $H^{(k)}$, singles out a suitable finite sequence of constant controls to be implemented along small time intervals. As for the feedback issue, see also Remark 5.1 below.

5.2. Degree-k CLF as viscosity supersolutions. One is obviously tempted to refer to a degree-k control Lyapunov function as a *strict supersolution* of the Hamilton–Jacobi equation

$$-H^{(k)}(x, DU(x)) = 0.$$

Actually, this holds true as soon as we consider, for instance, the notion of viscosity solution. More precisely, one has the following.

Let $U : \overline{\mathbb{R}^n \setminus \mathcal{T}} \to \mathbb{R}$ be a continuous function which, furthermore, is locally semiconcave, positive, and proper on $\mathbb{R}^n \setminus \mathcal{T}$. Then U is a degree-k control Lyapunov function if and only for any $N > 0$ there is some continuous, strictly increasing *function* $\gamma : [0, +\infty) \to [0, +\infty)$ *such that* U *is a viscosity supersolution*[13] *of*

$$- H^{(k)}(x, DU(x)) = \gamma(U(x)) \quad \textit{in } U^{-1}((0, N)). \tag{41}$$

Indeed, in the case of a locally semiconcave function U, at every $x \in DIFF(U)$ the subdifferential $D^-U(x) = \{\nabla U(x)\}$ coincides with $D^*U(x)$, while $D^-U(x)$ is empty if $x \notin D^-U(x)$. Therefore, thanks to Proposition 3.1, (41) follows from the inequality

$$H^{(k)}(x, D^*U(x)) < 0, \tag{42}$$

which defines the notion of degree-k CLF. To obtain the converse implication for any $x \in U^{-1}((0, N))$ and $p \in D^*U(x)$, let $(x_n)_n \subset U^{-1}((0, N)) \cap DIFF(U)$ be such that $\lim_n(x_n, \nabla U(x_n)) = (x, p)$. Then, by hypothesis (41), one has

$$-H^{(k)}\left(x_n, \nabla U(x_n)\right) \geq \gamma(U(x_n)) \qquad \forall n \in \mathbb{N},$$

so, passing to the limit, one gets (42).

5.3. Generalizations to larger classes of systems. As it is well known, the lack of symmetry poses nontrivial problems for controllability. The same kind of difficulty is therefore encountered in the attempt to define a reasonable notion of degree-k CLF for systems

$$\begin{cases} \dot{y} = f_0(y) + \displaystyle\sum_{i=1,\dots,m} a_i f_i(y) \\ \\ y(0) = x \in \mathbb{R}^n \backslash \mathcal{T} \end{cases} \qquad a \in \{0, \pm e_1, \dots, e_m\} \tag{43}$$

having a *nonzero drift* f_0. Let us assume that $f_0, \dots, f_m$ belong to $C^1_b(\Omega \setminus \mathcal{T})$ for any open, bounded set $\Omega \subset \mathbb{R}^n$.

Let us examine the case $k = 2$. Intuition coming from controllability literature suggests that a notion of degree-2 control Lyapunov function should be shaped in such a way that it would be allowed to violate the standard dissipative inequality only at the points where the drift f_0 vanishes. Accordingly, let us redefine the classes of vector fields

$$\mathcal{F}^{(1)} := \Big\{ f_0, f_0 - f_i, f_0 + f_i \quad i = 1 \dots, m \Big\},$$

$$\mathcal{F}^{(2)} := \mathcal{F}^{(1)} \cup \Big\{ \pm [f_0, f_i] \cdot \chi_{\{f_0 = 0\}}, [f_j, f_\ell] \cdot \chi_{\{f_0 = 0\}} \quad i, j, \ell = 1 \dots, m \Big\}$$

and the Hamiltonians

$$H^{(h)}(x, p) := \inf_{g \in \mathcal{F}^{(h)}(x)} \big\langle p, \, g \big\rangle \quad h = 1, 2.$$

Accordingly, one might call *degree-2 control Lyapunov function* any continuous function $U : \overline{\mathbb{R}^n \backslash \mathcal{T}} \to \mathbb{R}$ such that its restriction to $\mathbb{R}^n \backslash \mathcal{T}$ is locally semiconcave, positive definite and proper and verifies

$$H^{(2)}(x, D^*U(x)) < 0 \quad \forall x \in \mathbb{R}^n \backslash \mathcal{T}.$$

In particular, U is a degree-2 CLF if, at each point $x \in \mathbb{R}^n \backslash \mathcal{T}$, either

$$\min \Big\{ \langle D^*U(x), f_0(x) \rangle, \langle D^*U(x), (f_0 - f_i)(x) \rangle, \langle D^*U(x), (f_0 + f_j)(x) \rangle, \; i, j = 1, \dots, m \Big\} < 0$$

[13] Namely, $-H^{(k)}(x, p) \geq \gamma(U(x))$ for all $x \in \mathbb{R}^n \setminus \mathcal{T}$ and p in the subgradient $DU^-(x)$ of U at x (see [BCD]).

or

$$\begin{cases} f_0(x) = 0, \\ \min\Big\{ -|\langle D^*U(x), [f_0, f_i](x)\rangle|, -|\langle D^*U(x), [f_j, f_\ell(x)\rangle| \quad i, j, \ell = 1, \ldots, m\Big\} < 0. \end{cases}$$

With these settings, we get the following result.

THEOREM 5.1. *Let a degree-2 control Lyapunov function exist. Then system* (43) *is GAC to* $\mathcal{T}$.

A proof of this result can be deduced by first observing that Lemma 3.2 has a counterpart in an asymptotic formula for brackets of the form $[f_0, f_i]$, $[f_j, f_\ell]$ valid at all points x where $f_0(x) = 0$. Precisely, through standard Taylor expansions one can prove the following result.

LEMMA 5.1. *If* $f_0, f_1, \ldots, f_m \in C^1_b(\mathbb{R}^n \setminus \mathcal{T})$, *there exists a constant* $c > 0$ *such that for any* $x \in \mathbb{R}^n \setminus \mathcal{T}$ *where* $f_0(x) = 0$, *any* $j = 1, \ldots, m$, *and any* $t > 0$, *the following estimates hold true:*

$$d\left(y_x(t, \alpha_t) - x\,,\ [f_0, f_i](x)\,\frac{t^2}{4}\right) \le \frac{c}{4}\, t^3 \quad \forall s \in [0, t] \tag{44}$$

with

$$\alpha_t(s) := e_i \chi_{[0,t/2[}(s) - e_i \chi_{[t/2,t[}(s);$$

$$d\left(y_x(t, \hat\alpha_t) - x\,,\ [f_i, f_0](x)\,\frac{t^2}{4}\right) \le \frac{c}{4}\, t^3 \quad \forall s \in [0, t] \tag{45}$$

with

$$\hat\alpha_t(s) := -e_i \chi_{[0,t/2[}(s) + e_i \chi_{[t/2,t[}(s);$$

and

$$d\left(y_x(t, \tilde\alpha_t) - x\,,\ [f_j, f_\ell](x)\,\frac{t^2}{16}\right) \le \frac{c}{16}\, t^3, \quad s \in [0, t] \tag{46}$$

with

$$\tilde\alpha_t(s) = e_j \chi_{[0,t/4[}(s) + e_\ell \chi_{[t/4,t/2[}(s) - e_j \chi_{[t/2,3t/4[}(s) - e_\ell \chi_{[3t/4,t]}(s).$$

Hence, provided one gives a (obvious) notion of degree-2 feedback at the points x where $f_0(x) = 0$, the proof of Theorem 5.1 can be easily achieved by means of the same arguments as in the proof of Theorem 2.1.

Example 5.1. Consider the so-called *soft landing* problem, $\dot y_1 = y_2$, $\dot y_2 = a$, $a \in \{0, -1, 1\}$, $\mathcal{T} = \{(0, 0)\}$. The distance function $U(x) = |x|$—as well as $|x|^\alpha$, $\alpha > 1$—fails to be a degree-1 CLF (because the required inequality is not strict on the x_1 axis). However, $U(x)$ is a degree-2 CLF.

A reasonable and useful notion of degree-k CLF can be likely given for the general case $k \ge 1$ provided that only suitable subsets of brackets are allowed.[14]

[14]For instance, one can consider "good" brackets (see, e.g., [Co]), of which $[f_0, f_i]$, $[f_j, f_\ell]$ are degree-2 instances. Among degree-3 brackets, $[f_0, [f_0, f_i]]$ is good for every $i = 1, \ldots, m$, while $[f_0, [f_j, f_\ell]]$ is not good for all $j, \ell = 1, \ldots, m$.

Remark 5.1. An interesting line of research in the case of systems with drift can be found, e.g., in [MTT], [T], [TT]. In particular, in [TT, Proposition 3] a general hypothesis—akin to the one considered in Theorem 5.1 above in the special case of degree-2 control Lyapunov functions—is shown to imply not only asymptotic controllability but also some forms of feedback stabilizability. Actually, it might be interesting to extend such notions to the case of a general target $\mathcal{T}$ and investigate the relation with our results on GAC to $\mathcal{T}$, which are based on the construction of a rate function β. Let us remark that, in the quoted papers, Lyapunov-like functions are always assumed to be regular. Therefore, such an extension should address the regularity issue, the results of which are unavoidable because of both the target-shape issue and the "direction-shortage" argument mentioned in the Introduction.

5.4. Generalization to unbounded closed targets. Theorem 2.1 can be adapted to the case of a closed target set $\mathcal{T}$ with unbounded boundary. This requires some care, for in the proof of Theorem 2.1 one crucially relies on both the properness of the function U and the fact that U is uniformly Lipschitz and semiconcave on the level strata $U^{-1}([\eta_1, \eta_2])$, $\eta_1 > 0$.

To begin with, for every closed set C, let us introduce the notion of *near-C Lipschitz-semiconcavity.*

DEFINITION 5.1. *Let $U : \mathbb{R}^n \setminus C \to \mathbb{R}$ be a locally semiconcave function. We say that U is* near-C Lipschitz-semiconcave *if for all $\nu_1, \nu_2 \in \mathbb{R}$ such that $\nu_2 > \nu_1 > 0$, there exist some constants $L > 0$ and $\rho > 0$ such that*

$$
\begin{aligned}
&|p| \le L \quad \forall p \in D^*U(x), \\
&U(\hat{x}) - U(x) \le \langle p, \hat{x} - x \rangle + \rho |\hat{x} - x|^2
\end{aligned}
\tag{47}
$$

for all x, $\hat{x} \in \mathcal{K}$ such that $[x, \hat{x}] \subset \mathcal{K}$, where we have set

$$
\mathcal{K} := \{x \in \Omega : \ \nu_1 \le \mathbf{d}(x) \le \nu_2\}.
$$

Let us remark that if $\mathbf{d}$ denotes the distance function from a closed set C, then $\mathbf{d}$ is near-C Lipschitz-semiconcave (with $L = 1$ and $\rho = 1/\nu_1$). Moreover, if the set C has compact boundary, in view of Lemma 3.1, a map $U : \mathbb{R}^n \setminus C \to \mathbb{R}$ is near-C Lipschitz-semiconcave if (and only if) it is locally semiconcave on $\mathbb{R}^n \setminus C$.

In Theorem 5.2 below we replace the properness assumption of Theorem 2.1 with hypothesis (ii). Moreover, the uniform degree-k Lyapunov relation is directly assumed in (iii) (while in the bounded case it was a consequence of the simple degree-k Lyapunov relation).

THEOREM 5.2. *Assume that for some $k \ge 1$, the vector fields $f_1, \dots, f_n$ and their derivatives up to the order k are globally bounded and there exists a continuous function $U : \overline{\mathbb{R}^n \setminus \mathcal{T}} \to \mathbb{R}$ such that*

(i) *the restriction of U to $\mathbb{R}^n \setminus \mathcal{T}$ is near-$\mathcal{T}$ Lipschitz-semiconcave and positive definite;*

(ii) *there is a continuous, strictly increasing, surjective function $\eta : [0, +\infty) \to [0, +\infty)$ such that*

$$
U(x) \ge \eta(\mathbf{d}(x)) \quad \forall x \in \mathbb{R}^n \setminus \mathcal{T};
$$

(iii) *for every $\sigma > 0$ there is a continuous, strictly increasing function $\gamma : [0, +\infty) \to [0, +\infty)$ such that*

$$
H^{(k)}(x, D^*U(x)) \le -\gamma(U(x)) \quad \textit{for all } x \in U^{-1}((0, 2\sigma]).
$$

Then system (3) *is GAC to $\mathcal{T}$.*

REFERENCES

[AB] F. Ancona and A. Bressan, *Patchy vector fields and asymptotic stabilization*, ESAIM Control Optim. Calc. Var., 4 (1999), pp. 445–471.

[AgSa] A. A. Agrachev and Y. L. Sachkov, *Control theory from the geometric viewpoint*, in Encyclopaedia of Mathematical Sciences 87, Control Theory and Optimization, II. Springer-Verlag, Berlin, 2004.

[AC] J. P. Aubin and A. Cellina, *Differential Inclusions: Set-Valued Maps and Viability Theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] 264, Springer-Verlag, Berlin, 1984.

[BR] A. Bacciotti and L. Rosier, *Lyapunov Functions and Stability in Control Theory*, 2nd ed., Springer, New York, 2005.

[BCD] M. Bardi and I. Capuzzo Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Birkhäuser, Boston, MA, 1997.

[BP] A. Bressan and B. Piccoli, *Introduction to the Mathematical Theory of Control*, AIMS Series on Applied Mathematics 2, American Institute of Mathematical Sciences (AIMS), Springfield, MO, 2007.

[CS] P. Cannarsa and C. Sinestrari, *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*, in Progress in Nonlinear Differential Equations and Their Applications 58, Birkhäuser Boston, Inc., Boston, MA, 2004.

[CLSW] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, and P. R. Wolenski, *Nonsmooth Analysis and Control Theory*, Graduate Texts in Mathematics 178, Springer-Verlag, New York, 1998.

[CLSS] F. Clarke, Y. Ledyaev, E. Sontag, and A. Subbotin, *Asymptotic controllability implies feedback stabilization*, IEEE Trans. Automat. Control 42 (1997), pp. 1394–1407.

[CLRS] F. Clarke, Y. Ledyaev, L. Rifford, and R. J. Stern, *Feedback stabilization and Lyapunov functions*, SIAM J. Control Optim., 39 (2000), pp. 25–48.

[Co] J.-M. Coron, *Control and Nonlinearity*, Mathematical Surveys and Monographs 136, American Mathematical Society, Providence, RI, 2007.

[FR1] E. Feleqi and F. Rampazzo, *Integral representations for bracket-generating multi-flows*, Discrete Contin. Dyn. Syst., 35 (2015), pp. 4345–4366.

[FR2] E. Feleqi and F. Rampazzo, *Iterated Lie Brackets for Nonsmooth Vector Fields*, preprint, 2016.

[FHT] H. Frankowska, D. Hoehener, and D. Tonon, *A second-order maximum principle in optimal control under state constraints*, Serdica Math. J., 39 (2013), pp. 233–270.

[K] H.-W. Knobloch, *Higher Order Necessary Conditions in Optimal Control Theory*, Lecture Notes in Computer Science 34, Springer-Verlag, Berlin, 1981.

[Kr] M. Krastanov, *High-order variations and small-time local attainability*, Control Cybernet., 38 (2009), pp. 1411–1427.

[KrQ] M. Krastanov and M. Quincampoix, *Local small time controllability and attainability of a set for nonlinear control system*, ESAIM Control Optim. Calc. Var., 6 (2001), pp. 499–516.

[MaRS] M. Malisoff, L. Rifford, and E. Sontag, *Global asymptotic controllability implies input-to-state stabilization*, SIAM J. Control Optim., 42 (2004), pp. 2221–2238.

[Mar] A. Marigonda and S. Rigo, *Controllability of some nonlinear systems with drift via generalized curvature properties*, SIAM J. Control Optim., 53 (2015), pp. 434–474.

[MTT] H. Michalska and M.Torres-Torriti, *A geometric approach to feedback stabilization of nonlinear systems with drift*, Systems Control Lett., 50 (2003), pp. 303–318.

[MR] M. Motta and F. Rampazzo, *Asymptotic controllability and optimal control,* J. Differential Equations, 254 (2013), pp. 2744–2763.

[RS1] F. Rampazzo and H. J. Sussmann, *Set-valued differentials and a nonsmooth version of Chow's theorem,* in Proceedings of the 40th IEEE Conference on Decision and Control, Orlando, FL, Vol. 3, IEEE Publications, New York, 2001, pp. 2613–2618.

[RS2] F. Rampazzo and H. J. Sussmann, *Commutators of flow maps of nonsmooth vector fields*, J. Differential Equations, 232 (2007), pp. 134–175.

[Ri] L. Rifford, *Existence of Lipschitz and semiconcave control-Lyapunov functions*, SIAM J. Control Optim., 39 (2000), pp. 1043–1064.

[So] P. Soravia, *Hölder continuity of the minimum-time function for $C1$-manifold targets*, J. Optim. Theory Appl., 75 (1992), pp. 401–421.

[Su] H. J. Sussmann, *Geometry and Optimal Control,* in Mathematical Control Theory, J. B. Baillieul and J. C. Willems, eds., Springer Science and Business Media, New York, 2012.

[S1] E. Sontag, *Mathematical Control Theory: Deterministic Finite-Dimensional Systems*, 2nd ed., Texts in Applied Mathematics 6, Springer-Verlag, New York, 1998.

[S2] E. Sontag, *A Lyapunov-like characterization of asymptotic controllability*, SIAM J. Control Optim., 21 (1983), pp. 462–471.

[T] J. Tsinias, *New results on sampled-data feedback stabilization for autonomous nonlinear systems*, Systems Control Lett., 61 (2012), pp. 1032–1040.

[TT] J. Tsinias and D. Theodosis, *Sufficient Lie algebraic conditions for sampled-data feedback stabilizability of affine in the control nonlinear systems*, IEEE Trans. Automat. Control, 61 (2016), pp. 1334–1339.