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# CMC Spheres in the Heisenberg Group

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**Abstract:** We study a family of spheres with constant mean curvature (CMC) in the Riemannian Heisenberg group $H^1$. These spheres are conjectured to be the isoperimetric sets of $H^1$. We prove several results supporting this conjecture. We also focus our attention on the sub-Riemannian limit.

**Keywords:** Constant mean curvature surfaces; Heisenberg group; isoperimetric problem

**MSC:** 49Q10, 53A10

## 1 Introduction

In this paper, we study a family of spheres with constant mean curvature (CMC) in the Riemannian Heisenberg group $H^1$. We introduce in $H^1$ two real parameters that can be used to deform $H^1$ to the sub-Riemannian Heisenberg group, on the one hand, and to the Euclidean space, on the other hand. Even though we are not able to prove that these CMC spheres are in fact isoperimetric sets, we obtain several partial results in this direction. Our motivation comes from the sub-Riemannian Heisenberg group, where it is conjectured that the solution of the isoperimetric problem is obtained rotating a Carnot-Carathéodory geodesic around the center of the group, see [19]. This set is known as Pansu's sphere. The conjecture is proved only assuming some regularity ($C^2$-regularity, convexity) or symmetry, see [4, 10, 16, 17, 20, 21].

Given a real parameter $\tau \in \mathbb{R}$, let $\mathfrak{h} = \operatorname{span}\{X, Y, T\}$ be the three-dimensional real Lie algebra spanned by three elements $X, Y, T$ satisfying the relations $[X, Y] = -2\tau T$ and $[X, T] = [Y, T] = 0$. When $\tau \neq 0$, this is the Heisenberg Lie algebra and we denote by $H^1$ the corresponding Lie group. We will omit reference to the parameter $\tau \neq 0$ in our notation. In suitable coordinates, we can identify $H^1$ with $\mathbb{C} \times \mathbb{R}$ and assume that $X, Y, T$ are left-invariant vector fields in $H^1$ of the form

$$X = \frac{1}{\varepsilon}\Big(\frac{\partial}{\partial x} + \sigma y \frac{\partial}{\partial t}\Big), \quad Y = \frac{1}{\varepsilon}\Big(\frac{\partial}{\partial y} - \sigma x \frac{\partial}{\partial t}\Big), \quad \text{and} \quad T = \varepsilon^2 \frac{\partial}{\partial t}, \tag{1.1}$$

where $(z, t) \in \mathbb{C} \times \mathbb{R}$ and $z = x + iy$. The real parameters $\varepsilon > 0$ and $\sigma \neq 0$ are such that

$$\tau\varepsilon^4 = \sigma. \tag{1.2}$$

Let $\langle\cdot,\cdot\rangle$ be the scalar product on $\mathfrak{h}$ making $X, Y, T$ orthonormal, that is extended to a left-invariant Riemannian metric $g = \langle\cdot,\cdot\rangle$ in $H^1$. The Riemannian volume of $H^1$ induced by this metric coincides with the Lebesgue measure $\mathscr{L}^3$ on $\mathbb{C} \times \mathbb{R}$ and, in fact, it turns out to be independent of $\varepsilon$ and $\sigma$ (and hence of $\tau$). When $\varepsilon = 1$ and $\sigma \to 0$, the Riemannian manifold $(H^1, g)$ converges to the Euclidean space. When $\sigma \neq 0$ and $\varepsilon \to 0^+$, then $H^1$ endowed with the distance function induced by the rescaled metric $\varepsilon^{-2}\langle\cdot,\cdot\rangle$ converges to the sub-Riemannian Heisenberg group.

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The boundary of an isoperimetric region is a surface with constant mean curvature. In this paper, we study a family of CMC spheres $\Sigma_R \subset H^1$, with $R > 0$, that foliate $H^1_\star = H^1 \setminus \{0\}$, where 0 is the neutral element of $H^1$. Each sphere $\Sigma_R$ is centered at 0 and can be described by an explicit formula that was first obtained by Tomter [22]. We conjecture that, within its volume class and up to left translations, the sphere $\Sigma_R$ is the unique solution of the isoperimetric problem in $H^1$. When $\varepsilon = 1$ and $\sigma \to 0$, the spheres $\Sigma_R$ converge to the standard sphere of the Euclidean space. When $\sigma \neq 0$ is fixed and $\varepsilon \to 0^+$, the spheres $\Sigma_R$ converge to the Pansu's sphere.

In Section 3, we study some preliminary properties of $\Sigma_R$, its second fundamental form and principal curvatures. A central object in this setting is the left-invariant 1-form $\vartheta \in \Gamma(T^\star H^1)$ defined by

$$\vartheta(V) = \langle V, T\rangle \quad \text{for any } V \in \Gamma(TH^1). \tag{1.3}$$

The kernel of $\vartheta$ is the horizontal distribution. Let $N$ be the north pole of $\Sigma_R$ and $S = -N$ its south pole. In $\Sigma_R^\star = \Sigma_R \setminus \{\pm N\}$ there is an orthonormal frame of vector fields $X_1, X_2 \in \Gamma(T\Sigma_R^\star)$ such that $\vartheta(X_1) = 0$, i.e., $X_1$ is a linear combination of $X$ and $Y$. In Theorem 3.1, we compute the second fundamental form of $\Sigma_R$ in this frame. We show that the principal directions of $\Sigma_R$ are given by a rotation of the frame $X_1, X_2$ by a *constant* angle depending on the mean curvature of $\Sigma_R$.

In Section 4, we link in a continuous fashion the foliation property of the Pansu's sphere with the foliation by meridians of the round sphere in the Euclidean space. The foliation $H^1_\star = \bigcup_{R>0} \Sigma_R$ determines a unit vector field $\mathscr{N} \in \Gamma(TH^1_\star)$ such that $\mathscr{N}(p) \perp T_p\Sigma_R$ for any $p \in \Sigma_R$ and $R > 0$. The covariant derivative $\nabla_{\mathscr{N}}\mathscr{N}$, where $\nabla$ denotes the Levi-Civita connection induced by the metric $g$, measures how far the integral lines of $\mathscr{N}$ are from being geodesics of $H^1$ (i.e., how far the CMC spheres $\Sigma_R$ are from being metric spheres). In space forms, we would have $\nabla_{\mathscr{N}}\mathscr{N} = 0$, identically. Instead, in $H^1$ the normalized vector field

$$\mathscr{M}(z, t) = \operatorname{sgn}(t)\frac{\nabla_{\mathscr{N}}\mathscr{N}}{|\nabla_{\mathscr{N}}\mathscr{N}|}, \qquad (z, t) \in \Sigma_R^\star,$$

is well-defined and smooth outside the center of $H^1$. In Theorem 4.3, we prove that for any $R > 0$ we have

$$\nabla^{\Sigma_R}_{\mathscr{M}}\mathscr{M} = 0 \quad \text{on } \Sigma_R^\star,$$

where $\nabla^{\Sigma_R}$ denotes the restriction of $\nabla$ to $\Sigma_R$. This means that the integral lines of $\mathscr{M}$ are Riemannian geodesics of $\Sigma_R$. In the coordinates associated with the frame (1.1), when $\varepsilon = 1$ and $\tau = \sigma \to 0$ the integral lines of $\mathscr{M}$ converge to the meridians of the Euclidean sphere. When $\sigma \neq 0$ is fixed and $\varepsilon \to 0^+$, the vector field $\mathscr{M}$ properly normalized converges to the line flow of the geodesic foliation of the Pansu's sphere, see Remark 4.5.

In Section 5, we prove a stability result for the spheres $\Sigma_R$. Let $E_R \subset H^1$ be the region bounded by $\Sigma_R$ and let $\Sigma \subset H^1$ be the boundary of a smooth open set $E \subset H^1$, $\Sigma = \partial E$, such that $\mathscr{L}^3(E) = \mathscr{L}^3(E_R)$. Denoting by $\mathscr{A}(\Sigma)$ the Riemannian area of $\Sigma$, we conjecture that

$$\mathscr{A}(\Sigma) - \mathscr{A}(\Sigma_R) \geq 0. \tag{1.4}$$

We also conjecture that a set $E$ is isoperimetric (i.e., equality holds in (1.4)) if and only if it is a left translation of $E_R$. If isoperimetric sets are topological spheres, this statement would follow from Theorem A.10.

Isoperimetric sets are stable for perturbations fixing the volume: the second variation of the area is nonnegative. The spheres $\Sigma_R$ are in fact stable, this is proved in [24, Theorem 2.3] using Koiso's stability criterium [14]. The stability of $\Sigma_R$ in the northern and southern hemispheres can be obtained in a more elementary way using Jacobi fields arising from right-invariant vector fields of $H^1$. In these hemispheres, we can actually prove a stronger form of stability.

Using the coordinates associated with the frame (1.1), for $R > 0$ and $0 < \delta < R$ we consider the cylinder

$$C_{\delta,R} = \big\{(z, t) \in H^1 : |z| < R, t > f(R - \delta; R)\big\},$$

where $f(\cdot; R)$ is the profile function of $\Sigma_R$, see (2.1). Assume that the closure of $E\Delta E_R = E_R \setminus E \cup E \setminus E_R$ is a compact subset of $C_{\delta,R}$. In Theorem 5.1, we prove that there exists a positive constant $C_{R\tau\varepsilon} > 0$ such that the following quantitative isoperimetric inequality holds:

$$\mathscr{A}(\Sigma) - \mathscr{A}(\Sigma_R) \geq \sqrt{\delta} C_{R\tau\varepsilon} \mathscr{L}^3(E\Delta E_R)^2. \tag{1.5}$$

The proof relies on a sub-calibration argument. This provides further evidence on the conjecture that isoperimetric sets are precisely left translations of $\Sigma_R$. When $\varepsilon = 1$ and $\sigma \to 0$, inequality (1.5) becomes a restricted form of the quantitative isoperimetric inequality in [11]. For fixed $\sigma \neq 0$ and $\varepsilon \to 0^+$ the rescaled area $\varepsilon\mathscr{A}$ converges to the sub-Riemannian Heisenberg perimeter and $\varepsilon C_{R\tau\varepsilon}$ converges to a positive constant. Thus inequality (1.5) reduces to the isoperimetric inequality proved in [10].

In Appendix A, we give a self-contained proof of a known result that is announced in [2, Theorem 6] in the setting of three-dimensional homogeneous spaces with at least 4-dimensional isometry group. Namely, we show that any topological sphere with constant mean curvature in $H^1$ is isometric to a CMC sphere $\Sigma_R$. This result can be deduced by combining [1] and Daniel's correspondence theorem [6]. An alternative proof can be implicitly obtained collecting various results spread in the literature, starting from the Abresch-Rosenberg differential computed in [23], [8, Theorem 2.3], or [3] and then using the rigidity theorem of [6, Theorem 4.3]. A more self-contained proof can be found in [7, Lemma 6.1]. We remark that our proof, that follows the scheme of the fundamental paper [1], does not rely on the fact that the isometry group of $H^1$ is four-dimensional.

## 2 Foliation of $H^1_\star$ by concentric stationary spheres

We start by recalling a result by Tomter [22, Theorem 3]. In what follows, we work in the coordinates associated with the frame (1.1), where the parameters $\varepsilon > 0$ and $\sigma \in \mathbb{R}$ are related by (1.2). For any point $(z, t) \in H^1$, we set $r = |z| = \sqrt{x^2 + y^2}$.

**Theorem 2.1** (Tomter)**.** *For any $R > 0$ there exists a unique compact smooth embedded surface $\Sigma_R \subset H^1$ that is area stationary under volume constraint and such that*

$$\Sigma_R = \{(z, t) \in H^1 : |t| = f(|z|; R)\}$$

*for a function $f(\cdot; R) \in C^\infty([0, R))$ continuous at $r = R$ with $f(R) = 0$. Namely, for any $0 \leq r \leq R$ the function is given by*

$$f(r; R) = \varepsilon^3 \int_r^R \sqrt{\frac{1 + \tau^2\varepsilon^2 s^2}{R^2 - s^2}}\, s ds = \frac{\varepsilon^2}{2\tau}\left[\omega(R)^2 \arctan(p(r; R)) + \omega(r)^2 p(r; R)\right], \tag{2.1}$$

*where*

$$\omega(r) = \sqrt{1 + \tau^2\varepsilon^2 r^2} \quad \text{and} \quad p(r; R) = \tau\varepsilon\frac{\sqrt{R^2 - r^2}}{\omega(r)}.$$

For a proof of Theorem 2.1, we refer to [22, Theorem 3].

**Remark 2.2.** *The function $f(\cdot; R) = f(\cdot; R; \tau; \varepsilon)$ depends also on the parameters $\tau$ and $\varepsilon$, that are omitted in our notation. With $\varepsilon = 1$, we find*

$$\lim_{\tau\to 0} f(r; R; \tau; 1) = \sqrt{R^2 - r^2}.$$

*When $\tau \to 0$, the spheres $\Sigma_R$ converge to Euclidean spheres with radius $R > 0$ in the three-dimensional space.*

*With $\tau = \sigma/\varepsilon^4$ as in (1.2), we find the asymptotic*

$$\begin{aligned}\lim_{\varepsilon\to 0} f(r; R; \sigma/\varepsilon^4; \varepsilon) &= \frac{\sigma}{2}\left[R^2 \arctan\left(\frac{\sqrt{R^2 - r^2}}{r}\right) + r\sqrt{R^2 - r^2}\right] \\ &= \frac{\sigma}{2}\left[R^2 \arccos\left(\frac{r}{R}\right) + r\sqrt{R^2 - r^2}\right],\end{aligned}$$

*which gives the profile function of the Pansu's sphere, the conjectured solution to the sub-Riemannian Heisenberg isoperimetric problem, see e.g. [17] or [16], with $R = 1$ and $\sigma = 2$.*

**Remark 2.3.** *Starting from formula* (2.1), *we can compute the derivatives of* $f(\cdot; R)$ *in the variable* $R$. *The first order derivative is given by*

$$f_R(r;R) = \tau\varepsilon^4 R\Big[\arctan\big(p(r;R)\big) + \frac{1}{p(r;R)}\Big] = \frac{\sigma R}{p(r;R)\ell(p(r;R))}, \tag{2.2}$$

*where* $\ell : [0,\infty) \to \mathbb{R}$ *is the function defined as*

$$\ell(p) = \frac{1}{1 + p\arctan(p)}. \tag{2.3}$$

*The geometric meaning of* $\ell$ *will be clear in formula* (4.1).

We now show that $H^1_\star = H^1 \setminus \{0\}$ is foliated by the family $\{\Sigma_R\}_{R>0}$, i.e.,

$$H^1_\star = \bigcup_{R>0} \Sigma_R. \tag{2.4}$$

**Proposition 2.4.** *For any nonzero* $(z,t) \in H^1$ *there exists a unique* $R > 0$ *such that* $(z,t) \in \Sigma_R$.

*Proof.* Without loss of generality we can assume that $t \geq 0$. After an integration by parts in (2.1), we obtain the formula

$$f(r;R) = \varepsilon^3\Bigg\{\sqrt{R^2 - r^2}\,\omega(r) + \int_r^R \sqrt{R^2 - s^2}\,\omega_r(s)ds\Bigg\}, \quad 0 \leq r \leq R.$$

Since $\omega_r(r) > 0$ for $r > 0$, we deduce that the function $R \mapsto f(r;R)$ is strictly increasing for $R \geq r$. Moreover, we have

$$\lim_{R\to\infty} f(r;R) = \infty,$$

and hence for any $r \geq 0$ there exists a unique $R \geq r$ such that $f(r;R) = t$. □

**Remark 2.5.** *By Proposition 2.4, we can define the function* $R : H^1 \to [0,\infty)$ *by letting* $R(0) = 0$ *and* $R(z,t) = R$ *if and only if* $(z,t) \in \Sigma_R$ *for* $R > 0$. *The function* $R(z,t)$, *in fact, depends on* $r = |z|$ *and thus we may consider* $R(z,t) = R(r,t)$ *as a function of* $r$ *and* $t$. *This function is implicitly defined by the equation* $|t| = f(r; R(r,t))$. *Differentiating this equation, we find the derivatives of* $R$, *i.e.,*

$$R_r = -\frac{f_r}{f_R} \quad \textit{and} \quad R_t = \frac{\operatorname{sgn}(t)}{f_R}, \tag{2.5}$$

*where* $f_R$ *is given by* (2.2).

# 3 Second fundamental form of $\Sigma_R$

In this section, we compute the second fundamental form of the spheres $\Sigma_R$. In fact, we will see that $H = 1/(\varepsilon R)$ is the mean curvature of $\Sigma_R$. Let $N = (0, f(0;R)) \in \Sigma_R$ be the north pole of $\Sigma_R$ and let $S = -N = (0, -f(0;R))$ be its south pole. In $\Sigma_R^\star = \Sigma_R \setminus \{\pm N\}$ there is a frame of tangent vector fields $X_1, X_2 \in \Gamma(T\Sigma_R^\star)$ such that

$$|X_1| = |X_2| = 1, \quad \langle X_1, X_2\rangle = 0, \quad \vartheta(X_1) = 0, \tag{3.1}$$

where $\vartheta$ is the left-invariant 1-form introduced in (1.3). Explicit expressions for $X_1$ and $X_2$ are given in formula (3.9) below. This frame is unique up to the sign $\pm X_1$ and $\pm X_2$. Here and in the rest of the paper, we denote by $\mathscr{N}$ the exterior unit normal to the spheres $\Sigma_R$.

The second fundamental form $h$ of $\Sigma_R$ with respect to the frame $X_1, X_2$ is given by

$$h = (h_{ij})_{i,j=1,2}, \qquad h_{ij} = \langle \nabla_{X_i}\mathscr{N}, X_j\rangle, \quad i,j = 1,2,$$

where $\nabla$ denotes the Levi-Civita connection of $H^1$ endowed with the left-invariant metric $g$. The linear connection $\nabla$ is represented by the linear mapping $\mathfrak{h}\times\mathfrak{h}\mapsto\mathfrak{h}$, $(V,W)\mapsto\nabla_V W$. Using the fact that the connection is torsion free and metric, it can be seen that $\nabla$ is characterized by the following relations:

$$\begin{aligned}
&\nabla_X X = \nabla_Y Y = \nabla_T T = 0,\\
&\nabla_Y X = \tau T \quad\text{and}\quad \nabla_X Y = -\tau T,\\
&\nabla_T X = \nabla_X T = \tau Y,\\
&\nabla_T Y = \nabla_Y T = -\tau X.
\end{aligned}\tag{3.2}$$

Here and in the rest of the paper, we use the coordinates associated with the frame (1.1). For $(z,t)\in H^1$, we set $r=|z|$ and use the short notation

$$\varrho = \tau\varepsilon r.\tag{3.3}$$

**Theorem 3.1.** *For any $R>0$, the second fundamental form $h$ of $\Sigma_R$ with respect to the frame $X_1, X_2$ in* (3.1) *at the point $(z,t)\in\Sigma_R$ is given by*

$$h = \frac{1}{1+\varrho^2}\begin{pmatrix} H(1+2\varrho^2) & \tau\varrho^2\\ \tau\varrho^2 & H\end{pmatrix},\tag{3.4}$$

*where $R = 1/H\varepsilon$ and $H$ is the mean curvature of $\Sigma_R$. The principal curvatures of $\Sigma_R$ are given by*

$$\begin{aligned}
\kappa_1 &= H + \frac{\varrho^2}{1+\varrho^2}\sqrt{H^2+\tau^2},\\
\kappa_2 &= H - \frac{\varrho^2}{1+\varrho^2}\sqrt{H^2+\tau^2}.
\end{aligned}\tag{3.5}$$

*Outside the north and south poles, principal directions are given by*

$$\begin{aligned}
K_1 &= \cos\beta X_1 + \sin\beta X_2,\\
K_2 &= -\sin\beta X_1 + \cos\beta X_2,
\end{aligned}\tag{3.6}$$

*where $\beta=\beta_H\in(-\pi/4,\pi/4)$ is the angle*

$$\beta_H = \arctan\left(\frac{\tau}{H+\sqrt{H^2+\tau^2}}\right).\tag{3.7}$$

*Proof.* Let $a,b:\Sigma_R^\star\to\mathbb{R}$ and $c,p:\Sigma_R\to\mathbb{R}$ be the following functions depending on the radial coordinate $r=|z|$:

$$\begin{aligned}
a &= a(r;R) = \frac{\omega(r)}{r\omega(R)}, \qquad b = b(r;R) = \pm\frac{\sqrt{R^2-r^2}}{rR\omega(R)},\\
c &= c(r;R) = \frac{r\omega(R)}{R\omega(r)}, \qquad p = p(r;R) = \pm\tau\varepsilon\frac{\sqrt{R^2-r^2}}{\omega(r)}.
\end{aligned}\tag{3.8}$$

In fact, $b$ and $p$ also depend on the sign of $t$. Namely, in $b$ and $p$ we choose the sign + in the northern hemisphere, that is for $t\geq 0$, while we choose the sign – in the southern hemisphere, where $t\leq 0$. Our computations are in the case $t\geq 0$.

One can check that the vector fields

$$\begin{aligned}
X_1 &= -a\big((y-xp)X-(x+yp)Y\big),\\
X_2 &= -b\big((x+yp)X+(y-xp)Y\big)+cT
\end{aligned}\tag{3.9}$$

form an orthonormal frame for $T\Sigma_R^\star$ satisfying (3.1). The the outer unit normal to $\Sigma_R$ is given by the following formula (which is well defined also at the poles):

$$\mathcal{N} = \frac{1}{R}\Big\{(x+yp)X+(y-xp)Y+\frac{p}{\tau\varepsilon}T\Big\}.\tag{3.10}$$

We compute the entries $h_{11}$ and $h_{12}$. Using $X_1R = 0$, we find

$$\nabla_{X_1}\mathcal{N} = \frac{1}{R}\Big\{X_1(x+yp)X + X_1(y-xp)Y + X_1\Big(\frac{p}{\tau\varepsilon}\Big)T + (x+yp)\nabla_{X_1}X + (y-xp)\nabla_{X_1}Y + \frac{p}{\tau\varepsilon}\nabla_{X_1}T\Big\}. \tag{3.11}$$

Using the formulas $X_1x = -a(y-xp)/\varepsilon$ and $X_1y = a(x+yp), /\varepsilon$ we find the derivatives

$$\begin{aligned} X_1(x+yp) &= \frac{a}{\varepsilon}\big(2xp + y(p^2-1)\big) + yX_1p, \\ X_1(y-xp) &= \frac{a}{\varepsilon}\big(2yp + x(1-p^2)\big) - xX_1p. \end{aligned} \tag{3.12}$$

Inserting the latter into (3.11), together with the fundamental relations (3.2), we obtain

$$\nabla_{X_1}\mathcal{N} = \frac{1}{R}\Big\{\Big[-\frac{a}{\varepsilon}(y-xp) + yX_1p\Big]X + \Big[\frac{a}{\varepsilon}(x+yp) - xX_1p\Big]Y + \Big[\frac{X_1p}{\tau\varepsilon} + \tau r^2a(p^2+1)\Big]T\Big\}, \tag{3.13}$$

thus implying

$$h_{11} = \langle\nabla_{X_1}\mathcal{N}, X_1\rangle = \frac{r^2a}{R\varepsilon}\big\{a(p^2+1) - \varepsilon X_1p\big\},$$

where $p^2+1 = \omega(R)^2/\omega(r)^2$ and $X_1p$ can be computed starting from

$$p_r(r;R) = -\tau\varepsilon r\frac{\omega(R)^2}{\sqrt{R^2 - r^2\omega(r)^3}}. \tag{3.14}$$

Namely, also using the formula for $a$ and $p$ in (3.8), we have

$$X_1p = \frac{ra}{\varepsilon}pp_r = -\tau^2\varepsilon r\frac{\omega(R)}{\omega(r)^3}.$$

By (3.3) and the fact that $\varepsilon HR = 1$, we finally find

$$h_{11} = \frac{1}{R\varepsilon}\Big(1 + \frac{\tau^2\varepsilon^2r^2}{\omega(r)^2}\Big) = H\Big(1 + \frac{\varrho^2}{1+\varrho^2}\Big).$$

From (3.13) we also deduce

$$h_{12} = \langle\nabla_{X_1}\mathcal{N}, X_2\rangle = -\frac{b}{R}r^2pX_1p + \frac{c}{R}\Big\{\frac{X_1p}{\tau\varepsilon} + \tau r^2a(1+p^2)\Big\},$$

and using the formula for $X_1p$ and the formulas in (3.8) we obtain

$$h_{12} = \frac{\tau\varrho^2}{1+\varrho^2}.$$

To compute the entry $h_{22}$, we proceed in an equivalent way, starting from

$$\nabla_{X_2}\mathcal{N} = \frac{1}{R}\Big\{X_2(x+yp)X + X_2(y-xp)Y + \frac{X_2(p)}{\tau\varepsilon}T + (x+yp)\nabla_{X_2}X + (y-xp)\nabla_{X_2}Y + \frac{p}{\tau\varepsilon}\nabla_{X_2}T\Big\},$$

yielding $h_{22} = H/(1+\varrho^2)$.

The principal curvatures $\kappa_1, \kappa_2$ of $\Sigma_R$ are the solutions to the system

$$\begin{cases} \kappa_1 + \kappa_2 = \mathrm{tr}(h) = 2H \\ \kappa_1\kappa_2 = \det(h) = \dfrac{H^2(1+2\varrho^2) - \tau^2\varrho^4}{(1+\varrho^2)^2}. \end{cases}$$

They are given explicitly by the formulas (3.5).

Now let $K_1, K_2$ be tangent vectors as in (3.6). We identify $h$ with the shape operator $h \in \mathrm{Hom}(T_p\Sigma_R; T_p\Sigma_R)$, $h(K) = \nabla_K\mathcal{N}$, at any point $p \in \Sigma_R$ and $K \in T_p\Sigma_R$. When $\varrho \neq 0$ (i.e., outside the north and south poles), the system of equations

$$h(K_1) = \kappa_1K_1 \quad \text{and} \quad h(K_2) = \kappa_2K_2$$

is satisfied if and only if the angle $\beta = \beta_H$ is chosen as in (3.7). The argument of arctan in (3.7) is in the interval $(-1, 1)$ and thus $\beta_H \in (-\pi/4, \pi/4)$. □

**Remark 3.2.** *The convergence of the Riemannian second fundamental form towards its sub-Riemannian counterpart is studied in [5], in the setting of Carnot groups. See also [18].*

# 4 Geodesic foliation of $\Sigma_R$

We prove that each CMC sphere $\Sigma_R$ is foliated by a family of geodesics of $\Sigma_R$ joining the north to the south pole. In fact, we show that the foliation is governed by the normal $\mathcal{N}$ to the foliation $H^1_* = \bigcup_{R>0} \Sigma_R$. In the sub-Riemannian limit, we recover the foliation property of the Pansu's sphere. In the Euclidean limit, we find the foliation of the round sphere with meridians.

We need two preliminary lemmas. We define a function $R : H^1 \to [0, \infty)$ by letting $R(0) = 0$ and $R(z, t) = R$ if and only if $(z, t) \in \Sigma_R$. In fact, $R(z, t)$ depends on $r = |z|$ and $t$. The function $p$ in (3.8) is of the form $p = p(r, R(r, t))$.

Now, we compute the derivative of these functions in the normal direction $\mathcal{N}$.

**Lemma 4.1.** *The derivative along $\mathcal{N}$ of the functions R and p are, respectively,*

$$\mathcal{N} R = \frac{\ell(p)}{\varepsilon}, \tag{4.1}$$

*and*

$$\mathcal{N} p = \varepsilon\tau^2 \frac{R^2\omega(r)^2\ell(p) - r^2\omega(R)^2}{R\omega(r)^4 p}, \tag{4.2}$$

*where $\ell(p) = (1 + p\arctan p)^{-1}$, as in* (2.3).

*Proof.* We start from the following expression for the unit normal (in the coordinates $(x, y, t)$):

$$\mathcal{N} = \frac{1}{R}\left\{\frac{r}{\varepsilon}\partial_r + \frac{p}{\varepsilon}(y\partial_x - x\partial_y) + \operatorname{sgn}(t)\varepsilon^2\omega(r)\sqrt{R^2 - r^2}\partial_t\right\}.$$

We just consider the case $t \geq 0$. Using (2.5), we obtain

$$\mathcal{N} R = \frac{1}{R}\left\{\frac{r}{\varepsilon}R_r + \varepsilon^2\omega(r)\sqrt{R^2 - r^2}R_t\right\} = \frac{1}{Rf_R}\left\{\varepsilon^2\omega(r)\sqrt{R^2 - r^2} - \frac{r}{\varepsilon}f_r\right\}.$$

Inserting into this formula the expression for $f_r$ computed from (2.1), we get

$$\mathcal{N} R = \frac{\varepsilon^2 R\omega(r)}{f_R\sqrt{R^2 - r^2}},$$

and using formula (2.2) for $f_R$, namely,

$$f_R = \tau\varepsilon^4 R\left[\arctan(p) + \frac{1}{p}\right] = \frac{\tau\varepsilon^4 R}{p\ell(p)},$$

we obtain formula (4.1).

To compute the derivatives of $p$ in $r$ and $t$, we have to consider $p = p(r; R)$ and $R = R(r, t)$. Using the formula in (3.8) for $p$ and the expression (2.5) for $R_r$ yields

$$p_r = -\frac{\tau\varepsilon r\omega(R)^2}{\omega(r)^3\sqrt{R^2 - r^2}}, \quad p_R = \frac{\tau\varepsilon R}{\omega(r)\sqrt{R^2 - r^2}}, \quad R_r = -\frac{f_r}{f_R} = \frac{\varepsilon^3 r\omega(r)}{\sqrt{R^2 - r^2}f_R},$$

and thus

$$\begin{aligned}\frac{\partial}{\partial r}p(r, R(r, t)) &= p_r(r, R(r, t)) + p_R(r, R(r, t))R_r(r, t)\\ &= \frac{\tau\varepsilon r}{\omega(r)^3\sqrt{R^2 - r^2}}\left[\omega(r)^2\ell(p) - \omega(R)^2\right].\end{aligned}$$

Similarly, we compute

$$\frac{\partial}{\partial t}p(r; R(r, t)) = p_R(r; R(r, t))R_t(r, t) = \frac{\tau\ell(p)}{\varepsilon^2\omega(r)^2}.$$

The derivative of $p$ along $\mathcal{N}$ is thus as in (4.2), when $t \geq 0$. The case $t < 0$ is analogous. □

In the next lemma, we compute the covariant derivative $\nabla_{\mathcal{N}}\mathcal{N}$. The resulting vector field in $H^1_\star$ is tangent to each CMC sphere $\Sigma_R$, for any $R > 0$.

**Lemma 4.2.** *At any point in $(z, t) \in H^1_\star$ we have*

$$\nabla_{\mathcal{N}}\mathcal{N}(z,t) = \mathcal{N}\Big(\frac{p}{R}\Big)\Big[(y + x\Phi)X - (x - y\Phi)Y + \frac{1}{\tau\varepsilon}T\Big], \tag{4.3}$$

*where $\Phi = \Phi(r; R)$ is the function defined as*

$$\Phi = -\frac{\omega(r)^2 p}{\tau^2\varepsilon^2 r^2},$$

*and the derivative $\mathcal{N}(p/R)$ is given by*

$$\mathcal{N}\Big(\frac{p}{R}\Big) = -\frac{\varepsilon\tau^2 r^2\big(\omega(R)^2 - \ell(p)\omega(r)^2\big)}{R^2\omega(r)^4 p},$$

*with $\ell$ as in* (2.3).

*Proof.* Starting from formula (3.10) for $\mathcal{N}$, we find that

$$\begin{aligned}\nabla_{\mathcal{N}}\mathcal{N} &= \mathcal{N}\Big(\frac{x + yp}{R}\Big)X + \mathcal{N}\Big(\frac{y - xp}{R}\Big)Y + \mathcal{N}\Big(\frac{p}{\tau\varepsilon R}\Big)T \\ &\quad + \frac{1}{R}\Big((x + yp)\nabla_{\mathcal{N}}X + (y - xp)\nabla_{\mathcal{N}}Y + \frac{p}{\tau\varepsilon}\nabla_{\mathcal{N}}T\Big),\end{aligned} \tag{4.4}$$

where, by the fundamental relations (3.2), we have

$$(x + yp)\nabla_{\mathcal{N}}X + (y - xp)\nabla_{\mathcal{N}}Y + \frac{p}{\tau\varepsilon}\nabla_{\mathcal{N}}T = \frac{2p}{\varepsilon R}\Big(-(y - xp)X + (x + yp)Y\Big). \tag{4.5}$$

From the elementary formulas

$$\mathcal{N}x = \frac{1}{R\varepsilon}(x + yp) \quad \text{and} \quad \mathcal{N}y = \frac{1}{R\varepsilon}(y - xp),$$

we find

$$\begin{aligned}\mathcal{N}(x + yp) &= \frac{1}{\varepsilon R}\big(x(1 - p^2) + 2yp\big) + y\mathcal{N}p, \\ \mathcal{N}(y - xp) &= \frac{1}{\varepsilon R}\big(y(1 - p^2) - 2xp\big) - x\mathcal{N}p.\end{aligned} \tag{4.6}$$

Inserting (4.5) and (4.6) into (4.4) we obtain the following expression

$$\begin{aligned}\nabla_{\mathcal{N}}\mathcal{N} = \frac{1}{R^2}\Big[&\Big\{x(\varepsilon^{-1}(1 + p^2) - \mathcal{N}R) + y(R\mathcal{N}p - p\mathcal{N}R)\Big\}X \\ &+\Big\{y(\varepsilon^{-1}(1 + p^2) - \mathcal{N}R) - x(R\mathcal{N}p - p\mathcal{N}R)\Big\}Y + \frac{1}{\tau\varepsilon}(R\mathcal{N}p - p\mathcal{N}R)T\Big].\end{aligned} \tag{4.7}$$

From (4.1) and (4.2) we compute

$$R\mathcal{N}p - p\mathcal{N}R = -\frac{\varepsilon\tau^2 r^2}{\omega(r)^4 p}\big[\omega(R)^2 - \ell(p)\omega(r)^2\big].$$

Inserting this formula into (4.7) and using $1 + p^2 = \omega(R)^2/\omega(r)^2$ yields the claim. □

Let $\mathcal{N} \in \Gamma(TH^1_\star)$ be the exterior unit normal to the family of CMC spheres $\Sigma_R$ centered at $0 \in H^1$. The vector field $\nabla_{\mathcal{N}}\mathcal{N}$ is tangent to $\Sigma_R$ for any $R > 0$, and for $(z, t) \in \Sigma_R$ we have

$$\nabla_{\mathcal{N}}\mathcal{N}(z,t) = 0 \quad \text{if and only if} \quad z = 0 \text{ or } t = 0.$$

However, it can be checked that the normalized vector field

$$\mathscr{M}(z,t) = \operatorname{sgn}(t)\frac{\nabla_{\mathscr{N}}\mathscr{N}}{|\nabla_{\mathscr{N}}\mathscr{N}|} \in \Gamma(T\Sigma_R^\star)$$

is smoothly defined also at points $(z,t) \in \Sigma_R$ at the equator, where $t = 0$. We denote by $\nabla^{\Sigma_R}$ the restriction of the Levi-Civita connection $\nabla$ to $\Sigma_R$.

**Theorem 4.3.** *Let $\Sigma_R \subset H^1$ be the CMC sphere with mean curvature $H > 0$. Then the vector field $\nabla_{\mathscr{M}}\mathscr{M}$ is smoothly defined on $\Sigma_R$ and for any $(z,t) \in \Sigma_R$ we have*

$$\nabla_{\mathscr{M}}\mathscr{M}(z,t) = -\frac{H}{\omega(r)^2}\mathscr{N}. \tag{4.8}$$

*In particular, $\nabla^{\Sigma_R}_{\mathscr{M}}\mathscr{M} = 0$ and the integral curves of $\mathscr{M}$ are Riemannian geodesics of $\Sigma_R$ joining the north pole N to the south pole S. (See Figure 1.)*

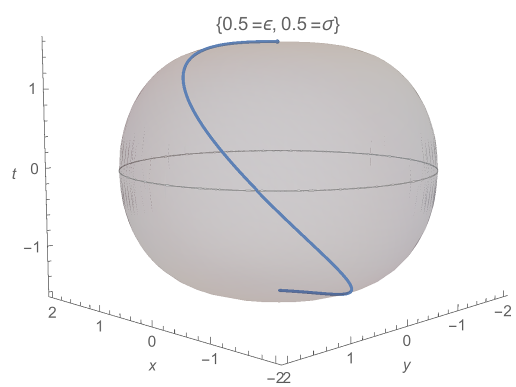


**Figure 1:** The plotted curve is an integral curve of the vector field $\mathscr{M}$ for $R = 2$, $\varepsilon = 0.5$, and $\sigma = 0.5$.

*Proof.* From (4.3) we obtain the following formula for $\mathscr{M}$:

$$\mathscr{M} = (x\lambda - y\mu)X + (y\lambda + x\mu)Y - \frac{\mu}{\tau\varepsilon}T, \tag{4.9}$$

where $\lambda, \mu : \Sigma_R^\star \to \mathbb{R}$ are the functions

$$\lambda = \lambda(r) = \pm\frac{\sqrt{R^2 - r^2}}{rR} \quad \text{and} \quad \mu = \mu(r) = \frac{\tau\varepsilon r}{R\omega(r)}, \tag{4.10}$$

with $r = |z|$ and $R = 1/(\varepsilon H)$. The functions $\lambda$ and $\mu$ are radially symmetric in $z$. In defining $\lambda$ we choose the sign $+$, when $t \geq 0$, and the sign $-$, when $t < 0$. In the coordinates $(x,y,t)$, the vector field $\mathscr{M}$ has the following expression

$$\mathscr{M} = \frac{1}{\varepsilon}\Big(\lambda r\partial_r + \mu(x\partial_y - y\partial_x) - \mu\frac{\varepsilon^2\omega(r)^2}{\tau}\partial_t\Big), \tag{4.11}$$

where $r\partial_r = x\partial_x + y\partial_y$, and so we have

$$\nabla_{\mathscr{M}}\mathscr{M} = (x\lambda - y\mu)\nabla_{\mathscr{M}}X + (y\lambda + x\mu)\nabla_{\mathscr{M}}Y - \frac{\mu}{\tau\varepsilon}\nabla_{\mathscr{M}}T + \mathscr{M}(x\lambda - y\mu)X + \mathscr{M}(y\lambda + x\mu)Y - \mathscr{M}\Big(\frac{\mu}{\tau\varepsilon}\Big)T. \tag{4.12}$$

Using (4.11), we compute

$$\mathscr{M}x = \frac{1}{\varepsilon}(x\lambda - y\mu) \quad \text{and} \quad \mathscr{M}y = \frac{1}{\varepsilon}(y\lambda + x\mu), \tag{4.13}$$

and so we find

$$\begin{aligned}\mathscr{M}(x\lambda - y\mu) &= \frac{1}{\varepsilon}(x\lambda - y\mu)\lambda + x\mathscr{M}\lambda - \frac{1}{\varepsilon}(y\lambda + x\mu)\mu - y\mathscr{M}\mu,\\ \mathscr{M}(y\lambda + x\mu) &= \frac{1}{\varepsilon}(y\lambda + x\mu)\lambda + y\mathscr{M}\lambda + \frac{1}{\varepsilon}(x\lambda - y\mu)\mu + x\mathscr{M}\mu.\end{aligned} \tag{4.14}$$

Now, inserting (4.13) and (4.14) into (4.12), we get

$$\nabla_{\mathscr{M}}\mathscr{M} = \Big(\frac{x}{\varepsilon}(\lambda^2 + \mu^2) + x\mathscr{M}\lambda - y\mathscr{M}\mu\Big)X\Big(\frac{y}{\varepsilon}(\lambda^2 + \mu^2) + y\mathscr{M}\lambda + x\mathscr{M}\mu\Big)Y - \frac{1}{\tau\varepsilon}\mathscr{M}\mu T.$$

The next computations are for the case $t \geq 0$. Again from (4.11), we get

$$\mathscr{M}\lambda = \frac{\lambda r}{\varepsilon}\partial_r\lambda = -\frac{R\lambda}{\varepsilon r\sqrt{R^2 - r^2}}, \quad \text{and} \quad \mathscr{M}\mu = \frac{\lambda r}{\varepsilon}\partial_r\mu = \frac{\tau r\lambda}{R\omega(r)^3}. \tag{4.15}$$

From (4.10) and (4.15) we have

$$\frac{1}{\varepsilon}(\lambda^2 + \mu^2) + \mathscr{M}\lambda = -\frac{1}{\varepsilon R^2\omega(r)^2},$$

and so we finally obtain

$$\nabla_{\mathscr{M}}\mathscr{M} = (x\Lambda - yM)X + (y\Lambda + xM)Y - \frac{M}{\tau\varepsilon}T, \tag{4.16}$$

where we have set

$$\Lambda = -\frac{1}{\varepsilon R^2\omega(r)^2}, \qquad M = \tau\frac{\sqrt{R^2 - r^2}}{R^2\omega(r)^3}. \tag{4.17}$$

Comparing with (3.10), we deduce that

$$\nabla_{\mathscr{M}}\mathscr{M} = -\frac{1}{\varepsilon R\omega(r)^2}\mathscr{N}.$$

The claim $\nabla^{\Sigma_R}_{\mathscr{M}}\mathscr{M} = 0$ easily follows from the last formula. □

**Remark 4.4.** *We compute the pointwise limit of $\mathscr{M}$ in (4.9) when $\sigma \to 0$, for $t \geq 0$. In the southern hemisphere the situation is analogous. By (4.11), the vector field $\mathscr{M}$ is given by*

$$\mathscr{M} = \frac{1}{\varepsilon R}\Big(\frac{\sqrt{R^2 - r^2}}{r}(x\partial_x + y\partial_y) + \frac{\sigma r}{\sqrt{\varepsilon^6 + \sigma^2 r^2}}(x\partial_y - y\partial_x) - r\sqrt{\varepsilon^6 + \sigma^2 r^2}\partial_t\Big).$$

*With $\varepsilon = 1$ we have*

$$\widehat{\mathscr{M}} = \lim_{\sigma\to 0}\mathscr{M} = \frac{\sqrt{R^2 - r^2}}{rR}(x\partial_x + y\partial_y) - \frac{r}{R}\partial_t.$$

*Clearly, the vector field $\widehat{\mathscr{M}}$ is tangent to the round sphere of radius $R > 0$ in the three-dimensional Euclidean space and its integral lines turn out to be the meridians from the north to the south pole.*

**Remark 4.5.** *We study the limit of $\varepsilon\mathscr{M}$ when $\varepsilon \to 0$, in the northern hemisphere.*

*The frame of left-invariant vector fields $\bar{X} = \varepsilon X$, $\bar{Y} = \varepsilon Y$ and $\bar{T} = \varepsilon^{-2}T$ is independent of $\varepsilon$. Moreover, the linear connection $\nabla$ restricted to the horizontal distribution spanned by $\bar{X}$ and $\bar{Y}$ is independent of the parameter $\varepsilon$. Indeed, from the fundamental relations (3.2) and from (1.2) we find*

$$\begin{aligned}&\nabla_{\bar{X}}\bar{X} = \nabla_{\bar{Y}}\bar{Y} = 0,\\ &\nabla_{\bar{X}}\bar{Y} = -\sigma\bar{T} \quad \textit{and} \quad \nabla_{\bar{Y}}\bar{X} = \sigma\bar{T}.\end{aligned}$$

*Now, it turns out that*

$$\begin{aligned}\bar{\mathscr{M}} = \lim_{\varepsilon\to 0}\varepsilon\mathscr{M} &= \frac{1}{R}\Big[\Big(x\frac{\sqrt{R^2 - r^2}}{r} - y\Big)\partial_x + \Big(y\frac{\sqrt{R^2 - r^2}}{r} + x\Big)\partial_y - \sigma r^2\partial_t\Big]\\ &= (x\bar{\lambda} - y\bar{\mu})\bar{X} + (y\bar{\lambda} + x\bar{\mu})\bar{Y},\end{aligned}$$

*where*

$$\bar{\lambda} = \lambda = \frac{\sqrt{R^2 - r^2}}{rR}, \qquad \bar{\mu} = \frac{1}{R}.$$

*The vector field $\bar{\mathscr{M}}$ is horizontal and tangent to the Pansu's sphere.*

*We denote by $J$ the complex structure $J(\bar{X}) = \bar{Y}$ and $J(\bar{Y}) = -\bar{X}$. A computation similar to the one in the proof of Theorem 4.3 shows that*

$$\nabla_{\bar{\mathscr{M}}}\bar{\mathscr{M}} = \frac{2}{R}J(\bar{\mathscr{M}}). \tag{4.18}$$

*This is the equation for Carnot-Carathéodory geodesics in $H^1$ for the sub-Riemannian metric making $\bar{X}$ and $\bar{Y}$ orthonormal, see [21, Proposition 3.1].*

*Thus, we reached the following conclusion. The integral curves of $\mathscr{M}$ are Riemannian geodesics of $\Sigma_R$ and converge to the integral curves of $\bar{\mathscr{M}}$. These curves foliate the Pansu's sphere and are Carnot-Carathéodory geodesics (not only of the Pansu's sphere but also) of $H^1$.*

*Using* (4.18) *we can pass to the limit as $\varepsilon \to 0$ in equation* (4.8)*, properly scaled. An inspection of the right hand side in* (4.16) *shows that the right hand side of* (4.8) *is asymptotic to $\varepsilon^4$. In fact, starting from* (4.17) *we get*

$$-\lim_{\varepsilon\to 0}\frac{H}{\varepsilon^4\omega(r)^2}\mathscr{N} = \frac{1}{R\sigma^2 r^2}\big[-(x\bar{\mu} + y\bar{\lambda})\bar{X} + (x\bar{\lambda} - y\bar{\mu})\bar{Y}\big] = \frac{1}{R\sigma^2 r^2}J(\bar{\mathscr{M}}). \tag{4.19}$$

*From* (4.8), (4.18), *and* (4.19) *we deduce that*

$$\lim_{\varepsilon\to 0}\varepsilon^{-4}\nabla_{\mathscr{M}}\mathscr{M} = \frac{1}{2\sigma^2 r^2}\nabla_{\bar{\mathscr{M}}}\bar{\mathscr{M}}.$$

## 5 Quantitative stability of $\Sigma_R$ in vertical cylinders

In this section, we prove a quantitative isoperimetric inequality for the CMC spheres $\Sigma_R$ with respect to compact perturbations in vertical cylinders, see Theorem 5.1. This is a strong form of stability of $\Sigma_R$ in the northern and southern hemispheres.

A CMC surface $\Sigma$ in $H^1$ with normal $\mathscr{N}$ is stable in an open region $A \subset \Sigma$ if for any function $g \in C_c^\infty(A)$ with $\int_\Sigma g\,d\mathscr{A} = 0$, where $\mathscr{A}$ is the Riemannian area measure of $\Sigma$, we have

$$\mathscr{S}(g) = \int_\Sigma \big\{|\nabla g|^2 - \big(|h|^2 + \mathrm{Ric}(\mathscr{N})\big)g^2\big\}d\mathscr{A} \geq 0.$$

The functional $\mathscr{S}(g)$ is the second variation, with fixed volume, of the area of $\Sigma$ with respect to the infinitesimal deformation of $\Sigma$ in the direction $g\mathscr{N}$. Above, $|\nabla g|$ is the length of the tangential gradient of $g$, $|h|^2$ is the squared norm of the second fundamental form of $\Sigma$ and $\mathrm{Ric}(\mathscr{N})$ is the Ricci curvature of $H^1$ in the direction $\mathscr{N}$.

The Jacobi operator associated with the second variation functional $\mathscr{S}$ is

$$\mathscr{L}g = \Delta g + (|h|^2 + \mathrm{Ric}(\mathscr{N}))g,$$

where $\Delta$ is the Laplace-Beltrami operator of $\Sigma$. As a consequence of Theorem 1 in [9], if there exists a strictly positive solution $g \in C^\infty(A)$ to equation $\mathscr{L}g = 0$ on $A$, then $\Sigma$ is stable in $A$ (even without the restriction $\int_A g\,d\mathscr{A} = 0$).

Now consider in $H^1$ the right-invariant vector fields

$$\widehat{X} = \frac{1}{\varepsilon}\Big(\frac{\partial}{\partial x} - \sigma y\frac{\partial}{\partial t}\Big), \quad \widehat{Y} = \frac{1}{\varepsilon}\Big(\frac{\partial}{\partial y} + \sigma x\frac{\partial}{\partial t}\Big), \quad \text{and} \quad \widehat{T} = \varepsilon^2\frac{\partial}{\partial t}.$$

These are generators of left-translations in $H^1$, and the functions

$$g_{\widehat{X}} = \langle\widehat{X}, \mathscr{N}\rangle, \quad g_{\widehat{Y}} = \langle\widehat{Y}, \mathscr{N}\rangle, \quad g_{\widehat{T}} = \langle\widehat{T}, \mathscr{N}\rangle$$

are solutions to $\mathscr{L}g = 0$. By the previous discussion, the CMC sphere $\Sigma_R$ is stable in the hemispheres

$$A_{\widehat{X}} = \{(z,t) \in \Sigma_R : g_{\widehat{X}} > 0\},$$
$$A_{\widehat{Y}} = \{(z,t) \in \Sigma_R : g_{\widehat{Y}} > 0\},$$
$$A_{\widehat{T}} = \{(z,t) \in \Sigma_R : g_{\widehat{T}} > 0\}.$$

In particular, $\Sigma_R$ is stable in the northern hemisphere $A_{\widehat{T}} = \{(z,t) \in \Sigma_R : t > 0\}$.

In fact, the whole $\Sigma_R$ is stable. This is shown in [24] producing a function $v$ un $\Sigma_R$ orthogonal to the kernel of $\mathscr{L}$, solving $\mathscr{L}v = 1$, and with nonnegative integral on $\Sigma_R$.

In the case of the northern hemisphere, we can prove the following quantitative stability. For $R > 0$, let $E_R \subset H^1$ be the open domain bounded by the CMC sphere $\Sigma_R$,

$$E_R = \{(z,t) \in H^1 : |t| < f(|z|;R),\ |z| < R\},$$

where $f(\cdot;R)$ is the profile function of $\Sigma_R$ in (2.1). For $0 \le \delta < R$, we define the half-cylinder

$$C_{R,\delta} = \{(z,t) \in H^1 : |z| < R \text{ and } t > t_{R,\delta}\},$$

where $t_{R,\delta} = f(r_{R,\delta};R)$ and $r_{R,\delta} = R - \delta$. In the following, we use the short notation

$$\begin{aligned} k_{R\varepsilon\tau} &= \varepsilon^3\omega(R)\sqrt{R}, \\ C_{R\varepsilon\tau} &= \frac{1}{4\pi\varepsilon R^3(Rk_{R\varepsilon\tau} + f(0;R))}, \\ D_{R\varepsilon\tau} &= \frac{1}{12\varepsilon\pi^2R^5(4Rk^2_{R\varepsilon\tau} + f(0;R)^2)}. \end{aligned} \tag{5.1}$$

We denote by $\mathscr{A}$ the Riemannian surface-area measure in $H^1$.

**Theorem 5.1.** *Let $R > 0$, $0 \le \delta < R$, $\varepsilon > 0$, and $\tau \in \mathbb{R}$ be as in (1.2). Let $E \subset H^1$ be a smooth open set such that $\mathscr{L}^3(E) = \mathscr{L}^3(E_R)$ and $\Sigma = \partial E$.*

*(i) If $E\Delta E_R \subset\subset C_{R,\delta}$ with $0 < \delta < R$ then we have*

$$\mathscr{A}(\Sigma) - \mathscr{A}(\Sigma_R) \ge \sqrt{\delta}C_{R\varepsilon\tau}\mathscr{L}^3(E\Delta E_R)^2. \tag{5.2}$$

*(ii) If $E\Delta E_R \subset\subset C_{R,0}$ then we have*

$$\mathscr{A}(\Sigma) - \mathscr{A}(\Sigma_R) \ge D_{R\varepsilon\tau}\mathscr{L}^3(E\Delta E_R)^3. \tag{5.3}$$

The proof of Theorem 5.1 is based on the foliation of the cylinder $C_{R,\delta}$ by a family of CMC surfaces (see Proposition 2.4) with quantitative estimates on the mean curvature.

**Theorem 5.2.** *For any $R > 0$ and $0 \le \delta < R$, there exists a continuous function $u : C_{R,\delta} \to \mathbb{R}$ with level sets $S_\lambda = \{(z,t) \in C_{R,\delta} : u(z,t) = \lambda\}$, $\lambda \in \mathbb{R}$, such that the following claims hold:*

*(i) $u \in C^1(C_{R,\delta} \cap E_R) \cap C^1(C_{R,\delta} \setminus E_R)$ and the normalized Riemannian gradient $\nabla u/|\nabla u|$ is continuously defined on $C_{R,\delta}$.*

*(ii) $\bigcup_{\lambda > R} S_\lambda = C_{R,\delta} \cap E_R$ and $\bigcup_{\lambda \le R} S_\lambda = C_{R,\delta} \setminus E_R$.*

*(iii) Each $S_\lambda$ is a smooth surface with constant mean curvature $H_\lambda = 1/(\varepsilon\lambda)$ for $\lambda > R$ and $H_\lambda = 1/(\varepsilon R)$ for $\lambda \le R$.*

*(iv) For any point $(z, f(|z|;R) - t) \in S_\lambda$ with $\lambda > R$ we have*

$$1 - \varepsilon RH_\lambda(z, f(|z|;R) - t) \ge \frac{t^2}{4Rk^2_{R\varepsilon\tau} + f(0;R)^2}, \quad \text{when } \delta = 0, \tag{5.4}$$

*and*

$$1 - \varepsilon RH_\lambda(z, f(|z|;R) - t) \ge \frac{\sqrt{\delta}t}{Rk_{R\varepsilon\tau} + f(0;R)}, \quad \text{when } 0 < \delta < R. \tag{5.5}$$

*Proof of Theorem 5.2.* For points $(z,t) \in C_{R,\delta} \setminus E_R$ we let

$$u(z,t) = f(|z|;R) - t + R.$$

Then $u$ satisfies $u(z,t) \le R$ for $t \ge f(|z|;R)$ and $u(z,t) = R$ if $t = f(|z|;R)$. In order to define $u$ in the set $C_{R,\delta} \cap E_R$, for $0 \le r < r_{R,\delta}$, $t_{R,\delta} < t < f(r;R)$, and $\lambda > R$ we consider the function

$$F(r,t,\lambda) = f(r;\lambda) - f(r_{R,\delta};\lambda) + t_{R,\delta} - t. \tag{5.6}$$

The function $F$ also depends on $\delta$. We claim that for any point $(z,t) \in C_{R,\delta} \cap E_R$ there exists a unique $\lambda > R$ such that $F(|z|,t,\lambda) = 0$. In this case, we can define

$$u(z,t) = \lambda \quad \text{if and only if} \quad F(|z|,t,\lambda) = 0. \tag{5.7}$$

We prove the previous claim. Let $(z,t) \in C_{R,\delta} \cap E_R$ and use the notation $r = |z|$. First of all, we have

$$\lim_{\lambda \to R^+} F(r,t,\lambda) = f(r;R) - t > 0. \tag{5.8}$$

We claim that we also have

$$\lim_{\lambda \to \infty} F(r,t,\lambda) = t_{R,\delta} - t < 0. \tag{5.9}$$

To prove this, we let $f(r;\lambda) - f(r_{R,\delta};\lambda) = \frac{\varepsilon^2}{2\tau}[f_1(\lambda) + f_2(\lambda)]$, where

$$\begin{aligned} f_1(\lambda) &= \omega(\lambda)^2 \Big[\arctan(p(r;\lambda)) - \arctan(p(r_{R,\delta};\lambda))\Big], \\ f_2(\lambda) &= \omega(r)^2 \Big(p(r;\lambda) - p(r_{R,\delta};\lambda)\Big). \end{aligned}$$

Using the asymptotic approximation

$$\arctan(s) = \frac{\pi}{2} - \frac{1}{s} + \frac{1}{3s^3} + o\Big(\frac{1}{s^3}\Big), \quad \text{as } s \to \infty,$$

we obtain for $\lambda \to \infty$

$$\begin{aligned} f_1(\lambda) &= \lambda\varepsilon\tau(\omega(r_{R,\delta}) - \omega(r))) + o(1), \\ f_2(\lambda) &= \lambda\varepsilon\tau(\omega(r) - \omega(r_{R,\delta})) + o(1), \end{aligned}$$

and thus $f(r;\lambda) - f(r_{R,\delta};\lambda) = o(1)$, where $o(1) \to 0$ as $\lambda \to \infty$. Since $\lambda \mapsto F(r,t,\lambda)$ is continuous, (5.8) and (5.9) imply the existence of a solution $\lambda$ of $F(r,t,\lambda) = 0$. The uniqueness follows from $\partial_\lambda F(r,t,\lambda) < 0$. This inequality can be proved starting from (2.2) and we skip the details. This finishes the proof of our initial claim.

Claims (i) and (ii) can be checked from the construction of $u$. Claim (iii) follows, by Theorem 3.1, from the fact that $S_\lambda$ for $\lambda > R$ is a vertical translation (this is an isometry of $H^1$) of the $t$-graph of $z \mapsto f(z;\lambda)$.

We prove Claim (iv). For any $(z,t) \in H^1$ such that $r = |z| < r_{R,\delta}$ and $0 \le t < f(r;R) - t_{R,\delta}$, we define

$$g_z(t) = u(z, f(r;R) - t) = \lambda, \tag{5.10}$$

where $\lambda \ge R$ is uniquely determined by the condition $(z, f(r;R) - t) \in S_\lambda$. Notice that $g_z(0) = u(z, f(r;R)) = R$. We estimate the derivative of the function $t \mapsto g_z(t)$. From the identity $F(r,t,u(z,t)) = 0$, see (5.7), we compute $\partial_t u(z,t) = (\partial_\lambda F(r,t,u(z,t)))^{-1}$ and so, also using (5.6), we find

$$g_z'(t) = -\partial_t u(z, f(r;R) - t) = \frac{-1}{\partial_\lambda F(r, f(r;R) - t, g_z(t))}. \tag{5.11}$$

Now from (2.1) we compute

$$\begin{aligned}\partial_\lambda F(r,t,\lambda) &= -\varepsilon^3\lambda \int_r^{r_{R,\delta}} \frac{s\omega(s)}{(\lambda^2-s^2)^{3/2}}\,ds \\ &\geq -\varepsilon^3\lambda\omega(r_{R,\delta}) \int_0^{r_{R,\delta}} \frac{s}{(\lambda^2-s^2)^{3/2}}\,ds \\ &= -\varepsilon^3\omega(r_{R,\delta})\left[\frac{\lambda}{\sqrt{\lambda^2-r_{R,\delta}^2}}-1\right] \\ &\geq -\varepsilon^3\omega(R)\frac{\sqrt{R}}{\sqrt{\lambda-r_{R,\delta}}}.\end{aligned} \tag{5.12}$$

In the last inequality, we used $r_{R,\delta} < R \leq \lambda$. From (5.11), (5.12) and with $k_{R\varepsilon\tau}$ as in (5.1), we deduce that

$$g_z'(t) \geq \frac{1}{k_{R\varepsilon\tau}}\sqrt{g_z(t)-r_{R,\delta}}. \tag{5.13}$$

In the case $\delta = 0$, (5.13) reads $g_z'(t) \geq \sqrt{g_z(t)-R}/k_{R\varepsilon\tau}$. Integrating this differential inequality we obtain $g_z(t) \geq R + t^2/(4k_{R\varepsilon\tau}^2)$, and thus

$$1-\varepsilon RH_\lambda(z,f(r;R)-t) = 1-\frac{R}{g_z(t)} \geq \frac{t^2}{4Rk_{R\varepsilon\tau}^2+f(0;R)^2},$$

that is Claim (5.4).

If $0<\delta<R$, (5.13) implies $g_z'(t) \geq \sqrt{\delta}/k_{R\varepsilon\tau}$ and an integration gives $g_z(t) \geq \sqrt{\delta}\,t + R/k_{R\varepsilon\tau}$. Then we obtain

$$1-\varepsilon RH_\lambda(z,f(r;R)-t) = 1-\frac{R}{g_z(t)} \geq \frac{\sqrt{\delta}}{Rk_{R\varepsilon\tau}+f(0;R)}t,$$

that is Claim (5.5). □

We can now prove Theorem 5.1.

*Proof of Theorem 5.1.* Let $u : C_{R,\delta} \to \mathbb{R}$, $0 \leq \delta < 1$, be the function constructed in Theorem 5.2 and let $S_\lambda = \{(z,t) \in C_{R,\delta} : u(z,t) = \lambda\}$, $\lambda \in \mathbb{R}$, be the leaves of the foliation. Let $\nabla u$ be the Riemannian gradient of $u$. The vector field

$$V(z,t) = -\frac{\nabla u(z,t)}{|\nabla u(z,t)|}, \quad (z,t) \in C_{R,\delta},$$

satisfies the following properties:

i) $|V| = 1$.
ii) For $(z,t) \in \Sigma_R \cap C_{R,\delta}$ we have $V(z,t) = \nu_{\Sigma_R}(z,t)$, where $\nu_{\Sigma_R} = \mathscr{N}$ is the exterior unit normal to $\Sigma_R$.
iii) For any point $(z,t) \in S_\lambda$, $\lambda \in \mathbb{R}$, the Riemannian divergence of $V$ satisfies

$$\begin{aligned}&\frac{1}{2}\operatorname{div}V(z,t) = H_\lambda(z,t) \leq \frac{1}{\varepsilon R} \quad \text{for } \lambda > R, \\ &\frac{1}{2}\operatorname{div}V(z,t) = H_\lambda(z,t) = \frac{1}{\varepsilon R} \quad \text{for } 0 < \lambda \leq R.\end{aligned} \tag{5.14}$$

Let $\nu_\Sigma$ be the exterior unit normal to the surface $\Sigma = \partial E$. By the Gauss-Green formula and (5.14) it follows that

$$\begin{aligned}\mathscr{L}^3(E_R \setminus E) &\geq \frac{\varepsilon R}{2}\int_{E_R\setminus E} \operatorname{div}V\,d\mathscr{L}^3 \\ &= \frac{\varepsilon R}{2}\Big(\int_{\Sigma_R\setminus\bar{E}} \langle V,\nu_{\Sigma_R}\rangle\,d\mathscr{A} - \int_{\Sigma\cap E_R} \langle V,\nu_\Sigma\rangle\,d\mathscr{A}\Big) \\ &\geq \frac{\varepsilon R}{2}\big(\mathscr{A}(\Sigma_R\setminus\bar{E}) - \mathscr{A}(\Sigma\cap E_R)\big).\end{aligned}$$

In the last inequality we used the Cauchy-Schwarz inequality and the fact that $\langle V, \nu_{\Sigma_R}\rangle = 1$ on $\Sigma_R \setminus \bar{E}$. By a similar computation we also have

$$\begin{aligned}\mathscr{L}^3(E \setminus E_R) &= \frac{\varepsilon R}{2} \int_{E\setminus E_R} \operatorname{div} V \, d\mathscr{L}^3 \\ &= \frac{\varepsilon R}{2} \left\{ \int_{\Sigma\setminus \bar{E}_R} \langle V, \nu_\Sigma\rangle d\mathscr{A} - \int_{\Sigma_R \cap E} \langle V, \nu_{\Sigma_R}\rangle d\mathscr{A} \right\} \\ &\le \frac{\varepsilon R}{2} \big(\mathscr{A}(\Sigma \setminus \bar{E}_R) - \mathscr{A}(\Sigma_R \cap E)\big).\end{aligned}$$

Using the inequalities above and the fact that $\mathscr{L}^3(E) = \mathscr{L}^3(E_R)$, it follows that:

$$\begin{aligned}\frac{\varepsilon R}{2}\big(\mathscr{A}(\Sigma_R \setminus \bar{E}) - \mathscr{A}(\Sigma \cap E_R)\big) &\le \frac{\varepsilon R}{2} \int_{E_R\setminus E} \operatorname{div} V \, d\mathscr{L}^3 \\ &= \mathscr{L}^3(E \setminus E_R) - \int_{E_R\setminus E} \left(1 - \frac{\varepsilon R}{2}\operatorname{div} V\right) d\mathscr{L}^3 \\ &\le \frac{\varepsilon R}{2}\big(\mathscr{A}(\Sigma \setminus \bar{E}_R) - \mathscr{A}(\Sigma_R \cap E)\big) - \mathscr{G}(E_R \setminus E),\end{aligned}$$

where we let

$$\mathscr{G}(E_R \setminus E) = \int_{E_R\setminus E} \left(1 - \frac{\varepsilon R}{2}\operatorname{div} V\right) d\mathscr{L}^3.$$

Hence, we obtain

$$\mathscr{A}(\Sigma) - \mathscr{A}(\Sigma_R) \ge \frac{2}{\varepsilon R}\mathscr{G}(E_R \setminus E). \tag{5.15}$$

For any $z$ with $|z| < R - \delta$, we define the vertical sections $E_R^z = \{t \in \mathbb{R} : (z, t) \in E_R\}$ and $E^z = \{t \in \mathbb{R} : (z, t) \in E\}$. By Fubini-Tonelli theorem, we have

$$\mathscr{G}(E_R \setminus E) = \int_{\{|z|<R\}} \int_{E_R^z\setminus E^z} \left(1 - \frac{\varepsilon R}{2}\operatorname{div} V(z,t)\right) dt\, dz.$$

The function $t \mapsto \operatorname{div} V(z, t)$ is increasing, and thus letting $m(z) = \mathscr{L}^1(E_R^z \setminus E^z)$, by monotonicity we obtain

$$\begin{aligned}\mathscr{G}(E_R \setminus E) &\ge \int_{\{|z|<1\}} \int_{f(|z|;R)-m(z)}^{f(|z|;R)} \left(1 - \frac{\varepsilon R}{2}\operatorname{div} V(z,t)\right) dt\, dz \\ &= \int_{\{|z|<1\}} \int_0^{m(z)} \left(1 - \frac{R}{g_z(t)}\right) dt\, dz,\end{aligned}$$

where $g_z(t) = u(z, f(|z|; R) - t)$ is the function introduced in (5.10).

When $\delta = 0$, by the inequality (5.4) and by Hölder inequality we find

$$\begin{aligned}\mathscr{G}(E_R \setminus E) &\ge \frac{1}{4Rk_{R\varepsilon\tau}^2 + f(0; R)^2} \int_{\{|z|<R\}} \int_0^{m(z)} t^2 dt\, dz \\ &\ge \frac{1}{24\pi^2 R^4 (4Rk_{R\varepsilon\tau}^2 + f(0; R)^2)} \mathscr{L}^3(E\Delta E_R)^3.\end{aligned} \tag{5.16}$$

From (5.16) and (5.15) we obtain (5.3).

By (5.5), when $0 < \delta < 1$ the function $g_z$ satisfies the estimate $1 - 1/g_z(t) \geq (\sqrt{\delta}/(k_{R\varepsilon\tau} + f(0;R)))t$ and we find

$$\begin{aligned}\mathscr{G}(E_R \setminus E) &\geq \frac{\sqrt{\delta}}{Rk_{R\varepsilon\tau} + f(0;R)} \int\limits_{\{|z|<R\}} \int\limits_0^{m(z)} t\,dt\,dz \\ &\geq \frac{\sqrt{\delta}}{8\pi R^2(Rk_{R\varepsilon\tau} + f(0;R))} \mathscr{L}^3(E\Delta E_R)^2.\end{aligned} \tag{5.17}$$

From (5.17) and (5.15) we obtain Claim (5.2). □

# A Topological CMC spheres are left translations of $\Sigma_R$

In this Appendix we give a self-contained proof of the rotational symmetry of CMC spheres in the Heisenberg group. Our proof follows the scheme of the fundamental paper [1]. We remark that the same result can be obtained, for instance by combining [1] and Daniel's correspondence theorem [6, Theorem 5.2] applied to the Heisenberg case [6, Example 5.7]. Nonetheless, our proof does not rely on the fact that the isometry group of $H^1$ is 4-dimensional.

We introduce the following notation. For an oriented surface $\Sigma$ in $H^1$ with unit normal vector $\mathscr{N}$, we denote by $h \in \mathrm{Hom}(T_p\Sigma; T_p\Sigma)$ the shape operator $h(W) = \nabla_W \mathscr{N}$, at any point $p \in \Sigma$. The 1-form $\vartheta$ in $H^1$, defined by $\vartheta(W) = \langle W, T\rangle$ for $W \in \Gamma(TH^1)$, can be restricted to the tangent bundle $T\Sigma$. The tensor product $\vartheta \otimes \vartheta \in \mathrm{Hom}(T_p\Sigma; T_p\Sigma)$ is defined, as a linear operator, by the formula

$$(\vartheta \otimes \vartheta)(W) = \vartheta(W)(\vartheta(X_1)X_1 + \vartheta(X_2)X_2), \qquad W \in \Gamma(T\Sigma),$$

where $X_1, X_2$ is any (local) orthonormal frame of $T\Sigma$. Finally, for any $H \in \mathbb{R}$ with $H \neq 0$, let $\alpha_H \in (-\pi/4, \pi/4)$ be the angle

$$\alpha_H = \frac{1}{2} \arctan\left(\frac{\tau}{H}\right), \tag{A.1}$$

and let $q_H \in \mathrm{Hom}(T_p\Sigma; T_p\Sigma)$ be the (counterclockwise) rotation by the angle $\alpha_H$ of each tangent plane $T_p\Sigma$ with $p \in \Sigma$.

**Definition A.1.** *Let $\Sigma$ be an (immersed) surface in $H^1$ with constant mean curvature $H \neq 0$. At any point $p \in \Sigma$, we define the linear operator $k \in \mathrm{Hom}(T_p\Sigma; T_p\Sigma)$ by*

$$k = h + \frac{2\tau^2}{\sqrt{H^2 + \tau^2}} q_H \circ (\vartheta \otimes \vartheta) \circ q_H^{-1}. \tag{A.2}$$

The operator $k$ is symmetric, i.e., $\langle k(V), W\rangle = \langle V, k(W)\rangle$. The trace-free part of $k$ is $k_0 = k - \frac{1}{2}\mathrm{tr}(k)\mathrm{Id}$. In fact, we have

$$k_0 = h_0 + \frac{2\tau^2}{\sqrt{H^2 + \tau^2}} q_H \circ (\vartheta \otimes \vartheta)_0 \circ q_H^{-1}. \tag{A.3}$$

In the following, we identify the linear operators $h, k, \vartheta \otimes \vartheta$ with the corresponding bilinear forms $(V, W) \mapsto h(V, W) = \langle h(V), W\rangle$, and so on.

The structure of $k$ in (A.2) can be established in the following way. Let $\Sigma_R$ be the CMC sphere with $R = 1/\varepsilon H$. From the formula (3.4), we deduce that, in the frame $X_1, X_2$ in (3.1), the trace-free shape operator at the point $(z, t) \in \Sigma_R$ is given by

$$h_0 = \frac{\varrho^2}{1 + \varrho^2} \begin{pmatrix} H & \tau \\ \tau & -H \end{pmatrix},$$

where $\varrho = \tau\varepsilon|z|$. On the other hand, from (3.9) and (3.8), we get

$$\vartheta(X_1) = 0 \quad \text{and} \quad \vartheta(X_2) = \frac{\varrho\sqrt{\tau^2 + H^2}}{\tau\sqrt{1 + \varrho^2}},$$

and we therefore obtain the following formula for the trace-free tensor $(\vartheta \otimes \vartheta)_0$ in the frame $X_1, X_2$:

$$(\vartheta \otimes \vartheta)_0 = -\frac{(\tau^2 + H^2)}{2\tau^2} \frac{\varrho^2}{1+\varrho^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now, in the unknowns $c \in \mathbb{R}$ and $q$ (that is a rotation by an angle $\beta$), the system of equations $h_0 + cq(\vartheta \otimes \vartheta)_0 q^{-1} = 0$ holds independently of $\varrho$ if and only if $c = 2\tau^2/\sqrt{H^2 + \tau^2}$ and $\beta$ is the angle in (A.1). We record this fact in the next:

**Proposition A.2.** *The linear operator $k$ on the sphere $\Sigma_R$ with mean curvature $H$, at the point $(z, t) \in \Sigma_R$, is given by*

$$k = \Big(H + \frac{\varrho^2}{1+\varrho^2}\sqrt{\tau^2 + H^2}\Big)\mathrm{Id}.$$

*In particular, $\Sigma_R$ has vanishing $k_0$ (i.e., $k_0 = 0$).*

**Remark A.3.** *Formula* (A.2) *is analogous to the one discovered in the product spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ in [1]. In conformal parameters, the trace-less part of* (A.2) *coincides, up to the sign, with the formula in [8].*

In Theorem A.8, we prove that *any* topological sphere in $H^1$ with constant mean curvature has vanishing $k_0$. We need to work in a conformal frame of tangent vector fields to the surface.

Let $z = x_1 + ix_2$ be the complex variable. Let $D \subset \mathbb{C}$ be an open set and, for a given map $F \in C^\infty(D; H^1)$, consider the immersed surface $\Sigma = F(D) \subset H^1$. The parametrization $F$ is conformal if there exists a positive function $E \in C^\infty(D)$ such that, at any point in $D$, the vector fields $V_1 = F_\star \frac{\partial}{\partial x_1}$ and $V_2 = F_\star \frac{\partial}{\partial x_2}$ satisfy:

$$|V_1|^2 = |V_2|^2 = E, \quad \langle V_1, V_2\rangle = 0. \tag{A.4}$$

We call $V_1, V_2$ a conformal frame for $\Sigma$ and we denote by $\mathscr{N}$ the normal vector field to $\Sigma$ such that triple $V_1, V_2, \mathscr{N}$ forms a positively oriented frame, i.e.,

$$\mathscr{N} = \frac{1}{E} V_1 \wedge V_2. \tag{A.5}$$

The second fundamental form of $\Sigma$ in the frame $V_1, V_2$ is denoted by

$$h = (h_{ij})_{i,j=1,2} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}, \quad h_{ij} = \langle \nabla_i \mathscr{N}, V_j\rangle, \tag{A.6}$$

where $\nabla_i = \nabla_{V_i}$ for $i = 1, 2$. This notation differs from (3.4), where the fixed frame is $X_1, X_2, \mathscr{N}$. Finally, the *mean curvature* of $\Sigma$ is

$$H = \frac{L+N}{2E} = \frac{h_{11} + h_{22}}{2E}. \tag{A.7}$$

By Hopf's technique on holomorphic quadratic differentials, the validity of the equation $k_0 = 0$ follows from the Codazzi's equations, which involve curvature terms. An interesting relation between the 1-form $\vartheta$ and the Riemann curvature operator, defined as

$$R(U, V)W = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W$$

for any $U, V, W \in \Gamma(TH^1)$, is described in the following:

**Lemma A.4.** *Let $V_1, V_2$ be a conformal frame of an immersed surface $\Sigma$ in $H^1$ with conformal factor $E$ and unit normal $\mathscr{N}$. Then, we have*

$$\langle R(V_2, V_1)\mathscr{N}, V_2\rangle = 4\tau^2 E \vartheta(V_1)\vartheta(\mathscr{N}). \tag{A.8}$$

*Proof.* We use the notation

$$\begin{aligned} V_i &= V_i^X X + V_i^Y Y + V_i^T T, \qquad i = 1, 2, \\ \mathscr{N} &= \mathscr{N}^X X + \mathscr{N}^Y Y + \mathscr{N}^T T. \end{aligned} \tag{A.9}$$

Using the fundamental relations (3.2) to write $\langle R(V_2, V_1)\mathscr{N}, V_2\rangle$, a direct computation based on the fact that $\langle V_1, \mathscr{N}\rangle = \langle V_2, \mathscr{N}\rangle = \langle V_1, V_2\rangle = 0$, and $\langle V_2, V_2\rangle = E$ yields the claim. □

For an immersed surface with conformal frame $V_1, V_2$, we use the notation $V_iE = E_i$, $V_iH = H_i$, $V_iN = N_i$, $V_iM = M_i$, $V_iL = L_i$, $i = 1, 2$.

**Theorem A.5** (Codazzi's Equations). *Let $\Sigma = F(D)$ be an immersed surface in $H^1$ with conformal frame $V_1, V_2$, conformal factor $E$ and unit normal $\mathscr{N}$. Then, we have*

$$H_1 = \frac{1}{E}\Big\{\frac{L_1 - N_1}{2} + M_2 - 4\tau^2 E\vartheta(V_1)\vartheta(\mathscr{N})\Big\}, \tag{A.10}$$

$$H_2 = \frac{1}{E}\Big\{\frac{N_2 - L_2}{2} + M_1 - 4\tau^2 E\vartheta(V_2)\vartheta(\mathscr{N})\Big\}, \tag{A.11}$$

*where $L, M, N, H$ are as in* (A.6) *and* (A.7).

*Proof.* We start from the following well-known formulas

$$H_1 = \frac{1}{E}\Big\{\frac{L_1 - N_1}{2} + M_2 + \langle R(V_1, V_2)\mathscr{N}, V_2\rangle\Big\}, \tag{A.12}$$

$$H_2 = \frac{1}{E}\Big\{\frac{N_2 - L_2}{2} + M_1 + \langle R(V_2, V_1)\mathscr{N}, V_1\rangle\Big\}. \tag{A.13}$$

Our claims (A.10) and (A.11) follow from these formulas and Lemma A.4, see e.g. [13] for the flat case. □

Now we switch to the complex variable $z = x_1 + ix_2 \in D$ and define the complex vector fields

$$Z = \frac{1}{2}(V_1 - iV_2) = F_\star\Big(\frac{\partial}{\partial z}\Big),$$

$$\bar{Z} = \frac{1}{2}(V_1 + iV_2) = F_\star\Big(\frac{\partial}{\partial \bar{z}}\Big).$$

Equations (A.10)-(A.11) can be transformed into one single equation:

$$E(ZH) = \bar{Z}\Big(\frac{L - N}{2} - iM\Big) - 4\tau^2 E\vartheta(\mathscr{N})\vartheta(Z). \tag{A.14}$$

Consider the trace-free part of $b = k - h$, i.e.,

$$b_0 = \frac{2\tau^2}{\sqrt{H^2 + \tau^2}} q_H \circ (\vartheta \otimes \vartheta)_0 \circ q_H^{-1}$$

The entries of $b_0$ as a quadratic form in the conformal frame $V_1, V_2$, with $\vartheta_i = \vartheta(V_i)$ and $c_H = \frac{2\tau^2}{H^2+\tau^2}$, are given by

$$\begin{aligned} A = b_0(V_1, V_1) &= c_H\Big(H\frac{\vartheta_1^2 - \vartheta_2^2}{2} - \tau\vartheta_1\vartheta_2\Big), \\ B = b_0(V_1, V_2) &= c_H\Big(H\vartheta_1\vartheta_2 + \tau\frac{\vartheta_1^2 - \vartheta_2^2}{2}\Big). \end{aligned} \tag{A.15}$$

These entries can be computed starting from $q_H(\vartheta \otimes \vartheta)_0 q_H^{-1} = q_H^2(\vartheta \otimes \vartheta)_0$, where $q_H^2$ is the rotation by the angle $2\alpha_H$ that, by (A.1), satisfies $\cos(2\alpha_H) = H/\sqrt{H^2 + \tau^2}$ and $\sin(2\alpha_H) = \tau/\sqrt{H^2 + \tau^2}$.

**Lemma A.6.** *Let $\Sigma$ be an immersed surface in $H^1$ with constant mean curvature $H$ and unit normal $\mathscr{N}$ such that $V_1, V_2, \mathscr{N}$ is positively oriented. Then, on $\Sigma$ we have*

$$\bar{Z}(A - iB) = -4\tau^2 E\vartheta(\mathscr{N})\vartheta(Z). \tag{A.16}$$

*Proof.* The complex equation (A.16) is equivalent to the system of real equations

$$\begin{aligned} A_1 + B_2 &= -4\tau^2 E\vartheta(\mathscr{N})\vartheta(V_1), \\ A_2 - B_1 &= 4\tau^2 E\vartheta(\mathscr{N})\vartheta(V_2), \end{aligned} \tag{A.17}$$

where $A_i = V_iA$ and $B_i = V_iB$, $i = 1, 2$. In order to verify (A.17), we proceed by direct computations. □

Let $\Sigma$ be an immersed surface in $H^1$ defined in terms of a conformal parametrization $F \in C^\infty(D; H^1)$. Let $f \in C^\infty(D; \mathbb{C})$ be the function of the complex variable $z \in D$ given by

$$f(z) = \frac{L - N}{2} - iM + A - iB, \tag{A.18}$$

where $L, M, M, A, B$ are defined as in (A.6) and (A.15) via the conformal frame $V_1, V_2$ and are evaluated at the point $F(z)$.

**Proposition A.7.** *If $\Sigma$ has constant mean curvature $H$ then the function $f$ in* (A.18) *is holomorphic in $D$.*

*Proof.* From (A.14) with $ZH = 0$ and (A.16), we obtain the equation on $\Sigma = F(D)$

$$\bar{Z}\Big(\frac{L - N}{2} - iM + A - iB\Big) = 0,$$

that is equivalent to $\partial_{\bar{z}} f = 0$ in $D$. □

Now, by a standard argument of Hopf, see [12] Chapter VI, for topological spheres the function $f$ is identically zero. By Liouville's theorem, this follows from the estimate

$$|f(z)| \le \frac{C}{|z|^4}, \quad z \in \mathbb{C},$$

that can be obtained expressing the second fundamental forms in two different charts without the north and south pole, respectively. We skip the details of the proof of the next:

**Theorem A.8.** *A topological sphere $\Sigma$ immersed in $H^1$ with constant mean curvature has vanishing $k_0$.*

In the rest of this section, we show how to deduce from the equation $k_0 = 0$ that any topological sphere is congruent to a sphere $\Sigma_R$. In fact, unlike the theory of holomorphic quadratic differentials in three-dimensional manifolds, we do not use the fact that the isometry group of $H^1$ is four-dimensional.

Let $\mathfrak{h}$ be the Lie algebra of $H^1$ and let $\langle\cdot,\cdot\rangle$ be the scalar product making $X, Y, T$ orthonormal. We denote by $S^2 = \{v \in \mathfrak{h} : |v| = \sqrt{\langle v, v\rangle} = 1\}$ the unit sphere in $\mathfrak{h}$. For any $p \in H^1$, let $\tau^p : H^1 \to H^1$ be the left-translation $\tau^p(q) = p^{-1} \cdot q$ by the inverse of $p$, where $\cdot$ is the group law of $H^1$, and denote by $\tau^p_\star \in \mathrm{Hom}(T_pH^1; \mathfrak{h})$ its differential.

For any point $(p, v) \in H^1 \times S^2$ there is a unique $\mathscr{N} \in T_pH^1$ such that $v = \tau^p_\star \mathscr{N}$ and we define $T^v_pH^1 = \{W \in T_pH^1 : \langle W, \mathscr{N}\rangle = 0\}$. Depending on the point $(p, v)$ and on the parameters $H, \tau \in \mathbb{R}$, with $H^2 + \tau^2 \neq 0$, below we define the linear operator $\mathscr{L}_H \in \mathrm{Hom}(T^v_pH^1; T_vS^2)$. The definition is motivated by the proof of Proposition A.9. For any $W \in T^v_pM$, we let

$$\mathscr{L}_H W = \tau^p_\star\Big(HW - \frac{2\tau^2}{\sqrt{H^2 + \tau^2}} q_H(\vartheta \otimes \vartheta)_0 q_H^{-1} W\Big) + (\nabla_W \tau^p_\star)(\mathscr{N}),$$

where $\nabla_W \tau^p_\star \in \mathrm{Hom}(T_pH^1; \mathfrak{h})$ is the covariant derivative of $\tau^p_\star$ in the direction $W$ and the trace-free operator $(\vartheta \otimes \vartheta)_0 \in \mathrm{Hom}(T^v_pH^1; T^v_pH^1)$ is

$$(\vartheta \otimes \vartheta)_0 = \vartheta \otimes \vartheta - \frac{1}{2}\mathrm{tr}(\vartheta \otimes \vartheta)\mathrm{Id}.$$

The operator $q_H \in \mathrm{Hom}(T^v_pH^1; T^v_pH^1)$ is the rotation by the angle $\alpha_H$ in (A.1). The operator $\mathscr{L}_H$ is well-defined, i.e., $\mathscr{L}_H W \in \mathfrak{h}$ and $\langle \mathscr{L}_H W, v\rangle = 0$ for any $W \in T^v_pH^1$. This can be checked using the identity $|\mathscr{N}| = 1$ and working with the formula

$$(\nabla_W \tau^p_\star)(\mathscr{N}) = \sum_{i=1}^{3} \langle \mathscr{N}, \nabla_W Y_i\rangle Y_i(0),$$

where $Y_1, Y_2, Y_3$ is any frame of orthonormal left-invariant vector fields.

Finally, for any point $(p, v) \in H^1 \times S^2$, define

$$\mathscr{E}_H(p, v) = \big\{(W, \mathscr{L}_H W) : W \in T^v_pH^1\big\} \subset T_pH^1 \times T_vS^2.$$

Then $(p, v) \mapsto \mathscr{E}_H(p, v)$ is a distribution of two-dimensional planes in $H^1 \times S^2$. The distribution $\mathscr{E}_H$ origins from CMC surfaces with mean curvature $H$ and vanishing $k_0$.

Let $\Sigma$ be a smooth oriented surface immersed in $H^1$ given by a parameterization $F \in C^\infty(D; H^1)$ where $D \subset \mathbb{C}$ is an open set. We denote by $\mathscr{N}(F(z)) \in T_pH^1$, with $p = F(z)$, the unit normal of $\Sigma$ at the point $z \in D$. The normal section is given by the mapping $G : D \to S^2$ defined by $G(z) = \tau_\star^{F(z)}\mathscr{N}(F(z))$, and we can define the Gauss section $\Phi : D \to H^1 \times S^2$ letting $\Phi(z) = (F(z), G(z))$. Then $\overline{\Sigma} = \Phi(D)$ is a two-dimensional immersed surface in $H^1 \times S^2$, called the *Gauss extension* of $\Sigma$.

**Proposition A.9.** *Let $\Sigma$ be an oriented surface immersed in $H^1$ with constant mean curvature $H$ and vanishing $k_0$. Then the Gauss extension $\overline{\Sigma}$ is an integral surface of the distribution $\mathscr{E}_H$ in $H^1 \times S^2$.*

*Proof.* Let $\mathscr{N}$ be the unit normal to $\Sigma$. For any tangent section $W \in \Gamma(T\Sigma)$, we have

$$\begin{aligned} W(\tau_\star^F(\mathscr{N})) &= \tau_\star^F(\nabla_W \mathscr{N}) + (\nabla_W \tau_\star^F)(\mathscr{N}) \\ &= \tau_\star^F(h(W)) + (\nabla_W \tau_\star^F)(\mathscr{N}), \end{aligned}$$

where $h(W) = \nabla_W \mathscr{N}$ is the shape operator. Therefore, the set of all sections of the tangent bundle of $\overline{\Sigma}$ is

$$\Gamma(T\overline{\Sigma}) = \Big\{ \Big( W, \tau_\star^F(h(W)) + (\nabla_W \tau_\star^F)(\mathscr{N}) \Big) : W \in \Gamma(T\Sigma) \Big\}.$$

The equation $k_0 = 0$ is equivalent to $h = H\mathrm{Id} - b_0$ where, by (A.3),

$$b_0 = \frac{2\tau^2}{\sqrt{H^2 + \tau^2}} q_H \Big( \vartheta \otimes \vartheta - \frac{\mathrm{tr}(\vartheta \otimes \vartheta)}{2} \mathrm{Id} \Big) q_H^{-1},$$

and thus the sections of $\overline{\Sigma}$ are of the form

$$(W, \mathscr{L}_H W) \in \Gamma(T\overline{\Sigma}) \quad \text{with} \quad W \in \Gamma(T\Sigma).$$

This concludes the proof. □

The proof of the next theorem follows the argument in [1, Proposition 4.3] with two minor differences. First, the construction of the distribution $\mathscr{E}_H$ is easier thanks to the Lie group structure of $H^1$. Moreover, we do not use the fact that the isometry group of $H^1$ is four dimensional. We instead observe that any topological sphere has $T$ as normal vector at some point.

**Theorem A.10.** *Let $\Sigma$ be a topological sphere in $H^1$ with constant mean curvature $H$. Then there exist a left translation $\iota$ and $R > 0$ such that $\iota(\Sigma) = \Sigma_R$.*

*Proof.* Let $H > 0$ be the mean curvature of $\Sigma$, let $R = 1/H\varepsilon$, and recall that the sphere $\Sigma_R$ has mean curvature $H$.

Let $T^\Sigma(p) \in T_p\Sigma$ be the orthogonal projection of the vertical vector field $T$ onto $T_p\Sigma$. Since $\Sigma$ is a topological sphere, there exists a point $p \in \Sigma$ such that $T^\Sigma(p) = 0$. This implies that either $T = \mathscr{N}$ or $T = -\mathscr{N}$ at the point $p$, where $\mathscr{N}$ is the outer normal to $\Sigma$ at $p$. Assume that $T = \mathscr{N}$.

Let $\iota$ be the left translation such that $\iota(p) = N$, where $N$ is the north pole of $\Sigma_R$. At the point $N$ the vector $T$ is the outer normal to $\Sigma_R$. Since $\iota_\star T = T$ (this holds for any isometry), we deduce that $\Sigma_R$ and $\iota(\Sigma)$ are two surfaces such that:

i) They have both constant mean curvature $H$.
ii) They have both vanishing $k_0$, by Proposition A.2 and Theorem A.8.
iii) $N \in \Sigma_R \cap \iota(\Sigma)$ with the same (outer) normal at $N$.

Let $M_1 = \overline{\Sigma}_R$ and $M_2 = \overline{\iota(\Sigma)}$ be the Gauss extensions of $\Sigma_R$ and $\iota(\Sigma)$, respectively. Let $v = \tau_\star^N \mathscr{N} \in S^2$. From i), ii) and Proposition A.9 it follows that $M_1$ and $M_2$ are both integral surfaces of the distribution $\mathscr{E}_H$. From iii), it follows that $(N, v) \in M_1 \cap M_2$. Being the two surfaces complete, this implies that $M_1 = M_2$ and thus $\Sigma_R = \iota(\Sigma)$. □

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