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# Regularity of the $\bar{\partial}$-Neumann problem and the Green operator

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CHAPTER 1

# Compactness

**Summary of Chapter 1.** Compactness estimates for the $\bar\partial-$Neumann problem hold whenever $\forall\,\epsilon\,\exists\,c_\epsilon$ such that: $\|u\|^2 \leq \epsilon Q(u,u) + C_\epsilon\|u\|_{-1}$, $\forall u \in Dom(\bar\partial^*)$. These yield regularity of the $\bar\partial-$Neumann problem; by taking $\epsilon = c_s$ where $c_s$ is a bound from above for the coefficients of $[D^s, \bar\partial]$ and $[D^s, \bar\partial^*]$ and by applying them for $u$ replaced by $D^s u$, one has $H^s-$regularity. A sufficient condition for compactness estimates is the celebrated $P-$property: the existence of an uniformly bounded family of weights which satisfies: $\partial\bar\partial\varphi_\epsilon > \epsilon^{-1}$. The same problem can be investigated over an abstract pseudoconvex oriented compact hypersurface-type manifold. Compactness is defined by $\|u\|^2 \leq \epsilon Q(u,u) + C_\epsilon\|u\|_{-1}$ and $P-$property is replaced by the (CR $P-$property), that is: $(\partial_b\bar\partial_b - \bar\partial_b\partial_b)\varphi_\epsilon \geq \epsilon^{-1}$ for $\varphi_\epsilon$ bounded. The approach consists of a tangential basic estimate in the formulation given by Khanh in his thesis which refines former work by Nicoara [**37**]. It has been proved by Raich[?] that if the CR manifold is embedded in the complex Euclidean space and orientable, property "$(CR-P)$" for $1 \leq q \leq \frac{n-1}{2}$ implies compactness estimates for the Kohn-Laplacian $\Box_b$ in any degree $k$ satisfying $1 \leq k \leq n-2$. The same result is stated by Straube[?] without the assumption of orientability. We regain these results by a simplified method and extend the conclusions to CR manifolds which are not necessarily embedded nor orientable. In this general setting, we also prove compactness estimates in degree $k = 0$ and $k = n-1$ under the assumption of $(CR-P)$ and, when $n = 2$, of closed range for $\bar\partial_b$. Notice that, if $M$ is embedded, this assumption can be dispensed[?] to a recent result by Baracco[?]. For $n \geq 3$, this refines former work by Raich and Straube and separatly by Straube. In fact, our setting is slighly more general when pseudoconvexity is replaced by $q-$pseudoconvexity.

## 1.1. Introduction

Definition 1.1.1. We say that $N$ is compact if for any bounded sequence $\{u_j\}$ the sequence $\{N(u_j)\}$ has a convergent subsequence.

DEFINITION 1.1.2. We say that a pseudoconvex domain $\Omega$ has global compactness estimates for the $\bar{\partial}-$Neumann problem if for every positive number $M$ and for any $u \in C^\infty(\Omega)^k \cap Dom(\bar{\partial}^*)$ there exists $C_M > 0$ such that

$$\|u\|^2 \leq \epsilon Q(u,u) + C_\epsilon \|u\|^2_{-1}. \tag{1.1.1}$$

REMARK 1.1.3. It is easy to observe that (1.1.1) implies for $u \in Dom(\Box)$:

$$\|u\|^2 \leq \epsilon \|\Box(u)\|^2 + C_\epsilon \|u\|^2_{-1}. \tag{1.1.2}$$

PROPOSITION 1.1.4. *For a pseudoconvex domain $\Omega$, the following are equivalent:*

(1) *the compactness of the Neumann operators $N_k$, for $1 \leq k \leq n-1$;*
(2) *the compactness of the embedding $j_k$ of the space $Dom(\bar{\partial})^k \cap Dom(\bar{\partial}^*)^k$, provided with the graph norm $\|u\| + \|\bar{\partial} u\| + \|\bar{\partial}^* u\|$, into $H^0(\Omega)^k$;*
(3) *the validity of global compactness estimates.*

PROOF. First we prove (3) $\Rightarrow$ (1). We want to prove that for any bounded sequence $\{u_n\} \subset H^0(\Omega)^k$ the sequence $\{N_k(u_n)\}$ admits a convergent subsequence. Since $N_k$ is a bounded operator in $H^0(\Omega)^k$ we observe that $\{N_k(u_n)\}$ is a bounded sequence in $H^0(\Omega)^k$. Hence there exists a subsequence $v_j = u_{n_j}$ such that $\{N(v_j)\}$ converges in $H^{-1}(\Omega)^k$ since $H^0(\Omega)^k$ is compactly embedded in $H^{-1}(\Omega)^k$. To conclude it is sufficient to prove that $\{N_k(v_j)\}$ is a Cauchy sequence. We observe that estimate (1.1.2) applied to $N_k(v_j - v_l)$ give us:

$$\|N_k(v_j - v_l)\| \leq \epsilon \|v_j - v_l\| + C_\epsilon \|N_k(v_j - v_l)\|_{-1}.$$

Fixed $\delta > 0$, we get the conclusion choosing $\epsilon$ such that $\epsilon\|v_j - v_l\| \leq \frac{\delta}{2}$ for any $j, l$ and $M \in \mathbb{N}$ such that $C_\epsilon \|N_k(v_j - v_l)\| \leq \frac{\delta}{2}$ for any $j, l \geq M$.

Now we prove that (1) $\Rightarrow$ (2). It is easy to observe that $N_k = j_k^*$, when the range $Dom(\bar{\partial}) \cap Dom(\bar{\partial}^*)$ is endowed with the graph norm. On the other hand, compactness is stable under adjuction.

Finally we prove (2) $\Rightarrow$ (3). If the compactness estimate does not hold we can choose a sequence $\{u_n\}$ such that $Q(u_n, u_n) = 1$ and

$$1 \geq \|u_n\|^2 \geq \epsilon + n\|u_n\|^2_{-1} \tag{1.1.3}$$

for any $n \in \mathbb{N}$. By compactness of the embedding there exists a subsequence $v_j = u_{n_j}$ which converges in $H^0(\Omega)^k$ and hence also in $H^{-1}(\Omega)^k$. From (1.1.3) the common limit is 0. But this contradicts the fact that, again by (1.1.3), $\|u_n\| \geq \epsilon$. $\Box$

LEMMA 1.1.5. *Let* $\{U_\nu\}_{\nu=1}^N$ *be a finite covering of* $b\Omega$ *by a local pathcing. If compactness estimates hold in each* $U_\nu$*:*

$$\|u\|^2 \le \epsilon Q(u,u) + C_\epsilon \|u\|_1 \quad \forall\, u \in C_c^\infty(\bar{\Omega} \cap U_\nu)$$

*then we have global compactness.*

PROOF. Let $\{\zeta_\nu\}_{\nu=0}^N$ be a partition of the unity such that $\zeta_0 \in C_c^\infty(\Omega)$, $\zeta_\nu \in C_c^\infty(U_\nu)$, $\nu = 1, \dots, N$ and

$$\sum_{\nu=0}^N \zeta_\nu^2 = 1 \quad \text{on } \bar{\Omega}.$$

For $u \in C^\infty(\bar{\Omega})^k \cap Dom(\bar{\partial}^*)$, we wish to prove (1.1.1). From the interior elliptic regularity of $Q$ we have $\|\zeta_0 u\|_1^2 \lesssim Q(\zeta_0 u, \zeta_0 u)$. On the other hand, by the interpolation estimates for Sobolev spaces, we have:

$$\|\zeta_0 u\| \lesssim \epsilon \|\zeta_0 u\|_1 + C_\epsilon \|\zeta_0 u\|_{-1}.$$

It follows

$$\begin{aligned} \|\zeta_0 u\| &\lesssim \epsilon Q(\zeta_0 u, \zeta_0 u) + C_\epsilon \|\zeta_0 u\|_{-1}^2 \\ &\lesssim \epsilon Q(u,u) + \epsilon \|[Q, \zeta_0]u\| + C_\epsilon \|u\|_{-1}. \end{aligned}$$

Similarly, for $\nu = 1, \dots, N$, using the hypothesis, we have

$$\begin{aligned} \|\zeta_\nu u\| &\lesssim \epsilon Q(\zeta_\nu u, \zeta_\nu u) + C_\epsilon \|\zeta_\nu u\|_{-1}^2 \\ &\lesssim \epsilon Q(u,u) + \epsilon \|[Q, \zeta_\nu]u\| + C_\epsilon \|u\|_{-1}. \end{aligned}$$

Summing up over $\nu$ and absorbing commutators in the left, we get the proof of the lemma. □

PROPOSITION 1.1.6. *Let* $\Omega$ *be a pseudoconvex domain. A compactness estimate implies boundedness of the Neumann operator* $N_k$ *in* $H^s(\Omega)^k$ *for any* $s > 0$ *and* $1 \le k \le n-1$*.*

PROOF. By a standard fact of elliptic regularization, we only have to prove the a priori estimates:

$$\|u\|_s \le \|\Box u\|_s + \|u\|_0 \tag{1.1.4}$$

for any $u \in C^\infty(\bar{\Omega})^k$ and for any positive integer $s$. We can prove the result in a patching $\{U_\nu\}_\nu$ in which there are well defined the pseudodifferential tangential operators $\Lambda^s$ and the related tangential norm $|||u|||_s = \|\Lambda^s u\|$. By non characteristicity it is enough to prove

$$\|\Lambda^s u\| \le \|\Lambda^s \Box u\| + s.c.\|u\|_s + \|u\|_0. \tag{1.1.5}$$

In fact, since $\|u\|_s \sim \sum_{m=0}^s |||\partial_r^m u|||_{s-m}$, it is sufficient to observe that the following two estimates hold:

$$|||\partial_r u|||_{s-1} \le C(|||\Box u|||_s + \|\Lambda^s u\|) \tag{1.1.6}$$

and

$$|||\partial_r^{k+2}u|||_{s-k-2} \lesssim \sum_{j=0}^{k} |||\partial_r^j \Box u|||_{s-j} + \|\Lambda^s u\| \tag{1.1.7}$$

for integer $k \geq 0$. The first inequality follows from

$$\begin{aligned} \|\partial_r u\|^2 &\leq \|\bar{L}_n u\|^2 + \|\Lambda u\|^2 \leq \\ &\lesssim Q(u,u) + |||u|||_1^2, \end{aligned}$$

and is then obtained by substituting $\Lambda^{s-1}u$ for $u$. The second inequality is obtained as follows. Since $\Box$ is elliptic we can solve the equation $\Box u = \alpha$ for the second derivatives with respect to $r$:

$$\partial_r^2 u_I = \sum_K a_I^K \alpha_K + \sum_{K,i,j} b_I^{Kij} \frac{\partial^2 u_K}{\partial_{x_i}\partial_{x_j}} + \sum_{K,i} c_I^{Ki} \frac{\partial^2 u_K}{\partial_{x_i}\partial_r} + \text{ first order.}$$

The second inequality is then obtained by applying $\Lambda^{s-k-2}\partial_r^k$ to the above equation and taking $L^2-$norm. Now we pass to (1.1.5); the idea of the proof is very simple. Using the compactness estimates for $\epsilon \sim c_s^{-1}$ where $c_s$ is an upper bound for the coefficients of $[\Lambda^s, \bar\partial]$, $[\Lambda^s, \bar\partial^*]$, $[\bar\partial^*, [\Lambda^s, \bar\partial]]$ and $[\bar\partial, [\Lambda^s, \bar\partial^*]]$ we have,

$$\begin{aligned} \|\Lambda^s u\|^2 &\leq \epsilon(\|\bar\partial\Lambda^s u\|^2 + \|\bar\partial^*\Lambda^s u\|^2) + C_\epsilon |||u|||_{s-1} \\ &= \epsilon(\|\Lambda^s\bar\partial u\|^2 + \|\Lambda^s\bar\partial^* u\|^2 + \|[\bar\partial, \Lambda^s]u\|^2 + \|[\bar\partial^*, \Lambda^s]u\|^2) \\ &\quad + C_\epsilon |||u|||_{s-1} \\ &\leq \epsilon\Big(\|\Lambda^s\bar\partial u\|^2 + \|\Lambda^s\bar\partial^* u\|^2 + c_s(|||u|||_s^2 + |||\partial_r u|||_{s-1}^2)\Big) \\ &\quad + C_\epsilon |||u|||_{s-1} \\ &\leq \epsilon\Big((\Lambda^s\Box u, \Lambda^s u) + \|[\bar\partial, \Lambda^s]u\|^2 + \|[\bar\partial^*, \Lambda^s]u\|^2 \\ &\quad + \|[\bar\partial, [\bar\partial^*, \Lambda^s]]u\|^2 + \|[\bar\partial^*, [\bar\partial, \Lambda^s]]u\|^2 \\ &\quad + c_s(|||u|||_s^2 + |||\partial_r u|||_{s-1}^2)\Big) + C_\epsilon |||u|||_{s-1} \qquad (1.1.8) \\ &\leq \|\Lambda^s\Box u\|^2 + \epsilon c_s(|||u|||_s^2 + |||\partial_r u|||_{s-1}^2 + |||\partial_r^2 u|||_{s-2}^2) \\ &\quad + C_\epsilon |||u|||_{s-1} \\ &\leq \|\Lambda^s\Box u\| + s.c.\|u\| + C_\epsilon \|u\|_{s-1} \end{aligned}$$

We then reduce $(s-1)$ to $0-$norm by iteration and get (1.1.5) $\Box$

DEFINITION 1.1.7. We say that a pseudoconvex domain $\Omega$ satisfies the $P-$property if there exists a family of weights $\{\psi_\epsilon\}_{\epsilon\in\mathbb{R}}$ such that:

(1) $\psi_\epsilon$ are bounded,

(2) for any $\epsilon > 0$ we have

$$\partial\bar{\partial}\psi_\epsilon(z)(u,u) \geq \frac{1}{\epsilon}|u|^2$$

for any $u \in \mathbb{C}^n$ and $z \in b\Omega$.

Remark 1.1.8. By the basic estimate it is obvious that $P-$property for $\Omega$ implies compactness; whether the opposite holds, is not known.

In the next proposition we present an obstruction to $P-$property.

Proposition 1.1.9. *If there exists an analytic disc in the boundary of $\Omega \subset \mathbb{C}^n$ then $P-$property does not hold.*

Proof. Let $A\colon \Delta \to \mathbb{C}^n$ be the parametrization of the analytic disc in the boundary of $\Omega$. By the $P-$property we have that $\Delta\psi_\epsilon(A(z)) > \frac{1}{\epsilon}$. In polar coordinate we have that

$$\Delta = \frac{1}{4}(\partial_r^2 + \frac{1}{2}\partial_r + \frac{1}{r^2}\partial_\theta).$$

Hence

$$\int_0^{2\pi} \frac{1}{4}(\partial_r^2 + \frac{1}{2}\partial_r + \frac{1}{r^2}\partial_\theta)\psi_\epsilon(A(r,\theta))\, d\theta \geq \frac{1}{\epsilon}2\pi$$

Note that the term under integration which contains the derivation in $\theta$ vanishes. By multiplication by $r$ in both sides of the above inequality we obtain:

$$\int_0^{2\pi} \frac{1}{4}(r\partial_r^2 + \frac{1}{2}\partial_r)\psi_\epsilon(A(r,\theta))\, d\theta \geq \frac{1}{\epsilon}2\pi r.$$

Hence with the notation $M_\epsilon(r) = \frac{1}{2\pi}\int_0^{2\pi} \psi_\epsilon(A(r,\theta))\, d\theta$:

$$rM_\epsilon^{''}(r) + M_\epsilon^{'}(r) \geq \frac{1}{\epsilon}r$$

i.e.

$$(rM_\epsilon^{'})^{'}(r) \geq \frac{1}{\epsilon}r.$$

Then by the monotonicity of the integration, we have:

$$rM_\epsilon^{'}(r) \geq \frac{1}{2\epsilon}r^2$$

and so

$$M_\epsilon(r) \geq M_\epsilon(0) + \frac{1}{4\epsilon}r^2.$$

The previous inequality contradicts the boundedness of $\psi_\epsilon$. $\square$

We pass to consider compactness estimates for $\bar\partial_b$ in an abstract CR manifold $M$. This is the main content at the present chapter. Let $M$ be a compact CR manifold endowed with the Cauchy-Riemann structure $T^{1,0}M$. By this, we mean $T^{1,0}M\cap T^{0,1}M=0$. $T^{1,0}$ and $T^{0,1}$ are involutive, that is, $[L_i,L_j]\in T^{1,0}M$ whenever $L_1,L_2\in T^{1,0}M$. We call $dim_{\mathbb{C}}(\frac{\mathbb{C}\otimes TM}{T^{1,0}M\oplus T^{0,1}M})$ the CR codimension of $M$. We say that $M$ is hypersurfice type whenever the codimension is 1. We choose a basis $L_1,...,L_{n-1}$ of $T^{1,0}M$, the conjugated basis $\bar L_1,...,\bar L_{n-1}$ of $T^{0,1}M$, and a transversal, purely imaginary, vector field $T$. We also take a hermitian metric on the complexified tangent bundle in which we get an orthogonal decomposition $\mathbb{C}TM=T^{1,0}M\oplus T^{0,1}M\oplus\mathbb{C}T$. We denote by $\omega_1,...,\omega_{n-1},\bar\omega_1,...,\bar\omega_{n-1},\gamma$ the dual basis of 1-forms. We denote by $\mathcal{L}_M$ the Levi form defined by $\mathcal{L}_M(L,\bar L'):=d\gamma(L,\bar L')$ for $L,\ L'\in T^{1,0}M$. The coefficients of the matrix $(c_{ij})$ of $\mathcal{L}_M$ in the above basis are described through Cartan formula as

$$c_{ij}=\langle\gamma,[L_i,\bar L_j]\rangle.$$

We denote by $\mathcal{B}^k$ the space of $(0,k)$-forms $u$ with $C^\infty$ coefficients; they are expressed, in the local basis, as $u=\sum'_{|J|=k}u_J\bar\omega_J$ for $\bar\omega_J=\bar\omega_{j_1}\wedge...\wedge\bar\omega_{j_k}$. Associated to the Riemaniann metric $\langle\cdot,\cdot\rangle_z,\ z\in M$ and to the element of volume $dV$, there is a $L^2$-inner product $(u,v)=\int_M\langle u,v\rangle_z dV$. We denote by $(L^2)^k$ the completion of $\mathcal{B}^k$ under this norm; we also use the notation $(H^s)^k$ for the completion under the Sobolev norm $H^s$. The de-Rham exterior derivative induces a complex $\bar\partial_b:\mathcal{B}^k\to\mathcal{B}^{k+1}$. This is defined as follows: $\bar\partial_b=\pi_{k+1}\circ d$, where $d$ is the exterior derivative on $M$ and $\pi_{k+1}$ is the projection of a $(k+1)-$form into $\mathcal{B}^{k+1}$. We denote by $\bar\partial_b^*:\mathcal{B}^k\to\mathcal{B}^{k-1}$ the adjoint and set $\Box_b=\bar\partial_b\bar\partial_b^*+\bar\partial_b^*\bar\partial_b$. Let $\varphi$ be a smooth function, denote by $(\varphi_{ij})$ the matrix of the Levi form $\mathcal{L}_\varphi=\frac12(\partial_b\bar\partial_b-\bar\partial_b\partial_b)(\varphi)$ in the basis above, and by $\lambda_1^{\varphi^\epsilon}\le...\le\lambda_{n-1}^{\varphi^\epsilon}$ the ordered eigenvalues of $\mathcal{L}_\varphi$. Let $L^2_\varphi$ be the $L^2$ space weighted by $e^{-\varphi}$ and, for $\varphi_j:=L_j(\varphi)$, denote by $L_j^\varphi=L_j-\varphi_j$ the $L^2_\varphi$-adjoint of $-\bar L_j$. The following is the tangential version of the celebrated Hörmander-Kohn-Morrey basic estimate. We present here the refinement by Khanh [**34**] of a former statement by Nicoara [**37**]. Le $z_o\in M$; for a suitable neighborhood $U$ of $z_o$ and a constant $c>0$, we have

$$
\begin{aligned}
&\|\bar\partial_b u\|^2_\varphi+\|\bar\partial^*_{b,\varphi}u\|^2_\varphi+c\|u\|^2_\varphi\\
&\qquad\geq \sum_{|K|=k-1}{}' \sum_{i,j}(\varphi_{ij}u_{iK},u_{jK})_\varphi-\sum_{|J|=k}{}'\sum_{j=1}^{q_o}(\varphi_{jj}u_J,u_J)_\varphi\\
&\qquad\quad+\sum_{|K|=k-1}{}' \sum_{i,j}(c_{ij}Tu_{iK},u_{jK})_\varphi-\sum_{|J|=k}{}'\sum_{j=1}^{q_o}(c_{jj}Tu_J,u_J)_\varphi\\
&\qquad\quad+\frac{1}{2}\Big(\sum_{j=1}^{q_o}\|L_j^\varphi u\|^2_\varphi+\sum_{j=q_o+1}^{n-1}\|\bar L_j u\|^2_\varphi\Big),
\end{aligned}
\tag{1.1.9}
$$

for any $u\in\mathbb{B}^k_c(U)$ where $q_o$ is any integer with $0\leq q_o\leq n-1$. We introduce now a potential-theoretical condition which is a variant of the "$P$-property" by Catlin [**31**]. In the present version it has been introduced by Raich [**40**].

Definition 1.1.10. Let $z_o$ be a point of $M$ and $q$ an index in the range $1\leq q\leq n-1$. We say that $M$ satisfies property $(CR-P_q)$ at $z_o$ if there is a family of weights $\{\varphi^\epsilon\}$ in a neighborhood $U$ of $z_o$ such that

$$
\begin{cases}
|\varphi^\epsilon(z)|\leq 1, & z\in U\\
\sum\limits_{j=1}^{q}\lambda_j^{\varphi^\epsilon}(z)\geq\epsilon^{-1}, & z\in U \text{ and } \ker\mathcal{L}_M(z)\neq\{0\}.
\end{cases}
\tag{1.1.10}
$$

It is obvious that $(CR-P_q)$ implies $(CR-P_k)$ for any $k\geq q$.

Remark 1.1.11. Outside a neighborhood $V_\epsilon$ of $\ker d\gamma$, the sum $\sum\limits_{j\leq q_o}\lambda_j^{\varphi^\epsilon}$ can get negative; let $-b_\epsilon$ be a bound from below. Now, if $c_\epsilon$ is a bound from below for $d\gamma$ outside $V_\epsilon$, by setting $a_\epsilon:=\frac{\epsilon^{-1}+b_\epsilon}{qc_\epsilon}$, we have,

$$
\sum_{j\leq q_o}\lambda_j^{\varphi^\epsilon}+a_\epsilon d\gamma=\sum_{j\leq q_o}\lambda_j^{\varphi^\epsilon}+qa_\epsilon c_\epsilon\geq\epsilon^{-1}\quad\text{on the whole } U.
\tag{1.1.11}
$$

Conversely, (1.1.11) readily yields the second of (1.1.10). This equivalence was already noticed in [**41**] and justifies our abuse of notation: in fact, (1.1.11) is named $(CR-P_q)$ by [**41**] in accordance with [**40**], whereas (1.1.10) is named "property $(P_q)$ in the nullspace of the Levi form".

Again, (1.1.11) for $q$ implies (1.1.11) for any $k\geq q$.

We state now one of the two main results of the paper

Theorem 1.1.12. *Let $M$ be a compact pseudoconconvex CR manifold of hypersurface type of dimension $2n-1$. Assume that $(CR-P_q)$ holds for a fixed $q$ with $1 \leq q \leq \frac{n-1}{2}$ over a covering $\{U\}$ of $M$. Then we have compactness estimates: given $\epsilon$ there is $C_\epsilon$ such that*

$$\|u\|^2 \leq \epsilon(\|\bar\partial_b u\|^2 + \|\bar\partial_b^* u\|^2) + C_\epsilon \|u\|_{-1}^2 \tag{1.1.12}$$

*for any $u \in D^k_{\bar\partial_b^*} \cap D^k_{\bar\partial_b}$ and $k \in [q, n-1-q]$, where $D^k_{\bar\partial_b^*}$ and $D^k_{\bar\partial_b}$ are the domains of $\bar\partial_b^*$ and $\bar\partial_b$ respectively.*

The argument of Proposition (1.1.6), adapted to the present situation, shows that compactness estimate imply $H^s$ regularity of the Green operator $G$, the inverse of $\Box_b$, in degree $q \leq k \leq n-1-q$ By Hodge duality between forms of complementary degree, we need the double constraint $k \geq q$ (for the positive microlocalization) and $k \leq n-1-q$ (for the negative one); this forces $q \leq \frac{n-1}{2}$. For $M$ embedded and orientable, Theorem 1.1.12 is contained in [**40**]. The same is proved in [**41**] without the assumption of orientability. The proof of this, as well as of the theorem which follows, is given in Section 1.2. Let $\mathcal{H}^k = \ker \bar\partial_b \cap \ker \bar\partial_b^*$ be the space of harmonic forms of degree $k$. As a consequence of (1.1.12), we have that for $q \leq k \leq n-1-q$, the space $\mathcal{H}^k$ is finite-dimensional, $\Box_b$ is invertible over $\mathcal{H}^{k\perp}$ (cf. [**37**] Lemma 5.3) and its inverse $G_k$ is a compact operator. When $k=0$ and $k=n-1$ it is no longer true that it is finite-dimensional. However, if $q=1$, we have a result analogous to (1.1.12) also in the critical degrees $k=0$ and $k=n-1$.

Theorem 1.1.13. *Let $M$ be a compact, pseudoconvex CR manifold of hypersurface type of dimension $2n-1$. Assume that property $(CR-P_q)$ holds for $q=1$ over a covering $\{U\}$ of $M$ and, in case $n=2$, make the additional hypothesis that $\bar\partial_b$ has closed range. Then for any $\epsilon$ there is $C_\epsilon$ such that*

$$\|u\|^2 \leq \epsilon(\|\bar\partial_b u\|^2 + \|\bar\partial_b^* u\|^2) + C_\epsilon \|u\|_{-1}^2 \tag{1.1.13}$$

*for any $u \in \mathcal{H}^{k\perp}$, $k=0$ and $k=n-1$. In particular, $G_k$ is compact for $k=0$ and $k=n-1$.*

For $n \geq 3$ and $M$ a boundary of a domain in $\mathbb{C}^n$, resp. embedded and orientable, Theorem 1.1.13 is contained in [**39**] (resp. [**41**]).

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## 1.2. Proofs

*Proof of Theorem 1.1.12.* We choose a local patch $U$ where a local frame of vector fields is found for which (1.1.9) is fulfilled. The key point is to specify the convenient choices of $q_o$ and $\varphi$ in (1.1.9). Let $1 = \psi^{+\,2} + \psi^{-\,2} + \psi^{0\,2}$ be a conic, smooth partition of the unity in the space $\mathbb{R}^{2n-1}$ dual to the space to which $U$ is identified in local coordinates. Let $\Psi^{\overset{\pm}{0}}$ be the pseudodifferential operators with symbols $\psi^{\overset{\pm}{0}}$ and let $\text{id} = \Psi^+\Psi^{+\,*} + \Psi^-\Psi^{-\,*} + \Psi^0\Psi^{0\,*}$ be the corresponding microlocal decomposition of the identity. For a cut off function $\zeta^1 \in C^\infty_c(U)$ we decompose a form $u$ as

$$u^{\overset{\pm}{0}} = \zeta^1\Psi^{\overset{\pm}{0}}u \quad u \in \mathbb{B}^k_c(U),\ \zeta^1|_{\text{supp}\,u} \equiv 1. \tag{1.2.1}$$

For $u^+$ we choose $q_o = 0$ and $\varphi = \varphi^\epsilon$. We also need to go back to Remark 1.1.11. Now, if $a_\epsilon$ has been chosen so that (1.1.11) is fulfilled, we remove $T$ from our scalar products observing that, for large $\xi$, we have $\xi_{2n+1} > a_\epsilon$ over $\text{supp}\,\psi^+$. In the same way as in Lemma 4.12 of [**37**], we conclude that for $k \geq q$

$$\begin{aligned}\sideset{}{'}\sum_{|K|=k-1}\sum_{ij}((c_{ij}T + \varphi^\epsilon_{ij})u^+_{iK}, u^+_{jK})_{\varphi^\epsilon} &\geq \sideset{}{'}\sum_{|K|=k-1}\sum_{ij}((a_\epsilon c_{ij} + \varphi^\epsilon_{ij})u^+_{iK}, u^+_{jK})_{\varphi^\epsilon} \\ &\quad - C\|u^+\|^2_{\varphi^\epsilon} - C_\epsilon\|u^+\|^2_{-1,\varphi^\epsilon} - C_\epsilon\|\zeta^2\tilde{\Psi}^0u^+\|^2_{\varphi^\epsilon} \\ &\geq \epsilon^{-1}\|u^+\|^2_{\varphi^\epsilon} - C_\epsilon\|u^+\|^2_{-1,\varphi^\epsilon} - C_\epsilon\|\zeta^2\tilde{\Psi}^0u^+\|^2_{\varphi^\epsilon},\end{aligned}$$

where $\tilde{\Psi}^0 \succ \Psi^0$ and $\zeta^2 \succ \zeta^1$ in the sense that $\tilde{\psi}^0|_{supp\,\psi^0} \equiv 1$ and $\zeta^2|_{\text{supp}\,\zeta^1} \equiv 1$ respectively. (Here $\|u^+\|_{-1,\varphi^\epsilon} = \|\Lambda^{-1}u^+\|_{\varphi^\epsilon}$ where $\Lambda^{-1}$ is the standard tangential pseudodifferential operator of order $-1$ in the local patch $U$.) Note that there is an additional term $-C_\epsilon\|u^+\|^2_{-1,\varphi^\epsilon}$ with respect to [**37**]. The reason is that $(c_{ij}\xi_{2n-1} + \varphi^\epsilon_{ij})$ can get negative values, even on $\text{supp}\,\psi^+$, when $\xi_{2n-1} < a_\epsilon$. Integration in this compact region, produces the above error term. It follows that for any $k = 1, ..., n-1$:

$$\|u^+\|^2_{\varphi^\epsilon} \leq \epsilon(\|\bar\partial_b u^+\|^2_{\varphi^\epsilon} + \|\bar\partial^*_{b,\varphi^\epsilon}u^+\|^2_{\varphi^\epsilon}) + C_\epsilon\|u^+\|^2_{-1,\varphi^\epsilon} + C_\epsilon\|\zeta^2\tilde{\Psi}^0u^+\|^2_{\varphi^\epsilon}. \tag{1.2.2}$$

By taking the composition $\chi(\varphi^\epsilon)$ where $\chi = \chi(t)$ is a smooth function on $\mathbb{R}^+$ satisfying $\dot\chi > 0$ and $\ddot\chi > 0$, we get

$$(\chi(\varphi^\epsilon))_{ij} = \dot\chi\varphi^\epsilon_{ij} + \ddot\chi|\varphi^\epsilon_j|^2\kappa_{ij},$$

where $\kappa_{ij}$ is the Kronecker symbol. We also notice that

$$|\bar\partial^*_{b,\chi(\varphi^\epsilon)}u|^2 \le 2|\bar\partial^*_b u|^2 + 2\dot\chi^2 \sum_{|K|=k-1}{}' \,|\sum_{j=1,\dots,n}\varphi^\epsilon_j u_{jK}|^2.$$

Remember that $\{\varphi^\epsilon\}$ are uniformly bounded by 1. Thus, if we choose $\chi = \frac{1}{4}e^{(t-1)}$, then we have that $\ddot\chi \ge 2\dot\chi^2$ for $t=\varphi^\epsilon$. For this reason, with this modified weight, we can replace the weighted adjoint $\bar\partial^*_{b,\varphi^\epsilon}$ by the unweighted $\bar\partial^*_b$ in (1.2.2). By the uniform boundedness of the weights, we can also remove them from the norms and end up with the estimate

$$\|u^+\|^2 \le \epsilon\left(\|\bar\partial^*_b u^+\|^2 + \|\bar\partial_b u^+\|^2\right) + C\epsilon\|u^+\|^2_{-1} + C_\epsilon\|\zeta^2\tilde\Psi^0 u\|^2, \tag{1.2.3}$$

for $k=q,\dots,n-1$. For $u^-$, we choose $q_o = n-1$ and $\varphi = -\varphi^\epsilon$. Observe that for $|\xi|$ large we have $-\xi_{2n-1}\ge a_\epsilon$ over $\operatorname{supp}\psi^-$ (cf. [**37**] Lemma 4.13); thus, we have in the current case, for $k \le n-1-q$

$$\begin{aligned}
&\sum_{|K|=k-1}{}' \sum_{ij}((c_{ij}T-\varphi^\epsilon_{ij})u^-_{iK}, u^-_{jK})_{-\varphi^\epsilon} - \sum_{|J|=k}{}' \sum_{j=1,\dots,n}((c_{jj}T-\varphi^\epsilon_{jj})u^-_J, u^-_J)_{-\varphi^\epsilon}\\
&\quad\ge -\sum_{|K|=k-1}{}' \sum_{ij}((a_\epsilon c_{ij}+\varphi^\epsilon_{ij})u^-_{iK}, u^-_{jK})_{-\varphi^\epsilon}\\
&\quad+\sum_{|J|=k}{}' \sum_{j=1,\dots,n}((a_\epsilon c_{jj}+\varphi^\epsilon_{jj})u^-_J, u^-_J)_{-\varphi^\epsilon} - C\|u^-\|^2_{\varphi^\epsilon} - C_\epsilon\|u^-\|^2_{-1,\varphi^\epsilon} - C_\epsilon\|\zeta^2\tilde\Psi^0 u^-\|^2_{\varphi^\epsilon}\\
&\quad\ge \epsilon^{-1}\|u^-\|^2_{\varphi^\epsilon} - C\|u^-\|^2_{\varphi^\epsilon} - C_\epsilon\|u^-\|^2_{-1,\varphi^\epsilon} - C_\epsilon\|\zeta^2\tilde\Psi^0 u^-\|^2_{\varphi^\epsilon}.
\end{aligned}$$

Thus, we get the analogous of (1.2.2) for $u^+$ replaced by $u^-$ and, removing again the weight from the adjoint $\bar\partial^*_{b,\varphi^\epsilon}$ and from the norms, we conclude

$$\|u^-\|^2 \le \epsilon(\|\bar\partial_b u^-\|^2 + \|\bar\partial^*_b u^-\|^2) + C_\epsilon\|u^-\|^2_{-1,\varphi^\epsilon} + C_\epsilon\|\zeta^2\tilde\Psi^0 u\|^2, \quad k=0,\dots,n-1-q. \tag{1.2.4}$$

In addition to (1.2.3) and (1.2.4), we have elliptic estimates for $u^0$

$$\|u^0\|^2_1 \lesssim \|\bar\partial u^0\|^2 + \|\bar\partial^*_b u^0\|^2 + \|u\|^2_{-1}. \tag{1.2.5}$$

The same estimate also holds for $u^0$ replaced by $\zeta^2\tilde\Psi^0 u$. We put together (1.2.3), (1.2.4) and (1.2.5) and notice that

$$\begin{aligned}
\|\bar\partial_b(\zeta^1\Psi^{\overset{\pm}{0}}u)\|^2 &\le \|\zeta^1\Psi^{\overset{\pm}{0}}\bar\partial_b u\|^2 + \|[\bar\partial_b, \zeta^1\Psi^{\overset{\pm}{0}}]u\|^2\\
&\le \|\zeta^1\Psi^{\overset{\pm}{0}}\bar\partial_b u\|^2 + \|\zeta^2\tilde\zeta\Psi^{\overset{\pm}{0}}u\|^2 + \|\zeta^2\tilde\Psi^0 u\|^2,
\end{aligned}\tag{1.2.6}$$

for $\zeta^2 \succ \zeta^1$ and $\tilde{\Psi}^0 \succ \Psi^0$. The similar estimate holds for $\bar\partial_b$ replaced by $\bar\partial_b^*$. Since $\zeta^1|_{\operatorname{supp} u} \equiv 1$, then

$$\begin{aligned}\|u\|^2 &\le \sum_{+,-,0} \|\zeta^1 \Psi^{\overset{\pm}{0}} u\|^2 + Op^{-\infty}(u)\\ &\le \epsilon \sum_{+,-,0} (\|(\bar\partial_b u)^{\overset{\pm}{0}}\|^2 + \|(\bar\partial_b^* u)^{\overset{\pm}{0}}\|^2) + C_\epsilon \|u\|^2_{-1},\end{aligned}$$

and therefore

$$\|u\|^2 \le \epsilon(\|\bar\partial_b u\|^2 + \|\bar\partial_b^* u\|^2) + C_\epsilon \|u\|^2_{-1}, \quad q \le k \le n-1-q. \tag{1.2.7}$$

We consider now $u$ globally defined on the whole $M$ instead of a local patch $U$. To pass from local to global compactness estimates is immediate (cf. e.g. [**30**]). We cover $M$ by $\{U_\nu\}$ so that in each patch there is a basis of forms in which the basic estimate holds. In the identification of $U_\nu$ to $\mathbb{R}^{2n-1}$, we suppose that the microlocal decomposition by the operators $\Psi^{\overset{\pm}{0}}$ which occur in (1.2.6) is well defined. We then get (1.2.7) and apply it to a decomposition $u = \sum_\nu \zeta_\nu u$ for a partition of the unity $\sum_\nu \zeta_\nu = 1$ on $M$. We point out that we first take summation over $+, -, 0$ on each patch $U_\nu$ and next summation over $\nu$; this is why orientability of $M$ is needless.

We observe that $[\bar\partial_b, \zeta_\nu]$ and $[\bar\partial_b^*, \zeta_\nu]$ are 0-order operators and, since they come with a factor of $\epsilon$, they are absorbed in the left side of (1.2.7); thus (1.2.7) holds for any $u \in \mathbb{B}^k$. Finally, by the density of smooth forms $\mathbb{B}^k$ into Sobolev forms $(H^1)^k$, (1.2.7) holds in fact for any $u \in D^k_{\bar\partial_b^*} \cap D^k_{\bar\partial_b}$. The proof is complete.

□

*Proof of Theorem 1.1.13.* We prove estimates in degree 0 (those in degree $n-1$ being similar). We first discuss the case $n > 2$. We make repeated use of (1.2.7) in degree 1. This first implies that $\bar\partial_b^*$ has closed range on 1-forms. In particular,

$$\begin{aligned}\mathcal{H}^{0\,\perp} &= (\ker \bar\partial_b)^\perp\\ &= \operatorname{range} \bar\partial_b^*.\end{aligned}$$

Thus, if $u \in \mathcal{H}^{0\,\perp}$, then there exists a solution $v \in (L^2)^1$ to the equation $\bar\partial_b^* v = u$. Moreover, we can choose $v$ belonging to $(\ker \bar\partial_b^*)^\perp$. In particular, to this $v$, the following estimate applies

$$\|v\|_0^2 \lesssim \|\bar\partial_b^* v\|_0^2 \qquad \text{for any } v \in (\ker \bar\partial_b^*)^\perp. \tag{1.2.8}$$

This can be proved by contradiction. If (1.2.8) is violated, there exists a sequence $v_\nu \in (\ker\,\bar\partial_b^*)^\perp$ such that $\|v_\nu\|_0^2 \equiv 1$ and $\|\bar\partial_b^* v_\nu\|_0 \to 0$. Any subsequential $L^2$-weak limit $v_o$ of $v_\nu$ is $\neq 0$ but satisfies $v_o \in \ker\bar\partial_b^* \cap (\ker\,\bar\partial_b^*)^\perp$, a contradiction. We also have

$$\|v\|_{-1}^2 \leq \epsilon\|\bar\partial_b^* v\|_0^2 + c_\epsilon\|\bar\partial_b^* v\|_{-1}^2, \qquad \text{for any } v \in (\ker\,\bar\partial_b^*)^\perp. \tag{1.2.9}$$

The argument is similar. If (1.2.9) is violated, then there is a sequence $v_\nu \in (\ker\,\bar\partial_b^*)^\perp$ such that $\|v_\nu\|_{-1} \equiv 1$, $\|\bar\partial_b^* v_\nu\|_{-1} \to 0$ and $\|\bar\partial_b^* v_\nu\|_0 \leq c$. But we also have from (1.2.8), $\|\bar\partial_b^* v_\nu\|_0 \underset{\sim}{>} \|v_\nu\|_0 \geq \|v_\nu\|_{-1} = 1$. Thus any subsequential $L^2$-weak limit of $\bar\partial_b^* v_\nu$ must be 0 and $\neq 0$.

We point out now that $(\ker\,\bar\partial_b^*)^\perp \subset \overline{\text{range}\,\bar\partial_b} \subset \ker\bar\partial_b$; in particular, our solution $v$ satisfies $\bar\partial_b v = 0$. We are ready to conclude the proof for $n > 2$. We use the notation lc and sc for a large and small constant respectively. We have for any function $u$

$$\begin{aligned}
\|u\|^2 &= (u, \bar\partial_b^* v)\\
&= (\bar\partial_b u, v)\\
&\leq \|\bar\partial_b u\|\|v\|\\
&\underset{(1.2.7)\text{ for } v}{\leq} \|\bar\partial_b u\|(\epsilon\|\bar\partial_b^* v\| + c_\epsilon\|v\|_{-1})\\
&\underset{(1.2.9)}{\underset{\sim}{<}} \|\bar\partial_b u\|(\epsilon\|u\| + c_\epsilon\|u\|_{-1})\\
&\leq lc_1\,\epsilon^2\|\bar\partial_b u\|^2 + sc_1\|u\|^2 + lc_2\,c_\epsilon^2\|u\|_{-1}^2 + sc_2\|\bar\partial_b u\|^2\\
&\leq \epsilon'\|\bar\partial_b u\|^2 + c_{\epsilon'}\|u\|_{-1}^2 + sc_1\|u\|^2,
\end{aligned} \tag{1.2.10}$$

for $\epsilon' = lc_1\,\epsilon^2 + sc_2$ and $c_{\epsilon'} = lc_2\,c_\epsilon^2$. By choosing $sc_1$ so that $sc_1\|u\|^2$ is absorbed in the left, (1.2.10) yields (1.2.7) for $u$ in degree 0. This concludes the proof of the case $n > 2$ for functions.

Let $n = 2$; we have only estimates for positively microlocalized 1-forms and for negatively microlocalized functions. We have to show how to get estimates for positively microlocalized functions (the argument for negative 1-forms being similar). We use our extra assumption of closed range for $\bar\partial_b$; thus for any $u \in (\ker\bar\partial_b)^\perp$ there is $v \in (\ker\,\bar\partial_b^*)^\perp$ such that $\bar\partial_b^* v = u$. Moreover, for this $v$, we have the estimates (1.2.8) and (1.2.9). On each $U_\nu$ we consider the positive microlocalization $\Psi^+$, take a pair of cut-off functions $\zeta_\nu$, $\zeta_\nu^1 \in C_c^\infty(U_\nu)$ with $\zeta_\nu^1|_{\text{supp}\,\zeta_\nu} \equiv 1$, and define $\Psi_\nu^+ := \zeta_\nu^1\Psi^+\zeta_\nu$. Note that the commutators $[\bar\partial_b^*, \Psi_\nu^+]$ and $[\bar\partial_b, \Psi_\nu^+]$ are operators with symbols of type $\dot\zeta_\nu^1\psi^+\zeta_\nu$, $\zeta_\nu^1\dot\psi^+\zeta_\nu$ and $\zeta_\nu^1\psi^+\dot\zeta_\nu$. All these symbols have support contained in the positive half-space $\xi_{2n-1} > 0$ and hence we have compactness estimates for 1-forms if their coefficients are subjected to the action of the corresponding pseudodifferential operators. We denote by a common symbol $\Phi^+_\nu$ all these operators coming from commutators. We have

$$
\begin{aligned}
\|\Psi^+_\nu v\| &\leq \epsilon\|\bar\partial^*_b \Psi^+_\nu v\| + c_\epsilon \|\Psi^+_\nu v\|_{-1} + c_\epsilon\|\zeta^2_\nu \tilde\Psi^0 \zeta_\nu v\| \\
&\leq \epsilon\|\Psi^+_\nu \bar\partial^*_b v\| + \epsilon\|\Phi^+_\nu v\| + c_\epsilon \|\Psi^+_\nu v\|_{-1} + c_\epsilon\|\zeta^2_\nu \tilde\Psi^0 \zeta_\nu v\| \\
&\leq \epsilon\|u\| + \epsilon\|v\| + c_\epsilon\|v\|_{-1} \\
&\underset{(1.2.8)\text{ and }(1.2.9)}{\leq} \epsilon\|u\| + c_\epsilon\|u\|_{-1}.
\end{aligned}
\tag{1.2.11}
$$

The same estimate also holds for $\|\Phi^+_\nu v\|$. It follows

(1.2.12)

$$
\begin{aligned}
\|\Psi^+_\nu u\|^2 &= (\Psi^+_\nu u, \Psi^+_\nu \bar\partial^*_b v) \\
&= (\Psi^+_\nu \bar\partial_b u, \Psi^+_\nu v) + (\Phi^+_\nu u, \Psi^+_\nu v) + (\Psi^+_\nu u, \Phi^+_\nu v) \\
&\leq (\|\Psi^+_\nu \bar\partial_b u\| + \|\Phi^+_\nu u\| + \|\Psi^+_\nu u\|)(\|\Phi^+_\nu v\| + \|\Psi^+_\nu v\|) \\
&\underset{(1.2.11)}{\leq} (\|\Psi^+_\nu \bar\partial_b u\| + \|u\|)(\epsilon\|u\| + c_\epsilon\|u\|_{-1}) \\
&\lesssim \epsilon\|\Psi^+_\nu \bar\partial_b u\|\|u\| + c_\epsilon\|\Psi^+_\nu \bar\partial_b u\|\|u\|_{-1} + \epsilon\|u\|^2 + c_\epsilon\|u\|_{-1}\|u\| \\
&\leq lc_1\,\epsilon^2\|\Psi^+_\nu \bar\partial_b u\|^2 + sc_1\|u\|^2 + sc_2\|\Psi^+_\nu \bar\partial_b u\|^2 + lc_2\,c^2_\epsilon\|u\|^2_{-1} + \epsilon\|u\|^2 + sc_3\|u\|^2 + lc_3\,c^2_\epsilon \\
&\leq \epsilon'\|\Psi^+_\nu \bar\partial_b u\|^2 + sc_4\|u\|^2 + c_{\epsilon'}\|u\|^2_{-1},
\end{aligned}
$$

where $\epsilon' = lc_1\,\epsilon^2 + sc_2$, $c_{\epsilon'} = lc_2\,c^2_\epsilon + lc_3\,c^2_\epsilon$ and $sc_4 = sc_1 + \epsilon + sc_3$. We have to recall now that the same estimate as (1.2.12) also holds for $\|\Psi^-_\nu u\|^2$ (the one for $\|\Psi^0_\nu u\|^2$ being trivial by ellipticity). Taking summation over $+$, $-$ and $0$ on each $U_\nu$, we get

$$\|\zeta_\nu u\|^2 \leq \epsilon\|\zeta^1_\nu \bar\partial_b u\|^2 + c_\epsilon\|u\|^2_{-1} + sc\|u\|^2.$$

We take now summation over $\nu$ and choose $sc$ so that the related term is absorbed by $\sum_\nu \|\zeta_\nu u\|^2 \sim \|u\|^2$ and end up with

$$\|u\|^2 \leq \epsilon\|\bar\partial_b u\|^2 + c_\epsilon\|u\|^2_{-1} \quad \text{for any function } u.$$

□

CHAPTER 2

# Global regularity

**Summary of Chapter 2.** We have seen in Chapter 1 that compactness implies the regularity of the $\bar{\partial}-$Neumann problem in the sobolev spaces $H^s$, that is, the $H^s$ continuity of the Neumann operator $N$. It is classical that regularity can hold under weaker conditions than compactness. The first approach to regularity in geometric terms has been done by Boas Straube[?] through the method of the "good vector field" $T$ or "good defining function" $r < 0$, condition. This consists in assuming, $\forall\,\epsilon$, the existence of $T^\epsilon$ purely immaginary tangential vector filed such that $|\langle[\bar{\partial}, T^\epsilon], L_n\rangle|_{|b\Omega} \leq \epsilon$, where $L_n$ is the $(1,0)$ vector field dual to $\partial r$. On the one hand, this condition yields regularity (?,?); on the other this condition is fulfilled, if there exists a plurisubharmonic defining function $r$. Notice that any circular complete domain satisfies this condition by the choice of $T$ as the angular vector field. However, in a Reinhardt domain the presence of a disc in the boundary prevents from compactness; this exhibits the essiest example of regularity without compactness.

The vector field condition has been weekened by Straube (?) to a multiplier condition: $\forall\,\epsilon\,\exists\,T^\epsilon$ such that $\|[\bar{\partial}^*, T^\epsilon]u\| \leq \epsilon Q_1(u,u) + C_\epsilon\|u\|$ (the original notation is slighly different). Thus is also refered to as "weak compactness" and it is sufficient for regularity. In Kohn[?] has given a quantitative version of this statement (not explicitly stated): if the multiplier conditions holds for a certain $\epsilon$, thus we have $s-$regularity for a related $s$ (under some additional condition of uniformity for exhaustion; this condition was latter bypassed by Straube). Also, what is mostly interesting in [?] is that Kohn is able to relate the constants $\epsilon$ to the number $\frac{1}{1-\delta}$ where $\delta$ is the Diederich Fornaess index. Strictly speaking, Kohn only proves regularity for the Bergmann projection on functions. The main purpouse of this section is to extend the conclusion for any degree of forms.

## 2.1. Introduction

It is well known that compactness is not a necessary condition for global regularity. We start from the following statement about failure of compactness estimate.

PROPOSITION 2.1.1 (cf. [**34**]). *Let $\Omega$ be a smooth bounded pseudoconvex domain of $\mathbb{C}^n$ with a $(n-1)-$"'Reinhardt flat"' piece of boundary. This means that, in some choice of coordinate*

$$b\Omega \supset b\Delta \times \Delta_\epsilon^{n-1}$$

*where $\Delta$ is the unit disc in $\mathbb{C}$ and $\Delta_\epsilon^{n-1}$ the $\epsilon-$polydisc in $\mathbb{C}^{n-1}$. Then compactness of $N_k$ does not hold for $k \geq 1$.*

REMARK 2.1.2. This result generalizes the one by Krantz [**38**] and is close to further developments by Boas Straube in [**?**]. In the original statement, Reinhardt complete domains having a flat portion of the boundary, are considered. Here the domain is not required to be fully Reinhardt.

PROOF. We prove the propisition for the case $n = 2$ and $k = 1$, since the general proof is identical. Let $\psi \in C_c^\infty(\mathbb{R})$ satisfies

$$\psi(t) = \begin{cases} 1 & \text{if} \quad t \leq \frac{\epsilon}{2} \\ 0 & \text{if} \quad t > \epsilon \end{cases}$$

For $m \in \mathbb{Z}^+$, set

$$u_m(z_1, z_2) = \sqrt{2(m+1)}\, z_1^m \psi(|z_2|^2) d\bar{z}_2.$$

Then, we have

$$r_{z_1}(u_m)_1 + r_{z_2}(u_m)_2 = 0$$

on the boundary, that is, $u_m \in Dom(\bar\partial^*)$. Moreover, we have

$$\begin{aligned} \|u_m\| &= 2(m+1)\int_\Omega |z_1|^{2m}\psi(|z_2|^2)\, dV \\ &\sim 2(m+1)\int_{D(0,1)\times D(0,\frac{\epsilon}{2m})} |z_1|^{2m}\, dV \sim 1. \end{aligned} \tag{2.1.1}$$

On the other hand, one checks readily that $\bar\partial u_m = 0$ and $\bar\partial^* u_m = \sqrt{2(m+1)}\, z_1^m \bar{z}_2 \partial_{z_2}\psi$; This yields $\|\bar\partial^* u_m\|^2 \lesssim 1$. In conclusion,

$$\|u_m\|_0 \gtrsim Q(u_m, u_m). \tag{2.1.2}$$

Since $\{u_m\}_{m\in\mathbb{Z}^+}$ is an $H_0-$bounded sequence and it converges pointwise to zero, we have that $u_m$ converges $H^0-$weakly to 0. Since $H^0$ is compactly embedded in $H^{-1}$ there exists a subsequence $u_{m_j}$ which is

$H^{-1}-$convergent; by uniqueness the limit must be 0. This, in combination with (2.1.2), violates compactness estimates. □

More generally, a disc contained in the boundary of a pseudoconvex domain in $\mathbb{C}^2$ is an obstruction to compactness (unpublished observation by Catlin). This fact can be generalized to the case of a pseudoconvex domain in $\mathbb{C}^n$ that contains in the boundary a $(n-1)-$complex manifold. However, when $n \geq 3$, whether an analytic disc in the boundary (say, of a smooth domain) is an obstruction to compactness is an open problem. This is known only in special cases: when $\Omega$ is convex or convexifiable.

For global regularity, there are several criteria which do not require compactness. The first is the so called Condition (T) (cf. [**33**] pag 129). We fix a defining function $r$ of $\Omega$ and of a normal vector field

$$L_n = \frac{1}{\sum_{j=1}^n |r_{z_j}|^2} \sum_{j=1}^n r_{\bar{z}_j} \partial_{z_j},$$

We then ask that for any $\epsilon$ there is a vector field $T = T_\epsilon$, tangent to the boundary of $\Omega$ and whose component along $L_n - \bar{L}_n$ has a uniform lower bound, such that

$$|\langle [T, S], L_n \rangle|_{|b\Omega} < \epsilon \tag{2.1.3}$$

for any $S \in \mathbb{C} \otimes T\mathbb{C}^n$. We refer to [**33**] Theorem 6.2.1 for the proof that condition $(T)$ implies global regularity. We recall briefly the idea which it is (?). In the proof of the estimate (1.1.8) we do not have $\epsilon$ for the full right hand side. However

$$[\bar{\partial}^*, T^s] = \epsilon T^s + \text{ terms containing } S \in \mathcal{S}$$

$$[\bar{\partial}, T^s] = \epsilon T^s + \text{ terms containing } S \in \mathcal{S}$$

These terms are absorbed as well.

An easy application of this result is:

Proposition 2.1.3. *Let $\Omega$ be a Reinhardt complete, smooth bounded, pseudoconvex domain of $\mathbb{C}^n$. Then, the $\bar{\partial}-$Neumann operator $N_k$ on $(0, k)-$forms is exactly regular in Sobolev norms, that is*

$$\|N_k \alpha\|_s \leq C_s \|\alpha\|_s \tag{2.1.4}$$

*for $s \geq 0$ and all $\alpha \in H_s(\Omega)^k$.*

PROOF. The proof goes through Condition (T). Since $\Omega$ is Reinhardt, then in particular it is circular, that is, invariant under multiplication by $e^{i\theta} \in S^1$ and therefore

$$T: \ = i\sum_{j=1}^{n} z_j\partial_{z_j} - i\sum_{j=1}^{n} \bar{z}_j\partial_{\bar{z}_j},$$

when restricted to $b\Omega$, is tangent to $b\Omega$ (since $T(z) = \pi_{z,*}\Big(\frac{\partial}{\partial\theta}|_{\theta=0}\Big)$ where $\pi$ is defined by:

$$\begin{aligned} \pi \ : \ & S^1 \times \Omega \to \Omega \\ & (e^{i\theta}, z) \to e^{i\theta} z \end{aligned} \tag{2.1.5}$$

where $S^1$ is the unitary circle). It is an easy exercise to check that in order that condition (T) is fullifiled, it is sufficent to show that $T$ is not complex tangential, that is,

$$(\sum_{j=1}^{n} z_j\partial_{z_j}(r))_{|b\Omega} \neq 0. \tag{2.1.6}$$

To prove (2.1.6) we reason by contradiction. If $(\sum_{j=1}^{n} z_j\partial_{z_j}(r))_{|b\Omega} = 0$ at some $z^0 \in b\Omega$, then the vector $z^0 - 0$ is orthogonal to $\partial r(z^0)$ and therefore, there are other points $z^1 \in \Omega$ such that $|z^1_j| > |z^0_j|$ for any $j$. Since $\Omega$ is Reinhardt complete, this is a contradiction which proves (2.1.6). Thus condition (T) is verified and the proposition is proved. □

On the other hand, We have already seen in Proposition (2.1.1) that there exists a complete Reinhardt domain that does not have compactness. In particular, compactness is not necessary for regularity.

DEFINITION 2.1.4. An exhaustion of a domain $\Omega$ is an increasing family of relatively compact subsets $\{\Omega_\rho\}_{\rho\in\mathbb{R}^+}$ of $\Omega$, such that:

$$\cup_\rho \Omega_\rho = \Omega.$$

PROPOSITION 2.1.5. *Let $\Omega$ be a pseudoconvex domain. If $\Omega$ admits a defining function $r$ such that $(\partial\bar{\partial}r)_{|b\Omega} \geq 0$ then there exists a strongly pseudoconvex $C^2-$bounded exhaustion.*

PROOF. It is sufficient to observe that $r_\rho(z) = r(z) + \rho \exp^{A|z|^2}$, $\rho \searrow 0$, is a family of defining function for a strictly pseudoconvex exhaustion, once $A$ is choosen to be large enough. □

For the converse we have:

PROPOSITION 2.1.6. *If there exists $C^{2+\epsilon}-$bounded strongly pseudoconvex exhaustion $\{\Omega_\rho\}$ of $\Omega$ then there exists a defining function $r$ for $\Omega$ such that $(\partial\bar{\partial}r)_{|b\Omega} \geq 0$.*

PROOF. Let $\{r_\rho\}$ be the $C^{2+\epsilon}-$bounded family of defining functions for $\{\Omega_\rho\}$. It is not restrictive suppose that the family $\{r_\rho\}$ is defined in a neighborhood of $\bar\Omega$. Since $C^{2+\epsilon}(\bar\Omega)$ is compactly embedded in $C^2(\bar\Omega)$, there exists a subsequence $r_{\rho_j}$ that converge in $C^2(\bar\Omega)$ to some function $r \in C^2(\bar\Omega)$. Since $r$ is a defining function for $\Omega$ and

$$\partial\bar\partial r_{\rho_j} \to \partial\bar\partial r \quad \text{in } C^0(\bar\Omega)-\text{norm}$$

we have the conclusion. □

There are always strongly pseudoconvex exhaustions. Problem is that they are not $C^2-$bounded in general. The regularity of the $\bar\partial-$Neumann problem is related to the existence of good totally real tangential vector field, or equivalently, to the existence of good defining functions. By this we mean, for any $\epsilon > 0$ the existence of a definig function $r^\epsilon$ such that

$$(2.1.7) \qquad |\sum_i r^\epsilon_{i,\bar j} r^\epsilon_{\bar i}\big|_{|b\Omega} \le \epsilon \quad \text{for any } j.$$

that is,

$$(2.1.8) \qquad \big|\langle[\partial_{\bar z_i}, T_\epsilon], \partial r\rangle\big|_{|b\Omega} \le \epsilon$$

writing $T_\epsilon = Im(\sum_i^n \partial_{z_i} r \partial_{\bar z_i})$. It is under these conditions that the regularity problem was pionered by Boas-Straube. In particular, they were able to prove that the existence of a plurisubharmonic functions implies the two equivalent conditions above. A more recent condition which weakens (2.1.7) is:

For any $\epsilon > 0$ there exists a defining function $r^\epsilon$ for $\Omega$ (with $\|r^\epsilon\|_{C^1} \sim 1$)

(2.1.9)

$$\|\sum r^\epsilon_{ij}\bar r_i{}^\epsilon u_j\| \le \epsilon Q(u,u) + C_\epsilon\|u\|_{-1} \quad \text{for any } u \in Dom(\bar\partial) \cap Dom(\bar\partial)$$

that is

$$(2.1.10) \qquad \|[\bar\partial^*, T_\epsilon]u\| \le \epsilon Q_1(u,u) + C_\epsilon\|u\|_{-1}.$$

We also remark that the tangentiality of $T_\epsilon$ in (2.1.8) and (2.1.10) can be replaced by "approximate tangentiality". We will discuss from a modified point of view how these conditions yield regularity and relate these to the Diederich-Fornaess index $\delta$ which approches 1. In all cases, we will give the quantified version of the result (that is, the precise relation between $s$, $\frac{1}{\epsilon}$ and $\frac{1}{1-\delta}$). A first way to enjoy the bigger flexibility of (2.1.9) with respect to (2.1.7) consists in the fact that the existence of a plurisubharmonic defining function readdly implies (2.1.9) for a single vector field $T$. Instead, Boas Straube prove that it implies (2.1.7); in this case a full family of $T_\epsilon$ is needed and the proof is much more involved. But the new point is that (2.1.9) covers indeed a reacher range of relations.

We deform the defining function $r$ to $r_\epsilon = g_\epsilon r$ and, accordingly, we deform the vector field $T = 2\mathsf{Im}\, \frac{\sum_i r_{\bar{i}}\partial_{z_i}}{\sum_i |r_i|^2}$ to $T_{g_\epsilon} = 2\mathsf{Im}\, \frac{\sum_i (r_\epsilon)_{\bar{i}}\partial_{z_i}}{\sum_i |(r_\epsilon)_i|^2}$. The condition of approximate tangentiality turns into $|\mathsf{Im}\, g_\epsilon| < \epsilon$. These two deformations are related by $[\bar\partial^*, T_{g_\epsilon}] \sim (\partial\bar\partial r_\epsilon \llcorner \bar\partial r_\epsilon)T_{g_\epsilon}$ modulo an error whose restriction to $bD$ belongs to $T^{1,0}bD \oplus T^{0,1}\mathbb{C}^n|_{bD}$; hence, the existence of $r_\epsilon$ such that

$$|\partial\bar\partial r_\epsilon \llcorner \bar\partial r_\epsilon| \leq \epsilon Q + c_\epsilon \Lambda^{-1}, \tag{2.1.11}$$

for $|\partial r_\epsilon| \sim 1$, implies (2.1.10). (Here $\Lambda$ is the standard elliptic operator of order 1.) This is indeed the assumption under which Straube proves in [**29**] $H^s$-regularity for any $s$. In particular, this condition is fulfilled when there is a smooth defining function $r$ such that $\partial\bar\partial r|_{bD} \geq 0$; in this case one takes, for any $\epsilon$, $r_\epsilon = r$ in (2.1.11) and $T_\epsilon = T$ in (2.1.10) respectively (cf. the proof of Theorem 2.2.4 below). Note that, historically, the conclusion was obtained, instead, through the “good vector fields” condition. However how this follows from the fact that there exists $r$ which is plurisubharmonic on $bD$ is not immediate (Remark 2.2.6 below). In any case, (2.1.8) calls into play a full family $\{T_\epsilon\}$ and the way of getting $T_\epsilon$ from the initial $T$ is involved. In [**27**], Kohn has given a quantitative result on regularity: he has specified, for given $s$, and by allowing a full flexibility in the choice of $g$, not necessarily $g \sim 1$, which is the constant $\mathcal{E}_{s,g}$ which is needed in (2.1.10) or (2.1.11) for $H^s$-regularity. This is not explicitly stated, but is entirely contained in [**27**] which, in turn, goes back to [**19**]. If this is separated from the body of the paper, as we do in Theorem 2.2.3, and under an additional assumption of uniformity under exhaustion, it gives $H^s$-estimates; this separation only requires minor modifications and yields a conclusion which naturally extends to forms of any degree $k \geq q$ on $q$-pseudoconvex domains.

It has been proved by Diederich-Fornaess in [**20**] that every domain possesses an index $\delta$ with $0 < \delta \leq 1$ such that $-(-r_\delta)^\delta$ is plurisubharmonic; this number $\delta$ is called the Diederich-Fornaess index. Again, $r_\delta$ is in the form $r_\delta = g_\delta r$ for some $g_\delta$. It is important to observe the following two facts:

(1) Locally $\delta \to 1$, because a possible choice of $g_\delta$ is $\exp(\frac{1}{1-\delta}|z|^2)$ and this satisfies $|\partial(\frac{1}{1-\delta}|z|^2)| \sim \frac{1}{1-\delta}|z| << 1$ near the origin;
(2) if $P-$property holds for $\Omega$ then $\delta \to 1$. In this case one uses $g_\delta(z) = \exp(\psi_\delta(z))$.

With $r_\delta$ in hands, we ca define a smooth bounded strictly plurisubharmonic exhaustion of $\Omega$ by

$$\{-(-r_\delta)^\delta + \rho^\delta \le 0\}_\rho.$$

On the other hand, it has been proved by Barret [**17**] that given a Sobolev index $s \searrow 0$, one can find a domain $D$ in which the Bergmann projection $B_k$ fails $H^s$-regularity; according to [**20**], for these domains, one has $\delta \searrow 0$. So the relation between the index of regularity $s$ and the Diederich-Fornaess index $\delta$ is an attractive problem. Indeed, what is explicitly stated by Kohn and is by far the most interesting content of [**27**], is the way of obtaining $\mathcal{E}_{s,g}$ out of $\delta$. This is described through the estimate of the Levi form

$$(-r_\delta)^{\frac{\delta}{2}}\Big|\partial\bar\partial(-(-r_\delta)^\delta)\llcorner\bar\partial r_\delta\Big| \lesssim (1-\delta)^{\frac12} Q_{(-r_\delta)^{\frac{\delta}{2}}}.$$

(For an operator Op, such as Op$=(-r_\delta)^\delta$, we define $Q_{\mathrm{Op}}$ by $Q_{\mathrm{Op}}(u,u) = \|\mathrm{Op}\bar\partial u\|^2 + \|\mathrm{Op}\bar\partial^* u\|^2$.) In this estimate, one enjois the presence of the factor $(1-\delta)^{\frac12}$. When $(1-\delta)^{\frac12} \le \mathcal{E}_{s,g}$, one expects $s$-regularity by what has been said above, but this is not given for free because one encounters the unpleasant factor $(-r_\delta)^{\frac{\delta}{2}}$. It is well known that $(-r_\delta)^{\frac{\delta}{2}} \sim (T^+)^{-\frac{\delta}{2}}$ when the action is restricted to harmonic functions. For this reason, Kohn can prove regularity for the projection $B_0$ on 0-forms, since this factorizes through the projection over harmonic functions. The main task of the present paper is to develop an accurate pseudodifferential calculus at the boundary which relates the action of $(-r_\delta)^{\frac{\delta}{2}}$ and $(T^+)^{-\frac{\delta}{2}}$ over general functions by describing the error terms by means of $\Delta$. In this way, when $(1-\delta)^{\frac12} \le \mathcal{E}_{s,g}$, we get $H^s$-regularity of $B_k$ in general degree $k \ge 0$ on a pseudoconvex domain.

Recent contribution to regularity of the Bergman projection by the method of the "multiplier" is given by Straube in the already mentioned paper [**29**] and Herbig-McNeal [**22**]; a combination of the "multiplier" and "potential" method (inspired to the "(P)-Property" by Catlin) is developed by Khanh [**34**] and Harrington [**21**].

## 2.2. Weak $s$-compactness and $H^s$-regularity

Let $D$ be a bounded smooth domain of $\mathbb{C}^n$ defined by $r < 0$ for $\partial r \neq 0$. We modify the defining function as $gr$ for $g \in C^\infty$ and use the notation $r_g$ or $r^g$ for $gr$. We use the lower scripts $i$ and $\bar j$ to denote derivative in $\partial_{z_i}$ and $\partial_{\bar z_j}$ respectively and work with various vector fields such as

$$(2.2.1)\quad N_g=\frac{\sum_i r^g_{\bar i}\partial_{z_i}}{\sum_i |r^g_i|^2},\qquad L^g_j=\partial_{z_j}-r^g_jN_g,\qquad T_g=-i(N_g-\bar N_g).$$

The $L^g_j$'s are complex-tangential; $T_g$ is the complementary real-tangential vector field. We consider an orthonormal basis $\bar D_1,...,\bar D_n$ of antiholomorphic 1-forms and general forms $u$ of degree $k$, that is, expressions of type $u=\sum'_{|J|=k} u_J\bar D_J$ where $J=j_1<...j_k$ are ordered multiindices and $\bar D_J=\bar D_1\wedge...\wedge\bar D_k$. We use the notations

$$\mathcal{S}=\text{Span}\{L^g_j,\ \partial_{\bar z_j},\ \text{for } j=1,...,n\},\qquad Q_s(u,u)=\|\bar\partial u\|^2_s+\|\bar\partial^* u\|^2_s.$$

We have (cf. [**19**] p. 83) for $u\in C^\infty(\bar D)$,

$$(2.2.2)\qquad \|Su\|^2_{s-1}\underset{\sim}{<} Q_{s-1}(u,u)+\|u\|_s\|u\|_{s-1}\quad \text{for any } S\in\mathcal{S}.$$

Since $\mathcal{S}\oplus\mathbb{C}T_g=\mathbb{C}\otimes T\mathbb{C}^n$, then (2.2.2) implies

$$(2.2.3)\qquad \|u\|^2_s\underset{\sim}{<} Q_{s-1}(u,u)+\|T^s_gu\|^2+\|u\|_s\|u\|_{s-1}.$$

With the notation $\bar\theta_j:=-\frac{1}{\sum_i|r^g_i|^2}\sum_i r^g_{i\bar j}r^g_{\bar i}$, we define

$$(2.2.4)\qquad \begin{cases}\bar\Theta_gu=\sum'_{|K|=k-1}\sum_{ij}\left(\bar\theta^g_ju_{iK}-\bar\theta^g_iu_{jK}\right)+\text{error},\\ \bar\Theta^*_gu=\sum'_{|K|=k-1}\sum_j\theta^g_ju_{jK}+\text{error}.\end{cases}$$

We have the crucial commutation relation between $T_g$ and the Euclidean derivatives ([**27**] Lemma 3.33)

$$(2.2.5)\qquad [\partial_{\bar z_j},T_g]=\bar\theta_jT_g\qquad \text{modulo } \mathcal{S}.$$

This implies

$$(2.2.6)\qquad [\bar\partial,T_g]=\bar\Theta_gT_g\qquad \text{modulo } \mathcal{S}.$$

As for the commutation of the adjoint $\bar\partial^*$, we need a modification of $T_g$ which preserves the condition of membership to $D_{\bar\partial^*}$. To this end, we define $\tilde T_g$ by

$$(2.2.7)\qquad (\tilde T_gu)_{jK}=T_gu_{jK}+\frac{r^g_{\bar j}}{\sum_i|r_{\bar i}|^2}\sum_i[T_g,r^g_i]u_{iK}.$$

Thus $u\in D_{\bar\partial^*}$ implies $\tilde T_gu\in D_{\bar{\bar\partial}^*}$. Note that $\tilde T_g$ differs from $T_g$ by a 0-order operator. With these preliminaries, (2.2.5) yields

$$(2.2.8)\qquad [\bar\partial^*,\tilde T_g]=\bar\Theta^*_g\tilde T_g\qquad \text{modulo } \mathcal{S}.$$

DEFINITION 2.2.1. Let $s$ be a positive integer and let $1 \leq q \leq n-1$. We say that $T^s_g$ well commutes with $\bar\partial^*$ in degree $\geq q$ when

$$\|\bar\Theta^*_g u\|^2 \leq \mathcal{E}_{s,g} Q(u,u) + c_g \|u\|^2_{-1}, \qquad \text{for any } u \text{ of degree } \geq q, \tag{2.2.9}$$

and for $\mathcal{E}_{s,g} \leq c_1^2 e^{-2c_2 s \operatorname{diam}^2 D} \inf \left(\frac{1}{|g|^s}\right)^{-1}$ or, alternatively, for $\mathcal{E}_{s,g} \leq c_1^2 e^{-2c_2 s \operatorname{diam}^2 D \sup(1+\frac{|g'|}{|g|})}$, where $c_1$ is a small constant and $c_2$ is controlled by the $C^2$ norm of $r_g$.

We introduce the notion of $q$-pseudoconvexity of $D$; this consists in the requirement that, for the ordered eigenvalues $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_{n-1}$ of the Levi form $\partial\bar\partial r|_{\partial r^\perp}$, we have $\sum_{j=1}^q \lambda_j \geq 0$. The basic estimates show that the complex Laplacian $\Box$ is invertible over $k$-forms for $k \geq q$. We denote by $N_k$ the inverse; we also denote by $B_k : L^{2,k} \to L^{2,k} \cap \ker \bar\partial$ the Bergman projection. Recall Kohn's formula $B_k = \mathrm{Id} - \bar\partial^*_{k+1} N_{k+1} \bar\partial_k$. We say that $B_k$ is regular, resp. $s$-exactly regular, when it preserves $C^\infty$, respectively $H^s$, the $s$-Sobolev space.

REMARK 2.2.2. Assume that for any $s$ there is $r_g$ with $|\partial r_g| \sim 1$, that is $|g| \sim 1$, such that $|\Theta^*_g u| \leq c_1 e^{-c_2 s \operatorname{diam}^2 D}$; then there is exact $s$-regularity for any $s$.

We recall from [**18**] that $s$-exact regularity of $N_k$ is equivalent to $s$-exact of the triplet $B_{k-1},\ B_k,\ B_{k+1}$.

THEOREM 2.2.3. *Let $D$ be $q$-peudoconvex and assume that for some $g$, $T^s_g$ well commutes with $\bar\partial^*$ in degree $\geq q$. Assume also that this property of good commutation holds, with a uniform constant $\mathcal{E}_{s,g}$, for a strongly $q$-pseudoconvex exhaustion of $D$. Then for any form $f \in H^s$ we have that $B_k f \in H^s$ and*

$$\|B_k f\|_s \leq c\|f\|_s, \qquad \text{for any } k \geq q-1. \tag{2.2.10}$$

The proof is intimately related to [**19**]. Formally, it follows the lines of [**27**] but also contains ideas taken from [**34**].

PROOF. We first assume that we already know that $B_k$ is regular for any $k \geq q-1$ and prove (2.2.10) for a constant $c$ which only depends on (2.2.9). In other terms, we show that (2.2.10) holds for $c$ if we knew from the beginning that it holds for some $c' >> c$. We reason by induction. An $n$ form is 0 at $bD$ ; thus $N_n$ "gains two derivatives" by elliptic regularity of $\Box$ in the interior and hence $B_{n-1}$ is regular. We assume now that $B_k$ is $s$-regular and prove that the same is true for $B_{k-1}$. We use the notation $f$ for the test form in our proof; the notation $u$, which occurs in (2.2.9), will be reserved to $\bar\partial N_k f$. It suffices to estimate $\|T_g^s B_{k-1}f\|$ since, by (2.2.3), this controls the full norm $\|B_{k-1}f\|_s$. We have

$$
\begin{aligned}
\|T_g^s B_{k-1}f\|^2 &= \big(T_g^s B_{k-1}f, T_g^s f\big) - \big(T_g^s B_{k-1}f, T_g^s \bar\partial^* N_k \bar\partial f\big)\\
&= \underbrace{\big(T_g^s B_{k-1}f, T_g^s f\big)}_{(a)} - \underbrace{\big(T_g^{s*}T_g^s \bar\partial B_{k-1}f, N_k \bar\partial f\big)}_{(b)}\\
&\quad - \underbrace{\big([\bar\partial, T_g^{s*}T_g^s]B_{k-1}f, N_k\bar\partial f\big)}_{(c)}.
\end{aligned}
\tag{2.2.11}
$$

Now, $(a) \leq sc\|T_g^s B_{k-1}f\|^2 + lc\|T_g^s f\|^2$, whereas $(b) = 0$. The term which comes with small constant can be absorbed because we know a-priori that $\|T_g^s B_{k-1}f\| < \infty$. As for the last term, we replace $T_g^s$ by $\tilde T_g^s$ modulo an operator of order $s-1$, that we regard as an error term, describe the commutator in the left of (c) by $\bar\Theta_g$ according to (2.2.6), switch it to the right as $\bar\Theta_g^*$ and end up with

$$
\begin{aligned}
|(c)| &\leq \Big|\Big(2s\bar\Theta_g \tilde T_g^s B_{k-1}f, \tilde T_g^s N_k\bar\partial f\Big)\Big| + \text{error}\\
&\leq sc\|T_g^s B_{k-1}f\|^2 + lc\,s\|\bar\Theta_g^* T_g^s N_k \bar\partial f\|^2 + \text{error}.
\end{aligned}
\tag{2.2.12}
$$

The error includes terms in $(s-1)$-norm and terms in which derivatives belonging to $\mathcal{S}$ occur (cf. (2.2.2)). We use the hypothesis (2.2.8) under the choice $\mathcal{E}_{s,g} \leq c_1^2 c^{-2c_2 s\,\mathrm{diam}^2 D}\sup\frac{1}{|g|^{2s}}$ and get, with the notation $u = N_k\bar\partial f$

(2.2.13)

$$
\begin{aligned}
\|\bar\Theta_g^*\tilde T_g^s u\|^2 &\leq \sup\frac{1}{|g|^{2s}}\|\bar\Theta_g^*\tilde T^s u\|^2\\
&\leq \mathcal{E}_{s,g}\sup\frac{1}{|g|^{2s}}Q(\tilde T^s u, \tilde T^s u) + \text{error}\\
&\leq \mathcal{E}_{s,g}\sup\frac{1}{|g|^{2s}}\Big(Q_{\tilde T^s}(u,u) + \|[\bar\partial, \tilde T^s]u\|^2 + \|[\bar\partial^*, \tilde T^s]u\|^2\Big) + \text{error}.
\end{aligned}
$$

(In case $\mathcal{E}_{s,g} \leq c_1^2 e^{-2c_2 s\,\mathrm{diam}^2 D(1+\sup\frac{|g'|}{|g|})}$ we have not to replace $\tilde T_g^s$ by $\tilde T^s$ and, instead, use the estimate

$$
|[\tilde T_g^s, \bar\partial]v| \underset{\sim}{<} c_2\sup(1+\frac{|g'|}{|g|})|\tilde T_g v| \quad \text{modulo } Sv \text{ for } S\in\mathcal{S},
\tag{2.2.14}
$$

and similarly for $\bar\partial$ replaced by $\bar\partial^*$; the proof will proceed similarly as below.)

Now,

$$Q_{\tilde{T}^s}(u,u) \lesssim \|T^s f\|^2 + \|T^s B_{k-1} f\|^2 + \text{error}.$$

Next,

$$\|[\bar\partial, \tilde{T}^s]u\|^2 \le c_2 s^2 \, \|T^s N_k \bar\partial f\|^2 + \text{error}.$$

We now observe that

$$\begin{aligned} N_k\bar\partial &= B_k N_k \bar\partial(\mathrm{Id} - B_{k-1}) \\ &= B_k e^{-\varphi_s} N_{k,\varphi_s} \bar\partial e^{\varphi_s}(\mathrm{Id} - B_{k-1}), \end{aligned} \tag{2.2.15}$$

where $N_{k,\varphi_s}$ is the $\bar\partial$-Neumann operator weighted by $e^{-\varphi_s} = e^{-c_2 s|z|^2}$. Since $[D^s, \bar\partial]$ is an operator of degree $s$ with coefficients controlled by $sc_2$ for $c_2 \sim \|r\|_{C^2}$, then $N_{k,\varphi_s}\bar\partial$ is continuous in $H^s_{\varphi_s}$ with a continuity constant that we can assume to be unitary. We use that $c_2 s^2\, e^{-2c_2 s \ \mathrm{diam}^2 D} \le \inf_{z\in D} e^{-2c_2 s|z|^2}$ (for different $c_2$) in order to remove weights from the norms. We also use the inductive assumption that (2.2.10) holds for $B_k$. In this way, we end up with

(2.2.16)

$$\begin{aligned} \mathcal{E}_{s,g} \sup \frac{1}{|g|^{2s}} c_2 s^2 \, \|T^s \bar\partial N_k f\|^2 &\le c_1^2 \left( \|T^s f\|^2 + \|T^s B_{k-1} f\|^2 \right) + \text{error} \\ &\le c_1^2 (\|T^s_g f\|^2 + \|T^s B_{k-1} f\|^2) + \text{error}, \end{aligned}$$

where the last inequality follows trivially from the fact that $T_g = \frac{1}{g} T$ for $\left|\frac{1}{g}\right| >> 1$. Here, $\mathcal{E}_{s,g}$ takes care of $\sup \frac{1}{|g|^{2s}}$ and also of the constant which arises from removing weights owing to $\mathcal{E}_{s,g} \le c_1^2 e^{-2sc_2 \mathrm{diam}^2 D} \sup \frac{1}{|g|^{2s}}$. Altogether, up to absorbable terms, $\|T^s_g B_{k-1} f\|^2$ has been estimated by $lc\|T^s_g f\|^2$ + error. This concludes the proof of Theorem 2.2.3 if we are able to remove the assumption that we already know that (2.2.10) holds for some $c' >> c$. For this, we recall that we are assuming that there is a strongly $q$-pseudoconvex exhaustion $D_\rho \nearrow D$ which satisfies (2.2.9) uniformly with respect to $\rho$. We observe that (2.2.10) holds over $D_\rho$ for $c' = c'_\rho$. What has been proved above shows that it holds in fact with $c$ independent of $\rho$. Passing to the limit over $\rho$ we get (2.2.10) for $D$.

□

THEOREM 2.2.4. *(Boas-Straube* **[19]***) If there is a defining function $r$ such that for the eigenvalues $\mu_1 \le \ldots \le \mu_n$ of the full Levi form $\partial\bar\partial r$ (not restricted to $\partial r^\perp$) we have $\sum_{j=1}^q \mu_j \ge 0$, then, $B_k$ is exactly $H^s$-regular for any $s$ and any $k \ge q-1$.*

PROOF. The proof consists in proving that (2.2.9) holds for any $\epsilon$ and uniformly over an exhaustion of $D$. More precisely, we will show that for any $\epsilon$, for $\bar{\Theta}^*$ independent of $\epsilon$ (associated to a normalized defining function $r$), and for suitable $c_\epsilon$, we have

$$\|\bar{\Theta}^* u\|^2 \leq \epsilon Q(u,u) + c_\epsilon |||u|||_{-1} \quad \text{for } u \text{ in degree } k \geq q; \tag{2.2.17}$$

moreover, we will prove that (2.2.17) holds for a strongly $q$-pseudoconvex exhaustion. (Here, the triplet $|||\cdot|||$ denotes the tangential norm (cf. [**26**]).)

**(a)** We begin by noticing that $\partial\bar{\partial}r + O(|r|)\mathrm{Id} \geq 0$ over $k$-forms for $k \geq q$. We can then apply Cauchy-Schwartz inequality and get

$$(r_{i\bar{j}})(u, \partial r) \leq (r_{i\bar{j}})(u,u)^{\frac{1}{2}} + O(|r|^{\frac{1}{2}})|u|. \tag{2.2.18}$$

**(b)** The Levi form is a "$\frac{1}{2}$-subelliptic multiplier" (cf. [**26**]), that is

$$|||\left((r_{i\bar{j}})(u,u)\right)^{\frac{1}{2}}|||^2_{\frac{1}{2}} \leq Q(u,u). \tag{2.2.19}$$

This can be proved from the basic estimate

$$\int_D (r_{i\bar{j}})(Tu,u)dV \leq Q(u,u),$$

by using the microlocalization $T^+$ and its decomposition $T^+ = (T^+)^{\frac{1}{2}}(T^+)^{\frac{1}{2}*}$. (Here $dV$ is the element of volume.) Also, by Sobolev interpolation, we have

$$\begin{aligned} \|(r_{i\bar{j}})(u,u)^{\frac{1}{2}}\|^2 &\leq \epsilon \|(r_{i\bar{j}})(u,u)^{\frac{1}{2}}\|^2_{\frac{1}{2}} + c_\epsilon \|u\|^2_{-1} \\ &\leq \epsilon Q(u,u) + c_\epsilon \|u\|^2_{-1}, \end{aligned} \tag{2.2.20}$$

where $c_\epsilon \sim \epsilon^{-1}||r||_{C^2}$. Finally, we estimate the norm of the last term in (2.2.18). We have

(2.2.21)
$$\begin{aligned} \|(-r)^{\frac{1}{2}}u\|^2 &\leq \epsilon \|\zeta_\epsilon u\|^2_0 + \|(1-\zeta_\epsilon)u\|^2_0 \\ &\leq \epsilon\|u\|^2_0 + \|(1-\zeta_\epsilon)u\|^2_0 \lesssim \epsilon Q(u,u) + \|(1-\zeta_\epsilon)u\|^2_0, \end{aligned}$$

where $\zeta_\epsilon$ is a cut-off outside of the $\epsilon$-strip such that $|\dot{\zeta}_\epsilon| \lesssim \frac{1}{\epsilon}$(with $\zeta_\epsilon \equiv 1$ at $bD$). Moreover, we have

$$\|(1-\zeta_\epsilon)u\|^2_0 \leq \epsilon^3 \|(1-\zeta_\epsilon)u\|^2_1 + c_\epsilon \|(1-\zeta_\epsilon)u\|^2_{-1}, \tag{2.2.22}$$

and,

$$\begin{aligned}\epsilon^3\|(1-\zeta_\epsilon)u\|_1^2 &\underset{(i)}{\lesssim} \epsilon^3 Q_0((1-\zeta_\epsilon)u,(1-\zeta_\epsilon)u)\\ &\lesssim \epsilon^3 Q_0(u,u)+\epsilon^3\|\dot{\zeta}_\epsilon u\|_0^2\\ &\lesssim \epsilon^3 Q_0(u,u)+\epsilon^3\epsilon^{-2}\|u\|_0^2\\ &\underset{(ii)}{\lesssim} 2\epsilon Q_0(u,u),\end{aligned} \tag{2.2.23}$$

where (i) is Garding inequality applied to $(1-\zeta_\epsilon)u|_{bD}\equiv 0$ and (ii) follows from applying the basic estimate to $\|u\|_0^2$. Putting together (2.2.18)–(2.2.23), we get (2.2.17).

**(c)** We consider the exhaustion of $D$ by the domains $D_\rho$ defined by $r_\rho<0$ for $r_\rho=r+\rho e^{A|z|^2}$; by a suitable choice of $A$, these domains are strongly $q$-pseudoconvex. We remark that $\partial\bar\partial r_\rho \gtrsim -\|r\|_{C^2}|r_\rho|\,\mathrm{Id}\geq -c|r_\rho|\,\mathrm{Id}$ over $k$ forms for $k\geq q$. By Cauchy-Schwarz inequality we get

$$(r^\rho_{i\bar j})(u,\partial r)\leq (r^\rho_{i\bar j})(u,u)^{\frac12}+c|r_\rho|^{\frac12}|u| \quad \text{for } u \text{ of degree } k\geq q. \tag{2.2.24}$$

The Levi form $(r^\rho_{i\bar j})$ is a $\frac12$-subelliptic multiplier (uniformly over $\rho$) and can be estimated as in (b) as well as the term with $O(|r_\rho|^{\frac12})$. Altogether, for fixed $\epsilon$ for any $\rho\leq\rho_\epsilon$ and for $\bar\Theta^*_\rho$ associated to the definng function $r_\rho$, we have got

$$\|\bar\Theta^*_\rho u\|^2\leq \epsilon Q_{D_\rho}(u,u)+c_\epsilon\|u\|^2_{-1}, \tag{2.2.25}$$

uniformly with respect to $\rho$. Passing to the limit over $\rho$, yields (2.2.17). □

Theorem 2.2.5. *Let $D$ be $q$-pseudoconvex and assume that for any $\epsilon$ there is $|g_\epsilon|\sim 1$ such that*

$$|\bar\Theta^*_{g_\epsilon}(u)|\leq\epsilon|u|^2 \quad \textit{on } bD \textit{ for } u \textit{ in degree } k\geq q. \tag{2.2.26}$$

*Then $B_k$ is exactly $H^s$-regular for any $s$ and any $k\geq q-1$.*

Proof. (2.2.26) readily implies

$$\|\bar\Theta^*_{g_\epsilon}u\|^2\lesssim \epsilon\|u\|^2+\|g_\epsilon r\|_{C^2}\|(1-\zeta_\epsilon)u\|^2 \quad \text{for } u \text{ in degree } k\geq q. \tag{2.2.27}$$

By plugging (2.2.26) with the basic estimate $\|u\|^2\lesssim Q(u,u)$ and the Garding inequality $\|g_\epsilon r\|_{C^2}\|(1-\zeta_\epsilon)u\|^2\lesssim \epsilon Q(u,u)+c_\epsilon\|u\|^2_{-1}$, we get

$$\|\bar\Theta^*_{g_\epsilon}u\|^2\lesssim \epsilon Q(u,u)+c_\epsilon\|u\|^2_{-1} \quad \text{for } u\in D_{\bar\partial^*} \text{ of degree } k\geq q. \tag{2.2.28}$$

This would give the $H^s$-regularity of $B_k$ if we were able to prove the stability of (2.2.26) under a strongly $q$-pseudoconvex exhaustion. For this, we fix $\epsilon_o$ and $g_{\epsilon_o}r$ and approximate $D$ by $D_\rho$ defined by $g_{\epsilon_o}r + \rho e^{A|z|^2}$; for suitable fixed $A$, these are strongly $q$-pseudoconvex for any $\rho$. Also, if we rewrite $g_{\epsilon_o}r + \rho e^{A|z|^2} = g_{\epsilon_o,\rho}r_\rho$ for a normalized equation $r_\rho$ of $D_\rho$, we have

$$\begin{cases} g_{\epsilon_o,\rho} \underset{C^2}{\to} g_{\epsilon_o}, \\ r_\rho \underset{C^2}{\to} r. \end{cases}$$

Hence

$$\bar{\Theta}^*_{\epsilon_o,\rho}(u) \to \bar{\Theta}^*_{\epsilon_o}(u) \quad \text{uniformly over } u.$$

We then apply Theorem 2.2.3 to each $\Omega_\rho$ and by uniformity of the estimate with respect to $\rho$ we get that $B_kf$ belongs to $H^s$ and satisfies (2.2.10).

□

Remark 2.2.6. We can give an alternative proof of Theorem 2.2.3 which uses Theorem 2.2.5. First, according to the lemma in [**19**], the existence of a plurisubharmonic defining function $r$ implies the vector fields condition (2.1.8). (If $r$ is only $q$-plurisubharmonic, (2.1.8) must be adpted by considering, similarly as in (2.2.26), the action over forms $u$ of degree $k \geq q$.) If we knew that the good vector fields $T_\epsilon$ are of type $T_{g_\epsilon} = -i(N_{g_\epsilon} - \bar{N}_{g_\epsilon})$, then, by (2.2.8) we would get (2.2.26) and reach the conclusion from Theorem 2.2.5. In the general case, by [**30**] Proposition 5.26, the condition of good vector fields implies (2.2.26). (In that proposition, it is proved a generalization of (2.2.8). For any tangential vector field $T_\epsilon$, not necessarily defined by (2.2.1), if we denote by $g_\epsilon$ its $(N - \bar{N})$-component, we have $[\bar{\partial}^*, T_\epsilon]|_{bD} = \bar{\Theta}^*_{g_\epsilon}|_{bD}T_\epsilon$ modulo elliptic multipliers ($r$ and $\partial r$) and $\frac{1}{2}$-subelliptic multipliers ($\partial\bar{\partial}r$).)

Remark 2.2.7. We point out that in [**29**], Straube proves that (2.2.28) suffices for exact $H^s$-regularity for any $s$. This requires heavy work since, differently from (2.2.26), (2.2.28) is not tranferred from $\Omega$ to $\Omega_\rho$.

## 2.3. Pseudodifferential calculus at the boundary

There is an important theory about the equivalence between $(-r)^\sigma$ and microlocal powers $T^{-\sigma}$ over harmonic functions; we need to develop this theory and allow the action over general functions controlling errors coming from the Laplacian. In this discussion, we do not modify $r$ to $r_g$ and $T$ nor $T_g$. Also, we still write $T$ but mean in fact its positive microlocalization $T^+$ which represents over $v^+$ the full elliptic standard operator $\Lambda$; for this reason, negative and fractionl powers of $T$ make sense. We denote by $U$ a neighborhood of $bD$,

Lemma 2.3.1. *We have*
(2.3.1)
$$\|(-r)^{\frac{\delta}{2}}r^\sigma T^\sigma v\| \underset{\sim}{<} lc\|(-r)^{\frac{\delta}{2}}v\|+sc\|T^{-\frac{\delta}{2}}v\|+sc\|-rT^{-1-\frac{\delta}{2}}\Delta v\| \quad \textit{for any } v\in C^\infty(\bar D\cap U) \textit{ and } \sigma$$

This is a generalization of [**27**] Lemma 2.6 in which the extra terms with power $\frac{\delta}{2}$ do not occur.

Proof. We have
$$\begin{aligned}\|(-r)^{\frac{\delta}{2}}r^\sigma T^\sigma v\|^2 &= ((-r)^{\delta+2\sigma}T^{2\sigma}v,v)\\ &= -(\partial_r(-r^{1+2\sigma+\delta})T^{2\sigma}v,v)\\ &= 2\mathsf{Re}\,((-r)^{1+2\sigma+\delta}\partial_rT^{2\sigma}v,v)\\ &\le lc\|(-r)^{\frac{\delta}{2}}v\|^2+sc\|(-r)^{1+2\sigma+\frac{\delta}{2}}\partial_rT^{2\sigma+\frac{\delta}{2}-\frac{\delta}{2}}v\|^2\\ &\underset{(*)}{\le} lc\|(-r)^{\frac{\delta}{2}}v\|^2+sc\|T^{-\frac{\delta}{2}}v\|^2+sc\|-rT^{-1-\frac{\delta}{2}}\Delta v\|^2,\end{aligned}$$
where $(*)$ follows from [**27**] (2.4) applied for $1+2\sigma+\frac{\delta}{2}>0$.

□

In [**27**] there is a result, Lemma 2.6, which applies to powers $>-\frac{1}{2}$ of $-r$; we need a variant, still for negative powers, for terms involving $\partial_r v$.

Lemma 2.3.2. *We have*
(2.3.2)
$$\|(-r)^\sigma\partial_rT^\sigma v\| \underset{\sim}{<} \|v\|+\|rT^{-1}\Delta v\|+\|T^{-2}\Delta v\|, \quad v\in C^\infty(\bar D\cap U),\ \sigma>-\frac{1}{2}.$$

Proof. We have
$$\Big(\partial_r(-r)^{2\sigma+1}\partial_rT^{2\sigma-2}v,\partial_rv\Big) = -2\mathsf{Re}\Big((-r)^{2\sigma+1}\partial_r^2T^{2\sigma-2}v,\partial_rv\Big).$$
Write $\partial_r^2=\Delta+Tan\partial_r+Tan^2\sim\Delta+T\partial_r+T^2$. For the three terms $\Delta$, $T^2$ and $T\partial_r$, we have the three relations below, respectively
$$\begin{cases}\Big(T^{-2}\Delta v,(-r)^{2\sigma+1}T^{2\sigma}\partial_rv\Big)\le\|T^{-2}\Delta v\|^2+\|v\|^2,\\ \Big((-r)^{2\sigma+1}T^{2\sigma}v,\partial_rv\Big)=\Big((-r)^{2\sigma+1}T^{2\sigma+1}v,\partial_rT^{-1}v\Big)\\ \qquad\qquad \underset{\sim}{<}\|v\|^2+\|-rT^{-1}\Delta v\|^2,\\ \Big((-r)^{2\sigma+1}\partial_rT^{2\sigma-1}v,\partial_rv\Big)=\Big((-r)^{2\sigma+1}\partial_rT^{(2\sigma+1)-1}v,\partial_rT^{-1}v\Big)\\ \qquad\qquad \le\|v\|^2+\|-rT^{-1}\Delta v\|^2,\end{cases}$$

where the three inequalities come from Cauchy-Schwartz inequality combined with repeated use of [**27**] (2.4) (always under the choice $s = 0$ with the notations therein). Finally, we have to estimate the error term

$$\Big((r)^{2\sigma+1}[\Delta, T^{2\sigma-2}]v, \partial_r v\Big). \tag{2.3.3}$$

We express the commutator in (2.3.3) as

$$[\Delta, T^{2\sigma-2}] = T^{2\sigma-1} + \partial_r T^{2\sigma-2}.$$

Thus (2.3.3) splits into two terms to which the two inequalities below apply

$$\begin{cases} \Big((-r)^{2\sigma+1}T^{2\sigma-1}v, \partial_r v\Big) = \Big((-r)^{2\sigma+1}T^{(2\sigma+1)-1}v, T^{-1}\partial_r v\Big) \\ \qquad\qquad \le \|v\|^2 + \|-rT^{-1}\Delta v\|^2, \\ \Big((-r)^{2\sigma+1}\partial_r T^{2\sigma-2}v, \partial_r v\Big) = \Big((-r)^{2\sigma+1}\partial_r T^{2\sigma-1}v, T^{-1}\partial_r v\Big) \\ \qquad\qquad \le \|v\|^2 + \|-rT^{-1}\Delta v\|^2. \end{cases}$$

□

We are ready for the main technical tool in interchanging powers of $-r$ and $T$.

PROPOSITION 2.3.3. *We have*

$$\|T^{-\frac{\delta}{2}}v\| \lesssim \|(-r)^{\frac{\delta}{2}}v\| + \|-rT^{-1-\frac{\delta}{2}}\Delta v\| + \|(-r)^{\frac{\delta}{2}}T^{-2}\Delta v\|. \tag{2.3.4}$$

PROOF. We start from [**27**] Lemma 2.11

$$\begin{aligned} \|T^{-\frac{\delta}{2}}v\| &\lesssim \|(-r_\delta)^{\frac{\delta}{2}}v\| + \|-rT^{-1-\frac{\delta}{2}}\Delta v\| \\ &\quad + \sum_j \|(-r_\delta)^{\frac{\delta}{2}}\partial_{\bar z_j}T^{-1}v\|. \end{aligned}$$

Now, the first and second terms in the right are good (in the right side of the estimate we wish to end with). As for the last, we have

(2.3.5)

$$\begin{aligned} \sum_j \Big((-r_\delta)^{\frac{\delta}{2}}\partial_{\bar z_j}T^{-1}v, (-r_\delta)^{\frac{\delta}{2}}\partial_{\bar z_j}T^{-1}v\Big) &\le \Big|\Big((-r_\delta)^{\frac{\delta}{2}}\Delta T^{-2}v, (-r_\delta)^{\frac{\delta}{2}}v\Big)\Big| \\ &\quad + 2\sum_j \Big|\mathsf{Re}\Big([(-r_\delta)^{\delta}, \partial_{z_j}]\partial_{\bar z_j}T^{-1}v, T^{-1}v\Big)\Big|. \end{aligned}$$

The first term in the right is estimated by

$$\begin{aligned}\Big|\Big((-r_\delta)^{\frac{\delta}{2}}\Delta T^{-2}v,(-r_\delta)^{\frac{\delta}{2}}v\Big)\Big| &\le lc\|(-r)^{\frac{\delta}{2}}v\|+sc\|(-r)^{\frac{\delta}{2}}(\partial_r^2+\partial_rT+T^2)T^{-2}v\|\\ &\le lc\|(-r)^{\frac{\delta}{2}}v\|+sc\Big(\|(-r)^{\frac{\delta}{2}}T^{-2}\partial_r^2v\|+\|(-r)^{\frac{\delta}{2}}\partial_rT^{-1}v\|\Big)\\ &\le lc\|(-r)^{\frac{\delta}{2}}v\|+sc\Big(\|(-r)^{\frac{\delta}{2}}T^{-2}\Delta v\|+\|T^{-\frac{\delta}{2}}v\|\Big).\end{aligned}$$

The second term in the right of (2.3.5) has the estimate

$$\begin{aligned}\Big|\mathsf{Re}\Big([(-r_\delta)^\delta,\partial_{z_j}]T^{-1}v,T^{-1}v\Big)\Big| &\lesssim \underbrace{\Big|\Big((-r)^{-1+\delta+\epsilon}T^{-1+\frac{\delta}{2}+\epsilon}v,(-r)^{-\epsilon}T^{-\frac{\delta}{2}-\epsilon}v\Big)\Big|}_{(i)}\\ &\quad+\underbrace{\Big|\Big((-r)^{-1+\delta}\partial_rT^{-1}v,T^{-1}v\Big)\Big|}_{(ii)}.\end{aligned}$$

To estimate (i), we write $-1+\delta+\epsilon=\frac{\delta}{2}+(-1+\frac{\delta}{2}+\epsilon)=\frac{\delta}{2}+\sigma$ under the choice of $\epsilon>\frac{1}{2}-\frac{\delta}{2}$ so that $-1+\frac{\delta}{2}+\epsilon>-\frac{1}{2}$. We then apply Lemma 2.3.1 and get the estimate of (i)

$$(i)\le lc\|(-r)^{\frac{\delta}{2}}v\|^2+sc(\|T^{-\frac{\delta}{2}}v\|^2+\|-rT^{-1-\frac{\delta}{2}}\Delta v\|).$$

As for (ii) we have

$$\begin{aligned}(ii)&=\Big|\Big((-r)^{-1+\delta+(1-\frac{\delta}{2}-\epsilon)}\partial_rT^{-1-\epsilon}v,(-r)^{-1+\frac{\delta}{2}+\epsilon}T^{-1+\epsilon}v\Big)\Big|\\ &\lesssim sc\Big(\|T^{-\frac{\delta}{2}}v\|^2+\|-rT^{-1}\Delta v\|+\|T^{-2}\Delta v\|\Big)\\ &\quad+lc\Big(\|(-r)^{\frac{\delta}{2}}v\|^2+\|-rT^{-1-\frac{\delta}{2}}\Delta v\|\Big).\end{aligned}$$

In fact, the term with lc in the last line comes from Lemma 2.3.1 applied for $\sigma=-1+\epsilon$ (which requires $\epsilon>\frac{1}{2}$). The term with sc is estimated by the aid of Lemma 2.3.2

$$\begin{aligned}\|(-r)^{-1+\delta+(1-\frac{\delta}{2}-\epsilon)}\partial_rT^{-1-\epsilon}v\| &=\|(-r)^{\frac{\delta}{2}-\epsilon}\partial_rT^{-1+(\frac{\delta}{2}-\epsilon)-\frac{\delta}{2}}v\|\\ &\underset{(2.3.2)}{\lesssim}\|T^{-\frac{\delta}{2}}v\|+\|-rT^{-1}\Delta v\|+\|T^{-2}\Delta v\|.\end{aligned}$$

□

We decompose now $v=v^{(h)}+v^{(0)}$ where $v^{(h)}$ is the harmonic extension and $v^{(0)}:=v-v^{(h)}$; note that $v^{(0)}|_{bD}\equiv0$. We also recall the modification $\tilde{T}$ of $T$ defined by (2.2.7) and designed to preserve $D_{\bar\partial^*}$.

PROPOSITION 2.3.4. *We have*

$$\|[\tilde{T}^{s-\frac{\delta}{2}},\bar{\partial}^*]v^{(h)}\| \underset{\sim}{<} \|(-r)^{\frac{\delta}{2}}[\tilde{T}^s,\bar{\partial}^*]v^{(h)}\|,\quad v\in C^\infty(\bar{D}\cap U). \tag{2.3.6}$$

REMARK 2.3.5. In turn, by (2.2.8), we have $[\tilde{T}^s,\bar{\partial}^*]=s\bar{\Theta}\tilde{T}^s$, and therefore (2.3.6) implies

$$\|[\tilde{T}^{s-\frac{\delta}{2}},\bar{\partial}^*]v^{(h)}\| \underset{\sim}{<} s\|(-r)^{\frac{\delta}{2}}\bar{\Theta}\tilde{T}^s v^{(h)}\|. \tag{2.3.7}$$

PROOF. In fact, Jacobi identity yields

$$[\tilde{T}^s,\bar{\partial}^*]=-\tilde{T}^{s-\frac{\delta}{2}}[\tilde{T}^{\frac{\delta}{2}},\bar{\partial}^*]+\tilde{T}^{\frac{\delta}{2}}[\tilde{T}^{s-\frac{\delta}{2}},\bar{\partial}^*]+[\tilde{T}^{s-\frac{\delta}{2}}[\tilde{T}^{\frac{\delta}{2}},\bar{\partial}^*]].$$

It follows

$$\tilde{T}^{\frac{\delta}{2}}[\tilde{T}^{s-\frac{\delta}{2}},\bar{\partial}^*]=[\tilde{T}^s,\bar{\partial}^*]+\tilde{T}^{s-\frac{\delta}{2}}[\tilde{T}^{\frac{\delta}{2}},\bar{\partial}^*]-[\tilde{T}^{s-\frac{\delta}{2}}[\tilde{T}^{\frac{\delta}{2}},\bar{\partial}^*]]. \tag{2.3.8}$$

We apply $\tilde{T}^{-\frac{\delta}{2}}$ to both sides of (2.3.8) and use Proposition 2.3.3. The conclusion will follow once we are able to show that $-r\tilde{T}^{-1-\frac{\delta}{2}}[\Delta,[\tilde{T}^s,\bar{\partial}^*]]$ and $(-r)^{\frac{\delta}{2}}T^2[\Delta,T^s\bar{\partial}^*]$ are error terms. In fact, we write

$$\begin{aligned}[\Delta,[\tilde{T}^s,\bar{\partial}^*]]&=[\partial_r^2+\partial_r Tan+Tan^2,Tan^s+\partial_r Tan^{s-1}]\\&=Tan^{s-1}+\partial_r Tan^s\underset{\sim}{<}\tilde{T}^{s+1}+\partial_r\tilde{T}^s\qquad\text{modulo }\mathcal{S}.\end{aligned}$$

It follows

$$\begin{cases}\|{-r}T^{-1\frac{\delta}{2}}[\Delta,[\tilde{T}^s,\bar{\partial}^*]]v^{(h)}\|\underset{\sim}{<}\|{-r}T^{s-\frac{\delta}{2}}v^{(h)}\|+\|{-r}\partial_r T^{s-1-\frac{\delta}{2}}v^{(h)}\| \underset{\mathbf{[27]}\ (2.4)}{\underset{\sim}{<}} \|T^{s-1-\frac{\delta}{2}}v^{(h)}\|,\\ \|(-r)^{\frac{\delta}{2}}T^{-2}[\Delta,[\tilde{T}^s,\bar{\partial}^*]]v^{(h)}\|\underset{\sim}{<}\|(-r)^{\frac{\delta}{2}}T^{s-1}v^{(h)}\|+\|(-r)^{\frac{\delta}{2}}\partial_r T^{s-2}v^{(h)}\| \underset{\mathbf{[27]}\ (2.4)}{\underset{\sim}{<}} \|T^{s-1-\frac{\delta}{2}}v^{(h)}\|\end{cases}$$

□

## 2.4. Non-smooth plurisubharmonic defing functions

DEFINITION 2.4.1. We say that $D$ has a Diederich-Fornaess index $\delta=\delta_s$ for $0<\delta\le 1$ which controls the commutators of $\bar{\partial}$ and $\bar{\partial}^*$ with $D^s$ over forms in degree $k\ge q$, when there is $r_\delta=g_\delta r$ for $g_\delta\in C^\infty$, $g_\delta\neq 0$, such that

$$\begin{cases}-(-r_\delta)^\delta \text{ is } q\text{-plurisubharmonic, that is, the sum of the first}\\ \qquad q \text{ eigenvalues of } \partial\bar{\partial}(-(-r_g)^\delta) \text{ is non-negative}\\ (1-\delta_s)\le\mathcal{E}_{s,g},\end{cases} \tag{2.4.1}$$

where $\mathcal{E}_{s,g}$ can be chosen so that $\mathcal{E}_{s,g}\le c_1e^{-c_2s\ \mathrm{diam}^2D}\sup\left(\frac{1}{|g|^s}\right)^{-1}$ or, alternatively, $\mathcal{E}_{s,g}\le c_1e^{-c_2s\ \mathrm{diam}^2D\,\sup(1+\frac{|g'|}{|g|})}$.

Related to the above notion, is the condition

$$\|(-r_\delta)^{\frac{\delta}{2}}\bar{\Theta}^*_g u\|^2 \leq \mathcal{E}_{s,g} Q_{(-r_\delta)^{\frac{\delta}{2}}}(u,u), \tag{2.4.2}$$

for $\delta \leq 1$.

THEOREM 2.4.2. *If $D$ is $q$-pseudoconvex and has a Diederich-Fornaess index $\delta = \delta_s$ which controls the commutators of $(\bar\partial, \bar\partial^*)$ with $D^s$ in degree $k \geq q$, then $B_k$ is $s$-regular for $k \geq q$.*

REMARK 2.4.3. The proof consists in showing that (2.4.1) implies (2.4.2) (points (a) and (b) below) and then showing that (2.4.2) implies the conclusion. Note that, when $\delta = 1$, we have in fact the better conclusion contained in Theorem 2.2.4.

PROOF. We decompose a form as $u = u^\tau + u^\nu$ where $u^\tau$ and $u^\nu$ are the tangential and normal component respectively. We have

$$\begin{cases} \|u^\nu\|_1^2 \leq \sum_i \|\partial_{\bar z_i} u^\nu\|_0^2 \lesssim Q(u,u) \\ Q(u^\tau, u^\tau) \leq Q(u,u) + Q(u^\nu, u^\nu) \\ \qquad\qquad \lesssim Q(u,u) + \|u^\nu\|_1^2 \\ \qquad\qquad \lesssim Q(u,u). \end{cases} \tag{2.4.3}$$

Hence it suffices to prove (2.4.2). The same conclusion also applies to the decompositin $u = u^{(h)} + u^{(0)}$ and, in general, to any decomposition in which either of the two terms is 0 at $bD$.

**(a)** We have

$$\left|\partial\bar\partial r_\delta(u^\tau, \partial r_\delta)\right| \lesssim (1-\delta)^{\frac{1}{2}}(-r_\delta)^{-\frac{\delta}{2}}\Big(\partial\bar\partial(-(-r_\delta)^\delta)(u^\tau, u^\tau)\Big)^{\frac{1}{2}}. \tag{2.4.4}$$

To see it, we start from

$$\partial\bar\partial(-(-r_\delta)^\delta) = \delta(-r_\delta)^{\delta-1}\partial\bar\partial r_\delta + (-r_\delta)^{\delta-2}\delta(1-\delta)\partial r \otimes \bar\partial r.$$

In particular,

$$\partial\bar\partial r_\delta = \frac{1}{\delta}(-r_\delta)^{1-\delta}\partial\bar\partial(-(-r_\delta)^\delta) - (-r_\delta)^{-1}(1-\delta)\partial r \otimes \bar\partial r.$$

We suppose that $\delta$ is bounded away from 0 and, indeed, that it approaches 1; thus we disregard it in the following. We have

$$\begin{aligned}\partial\bar\partial r_\delta(u,\partial r_\delta) &\sim (-r_\delta)^{1-\delta}\partial\bar\partial(-(-r_\delta)^\delta)(u,\partial r_\delta) - (-r_\delta)^{-1}(1-\delta)\partial r_\delta\otimes\bar\partial r_\delta(u,\partial r_\delta)\\ &\leq (-r_\delta)^{1-\delta}\Big(\partial\bar\partial(-(-r_\delta)^\delta)(u,u)\Big)^{\frac12}\Big((-r_\delta)^{-2+\delta}(1-\delta)|\partial r_\delta|^2+O((-r_\delta)^{-1+\delta})\Big)^{\frac12}\\ &\qquad + (1-\delta)|\partial r_\delta|^2(-r_\delta)^{-1}|\partial r_\delta\cdot u|\\ &\underset{\sim}{<} \Big((1-\delta)^{\frac12}(-r_\delta)^{-\frac{\delta}{2}}+O(-r_\delta)^{\frac12-\frac{\delta}{2}}\Big)\Big(\partial\bar\partial(-(r_\delta)^\delta)(u,u)\Big)^{\frac12}+(1-\delta)|\partial r_\delta|^2(-r_\delta)^{-1}|\partial r_\delta\cdot u|.\end{aligned}$$

Evaluation for $u=u^\tau$, yields (2.4.4).

**(b)** We prove now (2.4.2) using the basic estimates. Generally, these apply to smooth plurisubharmonic defining functions. However, in [**27**], Kohn has a version for Hölder continuous plurisubharmonic functions such as $-(-r_\delta)^\delta$. This implies the inequality $(*)$ below

$$\begin{aligned}\|(-r_\delta)^{\frac{\delta}{2}}\bar\Theta^*_g u^\tau\|^2 &\simeq \int_D(-r_\delta)^\delta\Big|\partial\bar\partial r_\delta(u^\tau,\partial r_\delta)\Big|^2dV\\ &\underset{(2.4.4)}{\underset{\sim}{<}} (1-\delta)\int_D\partial\bar\partial(-(-r_\delta)^\delta)(u^\tau,u^\tau)dV\\ &\underset{(*)}{\underset{\sim}{<}} (1-\delta)Q_{(-r_\delta)^{\frac{\delta}{2}}}(u^\tau,u^\tau)\\ &\underset{(2.4.1)}{\underset{\sim}{<}} \mathcal{E}_{s,g}Q_{(-r_\delta)^{\frac{\delta}{2}}}(u^\tau,u^\tau).\end{aligned}\tag{2.4.5}$$

This proves (2.4.2)

**(c)** We are therefore in the same situation as in Definition 2.2.1 apart from the term $(-r_\delta)^\delta$ which occurs in the integral in the left of (2.4.5) and in $Q_{(-r_\delta)^{\frac{\delta}{2}}}$. As above, we continue to write $T$ but take in fact its positive microlocalization $T^+$ which represents the full action of $\Lambda$ over $u^+$. To carry on the proof, we suppose from now on that $f\in C^\infty(\bar D)$ and that $B_k$ is $H^s$ regular for some continuity constant $c'$; we prove that this implies continuity for a constant $c$ which is solely related to the constants which occur in (2.4.1). An exhaustion by domains endowed with $H^s$-regular projections $B_k$, $k\geq q$, will be discussed only at the end. We start from (2.2.11)

$$\begin{aligned}\|T_g^{s-\frac{\delta}{2}}B_{k-1}f\| &\underset{\sim}{<} sc\|T_g^{s-\frac{\delta}{2}}B_{k-1}f\|^2+lc\|T_g^{s-\frac{\delta}{2}}f\|\\ &\qquad + lc\|[\bar\partial^*,T_g^{s-\frac{\delta}{2}}]N_k\bar\partial f\|.\end{aligned}\tag{2.4.6}$$

At this point, we need to convert $T_g^{s-\frac{\delta}{2}}$ into $(-r_\delta)^{\frac{\delta}{2}}T^s_g$ in the last term of (2.4.6) in order to enjoy (2.4.2). We also replace $N_k\bar\partial f$ by $(N_k\bar\partial f)^{(h)}$ where the supscript $(h)$ denotes the harmonic extension. We apply the crucial estimate (2.3.6) to the last term in (2.4.6), regard as errors the terms which come in $(s-1)$-norm or in which vector fields of $\mathcal{S}$ occur, and get

(2.4.7)

$$
\begin{aligned}
|||[\bar\partial^*, \tilde T_g^{s-\frac{\delta}{2}}](\bar\partial N_k f)^{(h)} \underset{(2.3.6)}{\le} & \|(-r_\delta)^{\frac{\delta}{2}}[\tilde T^s_g, \bar\partial^*](\bar\partial N_k f)^{(h)}\|^2 \\
\lesssim\ & s^2\|(-r_\delta)^{\frac{\delta}{2}}\bar\Theta^*_g\tilde T^s_g(\bar\partial N_k f)^{(h)}\|^2 + \text{error} \\
\lesssim\ & s^2\|(-r_\delta)^{\frac{\delta}{2}}\bar\Theta^*_g\tilde T^s_g(\bar\partial N_k f)^{(h)\,\tau}\|^2 + \text{error} \\
\lesssim\ & s^2 \sup\frac{1}{|g|^{2s}}\|(-r_\delta)^{\frac{\delta}{2}}\bar\Theta^*_g\tilde T^s(\bar\partial N_k f)^{\tau}\|^2 + \mathcal{E}^{(0)} + \text{error} \\
\lesssim\ & s^2\mathcal{E}_{s,g}\sup\frac{1}{|g|^{2s}}\Big(Q_{(-r_\delta)^{\frac{\delta}{2}}\tilde T^s}((\bar\partial N_k f)^\tau, (\bar\partial N_k f)^\tau) \\
& \quad + \|(-r_\delta)^{\frac{\delta}{2}}[\bar\partial, \tilde T^s](\bar\partial N_k f)^\tau\|^2 + \|(-r_\delta)^{\frac{\delta}{2}}[\bar\partial^*, \tilde T^s](\bar\partial N_k f)^\tau\|^2\Big) + \mathcal{E}^{(0)} + \text{error} \\
\lesssim\ & s^2\mathcal{E}_{s,g}\sup\frac{1}{|g|^{2s}}\Big(Q_{(-r_\delta)^{\frac{\delta}{2}}\tilde T^s}(\bar\partial N_k f, \bar\partial N_k f) \\
& \quad + \|(-r_\delta)^{\frac{\delta}{2}}[\bar\partial, \tilde T^s]\bar\partial N_k f\|^2 + \|(-r_\delta)^{\frac{\delta}{2}}[\bar\partial^*, \tilde T^s]\bar\partial N_k f\|^2\Big) + \mathcal{E}^{(0)} + \text{error} \\
\lesssim\ & s^2\mathcal{E}_{s,g}\sup\frac{1}{|g|^{2s}}\Big(\|(-r_\delta)^{\frac{\delta}{2}}T^s\bar\partial^*\bar\partial N_k f\|^2 + c_2 s^2\|(-r_\delta)^{\frac{\delta}{2}}T^s\bar\partial N_k f\|^2\Big) + \mathcal{E}^{(0)} + \text{error} \\
\lesssim\ & s^2\mathcal{E}_{s,g}\sup\frac{1}{|g|^{2s}}\Big(\|(-r_\delta)^{\frac{\delta}{2}}T^s\bar\partial^*\bar\partial N_k f\|^2 + e^{2c_2\, s\, \mathrm{diam}^2 D}c_2 s^2\|(-r_\delta)^{\frac{\delta}{2}}T^s\bar\partial^*\bar\partial N_k f\|^2\Big) + \\
\underset{(2.4.1)}{\lesssim}\ & sc\|(-r_\delta)^{\frac{\delta}{2}}T^s\bar\partial^*\bar\partial N_k f\|^2 + \mathcal{E}^{(0)} + \text{error},
\end{aligned}
$$

where we have used the notation $\mathcal{E}^{(0)} := \|(-r_\delta)^{\frac{\delta}{2}}\bar\Theta^*_g\tilde T^s_g(\bar\partial N_k f)^{(0)\,\tau}\|^2$. Here in (2.4.1) we have chosen the first alternative $s^2\mathcal{E}_{s,g}e^{c_2 s\ \mathrm{diam}^2 D}\sup\left(\frac{1}{|g|^s}\right) \le c_1 = sc$ (for a new $c_2$). (The other alternative $\mathcal{E}_{s,g}e^{c_2 s\ \mathrm{diam}^2 D\sup(1+\frac{|g'|}{|g|}} \le c_1 = sc$ can be handled similarly as in Theorem 2.2.3 without replacing $T_g$ by $T$. It is at this point, where the continuity of $B_k$ in $H^s$, not just in $C^\infty$, is needed; in fact, in formula (2.2.15) $N_{\varphi_s}$ is $H^s$, not $C^\infty$, continuous. We have to reconvert now $(-r_\delta)^{\frac{\delta}{2}}$ into $T^{-\frac{\delta}{2}}$. We first suppose that we had started from $f^{(h)}$ and wished to prove the regularity for

$B_{k-1}f^{(h)}$. We have

$$\|(-r_\delta)^{\frac{\delta}{2}}T^s\bar\partial^*\bar\partial N_k(f^{(h)})\| \underset{[27]\,(2.4)}{\lesssim} \underbrace{\|T^{s-\frac{\delta}{2}}\bar\partial^*\bar\partial N_k f^{(h)}\|}_{(i)}+\underbrace{\|-rT^{s-\frac{\delta}{2}-1}\Delta\bar\partial^*\bar\partial N_k f^{(h)}\|}_{(ii)}.$$

Now,

$$(i)\lesssim \|T^{s-\frac{\delta}{2}}f^{(h)}\|^2+\|T^{s-\frac{\delta}{2}}B_{k-1}f^{(h)}\|^2,$$

where the first term in the right is good and the second can be absorbed since it comes, inside (2.4.7), with sc. As for (ii),

$$\begin{aligned}(ii)&=\|-rT^{s-\frac{\delta}{2}-1}(\bar\partial^*\bar\partial+\bar\partial\bar\partial^*)\bar\partial^*\bar\partial N_k f^{(h)}\|+\text{error}\\ &=\|-rT^{s-\frac{\delta}{2}-1}(\bar\partial^*\bar\partial(\bar\partial\bar\partial^*+\bar\partial^*\bar\partial)N_k f^{(h)}\|+\text{error}\\ &=\|-rT^{s\frac{\delta}{2}-1}\bar\partial^*\bar\partial f^{(h)}\|+\text{error}.\end{aligned}$$

We have

$$\begin{cases}\bar\partial^*\bar\partial=Tan^2+\partial_rTan+\partial_r^2\sim T^2+\partial_rT+\partial_r^2,\\ \partial_r^2=\Delta+Tan^2+\partial_rTan\sim\Delta+T^2+\partial_rT,\end{cases}$$

which implies

$$\bar\partial^*\bar\partial\sim T^2+\partial_rT+\Delta.$$

It follows

$$\begin{aligned}\|-rT^{s-\frac{\delta}{2}-1}\bar\partial^*\bar\partial f^{(h)}\|&=\|-rT^{s-\frac{\delta}{2}-1}(T^2+\partial_rT+\Delta)f^{(h)}\|\\ &\le\|-rT^{s-\frac{\delta}{2}+1}f^{(h)}\|+\|-rT^{s-\frac{\delta}{2}}\partial_rf^{(h)}\|\\ &\underset{[27]\,(2.4)}{\lesssim}\|T^{s-\frac{\delta}{2}}f^{(h)}\|,\end{aligned}\tag{2.4.8}$$

which is good. As for the term $f^{(0)}$, the regularity of $B_{k-1}f^{(0)}$ follows readily, without using the machinery (a)–(c) above, from elliptic regularity

$$\|T^sN_{k-1}f^{(0)}\|\lesssim\|T^{s-2}f^{(0)}\|.\tag{2.4.9}$$

(Note that $N_{k-1}$ makes sense even for $k-1=0$ when acting on $f^{(0)}|_{bD}\equiv 0$ because $\Box$ is, under this restriction, invertible.)

We pass to the term which has been omitted in the estimate of $\bar\Theta_g^*$, that is, $\mathcal{E}^{(0)}$. The use of elliptic regularity is different here and applies to $(\bar\partial N_kf)^{(0)}$ instead of $f^{(0)}$; it then passes though $Q$ instead of $\Box$ and through Boas-Straube formula. We have

(2.4.10)

$$
\begin{aligned}
||(-r_\delta)^{\frac{\delta}{2}}\bar{\Theta}^*_g\tilde{T}^s_g(\bar\partial N_kf)^{(0)\,\tau}||^2 &\lesssim \sup\frac{1}{|g|^{2s}}\|(-r_\delta)^{\frac{\delta}{2}}\bar{\Theta}^*_g\tilde{T}^s(\bar\partial N_kf)^{(0)\tau}\|^2\\
&\lesssim \mathcal{E}_{s,g}\sup\frac{1}{|g|^{2s}}\Big(Q_{(-r_\delta)^{\frac{\delta}{2}}\tilde{T}^s}((\bar\partial N_kf)^{(0)\tau},(\bar\partial N_kf)^{(0)\tau}))+\text{error}\\
&\qquad+\|(-r_\delta)^{\frac{\delta}{2}}[\bar\partial,\tilde{T}^s](\bar\partial N_kf)^{(0)\tau}\|^2+\|(-r_\delta)^{\frac{\delta}{2}}[\bar\partial^*,\tilde{T}^s](\bar\partial N_kf)^{(0)\tau}\|^2\Big)+\text{error}\\
&\lesssim \mathcal{E}_{s,g}\sup\frac{1}{|g|^{2s}}\Big(Q_{(-r_\delta)^{\frac{\delta}{2}}\tilde{T}^s}(\bar\partial N_kf,\bar\partial N_kf)+\text{error}\\
&\qquad+\|(-r_\delta)^{\frac{\delta}{2}}[\bar\partial,\tilde{T}^s]\bar\partial N_kf\|^2+\|(-r_\delta)^{\frac{\delta}{2}}[\bar\partial^*,\tilde{T}^s]\bar\partial N_kf\|^2\Big)+\text{error}
\end{aligned}
$$

This is the same as (2.4.7) with the advantage that in the last line the Sobolev indices have decreased by $-1$ since terms with superscript (0) vanish at $bD$; these are therefore error terms. Also there remain to control $\|T^{-\frac{\delta}{2}}\bar{\Theta}^*_g\tilde{T}^s_g(\bar\partial N_kf)^{(0)}\|$ and $\|-r^{\frac{\delta}{2}}\bar{\Theta}^*_g\tilde{T}^s_g(\bar\partial N_kf)^{\nu}\|$; but these are controlled by elliptic regularity as in (2.4.10). Summarizing up, we have proved that for a suitable $c$, only related to the constants in (2.4.1), we have

$$\|B_kf\|_s\le c\|f\|_s \tag{2.4.11}$$

if we knew that it holds for some $c' >> c$. We show now that we can exhaust $D$ by domains $D_\rho$ endowed with continuous projections $B_k,\ k\ge q-1$ for some $c'$ and which inherit the assumption of Theorem 2.4.2 with uniform constants with respect to $\rho$. For this, we define $D_\rho=\{z:\ r_\delta(z)+\rho<0\}$. We first notice that, $bD_\rho$ being also defined by $-(-r_\delta)^\delta+\rho^\delta<0$, it has a smooth $q$-plurisubharmonic defining function. Hence, by Theorem 2.2.4, $B_k$ is $H^s$-regular for any $k\ge q-1$. Coming back to the initial defining function $r_\delta+\rho$, this satisfies $\partial\bar\partial(-(-r_\delta-\rho)^\delta)\ge\partial\bar\partial(-(-r_\delta)^\delta$; thus the Diederich-Fornaess index of $D_\rho$ is $\ge\delta$. Also, if for the new boundary we rewrite $r_\delta+\rho=g_{\delta,\rho}r_\delta$, for a normalized equation $r_\rho$ of $D_\rho$, and if $\mathcal{E}_{s,g,\rho}$ are the constants which occur in (2.4.1), then

$$
\begin{cases}
g_{\delta,\rho}\underset{C^2}{\to}g_\delta,\\
\mathcal{E}_{s,g,\rho}\underset{C^2}{\to}\mathcal{E}_{s,g}.
\end{cases}
$$

Thus, the estimate (2.4.11) passes from the $D_\rho$'s (in which it has been proved thanks to the regularity of the $B_k$ (for a different $c'$)) to the initial domain $D$.

The proof is complete. □

CHAPTER 3

# Hypoellipticity and loss of derivatives

**Summary of Chapter 3.** In this chapter, we discuss some a-priori localized estimates in Sobolev spaces for various systems of complex vector fields in $\mathbb{R}^{2n-1}$ for $n \geq 2$ with particular care for the case $n = 2$. A complex vector field in $\mathbb{R}^{2n-1}$ is a partial differential operator of degree one of the type: $L(x) = \sum_j^{2n-1} a_j(x)\partial_{x_j}$ where the $a_j(x)$'s are smooth, complex valued, functions in $\mathbb{R}^{2n-1}$. An a-priori localized Sobolev estimate for a system of complex vector fields: $\{L_1, \dots, L_n\}$, is meant to be an estimate of type: $\|\zeta_0 u\|_s \leq \|\zeta_1 L_1 u\|_{s+l} + \cdots + \|\zeta_1 L_n u\|_{s+l} + \|u\|_0$ for any $s \in \mathbb{R}^+$, where $\|\cdot\|_s$ is the $s-$Sobolev norm, $u \in C^\infty(\mathbb{R}^{2n-1})$ and $\zeta_0$, $\zeta_1$ are cutoff functions with support in a neighborhood $U$ of some point $p$ such that $\zeta_{1|_{supp(\zeta_0)}} \equiv 1$ and $\zeta_{0|_{U'}} \equiv 1$ where $p \in U' \Subset U$; in this situation, we write $\zeta_0 \prec \zeta_1$. If $l$ is positive we say that the system has a loss of regularity; instead, if $l$ is negative we say that the system has a gain of regularity.

Using the method of elliptic regularization [**24**], it is a well known fact that these estimates, both for the case of gain or loss, imply hypoellipticity (i.e. if $L_i(u) = f_i$ for $i = 1, \dots, n$ and $f_{i|_U} \in C^\infty(U)$ then $u \in C^\infty(U')$).

The vector fields considered in this chapter are modifications of vector fields that satisfy certain properties. These modifications are obtained by multiplying the vector fields by smooth functions which vanish at a certain order at 0. In Section (3.1), they satisfy the finite type condition and have therefore subelliptic estimates; the related loss in the estimates is a balance between the vanishing order and the type. We also consider in that Section the problem of local hypoellipticity for sums of squares, that is, second order differential operators $\Box^k = \sum_{j=1,\dots,n} L_j^* L_j$, where $L_j^*$ are the $L^2-$adjoints of $L_j$.

In Section (3.3) we consider, instead, vector fields of infinite type and point our attention to the exponential type with related logarithmic estimates; in particular, we focus our attention to the case of superlogarithmic estimates. For the modified system, we prove estimates with arbitrarily small fractional loss.

## 3.1. Introduction

A system of real vector fields $\{X_j\}$ in $T\mathbb{R}^n$ is said to satisfy the bracket finite type condition if

(3.1.1)

commutators of order $\leq h-1$ of the $X_j$'s span the whole $T\mathbb{R}^n$.

Explicitly: $\text{Span}\{X_j,\, [X_{j_1}, X_{j_2}], ..., [X_{j_1}, [X_{j_2}, ..., [X_{j_{h-1}}, X_{j_h}]]...]\} = T\mathbb{R}^n$. This system enjoys $\delta$-subelliptic estimates for $\delta = \frac{1}{h}$ and therefore it is hypoelliptic according to Hörmander [**6**]. (See also [**5**] and [**24**] for elliptic regularization which yields regularity from estimates.) This remains true for systems of complex vector fields $\{L_j\}$ stable under conjugation (both in $\mathbb{C}\otimes T\mathbb{R}^n$ or $\mathbb{C}\otimes T\mathbb{C}^n$) once one applies Hörmander's result to $\{\mathsf{Re}\, L_j,\, \mathsf{Im}\, L_j\}$. Stability under conjugation can be artificially achieved by adding $\{\epsilon\bar{L}_j\}$ in order to apply Hörmander's theorem $\|u\|^2_\delta \leq \sum_j (c_\epsilon \|L_j u\|^2 + \epsilon\|\bar{L}_j u\|^2) + c_\epsilon\|u\|^2$, $u \in C^\infty_c$. (Precision about $\epsilon$ and $c_\epsilon$ is not in the statement but transparent from the proof.) On the other hand, by integration by parts $\|\bar{L}_j u\|^2 \lesssim \|L_j u\|^2 + |([L_j, \bar{L}_j]u, u)| + \|u\|^2 \lesssim \|L_j u\|^2 + \|u\|^2_{\frac{1}{2}} + \|u\|^2$. Thus if the type is $h=2$, and hence $\delta = \frac{1}{2}$, the $\frac{1}{2}$-norm is absorbed in the left: $\{\epsilon\bar{L}_j\}$ can be taken back and one has $\frac{1}{2}$-subelliptic estimates for $\{L_j\}$. The restraint $h=2$ is substantial and in fact Kohn discovered in [**9**] a pair of vector fields $\{L_1,\, L_2\}$ in $\mathbb{R}^3$ of finite type $k+1$ (any fixed $k$) which are not subelliptic but, nonetheless, are hypoelliptic. Precisely, in the terminology of [**9**], they loose $\frac{k-1}{2}$ derivatives and the related sum of squares $\bar{L}_1L_1 + \bar{L}_2L_2$ looses $k-1$ derivatives. The vector fields in question are $L_1 = \partial_{\bar{z}} + iz\partial_t$ and $L_2 = \bar{z}^k(\partial_z - i\bar{z}\partial_t)$ in $\mathbb{C}\times\mathbb{R}$. Writing $t = \mathsf{Im}\, w$, they are identified to $\bar{L}$ and $\bar{z}^kL$ for the CR vector field $\bar{L}$ tangential to the strictly pseudoconvex hypersurface $\mathsf{Re}\, w = |z|^2$ of $\mathbb{C}^2$. Consider a more general hypersurface $M \subset \mathbb{C}^2$ defined by $\mathsf{Re}\, w = g(z)$ for $g$ real, and use the notations $g_1 = \partial_z g$, $g_{1\bar{1}} = \partial_z\partial_{\bar{z}} g$ and $g_{1\bar{1}\bar{1}} = \partial_z\partial_{\bar{z}}\partial_{\bar{z}} g$. Suppose that $M$ is pseudoconvex, that is, $g_{1\bar{1}} \geq 0$ and denote by $2m$ the vanishing order of $g$ at 0, that is, $g = 0^{2m}$. Going further in the analysis of loss of derivatives, Bove, Derridj, Kohn and Tartakoff have considered the case where

$$g_1 = \bar{z}|z|^{2(m-1)}h(z) \text{ and } g_{1\bar{1}} = |z|^{2(m-1)}f(z) \text{ for } f > 0. \tag{3.1.2}$$

If $L = \partial_z - ig_1\partial_t$ is the $(1,0)$ vector field tangential to $Rew = g$ for $g$ satisfying (3.1.2), they have proved loss of $\frac{k-1}{m}$ derivatives for the operator $L\bar{L} + \bar{L}|z|^{2k}L$.

We consider here a general pseudoconvex hypersurface $M \subset \mathbb{C}^2$; $\zeta$ and $\zeta'$ will denote cut-off functions in a neigborhood of 0 such that $\zeta'|_{\operatorname{supp}\zeta} \equiv 1$.

THEOREM 3.1.1. *Let* $\{L, \bar{L}\}$ *(or better* $\{\mathrm{Re}\, L, \mathrm{Im}\, L\}$*) have type* $2m$*; then the system* $\{\bar{L}, \bar{z}^k L\}$ *looses* $l := \frac{k-1}{2m}$ *derivatives. More precisely*

$$\begin{aligned}\|\zeta u\|_s^2 \lesssim \|\zeta' \bar{L} u\|^2_{s-\frac{1}{2m}} + \|\zeta' \bar{z}^k \bar{L} u\|^2_{s+l} \\ + \|\zeta' \bar{z}^k L u\|^2_{s+l} + \|u\|^2_{-\infty}.\end{aligned} \tag{3.1.3}$$

The estimate (3.1.3) says that the responsible of the loss $l$ is $\bar{z}^k L$ (plus the extra vector field $\bar{z}^k \bar{L}$) and not $\bar{L}$. The proof of this here, as well as the two theorems below, follows in Section 3.2. What underlies the whole technicality is the basic notion of subelliptic multiplier; also the stability of multipliers under radicals is crucial (hidden in the interpolation Lemma 3.2.2 below). We point out that though the coefficient of the vector field $\bar{L}$ gains much in generality ($+ig_{\bar{1}}$ instead of $+iz$ or $+iz|z|^{2(m-1)}$ as in [**9**] and [**1**] respectively), instead, the perturbation $\bar{z}^k$ of $L$ remains the same. This is substantial; only an antiholomorphic perturbation is allowed. We introduce a new notation for the perturbed Kohn-Laplacian

$$\Box^k = L\bar{L} + \bar{L}|z|^{2k} L \qquad \text{for } L = \partial_z - ig_1\partial_t. \tag{3.1.4}$$

THEOREM 3.1.2. *Let* $\{L, \bar{L}\}$ *have type* $2m$ *and assume moreover, that*

$$|g_1| \lesssim |z| g_{1\bar{1}} \quad \text{and} \quad |g_{1\bar{1}\bar{1}}| \lesssim |z|^{-1} g_{1\bar{1}}. \tag{3.1.5}$$

*Then* $\Box^k$ *looses* $l = \frac{k-1}{m}$ *derivatives, that is*

$$\|\zeta u\|_s^2 \lesssim \|\zeta' \Box^k u\|^2_{s+2l} + \|u\|^2_{-\infty}. \tag{3.1.6}$$

Differently from vector fields, loss for sums of squares requires the additional assumption (3.1.5); whether finite type suffices is an open question. Now, (3.1.3) and (3.1.6) yield hypoellipticity. Reason is that loss of derivatives takes place only in $\partial_t$ and, on the other hand, the coefficients of the vector fiels and of the sum of squares are constant in $t$. (In contrast, these vector fields and sum of squares are elliptic in $z$.) Thus, if we regularize with respect to $t$ the component $u^+$ of $u$ (positively microlocalized in $+t$ (cf. §3)) as $u^+_\nu \to u^+$ and use that $\bar{L} u^+_\nu = (\bar{L} u^+)_\nu$ (and the same for the other operators), then (3.1.3) and (3.1.4) applied to $u^+_\nu$

COROLLARY 3.1.3. *In the situation of Theorem 3.1.1 and 3.1.2, the system* $(\bar{L}, \bar{z}^k L)$*, resp. the operator* $\Box^k$*, are hypoelliptic with loss* *of $l$ (resp. $2l$) derivatives:* $(\bar{L}u, \bar{z}^k Lu) \in H^s$ *(resp.* $\Box^k u \in H^s$*) implies* $u \in H^{s-l}$ *(resp.* $u \in H^{s-2l}$*).*

EXAMPLE 3.1.4. Consider the boundary defined by $\mathsf{Re}\, w = g$ with $g(z) = 0^{2m}$ and assume

$$g_{1\bar{1}} \gtrsim |z|^{2(m-1)}. \tag{3.1.7}$$

This boundary is pseudoconvex, has bracket finite type $2m$ and (3.1.5) is satisfied. Thus Theorem 3.1.2 applies and we have (3.1.6). This is more general than [**1**] where it is assumed (3.1.2). Thus, for example, for the domain graphed by $g$ with

$$g = |z|^{2(m-1)} x^2 h(z) \qquad \text{for } h > 0 \text{ and } h_{1\bar{1}} > 0,$$

we have (3.1.7) though the second of (3.1.2) is never true, not even for $h \equiv 1$. For general $h$, neither of (3.1.2) is fulfilled.

There is a result for sum of squares which stays close to Theorem 3.1.1 and in particular only assumes finite type without the additional hypothesis (3.1.5). This requires to modify the Kohn-Laplacian as

$$\tilde{\Box}^k = \Lambda_t^{-2l} L\bar{L} + L|z|^{2k}\bar{L} + \bar{L}|z|^{2k}L,$$

where $\Lambda_t^{-2l}$ is the standard pseudodifferential operator of order $-2l$ in $t$.

THEOREM 3.1.5. *Let* $\{L, \bar{L}\}$ *have type* $2m$*; then*

$$\|\zeta u\|_s^2 \lesssim \|\zeta' \tilde{\Box}^k u\|_{s+2l}^2 + \|u\|_{-\infty}^2. \tag{3.1.8}$$

We restate in higher dimension the above results; in doing so we can better appreciate the different role which is played by the finite type with respect to (3.1.5). This discussion is a direct consequence of the results of Section 3.1 (plus ellipticity and maximal hypoellipticity related to microlocalization) and therefore it does not need a specific proof. In $\mathbb{C}^n \times \mathbb{R}_t$ we start from $L_1 = \partial_{z_1} - ig_1(z_1)\partial_t$ and complete $L_1$ to a system of smooth complex vector fields in a neighborhood of 0

$$L_j = \partial_{z_j} - ig_j(z)\partial_t,\ j = 1, ..., n \qquad \text{for } g_j|_0 = 0.$$

For a system of vector fields, we denote by $\mathcal{L}ie_{2m}$ the span of commutators of order $\leq 2m-1$ belonging to the system. We have $\|u^0\|_1^2 \lesssim \sum_{j=1,\dots,n} \|\bar{L}_j u^0\|_0^2 + \|u\|_0^2$ and, if for some index $j$, say $j = 1$, $\partial_t \in \mathcal{L}ie_{2m_1}\{L_1, \bar{L}_1\}$, then $\|u^-\|^2_{\frac{1}{2m_1}} \lesssim \sum_{j=1,\dots,n} \|\bar{L}_j u\|_0^2 + \|u\|_0^2$ (cf. the end

of Section 3.2). Summarizing up, if we only have (3.1.3) for $u^+$, we get, for the full $u$ and with $l$ replaced by $l_1=\frac{k_1}{2m_1}$:
(3.1.9)

$$\|\zeta u\|^2_s\underset{\sim}{<}\left(\|\zeta'\bar L_1u\|^2_{s-\frac{1}{2m_1}}+\|\zeta' z_1^{k_1}\bar L_1u\|^2_{s+l_1}+\|\zeta' z_1^{k_1}L_1u\|^2_{s+l_1}\right)+\sum_{j=2}^n\|\bar L_ju\|^2_{s-\frac{1}{2m_1}}+\|u\|^2_0.$$

We assume that each coefficient satisfy $g_j=\partial_{z_j}g$ for a real function $g=g(z),\ z=(z_1,...,z_n)\in\mathbb{C}^n$ and denote by $\mathbb{L}$ the bundle spanned by the $L_j$'s. We note that this defines a CR structure because, on account of $g_{i\bar j}=g_{j\bar i}$,

$$\mathbb{L}\text{ is involutive.}$$

Also, this structure is of hypersurface type in the sense that

$$T(\mathbb{C}^n_z\times\mathbb{R}_t)=\mathbb{L}\oplus\overline{\mathbb{L}}\oplus\mathbb{R}\partial_t.$$

Note that, in fact, the $L_j$'s commute; therefore, the Levi form is defined directly by $[L_i,\bar L_j]=g_{ij}\partial_t$, without passing to the quotient modulo $\mathbb{L}\oplus\overline{\mathbb{L}}$. We also assume that the Levi form $(g_{i\bar j})$ is positive semidefinite; in particular $g_{j\bar j}\geq 0$ for any $j$. (Geometrically, this means that the hypersurface $\mathsf{Im}\,w=g$ graphed by $g$, is pseudoconvex.) We choose $\kappa=(k_1,...,k_n)$ and define the perturbed Kohn-Laplacian

$$\Box^\kappa=\sum_{j=1,\dots,n}L_j\bar L_j+\bar L_j|z_j|^{2k_j}L_j.$$

THEOREM 3.1.6. *Assume that for any $j$, $\partial_t\in\mathcal{L}ie_{2m_j}\{L_j,\,\bar L_j\}$, and that*

$$|g_j|\underset{\sim}{<}|z_j|g_{j\bar j}\ \text{ and }\ |g_{j\bar j\bar j}|\underset{\sim}{<}|z_j|^{-1}g_{j\bar j}\quad\text{for any } j=1,...,n. \tag{3.1.10}$$

*Define $l_j:=\frac{k_j-1}{2m_j}$ and put $l=\max_j\frac{k_j-1}{2m_j}$; then*

$$\|\zeta u\|^2_s\underset{\sim}{<}\|\zeta'\Box^\kappa u\|^2_{s+2l}+\|u\|^2_0. \tag{3.1.11}$$

The proof of Theorem 3.1.6 and Theorem 3.1.7 below, are just a variation of those of the twin Theorems 3.1.2 and 3.1.5. We define now

$$\tilde\Box^\kappa=\sum_{j=1,\dots,n}\left(\Lambda_t^{-2l_j}L_j\bar L_j+\sum_{j=1,\dots,n}L_j|z_j|^{2k_j}\bar L_j+\bar L_j|z_j|^{2k_j}L_j\right).$$

THEOREM 3.1.7. *Assume that for any $j$, $\partial_t\in\mathcal{L}ie_{2m_j}\{L_j,\,\bar L_j\}$; then*

$$\|\zeta u\|^2_s\underset{\sim}{<}\|\zeta'\tilde\Box^k u\|^2_{s+2l}+\|u\|^2_0. \tag{3.1.12}$$

The material above will be developped in Section (3.2).

We pass to review the second half of this chapter, that is, the estimates of vector fields of the exponential type contained in Section (3.3). Our requirement is that the degenerancy is not too strong so that superlogarithmic estimates hold.

A system has a superlogarithmic estimate if it has logarithmic gain of derivative with an arbitrarily large constant, that is, for any $\delta$ and for suitable $c_\delta$

$$\|\log(\Lambda)u\|^2 \underset{\sim}{<} \delta \sum_j \|L_j u\|^2 + c_\delta \|u\|^2_{-1}, \qquad u \in C^\infty_c. \tag{3.1.13}$$

A system which satisfies (3.1.13) is "precisely $H^s$-hypoelliptic" for any $s$: $u$ is $H^s$ exactly where the $L_j u$'s are (Kohn [**8**]). In particular, the system is $C^\infty$-hypoelliptic. Let $L = \partial_z - ig_1(z)\partial_t$ for $g$ of infinite type but exponentially non-degenerate in the sense that

$$|z|^\alpha |\log g_{1\bar{1}}| \searrow 0 \text{ as } |z| \searrow 0 \text{ for } \alpha \leq 1. \tag{3.1.14}$$

Under this assumption, $\{L, \bar{L}\}$ enjoys a superlogarithmic estimate (cf. e.g. [**12**]). If we consider the perturbed system $\{\bar{L}, \bar{z}^k L\}$ (any fixed $k \geq 1$), the system has no more superlogarithmic estimate, in general; if $k > 1$, a logarithmic loss occurs (Proposition 3.1.11 below). However, notice that $\mathcal{L}ie\{\bar{L}, \bar{z}^k L\}$, the span of commutators of order $\leq k-1$, has a superlogarithmic estimate (since it gains $L$). We are able to prove here, in the terminology of Kohn [**9**], that $\{\bar{L}, \bar{z}^k L\}$ has an arbitrarily small loss of $\epsilon$ derivatives and thus, in particular, is $C^\infty$-, but not exactly $H^s$-, hypoelliptic. Let $\zeta_0$ and $\zeta_1$ be cut-off functions in a neighborhood of 0 with $\zeta_0 \prec \zeta_1$ in the sense that $\zeta_1|_{\text{supp}\zeta_0} \equiv 1$.

THEOREM 3.1.8. *Let $L = \partial_z - ig_1(z)\partial_t$ and assume that* 0 *be a point of infinite type, that is, $g_{1\bar{1}} = 0^\infty$ but not exponentially degenerate, that is,* (3.1.14) *be fulfilled. Then the system $\{\bar{L}, \bar{z}^k L\}$ (any $k$) has an arbitrarily small loss of $\epsilon$ derivatives, that is,*

$$\|\zeta_0 u\|^2_s \underset{\sim}{<} \|\zeta_1 \bar{L} u\|^2_{s+\epsilon} + \|\zeta_1 \bar{z}^k L u\|^2_{s+\epsilon} + \|\bar{z}^k u\|^2_\epsilon + \|u\|^2_0. \tag{3.1.15}$$

The proof of this, and the two theorems below, follows in Section 3.3. Generally, an estimate of type (3.1.15) for smooth $u$ does not yield finiteness of $\|\zeta_0 u\|_s$ for a $H^\epsilon$-solution $u$ of $\bar{L}u = f$, $z^k L u = g$ when $\zeta_1 f$ and $\zeta_1 g$ are in $H^{s+\epsilon}$. However, $L$ has coefficient $t$-independent. Then, since only the "positively microlocalized" component $u^+$ (cf. §2 below) must be controlled, Sobolev $t$-regularity is equivalent to full regularity. For this reason, if we use a sequence of pseudodifferential smoothing operators in $t$, $\chi_\nu(\partial_t) \to \text{id}$ as in [**9**] and [**1**], and remark that

$$\bar{L}(\chi_\nu(\partial_t)u^+) = \chi_\nu(\partial_t)(\bar{L}u^+) + \text{Order}_{-\infty},$$

then, (3.1.15) applied to $\Lambda^s\Big(\chi_\nu(\partial_t)u^+\Big) = \chi_\nu(\partial_t)\Big(\Lambda^s u^+\Big)$ yields

Corollary 3.1.9. *In the situation of Theorem 3.1.8, the system* $(\bar{L},\, \bar{z}^k Lu)$ *is hypoelliptic with loss of* $\epsilon$*-derivatives:* $(\bar{L}u, \bar{z}^k Lu) \in H^{s+\epsilon}$*,* $u \in H^\epsilon$ *implies* $u \in H^s$*.*

For $k = 1$ we have an estimate for local regularity without loss

Theorem 3.1.10. *In the situation above, assume in addition*

$$|g_1| \underset{\sim}{<} g_{1\bar{1}}^{\frac{1}{2}}; \qquad (3.1.16)$$

*then*

$$\|\zeta_0 u\|_s^2 \underset{\sim}{<} \|\zeta_1 \bar{L}u\|_s^2 + \|\zeta_1 \bar{z} Lu\|_s^2 + \|u\|_0^2. \qquad (3.1.17)$$

When $k > 1$, loss must occur

Proposition 3.1.11. *Assume that* $g = e^{-\frac{1}{|z|^\alpha}}$*. If*

$$\|\zeta_0 u\|_s^2 \underset{\sim}{<} \|(\log\Lambda)^r \zeta_1 \bar{L}u\|_s^2 + \|(\log\Lambda)^r \zeta_1 \bar{z}^k Lu\|_s^2 + \|\bar{z}^k u\|_\epsilon^2 + \|u\|_0^2, \qquad (3.1.18)$$

*then we must have* $r \underset{\sim}{>} \frac{k-(\alpha+1)}{\alpha}$*.*

Some references to current literature are in order. Hypoellipticity in presence of infinite degeneracy has been intensively discussed in recent years. The ultimate level to which the problem is ruled by a-priori estimates, are superlogarithmic estimates (Kusuoka and Strooke [**10**], Morimoto [**13**] and Kohn [**8**]). Related work is also by Bell and Mohammed [**2**] and Christ [**3**]. Beyond the level of estimates are the results by Kohn [**7**] which develop, in a geometric framework, an early result by Fedi [**4**]: the point here is that the degeneracy is confined to a real curve transversal to the system. This explains also why if the set of degeneracy is big, superlogarithmicity becomes in certain cases necessary ([**13**] and [**3**]). In all these results, however, there is somewhat a gain of derivatives (such as sublogarithmic). The simplest example of hypoellipticity without gain (nor loss) is $\Box_b + \lambda\,\mathrm{id}$, $\lambda > 0$ where $\Box_b$ is the Kohn-Laplacian of $\mathsf{Re}\, w = |z|^2$ (cf. Stein [**15**] where the bigger issue of the analytic-hypoellipticity is also addressed). Loss of derivatives for $L = \partial_z - i\bar{z}\partial_t$ was discovered by Kohn in [**9**]. In this case, $L$ is the $(1,0)$ vector field tangential to the strictly pseudoconvex hypersurface $\mathsf{Re}\, w = |z|^2$ and the loss amounts in $\frac{k-1}{2}$. The problem was further discussed by Bove, Derridj, Kohn and Tartakoff in [**1**] essentially for the vector field $L=\partial_z-i\bar{z}|z|^{2(m-1)}\partial_t$ tangential to the hypersurface $\mathsf{Re}\,w=|z|^{2m}$ and the corresponding loss is $\frac{k-1}{2m}$. In both cases the result extends to the sum of squares $L\bar{L}+\bar{L}|z|^{2k}L$ and the loss doubles to $\frac{k-1}{m}$. For vector fields $L=\partial_z-ig_1(z)\partial_t$ tangential to general pseudoconvex hypersurfaces of finite type (with $g_{1\bar{1}}$ vanishing at order $2(m-1)$), loss of $\frac{k-1}{2m}$ derivatives has been proved by the authors in [**11**]. Under some additional conditions, the result also extends to sums of squares (with doubled loss). In the limit position of type $\infty$, it was natural to expect for an arbitrarily small loss of $\epsilon$ derivatives. This is what we prove here for vector fields $\{\bar{L},\bar{z}^kL\}$ obtained from $L=\partial_z-ig_1(z)\partial_t$ of infinite type, that is, satisfying $g_{1\bar{1}}=0^\infty$, is considered. However as we have seen, some additional hypothesis such as (3.1.14), must be required. This guarantees superlogarithmic estimate ([**12**]), and in turn, hypoellipticity according to Kohn [**8**].

## 3.2. Estimates for vector fields in the subelliptic case and sum of squares

We identify $\mathbb{C}\times\mathbb{R}$ to $\mathbb{R}^3$ with coordinates $(z,\bar{z},t)$ or $(\mathsf{Re}\,z,\mathsf{Im}\,z,t)$. We denote by $\xi=(\xi_z,\xi_{\bar{z}},\xi_t)$ the variables dual to $(z,\bar{z},t)$, by $\Lambda^s_\xi$ the standard symbol $(1+|\xi|^2)^{\frac{s}{2}}$, and by $\Lambda^s$ (resp. $\Lambda^s_t$) the pseudodifferential operator with symbol $\Lambda^s_\xi$ (resp. $\Lambda^s_{\xi_t}$; this is defined by $\Lambda^s(u)=\mathcal{F}^{-1}(\Lambda^s_\xi\mathcal{F}(u))$ where $\mathcal{F}$ is the Fourier transform (and similarly for $\Lambda_t$). We consider the full (resp. totally real) $s$-Sobolev norm $\|u\|_s:=\|\Lambda^su\|_0$ (resp. $\|u\|_{\mathbb{R},s}:=\|\Lambda^s_tu\|_0$). In $\mathbb{R}^3_\xi$, we consider a conical partition of the unity $1=\psi^++\psi^++\psi^0$ where $\psi^\pm$ have support in a neighborhood of the axes $\pm\xi_t$ and $\psi^0$ in a neighborhood of the plane $\xi_t=0$, and introduce a decomposition of the identity $\mathrm{id}=\Psi^++\Psi^-+\Psi^0$ by means of $\Psi^{\overset{\pm}{0}}$, the pseudodifferential operators with symbols $\psi^{\overset{\pm}{0}}$; we accordingly write $u=u^++u^-+u^0$. Since $|\xi_z|+|\xi_{\bar{z}}|\underset{\sim}{<}\xi_t$ over $\operatorname{supp}\psi^+$, then $\|u^+\|_{\mathbb{R},s}=\|u^+\|_s$.

We carry on the discussion by describing the properties of commutation of the vector fields $L$ and $\bar{L}$ for $L=\partial_z-ig_1(z)\partial_t$. The crucial equality is

$$\|Lu\|^2=([L,\bar{L}]u,u)+\|\bar{L}u\|^2,\quad u\in C^\infty_c,\tag{3.2.1}$$

which is readily verified by integration by parts. Note here that errors coming from derivatives of coefficients do not occur since $g_1$ does not depend on $t$. Recall that $[L,\bar{L}]=g_{1\bar{1}}\partial_t$; this implies

$$|(g_{1\bar{1}}\partial_tu,u)|\underset{\sim}{<}s.c.\|\partial_tu\|^2+l.c.\|u\|^2.\tag{3.2.2}$$

We have

$$
\begin{aligned}
\|u^0\|_1^2 &\lesssim \|\bar{L}u^0\|^2 + \|Lu^0\|^2 + \|u\|^2 \\
&\leq 2\|\bar{L}u^0\|^2 + sc\|\partial_t u^0\|^2 + lc\|u\|^2.
\end{aligned} \tag{3.2.3}
$$

To check (3.2.3), we point our attention to the estimate for operator's symbols $(1+|\xi|^2)|\alpha|^2 \lesssim |\alpha|^2 + |\sigma(\bar{L})\alpha|^2 + |\sigma(L)\alpha|^2$ ($\alpha$ complex) over $U \times \operatorname{supp}\psi^0$ for a neighborhood $U$ of 0; in addition to the fact that $[L, \Psi^0]$ is of order 0, this yields the first inequality of (3.2.3). The second follows from (3.2.1) combined with (3.2.2). As for $u^-$, since $g_{11}\sigma(\partial_t) < 0$ over $\operatorname{supp}\psi^-$, then

$$
(g_{11}\partial_t u^-, u^-) = -\left|(g_{11}\Lambda_t u^-, u^-)\right|.
$$

Thus (3.2.1) implies $\|Lu^-\| \leq \|\bar{L}u^-\|$ (the second inequality in (3.2.4) below). Suppose now that $\{L, \bar{L}\}$ have type $2m$; this yields the first inequality below which, combined with the former, yields

$$
\begin{aligned}
\|u^-\|^2_{\frac{1}{2m}} &\lesssim \|Lu^-\|_0^2 + \|\bar{L}u^-\|_0^2 + \|u\|_0^2 \\
&\lesssim \|\bar{L}u^-\|_0^2 + \|u\|_0^2.
\end{aligned} \tag{3.2.4}
$$

In conclusion, only estimating $u^+$ is relevant. For this purpose, we have a useful statement

LEMMA 3.2.1. *Let $|[L, \bar{L}]|^{\frac{1}{2}}$ be the operator with symbol $|g_{11}|^{\frac{1}{2}}\Lambda_{\xi_t}^{\frac{1}{2}}$; then*

$$
\||[L, \bar{L}]|^{\frac{1}{2}}u^+\|^2 \leq \|Lu^+\|^2 + \|\bar{L}u^+\|^2. \tag{3.2.5}
$$

PROOF. From (3.2.1) we get

$$
|([L, \bar{L}]u, u)| \leq \|Lu\|^2 + \|\bar{L}u\|^2.
$$

The conclusion then follows from

$$
[L, \bar{L}] = |[L, \bar{L}]| \qquad \text{over } \operatorname{supp}\psi^+.
$$

□

We pass to a result about intepolation which plays a central role in our discussion.

LEMMA 3.2.2. *Let $f = f(z)$ be smooth and satisfy $f(0) = 0$. Then for any $\rho$, $r$, $n_1$ and $n_2$ with $0 < n_1 \leq r$, $n_2 > 0$*

$$
\|f^r u\|_0^2 \lesssim sc\|f^{r-n_1}u\|^2_{\mathbb{R},\,-n_1\rho} + lc\|f^{r+n_2}u\|^2_{\mathbb{R},\,n_2\rho}. \tag{3.2.6}
$$

PROOF. Set $A := \Lambda^\rho_t f$; interpolation for the pseudodifferential operator $A$ yields

$$\begin{aligned}\|f^r u\|_0^2 &= \|(\Lambda^\rho f)^r u\|^2_{\mathbb{R}\,-\rho r}\\ &= (\Lambda^{\rho(r-n_1)} f^{r-n_1}, \Lambda^{\rho(r+n_1)} f^{r+n_1})_{-\rho r}\\ &= (\Lambda^{-\rho n_1} f^{r-n_1}, \Lambda^{\rho n_1} f^{r+n_1})_0 \lesssim sc\|f^{r-n_1}\|^2_{\mathbb{R},\,-n_1\rho} + lc\|f^{r+n_1}u\|^2_{\mathbb{R},\,n_1\rho}.\end{aligned}$$

This proves the lemma for $n_2 = n_1$; the general conclusion is obtained by iteration.

□

We have now a result about factors in a scalar product.

LEMMA 3.2.3. *Let $h = h(z)$ satisfy $|h| \le |h_1||h_2|$ and take $f = f(z,t)$ and $g = g(z,t)$. Then*

$$|(f, hg)|_{\mathbb{R},\,s} \lesssim \|fh_1\|^2_{\mathbb{R},\,s} + \|gh_2\|^2_{\mathbb{R},\,s}. \tag{3.2.7}$$

PROOF. We use the notation $\mathcal{F}_t$ for the partial Fourier transform with respect to $t$ and $d\lambda$ for the element of volume in $\mathbb{C}_z \simeq \mathbb{R}^2_{\mathsf{Re}\,z,\mathsf{Im}\,z}$. The lemma follows from the following sequence of inequalities in which the crucial fact is that $h$, $h_1$ and $h_2$ are constant in the integration in $\xi_t$:

$$\begin{aligned}|(f, hg)_{\mathbb{R},\,s}| &= \left|\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^1_{\xi_t}} \Lambda^{2s}_{\xi_t}\mathcal{F}_t(f)h\mathcal{F}_t(g)d\xi_t\right)d\lambda\right|\\ &\le \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^1_{\xi_t}} \Lambda^{2s}_{\xi_t}|\mathcal{F}_t(f)h_1h_2\mathcal{F}_t(g)|d\xi_t\right)d\lambda\\ &\le \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^1_{\xi_t}} \Lambda^{2s}_{\xi_t}|\mathcal{F}_t(f)h_1|^2d\xi_t\right)d\lambda + \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^1_{\xi_t}} \Lambda^{2s}_{\xi_t}|\mathcal{F}_t(g)h_2|^2d\xi_t\right)d\lambda\\ &\underset{\text{Plancherel}}{=} \|fh_1\|^2_{\mathbb{R},\,s} + \|gh_2\|^2_{\mathbb{R},\,s}.\end{aligned}$$

□

We say a few words for the case of higher dimension. In $\mathbb{C}^n_{z_1,\dots,z_n}\times\mathbb{R}_t$, we consider a full system $L_j = \partial_{z_j} - ig_j\partial_t,\ j = 1, ..., n$ with $g_j|_0 = 0$. The same argument used in proving (3.2.3) yields

$$\|u^0\|_1^2 \lesssim \sum_{j=1,\dots,n} \|\bar{L}_j u^0\|^2 + \|u\|^2. \tag{3.2.8}$$

Similarly as above, we have $\|L_j u^-\|^2 \le \|\bar{L}_j u^-\|^2 + \|u\|^2$ for any $j$. Then, if at least one index $j$, say $j = 1$, the pair $\{L_1, \bar{L}_1\}$ has type $m = m_1$, we get, in the same way as in (3.2.4)

$$\|u^-\|^2_{\frac{1}{2m}} \underset{\sim}{<} \sum_{j=1,\dots,n} \|\bar{L}_j u^-\|^2 + \|u\|^2.$$

Again, only estimating $u^+$ is therefore relevant.

**Proof of Theorem 3.1.1, Theorem 3.1.2 and Theorem 3.1.5** In an estimate we call "good" a term in the right side (upper bound). We call "absorbable" a term that we encounter in the course of the estimate and which comes as a fraction (small constant or sc) of a former term. If cut-off are involved in the estimate, and in the right side the cut-off can be expanded, say passing from $\zeta$ to $\zeta'$, we call "neglectable" a term which comes with lower Sobolev index and possibly with a bigger cut-off. Neglectable is meant with respect to the initial (left-hand side) term of the estimate, to further terms that one encounters and even to extra terms provided that they can be estimated by "good". These latter are sometimes artificially added to expand the range of "neglectability".

*Proof of Theorem 3.1.1.* According to (3.2.3) and (3.2.4), it suffices to prove (3.1.3) for $u = u^+$; so, throughout the proof we write $u$ but mean $u^+$. Also, we use the equivalence, over $u^+$, between the totally real $\|\cdot\|_{\mathbb{R},s}-$ with the full $\|\cdot\|_s$-Sobolev norm; the specification of the norm will be omitted. Moreover, we can use a cut-off $\zeta = \zeta(t)$ in $t$ only. In fact, for a cut-off $\zeta = \zeta(z)$ we have $[L, \zeta(z)] = \dot{\zeta}$ and $\dot{\zeta} \equiv 0$ at $z = 0$. On the other hand, $z^k L \sim L$ outside $z = 0$ which yields (3.2.9) below (so that we have gain, instead of loss). Recall in fact that we are assuming that $M$ has type $2m$. It is classical that the tangential vector fields $L$ and $\bar{L}$ satisfy $\frac{1}{2m}$-subelliptic estimates, that is, the first inequality in the estimate below. In combination with (3.2.1) which implies the second inequality below, we get

$$\begin{aligned}\|\zeta u\|^2_s &\underset{\sim}{<} \|\zeta \bar{L} u\|^2_{s-\frac{1}{2m}} + \|\zeta L u\|^2_{s-\frac{1}{2m}} + \|\zeta' u\|^2_{s-\frac{1}{2m}} \\ &\underset{\sim}{<} \|\zeta \bar{L} u\|^2_{s-\frac{1}{2m}} + \|\zeta \left|[L, \bar{L}]\right|^{\frac{1}{2}} u\|^2_{s-\frac{1}{2m}} + \|\zeta' u\|^2_{s-\frac{1}{2m}}.\end{aligned} \tag{3.2.9}$$

Remark that $\|\zeta' u\|^2_{s-\frac{1}{m}}$ (for a new $\zeta'$) takes care of the error $\|\zeta' \bar{L} u\|^2_{s-\frac{1}{2m}-1}$ coming from $[\Lambda^{2s-\frac{1}{m}}, \zeta']$. Now, remember that $[L, \bar{L}] = g_{1\bar{1}} \partial_t$ without error terms, that is, combinations of $L$ and $\bar{L}$; recall also that $g_{1\bar{1}} \geq 0$.

We get

(3.2.10)

$$\begin{aligned}\|\zeta\left|[L,\bar{L}]\right|^{\frac{1}{2}}u\|^2_{s-\frac{1}{2m}} &\sim \|\zeta g_{1\bar{1}}^{\frac{1}{2}}\Lambda_t^{\frac{1}{2}}u\|^2_{s-\frac{1}{2m}}\\ &\lesssim sc\,\|\zeta u\|^2_s + lc\,\|\zeta g_{1\bar{1}}^{\frac{1}{2}+\frac{k}{2(m-1)}}\Lambda_t^{\frac{1}{2}}u\|^2_{s+l}\\ &\lesssim \text{absorbable} + \|\zeta g_{1\bar{1}}^{\frac{1}{2}}z^k\Lambda_t^{\frac{1}{2}}u\|^2_{s+l}\\ &= \text{absorbable} + \|\zeta\left|[L,\bar{L}]\right|^{\frac{1}{2}}z^k u\|^2_{s+l}\\ &\le \text{absorbable} + \|\zeta L(z^k u)\|^2_{s+l} + \|\zeta\bar{L}(z^k u)\|^2_{s+l} + \|\zeta' z^k u\|^2_{s+l},\end{aligned}$$

where the first "$\sim$" is a way of rewriting the commutator, the second "$\lesssim$" follows from Lemma 3.2.2 (under the choice $n_1 = m-1$, $n_2 = k$, $r = m-1$, $\rho = \frac{1}{2m}$ and $f = g_{1\bar{1}}^{\frac{1}{2(m-1)}}$), the third "$\lesssim$" follows from $|g_{1\bar{1}}| \lesssim |z|^{2(m-1)}$, the fourth "$=$" is obvious and the last "$\lesssim$" follows from Lemma 3.2.1. We go now to estimate, in the last line of (3.2.10), the two terms $\|\zeta\bar{L}(z^k u)\|^2_{s+l}$ and $\|\zeta' z^k u\|^2_{s+l}$. We start from

$$\|\zeta L(z^k u)\|^2_{s+l} \le \|\zeta z^k L u\|^2_{s+l} + \|\zeta z^{k-1}u\|^2_{s+l}, \tag{3.2.11}$$

where the last term is produced by the commutator $[L, z^k]$. By writing, in the scalar product, once $z^{k-1}$ and once $[L, z^k]$, we get

$$\begin{aligned}\|\zeta z^{k-1}u\|^2_{s+l} &= (\zeta z^{k-1}u, \zeta[L,z^k]u)_{s+l}\\ &= (\zeta z^{k-1}u, \zeta z^k L u)_{s+l} + (\zeta z^{k-1}u, \zeta L z^k u)_{s+l}.\end{aligned} \tag{3.2.12}$$

Now,

(3.2.13)

$$\begin{cases}(\zeta z^{k-1}u, \zeta z^k L u)_{s+l} \le \underbrace{sc\|\zeta z^{k-1}u\|^2_{s+l}}_{\text{absorbable}} + \underbrace{\|\zeta z^k L u\|^2_{s+l}}_{\text{good}}\\ (\zeta z^{k-1}u, \zeta L z^k u)_{s+l} = (\zeta z^{k-1}\bar{L}u, \zeta z^k u)_{s+l} + (\zeta z^{k-1}u, \zeta' z^k u)_{s+l}\\ \qquad\qquad \underbrace{\lesssim \|\zeta z^k \bar{L}u\|^2_{s+l}}_{\text{good}} + \underbrace{sc\|\zeta z^{k-1}u\|^2_{s+l}}_{\text{absorbable}} + \|\zeta' z^k u\|^2_{s+l}.\end{cases}$$

Thus $\|\zeta z^{k-1}u\|^2_{s+l}$ has been estimated by $\|\zeta' z^k u\|^2_{s+l}$. What we have obtained so far is

(3.2.14)

$$\|\zeta u\|^2_s \lesssim \|\zeta\bar{L}u\|^2_s + \|\zeta z^k\bar{L}u\|^2_{s+l} + \|\zeta z^k L u\|^2_{s+l} + \|\zeta' z^k u\|^2_{s+l} + \|\zeta' u\|^2_{s-\frac{1}{2m}}.$$

Note that in this estimate, the terms coming with $L$ and $\bar{L}$ carry the same cut-off $\zeta$ as the left side; it is in this form that Theorem 3.1.1

will be applied for the proof of Theorems 3.1.2 and 3.1.5. Instead, to conclude the proof of Theorem 3.1.1, we have to go further with the estimation of $\|\zeta' z^k u\|^2_{s+l}$ (which also provides the estimate of the last term in (3.2.10)). We have, by subelliptic estimates

(3.2.15)
$$\|\zeta' z^k u\|^2_{s+l} \underset{\sim}{<} \|\zeta' L z^k u\|^2_{s+l-\frac{1}{2m}} + \|\zeta' \bar{L} z^k u\|^2_{s+l-\frac{1}{2m}} + \|\zeta'' z^k u\|^2_{s+l-\frac{1}{2m}}.$$

To $\|\zeta' L z^k u\|^2_{s+l-\frac{1}{2m}}$ we apply (3.2.11) with $s+l$ replaced by $s+l-\frac{1}{2m}$. In turn, $\|\zeta' z^{k-1} u\|^2_{s+l-\frac{1}{2m}}$ can be estimated, by (3.2.12), (3.2.13) and (3.2.15) with Sobolev indices all lowered from $s+l$ to $s+l-\frac{1}{2m}$, by means of "good" + "absorbable" + $\|\zeta'' z^k u\|^2_{s+l-\frac{1}{2m}}$. (In fact, "good" even comes with lower index.) The conclusion (3.1.3) follows from induction over $j$ such that $\frac{j}{2m} \geq s+l$. This completes the proof of Theorem 3.1.1.

□

*Proof of Theorem 3.1.5.* We first prove Theorem 3.1.5 instead of Theorem 3.1.2 because it is by far easier. As it has already been remarked at the begining of the Section (3.2), it suffices to prove the theorem for $u = u^+$. Also, in this case, the full norm can be replaced by the totally real norm. So we write $u$ for $u^+$ and $\|\cdot\|_s$ for $\|\cdot\|_{\mathbb{R},s}$; however, in some crucial passage where Lemma 3.2.3 is on use, it is necessary to point attention to the kind of he norm. We start from (3.2.14); note that, for this estimate to hold, only finite type is required. We begin by noticing that the last term of (3.2.14) is neglectable. We then rewrite the third term in the right of (3.2.14) as

$$(3.2.16)\quad (\zeta z^k L u, \zeta z^k L u)_{s+l} = (\zeta \bar{L} |z|^{2k} L u, \zeta u)_{s+l} + (\zeta z^k L u, \zeta' z^k u)_{s+l},$$

where we recall that we are using the notation $l = \frac{k-1}{2m}$. (Note that the commutator $[L, \zeta]$ is not just $\zeta'$ but comes with an additional factor $g_1$, the coefficient of $L$; but we disregard this contribution here though it will play a crucial role in the proof of Theorem 3.1.2.) We keep the first term in the right of (3.2.16) as it stands and put together with the similar term coming from the first term in the right of (3.2.14) to form $\tilde{\Box}^\kappa$. We then apply Cauchy-Schwartz inequality and estimate the first term by $\|\zeta \tilde{\Box}^\kappa u\|^2_{s+2l} + sc\|\zeta u\|^2_s$. As for the second term in the right of (3.2.16), it can be estimated, via Cauchy-Schwartz, by $sc\|\zeta z^k L u\|^2_{s+l} + lc\|\zeta' z^k u\|_{s+l}$. To this latter, we apply subelliptic estimates

(3.2.17)
$$\|\zeta' z^k u\|^2_{s+l} \underset{\sim}{<} \|\zeta' z^k \bar{L} u\|^2_{s+l-\frac{1}{m}} + \|\zeta' z^k L u\|^2_{s+l-\frac{1}{2m}} + \|\zeta' z^{k-1} u\|^2_{s+l-\frac{1}{2m}} + \|\zeta'' z^k u\|^2_{s+l-\frac{1}{2m}}.$$

For the third term in the right, recalling (3.2.12) and (3.2.13), we get

$$\|\zeta' z^{k-1} u\|^2_{s+l-\frac{1}{2m}} \underset{\sim}{<} \text{neglectable} + \|\zeta'' z^k u\|^2_{s+l-\frac{1}{2m}}. \tag{3.2.18}$$

Thus $\|\zeta' z^k u\|^2_{s+l}$ is controlled by induction over $j$ with $\frac{j}{2m} \geq s+l$. (Recall, once more, that "good" is stable under passing from $\zeta'$ to $\zeta''$.) We notice that combination of (3.2.17) and (3.2.18) shows that $\|\zeta' z^k u\|^2_{s+l}$ is neglectable. We pass to $\|\zeta' z^k \bar{L} u\|^2_{s+l}$, the second term in the right of (3.2.14) and observe that it can be treated exactly in the same way as the third (with $L$ instead of $\bar{L}$). We end with the first which does not carry the loss $l$; we have

$$\begin{aligned} \|\zeta \bar{L} u\|^2_s &= (\zeta L \bar{L} u, \zeta u)_s + (\zeta \bar{L} u, \zeta' g_1 u)_s \\ &= (\Lambda^{2l}\Lambda^{-2l} L \bar{L} u, \zeta u)_s + (\zeta \bar{L} u, \zeta' g_1 u)_s. \end{aligned} \tag{3.2.19}$$

The first term in the right combines to form $\tilde{\Box}^k$. As for the second, we notice that $|g_1| \underset{\sim}{<} |z|$ and therefore applying Lemma 3.2.2 for $n_1 = k-1$ and $n_2 = 1$

$$(\zeta \bar{L} u, \zeta' g_1 u)_s \leq sc\|\zeta \bar{L} u\|^2_s + lc(\|\zeta' z^k u\|^2_{s+l} + \|\zeta' u\|^2_{s-\frac{1}{2m}}).$$

The first term in the right is absorbable, the last neglectable, the midle has already been proved to be neglectable by subelliptic estimates (3.2.17). This completes the proof.

□

*Proof of Theorem 3.1.2.* As before, we prove the theorem for $u = u^+$ and write $\|\cdot\|_s$ for $\|\cdot\|_{\mathbb{R},s}$ though, in some crucial passage, it is necessary to point the attention to the kind of the norm. Raising Sobolev indices, we rewrite (3.2.14) in a more symmetric fashion as

$$\|\zeta u\|^2_s \underset{\sim}{<} \|\zeta \bar{L} u\|^2_{s+l} + \|\zeta z^k L u\|^2_{s+l} + \|\zeta' u\|^2_{s-\frac{1}{2m}} + \|\zeta' z^k u\|^2_{s+l}. \tag{3.2.20}$$

We handle all terms in the right as in Theorem 3.1.2 except from the first which comes now with the loss $s+l$. We point out that to control these terms, only finite type has been used. Instead, to control the remaining term, we need the additional hypothesis (3.1.5). We have

$$\|\zeta \bar{L} u\|^2_{s+l} = (\zeta L \bar{L} u, \zeta u)_{s+l} + (\zeta \bar{L} u, \zeta' g_1 u)_{s+l}. \tag{3.2.21}$$

The first term combines to form $\Box^k$. As for the second, we recall the estimate $|g_1| \underset{\sim}{<} |z| g_{1\bar{1}}$ and apply Lemma 3.2.3 for $h = z g_{1\bar{1}}$, $h_1 = g_{1\bar{1}}^{\frac{1}{2}}$ and $h_2 = z g_{1\bar{1}}^{\frac{1}{2}}$ to get

$$|(\zeta \bar{L} u, \zeta' g_1 u)|_{s+l} \leq sc\|\zeta g_{1\bar{1}}^{\frac{1}{2}} \bar{L} u\|^2_{s+2l} + lc\|\zeta' z g_{1\bar{1}}^{\frac{1}{2}} u\|^2_s. \tag{3.2.22}$$

In the estimate above, we point our attention to the fact that the norms that we are considering are totally real norms (though we do not keep track in our notation) and therefore Lemma 3.2.3 can be applied. We start by estimating the second term in the right. By Lemma 3.2.1 and next, Lemma 3.2.2 for $n_1 = 1$, $n_2 = k - 1$

(3.2.23)
$$\begin{aligned}\|\zeta' g_{1\bar{1}}^{\frac{1}{2}} z u\|_s^2 &\underset{\sim}{<} \|\zeta' z L u\|_{s-\frac{1}{2}}^2 + \|\zeta' z \bar{L} u\|_{s-\frac{1}{2}}^2 + \text{neglectable} \\ &\leq \|z^k \zeta' L u\|_{s-\frac{1}{2}+l}^2 + \|\zeta' L u\|_{s-\frac{1}{2}-\frac{1}{2m}}^2 + \|\zeta' z \bar{L} u\|_{s-\frac{1}{2}}^2 + \text{neglectable},\end{aligned}$$

where neglectable comes from the commutators $[L, z]$ and $[L, \zeta']$. Also, the first term in the second line of (3.2.23) is neglectable. As for the second term, we have, by (3.2.1)

$$\|\zeta' L u\|_{s-\frac{1}{2}-\frac{1}{2m}}^2 \underset{\sim}{<} \|\zeta' g_{1\bar{1}}^{\frac{1}{2}} u\|_{s-\frac{1}{2m}}^2 + \|\zeta' \bar{L} u\|_{s-\frac{1}{2}-\frac{1}{2m}}^2 + \text{neglectable}. \tag{3.2.24}$$

Since both terms in the right of (3.2.24) are neglectable, we conclude that $\|\zeta' z g_{1\bar{1}}^{\frac{1}{2}} u\|_s^2$ itself is neglectable. From now on, we follow closely the track of [**1**]. We pass to consider the last and most difficult term to estimate, that is, the first in the right of (3.2.22). Along with this term, that we denote by (a), we introduce three additional terms; we set therefore

$$\begin{cases} (a) := \|\zeta g_{1\bar{1}} \bar{L} u\|_{s+2l}^2, & (b) := \|\zeta z^{2k-1} g_{1\bar{1}}^{\frac{1}{2}} u\|_{s+2l}, \\ (c) := \|\zeta z^{2k-1} L u\|_{s+2l-\frac{1}{2}}^2, & (d) := \|L \zeta \bar{L} u\|_{s+2l-\frac{1}{2}}^2. \end{cases}$$

Because of these additional terms, that we are able to estimate, "neglectable" and "absorbable" take an extended range. We first show that (b) is controlled by (c). This is apparently as in [**1**] first half of 5.3 but more complicated because our (b) and (c) are different from their $(LHS)_5$ and $(LHS)_6$ respectively. Now, by Lemma 3.2.1 we get

$$(b) \underset{\sim}{<} (c) + \|\zeta' z^{2k-1} g_1 u\|_{s+2l-\frac{1}{2}}^2 + \|\zeta z^{2k-2} u\|_{s+2l-\frac{1}{2}}^2 + \text{neglectable},$$

where the central terms in the right come from $[L, \zeta]$ and $[L, z^{2k-1}]$ respectively, and where neglectable, with respect to (a), is the term which involves $\bar{L} u$ and which comes lowered by $-\frac{1}{2}$. The first of the central terms is neglectable with respect to (b). As for the second, we have, using the notation $\# = s+2l-\frac{1}{2}-\frac{1}{2m}$

$$\underbrace{\|\zeta z^{2k-2}u\|^2_{s+2l-\frac{1}{2}}}_{(i)} \lesssim \underbrace{\|\zeta z^{2k-2}Lu\|_\#}_{(ii)} + \underbrace{\|\zeta z^{2k-2}\bar{L}u\|^2_\#}_{(iii)} + \underbrace{\|\zeta' z^{2k-2}g_1u\|^2_\#}_{(iv)} + \underbrace{\|\zeta z^{2k-3}u\|^2_\#}_{(v)},$$

where the two terms of the second line come from $[L,\zeta]$ and $[L, z^{2k-2}]$ respectively. First, (iv) is neglectable with respect to (i). Next, using Lemma 3.2.2 for $n_1 = 2k-2$ and $n_2 = 1$

$$(ii) \lesssim \underbrace{sc\|\zeta Lu\|^2_{\#-\frac{2k-2}{2m}}}_{(ii)_1} + \underbrace{lc\|\zeta z^{2k-1}Lu\|^2_{\#+\frac{1}{2m}}}_{(ii)_2}.$$

Note that $\# - \frac{2k-2}{2m} = s-\frac{1}{2}-\frac{1}{2m}$ and $\#+\frac{1}{2m} = s+2l-\frac{1}{2}$; thus $(ii)_1$ is absorbed by (3.2.24) and $(ii)_2$ is estimated by (c). Next, by Lemma 3.2.2 for $n_1 = 2k-3$ and $n_2 = 1$

$$(v) \lesssim \underbrace{lc\|\zeta u\|^2_{\#-\frac{2k-3}{2m}}}_{(v)_1} + \underbrace{sc\|\zeta z^{2k-2}u\|^2_{\#+\frac{1}{2m}}}_{(v)_2}.$$

We have $\#-\frac{2k-3}{2m} = s-\frac{1}{2}$ and, again, $\#+\frac{1}{2m} = s+2l-\frac{1}{2}$; thus $(v)_1$ is neglectable with respect to $\|\zeta u\|^2_s$, the term in the left of the estimate, and $(v)_2$ is absorbed by (i). Finally, by (3.2.1)

$$(iii) \le \underbrace{\|\zeta z^{2k-2}Lu\|_\#}_{(iii)_1} + \underbrace{\|\zeta z^{2k-2}g^{\frac{1}{2}}_{1\bar{1}}u\|^2_{\#+\frac{1}{2}}}_{(iii)_2}.$$

Now, applying Lemma 3.2.2 for $n_1 = k-2$, $n_2 = 1$ in the first line below and $n_1 = 2k-2$ and $n_2 = 1$ in the second respectively, we get

$$\begin{cases} (iii)_1 \lesssim \underbrace{\|\zeta z^{2k-1}Lu\|_{s+2l-\frac{1}{2}}}_{(c)} + \underbrace{\|\zeta z^k Lu\|_{s+l-\frac{1}{2}}}_{\text{neglectable w.r.to } \|\zeta z^k Lu\|^2_{s+l}} \\ (iii)_2 \lesssim sc\underbrace{\|\zeta z^{2k-1}g^{\frac{1}{2}}_{1\bar{1}}u\|^2_{s+2l}}_{(b)} + lc \underbrace{\|\zeta g^{\frac{1}{2}}_{1\bar{1}}u\|^2_{s-\frac{1}{2m}}}_{\text{neglectable w.r.to } \|\zeta u\|_s} . \end{cases}$$

Summarizing up,

$$(b) \lesssim (c) + \text{neglectable}.$$

We have to show now that

$$\begin{cases} (c) \lesssim \|\Box^k u\|^2_{s+2l-\frac{1}{2}} + \text{absorbable} + \text{neglectable}, \\ (a)+(d) \lesssim \|\Box^k u\|^2_{s+2l-\frac{1}{2}} + \text{absorbable} + \text{neglectable}. \end{cases}$$

The first inequality is proved in the same way as the second part of 5.3 of [**1**]. The second as in 5.4 of [**1**] with the relevant change that we do not have at our disposal their estimate $|[\bar{L}, |z|^{2k} g_{1\bar{1}}]| \underset{\sim}{<} |z|^{2k-1-2(m-1)}$. Instead, we have to use, as a consequence of our key assumption (3.1.3)

$$\begin{aligned}[\bar{L}, |z|^{2k} g_{1\bar{1}}] &\underset{\sim}{<} |z|^{2k-1} g_{1\bar{1}} + |z|^{2k} |g_{1\bar{1}\bar{1}} \\ &\underset{\sim}{<} |z|^{2k-1} g_{1\bar{1}}.\end{aligned}$$

Thus, when we arrive at the two error terms in the second displayed formula of p. 692 (second terms in the third and fourth lines), we have the factor $z^{2k-1} g_{1\bar{1}}$. With the notations of our Lemma 3.2.3, we split this factor as $h = h_1 h_2$ for $h_1 = z^{2k-1} g_{1\bar{1}}^{\frac{1}{2}}$ and $h_2 = g_{1\bar{1}}^{\frac{1}{2}}$ respectively and then control these error terms as sc (a) and lc (b). The proof is complete.

□

## 3.3. Loss of derivatives in the infinite type

We refer to the begining of section (3.2) for the notations which will be on use. In particular, we recall the standard decomposition $u = u^+ + u^- + u^0$ and the alliptic estimate $\|u^0\| \leq \|\bar{L}u^0\| + \|u^0\|$. As for $u^-$, recall that $[L, \bar{L}] = g_{1\bar{1}} \partial_t$ and hence $g_{1\bar{1}} \sigma(\partial_t) \leq 0$ over supp$\psi^-$. Thus (3.2.1) yields $\|Lu\|^2 \underset{\sim}{<} \|\bar{L}u\|^2$. It follows that, if $L$ and $\bar{L}$ have superlogarithmic estimate as in our application, then

$$\|\log(\Lambda) u^-\|^2 \leq \delta \|\bar{L}u^-\|^2 + c_\delta \|u\|^2.$$

In conclusion, only estimating $u^+$ is relevant. We note here that, over supp $\Psi^+$, we have $g_{1\bar{1}} \xi_t \geq 0$; thus

$$\begin{aligned}\|g_{1\bar{1}}^{\frac{1}{2}} u^+\|_{\frac{1}{2}}^2 &= |([L, \bar{L}]u^+, u^+)| \\ &\leq \|Lu^+\|^2 + \|\bar{L}u^+\|^2.\end{aligned} \tag{3.3.1}$$

Following Kohn [**8**], we introduce a microlocal modification of $\Lambda^s$, denoted by $R^s$; this is the pseudodifferential operator with symbol $R^s_\xi := (1+|\xi|^2)^{\frac{s\sigma(x)}{2}}$, $\sigma \in C^\infty_c$; often, what is used is in fact the partial operator in $t$, $R^s_t$ with symbol $R^s_{\xi_t}$. The relevant property of $R^s$ is

$$\|\Lambda^s \zeta_0 u\|^2 \underset{\sim}{<} \|R^s \zeta_0 u\|^2 + \|\zeta_0 u\|^2 \qquad \text{if } \zeta_0 \prec \sigma.$$

Thus, $R^s$ is equivalent to $\Lambda^s$ over functions supported in the region where $\sigma \equiv 1$. In addition, $\zeta R^s$ better behaves with respect to commutation with $L$; in fact, Jacobi equality yields

$$[\zeta R^s, L] \sim \dot{\zeta} R^s + \zeta \log(\Lambda) R^s. \tag{3.3.2}$$

Thus, on one hand we have the disadvantage of the additional $\log(\Lambda)$ in the second term, but we gain much in the cut-off because

$$\dot{\zeta} R^s \text{ is of order 0 if } \operatorname{supp} \dot{\zeta} \cap \operatorname{supp} \sigma = \emptyset. \tag{3.3.3}$$

Property (3.3.3) is crucial in localizing regularity in presence of super-logarithmic estimate.

*Proof of Theorem 3.1.8.* As it has already been noticed, it suffices to prove (3.1.15) only for $u^+$ and for $\|\cdot\|_{\mathbb{R},\, s}$; thus we write for simplicity $u$ and $\|\cdot\|_s$ but mean $u^+$ and $\|\cdot\|_{\mathbb{R},\, s}$. Moreover, we can use a cut-off $\zeta = \zeta(t)$ in $t$ only. In fact, for a cut-off $\zeta = \zeta(z)$ we have $[L, \zeta(z)] = \dot{\zeta}$ and $\dot{\zeta} \equiv 0$ at $z = 0$. On the other hand, $z^k L \sim L$ outside $z = 0$ which yields gain of derivatives, instead of loss. We call "good" a term in the right side (upper bound) of an estimate we wish to prove and "absorbable" a term which comes as a fraction (small constant or sc) of a formerly encountered term. We take cut-off functions in a neighborhood of 0: $\zeta_0 \prec \sigma \prec \zeta_1 \prec \zeta'$; we have for $u \in C^\infty$

$$\begin{aligned} \|\zeta_0 u\|_s^2 &= \|\zeta_0\zeta_1 u\|_s^2 \\ &\lesssim \|R^s\zeta_0\zeta_1 u\|^2 + \|u\|_0^2 \\ &\lesssim \|\zeta_0 R^s\zeta_1 u\|_0^2 + \|[R^s, \zeta_0]\zeta_1 u\|_0^2 + \|u\|_0^2 \\ &\lesssim \|R^s\zeta_1 u\|_0^2 + \|u\|_0^2 \\ &\lesssim \|\zeta' R^s\zeta_1 u\|_0^2 + \|u\|_0^2, \end{aligned} \tag{3.3.4}$$

where the inequality in the third line follows from interpolation in Sobolev spaces and the last from $\operatorname{supp}(1-\zeta') \cap \operatorname{supp}\sigma = \emptyset$. We have

$$\begin{aligned} \|\zeta_0 u\|_s^2 &\underset{\text{by (3.3.4)}}{\lesssim} \underbrace{\|\zeta' R^s\zeta_1 u\|^2}_{(a)} + \|u\|^2 \\ &\underset{\text{trivial}}{\lesssim} \underbrace{\|\log(\Lambda)\zeta' R^s\zeta_1 u\|^2}_{(b)} + \|u\|^2 \\ &\le \delta \left( \|L(\zeta' R^s\zeta_1)u\|^2 + \|\bar{L}(\zeta' R^s\zeta_1)u\|^2 \right) + c_\delta \|u\|^2, \end{aligned} \tag{3.3.5}$$

where the last inequality follows from superlogarithmic estimate. Using integration by parts, we estimate the first term in the last line

$$\|L(\zeta' R^s\zeta_1)u\|^2 \lesssim \|\bar{L}(\zeta' R^s\zeta_1)u\|^2 + \left\|[L,\bar{L}]^{\frac{1}{2}}(\zeta' R^s\zeta_1)u\right\|^2. \tag{3.3.6}$$

We rewrite the term with the commutator. For this we recall an easy result about interpolation in Sobolev spaces. For positive $\epsilon$, $r$, $n_1$, $n_2$ with $n_1$ and $n_2$ integers satisfying $0 < n_1 \le r$ and $n_2 > 0$,

$$\|f^r u\|^2_{\frac{1}{2}} \le sc\|f^{r-n_1}u\|^2_{\frac{1}{2}-n_1\epsilon} + lc\|f^{r+n_2}u\|^2_{\frac{1}{2}+n_2\epsilon}. \tag{3.3.7}$$

Thanks to the $\infty$ type of $g$ and the fact that $g = 0$ only at $z = 0$, it follows that $g_{1\bar{1}}^{\frac{1}{2r}}$ is a smooth function for any $r$ and is smaller than $|z^k|$ for any $k$. Thus, under the choice $f = g_{1\bar{1}}^{\frac{1}{2r}}$, $n_1 = r$, $\epsilon = \frac{1}{2r}$, $n_2 = 1$ we get

(3.3.8)

$$\begin{aligned}
\left\||[L,\bar{L}]|^{\frac{1}{2}}\zeta' R^s\zeta u\right\|^2 &= \left\|g_{1\bar{1}}^{\frac{1}{2}}\zeta' R^s\zeta u\right\|^2_{\frac{1}{2}} \\
&\lesssim \mathrm{sc}\|\zeta' R^s\zeta u\|_0^2 + \mathrm{lc}\left\|g_{1\bar{1}}^{\frac{1}{2}}g_{1\bar{1}}^{\frac{1}{2r}}u\right\|^2_{\frac{1}{2}+\frac{1}{2r}} \\
&\lesssim \mathrm{sc}\|\zeta' R^s\zeta u\|_0^2 + \mathrm{lc}\left\|g_{1\bar{1}}^{\frac{1}{2}}\Lambda^{\frac{1}{2}}z^k\zeta' R^s\zeta u\right\|^2_{\epsilon} \\
&= \mathrm{sc}\|\zeta' R^s\zeta u\|_0^2 + \mathrm{lc}\left\|[L,\bar{L}]^{\frac{1}{2}}z^k\zeta' R^s\zeta u\right\|^2_{\epsilon} \\
&\lesssim \|\zeta' R^s\zeta u\|_0^2 + \|Lz^k(\zeta' R^s\zeta_1)u\|^2_{\epsilon} + \|z^k\bar{L}(\zeta' R^s\zeta_1)u\|^2_{\epsilon}.
\end{aligned}$$

We wish to first discard the last term in the bottom of (3.3.8). For this, we recall Jacobi identity, observe that $[z^k, \zeta' R^s\zeta]$ has order arbitrarily close to $s-1$ (because of a logarithmic extra term), and get

(3.3.9)

$$\begin{aligned}
[z^k\bar{L}, \zeta' R^s\zeta_1] &= [\bar{L},\zeta']R^s\zeta_1 z^k + \zeta'[\bar{L},R^s]\zeta_1 z^k + \zeta' R^s[\bar{L},\zeta_1]z^k \\
&\sim \underbrace{\dot{\zeta}' R^s\zeta_1}_{\text{0-order by (3.3.3)}} z^k + \underbrace{\zeta'\log(\Lambda)R^s\zeta_1}_{\text{by (3.3.2)}} z^k + \underbrace{\zeta' R^s\dot{\zeta}_1}_{\text{0-order by (3.3.3)}} z^k.
\end{aligned}$$

Thus we can commutate $z^k\bar{L}$ with $\zeta' R^s\zeta_1$ in (3.3.8) up to an error as described in (3.3.9) which yields

$$\|z^k\bar{L}(\zeta' R^s\zeta_1)u\|^2_{\epsilon} \lesssim \|(\zeta' R^s\zeta_1)z^k\bar{L}u\|^2_{\epsilon} + \|(\zeta'\log(\Lambda)R^s\zeta_1)z^k u\|^2_{\epsilon} + \|z^k u\|^2_{\epsilon}.$$

On the other hand, since $[\zeta',\log(\Lambda)]R^s=0(\Lambda^{-1})$, then

$$\begin{aligned}\|(\zeta'\log(\Lambda)R^s\zeta_1)z^ku\|^2_\epsilon &\underset{\sim}{<} \|(\log(\Lambda)(\zeta' R^s\zeta_1)z^ku\|^2_\epsilon+\|\zeta_1z^ku\|^2_{-1+\epsilon}\\ &\underset{\text{suplog estimate}}{\underset{\sim}{<}} \underbrace{\delta\left(\|L(\zeta'R^s\zeta_1)z^ku\|^2_\epsilon+\|\bar L(\zeta'R^s\zeta_1)z^ku\|^2_\epsilon\right)}_{\text{absorbed by 2}^{\text{nd}}\text{ line of (3.3.8)}}+\|\zeta_1z^ku\|^2_{-1+\epsilon},\end{aligned}$$

where we are using the equality $[\Lambda^\epsilon_t,L]=0$ as well as $[\Lambda^\epsilon,\log(\Lambda)]=0$. In the same way, using again (3.3.9), we commutate $\bar L$ with $(\zeta'R^s\zeta_1)$ in (3.2.6) and (3.3.6). What is left, is to estimate the first term in the last line of (3.3.8). First, from Jacobi identity we get

$$[Lz^k,\zeta'R^s\zeta_1]\sim(\text{0-order})z^k+\zeta'\log(\Lambda)R^s\zeta_1z^k+(\text{0-order})z^k,$$

so that we are eventually reduced to estimate $\|(\zeta'R^s\zeta_1)Lz^ku\|^2$. This is the most difficult operation. We have (by the trivial identity $[L,z^k]=z^{k-1}$)

$$\|(\zeta'R^s\zeta_1)Lz^ku\|^2_\epsilon=\underbrace{\|(\zeta'R^s\zeta_1)z^kLu\|^2_\epsilon}_{\text{good}}+\|(\zeta'R^s\zeta_1)z^{k-1}u\|^2_\epsilon.$$

Next,

$$\begin{aligned}\underbrace{\|(\zeta'R^s\zeta_1)z^{k-1}u\|^2_\epsilon}_{(c)}&=(\underbrace{(\zeta'R^s\zeta_1)z^{k-1}u}_{*},(\zeta'R^s\zeta_1)[L,z^k]u)_\epsilon\\ &=-(*,(\zeta'R^s\zeta_1)z^kLu)_\epsilon+(*,(\zeta'R^s\zeta_1)Lz^ku)_\epsilon.\end{aligned}$$

Now,

$$\begin{cases}\left|(*,(\zeta'R^s\zeta_1)z^kLu)_\epsilon\right|\le sc\|*\|^2_\epsilon+\underbrace{\|(\zeta'R^s\zeta_1)z^kLu\|^2_\epsilon}_{\text{good}}\\ \left|(*,(\zeta'R^s\zeta_1)Lz^ku)_\epsilon\right|\le\left|(\underbrace{(\zeta'R^s\zeta_1)\bar Lz^{k-1}u}_{\text{good}},\underbrace{(\zeta'R^s\zeta_1)z^ku}_{\text{absorbed by (3.3.8)}})_\epsilon\right|\\ \qquad\qquad+2\left|(\underbrace{*}_{\text{absorbed by (c)}},\underbrace{[L,(\zeta'R^s\zeta_1)]z^ku}_{(d)})_\epsilon\right|.\end{cases}$$

We notice here that to absorbe a term by the last line of (3.3.8) we use compactness estimates which hold as a byproduct of superlogarithmic. We estimate (d). We notice that

$$[L,(\zeta'R^s\zeta_1)]\sim\zeta'\log(\Lambda)R^s\zeta_1+(\text{0-order}). \tag{3.3.10}$$

We also remark that

$$(3.3.11)\qquad \begin{cases} [\Lambda^\epsilon\zeta', \log(\Lambda)]R^s = 0(\Lambda^{-\epsilon}) \quad (i) \\ [\zeta', \Lambda^\epsilon]R^s \sim 0(\Lambda^{-\epsilon}) \quad (ii) \\ [L, \Lambda^\epsilon] = 0 \quad (iii). \end{cases}$$

Hence

(3.3.12)

$$\begin{aligned} \|(d)\|^2_\epsilon \underset{\text{by (3.3.10)}}{\lesssim} & \|(\zeta'\log(\Lambda)R^s\zeta_1)z^k u\|^2_\epsilon + \|z^k u\|^2_\epsilon \\ \underset{\text{by (3.3.11) (i) and (ii)}}{\leq} & \|(\log(\Lambda)\zeta'\Lambda^\epsilon R^s\zeta_1)z^k u\|^2_0 + \|z^k u\|^2_\epsilon + \|\zeta_1 z^k u\|^2_{-\epsilon} \\ \underset{\text{by suplog estimate}}{\leq} & \delta\left(\|L(\zeta'\Lambda^\epsilon R^s\zeta_1 z^k u\|^2 + \|\bar{L}(\zeta'\Lambda^\epsilon R^s\zeta_1)z^k u\|^2\right) + c_\delta\|z^k u\|^2_\epsilon. \end{aligned}$$

Now, the term with $\delta$ is absorbed by the last term in (3.3.8) (after we transform $\Lambda^\epsilon$ into $\|\cdot\|_\epsilon$ to fit into (3.3.8) and use the fact that $[L\zeta', \Lambda^\epsilon] \sim \Lambda^{1+\epsilon}$). This concludes the proof of (3.1.15).

□

*Proof of Theorem 3.1.10.* As above, we stay in the positive microlocal cone, the support of $\psi^+$, and consider only derivatives and cut-off with respect to $t$. From the trivial identity $[L, z] = 1$, and from $[L, \zeta_0] \sim \dot{\zeta}_0 g_1$, we get

$$\begin{aligned} \|\zeta_0 u\|^2_s &= ([L, z]\zeta_0 u, \zeta_0 u)_s \\ &\lesssim \|\bar{z}\zeta_0\bar{L}u\|^2_s + \|\bar{z}\zeta_0 L u\|^2_s + \|\bar{z}g_1\zeta_1 u\|^2_s + sc\|\zeta_0 u\|^2_s. \end{aligned}$$

Now, the last term is absorbed. As for the term before

$$\begin{aligned} \|\bar{z}g_1\zeta_1 u\|^2_s \underset{\text{by (3.1.16)}}{\leq} & \|\bar{z}g^{\frac{1}{2}}_{1\bar{1}}\Lambda^{\frac{1}{2}}\zeta_1 u\|^2_{s-\frac{1}{2}} \\ \underset{\text{by (3.3.1)}}{\leq} & \|\bar{z}L\zeta_1 u\|^2_{s-\frac{1}{2}} + \|\bar{z}\bar{L}\zeta_1 u\|^2_{s-\frac{1}{2}} + \|\bar{z}\zeta_1 u\|^2_{s-\frac{1}{2}} \\ \lesssim & \|\zeta_1\bar{z}Lu\|^2_{s-\frac{1}{2}} + \|\bar{z}\bar{L}\zeta_1 u\|^2_{s-\frac{1}{2}} + \|\bar{z}\zeta_2 u\|^2_{s-\frac{1}{2}} \quad \text{for } \zeta_2 \succ \zeta_1. \end{aligned}$$

Now, $\|\bar{z}\zeta_2 u\|^2_{s-\frac{1}{2}}$ is not absorbable by $\|\zeta_0 u\|^2_s$, but can be estimated by the 0-norm using induction over $j$ such that $\frac{j}{2} \geq s$.

□

*Proof of Proposition 3.1.11.* As ever, we stay in the positive microlocal cone and take derivatives and cut-off only in $t$. We prove the result for

$s$ replaced by 0 and $\epsilon$ replaced by $-\eta$. The conclusion for general $s$ follows from the fact that $\partial_t$ commutes with $L$ and $\bar{L}$. We define

$$v_\lambda = e^{-\lambda(e^{-\frac{1}{|z|^\alpha}}-it+(e^{-\frac{1}{|z|^\alpha}}-it)^2)} \qquad \lambda >> 0.$$

We denote by $-\lambda A$ the term at exponent and note that $\mathsf{Re}\,\lambda A \sim \lambda(e^{-\frac{1}{|z|^\alpha}} + t^2)$. For $L = \partial_z + ig_1(z)\partial_t$, we have $\bar{L}v_\lambda = 0$ (which is the key point) and moreover

$$|\bar{z}^k L v_\lambda| \sim \lambda |z|^{k-(\alpha+1)} e^{-\lambda(e^{-\frac{1}{|z|^\alpha}}+t^2)} e^{-\frac{1}{|z|^\alpha}}.$$

We set

$$\lambda(e^{-\frac{1}{|z|^\alpha}}, t) = (\theta_1, \frac{1}{\sqrt{\lambda}}\theta_2).$$

Under this change we have, over $\operatorname{supp}\zeta_0$ and $\operatorname{supp}\zeta_1$ which implies $\theta_1 << \lambda$,

$$|z|^{k-(\alpha+1)} = \frac{1}{(\log\lambda - \log\theta_1)^{\frac{k-(\alpha+1)}{\alpha}}}.$$

Hence we interchange

$$|\bar{z}^k L v_\lambda| \dashrightarrow \frac{1}{(\log\lambda)^{\frac{k-(\alpha+1)}{\alpha}}} \left( \frac{\theta_1 + \theta_2^2}{\left(1 - \frac{\log\theta_1}{\log\lambda}\right)^{\frac{k-(\alpha+1)}{\alpha}}} \right) e^{-(\theta_1+\theta_2^2)}.$$

Notice that $\theta_1 << \lambda$ and hence, for suitable positive $c_1$ and $c_2$, we have $c_1 < \frac{\theta_1+\theta_2^2}{\left(1-\frac{\log\theta_1}{\log\lambda}\right)^{\frac{k-(\alpha+1)}{\alpha}}} < c_2$, uniformly over $\lambda$. We also interchange

$$v_\lambda \dashrightarrow e^{-(\theta_1+\theta_2^2)}.$$

Taking $L^2$ norms yields

$$\|\bar{z}^k L v_\lambda\|^2 \sim \frac{1}{(\log\lambda)^{2\frac{k-(\alpha+1)}{\alpha}}} \|v_\lambda\|^2.$$

So, the effect on $L^2$ norm of the action of $\bar{z}^k L$ over $v_\lambda$ is comparable to $\frac{1}{(\log\lambda)^{\frac{k-(\alpha+1)}{\alpha}}}$. We describe now the effect of the pseudodifferential operator $\log(\Lambda_t)$. We claim that

$$\|\log(\Lambda_t)e^{-\lambda t^2}\|^2 \sim (\log\lambda)^2 \|e^{-\lambda t^2}\|^2. \tag{3.3.13}$$

This is a consequence of

$$\log(\Lambda_t)e^{-\lambda t^2} \sim \log\lambda e^{-\lambda t^2} + \left(\log(\Lambda_{\tilde{t}})e^{-\tilde{t}^2}\right)\Big|_{\tilde{t}=\sqrt{\lambda}t}, \tag{3.3.14}$$

that we go to prove now. Using the coordinate change $\tilde{\theta} = \sqrt{\lambda}\theta$, $\tilde{\xi} = \frac{\xi}{\sqrt{\lambda}}$, we get

$$\begin{aligned}\int e^{it\xi}\log(\Lambda_\xi)\Big(\int e^{-i\xi\theta}e^{-\lambda\theta^2}d\theta\Big)d\xi \\ = \int e^{it\sqrt{\lambda}\tilde{\xi}}\Big(\log(\frac{1}{\lambda}+|\tilde{\xi}|^2)^{\frac{1}{2}}+\log(\sqrt{\lambda})\Big)\Big(\int e^{i\tilde{\xi}\tilde{\theta}-\tilde{\theta}^2}d\tilde{\theta}\Big)d\tilde{\xi} \\ = \log(\sqrt{\lambda})e^{-\lambda t^2}+\Big(\log(\Lambda^{\lambda}_{\tilde{t}})e^{-\tilde{t}^2}\Big)\Big|_{\tilde{t}=\sqrt{\lambda}t},\end{aligned}$$

where $\log(\Lambda^{\lambda}_{\tilde{t}})$ is the operator with symbol $\log(\frac{1}{\lambda}+|\tilde{\xi}|^2)^{\frac{1}{2}}$. This proves (3.3.14) and in turn the claim (3.3.13). In the same way, we can check that $\|\Lambda_t^{-\eta}e^{-\lambda t^2}\|^2 \sim \lambda^{-2\eta}\|e^{-\lambda t^2}\|^2$.

We combine now the effect over $v_\lambda$ of $\bar{z}^k L$ with that of $\log(\Lambda_t)$. If

$$\|\zeta_0 v_\lambda\|^2 \lesssim \|\zeta_1(\log\Lambda_t)^r\bar{z}^k L v_\lambda\|^2 + \|v_\lambda\|^2_{-\eta},$$

then, since the right side is estimated from above by

$$\Big((\log\lambda)^{2r}(\log\lambda)^{-2\frac{k-(\alpha+1)}{\alpha}}+\lambda^{-2\eta}\Big)\|v_\lambda\|^2,$$

we must have that the logarithmic term is not infinitesimal which forces $r \geq \frac{k-(\alpha+1)}{\alpha}$.

□

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