Università degli Studi di Padova

UNIVERSITÀ DEGLI STUDI DI PADOVA

DIPARTIMENTO DI MATEMATICA

SCUOLA DI DOTTORATO DI RICERCA IN SCIENZE MATEMATICHE

INDIRIZZO: MATEMATICA COMPUTAZIONALE

CICLO XXIII

FOUR ESSAYS IN FINANCIAL MATHEMATICS

Direttore della Scuola: Ch.mo Prof. Paolo Dai Pra

Coordinatore d'indirizzo: Ch.mo Prof. Paolo Dai Pra

Supervisore: Ch.mo Prof. Wolfgang J. Runggaldier

Dottorando: Claudio Fontana

Four essays in financial mathematics:

An abstract and unifying approach to mean-variance optimization problems

Measure changes for reduced-form credit risk models

Diffusion-based models for financial markets without martingale measures

Weak no-arbitrage conditions: characterization, stability and hedging problems

Claudio Fontana

This version: August 29, 2011

Abstract

The first Chapter of the Thesis presents a general and abstract framework for the analysis of mean-variance portfolio optimization problems. Under a minimal no-arbitrage condition, we consider a whole range of quadratic optimization problems, which are solved in a unified way. We give general and model-independent characterizations of the optimal solutions as well as abstract generalizations of classical results from financial economics such as two-fund separation results, mean-variance efficiency and a *CAPM*-type formula. Finally, we apply our general results to the valuation of contingent claims according to several mean-variance indifference valuation rules.

The second Chapter considers a general reduced-form credit risk model, where the default time is modeled as a doubly stochastic random time with default intensity driven by a diffusion affine process. We characterize the family of all locally equivalent probability measures which preserve the affine structure of the model by giving necessary and sufficient conditions on their density process. We illustrate the usefulness of our results first in the context of a jump-to-default extension of the popular Heston (1993) stochastic volatility model and then in the context of a more general hybrid equity/credit risk multifactor model, providing applications of interest in view of risk management as well as pricing purposes.

The third Chapter deals with general diffusion-based models and shows that, even in the absence of an *Equivalent Local Martingale Measure*, the financial market may still be viable, in the sense that strong forms of arbitrage are ruled out. Relying partly on the recent literature, we provide necessary and sufficient conditions for market viability in terms of the *market price of risk* process and *martingale deflators*. Regardless of the existence of a martingale measure, we show that the financial market may still be complete and contingent claims can be valued under the original (*real-world*) probability measure, provided we use as numéraire the *Growth-Optimal Portfolio*.

Finally, the fourth Chapter deals with no-arbitrage conditions which are weaker than the classical *No Free Lunch with Vanishing Risk (NFLVR)* criterion, providing necessary and sufficient conditions for their validity in terms of the characteristics of the discounted price process. We study the stability of weak no-arbitrage conditions with respect to changes of numéraire, absolutely continuous changes of the reference probability measure and restrictions/enlargements of the reference filtration. In particular, we prove that weak no-arbitrage conditions, unlike the classical *No Arbitrage (NA)* and NFLVR criteria, are in general robust with respect to these changes. Finally, we provide a general characterization of attainable contingent claims and market completeness without relying on the NFLVR condition.

Sunto

Il primo Capitolo di questa Tesi contiene un approccio generale e astratto a problemi di ottimizzazione di portafoglio secondo un criterio media-varianza. In particolare, vengono studiati e risolti congiuntamente diversi problemi di ottimizzazione in media-varianza, assumendo unicamente una condizione minimale di non-arbitraggio. Le soluzioni ottime a tali problemi vengono descritte esplicitamente, senza alcuna ipotesi sulle caratteristiche del modello sottostante. Inoltre, vengono presentate generalizzazioni di risultati classici dell'economia finanziaria, come il teorema di separazione in due fondi, la frontiera efficiente media-varianza e una formula di tipo *CAPM*. Infine, i risultati generali ottenuti vengono applicati alla valutazione di strumenti finanziari.

Il secondo Capitolo è dedicato allo studio di un modello generale a forma ridotta per il rischio di credito, in cui il tempo di fallimento viene modellizzato come un tempo aleatorio doppiamente stocastico la cui intensità è funzione di un processo diffusivo di tipo affine. Si ottiene una caratterizzazione completa della famiglia di tutte le misure di probabilità localmente equivalenti che preservano la struttura affine del modello, formulando condizioni necessarie e sufficienti sul processo densità. L'utilità di questi risultati generali viene illustrata prima nel contesto di un modello a volatilità stocastica di Heston (1993) con l'aggiunta di un possibile fallimento e succesivamente nel contesto di un modello multi-fattoriale più generale che consente di modellizzare congiuntamente il rischio di credito e il rischio di mercato. Si considerano applicazioni di interesse per la valutazione di strumenti derivati come anche per il risk management.

Il terzo Capitolo è dedicato allo studio di modelli basati su processi diffusivi. In particolare, viene mostrato che, anche in assenza di una *Misura Martingala Locale Equivalente*, il mercato finanziario può essere privo di forme forti di arbitraggio. Basandoci in parte sulla letteratura recente, vengono fornite condizioni necessarie e sufficienti per l'assenza di forme forti di arbitraggio. Tali condizioni coinvolgono il *prezzo di mercato del rischio* e processi *martingale deflator*. Indipendentemente dall'esistenza di una misura martingala, si dimostra che il mercato finanziario può essere completo e strumenti derivati possono essere valutati rispetto alla misura di probabilità del *mondo reale*, utilizzando come numéraire il *Growth-Optimal Portfolio*.

Infine, il quarto Capitolo contiene uno studio delle condizioni di non-arbitraggio più deboli del classico criterio *No Free Lunch with Vanishing Risk (NFLVR)*. Vengono fornite condizioni necessarie e sufficienti per la validità di tali condizioni deboli di non-arbitraggio, espresse rispetto alle caratteristiche del processo che rappresenta il prezzo scontato degli asset. Viene anche studiata la stabilità delle condizioni deboli di non-arbitraggio rispetto a cambiamenti di numéraire, cambiamenti assolutamente continui della misura di probabilità di riferimento e restrizioni/allargamenti della filtrazione di riferimento. In particolare, si dimostra che le condizioni deboli di non-arbitraggio considerate nel presente lavoro godono di buone proprietà di stabilità, al contrario di quanto accade per le classiche condizioni di *Non Arbitraggio (NA)* e NFLVR. Infine, presentiamo una caratterizzazione generale dei titoli finanziari che possono essere replicati, dimostrando che il mercato finanziario può essere completo anche in assenza della condizione NFLVR.

Contents

Introduction **1**

1 An abstract and unifying approach to mean-variance optimization problems **5**
1.1 Introduction . . . 5
1.2 General setup and problem formulation . . . 6
1.3 Mathematical tools and general results . . . 8
1.4 Mean-variance problems and mean-variance efficiency . . . 15
1.5 Mean-variance indifference valuations . . . 21
1.6 Connections to the literature . . . 27

2 Measure changes for reduced-form affine credit risk models **31**
2.1 Introduction . . . 31
2.2 General setup and preliminaries . . . 34
2.3 The affine structure under a change of measure . . . 37
2.4 Application: the Heston with jump-to-default model . . . 49
2.4.1 Equivalent changes of measure . . . 50
2.4.2 Characterization of *Equivalent (Local) Martingale Measures* . . . 52
2.4.3 The Minimal Martingale Measure . . . 56
2.5 Application: a multifactor equity/credit risk model with stochastic volatility . . . 59
2.5.1 The modeling framework . . . 59
2.5.2 Risk management applications . . . 66
2.5.3 Risk-neutral measures and valuation of default-sensitive payoffs . . . 70
2.5.4 The incomplete information case . . . 80
2.5.5 Connections to the literature . . . 86
2.6 Conclusions and further developments . . . 87

3 Diffusion-based models for financial markets without martingale measures **89**
3.1 Introduction . . . 89
3.2 The general setting . . . 91
3.3 No-arbitrage conditions and the market price of risk . . . 93
3.4 The growth-optimal portfolio and the numéraire portfolio . . . 101
3.5 Replicating strategies and completeness of the financial market . . . 106

3.6 Contingent claim valuation without ELMMs . . . 109
3.6.1 Real-world pricing and the *benchmark approach* . . . 110
3.6.2 The *upper hedging price* approach . . . 111
3.6.3 Utility indifference valuation . . . 113
3.7 Conclusions, extensions and further developments . . . 117

4 Weak no-arbitrage conditions: characterization, stability and hedging problems 119
4.1 Introduction . . . 119
4.2 General setup and preliminaries . . . 122
4.3 Characterization of weak no-arbitrage conditions . . . 124
4.3.1 *No Unbounded Increasing Profit* . . . 124
4.3.2 *No Immediate Arbitrage Opportunity* . . . 128
4.3.3 *No Unbounded Profit with Bounded Risk* . . . 135
4.4 Stability properties of weak no-arbitrage conditions . . . 144
4.4.1 Changes of numéraire . . . 144
4.4.2 Absolutely continuous changes of measure . . . 150
4.4.3 Changes of the reference filtration . . . 155
4.5 General characterizations of hedgeable contingent claims . . . 170
4.5.1 Maximal elements . . . 171
4.5.2 Attainable contingent claims and market completeness . . . 174
4.6 Conclusions and further developments . . . 181
4.7 Appendix . . . 182

References 187

Introduction

As the title indicates, this Thesis consists of four essays dealing with some issues arising in mathematical finance. In the first chapter, we shall start with a general and abstract perspective on mean-variance portfolio optimization problems, which represent a classical, but still active, field of research in financial mathematics. Then, motivated by a popular approach to the modeling of credit risk which has emerged in the last decade, the second chapter deals with more theoretical questions concerning the stability of certain structural properties of a credit risk model under a change of measure. Finally, in the third and fourth chapters, we shall deal with some foundational issues which have recently attracted the attention of several researchers. More specifically, we shall be concerned with the analysis of financial market models which satisfy robust no-arbitrage conditions, which are in particular weaker than the classical no-arbitrage criteria usually adopted in the literature. We now present a brief description of the contents of the four chapters of the Thesis, referring to the introductory sections of each chapter for a more thorough discussion as well as for more references to the related literature.

Chapter 1, which is based on Fontana & Schweizer (2011), deals with classical mean-variance portfolio selection problems. We choose to work in a very general and abstract setting, similar to that originally introduced in Schweizer (1997),(2001a), which allows us to obtain model-independent results under the minimal no-arbitrage condition of *no approximate riskless profits in* L^2. By relying on projection techniques in Hilbert spaces, we provide general characterizations of the solutions to abstract versions of the classical Markowitz portfolio selection problems as well as of mean-variance hedging and utility maximization problems. In particular, our abstract framework allows to treat different mean-variance problems in a unified manner, showing in a clear way how the corresponding optimal solutions are related. Generalizing the classical two-fund-separation theorem, we show that the optimal solutions to all mean-variance problems can be decomposed into the sum of a fixed *minimum variance element* and a multiple of an additional fixed element given by the best L^2-approximation of the riskless payoff 1 onto the set of all cumulated (undiscounted) gains from trade. Furthermore, only the amount invested into the latter depends on the specific problem under consideration. We also derive abstract generalizations of classical results from financial economics such as a characterization of the mean-variance efficient frontier, a *CAPM*-type formula and a general solution to the problem of maximizing the Sharpe ratio. Finally, we also consider the problem of valuing square-integrable contingent claims according to several mean-variance indifference valuation rules, for which our abstract mean-variance theory yields general descriptions of the corresponding indifference values, thus extending some results of Mercurio

(2001), Møller (2001), Schweizer (2001a) and Sun & Wang (2005),(2006). In particular, under the simplifying assumption that a zero-coupon bond can be (approximately) attained in the abstract financial market, all mean-variance indifference valuation formulae admit a very natural economic interpretation. Related results can also be found in Chapter 1 of Fontana (2010b).

Chapter 2 starts by considering a general *reduced-form* credit risk model, where the default time is modeled as a doubly stochastic random time with default intensity driven by an affine diffusion process. In the first part of the Chapter, we characterize the family of all locally equivalent probability measures which preserve the affine structure of the model by giving necessary and sufficient conditions on their density process. In particular, this allows for a rigorous treatment of diffusive and jump-type risk premia and shows that the affine structure is preserved under rather general risk-premia specifications, which nest most of the specifications usually adopted in the literature. As an application, we consider a jump-to-default extension of the popular Heston (1993) stochastic volatility model, giving a complete description of the family of all risk-neutral measures which preserve the structure of the model, thus extending and sharpening the results of Wong & Heyde (2006). Always in the context of the Heston with jump-to-default model, our general results allow us to easily answer the question of whether discounted asset prices are true martingales or only local martingales under a given risk-neutral measure and the question of whether the so-called *Minimal Martingale Measure* exists. Finally, we illustrate the usefulness of our results on affine-preserving measure changes in the context of a hybrid equity/credit risk model, as considered in several recent papers (see e.g. Carr & Linetsky (2006), Campi et al. (2009), Carr & Wu (2010) and Cheridito & Wugalter (2011)). We propose a class of multifactor models, allowing for both stochastic volatility and stochastic default intensity. In particular, we ensure that the affine structure of the model is preserved under both the physical and the risk-neutral probability measure thus enabling us to explicitly compute several key quantities of interest for risk management as well as pricing applications. Finally, we shall also briefly consider the incomplete information case, where some components of the underlying driving process are not perfectly observable to market participants. As in Fontana (2010a) and Fontana & Runggaldier (2010), this will lead to the formulation of a suitable filtering problem.

In Chapter 3, which is based on Fontana & Runggaldier (2011), we consider a general class of diffusion-based models and show that, even in the absence of an *Equivalent Local Martingale Measure (ELMM)*, the financial market may still be viable, in the sense that strong forms of arbitrage are excluded and portfolio optimization problems can be meaningfully solved. Relying partly on the recent literature (see e.g. Karatzas & Kardaras (2007), Hulley & Schweizer (2010) and Kardaras (2010a)), we provide necessary and sufficient conditions for market viability in terms of the *market price of risk* process and the existence of a *martingale deflator*. In particular, a martingale deflator can be considered as a weaker counterpart to the density process of a traditional ELMM and inherits most, but not all, of the useful properties of the latter. We then explicitly compute the *growth-optimal portfolio (GOP)*, which is also shown to possess the *numéraire* property and to coincide with the reciprocal of a martingale deflator. Regardless of the existence of a well-defined ELMM, we show that the financial market may still be complete and contingent claims can be valued under the original (*real-world*) probability measure. In particular, we discuss three different but related valuation approaches: *real-world pricing*, *upper-hedging pricing* and *utility indifference pricing*. In the special case of a complete financial market, we show that these three valuation rules yield the same valuation formula, which amounts to taking the expectation (under the real-world probability measure) of the GOP-discounted payoff. We also discuss relations with *Stochastic Portfolio Theory* and with the *Benchmark Approach*, where financial market models not admitting an ELMM are typically encountered, see e.g. Fernholz & Karatzas (2009), Platen (2006),(2009) and Platen & Heath (2006).

Chapter 4 is concerned with the analysis of no-arbitrage conditions which are weaker than the classical *No Arbitrage (NA)* and *No Free Lunch with Vanishing Risk (NFLVR)* conditions considered in the seminal work of Delbaen & Schachermayer (1994). More specifically, we shall analyze the *No Unbounded Increasing Profit (NUIP)*, the *No Immediate Arbitrage Opportunity (NIAO)* and the *No Unbounded Profit with Bounded Risk (NUPBR)* conditions. We provide necessary and sufficient conditions for the validity of NUIP/NIAO/NUPBR in terms of the characteristics of the discounted price process of the risky assets. In particular, this allows us to generalize the results of Chapter 3 to more general financial market models based on continuous semimartingales. We then study the stability properties of the NUIP/NIAO/NUPBR no-arbitrage conditions with respect to several modifications of the structure of the underlying financial market model. More precisely, we analyze the impact of changes of numéraire, of absolutely continuous changes of the reference probability measure and of restrictions/enlargements of the reference filtration. The main message is that the weak NUIP/NIAO/NUPBR no-arbitrage conditions possess stronger stability properties than the classical NA and NFLVR criteria, thus confirming the economic soundness of the weak no-arbitrage conditions considered in the present work. Finally, assuming that the NUPBR condition holds, we provide an abstract and general characterization of attainable contingent claims, without relying on the full strength of the classical NFLVR condition. In particular, we obtain natural generalizations of the classical results of Ansel & Stricker (1994) and Delbaen & Schachermayer (1995c) on the attainability of contingent claims, with martingale deflators replacing the traditional ELMMs. Finally, we generalize the second fundamental theorem of asset pricing to the situation where the NFLVR condition fails to hold.

Chapter 1

An abstract and unifying approach to mean-variance optimization problems

1.1 Introduction

Mean-variance portfolio optimisation is one of the classical problems in financial economics. Many papers have been written on the subject, and many different settings and versions have been studied. So what is there left to be said?

We offer in this paper a new perspective that allows to treat mean-variance portfolio problems in a simple and yet general way. Our approach does not depend on any particular model and uses only simple mathematics. The key for this is a change of parametrisation.

Consider one standard formulation of the classical Markowitz problem (there are other versions and we discuss them all in the paper): Given a financial market, find a portfolio with maximal return (mean) given a constraint on its risk (variance). The familiar mathematical description is to search for a (self-financing) strategy ϑ whose resulting gains from trade $G_T(\vartheta)$ maximise $E[G_T(\vartheta)]$ over all allowed $\vartheta \in \Theta$ subject to $\mathrm{Var}[G_T(\vartheta)] \leq \sigma^2$ for some constant $\sigma^2 > 0$. The control variable is the strategy ϑ. In a one-period model with returns given by an $\mathbb{R}^d$-valued random variable ΔS, a strategy is simply a constant vector $\vartheta \in \mathbb{R}^d$, and trading gains are the scalar product $\vartheta^\top \Delta S$. In a continuous-time model with asset prices described by an $\mathbb{R}^d$-valued semimartingale $(S_t)_{0 \leq t \leq T}$, a strategy is an $\mathbb{R}^d$-valued predictable S-integrable process (satisfying some technical conditions), and $G_T(\vartheta)$ is given by the real-valued stochastic integral $\int_0^T \vartheta_u \, dS_u$.

The very simple idea of our approach is that we need not look at S and ϑ separately — all that matters for our problem is $G_T(\vartheta)$. Since this depends linearly on ϑ, the set of all possible gains from trade in a frictionless financial market is simply a linear space. (Of course, frictions or transaction costs will complicate this; but then we already leave the classical setting.) Moreover, that space $\mathcal{G}$ of gains from trade g should be a subset of $L^2(P)$ since our problem formulation involves mean and variance. In other words, we no longer look at trading *strategies* as control variables, but only at the resulting final *positions*. It turns out that this change of parametrisation from Θ to $\mathcal{G}$ makes everything very simple and tractable.

Of course, this idea is not completely new. It has been used (and, to the best of our knowledge, introduced) in Schweizer (1997) and has been picked up by other authors more recently. We give a detailed discussion of related literature in Section 1.6. However, the systematic exploitation for a whole range of four mean-variance optimisation criteria seems to be missing so far. We explicitly work out the connections between the four solutions, and we do all this carefully in an undiscounted framework. Our results include explicit formulas, two-fund separation results, a CAPM-type relation, and explicit indifference valuation rules.

The paper is structured as follows. Section 1.2 presents the general setup and the formulation of our four mean-variance optimisation problems. Section 1.3 contains the mathematics — it solves the four problems explicitly and provides a number of connections between their solutions. We even do this more generally than discussed above, by replacing g with $g - Y$ for some exogenous extra financial position. Section 1.4 starts on the financial economics; it determines the mean-variance efficient frontier, presents a CAPM-type relation, and derives two different but related two-fund separation results. Section 1.5 introduces mean-variance indifference valuation. Because we can compute the values of our optimisation problems explicitly, we can also explicitly obtain, for a suitable chosen Y, the value (financial amount) h at which an agent is indifferent, under a mean-variance criterion and at optimal investment, between either selling a contingent claim H for a compensation of h or not selling H and not getting extra money. Finally, Section 1.6 contains a detailed discussion of related work in the literature.

1.2 General setup and problem formulation

This section describes the abstract financial framework and introduces the main mean-variance portfolio optimisation problems we are interested in. For a given probability space $(\Omega, \mathcal{F}, P)$, denote by $L^2 := L^2(\Omega, \mathcal{F}, P)$ the space of all real-valued square-integrable random variables with the usual scalar product $(X, Y) = E[XY]$ and norm $\|X\|_{L^2} = (E[X^2])^{1/2}$. Let $\mathcal{G}$ be a given nonempty subset of L^2, denote by $\mathcal{G}^\perp := \{X \in L^2 \mid (X, Y) = 0 \text{ for all } Y \in \mathcal{G}\}$ its orthogonal complement in L^2, and write $\overline{\mathcal{G}}$ for its closure in L^2. Finally, let B be a real-valued random variable in L^2 such that $B > 0$ P-a.s.

The financial interpretation is as follows. Think of a time horizon $T \in (0, \infty)$ and let $t = 0$ be the initial time. Then $\mathcal{G}$ represents the set of all undiscounted cumulated gains from trade (evaluated at time T) generated by suitable self-financing trading strategies starting at $t = 0$ from zero initial capital. The element B represents the strictly positive value (at the final time T) of a *numéraire* asset and can, but need not, be interpreted as the final value of a *savings account*. We avoid calling B a “riskless” asset; in fact, investing one unit of money in this asset only guarantees that we end up at T with the strictly positive amount B, which is however random and can be strictly less than the initial investment of 1. The set $\{cB + g \mid c \in \mathbb{R}, g \in \mathcal{G}\} = \mathbb{R}B + \mathcal{G}$ then represents the set of all *attainable* undiscounted final wealths, i.e. all those square-integrable payoffs or contingent claims which can be replicated in the abstract financial market $(B, \mathcal{G})$ by following a self-financing strategy starting from some initial capital c. Note that we do not assume that $\mathcal{G}$ is closed in L^2. Square-integrability is imposed to ensure existence of means and variances, which is a basic necessary assumption when dealing with mean-variance problems. Finally, the Hilbert space structure of L^2 allows an easy and efficient derivation of general solutions to several mean-variance problems, as will be shown in Section 1.3.

Remark 1.2.1. It is worth emphasising that apart from the obvious requirement of square-integrability, the present setup for an abstract financial market does not rely on any underlying modelling structure. As a consequence, all the results we are going to present are model-independent, and in particular hold for both discrete- and continuous-time models. We refer the reader to Examples 1–3 in Schweizer (1999) for an illustration of how typical financial models can be embedded into the present abstract setting.

Let us now introduce a basic **standing assumption** for the rest of the paper.

Assumption I. The two following conditions hold:

(a) $\mathcal{G}$ is a linear subspace of L^2.

(b) There are *no approximate riskless profits in* L^2, meaning that $\overline{\mathcal{G}}$ does not contain 1.

Intuitively, part (a) of Assumption I amounts to considering a frictionless financial market without constraints or other restrictions on trading. The condition $1 \notin \overline{\mathcal{G}}$ of no approximate riskless profits in L^2 in part (b) represents an abstract and minimal no-arbitrage condition. It can be equivalently formulated as $\mathbb{R} \cap \overline{\mathcal{G}} = \{0\}$, and this amounts to excluding the undesirable situation where an agent is able to reach, or approximate in the L^2-sense, a deterministic riskless final wealth from zero initial capital. As will be shown in the next section, the condition of no approximate riskless profits in L^2 is necessary and sufficient for the solvability of the quadratic problems we are now going to introduce.

In the present paper, we shall be mainly concerned with four major mean-variance portfolio optimisation problems, denoted as Problems (A)–(D) and formulated in the following abstract terms. We let $Y \in L^2$ represent the final undiscounted value of a generic financial position/liability, $\alpha \in (0,\infty)$ a given risk-aversion coefficient, $\mu \in \mathbb{R}$ a target minimal expected value and $\sigma^2 \in (0,\infty)$ a target maximal variance. Then we consider

Problem (A') $$E[g-Y] - \alpha \operatorname{Var}[g-Y] = \max_{g\in\mathcal{G}}!$$

Problem (B') $$\operatorname{Var}[g-Y] = \min! \qquad \text{over all } g \in \mathcal{G} \text{ such that } E[g-Y] \geq \mu.$$

Problem (C') $$E[g-Y] = \max! \qquad \text{over all } g \in \mathcal{G} \text{ such that } \operatorname{Var}[g-Y] \leq \sigma^2.$$

Problem (D') $$\|Y-g\|_{L^2} = \min_{g\in\mathcal{G}}!$$

We shall argue below that each Problem (X') has the same optimal value as the corresponding Problem (X) where we optimise over $\overline{\mathcal{G}}$ instead of $\mathcal{G}$.

The financial interpretations of Problems (A')–(D') are rather obvious. In fact, (A') describes the portfolio optimisation problem faced by an agent with mean-variance preferences and risk-aversion coefficient α. (B') and (C') are abstract versions of the classical Markowitz portfolio selection problems, slightly extended by including the random liability Y. More specifically, in (B'), the agent is interested in minimising the variance of her/his final net position, given a minimal target level μ for its expected value. Symmetrically, in (C'), the agent wants to maximise the expected value of her/his final net position, given a maximal target level σ^2 for its variance. Finally, (D') consists of finding the optimal quadratic hedge for Y. As we illustrate at the end of Section 1.4, different investment situations can be represented via suitable choices of Y.

Remark 1.2.2. Part (b) of Assumption I excludes the case $1 \in \overline{\mathcal{G}}$, but not the case $1 \in \mathcal{G}^\perp$. However, the latter situation is neither particularly interesting from a mathematical point of view nor particularly realistic from an economic point of view. In fact, $1 \in \mathcal{G}^\perp$ mathematically means that $E[g] = (g, 1) = 0$ for all $g \in \mathcal{G}$. But then there is nothing to optimise in (C'), and the constraint in (B') is trivially always or never satisfied, depending on whether $E[Y] \leq -\mu$ or $E[Y] > -\mu$. Finally, (A)' reduces to the simpler problem of minimising the variance. In financial terms, the case $1 \in \mathcal{G}^\perp$ corresponds to the situation where all undiscounted cumulated gains have zero expectation under the original (real-world) probability measure P. In this case, there is no proper notion of a trade-off between risk (variance) and return (expected value), and so we cannot meaningfully consider mean-variance portfolio optimisation problems.

Due to Remark 1.2.2, there is no loss of generality in introducing the following additional **standing assumption** for the sequel.

Assumption II. $1 \notin \mathcal{G}^\perp$, or equivalently $\{g \in \mathcal{G} \mid E[g] \neq 0\} \neq \emptyset$.

1.3 Mathematical tools and general results

This section contains the mathematical ingredients for solving Problems (A)–(D). The Hilbert space structure of our framework makes the results both general and easy to obtain. We postpone to later sections all pertinent economic considerations and applications to financial problems.

Recall that the orthogonal complement $\mathcal{G}^\perp$ is a closed linear subspace of L^2, and denote by π the orthogonal projection in L^2 on $\mathcal{G}^\perp$. Since $\mathcal{G}$ is a linear subspace of L^2 by part (a) of Assumption I, we have $(\mathcal{G}^\perp)^\perp = \overline{\mathcal{G}}$. This yields the direct sum decomposition $L^2 = \overline{\mathcal{G}} \oplus \mathcal{G}^\perp$, meaning that any $Y \in L^2$ can be uniquely decomposed as

$$Y = g^Y + N^Y = g^Y + \pi(Y) \qquad \text{with } g^Y \in \overline{\mathcal{G}} \text{ and } N^Y = \pi(Y) \in \mathcal{G}^\perp. \tag{1.1}$$

Using this basic orthogonal decomposition, we can already tackle Problem (D'). Note first that

$$\inf_{g \in \mathcal{G}} \|Y - g\|_{L^2} = \inf_{g \in \overline{\mathcal{G}}} \|Y - g\|_{L^2}. \tag{1.2}$$

In fact, "$\geq$" is clear from $\mathcal{G} \subseteq \overline{\mathcal{G}}$, and conversely, any $\bar{g} \in \overline{\mathcal{G}}$ admits a sequence $(g_n)_{n \in \mathbb{N}} \subseteq \mathcal{G}$ converging in L^2 to $\bar{g}$. So

$$\inf_{g \in \mathcal{G}} \|Y - g\|_{L^2} \leq \|Y - g_n\|_{L^2} \longrightarrow \|Y - \bar{g}\|_{L^2} \qquad \text{as } n \to \infty,$$

and as $\bar{g} \in \overline{\mathcal{G}}$ is arbitrary, we also get "$\leq$" in (1.2). In other words, the optimal value of Problem (D') does not change if we replace $\mathcal{G}$ by its closure $\overline{\mathcal{G}}$ in L^2. The projection theorem then gives

$$\inf_{g\in\mathcal{G}} \|Y-g\|_{L^2} = \inf_{g\in\overline{\mathcal{G}}} \|Y-g\|_{L^2} = \min_{g\in\overline{\mathcal{G}}} \|Y-g\|_{L^2} = \|Y-g^Y\|_{L^2} = \|N^Y\|_{L^2}. \tag{1.3}$$

Optimising over the closed subspace $\overline{\mathcal{G}}$ ensures existence and uniqueness for the solution to Problem (D), where $\overline{\mathcal{G}}$ replaces $\mathcal{G}$ in (D'), and the solution is the projection in L^2 of Y on $\overline{\mathcal{G}}$,

$$g^Y = \operatorname*{argmin}_{g\in\overline{\mathcal{G}}} \|Y-g\|_{L^2}. \tag{1.4}$$

Remark 1.3.1. Also for Problems (A')–(C'), the optimal values do not depend on whether we optimise over $\mathcal{G}$ or $\overline{\mathcal{G}}$. This is easily checked by the same arguments as for (1.2), using that $g_n \to g$ in L^2 implies $E[g_n - Y] \to E[g-Y]$ and $\mathrm{Var}[g_n - Y] \to \mathrm{Var}[g-Y]$, for any $Y \in L^2$.

In view of Remark 1.3.1, we henceforth consider the Problems (A)–(D) instead of (A')–(D'), where the optimisation now goes over the closed linear subspace $\overline{\mathcal{G}}$ instead of $\mathcal{G}$. As a preliminary to the derivation of the solutions to Problems (A)–(C), let us introduce the following variance-minimisation problem.

Problem (MV) $$\mathrm{Var}[Y-g] = \min_{g\in\overline{\mathcal{G}}}!$$

The solution to Problem (MV) is given in the following result and denoted by g^Y_{mv}, where the subscript "mv" stands for "minimum variance". It is obtained by relying on the solution to Problem (D) we have just derived in (1.4).

Proposition 1.3.2. *For $Y \in L^2$, Problem (MV) admits in $\overline{\mathcal{G}}$ the unique solution*

$$g^Y_{\mathrm{mv}} := \operatorname*{argmin}_{g\in\overline{\mathcal{G}}} \mathrm{Var}[Y-g] = g^Y - a^*_Y\big(1-\pi(1)\big), \qquad \textit{where } a^*_Y := \frac{E[N^Y]}{E[\pi(1)]}. \tag{1.5}$$

Proof. The key idea allowing us to reduce Problem (MV) to Problem (D) is the simple fact that

$$\mathrm{Var}[Y-g] = \min_{a\in\mathbb{R}} E[(Y-g-a)^2].$$

Hence we can write

$$\min_{g\in\overline{\mathcal{G}}} \mathrm{Var}[Y-g] = \min_{g\in\overline{\mathcal{G}}} \min_{a\in\mathbb{R}} E[(Y-a-g)^2] = \min_{a\in\mathbb{R}} \min_{g\in\overline{\mathcal{G}}} E[(Y-a-g)^2].$$

The inner minimisation over $\overline{\mathcal{G}}$ corresponds to Problem (D) for $Y-a$ and is solved by g^{Y-a}. By linearity of the projection and (1.1), we have $g^{Y-a} = g^Y - ag^1 = g^Y - a(1-\pi(1))$ and so

$$\min_{g\in\overline{\mathcal{G}}} \mathrm{Var}[Y-g] = \min_{a\in\mathbb{R}} E[(Y-a-g^{Y-a})^2] = \min_{a\in\mathbb{R}} E\big[\big(N^Y - a\pi(1)\big)^2\big]. \tag{1.6}$$

Now observe that because $1-\pi(1)$ is in $\overline{\mathcal{G}}$, we have $(N^Y, 1-\pi(1)) = 0$ and $(\pi(1), 1-\pi(1)) = 0$. This gives $E[N^Y \pi(1)] = E[N^Y]$ and $E[\pi(1)] = E[(\pi(1))^2] = \|\pi(1)\|^2_{L^2} > 0$ since $1 \notin \overline{\mathcal{G}}$ by part (b) of Assumption I. Squaring out and completing the square therefore yields

$$E\big[\big(N^Y - a\pi(1)\big)^2\big] = E[\pi(1)]\left(a - \frac{E[N^Y]}{E[\pi(1)]}\right)^2 - \frac{(E[N^Y])^2}{E[\pi(1)]} + E[(N^Y)^2]. \tag{1.7}$$

So the optimal $a \in \mathbb{R}$ is uniquely given by

$$a^*_Y := \operatorname*{argmin}_{a\in\mathbb{R}} E\big[\big(N^Y - a\pi(1)\big)^2\big] = \frac{E[N^Y]}{E[\pi(1)]},$$

and we obtain

$$g^Y_{\text{mv}} := \operatorname*{argmin}_{g\in\overline{\mathcal{G}}} \operatorname{Var}[Y-g] = g^{Y-a^*_Y} = g^Y - a^*_Y\big(1-\pi(1)\big).$$

The uniqueness of the solution $g^Y_{\text{mv}} \in \overline{\mathcal{G}}$ follows from the projection theorem via the uniqueness of $g^{Y-a} \in \overline{\mathcal{G}}$ for all $a \in \mathbb{R}$. □

Let us now introduce the notation $R^Y_{\text{mv}} := g^Y_{\text{mv}} - Y$, where "$R$" stands for the final "result" of an abstract financial position. Then, for any $g \in \overline{\mathcal{G}}$, we can write

$$g - Y = R^Y_{\text{mv}} + g - g^Y_{\text{mv}}$$

and hence

$$E[g-Y] = E[R^Y_{\text{mv}}] + E[g - g^Y_{\text{mv}}]. \tag{1.8}$$

Furthermore, due to the optimality of $g^Y_{\text{mv}} \in \overline{\mathcal{G}}$ and the linearity of $\overline{\mathcal{G}}$, the first order condition for (MV) gives for the element R^Y_{mv} the fundamental zero-covariance property

$$\operatorname{Cov}(R^Y_{\text{mv}}, g) = 0 \qquad \text{for all } g \in \overline{\mathcal{G}}. \tag{1.9}$$

Since $g - g^Y_{\text{mv}} \in \overline{\mathcal{G}}$ for any $g \in \overline{\mathcal{G}}$, this implies that we have

$$\operatorname{Var}[g-Y] = \operatorname{Var}[R^Y_{\text{mv}} + g - g^Y_{\text{mv}}] = \operatorname{Var}[R^Y_{\text{mv}}] + \operatorname{Var}[g - g^Y_{\text{mv}}]. \tag{1.10}$$

Equations (1.8) and (1.10) show that in Problems (A)–(C), we can isolate the part coming from the minimum variance element R^Y_{mv}. Furthermore, since $g^Y_{\text{mv}} \in \overline{\mathcal{G}}$ and $\overline{\mathcal{G}}$ is a linear space, the mapping $g \mapsto g' := g - g^Y_{\text{mv}}$ is a bijection of $\overline{\mathcal{G}}$ to itself. These observations suggest that we can reduce the general versions of our abstract mean-variance problems to the particular case $Y \equiv 0$. This will be exploited in the proofs of the three following propositions.

Remark 1.3.3. For future use in later sections, we compute the mean and variance of the optimal position $R^Y_{\text{mv}} = g^Y_{\text{mv}} - Y = -N^Y - a^*_Y(1-\pi(1))$. From the expression for a^*_Y in (1.5), we get

$$E[R^Y_{\text{mv}}] = -\frac{E[N^Y]}{E[\pi(1)]}, \tag{1.11}$$

and using (1.6) and (1.7) yields

$$\operatorname{Var}[R^Y_{\text{mv}}] = E\big[(N^Y)^2\big] - \frac{(E[N^Y])^2}{E[\pi(1)]}. \tag{1.12}$$

We start with the solution to Problem (A), denoted by $g^Y_{\text{opt},A}(\gamma)$, where $\gamma := \frac{1}{\alpha}$ is the risk-tolerance coefficient corresponding to the risk-aversion coefficient α. We shall comment on the case $\gamma = 0$ below.

Proposition 1.3.4. *For $Y \in L^2$ and $\gamma \in [0, \infty)$, Problem (A) has a unique solution $g^Y_{\text{opt},A}(\gamma) \in \overline{\mathcal{G}}$. It is explicitly given by*

$$g^Y_{\text{opt},A}(\gamma) = \operatorname*{argmin}_{g \in \overline{\mathcal{G}}} \{\text{Var}[g - Y] - \gamma E[g - Y]\} = g^Y_{\text{mv}} + g^0_{\text{opt},A}(\gamma), \tag{1.13}$$

where $g^0_{\text{opt},A}(\gamma) \in \overline{\mathcal{G}}$ is the solution to Problem (A) for $Y \equiv 0$, explicitly given by

$$g^0_{\text{opt},A}(\gamma) = \operatorname*{argmin}_{g \in \overline{\mathcal{G}}} \{\text{Var}[g] - \gamma E[g]\} = \frac{\gamma}{2} \frac{1}{E[\pi(1)]} \big(1 - \pi(1)\big). \tag{1.14}$$

Proof. Notice first that with $\gamma = \frac{1}{\alpha}$, Problem (A) can be equivalently formulated as

$$\text{Var}[g - Y] - \gamma E[g - Y] = \min_{g \in \overline{\mathcal{G}}}!$$

Moreover, equations (1.8) and (1.10) allow us to write, for any $g \in \overline{\mathcal{G}}$,

$$\text{Var}[g - Y] - \gamma E[g - Y] = \text{Var}[R^Y_{\text{mv}}] - \gamma E[R^Y_{\text{mv}}] + \text{Var}[g - g^Y_{\text{mv}}] - \gamma E[g - g^Y_{\text{mv}}].$$

Since $\overline{\mathcal{G}}$ is linear and contains g^Y_{mv}, Problem (A) can thus be reduced to the basic problem

$$\text{Var}[g] - \gamma E[g] = \min_{g \in \overline{\mathcal{G}}}! \tag{1.15}$$

More precisely, if $g^0_{\text{opt},A}(\gamma) \in \overline{\mathcal{G}}$ denotes the solution to (1.15), then the solution $g^Y_{\text{opt},A}(\gamma) \in \overline{\mathcal{G}}$ to Problem (A) in its original formulation is given by (1.13). Hence it only remains to solve (1.15). Following the same idea as in the proof of Proposition 1.3.2, we write

$$\text{Var}[g] - \gamma E[g] = \min_{a \in \mathbb{R}} E[(g - a)^2] - \gamma E[g] = \min_{a \in \mathbb{R}} \Big(E\Big[\big(g - (a + \tfrac{\gamma}{2})\big)^2\Big] - \tfrac{\gamma^2}{4} - a\gamma \Big). \tag{1.16}$$

But for $Y \equiv a + \frac{\gamma}{2}$, the solution of Problem (D) is by (1.4) and linearity of the projection

$$g^{a + \frac{\gamma}{2}} = (a + \tfrac{\gamma}{2}) g^1 = (a + \tfrac{\gamma}{2})\big(1 - \pi(1)\big). \tag{1.17}$$

Combining this with (1.16) and completing the square gives

$$\begin{aligned}
\min_{g \in \overline{\mathcal{G}}} \{\text{Var}[g] - \gamma E[g]\} &= \min_{a \in \mathbb{R}} \Big\{ \min_{g \in \overline{\mathcal{G}}} E\Big[\big(g - (a + \tfrac{\gamma}{2})\big)^2\Big] - \tfrac{\gamma^2}{4} - a\gamma \Big\} \\
&= \min_{a \in \mathbb{R}} \Big\{ E\Big[\big((a + \tfrac{\gamma}{2})\pi(1)\big)^2\Big] - \tfrac{\gamma^2}{4} - a\gamma \Big\} \\
&= \min_{a \in \mathbb{R}} E[\pi(1)] \Big(a - \tfrac{\gamma}{2} \tfrac{1 - E[\pi(1)]}{E[\pi(1)]} \Big)^2 - \tfrac{\gamma^2}{4} \tfrac{E[1 - \pi(1)]}{E[\pi(1)]}.
\end{aligned} \tag{1.18}$$

Note that as in the proof of Proposition 1.3.2, Assumption I gives $E[\pi(1)] > 0$. The last expression in (1.18) is clearly minimised by the unique value

$$a^*_\gamma := \frac{\gamma}{2}\frac{E[1-\pi(1)]}{E[\pi(1)]},$$

and together with (1.17), this yields

$$g^0_{\mathrm{opt},A}(\gamma) = \operatorname*{argmin}_{g\in\overline{\mathcal{G}}}\{\mathrm{Var}[g] - \gamma E[g]\} = g^{a^*_\gamma+\frac{\gamma}{2}} = \frac{\gamma}{2}\frac{1}{E[\pi(1)]}\big(1-\pi(1)\big).$$

The uniqueness of the solution again follows from the projection theorem via the uniqueness of $g^{a+\frac{\gamma}{2}} \in \overline{\mathcal{G}}$ for all $a \in \mathbb{R}$ and $\gamma \in [0,\infty)$. □

Remark 1.3.5.

1. The proofs of Propositions 1.3.2 and 1.3.4 both rely on the elementary identity $\mathrm{Var}[X] = \min_{a\in\mathbb{R}} E[(X-a)^2]$ for $X \in L^2$. This allows us to reduce variance-minimisation problems to particular cases of Problem (D).

2. The above trick of expressing the variance as an optimal value for a minimisation problem over $\mathbb{R}$ is also at the root of the appearance of the quantity $1-\pi(1)$; in fact, this is simply the projection in L^2 of the constant $1 \in \mathbb{R}$ on $\overline{\mathcal{G}}$.

3. It is interesting to notice that the solution to Problem (MV) can be recovered from the solution to Problem (A). In fact, by letting $\gamma = 0$, Proposition 1.3.4 implies that

$$g^Y_{\mathrm{opt},A}(0) = \operatorname*{argmin}_{g\in\overline{\mathcal{G}}} \mathrm{Var}[g-Y] = g^Y_{\mathrm{mv}}.$$

   This simple relation is in line with intuition, because $\gamma = 0$ corresponds to infinite risk-aversion ($\alpha = \infty$), which means in (A) that one is only interested in minimising the risk.

4. As seen in the proof of Proposition 1.3.4, the condition $1 \notin \overline{\mathcal{G}}$ of no approximate riskless profits in L^2 is *sufficient* for ensuring the existence of a unique solution to Problem (A). But this condition is also *necessary* for the solvability of (A). In fact, suppose to the contrary that $\tilde{g} \in \overline{\mathcal{G}}$ solves Problem (A), but $1 \in \overline{\mathcal{G}}$. Then $g' := \tilde{g}+1 \in \overline{\mathcal{G}}$ satisfies $\mathrm{Var}[g'-Y] = \mathrm{Var}[\tilde{g}-Y]$ and $E[g'-Y] = 1 + E[\tilde{g}-Y] > E[\tilde{g}-Y]$, contradicting the optimality of $\tilde{g}$.

5. One can prove the uniqueness of the solution to (A) directly by using only its optimality. But the argument above via the projection theorem leads to a more compact proof.

The results obtained so far do not rely on Assumption II that $1 \notin \mathcal{G}^\perp$. It is easy to see from Proposition 1.3.4 that in the case $1 \in \mathcal{G}^\perp$, the solutions to Problem (MV) and Problem (A) coincide, since $1 \in \mathcal{G}^\perp$ implies $\pi(1) \equiv 1$. But for tackling Problems (B) and (C), we shall exploit Assumption II. The basic idea is well known; it is folklore that the solutions to (B) and (C) are obtained by choosing for the risk-aversion α in (A) a particular value, depending on the respective constraint in

(B) or (C). In more detail, this goes as follows. In analogy to $R^Y_{\text{mv}} = g^Y_{\text{mv}} - Y$, we first introduce the notation

$$R^Y_{\text{opt},A}(\gamma) := g^Y_{\text{opt},A}(\gamma) - Y = R^Y_{\text{mv}} + g^0_{\text{opt},A}(\gamma) = R^Y_{\text{mv}} + \frac{\gamma}{2}\frac{1}{E[\pi(1)]}\big(1-\pi(1)\big).$$

Using

$$\text{Var}[1-\pi(1)] = \text{Var}[\pi(1)] = E\Big[\big(\pi(1)\big)^2\Big] - (E[\pi(1)])^2 = E[\pi(1)](1-E[\pi(1)]) \tag{1.19}$$

and recalling from (1.9) the zero-covariance property of R^Y_{mv}, we then obtain

$$E\big[R^Y_{\text{opt},A}(\gamma)\big] = E[R^Y_{\text{mv}}] + \frac{\gamma}{2}\frac{E[1-\pi(1)]}{E[\pi(1)]}, \tag{1.20}$$

$$\text{Var}\left[R^Y_{\text{opt},A}(\gamma)\right] = \text{Var}[R^Y_{\text{mv}}] + \frac{\gamma^2}{4}\frac{E[1-\pi(1)]}{E[\pi(1)]}. \tag{1.21}$$

So for $1 \in \mathcal{G}^\perp$, we obtain $E[R^Y_{\text{opt},A}(\gamma)] = E[R^Y_{\text{mv}}]$ and $\text{Var}[R^Y_{\text{opt},A}(\gamma)] = \text{Var}[R^Y_{\text{mv}}]$ for all $\gamma \in [0,\infty)$. But if Assumption II holds, then we have $E[1-\pi(1)] = \|1-\pi(1)\|^2_{L^2} > 0$, and therefore the functions $\gamma \mapsto E[R^Y_{\text{opt},A}(\gamma)]$ from $[0,\infty)$ to $[E[R^Y_{\text{mv}}],\infty)$ and $\gamma \mapsto \text{Var}[R^Y_{\text{opt},A}(\gamma)]$ from $[0,\infty)$ to $[\text{Var}[R^Y_{\text{mv}}],\infty)$ are both surjective. This implies that for any $\mu \in [E[R^Y_{\text{mv}}],\infty)$, there exists $\gamma_\mu \in [0,\infty)$ such that $E[R^Y_{\text{opt},A}(\gamma_\mu)] = \mu$, and analogously, any $\sigma^2 \in [\text{Var}[R^Y_{\text{mv}}],\infty)$ admits some $\gamma_{\sigma^2} \in [0,\infty)$ such that $\text{Var}[R^Y_{\text{opt},A}(\gamma_{\sigma^2})] = \sigma^2$. Under the (standing) Assumptions I and II, this simple observation allows us to derive the solutions to Problems (B) and (C) from the solution to Problem (A), as shown in the next two results.

Proposition 1.3.6. *Let $Y \in L^2$ and $\mu \in \mathbb{R}$. If $\mu > E[R^Y_{\text{mv}}]$, then Problem (B) admits a unique solution $g^Y_{\text{opt},B}(\mu) \in \overline{\mathcal{G}}$. It is explicitly given by*

$$g^Y_{\text{opt},B}(\mu) = g^Y_{\text{mv}} + g^0_{\text{opt},B}\big(\mu - E[R^Y_{\text{mv}}]\big), \tag{1.22}$$

where $g^0_{\text{opt},B}(m)$ is the solution to Problem (B) for $Y \equiv 0$ and constraint m, explicitly given by

$$g^0_{\text{opt},B}(m) = \frac{m}{E[1-\pi(1)]}\big(1-\pi(1)\big). \tag{1.23}$$

If $\mu \le E[R^Y_{\text{mv}}]$, then Problem (B) has g^Y_{mv} as unique solution.

Proof. As in the proof of Proposition 1.3.4, Problem (B) can be reduced to the basic version

$$\text{Var}[g] = \min! \qquad \text{over all } g \in \overline{\mathcal{G}} \text{ such that } E[g] \ge m, \tag{1.24}$$

where m in (1.24) stands for $\mu - E[R^Y_{\text{mv}}]$. More precisely, if $g^0_{\text{opt},B}(m) \in \overline{\mathcal{G}}$ denotes the solution to (1.24), then the solution $g^Y_{\text{opt},B}(\mu) \in \overline{\mathcal{G}}$ to Problem (B) in its original formulation is given by (1.22), due to (1.10) and (1.8).

If $m \le 0$, then (1.24) is trivially solved by $g \equiv 0$, which proves the last assertion. On the other hand, there is for any $m > 0$ some $\gamma_m \in (0,\infty)$ with $m = E[R^0_{\text{opt},A}(\gamma_m)] = E[g^0_{\text{opt},A}(\gamma_m)]$; in fact, (1.20) gives due to $R^0_{\text{mv}} = 0$ that

$$\gamma_m = 2m\frac{E[\pi(1)]}{E[1-\pi(1)]}. \tag{1.25}$$

We claim that $g^0_{\mathrm{opt},B}(m) = g^0_{\mathrm{opt},A}(\gamma_m)$, i.e. that $g^0_{\mathrm{opt},A}(\gamma_m)$ solves (1.24). To see this, take $g' \in \overline{\mathcal{G}}$ with $E[g'] \geq m = E[g^0_{\mathrm{opt},A}(\gamma_m)]$. Because $g^0_{\mathrm{opt},A}(\gamma_m)$ solves (A) for γ_m and $Y \equiv 0$, we then get

$$m - \frac{\mathrm{Var}[g']}{\gamma_m} \leq E[g'] - \frac{\mathrm{Var}[g']}{\gamma_m} \leq E\big[g^0_{\mathrm{opt},A}(\gamma_m)\big] - \frac{\mathrm{Var}\left[g^0_{\mathrm{opt},A}(\gamma_m)\right]}{\gamma_m} = m - \frac{\mathrm{Var}\left[g^0_{\mathrm{opt},A}(\gamma_m)\right]}{\gamma_m}.$$

Since $\gamma_m > 0$, this implies $\mathrm{Var}[g'] \geq \mathrm{Var}[g^0_{\mathrm{opt},A}(\gamma_m)]$ which shows that $g^0_{\mathrm{opt},A}(\gamma_m)$ solves (1.24). The uniqueness of the solution to Problem (B) then follows from the uniqueness of the solution to Problem (A). Finally, the explicit expression (1.23) is obtained by inserting (1.25) into (1.14). □

The solution for (C) is derived next; the proof is symmetric to that of Proposition 1.3.6.

Proposition 1.3.7. *Let $Y \in L^2$ and $\sigma^2 \in [0,\infty)$. If $\sigma^2 > \mathrm{Var}[R^Y_{\mathrm{mv}}]$, then Problem (C) admits a unique solution $g^Y_{\mathrm{opt},C}(\sigma^2) \in \overline{\mathcal{G}}$. It is explicitly given by*

$$g^Y_{\mathrm{opt},C}(\sigma^2) = g^Y_{\mathrm{mv}} + g^0_{\mathrm{opt},C}\big(\sigma^2 - \mathrm{Var}[R^Y_{\mathrm{mv}}]\big), \tag{1.26}$$

where $g^0_{\mathrm{opt},C}(v)$ is the solution to Problem (C) for $Y \equiv 0$ and constraint v, explicitly given by

$$g^0_{\mathrm{opt},C}(v) = \sqrt{\frac{v}{\mathrm{Var}[1-\pi(1)]}}\big(1-\pi(1)\big). \tag{1.27}$$

If $\sigma^2 = \mathrm{Var}[R^Y_{\mathrm{mv}}]$, then Problem (C) admits g^Y_{mv} as unique solution. If $\sigma^2 < \mathrm{Var}[R^Y_{\mathrm{mv}}]$, Problem (C) cannot be solved.

Proof. As in the proof of Proposition 1.3.4, we use (1.8) and (1.10). In view of (1.10), the last two assertions are clear; so we focus on the case where $\sigma^2 > \mathrm{Var}[R^Y_{\mathrm{mv}}]$. Then Problem (C) can be reduced to the basic version

$$E[g] = \max! \qquad \text{over all } g \in \overline{\mathcal{G}} \text{ such that } \mathrm{Var}[g] \leq v, \tag{1.28}$$

where v stands for $\sigma^2 - \mathrm{Var}[R^Y_{\mathrm{mv}}]$. More precisely, if $g^0_{\mathrm{opt},C}(v) \in \overline{\mathcal{G}}$ denotes the solution to (1.28), then the solution $g^Y_{\mathrm{opt},C}(\sigma^2) \in \overline{\mathcal{G}}$ to Problem (C) in its original formulation is given by (1.26).

To solve (1.28), note that (1.19) and (1.21) with $Y \equiv 0$, hence $R^Y_{\mathrm{mv}} = 0$, give for $v > 0$ that

$$\gamma_v = 2\sqrt{v}\frac{E[\pi(1)]}{\sqrt{\mathrm{Var}[1-\pi(1)]}} \in (0,\infty) \tag{1.29}$$

satisfies $v = \mathrm{Var}[R^0_{\mathrm{opt},A}(\gamma_v)] = \mathrm{Var}[g^0_{\mathrm{opt},A}(\gamma_v)]$. We claim that $g^0_{\mathrm{opt},C}(v) = g^0_{\mathrm{opt},A}(\gamma_v)$, i.e. that $g^0_{\mathrm{opt},A}(\gamma_v)$ solves (1.28). Indeed, for any $g' \in \overline{\mathcal{G}}$ with $\mathrm{Var}[g'] \leq v = \mathrm{Var}[g^0_{\mathrm{opt},A}(\gamma_v)]$, we obtain from the fact that $g^0_{\mathrm{opt},A}(\gamma_v)$ solves Problem (A) for γ_v and $Y \equiv 0$ that

$$E[g'] - \frac{v}{\gamma_v} \leq E[g'] - \frac{\mathrm{Var}[g']}{\gamma_v} \leq E\big[g^0_{\mathrm{opt},A}(\gamma_v)\big] - \frac{\mathrm{Var}\left[g^0_{\mathrm{opt},A}(\gamma_v)\right]}{\gamma_v} = E\big[g^0_{\mathrm{opt},A}(\gamma_v)\big] - \frac{v}{\gamma_v}.$$

This yields $E[g'] \leq E[g^0_{\mathrm{opt},A}(\gamma_v)]$, showing that $g^0_{\mathrm{opt},A}(\gamma_v)$ solves (1.28). Uniqueness follows again from the uniqueness of the solution to Problem (A), and the explicit expression (1.27) is obtained by inserting (1.29) into (1.14). □

Remark 1.3.8.

1. Note that the solutions to Problems (B) and (C) both satisfy their constraints with equalities, at least in the genuinely interesting cases where $\mu \geq E[R^Y_{\mathrm{mv}}]$ and $\sigma^2 \geq \mathrm{Var}[R^Y_{\mathrm{mv}}]$. As a consequence, Problems (B) and (C) could equivalently be formulated with equality constraints. Alternatively, this could also be seen by checking directly that an element $g \in \overline{\mathcal{G}}$ satisfying the constraints with strict inequality cannot be optimal.

2. Propositions 1.3.4–1.3.7 show that the solutions to Problems (A)–(C) all have a very similar and simple structure — they all are linear combinations of the minimum variance element g^Y_{mv} and $1-\pi(1)$. If one knows a priori the key role played by the element $1-\pi(1)$, then the solutions to Problems (A)–(C) can be quickly derived as follows. Notice first that $\overline{\mathcal{G}} = \overline{\mathcal{G}} + g^Y_{\mathrm{mv}}$ since $\overline{\mathcal{G}}$ is a linear space and $g^Y_{\mathrm{mv}} \in \overline{\mathcal{G}}$. Furthermore, the space $\overline{\mathcal{G}}$ can be represented as

$$\overline{\mathcal{G}} = \mathbb{R}\big(1-\pi(1)\big) \oplus \mathcal{N}, \qquad \text{where } \mathcal{N} := \{g \in \overline{\mathcal{G}} \mid E[g] = 0\}. \tag{1.30}$$

Indeed, this direct sum decomposition is obtained by noting that $(\mathrm{span}\{1-\pi(1)\})^{\perp} \cap \overline{\mathcal{G}} = \mathcal{N}$, because $E[g] = E[g(1-\pi(1))]$ for $g \in \overline{\mathcal{G}}$. So we can write:

$$\overline{\mathcal{G}} = g^Y_{\mathrm{mv}} + \overline{\mathcal{G}} = g^Y_{\mathrm{mv}} + \mathbb{R}(1-\pi(1)) + \mathcal{N}$$

and hence all $g \in \overline{\mathcal{G}}$ admit the decomposition:

$$g = g^Y_{\mathrm{mv}} + w\big(1-\pi(1)\big) + n \qquad \text{for some } w \in \mathbb{R} \text{ and } n \in \mathcal{N}.$$

Because $\mathrm{Cov}(R^Y_{\mathrm{mv}}, g) = 0$ for all $g \in \overline{\mathcal{G}}$ by (1.9) and $\mathrm{Cov}(1-\pi(1), n) = E[(1-\pi(1))n] = 0$ for all $n \in \mathcal{N}$, we obtain for $R^Y_{\mathrm{mv}} = g^Y_{\mathrm{mv}} - Y$ that

$$\begin{aligned} E[g-Y] &= E[R^Y_{\mathrm{mv}}] + wE[1-\pi(1)], \\ \mathrm{Var}[g-Y] &= \mathrm{Var}[R^Y_{\mathrm{mv}}] + w^2\,\mathrm{Var}[1-\pi(1)] + \mathrm{Var}[n]. \end{aligned}$$

But then it is an easy exercise to check that optimising over $w \in \mathbb{R}$ and $n \in \mathcal{N}$ directly yields the solutions to Problems (A)–(C) as given in Propositions 1.3.4–1.3.7.

The above reasoning does not yet explain how the special element $1-\pi(1)$ comes up. For that, note that $1 \notin \overline{\mathcal{G}}$ by Assumption I and $1 \notin \mathcal{G}^{\perp}$ by Assumption II. So $1-\pi(1)$ is simply the projection of 1 on $\overline{\mathcal{G}}$, and (1.30) is the orthogonal decomposition of $\overline{\mathcal{G}}$ into the span of this element and its orthogonal complement. A similar comment appears in Remark 1.3.5.

1.4 Mean-variance problems and mean-variance efficiency

We now discuss the financial implications of the abstract results obtained in Section 1.3. In particular, we derive some properties of the solutions to Problems (A)–(D) which are abstract versions of classical results from mean-variance portfolio selection. Consider a fixed element $Y \in L^2$. In order to focus on the more interesting cases, we assume throughout this section that the parameters $\mu \in \mathbb{R}$ and $\sigma^2 \in (0, \infty)$ appearing in Problems (B) and (C) are such that

$$\mu > E[R^Y_{\mathrm{mv}}] \quad \text{and} \quad \sigma^2 > \mathrm{Var}[R^Y_{\mathrm{mv}}].$$

We first make a crucial observation. As can be seen from Propositions 1.3.2–1.3.7, the solutions to Problems (A)–(D) all have the same fundamental structure

$$g^Y_{\mathrm{opt},i} = g^Y_{\mathrm{mv}} + c^Y_{\mathrm{opt},i}\big(1 - \pi(1)\big) \qquad \text{for } i \in \{A, B, C, D\} \tag{1.31}$$

for suitable constants $c^Y_{\mathrm{opt},i} \in \mathbb{R}$ and where $g^Y_{\mathrm{opt},D} := g^Y$. This can be seen as an abstract generalisation of the classical *two-fund separation theorem*. In fact, the solutions to Problems (A)–(D) can all be decomposed into the sum of the "minimum variance payoff" g^Y_{mv} and an additional term proportional to $1 - \pi(1)$. Clearly, the latter represents the best L^2-approximation in $\overline{\mathcal{G}}$ of the constant payoff 1, and only the *amount* invested in that depends on the problem under consideration (and on the specific values of the parameters α, μ and σ^2). Alternatively, the element $1 - \pi(1)$ can be characterised as the unique element of $\overline{\mathcal{G}}$ in the Riesz representation of the continuous linear functional $E[\,\cdot\,]$ on $\overline{\mathcal{G}}$; in fact, $E[g] = E[g(1 - \pi(1) + \pi(1))] = (g, 1 - \pi(1))$ for all $g \in \overline{\mathcal{G}}$.

Using the notation $R^Y_{\mathrm{opt},i} := g^Y_{\mathrm{opt},i} - Y$ and omitting the dependence on α, μ and σ^2 gives

$$E[R^Y_{\mathrm{opt},i}] = E[R^Y_{\mathrm{mv}}] + c^Y_{\mathrm{opt},i} E[1 - \pi(1)], \tag{1.32}$$

$$\mathrm{Var}[R^Y_{\mathrm{opt},i}] = \mathrm{Var}[R^Y_{\mathrm{mv}}] + (c^Y_{\mathrm{opt},i})^2 \,\mathrm{Var}[1 - \pi(1)], \tag{1.33}$$

where (1.33) follows from the zero-covariance property of R^Y_{mv} in (1.9). Recall also that Assumption II implies $E[1 - \pi(1)] = \|1 - \pi(1)\|^2_{L^2} > 0$. Thus we can use (1.19) to solve (1.32) for $c^Y_{\mathrm{opt},i} = \frac{E[R^Y_{\mathrm{opt},i}] - E[R^Y_{\mathrm{mv}}]}{E[1-\pi(1)]}$ and insert this expression into (1.33) to get, for $i \in \{A, B, C, D\}$,

$$\mathrm{Var}[R^Y_{\mathrm{opt},i}] = \mathrm{Var}[R^Y_{\mathrm{mv}}] + \big(E[R^Y_{\mathrm{opt},i}] - E[R^Y_{\mathrm{mv}}]\big)^2 \frac{E[\pi(1)]}{E[1 - \pi(1)]}. \tag{1.34}$$

Similarly, solving for $c^Y_{\mathrm{opt},i}$ in (1.33) and plugging that into (1.32), we obtain

$$E[R^Y_{\mathrm{opt},i}] = E[R^Y_{\mathrm{mv}}] + \sqrt{\mathrm{Var}[R^Y_{\mathrm{opt},i}] - \mathrm{Var}[R^Y_{\mathrm{mv}}]}\,\sqrt{\frac{E[1 - \pi(1)]}{E[\pi(1)]}}. \tag{1.35}$$

Equations (1.34) and (1.35) represent abstract versions of the classical *mean-variance efficient frontier*, which provides a simple relationship between the mean ("return") and the variance ("risk") of any element $R^Y_{\mathrm{opt},i}$ which is an optimal outcome according to a mean-variance criterion. In particular, (1.35) shows a linear relationship between the "excess return", with respect to R^Y_{mv}, of a mean-variance optimal element $R^Y_{\mathrm{opt},i}$ and the square root of its "excess risk".

The coefficients $c^Y_{\mathrm{opt},i}$ appearing in (1.31) also admit an interesting characterisation as *"beta factors"*. To see this, notice first that the zero-covariance property of R^Y_{mv} in (1.9) yields

$$\mathrm{Cov}\big(R^Y_{\mathrm{opt},i}, 1 - \pi(1)\big) = \mathrm{Cov}\big(R^Y_{\mathrm{mv}} + c^Y_{\mathrm{opt},i}\big(1 - \pi(1)\big), 1 - \pi(1)\big) = c^Y_{\mathrm{opt},i}\,\mathrm{Var}[1 - \pi(1)].$$

Because $\mathrm{Var}[1-\pi(1)] > 0$ due to Assumption II, we thus obtain

$$c^Y_{\mathrm{opt},i} = \frac{\mathrm{Cov}\left(R^Y_{\mathrm{opt},i}, 1-\pi(1)\right)}{\mathrm{Var}[1-\pi(1)]}.$$

We have therefore proved for $i \in \{A, B, C, D\}$ the relation

$$E[g^Y_{\mathrm{opt},i}] - E[g^Y_{\mathrm{mv}}] = E[R^Y_{\mathrm{opt},i}] - E[R^Y_{\mathrm{mv}}] = \frac{\mathrm{Cov}\left(R^Y_{\mathrm{opt},i}, 1-\pi(1)\right)}{\mathrm{Var}[1-\pi(1)]} E[1-\pi(1)].$$

This can be regarded as an abstract version of the classical *CAPM relation*, with the element $1-\pi(1)$ playing the role of a "market portfolio" or reference asset. In fact, the excess expected value (with respect to g^Y_{mv}) of the solution to any of the Problems (A)–(D) is proportional to the expected value of the "market portfolio" $1-\pi(1)$, with a proportionality factor having the typical structure "$\beta = \mathrm{Cov}\,/\,\mathrm{Var}$". A similar abstract CAPM relation can be found in Proposition 1.29 of Fontana (2010b), which in turn generalises a result due to Courtault et al. (2004).

The zero-covariance property of the minimum variance element R^Y_{mv} also implies another interesting relation. For any $R^Y := g - Y$ with $g \in \overline{\mathcal{G}}$ and all $i \in \{A, B, C, D\}$, we have

$$\begin{aligned}
\mathrm{Cov}(R^Y_{\mathrm{opt},i}, R^Y) &= \mathrm{Cov}\left(R^Y_{\mathrm{mv}} + c^Y_{\mathrm{opt},i}\big(1-\pi(1)\big), R^Y_{\mathrm{mv}} - g^Y_{\mathrm{mv}} + g\right) \\
&= \mathrm{Var}[R^Y_{\mathrm{mv}}] + c^Y_{\mathrm{opt},i}\,\mathrm{Cov}\left(1-\pi(1), g - g^Y_{\mathrm{mv}}\right) \\
&= \mathrm{Var}[R^Y_{\mathrm{mv}}] + c^Y_{\mathrm{opt},i} E[\pi(1)] E[g - g^Y_{\mathrm{mv}}] \\
&= \mathrm{Var}[R^Y_{\mathrm{mv}}] + c^Y_{\mathrm{opt},i} E[\pi(1)]\big(E[R^Y] - E[R^Y_{\mathrm{mv}}]\big).
\end{aligned} \tag{1.36}$$

Let us specialise this to the case $i = B$ (with constraint μ), where $c^Y_{\mathrm{opt},B} = \frac{\mu - E[R^Y_{\mathrm{mv}}]}{E[1-\pi(1)]}$ due to (1.22) and (1.23). We then have

$$\mathrm{Cov}(R^Y_{\mathrm{opt},B}, R^Y) = \mathrm{Var}[R^Y_{\mathrm{mv}}] + \big(\mu - E[R^Y_{\mathrm{mv}}]\big)\big(E[R^Y] - E[R^Y_{\mathrm{mv}}]\big)\frac{E[\pi(1)]}{E[1-\pi(1)]}. \tag{1.37}$$

Now take any $\hat{g} \in \overline{\mathcal{G}}$ such that $\hat{R}^Y := \hat{g} - Y$ and $R^Y_{\mathrm{opt},B}$ are uncorrelated. Then (1.37) yields

$$E[\hat{R}^Y] = E[R^Y_{\mathrm{mv}}] - \frac{\mathrm{Var}[R^Y_{\mathrm{mv}}]}{\mu - E[R^Y_{\mathrm{mv}}]}\frac{E[1-\pi(1)]}{E[\pi(1)]}. \tag{1.38}$$

Solving (1.36) for $E[R^Y]$, plugging in (1.38), using (1.34) for $i = B$ and again (1.38) give

$$\begin{aligned}
E[R^Y] &= E[R^Y_{\mathrm{mv}}] + \frac{\mathrm{Cov}(R^Y_{\mathrm{opt},B}, R^Y) - \mathrm{Var}[R^Y_{\mathrm{mv}}]}{\mu - E[R^Y_{\mathrm{mv}}]}\frac{E[1-\pi(1)]}{E[\pi(1)]} \\
&= E[\hat{R}^Y] + \frac{\mathrm{Cov}(R^Y_{\mathrm{opt},B}, R^Y)}{\mu - E[R^Y_{\mathrm{mv}}]}\frac{E[1-\pi(1)]}{E[\pi(1)]} \\
&= E[\hat{R}^Y] + \frac{\mathrm{Cov}(R^Y_{\mathrm{opt},B}, R^Y)}{\mathrm{Var}[R^Y_{\mathrm{opt},B}]}\frac{\mathrm{Var}[R^Y_{\mathrm{mv}}] + (\mu - E[R^Y_{\mathrm{mv}}])^2\frac{E[\pi(1)]}{E[1-\pi(1)]}}{\mu - E[R^Y_{\mathrm{mv}}]}\frac{E[1-\pi(1)]}{E[\pi(1)]} \\
&= E[\hat{R}^Y] + \frac{\mathrm{Cov}(R^Y_{\mathrm{opt},B}, R^Y)}{\mathrm{Var}[R^Y_{\mathrm{opt},B}]}(\mu - E[\hat{R}^Y])
\end{aligned}$$

for any $g \in \overline{\mathcal{G}}$. This shows that for any fixed $\mu \in \mathbb{R}$, the expected value of an arbitrary $R^Y := g - Y$ can be written as a generalised convex combination of $\mu = E[R^Y_{\mathrm{opt},B}] = E[g^Y_{\mathrm{opt},B} - Y]$ and $E[\hat{R}^Y]$, where $g^Y_{\mathrm{opt},B}$ is the solution to Problem (B) with constraint μ, and $\hat{R}^Y = \hat{g} - Y$ is an element having zero correlation with $R^Y_{\mathrm{opt},B}$. We emphasise that this holds for *any* $R^Y := g - Y$ with $g \in \overline{\mathcal{G}}$. In particular, R^Y need not be optimal according to any of our mean-variance criteria.

Let us now consider a related mean-variance portfolio optimisation problem, namely

Problem (SR) $\quad \dfrac{E[g-Y]}{\sqrt{\mathrm{Var}[g-Y]}} = \max! \qquad$ over all $g \in \overline{\mathcal{G}}$ such that $\mathrm{Var}[g-Y] > 0$.

Observe that the quantity to be maximised in Problem (SR) is an abstract counterpart of the classical *Sharpe ratio*, a typical measure for the trade-off between risk and return. The solution to Problem (SR) is given in the following result.

Proposition 1.4.1. *Let $Y \in L^2$ and suppose that $E[R^Y_{\mathrm{mv}}] > 0$ and $\mathrm{Var}[R^Y_{\mathrm{mv}}] > 0$. Then Problem (SR) admits a unique solution $g^Y_{\mathrm{sr}} \in \overline{\mathcal{G}}$, explicitly given by*

$$g^Y_{\mathrm{sr}} = g^Y_{\mathrm{mv}} + \frac{\mathrm{Var}[R^Y_{\mathrm{mv}}]}{E[R^Y_{\mathrm{mv}}]} \frac{1}{E[\pi(1)]} \big(1 - \pi(1)\big). \tag{1.39}$$

Proof. Since $\mathrm{Var}[R^Y_{\mathrm{mv}}] > 0$, the same reasoning via (1.8) and (1.10) as in the proofs of Propositions 1.3.4–1.3.7 allows us to reduce Problem (SR) to the basic version

$$\frac{E[R^Y_{\mathrm{mv}}] + E[g]}{\sqrt{\mathrm{Var}[R^Y_{\mathrm{mv}}] + \mathrm{Var}[g]}} = \max_{g \in \overline{\mathcal{G}}}! \tag{1.40}$$

If we denote by $g^* \in \overline{\mathcal{G}}$ the solution to (1.40), then the solution to the original problem is $g^Y_{\mathrm{sr}} = g^* + g^Y_{\mathrm{mv}}$, where g^Y_{mv} is the solution to Problem (MV). To solve (1.40), we proceed in two steps. We first fix $\mu \in (0, \infty)$ and want to minimise $\mathrm{Var}[g]$ over all $g \in \overline{\mathcal{G}}$ satisfying the extra constraint $E[g] = \mu$. Due to Proposition 1.3.6, this problem is uniquely solved by

$$g^0_{\mathrm{opt},B}(\mu) = \frac{\mu}{E[1 - \pi(1)]} \big(1 - \pi(1)\big),$$

and so we get

$$\max_{g \in \overline{\mathcal{G}} : E[g] = \mu} \frac{E[R^Y_{\mathrm{mv}}] + E[g]}{\sqrt{\mathrm{Var}[R^Y_{\mathrm{mv}}] + \mathrm{Var}[g]}} = \frac{E[R^Y_{\mathrm{mv}}] + \mu}{\sqrt{\mathrm{Var}[R^Y_{\mathrm{mv}}] + \mathrm{Var}\left[g^0_{\mathrm{opt},B}(\mu)\right]}} = \frac{E[R^Y_{\mathrm{mv}}] + \mu}{\sqrt{\mathrm{Var}[R^Y_{\mathrm{mv}}] + \mu^2 \frac{E[\pi(1)]}{E[1-\pi(1)]}}}.$$

Since $E[R^Y_{\mathrm{mv}}] > 0$, it can be readily checked that the last expression is maximised over μ by

$$\mu_* = \frac{\mathrm{Var}[R^Y_{\mathrm{mv}}]}{E[R^Y_{\mathrm{mv}}]} \frac{E[1 - \pi(1)]}{E[\pi(1)]} =: c^Y_{\mathrm{sr}} E[1 - \pi(1)], \tag{1.41}$$

and so (1.40) is uniquely solved by $g^* = g^0_{\mathrm{opt},B}(\mu_*)$. Problem (SR) is therefore uniquely solved by $g^Y_{\mathrm{sr}} = g^Y_{\mathrm{mv}} + g^0_{\mathrm{opt},B}(\mu_*)$. □

Remark 1.4.2. It can be checked that if $E[R^Y_{\mathrm{mv}}] < 0$, the element $g^Y_{\mathrm{sr}} \in \overline{\mathcal{G}}$ given in Proposition 1.4.1 can be characterised as the unique *minimiser* of the ratio $E[g-Y]/\sqrt{\mathrm{Var}[g-Y]}$.

Combining (1.31) (or Propositions 1.3.4–1.3.7) with Proposition 1.4.1 yields an alternative formulation of a *two-fund separation* result. In fact, writing (1.31) and (1.39) via (1.41) as

$$\begin{aligned} g^Y_{\mathrm{opt},i} &= g^Y_{\mathrm{mv}} + c^Y_{\mathrm{opt},i}\big(1-\pi(1)\big), \\ g^Y_{\mathrm{sr}} &= g^Y_{\mathrm{mv}} + c^Y_{\mathrm{sr}}\big(1-\pi(1)\big) \end{aligned}$$

allows us to solve for $1-\pi(1)$ and obtain, for $i \in \{A,B,C,D\}$,

$$g^Y_{\mathrm{opt},i} = g^Y_{\mathrm{mv}} + \frac{c^Y_{\mathrm{opt},i}}{c^Y_{\mathrm{sr}}}(g^Y_{\mathrm{sr}} - g^Y_{\mathrm{mv}}) = \frac{c^Y_{\mathrm{opt},i}}{c^Y_{\mathrm{sr}}} g^Y_{\mathrm{sr}} + \left(1 - \frac{c^Y_{\mathrm{opt},i}}{c^Y_{\mathrm{sr}}}\right) g^Y_{\mathrm{mv}}.$$

So the solutions to Problems (A)–(D) can all be written as generalised convex combinations of g^Y_{mv} and g^Y_{sr}, the solutions of *minimising the variance* and of *maximising the Sharpe ratio* for $g-Y$, respectively.

In preparation for the next section, we now specialise the abstract results from Section 1.3 to a more concrete financial situation. We replace the abstract random variable $Y \in L^2$ by

$$Y = -cB + (H - hB) - H_0 \qquad \text{with } c, h \in \mathbb{R} \text{ and } H, H_0 \in L^2. \tag{1.42}$$

This describes the net financial balance (outflows minus incomes) at the final time T faced by an agent who is endowed with initial capital c at the starting time $t = 0$ and sells the contingent claim H, to be paid at T, for a compensation of h, obtained at $t = 0$. In addition, the agent has a position H_0 (evaluated at T), which can be interpreted as an existing book of options or as a random endowment. We can then study what happens if the agent trades in the market by choosing an optimal $g \in \overline{\mathcal{G}}$ according to one of the mean-variance rules formalised as Problems (A)–(D). Of course, this includes “pure investment” problems without trading the contingent claim H by simply letting $H \equiv 0$ and $h = 0$.

For later use in solving mean-variance indifference valuation problems, we now give explicit formulas for the optimal values of Problems (A)–(D) for the specific Y given in (1.42). Recall that $R^Y_x = g^Y_x - Y$ and note that (1.42) yields $N^Y = N^H - N^{H_0} - (c+h)\pi(B)$. First, (1.11) and (1.12) in Remark 1.3.3 give for the minimum variance result R^Y_{mv} the mean and variance as

$$\begin{aligned} \mu_{\mathrm{mv}}(c,H,h,H_0) &:= E\big[R^{-cB+(H-hB)-H_0}_{\mathrm{mv}}\big] \\ &= (c+h)\frac{E[\pi(B)]}{E[\pi(1)]} - \frac{E[N^H] - E[N^{H_0}]}{E[\pi(1)]}, \end{aligned} \tag{1.43}$$

$$\begin{aligned} \sigma^2_{\mathrm{mv}}(c,H,h,H_0) &:= \mathrm{Var}\,\big[R^{-cB+(H-hB)-H_0}_{\mathrm{mv}}\big] \\ &= E\Big[\big((c+h)\pi(B) - N^H + N^{H_0}\big)^2\Big] \\ &\quad - \frac{\big((c+h)E[\pi(B)] - E[N^H] + E[N^{H_0}]\big)^2}{E[\pi(1)]}. \end{aligned} \tag{1.44}$$

Next, the optimal value of Problem (A) with a risk-aversion coefficient $\alpha \in (0,\infty)$ is given from (1.20) and (1.21) by

$$\begin{aligned} v_*(c,H,h,H_0;\alpha) &:= E\big[R_{\mathrm{opt},A}^{-cB+(H-hB)-H_0}(1/\alpha)\big] - \alpha \operatorname{Var}\big[R_{\mathrm{opt},A}^{-cB+(H-hB)-H_0}(1/\alpha)\big] \\ &= \mu_{\mathrm{mv}}(c,H,h,H_0) - \alpha\sigma_{\mathrm{mv}}^2(c,H,h,H_0) + \frac{1}{4\alpha}\frac{E[1-\pi(1)]}{E[\pi(1)]}. \end{aligned} \tag{1.45}$$

The Markowitz problem (B) of minimising the variance given a constraint $\mu \in \mathbb{R}$ on the mean leads via (1.33), (1.19), (1.22) and (1.23) to the optimal variance

$$\begin{aligned} \sigma_*^2(c,H,h,H_0;\mu) &:= \operatorname{Var}\big[R_{\mathrm{opt},B}^{-cB+(H-hB)-H_0}(\mu)\big] \\ &= \sigma_{\mathrm{mv}}^2(c,H,h,H_0) + \Big(\big(\mu-\mu_{\mathrm{mv}}(c,H,h,H_0)\big)^+\Big)^2 \frac{E[\pi(1)]}{E[1-\pi(1)]}. \end{aligned} \tag{1.46}$$

Finally, the optimal mean in Problem (C), given a constraint $\sigma^2 \in (0,\infty)$ on the variance with $\sigma^2 \geq \sigma_{\mathrm{mv}}^2(c,H,h,H_0)$, is due to (1.32), (1.26), (1.27) and (1.19) given by

$$\begin{aligned} \mu_*(c,H,h,H_0;\sigma^2) &:= E\big[R_{\mathrm{opt},C}^{-cB+(H-hB)-H_0}(\sigma^2)\big] \\ &= \mu_{\mathrm{mv}}(c,H,h,H_0) + \sqrt{\sigma^2-\sigma_{\mathrm{mv}}^2(c,H,h,H_0)}\sqrt{\frac{E[1-\pi(1)]}{E[\pi(1)]}}. \end{aligned} \tag{1.47}$$

Remark 1.4.3. The mean-variance hedging problem for an initial capital $c \in \mathbb{R}$ and a contingent claim $H \in L^2$ is usually written as

$$\|H-cB-g\|_{L^2} = \min_{g\in\overline{\mathcal{G}}}!$$

see e.g. Schweizer (1996),(2001a). In our notation, this is Problem (D) for $Y := H - cB$. The corresponding minimal value is due to (1.3) given by

$$\min_{g\in\overline{\mathcal{G}}} \|H-cB-g\|_{L^2} = \|N^H - c\pi(B)\|_{L^2} \tag{1.48}$$

Instead of fixing c, we could optimise with respect to the initial capital as well and consider

$$\|H-cB-g\|_{L^2} = \min_{(c,g)\in\mathbb{R}\times\overline{\mathcal{G}}}!$$

If $B \notin \overline{\mathcal{G}}$ so that $E[B\pi(B)] \neq 0$, the optimal initial capital $c_*(H) \in \mathbb{R}$ is due to (1.48) given by

$$c_*(H) := \operatorname*{argmin}_{c\in\mathbb{R}} \big\{\|N^H - c\pi(B)\|_{L^2}\big\} = \frac{E[N^H\pi(B)]}{E[B\pi(B)]} = \frac{E[H\pi(B)]}{E[B\pi(B)]} = \left(\frac{d\widetilde{P}}{dP}, \frac{H}{B}\right), \tag{1.49}$$

where $\widetilde{P}$ denotes the so-called *variance-optimal signed* $(\mathcal{G},B)$*-martingale measure*; see Schweizer (2001a). The value $c_*(H)$ is also called the L^2*-approximation value* of the payoff H.

1.5 Mean-variance indifference valuations

In the last section, we have introduced a financial position of the form $Y = -cB + (H - hB) - H_0$, where $H \in L^2$ represents a contingent claim sold by our agent for a compensation h. However, h has been considered as exogenously given. In the present section, we study how a value for h can be determined endogenously. As an application of the mean-variance theory developed so far, we analyse several *mean-variance indifference valuation rules*, i.e. we determine the value h at which an agent is indifferent, in terms of optimal value according to a mean-variance criterion, between the two following alternatives:

1. Sell the contingent claim H, receive the compensation h and optimise the final net position $(c + h)B + g - H + H_0$ over $g \in \overline{\mathcal{G}}$.
2. Ignore the contingent claim H and just optimise the final position $cB + g + H_0$ over $g \in \overline{\mathcal{G}}$.

In order to derive explicit results for this approach, we need some preliminaries. We first introduce the set $\mathcal{A} := \mathbb{R}B + \mathcal{G}$ and its L^2-closure $\overline{\mathcal{A}}$. Intuitively, $\mathcal{A}$ contains all undiscounted final wealths generated by a trading strategy for some $g \in \mathcal{G}$ starting from some initial capital $c \in \mathbb{R}$. So $\overline{\mathcal{A}}$ consists of those undiscounted payoffs which can be approximately attained in the financial market $(B, \mathcal{G})$, in the sense that they are L^2-limits of a sequence of attainable final wealths. Then we introduce

Assumption III. There exist a constant $\delta \neq 0$ and $\bar{g} \in \overline{\mathcal{G}}$ such that $\delta B + \bar{g} = 1$ P-a.s.

With the above interpretation of $\overline{\mathcal{A}}$, Assumption III is equivalent to saying that a riskless *zero-coupon bond* can be approximately attained in the abstract financial market $(B, \mathcal{G})$ (from an initial investment of δ).

Remark 1.5.1. 1) An easy extension (taking into account both the cases $B \notin \overline{\mathcal{G}}$ and $B \in \overline{\mathcal{G}}$) of the arguments used in Lemma 2 of Schweizer (2001a) allows to show that $\overline{\mathcal{A}} = \mathbb{R}B + \overline{\mathcal{G}}$. Hence Assumption III can be equivalently formulated as $1 \in \overline{\mathcal{A}}$ (or, equivalently, $\mathbb{R} \cap \overline{\mathcal{A}} \neq \{0\}$).

2) Due to the linearity of $\overline{\mathcal{G}}$, it is easy to check that Assumption III is equivalent to the condition $\mathbb{R}B + \overline{\mathcal{G}} = \mathbb{R} + \overline{\mathcal{G}}$.

In this section, we always suppose that Assumption III is satisfied, with $\delta > 0$ (in fact, the case $\delta < 0$ can be seen as a pathological *arbitrage* situation). This is motivated on the one hand by the fact that it makes the theory particularly simple and elegant, as we shall see below. On the other hand, it is also reasonable to expect that such an assumption will be satisfied in many financial markets. One could still solve mean-variance indifference valuation problems without Assumption III, but this would lead to more involved formulae without a clear economic interpretation. Hence we omit the details.

It is interesting to note that Assumption III is related to the notion of *no approximate profits in* L^2, formally defined as the condition $B \notin \overline{\mathcal{G}}$; see Schweizer (1999),(2001a).

Lemma 1.5.2. *If Assumption III holds, then the conditions of* "no approximate riskless profits in L^2" *and* "no approximate profits in L^2" *are equivalent, i.e. we have* $1 \notin \overline{\mathcal{G}}$ *if and only if* $B \notin \overline{\mathcal{G}}$.

Proof. This follows directly from the linearity of $\overline{\mathcal{G}}$, since $1 = \delta B + \bar{g}$ can be rewritten as $B = \frac{1}{\delta}(1 - \bar{g})$. □

Lemma 1.5.2 implies that as soon as Assumption III is satisfied, we can equivalently work with any of the two no-arbitrage conditions $1 \notin \overline{\mathcal{G}}$ and $B \notin \overline{\mathcal{G}}$. Moreover, the condition $B \notin \overline{\mathcal{G}}$ can be shown to be equivalent to an abstract version of the classical *law of one price*; see Courtault et al. (2004) and Fontana (2010b), Section 1.4.1. Finally, we can use Assumption III to obtain a more detailed version of the orthogonal decomposition (1.1), as follows.

Lemma 1.5.3. *Under Assumption III, the terms $g^Y \in \overline{\mathcal{G}}$ and $N^Y \in \mathcal{G}^\perp$ in the decomposition* (1.1) *of $Y \in L^2$ can be uniquely represented as*

$$g^Y = \tilde{g}^Y + c^Y(B - \pi(B)) \quad \textit{and} \quad N^Y = c^Y \pi(B) + L^Y, \tag{1.50}$$

where $c^Y = \frac{E[Y\pi(B)]}{E[B\pi(B)]}$, the element $\tilde{g}^Y \in \overline{\mathcal{G}}$ is the orthogonal projection in L^2 of $Y - c^Y B$ on $\overline{\mathcal{G}}$, and $L^Y \in \overline{\mathcal{A}}^\perp$ is given by $L^Y = Y - c^Y B - \tilde{g}^Y$. Furthermore, we have $E[L^Y] = 0$.

Proof. Because $L^2 = \overline{\mathcal{A}} \oplus \overline{\mathcal{A}}^\perp$, any $Y \in L^2$ can be uniquely decomposed as

$$Y = a^Y + L^Y, \qquad \text{where } a^Y \in \overline{\mathcal{A}} \text{ and } L^Y \in \overline{\mathcal{A}}^\perp.$$

Moreover, $a^Y \in \overline{\mathcal{A}} = \mathbb{R}B + \overline{\mathcal{G}}$ gives $a^Y = c^Y B + \tilde{g}^Y$ with $c^Y \in \mathbb{R}$ and $\tilde{g}^Y \in \overline{\mathcal{G}}$ and therefore

$$Y = c^Y(B - \pi(B)) + \tilde{g}^Y + L^Y + c^Y \pi(B). \tag{1.51}$$

Note that $c^Y(B - \pi(B)) + \tilde{g}^Y \in \overline{\mathcal{G}}$ and $L^Y \in \mathcal{G}^\perp$, since $L^Y \in \overline{\mathcal{A}}^\perp$ and $\mathcal{G} \subseteq \mathcal{A}$. The assertion (1.50) thus follows from the uniqueness of the decomposition (1.1), and we have $E[L^Y] = (L^Y, 1) = 0$ since $L^Y \in \overline{\mathcal{A}}^\perp$ and $1 \in \overline{\mathcal{A}}$. Finally, because $B = \frac{1}{\delta}(1 - \bar{g})$ is in $\overline{\mathcal{A}}$, $L^Y \in \overline{\mathcal{A}}^\perp$ implies that $(L^Y, B) = 0$. Since also $L^Y \in \mathcal{G}^\perp$, we get $E[L^Y \pi(B)] = (L^Y, B) - (L^Y, B - \pi(B)) = 0$ and therefore $E[Y\pi(B)] = c^Y E[B\pi(B)]$ due to (1.51). Because $B \notin \overline{\mathcal{G}}$ by Lemma 1.5.2, we have $E[B\pi(B)] > 0$, and solving for c^Y thus completes the proof. □

Remark 1.5.4. If we think of $Y := H$ as a contingent claim, the term c^H in Lemma 1.5.3 represents in financial terms the "replication price" of the attainable part $a^H \in \overline{\mathcal{A}}$ of H. Moreover, c^H also coincides with the quantity $c_*(H)$ in (1.49) because

$$c^H = \frac{E[H\pi(B)]}{E[B\pi(B)]} = \frac{E[N^H \pi(B)]}{E[B\pi(B)]} = c_*(H).$$

Thus the constant c^H can also be interpreted as the L^2-*approximation value* of H.

Using Lemma 1.5.3, we can obtain more explicit expressions for the optimal values of our mean-variance problems. Since Assumption III gives $\pi(B) = \frac{\pi(1)}{\delta}$ and we have by (1.19) that $E[1 - \pi(1)] = \frac{\mathrm{Var}[\pi(1)]}{E[\pi(1)]}$, we can rewrite (1.43)–(1.47) by simple computations as

$$\mu_{\mathrm{mv}}(c, H, h, H_0) = \frac{c + h - c^H + c^{H_0}}{\delta}, \tag{1.52}$$

$$\sigma^2_{\text{mv}}(c,H,h,H_0) = \operatorname{Var}[L^H - L^{H_0}] = \operatorname{Var}[L^H] + \operatorname{Var}[L^{H_0}] - 2\operatorname{Cov}(L^H, L^{H_0}), \tag{1.53}$$

$$v_*(c,H,h,H_0;\alpha) = \frac{c+h-c^H+c^{H_0}}{\delta} - \alpha\operatorname{Var}[L^H - L^{H_0}] + \frac{1}{4\alpha}\frac{\operatorname{Var}[\pi(B)]}{(E[\pi(B)])^2}, \tag{1.54}$$

$$\sigma^2_*(c,H,h,H_0;\mu) = \left(\left(\mu - \frac{c+h-c^H+c^{H_0}}{\delta}\right)^+\right)^2 \frac{(E[\pi(B)])^2}{\operatorname{Var}[\pi(B)]} + \operatorname{Var}[L^H - L^{H_0}], \tag{1.55}$$

$$\mu_*(c,H,h,H_0;\sigma^2) = \frac{c+h-c^H+c^{H_0}}{\delta} + \sqrt{\sigma^2 - \operatorname{Var}[L^H - L^{H_0}]}\,\frac{\sqrt{\operatorname{Var}[\pi(B)]}}{E[\pi(B)]}. \tag{1.56}$$

For the "pure investment case" $H \equiv 0$ and $h = 0$, this simplifies to

$$\mu_{\text{mv}}(c,0,0,H_0) = \frac{c+c^{H_0}}{\delta}, \tag{1.57}$$

$$\sigma^2_{\text{mv}}(c,0,0,H_0) = \operatorname{Var}[L^{H_0}], \tag{1.58}$$

$$v_*(c,0,0,H_0;\alpha) = \frac{c+c^{H_0}}{\delta} - \alpha\operatorname{Var}[L^{H_0}] + \frac{1}{4\alpha}\frac{\operatorname{Var}[\pi(B)]}{(E[\pi(B)])^2}, \tag{1.59}$$

$$\sigma^2_*(c,0,0,H_0;\mu) = \left(\left(\mu - \frac{c+c^{H_0}}{\delta}\right)^+\right)^2 \frac{(E[\pi(B)])^2}{\operatorname{Var}[\pi(B)]} + \operatorname{Var}[L^{H_0}], \tag{1.60}$$

$$\mu_*(c,0,0,H_0;\sigma^2) = \frac{c+c^{H_0}}{\delta} + \sqrt{\sigma^2 - \operatorname{Var}[L^{H_0}]}\,\frac{\sqrt{\operatorname{Var}[\pi(B)]}}{E[\pi(B)]}. \tag{1.61}$$

We now formally introduce the mean-variance valuation rules we analyse in this section.

Definition 1.5.5. *Let $c \in \mathbb{R}$ and $H, H_0 \in L^2$. For a given risk-aversion coefficient $\alpha \in (0,\infty)$, the* (A)-indifference value *of H is defined by*

$$h_A(H;c,H_0,\alpha) := \inf\{h \in \mathbb{R} \mid v_*(c,H,h,H_0;\alpha) \geq v_*(c,0,0,H_0;\alpha)\}. \tag{1.62}$$

For $\mu \in \mathbb{R}$, the (B)-indifference value *of H is defined by*

$$h_B(H;c,H_0,\mu) := \inf\{h \in \mathbb{R} \mid \sigma^2_*(c,H,h,H_0;\mu) \leq \sigma^2_*(c,0,0,H_0;\mu)\}. \tag{1.63}$$

For $\sigma^2 \in (0,\infty)$, the (C)-indifference value *of H is defined by*

$$h_C(H;c,H_0,\sigma^2) := \inf\{h \in \mathbb{R} \mid \mu_*(c,H,h,H_0;\sigma^2) \geq \mu_*(c,0,0,H_0;\sigma^2)\}. \tag{1.64}$$

We use here the notation introduced in (1.45)–(1.47) *and the convention* $\inf\emptyset = \infty$.

Remark 1.5.6.

1. We emphasise that the mean-variance indifference *values* introduced above should not be regarded as market *prices* for the contingent claim H, since they are outcomes of subjective valuation mechanisms.

2. As can be seen from (1.54), the function $h \mapsto v_*(c,H,h,H_0;\alpha)$ is continuous (even affine) and strictly increasing, since $\delta > 0$. Consequently, the (A)-indifference value $h_A(H;c,H_0,\alpha)$ satisfies the relation

$$v_*\big(c,H,h_A(H;c,H_0,\alpha),H_0;\alpha\big) = v_*(c,0,0,H_0;\alpha). \tag{1.65}$$

This means that $h_A(H; c, H_0, \alpha)$ could be defined by the implicit requirement that it makes the agent indifferent, in terms of maximal values for Problem (A), between the two alternatives of selling or not selling H, as explained at the beginning of this section. An analogous result holds true for the (B)- and (C)-indifference values, at least in the more interesting cases where the functions $h \mapsto \sigma_*^2(c, H, h, H_0; \mu)$ and $h \mapsto \mu_*(c, H, h, H_0; \sigma^2)$ are continuous and strictly monotonic. See the proofs of Propositions 1.5.8 and 1.5.9 for more details.

3. We have defined all our indifference values from the point of view of a *seller* of the contingent claim H. One can also consider the *buyer* versions by simply replacing H and h with $-H$ and $-h$, respectively, and "inf" with "sup" in the definitions. In the case of the (A)-indifference value, we have for instance

$$h_A^{\text{buyer}}(H; c, H_0, \alpha) := \sup\{h \in \mathbb{R} \mid v_*(c, -H, -h, H_0; \alpha) \geq v_*(c, 0, 0, H_0; \alpha)\}.$$

It is easy to check that one has between the seller and buyer versions the intuitive relation

$$h_i^{\text{seller}}(H) := h_i(H) = -h_i^{\text{buyer}}(-H) \qquad \text{for } i \in \{A, B, C\}.$$

4. Let us briefly consider the case where $1 \notin \overline{\mathcal{G}}$, but $B \in \overline{\mathcal{G}}$. In particular, due to Lemma 1.5.2, Assumption III cannot hold. Since $B \in \overline{\mathcal{G}}$ implies that $\pi(B) \equiv 0$, (1.43)–(1.47) show that μ_{mv} and σ^2_{mv} and hence also the optimal values of Problems (A)–(C) do not depend on h. In this case, the mean-variance indifference valuation problems formulated above are not well-posed and we always have $h_i(H) \in \{-\infty, +\infty\}$ for any $H \in L^2$ and $i \in \{A, B, C\}$.

We are now ready to solve the mean-variance indifference valuation problems explicitly. To focus on the financially meaningful cases, we always impose Assumption II that $1 \notin \mathcal{G}^\perp$. With all the work done so far, the proofs of the next three results are very simple; we just use the explicit expressions for the optimal values of Problems (A)–(C) given in (1.52)–(1.61).

Proposition 1.5.7. *Let $c \in \mathbb{R}$ and $H_0 \in L^2$. For any risk-aversion coefficient $\alpha \in (0, \infty)$ and any $H \in L^2$, the (A)-indifference value is explicitly given by*

$$h_A(H; c, H_0, \alpha) = c^H + \delta\alpha\big(\operatorname{Var}[L^H] - 2\operatorname{Cov}(L^H, L^{H_0})\big),$$

where c^H, L^H and L^{H_0} are from Lemma 1.5.3.

Proof. Use (1.62) and compare (1.54) and (1.59). □

Proposition 1.5.8. *Let $c \in \mathbb{R}$ and $H_0 \in L^2$. For $\mu \in \mathbb{R}$ and $H \in L^2$, the (B)-indifference value is explicitly given by*

$$h_B(H; c, H_0, \mu) = \begin{cases} \infty & \text{if } \operatorname{Var}[L^H - L^{H_0}] > \sigma_*^2(c, 0, 0, H_0; \mu), \\ h_*(c, H, H_0; \mu) & \text{if } \operatorname{Var}[L^H - L^{H_0}] \leq \sigma_*^2(c, 0, 0, H_0; \mu), \end{cases}$$

where

$$h_*(c, H, H_0; \mu) := (\mu\delta - c + c^H - c^{H_0}) - \delta\sqrt{\sigma_*^2(c, 0, 0, H_0; \mu) - \operatorname{Var}[L^H - L^{H_0}]}\,\frac{\sqrt{\operatorname{Var}[\pi(B)]}}{E[\pi(B)]}.$$

Proof. Comparing (1.55) and (1.60) shows that we have $\sigma_*^2(c,H,h,H_0;\mu) > \sigma_*^2(c,0,0,H_0;\mu)$ for all $h \in \mathbb{R}$ if $\mathrm{Var}[L^H - L^{H_0}] > \sigma_*^2(c,0,0,H_0;\mu)$; so (1.63) then gives $h_B(H;c,H_0,\mu) = \infty$. On the other hand, if $\mathrm{Var}[L^H - L^{H_0}] \leq \sigma_*^2(c,0,0,H_0;\mu)$, then $h_*(c,H,H_0;\mu)$ above is well defined and due to (1.55) and (1.60) satisfies $\sigma_*^2(c,H,h_*(c,H,H_0;\mu),H_0;\mu) = \sigma_*^2(c,0,0,H_0;\mu)$. This implies $h_B(H;c,H_0,\mu) = h_*(c,H,H_0;\mu)$. □

Proposition 1.5.9. *Let $c \in \mathbb{R}$ and $H_0 \in L^2$. For $\sigma^2 \geq \mathrm{Var}[L^{H_0}]$ and $H \in L^2$, the (C)-indifference value is explicitly given by*

$$h_C(H;c,H_0,\sigma^2) = \begin{cases} \infty & \text{if } \mathrm{Var}[L^H - L^{H_0}] > \sigma^2, \\ h_*(c,H,H_0;\sigma^2) & \text{if } \mathrm{Var}[L^H - L^{H_0}] \leq \sigma^2, \end{cases}$$

where

$$h_*(c,H,H_0;\sigma^2) = c^H - \delta\Big(\sqrt{\sigma^2 - \mathrm{Var}[L^H - L^{H_0}]} - \sqrt{\sigma^2 - \mathrm{Var}[L^{H_0}]}\Big)\frac{\sqrt{\mathrm{Var}[\pi(B)]}}{E[\pi(B)]}.$$

Proof. Proposition 1.3.7 and (1.53) show that if $\sigma^2 < \sigma_{\mathrm{mv}}^2(c,H,h,H_0) = \mathrm{Var}[L^H - L^{H_0}]$, Problem (C) for $Y = -cB + (H - hB) - H_0$ cannot be solved and hence $h_C(H;c,H_0,\sigma^2) = \infty$ by (1.64). On the other hand, if $\sigma^2 \geq \mathrm{Var}[L^H - L^{H_0}]$, then $h_*(c,H,H_0;\sigma^2)$ above is well defined and satisfies $\mu_*(c,H,h_*(c,H,H_0;\sigma^2),H_0;\sigma^2) = \mu_*(c,0,0,H_0;\sigma^2)$ due to (1.56) and (1.61). This implies $h_C(H;c,H_0,\sigma^2) = h_*(c,H,H_0;\sigma^2)$. □

The next result shows that in the nontrivial cases, all mean-variance indifference values share the same fundamental structure. For ease of notation, we omit most arguments of the h_i.

Corollary 1.5.10. *Let $c \in \mathbb{R}$ and $H_0 \in L^2$. For any $\alpha \in (0,\infty)$, $\mu > \frac{c + c^{H_0}}{\delta}$ and $\sigma^2 > \mathrm{Var}[L^{H_0}]$ and any $H \in L^2$ such that $\mathrm{Var}[L^H - L^{H_0}] \leq \sigma_*^2(c,0,0,H_0;\mu)$ and $\mathrm{Var}[L^H - L^{H_0}] \leq \sigma^2$, we have for some $\alpha_i \in (0,\infty)$ that*

$$h_i(H) = c^H + \delta\alpha_i\big(\mathrm{Var}[L^H] - 2\,\mathrm{Cov}(L^H, L^{H_0})\big) =: c^H + \varrho_i(H) \qquad \text{for } i \in \{A,B,C\}. \quad (1.66)$$

Note, however, that α_i can depend on H via L^H.

Proof. For $i = A$, this is immediate from Proposition 1.5.7 with $\alpha_A := \alpha$. For $i = B$ and $i = C$, one simply checks by direct computation that (1.66) holds with

$$\alpha_B := \Big(\sqrt{\sigma_*^2(c,0,0,H_0;\mu) - \mathrm{Var}[L^{H_0}]} + \sqrt{\sigma_*^2(c,0,0,H_0;\mu) - \mathrm{Var}[L^H - L^{H_0}]}\Big)^{-1}\frac{\sqrt{\mathrm{Var}[\pi(B)]}}{E[\pi(B)]}$$

and

$$\alpha_C := \Big(\sqrt{\sigma^2 - \mathrm{Var}[L^{H_0}]} + \sqrt{\sigma^2 - \mathrm{Var}[L^H - L^{H_0}]}\Big)^{-1}\frac{\sqrt{\mathrm{Var}[\pi(B)]}}{E[\pi(B)]},$$

respectively. □

The representation in Corollary 1.5.10 has an interesting financial interpretation. Indeed, (1.66) shows that all mean-variance indifference values can be written as the sum of c^H and an additional risk premium $\varrho_i(H)$. By Remark 1.5.4, the term c^H is the replication price for the attainable part of the contingent claim H, or the L^2-approximation value of H. The risk premium depends on H only via L^H, which represents the unhedgeable part of H, and it also takes into account the covariance between L^H and the unhedgeable part L^{H_0} of the existing position H_0. The indifference value $h_i(H)$ itself is always increasing with respect to the difference $\operatorname{Var}[L^H] - 2\operatorname{Cov}(L^H, L^{H_0})$. So an agent might be willing to *pay* for selling a payoff H if in his view, its unhedgeable part L^H has a *diversification* or *insurance* effect on his overall position.

In view of part 3) in Remark 1.5.6, Corollary 1.5.10 also yields an explicit expression for the *bid-ask spread* s_A between the seller and buyer versions of the (A)-indifference value; we have

$$s_A(H) := h_A^{\text{seller}}(H) - h_A^{\text{buyer}}(H) = h_A(H) + h_A(-H) = 2\delta\alpha_A \operatorname{Var}[L^H].$$

It is interesting to observe that the bid-ask spread depends only on the risk associated to the unhedgeable part L^H of the contingent claim H; the existing position H_0 plays no role. We remark that $\operatorname{Var}[L^H]$ also represents the remaining risk in the quadratic hedging problem for the claim H, because Lemma 1.5.3, Remark 1.5.4 and Remark 1.4.3 yield

$$\operatorname{Var}[L^H] = \|L^H\|_{L^2}^2 = \|N^H - c^H\pi(B)\|_{L^2}^2 = \|N^H - c_*(H)\pi(B)\|_{L^2}^2 = \min_{(c,g)\in\mathbb{R}\times\overline{\mathcal{G}}} \|H - cB - g\|_{L^2}^2.$$

Remark 1.5.11.

1. For $i = B$ and $i = C$, the bid-ask spread $s_i(H)$ has a more complicated form because then $\alpha_i(H) \neq \alpha_i(-H)$. We do not write out the formula, but we mention that we do obtain $\alpha_i(H) = \alpha_i(-H)$, and hence $s_i(H) = 2\delta\alpha_i(H)\operatorname{Var}\left[L^H\right]$, if we have $\operatorname{Cov}\left(L^H, L^{H_0}\right) = 0$.

2. It is worth pointing out that the indifference values satisfy the following very intuitive *iterativity property*: For any $H_1, H_2 \in L^2$ and $i \in \{A, B, C\}$, we have

$$h_i(H_1 + H_2; c, H_0) = h_i(H_1; c, H_0) + h_i\big(H_2; c + h_i(H_1; c, H_0), H_0 - H_1\big), \qquad (1.67)$$

at least in the nontrivial cases. This can be shown by the same arguments as in Section 5.3 of Schweizer (2001a). The reason why this holds is the description of h_i via an implicit equation as in part 2) of Remark 1.5.6; see (1.65) for the example case $i = A$. In financial terms, (1.67) says that the value for selling the sum claim $H_1 + H_2$ equals the sum of the value for first selling the claim H_1 plus the value for then selling the claim H_2, if we adjust before the second sale both initial capital and initial position to take into proper account the effect of the first sale.

3. Now consider the case where $H \in \overline{\mathcal{A}}$ so that $H = c^H B + \tilde{g}^H$ for some $c^H \in \mathbb{R}$ and $\tilde{g}^H \in \overline{\mathcal{G}}$. Intuitively, this means that the contingent claim H is (approximately) attainable with initial capital c^H. Under the assumptions of Corollary 1.5.10, all mean-variance indifference values then coincide with the replication price c^H because $L^H \equiv 0$. This is of course an expected result — the value of an attainable payoff does not depend on preferences, but is determined by arbitrage arguments alone.

4. Suppose $c \geq 0$ (and not only $c \in \mathbb{R}$), so that the initial capital is nonnegative, and also that Problem (B) has a constraint $\mu \geq 0$. Let us also restrict the definitions of all indifference values to the interval $[-c, \infty)$ and denote by $h_i^c(H)$ the resulting version of $h_i(H)$, for $i \in \{A, B, C\}$. Intuitively, this amounts to excluding the undesirable situation where an agent is allowed to start with $c + h_i(H) < 0$, i.e. in a debt position. It is then easy to verify that we have the natural relation $h_i^c(H) = \max(-c, h_i(H))$.

1.6 Connections to the literature

As already mentioned in the introduction, mean-variance portfolio optimisation problems have always represented a classical topic in financial economics. In the traditional and simplest formulation, beginning with the seminal work of Markowitz (1952), one considers a single-period model with a random vector in $\mathbb{R}^d$ representing the returns on a finite number of assets. One then derives the mean-variance optimal strategy, represented by a deterministic vector in $\mathbb{R}^d$, and the equations describing the mean-variance efficient frontier. For standard textbook accounts, we refer the reader to Chapter 4 of Ingersoll (1987), the book by Markowitz (1987), Chapter 3 of Huang & Litzenberger (1988) or Chapter 6 of Luenberger (1998). The survey by Steinbach (2001) contains a more detailed treatment and an extensive bibliography.

In the last two decades, quadratic portfolio optimisation problems have also drawn the attention of researchers in the mathematical finance community. Typically, one considers more or less general continuous-time semimartingale models and uses the powerful tools of stochastic calculus to characterise the optimal strategy, which is here represented by a predictable process (satisfying suitable technical conditions). We do not attempt here a detailed survey of the extensive relevant literature, but only refer to Schweizer (2010). We just mention that a large body of literature on mean-variance hedging is based on projection techniques and martingale methods; see for instance the survey papers by Pham (2000) and Schweizer (2001b). In addition, stochastic control techniques and backward stochastic differential equations have been used to solve Markowitz problems in continuous-time models; see for instance Zhou (2003) for an overview of the Itô process case, or Czichowsky & Schweizer (2011) for some recent results in a general semimartingale framework. In the context of discrete-time multiperiod models, Markowitz problems and mean-variance optimal strategies have been studied in Li & Ng (2000) via recursive techniques, and in Leippold et al. (2004) by a geometric approach. Duality methods have also been employed, to obtain characterisations of mean-variance optimal strategies in terms of optimal (signed) martingale measures; see for instance Leitner (2000), Hou & Karatzas (2004), Xia & Yan (2006) and Czichowsky & Schweizer (2010). All these authors work in general semimartingale settings which do not assume specific modelling structures. In that respect, and also in some of its techniques, this strand of literature is rather close to our abstract approach.

The setup of the present paper lies in some ways on a middle ground between the classical approach outlined in the first paragraph and the more sophisticated semimartingale models surveyed in the second one. On the one hand, our setting is essentially a one-period model. On the other hand, we avoid any description of the underlying financial market because we do not model assets, returns and strategies, but only work with the abstract space $\mathcal{A} := \mathbb{R}B + \mathcal{G} \subseteq L^2$ of attainable final wealths. Put differently, we *parametrise* our variables not via strategies, but directly via the resulting final *positions*. The key advantage of this approach is that it allows to describe in a simple way the general structure of all mean-variance optimal portfolios, together with their fundamental economic properties. Moreover, our results are by construction completely model-independent and hence hold for any semimartingale financial model. But of course, there is also a price to pay: We can describe the optimal wealth positions and their general properties, but we cannot give the corresponding trading strategies — there are no strategies in our setup because these depend on the financial market model.

In the economic literature, the introduction of Hilbert space techniques in the context of mean-variance problems goes back to Chamberlain & Rothschild (1983) and Hansen & Richard (1987); see also Chapters 5 and 6 of Cochrane (2005) for a textbook account. A related approach can be found in Luenberger (2001), under the standard assumption that the market is generated by a finite number of assets. Coming closer to our work, the abstract L^2-framework adopted in this paper has been first introduced in Schweizer (1996),(1997) and then used in several related works; see for instance Schweizer (1999),(2001a), Møller (2001) and Sun & Wang (2005),(2006).

In comparison with the last group of papers, our results here provide two main innovations. One is that we systematically tackle and solve a whole range of quadratic optimisation problems in a unified way, including connections between the different problems and their solutions. The second major point is that we systematically deal with *undiscounted* quantities. This contrasts with the standard mathematical finance literature where one typically ("without loss of generality") works from the beginning with already discounted quantities. If we interpret B as the final value of a savings account, discounting corresponds to letting $B \equiv 1$. As a consequence, several well-known results for the discounted case (see for instance Møller (2001), Theorem 4.3) can be recovered by just specialising our general results to the case $B \equiv 1$. However, using undiscounted terms seems to us more natural from a financial economics point of view.

Earlier work on abstract financial markets with stochastic interest rates by Schweizer (2001a) and Sun & Wang (2005),(2006) has interpreted B as the final value of a savings account and then considered mean-variance problems in terms of B-discounted quantities, under the no-arbitrage condition $B \notin \overline{\mathcal{G}}$ of no approximate profits in L^2. For related work, compare also Section 3.5 in Rheinländer (1999) and Chapter 1 in Fontana (2010b). Because we do not discount and give no specific interpretation to B, we impose instead the no-arbitrage condition $1 \notin \overline{\mathcal{G}}$ of no approximate *riskless* profits in L^2. As we have seen, only the latter is necessary for solving our general mean-variance problems. Of course, the distinction only matters if B is random.

Remark 1.6.1. The issue of discounting is here actually a bit more subtle than "without loss of generality" suggests. Several papers introduce B-discounted quantities and then solve mean-variance portfolio optimisation problems with respect to the measure P^B defined by $\frac{dP^B}{dP} := \frac{B^2}{E[B^2]}$, instead of the original measure P. For mean-variance hedging, this is fine because $\|\frac{g}{B}\|_{L^2(P^B)}$, the second moment of B-discounted gains with respect to the measure P^B, corresponds (up to a normalising factor) to $\|g\|_{L^2(P)}$. But this does not hold for the mean, since

$$E^B\left[\frac{g}{B}\right] := E\left[\frac{dP^B}{dP}\frac{g}{B}\right] = E\left[\frac{B^2}{E[B^2]}\frac{g}{B}\right] = \frac{1}{E[B^2]}E[Bg].$$

It seems not clear if this quantity has a meaningful economic interpretation under the original measure P, nor why an agent with mean-variance preferences should be interested in it. In that sense, the approach first suggested in Schweizer (1997) and later followed by Sun & Wang (2005),(2006), among others, is mathematically elegant but seems economically flawed. Our current approach does not suffer from this inconsistency.

Let us briefly return to the discounted case $B \equiv 1$. As can be seen from Schweizer (1996), (2001b), Pham (2000) and Møller (2001), mean-variance optimisation problems are via duality deeply linked to the so-called *variance-optimal (signed) martingale measure* $\widetilde{P}$. In our abstract terms, this is defined (for $B \equiv 1$) by $\frac{d\widetilde{P}}{dP} := \widetilde{D}$, where $\widetilde{D} \in \mathcal{G}^\perp$ denotes the element which minimises $\|D\|_{L^2}$ over all $D \in \mathcal{G}^\perp$ such that $E[D] = 1$. The following result is known; but the proof we give here, and especially the insight behind it, is much more elegant than previous ones (e.g. in Schweizer (1996)).

Corollary 1.6.2. *Let $B \equiv 1$. If Assumption I holds, the variance-optimal (signed) martingale measure $\widetilde{P}$ can be uniquely characterised by*

$$\frac{d\widetilde{P}}{dP} = \frac{\pi(1)}{E[\pi(1)]}.$$

Proof. Equivalently to the definition, $\widetilde{D}$ minimises $\mathrm{Var}[D]$ over all $D \in \mathcal{G}^\perp$ such that $E[D] = 1$. But this is simply a particular case of Problem (B), with $Y \equiv 0$ and $\overline{\mathcal{G}}$ exchanged for $\mathcal{G}^\perp$. In Proposition 1.3.6, we thus have to replace π by $\mathrm{Id} - \pi$, hence $1 - \pi(1)$ by $\pi(1)$, and so the result follows directly from (1.23) with $m = 1$. □

We conclude this section with a brief literature review for Section 1.5. Utility-based indifference valuation rules were introduced in the mathematical finance literature by Hodges & Neuberger (1989) and then studied in a variety of settings; see for instance Henderson & Hobson (2009) and Becherer (2010) for recent overviews. However, explicit results are available only in a handful of cases; this mainly includes exponential utility as in Becherer (2003),(2006) and mean-variance preferences as in the present paper. More specifically, the indifference valuation rules analysed in Section 1.5 are closely related to the *utility indifference prices* under mean-variance preferences used in Mercurio (2001), Møller (2001), Schweizer (2001a), Sun & Wang (2005) and Section 1.3 of Fontana (2010b). By letting $B \equiv 1$ throughout Section 1.5, we easily obtain mean-variance indifference values with respect to discounted quantities, recovering the case studied in Mercurio (2001) and Møller (2001). In particular, if $B \equiv 1$, Assumption III is automatically satisfied with $\delta = 1$ and $\bar{g} \equiv 0$. Definition 1.5.5 is inspired by the notion of *mean-variance price* introduced by Bielecki et al. (2004) in the context of credit risk modelling, and our Proposition 1.5.8 can be regarded as a generalised and abstract counterpart to their Proposition 18.

Chapter 2

Measure changes for reduced-form affine credit risk models, with applications to hybrid equity/credit risk models

2.1 Introduction

Credit risk is one of the main constituents of financial risk in general. Historically, two main approaches to credit risk modeling have been proposed in the literature. The first one consists in the class of *structural models* and goes back to the seminal work of Merton (1974). Structural models allow for the joint modeling of equity and credit risk and lead to a clear economic explanation of the occurrence of the default event, which happens if the assets' value process hits a default-triggering barrier. However, it has been amply demonstrated that structural models suffer from severe shortcomings when they have to be applied to the valuation of default-sensitive payoffs, see e.g. Section 9.3 of Schönbucher (2003a). In fact, the fundamental value of the assets of a firm cannot be easily observed in general, the determination of a suitable default-triggering barrier is rather arbitrary and, moreover, significant short-term credit spreads cannot be easily reproduced.

To overcome some of the drawbacks inherent in structural models, *reduced-form* (also known as *intensity-based*) models have been more recently proposed. According to the reduced-form modeling paradigm, the occurrence of the default event is represented by an exogenously given random time with an intensity process and there is no attempt at explaining the precise mechanism leading to the default event. Among reduced-form credit risk models, a typical and convenient formulation consists in letting the default intensity be a linear function of an affine diffusion process, thus giving rise to the class of *reduced-form affine credit risk models*, first proposed by Lando (1998), Duffee (1999) and Duffie & Singleton (1999) (for textbook accounts, see Schönbucher (2003a), Chapter 7, and McNeil et al. (2005), Section 9.5). The main advantage of reduced-form affine credit risk models consists in the possibility of relying on the well-known and powerful machinery of affine processes, originally introduced in the context of interest rate term-structure modeling, for the purpose of evaluating default-sensitive quantities.

In the context of credit risk modeling, the two most basic tasks consist in the computation of default/survival probabilities and in the arbitrage-free valuation of credit-risky financial derivatives. In fact, on the one hand, the knowledge of the survival probability of a given firm up to some time horizon is of fundamental importance for risk management purposes, e.g. for the computation of *Value-at-Risk* and other risk measures. On the other hand, the increasing number of credit derivatives traded in modern financial markets requires the development of reliable pricing techniques. At this point, a rather delicate issue arises. In fact, survival probabilities and related quantities need to be computed with respect to the physical/real-world probability measure, while arbitrage-free prices of financial products are expressed with respect to some risk-neutral probability measure. Thus, in order to cover both risk-management as well as pricing applications, one needs to jointly consider the physical and the risk-neutral probability measures, thereby requiring a precise knowledge of the corresponding Radon-Nikodym density process. Furthermore, the underlying structure of the model can be profoundly altered by a change of the reference probability measure and, therefore, special care must be taken in order not to destroy the nice features of the model if one aims at preserving a sufficient analytical tractability of the latter.

We provide a complete characterization of the family of all locally equivalent probability measures which preserve the affine structure of a reduced-form credit risk model. More precisely, we formulate necessary and sufficient conditions on the density process ensuring that the default time is a doubly stochastic random time with respect to both probability measures and the diffusion process driving the default intensity maintains its affine structure under both probability measures. It turns out that these questions are also related to the preservation of the so-called *immersion property* under a change of measure, i.e. the preservation of the property that any martingale with respect to the original filtration $\mathbb{F}$ is also a martingale in the filtration $\mathbb{G} \supseteq \mathbb{F}$ obtained as the progressive enlargement of $\mathbb{F}$ with respect to the default time. In a general semimartingale (default-free) setting, sufficient conditions for the preservation of the affine structure under a change of measure have been recently obtained by Kallsen & Muhle-Karbe (2010). However, in the case of an affine diffusion framework, our results allow to consider more general density processes for the change of measure. In the special default-free case, our density process specifications correspond to those studied by Cheridito et al. (2007) in the context of term-structure modeling. In turn, our results are also related to the question of whether a positive exponential local martingale is a true martingale and, hence, can be used as the density process for an equivalent change of measure. This represents a classical issue in stochastic calculus (see e.g. Protter (2005), Section III.8, and Revuz & Yor (1999), Chapter VIII) which has also attracted new interest in recent years, especially in view of its applications in mathematical finance, see e.g. Kallsen & Shiryaev (2002), Cheridito et al. (2005), Protter & Shimbo (2008), Blei & Engelbert (2009), Kallsen & Muhle-Karbe (2010), Mijatović & Urusov (2010a) and Mayerhofer et al. (2011).

As an application of our general results on measure changes for reduced-form affine credit risk models, we consider hybrid equity/credit risk models, i.e. models that allow for a unified treatment of market and default risk. In the last years, researchers in financial mathematics have been paying increasing attention to hybrid equity/credit models. In particular, one of the most appealing features of such models is represented by their capability of linking the stochastic behavior of the stock price (and/or of its volatility) with the random occurrence of the default event and, as a consequence, with the level of credit spreads. There is strong empirical evidence showing that equity and credit risk are deeply related and, moreover, several studies also document significant relationships between stock price volatility and typical measures of default risk such as *credit spreads* of corporate bonds and spreads of *Credit Default Swaps (CDS)*. For an overview of the related literature we refer the reader to the introductory section of Carr & Linetsky (2006) (see also Section 2.5.5 for more references to the literature). Furthermore, phenomena such as the *leverage effect*, *volatility smiles* and *volatility skews* are nowadays widely acknowledged in the literature dealing with equity risk modeling: see for instance Gatheral (2006) for a general account.

We shall first consider a rather simple model, which extends the classical Heston (1993) stochastic volatility model by introducing a jump-to-default which kills the stock price process as soon as the default event occurs, as in Carr & Schoutens (2008). By specializing our general results on the preservation of the affine structure of the model under a change of measure, we are able to provide in a simple way a complete characterization of the family of all *Equivalent Local Martingale Measures* (ELMMs) which preserve the Heston with jump-to-default structure of the model. Furthermore, we also show that, under any ELMM, the discounted defaultable stock price process is a true martingale and not only a local martingale. Of course, these results also cover the classical default-free Heston (1993) stochastic volatility model. This allows us to immediately obtain a significant extension of the results of Wong & Heyde (2006) on equivalent changes of measure in stochastic volatility models.

Extending the Heston with jump-to-default model, we propose a general framework for the joint modeling of equity and credit risk which allows for a flexible modeling of the interdependences between stock price, stochastic volatility and default intensity, thus being consistent with several empirical observations. At the same time, this framework preserves a remarkable analytical tractability, since it relies on the technology of affine processes. More specifically, we jointly model the logarithm of the pre-default stock price, its volatility and an additional factor process via a multivariate affine process. The random default time is modeled as the first jump time of a Poisson process with stochastic intensity, the latter being given by an affine function of the volatility and of the factor process. We also specify the interest rate as an affine function of the joint vector affine process. This framework nests several stochastic volatility models which have been proposed in the literature and extends their analysis to a defaultable setting. The stochastic factors themselves allow for a very general economic interpretation as macroeconomic, idiosyncratic and also latent factors. The affine structure of the model allows us to obtain an explicit expression for the conditional characteristic function of the joint factor process, under both the physical and the risk-neutral probability measure. Furthermore, we are able to explicitly derive the conditional characteristic function of the joint factor process with respect to suitable *survival measures*, which turn out to be useful for the valuation of default-sensitive payoffs. We will also briefly consider the incomplete information case, i.e. the case where some of the components of the joint factor process cannot be perfectly observed. This situation is of particular interest, since it allows us to capture unmeasurable variables and unobservable *frailty* effects driving the default intensity, as in Fontana & Runggaldier (2010) and Fontana (2010a). A side advantage of such an incomplete information framework is that we can also let the stochastic volatility be an unobservable process. On the basis of noisy observations of market data, we then set up a state-observation system from which one can formulate a filtering problem.

This Chapter is structured as follows. Section 2.2 describes the general setup of a reduced-form credit risk model based on an affine diffusion process which drives the default intensity. Section 2.3 contains our main theoretical results and provides necessary and sufficient conditions for the preservation of the affine structure of an intensity-based default risk model with respect to a locally equivalent change of the reference probability measure. In Section 2.4 and 2.5, we consider two applications of the results of Section 2.3. More precisely, Section 2.4 deals with the Heston (1993) stochastic volatility model, both in its typical default-free formulation as well as in a jump-to-default extended version, and gives a complete characterization of the family of all risk-neutral measures which preserve the structure of the model, thus sharpening the results of Wong & Heyde (2006). Section 2.5 generalizes the Heston with jump-to-default model to a multifactor framework, thus allowing the stock price, its volatility, the interest rate and the default intensity to depend on a stochastic factor process. By relying on the affine technology and on the general results of Section 2.3, we are able to compute several quantities of interest for both risk management as well as pricing applications. Finally, Section 2.6 concludes by pointing out some further developments which are currently under investigation.

2.2 General setup and preliminaries

This Section describes the key mathematical features of a general *reduced-form* credit risk model. The basic framework we are going to introduce is characterized by a random time τ, which models the random occurrence of a credit event (for instance, the default of a firm), and by a multivariate affine diffusion process, which represents the stochastic evolution of the market situation. As usual in the context of interest rate and credit risk modeling via affine processes, we shall restrict some of the components of the multivariate affine process to be strictly positive. We postpone to Sections 2.4 and 2.5 all pertinent economic considerations and applications to specific financial models.

Let $(\Omega, \mathcal{G}, P)$ be the reference probability space, with P denoting the physical/real-world probability measure. Let $W = (W_t)_{t\geq 0}$ be a d-dimensional Brownian motion on $(\Omega, \mathcal{G}, P)$, with $d \in \mathbb{N}$, and denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the right-continuous P-augmented natural filtration of W. To allow for greater generality, we consider an infinite time horizon[1]. On the filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, P)$, let us consider the following SDE:

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t \qquad X_0 = \bar{x} \in \mathbb{R}^d \tag{2.1}$$

where $\mu : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are measurable continuous functions such that:

$$\mu(x) = b + Ax \qquad \text{and} \qquad \sigma(x)\,\sigma(x)' = \Sigma_0 + \sum_{i=1}^{d} \Sigma_i x_i \tag{2.2}$$

[1]Clearly, one can deal with a finite time horizon $T \in (0, \infty)$ by simply considering the relevant processes stopped at the fixed time T.

for $b \in \mathbb{R}^d$ and $A, \Sigma_0, \Sigma_1, \ldots, \Sigma_d \in \mathbb{R}^{d \times d}$ and where the superscript $'$ denotes transposition. Henceforth, we call *affine SDE* an SDE of the type (2.1), with parameters μ and σ satisfying (2.2).

Remark 2.2.1 (Notation). Throughout this Chapter, we shall often deal with more than one probability measure defined on the same measurable space $(\Omega, \mathcal{G})$. Hence, in order to distinguish expectations with respect to different measures, we shall denote by $E[\cdot]$ the expectation with respect to the original probability measure P and by $E^Q[\cdot]$ the expectation with respect to any other probability measure Q on $(\Omega, \mathcal{G})$. Analogously, W denotes the Brownian motion with respect to P, while W^Q denotes a Brownian motion with respect to any other probability measure Q on $(\Omega, \mathcal{G})$.

For the time being, we can think of the process $X = (X_t)_{t \geq 0}$ as an abstract factor process (financial applications will be considered in detail in Sections 2.4 and 2.5). For a fixed $m \in \{1, \ldots, d\}$, we restrict our attention to solutions to the SDE (2.1) taking values in the *canonical state space* $\mathbb{R}^m_{++} \times \mathbb{R}^{d-m}$, where $\mathbb{R}^m_{++} := \{x \in \mathbb{R}^m : x^i > 0, \forall\, i = 1, \ldots, m\}$. In order to ensure existence and uniqueness of a strong solution to the SDE (2.1) on the canonical state space $\mathbb{R}^m_{++} \times \mathbb{R}^{d-m}$, let us introduce the following set of conditions.

Condition A.

(i) $A^{ij} = 0$, *for* $i \in \{1, \ldots, m\}$ *and* $j \in \{m+1, \ldots, d\}$, *and* $A^{ij} \geq 0$, *for* $i, j \in \{1, \ldots, m\}$ *with* $i \neq j$*;*

(ii) $\Sigma_i = 0 \in \mathbb{R}^{d \times d}$, *for* $i \in \{m+1, \ldots, d\}$, *and* $\Sigma_0, \Sigma_1, \ldots, \Sigma_m$ *are symmetric positive semi-definite, with* Σ_i *positive definite for at least one* $i \in \{0, 1, \ldots, m\}$*;*

(iii) $\sigma^{ij}(x) = \delta_{ij} \sqrt{\Sigma_i^{ii} x^i}$, *for* $i \in \{1, \ldots, m\}$ *and* $j \in \{1, \ldots, d\}$, *where* $\delta_{ij} = 1$ *if* $i = j$ *and* 0 *otherwise;*

(iv) $b^i \geq \frac{1}{2} \Sigma_i^{ii}$, *for* $i \in \{1, \ldots, m\}$.

Proposition 2.2.2. *Suppose that Condition (A) holds. Then, for any* $\bar{x} \in \mathbb{R}^m_{++} \times \mathbb{R}^{d-m}$*, there exists an unique strong solution* $X = (X_t)_{t \geq 0}$ *on* $(\Omega, \mathcal{G}, \mathbb{F}, P)$ *to the SDE* (2.1)*, taking values in* $\mathbb{R}^m_{++} \times \mathbb{R}^{d-m}$ *and with* $X_0 = \bar{x}$*. Furthermore,* 0 *is an entrance boundary (that is, never hit) for* X^i*, for all* $i \in \{1, \ldots, m\}$*, and the matrix* $\sigma(X_t)$ *is* P*-a.s. non-singular, for all* $t \geq 0$*.*

Proof. The first assertion can be proved as in Lemma 10.6 of Filipović (2009), since $(X^1, \ldots, X^m)'$ satisfies an autonomous square-root SDE. The fact that 0 is an entrance boundary for X^i, for all $i \in \{1, \ldots, m\}$, follows from Duffie & Kan (1996), due to part *(iv)* of Condition A. Due to part *(ii)* of Condition A, this also implies that the matrix $\sigma(X_t)$ is P-a.s. non-singular for all $t \geq 0$. □

Let us now denote by τ the random default time[2], with $\tau > 0$ P-a.s. We assume that τ is a *doubly stochastic random time* with respect to $(P, \mathbb{F})$, in the sense of Definition 9.11 of McNeil et

[2]We want to point out that, for the time being, the random time τ does not need to be necessarily linked to the random occurrence of a default event. Indeed, the theoretical results of Sections 2.2-2.3 hold true for any doubly stochastic random time τ.

al. (2005). This means that there exists a P-a.s. strictly positive $\mathbb{F}$-adapted process $\lambda^P = \left(\lambda_t^P\right)_{t\geq 0}$ such that:

$$P(\tau > t|\mathcal{F}_\infty) = P(\tau > t|\mathcal{F}_t) = e^{-\int_0^t \lambda_u^P du} \qquad \text{for all } t \geq 0 \tag{2.3}$$

with $\int_0^t \lambda_u^P du < \infty$ P-a.s. for all $t \geq 0$. In particular, this implies that the random default time τ is not an $\mathbb{F}$-stopping time. We call the process λ^P the *P-intensity* of τ, thus emphasizing the role of the reference probability measure P. Equivalently, according to the terminology adopted in Bielecki & Rutkowski (2002) and Coculescu & Nikeghbali (2010), the process $\int_0^\cdot \lambda_u^P du = \left(\int_0^t \lambda_u^P du\right)_{t\geq 0} = \left(-\log P(\tau > t|\mathcal{F}_t)\right)_{t\geq 0}$ represents the *P-hazard process* of the random time τ. Let us now introduce the enlarged filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$, formally defined as $\mathcal{G}_t := \mathcal{G}_{t+}^0$, for all $t \geq 0$, where $\mathcal{G}_t^0 := \mathcal{F}_t \vee \sigma\{\tau \wedge t\}$. It is well-known that $\mathbb{G}$ is the smallest filtration satisfying the usual conditions which makes τ a stopping time and contains $\mathbb{F}$, i.e. $\mathcal{F}_t \subseteq \mathcal{G}_t$ for all $t \geq 0$ (see e.g. Protter (2005), Section VI.3). We assume that $\mathcal{G} = \mathcal{G}_\infty = \bigvee_{t\geq 0} \mathcal{G}_t$ and we denote by $H = (H_t)_{t\geq 0}$ the default indicator process, defined as $H_t := \mathbf{1}_{\{\tau \leq t\}}$ for $t \geq 0$. We summarize in the following Lemma two key properties of doubly stochastic random times.

Lemma 2.2.3. *Let τ be a doubly stochastic random time with respect to $(P, \mathbb{F})$ with P-intensity process $\lambda^P = \left(\lambda_t^P\right)_{t\geq 0}$. Then the following hold:*

(a) the process $M^P = \left(M_t^P\right)_{t\geq 0}$, defined by:

$$M_t^P := H_t - \int_0^{t\wedge\tau} \lambda_u^P du \qquad t \geq 0 \tag{2.4}$$

is a $(P, \mathbb{G})$-martingale;

(b) every $(P, \mathbb{F})$-martingale is also a $(P, \mathbb{G})$-martingale.

Proof. Part *(a)* follows from Proposition 9.15 of McNeil et al. (2005). Due to the first equality in (2.3), part *(b)* follows from Lemma 5.9.4.2 of Jeanblanc et al. (2009). □

Remark 2.2.4.

1. The $(P, \mathbb{F})$*-martingale hazard process* of a random time τ is formally defined as an $\mathbb{F}$-predictable right-continuous increasing process $L = (L_t)_{t\geq 0}$ such that $H - L^\tau$ is a $(P, \mathbb{G})$-martingale, where L^τ denotes the process L stopped at τ. Hence, part *(a)* of Lemma 2.2.3 shows that the P-hazard process $\int_0^\cdot \lambda_u^P du$ represents also the $(P, \mathbb{F})$-martingale hazard process of τ. See also Bielecki & Rutkowski (2002) and Coculescu & Nikeghbali (2010).

2. Part *(b)* of Lemma 2.2.3 asserts that, under the probability measure P, the so-called *immersion property* (also known as *martingale invariance property*, see Bielecki & Rutkowski (2002), Section 6.1.1) holds between the filtrations $\mathbb{F}$ and $\mathbb{G}$. It is easy to show that this property can be equivalently formulated as follows: every $(P, \mathbb{F})$-local martingale is a $(P, \mathbb{G})$-local martingale. Furthermore, the immersion property between the filtrations $\mathbb{F}$ and $\mathbb{G}$ holds if and only if $P(\tau \leq t|\mathcal{F}_\infty) = P(\tau \leq t|\mathcal{F}_t)$ for all $t \geq 0$ (see e.g. Jeanblanc et al. (2009), Lemma 5.9.4.2).

The diffusion process X and the random default time τ represent the two fundamental ingredients of a general reduced-form (or *intensity-based*) model, where the default intensity is driven by the factor process X. As explained in the Introduction, a convenient specification consists in letting the default intensity be given as a linear function of X_t. We formalize this modeling approach in the following Definition.

Definition 2.2.5. *Let Q be a probability measure on $(\Omega, \mathcal{G})$. We say that the pair (X, τ) has an* affine structure with respect to Q *if the following hold:*

(i) the process $X = (X_t)_{t \geq 0}$ satisfies an affine SDE of the type (2.1) *on $(\Omega, \mathcal{G}, \mathbb{F}, Q)$, with respect to a $(Q, \mathbb{F})$-Brownian motion W^Q and with parameters satisfying Condition A;*

(ii) the random default time τ is a doubly stochastic random time with respect to $(Q, \mathbb{F})$ and admits a Q-intensity process $\lambda^Q = \left(\lambda_t^Q\right)_{t \geq 0}$ with the following affine structure:

$$\lambda_t^Q = \bar{\lambda}^Q + \left(\Lambda^Q\right)' X_t \qquad \textit{for all } t \geq 0 \tag{2.5}$$

for $\bar{\lambda}^Q \in \mathbb{R}_+$ and $\Lambda^Q \in \mathbb{R}_+^d$ with $\Lambda^{Q,i} = 0$ for all $i \in \{m+1, \ldots, d\}$ and $\bar{\lambda}^Q + \sum_{i=1}^m \Lambda^{Q,i} > 0$.

2.3 The affine structure under a change of measure

This Section studies the effects of a locally equivalent change of measure on the affine structure of a general reduced-form credit risk model. Financial applications will be considered in Sections 2.4 and 2.5. In this Section, we shall always work under the **standing assumption** that (X, τ) has an affine structure with respect to the original probability measure P, in the sense of Definition 2.2.5. Our main goal consists in characterizing the family of all probability measures Q on $(\Omega, \mathcal{G})$ locally equivalent to P which preserve the affine structure of (X, τ). Recall that, according to Definition III.3.2 of Jacod & Shiryaev (2003), a probability measure Q on $(\Omega, \mathcal{G})$ is said to be locally equivalent to P if $Q|_{\mathcal{G}_t} \sim P|_{\mathcal{G}_t}$ for all $t \geq 0$, where $Q|_{\mathcal{G}_t}$ and $P|_{\mathcal{G}_t}$ denote the restrictions of Q and P, respectively, to the σ-field $\mathcal{G}_t$.

Definition 2.3.1. *Suppose that (X, τ) has an affine structure with respect to P. Let Q be a probability measure on $(\Omega, \mathcal{G})$ such that $Q \overset{loc}{\sim} P$. We say that Q* preserves the affine structure of (X, τ) *if (X, τ) has an affine structure with respect to Q as well.*

Let Q be a probability measure on $(\Omega, \mathcal{G})$ locally equivalent to P, i.e. $Q \overset{loc}{\sim} P$. Due to Theorem III.3.4 of Jacod & Shiryaev (2003), there exists an unique $(P, \mathbb{G})$-martingale $Z^{Q,\mathbb{G}} = \left(Z_t^{Q,\mathbb{G}}\right)_{t \geq 0}$ such that $Z_t^{Q,\mathbb{G}} = \frac{dQ|_{\mathcal{G}_t}}{dP|_{\mathcal{G}_t}}$ for all $t \geq 0$. Furthermore, since the filtration $\mathbb{G}$ satisfies the usual conditions, there is no loss of generality in assuming that $Z^{Q,\mathbb{G}}$ is right-continuous. We call $Z^{Q,\mathbb{G}}$ the density process of the measure Q with respect to $(P, \mathbb{G})$. However, note that $Q \overset{loc}{\sim} P$ does not necessarily imply that $Q \sim P$ on $\mathcal{G} = \mathcal{G}_\infty$ and, equivalently, the $(P, \mathbb{G})$-martingale $Z^{Q,\mathbb{G}}$ may fail to be uniformly integrable.

Lemma 2.3.2. *Let $Q \overset{loc}{\sim} P$. Then the following hold:*

$$P\left(Z_t^{Q,\mathbb{G}} > 0 \text{ and } Z_{t-}^{Q,\mathbb{G}} > 0 \text{ for all } t > 0\right) = Q\left(Z_t^{Q,\mathbb{G}} > 0 \text{ and } Z_{t-}^{Q,\mathbb{G}} > 0 \text{ for all } t > 0\right) = 1$$

Proof. Define the increasing sequence $(T_n)_{n\in\mathbb{N}}$ of $\mathbb{G}$-stopping times by $T_n{:=}\inf\{t > 0 : Z_t^{Q,\mathbb{G}} < 1/n\}$, for $n \in \mathbb{N}$, and let $T := \inf\{t > 0 : Z_t^{Q,\mathbb{G}} = 0 \text{ or } Z_{t-}^{Q,\mathbb{G}} = 0\} = \lim_{n\to\infty} T_n$. Part *(ii)* of Theorem III.3.4 of Jacod & Shiryaev (2003) implies that, for all $n \in \mathbb{N}$, we have $Q|_{\mathcal{G}_{T_n}} \sim P|_{\mathcal{G}_{T_n}}$ on the set $\{T_n < \infty\}$, with $Z_{T_n}^{Q,\mathbb{G}} = \frac{dQ|_{\mathcal{G}_{T_n}}}{dP|_{\mathcal{G}_{T_n}}}$. Due to the right-continuity of $Z^{Q,\mathbb{G}}$, we have then, for all $n \in \mathbb{N}$ and $t > 0$:

$$Q\left(T \le t\right) \le Q\left(T_n \le t\right) = E\left[Z_{T_n}^{Q,\mathbb{G}} \mathbf{1}_{\{T_n \le t\}}\right] \le \frac{1}{n}$$

Taking the limit for $n \to \infty$ we get $Q\left(T \le t\right) = 0$ for all $t > 0$. Noting that $\{T \le t\} \in \mathcal{G}_t$ for all $t > 0$, being T a $\mathbb{G}$-stopping time (see e.g. Ethier & Kurtz (1986), Proposition 1.1.2), and that $Q|_{\mathcal{G}_t} \sim P|_{\mathcal{G}_t}$ for all $t > 0$, this also implies that $P\left(T \le t\right) = 0$ for all $t > 0$, thus proving the claim. □

For any probability measure Q on $(\Omega, \mathcal{G})$ with $Q \overset{loc}{\sim} P$, the following Lemma gives a general representation of the density process $Z^{Q,\mathbb{G}}$ with respect to $(P, \mathbb{G})$. A similar result can also be found in Kusuoka (1999).

Lemma 2.3.3. *Let $Q \overset{loc}{\sim} P$. Then the density process $Z^{Q,\mathbb{G}} = \left(Z_t^{Q,\mathbb{G}}\right)_{t\ge 0}$ of Q with respect to $(P, \mathbb{G})$ can be represented as follows, for all $t \ge 0$:*

$$\begin{aligned} Z_t^{Q,\mathbb{G}} &= \mathcal{E}\left(\int \theta' dW + \int \gamma\, dM^P\right)_t \\ &= \exp\left(\sum_{i=1}^d \int_0^t \theta_u^i\, dW_u^i - \frac{1}{2}\sum_{i=1}^d \int_0^t \left(\theta_u^i\right)^2 du - \int_0^{\tau\wedge t} \gamma_u \lambda_u^P du\right)\left(1 + \gamma_\tau H_t\right) \end{aligned} \tag{2.6}$$

where $\mathcal{E}(\cdot)$ denotes the stochastic exponential and M^P is the $(P, \mathbb{G})$-martingale defined in (2.4) *and where $\theta = (\theta_t)_{t\ge 0}$ is an $\mathbb{R}^d$-valued $\mathbb{G}$-predictable process such that $\int_0^t \|\theta_u\|^2 du < \infty$ P-a.s. for all $t \ge 0$, i.e. $\theta \in L^2_{loc}(W)$, and $\gamma = (\gamma_t)_{t\ge 0}$ is a real-valued $\mathbb{G}$-predictable process such that $\gamma_t > -1$ and $\int_0^t |\gamma_u| \lambda_u^P du < \infty$ P-a.s. for all $t \ge 0$.*

Proof. Let $Q \overset{loc}{\sim} P$ and let $Z^{Q,\mathbb{G}}$ be its density process with respect to $(P, \mathbb{G})$. Lemma 2.3.2 shows that $Z_-^{Q,\mathbb{G}} > 0$ P-a.s. Furthermore, being $\mathbb{G}$-adapted and left-continuous, the process $Z_-^{Q,\mathbb{G}}$ is also predictable and locally bounded with respect to the filtration $\mathbb{G}$. Hence, due to Theorem IV.29 of Protter (2005), the stochastic integral process $L^{Q,\mathbb{G}} := \int \left(Z_-^{Q,\mathbb{G}}\right)^{-1} dZ^{Q,\mathbb{G}}$ is well-defined as a $(P, \mathbb{G})$-local martingale with $L_0^{Q,\mathbb{G}} = 0$. It is well-known (see e.g. Runggaldier (2003), Theorem 2.3, and Jeanblanc et al. (2009), Proposition 8.8.6.1) that $\left(W, M^P\right)$ has the *representation property* with respect to $(P, \mathbb{G})$, in the sense that any $(P, \mathbb{G})$-local martingale can be written as a stochastic integral of $\left(W, M^P\right)$. This implies that we can represent $L^{Q,\mathbb{G}}$ as follows:

$$L^{Q,\mathbb{G}} = \int \theta' dW + \int \gamma\, dM^P$$

for some $\mathbb{R}^d$-valued $\mathbb{G}$-predictable process $\theta = (\theta_t)_{t\geq 0}$ belonging to $L^2_{loc}(W)$ and some real-valued $\mathbb{G}$-predictable process $\gamma = (\gamma_t)_{t\geq 0}$ such that $\int_0^t |\gamma_u| \lambda_u^P du < \infty$ P-a.s. for all $t \geq 0$. Since $Z^{Q,\mathbb{G}} = 1 + \int Z_-^{Q,\mathbb{G}} dL^{Q,\mathbb{G}}$, the process $Z^{Q,\mathbb{G}}$ can be represented as $Z^{Q,\mathbb{G}} = \mathcal{E}\left(L^{Q,\mathbb{G}}\right)$, thus showing the first equality in (2.6). The explicit expression in second line of (2.6) follows from Theorem II.37 of Protter (2005). Finally, Lemma 2.3.2 shows that $Z^{Q,\mathbb{G}} > 0$ P-a.s. and thus, due to (2.6), we have $\gamma_t > -1$ P-a.s. for all $t \geq 0$. □

Remark 2.3.4. Note that part *(b)* of Lemma 2.2.3 implies that the $(P, \mathbb{F})$-Brownian motion W is also a continuous $(P, \mathbb{G})$-martingale and, due to Lévy's characterization of Brownian motion (see e.g. Protter (2005), Theorem II.39), also a $(P, \mathbb{G})$-Brownian motion. Furthermore, the $(P, \mathbb{G})$-martingales W and M^P are orthogonal. In fact, we have $\left[W, M^P\right] = \langle W, M^P\rangle \equiv 0$, since W is continuous and M^P is a pure jump process. Due to Yor's formula (see Protter (2005), Theorem II.38), equation (2.6) can then be rewritten as follows, for all $t \geq 0$:

$$Z_t^{Q,\mathbb{G}} = \mathcal{E}\left(\int \theta' dW\right)_t \mathcal{E}\left(\int \gamma \, dM^P\right)_t \tag{2.7}$$

The goal of the remaining part of this Section is to show that a probability measure Q on $(\Omega, \mathcal{G})$ with $Q \overset{loc}{\sim} P$ preserves the affine structure of (X, τ) if and only if the processes θ and γ appearing in the representation (2.6) of its density process $Z^{Q,\mathbb{G}}$ satisfy the following Condition.

Condition B.

(i) The $\mathbb{R}^d$-valued process $\theta = (\theta_t)_{t\geq 0}$ has the following form, for all $t \geq 0$:

$$\theta_t = \theta(X_t) := \sigma(X_t)^{-1}\left(\hat{\theta} + \Theta X_t\right) \tag{2.8}$$

for some $\hat{\theta} \in \mathbb{R}^d$ such that $\hat{\theta}^i \geq \frac{1}{2}\Sigma_i^{ii} - b^i$, for all $i \in \{1, \ldots, m\}$, and for some $\Theta \in \mathbb{R}^{d\times d}$ such that $\Theta^{ij} = 0$ for all $i \in \{1, \ldots, m\}$ and $j \in \{m+1, \ldots, d\}$, and $\Theta^{ij} \geq -A^{ij}$ for all $i, j \in \{1, \ldots, m\}$ with $i \neq j$;

(ii) the real-valued process $\gamma = (\gamma_t)_{t\geq 0}$ has the following form, for all $t \geq 0$:

$$\gamma_t = \gamma(X_t) := \frac{\left(\bar{\lambda}^Q - \bar{\lambda}^P\right) + \left(\Lambda^Q - \Lambda^P\right)' X_t}{\bar{\lambda}^P + \left(\Lambda^P\right)' X_t} \tag{2.9}$$

for some $\bar{\lambda}^Q \in \mathbb{R}_+$ and $\Lambda^Q \in \mathbb{R}^d_+$ with $\Lambda^{Q,i} = 0$ for all $i \in \{m+1, \ldots, d\}$ and $\bar{\lambda}^Q + \sum_{i=1}^m \Lambda^{Q,i} > 0$.

Note that, if (X, τ) has an affine structure with respect to P, in the sense of Definition 2.2.5, the left-hand sides of (2.8) and (2.9) are well-defined, since the matrix $\sigma(X_t)$ is P-a.s. invertible for all $t \geq 0$ and $X_t \in \mathbb{R}^m_{++} \times \mathbb{R}^{d-m}$ for all $t \geq 0$ (see Proposition 2.2.2). Furthermore, observe that Condition B implies that the processes $\theta = (\theta_t)_{t\geq 0}$ and $\gamma = (\gamma_t)_{t\geq 0}$ are $\mathbb{F}$-adapted. As in Section 3 of Björk et al. (1997), let us introduce the following simplifying Assumption.

Assumption 2.3.5. *For any non-negative $(P, \mathbb{G})$-martingale $Z = (Z_t)_{t\geq 0}$ with $Z_0 = 1$ there exists a probability measure Q on $(\Omega, \mathcal{G})$ such that $Z_t = \frac{dQ|_{\mathcal{G}_t}}{dP|_{\mathcal{G}_t}}$ for all $t \geq 0$.*

Remark 2.3.6. Let us briefly comment on Assumption 2.3.5. Suppose that $Z = (Z_t)_{t\geq 0}$ is a non-negative $(P, \mathbb{G})$-martingale with $Z_0 = 1$. Then, for any $T \geq 0$, we can define a probability measure $Q^T \ll P$ on $(\Omega, \mathcal{G})$ by letting $\frac{dQ^T}{dP} := Z_T$. Furthermore, the family $\left(Q^T\right)_{T\geq 0}$ has the following *consistency* property: for all $0 \leq S \leq T$, the restriction $Q^T|_{\mathcal{G}_S}$ of Q^T to the σ-field $\mathcal{G}_S$ coincides with Q^S on $(\Omega, \mathcal{G}_S)$. However, nothing ensures that there exists a probability measure Q on $(\Omega, \mathcal{G})$ such that, for every $T \geq 0$, its restriction $Q|_{\mathcal{G}_T}$ coincides with Q^T on $(\Omega, \mathcal{G}_T)$, since Z may fail to be uniformly integrable. Assumption 2.3.5 is meant to avoid this awkward situation. One can formulate precise technical conditions on the reference filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, P)$ which ensure that Assumption 2.3.5 is satisfied. For instance, Assumption 2.3.5 is satisfied if one works on the Skorohod canonical space $\mathbb{D}$ of all càdlàg functions (see Jacod & Shiryaev (2003), Section VI.1), equipped with the natural filtration $(\mathcal{D}^0_t)_{t\geq 0}$ generated by the coordinate process: see Proposition 3.9.17 of Bichteler (2002). We warn the reader that there is a potential conflict between imposing the usual conditions on the reference filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, P)$ and Assumption 2.3.5. A possible way out of this problem consists in replacing the condition of P-completion of the filtration $\mathbb{G}$ with the slightly weaker assumption that, for all $t \geq 0$, the σ-field $\mathcal{G}_t$ contains all countable unions of P-null-sets of $\bigcup_{t\geq 0} \mathcal{G}_t$ together with their subsets. For more details, we refer the interested reader to Bichteler (2002) and to the recent paper Najnudel & Nikeghbali (2011). Finally, note also that it may well be that Q is not equivalent (or even absolutely continuous) with respect to P on $(\Omega, \mathcal{G})$, even if its restriction $Q|_{\mathcal{G}_T}$ coincides with Q^T on $(\Omega, \mathcal{G}_T)$, for every $T \geq 0$.

Theorem 2.3.7. *Let θ and γ be two processes satisfying Condition B. Then the process $Z = (Z_t)_{t\geq 0}$ defined as $Z := \mathcal{E}\left(\int \theta' dW + \int \gamma\, dM^P\right)$ is a P-a.s. strictly positive $(P, \mathbb{G})$-martingale. If Assumption 2.3.5 holds, then Z is the density process with respect to $(P, \mathbb{G})$ of a probability measure Q on $(\Omega, \mathcal{G})$ such that $Q \overset{loc}{\sim} P$ and the following two properties hold:*

- *(a) the process X satisfies on $(\Omega, \mathcal{G}, \mathbb{F}, Q)$ an affine SDE of the type* (2.1)*, with parameters satisfying Condition A and state space $\mathbb{R}^m_{++} \times \mathbb{R}^{d-m}$;*
- *(b) the $(Q, \mathbb{F})$-martingale hazard process of the random default time τ is given by $\int_0^\cdot \lambda^Q_u du$, where $\lambda^Q_t := \bar{\lambda}^Q + \left(\Lambda^Q\right)' X_t$ for all $t \geq 0$.*

Proof. Suppose that we are given two processes $\theta = (\theta_t)_{t\geq 0}$ and $\gamma = (\gamma_t)_{t\geq 0}$ satisfying Condition B. Since X is a continuous $\mathbb{F}$-adapted process, θ and γ are continuous and $\mathbb{F}$-adapted as well and, hence, also predictable and locally bounded with respect to the filtration $\mathbb{F}$. Since $\mathbb{F} \subseteq \mathbb{G}$, the same holds true for the enlarged filtration $\mathbb{G}$. Note also that, as explained in Remark 2.3.4, the $(P, \mathbb{F})$-Brownian motion W is also a $(P, \mathbb{G})$-Brownian motion. These observations, together with Theorem IV.29 of Protter (2005), imply that the stochastic integrals $\int \theta' dW$ and $\int \gamma\, dM^P$ are well-defined as $(P, \mathbb{G})$-local martingales. It follows that the process $Z := \mathcal{E}\left(\int \theta' dW + \int \gamma\, dM^P\right)$ is also a $(P, \mathbb{G})$-local martingale. Furthermore, since $\gamma_t > -1$ P-a.s. for all $t \geq 0$, we have

$Z_t = Z_{t-} + \Delta Z_t = Z_{t-}\,(1+\gamma_\tau H_t) > 0$ P-a.s. for all $t \geq 0$. Fatou's lemma implies that Z is also a $(P,\mathbb{G})$-supermartingale, being a positive $(P,\mathbb{G})$-local martingale.

We now show that $Z = (Z_t)_{t\geq 0}$ is a true $(P,\mathbb{G})$-martingale, extending to the present context some of the arguments used in the proof of Theorem 1 of Cheridito et al. (2007) (see also Cheridito et al. (2005), Section 4). Let us first introduce the following notation:

$$A^Q := A + \Theta \qquad \text{and} \qquad b^Q := b + \hat{\theta} \tag{2.10}$$

and consider the following SDE on the filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, P)$:

$$d\widetilde{X}_t = \left(b^Q + A^Q \widetilde{X}_t\right) dt + \sigma\big(\widetilde{X}_t\big) dW_t \qquad \tilde{X}_0 = \bar{x} \in \mathbb{R}^m_{++} \times \mathbb{R}^{d-m} \tag{2.11}$$

Since θ and γ satisfy Condition B, it is easy to check that the pair $\left(A^Q, b^Q\right)$ satisfies items *(i)* and *(iv)* of Condition A. Hence, Proposition 2.2.2 implies that there exists an unique strong solution $\widetilde{X} = \big(\widetilde{X}_t\big)_{t\geq 0}$ to (2.11) on $(\Omega, \mathcal{G}, \mathbb{F}, P)$ taking values in $\mathbb{R}^m_{++} \times \mathbb{R}^{d-m}$. This also implies that the processes $\tilde{\theta} = \big(\tilde{\theta}_t\big)_{t\geq 0}$ and $\tilde{\gamma} = \left(\tilde{\gamma}_t\right)_{t\geq 0}$ defined as $\tilde{\theta}_t := \theta\big(\widetilde{X}_t\big)$ and $\tilde{\gamma}_t := \gamma\big(\widetilde{X}_t\big)$ for all $t \geq 0$, with the functions $\theta(\cdot)$ and $\gamma(\cdot)$ being given in (2.8)-(2.9), are well-defined. Fix now some $T \in (0,\infty)$ and define the sequences $(\tau_n)_{n\in\mathbb{N}}$ and $(\tilde{\tau}_n)_{n\in\mathbb{N}}$ of $\mathbb{F}$-stopping times as follows:

$$\begin{aligned}
\tau_n &:= \inf\left\{t > 0 : \|\theta_t\| \geq n \text{ or } \gamma_t \notin \left(\frac{1}{n} - 1, n\right)\right\} \wedge T \\
\tilde{\tau}_n &:= \inf\left\{t > 0 : \big\|\tilde{\theta}_t\big\| \geq n \text{ or } \tilde{\gamma}_t \notin \left(\frac{1}{n} - 1, n\right)\right\} \wedge T
\end{aligned}$$

Since the processes θ, γ and $\tilde{\theta}$, $\tilde{\gamma}$ are P-a.s. finite-valued and 0 is an unattainable boundary for X^i and $\widetilde{X}^i$, for all $i \in \{1,\ldots,m\}$ (and, hence, -1 is an unattainable boundary for γ and $\tilde{\gamma}$), it is clear that both $(\tau_n)_{n\in\mathbb{N}}$ and $(\tilde{\tau}_n)_{n\in\mathbb{N}}$ increase P-a.s. to T as $n \to \infty$. For each $n \in \mathbb{N}$, define now the stopped processes $\theta^n := \theta\mathbf{1}_{]\!]0,\tau_n]\!]}$ and $\gamma^n := \gamma\mathbf{1}_{]\!]0,\tau_n]\!]}$ and the process $Z^n = (Z^n_t)_{0\leq t\leq T}$ as follows:

$$Z^n := \mathcal{E}\left(\int \theta^{n\,\prime} dW + \int \gamma^n \, dM^P\right)$$

Due to the definition of θ^n and γ^n it is easy to see that:

$$E\left[\exp\left(\frac{1}{2}\int_0^T \|\theta^n_u\|^2\, du\right)(1+\gamma^n_\tau H_t)\exp\left(-\frac{\gamma^n_\tau H_t}{1+\gamma^n_\tau}\right)\right] \leq (1+n)\exp\left(n - 1 + \frac{n^2\, T}{2}\right) < \infty$$

For each $n \in \mathbb{N}$, the results of Lepingle & Mémin (1978) imply that Z^n is a P-a.s. strictly positive uniformly integrable $(P,\mathbb{G})$-martingale. Hence, we can define a probability measure Q^n on $(\Omega,\mathcal{G})$ by letting $Z^n_T =: \frac{dQ^n}{dP}$. Due to Girsanov's theorem (see e.g. Protter (2005), Theorem III.40), the $\mathbb{R}^d$-valued process $W^{Q^n} = \big(W^{Q^n}_t\big)_{0\leq t\leq T}$ defined as:

$$W^{Q^n}_t := W_t - \int_0^t \frac{1}{Z^n_{u-}} d\langle Z^n, W\rangle_u = W_t - \left\langle \int \theta^{n\,\prime} dW + \int \gamma^n \, dM^P, W\right\rangle_t = W_t - \int_0^t \theta^n_u du$$

is a $(Q^n, \mathbb{G})$-Brownian motion, for every $n \in \mathbb{N}$. As a consequence, for all $t \geq 0$, we can write as follows:

$$\begin{aligned} X_t^{\tau_n} &= \bar{x} + \int_0^{t\wedge\tau_n} \left(b + AX_u^{\tau_n}\right) du + \int_0^{t\wedge\tau_n} \sigma\left(X_u^{\tau_n}\right) dW_u \\ &= \bar{x} + \int_0^{t\wedge\tau_n} \left(b + AX_u^{\tau_n} + \sigma\left(X_u^{\tau_n}\right)\theta_u^n\right) du + \int_0^{t\wedge\tau_n} \sigma\left(X_u^{\tau_n}\right)\left(dW_u - \theta_u^n du\right) \\ &= \bar{x} + \int_0^{t\wedge\tau_n} \left(b^Q + A^Q X_u^{\tau_n}\right) du + \int_0^{t\wedge\tau_n} \sigma\left(X_u^{\tau_n}\right) dW_u^{Q^n} \end{aligned} \tag{2.12}$$

Recall now that, due to Proposition 2.2.2, there exists an unique strong solution $\widetilde{X}$ to the SDE (2.11). Due to Theorem 5.3.6 of Ethier & Kurtz (1986), existence of a unique strong solution implies uniqueness in law for the solution to the SDE (2.11) and, furthermore, due to Corollary 5.3.4 of Ethier & Kurtz (1986), uniqueness in law is equivalent to the uniqueness of the solution to the corresponding *martingale problem*. This shows that, according to the terminology of Ethier & Kurtz (1986), the martingale problem corresponding to the SDE (2.11) is *well-posed*. Then, due to Theorem 4.6.1 of Ethier & Kurtz (1986) (see also Jacod & Shiryaev (2003), Theorem III.2.40), there exists an unique solution to the *stopped* martingale problem corresponding to the SDE (2.11). Note that $\{\tilde{\tau}_n \leq t\} \in \mathcal{F}_t^{\widetilde{X}}$ for all $n \in \mathbb{N}$ and $t \geq 0$, where $\mathcal{F}_t^{\widetilde{X}} := \sigma\{\widetilde{X}_s : s \leq t\}$. Analogously, we have $\{\tau_n \leq t\} \in \mathcal{F}_t^X$ for all $n \in \mathbb{N}$ and $t \geq 0$, where $\mathcal{F}_t^X := \sigma\{X_s : s \leq t\}$. Due to Theorem 4.6.1 and Lemma 4.5.16 of Ethier & Kurtz (1986) together with equation (2.12), this implies that the law of the pair $\left(\widetilde{X}^{\tilde{\tau}_n}, \tilde{\tau}_n\right)$ under the measure P coincides with the law of the pair (X^{τ_n}, τ_n) under the measure Q^n, for all $n \in \mathbb{N}$. Hence, recalling that $\tau_n \nearrow T$ P-a.s. and $\tilde{\tau}_n \nearrow T$ P-a.s. as $n \to \infty$:

$$\begin{aligned} E[Z_T] &= \lim_{n\to\infty} E\left[Z_T \mathbf{1}_{\{\tau_n \geq T\}}\right] = \lim_{n\to\infty} E\left[Z_T^n \mathbf{1}_{\{\tau_n \geq T\}}\right] = \lim_{n\to\infty} Q^n(\tau_n \geq T) \\ &= \lim_{n\to\infty} P(\tilde{\tau}_n \geq T) = 1 \end{aligned}$$

where the first equality follows from the monotone convergence theorem, the second uses the definition of the process Z^n, the third follows from the definition of the measure Q^n and the fourth uses the fact that, for all $n \in \mathbb{N}$, the law of τ_n under Q^n coincides with the law of $\tilde{\tau}_n$ under P. Since $T \in (0, \infty)$ is arbitrary, this shows that the process $Z = (Z_t)_{t\geq 0}$ is a $(P, \mathbb{G})$-martingale, being a $(P, \mathbb{G})$-supermartingale with constant expectation.

For any $T \geq 0$, we can define a probability measure $Q^T \sim P$ on $(\Omega, \mathcal{G})$ by letting $\frac{dQ^T}{dP} := Z_T$. Furthermore, the family $\left(Q^T\right)_{T\geq 0}$ is consistent, in the sense that, for any $S \leq T$, the restriction $Q^T|_{\mathcal{G}_S}$ coincides with Q^S on $(\Omega, \mathcal{G}_S)$. Assumption 2.3.5 implies that there exists a probability measure Q on $(\Omega, \mathcal{G})$ such that its restriction $Q|_{\mathcal{G}_T}$ coincides with Q^T on $(\Omega, \mathcal{G}_T)$, for every $T \geq 0$. Since $Q^T \sim P$ for all $T \geq 0$, this implies that $Q \overset{loc}{\sim} P$. Properties *(a)* and *(b)* now follow by simple applications of Girsanov's theorem for locally equivalent changes of measure (see Jacod & Shiryaev (2003), Theorem III.3.11). In fact, the process $W^Q = \left(W_t^Q\right)_{t\geq 0}$, defined as $W_t^Q := W_t - \int_0^t \theta_u du$ for all $t \geq 0$, is a $(Q, \mathbb{G})$-Brownian motion and, being $\mathbb{F}$-adapted, also a $(Q, \mathbb{F})$-Brownian motion (moreover, we have $W^Q = W^n$ on $[\![0, \tau_n]\!]$, for all $n \in \mathbb{N}$). Hence, we can write as follows:

$$\begin{aligned} dX_t &= (b + AX_t)\, dt + \sigma(X_t)\, dW_t \\ &= (b + AX_t)\, dt + \sigma(X_t) \left(dW_t - \frac{1}{Z_{t-}} d\langle Z, W\rangle_t + \frac{1}{Z_{t-}} d\langle Z, W\rangle_t \right) \\ &= (b + AX_t + \sigma(X_t)\,\theta_t)\, dt + \sigma(X_t)\,(dW_t - \theta_t dt) \\ &= \left(b + AX_t + \hat{\theta} + \Theta X_t\right) dt + \sigma(X_t)\,(dW_t - \theta_t dt) \\ &= \left(b^Q + A^Q X_t\right) dt + \sigma(X_t)\, dW_t^Q \end{aligned} \tag{2.13}$$

This shows property *(a)*. For property *(b)*, we have that the process $M^Q = \left(M_t^Q\right)_{t \geq 0}$ is a $(Q, \mathbb{G})$-local martingale, where[3]:

$$\begin{aligned} M_t^Q &:= M_t^P - \int_0^t \frac{1}{Z_{u-}} d\langle Z, M^P\rangle_u = M_t^P - \int_0^{t\wedge\tau} \lambda_u^P \gamma_u\, du = H_t - \int_0^{t\wedge\tau} \lambda_u^P (1 + \gamma_u)\, du \\ &= H_t - \int_0^{t\wedge\tau} \lambda_u^Q du \end{aligned} \tag{2.14}$$

with the $\mathbb{F}$-adapted continuous process $\lambda^Q = \left(\lambda_t^Q\right)_{t \geq 0}$ being defined as $\lambda_t^Q := \lambda_t^P (1 + \gamma_t) = \bar{\lambda}^Q + \left(\Lambda^Q\right)' X_t$ for all $t \geq 0$. Equation (2.14) shows that the process $\int_0^{\cdot\wedge\tau} \lambda_u^Q du$ is the $(P, \mathbb{G})$-compensator of the increasing process H, being $\mathbb{G}$-predictable and of finite variation. Since H is bounded, the Doob-Meyer decomposition theorem (see Protter (2005), Theorem III.11) implies that the process $M^Q = \left(M_t^Q\right)_{t \geq 0}$ is a uniformly integrable $(Q, \mathbb{G})$-martingale. According to part 1 of Remark 2.2.4, this shows that the process $\int_0^{\cdot} \lambda_u^Q du$ is the $(Q, \mathbb{F})$-martingale hazard process of the random default time τ. □

Remark 2.3.8. Let us suppose for a moment that $P(\tau = \infty) = 1$, thus reducing our analysis to the default-free case, and let $\theta = (\theta_t)_{t \geq 0}$ be an $\mathbb{R}^d$-valued process satisfying part *(i)* of Condition B. Under Assumption 2.3.5, Theorem 2.3.7 implies that there exists a probability measure Q on $(\Omega, \mathcal{G})$ with $Q \overset{loc}{\sim} P$ such that X is a well-defined affine diffusion process with respect to Q as well. In the more general context of affine semimartingales, an analogous problem has been recently studied in Kallsen & Muhle-Karbe (2010). In particular, Corollary 4.2 of Kallsen & Muhle-Karbe (2010) implies that property *(a)* of our Theorem 2.3.7 holds if $\theta_t = \sigma(X_t)' h$ for all $t \geq 0$, for some $h \in \mathbb{R}^d$. However, this specification for the process θ is just a particular case of our specification (2.8). In fact, as can be easily checked, we can recover the specification $\theta_t = \sigma(X_t)' h$ by letting $\hat{\theta} = \Sigma_0 h$ and $\Theta^{ij} = \sum_{k=1}^d \Sigma_j^{ik} h^k$ for $i, j \in \{1, \ldots, d\}$ in (2.8). This shows that, in the case where X is an affine diffusion, our Theorem 2.3.7 allows for more general specifications of the process θ than Corollary 4.2 of Kallsen & Muhle-Karbe (2010).

Theorem 2.3.7 implies that, as soon as the processes θ and γ satisfy Condition B and Assumption 2.3.5 holds, there exists a probability measure $Q \overset{loc}{\sim} P$ such that the process $M^Q := H - \int_0^{\cdot\wedge\tau} \lambda_u^Q du$ is a $(Q, \mathbb{G})$-martingale. However, this does not necessarily imply that the $(Q, \mathbb{F})$-*martingale hazard process* $\int_0^{\cdot} \lambda_u^Q du$ of the random time τ coincides with the *Q-hazard process*

[3] Note that the predictable covariation $\langle Z, M^P\rangle$ always exists since the $(P, \mathbb{G})$-martingale M^P is locally bounded.

$\left(-\log Q\left(\tau>t|\mathcal{F}_t\right)\right)_{t\geq 0}$ of the random default time τ. In other words, it does not automatically follow from Theorem 2.3.7 that $Q\left(\tau>t|\mathcal{F}_t\right)=\exp\left(-\int_0^t \lambda_u^Q du\right)$ for all $t\geq 0$. This means that a priori we do not know whether $\lambda^Q=\left(\lambda_t^Q\right)_{t\geq 0}$ is the *Q-intensity process* of τ (see also the discussion at the end of Section 5.3 of Bielecki & Rutkowski (2002)). We now show that, under suitable conditions on the processes θ and γ, this is indeed the case. To this effect, we need the following preliminary Lemma.

Lemma 2.3.9. *Let $Q\overset{loc}{\sim}P$ and suppose that the processes θ and γ appearing in* (2.6) *are $\mathbb{F}$-adapted and suppose furthermore that the process γ is $\mathbb{F}$-locally bounded. Then, the density process $Z^{Q,\mathbb{F}}=\left(Z_t^{Q,\mathbb{F}}\right)_{t\geq 0}$ of Q with respect to $(P,\mathbb{F})$ is given as follows:*

$$Z_t^{Q,\mathbb{F}}=E\left[Z_t^{Q,\mathbb{G}}|\mathcal{F}_t\right]=\mathcal{E}\left(\int\theta' dW\right)_t \qquad \textit{for all } t\geq 0$$

Proof. Note first that, for any $\mathbb{F}$-stopping time ρ and for any $t\geq 0$:

$$Z_{t\wedge\rho}^{Q,\mathbb{F}}=E\left[Z_{t\wedge\rho}^{Q,\mathbb{G}}|\mathcal{F}_{t\wedge\rho}\right]=\mathcal{E}\left(\int\theta' dW\right)_{t\wedge\rho}E\left[\mathcal{E}\left(\int\gamma\, dM^P\right)_{t\wedge\rho}\Big|\mathcal{F}_{t\wedge\rho}\right] \tag{2.15}$$

where the second equality follows from (2.7) and from the fact that the process $\mathcal{E}\left(\int\theta' dW\right)$ is $\mathbb{F}$-adapted, since both W and θ are $\mathbb{F}$-adapted. Observe that M^P is $\mathbb{F}$-locally bounded (since the process H is bounded between 0 and 1 and the process λ^P is continuous and $\mathbb{F}$-adapted, hence $\mathbb{F}$-locally bounded) and the integrand γ is also $\mathbb{F}$-locally bounded by assumption. It follows that there exists a sequence $(\rho_n)_{n\in\mathbb{N}}$ of $\mathbb{F}$-stopping times such that $\rho_n\nearrow\infty$ P-a.s. as $n\to\infty$ and the stopped process $\mathcal{E}\left(\int\gamma\, dM^P\right)^{\rho_n}$ is bounded, for any $n\in\mathbb{N}$. For any $n\in\mathbb{N}$ and $t\geq 0$, take now an arbitrary set $A\in\mathcal{F}_{t\wedge\rho_n}$ and consider the random variable $\mathbf{1}_A$. Since the filtration $\mathbb{F}$ is the (right-continuous P-augmentation of the) filtration generated by W, Itô's representation theorem (see e.g. Protter (2005), Theorem IV.43) implies that $\mathbf{1}_A$ can be written as follows:

$$\mathbf{1}_A=c^A+\int_0^{t\wedge\rho_n}\left(\xi_u^A\right)' dW_u \tag{2.16}$$

for some $c^A\in\mathbb{R}$ and where the process $\xi^A=\left(\xi_t^A\right)_{t\geq 0}$ is an $\mathbb{F}$-predictable $\mathbb{R}^d$-valued process such that $E\left[\int_0^{t\wedge\rho_n}\left\|\xi_u^A\right\|^2 du\right]<\infty$. Hence, we can write as follows:

$$\begin{aligned}E\left[\mathbf{1}_A\,\mathcal{E}\left(\int\gamma\, dM^P\right)_{t\wedge\rho_n}\right]&=c^A E\left[\mathcal{E}\left(\int\gamma\, dM^P\right)_{t\wedge\rho_n}\right]+E\left[\int_0^{t\wedge\rho_n}(\xi_u^A)' dW_u\,\mathcal{E}\left(\int\gamma\, dM^P\right)_{t\wedge\rho_n}\right]\\&=c^A=c^A+E\left[\int_0^{t\wedge\rho_n}\left(\xi_u^A\right)' dW_u\right]=E\left[\mathbf{1}_A\right]\end{aligned} \tag{2.17}$$

In (2.17), the first equality follows from (2.16), the second is due to the martingale property of the product $\left(\int(\xi^A)' dW\right)^{\rho_n}\mathcal{E}\left(\int\gamma\, dM^P\right)^{\rho_n}$, which follows from the fact that the stopped process $\mathcal{E}\left(\int\gamma\, dM^P\right)^{\rho_n}$ is a bounded $(P,\mathbb{G})$-martingale orthogonal to the $(P,\mathbb{F})/(P,\mathbb{G})$-martingale $\int\left(\xi^A\right)' dW$, and, finally, the third equality uses the fact that $E\left[\int_0^{t\wedge\rho_n}\left(\xi_u^A\right)' dW_u\right]=0$ due to the martingale property of $\int \left(\xi^A\right)' dW$. Since the $\mathcal{F}_{t\wedge\rho_n}$-measurable set A was arbitrary, equation (2.17) shows that:

$$E\left[\mathcal{E}\left(\int \gamma\, dM^P\right)_{t\wedge\rho_n}\middle|\mathcal{F}_{t\wedge\rho_n}\right] = 1$$

and hence, due to equation (2.15) with $\rho = \rho_n$, we get $Z^{Q,\mathbb{F}}_{t\wedge\rho_n} = \mathcal{E}\left(\int \theta' dW\right)_{t\wedge\rho_n}$, for every $n \in \mathbb{N}$. Since $\mathbb{F}$ is the (right-continuous P-augmented) filtration generated by the $(P,\mathbb{F})$-Brownian motion W, the $(P,\mathbb{F})$-martingale $Z^{Q,\mathbb{F}}$ is continuous (see e.g. Protter (2005), Corollary 1 to Theorem IV.43). Since $\rho_n \nearrow \infty$ P-a.s. as $n \to \infty$, we can take the limit for $n \to \infty$ and conclude the proof. □

Theorem 2.3.10. *Let θ and γ be two processes satisfying Condition B. Then, if Assumption 2.3.5 holds, there exists a probability measure Q on $(\Omega, \mathcal{G})$ with $Q \overset{loc}{\sim} P$ and density process given by $Z^{Q,\mathbb{G}} = \mathcal{E}\left(\int \theta' dW + \int \gamma\, dM^P\right)$. Moreover, the following hold:*

(a) every $(Q,\mathbb{F})$-martingale is a $(Q,\mathbb{G})$-martingale, i.e. the immersion property between the filtrations $\mathbb{F}$ and $\mathbb{G}$ holds under the measure Q;

(b) τ is a doubly stochastic random time with respect to $(Q,\mathbb{F})$ and the Q-intensity of τ coincides with the $(Q,\mathbb{F})$-martingale hazard rate $\lambda^Q_t = \bar{\lambda}^Q + \left(\Lambda^Q\right)' X_t$, for all $t \geq 0$, i.e.:

$$Q\left(\tau > t | \mathcal{F}_t\right) = \exp\left(-\int_0^t \lambda^Q_u du\right) \qquad \text{for all } t \geq 0 \tag{2.18}$$

Proof. For θ and γ satisfying Condition B, the existence of a probability measure Q on $(\Omega, \mathcal{G})$ with $Q \overset{loc}{\sim} P$ and $Z^{Q,\mathbb{G}} = \mathcal{E}\left(\int \theta' dW + \int \gamma\, dM^P\right)$ follows from Theorem 2.3.7. To prove part *(a)*, recall that, due to part 2 of Remark 2.2.4, the immersion property between $\mathbb{F}$ and $\mathbb{G}$ can be equivalently formulated in terms of local martingales. Hence, let $N^{Q,\mathbb{F}} = \left(N^{Q,\mathbb{F}}_t\right)_{t\geq 0}$ be an arbitrary $(Q,\mathbb{F})$-local martingale. Due to Girsanov's theorem (in the version of Protter (2005), Theorem III.40, with respect to the filtration $\mathbb{F}$), we have that $N^{Q,\mathbb{F}}_t$ can be written as follows, for any $t \geq 0$:

$$N^{Q,\mathbb{F}}_t = N^P_t - \int_0^t \frac{1}{Z^{Q,\mathbb{F}}_{u-}}\, d\left\langle Z^{Q,\mathbb{F}}, N^P\right\rangle_u \tag{2.19}$$

where $N^P = \left(N^P_t\right)_{t\geq 0}$ is a suitable continuous $(P,\mathbb{F})$-local martingale and $Z^{Q,\mathbb{F}}$ is the density process of Q with respect to $(P,\mathbb{F})$. Note that $\langle Z^{Q,\mathbb{F}}, N^P\rangle$ in (2.19) is well-defined, since both $Z^{Q,\mathbb{F}}$ and N^P are continuous, being $(P,\mathbb{F})$-local martingales, and hence locally bounded. Recall now that part *(b)* of Lemma 2.2.3 implies that the immersion property holds under P, hence N^P is also a continuous $(P,\mathbb{G})$-local martingale. Again due to Girsanov's theorem (now with respect to the filtration $\mathbb{G}$), this implies that N^P can be decomposed as follows:

$$N^P_t = N^{Q,\mathbb{G}}_t + \int_0^t \frac{1}{Z^{Q,\mathbb{G}}_{u-}}\, d\left\langle Z^{Q,\mathbb{G}}, N^P\right\rangle_u \tag{2.20}$$

where $N^{Q,\mathbb{G}} = \left(N_t^{Q,\mathbb{G}}\right)_{t\geq 0}$ is a suitable $(Q,\mathbb{G})$-local martingale. By combining (2.19) with (2.20) we get:

$$\begin{aligned} N_t^{Q,\mathbb{F}} &= N_t^{Q,\mathbb{G}} + \int_0^t \frac{1}{Z_{u-}^{Q,\mathbb{G}}}\, d\langle Z^{Q,\mathbb{G}}, N^P\rangle_u - \int_0^t \frac{1}{Z_{u-}^{Q,\mathbb{F}}}\, d\langle Z^{Q,\mathbb{F}}, N^P\rangle_u \\ &= N_t^{Q,\mathbb{G}} + \left\langle \int \gamma\, dM^P, N^P \right\rangle_t = N_t^{Q,\mathbb{G}} \end{aligned}$$

where the second equality follows from equations (2.6) and Lemma 2.3.9 and the last equality is due to the fact that the $(P,\mathbb{G})$-martingales N^P and $\int \gamma\, dM^P$ are orthogonal, being continuous and purely discontinuous, respectively. This shows that any $(Q,\mathbb{F})$-local martingale is also a $(Q,\mathbb{G})$-local martingale, i.e. the immersion property between the filtrations $\mathbb{F}$ and $\mathbb{G}$ holds with respect to the measure Q.

To prove part *(b)*, recall first that, since $Q \stackrel{loc}{\sim} P$, any $(Q,\mathbb{F})$-local martingale can be represented as a stochastic integral of W^Q (see e.g. Jeanblanc et al. (2009), Proposition 1.7.7.1). Then, part *(a)* and the arguments used in the proof of Lemma 2.3.9 (now with respect to the measure Q) allow to show that $E^Q\left[M_t^Q|\mathcal{F}_t\right] = 0$, for all $t \geq 0$, where $M^Q := H - \int_0^{\cdot\wedge\tau} \lambda_u^Q du$ (see also Coculescu et al. (2008), Lemma 5.1). Thus:

$$\begin{aligned} 0 = E^Q\left[H_t - \int_0^{t\wedge\tau} \lambda_u^Q du \Big| \mathcal{F}_t\right] &= Q\left(\tau \leq t|\mathcal{F}_t\right) - E^Q\left[\int_0^t \mathbf{1}_{\{\tau > u\}} \lambda_u^Q du \Big| \mathcal{F}_t\right] \\ = Q\left(\tau \leq t|\mathcal{F}_t\right) - \int_0^t Q\left(\tau > u|\mathcal{F}_t\right)\lambda_u^Q du &= 1 - Q\left(\tau > t|\mathcal{F}_t\right) - \int_0^t Q\left(\tau > u|\mathcal{F}_u\right)\lambda_u^Q du \end{aligned}$$

where the third equality uses Tonelli's theorem (together with the $\mathbb{F}$-adaptedness of λ^Q) and the fourth equality follows from the fact that $Q\left(\tau > u|\mathcal{F}_t\right) = Q\left(\tau > u|\mathcal{F}_u\right)$ for any $t \geq u$, since the immersion property between $\mathbb{F}$ and $\mathbb{G}$ holds under Q (see part 2 of Remark 2.2.4). We have thus shown that, for all $t \geq 0$:

$$Q\left(\tau > t|\mathcal{F}_t\right) = 1 - \int_0^t Q\left(\tau > u|\mathcal{F}_u\right)\lambda_u^Q du = \exp\left(-\int_0^t \lambda_u^Q du\right)$$

Since, due to Theorem 2.3.7, the process λ^Q is Q-a.s. strictly positive, continuous and $\mathbb{F}$-adapted, this shows that $\lambda^Q = \left(\lambda_t^Q\right)_{t\geq 0}$ is the Q-intensity of the random default time τ. Furthermore, as argued in part 2 of Remark 2.2.4, the immersion property between the filtrations $\mathbb{F}$ and $\mathbb{G}$ holds under the measure Q if and only if $Q\left(\tau \leq t|\mathcal{F}_\infty\right) = Q\left(\tau \leq t|\mathcal{F}_t\right)$ for all $t \geq 0$. Hence, due to part *(a)* of the Theorem, this implies that τ is a doubly stochastic random time with respect to $(Q,\mathbb{F})$, in the sense of Definition 9.11 of McNeil et al. (2005). □

Remark 2.3.11.

1. It is well-known that the immersion property between $\mathbb{F}$ and $\mathbb{G}$ is preserved by an equivalent change of measure if the density of the new measure with respect to the old one is $\mathcal{F}_\infty$-measurable, see e.g. Coculescu et al. (2008), Proposition 4.3, and Jeanblanc et al. (2009), Proposition 5.9.1.2. However, due to the presence of the term $\int \gamma\, dM^P$ appearing in (2.6), this is not our case, which is therefore more general. Furthermore, we explicitly consider

locally equivalent changes of measure. The proof of Theorem 2.3.10 is rather simple and only uses a general version of Girsanov's theorem and the structure of the density processes of a probability measure $Q \overset{loc}{\sim} P$ with respect to the two filtrations $\mathbb{F}$ and $\mathbb{G}$. Under the additional assumption that $Q \sim P$ (or, equivalently, that the $(P,\mathbb{G})$-martingale $Z^{Q,\mathbb{G}}$ is uniformly integrable) part *(a)* of Theorem 2.3.10 can also be deduced from Theorem 6.4 of Coculescu et al. (2008).

2. Note that, once we know that the immersion property between $\mathbb{F}$ and $\mathbb{G}$ holds under the measure Q, part *(b)* of Theorem 2.3.10 can be deduced from the general results of Coculescu & Nikeghbali (2010), in particular from their Theorem 3.8. In fact, since $\mathbb{F}$ is immersed in $\mathbb{G}$ under Q, the random time τ is a *pseudo* $(Q,\mathbb{F})$-*stopping time*. Using the notation of Coculescu & Nikeghbali (2010), we have $Z_t^\tau := -\log Q\left(\tau > t|\mathcal{F}_t\right) > 0$ Q-a.s. for all $t \geq 0$, since $P\left(\tau > t|\mathcal{F}_t\right) > 0$ for all $t \geq 0$ and $Q \overset{loc}{\sim} P$. Furthermore, the filtration $\mathbb{F}$ supports only continuous martingales and the $(Q,\mathbb{F})$-martingale hazard process $\int_0^\cdot \lambda_u^Q du$ is continuous, due to Theorem 2.3.7. Summing up, all the assumptions of Theorem 3.8 of Coculescu & Nikeghbali (2010) are satisfied, thus implying that the $(Q,\mathbb{F})$-martingale hazard rate $\lambda^Q = \left(\lambda_t^Q\right)_{t\geq 0}$ coincides with the Q-intensity of τ.

We can now prove the following Theorem, which characterizes the family of all probability measures Q on $(\Omega,\mathcal{G})$ with $Q \overset{loc}{\sim} P$ which preserve the affine structure of (X,τ), in the sense of Definition 2.3.1.

Theorem 2.3.12. *Let Q be a probability measure on $(\Omega,\mathcal{G})$ with $Q \overset{loc}{\sim} P$. Then Q preserves the affine structure of (X,τ) if and only if its density process $Z^{Q,\mathbb{G}}$ with respect to $(P,\mathbb{G})$ can be represented as in* (2.6) *for some processes θ and γ satisfying Condition B.*

Proof. Let $Z^{Q,\mathbb{G}} = \mathcal{E}\left(\int \theta' dW + \int \gamma\, dM^P\right)$, where the processes θ and γ satisfy Condition B. Part *(a)* of Theorem 2.3.7 shows that X satisfies on $(\Omega,\mathcal{G},\mathbb{F},Q)$ an affine SDE of the type (2.1), with respect to a $(Q,\mathbb{F})$-Brownian motion W^Q and with parameters satisfying Condition A. Part *(b)* of Theorem 2.3.10 implies that τ is a doubly stochastic random time with Q-intensity $\lambda^Q = \left(\lambda_t^Q\right)_{t\geq 0}$ given by $\lambda_t^Q = \bar{\lambda}^Q + \left(\Lambda^Q\right)' X_t$ for all $t \geq 0$, for some $\bar{\lambda}^Q \in \mathbb{R}_+$ and $\Lambda^Q \in \mathbb{R}_+^d$ with $\Lambda^{Q,i} = 0$ for all $i \in \{m+1,\dots,d\}$ and $\bar{\lambda}^Q + \sum_{i=1}^m \Lambda^{Q,i} > 0$. According to Definition 2.2.5, this means that (X,τ) has an affine structure with respect to Q.

Conversely, let Q be a probability measure on $(\Omega,\mathcal{G})$ with $Q \overset{loc}{\sim} P$ and such that (X,τ) has an affine structure with respect to Q. Due to Definitions 2.2.5 and 2.3.1, this means that the process $X = (X_t)_{t\geq 0}$ satisfies on $(\Omega,\mathcal{G},\mathbb{F},Q)$ the following SDE:

$$dX_t = \left(b^Q + A^Q X_t\right) dt + \sigma\left(X_t\right) dW_t^Q \qquad X_0 = \bar{x} \in \mathbb{R}_{++}^m \times \mathbb{R}^{d-m}$$

where the parameters $\left(A^Q, b^Q\right)$ satisfy items *(i)* and *(iv)* of Condition A and where the process W^Q is a $(Q,\mathbb{F})$-Brownian motion. Since τ is a doubly stochastic random time with respect to $(Q,\mathbb{F})$, we have $Q\left(\tau \leq t|\mathcal{F}_\infty\right) = Q\left(\tau \leq t|\mathcal{F}_t\right)$ for all $t \geq 0$. Due to part 2 of Remark 2.2.4, this implies that the immersion property between the filtrations $\mathbb{F}$ and $\mathbb{G}$ holds under the measure Q. Hence, the

$(Q, \mathbb{F})$-Brownian motion W^Q is also a $(Q, \mathbb{G})$-Brownian motion and is related to the $(P, \mathbb{F})$/$(P, \mathbb{G})$-Brownian motion W by $W^Q = W - \int \frac{1}{Z_-^{Q,\mathbb{G}}} d\langle W, Z^{Q,\mathbb{G}} \rangle = W - \int_0^{\cdot} \theta_u du$, due to Girsanov's theorem and Lemma 2.3.3. Hence, we can write:

$$\begin{aligned} dX_t &= \left(b^Q + A^Q X_t - \sigma(X_t)\, \theta_t \right) dt + \sigma(X_t)\, dW_t \\ &= (b + AX_t)\, dt + \sigma(X_t)\, dW_t \end{aligned}$$

where the second equality follows from the standing assumption that (X, τ) has an affine structure with respect to P. This implies that the following identity holds up to a nullset of $\Omega \times [0, \infty)$:

$$b^Q + A^Q X_t - \sigma(X_t)\, \theta_t = b + AX_t$$

from which we get, using the fact that the matrix $\sigma(X_t)$ is P-a.s. invertible for all $t \geq 0$ (see Proposition 2.2.2):

$$\theta_t = \sigma(X_t)^{-1} \left(\left(b^Q - b \right) + \left(A^Q - A \right) X_t \right)$$

Letting $\hat{\theta} := b^Q - b$ and $\Theta := A^Q - A$, this shows that the process $\theta = (\theta_t)_{t \geq 0}$ satisfies part *(i)* of Condition B, since the parameters b^Q and A^Q satisfy items *(i)* and *(iv)* of Condition A. It remains to show that the process $\gamma = (\gamma_t)_{t \geq 0}$ appearing in (2.6) satisfies part *(ii)* of Condition B. Since (X, τ) has an affine structure with respect to Q, the Q-intensity $\lambda^Q = \left(\lambda_t^Q \right)_{t \geq 0}$ of the random default time τ is of the form $\lambda_t^Q = \bar{\lambda}^Q + \left(\Lambda^Q \right)' X_t$ for some $\bar{\lambda}^Q \in \mathbb{R}_+$ and $\Lambda^Q \in \mathbb{R}_+^d$ with $\Lambda^{Q,i} = 0$ for all $i \in \{m+1, \ldots, d\}$ and $\bar{\lambda}^Q + \sum_{i=1}^m \Lambda^{Q,i} > 0$. Furthermore, part *(a)* of Lemma 2.2.3 (with respect to Q) shows that the $(Q, \mathbb{F})$-martingale hazard process of τ is given by $\int_0^{\cdot} \lambda_u^Q du$, i.e. the process $M^Q := H - \int_0^{\cdot \wedge \tau} \lambda_u^Q du$ is a $(Q, \mathbb{G})$-martingale. Then, for all $t \geq 0$:

$$\bar{\lambda}^Q + \left(\Lambda^Q \right)' X_t = \lambda_t^Q = \lambda_t^P (1 + \gamma_t) = \left(\bar{\lambda}^P + \left(\Lambda^P \right)' X_t \right) (1 + \gamma_t)$$

where the first and the third equalities follow from the assumption that (X, τ) has an affine structure (see part *(ii)* of Definition 2.2.5) with respect to Q and P, respectively, and the second equality follows from Girsanov's theorem and Lemma 2.3.3, as in (2.14). Since $\lambda_t^P > 0$ P-a.s. for all $t \geq 0$, we have then:

$$\gamma_t = \frac{\left(\bar{\lambda}^Q - \bar{\lambda}^P \right) + \left(\Lambda^Q - \Lambda^P \right)' X_t}{\bar{\lambda}^P + (\Lambda^P)' X_t}$$

thus showing that γ satisfies part *(ii)* of Condition B. □

Remark 2.3.13. Observe that if a probability measure Q on $(\Omega, \mathcal{G})$ with $Q \overset{loc}{\sim} P$ preserves the affine structure of (X, τ) then it also preserves the immersion property between the filtrations $\mathbb{F}$ and $\mathbb{G}$. This follows from part *(a)* of Theorem 2.3.10 or also from the fact that, for all $t \geq 0$, we have $Q(\tau \leq t | \mathcal{F}_\infty) = Q(\tau \leq t | \mathcal{F}_t)$, since τ is a doubly stochastic random time with respect to $(Q, \mathbb{F})$, together with Lemma 5.9.4.2 of Jeanblanc et al. (2009).

2.4 Application: the Heston with jump-to-default model

We now present an application of the general results of Section 2.3 on the preservation of the affine structure under a change of measure. More specifically, we consider a version of the popular stochastic volatility model proposed by Heston (1993), here extended by allowing the stock price process to be killed by a *jump-to-default* event. We shall first formulate the model with respect to the original probability measure P and then characterize the set of all probability measures $Q \sim P$ which preserve the key features of the model.

Using the notations introduced in Section 2.2, we let $d = 2$ and $m = 1$ and specify the functions $\mu(\cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ and $\sigma(\cdot) : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ appearing in the SDE (2.1) as follows, for all $x = (x^1, x^2)' \in (0,\infty) \times \mathbb{R}$:

$$\mu(x) := b + Ax \qquad \text{and} \qquad \sigma(x) := \sqrt{x^1}\begin{pmatrix} k & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \tag{2.21}$$

where $b = \big(\beta, \mu + \bar{\lambda}^P\big)'$ with $\beta \geq \frac{k^2}{2}$ and $\mu \in \mathbb{R}$, $A = \left(\begin{smallmatrix} \alpha & 0 \\ \Lambda^P - 1/2 & 0 \end{smallmatrix}\right)$ with $\alpha \in \mathbb{R}$, $\big(\bar{\lambda}^P, \Lambda^P\big) \in \mathbb{R}^2_+ \setminus \{0\}$, $k > 0$ and $\rho \in (-1,1)$. It is easy to check that this specification of the model parameters satisfies Condition A. Hence, due to Proposition 2.2.2, for any $\bar{x} \in (0,\infty) \times \mathbb{R}$, there exists an unique strong solution $X = (X_t)_{t\geq 0}$ to the SDE (2.1) on $(\Omega, \mathcal{G}, \mathbb{F}, P)$ taking values in $(0,\infty) \times \mathbb{R}$ with $X_0 = \bar{x}$. Let us denote the components of the $\mathbb{R}^2$-valued process X as $v := X^1$ and $L := X^2$. Let the random default time τ be a doubly stochastic random time with respect to $(P, \mathbb{F})$, with P-default intensity $\lambda^P_t = \bar{\lambda}^P + \Lambda^P v_t$ for all $t \geq 0$. We restrict our attention to the finite time horizon $[0,T]$, for some fixed $T \in (0,\infty)$ and we let the filtrations $\mathbb{F}$ and $\mathbb{G}$ be as in Section 2.2, with $\mathcal{G} = \mathcal{G}_T$.

The random time τ models the random occurrence of the default event of a given firm and we let the process $S = (S_t)_{0\leq t\leq T}$ represent the price of one share issued by that firm. We assume that the firm goes bankrupt as soon as the default event occurs, in which case the process S jumps to zero and remains thereafter frozen at zero. Formally, the defaultable stock price process $S = (S_t)_{0\leq t\leq T}$ is defined as $S_t := (1 - H_t)\,\widetilde{S}_t$ for all $t \in [0,T]$, where $\widetilde{S} = \big(\widetilde{S}_t\big)_{0\leq t\leq T}$ is a continuous $\mathbb{F}$-adapted process which represents the pre-default value of the stock[4]. The latter is defined as $\widetilde{S}_t := \exp(L_t)$ for all $t \in [0,T]$. A simple application of Itô's formula shows that the defaultable stock price process S satisfies the following SDE:

$$\begin{aligned} dS_t &= (1 - H_{t-})\,d\widetilde{S}_t - \widetilde{S}_t\,dH_t - d\big[\widetilde{S}, H\big]_t = (1 - H_{t-})\,d\exp(L_t) - \widetilde{S}_t\,dH_t \\ &= (1 - H_{t-})\exp(L_t)\Big(dL_t + \frac{1}{2}\,d\langle L\rangle_t\Big) - (1 - H_{t-})\,\widetilde{S}_t\,dH_t \\ &= S_{t-}\Big(dL_t + \frac{1}{2}\,d\langle L\rangle_t\Big) - S_{t-}\,dH_t \end{aligned} \tag{2.22}$$

where we have used the integration by parts formula, the fact that $\big[\widetilde{S}, H\big] \equiv 0$, since $\widetilde{S}$ is continuous and H is a pure jump process, and the definition of the process S. More explicitly, the pair (S, v)

[4]Note that, if the process S is continuous before default and jumps to zero at default, there is no loss of generality in assuming that the pre-default price process $\widetilde{S}$ is $\mathbb{F}$-adapted. This is due to the fact that every $\mathbb{G}$-predictable process is equal on the set $\{\tau \geq t\}$ to an $\mathbb{F}$-predictable process, see e.g. Proposition 5.9.4.1 of Jeanblanc et al. (2009).

satisfies the following system of SDEs:

$$\begin{aligned} dS_t &= S_{t-}\left(\mu+\lambda_t^P\right)dt + S_{t-}\sqrt{v_t}\left(\rho\, dW_t^1 + \sqrt{1-\rho^2}\, dW_t^2\right) - S_{t-}dH_t \\ &= S_{t-}\mu\, dt + S_{t-}\sqrt{v_t}\left(\rho\, dW_t^1 + \sqrt{1-\rho^2}\, dW_t^2\right) - S_{t-}dM_t^P \\ dv_t &= (\beta+\alpha v_t)\, dt + k\sqrt{v_t}\, dW_t^1 \end{aligned} \tag{2.23}$$

where, recalling part *(a)* of Lemma 2.2.3, the process $M^P := H - \int_0^{\cdot\wedge\tau}\lambda_u^P du$ is a $(P,\mathbb{G})$-martingale. Clearly, the system of SDEs (2.23) recalls the classical stochastic volatility model proposed by Heston (1993), here extended with a *jump-to-default*, as in Carr & Schoutens (2008). However, unlike in Carr & Schoutens (2008), the model is here formulated with respect to the original probability measure P and not directly with respect to a risk-neutral measure Q, the existence of which has yet to be properly studied. Furthermore, unlike in Carr & Schoutens (2008), the P-default intensity $\lambda^P = \left(\lambda_t^P\right)_{0\leq t\leq T}$ of the random default time τ is allowed to be stochastic, due to its dependence on the volatility process v.

2.4.1 Equivalent changes of measure

The process $X = (v,L)'$ satisfies an affine SDE of the type (2.1), where $v = (v_t)_{0\leq t\leq T}$ and $L = (L_t)_{0\leq t\leq T}$ denote the stochastic volatility and the logarithm of the pre-default stock price, respectively. The results of Section 2.3 allow us to characterize the family of all probability measures Q on $(\Omega,\mathcal{G})$ with $Q\sim P$ which preserve the affine structure of (X,τ). Note that, since we consider a finite time horizon, all martingales are automatically uniformly integrable[5] and, therefore, there is no need to enforce Assumption 2.3.5 (see also Remark 2.3.6). The following Proposition is an immediate consequence of Theorem 2.3.12.

Proposition 2.4.1. *Let the processes $\theta = (\theta_t)_{0\leq t\leq T}$ and $\gamma = (\gamma_t)_{0\leq t\leq T}$ be defined as follows:*

$$\begin{aligned} \theta_t := \theta(X_t) &= \frac{1}{\sqrt{v_t}}\begin{pmatrix} 1/k & 0 \\ -\frac{\rho}{k\sqrt{1-\rho^2}} & \frac{1}{\sqrt{1-\rho^2}} \end{pmatrix}\left(\hat\theta + \Theta\begin{pmatrix} v_t \\ L_t \end{pmatrix}\right) \\ &= \begin{pmatrix} \frac{1}{k}\left(\frac{\hat\theta^1}{\sqrt{v_t}} + \Theta^{11}\sqrt{v_t}\right) \\ \frac{1}{\sqrt{1-\rho^2}}\left(\frac{\hat\theta^2+\Theta^{22}L_t}{\sqrt{v_t}} + \Theta^{21}\sqrt{v_t}\right) - \frac{\rho}{\sqrt{1-\rho^2}}\theta_t^1 \end{pmatrix} \end{aligned} \tag{2.24}$$

$$\gamma_t := \gamma(X_t) = \frac{\left(\bar\lambda^Q - \bar\lambda^P\right) + \left(\Lambda^Q - \Lambda^P\right)v_t}{\bar\lambda^P + \Lambda^P v_t} \tag{2.25}$$

for $\hat\theta\in\mathbb{R}^2$ with $\hat\theta^1 \geq \frac{k^2}{2} - \beta$ and $\Theta\in\mathbb{R}^{2\times 2}$ with $\Theta^{12} = 0$ and $\left(\bar\lambda^Q,\Lambda^Q\right)\in\mathbb{R}_+^2\setminus\{0\}$. Then we can define a probability measure $Q\sim P$ on $(\Omega,\mathcal{G})$ by letting $\frac{dQ}{dP} := \mathcal{E}\left(\int\theta' dW + \int\gamma\, dM^P\right)_T$ and (X,τ) has an affine structure with respect to Q. Conversely, any probability measure Q on $(\Omega,\mathcal{G})$ with $Q\sim P$ which preserves the affine structure of (X,τ) has a density process with respect

[5] This is due to the fact that any martingale $M = (M_t)_{0\leq t\leq T}$ is *closed* by its terminal value M_T, meaning that we have $M_t = E\left[M_T|\mathcal{F}_t\right]$ P-a.s. for all $t\in[0,T]$ (compare also with Protter (2005), Theorem I.13).

to $(P,\mathbb{G})$ *of the form* $Z^{Q,\mathbb{G}} = \mathcal{E}\left(\int \theta' dW + \int \gamma\, dM^P\right)$*, where the processes* θ *and* γ *are as in* (2.24)-(2.25) *and satisfy the above conditions.*

Let Q be a probability measure on $(\Omega,\mathcal{G})$ with $Q \sim P$ such that the affine structure of (X,τ) is preserved. By relying on Proposition 2.4.1 and Girsanov's theorem, we can write as follows the dynamics of (S,v) with respect to Q:

$$\begin{aligned} dS_t &= S_{t-}\left(\mu + \bar{\lambda}^P - \bar{\lambda}^Q + \hat{\theta}^2 + \left(\Theta^{21} + \Lambda^P - \Lambda^Q\right) v_t + \Theta^{22} L_t\right) dt \\ &\quad + S_{t-}\sqrt{v_t}\left(\rho\, dW_t^{Q,1} + \sqrt{1-\rho^2}\, dW_t^{Q,2}\right) - S_{t-} dM_t^Q \\ dv_t &= \left(\beta + \hat{\theta}^1 + \left(\alpha + \Theta^{11}\right) v_t\right) dt + k\sqrt{v_t}\, dW_t^{Q,1} \end{aligned} \tag{2.26}$$

where, for $i = 1,2$, the process $W^{Q,i} = \left(W_t^{Q,i}\right)_{0\leq t\leq T}$ is the $(Q,\mathbb{F})$/$(Q,\mathbb{G})$-Brownian motion defined as $W^{Q,i} := W^i - \int_0^{\cdot} \theta_u^i du$, and the process $M^Q = \left(M_t^Q\right)_{0\leq t\leq T}$ is the $(Q,\mathbb{G})$-martingale defined as $M^Q := M^P - \int_0^{\cdot\wedge\tau} \gamma_u \lambda_u^P du = H - \int_0^{\cdot\wedge\tau} \lambda_u^Q du$, with $\lambda_t^Q = \lambda_t^P(1+\gamma_t) = \bar{\lambda}^Q + \Lambda^Q v_t$ for all $t \in [0,T]$.

Note that, even if (X,τ) maintains the affine structure, the equivalent change of measure from P to Q may introduce a dependence on v and L in the drift term of the defaultable stock price process S, as can be seen from (2.26). Hence, in view of practical applications, one could be interested in the set of all equivalent probability measures which preserve the original structure (2.23) of the Heston with jump-to-default model. More precisely, let us give the following Definition.

Definition 2.4.2. *Let* Q *be a probability measure on* $(\Omega,\mathcal{G})$ *with* $Q \sim P$*. We say that* Q preserves the Heston with jump-to-default structure *if* Q *preserves the affine structure of* (X,τ)*, in the sense of Definition 2.3.1, and the pair* (S,v) *satisfies a system of SDEs of the type* (2.23) *also with respect to* Q*.*

By looking at (2.23) and (2.26), we can rephrase Definition 2.4.2 by saying that a probability measure Q on $(\Omega,\mathcal{G})$ with $Q \sim P$ preserves the Heston with jump-to-default structure if it preserves the affine structure of (X,τ) and the drift term of the SDE satisfied by the process S under Q is of the form $S_{t-}\mu^Q dt$ for some $\mu^Q \in \mathbb{R}$. We have then the following simple Corollary.

Corollary 2.4.3. *A probability measure* Q *on* $(\Omega,\mathcal{G})$ *with* $Q \sim P$ *preserves the Heston with jump-to-default structure if and only if its density process with respect to* $(P,\mathbb{G})$ *is of the form* $Z^{Q,\mathbb{G}} = \mathcal{E}\left(\int \theta' dW + \int \gamma\, dM^P\right)$*, where the processes* θ *and* γ *are as in Proposition 2.4.1 with* $\Theta^{22} = 0$ *and* $\Theta^{21} = \Lambda^Q - \Lambda^P$*.*

Proof. If the processes θ and γ are as in (2.24)-(2.25) with $\Theta^{22} = 0$ and $\Theta^{21} = \Lambda^Q - \Lambda^P$, Proposition 2.4.1 and (2.26) imply that the Heston with jump-to-default structure is preserved under the probability measure $Q \sim P$ defined by $\frac{dQ}{dP} := \mathcal{E}\left(\int \theta' dW + \int \gamma\, dM^P\right)_T$. Conversely, let Q be a probability measure on $(\Omega,\mathcal{G})$ with $Q \sim P$ which preserves the Heston with jump-to-default structure. Then, Proposition 2.4.1 implies that $Z^{Q,\mathbb{G}} = \mathcal{E}\left(\int \theta' dW + \int \gamma\, dM^P\right)$, where θ and γ are as in

(2.24)-(2.25). Furthermore, due to (2.26), the following identity must hold P-a.s. for all $t \in [0,T]$, for some $\mu^Q \in \mathbb{R}$:

$$\mu^Q = \mu + \bar{\lambda}^P - \bar{\lambda}^Q + \hat{\theta}^2 + \left(\Theta^{21} + \Lambda^P - \Lambda^Q\right) v_t + \Theta^{22} L_t$$

Since the process $X = (v, L)'$ takes values in $(0, \infty) \times \mathbb{R}$, it can be easily checked that this implies $\Theta^{22} = 0$ and $\Theta^{21} = \Lambda^Q - \Lambda^P$. □

2.4.2 Characterization of *Equivalent (Local) Martingale Measures*

Let us denote the discounted defaultable stock price process as $\bar{S} = \left(\bar{S}_t\right)_{0 \leq t \leq T}$, with $\bar{S}_t := e^{-rt} S_t$ for all $t \in [0,T]$ and where r is the risk-free interest rate, which is supposed to be constant in the time interval $[0,T]$. Clearly, the $\mathbb{G}$-adapted process $\bar{S}$ is locally bounded, since it is continuous before the default time τ and goes to the cemetery state 0 as soon as the default event occurs. Due to the fundamental theorem of asset pricing in the version of Delbaen & Schachermayer (1994), the *No Free Lunch with Vanishing Risk* no-arbitrage condition holds if and only if there exists an *Equivalent Local Martingale Measure* (ELMM) for S, i.e. a probability measure Q on $(\Omega, \mathcal{G})$ with $Q \sim P$ such that $\bar{S}$ is a $(Q, \mathbb{G})$-local martingale. Hence, in order to exclude arbitrage opportunities, we want to be sure that the set of all ELMMs for S is non-empty. Furthermore, especially in view of practical applications, one could be interested in preserving the Heston with jump-to-default structure under an ELMM. The following simple Corollary characterizes the set of all ELMMs for S which preserves the Heston with jump-to-default structure.

Corollary 2.4.4. *A probability measure Q on $(\Omega, \mathcal{G})$ with $Q \sim P$ is an ELMM for S which preserves the Heston with jump-to-default structure if and only if its density process $Z^{Q,\mathbb{G}}$ with respect to $(P, \mathbb{G})$ is of the form $Z^{Q,\mathbb{G}} = \mathcal{E}\left(\int \theta' dW + \int \gamma \, dM^P\right)$, where the processes θ and γ are given as follows, for all $t \in [0,T]$:*

$$\theta_t = \begin{pmatrix} \theta_t^1 \\ \theta_t^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{k}\left(\frac{\hat{\theta}^1}{\sqrt{v_t}} + \Theta^{11}\sqrt{v_t}\right) \\ \frac{1}{\sqrt{1-\rho^2}}\left(\frac{\bar{\lambda}^Q - \bar{\lambda}^P - \mu + r}{\sqrt{v_t}} + \left(\Lambda^Q - \Lambda^P\right)\sqrt{v_t}\right) - \frac{\rho}{\sqrt{1-\rho^2}}\theta_t^1 \end{pmatrix} \tag{2.27}$$

$$\gamma_t = \frac{\left(\bar{\lambda}^Q - \bar{\lambda}^P\right) + \left(\Lambda^Q - \Lambda^P\right) v_t}{\bar{\lambda}^P + \Lambda^P v_t} \tag{2.28}$$

with $\hat{\theta}^1 \geq \frac{k^2}{2} - \beta$, $\Theta^{11} \in \mathbb{R}$ and $\left(\bar{\lambda}^Q, \Lambda^Q\right) \in \mathbb{R}_+^2 \setminus \{0\}$.

Proof. The result follows immediately from Proposition 2.4.1 and Corollary 2.4.3 together with (2.26), since the process S satisfies the following SDE under any ELMM Q:

$$dS_t = S_{t-} r dt + S_{t-}\sqrt{v_t}\left(\rho \, dW_t^{Q,1} + \sqrt{1-\rho^2}\, dW_t^{Q,2}\right) - S_{t-} dM_t^Q \tag{2.29}$$

□

Remark 2.4.5 (The default-free Heston (1993) model). Let us briefly consider the case of the Heston (1993) stochastic volatility model in its typical formulation, i.e. without the inclusion of the random default event, so that $\mathbb{F} = \mathbb{G}$. In this case, Corollary 2.4.4 implies that the density process $Z^{Q,\mathbb{F}}$ with respect to $(P, \mathbb{F})$ of any ELMM Q for S which preserves the Heston structure is of the form $Z^{Q,\mathbb{F}} = \mathcal{E}\left(\int \theta' dW\right)$, where the $\mathbb{R}^2$-valued process $\theta = (\theta_t)_{0 \leq t \leq T}$ is given as follows:

$$\theta_t = \begin{pmatrix} \theta_t^1 \\ \theta_t^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{k}\left(\frac{\hat{\theta}^1}{\sqrt{v_t}} + \Theta^{11}\sqrt{v_t}\right) \\ \frac{1}{\sqrt{1-\rho^2}}\frac{r-\mu}{\sqrt{v_t}} - \frac{\rho}{\sqrt{1-\rho^2}}\theta_t^1 \end{pmatrix} \tag{2.30}$$

for all $t \in [0, T]$, with $\hat{\theta}^1 \geq \frac{k^2}{2} - \beta$ and $\Theta^{11} \in \mathbb{R}$.

In mathematical finance, it is customary to call *market prices of risk* (or *risk premia*) the processes θ and γ characterizing the density process of an ELMM for S. In particular, the process θ^1 represents the market price of volatility risk, since it is associated to the Girsanov's transformation of the Brownian motion W^1 which drives the stochastic volatility process v. Similarly, the process θ^2 is linked to the market price of diffusive risk for the stock price, since it is associated to the Girsanov's transformation of the Brownian motion W^2 which only intervenes in the dynamics of S. Finally, the process γ represents the market price of risk associated to the jump-to-default risk, i.e. the risk of an unpredictable default event which kills the stock price process. We refer the reader to Section 2.5.3 for a more detailed discussion of the financial interpretation of the processes θ and γ.

We want to emphasize that the specification (2.27)-(2.28) for the risk premia processes θ and γ is very general and nests most of the market price of risk specifications usually encountered in the literature, as we are going to argue now. For simplicity, let us consider for the moment the default-free case, as in Remark 2.4.5. The typical choice for the market price of volatility risk in the Heston model assumes that $\hat{\theta}^1 = 0$, so that θ_t^1 is proportional to $\sqrt{v_t}$. This implies that the market price for volatility risk θ_t^1 becomes very small as the volatility approaches zero. By letting $\hat{\theta}^1 = 0$ in (2.30), we recover the specification of $\theta_t = (\theta_t^1, \theta_t^2)'$ considered in Wong & Heyde (2006). However, unlike in Wong & Heyde (2006), we do not need to impose any restriction on the parameter Θ^{11} in order to ensure that the process $\mathcal{E}\left(\int \theta' dW\right)$ is well-defined as a $(P, \mathbb{F})$-martingale. This generalizes Theorem 3.5 of Wong & Heyde (2006). The reason why we are able to obtain sharper results than those of Wong & Heyde (2006) is that, by relying on the general theory presented in Section 2.3, we avoid any model-specific computation and the use of Novikov conditions. Furthermore, our results can also cover the case where $\hat{\theta}^1 \neq 0$. In the latter case, the fact that θ_t^1 involves then the reciprocal of $\sqrt{v_t}$ and, hence, θ_t^1 can grow without bound as v_t approaches zero, creates no technical problems, due to the fact that Condition A ensures that zero is an unattainable boundary for v (see Proposition 2.2.2) and so θ_t^1 remains finite. Letting $\hat{\theta}^1 \neq 0$ allows all the parameters in the drift term of the process v to change from P to Q, with a consequent improvement of the flexibility of the model, as has been empirically documented by Cheridito et al. (2007) in the context of term structure modeling. Furthermore, in the jump-to-default case, also the default intensity parameters can change from $\left(\bar{\lambda}^P, \Lambda^P\right)$ to $\left(\bar{\lambda}^Q, \Lambda^Q\right)$. Unlike previous known results on the Heston model, the conditions on the processes θ and γ given in Corollary 2.4.4 are not only sufficient but also necessary if one wants the probability measure Q to be an ELMM for S and preserve the Heston with jump-to-default structure as well.

Suppose now that θ and γ satisfy (2.27)-(2.28). Corollary 2.4.4 then implies that the probability measure Q on $(\Omega, \mathcal{G})$ defined by $\frac{dQ}{dP} := \mathcal{E}\left(\int \theta' dW + \int \gamma \, dM^P\right)_T$ is an ELMM for S and preserves the Heston with jump-to-default structure. However, we do not know a priori if $\bar{S}$ is a true $(Q, \mathbb{G})$-martingale or only a $(Q, \mathbb{G})$-local martingale, i.e. we do not know if Q is an *Equivalent Martingale Measure* (EMM) for S or only an ELMM. We now provide an answer to this question, showing that any ELMM for S which preserves the Heston with jump-to-default structure is automatically an EMM for S. We first consider in the following Lemma the default-free case (with $\mathbb{F} = \mathbb{G}$), as in Remark 2.4.5.

Lemma 2.4.6. *Suppose that $\lambda^P \equiv 0$, i.e. the random default time τ satisfies $P(\tau = \infty) = 1$. Let Q be an ELMM for S which preserves the affine structure of $X = (v, L)'$, where $L = \log S$. Then the discounted stock price process $\bar{S}$ is a true $(Q, \mathbb{F})$-martingale, i.e. Q is an EMM for S.*

Proof. Let Q be an ELMM for S which preserves the affine structure of $X = (v, L)'$. This means that X satisfies an SDE of the form (2.1) with respect to Q and the process S satisfies the following SDE under Q:

$$dS_t = S_t r dt + S_t \sqrt{v_t}\left(\rho \, dW_t^{Q,1} + \sqrt{1-\rho^2}\, dW_t^{Q,2}\right) \qquad S_0 \in (0, \infty)$$

Define now the process $\widetilde{Z} = \left(\widetilde{Z}_t\right)_{0 \le t \le T}$ by $\widetilde{Z}_t := e^{-rt} S_t / S_0$ for all $t \in [0, T]$. A simple application of Itô's formula allows to show that the process $\widetilde{Z}$ can be represented as follows, for all $t \in [0, T]$:

$$\widetilde{Z}_t = \mathcal{E}\left(\rho \int \sqrt{v}\, dW^{Q,1} + \sqrt{1-\rho^2} \int \sqrt{v}\, dW^{Q,2}\right)_t = \mathcal{E}\left(\int \tilde{\theta}^1 \, dW^{Q,1} + \int \tilde{\theta}^2 \, dW^{Q,2}\right)_t$$

where $\tilde{\theta}_t^1 := \rho \sqrt{v_t}$ and $\tilde{\theta}_t^2 := \sqrt{1-\rho^2}\sqrt{v_t}$ for all $t \in [0, T]$. Observe now that we can represent $\left(\tilde{\theta}_t^1, \tilde{\theta}_t^2\right)'$ as follows:

$$\begin{pmatrix} \tilde{\theta}_t^1 \\ \tilde{\theta}_t^2 \end{pmatrix} = \frac{1}{\sqrt{v_t}} \begin{pmatrix} 1/k & 0 \\ -\frac{\rho}{k\sqrt{1-\rho^2}} & \frac{1}{\sqrt{1-\rho^2}} \end{pmatrix} \left(0 + \widetilde{\Theta} \begin{pmatrix} v_t \\ L_t \end{pmatrix}\right)$$

where $\widetilde{\Theta} = \left(\begin{smallmatrix} k\rho & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Proposition 2.4.1, applied with respect to the probability measure Q (and neglecting the process γ since $P(\tau = \infty) = 1$), implies then that the process $\widetilde{Z}$ is the density process with respect to $(Q, \mathbb{F})$ of the probability measure $\widetilde{Q} \sim Q$ on $(\Omega, \mathcal{G})$ defined by $\frac{d\widetilde{Q}}{dQ} := \widetilde{Z}_T = e^{-rT} S_T / S_0$. In particular, $\widetilde{Z}$ is a $(Q, \mathbb{F})$-martingale, thus proving the Lemma. □

We want to point out that Lemma 2.4.6 extends Theorem 3.6 of Wong & Heyde (2006), where the authors obtain an analogous result under more restrictive conditions on the parameters of the model. The following Proposition, the proof of which relies on Lemma 2.4.6, deals with the more general case of the Heston with jump-to-default model.

Proposition 2.4.7. *Let Q be an ELMM for S which preserves the Heston with jump-to-default structure. Then the discounted stock price process $\bar{S}$ is a true $(Q,\mathbb{G})$-martingale, i.e. Q is an EMM for S.*

Proof. Let Q be an ELMM for S which preserves the Heston with jump-to-default structure. As in the proof of Lemma 2.4.6, let us define the process $\widetilde{Z} = \big(\widetilde{Z}_t\big)_{0\leq t\leq T}$ by $\widetilde{Z}_t := e^{-rt}S_t/S_0$ for all $t \in [0,T]$. Recall that we always assume $\tau > 0$ P-a.s., so that $S_0 > 0$ P-a.s. and, hence, also Q-a.s. Equation (2.29) and a simple application of Itô's formula allow to represent $\widetilde{Z}$ as follows, for all $t \in [0,T]$:

$$\widetilde{Z}_t = \mathcal{E}\left(\rho\int\sqrt{v}\,dW^{Q,1} + \sqrt{1-\rho^2}\int\sqrt{v}\,dW^{Q,2} - M^Q\right)_t$$

Note that (see e.g. Jeanblanc et al. (2009), Proposition 8.4.4.1):

$$\mathcal{E}\left(-M^Q\right)_t = (1-H_t)\exp\left(\int_0^{t\wedge\tau}\lambda_u^Q du\right) = \mathbf{1}_{\{\tau>t\}}\exp\left(\int_0^t\lambda_u^Q du\right)$$

Clearly, the process $\widetilde{Z}$ jumps at 0 at the random default time τ and is a non-negative $(Q,\mathbb{G})$-local martingale. Therefore, $\widetilde{Z}$ is also a $(Q,\mathbb{G})$-supermartingale. To prove the Proposition, it suffices to show that $E^Q\big[\widetilde{Z}_T\big] = 1$, where $E^Q[\cdot]$ denotes the expectation under the measure Q.

$$\begin{aligned}
E^Q\big[\widetilde{Z}_T\big] &= E^Q\Big[E^Q\big[\widetilde{Z}_T|\mathcal{F}_T\big]\Big] \\
&= E^Q\left[\mathcal{E}\left(\rho\int\sqrt{v}\,dW^{Q,1} + \sqrt{1-\rho^2}\int\sqrt{v}\,dW^{Q,2}\right)_T Q\left(\tau>T|\mathcal{F}_T\right)\exp\left(\int_0^T\lambda_u^Q du\right)\right] \\
&= E^Q\left[\mathcal{E}\left(\rho\int\sqrt{v}\,dW^{Q,1} + \sqrt{1-\rho^2}\int\sqrt{v}\,dW^{Q,2}\right)_T\right] = 1
\end{aligned}$$

where the second equality uses the fact that the Q-intensity process λ^Q is $\mathbb{F}$-adapted, being an affine function of the $\mathbb{F}$-adapted process v, the third equality follows from (2.3) (with respect to Q) and, finally, the last equality follows as in the proof of Lemma 2.4.6. □

Lemma 2.4.6 and Proposition 2.4.7 show that, both for the classical Heston (1993) model and for the Heston with jump-to-default model, any ELMM for S which preserves the structure of the model is automatically an EMM for S. This excludes the situation where the discounted stock price process $\bar{S}$ is a strict local martingale, i.e. a local martingale which is not a true martingale.

Remark 2.4.8 (On the impossibility of stock price bubbles). In the recent literature, several authors have considered models that allow the discounted stock price process to follow a strict local martingale under an ELMM Q, see e.g. Cox & Hobson (2005), Heston et al. (2007), Jarrow et al. (2007) and Jarrow et al. (2010). This apparent anomaly does not contradict the NFLVR no-arbitrage condition and has been interpreted as the occurrence of a *stock price bubble*, since $S_u > E^Q\left[e^{-r(t-u)}S_t|\mathcal{G}_u\right]$ for all $0 < u < t \leq T$, due to the strict supermartingale property of $\bar{S}$. This means that the current price S_u of the stock exceeds its fundamental value, the latter being defined as the expectation (under the risk-neutral measure Q) of the discounted future value S_t.

Proposition 2.4.7 shows that, if the Heston with jump-to-default structure is preserved under the ELMM Q, there cannot exist stock price bubbles. However, note that the impossibility of stock price bubbles does not imply that *option bubbles* are automatically banned from the market, as shown in Example 1.3 of Heston et al. (2007).

Remark 2.4.9 (On the change of numéraire). Let us briefly return to the default-free case considered in Lemma 2.4.6 and let Q be an ELMM for S which preserves the affine structure $X = (v, L)'$. Let us define the process $\widetilde{Z} = \big(\widetilde{Z}_t\big)_{0 \leq t \leq T}$ as $\widetilde{Z}_t := e^{-rt} S_t / S_0$ for all $t \in [0, T]$. The proof of Lemma 2.4.6 shows that $\widetilde{Z}$ can be represented as $\widetilde{Z} = \mathcal{E}\left(\int \tilde{\theta}' dW^Q\right)$, where the $\mathbb{R}^2$-valued process $\tilde{\theta}$ is of the form (2.24). Hence, due to Proposition 2.4.1, the process $\widetilde{Z}$ is the density process of the probability measure $\widetilde{Q}$ on $(\Omega, \mathcal{G})$ defined by $\frac{d\widetilde{Q}}{dQ} := \widetilde{Z}_T = e^{-rT} S_T / S_0$. We have $\widetilde{Q} \sim Q$ and $\widetilde{Q}$ preserves the affine structure of $X = (v, L)'$. Clearly, this has important and useful implications if one wants to use *change of numéraire* techniques (see e.g. Jeanblanc et al. (2009), Section 2.4) with the stock price process S as numéraire. In fact, under the probability measure $\widetilde{Q}$ which uses S as numéraire, the preservation of the affine structure of the process $X = (v, L)'$ ensures a remarkable analytical tractability of the model. Of course, an analogous result holds true for the Heston with jump-to-default model if we restrict our attention to the stochastic interval $[\![0, \tau[\![$.

2.4.3 The Minimal Martingale Measure

This Section is devoted to another application of the general results presented in Section 2.3. More specifically, we shall be concerned with the study of the existence and the characterization of the *Minimal Martingale Measure* in the context of the Heston with jump-to-default model.

For simplicity, let us first consider the case of the default-free Heston model, where the stochastic volatility process $v = (v_t)_{0 \leq t \leq T}$ and the stock price process $S = (S_t)_{0 \leq t \leq T}$ satisfy the following system of SDEs on $(\Omega, \mathcal{G}, \mathbb{F}, P)$:

$$
\begin{aligned}
dv_t &= (\beta + \alpha v_t)\, dt + k\sqrt{v_t} dW_t^1 && v_0 = \bar{v} \in (0, \infty) \\
dS_t &= S_t \mu\, dt + S_t \sqrt{v_t}\left(\rho\, dW_t^1 + \sqrt{1-\rho^2} dW_t^2\right) && S_0 = \bar{s} \in (0, \infty)
\end{aligned}
$$

with $\alpha, \mu \in \mathbb{R}$, $\beta \geq \frac{k^2}{2}$, $k > 0$ and $\rho \in (-1, 1)$. Observe that the discounted stock price process $\bar{S} = \big(\bar{S}_t\big)_{0 \leq t \leq T}$ admits the following unique *canonical decomposition* (see e.g. Protter (2005), Section III.7), being a continuous $\mathbb{F}$-semimartingale:

$$
\bar{S} = \bar{s} + \bar{A} + \bar{M}
$$

where $\bar{A}$ is a continuous $\mathbb{F}$-adapted process of finite variation with $\bar{A}_0 = 0$ and $\bar{M}$ is a continuous $(P, \mathbb{F})$-local martingale with $\bar{M}_0 = 0$, explicitly given as:

$$
\bar{A} = \int_0^{\cdot} \bar{S}_u\, (\mu - r)\, du \qquad \text{and} \qquad \bar{M} = \int_0^{\cdot} \bar{S}_u \sqrt{v_u}\left(\rho\, dW_u^1 + \sqrt{1-\rho^2}\, dW_u^2\right)
$$

Denote also by $\langle \bar{M} \rangle = \int_0^\cdot \bar{S}_u^2 v_u du$ the predictable quadratic variation of $\bar{M}$. Clearly, we have $\bar{A} \ll \langle \bar{M} \rangle$. In fact, we can write, noting that the processes $\bar{S}$ and v are strictly positive:

$$\bar{A} = \int \phi \, d\langle \bar{M} \rangle \qquad \text{where } \phi_t = \frac{\mu - r}{\bar{S}_t \, v_t} \quad \forall t \in [0, T]$$

Note that, since the processes $\bar{S}$ and v are both continuous and $\mathbb{F}$-adapted, the process $\phi = (\phi_t)_{0 \leq t \leq T}$ is $\mathbb{F}$-predictable and locally bounded. Hence, we have $\phi \in L^2_{loc}(\bar{M})$, thus implying that the stochastic integral $\int \phi \, d\bar{M}$ is well-defined as a continuous $(P, \mathbb{F})$-local martingale. Let us now define the process $\widehat{Z} = (\widehat{Z}_t)_{0 \leq t \leq T}$ as follows:

$$\widehat{Z} := \mathcal{E}\left(- \int \phi \, d\bar{M} \right) = \mathcal{E}\left(- \int_0^\cdot \frac{\mu - r}{\sqrt{v_u}} \left(\rho \, dW_u^1 + \sqrt{1 - \rho^2} \, dW_u^2 \right) \right)$$

Since $\phi \in L^2_{loc}(\bar{M})$, the process $\widehat{Z}$ is a P-a.s. strictly positive $(P, \mathbb{F})$-local martingale and is the *candidate* density process of the so-called *Minimal Martingale Measure* $\widehat{Q}$, first introduced in Föllmer & Schweizer (1991). Intuitively, the candidate density process of the Minimal Martingale Measure is the "simplest" local martingale $\widehat{Z} = (\widehat{Z}_t)_{0 \leq t \leq T}$ with $\widehat{Z}_0 = 1$ which satisfies the property that the product $\widehat{Z}\bar{S}$ is a local martingale. The process $\widehat{Z}$ is a well-defined density process for $\widehat{Q}$ if and only if $\widehat{Z}$ is a $(P, \mathbb{F})$-martingale. However, we do not know if $\widehat{Z}$ is a true $(P, \mathbb{F})$-martingale or just a $(P, \mathbb{F})$-local martingale. In the case of the default-free Heston model, the following Proposition gives an affirmative answer. Furthermore, we show that the minimal martingale measure $\widehat{Q}$ is not only an ELMM for S but also an EMM which preserves the affine structure of $X = (v, L)'$, where $L = \log S$.

Proposition 2.4.10. *Suppose that $\beta - k\rho(\mu - r) \geq \frac{k^2}{2}$. Then the process $\widehat{Z}$ is a $(P, \mathbb{F})$-martingale and the measure $\widehat{Q}$ on $(\Omega, \mathcal{G})$ defined by $\frac{d\widehat{Q}}{dP} := \widehat{Z}_T$ is an EMM for S which preserves the affine structure of $X = (v, L)'$.*

Proof. Observe first that the process $\widehat{Z}$ can be represented as follows, for all $t \in [0, T]$:

$$\widehat{Z}_t = \mathcal{E}\left(\int \theta^1 \, dW^1 + \int \theta^2 \, dW^2 \right)_t \qquad \text{where } \theta_t^1 := -\rho \frac{\mu - r}{\sqrt{v_t}} \text{ and } \theta_t^2 := -\sqrt{1 - \rho^2} \frac{\mu - r}{\sqrt{v_t}}$$

Note that the processes $\theta^1 = (\theta_t^1)_{0 \leq t \leq T}$ and $\theta^2 = (\theta_t^2)_{0 \leq t \leq T}$ are of the form (2.24), with:

$$\hat{\theta} = \begin{pmatrix} k\rho(r - \mu) \\ r - \mu \end{pmatrix} \qquad \text{and} \qquad \Theta = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

If $\beta + \hat{\theta}^1 = \beta - k\rho(\mu - r) \geq \frac{k^2}{2}$, Corollary 2.4.4 and Remark 2.4.5 imply that the probability measure $\widehat{Q}$ on $(\Omega, \mathcal{G})$ defined by $\frac{d\widehat{Q}}{dP} := \widehat{Z}_T$ admits $\widehat{Z}$ as density process and is an ELMM for S which preserves the affine structure of $X = (v, L)'$. Furthermore, due to Lemma 2.4.6, the process $\bar{S}$ is a true $(\widehat{Q}, \mathbb{F})$-martingale and not only a $(\widehat{Q}, \mathbb{F})$-local martingale, meaning that $\widehat{Q}$ is an EMM for S. $\square$

Let us now move to the more general case of the Heston with jump-to-default model. Recall that, under the original probability measure P, the stochastic volatility process v and the defaultable stock price process S satisfy the following system of SDEs:

$$\begin{aligned} dv_t &= (\beta + \alpha v_t)\,dt + k\sqrt{v_t}dW_t^1 && v_0 = \bar{v} \in (0,\infty) \\ dS_t &= S_{t-}\mu\,dt + S_{t-}\sqrt{v_t}\left(\rho\,dW_t^1 + \sqrt{1-\rho^2}dW_t^2\right) - S_{t-}dM_t^P && S_0 = \bar{s} \in (0,\infty) \end{aligned}$$

Since the discounted stock price process $\bar{S} = \left(\bar{S}_t\right)_{0\leq t\leq T}$ is a locally bounded $\mathbb{G}$-semimartingale, it admits the following unique *canonical decomposition*:

$$\bar{S} = \bar{s} + \bar{A} + \bar{M}$$

where $\bar{A}$ is a $\mathbb{G}$-predictable process of finite variation with $\bar{A}_0 = 0$ and $\bar{M}$ is a $(P,\mathbb{G})$-local martingale with $\bar{M}_0 = 0$, explicitly given as:

$$\bar{A} = \int_0^\cdot \bar{S}_{u-}(\mu - r)\,du \quad \text{and} \quad \bar{M} = \int_0^\cdot \bar{S}_{u-}\sqrt{v_u}\left(\rho\,dW_u^1 + \sqrt{1-\rho^2}\,dW_u^2\right) - \int_0^\cdot \bar{S}_{u-}dM_u^P$$

Denote also by $\langle\bar{M}\rangle = \int_0^\cdot \bar{S}_{u-}^2\left(v_u + \lambda_u^P\right)du$ the predictable quadratic variation of $\bar{M}$. Note that $\langle\bar{M}\rangle$ exists since $\bar{M}$ is locally bounded. We then have $\bar{A} \ll \langle\bar{M}\rangle$. In fact, we can write:

$$\bar{A} = \int \phi\,d\langle\bar{M}\rangle \qquad \text{where } \phi_t = \frac{\mu - r}{\bar{S}_{t-}\left(v_t + \lambda_t^P\right)} \quad \forall t \in [0,T]$$

Note that, since the processes v and λ^P are both continuous and $\mathbb{F}$-adapted and the process $\bar{S}_-$ is left-continuous and $\mathbb{G}$-adapted, the process ϕ is $\mathbb{G}$-predictable and locally bounded. Hence, the stochastic integral $\int\phi\,d\bar{M}$ is well-defined as a $(P,\mathbb{G})$-local martingale (see e.g. Protter (2005), Theorem IV.29). Let us now define the process $\widehat{Z} = \left(\widehat{Z}_t\right)_{0\leq t\leq T}$ as follows:

$$\begin{aligned} \widehat{Z} &:= \mathcal{E}\left(-\int\phi\,d\bar{M}\right) \\ &= \mathcal{E}\left(-\int_0^\cdot \frac{\sqrt{v_u}\,(\mu - r)}{v_u + \lambda_u^P}\left(\rho\,dW_u^1 + \sqrt{1-\rho^2}\,dW_u^2\right) + \int_0^\cdot \frac{\mu - r}{v_u + \lambda_u^P}dM_u^P\right) \end{aligned}$$

As in the case of the default-free Heston model, the process $\widehat{Z}$ represents the candidate density process with respect to $(P,\mathbb{G})$ of the minimal martingale measure $\widehat{Q}$. In general, $\widehat{Z}$ is a $(P,\mathbb{G})$-local martingale, but we do not know a priori whether it is a P-a.s. strictly positive true $(P,\mathbb{G})$-martingale. The following Proposition gives some simple sufficient conditions for $\widehat{Z}$ to be a well-defined density process.

Proposition 2.4.11. *Let $\mu \geq r$ and suppose that $\bar{\lambda}^P = 0$ and $\beta - \frac{k\rho(\mu-r)}{1+\Lambda^P} \geq \frac{k^2}{2}$. Then the process $\widehat{Z}$ is a P-a.s. strictly positive $(P,\mathbb{G})$-martingale. Furthermore, the measure $\widehat{Q}$ on $(\Omega,\mathcal{G})$ defined by $\frac{d\widehat{Q}}{dP} := \widehat{Z}_T$ is an EMM for S and preserves the Heston with jump-to-default structure.*

Proof. Observe first that the process $\widehat{Z}$ can be represented as follows:

$$\widehat{Z} = \mathcal{E}\left(\int \theta^1 \, dW^1 + \int \theta^2 \, dW^2 + \int \gamma \, dM^P\right)$$

where, for all $t \in [0,T]$:

$$\theta_t^1 := -\rho \, \frac{\sqrt{v_t}\,(\mu - r)}{(1+\Lambda^P)\,v_t} \qquad \theta_t^2 := -\sqrt{1-\rho^2}\,\frac{\sqrt{v_t}\,(\mu - r)}{(1+\Lambda^P)\,v_t} \qquad \gamma_t := \frac{\mu - r}{(1+\Lambda^P)\,v_t}$$

Note first that $\mu \geq r$ implies that $\gamma_t \geq 0$ P-a.s. for all $t \in [0,T]$, since the process v is P-a.s. strictly positive. This implies that $\widehat{Z}$ is P-a.s. strictly positive. Then, note that the processes $\theta^1 = (\theta_t^1)_{0 \leq t \leq T}$, $\theta^2 = (\theta_t^2)_{0 \leq t \leq T}$ and $\gamma = (\gamma_t)_{0 \leq t \leq T}$ are of the form (2.24)-(2.25), with:

$$\hat{\theta} = \begin{pmatrix} \frac{k\rho(r-\mu)}{1+\Lambda^P} \\ \frac{r-\mu}{1+\Lambda^P} \end{pmatrix} \qquad \Theta = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \bar{\lambda}^{\widehat{Q}} = (\mu - r)\,\frac{\Lambda^P}{1+\Lambda^P} \qquad \Lambda^{\widehat{Q}} = \Lambda^P$$

If $\beta - \frac{k\rho(\mu-r)}{1+\Lambda^P} \geq \frac{k^2}{2}$, Corollary 2.4.4 implies that the probability measure $\widehat{Q}$ on $(\Omega, \mathcal{G})$ defined by $\frac{d\widehat{Q}}{dP} := \widehat{Z}_T$ admits $\widehat{Z}$ as density process and is an ELMM for S which preserves the Heston with jump-to-default structure. Furthermore, due to Proposition 2.4.7, the process $\bar{S}$ is a true $(\widehat{Q}, \mathbb{G})$-martingale and not only a $(\widehat{Q}, \mathbb{G})$-local martingale, meaning that $\widehat{Q}$ is an EMM for S. □

2.5 Application: a multifactor equity/credit risk model with stochastic volatility

This Section is meant to be an illustration of the benefits of preserving the affine structure of a reduced-form credit risk model under both the physical and the risk-neutral probability measure. More specifically, extending the Heston with jump-to-default model analyzed in Section 2.4, we shall consider a hybrid equity/credit risk model, where the stock price process can be killed by a jump-to-default event and stochastic volatility, interest rate and default intensity are all linked through a multivariate affine process. Section 2.5.1 illustrates the general features of this modeling framework, while Section 2.5.2 and Section 2.5.3 deal with several applications to risk management and risk-neutral valuation, respectively. In Section 2.5.4, we shall briefly consider the incomplete information case where some of the components of the underlying affine process cannot be directly observed. Finally, Section 2.5.5 contains a brief overview of the related literature.

2.5.1 The modeling framework

As in Section 2.2, let $(\Omega, \mathcal{G}, \mathbb{F}, P)$ be the underlying filtered probability space, with an $\mathbb{R}^{\bar{d}}$-valued Brownian motion $\bar{W} = \left(\bar{W}_t\right)_{0 \leq t \leq T^*}$, for some $\bar{d} \geq 2$, and where $T^* \in (0, \infty)$ denotes a finite time horizon. Let the default time τ be a doubly stochastic random time with respect to $(P, \mathbb{F})$, with P-default intensity $\lambda^P = \left(\lambda_t^P\right)_{0 \leq t \leq T^*}$, and let the filtration $\mathbb{G}$ be the progressive enlargement of

$\mathbb{F}$ with respect to τ, as in Section 2.2. The random time τ models the occurrence of the default event of a given firm and we assume that the price of the traded stock of that firm jumps to zero as soon as the default event occurs. Adopting the notations of Section 2.4, we let the processes $S = (S_t)_{0\leq t\leq T^*}$, $\widetilde{S} = \left(\widetilde{S}_t\right)_{0\leq t\leq T^*}$ and $L = (L_t)_{0\leq t\leq T^*}$ represent the defaultable stock price, the pre-default stock price and the logarithm of the pre-default stock price, respectively. Recall that these three processes are related as follows, for all $t \in [0, T^*]$:

$$S_t = (1 - H_t)\,\widetilde{S}_t = (1 - H_t)\,e^{L_t}$$

where $H_t := \mathbf{1}_{\{\tau\leq t\}}$ denotes the default indicator process. Furthermore, we let $v = (v_t)_{0\leq t\leq T^*}$ represent the stochastic volatility affecting the stock price process and we also introduce the $\mathbb{R}^d$-valued process $Y = (Y_t)_{0\leq t\leq T^*}$ representing a vector of economic factors, with $d \leq \bar{d}-2$. For convenience of notation, let us define the $\mathbb{R}^{d+2}$-valued process $X = (X_t)_{0\leq t\leq T^*}$ as $X_t := (v_t, Y_t', L_t)'$, for all $t \in [0, T^*]$. The processes v, Y and L are jointly specified via the following square-root SDE:

$$dX_t = (AX_t + b)\,dt + \Sigma\,\sqrt{R_t}\,dW_t \tag{2.31}$$

where $A \in \mathbb{R}^{(d+2)\times(d+2)}$, $b \in \mathbb{R}^{d+2}$, $\Sigma \in \mathbb{R}^{(d+2)\times(d+2)}$, R_t is a diagonal $(d+2)$-matrix with elements $R_t^{i,i} := R^{i,i}(X_t) = \alpha_i + \beta_i' X_t$, with $\alpha_i \in \mathbb{R}$ and $\beta_i \in \mathbb{R}^{d+2}$, for $i = 1, \ldots, d+2$, and $W = (W_t)_{0\leq t\leq T^*}$ is an $\mathbb{R}^{d+2}$-valued $(P, \mathbb{F})$-Brownian motion composed of the first $d+2$ elements of the $\mathbb{R}^{\bar{d}}$-valued $(P, \mathbb{F})$-Brownian motion $\bar{W}$. The last $\bar{d} - d - 2$ components of the original Brownian motion $\bar{W}$ will only appear in Section 2.5.4 to model noise factors affecting the observations of market data.

Remark 2.5.1. Under mild non-degeneracy conditions, Duffie & Kan (1996) have shown that (2.31) can be regarded as equivalent to the specification (2.1) for the dynamics of a diffusion affine process. Note that the matrices A and Σ in (2.31) are allowed to be non-diagonal and asymmetric, thus allowing to model flexible correlation patterns among all the components of X. We also want to point out that assuming the matrix Σ to be the identity matrix, as in the "canonical representation" of Dai & Singleton (2000), may lead to unnecessary restrictions on the model if $d > 1$, as has been recently pointed out by Cheridito et al. (2010). See also Remark 1 after Condition C for related comments.

Due to the presence of the square-root in the diffusion term, the SDE (2.31) can fail to have an unique strong solution on $(\Omega, \mathcal{G}, \mathbb{F}, P)$. Therefore, we impose the following parameter restrictions, where we denote by α the vector in $\mathbb{R}^{d+2}$ defined as $\alpha := (\alpha_1, \ldots, \alpha_{d+2})'$ and by β the square $(d+2)$-matrix defined as $\beta := (\beta_1, \ldots, \beta_{d+2})$.

Condition C. *Let $m \in \{1, \ldots, d+1\}$. The parameters $A \in \mathbb{R}^{(d+2)\times(d+2)}$, $b \in \mathbb{R}^{d+2}$, $\Sigma \in \mathbb{R}^{(d+2)\times(d+2)}$, $\alpha \in \mathbb{R}_+^{d+2}$ and $\beta \in \mathbb{R}_+^{(d+2)\times(d+2)}$ satisfy the following restrictions:*

- $b_i \geq 0$ *for* $i \in \{1, \ldots, m\}$;
- $A_{i,j} = 0$ *for* $i \in \{1, \ldots, m\}$ *and* $j \in \{m+1, \ldots, d+2\}$ *and* $A_{i,j} \geq 0$ *for* $i \in \{1, \ldots, m\}$ *and* $j \in \{1, \ldots, m\} \setminus \{i\}$;

- $\Sigma_{i,j} = 0$, *for* $i \in \{1, \ldots, m\}$ *and* $j \in \{1, \ldots, d+2\} \setminus \{i\}$*;*
- $\alpha_i = 0$ *for* $i \in \{1, \ldots, m\}$*;*
- $\beta_{j,i} = 0$, *for* $i \in \{1, \ldots, d+2\}$ *and* $j \in \{m+1, \ldots, d+2\}$, *and* $\beta_{j,i} = 0$ *for* $i \in \{1, \ldots, m\}$ *and* $j \in \{1, \ldots, m\} \setminus \{i\}$*;*

and $b_i \geq \frac{1}{2} \left(\Sigma_{i,i}\right)^2 \beta_{i,i}$ *for all* $i = 1, \ldots, m$.

Remark 2.5.2.

1. The parameter restrictions of Condition C are similar to the parametrization adopted in the "canonical representation" of Dai & Singleton (2000), except that here we allow for a richer correlation structure. More precisely, the canonical representation of Dai & Singleton (2000) requires each component of the vector process X to be driven by a single element of the vector Brownian motion W, while in our parametrization the last $d+2-m$ components of X can be driven by all the elements of W. This possibility has significant implications from the point of view of financial modeling. For instance, it has been shown that the absence of correlation between the Brownian motions driving the stock price process and the stochastic volatility process hinders the capability of the model of fitting the observed volatility surface, see e.g. Gatheral (2006) for a general account.

2. In order to facilitate the comparison with Section 2.2, we now formulate the model described in the present Section by using the notations introduced in Section 2.2, here denoted with the superscript *. The SDE (2.31) can be rewritten in the form (2.1) as follows:

$$dX_t = \mu^* \left(X_t\right) dt + \sigma^* \left(X_t\right) \, dW_t$$

where the function $\mu^* : \mathbb{R}^{d+2} \to \mathbb{R}^{d+2}$ is given by $\mu^*(x) = Ax + b$ and the function $\sigma^* : \mathbb{R}^{d+2} \to \mathbb{R}^{(d+2)\times(d+2)}$ is such that $\sigma^*(X_t)\,\sigma^*(X_t)' = \frac{d\langle X,X\rangle_t}{dt}$. More specifically, for any $t \in [0, T^*]$:

$$\sigma^* \left(X_t\right) \sigma^* \left(X_t\right)' = \Sigma \sqrt{R_t} \left(\Sigma \sqrt{R_t}\right)' = \Sigma \, R_t \, \Sigma'$$

Hence, for any $i, j \in \{1, \ldots, d+2\}$:

$$\left(\sigma^* \left(X_t\right) \sigma^* \left(X_t\right)'\right)_{i,j} = \sum_{k=1}^{d+2} \Sigma_{i,k} \, \Sigma_{j,k} \left(\alpha_k + \beta_k' X_t\right) \quad = \left(\Sigma_0^*\right)^{i,j} + \sum_{\ell=1}^{d+2} \left(\Sigma_\ell^*\right)^{i,j} X_t^\ell$$

where we have used the following notation, for $i, j, \ell \in \{1, \ldots, d+2\}$:

$$\left(\Sigma_0^*\right)^{i,j} := \sum_{k=1}^{d+2} \Sigma_{i,k} \, \Sigma_{j,k} \, \alpha_k \qquad \text{and} \qquad \left(\Sigma_\ell^*\right)^{i,j} := \sum_{k=1}^{d+2} \Sigma_{i,k} \, \Sigma_{j,k} \, \beta_{\ell,k} \tag{2.32}$$

It can be easily checked that Condition C implies that the parameter restrictions formulated in Condition A are satisfied. Hence, due to Proposition 2.2.2, there exists an unique strong solution

$X = (X_t)_{0 \leq t \leq T^*}$ on $(\Omega, \mathcal{G}, \mathbb{F}, P)$ taking values in $\mathbb{R}^m_{++} \times \mathbb{R}^{d+2-m}$ to the SDE (2.31). More explicitly, we have that, for $i \in \{1, \ldots, m\}$:

$$dX_t^i = \left(b_i + \sum_{j=1}^{m} A_{i,j}\, X_t^j \right) dt + \Sigma_{i,i} \sqrt{\beta_{i,i}\, X_t^i}\, dW_t^i \tag{2.33}$$

and, for $i \in \{m+1, \ldots, d+2\}$:

$$dX_t^i = \left(b_i + \sum_{j=1}^{d+2} A_{i,j}\, X_t^j \right) dt + \sum_{j=1}^{m} \Sigma_{i,j} \sqrt{\beta_{j,j}\, X_t^j}\, dW_t^j + \sum_{k=m+1}^{d+2} \Sigma_{i,k} \sqrt{\alpha_k + \sum_{\ell=1}^{m} \beta_{\ell,k}\, X_t^\ell}\, dW_t^k \tag{2.34}$$

By combining (2.22) and (2.34) (for $i = d+2$), we can obtain the following dynamics for the defaultable stock price process S:

$$\begin{aligned}
dS_t = S_{t-} \Bigg(& b_{d+2} + \frac{1}{2} \sum_{k=m+1}^{d+2} \left(\Sigma_{d+2,k}\right)^2 \alpha_k + A_{d+2,d+2} \log S_{t-} \\
& + \left(A_{d+2,1} + \frac{1}{2} \left(\Sigma_{d+2,1}\right)^2 \beta_{11} + \frac{1}{2} \sum_{k=m+1}^{d+2} \left(\Sigma_{d+2,k}\right)^2 \beta_{1,k} \right) v_t \\
& + \sum_{j=1}^{d} A_{d+2,j+1}\, Y_t^j + \frac{1}{2} \sum_{j=1}^{m-1} \left(\Sigma_{d+2,j+1}\right)^2 \beta_{j+1,j+1}\, Y_t^j \\
& + \frac{1}{2} \sum_{k=m+1}^{d+2} \sum_{\ell=1}^{m-1} \left(\Sigma_{d+2,k}\right)^2 \beta_{\ell+1,k}\, Y_t^\ell \Bigg) dt \\
& + S_{t-}\, \Sigma_{d+2,1} \sqrt{\beta_{1,1}\, v_t}\, dW_t^1 + S_{t-} \sum_{j=1}^{m-1} \Sigma_{d+2,j+1} \sqrt{\beta_{j+1,j+1}\, Y_t^j}\, dW_t^{j+1} \\
& + S_{t-} \sum_{k=m+1}^{d+2} \Sigma_{d+2,k} \sqrt{\alpha_k + \beta_{1,k}\, v_t + \sum_{\ell=1}^{m-1} \beta_{\ell+1,k}\, Y_t^\ell}\, dW_t^k - S_{t-}\, dH_t \\
=: {} & S_{t-} \mu_S\left(X_t\right) dt + S_{t-} \sum_{j=1}^{d+2} \Sigma_{d+2,j} \sqrt{R_t^{j,j}}\, dW_t^j - S_{t-}\, dH_t \\
= {} & S_{t-} \left(\mu_S\left(X_t\right) - \lambda_t^P \right) dt + S_{t-} \sum_{j=1}^{d+2} \Sigma_{d+2,j} \sqrt{R_t^{j,j}}\, dW_t^j - S_{t-}\, dM_t^P
\end{aligned} \tag{2.35}$$

where M^P is the $(P, \mathbb{G})$-martingale introduced in part *(a)* of Lemma 2.2.3. As can be immediately seen from (2.35), the defaultable stock price process S has rich and flexible stochastic dynamics. In particular, the SDE (2.35) allows for stochastic volatility and both the drift and the diffusion terms of S depend on the factor process Y. In addition, the drift term can depend on the volatility and on the pre-default stock price process itself. Observe that we have three distinct layers of interaction between S and the volatility v:

1. a direct interaction, since v appears explicitly in the dynamics of S;
2. a "semi-direct" interaction, since the Brownian motion W^1 driving the process v is also one of the drivers in the dynamics of S;
3. an indirect interaction, due to the common dependence of the processes S and v on the factor process Y.

The framework so far described is rather general and, in particular, can be regarded as an extension to a defaultable setting of several stochastic volatility models proposed in the literature, as shown in the following Examples.

Example 2.5.3 (The Heston with jump-to-default model). Let us consider the simple case where $d = 0$ and $m = 1$, meaning that we do not consider any factor process Y. Let the model parameters be specified as follows:

$$b = \begin{pmatrix} \bar{b} \\ \mu \end{pmatrix} \qquad A = \begin{pmatrix} a & 0 \\ -1/2 & 0 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} k & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \qquad \alpha = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

where $\bar{b} \geq k^2/2$, $\mu, a, k \in \mathbb{R}$ and $\rho \in [-1, 1]$. It is easy to check that this parametrization satisfies Condition C. Due to (2.35), the defaultable stock price process satisfies the following SDE:

$$\begin{aligned} dS_t &= S_{t-}\mu\,dt + S_{t-}\sqrt{v_t}\,\rho\,dW_t^1 + S_{t-}\sqrt{v_t}\sqrt{1-\rho^2}\,dW^2 - S_{t-}\,dH_t \\ &= S_{t-}\mu\,dt + S_{t-}\sqrt{v_t}\,d\widetilde{W}_t - S_{t-}\,dH_t \end{aligned} \tag{2.36}$$

where the process $\widetilde{W} = \big(\widetilde{W}_t\big)_{0\leq t\leq T^*}$ is a $(P, \mathbb{F})$-Brownian motion defined as $\widetilde{W}_t := \rho\,W_t^1 + \sqrt{1-\rho^2}\,W_t^2$, for $t \in [0, T^*]$. The volatility process satisfies the following SDE, as can be easily seen from (2.33):

$$dv_t = \big(\bar{b} + av_t\big)\,dt + k\sqrt{v_t}\,dW_t^1 \tag{2.37}$$

and $d\langle\widetilde{W}, W^1\rangle_t = \rho\,dt$. We have thus recovered the Heston with jump-to-default model considered in Section 2.4. Clearly, we can extend this model by introducing a non-trivial factor process Y, as will be shown in the next Examples.

Example 2.5.4 (A defaultable two-factor stochastic volatility model). Let $d = 1$ and $m = 2$ and consider the following specification for the model parameters:

$$A = \begin{pmatrix} -k & k & 0 \\ 0 & -k_0 & 0 \\ -1/2 & 0 & 0 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \bar{\sigma} & 0 & 0 \\ 0 & \sigma_0 & 0 \\ \rho & 0 & \sqrt{1-\rho^2} \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$b = \big(k\bar{v}, k_0\bar{y}, \mu - \bar{\zeta}\big)' \qquad \alpha = (0,0,0)'$$

with $k \in \mathbb{R}_+$, $k_0, \bar{v}, \bar{\sigma}, \sigma_0, \mu, \bar{\zeta} \in \mathbb{R}$, $k\bar{v} \geq \bar{\sigma}^2/2$, $k_0\bar{y} \geq \sigma_0^2/2$ and $\rho \in [-1, 1]$. It is easy to check that this parametrization satisfies Condition C. As can be deduced from (2.35), the defaultable stock price process satisfies the following SDE:

$$dS_t = S_{t-}\big(\mu - \bar{\zeta}\big)\,dt + S_{t-}\sqrt{v_t}\Big(\rho\,dW_t^1 + \sqrt{1-\rho^2}\,dW_t^3\Big) - S_{t-}\,dH_t \tag{2.38}$$

where the parameters μ and $\bar{\zeta}$ represent the expected return rate and the dividend yield, respectively. The stochastic volatility process v satisfies the following SDE:

$$dv_t = k\,(\bar{v} + Y_t - v_t)\,dt + \bar{\sigma}\,\sqrt{v_t}\,dW_t^1 \tag{2.39}$$

The factor Y can be thought of as a long-term stochastic trend component of the volatility process and satisfies the following SDE:

$$dY_t = k_0\,(\bar{y} - Y_t)\,dt + \sigma_0\sqrt{Y_t}\,dW_t^2 \tag{2.40}$$

Equations (2.38)-(2.40) are analogous to the two-factor stochastic volatility model considered in Section 4.3 of Duffie et al. (2000), with the additional feature that the stock price process jumps to zero as soon as the default event occurs.

Example 2.5.5 (A defaultable stochastic volatility - stochastic interest rate model). Let $d = 1$ and $m = 2$ and suppose that the real-valued process Y represents the stochastic evolution of the risk-free interest rate. Consider the following specification for the model parameters:

$$A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} k_1/\rho_1 & 0 & 0 \\ 0 & k_2/\rho_2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \qquad \beta = \begin{pmatrix} \rho_1^2 & 0 & 1-\rho_1^2 \\ 0 & \rho_2^2 & 1-\rho_2^2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$b = (b_1, b_2, 0)' \qquad \alpha = (0,0,0)'$$

with $a_1, a_2, k_1, k_2 \in \mathbb{R}$, $b_1 \geq k_1^2/2$, $b_2 \geq k_2^2/2$ and $\rho_1, \rho_2 \in [-1,1] \setminus \{0\}$. It is easy to check that this parametrization satisfies Condition C. As can be seen from (2.35), the defaultable stock price process satisfies the following SDE:

$$dS_t = S_{t-}\,Y_t\,dt + S_{t-}\left(\rho_1\sqrt{v_t}\,dW_t^1 + \rho_2\sqrt{Y_t}\,dW_t^2 + \sqrt{(1-\rho_1^2)\,v_t + (1-\rho_2^2)\,Y_t}\,dW_t^3\right) - S_{t-}\,dH_t \tag{2.41}$$

and the stochastic volatility process v and the interest rate process Y satisfy the following SDEs, respectively:

$$dv_t = (b_1 + a_1 v_t)\,dt + k_1\sqrt{v_t}\,dW_t^1 \tag{2.42}$$

$$dY_t = (b_2 + a_2 Y_t)\,dt + k_2\sqrt{Y_t}\,dW_t^2 \tag{2.43}$$

Observe that (2.41) can be equivalently written as follows:

$$dS_{t-} = S_{t-}\,Y_t\,dt + S_{t-}\left(\sqrt{v_t}\,d\widetilde{W}_t^1 + \sqrt{Y_t}\,d\widetilde{W}_t^2\right) - S_{t-}\,dH_t \tag{2.44}$$

where $\widetilde{W}^1$ and $\widetilde{W}^2$ are two Brownian motions such that $d\langle\widetilde{W}^1, W^1\rangle_t = \rho_1 dt$ and $d\langle\widetilde{W}^2, W^2\rangle_t = \rho_2 dt$. Equations (2.42)-(2.44) represent a defaultable version of the stochastic volatility - stochastic interest rate model proposed by Ahlip & Rutkowski (2009), here extended by allowing the stock price process to jump to zero as soon as the default event occurs.

Coming back to our general framework, note that the default intensity $\lambda^P = \left(\lambda_t^P\right)_{0\leq t\leq T^*}$ has been left so far unspecified. Following Section 2.2, we specify λ^P as follows:

$$\lambda_t^P = \bar{\lambda}^P + \left(\Lambda^P\right)' X_t \qquad \text{for all } t \in [0,T^*] \tag{2.45}$$

with $\bar{\lambda}^P \in \mathbb{R}_+$ and $\Lambda^P \in \mathbb{R}_+^{d+2}$ with $\Lambda^{P,i} = 0$ for all $i \in \{m+1,\ldots,d+2\}$ and $\bar{\lambda}^P + \sum_{i=1}^m \Lambda^{P,i} > 0$. The specification (2.45) ensures that the process λ^P remains P-a.s. strictly positive, since 0 is an unattainable boundary for the first m components of the process X (see Proposition 2.2.2). The reason why we impose a linear structure on λ^P lies in the possibility of obtaining analytically tractable formulae for several quantities of interest, as will be shown in Section 2.5.2. It is interesting to observe that (2.45) allows for a direct dependence of the default intensity on the stochastic volatility v. This is a very important feature, being consistent with several empirical observations which document a link between default risk and volatility risk (see for instance the introduction of Carr & Linetsky (2006) and Gatheral (2006), Chapter 6).

Remark 2.5.6 (**On the interaction between the stock price and the default intensity**). Note that the specification (2.45) for the default intensity does not allow for a direct dependence of λ^P on the logarithm of the pre-default stock price L, since the latter may take negative values[6]. However, we can still capture a stochastic interaction between the default intensity and the stock price. In fact, the dynamics of λ^P and L both depend on the factor process Y and, furthermore, the diffusion terms in the dynamics of L are also directly correlated with the diffusion terms in the dynamics of λ^P.

Let us also specify the risk-free spot interest rate $r = (r_t)_{0\leq t\leq T^*}$ as follows:

$$r_t = \bar{r} + \Upsilon' X_t \qquad \text{for all } t \in [0,T^*] \tag{2.46}$$

where $\bar{r} \in \mathbb{R}$ and $\Upsilon \in \mathbb{R}^{d+2}$. As in the case of the default intensity process, if we want to ensure non-negative interest rates, we can impose $\bar{r} \in \mathbb{R}_+$ and $\Upsilon \in \mathbb{R}_+^{d+2}$ with $\Upsilon^j = 0$ for all $j \in \{m+1,\ldots,d+2\}$. However, in order to cover the most general case, we do not impose this restriction on the interest rate process[7]. We want to point out that specifications of the type (2.46) are very common in the context of term structure modeling via affine processes: see for instance Duffie & Kan (1996), Dai & Singleton (2000), Duffee (2002) and Chapter 10 of Filipović (2009).

Remark 2.5.7.

1. The specifications (2.45)-(2.46) for the default intensity and the interest rate allow the processes λ^P and r to be correlated, due to their common dependence on the first m components

[6] Clearly, one can let the default intensity λ^P depend also on the last $d+2-m$ components of the vector process X, including the pre-default stock price process L, via quadratic or exponential functions in order to ensure that the process λ^P does not leave the positive orthant, see e.g. Fontana & Runggaldier (2010). However, as will be shown in the following Sections, the affine structure of equations (2.45)-(2.46) is crucial for obtaining explicit formulae for several key quantities useful for both risk management as well as pricing applications.

[7] Note that, even in the case where the interest rate process r is not restricted to the positive orthant, the probability of r_t taking negative values can be made almost negligible from a practical point of view by a suitable choice of the mean-reversion parameters in the SDE (2.31).

of the $\mathbb{R}^{d+2}$-valued process X. Furthermore, all the key elements appearing in our model enjoy a rich correlation structure, in the sense that the stock price, the stochastic volatility, the default intensity, the interest rate and the factor process Y are all mutually correlated. An analogous structure for both the interest rate and the default intensity has been adopted in Fontana (2010a) in a multi-firm setting and with the additional inclusion of a rating-based component in the default intensity specification.

2. The present framework can be extended to a multi-firm setting. In fact, suppose that the market comprises a number $p \in \mathbb{N}$ of firms and denote by τ^i the random default time of the i-th firm, for $i = 1, \ldots, p$. As in Section 9.6 of McNeil et al. (2005), we can model the random default times $\{\tau^i\}_{i=1,\ldots,p}$ as *conditionally independent doubly stochastic random times* with respect to $(P, \mathbb{F})$, with corresponding P-default intensities $\left\{\lambda^{P,i}\right\}_{i=1,\ldots,p}$, where the latter are specified as in (2.45). Let us also denote by L^i the logarithm of the pre-default stock price process of the i-th firm and, analogously, by v^i the stochastic volatility process associated to the i-th firm, for $i = 1, \ldots, p$. We can then define the $\mathbb{R}^{d+2p}$-valued process $X = (X_t)_{0 \leq t \leq T^*}$ as $X_t := (v_t^1, \ldots, v_t^p, Y_t', L_t^1, \ldots, L_t^p)'$, for all $t \in [0, T^*]$. It is easy to see that if X satisfies an SDE of the form (2.31) and if a suitable version of Condition C holds, then we can still rely on the powerful results made available by the affine technology. Clearly, this multi-firm framework can accommodate flexible stochastic interactions between the different stock price processes and their stochastic volatilities. In particular, the dynamics of L^i can also depend on v^j for $j \neq i$. Furthermore, the default intensities associated to the different firms are mutually correlated, due to the common dependence of $\left\{\lambda_t^{P,i}\right\}_{i=1,\ldots,p}$ on the vector $(v_t^1, \ldots, v_t^p, Y_t')'$.

2.5.2 Risk management applications

By relying on the affine structure of the general framework described in Section 2.5.1, we shall now be concerned with some simple computations (under the real-world probability measure P), which may be of interest in view of risk management applications. Let us start with the following Proposition, which will play a key role in the derivation of most of the results of this Section.

Proposition 2.5.8. *Suppose that Condition C holds. Then, there exists an unique solution* $\big(\Phi(\cdot, u),$ $\Psi(\cdot, u)\big) : [0, T^*] \to \mathbb{C} \times \mathbb{C}^{d+2}$ *to the following system of Riccati ordinary differential equations:*

$$\begin{aligned}
\partial_t \Phi(t,u) &= b' \Psi(t,u) + \frac{1}{2} \sum_{k=m+1}^{d+2} \left[\Sigma' \Psi(t,u)\right]_k^2 \alpha_k - \bar{\lambda}^P \\
\Phi(0,u) &= 0 \\
\partial_t \Psi_i(t,u) &= \sum_{k=1}^{d+2} A_{k,i} \Psi_k(t,u) + \frac{1}{2} \left[\Sigma' \Psi(t,u)\right]_i^2 \beta_{i,i} + \frac{1}{2} \sum_{k=m+1}^{d+2} \left[\Sigma' \Psi(t,u)\right]_k^2 \beta_{i,k} - \Lambda^{P,i} \qquad (2.47) \\
\partial_t \Psi_j(t,u) &= \sum_{k=m+1}^{d+2} A_{k,j} \Psi_k(t,u) \\
\Psi(0,u) &= u
\end{aligned}$$

for $i = 1, \ldots, m$ *and* $j = m+1, \ldots, d+2$ *for the initial condition* $u = 0$*. Furthermore, the following holds:*

$$E\left[e^{-\int_t^T \lambda_u^P du}\, e^{u' X_T} \middle| \mathcal{F}_t\right] = e^{\Phi(T-t,u)+\Psi(T-t,u)' X_t}$$

for any $t, T \in [0, T^*]$ *with* $t \leq T$ *and for any* $u \in \mathbb{C}^{d+2}$ *such that there exists an unique solution* $\Big(\Phi\big(\cdot, \Re(u)\big), \Psi\big(\cdot, \Re(u)\big)\Big) : \big[0, \bar{T}\big] \to \mathbb{R} \times \mathbb{R}^{d+2}$ *to system* (2.47) *for some* $\bar{T} \geq T$*.*

Proof. The proof relies on the same arguments used in the proof of Theorem 10.4 of Filipović (2009). Note that condition *(a)* in Theorem 10.4 of Filipović (2009) is always satisfied by the process λ^P, i.e. we have $E\big[\exp\big(-\int_0^T \lambda_u^P du\big)\big] < \infty$ for all $T \in [0, T^*]$, since λ^P is uniformly bounded from below by 0, due to (2.45) together with Proposition 2.2.2. □

From the risk-management perspective, one of the most important quantities we are interested in is represented by the $\mathcal{G}_t$-conditional survival probability over a given time horizon $[t, T]$, with $0 \leq t \leq T \leq T^*$. This is the content of the following Corollary, the proof of which relies on Proposition 2.5.8.

Corollary 2.5.9. *Suppose that Condition C holds. For any* $t, T \in [0, T^*]$ *with* $t \leq T$*, the* $\mathcal{G}_t$*-conditional survival probability over the time interval* $[t, T]$ *is explicitly given as follows:*

$$P\left(\tau > T \middle| \mathcal{G}_t\right) = \left(1 - H_t\right) e^{-A(T-t)-B(T-t)'\hat{X}_t}$$

where $\hat{X}_t := (X_t^1, \ldots, X_t^m)'$ *and the functions* $A : [0, T^*] \to \mathbb{R}$ *and* $B : [0, T^*] \to \mathbb{R}^m$ *are the unique solutions to the following system of Riccati ordinary differential equations:*

$$\begin{aligned}
\partial_t A\left(t\right) &= b' B\left(t\right) - \frac{1}{2}\sum_{k=m+1}^{d+2} \left[\Sigma' B\left(t\right)\right]_k^2 \alpha_k + \bar{\lambda}^P \\
A\left(0\right) &= 0 \\
\partial_t B_i\left(t\right) &= \sum_{k=1}^{d+2} A_{k,i}\, B_k\left(t\right) - \frac{1}{2}\left[\Sigma' B\left(t\right)\right]_i^2 \beta_{i,i} - \frac{1}{2}\sum_{k=m+1}^{d+2} \left[\Sigma' B\left(t\right)\right]_k^2 \beta_{i,k} + \Lambda^{P,i} \\
B\left(0\right) &= 0
\end{aligned} \tag{2.48}$$

for $i = 1, \ldots, m$*.*

Proof. Observe first that system (2.48) corresponds to (2.47) with initial condition $u = 0$ by letting $\Phi\left(t, 0\right) = -A\left(t\right)$ and $\Psi_i\left(t, 0\right) = -B_i\left(t\right)$ for $i = 1, \ldots, m$. Due to Proposition 2.5.8, there exists an unique solution $\big(\Phi\left(\cdot, u\right), \Psi\left(\cdot, u\right)\big) : [0, T^*] \to \mathbb{R} \times \mathbb{R}^{d+2}$ to system (2.47) for $u = 0$. It is also easy to see that we have $\Psi_j\left(t, 0\right) = 0$ for all $t \in [0, T^*]$ and $j = m+1, \ldots, d+2$. Hence, there exists an unique solution $\big(A\left(\cdot\right), B\left(\cdot\right)\big) : [0, T^*] \to \mathbb{R} \times \mathbb{R}^m$ to (2.48). We have then:

$$\begin{aligned}
P\left(\tau > T \middle| \mathcal{G}_t\right) &= \left(1 - H_t\right) \frac{P\left(\tau > T | \mathcal{F}_t\right)}{P\left(\tau > t | \mathcal{F}_t\right)} = \left(1 - H_t\right) E\left[e^{-\int_t^T \lambda_u^P du} \middle| \mathcal{F}_t\right] \\
&= \left(1 - H_t\right) e^{-A(T-t)-B(T-t)'\hat{X}_t}
\end{aligned}$$

where the first two equalities follow from Lemma 5.1.2 and formula (5.9) of Bielecki & Rutkowski (2002) and the last equality follows from Proposition 2.5.8 with $u = 0$. □

In particular, note that the $\mathcal{G}_t$-conditional survival probability depends only on the first m components of X_t, i.e. on the components of X which drive the default intensity itself. For $T \in [0, T^*]$, let us now introduce a probability measure P^T on $(\Omega, \mathcal{G})$, named T*-survival probability measure* and formally defined as follows[8]:

$$\frac{dP^T}{dP} := \frac{e^{-\int_0^T \lambda_u^P du}}{E\left[e^{-\int_0^T \lambda_u^P du}\right]} \tag{2.49}$$

The terminology T*-survival probability measure* is justified by the following property, which holds for any integrable $\mathcal{F}_{T^*}$-measurable random variable F and for any $t, T \in [0, T^*]$ with $t \leq T$:

$$\begin{aligned} E\left[F\,\mathbf{1}_{\{\tau>T\}}|\mathcal{G}_t\right] &= (1-H_t)\,\frac{E\left[F\,\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_t\right]}{P\left(\tau>t|\mathcal{F}_t\right)} = (1-H_t)\,\frac{E\Big[F\,E\left[\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_{T^*}\right]\Big|\mathcal{F}_t\Big]}{P\left(\tau>t|\mathcal{F}_t\right)} \\ &= (1-H_t)\,\frac{E\Big[F\,E\left[\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_T\right]\Big|\mathcal{F}_t\Big]}{P\left(\tau>t|\mathcal{F}_t\right)} = (1-H_t)\,\frac{E\left[e^{-\int_0^T \lambda_u^P du}\,F\Big|\mathcal{F}_t\right]}{P\left(\tau>t|\mathcal{F}_t\right)} \\ &= (1-H_t)\,E\left[e^{-\int_t^T \lambda_u^P du}\,F\Big|\mathcal{F}_t\right] = (1-H_t)\,E\left[e^{-\int_t^T \lambda_u^P du}\Big|\mathcal{F}_t\right]E^{P^T}\left[F|\mathcal{F}_t\right] \\ &= P\left(\tau>T|\,\mathcal{G}_t\right)E^{P^T}\left[F|\mathcal{F}_t\right] \end{aligned} \tag{2.50}$$

where $E^{P^T}[\cdot]$ denotes the expectation with respect to P^T. In (2.50), the first equality follows from Lemma 5.1.2 of Bielecki & Rutkowski (2002), the third equality uses the property that $P\left(\tau>t|\mathcal{F}_t\right) = P\left(\tau>t|F_s\right)$ for any $s \geq t$ (see part 2 of Remark 2.2.4), the sixth equality follows from the definition of the measure P^T and Bayes' formula and, finally, the last equality follows as in the proof of Corollary 2.5.9. Hence, we see that the computation of the $\mathcal{G}_t$-conditional expectation of the $\mathcal{F}_{T^*}$-measurable random variable F in the case of survival until T reduces to the computation of the $\mathcal{F}_t$-conditional expectation of the random variable F with respect to the measure P^T, while the term $P\left(\tau>T|\,\mathcal{G}_t\right)$ can be computed as in Corollary 2.5.9. The next Corollary gives an explicit formula for the $\mathcal{F}_t$-conditional characteristic function of the random vector X_T under the T*-survival measure* P^T, for any $t, T \in [0, T^*]$ with $t \leq T$.

Corollary 2.5.10. *Suppose that Condition C holds. Then, for any $t, T \in [0, T^*]$ with $t \leq T$, the $\mathcal{F}_t$-conditional characteristic function of the random vector X_T with respect to the measure P^T is explicitly given as follows, for all $u \in i\mathbb{R}^{d+2}$:*

$$E^{P^T}\left[e^{u'X_T}\Big|\mathcal{F}_t\right] = \frac{e^{\Phi(T-t,u)+\Psi(T-t,u)'X_t}}{e^{-A(T-t)-B(T-t)'\hat{X}_t}}$$

Proof. Due to the definition of the measure P^T we have that:

$$E^{P^T}\left[e^{u'X_T}\Big|\mathcal{F}_t\right] = \frac{E\left[e^{-\int_t^T \lambda_s^P ds}\,e^{u'X_T}\Big|\mathcal{F}_t\right]}{E\left[e^{-\int_t^T \lambda_s^P ds}\Big|\mathcal{F}_t\right]}$$

[8] The T*-survival probability measure* P^T bears resemblance to the *restricted defaultable forward measure* introduced in Bielecki & Rutkowski (2002), Section 15.2.2, except that here P^T is defined with respect to the physical probability measure P and not with respect to a risk-neutral T-forward probability measure. Compare also the definition of the T*-forward survival risk-neutral measure* $\widetilde{Q}^T$ in Section 2.5.3.

The claim then follows by applying Proposition 2.5.8 (and Corollary 2.5.9) with $u \in i\mathbb{R}^{d+2}$ and $u = 0$ to the numerator and to the denominator, respectively. $\square$

Due to (2.50), we can compute the $\mathcal{G}_t$-conditional expectation (under the real-world probability measure P) of arbitrary functions of the random vector X_T in the case of survival by relying on Corollaries 2.5.9 and 2.5.10 and employing standard Fourier inversion techniques (for a specific application, see Proposition 2.5.12).

Remark 2.5.11. For any $t, T \in [0, T^*]$ with $t \leq T$, we have derived in Corollary 2.5.10 the $\mathcal{F}_t$-conditional characteristic function of the random vector X_T with respect to the T-*survival measure* P^T. It is worth pointing out that the $\mathcal{F}_t$-conditional characteristic function also coincides with the $\mathcal{G}_t$-conditional characteristic function, i.e.:

$$E^{P^T}\left[e^{u'X_T}\middle|\mathcal{G}_t\right] = E^{P^T}\left[e^{u'X_T}\middle|\mathcal{F}_t\right] = \frac{e^{\Phi(T-t,u)+\Psi(T-t,u)'X_t}}{e^{-A(T-t)-B(T-t)'\hat{X}_t}}$$

To prove the first equality, recall that the immersion property holds under the measure P, due to part *(b)* of Lemma 2.2.3. Since the density $\frac{dP^T}{dP}$ of the measure P^T with respect to P is $\mathcal{F}_T$-measurable, being λ^P an $\mathbb{F}$-adapted process, Proposition 5.9.1.2 of Jeanblanc et al. (2009) implies that the immersion property holds also with respect to the measure P^T. Hence, due to Proposition 5.9.1.1 of Jeanblanc et al. (2009), we obtain that $E^{P^T}\left[e^{u'X_T}|\,\mathcal{G}_t\right] = E^{P^T}\left[e^{u'X_T}|\mathcal{F}_t\right]$, since X_T is $\mathcal{F}_T$-measurable.

Many risk management applications (like the computation of *Value-at-Risk* or similar risk measures) require the knowledge of quantiles of the conditional distribution (under the real-world measure P) of the defaultable stock price at a given future time. To this effect, we derive in the following Proposition the $\mathcal{G}_t$-conditional distribution function of the defaultable stock price at time T in the case of survival.

Proposition 2.5.12. *Suppose that Condition C holds. Let $x \in \mathbb{R}_+$ and $t, T \in [0, T^*]$ with $t \leq T$. Then the following hold:*

$$P\left(S_T \leq x, \tau > T|\,\mathcal{G}_t\right) = (1 - H_t)\, e^{-A(T-t)-B(T-t)'\hat{X}_t}\left(\frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\mathfrak{Im}\left(e^{-iy\log x}\,\varphi_t^T(y)\right)}{y}dy\right)$$

where $\mathfrak{Im}$ denotes the imaginary part and φ_t^T denotes the $\mathcal{F}_t$-conditional characteristic function of L_T under the measure P^T, explicitly given as follows:

$$\varphi_t^T(y) := E^{P^T}\left[e^{iyL_T}|\mathcal{F}_t\right] = \frac{e^{\Phi(T-t,iy_{d+2})+\Psi(T-t,iy_{d+2})'X_t}}{e^{-A(T-t)-B(T-t)'\hat{X}_t}}$$

for all $y \in \mathbb{R}$ and where y_{d+2} denotes the vector $(0, \ldots, 0, y)' \in \mathbb{R}^{d+2}$.

Proof. Observe first that:

$$\begin{aligned} P\left(S_T \leq x, \tau > T|\mathcal{G}_t\right) &= P\left(L_T \leq \log x, \tau > T|\mathcal{G}_t\right) = P\left(\tau > T|\mathcal{G}_t\right) P^T\left(L_T \leq \log x|\mathcal{F}_t\right) \\ &= (1 - H_t)\, e^{-A(T-t)-B(T-t)'\hat{X}_t}\, P^T\left(L_T \leq \log x|\mathcal{F}_t\right) \end{aligned}$$

where the second equality follows from (2.50) and the third equality from Corollary 2.5.9. By relying on standard Fourier inversion techniques, it can be shown that (see e.g. Paolella (2007), Section 1.2.6):

$$P^T\left(L_T \leq \log x|\mathcal{F}_t\right) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\mathfrak{Im}\left(e^{-iy\log x}\,\varphi_t^T\left(y\right)\right)}{y}dy$$

where $\mathfrak{Im}$ denotes the imaginary part and φ_t^T denotes the $\mathcal{F}_t$-conditional characteristic function of L_T under the measure P^T. The latter can be determined by relying on Corollary 2.5.10, with $u = i\left(0,\dots,0,y\right)' =: iy_{d+2} \in i\mathbb{R}^{d+2}$. □

2.5.3 Risk-neutral measures and valuation of default-sensitive payoffs

In Section 2.5.1, we have described the modeling framework with respect to the probability measure P, which represents the probability measure characterizing the physical world. Accordingly, we have considered in Section 2.5.2 several risk management applications, always under the real-world measure P. However, if we want to proceed with the arbitrage-free valuation of financial derivatives, we need to shift the model to a proper risk-neutral measure. As in Section 2.4.2, let us denote by $\bar{S} = \left(\bar{S}_t\right)_{0\leq t\leq T^*}$ the discounted stock price process, where $\bar{S}_t := e^{-\int_0^t r_u du}S_t$ for all $t \in [0,T^*]$. A probability measure Q on $(\Omega,\mathcal{G})$ with $Q \sim P$ is said to be an *Equivalent Local Martingale Measure* (ELMM) (or *risk-neutral* measure) for S if the process $\bar{S}$ is a $(Q,\mathbb{G})$-local martingale. It is well-known that the existence of an ELMM for S is equivalent to the *No Free Lunch with Vanishing Risk* no-arbitrage condition. This follows from the fundamental theorem of asset pricing in the version of Delbaen & Schachermayer (1994), since $\bar{S}$ is locally bounded[9].

It is important to be aware of the fact that most of the key features of the modeling framework described in Section 2.5.1 are not necessarily preserved by an equivalent change of measure. In particular, the affine structure of (X,τ), in the sense of Definition 2.2.5, may be lost. However, the general results of Section 2.3 allow us to formulate conditions on the density process of an ELMM so that the affine structure of (X,τ) is preserved.

Proposition 2.5.13. *Suppose that Condition C holds. Let $\theta = \left(\theta_t\right)_{0\leq t\leq T^*}$ be an $\mathbb{R}^{d+2}$-valued process satisfying the following condition, for all $t \in [0,T^*]$:*

$$\theta_t = \theta\left(X_t\right) := \left(\sqrt{R_t}\right)^{-1}\left(\hat{\theta} + \Theta X_t\right) \tag{2.51}$$

for some $\hat{\theta} \in \mathbb{R}^{d+2}$ and $\Theta \in \mathbb{R}^{(d+2)\times(d+2)}$ such that:

(i) $\sum_{k=1}^{d+2}\Sigma_{i,k}\hat{\theta}_k \geq \frac{1}{2}\left(\Sigma_{i,i}\right)^2\beta_{i,i} - b_i$ *for all* $i \in \{1,\dots,m\}$;

(ii) $\sum_{k=1}^{d+2}\Sigma_{i,k}\Theta_{k,j} = 0$, *for all* $i \in \{1,\dots,m\}$ *and* $j \in \{m+1,\dots,d+2\}$, *and* $\sum_{k=1}^{d+2}\Sigma_{i,k}\Theta_{k,j} \geq -A_{i,j}$, *for all* $i \in \{1,\dots,m\}$ *and* $j \in \{1,\dots,m\}\setminus\{i\}$.

[9] In fact, $\bar{S}$ is non-negative and bounded from above by the discounted pre-default stock price process $e^{-\int_0^{\cdot} r_u du}\,\widetilde{S}$, which, being continuous, is locally bounded.

Let $\gamma = (\gamma_t)_{0\leq t\leq T^*}$ *be a real-valued process satisfying the following condition, for all* $t \in [0, T^*]$*:*

$$\gamma_t = \gamma(X_t) := \frac{\left(\bar{\lambda}^Q - \bar{\lambda}^P\right) + \left(\Lambda^Q - \Lambda^P\right)' X_t}{\bar{\lambda}^P + (\Lambda^P)' X_t} \tag{2.52}$$

for some $\bar{\lambda}^Q \in \mathbb{R}_+$ *and* $\Lambda^Q \in \mathbb{R}^{d+2}_+$ *with* $\Lambda^{Q,i} = 0$ *for all* $i \in \{m+1, \ldots, d+2\}$ *and* $\bar{\lambda}^Q + \sum_{i=1}^m \Lambda^{Q,i} > 0$ *and where* $\bar{\lambda}^P$ *and* Λ^P *are as in* (2.45)*. Furthermore, suppose that the following identity holds* P*-a.s. on* $[0, T^* \wedge \tau]$*:*

$$\mu_S(X_t) - \lambda^P_t(1+\gamma_t) + \sum_{j=1}^{d+2} \Sigma_{d+2,j}\sqrt{R^{j,j}_t}\,\theta^j_t = r_t \tag{2.53}$$

where the function $\mu^S : \mathbb{R}^{d+2} \to \mathbb{R}$ *is as in* (2.35)*. Then the measure* Q *on* $(\Omega, \mathcal{G})$ *defined by* $\frac{dQ}{dP} := \mathcal{E}\left(\int \theta' dW + \int \gamma\, dM^P\right)_{T^*}$ *is an ELMM for* S *which preserves the affine structure of* (X, τ)*.*

Proof. Note that conditions (2.51)-(2.52) are analogous to conditions (2.8)-(2.9) for the affine process X as given in (2.31). Theorem 2.3.7 implies that $\mathcal{E}\left(\int \theta' dW + \int \gamma\, dM^P\right)$ is a P-a.s. strictly positive $(P, \mathbb{G})$-martingale on $[0, T^*]$ and, hence, we can define a measure Q on $(\Omega, \mathcal{G})$ with $Q \sim P$ by letting $\frac{dQ}{dP} := \mathcal{E}\left(\int \theta' dW + \int \gamma\, dM^P\right)_{T^*}$. Theorem 2.3.12 shows that Q preserves the affine structure of (X, τ). Finally, Girsanov's theorem and (2.53) imply that $\bar{S}$ is a $(Q, \mathbb{G})$-local martingale. In fact, recalling equation (2.35):

$$\begin{aligned}
d\bar{S}_t &= \bar{S}_{t-}\left(\mu_S(X_t) - \lambda^P_t - r_t\right)dt + \bar{S}_{t-}\sum_{j=1}^{d+2}\Sigma_{d+2,j}\sqrt{R^{j,j}_t}\,dW^j_t - \bar{S}_{t-}\,dM^P_t \\
&= \bar{S}_{t-}\left(\mu_S(X_t) - \lambda^P_t - r_t + \sum_{j=1}^{d+2}\Sigma_{d+2,j}\sqrt{R^{j,j}_t}\theta^j_t - \lambda^P_t\gamma_t\right)dt + \bar{S}_{t-}\sum_{j=1}^{d+2}\Sigma_{d+2,j}\sqrt{R^{j,j}_t}\,dW^{Q,j}_t \\
&\quad - \bar{S}_{t-}\,dM^Q_t \\
&= \bar{S}_{t-}\sum_{j=1}^{d+2}\Sigma_{d+2,j}\sqrt{R^{j,j}_t}\,dW^{Q,j}_t - \bar{S}_{t-}\,dM^Q_t
\end{aligned}$$

where $W^Q = \left(W^Q_t\right)_{0\leq t\leq T^*}$ is the $(Q,\mathbb{F})$/$(Q,\mathbb{G})$-Brownian motion defined by $W^Q := W - \int_0^{\cdot}\theta_u du$ and $M^Q = \left(M^Q_t\right)_{0\leq t\leq T^*}$ is the $(Q,\mathbb{G})$-martingale defined by $M^Q := M^P - \int_0^{\cdot\wedge\tau}\lambda^P_u\gamma_u du$. □

Remark 2.5.14 (On diffusive and jump-type risk premia). The processes θ and γ admit the financial interpretation of *risk premia* associated to the randomness generated by W and M^P, respectively. The $\mathbb{R}^{d+2}$-valued process θ represents the risk premium associated to the "diffusive risk" generated by the randomness driving the process X. Since the stock price, the volatility, the interest rate and the default intensity all depend on X, the risk premium θ can be considered as a market-wide non-diversifiable risk premium. The real-valued process γ represents the risk premium associated to the default event or, more precisely, the risk premium associated to the idiosyncratic component of the risk generated by the occurrence of the default event (see also El

Karoui & Martellini (2001) and Campi et al. (2009)). Assuming $\gamma \equiv 0$ means that the idiosyncratic component of the default risk can be diversified away in the market, as explained in Jarrow et al. (2005), and, therefore, market participants do not require a compensation for it. However, since we are considering a single firm, assuming $\gamma \equiv 0$ may be seen as an over-simplification of the model. Indeed, it is reasonable to expect the jump-type risk premium to be large when it is difficult to hedge the risk associated with the timing of the default event of a particular firm. As can be immediately seen from (2.51)-(2.52), the risk premia θ and γ are not independent, due to their common dependence on X. Finally, as we have pointed out at the beginning of Section 2.5.1, the last $\bar{d} - d - 2$ components of the $\mathbb{R}^{\bar{d}}$-valued Brownian motion $\bar{W}$ will only be used in Section 2.5.4 to model noise factors affecting the observations of market data. Consequently, it seems reasonable to assume that such "noise-related risk" does not carry any intrinsic financial risk and can be diversified away. Consequently, market participants do not require a compensation for being exposed to the randomness generated by the last $\bar{d} - d - 2$ components of the Brownian motion $\bar{W}$.

Until the end of this Section, we shall always assume that the hypotheses of Proposition 2.5.13 hold and, hence, there exists a risk-neutral measure Q which preserves the affine structure of (X, τ). This implies that the process X satisfies the following SDE on $(\Omega, \mathcal{G}, \mathbb{F}, Q)$:

$$dX_t = \left(b^Q + A^Q X_t\right) dt + \Sigma\sqrt{R_t}\, dW_t^Q \tag{2.54}$$

where $b^Q \in \mathbb{R}^{d+2}$ and $A^Q \in \mathbb{R}^{(d+2)\times(d+2)}$ are defined as follows, for $i, j \in \{1, \ldots, d+2\}$:

$$A_{i,j}^Q := A_{i,j} + \sum_{k=1}^{d+2} \Sigma_{i,k}\Theta_{k,j} \qquad \text{and} \qquad b_i^Q := b_i + \sum_{k=1}^{d+2} \Sigma_{i,k}\hat{\theta}_k \tag{2.55}$$

For the purpose of valuing default-sensitive payoffs, the *T-forward survival risk-neutral probability measure* $\widetilde{Q}^T$ turns out to be useful. For any $T \in [0, T^*]$, the measure $\widetilde{Q}^T$ is defined as follows[10]:

$$\frac{d\widetilde{Q}^T}{dQ} = \frac{e^{-\int_0^T \left(r_u + \lambda_u^Q\right) du}}{E^Q\left[e^{-\int_0^T \left(r_u + \lambda_u^Q\right) du}\right]} \tag{2.56}$$

where $E^Q[\cdot]$ denotes the expectation with respect to the risk-neutral measure Q. The measure $\widetilde{Q}^T$ bears resemblance to the *T-survival measure* P^T introduced in Section 2.5.2, except that $\widetilde{Q}^T$ is defined with respect to a risk-neutral measure Q and the density $\frac{d\widetilde{Q}^T}{dQ}$ also involves the discount factor.

In the following Proposition, we derive a general formula for the arbitrage-free price of a general payoff with maturity $T \in [0, T^*]$, paid only if the default event occurs after T. We call this situation the *zero recovery* case.

[10]We assume throughout this Section that $E^Q\left[e^{-\int_0^T r_u du}\right] < \infty$ for all $T \in [0, T^*]$. Clearly, under this assumption the expectation appearing in the denominator of (2.56) is finite.

Proposition 2.5.15. *Let $t, T \in [0, T^*]$ with $t \leq T$ and let $F : \mathbb{R}^{d+2} \to \mathbb{R}_+$. Then the arbitrage-free price at time t of the random payoff $F(X_T)$ at maturity T on the event $\{\tau > T\}$, with zero recovery in the case of default, is given by the following expression:*

$$E^Q\left[e^{-\int_t^T r_u du} F(X_T)(1-H_T)\big|\mathcal{G}_t\right] = (1-H_t)\, E^Q\left[e^{-\int_t^T \left(r_u+\lambda_u^Q\right)du}\big|\mathcal{F}_t\right] E^{\widetilde{Q}^T}\left[F(X_T)\,|\mathcal{F}_t\right] \tag{2.57}$$

Proof. Since the random variable $e^{-\int_t^T r_u du} F(X_T)$ is $\mathcal{F}_T$-measurable, Corollary 5.1.1 of Bielecki & Rutkowski (2002) implies that:

$$E^Q\left[e^{-\int_t^T r_u du} F(X_T)(1-H_T)\big|\mathcal{G}_t\right] = (1-H_t)\, E^Q\left[e^{-\int_t^T \left(r_u+\lambda_u^Q\right)du} F(X_T)\big|\mathcal{F}_t\right]$$

Equation (2.57) follows then immediately from the definition of $\widetilde{Q}^T$ and Bayes' rule. □

The next Proposition deals with the arbitrage-free valuation of a *recovery payment*, i.e. a payoff which is paid exactly at the time of default if the latter occurs before a fixed maturity $T \in [0, T^*]$.

Proposition 2.5.16. *Let $t, T \in [0, T^*]$ with $t \leq T$ and let $G : \mathbb{R}^{d+2} \to \mathbb{R}_+$. Then the arbitrage-free price at time t of the random payoff $G(X_\tau)$ at the default time τ on the event $\{\tau \leq T\}$ is given by the following expression:*

$$E^Q\left[e^{-\int_0^\tau r_u du} G(X_\tau)\,\mathbf{1}_{\{t<\tau\leq T\}}\big|\mathcal{G}_t\right] = (1-H_t)\int_t^T E^Q\left[e^{-\int_t^u \left(r_s+\lambda_s^Q\right)ds}\big|\mathcal{F}_t\right] E^{\widetilde{Q}^u}\left[\lambda_u^Q\, G(X_u)\,|\mathcal{F}_t\right] du \tag{2.58}$$

Proof. Corollary 5.1.3 of Bielecki & Rutkowski (2002) implies that:

$$E^Q\left[e^{-\int_0^\tau r_u du} G(X_\tau)\,\mathbf{1}_{\{t<\tau\leq T\}}\big|\mathcal{G}_t\right] = (1-H_t)\, E^Q\left[\int_t^T e^{-\int_t^u \left(r_s+\lambda_s^Q\right)ds}\,\lambda_u^Q\, G(X_u)\, du\Big|\mathcal{F}_t\right]$$

Equation (2.58) then easily follows by first applying Tonelli's theorem and then using the definition of the u-forward survival risk-neutral measure $\widetilde{Q}^u$ and Bayes' rule. □

Remark 2.5.17. In the particular case where $G \equiv \delta$ for some $\delta \in (0, \infty)$, one can obtain an alternative formula for the recovery payment in terms of *defaultable forward rates* and *forward hazard rates conditioned on survival*: see Lemma 2.1 of Fontana & Runggaldier (2010).

By combining Propositions 2.5.15 and 2.5.16 we can obtain the arbitrage-free price of a generic defaultable claim which pays a random amount at maturity in the case of survival and a random recovery at the time of default if the latter occurs before the maturity. Most (single-name) credit derivatives can be written as linear combinations of zero-recovery contingent claims and a pure recovery contingent claim, the latter being paid only in the case of default. Note that Propositions 2.5.15 and 2.5.16 do not rely on the affine structure of (X, τ). As can be seen from (2.57)-(2.58), if we want to obtain more explicit results, we need to be able to compute the two following quantities:

1. the $\mathcal{F}_t$-conditional expected value (under Q) of random variables of the form $\exp\left(-\int_t^T \left(\lambda_u^Q + r_u\right)du\right)$, for $t, T \in [0, T^*]$ with $t \leq T$;

2. the $\mathcal{F}_t$-conditional expected value (under $\widetilde{Q}^T$) of random variables of the form $\Gamma(X_T)$, for some function $\Gamma : \mathbb{R}^{d+2} \to \mathbb{R}_+$ and $t, T \in [0, T^*]$ with $t \leq T$.

Due to the affine structure of (X, τ) under the measure Q, the first of the above two elements can be computed in closed form. Furthermore, we are able to obtain an explicit expression for the $\mathcal{F}_t$-conditional characteristic function of X_T under the measure $\widetilde{Q}^T$ and, hence, we can compute $E^{\widetilde{Q}^T}[\Gamma(X_T)|\mathcal{F}_t]$ by relying on standard Fourier inversion techniques, see e.g. Carr & Madan (1999).

Proposition 2.5.18. *Let $t, T \in [0, T^*]$ with $t \leq T$. Then the following hold:*

(a) $E^Q\left[e^{-\int_t^T \left(r_u + \lambda_u^Q\right) du} \big| \mathcal{F}_t\right] = e^{\Phi^Q(T-t,0)+\Psi^Q(T-t,0)' X_t}$

(b) the $\mathcal{F}_t$-conditional characteristic function of X_T with respect to $\widetilde{Q}^T$ is explicitly given as follows, for all $u \in i\mathbb{R}^{d+2}$:

$$E^{\widetilde{Q}^T}\left[e^{u' X_T} \big| \mathcal{F}_t\right] = \frac{e^{\Phi^Q(T-t,u)+\Psi^Q(T-t,u)' X_t}}{e^{\Phi^Q(T-t,0)+\Psi^Q(T-t,0)' X_t}} \tag{2.59}$$

where the functions $\Phi^Q : [0, T^*] \times \mathbb{C}^{d+2} \to \mathbb{C}$ and $\Psi^Q : [0, T^*] \times \mathbb{C}^{d+2} \to \mathbb{C}^{d+2}$ solve the following system of Riccati ordinary differential equations:

$$\begin{aligned}
\partial_t \Phi^Q(t,u) &= \left(b^Q\right)' \Psi^Q(t,u) + \frac{1}{2} \sum_{k=m+1}^{d+2} \left[\Sigma' \Psi^Q(t,u)\right]_k^2 \alpha_k - \bar{r} - \bar{\lambda}^Q \\
\Phi^Q(0,u) &= 0 \\
\partial_t \Psi_i^Q(t,u) &= \sum_{k=1}^{d+2} A_{k,i}^Q \Psi_k^Q(t,u) + \frac{1}{2}\left[\Sigma' \Psi^Q(t,u)\right]_i^2 \beta_{i,i} + \frac{1}{2} \sum_{k=m+1}^{d+2} \left[\Sigma' \Psi^Q(t,u)\right]_k^2 \beta_{i,k} - \Upsilon_i - \Lambda^{Q,i} \\
\partial_t \Psi_j^Q(t,u) &= \sum_{k=m+1}^{d+2} A_{k,j}^Q \Psi_k^Q(t,u) - \Upsilon_j \\
\Psi^Q(0,u) &= u
\end{aligned} \tag{2.60}$$

for $i = 1, \ldots, m$ and $j = m+1, \ldots, d+2$, where the parameters A^Q and b^Q are defined as in (2.55)*, the parameters $\bar{r}$ and Υ as in* (2.46) *and the parameters $\bar{\lambda}^Q$ and Λ^Q as in* (2.52).

Proof. Recalling that in this Section we always assume that the hypotheses of Proposition 2.5.13 hold, the tuple $\left(A^Q, b^Q, \Sigma, \alpha, \beta\right)$ satisfies Condition C. Therefore, part *(a)* of the Proposition can be proved by relying on the same arguments used in the proof of Proposition 2.5.8 and Corollary 2.5.9, now with respect to the measure Q, and part *(b)* by relying on Proposition 2.5.8 and Corollary 2.5.10, using also the definition of the T-forward survival risk-neutral measure $\widetilde{Q}^T$. Analogous results are given in Theorem 10.4 and Corollary 10.2 of Filipović (2009). □

Risk-neutral valuation of fixed-income defaultable products

By relying on Propositions 2.5.15, 2.5.16 and 2.5.18, we can now proceed to the arbitrage-free valuation of some typical payoffs. The following Corollary deals with the valuation of *default-free* 0*-coupon bonds* and *defaultable* 0*-coupon* 0*-recovery bonds*, with unitary face value. It is well-known that these elementary financial products can be considered as the building blocks of more complex financial derivatives. In the following Corollary, the subscript "*rf*" stands for "risk-free" while "*df*" stands for "defaultable".

Corollary 2.5.19. *Let $t, T \in [0, T^*]$ with $t \leq T$. Then the following hold:*

(a) the arbitrage-free price at time t of a 0-coupon default-free bond with maturity T is given by the following expression:

$$\Pi_{rf}(t,T) := E^Q\left[e^{-\int_t^T r_u du}\middle|\mathcal{G}_t\right] = e^{\bar{\Phi}^Q(T-t,0)+\bar{\Psi}^Q(T-t,0)'X_t}$$

where the functions $\bar{\Phi}^Q : [0,T^*] \times \mathbb{C}^{d+2} \to \mathbb{C}$ and $\bar{\Psi}^Q : [0,T^*] \times \mathbb{C}^{d+2} \to \mathbb{C}^{d+2}$ satisfy a system of Riccati ODEs of the type (2.60) *but without the parameters $\bar{\lambda}^Q$ and Λ^Q;*

(b) the arbitrage-free price at time t of a 0-coupon 0-recovery defaultable bond with maturity T is given by the following expression:

$$\Pi_{df}(t,T) := E^Q\left[e^{-\int_t^T r_u du}(1-H_T)\middle|\mathcal{G}_t\right] = (1-H_t)\, e^{\Phi^Q(T-t,0)+\Psi^Q(T-t,0)'X_t}$$

where the functions $\Phi^Q : [0,T^*] \times \mathbb{C}^{d+2} \to \mathbb{C}$ and $\Psi^Q : [0,T^*] \times \mathbb{C}^{d+2} \to \mathbb{C}^{d+2}$ satisfy the system of Riccati ODEs (2.60).

Proof. To prove part *(a)*, note that $E^Q\left[e^{-\int_t^T r_u du}\middle|\mathcal{G}_t\right] = E^Q\left[e^{-\int_t^T r_u du}\middle|\mathcal{F}_t\right]$, due to the immersion property between the filtrations $\mathbb{F}$ and $\mathbb{G}$ under Q (see part *(a)* of Theorem 2.3.10) together with Proposition 5.9.1.1 of Jeanblanc et al. (2009). Since the interest rate process r is linear with respect to X, due to (2.46), and since X satisfies on $(\Omega, \mathcal{G}, \mathbb{F}, Q)$ the SDE (2.54), part *(a)* can be proved as the first part of Corollary 10.2 of Filipović (2009). To prove part *(b)* it suffices to use Proposition 2.5.15 (with $F \equiv 1$) together with part *(a)* of Proposition 2.5.18. □

The following Definition formalizes the well-known concepts of *yield* of a 0-coupon default-free bond and *credit spread* of a 0-coupon 0-recovery defaultable bond.

Definition 2.5.20. *Let $t, T \in [0, T^*]$ with $t < T$:*

- *the yield $YL(t,T)$ of a 0-coupon default-free bond is defined as follows:*

$$YL(t,T) := -\frac{1}{T-t}\log\big(\Pi_{rf}(t,T)\big)$$

- *the credit spread $CS(t,T)$ of a 0-coupon 0-recovery defaultable bond, computed with respect to a 0-coupon default-free bond with the same maturity T, is defined as follows on the set $\{\tau > t\}$:*

$$CS(t,T) := -\frac{1}{T-t}\log\left(\frac{\Pi_{df}(t,T)}{\Pi_{rf}(t,T)}\right)$$

It is obvious that, due to Corollary 2.5.19, yields and credit spreads take the following linear form:

$$
\begin{aligned}
YL(t,T) &= -\frac{\bar{\Phi}^Q(T-t,0)}{T-t} - \frac{\bar{\Psi}^Q(T-t,0)'}{T-t}X_t \\
&=: A_{YL}(T-t) + B_{YL}(T-t)' X_t
\end{aligned}
\tag{2.61}
$$

and, on the set $\{\tau > t\}$:

$$
\begin{aligned}
CS(t,T) &= \frac{\bar{\Phi}^Q(T-t,0) - \Phi^Q(T-t,0)}{T-t} + \frac{\left(\bar{\Psi}^Q(T-t,0) - \Psi^Q(T-t,0)\right)'}{T-t}X_t \\
&=: A_{CS}(T-t) + B_{CS}(T-t)' X_t
\end{aligned}
\tag{2.62}
$$

By relying on Corollary 2.5.19, we can easily compute the arbitrage-free prices of default-free and defaultable *coupon-bearing* bonds: see for instance Bielecki & Rutkowski (2002), Section 1.1.5. By combining Corollary 2.5.19 with Proposition 2.5.16, we can also value defaultable corporate bonds with non-zero recovery payments in the case of default. Furthermore, these results allow us to compute the *fair spread* of a *Credit Default Swap* (see e.g. Fontana & Runggaldier (2010), Section 2.2).

Risk-neutral valuation of equity-related defaultable products

While Corollary 2.5.19 deals with the valuation of fixed-income default-free and defaultable financial products, let us now consider the case of equity-related products. In particular, we now derive semi-explicit formulae for the arbitrage-free prices of *call* and *put* options written on the defaultable stock S. In the following Corollary we assume that the options are issued by the defaultable firm itself and written on its own stock S, with zero recovery in the case of default. This means that if the firm goes bankrupt before the maturity of the option then the latter becomes worthless.

Corollary 2.5.21. *Let $t,T \in [0,T^*]$ with $t \leq T$ and let $K \in (0,\infty)$ denote a fixed strike price. Then the following hold*[11]*:*

(a) the arbitrage-free price at time t of an European call option issued by the defaultable firm itself and written on S, with maturity T and strike price K, is given by the following expression:

$$
\begin{aligned}
\tilde{C}_K(t,T) &:= E^Q\left[e^{-\int_t^T r_u du}(S_T - K)^+ (1-H_T)\middle|\mathcal{G}_t\right] \\
&= (1-H_t)\, e^{\Phi^Q(T-t,0)+\Psi^Q(T-t,0)'X_t}\,\frac{1}{2\pi}\int_{-\infty}^{+\infty}\tilde{\varphi}_t^T(w+iu)\,\frac{K^{-(w-1+iu)}}{(w+iu)(w-1+iu)}du
\end{aligned}
\tag{2.63}
$$

for some $w \in (1,\infty)$ such that the system of Riccati ODEs (2.60) *has an unique solution with initial condition $u = (0,\ldots,0,w)' \in \mathbb{R}^{d+2}$;*

[11] Observe that the only difference between the risk-neutral valuation formulae (2.63) and (2.64) lies in the interval the fixed parameter w belongs to.

(b) *the arbitrage-free price at time* t *of an European put option issued by the defaultable firm itself and written on* S*, with maturity* T *and strike price* K*, is given by the following expression:*

$$\tilde{P}_K(t,T) := E^Q\left[e^{-\int_t^T r_u du}(K-S_T)^+(1-H_T)\big|\mathcal{G}_t\right]$$
$$= (1-H_t)\,e^{\Phi^Q(T-t,0)+\Psi^Q(T-t,0)'X_t}\,\frac{1}{2\pi}\int_{-\infty}^{+\infty}\tilde{\varphi}_t^T(w+iu)\,\frac{K^{-(w-1+iu)}}{(w+iu)(w-1+iu)}du \tag{2.64}$$

for some $w\in(-\infty,0)$ *such that the system of Riccati ODEs* (2.60) *has an unique solution with initial condition* $u=(0,\dots,0,w)'\in\mathbb{R}^{d+2}$.

Here, $\tilde{\varphi}_t^T$ *denotes the* $\mathcal{F}_t$*-conditional characteristic function of the logarithm of the pre-default stock price* L_T *with respect to the measure* $\widetilde{Q}^T$*, explicitly given as follows, for* $y\in\mathbb{C}$*:*

$$\tilde{\varphi}_t^T(y) := E^{\widetilde{Q}^T}\left[e^{yL_T}|\mathcal{F}_t\right] = \frac{e^{\Phi^Q(T-t,y_{d+2})+\Psi^Q(T-t,y_{d+2})'X_t}}{e^{\Phi^Q(T-t,0)+\Psi^Q(T-t,0)'X_t}} \tag{2.65}$$

where $y_{d+2} := (0,\dots,0,y)'\in\mathbb{C}^{d+2}$ *and the functions* $\Phi^Q:[0,T^*]\times\mathbb{C}^{d+2}\to\mathbb{C}$ *and* $\Psi^Q:[0,T^*]\times\mathbb{C}^{d+2}\to\mathbb{C}^{d+2}$ *solve the system of Riccati ODEs* (2.60).

Proof. Note first that, due to Proposition 2.5.15, we have the following:

$$E^Q\left[e^{-\int_t^T r_u du}(S_T-K)^+(1-H_T)\big|\mathcal{G}_t\right] = E^Q\left[e^{-\int_t^T r_u du}\left(e^{L_T}-K\right)^+(1-H_T)\big|\mathcal{G}_t\right]$$
$$= (1-H_t)\,E^Q\left[e^{-\int_t^T\left(r_u+\lambda_u^Q\right)du}\big|\mathcal{F}_t\right]E^{\widetilde{Q}^T}\left[\left(e^{L_T}-K\right)^+\big|\mathcal{F}_t\right]$$
$$= (1-H_t)\,e^{\Phi^Q(T-t,0)+\Psi^Q(T-t,0)'X_t}\,E^{\widetilde{Q}^T}\left[\left(e^{L_T}-K\right)^+\big|\mathcal{F}_t\right]$$

where the last equality follows from part *(a)* of Proposition 2.5.18. As in Carr & Madan (1999) and Filipović (2009), Lemma 10.2, it can be shown that, for any $w\in(1,\infty)$:

$$(e^x-K)^+ = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{(w+iu)x}\frac{K^{-(w-1+iu)}}{(w+iu)(w-1+iu)}du$$

Hence:

$$E^{\widetilde{Q}^T}\left[\left(e^{L_T}-K\right)^+|\mathcal{F}_t\right] = \frac{1}{2\pi}E^{\widetilde{Q}^T}\left[\int_{-\infty}^{\infty}e^{(w+iu)L_T}\frac{K^{-(w-1+iu)}}{(w+iu)(w-1+iu)}du\Big|\mathcal{F}_t\right]$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}E^{\widetilde{Q}^T}\left[e^{(w+iu)L_T}|\mathcal{F}_t\right]\frac{K^{-(w-1+iu)}}{(w+iu)(w-1+iu)}du$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\varphi}_t^T(w+iu)\,\frac{K^{-(w-1+iu)}}{(w+iu)(w-1+iu)}du$$

where the second equality follows by Fubini's theorem, which can be applied since the following holds (here we use the shorthand notation $\tilde{f}(u) := \frac{K^{-(w-1+iu)}}{(w+iu)(w-1+iu)}$, for $u\in\mathbb{R}$):

$$E^{\widetilde{Q}^T}\left[\int_{-\infty}^{\infty}\left|e^{(w+iu)L_T}\tilde{f}(u)\right|du\right] \leq E^{\widetilde{Q}^T}\left[e^{wL_T}\right]\int_{-\infty}^{\infty}\left|\tilde{f}(u)\right|du < \infty$$

due to the fact that the function $\tilde{f}$ is integrable and $w \in (1, \infty)$ is such that there exists an unique solution to the system of Riccati ODEs (2.60) with initial condition $u = (0, \ldots, 0, w)' \in \mathbb{R}^{d+2}$ (see also Filipović (2009), Theorem 10.5). The explicit expression given in equation (2.65) for the $\mathcal{F}_t$-conditional characteristic function of L_T under the measure $\widetilde{Q}^T$ follows from part *(b)* of Proposition 2.5.18. This completes the proof of part *(a)* of the Corollary. Part *(b)* can be proved in a similar way, using the fact that, for $w \in (-\infty, 0)$:

$$(K - e^x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(w+iu)x} \frac{K^{-(w-1+iu)}}{(w + iu)(w - 1 + iu)} du$$

□

By relying on techniques similar to those used in the proof of Corollary 2.5.21 and exploiting the knowledge of the $\mathcal{F}_t$-conditional characteristic function of X_T with respect to $\widetilde{Q}^T$, one can derive the arbitrage-free value of more complex payoffs, see e.g. Section 10.3.1 of Filipović (2009).

We want to remark that, since the defaultable stock price process jumps to zero as soon as the default event occurs and remains thereafter frozen at zero, we have that:

$$\begin{aligned}\tilde{C}_K(t, T) &= E^Q \left[e^{-\int_t^T r_u du} (S_T - K)^+ (1 - H_T) \big| \mathcal{G}_t \right] \\ &= E^Q \left[e^{-\int_t^T r_u du} (S_T - K)^+ \big| \mathcal{G}_t \right] =: C_K(t, T)\end{aligned} \tag{2.66}$$

As in Corollary 2.5.21, $\tilde{C}_K(t, T)$ denotes the arbitrage-free price of a call option issued by the defaultable firm itself. On the contrary, $C_K(t, T)$ denotes the arbitrage-free price of a call option written on the defaultable stock S but issued by a default-free third party[12]. Hence, (2.66) shows that the arbitrage-free price of an European call option written on S does not depend on whether it is issued by the defaultable firm itself or by a default-free third party. This is intuitively clear. In fact, both options are written on the same defaultable stock S and, since the process S remains frozen at zero as soon as the default event occurs, a call option becomes worthless as soon as the default event occurs regardless of the fact that the option is issued by the defaultable firm itself or by a default-free third party. However, this is not the case for a put option. In fact, part *(b)* of Corollary 2.5.21 gives the arbitrage-free price of an European put option, with strike price K and maturity T, issued by the defaultable firm itself. If we consider instead a put option (with the same strike price K and maturity T) written on the defaultable stock S but issued by a default-free third party and not by the defaultable firm itself, then it has to be valued as follows:

$$\begin{aligned}P_K(t, T) &:= E^Q \left[e^{-\int_t^T r_u du} (K - S_T)^+ \big| \mathcal{G}_t \right] \\ &= E^Q \left[e^{-\int_t^T r_u du} (K - S_T)^+ (1 - H_T) \big| \mathcal{G}_t \right] + K\, E^Q \left[e^{-\int_t^T r_u du} H_T \big| \mathcal{G}_t \right] \\ &= \tilde{P}_K(t, T) + K\, E^Q \left[e^{-\int_t^T r_u du} H_T \big| \mathcal{G}_t \right]\end{aligned} \tag{2.67}$$

[12] We distinguish arbitrage-free prices of options issued by a default-free third party and written on the defaultable stock S from arbitrage-free prices of options issued by the defaultable firm itself and written on its own stock S by using the tilde notation for the latter.

We can notice that the arbitrage-free price of a put option issued by a default-free third party can be decomposed into the sum of the arbitrage-free price of a put option issued by the defaultable firm itself and an additional term equal to the arbitrage-free value at time t of the payment of the strike price K at the maturity T if the firm defaults before time T. Let us focus on this last term:

$$\begin{aligned} E^Q\left[e^{-\int_t^T r_u du} H_T \middle| \mathcal{G}_t\right] &= E^Q\left[e^{-\int_t^T r_u du}\left(1-1+H_T\right) \middle| \mathcal{G}_t\right] \\ &= E^Q\left[e^{-\int_t^T r_u du} \middle| \mathcal{G}_t\right] - E^Q\left[e^{-\int_t^T r_u du}\left(1-H_T\right) \middle| \mathcal{G}_t\right] \\ &= \Pi_{rf}\left(t,T\right) - \Pi_{df}\left(t,T\right) > 0 \end{aligned} \tag{2.68}$$

where we have used the definition of $\Pi_{rf}\left(t,T\right)$ and $\Pi_{df}\left(t,T\right)$ (compare Corollary 2.5.19). We have thus shown that, in line with the economic intuition, the arbitrage-free price $P_K\left(t,T\right)$ of a put option issued by a default-free third party can be decomposed into the sum of the price $\tilde{P}_K\left(t,T\right)$ of a put option with identical characteristics (in terms of underlying, strike price and maturity) but issued by the defaultable firm itself and a "default-protection" term proportional to the difference $\left(\Pi_{rf}\left(t,T\right) - \Pi_{df}\left(t,T\right)\right)$ between the arbitrage-free prices of a 0-coupon default-free bond and of a 0-coupon 0-recovery defaultable bond.

Similarly as in Chapter 6 of Gatheral (2006), we can easily derive the following *put-call parity* relation between the arbitrage-free prices of European call and put options (with the same strike price K and the same maturity T) issued by the same defaultable firm on its own stock S:

$$\begin{aligned} \tilde{C}_K\left(t,T\right) - \tilde{P}_K\left(t,T\right) &= E^Q\left[e^{-\int_t^T r_u du}\left(\left(S_T-K\right)^+ - \left(K-S_T\right)^+\right)\left(1-H_T\right) \middle| \mathcal{G}_t\right] \\ &= E^Q\left[e^{-\int_t^T r_u du}\left(S_T-K\right)\left(1-H_T\right) \middle| \mathcal{G}_t\right] \\ &= E^Q\left[e^{-\int_t^T r_u du} S_T \middle| \mathcal{G}_t\right] - K\,\Pi_{df}\left(t,T\right) \end{aligned} \tag{2.69}$$

Analogously, if we consider call and put options written on the defaultable stock S and with the same characteristics but issued by a default-free third party we have that:

$$\begin{aligned} C_K\left(t,T\right) - P_K\left(t,T\right) &= E^Q\left[e^{-\int_t^T r_u du}\left(\left(S_T-K\right)^+ - \left(K-S_T\right)^+\right) \middle| \mathcal{G}_t\right] \\ &= E^Q\left[e^{-\int_t^T r_u du} S_T \middle| \mathcal{G}_t\right] - K\,\Pi_{rf}\left(t,T\right) \end{aligned} \tag{2.70}$$

Recall that, due to equation (2.66), we have that $\tilde{C}\left(t,T\right) = C\left(t,T\right)$. Hence, by combining equations (2.69) and (2.70) we obtain that:

$$\begin{aligned} P_K\left(t,T\right) - \tilde{P}_K\left(t,T\right) &= P_K\left(t,T\right) - C_K\left(t,T\right) - \left(\tilde{P}_K\left(t,T\right) - \tilde{C}_K\left(t,T\right)\right) \\ &= K\left(\Pi_{rf}\left(t,T\right) - \Pi_{df}\left(t,T\right)\right) \end{aligned}$$

which agrees with the result of equations (2.67) and (2.68). Furthermore, if the discounted stock price process $\bar{S}$ is a *true* $(Q,\mathbb{G})$-martingale and not only a $(Q,\mathbb{G})$-local martingale, equations (2.69) and (2.70) can be rewritten in the following classical versions:

$$\tilde{C}_K\left(t,T\right) - \tilde{P}_K\left(t,T\right) = S_t - K\,\Pi_{df}\left(t,T\right)$$

$$C_K\left(t,T\right) - P_K\left(t,T\right) = S_t - K\,\Pi_{rf}\left(t,T\right)$$

Remark 2.5.22 (On the supermartingale property of the discounted defaultable stock price process). Note that, due to Fatou's Lemma, the process $\bar{S}$ is a non-negative $(Q, \mathbb{G})$-supermartingale, being a non-negative $(Q, \mathbb{G})$-local martingale. This implies that, for all $t, T \in [0, T^*]$ with $t \le T$:

$$E^Q \left[e^{-\int_t^T r_u du} S_T \middle| \mathcal{G}_t \right] \le S_t$$

Hence, equations (2.69)-(2.70) imply the following:

$$\tilde{C}_K(t,T) - \tilde{P}_K(t,T) \le S_t - K\,\Pi_{df}(t,T)$$

$$C_K(t,T) - P_K(t,T) \le S_t - K\,\Pi_{rf}(t,T)$$

The above inequalities are strict in the case where the process $\bar{S}$ is a *strict* $(Q, \mathbb{G})$-local martingale, i.e. it is not a true $(Q, \mathbb{G})$-martingale (to this effect, compare also with Remark 2.4.8). This situation is connected to the existence of *stock price bubbles* and has been analyzed in a series of recent papers: see for instance Cox & Hobson (2005), Heston et al. (2007), Jarrow et al. (2007) and Jarrow et al. (2010).

2.5.4 The incomplete information case

So far, we have supposed that all market participants have access to the information contained in the filtration $\mathbb{G}$, to which the factor process Y is adapted, since it is $\mathbb{F}$-adapted and $\mathbb{F} \subseteq \mathbb{G}$. In this Section, we depart from this hypothesis and let some of the components of Y be unobservable. This implies that investors have no longer access to the *full information* filtration $\mathbb{G}$. In the context of credit risk modeling, the introduction of latent factors is particularly interesting, since it allows to represent unobserved *frailty* variables that impact on the likelihood of the occurrence of the default event (to this effect, see e.g. Schönbucher (2003b) and Duffie et al. (2009)). Furthermore, a model enriched with latent factors is less prone to inadequate specifications of the factor process Y and can also capture truly unmeasurable effects. For a more detailed discussion of these aspects, we refer the reader to the introductory sections of Fontana & Runggaldier (2010) and Fontana (2010a), which also contain several references to the pertinent literature.

In the present context, letting some of the driving processes be unobservable seems to be particularly appropriate. In fact, our modeling framework explicitly includes a stochastic volatility process. Clearly, the latter cannot be directly observed in real financial markets, since it represents a rather abstract quantity[13]. As a consequence, it seems reasonable to model stochastic volatility as an unobservable process. On the other hand, it seems also reasonable to assume that investors can

[13] We want to emphasize that one should distinguish between the different notions of *volatility*, *realized volatility* and *implied volatility*. While the last two concepts represent observable quantities, the concept of *volatility*, which denotes the instantaneous variance over an infinitesimal time interval, should not be assumed to be directly observable. It is well-known that the volatility of the stock price process can be approximated by the empirical quadratic variation. However, as can be seen from (2.35), the diffusion term of the stock price process S does not depend only on the stochastic volatility process v but also on several components of the factor process Y. Hence, in general we cannot directly disentangle the stochastic volatility component v_t from the observation of the empirical quadratic variation of the stock price process S.

observe the behavior of the stock price process, being the latter a traded security, and the occurrence of the default event. Note that, since we assume that the stock price jumps to zero as soon as the default event occurs, the observation of the default indicator process is embedded in the observation of the defaultable stock price process. Hence, let us make the following Assumption.

Assumption 2.5.23. *For a fixed $\check{d} \in \{1, \ldots, d\}$, the first $\check{d}$ components of the $\mathbb{R}^d$-valued factor process $Y = (Y_t)_{0 \leq t \leq T^*}$ are unobservable while the last $d - \check{d}$ components are observable.*

Note that there is no loss of generality in assuming that the *first* $\check{d}$ components of Y are unobservable, since otherwise we can simply reorder the components of Y. Clearly, the dynamics of each component of the vector process X as well as the affine structure of the model are not altered by a reordering of the components of Y. We denote by $\check{Y} = \big(\check{Y}_t\big)_{0 \leq t \leq T^*}$ the $\mathbb{R}^{\check{d}}$-valued unobservable process defined as $\check{Y}_t := \big(Y_t^1, \ldots, Y_t^{\check{d}}\big)'$ for $t \in [0, T^*]$ and by $\tilde{Y} = \big(\tilde{Y}_t\big)_{0 \leq t \leq T^*}$ the $\mathbb{R}^{d-\check{d}}$-valued observable process defined as $\tilde{Y}_t := \big(Y_t^{\check{d}+1}, \ldots, Y_t^d\big)'$ for $t \in [0, T^*]$. We also define $\check{X} := \big(v, \check{Y}'\big)'$ and similarly $\tilde{X} := \big(\tilde{Y}', L\big)'$, so that the $\mathbb{R}^{\check{d}+1}$-valued process $\check{X}$ collects all unobservable elements of X and the $\mathbb{R}^{d-\check{d}+1}$-valued process $\tilde{X}$ all observable elements of X. Due to (2.54), it is easy to see that the processes $\check{X}$ and $\tilde{X}$ satisfy the following dynamics under a risk-neutral measure Q:

$$d\check{X}_t = \left(\check{A}\check{X}_t + \tilde{A}\tilde{X}_t + \check{b}\right) dt + \check{\Sigma}\,\sqrt{R\big(\check{X}_t, \tilde{Y}_t\big)}\, dW_t^Q \tag{2.71}$$

$$d\tilde{X}_t = \left(\check{C}\check{X}_t + \tilde{C}\tilde{X}_t + \tilde{b}\right) dt + \tilde{\Sigma}\,\sqrt{R\big(\check{X}_t, \tilde{Y}_t\big)}\, dW_t^Q \tag{2.72}$$

for suitable matrices $\check{A}$, $\tilde{A}$, $\check{C}$, $\tilde{C}$, $\check{\Sigma}$, $\tilde{\Sigma}$ and vectors $\check{b}$ and $\tilde{b}$ such that:

$$A^Q = \begin{pmatrix} \check{A} & \tilde{A} \\ \check{C} & \tilde{C} \end{pmatrix} \in \mathbb{R}^{(d+2)\times(d+2)} \qquad b^Q = \begin{pmatrix} \check{b} \\ \tilde{b} \end{pmatrix} \in \mathbb{R}^{d+2} \qquad \Sigma = \begin{pmatrix} \check{\Sigma} \\ \tilde{\Sigma} \end{pmatrix} \in \mathbb{R}^{(d+2)\times(d+2)}$$

Similarly as in Fontana & Runggaldier (2010), we assume that all market participants are able to observe, besides the stock price process and the process $\tilde{Y}$ of observable factors, the following elements:

1. the interest rate process $r = (r_t)_{0 \leq t \leq T^*}$;
2. a vector composed of p *yields* computed on 0-coupon default-free bonds with respect to p different maturities T_i, $i = 1, \ldots, p$;
3. a vector composed of q *credit spreads* computed on 0-coupon 0-recovery defaultable bonds with respect to q different maturities T_j, $j = 1, \ldots, q$.

Of course, yields and credit spreads are computed on 0-coupon (0-recovery) default-free and defaultable bonds, respectively, which are rather stylized financial products and, as such, not liquidly traded in real financial markets. As a consequence, yields and credit spreads have to be reconstructed from more complex objects such as CDS spreads and prices of corporate bonds. Hence, we assume that investors are able to observe yields and credit spreads up to a noise factor, also due to liquidity and tax effects affecting CDS spreads and corporate bond prices. The noise factors are represented by the $\mathbb{R}^{\bar{d}-d-2}$-valued Brownian motion $W^* = (W^*_t)_{0\leq t\leq T^*}$, composed of the last $\bar{d}-d-2$ elements of the $\mathbb{R}^{\bar{d}}$-valued Brownian motion $\bar{W}$. Recall that we assume that there is no risk premium associated to the noise factor W^* (see Remark 2.5.14). Indeed, unlike the Brownian motion W driving the vector process X, the Brownian motion W^* does not represent a fundamental source of financial risk but only a small uncertainty affecting market prices, due to the presence of bid-ask spreads, transmission errors, liquidity and tax effects. This assumption implies that the process W^* is a Brownian motion with respect to both the physical and the risk-neutral probability measures[14].

For any $\{T_1,\ldots,T_p\}$, with $T_i \in (t,T^*]$ for all $i=1,\ldots,p$, let us denote by yl_t the vector in $\mathbb{R}^p$ composed of the p noisily-observed yields, corresponding to their theoretical values $YL\,(t,T_i)$, for $i=1,\ldots,p$:

$$yl_t := \big(YL\,(t,T_1)\,,\ldots,YL\,(t,T_p)\big)' + \varrho W^*_t \tag{2.73}$$

for some $\varrho \in \mathbb{R}^{p\times(\bar{d}-d-2)}$. Analogously, we denote by cs_t the vector in $\mathbb{R}^q$ composed of the q noisily-observed credit spreads, corresponding to their theoretical values $CS\,(t,T_j)$, for $j=1,\ldots,q$:

$$cs_t := \big(CS\,(t,T_1)\,,\ldots,CS\,(t,T_q)\big)' + \nu W^*_t \tag{2.74}$$

for some $\nu \in \mathbb{R}^{q\times(\bar{d}-d-2)}$. Equations (2.73)-(2.74), together with (2.61), (2.62) and (2.54), imply the following dynamics, for $i \in \{1,\ldots,p\}$:

$$\begin{aligned} d\,yl^i_t &= \big(\partial_t A_{YL}\,(T_i-t) + \partial_t B_{YL}\,(T_i-t)'\,X_t\big)\,dt + B_{YL}\,(T_i-t)\,\big(A^Q X_t + b^Q\big)\,dt \\ &\quad + B_{YL}\,(T_i-t)\,\Sigma\,\sqrt{R\,(X_t)}\,dW^Q_t + \varrho\,dW^*_t \\ &=: \Big(f\,(T_i-t) + \check{F}\,(T_i-t)'\,\check{X}_t + \tilde{F}\,(T_i-t)'\,\tilde{X}_t\Big)\,dt \\ &\quad + B_{YL}\,(T_i-t)\,\Sigma\,\sqrt{R\big(\check{X}_t,\tilde{Y}_t\big)}\,dW^Q_t + \varrho\,dW^*_t \end{aligned} \tag{2.75}$$

for suitable functions f, $\check{F}$ and $\tilde{F}$. Analogously, we have the following dynamics for the noisily-observed credit spread on $\{\tau > t\}$, for any $j \in \{1,\ldots,q\}$:

$$\begin{aligned} d\,cs^j_t &= \big(\partial_t A_{CS}\,(T_j-t) + \partial_t B_{CS}\,(T_j-t)'\,X_t\big)\,dt + B_{CS}\,(T_j-t)\,\big(A^Q X_t + b^Q\big)\,dt \\ &\quad + B_{CS}\,(T_j-t)\,\Sigma\,\sqrt{R\,(X_t)}\,dW^Q_t + \nu\,dW^*_t \\ &=: \Big(g\,(T_j-t) + \check{G}\,(T_j-t)'\,\check{X}_t + \tilde{G}\,(T_j-t)'\,\tilde{X}_t\Big)\,dt \\ &\quad + B_{CS}\,(T_j-t)\,\Sigma\,\sqrt{R\big(\check{X}_t,\tilde{Y}_t\big)}\,dW^Q_t + \nu\,dW^*_t \end{aligned} \tag{2.76}$$

for suitable functions g, $\check{G}$ and $\tilde{G}$. Due to equations (2.46) and (2.71)-(2.72), we have the following

[14] Furthermore, under the standing assumption that the risk-neutral measure Q preserves the affine structure of (X,τ), the process W^* is also a Brownian motion with respect to the enlarged filtration $\mathbb{G}$ under both P and Q, since the immersion property between $\mathbb{F}$ and $\mathbb{G}$ holds under both P and Q (see Proposition 2.5.13).

dynamics for the interest rate process:

$$\begin{aligned} dr_t = &\Big(\big(\Upsilon'_{[1:\check{d}+1]}\check{A} + \Upsilon'_{[\check{d}+2:d+2]}\check{C}\big)\check{X}_t + \big(\Upsilon'_{[1:\check{d}+1]}\tilde{A} + \Upsilon'_{[\check{d}+2:d+2]}\tilde{C}\big)\tilde{X}_t + \Upsilon' b^Q\Big)\, dt \\ &+ \Upsilon'\, \Sigma \sqrt{R\big(\check{X}_t, \tilde{Y}_t\big)}\, dW_t^Q \end{aligned} \tag{2.77}$$

where $\Upsilon_{[j:k]}$ denotes the vector $(\Upsilon_j, \ldots, \Upsilon_k)' \in \mathbb{R}^{k-j+1}$, for any $1 \le j \le k \le d+2$.

Let us now introduce the $\mathbb{R}^{d-\check{d}+p+q+2}$-valued vector process $V = (V_t)_{0 \le t < T^* \wedge \tau}$ defined as $V_t := \big(\tilde{X}'_t, r_t, yl'_t, cs'_t\big)'$ for $t \in [0, T^* \wedge \tau]$. We call V the *observations' process*. As can be seen from (2.72) and (2.75)-(2.77), the observations' process V has a drift term which is linear with respect to $\check{X}$ and V itself and a diffusion term proportional to the square root of a linear function of $\check{X}$ and V. Due to (2.71), the same holds true for the *unobservable state process* $\check{X}$. Hence, considering the couple $\big(\check{X}, V\big)$, we have obtained a *state-observation filtering system*, the dynamics of which preserve the fundamental affine structure of our framework. We do not discuss here possible approaches to the actual solution of the filtering problem: we just mention that in Fontana & Runggaldier (2010) the authors consider a similar filtering problem, which is tackled by introducing an auxiliary state process of lower dimension and relying on the *Extended Kalman Filter*.

Remark 2.5.24.

1. Note that, since we are considering a single defaultable firm, it is reasonable to restrict our attention to the set $\{\tau > t\}$, due to the fact that after the random default time τ the stock price remains frozen at zero and the firm cannot exit from the default state.

2. We have so far assumed that investors can observe, besides the interest rate and the defaultable stock price process, a vector of yields computed on default-free bonds and a vector of credit spreads computed on defaultable bonds. However, the present setting can also be extended to the case where investors are able to observe market prices of liquidly traded derivatives written on the stock S. As we have seen in Section 2.5.3, the arbitrage-free price $\pi_t^{F,T}$ of a derivative paying the random amount $F(S_T)$ at the maturity T (in the case of survival until T), for $t, T \in [0, T^*]$ with $t \le T$, can be expressed as a functional of $X_t = \big(\check{X}', \tilde{X}'\big)' \in \mathbb{R}^{d+2}$:

$$\pi_t^{F,T} := E^Q\left[e^{-\int_t^T r_u du}\, F(S_T)\,(1 - H_T)\big|\mathcal{G}_t\right] = (1 - H_t)\, e^{\Phi^Q(T-t,0) + \Psi^Q(T-t,0)' X_t}\, \bar{F}(X_t)$$

where $\bar{F}(X_t) := E^{\widetilde{Q}^T}\left[F\left(e^{L_T}\right)|X_t\right] = E^{\widetilde{Q}^T}\left[F\left(e^{L_T}\right)|\mathcal{F}_t\right]$, where we have used the Markov property of X (under the measure $\widetilde{Q}^T$) and the fact that $\sigma\{X_s : s \le t\} = \sigma\{W_s : s \le t\}$, for all $t \in [0, T^*]$. If the function $\bar{F}$ satisfies suitable smoothness conditions (i.e. $\bar{F} \in \mathcal{C}^2$), we can apply Itô's formula and compute its stochastic differential, which will involve the processes $\check{X}$ and $\tilde{X}$. Since most market prices are observed through a bid-ask spread, we can assume that the theoretical arbitrage-free price $\pi_t^{F,T}$ is noisily observed as well, as in (2.73)-(2.74). We can then enlarge the observations' process V by adding noisily observed derivatives' prices. However, depending on the model's specification, the function $\bar{F}$ may be rather complicated (see for instance Corollary 2.5.21) and we should not expect that the dynamics of noisily observed derivative prices enjoy the affine structure shown in (2.72) and

(2.75)-(2.77). Depending on the model's specification, this might be a more or less serious drawback.

3. Due to Corollary 2.5.9, on the set $\{\tau > t\}$, the logarithm of the conditional survival probability has a linear structure with respect to X. Hence, if we compute the stochastic differential of $\log P(\tau > t|\mathcal{G}_t)$ we obtain an SDE with a linear drift term and a diffusion term which is proportional to the square root of a linear function of X_t, where the driving Brownian motion is with respect to the physical probability measure P. We can then apply Girsanov's theorem in order to obtain the stochastic dynamics of $\log P(\tau > t|\mathcal{G}_t)$ under the risk-neutral measure Q. Due to the specific form of the risk premia considered in Proposition 2.5.13, such Q-dynamics will still be characterized by an affine structure of the type shown in equations (2.75)-(2.77). Hence, we can assume that investors can also observe noisy proxies of the (logarithms of) true survival probabilities and include them in the observations' process V. For more details on this approach, we refer the interested reader to Fontana (2010a).

Assuming that we are able to solve the filtering problem for the couple $\left(\check{X}, V\right)$, we can then easily deal with general pricing problems in the incomplete information case. Let us denote by $\mathbb{V} = (\mathcal{V}_t)_{0\leq t\leq T^*}$ the *investors' filtration*, i.e. the right-continuous P-augmentation of the filtration $\mathbb{V}^0 = \left(\mathcal{V}_t^0\right)_{0\leq t\leq T^*}$, where $\mathcal{V}_t^0 := \sigma\{V_s : s \leq t\} \vee \mathcal{H}_t$ for $t \in [0, T^*]$. We have of course $\mathbb{V} \subseteq \mathbb{G}$, i.e. $\mathcal{V}_t \subseteq \mathcal{G}_t$ for all $t \in [0, T^*]$. Clearly, since the process r is one of the elements of the observations' process V, the *savings account* process $\left(e^{\int_0^t r_u du}\right)_{0\leq t\leq T^*}$ is $\mathbb{V}$-adapted. As a consequence, $e^{\int_0^\cdot r_u du}$ is not only a *numéraire* with respect to $(Q, \mathbb{G})$, as in Section 2.5.3, but also with respect to $(Q, \mathbb{V})$. Let us denote by Q an ELMM for S with respect to the full-information filtration $\mathbb{G}$. The following simple Lemma shows that the measure Q is also an ELMM for S with respect to the investors' filtration $\mathbb{V}$.

Lemma 2.5.25. *Let Q be an ELMM for S with respect to the full-information filtration $\mathbb{G}$. Then, Q is also an ELMM for S with respect to the investors' filtration $\mathbb{V}$.*

Proof. Note first that the discounted defaultable stock price process $\bar{S} = e^{-\int_0^\cdot r_u du} S$ is $\mathbb{V}$-adapted. Since $\bar{S}$ is continuous before τ and jumps to zero at τ, it is locally bounded. More precisely, there exists a sequence $(\tau_n)_{n\in\mathbb{N}}$ of $\mathbb{V}$-stopping times such that $\tau_n \nearrow \infty$ P-a.s. as $n \to \infty$ and $\bar{S}^{\tau_n} \leq K(n)$ P-a.s., for some $K(n) \in (0, \infty)$. Hence, we can write as follows, for any $s, t \in [0, T^*]$ with $s \leq t$ and $n \in \mathbb{N}$:

$$E^Q\left[\bar{S}_{t\wedge\tau_n}|\mathcal{V}_s\right] = E^Q\left[E^Q\left[\bar{S}_{t\wedge\tau_n}|\mathcal{G}_s\right]\big|\mathcal{V}_s\right] = E^Q\left[\bar{S}_{s\wedge\tau_n}|\mathcal{V}_s\right] = \bar{S}_{s\wedge\tau_n}$$

where the second equality follows from the fact that the process $\bar{S}$ is a $(Q, \mathbb{G})$-local martingale, since Q is assumed to be an ELMM for S with respect to $\mathbb{G}$. This shows that $\bar{S}$ is a $(Q, \mathbb{V})$-local martingale, meaning that Q is an ELMM for S with respect to $\mathbb{V}$. □

Lemma 2.5.25 implies that, in the incomplete information case, we can obtain an arbitrage-free price system with respect to the investors' filtration $\mathbb{V}$ by computing conditional expectations of discounted payoffs with respect to $(Q, \mathbb{V})$. Furthermore, this $(Q, \mathbb{V})$-arbitrage-free pricing system is linked in an intuitive way to the original *full information* $(Q, \mathbb{G})$-arbitrage-free pricing system[15]. To see this, for $t, T \in [0, T^*]$ with $t \leq T$, let us denote by $\Pi(t, T; F)$ the $(Q, \mathbb{G})$-arbitrage-free price at time t of an integrable contingent claim F with maturity T. We have then[16]:

$$\hat{\Pi}(t, T; F) := E^Q\left[e^{-\int_t^T r_u du} F \middle| \mathcal{V}_t\right] = E^Q\left[E^Q\left[e^{-\int_t^T r_u du} F \middle| \mathcal{G}_t\right] \middle| \mathcal{V}_t\right] = E^Q\left[\Pi(t, T; F) | \mathcal{V}_t\right] \tag{2.78}$$

where we have denoted by $\hat{\Pi}(t, T; F)$ the $(Q, \mathbb{V})$-arbitrage-free price at time t of the contingent claim F.

The following Proposition deals with the general problem of computing the $(Q, \mathbb{V})$-arbitrage-free price of a credit-risky derivative which pays on the set $\{\tau > t\}$ a random amount $F(X_T)$ at the maturity T, for $F : \mathbb{R}^{d+2} \to \mathbb{R}_+$.

Proposition 2.5.26. *Let $t, T \in [0, T^*]$ with $t \leq T$ and $F : \mathbb{R}^{d+2} \to \mathbb{R}_+$. Then the $(Q, \mathbb{V})$-arbitrage-free price at time t of the random payoff $F(X_T)$ at maturity T on the event $\{\tau > T\}$, with zero recovery in the case of default, is given by the following expression:*

$$E^Q\left[e^{-\int_t^T r_u du} F(X_T)(1 - H_T) \middle| \mathcal{V}_t\right] = \hat{\Pi}_{df}(t, T)\, E^{\hat{Q}^T}\left[F(X_T) | \mathcal{V}_t\right] \tag{2.79}$$

where the measure $\hat{Q}^T \ll Q$ on $(\Omega, \mathcal{G})$ is defined as follows:

$$\frac{d\hat{Q}^T}{dQ} := \frac{e^{-\int_0^T r_u du}(1 - H_T)}{E^Q\left[e^{-\int_0^T r_u du}(1 - H_T)\right]}$$

and $\hat{\Pi}_{df}(t, T)$ denotes the $(Q, \mathbb{V})$-arbitrage-free price at time t of a 0-coupon 0-recovery defaultable bond with maturity T, explicitly given as follows:

$$\hat{\Pi}_{df}(t, T) = (1 - H_t)\, e^{\Phi^Q(T-t,0)}\, E^Q\left[e^{\Psi^Q(T-t,0)' X_t} \middle| \mathcal{V}_t\right] \tag{2.80}$$

Furthermore, the $\mathcal{V}_t$-conditional characteristic function of X_T with respect to the measure $\hat{Q}^T$ is given as follows on the set $\{\tau > t\}$, for all $u \in i\mathbb{R}^{d+2}$:

$$\hat{\varphi}_t^T(u) := E^{\hat{Q}^T}\left[e^{u' X_T} | \mathcal{V}_t\right] = \frac{e^{\Phi^Q(T-t,u)}\, E^Q\left[e^{\Psi^Q(T-t,u)' X_t} \middle| \mathcal{V}_t\right]}{e^{\Phi^Q(T-t,0)}\, E^Q\left[e^{\Psi^Q(T-t,0)' X_t} \middle| \mathcal{V}_t\right]} \tag{2.81}$$

where the functions $\Phi^Q : [0, T^*] \times \mathbb{C}^{d+2} \to \mathbb{C}$ and $\Psi^Q : [0, T^*] \times \mathbb{C}^{d+2} \to \mathbb{C}^{d+2}$ are the unique solutions to the system of Riccati ODEs (2.60).

Proof. Equation (2.79) follows directly by using the definition of the measure $\hat{Q}^T$ together with Bayes' rule and equation (2.78), here applied to the arbitrage-free price of a 0-coupon 0-recovery defaultable bond with maturity T. To prove equation (2.80) it suffices to combine (2.78) with part *(b)* of Corollary 2.5.19. Hence, it remains to show that the $\mathcal{V}_t$-conditional characteristic function of

[15] We refer the reader to Gombani et al. (2007) for a more detailed analysis of the *consistency* of arbitrage-free pricing systems with respect to different filtrations.

[16] An analogous result can be found in Lemma 3.1 of Fontana & Runggaldier (2010).

the random vector X_T under the measure $\hat{Q}^T$ can be expressed as in (2.81). On the set $\{\tau > t\} = \{H_t = 0\}$, this can be shown as follows, for any $u \in i\mathbb{R}^{d+2}$:

$$
\begin{aligned}
E^{\hat{Q}^T}\left[e^{u'X_T}|\mathcal{V}_t\right] &= \frac{E^Q\left[e^{-\int_t^T r_u du}\left(1-H_T\right)e^{u'X_T}\big|\mathcal{V}_t\right]}{E^Q\left[e^{-\int_t^T r_u du}\left(1-H_T\right)\big|\mathcal{V}_t\right]} \\
&= \frac{E^Q\left[E^Q\left[e^{-\int_t^T r_u du}\left(1-H_T\right)e^{u'X_T}\big|\mathcal{G}_t\right]\big|\mathcal{V}_t\right]}{E^Q\left[E^Q\left[e^{-\int_t^T r_u du}\left(1-H_T\right)\big|\mathcal{G}_t\right]\big|\mathcal{V}_t\right]} \\
&= \frac{E^Q\left[E^Q\left[e^{-\int_t^T\left(r_u+\lambda_u^Q\right)du}\big|\mathcal{F}_t\right]E^{\tilde{Q}^T}\left[e^{u'X_T}|\mathcal{F}_t\right]\big|\mathcal{V}_t\right]}{e^{\,\Phi^Q(T-t,0)}\,E^Q\left[e^{\,\Psi^Q(T-t,0)'X_t}\big|\mathcal{V}_t\right]} \\
&= \frac{e^{\,\Phi^Q(T-t,u)}\,E^Q\left[e^{\,\Psi^Q(T-t,u)'X_t}\big|\mathcal{V}_t\right]}{e^{\,\Phi^Q(T-t,0)}\,E^Q\left[e^{\,\Psi^Q(T-t,0)'X_t}\big|\mathcal{V}_t\right]}
\end{aligned}
$$

where the first equality follows from the definition of the measure $\hat{Q}^T$, the third equality (for the numerator) is due to Proposition 2.5.15 applied to the function $F(x) = e^{u'x}$ and (for the denominator) to part *(b)* of Corollary 2.5.19 and the fourth equality uses Proposition 2.5.18. □

Remark 2.5.27.

1. As can be seen from Proposition 2.5.26 and (2.81), the key ingredient for the computation of arbitrage-free prices at time t with respect to the investors' filtration $\mathbb{V}$ turns out to be the $\mathcal{V}_t$-conditional characteristic function of the random vector X_t under the risk-neutral measure Q. This can be easily computed as soon as we manage to solve the filtering problem for the couple $\left(\check{X}, V\right)$ under the measure Q.

2. It is interesting to observe that the probability measure $\hat{Q}^T$ introduced in Proposition 2.5.26 is similar to the T-forward survival risk-neutral measure $\widetilde{Q}^T$ introduced at the beginning of Section 2.5.3. The only difference is that the density $\frac{d\hat{Q}^T}{dQ}$ involves the default indicator term $(1 - H_T)$, which may be equal to 0 in the case of default, unlike the term $e^{-\int_t^T \lambda_u^Q du}$. As a consequence, the measure $\hat{Q}^T$ is absolutely continuous with respect to Q but is not equivalent to Q. However, this has no implications for the arbitrage-free properties of the model, since we only use the measure $\hat{Q}^T$ as a computational tool.

3. Clearly, by following the arguments used in the proof of Proposition 2.5.16, one can extend Proposition 2.5.26 to the arbitrage-free valuation of recovery payments with respect to the investors' filtration $\mathbb{V}$.

2.5.5 Connections to the literature

We give here a very brief overview of the related literature on the joint modeling of equity and credit risk, which has been the subject of several recent papers. In particular, in Campi et al. (2009), Carr

& Linetsky (2006) and Carr & Madan (2010) the authors consider *local volatility* models (more specifically, Campi et al. (2009) and Carr & Linetsky (2006) work with a *CEV* process) for the stock price process and introduce a random default time modeled as the first jump time of a Poisson process with stochastic intensity, the latter being given by a deterministic function of the pre-default stock price. Moving from a local volatility to a stochastic volatility approach, Carr & Schoutens (2008) and Carr & Wu (2010) consider the classical stochastic volatility model of Heston (1993) with the inclusion of an unpredictable jump to default. In the present Chapter, we embed these models into a rather general class of multivariate affine diffusion models.

The very recent paper by Cheridito & Wugalter (2011) is rather close to our framework. Indeed, the authors jointly model the stock price process, its volatility and a vector of stochastic factors via a multivariate affine process, allowing also for jumps in the factors' dynamics, and then proceed to the analysis of pricing and hedging problems for several typical payoff structures. However, they do only consider the model under an a priori chosen risk-neutral probability measure, while the major focus of this Chapter consists in studying the model under both the physical and the risk-neutral probability measure, ensuring that the structure of the model is preserved after the change of measure. Indeed, the common approach in the literature surveyed so far is to formulate a model with respect to an exogenously fixed risk-neutral probability measure, the only exception being the papers by Campi et al. (2009) and, to a lesser extent, Carr & Wu (2010). Even in the last two papers, the authors adopt very particular specifications of the risk premium process and, therefore, our more general results could be of interest.

Concerning the financial interpretation as *risk premia* of the processes θ and γ appearing in density process of a risk-neutral probability measure, we refer the interested reader to El Karoui & Martellini (2001), Driessen (2005), Jarrow et al. (2005), Section 9.3 of McNeil et al. (2005) and Giesecke & Goldberg (2008). In particular, Driessen (2005) points out the importance of explicitly distinguishing between the risk of credit spread changes, if no default occurs, from the risk of the default event itself. A positive jump-type risk premium indicates that not enough corporate bonds (or analogous default-sensitive products) are traded in order to fully diversify the default jump risk. Furthermore, if one does not allow for a jump-type risk premium, one then incurs into the risk of misspecifying the model, thus leading to an inaccurate fit of the observed default rates. In Giesecke & Goldberg (2008), the authors propose a different interpretation of the risk premium process γ in terms of a *transparency premium* required by investors to compensate for their imperfect knowledge of the threshold which triggers the default event in the context of a structural model.

2.6 Conclusions and further developments

In this Chapter, we have studied the effects of (locally) equivalent changes of measure on reduced-form credit risk models, where the default intensity is driven by a multivariate diffusion affine process. In particular, we have established necessary and sufficient conditions for the preservation of the affine structure of the model under a locally equivalent change of measure. As an application, we have considered a defaultable extension of the popular Heston (1993) stochastic volatility model. In that context, we have characterized the family of risk-neutral measures which preserve the Heston with jump-to-default structure, thus generalizing the results obtained by Wong & Heyde (2006). Then, extending the Heston with jump-to-default model, we have shown how our techniques can be applied in a general hybrid equity/credit risk modeling framework allowing for stochastic volatility and multiple (possibly unobserved) stochastic factors which affect the interest rate and the default intensity.

Among the future developments of this Chapter, the extension of the results of Section 2.3 to general semimartingale affine processes seems of particular interest. Clearly, many of the results of the present Chapter rely on the continuity of the underlying affine process and, hence, the extension to the discontinuous case requires different strategies and techniques. However, we are rather confident that one can obtain an analogous characterization of the family of all locally equivalent probability measures which preserve the affine structure of the model. In that direction, we already have some preliminary results which shall be presented in a future work. In a default-free context, related questions have also been studied in the recent paper Kallsen & Muhle-Karbe (2010).

In the final part of this Chapter, we have briefly considered the case where some of the components of the underlying affine process represent latent factors which cannot be directly observed. Section 2.5.4 only lays the foundations for the analysis of the incomplete information situation, together with the ensuing filtering problem. In particular, it could be of interest to apply the filtering techniques proposed in Fontana & Runggaldier (2010) to a hybrid equity/credit risk model allowing for both stochastic volatility and incomplete information, under both the real-world and the risk-neutral probability measure. Knowing that we can preserve the affine structure of the model under an equivalent change of measure, we can take advantage of the analytical tractability ensured by the affine framework and compute risk-neutral prices of several derivatives, in particular call/put options written on the defaultable stock. Then, an interesting task would consist in studying the different impacts of default risk, stochastic volatility and incomplete information on the shape of the implied volatility surface obtained from those option prices.

Chapter 3

Diffusion-based models for financial markets without martingale measures

3.1 Introduction

The concepts of *Equivalent (Local) Martingale Measure* (E(L)MM), *no-arbitrage* and *risk-neutral pricing* can be rightfully considered as the cornerstones of modern mathematical finance. It seems to be almost folklore that such concepts can be regarded as mutually equivalent. In fact, most practical applications in quantitative finance are directly formulated under suitable assumptions which ensure that those concepts are indeed equivalent.

In recent years, maybe due to the dramatic turbulences raging over financial markets, an increasing attention has been paid to models that allow for financial market anomalies. More specifically, several authors have studied market models where stock price bubbles may occur: see e.g. Cox & Hobson (2005), Heston et al. (2007), Hulley (2010), Jarrow et al. (2007), Jarrow et al. (2010). It has been shown that bubble phenomena are consistent with the classical no-arbitrage theory based on the notion of *No Free Lunch with Vanishing Risk* (NFLVR), as developed in Delbaen & Schachermayer (1994) and Delbaen & Schachermayer (2006). However, in the presence of a bubble, discounted prices of risky assets follow, under a risk-neutral measure, a *strict* local martingale, i.e. a local martingale which is not a true martingale. This fact already implies that several well-known and classical results (for instance *put-call parity*, see e.g. Cox & Hobson (2005)) of mathematical finance do not hold anymore and must be modified accordingly.

A decisive step towards enlarging the scope of financial models has been represented by the study of models which do not fit at all into the classical no-arbitrage theory based on (NFLVR). Indeed, several authors (see e.g. Christensen & Larsen (2007), Delbaen & Schachermayer (1995a), Hulley (2010), Karatzas & Kardaras (2007), Loewenstein & Willard (2000)) have studied instances where an ELMM may fail to exist. More specifically, financial models that do not admit an ELMM appear in the context of *Stochastic Portfolio Theory* (see Fernholz & Karatzas (2009) for a recent overview) and in the *Benchmark Approach* (see the monograph Platen & Heath (2006) for a detailed account). In the absence of a well-defined ELMM, many of the usual results of mathematical finance seem to break down and one is led to ask whether there is still a meaningful way to proceed in order to solve the crucial problems of portfolio optimisation and contingent claim valuation. It is then a remarkable result that a satisfactory theory can be developed even in the absence of an ELMM, especially in the case of a complete financial market model, as we are going to illustrate.

The present Chapter aims at carefully analysing a general class of diffusion-based financial models, without relying on the existence of an ELMM. More specifically, we discuss several notions of no-arbitrage that are weaker than the traditional (NFLVR) condition and we study necessary and sufficient conditions for their validity. We show that the financial market may still be viable, in a sense to be made precise, even in the absence of an ELMM. In particular, it turns out that the viability of the financial market is fundamentally linked to a square-integrability property of the *market price of risk* process. Some of the results that we are going to present have already been obtained, also in more general settings (see e.g. Christensen & Larsen (2007), Hulley & Schweizer (2010), Karatzas & Kardaras (2007), Kardaras (2010b)). However, by exploiting the Itô-process structure of our model, we are able to provide simple and transparent proofs, highlighting the key ideas behind the general theory. We also discuss the connections to the *Growth-Optimal Portfolio* (GOP), which is shown to be the unique portfolio possessing the *numéraire* property. In similar diffusion-based settings, related works that study the question of market viability in the absence of an ELMM include Fernholz & Karatzas (2009), Galesso & Runggaldier (2010), Heston et al. (2007), Loewenstein & Willard (2000), Londono (2004), Platen (2002) and Runggaldier (2003).

Besides studying the question of market viability, a major focus of this Chapter is on the valuation and hedging of contingent claims in the absence of an ELMM. In particular, we argue that the concept of *market completeness*, namely the capability to replicate every contingent claim, must be kept distinct from the existence of an ELMM. Indeed, we prove that the financial market may be viable and complete regardless of the existence of an ELMM. We then show that, in the context of a complete financial market, there is a unique natural candidate for the price of an arbitrary contingent claim, given by its GOP-discounted expected value under the original (*real-world*) probability measure. To this effect, we revisit some ideas originally appeared in the context of the *Benchmark Approach*, providing more careful proofs and extending some previous results.

The present Chapter is strongly linked to Chapter 4. In fact, on the one hand, many of the results of the present Chapter can be recovered by specializing the more general results of Chapter 4 to the case of an Itô-process-based model for a financial market. On the other hand, since the results of Chapter 4 are formulated in general and abstract terms, the present Chapter can also serve as an illustration of the concepts and of the main results contained in Chapter 4, avoiding some of the technicalities which arise in the more general context by restricting our attention to a diffusion-based financial market model.

The present Chapter is structured as follows. Section 3.2 introduces the general setting, which consists of a class of Itô-process models satisfying minimal technical conditions. We introduce a basic standing Assumption and we carefully describe the set of admissible trading strategies. The question of whether (properly defined) arbitrage opportunities do exist or not is dealt with in Section 3.3. In particular, we explore the notions of *increasing profit* and *arbitrage of the first kind*, giving necessary and sufficient conditions for their absence from the financial market. In turn, this lead us to introduce the concept of *martingale deflators*, which can be regarded as weaker counterparts to the traditional martingale measures. Section 3.4 proves the existence of an unique *Growth-Optimal* strategy, which admits an explicit characterization and also generates the *numéraire portfolio*. In turn, the latter is shown to be the reciprocal of a martingale deflator, thus linking the numéraire portfolio to the no-arbitrage criteria discussed in Section 3.3. Section 3.5 starts with the hedging and valuation of contingent claims, showing that the financial market may be complete even in the absence of an ELMM. Section 3.6 deals with contingent claim valuation according to three alternative approaches: *real-world pricing*, *upper-hedging pricing* and *utility indifference valuation*. In the particular case of a complete market, we show that they yield the same valuation formula. Section 6 concludes by pointing out possible extensions and further developments.

3.2 The general setting

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. For a fixed time horizon $T \in (0, \infty)$, let $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ be a filtration on $(\Omega, \mathcal{F}, P)$ satisfying the usual conditions of right-continuity and completeness. Let $W = (W_t)_{0 \leq t \leq T}$ be an $\mathbb{R}^d$-valued Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. To allow for greater generality we do not assume from the beginning that $\mathbb{F} = \mathbb{F}^W$, meaning that the filtration $\mathbb{F}$ may be strictly larger than the P-augmented Brownian filtration $\mathbb{F}^W$. Also, the initial σ-field $\mathcal{F}_0$ may be strictly larger than the trivial σ-field.

We consider a *financial market* composed of $N+1$ securities S^i, for $i = 0, 1, \ldots, N$, with $N \leq d$. As usual, we let S^0 represent a locally riskless asset, which we name *savings account*, and we define the process $S^0 = (S^0_t)_{0 \leq t \leq T}$ as follows:

$$S^0_t := \exp\left(\int_0^t r_u \, du\right) \qquad \text{for } t \in [0, T] \tag{3.1}$$

where the *interest rate* process $r = (r_t)_{0 \leq t \leq T}$ is a real-valued progressively measurable process such that $\int_0^T |r_t| \, dt < \infty$ P-a.s. The remaining assets S^i, for $i = 1, \ldots, N$, are supposed to be *risky assets*. For $i = 1, \ldots, N$, the process $S^i = (S^i_t)_{0 \leq t \leq T}$ is given by the solution to the following SDE:

$$dS^i_t = S^i_t \, \mu^i_t \, dt + \sum_{j=1}^d S^i_t \, \sigma^{i,j}_t \, dW^j_t \qquad S^i_0 = s^i \tag{3.2}$$

where:

(i) $s^i \in (0, \infty)$ for all $i = 1, \ldots, N$;

(ii) $\mu = (\mu_t)_{0 \leq t \leq T}$ is an $\mathbb{R}^N$-valued progressively measurable process with $\mu_t = \left(\mu^1_t, \ldots, \mu^N_t\right)'$ and satisfying $\int_0^T \|\mu_t\| \, dt < \infty$ P-a.s.;

(iii) $\sigma = (\sigma_t)_{0 \leq t \leq T}$ is an $\mathbb{R}^{N \times d}$-valued progressively measurable process with $\sigma_t = \left\{\sigma^{i,j}_t\right\}_{\substack{i=1,\ldots,N \\ j=1,\ldots,d}}$ and satisfying $\sum_{i=1}^N \sum_{j=1}^d \int_0^T \left(\sigma^{i,j}_t\right)^2 dt < \infty$ P-a.s.

The SDE (3.2) admits the following explicit solution, for every $i = 1, \dots, N$ and $t \in [0, T]$:

$$S_t^i = s^i \exp\left(\int_0^t \left(\mu_u^i - \frac{1}{2}\sum_{j=1}^d \left(\sigma_u^{i,j}\right)^2\right) du + \sum_{j=1}^d \int_0^t \sigma_u^{i,j}\, dW_u^j\right) \tag{3.3}$$

Note that conditions *(ii)-(iii)* above represent minimal conditions in order to have a meaningful definition of the ordinary and stochastic integrals appearing in (3.3). Apart from these technical requirements, we leave the stochastic processes μ and σ fully general. For $i = 0, 1, \dots, N$, we denote by $\bar{S}^i = \left(\bar{S}_t^i\right)_{0 \leq t \leq T}$ the *discounted price* process of the i-th asset, defined as $\bar{S}_t^i := S_t^i / S_t^0$ for $t \in [0, T]$.

Let us now introduce the following standing Assumption, which we shall always assume to be satisfied without any further mention.

Assumption 3.2.1. *For all $t \in [0, T]$, the $(N \times d)$-matrix σ_t has P-a.s. full rank.*

Remark 3.2.2. From a financial perspective, Assumption 3.2.1 means that the financial market does not contain redundant assets, i.e. there does not exist a non-trivial linear combination of $\left(S^1, \dots, S^N\right)$ that is locally riskless, in the sense that its dynamics are not affected by the Brownian motion W. However, we want to point out that Assumption 3.2.1 is only used in the following for proving uniqueness properties of trading strategies and, hence, could also be relaxed.

In order to rigorously describe the activity of trading in the financial market, we now introduce the concepts of *trading strategy* and *discounted portfolio process*. In the following Definition we only consider *self-financing* trading strategies which generate positive portfolio processes.

Definition 3.2.3.

(a) An $\mathbb{R}^N$-valued progressively measurable process $\pi = (\pi_t)_{0 \leq t \leq T}$ is an admissible trading strategy *if $\int_0^T \left\|\sigma_t' \pi_t\right\|^2 dt < \infty$ P-a.s. and $\int_0^T \left|\pi_t' \left(\mu_t - r_t \mathbf{1}\right)\right| dt < \infty$ P-a.s., where $\mathbf{1} := (1, \dots, 1)' \in \mathbb{R}^N$. We denote by $\mathcal{A}$ the set of all admissible trading strategies.*

(b) For any $(v, \pi) \in \mathbb{R}_+ \times \mathcal{A}$, the associated discounted portfolio process $\bar{V}^{v,\pi} = \left(\bar{V}_t^{v,\pi}\right)_{0 \leq t \leq T}$ *is defined by:*

$$\begin{aligned} \bar{V}_t^{v,\pi} &:= v\, \mathcal{E}\left(\sum_{i=1}^N \int \pi^i \frac{d\bar{S}^i}{\bar{S}^i}\right)_t \\ &= v \exp\left(\int_0^t \pi_u' \left(\mu_u - r_u \mathbf{1}\right) du - \frac{1}{2}\int_0^t \left\|\sigma_u' \pi_u\right\|^2 du + \int_0^t \pi_u' \sigma_u\, dW_u\right) \end{aligned} \tag{3.4}$$

for all $t \in [0, T]$, where $\mathcal{E}(\cdot)$ denotes the stochastic exponential.

The integrability conditions in part *(a)* of Definition 3.2.3 ensure that both the ordinary and the stochastic integrals appearing in (3.4) are well-defined. For all $i = 1, \dots, N$ and $t \in [0, T]$, π_t^i represents the proportion of wealth invested in the i-th risky asset S^i at time t. Consequently, $1 - \pi_t' \mathbf{1}$ represents the proportion of wealth invested in the savings account S^0 at time t. Note that part *(b)* of Definition 3.2.3 corresponds to requiring the trading strategy π to be *self-financing*. Observe that Definition 3.2.3 implies that, for any $(v, \pi) \in \mathbb{R}_+ \times \mathcal{A}$, we have $V_t^{v,\pi} = v\, V_t^{1,\pi}$, for all $t \in [0, T]$. Due to this scaling property, we shall often let $v = 1$ without loss of generality, denoting $V^{\pi} := V^{1,\pi}$ for any $\pi \in \mathcal{A}$. By definition, the discounted portfolio process $\bar{V}^{\pi}$ satisfies the following dynamics:

$$d\bar{V}_t^{\pi} = \bar{V}_t^{\pi} \sum_{i=1}^{N} \pi_t^i \frac{d\bar{S}_t^i}{\bar{S}_t^i} = \bar{V}_t^{\pi}\, \pi_t' \left(\mu_t - r_t \mathbf{1}\right) dt + \bar{V}_t^{\pi}\, \pi_t' \,\sigma_t \, dW_t \tag{3.5}$$

Remark 3.2.4. The fact that admissible portfolio processes are uniformly bounded from below by zero excludes pathological *doubling strategies* (see e.g. Karatzas & Shreve (1998), Section 1.1.2). Moreover, an economic motivation for focusing on positive portfolios only is given by the fact that market participants have *limited liability* and, therefore, are not allowed to trade anymore if their total tradeable wealth reaches zero. See also Section 2 of Christensen & Larsen (2007), Section 6 of Platen (2009) and Section 10.3 of Platen & Heath (2006) for an amplification of this point.

3.3 No-arbitrage conditions and the market price of risk

In order to ensure that the model introduced in the previous Section represents a viable financial market, in a sense to be made precise (see Definition 3.3.9), we need to carefully answer the question of whether properly defined arbitrage opportunities are excluded. We start by giving the following Definition.

Definition 3.3.1. *A trading strategy $\pi \in \mathcal{A}$ is said to yield an* increasing profit *if the corresponding discounted portfolio process $\bar{V}^{\pi} = \left(\bar{V}_t^{\pi}\right)_{0 \le t \le T}$ satisfies the following two conditions:*

(a) $\bar{V}^{\pi}$ is P-a.s. increasing, in the sense that $P\left(\bar{V}_s^{\pi} \le \bar{V}_t^{\pi} \text{ for all } s, t \in [0, T] \text{ with } s \le t\right) = 1$;

(b) $P\left(\bar{V}_T^{\pi} > 1\right) > 0$.

The notion of increasing profit represents the most glaring type of arbitrage opportunity and, hence, it is of immediate interest to know whether it is allowed or not in the financial market. As a preliminary, the following Lemma gives an equivalent characterization of the notion of increasing profit. We denote by ℓ the Lebesgue measure on $[0, T]$.

Lemma 3.3.2. *There exists an increasing profit if and only if there exists a trading strategy $\pi \in \mathcal{A}$ satisfying the following two conditions:*

(a) $\pi_t' \sigma_t = 0$ $P \otimes \ell$-a.e.;

(b) $\pi_t' \left(\mu_t - r_t \mathbf{1}\right) \neq 0$ on some subset of $\Omega \times [0, T]$ with positive $P \otimes \ell$-measure.

Proof. Let $\pi \in \mathcal{A}$ be a trading strategy yielding an increasing profit. Due to Definition 3.3.1, the process $\bar{V}^{\pi}$ is P-a.s. increasing, hence of finite variation. Equation (3.5) implies then that the continuous local martingale $\left(\int_0^t \bar{V}_u^\pi \, \pi'_u \, \sigma_u \, dW_u\right)_{0 \leq t \leq T}$ is also of finite variation. This fact in turn implies that $\pi'_t \, \sigma_t = 0$ $P \otimes \ell$-a.e. (see e.g. Karatzas & Shreve (1991), Section 1.5). Since $P\left(\bar{V}_T^\pi > 1\right) > 0$, we must have $\pi'_t \, (\mu_t - r_t \mathbf{1}) \neq 0$ on some subset of $\Omega \times [0,T]$ with positive $P \otimes \ell$-measure.

Conversely, let $\pi \in \mathcal{A}$ be a trading strategy satisfying conditions *(a)-(b)*. Define then the process $\bar{\pi} = (\bar{\pi}_t)_{0 \leq t \leq T}$ as follows, for $t \in [0,T]$:

$$\bar{\pi}_t := \text{sign}\big(\pi'_t \, (\mu_t - r_t \mathbf{1})\big) \pi_t$$

It is clear that $\bar{\pi} \in \mathcal{A}$ and $\bar{\pi}'_t \, \sigma_t = 0$ $P \otimes \ell$-a.e. and hence, due to (3.4), for all $t \in [0,T]$:

$$\bar{V}_t^{\bar{\pi}} = \exp\left(\int_0^t \bar{\pi}'_u \, (\mu_u - r_u \mathbf{1}) \, du\right)$$

Furthermore, we have that $\bar{\pi}'_t \, (\mu_t - r_t \mathbf{1}) \geq 0$, with strict inequality holding on some subset of $\Omega \times [0,T]$ with positive $P \otimes \ell$-measure. This implies that the process $\bar{V}^{\bar{\pi}} = \left(\bar{V}_t^{\bar{\pi}}\right)_{0 \leq t \leq T}$ is P-a.s. increasing and satisfies $P\left(\bar{V}_T^{\bar{\pi}} > 1\right) > 0$, meaning that $\bar{\pi}$ yields an increasing profit. □

The following Proposition gives a necessary and sufficient condition in order to exclude the existence of increasing profits (compare also with Theorem 4.3.2 in Chapter 4).

Proposition 3.3.3. *There are no increasing profits if and only if there exists an $\mathbb{R}^d$-valued progressively measurable process $\gamma = (\gamma_t)_{0 \leq t \leq T}$ such that the following condition holds:*

$$\mu_t - r_t \mathbf{1} = \sigma_t \gamma_t \qquad P \otimes \ell\text{-a.e.} \tag{3.6}$$

Proof. Suppose there exists an $\mathbb{R}^d$-valued progressively measurable process $\gamma = (\gamma_t)_{0 \leq t \leq T}$ such that (3.6) is satisfied and let $\pi \in \mathcal{A}$ be such that $\pi'_t \, \sigma_t = 0$ $P \otimes \ell$-a.e. Then we have:

$$\pi'_t \, (\mu_t - r_t \mathbf{1}) = \pi'_t \, \sigma_t \gamma_t = 0 \qquad P \otimes \ell\text{-a.e.}$$

meaning that there cannot exist a trading strategy $\pi \in \mathcal{A}$ satisfying conditions *(a)-(b)* of Lemma 3.3.2. Due to the equivalence result of Lemma 3.3.2, this implies that there are no increasing profits.

Conversely, suppose that there exists no trading strategy in $\mathcal{A}$ yielding an increasing profit. Let us first introduce the following linear spaces, for every $t \in [0,T]$:

$$\mathcal{R}\left(\sigma_t\right) := \left\{\sigma_t y : y \in \mathbb{R}^d\right\} \qquad \mathcal{K}\left(\sigma'_t\right) := \left\{y \in \mathbb{R}^N : \sigma'_t y = 0\right\}$$

Denote by $\Pi_{\mathcal{K}(\sigma'_t)}$ the orthogonal projection on $\mathcal{K}\left(\sigma'_t\right)$. As in Lemma 1.4.6 of Karatzas & Shreve (1998), we define the process $p = (p_t)_{0 \leq t \leq T}$ by:

$$p_t := \Pi_{\mathcal{K}(\sigma'_t)} \left(\mu_t - r_t \mathbf{1}\right)$$

Define then the process $\hat{\pi} = (\hat{\pi}_t)_{0 \leq t \leq T}$ by:

$$\hat{\pi}_t := \begin{cases} \frac{p_t}{\|p_t\|} & \text{if } p_t \neq 0, \\ 0 & \text{if } p_t = 0. \end{cases}$$

Since the processes μ and r are progressively measurable, Corollary 1.4.5 of Karatzas & Shreve (1998) ensures that $\hat{\pi}$ is progressively measurable. Clearly, we have then $\hat{\pi} \in \mathcal{A}$ and, by definition, $\hat{\pi}$ satisfies condition *(a)* of Lemma 3.3.2. Since there are no increasing profits, Lemma 3.3.2 implies that the following identity holds $P \otimes \ell$-a.e.:

$$\|p_t\| = \frac{p_t'}{\|p_t\|} \left(\mu_t - r_t \mathbf{1}\right) \mathbf{1}_{\{p_t \neq 0\}} = \hat{\pi}_t' \left(\mu_t - r_t \mathbf{1}\right) \mathbf{1}_{\{p_t \neq 0\}} = 0 \tag{3.7}$$

where the first equality uses the fact that $\mu_t - r_t \mathbf{1} - p_t \in \mathcal{K}^{\perp}\left(\sigma_t'\right)$, for all $t \in [0,T]$, with the superscript $\perp$ denoting the orthogonal complement. From (3.7) we have $p_t = 0$ $P \otimes \ell$-a.e., meaning that $\mu_t - r_t \mathbf{1} \in \mathcal{K}^{\perp}\left(\sigma_t'\right) = \mathcal{R}\left(\sigma_t\right)$ $P \otimes \ell$-a.e. This amounts to saying that we have:

$$\mu_t - r_t \mathbf{1} = \sigma_t \gamma_t \qquad P \otimes \ell\text{-a.e.}$$

for some $\gamma_t \in \mathbb{R}^d$. Taking care of the measurability issues, it can be shown that we can take $\gamma = \left(\gamma_t\right)_{0 \leq t \leq T}$ as a progressively measurable process (compare Karatzas & Shreve (1998), proof of Theorem 1.4.2). □

Let us now introduce one of the crucial objects in our analysis: the *market price of risk* process.

Definition 3.3.4. *The $\mathbb{R}^d$-valued progressively measurable* market price of risk *process $\theta = \left(\theta\right)_{0 \leq t \leq T}$ is defined as follows, for $t \in [0,T]$:*

$$\theta_t := \sigma_t' \left(\sigma_t \, \sigma_t'\right)^{-1} \left(\mu_t - r_t \mathbf{1}\right)$$

The standing Assumption 3.2.1 ensures that the market price of risk process θ is well-defined[1]. From a financial perspective, θ_t measures the excess return $\left(\mu_t - r_t \mathbf{1}\right)$ of the risky assets (with respect to the savings account) in terms of their volatility.

Remark 3.3.5 (***Absence of increasing profits***)**.** Note that, by definition, the market price of risk process θ satisfies condition (3.6). Proposition 3.3.3 then implies that, under the standing Assumption 3.2.1, there are no increasing profits. Note however that θ may not be the unique process satisfying condition (3.6).

Let us now introduce the following integrability condition on the market price of risk process.

Assumption 3.3.6. *The market price of risk process $\theta = \left(\theta_t\right)_{0 \leq t \leq T}$ belongs to $L^2_{loc}\left(W\right)$, meaning that $\int_0^T \|\theta_t\|^2 \, dt < \infty$ P-a.s.*

Remark 3.3.7. Let $\gamma = \left(\gamma_t\right)_{0 \leq t \leq T}$ be an $\mathbb{R}^d$-valued progressively measurable process satisfying condition (3.6). Using the notation $\mathcal{R}\left(\sigma_t'\right) = \left\{\sigma_t' \, x : x \in \mathbb{R}^N\right\}$ and $\mathcal{R}^{\perp}\left(\sigma_t'\right) = \mathcal{K}\left(\sigma_t\right) = \left\{x \in \mathbb{R}^d : \sigma_t \, x = 0\right\}$, we have the orthogonal decomposition $\gamma_t = \Pi_{\mathcal{R}(\sigma_t')}\left(\gamma_t\right) + \Pi_{\mathcal{K}(\sigma_t)}\left(\gamma_t\right)$, for $t \in [0,T]$. Under Assumption 3.2.1, elementary linear algebra shows that:

$$\Pi_{\mathcal{R}(\sigma_t')}\left(\gamma_t\right) = \sigma_t' \left(\sigma_t \sigma_t'\right)^{-1} \sigma_t \gamma_t = \sigma_t' \left(\sigma_t \sigma_t'\right)^{-1} \left(\mu_t - r_t \mathbf{1}\right) = \theta_t$$

[1] It is worth pointing out that, if Assumption 3.2.1 does not hold, then the market price of risk process θ can still be defined by replacing $\sigma_t' \left(\sigma_t \, \sigma_t'\right)^{-1}$ with the *Moore-Penrose pseudoinverse* of the matrix σ_t.

thus giving $\|\gamma_t\| = \|\theta_t\| + \big\|\Pi_{\mathcal{K}(\sigma_t)}(\gamma_t)\big\| \geq \|\theta_t\|$, for all $t \in [0,T]$. This implies that, as soon as there exists *some* $\mathbb{R}^d$-valued progressively measurable process γ satisfying (3.6) and such that $\gamma \in L^2_{loc}(W)$, then the market price of risk process θ satisfies Assumption 3.3.6.

The key relation between Assumption 3.3.6 and no-arbitrage was first discovered in Delbaen & Schachermayer (1995b) and Levental & Skorohod (1995) and will play a crucial role in deriving many of our results. We now introduce a fundamental local martingale associated to the market price of risk process θ. Let us define the process $\widehat{Z} = \big(\widehat{Z}_t\big)_{0\leq t\leq T}$ as follows, for $t \in [0,T]$:

$$\widehat{Z}_t := \mathcal{E}\left(-\int \theta' dW\right)_t = \exp\left(-\sum_{j=1}^{d}\int_0^t \theta^j_u\, dW^j_u - \frac{1}{2}\sum_{j=1}^{d}\int_0^t \left(\theta^j_u\right)^2 du\right) \tag{3.8}$$

Note that Assumption 3.3.6 ensures that the stochastic integral $\int \theta' dW$ is well-defined as a continuous local martingale. It is well-known that $\widehat{Z} = \big(\widehat{Z}_t\big)_{0\leq t\leq T}$ is a strictly positive continuous local martingale with $\widehat{Z}_0 = 1$. Hence, due to Fatou's Lemma, the process $\widehat{Z}$ is also a supermartingale (see e.g. Karatzas & Shreve (1991), Problem 1.5.19) and we have $E\big[\widehat{Z}_T\big] \leq E\big[\widehat{Z}_0\big] = 1$. It is easy to show that the process $\widehat{Z}$ is a true martingale, and not only a local martingale, if and only if $E\big[\widehat{Z}_T\big] = E\big[\widehat{Z}_0\big] = 1$. However, it may happen that the process $\widehat{Z}$ is a *strict* local martingale, i.e. a local martingale which is not a true martingale. In any case, the following Proposition shows the basic property of the process $\widehat{Z}$.

Proposition 3.3.8. *Suppose that Assumption 3.3.6 holds and let $\widehat{Z} = \big(\widehat{Z}_t\big)_{0\leq t\leq T}$ be defined as in* (3.8)*. Then the following hold:*

(a) for all $i = 1,\ldots,N$, the process $\widehat{Z}\,\bar{S}^i = \big(\widehat{Z}_t\,\bar{S}^i_t\big)_{0\leq t\leq T}$ is a local martingale;

(b) for any trading strategy $\pi \in \mathcal{A}$ the process $\widehat{Z}\,\bar{V}^\pi = \big(\widehat{Z}_t\,\bar{V}^\pi_t\big)_{0\leq t\leq T}$ is a local martingale.

Proof. For any $i = 1,\ldots,N$, part *(a)* follows from part *(b)* by taking $\pi \in \mathcal{A}$ with $\pi^i \equiv 1$ and $\pi^j \equiv 0$ for $j \neq i$. Hence, it suffices to prove part *(b)*. Recalling equation (3.5), an application of the Itô product rule gives:

$$\begin{aligned} d\big(\widehat{Z}_t\,\bar{V}^\pi_t\big) &= \bar{V}^\pi_t\, d\widehat{Z}_t + \widehat{Z}_t\, d\bar{V}^\pi_t + d\langle \bar{V}^\pi, \widehat{Z}\rangle_t \\ &= -\bar{V}^\pi_t\,\widehat{Z}_t\,\theta'_t\, dW_t + \widehat{Z}_t\,\bar{V}^\pi_t\,\pi'_t\,(\mu_t - r_t\mathbf{1})\,dt + \widehat{Z}_t\,\bar{V}^\pi_t\,\pi'_t\,\sigma_t\,dW_t - \widehat{Z}_t\,\bar{V}^\pi_t\,\pi'_t\,\sigma_t\,\theta_t\,dt \\ &= \widehat{Z}_t\,\bar{V}^\pi_t\,(\pi'_t\,\sigma_t - \theta'_t)\,dW_t \end{aligned} \tag{3.9}$$

Since $\sigma'\pi \in L^2_{loc}(W)$ and $\theta \in L^2_{loc}(W)$, this shows the local martingale property of $\widehat{Z}\,\bar{V}^\pi$. □

Under the standing Assumption 3.2.1, we have seen that the diffusion-based financial market described in Section 3.2 does not allow for increasing profits (see Remark 3.3.5). However, the concept of increasing profit represents an almost pathological notion of arbitrage opportunity. Hence, we would like to known whether weaker and more economically meaningful types of arbitrage opportunities can exist. To this effect, let us give the following Definition, adapted from Kardaras (2010b).

Definition 3.3.9. *An $\mathcal{F}$-measurable non-negative random variable ξ is called an* arbitrage of the first kind *if $P(\xi > 0) > 0$ and, for all $v \in (0,\infty)$, there exists a trading strategy $\pi^v \in \mathcal{A}$ such that $\bar{V}_T^{v,\pi^v} \geq \xi$ P-a.s. We say that the financial market is* viable *if there are no arbitrages of the first kind.*

The following Proposition shows that the existence of an increasing profit implies the existence of an arbitrage of the first kind. Due to the Itô-process framework considered in this Chapter, we are able to provide a simple proof (in a more general context, see also Section 4.3.3).

Proposition 3.3.10. *Let $\pi \in \mathcal{A}$ be a trading strategy yielding an increasing profit. Then there exists an arbitrage of the first kind.*

Proof. Let $\pi \in \mathcal{A}$ yield an increasing profit and define $\xi := \bar{V}_T^{\pi} - 1$. Due to Definition 3.3.1, we have $P(\xi \geq 0) = 1$ and $P(\xi > 0) > 0$. Then, for any $v \in [1,\infty)$, we have $\bar{V}_T^{v,\pi} = v\bar{V}_T^{\pi} > v\,\xi \geq \xi$ P-a.s. For any $v \in (0,1)$, let us define $\pi_t^v := -\frac{\log(v)+\log(1-v)}{v}\,\pi_t$. Clearly, for any $v \in (0,1)$, the process $\pi^v = (\pi_t^v)_{0\leq t\leq T}$ satisfies $\pi^v \in \mathcal{A}$ and, due to Lemma 3.3.2, $(\pi_t^v)'\,\sigma_t = 0$ $P \otimes \ell$-a.e. We have then:

$$\bar{V}_T^{v,\pi^v} = v\exp\left(\int_0^T (\pi_t^v)'\,(\mu_t - r_t\mathbf{1})\,dt\right) = v\left(\bar{V}_T^{\pi}\right)^{-\frac{\log(v)+\log(1-v)}{v}} > \bar{V}_T^{\pi} - 1 = \xi \qquad P\text{-a.s.}$$

where the second equality follows from the elementary identity $\exp(\alpha x) = (\exp x)^{\alpha}$ and the last inequality follows since $vx^{-\frac{\log(v)+\log(1-v)}{v}} > x - 1$ for $x \geq 1$ and for every $v \in (0,1)$. We have thus shown that, for every $v \in (0,\infty)$, there exists a trading strategy $\pi^v \in \mathcal{A}$ such that $\bar{V}_T^{v,\pi^v} \geq \xi$ P-a.s., meaning that the random variable ξ is an arbitrage of the first kind. □

We now proceed with the question of whether arbitrages of the first kind are allowed in our financial market model. To this effect, let us first give the following Definition (compare also with Definition 4.3.11 in Chapter 4).

Definition 3.3.11. *A real-valued non-negative adapted process $D = (D_t)_{0\leq t\leq T}$ with $D_0 = 1$ and $D_T > 0$ P-a.s. is said to be a* martingale deflator *if the product $D\bar{V}^{\pi} = \left(D_t\bar{V}_t^{\pi}\right)_{0\leq t\leq T}$ is a local martingale for every $\pi \in \mathcal{A}$. We denote by $\mathcal{D}$ the set of all martingale deflators.*

Remark 3.3.12. Let $D \in \mathcal{D}$. Then, taking $\pi \equiv 0$, Definition 3.3.11 implies that D is a non-negative local martingale and hence, due to Fatou's Lemma, also a supermartingale. Since $D_T > 0$ P-a.s., the minimum principle for non-negative supermartingales (see e.g. Revuz & Yor (1999), Proposition II.3.4) implies that $P\big(D_t > 0 \text{ for all } t \in [0,T]\big) = 1$.

The following Proposition shows that the existence of a martingale deflator is a sufficient condition for the absence of arbitrages of the first kind.

Proposition 3.3.13. *If $\mathcal{D} \neq \emptyset$ then there cannot exist arbitrages of the first kind.*

Proof. Let $D \in \mathcal{D}$ and suppose that there exists a random variable ξ yielding an arbitrage of the first kind. Then, for every $n \in \mathbb{N}$, there exists a strategy $\pi^n \in \mathcal{A}$ such that $\bar{V}_T^{1/n,\pi^n} \geq \xi$ P-a.s. For every $n \in \mathbb{N}$, the process $D\bar{V}^{1/n,\pi^n} = \left(D_t\bar{V}_t^{1/n,\pi^n}\right)_{0\leq t\leq T}$ is a positive local martingale and, hence, a supermartingale. So, for every $n \in \mathbb{N}$:

$$E\left[D_T\,\xi\right] \leq E\left[D_T\bar{V}_T^{1/n,\pi^n}\right] \leq E\left[D_0\bar{V}_0^{1/n,\pi^n}\right] = \frac{1}{n}$$

Letting $n \to \infty$ gives $E\left[D_T\,\xi\right] = 0$ and hence $D_T\,\xi = 0$ P-a.s. Since, due to Definition 3.3.11, we have $D_T > 0$ P-a.s. this implies that $\xi = 0$ P-a.s., which contradicts the assumption that ξ is an arbitrage of the first kind. □

It is worth pointing out that one can also prove a converse result to Proposition 3.3.13, showing that if there are no arbitrages of the first kind then there exists at least one martingale deflator. In a general semimartingale setting, this has been recently shown in Kardaras (2010b) (compare also Hulley & Schweizer (2010) in the context of continuous path processes). Furthermore, Proposition 1 of Kardaras (2010a) shows that the absence of arbitrages of the first kind is equivalent to the condition of *No Unbounded Profit with Bounded Risk* (NUPBR), formally defined as the condition that the set $\left\{\bar{V}_T^\pi : \pi \in \mathcal{A}\right\}$ be bounded in probability. By relying on these facts, we can state the following Theorem, the second part of which follows from Proposition 4.19 of Karatzas & Kardaras (2007).

Theorem 3.3.14. *The following are equivalent:*

(a) $\mathcal{D} \neq \emptyset$*;*

(b) there are no arbitrages of the first kind;

(c) $\left\{\bar{V}_T^\pi : \pi \in \mathcal{A}\right\}$ *is bounded in probability, i.e. the condition (NUPBR) holds.*

Moreover, for every concave and strictly increasing utility function $U : [0,\infty) \to \mathbb{R}$*, the utility optimisation problem of finding an element* $\pi^* \in \mathcal{A}$ *such that*

$$E\left[U\left(\bar{V}_T^{\pi^*}\right)\right] = \sup_{\pi\in\mathcal{A}} E\left[U\left(\bar{V}_T^{\pi}\right)\right]$$

either does not have a solution or has infinitely many solutions when any of the conditions (a)-(c) fails.

We want to remark that an analogous result has already been shown in Theorem 2 of Loewenstein & Willard (2000) under the assumption of a complete financial market. In view of the second part of the above Theorem, the condition of absence of arbitrages of the first kind can be seen as the minimal no-arbitrage condition in order to be able to meaningfully solve portfolio optimisation problems. It is now straightforward to show that, as soon as Assumption 3.3.6 holds, the diffusion-based model introduced in Section 3.2 satisfies the equivalent conditions of Theorem 3.3.14. In fact, due to Proposition 3.3.8, the process $\widehat{Z}$ defined in (3.8) is a martingale deflator for the financial market $\left(S^0, S^1, \ldots, S^N\right)$ as soon as Assumption 3.3.6 is satisfied. Conversely, if Assumption 3.3.6 fails to hold, then there cannot exist any martingale deflator (see Remark 3.3.7). In a more general context, compare also with Theorem 4.3.23 in Chapter 4.

Corollary 3.3.15. *The financial market* $\left(S^0, S^1, \ldots, S^N\right)$ *is viable, i.e. it does not admit arbitrages of the first kind (see Definition 3.3.9), if and only if Assumption 3.3.6 holds.*

We want to emphasise that, due to Theorem 3.3.14, the diffusion-based model introduced in Section 3.2 allows us to meaningfully consider portfolio optimisation problems as soon as Assumption 3.3.6 holds. However, nothing guarantees that an *Equivalent Local Martingale Measure* (ELMM) exists, as shown in the following classical example, already considered in Delbaen & Schachermayer (1995a), Hulley (2010) and Karatzas & Kardaras (2007). Other instances of models for which an ELMM does not exist arise in the context of *diverse* financial markets, see Chapter II of Fernholz & Karatzas (2009).

Example 3.3.16. Let us suppose that $\mathbb{F} = \mathbb{F}^W$, where W is a standard Brownian motion ($d = 1$), and let $N = 1$. Assume that $S_t^0 \equiv 1$ for all $t \in [0, T]$ and that the real-valued process $S = (S_t)_{0 \le t \le T}$ is given as the solution to the following SDE:

$$dS_t = \frac{1}{S_t}\, dt + dW_t \qquad S_0 = s \in (0, \infty) \tag{3.10}$$

It is well-known that the process S is a Bessel process of dimension three (see e.g. Revuz & Yor (1999), Section XI.1). So, S_t is P-a.s. strictly positive and finite for all $t \in [0, T]$. Furthermore, the market price of risk process θ is given by $\theta_t = \sigma_t^{-1} \mu_t = \frac{1}{S_t}$, for $t \in [0, T]$. Since S is continuous, we clearly have $\int_0^T \theta_t^2 dt < \infty$ P-a.s., meaning that Assumption 3.3.6 is satisfied. Hence, due to Corollary 3.3.15, there are no arbitrages of the first kind.

However, for this particular financial market model there exists no ELMM. We prove this claim arguing by contradiction. Suppose that Q is an ELMM for S and denote by $Z^Q = \left(Z_t^Q\right)_{0 \le t \le T}$ its density process. Then, due to the martingale representation theorem (see Karatzas & Shreve (1991), Theorem 3.4.15 and Problem 3.4.16), we can represent Z^Q as follows:

$$Z_t^Q = \mathcal{E}\left(-\int \lambda\, dW\right)_t \qquad \text{for } t \in [0, T]$$

where $\lambda = (\lambda_t)_{0 \le t \le T}$ is a progressively measurable process such that $\int_0^T \lambda_t^2\, dt < \infty$ P-a.s. Due to Girsanov Theorem, the process $W^Q = \left(W_t^Q\right)_{0 \le t \le T}$ defined by $W_t^Q := W_t + \int_0^t \lambda_u\, du$, for $t \in [0, T]$, is a Brownian motion under Q. Hence, the process S satisfies the following SDE under Q:

$$dS_t = \left(\frac{1}{S_t} - \lambda_t\right) dt + dW_t^Q \qquad S_0 = s \tag{3.11}$$

Since Q is an ELMM for S, the SDE (3.11) must have a zero drift coefficient, i.e. it must be $\lambda_t = \frac{1}{S_t} = \theta_t$ for all $t \in [0, T]$. Then, a simple application of Itô's formula gives:

$$Z_t^Q = \mathcal{E}\left(-\int \frac{1}{S}\, dW\right)_t = \exp\left(-\int_0^t \frac{1}{S_u}\, dW_u - \frac{1}{2}\int_0^t \frac{1}{S_u^2}\, du\right) = \frac{1}{S_t}$$

However, since S is a Bessel process of dimension three, it is well-known that the process $1/S = (1/S_t)_{0 \le t \le T}$ is a strict local martingale, i.e. it is a local martingale but not a true martingale (see e.g. Revuz & Yor (1999), Exercise XI.1.16). Clearly, this contradicts the fact that Q is a well-defined probability measure, thus showing that there cannot exist an ELMM for S.

As the above Example shows, Assumption 3.3.6 does not guarantee the existence of an ELMM for the financial market $\left(S^0, S^1, \ldots, S^N\right)$. It is well-known that, in the case of continuous-path processes, the existence of an ELMM is equivalent to the no-arbitrage condition of *No Free Lunch with Vanishing Risk* (NFLVR), see Delbaen & Schachermayer (1994) and Delbaen & Schachermayer (2006). Furthermore, (NFLVR) holds if and only if both (NUPBR) and (NA) hold (see Section 3 of Delbaen & Schachermayer (1994) and Proposition 4.2 of Karatzas & Kardaras (2007)), where, recalling that $\bar{V}_0^\pi = 1$, the classical *no-arbitrage* condition (NA) precludes the existence of a trading strategy $\pi \in \mathcal{A}$ such that $P\left(\bar{V}_T^\pi \geq 1\right) = 1$ and $P\left(\bar{V}_T^\pi > 1\right) > 0$. This implies that, even if Assumption 3.3.6 holds, the well-known no-arbitrage condition (NFLVR) may fail to hold. However, due to Theorem 3.3.14, the financial market may still be viable.

Remark 3.3.17 (***On the martingale property of*** $\boldsymbol{\widehat{Z}}$**).** It is important to note that Assumption 3.3.6 does not suffice to ensure that $\widehat{Z}$ is a true martingale. Well-known sufficient conditions for this to hold include the Novikov and Kazamaki criteria, see e.g. Revuz & Yor (1999), Section VIII.1. If $\widehat{Z}$ is a true martingale we have then $E\big[\widehat{Z}_T\big] = 1$ and we can define a probability measure $\widehat{Q} \sim P$ by letting $\frac{d\widehat{Q}}{dP} := \widehat{Z}_T$. The martingale $\widehat{Z}$ represents then the *density process* of $\widehat{Q}$ with respect to P, i.e. $\widehat{Z}_t = E\Big[\frac{d\widehat{Q}}{dP}\big|\mathcal{F}_t\Big]$ P-a.s. for all $t \in [0,T]$. Recall that a process $M = (M_t)_{0\leq t\leq T}$ is a local $\widehat{Q}$-martingale if and only if the process $\widehat{Z}M = \big(\widehat{Z}_t M_t\big)_{0\leq t\leq T}$ is a local P-martingale. Due to Proposition 3.3.8-*(a)*, this implies that if $E\big[\widehat{Z}_T\big] = 1$ then the process $\bar{S} := \left(\bar{S}^1, \ldots, \bar{S}^N\right)'$ is a local $\widehat{Q}$-martingale or, in other words, the probability measure $\widehat{Q}$ is an ELMM. Girsanov's theorem implies then that the process $\widehat{W} = \big(\widehat{W}_t\big)_{0\leq t\leq T}$ defined by $\widehat{W}_t := W_t + \int_0^t \theta_u\, du$ for $t \in [0,T]$ is a Brownian motion under $\widehat{Q}$. Since the dynamics of $S := \left(S^1, \ldots, S^N\right)'$ in (3.2) can be rewritten as:

$$dS_t = \operatorname{diag}(S_t)\,\mathbf{1}\, r_t\, dt + \operatorname{diag}(S_t)\,\sigma_t\big(\theta_t\, dt + dW_t\big) \qquad S_0 = s$$

the process $\bar{S} := \left(\bar{S}^1, \ldots, \bar{S}^N\right)'$ satisfies the following SDE under the measure $\widehat{Q}$:

$$d\bar{S}_t = \operatorname{diag}\left(\bar{S}_t\right)\sigma_t\, d\widehat{W}_t \qquad \bar{S}_0 = s$$

We want to point out that the process $\widehat{Z} = \big(\widehat{Z}_t\big)_{0\leq t\leq T}$ represents the density process with respect to P of the *minimal martingale measure*, when the latter exists, see e.g. Hulley & Schweizer (2010). Again, we emphasise that in this Chapter we do not assume neither that $E\big[\widehat{Z}_T\big] = 1$ nor the existence of an ELMM.

We close this Section with a simple technical result which turns out to be useful in the following.

Lemma 3.3.18. *Suppose that Assumption 3.3.6 holds. Then an* $\mathbb{R}^N$*-valued progressively measurable process* $\pi = (\pi_t)_{0\leq t\leq T}$ *belongs to* $\mathcal{A}$ *if and only if* $\int_0^T \|\sigma_t'\,\pi_t\|^2\, dt < \infty$ *P-a.s.*

Proof. We only need to show that Assumption 3.3.6 and $\int_0^T \|\sigma_t'\,\pi_t\|^2\, dt < \infty$ P-a.s. together imply that $\int_0^T |\pi_t'(\mu_t - r_t\mathbf{1})|\, dt < \infty$ P-a.s. This follows easily from the Cauchy-Schwarz inequality, in fact:

$$\int_0^T |\pi_t'(\mu_t - r_t\mathbf{1})|\, dt = \int_0^T |\pi_t'\,\sigma_t\,\theta_t|\, dt \leq \left(\int_0^T \|\sigma_t'\,\pi_t\|^2\, dt\right)^{\frac{1}{2}} \left(\int_0^T \|\theta_t\|^2\, dt\right)^{\frac{1}{2}} < \infty \quad P\text{-a.s.}$$

□

3.4 The growth-optimal portfolio and the numéraire portfolio

As we have seen in the last Section, the diffusion-based model introduced in Section 3.2 can represent a viable financial market even if the traditional no-arbitrage condition (NFLVR) fails to hold or, equivalently, if an ELMM for $\left(S^0, S^1, \dots, S^N\right)$ fails to exist. Let us now consider an interesting portfolio optimisation problem, namely the problem of maximising the *growth rate*, formally defined as follows (compare Fernholz & Karatzas (2009), Platen (2006) and Platen & Heath (2006), Section 10.2).

Definition 3.4.1. *For a trading strategy $\pi \in \mathcal{A}$ the* growth rate *process $g^\pi = (g^\pi_t)_{0\leq t\leq T}$ is defined as the drift term in the SDE satisfied by the process $\log V^\pi = (\log V_t^\pi)_{0\leq t\leq T}$, i.e. the term g^π_t in the SDE:*

$$d\log V_t^\pi = g^\pi_t\,dt + \pi_t'\sigma_t dW_t \tag{3.12}$$

A trading strategy $\pi^* \in \mathcal{A}$ (and the corresponding portfolio process V^{π^*}) is said to be growth-optimal *if $g^{\pi^*}_t \geq g^\pi_t$ P-a.s. for all $t \in [0,T]$ for any trading strategy $\pi \in \mathcal{A}$.*

The terminology "growth rate" is motivated by the fact that:

$$\lim_{T\to\infty}\frac{1}{T}\left(\log V_T^\pi - \int_0^T g^\pi_t dt\right) = 0 \qquad P\text{-a.s.}$$

under "controlled growth" of $\sigma\sigma'$, i.e. $\lim_{T\to\infty}\left(\frac{\log\log T}{T^2}\int_0^T a^{i,i}_t dt\right) = 0$ P-a.s. where $a_t := \sigma_t\sigma_t'$ for $t \in [0,T]$ (see Fernholz & Karatzas (2009), Section 1). In the context of the diffusion-based financial market described in Section 3.2, the following Theorem gives an explicit description of the growth-optimal strategy $\pi^* \in \mathcal{A}$.

Theorem 3.4.2. *Suppose that Assumption 3.3.6 holds. Then there exists an unique growth-optimal strategy $\pi^* \in \mathcal{A}$, explicitly given by:*

$$\pi^*_t = \left(\sigma_t\,\sigma_t'\right)^{-1}\sigma_t\,\theta_t \tag{3.13}$$

where the process $\theta = (\theta_t)_{0\leq t\leq T}$ is the market price of risk introduced in Definition 3.3.4. The corresponding Growth-Optimal Portfolio (GOP) $V^{\pi^*} = \left(V_t^{\pi^*}\right)_{0\leq t\leq T}$ satisfies the following dynamics:

$$\frac{dV_t^{\pi^*}}{V_t^{\pi^*}} = r_t\,dt + \theta_t'\left(\theta_t\,dt + dW_t\right) \tag{3.14}$$

Proof. Let $\pi \in \mathcal{A}$ be a trading strategy. A simple application of Itô's formula gives that:

$$d\log V_t^\pi = g^\pi_t\,dt + \pi_t'\,\sigma_t\,dW_t \tag{3.15}$$

where $g^\pi_t := r_t + \pi_t'\left(\mu_t - r_t\mathbf{1}\right) - \frac{1}{2}\,\pi_t'\,\sigma_t\,\sigma_t'\,\pi_t$, for $t \in [0,T]$. Since the matrix $\sigma_t\sigma_t'$ is P-a.s. positive definite for all $t \in [0,T]$, due to Assumption 3.2.1, a trading strategy $\pi^* \in \mathcal{A}$ maximises the growth rate if and only if, for every $t \in [0,T]$, π^*_t solves the first order condition obtained by differentiating g^π_t with respect to π_t. This means that π^*_t must satisfy the following equation, for every $t \in [0,T]$:

$$\mu_t - r_t\mathbf{1} - \sigma_t\sigma_t'\pi^*_t = 0$$

Recall that, due to Assumption 3.2.1, the matrix $\sigma_t\sigma_t'$ is P-a.s. positive definite for all $t \in [0,T]$. So, using Definition 3.3.4, we get the following unique optimiser π_t^*:

$$\pi_t^* = (\sigma_t\,\sigma_t')^{-1}\,(\mu_t - r_t\mathbf{1}) = (\sigma_t\,\sigma_t')^{-1}\,\sigma_t\,\theta_t \qquad \text{for } t \in [0,T]$$

We now need to verify that $\pi^* = (\pi_t^*)_{0\le t\le T} \in \mathcal{A}$. Due to Lemma 3.3.18, it suffices to check that $\int_0^T \|\sigma_t'\pi_t^*\|^2\,dt < \infty$ P-a.s. To show this, it is enough to notice that:

$$\int_0^T \|\sigma_t'\,\pi_t^*\|^2\,dt = \int_0^T (\mu_t - r_t\mathbf{1})'\,(\sigma_t\,\sigma_t')^{-1}\,(\mu_t - r_t\mathbf{1})\,dt = \int_0^T \|\theta_t\|^2\,dt < \infty \qquad P\text{-a.s.}$$

due to Assumption 3.3.6. We have thus shown that π^* maximises the growth rate and is an admissible trading strategy. Finally, note that equation (3.15) leads to:

$$\begin{aligned} d\log V_t^{\pi^*} &= g_t^{\pi^*}\,dt + (\pi_t^*)'\,\sigma_t\,dW_t \\ &= r_t\,dt + \theta_t'\,\sigma_t'\,(\sigma_t\,\sigma_t')^{-1}\,(\mu_t - r_t\mathbf{1})\,dt - \frac{1}{2}\,\theta_t'\,\sigma_t'\,(\sigma_t\,\sigma_t')^{-1}\,\sigma_t\,\sigma_t'\,(\sigma_t\,\sigma_t')^{-1}\,\sigma_t\,\theta_t\,dt \\ &\quad + \theta_t'\,\sigma_t'\,(\sigma_t\,\sigma_t')^{-1}\,\sigma_t\,dW_t \\ &= \Big(r_t + \frac{1}{2}\,\|\theta_t\|^2\Big)dt + \theta_t' dW_t \end{aligned}$$

where the last equality is obtained by replacing θ_t with its expression as given in Definition 3.3.4 and from which (3.14) follows by a simple application of Itô's formula. □

Remark 3.4.3.

1. Results analogous to Theorem 3.4.2 can be found in Section 2 of Galesso & Runggaldier (2010), Example 3.7.9 of Karatzas & Shreve (1998), Section 2.7 of Platen (2002), Section 3.2 of Platen (2006), Section 10.2 of Platen & Heath (2006) and Proposition 2 of Platen & Runggaldier (2007). However, in all these works the growth-optimal strategy has been derived for the specific case of a complete financial market, i.e. under the additional assumptions that $d = N$ and $\mathbb{F} = \mathbb{F}^W$. Here, we have instead chosen to deal with the more general situation described in Section 3.2, i.e. with a general incomplete market. Furthermore, we rigorously check the admissibility of the candidate growth-optimal strategy.

2. Due to Corollary 3.3.15, Assumption 3.3.6 ensures that there are no arbitrages of the first kind. However, it is worth emphasising that Theorem 3.4.2 does not rely on the existence of an ELMM for the financial market $\big(S^0, S^1, \dots, S^N\big)$.

3. Due to equation (3.14), the discounted GOP process $\bar{V}^{\pi^*} = \big(\bar{V}_t^{\pi^*}\big)_{0\le t\le T}$ satisfies the following SDE:
$$\frac{d\bar{V}_t^{\pi^*}}{\bar{V}_t^{\pi^*}} = \|\theta_t\|^2\,dt + \theta_t'\,dW_t \tag{3.16}$$
We can immediately observe that the drift coefficient is the "square" of the diffusion coefficient, thus showing that there is a strong link between rate of return and volatility in the

GOP dynamics. Moreover, the market price of risk plays a key role in the GOP dynamics. To this effect, compare the discussion in Chapter 13 of Platen & Heath (2006). Observe also that Assumption 3.3.6 is equivalent to requiring that the solution $\bar{V}^{\pi^*}$ to the SDE (3.16) is well-defined and P-a.s. finite valued, meaning that the discounted GOP does not explode in the finite time interval $[0,T]$. It can also be shown, and this holds true in general semimartingale models, that the existence of a non-explosive GOP is in fact *equivalent* to the absence of arbitrages of the first kind, as can be deduced by combining Theorem 3.3.14 and Karatzas & Kardaras (2007), Theorem 4.12 (see also Christensen & Larsen (2007)).

Example 3.4.4 (***The classical Black-Scholes model***)**.** In order to develop an intuitive feeling for some of the concepts introduced in this Section, let us briefly consider the case of the classical Black-Scholes model, i.e. a financial market represented by (S^0, S), with $r_t \equiv r$ for some $r \in \mathbb{R}$ for all $t \in [0,T]$ and $S = (S_t)_{0 \le t \le T}$ a real-valued process satisfying the following SDE:

$$dS_t = S_t \, \mu \, dt + S_t \, \sigma \, dW_t \qquad S_0 = s \in (0, \infty)$$

with $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R} \setminus \{0\}$. The market price of risk process $\theta = (\theta_t)_{0 \le t \le T}$ is then given by $\theta_t \equiv \theta := \frac{\mu - r}{\sigma}$ for all $t \in [0,T]$. By Theorem 3.4.2, the GOP strategy $\pi^* = (\pi^*_t)_{0 \le t \le T}$ is then given by $\pi^*_t \equiv \pi^* := \frac{\mu - r}{\sigma^2}$, for all $t \in [0,T]$. In this special case, Novikov's condition implies that $\widehat{Z}$ is a true martingale, yielding the density process of the martingale measure $\widehat{Q}$.

The remaining part of this Section is devoted to the derivation of some basic but fundamental properties of the GOP. Let us start with the following simple Proposition.

Proposition 3.4.5. *Suppose that Assumption 3.3.6 holds. Then the discounted GOP process* $\bar{V}^{\pi^*} = \big(\bar{V}_t^{\pi^*}\big)_{0 \le t \le T}$ *is related to the process* $\widehat{Z} = \big(\widehat{Z}_t\big)_{0 \le t \le T}$ *as follows, for all* $t \in [0,T]$*:*

$$\bar{V}_t^{\pi^*} = \frac{1}{\widehat{Z}_t}$$

Proof. Assumption 3.3.6 ensures that the process $\widehat{Z} = \big(\widehat{Z}_t\big)_{0 \le t \le T}$ is well-defined and P-a.s. strictly positive. Furthermore, due to Theorem 3.4.2, the growth-optimal strategy $\pi^* \in \mathcal{A}$ exists and is explicitly given by (3.13). Now it suffices to observe that, due to equations (3.16) and (3.8):

$$\bar{V}_t^{\pi^*} = \exp\left(\int_0^t \theta'_u dW_u + \frac{1}{2} \int_0^t \|\theta_u\|^2 \, du \right) = \frac{1}{\widehat{Z}_t}$$

□

We then immediately obtain the following Corollary.

Corollary 3.4.6. *Suppose that Assumption 3.3.6 holds. Then, for any trading strategy* $\pi \in \mathcal{A}$*, the process* $\hat{V}^{\pi} = \big(\hat{V}_t^{\pi}\big)_{0 \le t \le T}$ *defined by* $\hat{V}_t^{\pi} := \frac{V_t^{\pi}}{V_t^{\pi^*}}$*, for* $t \in [0,T]$*, is a non-negative local martingale and, hence, a supermartingale.*

Proof. Passing to discounted quantities, we have $\hat{V}_t^{\pi} = \frac{V_t^{\pi}}{V_t^{\pi^*}} = \frac{\bar{V}_t^{\pi}}{\bar{V}_t^{\pi^*}}$. The claim then follows by combining Proposition 3.4.5 with part *(b)* of Proposition 3.3.8. □

In order to give a better interpretation to the preceding Corollary, let us give the following Definition, adapted from Becherer (2001), Karatzas & Kardaras (2007) and Platen (2009).

Definition 3.4.7. *An admissible portfolio process* $V^{\tilde{\pi}} = \big(V_t^{\tilde{\pi}}\big)_{0\leq t\leq T}$ *has the* numéraire property *if all admissible portfolio processes* $V^{\pi} = (V_t^{\pi})_{0\leq t\leq T}$ *, when denominated in units of* $V^{\tilde{\pi}}$*, are supermartingales, i.e. if the process* $\frac{V^{\pi}}{V^{\tilde{\pi}}} = \left(\frac{V_t^{\pi}}{V_t^{\tilde{\pi}}}\right)_{0\leq t\leq T}$ *is a supermartingale for all* $\pi \in \mathcal{A}$*.*

The following Proposition shows that if a numéraire portfolio exists then it is also unique.

Proposition 3.4.8. *The numéraire portfolio process* $V^{\tilde{\pi}} = \big(V_t^{\tilde{\pi}}\big)_{0\leq t\leq T}$ *is unique. Furthermore, if Assumption 3.3.6 holds, there exists an unique trading strategy* $\tilde{\pi} \in \mathcal{A}$ *such that* $V^{\tilde{\pi}}$ *is the numéraire portfolio, up to a null subset of* $\Omega \times [0, T]$*.*

Proof. Let us first prove that if $M = (M_t)_{0\leq t\leq T}$ is a P-a.s. strictly positive supermartingale such that $\frac{1}{M}$ is also a supermartingale then $M_t = M_0$ P-a.s. for all $t \in [0, T]$. In fact, for any $0 \leq s \leq t \leq T$:

$$1 = \frac{M_s}{M_s} \geq \frac{1}{M_s} E\left[M_t|\mathcal{F}_s\right] \geq E\left[\frac{1}{M_t}\middle|\mathcal{F}_s\right] E\left[M_t|\mathcal{F}_s\right] \geq \frac{1}{E\left[M_t|\mathcal{F}_s\right]} E\left[M_t|\mathcal{F}_s\right] = 1 \qquad P\text{-a.s.}$$

where the first inequality follows from the supermartingale property of M, the second from the supermartingale property of $\frac{1}{M}$ and the third from Jensen's inequality. Hence, both M and $\frac{1}{M}$ are martingales. Furthermore, since we have $E\left[\frac{1}{M_t}\middle|\mathcal{F}_s\right] = \frac{1}{E[M_t|\mathcal{F}_s]}$ and the function $x \mapsto x^{-1}$ is strictly convex on $(0, \infty)$, again Jensen's inequality implies that M_t is $\mathcal{F}_s$-measurable, for all $0 \leq s \leq t \leq T$. For $s = 0$, this implies that $M_t = E\left[M_t|\mathcal{F}_0\right] = M_0$ P-a.s. for all $t \in [0, T]$.

Suppose now there exist two elements $\tilde{\pi}^1, \tilde{\pi}^2 \in \mathcal{A}$ such that both $V^{\tilde{\pi}^1}$ and $V^{\tilde{\pi}^2}$ have the numéraire property. By Definition 3.4.7, both $\frac{V^{\tilde{\pi}^1}}{V^{\tilde{\pi}^2}}$ and $\frac{V^{\tilde{\pi}^2}}{V^{\tilde{\pi}^1}}$ are P-a.s. strictly positive supermartingales. Hence, it must be $V_t^{\tilde{\pi}^1} = V_t^{\tilde{\pi}^2}$ P-a.s. for all $t \in [0, T]$, due to the general result just proved. In order to show that the two trading strategies $\tilde{\pi}^1$ and $\tilde{\pi}^2$ coincide, let us write as follows:

$$\begin{aligned}
&E\left[\int_0^T \big(\widehat{Z}_t\,\bar{V}_t^{\tilde{\pi}^1}\tilde{\pi}_t^1 - \widehat{Z}_t\,\bar{V}_t^{\tilde{\pi}^2}\tilde{\pi}_t^2\big)'\sigma_t\,\sigma_t'\big(\widehat{Z}_t\,\bar{V}_t^{\tilde{\pi}^1}\tilde{\pi}_t^1 - \widehat{Z}_t\,\bar{V}_t^{\tilde{\pi}^2}\tilde{\pi}_t^2\big)dt\right] \\
&= E\left[\left\langle \int \widehat{Z}\big(\bar{V}^{\tilde{\pi}^1}\tilde{\pi}^1 - \bar{V}^{\tilde{\pi}^2}\tilde{\pi}^2\big)'\sigma\, dW \right\rangle_T\right] \\
&= E\left[\left\langle \int \widehat{Z}\,\bar{V}^{\tilde{\pi}^1}\big((\tilde{\pi}^1)'\sigma - \theta'\big)dW - \int \widehat{Z}\,\bar{V}^{\tilde{\pi}^2}\big((\tilde{\pi}^2)'\sigma - \theta'\big)dW \right\rangle_T\right] \\
&= E\left[\left\langle \widehat{Z}\big(\bar{V}^{\tilde{\pi}^1} - \bar{V}^{\tilde{\pi}^2}\big) \right\rangle_T\right] \leq C\, E\left[\sup_{t\in[0,T]}\left|\,\widehat{Z}_t\big(\bar{V}_t^{\tilde{\pi}^1} - \bar{V}_t^{\tilde{\pi}^2}\big)\,\right|^2\right] = 0
\end{aligned}$$

for some $C > 0$, where we have used the fact that $\bar{V}_t^{\tilde{\pi}^1} = \bar{V}_t^{\tilde{\pi}^2}$ P-a.s. for all $t \in [0, T]$, equation (3.9) and the Burkholder-Davis-Gundy inequality (see Karatzas & Shreve (1991), Theorem 3.3.28) applied to the local martingale $\widehat{Z}\big(\bar{V}^{\tilde{\pi}^1} - \bar{V}^{\tilde{\pi}^2}\big)$. Since, due to the standing Assumption 3.2.1, the matrix $\sigma_t\sigma_t'$ is P-a.s. positive definite for all $t \in [0, T]$, this implies that it must be $\tilde{\pi}_t := \tilde{\pi}_t^1 = \tilde{\pi}_t^2$ $P \otimes \ell$-a.e., thus showing the uniqueness of the strategy $\tilde{\pi} \in \mathcal{A}$. □

Remark 3.4.9. Note that the first part of Proposition 3.4.8 does not rely on any modelling assumption and, hence, is valid in full generality for any semimartingale model (compare also Becherer (2001), Section 4).

From Corollary 3.4.6 and Definition 3.4.7 we have that the GOP possesses the numéraire property. Proposition 3.4.8 then immediately yields the following Corollary.

Corollary 3.4.10. *Suppose that Assumption 3.3.6 holds. Then, the growth-optimal portfolio* V^{π^*} *coincides with the numéraire portfolio* $V^{\tilde{\pi}}$. *Furthermore, the corresponding trading strategies* $\pi^*, \tilde{\pi} \in \mathcal{A}$ *coincide, up to a null subset of* $\Omega \times [0,T]$.

We emphasize again that all these results hold true even in the absence of an ELMM. For further comments on the relations between the GOP and the numéraire portfolio in a general semimartingale setting, we refer to Section 3 of Karatzas & Kardaras (2007).

Remark 3.4.11 (***On the GOP-denominated market***)**.** Due to Corollary 3.4.10, the GOP coincides with the numéraire portfolio. Moreover, Corollary 3.4.6 shows that all portfolio processes V^{π}, for $\pi \in \mathcal{A}$, are local martingales when denominated in units of the GOP V^{π^*}. This means that, if we express all price processes in terms of the GOP, then the original probability measure P becomes an ELMM for the GOP-denominated market. Hence, due to the fundamental theorem of asset pricing (see Delbaen & Schachermayer (1994)), the classical no-arbitrage condition (NFLVR) holds for the GOP-denominated market. This observation suggests that the GOP-denominated market may be regarded as the minimal and natural setting for dealing with valuation and portfolio optimisation problems, even when there does not exist an ELMM for the original market $(S^0, S^1, \ldots, S^N)$. To this effect, compare also with Christensen & Larsen (2007).

Following Platen (2002),(2006),(2009) and Platen & Heath (2006), let us give the following Definition.

Definition 3.4.12. *For any portfolio process* V^{π}, *the process* $\hat{V}^{\pi} = \left(\hat{V}_t^{\pi}\right)_{0 \le t \le T}$, *defined as* $\hat{V}_t^{\pi} := \frac{V_t^{\pi}}{V_t^{\pi^*}}$ *for* $t \in [0,T]$, *is called* benchmarked *portfolio process. A portfolio process* V^{π} *and the associated trading strategy* $\pi \in \mathcal{A}$ *are said to be* fair *if the benchmarked portfolio process* $\hat{V}^{\pi}$ *is a martingale. We denote by* $\mathcal{A}^F$ *the set of all fair trading strategies in* $\mathcal{A}$.

According to Definition 3.4.12, the result of Corollary 3.4.6 amounts to saying that all benchmarked portfolio processes are non-negative supermartingales. Note that every benchmarked portfolio process is a local martingale but not necessarily a true martingale. This amounts to saying that there may exist *unfair* portfolios, namely portfolios for which the benchmarked value process is a strict local martingale. The concept of benchmarking will become relevant in Section 3.6.1, where we shall discuss its role for valuation purposes.

Remark 3.4.13 (***Other optimality properties of the GOP***)**.** Besides maximising the growth-rate, the GOP enjoys several other optimality properties, many of which are illustrated in the monograph Platen & Heath (2006). In particular, it has been shown that the GOP maximises the long-term growth rate among all admissible portfolios, see e.g. Platen (2009). It is also well-known that the GOP is the solution to the problem of maximising an expected logarithmic utility function, see Section 3.6.3 and also Karatzas & Kardaras (2007). Other interesting properties of the GOP include the impossibility of *relative arbitrages* (or *systematic outperformance*) with respect to it, see Fernholz & Karatzas (2009) and Platen (2009), and, under suitable assumptions on the behavior of market participants, *two-fund separation* results and connections with mean-variance efficiency, see e.g. Platen (2002),(2006). Other properties of the growth-optimal strategy are also described in the recent paper MacLean et al. (2010).

3.5 Replicating strategies and completeness of the financial market

Without relying on the existence of an ELMM for the financial market $\left(S^0, S^1, \dots, S^N\right)$, in this Section we start laying the foundations for the valuation of arbitrary contingent claims. More specifically, in this Section we shall be concerned with the study of replicating (or hedging) strategies, formally defined as follows.

Definition 3.5.1. *Let H be a positive $\mathcal{F}$-measurable contingent claim (i.e. random variable) such that $E\left[\frac{\widehat{Z}_T}{S_T^0}H\right] < \infty$. If there exists a couple $\left(v^H, \pi^H\right) \in (0,\infty) \times \mathcal{A}$ such that $V_T^{v^H,\pi^H} = H$ P-a.s., then we say that π^H is a replicating strategy for H.*

The following Proposition deals with the issue of the uniqueness of a replicating strategy.

Proposition 3.5.2. *Suppose that Assumption 3.3.6 holds. Let H be a positive $\mathcal{F}$-measurable contingent claim such that $E\left[\frac{\widehat{Z}_T}{S_T^0}H\right] < \infty$ and suppose there exists a trading strategy $\pi^H \in \mathcal{A}$ such that $V_T^{v^H,\pi^H} = H$ P-a.s. for $v^H = E\left[\frac{\widehat{Z}_T}{S_T^0}H\right]$. Then the following hold:*

(a) the strategy π^H is fair, in the sense of Definition 3.4.12;

(b) the strategy π^H is unique, up to a null subset of $\Omega \times [0,T]$.

Moreover, for every $(v,\pi) \in (0,\infty) \times \mathcal{A}$ such that $V_T^{v,\pi} = H$ P-a.s., we have $V_t^{v,\pi} \geq V_t^{v^H,\pi^H}$ P-a.s. for all $t \in [0,T]$. In particular, there cannot exist an element $\bar{\pi} \in \mathcal{A}$ such that $V_T^{\bar{v},\bar{\pi}} = H$ P-a.s. for some $\bar{v} < v^H$.

Proof. Corollary 3.4.6 implies that the benchmarked portfolio process $\hat{V}^{v^H,\pi^H} = \left(V_t^{v^H,\pi^H}/V_t^{\pi^*}\right)_{0\leq t\leq T}$ is a supermartingale. Moreover, it is also a martingale, due to the fact that:

$$\hat{V}_0^{v^H,\pi^H} = v^H = E\left[\frac{\widehat{Z}_T}{S_T^0}H\right] = E\left[\frac{V_T^{v^H,\pi^H}}{V_T^{\pi^*}}\right] = E\left[\hat{V}_T^{v^H,\pi^H}\right] \tag{3.17}$$

Part *(a)* then follows from Definition 3.4.12. To prove part *(b)*, let $\hat{\pi} \in \mathcal{A}$ be a trading strategy such that $V_T^{v^H,\hat{\pi}} = H$ P-a.s. for $v^H = E\left[\frac{\widehat{Z}_T}{S_T^0}H\right]$. Reasoning as in (3.17), the benchmarked portfolio process $\hat{V}^{v^H,\hat{\pi}} = \left(V_t^{v^H,\hat{\pi}}/V_t^{\pi^*}\right)_{0\leq t\leq T}$ is a martingale. Together with the fact that $\hat{V}_T^{v^H,\hat{\pi}} =$

$\frac{\widehat{Z}_T}{S_T^0} H = \hat{V}_T^{v^H,\pi^H}$ P-a.s., this implies that $V_t^{v^H,\pi^H} = V_t^{v^H,\hat{\pi}}$ P-a.s. for all $t \in [0,T]$. Part *(b)* then follows by the same arguments as in the second part of the proof of Proposition 3.4.8. To prove the last assertion let $(v,\pi) \in (0,\infty) \times \mathcal{A}$ be such that $V_T^{v,\pi} = H$ P-a.s. Due to Corollary 3.4.6, the benchmarked portfolio process $\hat{V}^{v,\pi} = \left(V_t^{v,\pi}/V_t^{\pi^*}\right)_{0\leq t\leq T}$ is a supermartingale. So, for any $t \in [0,T]$, due to part *(a)*:

$$\hat{V}_t^{v^H,\pi^H} = E\left[\hat{V}_T^{v^H,\pi^H}\big|\mathcal{F}_t\right] = E\left[\frac{\widehat{Z}_T}{S_T^0} H\Big|\mathcal{F}_t\right] = E\left[\hat{V}_T^{v,\pi}\big|\mathcal{F}_t\right] \leq \hat{V}_t^{v,\pi} \qquad P\text{-a.s.}$$

and, hence, $V_t^{v^H,\pi^H} \leq V_t^{v,\pi}$ P-a.s. for all $t \in [0,T]$. For $t = 0$, this implies that $v \geq v^H$, thus completing the proof. □

Remark 3.5.3. Notice that Proposition 3.5.2 does not exclude the existence of a trading strategy $\breve{\pi} \in \mathcal{A}$ such that $V_T^{\breve{v},\breve{\pi}} = H$ P-a.s. for some $\breve{v} > v^H$. However, one can argue that it may not be optimal to invest in such an unfair strategy in order to replicate H, since it requires a larger initial investment. In fact, Proposition 3.5.2 shows that $v^H = E\left[\frac{\widehat{Z}_T}{S_T^0} H\right]$ is the minimal initial capital starting from which one can replicate the contingent claim H (compare also with Karatzas & Shreve (1998), Remark 1.6.4).

A particularly nice and interesting situation arises when the financial market is *complete*, meaning that every *contingent claim* (i.e. every positive random variable) can be perfectly replicated from some initial investment by trading in the market according to some admissible self-financing trading strategy.

Definition 3.5.4. *The financial market $\left(S^0, S^1, \ldots, S^N\right)$ is said to be* complete *if for any positive $\mathcal{F}$-measurable contingent claim H such that $E\left[\frac{\widehat{Z}_T}{S_T^0} H\right] < \infty$ there exists a couple $\left(v^H, \pi^H\right) \in (0,\infty) \times \mathcal{A}$ such that $V_T^{v^H,\pi^H} = H$ P-a.s.*

In general, the financial market described in Section 3.2 is incomplete and, hence, not all contingent claims can be perfectly replicated. The following Theorem gives a sufficient condition for the financial market to be complete. The proof is similar to that of Theorem 1.6.6 in Karatzas & Shreve (1998), except that we avoid the use of any ELMM, since the latter may fail to exist in our general context. This allows us to show that the concept of market completeness does not depend on the existence of an ELMM. In a more general context, see also Theorem 4.5.13 in Chapter 4.

Theorem 3.5.5. *Suppose that Assumption 3.3.6 holds. Assume furthermore that $\mathbb{F} = \mathbb{F}^W$, where $\mathbb{F}^W$ is the P-augmented Brownian filtration associated to W, and that $d = N$. Then the financial market $\left(S^0, S^1, \ldots, S^N\right)$ is complete. More precisely, any positive $\mathcal{F}$-measurable contingent claim H with $E\left[\frac{\widehat{Z}_T}{S_T^0} H\right] < \infty$ can be replicated by a fair portfolio process V^{v^H,π^H}, with $v^H = E\left[\frac{\widehat{Z}_T}{S_T^0} H\right]$ and $\pi^H \in \mathcal{A}^F$.*

Proof. Let H be a positive $\mathcal{F} = \mathcal{F}_T^W$-measurable random variable such that $E\left[\frac{\widehat{Z}_T}{S_T^0} H\right] < \infty$ and define the martingale $M = (M_t)_{0\leq t\leq T}$ by $M_t = E\left[\frac{\widehat{Z}_T}{S_T^0} H\big|\mathcal{F}_t\right]$, for $t \in [0,T]$. According to the martingale representation theorem (see Karatzas & Shreve (1991), Theorem 3.4.15 and Problem 3.4.16) there exists an $\mathbb{R}^N$-valued progressively measurable process $\varphi = (\varphi_t)_{0\leq t\leq T}$ such that $\int_0^T \|\varphi_t\|^2\,dt < \infty$ P-a.s. and:

$$M_t = M_0 + \int_0^t \varphi'_u\,dW_u \qquad \text{for all } t \in [0,T] \tag{3.18}$$

Define then the positive process $V = (V_t)_{0\leq t\leq T}$ by $V_t := \frac{S^0_t}{\widehat{Z}_t} M_t$, for $t \in [0,T]$. Recalling that $S^0_0 = 1$, we have $v^H := V_0 = M_0 = E\left[\frac{\widehat{Z}_T}{S^0_T}\,H\right]$. The standing Assumption 3.2.1, together with the fact that $d = N$, then implies that the matrix σ_t is P-a.s. invertible for all $t \in [0,T]$. Then, an application of the product rule together with equations (3.8) and (3.18), gives:

$$\begin{aligned}
d\left(\frac{V_t}{S^0_t}\right) = d\left(\frac{M_t}{\widehat{Z}_t}\right) &= M_t\,d\frac{1}{\widehat{Z}_t} + \frac{1}{\widehat{Z}_t}\,dM_t + d\Big\langle M, \frac{1}{\widehat{Z}}\Big\rangle_t \\
&= \frac{M_t}{\widehat{Z}_t}\,\theta'_t\,dW_t + \frac{M_t}{\widehat{Z}_t}\,\|\theta_t\|^2\,dt + \frac{1}{\widehat{Z}_t}\,\varphi'_t\,dW_t + \frac{1}{\widehat{Z}_t}\,\varphi'_t\,\theta_t\,dt \\
&= \frac{V_t}{S^0_t}\left(\theta_t + \frac{\varphi_t}{M_t}\right)'\theta_t\,dt + \frac{V_t}{S^0_t}\left(\theta_t + \frac{\varphi_t}{M_t}\right)'dW_t \\
&= \frac{V_t}{S^0_t}\left(\theta_t + \frac{\varphi_t}{M_t}\right)'\sigma_t^{-1}\left(\mu_t - r_t\mathbf{1}\right)dt + \frac{V_t}{S^0_t}\left(\theta_t + \frac{\varphi_t}{M_t}\right)'\sigma_t^{-1}\,\sigma_t\,dW_t \\
&= \frac{V_t}{S^0_t}\sum_{i=1}^N \pi^{H,i}_t\,\frac{d\bar{S}^i_t}{\bar{S}^i_t}
\end{aligned} \tag{3.19}$$

where $\pi^H_t = \left(\pi^{H,1}_t, \ldots, \pi^{H,N}_t\right)' := (\sigma'_t)^{-1}\left(\theta_t + \frac{\varphi_t}{M_t}\right)$, for all $t \in [0,T]$. The last line of (3.19) shows that the process $\bar{V} := V/S^0 = (V_t/S^0_t)_{0\leq t\leq T}$ can be represented as a stochastic exponential as in part *(b)* of Definition 3.2.3. Hence, it remains to check that the process π^H satisfies the integrability conditions of part *(a)* of Definition 3.2.3. Due to Lemma 3.3.18, it suffices to verify that $\int_0^T \left\|\sigma'_t\,\pi^H_t\right\|^2 dt < \infty$ P-a.s. This can be shown as follows:

$$\int_0^T \left\|\sigma'_t\,\pi^H_t\right\|^2 dt = \int_0^T \left\|\theta_t + \frac{\varphi_t}{M_t}\right\|^2 dt \leq 2\int_0^T \|\theta_t\|^2\,dt + 2\left\|\frac{1}{M}\right\|_\infty \int_0^T \|\varphi_t\|^2\,dt < \infty \qquad P\text{-a.s.}$$

due to Assumption 3.3.6 and because $\left\|\frac{1}{M}\right\|_\infty := \max_{t\in[0,T]}\left|\frac{1}{M_t}\right| < \infty$ P-a.s. due to the continuity of M. We have thus shown that π^H is an admissible trading strategy, i.e. $\pi^H \in \mathcal{A}$, and the associated portfolio process $V^{v^H,\pi^H} = \left(V^{v^H,\pi^H}_t\right)_{0\leq t\leq T}$ satisfies $V^{v^H,\pi^H}_T = V_T = H$ P-a.s. with $v^H = E\left[\frac{\widehat{Z}_T}{S^0_T}\,H\right]$. Furthermore, since $\hat{V}^{v^H,\pi^H}_t = V^{v^H,\pi^H}_t / V^{\pi^*}_t = V_t\,\widehat{Z}_t/S^0_t = M_t$, we also have $\pi^H \in \mathcal{A}^F$. □

We conclude this Section with some important Remarks on the result of Theorem 3.5.5.

Remark 3.5.6.

1. We want to emphasise that Theorem 3.5.5 does not require the existence of an ELMM for the financial market $\left(S^0, S^1, \ldots, S^N\right)$. This amounts to saying that the completeness of a financial market does not necessarily imply that some mild forms of arbitrage opportunities are *a priori* excluded. Typical "textbook versions" of the so-called *second Fundamental Theorem of Asset Pricing* state that the completeness of the financial market is equivalent to the uniqueness of the *Equivalent (Local) Martingale Measure*, loosely speaking. However, Theorem 3.5.5 shows that we can have a complete financial market even when no E(L)MM exists at all. The fact that absence of arbitrage opportunities and market completeness should be regarded as distinct concepts has been already pointed out in Jarrow & Madan (1999). The completeness of the financial market model will play a crucial role in Section 3.6, where we shall be concerned with valuation and hedging problems in the absence of an ELMM.

2. Following the reasoning in the proof of Theorem 1.6.6 of Karatzas & Shreve (1998), but avoiding the use of an ELMM (which in our context may fail to exist), it is possible to prove a converse result to Theorem 3.5.5. More precisely, if we assume that $\mathbb{F} = \mathbb{F}^W$ and that every $\mathcal{F}$-measurable positive random variable H with $v^H := E\left[\frac{\widehat{Z}_T}{S_T^0} H\right] < \infty$ admits a trading strategy $\pi^H \in \mathcal{A}$ such that $V_T^{v^H, \pi^H} = H$ P-a.s., then we necessarily have $d = N$. In a more general context, compare also with Section 4.5.2 in Chapter 4.

3.6 Contingent claim valuation without ELMMs

The main goal of this Section is to show how one can proceed to the valuation of contingent claims in financial market models which may not necessarily admit an ELMM. Since the non-existence of a properly defined martingale measure precludes the whole machinery of risk-neutral pricing, this appears as a non-trivial issue. Here we concentrate on the situation of a complete financial market, as considered at the end of the last Section (see Section 3.7 for possible extensions to incomplete markets). A major focus of this Section is on providing a mathematical justification for the so-called *real-world pricing approach*, according to which the valuation of contingent claims is performed under the original (or *real-world*) probability measure P using the GOP as the natural numéraire.

Remark 3.6.1. In this Section we shall be concerned with the problem of *pricing* contingent claims. However, one should be rather careful with the terminology and distinguish between a *value* assigned to a contingent claim and its prevailing *market price*. Indeed, the former represents the outcome of an a priori chosen valuation rule, while the latter is the price determined by supply and demand forces in the financial market. Since the choice of the valuation criterion is a subjective one, the two concepts of *value* and *market price* do not necessarily coincide. This is especially true when arbitrage opportunities and/or bubble phenomena are not excluded from the financial market. In this Section, we use the word "price" in order to be consistent with the standard terminology in the literature.

3.6.1 Real-world pricing and the *benchmark approach*

We start by introducing the concept of *real-world price*, which is at the core of the so-called *benchmark approach* to the valuation of contingent claims.

Definition 3.6.2. *Let H be a positive $\mathcal{F}$-measurable contingent claim such that $E\left[\frac{\widehat{Z}_T}{S_T^0}\,H\right] < \infty$. If there exists a fair portfolio process $V^{v^H,\pi^H} = \left(V_t^{v^H,\pi^H}\right)_{0\leq t\leq T}$ such that $V_T^{v^H,\pi^H} = H$ P-a.s., for some $v^H, \pi^H \in (0,\infty)\times\mathcal{A}^F$, then the* real-world price *of H at time t, denoted as Π_t^H, is defined as follows:*

$$\Pi_t^H := V_t^{\pi^*}\,E\left[\frac{H}{V_T^{\pi^*}}\Big|\mathcal{F}_t\right] \tag{3.20}$$

for every $t\in[0,T]$ and where $V^{\pi^*} = \left(V_t^{\pi^*}\right)_{0\leq t\leq T}$ denotes the GOP.

The terminology *real-world price* is used to indicate that, unlike in the traditional setting, all contingent claims are valued under the original real-world probability measure P and not under an equivalent risk-neutral measure. This allows us to extend the valuation of contingent claims to financial markets for which no ELMM may exist. The concept of *real-world price* gives rise to the so-called *benchmark approach* to the valuation of contingent claims in view of the fact that the GOP plays the role of the natural numéraire portfolio (compare Remark 3.4.11). For this reason we shall refer to it as the *benchmark* portfolio. We refer the reader to Platen (2006),(2009) and Platen & Heath (2006) for a thorough presentation of the benchmark approach.

Clearly, if there exists a fair portfolio process V^{v^H,π^H} such that $V_T^{v^H,\pi^H} = H$ P-a.s. for $\left(v^H,\pi^H\right)\in(0,\infty)\times\mathcal{A}^F$, then the real-world price coincides with the value of the fair portfolio. In fact, for all $t\in[0,T]$:

$$\Pi_t^H = V_t^{\pi^*}E\left[\frac{H}{V_T^{\pi^*}}\Big|\mathcal{F}_t\right] = V_t^{\pi^*}E\left[\frac{V_T^{v^H,\pi^H}}{V_T^{\pi^*}}\Big|\mathcal{F}_t\right] = V_t^{v^H,\pi^H} \qquad P\text{-a.s.}$$

where the last equality is due to the fairness of V^{v^H,π^H}, see Definition 3.4.12. Moreover, the second part of Proposition 3.5.2 gives an economic rationale for the use of the real-world pricing formula (3.20), since it shows that the latter gives the value of the least expensive replication portfolio. This property has been called the *law of the minimal price* (see Platen (2009), Section 4). The following simple Proposition immediately follows from Theorem 3.5.5.

Proposition 3.6.3. *Suppose that Assumption 3.3.6 holds. Let H be a positive $\mathcal{F}$-measurable contingent claim such that $E\left[\frac{\widehat{Z}_T}{S_T^0}\,H\right] < \infty$. Then, under the assumptions of Theorem 3.5.5, the following hold:*

(a) there exists a fair portfolio process $V^{v^H,\pi^H} = \left(V_t^{v^H,\pi^H}\right)_{0\leq t\leq T}$ such that $V_T^{v^H,\pi^H} = H$ P-a.s.;

(b) the real-world price is given by $\Pi_0^H = E\left[\frac{H}{V_T^{\pi^*}}\right] = E\left[\frac{\widehat{Z}_T}{S_T^0}H\right] = v^H$.

Remark 3.6.4.

1. Notice that, due to Proposition 3.4.5, the real-world pricing formula (3.20) can be rewritten as follows, for any $t \in [0,T]$:

$$\Pi_t^H = \frac{S_t^0}{\widehat{Z}_t} E\left[\frac{\widehat{Z}_T}{S_T^0} H \Big| \mathcal{F}_t\right] \tag{3.21}$$

Suppose now that $E\big[\widehat{Z}_T\big] = 1$, so that $\widehat{Z}$ is the density process of the ELMM $\widehat{Q}$ (see Remark 3.3.17). Due to the Bayes formula, equation (3.21) can then be rewritten as follows:

$$\Pi_t^H = S_t^0 \, E^{\widehat{Q}}\left[\frac{H}{S_T^0}\Big|\mathcal{F}_t\right]$$

and we recover the usual risk-neutral pricing formula (compare also Platen (2009), Section 5, and Platen & Heath (2006), Section 10.4). In this sense, the real-world pricing approach can be regarded as a consistent extension of the usual risk-neutral valuation approach to a financial market for which an ELMM may fail to exist.

2. Let us briefly suppose that H and the final value of the GOP $V_T^{\pi^*}$ are conditionally independent given the σ-field $\mathcal{F}_t$, for all $t \in [0,T]$. The real-world pricing formula (3.20) can then be rewritten as follows:

$$\Pi_t^H = V_t^{\pi^*} E\left[\frac{1}{V_T^{\pi^*}}\Big|\mathcal{F}_t\right] E\left[H|\mathcal{F}_t\right] =: P\left(t,T\right) E\left[H|\mathcal{F}_t\right] \tag{3.22}$$

where $P\left(t,T\right)$ denotes the *fair value* of a zero coupon T-bond (i.e. a contingent claim which pays the deterministic amount 1 at maturity T). This shows that, under the (rather strong) assumption of conditional independence, one can recover the well-known *actuarial pricing formula* (compare also Platen (2006), Corollary 3.4, and Platen (2009), Section 5).

3. We want to point out that part *(b)* of Proposition 3.6.3 can be easily generalised to any time $t \in [0,T]$: compare for instance Proposition 10 in Galesso & Runggaldier (2010).

In view of the above Remarks, it is interesting to observe how several different valuation approaches which have been widely used in finance and insurance, such as risk-neutral pricing and actuarial pricing, are both generalised and unified under the concept of real-world pricing. We refer to Section 10.4 of Platen & Heath (2006) for related comments on the unifying aspects of the benchmark approach.

3.6.2 The *upper hedging price* approach

The *upper hedging price* (or *super-hedging price*) is a classical approach to the valuation of contingent claims (see e.g. Karatzas & Shreve (1998), Section 5.5.3). The intuitive idea is to find the smallest initial capital which allows one to obtain a final wealth which is greater or equal than the payoff at maturity of a given contingent claim.

Definition 3.6.5. *Let H be a positive $\mathcal{F}$-measurable contingent claim. The* upper hedging price *$\mathcal{U}(H)$ of H is defined as follows:*

$$\mathcal{U}(H) := \inf\big\{v \in [0,\infty) : \exists\, \pi \in \mathcal{A} \text{ such that } V_T^{v,\pi} \geq H \ P\text{-a.s.}\big\}$$

with the usual convention $\inf \emptyset = \infty$.

The next Theorem shows that, in a complete diffusion-based financial market, the upper hedging price takes a particularly simple and natural form.

Theorem 3.6.6. *Let H be a positive $\mathcal{F}$-measurable contingent claim such that $E\left[\frac{\widehat{Z}_T}{S_T^0} H\right] < \infty$. Then, under the assumptions of Theorem 3.5.5, the upper hedging price of H is explicitly given by:*

$$\mathcal{U}(H) = E\left[\frac{\widehat{Z}_T}{S_T^0} H\right] \tag{3.23}$$

Proof. In order to prove (3.23), we show both directions of inequality.

($\geq$): If $\{v \in [0,\infty) : \exists\, \pi \in \mathcal{A}$ such that $V_T^{v,\pi} \geq H$ P-a.s.$\} = \emptyset$ then we have $E\left[\frac{\widehat{Z}_T}{S_T^0} H\right] < \mathcal{U}(H) = \infty$. So, let us assume there exists a couple $(v,\pi) \in [0,\infty) \times \mathcal{A}$ such that $V_T^{v,\pi} \geq H$ P-a.s. Under Assumption 3.3.6, due to Corollary 3.4.6, the benchmarked portfolio process $\hat{V}^{v,\pi} = \big(V_t^{v,\pi}/V_t^{\pi^*}\big)_{0\leq t\leq T}$ is a supermartingale and so, recalling also Proposition 3.4.5:

$$v = \hat{V}_0^{v,\pi} \geq E\big[\hat{V}_T^{v,\pi}\big] = E\left[\frac{\widehat{Z}_T}{S_T^0} V_T^{v,\pi}\right] \geq E\left[\frac{\widehat{Z}_T}{S_T^0} H\right]$$

This implies that $\mathcal{U}(H) \geq E\left[\frac{\widehat{Z}_T}{S_T^0} H\right]$.

($\leq$): Under the present assumptions, Theorem 3.5.5 yields the existence of a couple $\big(v^H, \pi^H\big) \in (0,\infty) \times \mathcal{A}^F$ such that $V_T^{v^H,\pi^H} = H$ P-a.s. and where $v^H = E\left[\frac{\widehat{Z}_T}{S_T^0} H\right]$. Hence:

$$E\left[\frac{\widehat{Z}_T}{S_T^0} H\right] = v^H \in \big\{v \in [0,\infty) : \exists\, \pi \in \mathcal{A} \text{ such that } V_T^{v,\pi} \geq H \ P\text{-a.s.}\big\}$$

This implies that $\mathcal{U}(H) \leq E\left[\frac{\widehat{Z}_T}{S_T^0} H\right]$.

□

An analogous result can be found in Proposition 5.3.2 of Karatzas & Shreve (1998) (compare also Fernholz & Karatzas (2009), Section 10). We want to point out that Definition 3.6.5 can be easily generalised to an arbitrary time point $t \in [0,T]$ in order to define the upper hedging price at $t \in [0,T]$. The result of Theorem 3.6.6 carries over to this slightly generalised setting with essentially the same proof, compare with Theorem 3 in Galesso & Runggaldier (2010).

Remark 3.6.7.

1. Notice that, due to Proposition 3.4.5, equation (3.23) can be rewritten as follows:

$$\mathcal{U}(H) = E\left[\frac{\widehat{Z}_T}{S_T^0} H\right] = E\left[\frac{H}{V_T^{\pi^*}}\right]$$

   This shows that the upper hedging price can be obtained by computing the expectation of the benchmarked value (in the sense of Definition 3.4.12) of the contingent claim H under the real-world probability measure P and thus coincides with the real-world price (evaluated at $t = 0$), compare part *(b)* of Proposition 3.6.3.

2. Suppose that $E\big[\widehat{Z}_T\big] = 1$. As explained in Remark 3.3.17, the process $\widehat{Z}$ represents then the density process of the ELMM $\widehat{Q}$. In this case, the upper hedging price $\mathcal{U}(H)$ yields the usual risk-neutral valuation formula, i.e. we have $\mathcal{U}(H) = E^{\widehat{Q}}\left[H/S_T^0\right]$.

3.6.3 Utility indifference valuation

The real-world pricing formula has been justified so far on the basis of replication arguments, as can be seen from Proposition 3.6.3. We now present a different approach which uses the idea of *utility indifference valuation*. To this effect, let us first consider the problem of maximising an expected utility function of the discounted final wealth. Recall that, due to Theorem 3.3.14, we can meaningfully consider portfolio optimisation problems even in the absence of an ELMM for $\left(S^0, S^1, \ldots, S^N\right)$.

Definition 3.6.8. *A* utility function U *is a function* $U : [0,\infty) \to [0,\infty)$ *such that:*

1. *U is strictly increasing and strictly concave, continuously differentiable;*

2. $\lim_{x\to\infty} U'(x) = 0$ *and* $\lim_{x\to 0} U'(x) = \infty$.

Problem (expected utility maximisation). *Let U be as in Definition 3.6.8 and let $v \in (0,\infty)$. The problem of expected utility maximisation consists in the following:*

$$\textit{maximise } E\left[U\left(\bar{V}_T^{v,\pi}\right)\right] \textit{ over all } \pi \in \mathcal{A} \tag{3.24}$$

The following Lemma shows that, in the case of a complete financial market, there is no loss of generality in restricting our attention to fair strategies only. Recall that, due to Definition 3.4.12, $\mathcal{A}^F$ denotes the set of all fair trading strategies in $\mathcal{A}$.

Lemma 3.6.9. *Under the assumptions of Theorem 3.5.5, for any utility function U and for any $v \in (0,\infty)$, the following holds:*

$$\sup_{\pi\in\mathcal{A}} E\left[U\left(\bar{V}_T^{v,\pi}\right)\right] = \sup_{\pi\in\mathcal{A}^F} E\left[U\left(\bar{V}_T^{v,\pi}\right)\right] \tag{3.25}$$

Proof. It is clear that "$\geq$" holds in (3.25), since $\mathcal{A}^F \subseteq \mathcal{A}$. To show the reverse inequality, let us consider an arbitrary element $\pi \in \mathcal{A}$. The benchmarked portfolio process $\hat{V}^{v,\pi} = \left(V_t^{v,\pi}/V_t^{\pi^*}\right)_{0\leq t\leq T}$ is a supermartingale, due to Corollary 3.4.6, and hence:

$$v' := E\left[\frac{\widehat{Z}_T}{S_T^0} V_T^{v,\pi}\right] = E\left[\frac{V_T^{v,\pi}}{V_T^{\pi^*}}\right] \leq v$$

with equality holding if and only if $\pi \in \mathcal{A}^F$. Let $\bar{v} := v - v' \geq 0$. It is clear that the positive $\mathcal{F}$-measurable random variable $\bar{H} := \bar{V}_T^{v,\pi} + \bar{v}/\widehat{Z}_T$ satisfies $E\big[\widehat{Z}_T \bar{H}\big] = v$ and so, due to Theorem 3.5.5, there exists an element $\pi^H \in \mathcal{A}^F$ such that $\bar{V}_T^{v,\pi^H} = \bar{H} \geq \bar{V}_T^{v,\pi}$ P-a.s., with equality holding if and only if the strategy π is fair. We then have, due to the monotonicity of U:

$$E\big[U\left(\bar{V}_T^{v,\pi}\right)\big] \leq E\big[U\left(\bar{H}\right)\big] = E\big[U\big(\bar{V}_T^{v,\pi^H}\big)\big] \leq \sup_{\pi\in\mathcal{A}^F} E\big[U\left(\bar{V}_T^{v,\pi}\right)\big]$$

Since $\pi \in \mathcal{A}$ was arbitrary, this shows the "$\leq$" inequality in (3.25). $\square$

In particular, Lemma 3.6.9 shows that, in the context of portfolio optimisation problems, restricting the class of admissible trading strategies to *fair* admissible strategies is not only "reasonable", as argued in Chapter 11 of Platen & Heath (2006), but exactly yields the same optimal value. The following Theorem gives the solution to problem (3.24), in the case of a complete financial market. Related results can be found in Lemma 5 of Galesso & Runggaldier (2010) and Theorem 3.7.6 of Karatzas & Shreve (1998).

Theorem 3.6.10. *Suppose that Assumption 3.3.6 holds. Let U be a utility function and $v \in (0,\infty)$. Assume that the function $\mathcal{W}(y) := E\left[\widehat{Z}_T\, I\big(y/\bar{V}_T^{v,\pi^*}\big)\right]$ is finite for every $y \in (0,\infty)$, where I is the inverse function of U'. The function $\mathcal{W}$ is invertible and, under the assumptions of Theorem 3.5.5, the optimal discounted final wealth $\bar{V}_T^{v,\pi^U}$ for Problem* (3.24) *is explicitly given as follows:*

$$\bar{V}_T^{v,\pi^U} = I\left(\frac{\mathcal{Y}(v)}{\bar{V}_T^{v,\pi^*}}\right) \tag{3.26}$$

where $\mathcal{Y}$ denotes the inverse function of $\mathcal{W}$. The optimal strategy $\pi^U \in \mathcal{A}^F$ is given by the replicating strategy for the right hand side of (3.26).

Proof. Note first that, due to Definition 3.6.8, the function U' admits a strictly decreasing continuous inverse function $I : [0,\infty] \to [0,\infty]$ with $I(0) = \infty$ and $I(\infty) = 0$. We have then the following well-known result from convex analysis (see e.g. Karatzas & Shreve (1998), Section 3.4):

$$U\big(I(y)\big) - yI(y) \geq U(x) - xy \qquad \text{for} \quad 0 \leq x < \infty,\ 0 < y < \infty \tag{3.27}$$

As in Lemma 3.6.2 of Karatzas & Shreve (1998), it can be shown that the function $\mathcal{W} : [0,\infty] \to [0,\infty]$ is strictly decreasing and continuous and, hence, it admits an inverse function $\mathcal{Y} : [0,\infty] \to [0,\infty]$. Since $\mathcal{W}\big(\mathcal{Y}(v)\big) = v$, for any $v \in (0,\infty)$, Theorem 3.5.5 shows that there exists a fair strategy $\pi^U \in \mathcal{A}^F$ such that $\bar{V}_T^{v,\pi^U} = I\big(\mathcal{Y}(v)/\bar{V}_T^{v,\pi^*}\big)$ P-a.s. Furthermore, for any $\pi \in \mathcal{A}^F$, equation (3.27) with $y = \mathcal{Y}(v)/\bar{V}_T^{v,\pi^*}$ and $x = \bar{V}_T^{v,\pi}$ gives that:

$$E\left[U\left(\bar{V}_T^{v,\pi^U}\right)\right] = E\left[U\left(I\left(\frac{\mathcal{Y}(v)}{\bar{V}_T^{v,\pi^*}}\right)\right)\right] \geq E\big[U\left(\bar{V}_T^{v,\pi}\right)\big] + \mathcal{Y}(v)\,E\left[\frac{1}{\bar{V}_T^{v,\pi^*}}\left(I\left(\frac{\mathcal{Y}(v)}{\bar{V}_T^{v,\pi^*}}\right) - \bar{V}_T^{v,\pi}\right)\right]$$

$$= E\big[U\left(\bar{V}_T^{v,\pi}\right)\big] + \mathcal{Y}(v)\,E\left[\frac{1}{\bar{V}_T^{v,\pi^*}}\left(\bar{V}_T^{v,\pi^U} - \bar{V}_T^{v,\pi}\right)\right] = E\big[U\left(\bar{V}_T^{v,\pi}\right)\big]$$

thus showing that, based also on Lemma 3.6.9, $\pi^U \in \mathcal{A}^F$ solves Problem (3.24). □

Remark 3.6.11.

1. It is important to observe that Theorem 3.6.10 does not rely on the existence of an ELMM. This amounts to saying that we can meaningfully solve expected utility maximisation problems even when no ELMM exists or, equivalently, when the traditional no-arbitrage condition (NFLVR) fails to hold. The crucial assumption for the validity of Theorem 3.6.10 is Assumption 3.3.6, which ensures that there are no arbitrages of the first kind (compare Theorem 3.3.14 and Corollary 3.3.15).

2. The assumption that the function $\mathcal{W}(y) := E\left[\widehat{Z}_T\, I\big(y/\bar{V}_T^{v,\pi^*}\big)\right]$ be finite for every $y \in (0,\infty)$ can be replaced by suitable technical conditions on the utility function U and on the processes μ and σ (see Remarks 3.6.8 and 3.6.9 in Karatzas & Shreve (1998) for more details).

Having solved the general utility maximisation problem, we are now in a position to give the definition of the *utility indifference price*, in the spirit of Davis (1997) (compare also Galesso & Runggaldier (2010), Section 4.2, Platen & Heath (2006), Definition 11.4.1, and Platen & Runggaldier (2007), Definition 10)[2]. Until the end of this Section, we let U be a utility function, in the sense of Definition 3.6.8, such that all expected values below exist and are finite.

Definition 3.6.12. *Let H be a positive $\mathcal{F}$-measurable contingent claim and $v \in (0,\infty)$. For $p \geq 0$, let us define, for a given utility function U, the function $W_p^U : [0,1] \to [0,\infty)$ as follows:*

$$W_p^U(\varepsilon) := E\left[U\left((v - \varepsilon p)\,\bar{V}_T^{\pi^U} + \varepsilon \bar{H}\right)\right] \tag{3.28}$$

where $\pi^U \in \mathcal{A}^F$ solves Problem (3.24) *for the utility function U. The* utility indifference price *of the contingent claim H is defined as the value $p(H)$ which satisfies the following condition:*

$$\lim_{\varepsilon \to 0} \frac{W_{p(H)}^U(\varepsilon) - W_{p(H)}^U(0)}{\varepsilon} = 0 \tag{3.29}$$

Definition 3.6.12 is based on a "marginal rate of substitution" argument, as first pointed out in Davis (1997). In fact, $p(H)$ can be thought of as the price at which an investor is marginally indifferent between the two following alternatives:

[2] In Galesso & Runggaldier (2010) and Platen & Runggaldier (2007) the authors generalise Definition 3.6.12 to an arbitrary time $t \in [0,T]$. However, since the results and the techniques remain essentially unchanged, we only consider the basic case $t = 0$.

- invest an infinitesimal part $\varepsilon p(H)$ of the initial endowment v into the contingent claim H and invest the remaining wealth $\big(v - \varepsilon p(H)\big)$ according to the optimal trading strategy π^U;
- ignore the contingent claim H and simply invest the whole initial endowment v according to the optimal trading strategy π^U.

The following simple result, essentially due to Davis (1997) (compare also Platen & Heath (2006), Section 11.4), gives a general representation of the utility indifference price $p(H)$.

Proposition 3.6.13. *Let U be a utility function and H a positive $\mathcal{F}$-measurable contingent claim. The utility indifference price $p(H)$ can be represented as follows:*

$$p(H) = \frac{E\left[U'\left(\bar{V}_T^{v,\pi^U}\right)\bar{H}\right]}{E\left[U'\left(\bar{V}_T^{v,\pi^U}\right)\bar{V}_T^{\pi^U}\right]} \tag{3.30}$$

Proof. Using equation (3.28), let us write the following Taylor's expansion:

$$\begin{aligned} W_p^U(\varepsilon) &= E\left[U\left(\bar{V}_T^{v,\pi^U}\right) + \varepsilon\, U'\left(\bar{V}_T^{v,\pi^U}\right)\left(\bar{H} - p\,\bar{V}_T^{\pi^U}\right) + o(\varepsilon)\right] \\ &= W_p^U(0) + \varepsilon\, E\left[U'\left(\bar{V}_T^{v,\pi^U}\right)\left(\bar{H} - p\,\bar{V}_T^{\pi^U}\right)\right] + o(\varepsilon) \end{aligned} \tag{3.31}$$

If we insert (3.31) into (3.29) we get:

$$E\left[U'\left(\bar{V}_T^{v,\pi^U}\right)\left(\bar{H} - p(H)\,\bar{V}_T^{\pi^U}\right)\right] = 0$$

from which (3.30) immediately follows. □

By combining Theorem 3.6.10 with Proposition 3.6.13, we can easily prove the following Corollary, which yields an explicit and "universal" representation of the utility indifference price $p(H)$ (compare also Galesso & Runggaldier (2010), Theorem 8, Platen & Heath (2006), Section 11.4, and Platen & Runggaldier (2007), Proposition 11).

Corollary 3.6.14. *Let H be a positive $\mathcal{F}$-measurable contingent claim. Then, under the assumptions of Theorem 3.6.10, the utility indifference price coincides with the real-world price (at $t = 0$) for any utility function U, namely:*

$$p(H) = E\left[\frac{H}{V_T^{\pi^*}}\right]$$

Proof. The present assumptions imply that, due to (3.26), we can rewrite (3.30) as follows:

$$p(H) = \frac{E\left[U'\left(I\left(\frac{\mathcal{Y}(v)}{\bar{V}_T^{v,\pi^*}}\right)\right)\bar{H}\right]}{E\left[U'\left(I\left(\frac{\mathcal{Y}(v)}{\bar{V}_T^{v,\pi^*}}\right)\right)\bar{V}_T^{\pi^U}\right]} = \frac{E\left[\frac{\mathcal{Y}(v)}{\bar{V}_T^{v,\pi^*}}\,\bar{H}\right]}{E\left[\frac{\mathcal{Y}(v)}{\bar{V}_T^{v,\pi^*}}\,\bar{V}_T^{\pi^U}\right]} = \frac{\frac{1}{v}\,E\left[\frac{\bar{H}}{\bar{V}_T^{\pi^*}}\right]}{\frac{1}{v}\,\frac{\bar{V}_0^{\pi^U}}{\bar{V}_0^{\pi^*}}} = E\left[\frac{H}{V_T^{\pi^*}}\right] \tag{3.32}$$

where the third equality uses the fact that $\pi^U \in \mathcal{A}^F$. □

Remark 3.6.15. As follows from Definition 3.6.12, the utility indifference price $p(H)$ depends a priori both on the initial endowment v and on the chosen utility function U. The remarkable result of Corollary 3.6.14 consists in the fact that, under the hypotheses of Theorem 3.6.10, the utility indifference price $p(H)$ represents an "universal" pricing rule, since it does not depend neither on v nor on the utility function U and, furthermore, it coincides with the real-world pricing formula.

3.7 Conclusions, extensions and further developments

In this Chapter, we have studied a general class of diffusion-based models for financial markets, weakening the traditional assumption that the no-arbitrage condition (NFLVR) holds or, equivalently, that there exists an ELMM. We have shown that the financial market may still be viable, in the sense that arbitrages of the first kind are not permitted, as soon as the market price of risk process satisfies a crucial square-integrability condition. In particular, we have shown that the failure of the existence of an ELMM does not preclude the completeness of the financial market and the solvability of portfolio optimisation problems. Furthermore, in the context of a complete market, contingent claims can be consistently evaluated by relying on the real-world pricing formula.

The results of Section 3.6 on the valuation of contingent claims have been obtained under the assumption of a complete financial market. These results, namely that the real-world pricing formula (3.20) coincides with the utility indifference price, can be extended to the more general context of an incomplete financial market, provided that we choose a logarithmic utility function.

Proposition 3.7.1. *Suppose that Assumption 3.3.6 holds. Let H be a positive $\mathcal{F}$-measurable contingent claim such that $E\left[\frac{\widehat{Z}_T}{S_T^0} H\right] < \infty$ and let $U(x) = \log(x)$. Then, the log-utility indifference price $p_{\log}(H)$ is explicitly given as follows:*

$$p_{\log}(H) = E\left[\frac{H}{V_T^{\pi^*}}\right]$$

Proof. Note first that $U(x) = \log(x)$ is a well-defined utility function in the sense of Definition 3.6.8. Let us first consider Problem (3.24) for $U(x) = \log(x)$. Using the notations introduced in the proof of Theorem 3.6.10, the function I is now given by $I(x) = x^{-1}$, for $x \in (0,\infty)$. Due to Proposition 3.4.5, we have $\mathcal{W}(y) = v/y$ for all $y \in (0,\infty)$ and, hence, $\mathcal{Y}(v) = 1$. Then, equation (3.26) implies that $\bar{V}_T^{v,\pi^U} = \bar{V}_T^{v,\pi^*}$, meaning that the growth-optimal strategy $\pi^* \in \mathcal{A}^F$ solves Problem (3.24) for logarithmic utility. The same computations as in (3.32) imply then the following:

$$p_{\log}(H) = \frac{E\left[\frac{\bar{H}}{\bar{V}_T^{v,\pi^*}}\right]}{E\left[\frac{1}{\bar{V}_T^{v,\pi^*}}\bar{V}_T^{\pi^*}\right]} = E\left[\frac{H}{V_T^{\pi^*}}\right]$$

□

The interesting feature of Proposition 3.7.1 is that the claim H does not need to be replicable. However, Proposition 3.7.1 depends on the choice of the logarithmic utility function and does not hold for a generic utility function U, unlike the "universal" result shown in Corollary 3.6.14. Of course, the result of Proposition 3.7.1 is not surprising, due to the well-known fact that the growth-optimal portfolio solves the log-utility maximisation problem, see e.g. Becherer (2001), Christensen & Larsen (2007) and Karatzas & Kardaras (2007).

Remark 3.7.2. Following Section 11.3 of Platen & Heath (2006), let us suppose that the discounted GOP process $\bar{V}^{\pi^*} = \big(\bar{V}_t^{\pi^*}\big)_{0\leq t\leq T}$ has the Markov property under P. Under such an assumption, one can obtain an analogous version of Theorem 3.6.10 also in the case of an incomplete financial market model (see Platen & Heath (2006), Theorem 11.3.3). In fact, the first part of the proof of Theorem 3.6.10 remains unchanged. One then proceeds by considering the martingale $M = (M_t)_{0\leq t\leq T}$ defined by $M_t := E\big[\widehat{Z}_T\, I\big(\mathcal{Y}(v)\,/\bar{V}_T^{v,\pi^*}\big)\big|\mathcal{F}_t\big] = E\big[1/\bar{V}_T^{\pi^*}\, I\big(\mathcal{Y}(v)\,/\bar{V}_T^{v,\pi^*}\big)\big|\mathcal{F}_t\big]$, for $t \in [0,T]$. Due to the Markov property, M_t can be represented as $g\big(t, \bar{V}_t^{\,v,\pi^*}\big)$, for every $t \in [0,T]$. If the function g is sufficiently smooth one can apply Itô's formula and express M as the value process of a benchmarked fair portfolio. If one can shown that the resulting strategy satisfies the admissibility conditions, Proposition 3.6.13 and Corollary 3.6.14 can then be applied to show that the real-world pricing formula coincides with the utility indifference price (for any utility function!). Always in a diffusion-based Markovian context, a detailed analysis to this effect can also be found in the recent paper Ruf (2011a).

We want to point out that the modeling framework considered in this Chapter is not restricted to stock markets, but can also be applied to the valuation of fixed income products. In particular, in Bruti-Liberati et al. (2010) and Platen & Heath (2006), Section 10.4, the authors develop a version of the Heath-Jarrow-Morton approach to the modeling of the term structure of interest rates without relying on the existence of a martingale measure. In this context, they derive a real-world version of the classical Heath-Jarrow-Morton drift condition, relating the drift and diffusion terms in the system of SDEs describing the evolution of forward interest rates. Unlike in the traditional setting, this real-world drift condition explicitly involves the market price of risk process.

Finally, we want to mention that the concept of real-world pricing has also been studied in the context of incomplete information models, meaning that investors are supposed to have access only to the information contained in a sub-filtration of the original full-information filtration $\mathbb{F}$, see Galesso & Runggaldier (2010), Platen & Runggaldier (2005) and Platen & Runggaldier (2007).

Chapter 4

Weak no-arbitrage conditions: characterization, stability and hedging problems

4.1 Introduction

Modern mathematical finance is based on the concept of *no-arbitrage*. In a nutshell, the no-arbitrage condition amounts at excluding the possibility of making money out of nothing by trading in the financial market according to a well-chosen strategy. Obviously, the existence of arbitrage strategies leads to inconsistencies in the price system and conflicts with the existence of a financial market equilibrium. As a consequence, any mathematical model which aims at describing the functioning of a financial market needs to satisfy a suitable no-arbitrage condition, in the absence of which one cannot draw reliable conclusions concerning market prices and investors' behavior.

The search for a satisfactory no-arbitrage condition has a rather long history, which has developed on the border between financial economics and mathematics, see Delbaen & Schachermayer (2006) for a detailed account. A decisive step towards the establishment of a general and economically sound notion of no-arbitrage was marked by the classical paper Delbaen & Schachermayer (1994), where the authors proved (in the case of locally bounded processes) the equivalence of the *No Free Lunch with Vanishing Risk (NFLVR)* condition (a condition slightly stronger than the classical *No Arbitrage (NA)* condition) with the existence of a probability measure Q equivalent to the original probability measure P which makes the discounted price process S a local Q-martingale, i.e. an *Equivalent Local Martingale Measure (ELMM)* for S. This result was later extended to general (possibly non-locally bounded) semimartingales in Delbaen & Schachermayer (1998b).

By now, the NFLVR condition has become a sort of "golden standard" in mathematical finance and the vast majority of models proposed in the literature satisfies the NFLVR condition. However, especially in the last couple of years, financial market models which do not admit an ELMM (and, hence, do not satisfy the NFLVR condition) have also appeared in the literature. For instance, in the context of *Stochastic Portfolio Theory* (see Fernholz & Karatzas (2009) for an overview), the

NFLVR condition is not imposed as a normative assumption and, indeed, it is shown that arbitrage opportunities naturally arise under some realistic market conditions. Financial market models which do not satisfy the NFLVR condition are also encountered in the context of the *Benchmark Approach* (see Platen (2006),(2009) and Platen & Heath (2006) for a complete account), the main goal of which consists in the development of an asset pricing theory which does not rely on the existence of an ELMM. Related works which explicitly consider market models not admitting an ELMM are Loewenstein & Willard (2000), Cassese (2005), Christensen & Larsen (2007), Hulley (2010) and Ruf (2011b). Somewhat surprisingly, it has also been shown that the NFLVR condition is not indispensable for the analysis of classical issues in financial mathematics such as portfolio optimization problems and valuation and hedging problems, provided suitable strong forms of arbitrage are banned from the market. Hence, we can conclude that there is some evidence pointing at the fact that the full strength of the NFLVR condition may not be needed in order to formulate a coherent modeling framework for financial markets.

In the first part of the present Chapter, we shall deal with three no-arbitrage conditions which are strictly weaker than the classical NFLVR condition, namely the *No Unbounded Increasing Profit (NUIP)* condition, the *No Immediate Arbitrage Opportunity (NIAO)* condition and the *No Unbounded Profit with Bounded Risk (NUPBR)* condition. These three no-arbitrage conditions are not new in the literature. In fact, the NUIP condition has been introduced in Karatzas & Kardaras (2007), the NIAO condition goes back to Delbaen & Schachermayer (1995b) and the NUPBR condition has been introduced under that name in Karatzas & Kardaras (2007), but its importance was already recognized in Delbaen & Schachermayer (1994) and Kabanov (1997). However, a single reference discussing the precise connections among the NUIP/NIAO/NUPBR no-arbitrage conditions seems to be missing so far. We provide a detailed analysis of the NUIP/NIAO/NUPBR conditions, providing necessary and sufficient conditions for their validity as well as equivalent characterizations. In particular, we show that the validity of the NUIP/NIAO/NUPBR conditions can be directly verified by looking at the characteristics of the discounted price process. As shown by an explicit counterexample in Karatzas & Kardaras (2007), this is not possible for the NFLVR condition. We show that the NUIP condition is the minimal no-arbitrage condition, in the sense that if NUIP fails then also the stronger NIAO/NUPBR/NFLVR conditions fail. The NIAO condition is slightly stronger than the NUIP condition, but still weaker than the NUPBR/NFLVR conditions. Finally, the NUPBR condition is situated on a middle ground between the NIAO and the stronger NFLVR condition. We discuss these relations by providing explicit examples and counterexamples and we also provide connections with other notions of arbitrage which have appeared in the literature. In particular, we show that at the level of the NIAO condition we are essentially excluding pathological forms or arbitrage, in the presence of which the situation seems to be utterly hopeless from the point of view of financial modeling. In contrast, the NUPBR has a real economic content, as has been recently shown in Karatzas & Kardaras (2007), Kardaras (2010a) and Kardaras (2010b). Furthermore, the NUPBR condition ensures the existence of a *martingale deflator*, which inherits some (but not all) of the properties of the density process of an ELMM.

From an economic point of view, it is important for a no-arbitrage condition to be robust with respect to reasonable modifications of the underlying financial market model. Unfortunately, the classical NFLVR condition is not very robust with respect to changes in the original market configuration. For instance, the NFLVR condition can be easily destroyed by a change of numéraire, as shown in Delbaen & Schachermayer (1995c), thus leading to serious inconsistencies in financial market models based on several currencies. This motivates our analysis of the stability properties of the NUIP/NIAO/NUPBR conditions with respect to changes in the original financial market model. More precisely, we shall analyze the effects of changes of numéraire, absolutely continuous changes of the reference probability measure and restrictions/enlargements of the reference filtration. We shall prove that, unlike the classical NFLVR condition, the weaker NUIP/NIAO/NUPBR conditions are in general preserved under these modifications of the model. To the best of our knowledge, these results are new and confirm the economic soundness of the NUIP/NIAO/NUPBR weak no-arbitrage conditions. The reason why the stability properties of the NUIP/NIAO/NUPBR conditions are easier to ascertain than those of the NFLVR condition is that the former can be expressed in terms of the characteristics of the discounted price process S. Hence, by studying the impact of changes of numéraire/measure/filtration on the characteristics of S, we can directly see whether any of the NUIP/NIAO/NUPBR conditions is preserved or not.

One of the key issues in financial mathematics is represented by the valuation and hedging of contingent claims. Under the classical NFLVR condition, one can rely on the well-known machinery of *risk neutral* pricing. In that context, general results on the attainability of contingent claims have been obtained by Jacka (1992), Ansel & Stricker (1994) and Delbaen & Schachermayer (1994),(1995c),(1998b). Always under the NFLVR condition, the so-called second fundamental theorem of asset pricing asserts that the market is complete, meaning that every (sufficiently integrable) contingent claim can be perfectly replicated by trading in the market, if and only if there exists an unique ELMM. Motivated by these classical results, we aim at understanding what one can say when the NFLVR condition fails but the weaker NUPBR condition holds. By replacing density processes of ELMMs with martingale deflators (which represent the weaker counterparts of the former), we show that there still exists a general characterization of attainable claims. Furthermore, the financial market is complete if and only if there exists an unique martingale deflator, which can be explicitly computed in terms of the canonical decomposition of the discounted price process S, thus providing a natural generalization of the second fundamental theorem of asset pricing. In the special case of diffusion-based financial market models, related (but only partial) results have been shown in the recent papers Fernholz & Karatzas (2009), Fontana & Runggaldier (2011) and Ruf (2011b). Precursors to our results, based on an entirely different approach, can also be found in Stricker & Yan (1998). The key idea underlying our proofs consists in applying a suitable change of numéraire so that the classical results of Delbaen & Schachermayer (1995c) can be applied to the price process expressed in terms of the new numéraire.

This Chapter is structured as follows. Section 4.2 contains the general description of our abstract financial market model. In Section 4.3, we study three weak no-arbitrage notions, namely the *No Unbounded Increasing Profit (NUIP)* condition (Section 4.3.1), the *No Immediate Arbitrage Opportunity (NIAO)* condition (Section 4.3.2) and the *No Unbounded Profit with Bounded Risk (NUPBR)* condition (Section 4.3.3). In the context of a financial market model based on continuous semimartingales, we provide necessary and sufficient conditions for their validity as well as equivalent characterizations. We also discuss the relations with the classical *No Arbitrage (NA)* and *No Free Lunch with Vanishing Risk (NFLVR)* conditions. Section 4.4 is devoted to the study of the stability properties of weak no-arbitrage conditions with respect to changes in the structure of the underlying financial market model. More specifically, we shall consider the effects of changes of numéraire (Section 4.4.1), absolutely continuous changes of the reference probability measure (Section 4.4.2) and restrictions/enlargements of the reference filtration (Section 4.4.3). The main message is that, unlike the classical NA and NFLVR conditions, the weak NUIP/NIAO/NUPBR conditions are in general robust with respect to these modifications of the underlying financial market model. Section 4.5 deals with the characterization of attainable contingent claims under the assumption that the NUPBR condition holds, without assuming the full strength of the NFLVR condition. In particular, by replacing *Equivalent Local Martingale Measures* with the weaker concept of *martingale deflators*, we shall extend classical results on the attainability of contingent claims, including the so-called second fundamental theorem of asset pricing. Finally, Section 4.6 concludes and discusses some further developments of the present Chapter.

4.2 General setup and preliminaries

Let us start from a given filtered probability space $(\Omega,\mathcal{F},\mathbb{F},P)$, where the filtration $\mathbb{F}=(\mathcal{F}_t)_{0\leq t\leq T}$ satisfies the usual conditions of right-continuity and P-completeness and $T\in(0,\infty)$ denotes a finite time horizon. We consider a general financial market comprising $d+1$ assets, the prices of which are described by the $\mathbb{R}^{d+1}$-valued process $\widetilde{S}=\bigl(\widetilde{S}_t\bigr)_{0\leq t\leq T}$, with $\widetilde{S}_t=\bigl(\widetilde{S}^0_t,\widetilde{S}^1_t,\ldots,\widetilde{S}^d_t\bigr)'$, and we assume that $\widetilde{S}^0_t$ is P-a.s. strictly positive for all $t\in[0,T]$. Without loss of generality, we take asset 0 as *numéraire* and express all quantities in terms of $\widetilde{S}^0$. This means that the ($\widetilde{S}^0$-discounted) price of asset 0 is constant and equal to 1 and the remaining d risky assets have ($\widetilde{S}^0$-discounted) prices described by the $\mathbb{R}^d$-valued process $S=(S_t)_{0\leq t\leq T}$, where $S^i_t:=\widetilde{S}^i_t/\widetilde{S}^0_t$ for all $t\in[0,T]$ and $i=1,\ldots,d$. We assume that the process S is a continuous $\mathbb{R}^d$-valued semimartingale on $(\Omega,\mathcal{F},\mathbb{F},P)$. In particular, this implies that S is a *special* semimartingale, admitting an unique canonical decomposition $S=S_0+A+M$, where A is a continuous $\mathbb{R}^d$-valued predictable process of finite variation with $A_0=0$ and M is a continuous $\mathbb{R}^d$-valued local martingale with $M_0=0$ (see e.g. Jacod & Shiryaev (2003), Lemma I.4.24). Recall also that, due to Proposition II.2.9 of Jacod & Shiryaev (2003), we can write as follows, for all $i,j=1,\ldots,d$:

$$A^i=\int a^i dB \qquad \text{and} \qquad \langle S^i,S^j\rangle=\langle M^i,M^j\rangle=\int c^{ij}dB \tag{4.1}$$

where B is a continuous real-valued predictable increasing process of locally integrable variation, $a=\bigl(a^1,\ldots,a^d\bigr)'$ is an $\mathbb{R}^d$-valued predictable process and $c=\bigl((c^{i1})_{1\leq i\leq d},\ldots,\bigl(c^{id}\bigr)_{1\leq i\leq d}\bigr)$ is a predictable process taking values in the cone of symmetric nonnegative $d\times d$ matrices. The processes a, c and B satisfying (4.1) are not unique in general, but our results do not depend on the specific choice we make. Note also that we do not necessarily assume that the process S takes values in the positive orthant of $\mathbb{R}^d$.

We suppose that our general financial market is frictionless, meaning that we do not consider trading restrictions, transaction costs, liquidity effects or other market imperfections. In order to mathematically represent the activity of trading in the financial market, we need to properly define the concept of *admissible trading strategy*. To this effect, let us first introduce the set $L(S)$ of all $\mathbb{R}^d$-valued S-integrable predictable processes and denote by $H \cdot S$ the stochastic integral process $\int H dS = \left(\int_0^t H_u dS_u\right)_{0 \leq t \leq T}$, for $H \in L(S)$. Since S is a continuous semimartingale, Proposition III.6.22 of Jacod & Shiryaev (2003) implies that $L(S) = L^2_{loc}(M) \cap L^0(A)$, where $L^2_{loc}(M)$ is the set of all $\mathbb{R}^d$-valued predictable processes $H = (H_t)_{0 \leq t \leq T}$ such that $\int_0^T H'_t d\langle M, M\rangle_t H_t < \infty$ P-a.s. and $L^0(A)$ is the set of all $\mathbb{R}^d$-valued predictable processes $H = (H_t)_{0 \leq t \leq T}$ such that $\int_0^T |H'_t dA_t| < \infty$ P-a.s. For $H \in L(S)$, the continuous semimartingale $H \cdot S$ admits the unique canonical decomposition $H \cdot S = H \cdot A + H \cdot M$. Hence, recalling also (4.1), we have that an $\mathbb{R}^d$-valued predictable process $H = (H_t)_{0 \leq t \leq T}$ belongs to $L(S)$ if and only if (see also Jacod & Shiryaev (2003), Theorem III.6.30):

$$\int_0^T v(H)_t \, dB_t < \infty \; P\text{-a.s.} \qquad \text{where} \qquad v(H)_t := \sum_{i,j=1}^d H^i_t c^{ij}_t H^j_t + \Big|\sum_{i=1}^d H^i_t a^i_t\Big|, \; t \in [0,T]$$

Remark 4.2.1 (On the set $L(S)$). As pointed out in Sections III.4a and III.6c of Jacod & Shiryaev (2003), the set $L(S)$ represents the most general class of predictable integrands with respect to the semimartingale S. In particular, we also allow for non-locally bounded integrands, as in Chou et al. (1980). Note also that, for $H \in L(S)$, the process $H \cdot M$ is a continuous local martingale, since we have $L(S) \subseteq L^2_{loc}(M)$ and M is continuous. Furthermore, we want to emphasize that $H \cdot S$ has to be understood as the vector stochastic integral of H with respect to S and is in general different from the sum of the componentwise stochastic integrals $\sum_{i=1}^d \int H^i dS^i$; see for instance Jacod (1980).

We are now in a position to formulate the following classical Definition.

Definition 4.2.2. *Let $a \in \mathbb{R}_+$. An element $H \in L(S)$ is said to be an a-*admissible strategy *if $H_0 = 0$ and $(H \cdot S)_t \geq -a$ P-a.s. for all $t \in [0,T]$. An element $H \in L(S)$ is said to be an* admissible strategy *if it is an a-admissible strategy for some $a \in \mathbb{R}_+$.*

For $a \in \mathbb{R}_+$, we denote by $\mathcal{A}_a$ the set of all a-admissible strategies and by $\mathcal{A}$ the set of all admissible strategies. Clearly, $\mathcal{A} = \bigcup_{a \in \mathbb{R}_+} \mathcal{A}_a$. Let H be an admissible strategy. As usual, H^i_t is assumed to represent the number of units of asset i held in the portfolio at time t, for all $t \in [0,T]$ and $i = 1, \ldots, d$. The condition $H_0 = 0$ in Definition 4.2.2 amounts to requiring that the initial holdings (at time $t = 0$) of the risky assets are zero and, hence, the initial endowment is expressed in terms of the numéraire asset only. We define the *gains from trading* process $G(H) = \left(G_t(H)\right)_{0 \leq t \leq T}$ as $G_t(H) := (H \cdot S)_t$, for all $t \in [0,T]$. According to Definition 4.2.2, the process $G(H)$ corresponding to an admissible strategy $H \in \mathcal{A}$ is uniformly bounded from below by some constant. This restriction is needed since the set $L(S)$ is too large for the purpose of modeling reasonable trading strategies and may also contain pathological doubling strategies (see Delbaen & Schachermayer (2006) for a more detailed discussion). However, this sort of strategies is automatically ruled out from the market if we impose an upper bound on the line of credit which may be granted to any market participant, as in Definition 4.2.2.

Remark 4.2.3 (On self-financing strategies). Let $x \in \mathbb{R}$ represent the initial endowment and let H be an admissible strategy. Let us introduce the real-valued process $H^0 = (H^0_t)_{0 \leq t \leq T}$, with H^0_t representing the units of the numéraire asset held in the portfolio at time t, for $t \in [0,T]$. The value at time t of the portfolio corresponding to the pair (H^0, H) is given by $V_t(x; H^0, H) := H^0_t + \sum_{i=1}^d H^i_t S^i_t$, for all $t \in [0,T]$, with initial endowment $x = V_0(x; H^0, H) = H^0_0$, since $H_0 = 0$. The pair (H^0, H) is said to be *self-financing* if $V_t(x; H^0, H) = x + G_t(H)$, for all $t \in [0,T]$. Note that, if the pair (H^0, H) is self-financing, the process $H^0 = (H^0_t)_{0 \leq t \leq T}$ is predictable. This is clear if S is continuous, as we have assumed above, since any continuous adapted process is predictable. More generally, we can argue as follows:

$$\begin{aligned} H^0_t &= x + G_t(H) - \sum_{i=1}^d H^i_t S^i_t = x + G_{t-}(H) + \Delta G_t(H) - \sum_{i=1}^d H^i_t S^i_t \\ &= x + G_{t-}(H) + \sum_{i=1}^d H^i_t \Delta S^i_t - \sum_{i=1}^d H^i_t S^i_t = x + G_{t-}(H) - \sum_{i=1}^d H^i_t S^i_{t-} \end{aligned} \tag{4.2}$$

where the first equality is due to the self-financing property of (H^0, H) and third equality follows from Theorem IV.18 of Protter (2005). The processes $G_-(H)$ and S_- are predictable, being adapted and left-continuous. Since the process H is predictable, it follows that H^0 is predictable as well. Conversely, given an initial endowment $x \in \mathbb{R}$ and an admissible strategy $H \in \mathcal{A}$, we can always construct a self-financing pair (H^0, H) such that $V_t(x; H^0, H) = x + G_t(H)$ for all $t \in [0,T]$ by simply defining H^0 as the right-hand side of (4.2).

4.3 Characterization of weak no-arbitrage conditions

Having described the general setting of our abstract financial market, we are now in a position to begin our journey through several no-arbitrage criteria. In particular, as mentioned in the Introduction, our attention shall be focused on no-arbitrage conditions which are weaker than the classical *No Free Lunch with Vanishing Risk (NFLVR)* condition. We postpone to the last part of Section 4.3.3 a discussion of the relations between the weak no-arbitrage conditions considered in the present Chapter and the classical no-arbitrage theory based on the NFLVR condition.

4.3.1 *No Unbounded Increasing Profit*

The *No Unbounded Increasing Profit* condition has been first introduced in Karatzas & Kardaras (2007) and is formally defined as follows.

Definition 4.3.1. *Let $H \in \mathcal{A}_0$. We say that the strategy H generates an* unbounded increasing profit *if $P\big(G_s(H) \leq G_t(H)\,,$ for all $0 \leq s \leq t \leq T\big) = 1$ and $P\big(G_T(H) > 0\big) > 0$. If there exists no such $H \in \mathcal{A}_0$ we say that the* No Unbounded Increasing Profit (NUIP) *condition holds.*

Intuitively, an unbounded increasing profit corresponds to a trading strategy which does not require any initial investment nor any amount of credit at intermediate times and, moreover, generates an increasing wealth process, obtaining a strictly positive wealth at the final time T with non-zero probability. From a financial point of view, the notion of unbounded increasing profit represents the most egregious form of arbitrage and, therefore, should be banned from any reasonable financial market model. The following Theorem gives a necessary and sufficient condition for the validity of the NUIP condition.

Theorem 4.3.2. *The NUIP condition holds if and only if there exists an $\mathbb{R}^d$-valued predictable process $\lambda = (\lambda_t)_{0 \leq t \leq T}$ such that $a = c\lambda$ holds $P \otimes B$-a.e., where the processes a, c and B are as in* (4.1).

Proof. Suppose that there exists an $\mathbb{R}^d$-valued predictable process $\lambda = (\lambda_t)_{0 \leq t \leq T}$ such that $a = c\lambda$ holds $P \otimes B$-a.e. and let $H \in \mathcal{A}_0$ be a 0-admissible strategy generating an unbounded increasing profit. By Definition 4.3.1, the gains from trading process $G(H) = H \cdot S$ is increasing, hence of finite variation. This implies that $H \cdot M = H \cdot S - H \cdot A$ is also of finite variation, being the difference of two processes of finite variation. Corollary 1 to Theorem II.27 of Protter (2005) implies then that the process $H \cdot M$ is constant and equal to $(H \cdot M)_0 = 0$, being a continuous local martingale of finite variation. This also gives $\langle H \cdot M \rangle \equiv 0$. Hence, due to the Kunita-Watanabe inequality (see e.g. Protter (2005), Theorem II.25), for all $t \in [0,T]$ and $i = 1, \ldots, d$:

$$\left| \langle H \cdot M, M^i \rangle_t \right| \leq \big(\langle H \cdot M \rangle_t \big)^{1/2} \big(\langle M^i \rangle_t \big)^{1/2} = 0 \qquad P\text{-a.s.}$$

since $\langle M^i \rangle_t < \infty$ P-a.s. for all $t \in [0,T]$ and $i = 1, \ldots, d$. Hence, for all $t \in [0,T]$:

$$\begin{aligned} (H \cdot S)_t = (H \cdot A)_t &= \int_0^t H'_u a_u dB_u = \int_0^t H'_u c_u \lambda_u dB_u = \int_0^t H'_u d\langle M, M \rangle_u \lambda_u \\ &= \int_0^t d\langle H \cdot M, M \rangle_u \lambda_u = \sum_{i=1}^d \int_0^t d\langle H \cdot M, M^i \rangle_u \lambda^i_u = 0 \qquad P\text{-a.s.} \end{aligned}$$

Clearly, this contradicts the assumption that $P\big((H \cdot S)_T > 0\big) > 0$ and, hence, H cannot yield an unbounded increasing profit.

Conversely, suppose that the NUIP condition holds. As in the proof of Theorem 3.5 of Delbaen & Schachermayer (1995b), take an $\mathbb{R}^d$-valued predictable process $H = (H_t)_{0 \leq t \leq T}$ such that $\|H_t(\omega)\| \in \{0, 1\}$ for all $(\omega, t) \in \Omega \times [0,T]$. Clearly, H belongs to $L(S)$, being a bounded predictable process. Suppose that $H \cdot \langle M, M \rangle \equiv 0$ (so that $H \cdot M \equiv 0$) but $H \cdot A \not\equiv 0$. By the Hahn-Jordan decomposition (see Delbaen & Schachermayer (1995b), Theorem 2.1), we can write $H \cdot A = \int (\mathbf{1}_{D^+} - \mathbf{1}_{D^-})\, dV$, where D^+ and D^- are two disjoint predictable subsets of $\Omega \times [0,T]$ such that $D^+ \cup D^- = \Omega \times [0,T]$ and V denotes the total variation of $H \cdot A$. Let $\psi := \mathbf{1}_{D^+} - \mathbf{1}_{D^-}$ and define the $\mathbb{R}^d$-valued predictable process $\tilde{H} := \psi H$. It is easy to see that $\tilde{H} \in L(S)$ and $\tilde{H} \cdot M \equiv 0$. Thus:

$$\tilde{H} \cdot S = \tilde{H} \cdot A = (\psi H) \cdot A = \psi \cdot (H \cdot A) = \psi^2 \cdot V = V$$

where the third equality follows from the associativity of the stochastic integral (see Protter (2005), Theorem IV.21). The process V is non-negative and increasing and satisfies $P(V_T > 0) = 1$, being the total variation of $H \cdot A$, which is supposed to be not identically zero. Clearly, this amounts to saying that the strategy $\tilde{H}$ yields an unbounded increasing profit, thus contradicting the assumption that the NUIP condition holds. Hence, it must be $H \cdot A \equiv 0$. Due to Theorem 2.3 of Delbaen & Schachermayer (1995b), this implies that there exists an $\mathbb{R}^d$-valued predictable process $\lambda = (\lambda_t)_{0 \leq t \leq T}$ such that the identity $a = c\lambda$ holds $P \otimes B$-a.e. □

Clearly, the existence of an $\mathbb{R}^d$-valued predictable process $\lambda = (\lambda_t)_{0 \leq t \leq T}$ such that $a = c\lambda$ holds $P \otimes B$-a.e. implies that $dA \ll d\langle M, M\rangle$. Indeed, we have $A_t^i = \sum_{j=1}^d \int_0^t \lambda_u^j d\langle M^i, M^j\rangle_u$ for all $t \in [0,T]$ and $i = 1, \dots, d$. The condition $dA \ll d\langle M, M\rangle$ is known in the literature as the *weak structure condition* and the process λ is usually referred to as the *instantaneous market price of risk* (see e.g. Hulley & Schweizer (2010), Section 3). We want to point out that results similar to Theorem 4.3.2 have already appeared in the literature, albeit under stronger assumptions. In particular, Theorem 3.5 of Delbaen & Schachermayer (1995b) shows that the weak structure condition $dA \ll d\langle M, M\rangle$ holds under the classical *No Arbitrage (NA)* condition $\big\{G_T(H) : H \in \mathcal{A}\big\} \cap L_+^0 = \{0\}$, where we denote by L_+^0 the set of all non-negative $\mathcal{F}$-measurable random variables. However, the NA condition is strictly stronger than the NUIP condition, as we shall argue at the end of Section 4.3.3. Somewhat more generally, Kabanov & Stricker (2005) prove that the weak structure condition holds under the assumption $\big\{G_T(H) : H \in \mathcal{A}_0\big\} \cap L_+^0 = \{0\}$, which is also strictly stronger than the NUIP condition (see Definition 4.3.7 and Lemma 4.3.8 in Section 4.3.2). Our Theorem 4.3.2 shows that the weak structure condition is *equivalent* to the NUIP condition[1], which therefore represents the *minimal* no-arbitrage condition for any reasonable financial market model. To this effect, compare also Section 3.4 of Karatzas & Kardaras (2007). Note also that, in the special case where S is modeled as an Itô process, Theorem 4.3.2 allows to recover the result of Proposition 2.4 of Fontana & Runggaldier (2011).

Remark 4.3.3 (**Extension to discontinuous semimartingales**). Let us briefly consider the case where the semimartingale S is assumed to be not necessarily continuous but only locally square-integrable, in the sense of Definition II.2.27 of Jacod & Shiryaev (2003). Note that $\langle M, M\rangle$ is still well-defined, since the local martingale part M in the canonical decomposition of S is a locally square-integrable local martingale. In this case, it has been shown in Protter & Shimbo (2008) that the weak structure condition $dA \ll d\langle M, M\rangle$ holds if there exists a probability measure $Q \sim P$ on $(\Omega, \mathcal{F})$ such that S is a local Q-martingale. More generally, we can prove that the weak structure condition holds under a much weaker assumption. In fact, it can be shown that the result of Theorem 4.3.2 still holds true for S locally square-integrable, provided we add in Definition 4.3.1 the requirement that the process $G(H)$ associated to the strategy $H \in \mathcal{A}_0$ generating an unbounded increasing profit is predictable. Indeed, the continuity of S is only used in the first part

[1]More precisely, as can be seen by inspecting the proof of Theorem 4.3.2, the weak structure condition is also equivalent to the absence of a strategy $H \in \mathcal{A}_0$ such that the gains from trading process $G(H)$ is of finite variation (not necessarily increasing) and satisfies $P\big(G_T(H) > 0\big) > 0$.

of the proof of Theorem 4.3.2 to invoke Corollary 1 to Theorem II.27 of Protter (2005)[2]. In the more general case where S is discontinuous but $G(H)$ is predictable and of finite variation, we can instead apply Theorem III.15 of Protter (2005) to get $H \cdot M \equiv 0$. A similar result is also stated (without proof) in Theorem 2.2 of Strasser (2005).

Example 4.3.4. We now give an explicit example of a model allowing for unbounded increasing profits. This example uses the concept of *local time* (at the level 0) of a continuous local martingale (see e.g. Protter (2005), Section IV.7, and Revuz & Yor (1999), Chapter VI). Let $N = (N_t)_{0 \le t \le T}$ be a real-valued continuous local martingale on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ and define the discounted price process S of a single risky asset as $S := |N|$. The Meyer-Tanaka formula (see Protter (2005), Corollary 3 to Theorem IV.70) gives the following representation, for all $t \in [0, T]$:

$$S_t = |N_0| + \int_0^t \operatorname{sign}(N_u)\, dN_u + L_t^0 \tag{4.3}$$

where the process $L^0 = (L_t^0)_{0 \le t \le T}$ is the local time at the level 0 of the continuous local martingale N. Equation (4.3) gives the canonical decomposition of the continuous semimartingale S into a continuous local martingale and a continuous finite variation predictable process. Indeed, the stochastic integral process $\int \operatorname{sign}(N)\, dN$ is a continuous local martingale with initial value 0 and the local time process L^0 is continuous non-decreasing and adapted, hence predictable and of finite variation. Hence, using the notations introduced at the beginning of Section 4.2, we get the canonical decomposition $S = S_0 + A + M$, with $A = L^0$ and $M = \int \operatorname{sign}(N)\, dN$. We now show that we cannot have $dA \ll d\langle M, M\rangle$, where $\langle M, M\rangle = \left\langle \int \operatorname{sign}(N)\, dN \right\rangle = \int \left(\operatorname{sign}(N)\right)^2 d\langle N\rangle = \langle N\rangle$. In fact, Theorem IV.69 of Protter (2005) shows that, for almost all $\omega \in \Omega$, the measure (in t) $dL_t^0(\omega)$ is carried by the set $\{t : N_t(\omega) = 0\}$. However, for almost all $\omega \in \Omega$, the set $\{t : N_t(\omega) = 0\}$ has zero measure with respect to $d\langle N\rangle_t(\omega)$. In fact, due to the occupation time formula (see Protter (2005), Corollary 1 to Theorem IV.70), we have $\int_0^t \mathbf{1}_{\{N_u=0\}} d\langle N\rangle_u = \int_{-\infty}^{\infty} L_t^x \mathbf{1}_{\{x=0\}} dx = 0$ P-a.s. for all $t \in [0, T]$. We have thus shown that L^0 induces a measure which is singular with respect to the measure induced by $\langle N\rangle$, thus showing that there cannot exist a predictable process $\lambda = (\lambda_t)_{0 \le t \le T}$ such that $dL_t^0 = \lambda_t d\langle N\rangle_t$. Theorem 4.3.2 then implies that the NUIP condition fails to hold.

In the framework of the above example, we can also explicitly construct a trading strategy yielding an unbounded increasing profit. For simplicity, let us suppose that $N_0 = 0$ P-a.s. Let us define the process $H = (H_t)_{0 \le t \le T}$ by $H_0 := 0$ and $H_t := \mathbf{1}_{\{N_t=0\}}$, for all $t \in (0, T]$. Clearly, H is a bounded predictable process and so $H \in L(S)$. Furthermore, we have $(H \cdot M)_t = \int_0^t H_u \operatorname{sign}(N_u)\, dN_u = 0$ P-a.s. for all $t \in [0, T]$, due to the fact that:

$$\left\langle \int H \operatorname{sign}(N)\, dN \right\rangle_T = \int_0^T H_u^2 \left(\operatorname{sign}(N_u)\right)^2 d\langle N\rangle_u = \int_{0+}^T \mathbf{1}_{\{N_u=0\}} d\langle N\rangle_u = 0 \qquad P\text{-a.s.}$$

[2]Observe that, if the gains from trading process $G(H) = H \cdot S$ associated to a strategy $H \in \mathcal{A}_0$ generating an unbounded increasing profit is predictable, then we can always write $H \cdot S = H \cdot M + H \cdot A$, due to Proposition 2 of Jacod (1980). In fact, the stochastic integral process $G(H) = H \cdot S$ is a special semimartingale, being predictable and of finite variation, and S is also special, being a locally square-integrable semimartingale.

Note also that $\int H dL^0 = L^0$, since the measure $dL_t^0(\omega)$ is carried by the set $\{t : N_t(\omega) = 0\}$ for almost all $\omega \in \Omega$. Hence, for all $t \in [0, T]$:

$$(H \cdot S)_t = \int_0^t H_u \operatorname{sign}(N_u)\, dN_u + \int_0^t H_u dL_u^0 = L_t^0 = \sup_{s \le t}\left(-\int_0^s \operatorname{sign}(N_u)\, dN_u\right)$$

where the last equality follows from Skorohod's Lemma (see e.g. Protter (2005), Exercise IV.30). This shows that the gains from trading process $G(H) = H \cdot S$ starts from 0 and is P-a.s. non-decreasing. In particular, this also implies $H \in \mathcal{A}_0$. Finally, if we assume that the local martingale N is not trivial, we also have $P\big(G_T(H) > 0\big) > 0$. Indeed, suppose on the contrary that $P\big(G_T(H) > 0\big) = 0$, so that $\sup_{s \le T}\left(-\int_0^s \operatorname{sign}(N_u)\, dN_u\right) = 0$ P-a.s. and, hence, $\int_0^t \operatorname{sign}(N_u)\, dN_u \ge 0$ P-a.s. for all $t \in [0, T]$. Due to Fatou's Lemma, this implies that the process $\int \operatorname{sign}(N)\, dN$ is a non-negative supermartingale, being a non-negative continuous local martingale. Since $\int \operatorname{sign}(N)\, dN$ has initial value zero, the supermartingale property gives $\int \operatorname{sign}(N)\, dN \equiv 0$, which in turn implies that $\langle N \rangle = \left\langle \int \operatorname{sign}(N)\, dN \right\rangle \equiv 0$, thus contradicting the assumption that the continuous local martingale N is not trivial.

Remark 4.3.5. An interesting interpretation of the arbitrage opportunities arising with local times can be found in Jarrow & Protter (2005), where the authors show that the existence of large traders (whose orders affect market prices) can introduce "hidden" arbitrage opportunities for the small traders, who act as price-takers. These arbitrage opportunities are "hidden" since they occur on time sets of Lebesgue measure zero, being related to the local time of Brownian motion.

4.3.2 *No Immediate Arbitrage Opportunity*

In this Section we shall be concerned with another notion of no-arbitrage, slightly stronger than the NUIP condition studied in Section 4.3.1 but still weaker than the classical *No Arbitrage (NA)* and *No Free Lunch with Vanishing Risk (NFLVR)* conditions. The following Definition has appeared for the first time in Delbaen & Schachermayer (1995b).

Definition 4.3.6. *Let $H \in \mathcal{A}_0$. We say that the strategy H generates an* immediate arbitrage opportunity *if there exists a stopping time τ such that $P(\tau < T) > 0$ and $H = H\mathbf{1}_{]\!]\tau,T]\!]}$ and $G_t(H) = (H \cdot S)_t > 0$ P-a.s. for all $t \in (\tau, T]$. If there exists no such $H \in \mathcal{A}_0$ we say that the* No Immediate Arbitrage Opportunity (NIAO) *condition holds.*

Suppose that $H \in \mathcal{A}_0$ generates an immediate arbitrage opportunity. Then, on the set $\{\tau < T\}$, we have $(H \cdot S)_{\tau+t} > 0$ P-a.s. for all $t \in (0, T - \tau]$, meaning that we can realize an arbitrage opportunity immediately after the stopping time τ has occurred. This explains the terminology *immediate* arbitrage opportunity. It is easy to see that the existence of an unbounded increasing profit, in the sense of Definition 4.3.1, implies the existence of an immediate arbitrage opportunity. Indeed, suppose that the strategy $H \in \mathcal{A}_0$ yields an unbounded increasing profit and let $\tau := \inf\big\{t > 0 : G_t(H) > 0\big\} \wedge T$. Then it must be $P(\tau < T) > 0$, since we would otherwise have $\tau = T$ P-a.s. and, by continuity, $G_T(H) = 0$ P-a.s., thus contradicting the assumption that $P\big(G_T(H) > 0\big) > 0$. Since the gains from trading process $G(H)$ is P-a.s. increasing, it follows that the strategy $\tilde{H} := H\mathbf{1}_{]\![\tau,T]\!]} \in \mathcal{A}_0$ realizes an immediate arbitrage opportunity. An example of a model which satisfies the NUIP condition but which allows for immediate arbitrage opportunities will be presented at the end of this Section, thus showing that the NIAO condition is strictly stronger than the NUIP condition.

Let us also give the following Definition, which formalizes an economically sound notion of arbitrage opportunity.

Definition 4.3.7. *Let $H \in \mathcal{A}_0$. We say that the strategy H generates a* strong arbitrage opportunity *if $P\big(G_T(H) > 0\big) > 0$. If there exists no such $H \in \mathcal{A}_0$, i.e. $\big\{G_T(H) : H \in \mathcal{A}_0\big\} \cap L^0_+ = \{0\}$, we say that the* No Strong Arbitrage (NSA) *condition holds.*

The NSA condition introduced in Definition 4.3.7 corresponds to the NA^+ property considered in Strasser (2005). Intuitively, a strong arbitrage opportunity consists in a trading strategy which does not require any initial capital nor any amount of credit at intermediate times and leads (with non-zero probability) to a strictly positive final wealth. Of course, this sort of strategy should be banned from any reasonable financial market model. We have the following simple Lemma.

Lemma 4.3.8. *The NIAO condition and the NSA condition are equivalent.*

Proof. By following the arguments used in the first part of the proof of Lemma 3.1 in Delbaen & Schachermayer (1995b), it can be shown that a strong arbitrage opportunity implies the existence of an immediate arbitrage opportunity. Hence, NIAO implies NSA. Conversely, it is easy to see directly from Definitions 4.3.6 and 4.3.7 that an immediate arbitrage opportunity is also a strong arbitrage opportunity. Hence, NSA implies NIAO. □

Remark 4.3.9 (On the NSA condition). Our definition of strong arbitrage opportunity is similar to the notion of *arbitrage opportunity* adopted in Loewenstein & Willard (2000). In particular, in the context of a complete diffusion-based financial market model, Theorem 1 of Loewenstein & Willard (2000) shows that the absence of arbitrage opportunities should be regarded as a minimal condition, in the absence of which there is no solution to any utility maximization problem. The NSA condition has been also adopted as a minimal condition in Christensen & Larsen (2007), in lieu of the stronger classical *No Arbitrage (NA)* condition $\big\{G_T(H) : H \in \mathcal{A}\big\} \cap L^0_+ = \{0\}$. Furthermore, the notion of strong arbitrage opportunity corresponds to the notion of arbitrage adopted in the context of the *Benchmark Approach*, see e.g. Section 10.3 of Platen & Heath (2006) and Section 7 of Platen (2009). However, we want to point out that typical applications of the benchmark approach require stronger assumptions than the mere absence of strong arbitrage opportunities (or, equivalently, immediate arbitrage opportunities), namely the existence of the *growth-optimal portfolio*. In Karatzas & Kardaras (2007), the authors show that the existence of a non-exploding growth-optimal portfolio is equivalent to the *No Unbounded Profit with Bounded Risk (NUPBR)* condition. As we are going to show in Section 4.3.3, the NUPBR condition is strictly stronger than the NSA condition. This means that, in the context of the benchmark approach, not only strong arbitrage opportunities but also slightly weaker forms of arbitrage are ruled out from the market.

As we have seen, the NIAO condition (or, equivalently, the NSA condition) is stronger than the NUIP condition. Theorem 4.3.2 shows that the NUIP condition is equivalent to the existence of an $\mathbb{R}^d$-valued predictable process $\lambda = (\lambda_t)_{0 \leq t \leq T}$ such that $a = c\lambda$ holds $P \otimes B$-a.e., where the processes a, c and B are as in (4.1). Hence, it seems natural to conjecture that the NIAO condition is equivalent to the existence of a process λ as above satisfying some additional integrability property. This conjecture is correct, as we are going to show in the following Theorem. As a preliminary, let us define the *mean-variance tradeoff process* $\widehat{K} = \big(\widehat{K}_t\big)_{0 \leq t \leq T}$ as follows, for $t \in [0, T]$:

$$\widehat{K}_t := \int_0^t \lambda'_u d\langle M, M\rangle_u \lambda_u = \sum_{i,j=1}^d \int_0^t \lambda^i_u c^{ij}_u \lambda^j_u dB_u \tag{4.4}$$

where the $\mathbb{R}^d$-valued process $\lambda = (\lambda_t)_{0 \leq t \leq T}$ is as in Theorem 4.3.2. Let also $\widehat{K}^t_s := \widehat{K}_t - \widehat{K}_s$, for all $s, t \in [0, T]$ with $s \leq t$. As in Levental & Skorohod (1995) and Strasser (2005), let us also define the following stopping time:

$$\alpha := \inf\Big\{t > 0 : \widehat{K}^{t+h}_t = \infty, \forall h \in (0, T - t]\Big\} \tag{4.5}$$

with the usual convention $\inf \emptyset = \infty$. The result of the following Theorem has been already obtained in Strasser (2005), albeit with a slightly different proof.

Theorem 4.3.10. *The NIAO condition holds if and only if there exists an $\mathbb{R}^d$-valued predictable process $\lambda = (\lambda_t)_{0 \leq t \leq T}$ such that $a = c\lambda$ holds $P \otimes B$-a.e. and the stopping time α satisfies $\alpha = \infty$ P-a.s., meaning that the process $\widehat{K}$ does not jump to infinity.*

Proof. We have already seen that the NIAO condition implies the NUIP condition. Hence, the existence of an $\mathbb{R}^d$-valued predictable process $\lambda = (\lambda_t)_{0 \leq t \leq T}$ such that $a = c\lambda$ holds $P \otimes B$-a.e. follows from Theorem 4.3.2. The fact that the NIAO condition implies $\alpha = \infty$ P-a.s. is shown in Theorem 3.6 of Delbaen & Schachermayer (1995b) (compare also with Kabanov & Stricker (2005), Section 3, and Strasser (2005), Theorem 3.5).

Conversely, suppose that there exists an $\mathbb{R}^d$-valued predictable process $\lambda = (\lambda_t)_{0 \leq t \leq T}$ such that $a = c\lambda$ holds $P \otimes B$-a.e. and $\alpha = \infty$ P-a.s. and let the strategy $H \in \mathcal{A}_0$ yield an immediate arbitrage opportunity with respect to a stopping time τ with $P(\tau < T) > 0$. By assumption, since $\alpha = \infty$ P-a.s., we have $P\Big(\widehat{K}^{\tau+h}_\tau = \infty, \forall h \in (0, T - \tau]\Big) = 0$. For every $n \in \mathbb{N}$, define the stopping time $\rho_n := \inf\big\{t > \tau : \widehat{K}^t_\tau \geq n\big\} \wedge T$. Since $\widehat{K}$ is continuous and does not jump to infinity, it is clear that $\rho_n > \tau$ P-a.s. on the set $\{\tau < T\}$, for all $n \in \mathbb{N}$. Let us define the process $\lambda^{\tau,n} := \lambda \mathbf{1}_{\rrbracket \tau, \rho_n \rrbracket}$, for every $n \in \mathbb{N}$. Then, on the set $\{\tau < T\}$:

$$\int_0^T \left(\lambda^{\tau,n}_t\right)' d\langle M, M\rangle_t \lambda^{\tau,n}_t = \int_0^T \mathbf{1}_{\rrbracket \tau, \rho_n \rrbracket} \lambda'_t d\langle M, M\rangle_t \lambda_t = \widehat{K}^{\rho_n}_\tau \leq n \qquad P\text{-a.s.}$$

This implies that $\lambda^{\tau,n} \in L^2(M)$, for all $n \in \mathbb{N}$, and we can meaningfully define the stochastic exponential $\widehat{Z}^{\tau,n} := \mathcal{E}\left(-\lambda^{\tau,n} \cdot M\right)$. The process $\widehat{Z}^{\tau,n}$ is a strictly positive continuous local martingale (due to Novikov's condition it is also a uniformly integrable martingale, see e.g. Protter (2005), Theorem III.45). It is also obvious that $\widehat{Z}^{\tau,n} \equiv 1$ on the stochastic interval $[\![0,\tau]\!]$ and $\widehat{Z}^{\tau,n} \equiv \widehat{Z}^{\tau,n}_{\rho_n}$ on the stochastic interval $]\!]\rho_n, T]\!]$. Let us now compute the stochastic differential of the product $\widehat{Z}^{\tau,n}\,(H\cdot S)^{\rho_n}$, where $(H\cdot S)^{\rho_n}$ denotes the stochastic integral process $H\cdot S$ stopped at ρ_n:

$$\begin{aligned} d\big(\widehat{Z}^{\tau,n}_t\,(H\cdot S)^{\rho_n}_t\big) &= \widehat{Z}^{\tau,n}_t d\,(H\cdot S)^{\rho_n}_t + (H\cdot S)^{\rho_n}_t\, d\widehat{Z}^{\tau,n}_t + d\big[\widehat{Z}^{\tau,n},(H\cdot S)^{\rho_n}\big]_t \\ &= \widehat{Z}^{\tau,n}_t d\,(H\cdot M)^{\rho_n}_t + \widehat{Z}^{\tau,n}_t d\,(H\cdot A)^{\rho_n}_t + (H\cdot S)^{\rho_n}_t\, d\widehat{Z}^{\tau,n}_t - \widehat{Z}^{\tau,n}_t H'_t d\langle M,M\rangle_t \lambda^{\tau,n}_t \\ &= \widehat{Z}^{\tau,n}_t d\,(H\cdot M)^{\rho_n}_t + (H\cdot S)^{\rho_n}_t\, d\widehat{Z}^{\tau,n}_t + \widehat{Z}^{\tau,n}_t H'_t\Big(dA^{\rho_n}_t - d\langle M,M\rangle_t \lambda^{\tau,n}_t\Big) \\ &= \widehat{Z}^{\tau,n}_t d\,(H\cdot M)^{\rho_n}_t + (H\cdot S)^{\rho_n}_t\, d\widehat{Z}^{\tau,n}_t \end{aligned} \tag{4.6}$$

where the first equality follows from the integration by parts formula (see e.g. Protter (2005), Corollary 2 to Theorem II.22), the third equality uses the fact that $\widehat{Z}^{\tau,n}_t d\,(H\cdot A)^{\rho_n}_t = \widehat{Z}^{\tau,n}_t H'_t dA^{\rho_n}_t$, since we have $\widehat{Z}^{\tau,n} \in L(H\cdot A)$ (being adapted and continuous, hence predictable and locally bounded) and $H \in L(S)$, and, finally, the last equality is due to the fact that $dA = d\langle M,M\rangle\lambda$ and $H = H\mathbf{1}_{]\!]\tau,T]\!]}$. Due to Theorem IV.29 of Protter (2005), equation (4.6) shows that the product $\widehat{Z}^{\tau,n}\,(H\cdot S)^{\rho_n}$ is a non-negative local martingale and, due to Fatou's Lemma, also a supermartingale, for all $n \in \mathbb{N}$. Since $\widehat{Z}^{\tau,n}_0\,(H\cdot S)^{\rho_n}_0 = 0$, the supermartingale property implies that $\widehat{Z}^{\tau,n}_t\,(H\cdot S)^{\rho_n}_t = 0$ for all $t \in [0,T]$ P-a.s. Recall that, on the set $\{\tau < T\}$, we have $\rho_n > \tau$ P-a.s. and so $\widehat{Z}^{\tau,n} > 0$ P-a.s., for all $n \in \mathbb{N}$. This implies that $(H\cdot S)^{\rho_n} \equiv 0$ on the set $\{\tau < T\}$, for all $n \in \mathbb{N}$. Hence, we can conclude that $H\cdot S \equiv 0$ on $\bigcup_{n\in\mathbb{N}}]\!]\tau,\rho_n]\!]$. Since $P(\tau < T) > 0$, this contradicts the fact that $(H\cdot S)_t > 0$ P-a.s. for all $t \in (\tau, T]$, thus showing that there cannot exist an immediate arbitrage opportunity. □

In particular, it is important to note that Theorem 4.3.10 shows that we can check whether a financial market model allows for immediate arbitrage opportunities by looking directly at the characteristics of the process S representing the discounted price of the risky assets.

Let us now formalize in the following Definition the concept of *(weak) martingale deflator*, which plays the role of a weaker counterpart to the classical concept of martingale measure (see the last part of Section 4.3.3 for more details on this point) and bears a close similarity to the notion of *martingale density* introduced in Schweizer (1992),(1995).

Definition 4.3.11. *Let $Z = (Z_t)_{0\le t\le T}$ be a non-negative local martingale with $Z_0 = 1$. We say that Z is a* weak martingale deflator *if the product ZS^i is a local martingale, for all $i = 1,\ldots,d$. If Z satisfies in addition $Z_T > 0$ P-a.s. we say that Z is a* martingale deflator.

Remark 4.3.12. Let the process $Z = (Z_t)_{0\le t\le T}$ be a weak martingale deflator. Since Z is a non-negative local martingale, Fatou's Lemma implies that Z is also a supermartingale (and, hence, a true martingale if and only if $E[Z_T] = E[Z_0] = 1$). Furthermore, if Z is a martingale deflator, so that $Z_T > 0$ P-a.s., the minimum principle for non-negative supermartingales (see e.g. Revuz & Yor (1999), Proposition II.3.4) implies that $P\big(Z_t > 0 \text{ and } Z_{t-} > 0 \text{ for all } t \in [0,T]\big) = 1$.

The following Lemma shows the fundamental property of weak martingale deflators. For the definition and the main properties of σ-martingales we refer the reader to Section 2 of Delbaen &

Schachermayer (1998b), Section III.6e of Jacod & Shiryaev (2003), Kallsen (2004) and Section IV.9 of Protter (2005). Observe that the following Lemma does not rely on the continuity of S but holds true also in the more general context where S is a general (possibly discontinuous and non-locally bounded) semimartingale.

Lemma 4.3.13. *Let the process $Z = (Z_t)_{0\leq t\leq T}$ be a weak martingale deflator. Then, for any $H \in L(S)$, the product $Z(H \cdot S)$ is a σ-martingale. If in addition $H \in \mathcal{A}$, then the product $Z(H \cdot S)$ is a local martingale.*

Proof. Let $Z = (Z_t)_{0\leq t\leq T}$ be a weak martingale deflator and let $H \in L(S)$. Define the $\mathbb{R}^{d+1}$-valued local martingale $Y = (Y_t)_{0\leq t\leq T}$ by $Y_t := \bigl(Z_tS^1_t, \ldots, Z_tS^d_t, Z_t\bigr)'$ and let $L(Y)$ be the set of all $\mathbb{R}^{d+1}$-valued predictable Y-integrable processes, in the sense of Definition III.6.17 of Jacod & Shiryaev (2003). For all $n \in \mathbb{N}$, define also $H^n := H\mathbf{1}_{\{\|H\|\leq n\}}$. Then, using twice the integration by parts formula:

$$\begin{aligned}
Z\bigl(H(n)\cdot S\bigr) &= Z_- \cdot \bigl(H(n)\cdot S\bigr) + \bigl(H(n)\cdot S\bigr)_- \cdot Z + \bigl[Z, H(n)\cdot S\bigr] \\
&= \bigl(Z_-H(n)\bigr)\cdot S + \bigl(H(n)\cdot S\bigr)_- \cdot Z + H(n)\cdot [S,Z] \\
&= H(n)\cdot (ZS - S_-\cdot Z) + \bigl(H(n)\cdot S\bigr)_- \cdot Z \\
&= H(n)\cdot (ZS) + \Bigl(\bigl(H(n)\cdot S\bigr)_- - H(n)'S_-\Bigr)\cdot Z = K(n)\cdot Y
\end{aligned}$$

where, for every $n \in \mathbb{N}$, the $\mathbb{R}^{d+1}$-valued predictable process $K(n)$ is defined as $K(n)^i := H(n)^i$, for all $i = 1, \ldots, d$, and $K(n)^{d+1} := \bigl(H(n)\cdot S\bigr)_- - H(n)'S_-$. Clearly, we have $K(n) \in L(Y)$, since $K(n)$ is a predictable and locally bounded process, for all $n \in \mathbb{N}$. Define also the $\mathbb{R}^{d+1}$-valued predictable process K by $K^i := H^i$, for all $i = 1, \ldots, d$, and $K^{d+1} := (H\cdot S)_- - H'S_-$. Arguing as in the proof of Proposition 8 of Rheinländer & Schweizer (1997), note that $H(n)\cdot S$ converges to $H\cdot S$ in the semimartingale topology as $n \to \infty$, since $H \in L(S)$. This implies that $K(n)\cdot Y = Z\bigl(H(n)\cdot S\bigr)$ also converges in the semimartingale topology, since the multiplication with the fixed semimartingale Z is a continuous operation. Since the space $\bigl\{K\cdot Y : K \in L(Y)\bigr\}$ is closed in the semimartingale topology (see e.g. Jacod & Shiryaev (2003), Proposition III.6.26), we can conclude that $Z(H\cdot S) = \bar{K}\cdot Y$ for some $\bar{K} \in L(Y)$. But since $K(n)$ converges to K (P-a.s. uniformly in t, at least along a subsequence) as $n \to \infty$, we can conclude that $\bar{K} = K$; see Mémin (1980). This shows that $K \in L(Y)$. Since the process Y is a local martingale and $K \in L(Y)$, Theorem IV.90 of Protter (2005) implies that $Z(H\cdot S) = K\cdot Y$ is a σ-martingale. Now, to prove the second assertion of the Lemma, suppose that we also have $H \in \mathcal{A}$. Then, due to Definition 4.2.2, there exists a positive constant a such that $(H\cdot S)_t \geq -a$ P-a.s. for all $t \in [0,T]$. The process $Z(a + H\cdot S)$ is a σ-martingale, being the sum of a local martingale and a σ-martingale. Furthermore, Exercise IV.43 of Protter (2005) (see also Kallsen (2004), Proposition 3.1) implies that $Z(a + H\cdot S)$ is a local martingale and also a supermartingale, being a non-negative σ-martingale. In turn, this implies that the process $Z(H\cdot S)$ is a local martingale, being the difference of two local martingales. □

If the process $Z = (Z_t)_{0\leq t\leq T}$ is a weak martingale deflator and $H \in \mathcal{A}_1$, Lemma 4.3.13 implies that $Z(1 + H\cdot S)$ is a non-negative local martingale and, hence, a supermartingale. According to the terminology adopted in Becherer (2001), this means that the process $Z = (Z_t)_{0 \leq t \leq T}$ is a P-*supermartingale density*. Similarly, if Z is a martingale deflator, Lemma 4.3.13 implies that Z is an *equivalent supermartingale deflator* in the sense of Definition 4.9 of Karatzas & Kardaras (2007). The importance of supermartingale densities/deflators has been first recognized by Kramkov & Schachermayer (1999).

Our next goal consists in showing that the NIAO condition ensures the existence of a *weak martingale deflator*. This can already be guessed by following the arguments used in the second part of the proof of Theorem 4.3.10, but we prefer to give full details. So, let us suppose that the NIAO condition holds and define the stopping time $\tau := \inf\{t \in [0,T] : \widehat{K}_t = \infty\}$. Then, due to Theorem 4.3.10, we have $\alpha = \infty$ P-a.s. and, hence, $\tau > 0$ P-a.s. (see also Delbaen & Schachermayer (1995b), Theorem 3.6). Let the process $\lambda = (\lambda_t)_{0 \leq t \leq T}$ be as in Theorem 4.3.10. Then, on the stochastic interval $[\![0,\tau[\![$, the stochastic integral $\lambda \cdot M$ is well-defined as a continuous local martingale. This allows us to define the process $\widehat{Z} := \mathcal{E}(-\lambda \cdot M)\mathbf{1}_{[0,\tau)} = \exp\left(-\lambda \cdot M - \frac{1}{2}\int \lambda' d\langle M, M\rangle \lambda\right)\mathbf{1}_{[0,\tau)}$. On the set $\{\tau \leq T\}$, the process $\widehat{Z}$ hits zero not by a jump and we have $\{\widehat{Z}_{\tau-} = 0\} = \{\widehat{K}_\tau = \infty\}$, see also Section 4 of Kabanov & Stricker (2005). The process $\widehat{Z}$ is well-defined as a continuous local martingale (and furthermore, being non-negative, Fatou's Lemma implies that it is a supermartingale). On the stochastic interval $[\![0,\tau[\![$, the same calculations as in (4.6) allow to show the following, for any $H \in L(S)$:

$$\widehat{Z}(H \cdot S) = \left(\widehat{Z}H\right) \cdot M + (H \cdot S) \cdot \widehat{Z}$$

where we have also used the continuity of $\widehat{Z}$ and S. Since $H \in L(S) \subseteq L^2_{loc}(M)$ and $\widehat{Z}$ is adapted and continuous, hence predictable and locally bounded, we have $\widehat{Z}H \in L^2_{loc}(M)$. Similarly, we also have $H \cdot S \in L^2_{loc}(\widehat{Z})$. This shows that the process $\widehat{Z}(H \cdot S)$ is a continuous local martingale on the stochastic interval $[\![0,\tau[\![$. Since $\widehat{Z}$ remains frozen at zero after τ and $\widehat{Z}_\tau = \widehat{Z}_{\tau-}$ on $\{\tau \leq T\}$, this implies that the process $\widehat{Z}(H \cdot S)$ is a continuous local martingale on the whole interval $[0,T]$. In a similar way, using the fact that, as a consequence of Theorem 4.3.10, we have $dA = d\langle M, M\rangle\lambda$, we can easily show that $\widehat{Z}S^i$ is also a continuous local martingale, for all $i = 1, \ldots, d$. Furthermore, the process $\widehat{Z}$ is such that, for any local martingale $N = (N_t)_{0 \leq t \leq T}$ strongly orthogonal (in the sense of Jacod & Shiryaev (2003), Definition I.4.11) to the local martingale part M in the canonical decomposition of S, the product $\widehat{Z}N$ is a local martingale. This can be easily shown as follows:

$$\widehat{Z}N = \widehat{Z}_- \cdot N + N_- \cdot \widehat{Z} + \left[\widehat{Z}, N\right] = \widehat{Z} \cdot N + N_- \cdot \widehat{Z} - \widehat{Z}\lambda \cdot \langle M, N\rangle = \widehat{Z} \cdot N + N_- \cdot \widehat{Z}$$

where we have used the continuity of M and the orthogonality of M and N. Since the processes $\widehat{Z}$ and N_- are predictable and locally bounded, being adapted and left-continuous, and since $\widehat{Z}$ and N are both local martingales, Theorem IV.29 of Protter (2005) implies that $\widehat{Z}N$ is a local martingale. Summing up, we have proved the following Proposition.

Proposition 4.3.14. *Suppose that the NIAO condition holds and let* $\tau := \inf\{t \in [0,T] : \widehat{K}_t = \infty\}$. *Then the process* $\widehat{Z} := \mathcal{E}(-\lambda \cdot M)\mathbf{1}_{[0,\tau)}$ *is a weak martingale deflator. Furthermore, for any local martingale* $N = (N_t)_{0 \leq t \leq T}$ *strongly orthogonal to the local martingale part* M *in the canonical decomposition of* S*, the product* $\widehat{Z}N$ *is a local martingale.*

Remark 4.3.15 (On the minimal martingale measure). The process $\widehat{Z} := \mathcal{E}\left(-\lambda \cdot M\right)$ represents the candidate density process of the so-called *minimal martingale measure*, originally introduced by Föllmer & Schweizer (1991) and formally defined as a probability measure $\widehat{Q}$ on $(\Omega, \mathcal{F})$ such that $\widehat{Q} = P$ on $\mathcal{F}_0$ and $\widehat{Q} \sim P$, such that S is a local $\widehat{Q}$-martingale and such that $\widehat{Q}$ preserves orthogonality, in the sense that every local P-martingale which is strongly P-orthogonal to the martingale part M in the canonical decomposition of S (with respect to P) is also a local $\widehat{Q}$-martingale. Proposition 4.3.14 shows that the process $\widehat{Z}$ is a weak martingale deflator and possesses such an orthogonality-preserving property. However, the process $\widehat{Z}$ can fail to be the well-defined density process of the minimal martingale measure $\widehat{Q}$ for two reasons. First, it could be that $P\big(\widehat{Z}_T > 0\big) < 1$, so that the measure $\widehat{Q}$ on $(\Omega, \mathcal{F})$ defined by $\frac{d\widehat{Q}}{dP} := \widehat{Z}_T$ would fail to be equivalent to P, being only absolutely continuous to P. Second, $\widehat{Z}$ may fail to be a true martingale, being instead a *strict* local martingale (and a strict supermartingale), i.e. a local martingale which is not a true martingale, meaning that $E\big[\widehat{Z}_T\big] < E\big[\widehat{Z}_0\big] = 1$. In the latter case, $\widehat{Q}$ would fail to be a probability measure, since $\widehat{Q}\left(\Omega\right) = E\big[\widehat{Z}_T\big] < 1$.

Remark 4.3.16 (On the tradeability of $\widehat{Z}$). The weak martingale deflator $\widehat{Z}$ introduced in Proposition 4.3.14 has another interesting property. To see that, let us define the process $X := 1/\widehat{Z} - 1$ on the stochastic interval $[\![0, \tau[\![$. On the set $\big\{\widehat{Z}_\tau = 0\big\}$ we let $X_{\tau-} = X_\tau = \infty$ and, since after τ the process $\widehat{Z}$ remains frozen at 0, we extend X in a natural way on the stochastic interval $[\![\tau, T]\!]$ by letting $X = \infty$. Note that, as soon as the NIAO condition holds, the stochastic integrals $H \cdot M$ and $H \cdot S$ are well-defined on the stochastic interval $[\![0, \tau[\![$ and, hence, the process X satisfies the following:

$$dX_t = -\widehat{Z}_t^{-2} d\widehat{Z}_t + \widehat{Z}_t^{-3} d\big\langle \widehat{Z}\big\rangle_t = \widehat{Z}_t^{-1}\lambda_t dM_t + \widehat{Z}_t^{-1}\lambda_t' d\langle M, M\rangle_t \lambda_t = \widehat{H}_t dS_t$$

where the $\mathbb{R}^d$-valued predictable process $\widehat{H} \in L\left(S\right)$ is defined as $\widehat{H} := \widehat{Z}^{-1}\lambda$. Since $X \geq -1$ P-a.s., we also have that the strategy $\widehat{H}$ is 1-admissible, i.e. $\widehat{H} = \widehat{Z}^{-1}\lambda \in \mathcal{A}_1$. This means that, using the terminology introduced in Section 4.4 of Karatzas & Kardaras (2007), the weak martingale deflator $\widehat{Z}$ is *tradeable*, since it can be represented as the wealth process associated to a suitable 1-admissible trading strategy. Related results can also be found in Delbaen & Schachermayer (1995b) and Kabanov & Stricker (2005).

We conclude this Section by showing an example of a model which satisfies the NUIP condition but which allows for immediate arbitrage opportunities. In view of Theorem 4.3.10, the example below exhibits a model which satisfies the weak structure condition $dA \ll d\langle M, M\rangle$ but for which $P\left(\alpha < T\right) > 0$.

Example 4.3.17. This Example is a generalization of Example 3.4 of Delbaen & Schachermayer (1995b). Let $M = \left(M_t\right)_{0 \leq t \leq T}$ be a real-valued continuous local martingale with $M_0 = 0$ on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ and define the discounted price process $S = \left(S_t\right)_{0 \leq t \leq T}$ of a single risky asset as $S := M + \langle M\rangle^\gamma$, for some $\gamma \leq 1/2$. Then, due to Itô's formula:

$$dS_t = dM_t + \gamma\langle M\rangle_t^{\gamma-1} d\langle M\rangle_t$$

Of course, we see that the finite variation part $\langle M\rangle^\gamma$ in the canonical decomposition of S is absolutely continuous with respect to $\langle M\rangle$. Hence, Theorem 4.3.2 implies that the NUIP condition holds. However, for any $\varepsilon > 0$ we have that:

$$\widehat{K}_\varepsilon = \int_0^\varepsilon \left(\gamma\langle M\rangle_t^{\gamma-1}\right)^2 d\langle M\rangle_t = \gamma^2 \int_0^\varepsilon \langle M\rangle_t^{2(\gamma-1)} d\langle M\rangle_t = \begin{cases} \gamma^2 \log\big(\langle M\rangle_t\big)\Big|_0^\varepsilon & \text{if } \gamma = 1/2 \\ \frac{\gamma^2}{2\gamma-1}\langle M\rangle_t^{2\gamma-1}\Big|_0^\varepsilon & \text{if } \gamma < 1/2 \end{cases} = \infty$$

This shows that in the present Example we have $\alpha = 0$ P-a.s. Hence, due to Theorem 4.3.10, the NIAO condition fails to hold. By letting $M = W$, for W a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, P)$, and $\gamma = 1/2$, we recover the situation considered in Example 3.4 of Delbaen & Schachermayer (1995b).

4.3.3 *No Unbounded Profit with Bounded Risk*

In this Section we shall study a crucial no-arbitrage condition, named *No Unbounded Profit with Bounded Risk (NUPBR)*, slightly stronger than the NUIP and NIAO conditions discussed in Sections 4.3.1 and 4.3.2, respectively, but still weaker than the classical *No Free Lunch with Vanishing Risk (NFLVR)* condition (see later in this Section for more details on the relations between NUPBR and NFLVR). Let us start with the following Definition.

Definition 4.3.18. *Let $(X^n)_{n\in\mathbb{N}}$ be a sequence in $\{1 + G_T(H) : H \in \mathcal{A}_1\}$. We say that $(X^n)_{n\in\mathbb{N}}$ generates an* unbounded profit with bounded risk *if the collection $(X^n)_{n\in\mathbb{N}}$ is unbounded in probability, i.e. $\lim_{m\to\infty} \sup_{n\in\mathbb{N}} P(X^n > m) > 0$. If there exists no such sequence we say that the* No Unbounded Profit with Bounded Risk (NUPBR) *condition holds.*

In words, the NUPBR condition requires the set of terminal wealths generated by 1-admissible self-financing trading strategies to be bounded in probability. The NUPBR condition has been first introduced under that name in Karatzas & Kardaras (2007). However, the same condition plays also a key role in the seminal work Delbaen & Schachermayer (1994), where the authors show that it is a necessary condition for the validity of the stronger classical NFLVR condition (see also Kabanov (1997) and later in this Section for more details).

Remark 4.3.19.

1. Note that in Definition 4.3.18 there is no loss of generality in considering 1-admissible strategies only. In fact, we have $\{a + G_T(H) : H \in \mathcal{A}_a\} = a\{1 + G_T(H) : H \in \mathcal{A}_1\}$, for any $a > 0$. This implies that, for all $a > 0$, the set of all final wealths generated by a-admissible self-financing trading strategies is bounded in probability if and only if the set of all final wealths generated by 1-admissible self-financing trading strategies is bounded in probability.

2. At first sight, the content of Definition 4.3.18 may seem rather technical and the economic meaning not so clear. However, there exists an alternative characterization of the NUPBR condition in terms of the absence of *arbitrages of the first kind*, as has been recently shown

in Kardaras (2010a). Formally, an arbitrage of the first kind is defined as a non-negative $\mathcal{F}$-measurable random variable ξ with $P(\xi > 0) > 0$ such that for all $x > 0$ there exists an x-admissible strategy $H^x \in \mathcal{A}_x$ satisfying $x + G_T(H^x) \geq \xi$ P-a.s.

As we have seen in Sections 4.3.1 and 4.3.2, the NUIP and NIAO conditions exclude rather blatant forms of arbitrage, which can be seen as pathologies of a given financial market model. The NUIP and NIAO conditions can therefore be regarded as indispensable "sanity checks" for any financial market model. However, they do not guarantee the solvability of fundamental problems in financial mathematics such as portfolio optimization, valuation and hedging problems. The result of Proposition 4.19 of Karatzas & Kardaras (2007) is therefore of great interest, since it shows that the failure of the NUPBR condition precludes the solvability of any utility maximization problem. In the particular case of a complete diffusion-based financial market model, an analogous result has already been shown in Loewenstein & Willard (2000). Furthermore, as soon as the NUPBR condition holds, the benchmark approach proposed by Eckhard Platen and co-authors provides a coherent framework for dealing with valuation and hedging problems for non-negative contingent claims, see e.g. Platen (2006), Platen & Heath (2006) and Platen (2009). Under the NUPBR condition, but not requiring the full strength of the classical NFLVR condition, valuation and hedging problems are also dealt with in Fernholz & Karatzas (2009), Fontana & Runggaldier (2011) and Ruf (2011b) in the context of diffusion-based financial market models. Summing up, these observations imply that the NUPBR condition not only excludes strong forms of arbitrage from the market but also opens the doors to the solution of classical problems of mathematical finance and, hence, can be regarded as *the* fundamental no-arbitrage condition.

It is easy to show that if there is an immediate arbitrage opportunity, in the sense of Definition 4.3.6, then there is also an unbounded profit with bounded risk, meaning that the NUPBR condition is stronger than the NIAO condition. In fact, suppose that the strategy $H \in \mathcal{A}_0$ generates an immediate arbitrage opportunity with respect to a stopping time τ with $P(\tau < T) > 0$. Let $\xi := G_T(H) = \mathbf{1}_{\{\tau<T\}}(H \cdot S)_T$, so that $P(\xi \geq 0) = 1$ and $P(\xi > 0) > 0$. For all $n \in \mathbb{N}$, let us define the process $H^n := nH \in \mathcal{A}_0 \subseteq \mathcal{A}_1$ and the $\mathcal{F}$-measurable random variable $X^n := 1+(H^n \cdot S)_T = 1 + n\xi$. Clearly, the collection $(X^n)_{n\in\mathbb{N}} \subseteq \{1 + G_T(H) : H \in \mathcal{A}_1\}$ is not bounded in probability, meaning that $(X^n)_{n\in\mathbb{N}}$ generates an unbounded profit with bounded risk. Equivalently (see part 2 of Remark 4.3.19), the $\mathcal{F}$-measurable random variable ξ is an arbitrage of the first kind. In fact, for all $x > 0$, we have $x + G_T(H) = x + \xi > \xi$. An example of a model which satisfies the NIAO condition but which allows for an unbounded profit with bounded risk will be presented later in this Section, thus showing that the NUPBR condition is strictly stronger than the NIAO condition.

We now continue the study of the NUPBR condition by giving an equivalent criterion for its validity. To this effect, let us first introduce the following Definition[3], which is adapted from Christensen & Larsen (2007).

Definition 4.3.20. *Let $(H^n)_{n\in\mathbb{N}}$ be a sequence in $L(S)$ with $H^n \cdot S \geq -1/n$ for all $n \in \mathbb{N}$. We say that $(H^n)_{n\in\mathbb{N}}$ generates an* approximate arbitrage opportunity *if $G_T(H^n) \to f$ in probability*

[3]We warn the reader that in Levental & Skorohod (1995) the term *approximate arbitrage* is used with a different meaning.

as $n \to \infty$, for some $\mathcal{F}$-measurable non-negative random variable f such that $P(f > 0) > 0$.

Lemma 4.3.21. *The NUPBR condition holds if and only if there do not exist approximate arbitrage opportunities.*

Proof. It is clear that the set $\{1 + (H \cdot S)_T : H \in \mathcal{A}_1\}$ cannot be bounded in probability if there exists a sequence $(H^n)_{n\in\mathbb{N}}$ which generates an approximate arbitrage opportunity. In fact, the strategy nH^n is 1-admissible, for every $n \in \mathbb{N}$, and the collection $\{1 + n(H^n \cdot S)_T : n \in \mathbb{N}\}$ is unbounded in probability (compare also with Delbaen & Schachermayer (1994), Proposition 3.6). Conversely, suppose that the NUPBR condition fails to hold. Then there exists a sequence $(H^n)_{n\in\mathbb{N}} \subseteq \mathcal{A}_1$ such that $P\big((H^n \cdot S)_T \geq n\big) > \beta$ for all $n \in \mathbb{N}$ and for some $\beta > 0$. For every $n \in \mathbb{N}$, let $\widetilde{H}^n := \frac{1}{n}H^n$. Clearly, we have $\widetilde{H}^n \in \mathcal{A}_{1/n}$ and $P\big((\widetilde{H}^n \cdot S)_T \geq 1\big) > \beta$, for all $n \in \mathbb{N}$. Let $g_n := \frac{1}{n} + \big(\widetilde{H}^n \cdot S\big)_T \geq 0$ P-a.s., for all $n \in \mathbb{N}$. Due to Lemma A1.1 of Delbaen & Schachermayer (1994), there exists a sequence $(f_n)_{n\in\mathbb{N}}$, with $f_n \in \text{conv}\{g_n, g_{n+1}, \ldots\}$, such that $(f_n)_{n\in\mathbb{N}}$ converges P-a.s. to a non-negative $\mathcal{F}$-measurable random variable f. For all $n \in \mathbb{N}$, let K^n be the convex combination of the strategies $\big(\widetilde{H}^m\big)_{m\geq n}$ corresponding to f_n. It is easy to see that $K^n \in \mathcal{A}_{1/n}$, for all $n \in \mathbb{N}$. Furthermore, we have $G_T(K^n) = (K^n \cdot S)_T = f_n + O(n^{-1})$, so that $G_T(K^n) \to f$ P-a.s. and, hence, also in probability, as $n \to \infty$. Furthermore, due to the last part of Lemma A1.1 of Delbaen & Schachermayer (1994), we have that $P(f > 0) > 0$, thus showing that the sequence $(K^n)_{n\in\mathbb{N}}$ generates an approximate arbitrage opportunity. □

Remark 4.3.22 (On approximate arbitrage opportunities and cheap thrills). The notion of approximate arbitrage opportunity is closely related to the concept of *cheap thrill* introduced in Loewenstein & Willard (2000). Formally, a cheap thrill is defined as a sequence $(H^n)_{n\in\mathbb{N}} \subseteq L(S)$ such that $H^n \in \mathcal{A}_{x_n}$, for some $x_n \geq 0$, for all $n \in \mathbb{N}$, with $x_n \searrow 0$ as $n \to \infty$ and $x_n + G_T(H^n) \to \infty$ P-a.s. as $n \to \infty$ on some set with non-zero probability. By reasoning as in the first part of the proof of Lemma 4.3.21, it can be easily shown that the set $\{1 + G_T(H) : H \in \mathcal{A}_1\}$ cannot be bounded in probability if there exists a cheap thrill and, hence, due to Lemma 4.3.21, there exists an immediate arbitrage opportunity. Conversely, one can easily construct a cheap thrill from a sequence $(H^n)_{n\in\mathbb{N}} \subseteq L(S)$ which generates an approximate arbitrage opportunity. In fact, for all $n \in \mathbb{N}$, let $x_n := \frac{\log(n)}{n}$ and $\widetilde{H}^n := \log(n) H^n$. Then $x_n + \widetilde{H}^n \cdot S = \log(n)\big(\frac{1}{n} + H^n \cdot S\big) \geq 0$ P-a.s., since $H^n \in \mathcal{A}_{1/n}$, so that we have $\widetilde{H}^n \in \mathcal{A}_{x_n}$, for all $n \in \mathbb{N}$. Furthermore, we have $x_n \searrow 0$ as $n \to \infty$. By assumption, $G_T(H^n) = (H^n \cdot S)_T \to f$ in probability as $n \to \infty$, for some $\mathcal{F}$-measurable non-negative random variable f with $P(f > 0) > 0$. Passing to a subsequence, we can assume that the convergence takes place P-a.s. This implies $G_T\big(\widetilde{H}_n\big) = \log(n)\, G_T(H^n) \to \infty$ P-a.s. as $n \to \infty$ on some set with strictly positive probability, thus showing that there exists a cheap thrill.

Summing up, by combining part 2 of Remark 4.3.19, Lemma 4.3.21 and Remark 4.3.22, we have shown the equivalence between the following types of arbitrage opportunities:

- *unbounded profit with bounded risk*;
- *arbitrage of the first kind*;

- *approximate arbitrage*;
- *cheap thrill*.

We want to emphasize that all the above forms of arbitrage opportunities are stronger than the classical *Free Lunch with Vanishing Risk* notion adopted in the context of the classical no-arbitrage theory as formulated in Delbaen & Schachermayer (1994),(1998b),(2006).

The following Theorem gives necessary and sufficient conditions for the validity of NUPBR in the context of general financial market models based on continuous semimartingales.

Theorem 4.3.23. *The following are equivalent:*

(a) the NUPBR condition holds;

(b) there exists an $\mathbb{R}^d$-valued predictable process $\lambda = (\lambda_t)_{0\leq t\leq T}$ such that $a = c\lambda$ holds $P \otimes B$-a.e. and $\widehat{K}_T < \infty$ P-a.s., i.e. $\lambda \in L^2_{loc}(M)$;

(c) there exists a martingale deflator (in the sense of Definition 4.3.11).

Proof. We first show that *(a)* implies *(b)*. We already know that the NUPBR condition is stronger than the NIAO condition. Hence, due to Theorem 4.3.10, there exists an $\mathbb{R}^d$-valued predictable process $\lambda = (\lambda_t)_{0\leq t\leq T}$ such that $a = c\lambda$ holds $P \otimes B$-a.e. and $\alpha := \inf\{t > 0 : \widehat{K}_t^{t+h} = \infty, \forall h \in (0, T-t]\} = \infty$ P-a.s. It remains to show that $\widehat{K}_T < \infty$ P-a.s. Suppose that, on the contrary, we have $P(\tau \leq T) > 0$, where $\tau := \inf\{t \in [0,T] : \widehat{K}_t = \infty\}$, so that $P(\widehat{Z}_T = 0) = P(\widehat{K}_T = \infty) > 0$, where the process $\widehat{Z}$ is defined as in Proposition 4.3.14. Define the sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ by $\tau_n := \inf\{t \in [0,T] : \widehat{K}_t \geq n\}$, for every $n \in \mathbb{N}$. Clearly, we have $\tau_n \nearrow \tau$ P-a.s. as $n \to \infty$. For all $n \in \mathbb{N}$, define the $\mathbb{R}^d$-valued predictable process $\lambda^n := \lambda \mathbf{1}_{\rrbracket 0,\tau_n\rrbracket}$ and note that:

$$\int_0^T (\lambda_t^n)' \, d\langle M, M\rangle_t \lambda_t^n = \int_0^T \lambda_t \mathbf{1}_{\rrbracket 0,\tau_n\rrbracket} d\langle M, M\rangle_t \lambda_t = \widehat{K}_{T\wedge\tau_n} \leq n \qquad P\text{-a.s.}$$

This implies that $\lambda^n \in L^2(M)$ and, since $dA = d\langle M, M\rangle\lambda$ (due to Theorem 4.3.2), we also have $\lambda^n \in L^0(A)$, for all $n \in \mathbb{N}$. Hence, we can conclude that $\lambda^n \in L(S)$, for all $n \in \mathbb{N}$. Furthermore, $\widehat{K}_{T\wedge\tau_n} < \infty$ P-a.s. implies that the stopped process $\widehat{Z}^{\tau_n}$ satisfies $\widehat{Z}^{\tau_n} > 0$ P-a.s. and, hence, the process $(\widehat{Z}^{\tau_n})^{-1}$ is well-defined for all $n \in \mathbb{N}$. Being adapted and continuous, the process $(\widehat{Z}^{\tau_n})^{-1}$ is also predictable and locally bounded, for all $n \in \mathbb{N}$. Thus, we have that $(\widehat{Z}^{\tau_n})^{-1}\lambda^n \in L(S)$ for all $n \in \mathbb{N}$. As in Remark 4.3.16, a simple application of Itô's formula gives the following, for every $n \in \mathbb{N}$:

$$\left(\widehat{Z}^{\tau_n}\right)^{-1} - 1 = \left(\left(\widehat{Z}^{\tau_n}\right)^{-1}\lambda^n\right) \cdot S \tag{4.7}$$

Since $\widehat{Z}^{\tau_n} > 0$ P-a.s., this also gives $(\widehat{Z}^{\tau_n})^{-1}\lambda^n \in \mathcal{A}_1$, meaning that the strategy $H^n := \widehat{Z}^{-1}\lambda\mathbf{1}_{\rrbracket 0,\tau_n\rrbracket} = (\widehat{Z}^{\tau_n})^{-1}\lambda^n \in L(S)$ is 1-admissible for all $n \in \mathbb{N}$. Furthermore:

$$1 + (H^n \cdot S)_T = \frac{1}{\widehat{Z}_{T\wedge\tau_n}} \to \frac{1}{\widehat{Z}_{T\wedge\tau}} \qquad P\text{-a.s. as } n \to \infty$$

Since $\widehat{Z}_{T\wedge\tau} = 0$ on the set $\{\widehat{K}_T = \infty\}$, which is supposed to have strictly positive probability, this shows that the collection $\{1 + G_T(H^n) : n \in \mathbb{N}\}$ is not bounded in probability, thus contradicting the assumption that NUPBR holds.

Let us now show that *(b)* implies *(c)*. If $\lambda \in L^2_{loc}(M)$ the stochastic integral $\lambda \cdot M$ is well-defined as a continuous local martingale. Moreover, the stochastic exponential $\widehat{Z} := \mathcal{E}(-\lambda \cdot M)$ is well-defined on the whole interval $[0, T]$ as a P-a.s. strictly positive continuous local martingale with $\widehat{Z}_0 = 1$. Similarly to (4.6), the product rule, together with the fact that $dA = d\langle M, M\rangle\lambda$, allows to show that, for all $i = 1, \ldots, d$:

$$d\big(\widehat{Z}_t S^i_t\big) = \widehat{Z}_t dS^i_t + S^i_t d\widehat{Z}_t + d\big[S^i, \widehat{Z}\big]_t = \widehat{Z}_t dM^i_t + S^i_t d\widehat{Z}_t + \widehat{Z}_t\big(dA^i_t - d\langle M^i, M\rangle_t \lambda_t\big) = \widehat{Z}_t dM^i_t + S^i_t d\widehat{Z}_t$$

where we have also used the continuity of $\widehat{Z}$ and S^i. This shows that $\widehat{Z}S^i$ is a local martingale for all $i = 1, \ldots, d$, meaning that $\widehat{Z}$ is a martingale deflator.

Finally, it remains to show that *(c)* implies *(a)*. Let the process $Z = (Z_t)_{0\leq t\leq T}$ be a martingale deflator and let $H \in \mathcal{A}_1$. Then, due to Lemma 4.3.13, the product $Z\big(1 + (H \cdot S)\big)$ is a non-negative continuous local-martingale and, hence, also a supermartingale, so that $E\left[Z_T\big(1 + (H \cdot S)_T\big)\right] \leq E\left[Z_0\big(1 + (H \cdot S)_0\big)\right] = 1$, for all $H \in \mathcal{A}_1$. This shows that the set $\{Z_T\big(1 + (H \cdot S)_T\big) : H \in \mathcal{A}_1\}$ is bounded in $L^1(P)$ and, therefore, it is also bounded in probability. Since the multiplication by the finite random variable Z_T does not affect the boundedness in probability, this implies that the set $\{1 + G_T(H) : H \in \mathcal{A}_1\}$ is bounded in probability, meaning that the NUPBR condition holds. □

We want to point out that results analogous to Theorem 4.3.23 have already been obtained in Section 4 of Kardaras (2010a) and Section 3 of Hulley & Schweizer (2010), always in the context of general financial market models driven by continuous sermimartingales. However, our proof of Theorem 4.3.23 emphasizes the role of the process $\widehat{Z} = \mathcal{E}(-\lambda \cdot M)$ as the natural martingale deflator. Furthermore, our proof of Theorem 4.3.23 highlights the fact that the process $\widehat{Z}$ is tradeable, in the sense of Remark 4.3.16, and, hence, the event $\{\widehat{Z}_T = 0\}$ corresponds to the explosion of the final wealth generated by a suitable 1-admissible self-financing trading strategy. For a related discussion, see also Section 6 of Christensen & Larsen (2007).

Note that, due to the equivalence between conditions *(a)* and *(b)* in Theorem 4.3.23, as soon as the NUPBR condition holds we have $\int_0^T \lambda'_t d\langle M, M\rangle\lambda_t = \widehat{K}_T < \infty$ P-a.s. and so $\widehat{Z}_T = \mathcal{E}(-\lambda \cdot M)_T > 0$ P-a.s. Due to the supermartingale property, this also implies $P\big(\widehat{Z}_t > 0 \text{ for all } t \in [0, T]\big) = 1$. If compared with the previous results concerning the NIAO condition (see Remark 4.3.15), this implies that the process $\widehat{Z} = \big(\widehat{Z}_t\big)_{0\leq t\leq T}$ can fail to be the well-defined density process of a probability measure Q on $(\Omega, \mathcal{F})$ with $Q \sim P$ and such that S is a local Q-martingale for only one reason: $\widehat{Z}$ can fail to be a true martingale, being instead a strict local martingale. In the latter case we would have $E\big[\widehat{Z}_T\big] < E\big[\widehat{Z}_0\big] = 1$ and so letting $\frac{d\widehat{Q}}{dP} := \widehat{Z}_T$ would not define a probability measure on $(\Omega, \mathcal{F})$. In the recent paper Kardaras (2010a) it is shown that the existence of a martingale deflator is equivalent to the existence of a finitely additive measure Q on $(\Omega, \mathcal{F})$, weakly equivalent to P and only locally countably additive (hence, Q is not a true probability measure), under which the discounted price process S of the risky assets has a sort of local martingale behavior (see also Section 5 of Cassese (2005) for some related results). The following Proposition shows the general structure of all martingale deflators. Note that a martingale deflator does not necessarily have to be continuous.

Proposition 4.3.24. *Suppose that the NUPBR condition holds. Then a process $Z = (Z_t)_{0 \leq t \leq T}$ is a martingale deflator, in the sense of Definition 4.3.11, if and only if it can be written as follows:*

$$Z = \mathcal{E}\left(-\lambda \cdot M + N\right) = \widehat{Z}\,\mathcal{E}\left(N\right) \tag{4.8}$$

for some local martingale $N = (N_t)_{0 \leq t \leq T}$ strongly orthogonal to M with $N_0 = 0$ and $\Delta N > -1$ P-a.s.

Proof. Let $Z = (Z_t)_{0 \leq t \leq T}$ be a martingale deflator. Due to Remark 4.3.12, we know that Z is a local martingale with $P\big(Z_t > 0 \text{ and } Z_{t-} > 0 \text{ for all } t \in [0,T]\big) = 1$. Since the process Z_- is adapted and left-continuous, it is also predictable and locally bounded. These observations, together with Theorem IV.29 of Protter (2005), imply that the stochastic integral process $L := Z_-^{-1} \cdot Z$ is well-defined as a local martingale with $L_0 = 0$. Clearly, the local martingale L is the stochastic logarithm of Z, meaning that we can write $Z = \mathcal{E}\left(L\right)$, see e.g. Theorem II.8.3 of Jacod & Shiryaev (2003). Since the martingale part M in the canonical decomposition of S is continuous, L admits a Galtchouk-Kunita-Watanabe decomposition with respect to M, see Ansel & Stricker (1993). So, we can write:

$$L = \psi \cdot M + N$$

for some $\mathbb{R}^d$-valued predictable process $\psi = (\psi_t)_{0 \leq t \leq T}$ such that $\psi \in L^2_{loc}\left(M\right)$ and a local martingale $N = (N_t)_{0 \leq t \leq T}$ strongly orthogonal to M with $N_0 = 0$. Then, for all $i = 1, \ldots, d$:

$$\begin{aligned} ZS^i &= Z_- \cdot S^i + S^i_- \cdot Z + \left[Z, S^i\right] = Z_- \cdot A^i + Z_- \cdot M^i + S^i_- \cdot Z + \left[Z, M^i\right] + \left[Z, A^i\right] \\ &= Z_- \cdot \left(\int d\langle M^i, M\rangle \lambda\right) + Z_- \cdot M^i + S^i_- \cdot Z + Z_- \cdot \left[\psi \cdot M + N, M^i\right] + \left[Z, A^i\right] \\ &= Z_- \cdot \left(\int d\langle M^i, M\rangle \left(\lambda + \psi\right)\right) + Z_- \cdot M^i + S^i_- \cdot Z + Z_- \cdot \left[N, M^i\right] + \left[Z, A^i\right] \end{aligned} \tag{4.9}$$

Observe now that $Z_- \cdot M^i$ and $S^i_- \cdot Z$ are both local martingales (see e.g. Protter (2005), Theorem IV.29). Furthermore, $Z_- \cdot [N, M^i]$ is a local martingale, since $[N, M^i]$ is a local martingale due to the strong orthogonality of N and M, and $[Z, A^i]$ is also a martingale due to Yoeurp's Lemma (see e.g. Jacod & Shiryaev (2003), Proposition I.4.49). Hence, equation (4.9) implies that the process $Z_- \cdot \left(\int d\langle M^i, M\rangle \left(\lambda + \psi\right)\right)$ is a local martingale, for all $i = 1, \ldots, d$. Being a predictable process of finite variation, Theorem III.15 of Protter (2005) implies that it is also constant and P-a.s. equal to 0. Since $Z_- > 0$ P-a.s., this means that $\int d\langle M^i, M\rangle \left(\lambda + \psi\right) \equiv 0$ for all $i = 1, \ldots, d$, which in turn gives $\int_0^T \left(\lambda_t + \psi_t\right)' d\langle M, M\rangle_t \left(\lambda_t + \psi_t\right) = 0$ P-a.s. This implies that the stochastic integral $\psi \cdot M$ is indistinguishable from $-\lambda \cdot M$, thus yielding the following representation:

$$Z = \mathcal{E}\left(\psi \cdot M + N\right) = \mathcal{E}\left(-\lambda \cdot M + N\right) = \widehat{Z}\,\mathcal{E}\left(N\right)$$

where the last equality follows by using the definition of $\widehat{Z}$ and Yor's formula (see Protter (2005), Theorem II.38), since $[M, N] = \langle M, N\rangle = 0$ due to the continuity of M and the strong orthogonality of M and N. Since $Z > 0$ and $\widehat{Z} > 0$ P-a.s., we also have $\mathcal{E}\left(N\right) > 0$ P-a.s., meaning that

$\Delta N > -1$ P-a.s. Conversely, if $Z = (Z_t)_{0 \leq t \leq T}$ is a process which admits the representation (4.8), then Z is obviously a P-a.s. strictly positive local martingale with $Z_0 = 1$, due to the definition of stochastic exponential. Furthermore, analogous computations as in (4.9) allow to show that ZS^i is a local martingale for all $i = 1, \ldots, d$. Due to Definition 4.3.11, we can conclude that Z is a martingale deflator. □

Theorem 4.3.23 and Proposition 4.3.24 show that the process $\widehat{Z} = \mathcal{E}(-\lambda \cdot M)$ can be rightfully considered as the *minimal* and *natural* martingale deflator. Indeed, if $\widehat{Z}$ fails to be a well-defined martingale deflator then there cannot exist any martingale deflator.

Remark 4.3.25 (**Extension to discontinuous semimartingales**). We want to point out that the continuity of S does not play a crucial role in the proof of Proposition 4.3.24. In fact, suppose that the semimartingale S is not necessarily continuous but only locally square-integrable, in the sense of Definition II.2.27 of Jacod & Shiryaev (2003). Then, any martingale deflator $Z = (Z_t)_{0 \leq t \leq T}$ such that Z is locally square-integrable can be represented as $Z = \mathcal{E}(-\lambda \cdot M + N)$ for some locally square-integrable local martingale $N = (N_t)_{0 \leq t \leq T}$ strongly orthogonal to M with $N_0 = 0$. In fact, the process $L := Z_-^{-1} \cdot Z$ is well-defined as a locally square-integrable local martingale (see Protter (2005), Theorem IV.28) with $L_0 = 0$ and, as such, it admits a Galtchouk-Kunita-Watanabe decomposition with respect to M, since M is also a locally square-integrable local martingale, see Ansel & Stricker (1993). Then, the computations in (4.9) and the remaining part of the proof of Proposition 4.3.24 do not rely on the continuity of M. However, note that under the present assumptions we cannot write $Z = \widehat{Z}\,\mathcal{E}(N)$, since the process $[M, N]$ is not necessarily equal to zero. Analogous results have been already obtained in Theorem 1 of Schweizer (1995) and Theorem 2.2 and Corollary 2.3 of Choulli & Stricker (1996) (compare also with Christensen & Larsen (2007), Lemma 6.3). We also want to point out that the equivalence between *(a)* and *(b)* in Theorem 4.3.23 holds true also in the more general context where the discounted price process S of the risky assets is a general (possibly discontinuous and non-locally bounded) semimartingale, as has been recently shown in Kardaras (2010b) and Takaoka (2010).

Let us now exhibit a simple example of a model which satisfies the NIAO condition but for which the NUPBR condition fails to hold. A similar example can also be found in Section 3.1 of Loewenstein & Willard (2000).

Example 4.3.26. Let $W = (W_t)_{0 \leq t \leq T}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ and define the discounted price process $S = (S_t)_{0 \leq t \leq T}$ of a single risky asset as the solution to the following SDE, for some fixed $K > 0$:

$$dS_t = S_t \left(\frac{K - \log(S_t)}{T - t} + \frac{1}{2} \right) dt + S_t\, dW_t \qquad S_0 = 1 \tag{4.10}$$

The SDE (4.10) admits an unique strong solution S, which also satisfies $S_t > 0$ P-a.s. for all $t \in [0, T]$. If we let $X_t := \log(S_t)$, Itô's formula gives the following:

$$dX_t = \frac{K - X_t}{T - t} dt + dW_t \qquad X_0 = 0$$

This shows that X is a *Brownian bridge* (see e.g. Revuz & Yor (1999), Exercise IX.2.12) starting at time $t = 0$ at the level 0 and ending at time $t = T$ at the level $K > 0$. From (4.10) it is easy to see that the conditions of Theorem 4.3.10 are satisfied and, hence, there are no immediate arbitrage opportunities, thus implying that the process $\widehat{K} = \int_0^{\cdot} \left(\frac{K-\log(S_u)}{T-u} + \frac{1}{2}\right)^2 du$ does not jump to infinity. However, we have $\widehat{K}_t < \infty$ P-a.s. for all $t \in [0,T)$ but $\widehat{K}_T = \infty$ P-a.s. Theorem 4.3.23 then implies that the NUPBR condition fails to hold.

We close this Section by discussing the relations between the NUPBR condition and the classical *No Arbitrage (NA)* and *No Free Lunch with Vanishing Risk (NFLVR)* conditions, formally defined as follows.

Definition 4.3.27. *Let* $\mathcal{C} := \left(\left\{G_T(H) : H \in \mathcal{A}\right\} - L^0_+\right) \cap L^\infty$, *where* L^∞ *denotes the set of all* P*-a.s. bounded* $\mathcal{F}$*-measurable random variables. We say that the* No Arbitrage (NA) *condition holds if* $\mathcal{C} \cap L^\infty_+ = \{0\}$. *We say that the* No Free Lunch with Vanishing Risk (NFLVR) *condition holds if* $\overline{\mathcal{C}} \cap L^\infty_+ = \{0\}$, *where the bar denotes the closure in the norm topology of* L^∞.

Observe that the NA condition can be equivalently formulated as $\left\{G_T(H) : H \in \mathcal{A}\right\} \cap L^0_+ = \{0\}$. It is easy to see that neither the NA nor the NFLVR condition can hold if there exist unbounded increasing profits or immediate arbitrage opportunities. As pointed out in Karatzas & Kardaras (2007), there is no general relation between the NUPBR condition and the NA condition, in the sense that none of the two implies the other and they are not mutually exclusive. However, the fundamental results of Delbaen & Schachermayer (1994),(1998b) allow to state the following Proposition.

Proposition 4.3.28. *Let the discounted price process* S *of the risky assets be a general (possibly discontinuous and non-locally bounded) semimartingale. Then the following hold:*

(a) the NFLVR condition holds if and only if both the NUPBR and the NA conditions hold;

(b) the NFLVR condition holds if and only if there exists an Equivalent σ-Martingale Measure (EσMM) Q, *i.e. a probability measure* Q *on* $(\Omega, \mathcal{F})$ *with* $Q \sim P$ *such that* S *is a* σ*-martingale with respect to* Q*;*

(c) if S *is in addition locally bounded, the NFLVR condition holds if and only if there exists an* Equivalent Local Martingale Measure (ELMM) Q, *i.e. a probability measure* Q *on* $(\Omega, \mathcal{F})$ *with* $Q \sim P$ *such that* S *is a local martingale with respect to* Q.

Proof. Part *(a)* follows from Corollaries 3.4 and 3.8 of Delbaen & Schachermayer (1994), see also Lemma 2.2 of Kabanov (1997) and Proposition 4.2 of Karatzas & Kardaras (2007). Part *(b)* is the main result of Delbaen & Schachermayer (1998b). Finally, part *(c)* corresponds to Theorem 1.1 of Delbaen & Schachermayer (1994). □

There exists a close relationship between the density process of an equivalent local martingale measure and the concept of martingale deflator, introduced in Definition 4.3.11. More precisely, assuming that $\mathcal{F}_0$ is trivial for simplicity, the density process of any ELMM defines a martingale deflator. In fact, let Q be an ELMM for S and denote by $Z^Q = \left(Z_t^Q\right)_{0\leq t\leq T}$ its density process with respect to P, i.e. $Z_t^Q = \frac{dQ|_{\mathcal{F}_t}}{dP|_{\mathcal{F}_t}}$ for all $t \in [0,T]$. Since $Q \sim P$, we have $P\big(Z_t^Q > 0$ and $Z_{t-}^Q > 0$ for all $t \in [0,T]\big) = 1$, see e.g. Proposition VIII.1.2 of Revuz & Yor (1999). Furthermore, Exercise IV.21 of Protter (2005) implies that the process $Z^Q S^i$ is a local P-martingale, for all $i = 1,\ldots,d$, since S^i is a local Q-martingale, for all $i = 1,\ldots,d$. Conversely, if the process $Z = (Z_t)_{0\leq t\leq T}$ is a martingale deflator with $E\left[Z_T\right] = 1$, we can define an ELMM Q for S by letting $\frac{dQ}{dP} := Z_T$. If we replace in Definition 4.3.11 the term "local martingale" with the term "σ-martingale", then an analogous relationships holds true between density processes of EσMMs and martingale deflators.

Part *(a)* of Proposition 4.3.28 shows that the NFLVR condition is stronger than the NUPBR condition. In fact, it can be directly shown that if there exists an unbounded profit with bounded risk then there also exists a free lunch with vanishing risk, see e.g. Propositions 3.1 and 3.6 of Delbaen & Schachermayer (1994) and Section 2 of Kabanov (1997). Moreover, the NFLVR condition is strictly stronger than NUPBR, meaning that there are situations where the NUPBR condition is satisfied but the NA condition fails to hold and, hence (see part *(a)* of Proposition 4.3.28), also the NFLVR condition fails to hold. Instances of such situations have appeared in the context of *Stochastic Portfolio Theory*, see e.g. Fernholz & Karatzas (2009) for an overview, and in the context of the *Benchmark Approach*, see e.g. Platen (2006), Platen & Heath (2006) and Platen (2009). Classical examples of models for which the NUPBR condition holds but NFLVR fails involve Bessel processes, see e.g. Corollary 2.10 of Delbaen & Schachermayer (1995c), Example 4.6 of Karatzas & Kardaras (2007), Section 12.1 of Fernholz & Karatzas (2009) and Hulley (2010). See also Fontana & Runggaldier (2011) for a related discussion in the context of diffusion-based financial market models.

As can be seen from Theorems 4.3.2, 4.3.10 and 4.3.23, the validity of the NUIP, NIAO and NUPBR conditions can be directly ascertained by looking at the characteristics (and at their integrability properties) of the discounted price process S of the risky assets. On the contrary, there is no general way of assessing the validity of the NFLVR condition in terms of the characteristics of S alone, even when S is a continuous semimartingale, as shown by an explicit counterexample in Section 4.3.2 of Karatzas & Kardaras (2007). This fact represents a severe limitation of the classical NFLVR condition, also from an economic point of view, since it implies that there is no practical and direct way of detecting the existence of arbitrages (in the sense of Definition 4.3.27) in a given financial market model[4].

[4]For the sake of completeness, we want to point out that in some very special cases there exist characterizations of the NFLVR condition in terms of the characteristics of the discounted price process of the risky assets. Indeed, suppose that the discounted price process $S = (S_t)_{0\leq t\leq T}$ of a single risky asset satisfies the SDE $dS_t = \mu\left(S_t\right)dt + \sigma\left(S_t\right)dW_t$, $S_0 > 0$, with respect to a standard Brownian motion $W = (W_t)_{0\leq t\leq T}$ and where μ and σ are measurable functions satisfying the Engelbert-Schmidt conditions (see e.g. Karatzas & Shreve (1991), Section 5.5). In this particular case, a deterministic necessary and sufficient condition (involving only integrability properties of the functions μ and σ) for the validity of NFLVR has been recently obtained in Mijatović & Urusov (2010b).

Remark 4.3.29 (On the martingale property of $\widehat{Z}$). As we have seen, as soon as the NUPBR condition holds, the process $\widehat{Z} = \left(\widehat{Z}_t\right)_{0\leq t\leq T}$ defined as $\widehat{Z} := \mathcal{E}\left(-\lambda \cdot M\right)$ is a martingale deflator, in the sense of Definition 4.3.11. Suppose now that the NFLVR condition is also satisfied. Then, due to part *(c)* of Proposition 4.3.28, there exists an ELMM Q for S. However, it is important to note that this does not ensure that the measure $\widehat{Q}$ on $(\Omega, \mathcal{F})$ defined by $\frac{d\widehat{Q}}{dP} := \widehat{Z}_T$ is an ELMM for S. In fact, even when the NFLVR condition holds, there are situations where the process $\widehat{Z} = \left(\widehat{Z}_t\right)_{0\leq t\leq T}$ is a strict local martingale, i.e. a local martingale which is not a true martingale, as shown in Delbaen & Schachermayer (1998a). In other words, recalling Remark 4.3.15, this amounts to saying that the NFLVR condition does not ensure the existence of the minimal martingale measure.

We end this Section with the following Table, which summarizes the weak no-arbitrage conditions studied so far, together with their characterizations (see Theorems 4.3.2, 4.3.10 and 4.3.23) and their equivalent formulations (see Lemma 4.3.8, part 2 of Remark 4.3.19, Lemma 4.3.21 and Remark 4.3.22).

CONDITION	PROBABILISTIC CHARACTERIZATION	EQUIVALENT FORMULATIONS
No Unbounded Increasing Profit (NUIP)	$\exists\ \mathbb{R}^d$-valued predictable process λ such that $a = c\lambda\ P \otimes B$-a.e.	-
No Immediate Arbitrage Opportunity (NIAO)	$\exists\ \mathbb{R}^d$-valued predictable process λ such that $a = c\lambda\ P \otimes B$-a.e. and $\widehat{K}$ does not jump to infinity ($\alpha = \infty$ P-a.s.)	*No Strong Arbitrage Opportunity*
No Unbounded Profit with Bounded Risk (NUPBR)	$\exists\ \mathbb{R}^d$-valued predictable process λ such that $a = c\lambda\ P \otimes B$-a.e. and $\widehat{K}_T < \infty$ P-a.s.	*No Arbitrage of the First Kind* *No Approximate Arbitrage Opportunity* *No Cheap Thrill*

4.4 Stability properties of weak no-arbitrage conditions

This Section is devoted to the study of the robustness properties of the weak no-arbitrage conditions discussed in Section 4.3. More specifically, we shall be concerned with the behavior of weak no-arbitrage conditions with respect to changes of *numéraire*, absolutely continuous changes of the reference probability measure and, finally, changes of the reference filtration. As we shall see throughout this Section, the main message is that the weak no-arbitrage conditions considered in Section 4.3 enjoy stronger stability properties than the classical *No Arbitrage (NA)* and *No Free Lunch with Vanishing Risk (NFLVR)* conditions.

4.4.1 Changes of numéraire

We continue to work within the general setting described in Section 4.2, where the discounted price process of the risky assets is given by the $\mathbb{R}^d$-valued continuous semimartingale $S = (S_t)_{0\leq t\leq T}$, with canonical decomposition $S = S_0 + A + M$. A *numéraire* asset is a traded asset (or, more generally, a portfolio composed of traded assets) with strictly positive price and in terms of which the prices of all other assets can be expressed. More formally, we have the following general Definition.

Definition 4.4.1. *Let H be a 1-admissible self-financing strategy, i.e. $H \in \mathcal{A}_1$, and let $V := 1 + H \cdot S$ be its corresponding wealth process. The process $V = (V_t)_{0 \le t \le T}$ is said to be a* numéraire *for S if $V_t > 0$ P-a.s. for all $t \in [0,T]$.*

Recall that in Section 4.2 we have taken asset 0 as numéraire and we have expressed the prices of all assets in terms of $\widetilde{S}^0$. Of course, passing to $\widetilde{S}^0$-discounted quantities, this means that the numéraire has been implicitly assumed to be the constant 1 and, hence, the resulting price system is fully characterized by the pair $(S,1)$. Now, let V be a numéraire for S, in the sense of Definition 4.4.1. If we change the numéraire from the constant 1 to the non-trivial process V, we then need to rescale the pair $(S,1)$. Indeed, if we express all quantities in terms of the new numéraire V, the resulting price system will then be described by the pair $\left(\frac{S}{V}, \frac{1}{V}\right)$, where $\frac{S}{V}$ represents the V-discounted price process of the risky assets and $\frac{1}{V}$ represents the price process of the old numéraire in terms of the new one. In particular, note that the V-discounted price system is characterized by the $\mathbb{R}^{d+1}$-valued process $\left(\frac{S}{V}, \frac{1}{V}\right)$, unlike the original price system which can be fully characterized by the $\mathbb{R}^d$-valued process S. Now, the main question we shall answer in this Section can be formulated in the following terms. Suppose that the pair $(S,1)$ satisfies one of the weak no-arbitrage conditions discussed in Section 4.3 and let V be a numéraire for S, in the sense of Definition 4.4.1. Does the pair $\left(\frac{S}{V}, \frac{1}{V}\right)$ satisfy the same no-arbitrage condition? In other words, how are the weak no-arbitrage conditions studied in Section 4.3 affected by a change of numéraire? This question is not only of theoretical interest. In fact, it comes up naturally when one considers currency markets, where different numéraires correspond to exchange ratios between different currencies. In that context, one needs to ensure that the no-arbitrage properties of the model do not depend on the chosen currency, otherwise one would have inconsistencies within the financial market model.

Let us first introduce some notations. Suppose that $V = 1 + H^V \cdot S$ is a numéraire for S, in the sense of Definition 4.4.1, and define the $\mathbb{R}^{d+1}$-valued process $\bar{S} = \left(\bar{S}_t\right)_{0 \le t \le T}$ as $\bar{S} := \left(\frac{S}{V}, \frac{1}{V}\right)$. Furthermore, let us denote by $L\left(\bar{S}\right)$ the set of all $\mathbb{R}^{d+1}$-valued predictable $\bar{S}$-integrable processes and by $\bar{\mathcal{A}}_a$ the set of all processes $\bar{H} = \left(\bar{H}_t\right)_{0 \le t \le T}$ in $L\left(\bar{S}\right)$ such that $\left(\bar{H} \cdot \bar{S}\right)_t \ge -a$ P-a.s. for all $t \in [0,T]$. We also let $\bar{\mathcal{A}} := \bigcup_{a>0} \bar{\mathcal{A}}_a$ denote the set of all admissible trading strategies with respect to $\bar{S}$. The following Lemma shows the relation between the gains from trading processes generated by admissible trading strategies with respect to different numéraires. Note that the proof of the following Lemma does not rely on the continuity of S and, hence, the result holds true also for more general financial market models based on general (possibly discontinuous and non-locally bounded) semimartingales. As a preliminary, note that, if $V = 1 + H^V \cdot S$ is a numéraire for S (in the sense of Definition 4.4.1), then the process $1/V$ is a numéraire for $\bar{S}$. In fact, by defining the constant process $\bar{H}^V := (0, \ldots, 0, 1)' \in \mathbb{R}^{d+1}$, which trivially belongs to $L\left(\bar{S}\right)$, we can write $\frac{1}{V} = 1 + \bar{H}^V \cdot \bar{S}$ and, since $V > 0$ P-a.s., we also have $\bar{H}^V \in \bar{\mathcal{A}}_1$.

Lemma 4.4.2. *Let V be a numéraire for S, with $V = 1 + H^V \cdot S$. Then the following hold:*

$$\Big\{H \cdot S : H \in L(S)\Big\} = V\Big\{\bar{H} \cdot \bar{S} : \bar{H} \in L\left(\bar{S}\right)\Big\} \tag{4.11}$$

Furthermore, for any $a > 0$:

$$\begin{aligned} \left\{\bar{H} \cdot \bar{S} : \bar{H} \in \bar{\mathcal{A}}_a\right\} &= \frac{1}{V}\left\{\left(H - aH^V\right) \cdot S : H \in \mathcal{A}_a\right\} \\ \left\{H \cdot S : H \in \mathcal{A}_a\right\} &= V\left\{\left(\bar{H} - a\bar{H}^V\right) \cdot \bar{S} : \bar{H} \in \bar{\mathcal{A}}_a\right\} \end{aligned} \tag{4.12}$$

where the $\bar{H}^V := (0, \ldots, 0, 1)' \in \mathbb{R}^{d+1}$.

Proof. The first assertion can be proved by the same arguments used in the proof of Lemma 4.3.13. More precisely, by replacing Z with $1/V$ and Y with $\bar{S}$, the first part of the proof of Lemma 4.3.13 gives that:

$$\frac{1}{V}\left\{H \cdot S : H \in L(S)\right\} \subseteq \left\{\bar{H} \cdot \bar{S} : \bar{H} \in L\left(\bar{S}\right)\right\}$$

Conversely, let $\bar{H} \in L\left(\bar{S}\right)$. Then, using twice the integration by parts formula and the associativity of the stochastic integral (see Protter (2005), Theorem IV.21):

$$\begin{aligned} V\left(\bar{H} \cdot \bar{S}\right) &= V_- \cdot \left(\bar{H} \cdot \bar{S}\right) + \left(\bar{H} \cdot \bar{S}\right)_- \cdot V + \left[\bar{H} \cdot \bar{S}, V\right] \\ &= \left(V_- \bar{H}\right) \cdot \bar{S} + \left(\bar{H} \cdot \bar{S}\right)_- \cdot V + \bar{H} \cdot \left[\bar{S}, V\right] \\ &= \bar{H} \cdot \Big(V_- \cdot \bar{S} + \left[\bar{S}, V\right]\Big) + \left(\bar{H} \cdot \bar{S}\right)_- \cdot V \\ &= \bar{H} \cdot \left(V\bar{S} - \bar{S}_- \cdot V\right) + \left(\bar{H} \cdot \bar{S}\right)_- \cdot V \\ &= \bar{H} \cdot \left(V\bar{S}\right) + H^V\Big(\left(\bar{H} \cdot \bar{S}\right)_- - \bar{H}'\bar{S}_-\Big) \cdot S = K \cdot S \end{aligned}$$

where the $\mathbb{R}^d$-valued predictable process $K = (K_t)_{0 \leq t \leq T}$ is defined as follows:

$$K^i := \bar{H}^i + \left(H^V\right)^i \Big(\left(\bar{H} \cdot \bar{S}\right)_- - \bar{H}'\bar{S}_-\Big) \qquad i = 1, \ldots, d$$

Now, the same arguments used in the proof of Lemma 4.3.13 (compare also Rheinländer & Schweizer (1997), proof of Proposition 8), based on the notion of convergence in the semimartingale topology, allow to show that $K \in L(S)$, thus implying the reverse inclusion:

$$\left\{H \cdot S : H \in L(S)\right\} \supseteq V\left\{\bar{H} \cdot \bar{S} : \bar{H} \in L\left(\bar{S}\right)\right\}$$

For any $a > 0$, equation (4.12) can then be easily shown as follows, using the linearity of the space $L(S)$:

$$\begin{aligned} \left\{\bar{H} \cdot \bar{S} : \bar{H} \in \bar{\mathcal{A}}_a\right\} &= \frac{1}{V}\left\{H \cdot S : H \in L(S), H \cdot S \geq -aV \ P\text{-a.s.}\right\} \\ &= \frac{1}{V}\left\{H \cdot S : H \in L(S), \left(H + aH^V\right) \cdot S \geq -a \ P\text{-a.s.}\right\} \\ &= \frac{1}{V}\left\{\left(H - aH^V\right) \cdot S : H \in L(S), H \cdot S \geq -a \ P\text{-a.s.}\right\} \end{aligned}$$

Similarly, by using the representation $\frac{1}{V} = 1 + \bar{H}^V \cdot \bar{S}$, where $\bar{H}^V := (0, \ldots, 0, 1)' \in \mathbb{R}^{d+1}$, we obtain the following relation, for any $a > 0$:

$$\begin{aligned}\Big\{H \cdot S : H \in \mathcal{A}_a\Big\} &= V\Big\{\bar{H} \cdot \bar{S} : \bar{H} \in L\left(\bar{S}\right), \bar{H} \cdot \bar{S} \geq -\frac{a}{V}\ P\text{-a.s.}\Big\} \\ &= V\Big\{\bar{H} \cdot \bar{S} : \bar{H} \in L\left(\bar{S}\right), \left(\bar{H} + a\bar{H}^V\right) \cdot \bar{S} \geq -a\ P\text{-a.s.}\Big\} \\ &= V\Big\{\left(\bar{H} - a\bar{H}^V\right) \cdot \bar{S} : \bar{H} \in L\left(\bar{S}\right), \bar{H} \cdot \bar{S} \geq -a\ P\text{-a.s.}\Big\}\end{aligned}$$

□

The following Theorem deals with the behavior of the *No Unbounded Increasing Profit (NUIP)* condition under a change of numéraire. The proof is based on some simple but lengthy computations and is given in the Appendix.

Theorem 4.4.3. *Let $V = 1 + H \cdot S$ be a numéraire for S. If S satisfies the NUIP condition, then $\bar{S}$ admits the following canonical decomposition:*

$$\bar{S} = S_0 + \bar{A} + \bar{M}$$

where $\bar{M}$ is an $\mathbb{R}^{d+1}$-valued continuous local martingale with $\bar{M}_0 = 0$ and $\bar{A}$ is an $\mathbb{R}^{d+1}$-valued continuous predictable process of finite variation with $\bar{A}_0 = 0$ and such that, for all $t \in [0, T]$:

$$\bar{A}^i_t = \sum_{j=1}^{d+1} \int_0^t \bar{\lambda}^j_u d\big\langle \bar{M}^i, \bar{M}^j \big\rangle_u \qquad i = 1, \ldots, d+1 \tag{4.13}$$

where $\bar{\lambda}^j_t := V_t \lambda^j_t$, for all $j = 1, \ldots, d$, and $\bar{\lambda}^{d+1}_t := V_t\left(1 - \sum_{k=1}^d S^k_t \lambda^k_t\right)$, for all $t \in [0, T]$, and where the process $\lambda = (\lambda_t)_{0 \leq t \leq T}$ is as in Theorem 4.3.2.
Conversely, if $\bar{S}$ satisfies the NUIP condition, then the following hold for all $t \in [0, T]$:

$$A^i_t = \sum_{j=1}^{d} \int_0^t \lambda^j_u d\langle M^i, M^j \rangle_u,\ i = 1, \ldots, d \quad \text{where} \quad \lambda^j_t := \frac{\bar{\lambda}^j_t}{V_t} + \left(1 - \sum_{k=1}^{d+1} \frac{S^k_t \bar{\lambda}^k_t}{V_t}\right) \frac{H^j_t}{V_t},\ j = 1, \ldots, d$$

where A and M denote the finite variation and the local martingale part, respectively, in the canonical decomposition of the $\mathbb{R}^d$-valued semimartingale S and where the process $\bar{\lambda} = \left(\bar{\lambda}_t\right)_{0 \leq t \leq T}$ is the $\mathbb{R}^{d+1}$-valued predictable process which satisfies $d\bar{A} = d\langle \bar{M}, \bar{M} \rangle \bar{\lambda}$.
In particular, the NUIP condition holds for S if and only if the NUIP condition holds for $\bar{S} = \left(\frac{S}{V}, \frac{1}{V}\right)$.

From an economic point of view, the result of Theorem 4.4.3 is somewhat expected. In fact, as we argued in Section 4.3.1, the NUIP condition excludes only very strong (pathological) forms of arbitrage. Therefore, it is natural to conjecture that the validity of the NUIP condition should not depend on the choice of the reference asset in terms of which we express all market prices. It is worth pointing out that Theorem 4.4.3 not only shows that the NUIP condition is stable under a change of numéraire, but also gives the explicit canonical decomposition of the price process under the new numéraire. Furthermore, we can also explicitly compute the mean-variance tradeoff process of $\bar{S}$ in terms of the original price process S, as shown in the following Corollary, the proof of which is given in the Appendix.

Corollary 4.4.4. *Let $V = 1 + H \cdot S$ be a numéraire for S and suppose that S satisfies the NUIP condition. Then the mean-variance tradeoff process $\bar{K} = \left(\bar{K}_t\right)_{0\leq t\leq T}$ for $\bar{S} = \left(\frac{S}{V}, \frac{1}{V}\right)$ is explicitly given as follows, for all $t \in [0, T]$:*

$$\bar{K}_t := \sum_{i,j=1}^{d+1} \int_0^t \bar{\lambda}_u^i \bar{\lambda}_u^j d\langle \bar{M}^i, \bar{M}^j \rangle_u = \int_0^t \left(\frac{H_u}{V_u} - \lambda_u \right)' d\langle M, M \rangle_u \left(\frac{H_u}{V_u} - \lambda_u \right) \tag{4.14}$$

Having shown the stability of the NUIP condition with respect to a change of numéraire, we now move to the *No Immediate Arbitrage Opportunity (NIAO)* condition, analyzed in Section 4.3.2. The following Theorem shows that also the NIAO condition is not affected by a change of numéraire. The proof of the following Theorem uses only Definition 4.3.6 and Lemma 4.4.2 and, hence, it holds true also for general (possibly discontinuous and non-locally bounded) semimartingales.

Theorem 4.4.5. *Let $V = 1 + H \cdot S$ be a numéraire for S. Then the NIAO condition holds for S if and only if the NIAO condition holds for $\bar{S} = \left(\frac{S}{V}, \frac{1}{V}\right)$.*

Proof. We shall argue by contradiction. Suppose first that the NIAO condition holds for S but there exists a strategy $\bar{K} \in \bar{\mathcal{A}}_0$ which generates an immediate arbitrage opportunity for $\bar{S}$, with respect to a stopping time τ with $P(\tau < T) > 0$. Due to Lemma 4.4.2, there exists a strategy $K \in L(S)$ such that $K \cdot S = V\left(\bar{K} \cdot \bar{S}\right)$. Since $\bar{K} \cdot \bar{S} = 0$ on the stochastic interval $[\![0, \tau]\!]$, we can assume that $K = K\mathbf{1}_{]\!]\tau,T]\!]}$. On the set $\{\tau < T\}$, we have $(K \cdot S)_t = V_t \left(\bar{K} \cdot \bar{S}\right)_t > 0$ P-a.s. for all $t \in (\tau, T]$, since $V > 0$ P-a.s. Due to Definition 4.3.6, this shows that the strategy $K \in \mathcal{A}_0$ generates an immediate arbitrage opportunity for S, thus contradicting the assumption that NIAO holds for S. The converse implication can be shown in analogous way. Indeed, suppose that the NIAO condition holds for $\bar{S}$ and let $K \in \mathcal{A}_0$ generate an immediate arbitrage opportunity for S, with respect to a stopping time τ with $P(\tau < T) > 0$. Then, due to Lemma 4.4.1, there exists an element $\bar{K} \in L\left(\bar{S}\right)$ such that $\bar{K} \cdot \bar{S} = \frac{1}{V}(K \cdot S)$. Since $K \cdot S = 0$ on the stochastic interval $[\![0, \tau]\!]$, we can assume that $\bar{K} = \bar{K}\mathbf{1}_{]\!]\tau,T]\!]}$. On the set $\{\tau < T\}$, we have $\left(\bar{K} \cdot \bar{S}\right)_t = \frac{1}{V_t}(K \cdot S)_t > 0$ P-a.s. for all $t \in (\tau, T]$, since $V > 0$ P-a.s., which contradicts the assumption that the NIAO condition holds for $\bar{S}$. □

Remark 4.4.6 (An alternative proof to Theorem 4.4.5). We want to point out that Theorem 4.4.5 can also be proved by relying on Theorem 4.3.10 and Corollary 4.4.4. In fact, due to Corollary 4.4.4 and to the Cauchy-Schwarz inequality, the mean-variance tradeoff process $\bar{K}$ for $\bar{S}$ satisfies the following inequalities, for all $t \in [0, T]$:

$$\begin{aligned} \widehat{K}_t - \int_0^t \frac{1}{V_u^2} H_u' d\langle M, M \rangle_u H_u &\leq \int_0^t \left(\frac{H_u}{V_u} - \lambda_u \right)' d\langle M, M \rangle_u \left(\frac{H_u}{V_u} - \lambda_u \right) \\ = \bar{K}_t = \int_0^t \left(\frac{H_u}{V_u} - \lambda_u \right)' d\langle M, M \rangle_u \left(\frac{H_u}{V_u} - \lambda_u \right) &\leq \int_0^t \frac{1}{V_u^2} H_u' d\langle M, M \rangle_u H_u + \widehat{K}_t \end{aligned} \tag{4.15}$$

where the process $\widehat{K}$ is as in (4.4). Due to Definition 4.4.1, we have $H \in L(S) \subseteq L^2_{loc}(M)$, thus implying that $\int_0^t \frac{1}{V_u^2} H'_u d\langle M, M\rangle_u H_u < \infty$ P-a.s. for all $t \in [0,T]$, since the process $1/V$ is continuous and, hence, locally bounded. Thus, as can be seen from (4.15), we have that the process $\bar{K}$ jumps to infinity if and only if the process $\widehat{K}$ jumps to infinity. Due to Theorem 4.3.10, this means that the NIAO condition holds for $\bar{S}$ if and only if the NIAO condition holds for S.

Let us now study the behavior of the crucial *No Unbounded Profit with Bounded Risk (NUPBR)* condition under a change of numéraire. The proof of the following Theorem is surprisingly simple and follows easily from Lemma 4.4.2. As for Theorem 4.4.5, observe that the proof of the following Theorem does not rely on the continuity of the semimartingale S. Hence, the result of Theorem 4.4.7 holds true also in financial market models based on general (possibly discontinuous and non-locally bounded) semimartingales.

Theorem 4.4.7. *Let $V = 1 + H \cdot S$ be a numéraire for S. Then the NUPBR condition holds for S if and only if the NUPBR condition holds for $\bar{S}$.*

Proof. The second part of Lemma 4.4.2, for $a = 1$, implies that:

$$\begin{aligned}\Big\{(H \cdot S)_T : H \in \mathcal{A}_1\Big\} &= V_T \Big\{\big(\bar{H} \cdot \bar{S}\big)_T : \bar{H} \in \bar{\mathcal{A}}_1\Big\} - V_T\left(\bar{H}^V \cdot \bar{S}\right)_T \\ &= V_T \Big\{\big(\bar{H} \cdot \bar{S}\big)_T : \bar{H} \in \bar{\mathcal{A}}_1\Big\} - 1 + V_T\end{aligned}$$

where the second equality uses the fact that $\frac{1}{V} = 1 + \bar{H}^V \cdot \bar{S}$ and the fact that $L\left(\bar{S}\right)$ is a linear space. Since the random variable V_T is finite and P-a.s. strictly positive and since boundedness in probability is not affected by the multiplication and the addition of V_T, this shows that the set $\{(H \cdot S)_T : H \in \mathcal{A}_1\}$ is bounded in probability if and only if the set $\left\{\left(\bar{H} \cdot \bar{S}\right)_T : \bar{H} \in \bar{\mathcal{A}}_1\right\}$ is bounded in probability. □

We want to point out that the necessity part of Theorem 4.4.7 has been shown also in the recent paper Takaoka (2010). Furthermore, the result of Theorem 4.4.7 can be also obtained by combining our Theorem 4.3.23 with Corollary 2.8 of Choulli & Stricker (1996). However, the continuity of S is essential in the proof of Corollary 2.8 of Choulli & Stricker (1996), unlike the simple proof of Theorem 4.4.7 here provided.

Remark 4.4.8 (An alternative proof to Theorem 4.4.7). Similarly as in Remark 4.4.6, we can also give an alternative proof of Theorem 4.4.7 by relying on Theorem 4.3.23 and Corollary 4.4.4. In fact, equation (4.15) shows that we have $\widehat{K}_T < \infty$ P-a.s. if and only if $\bar{K}_T < \infty$ P-a.s. Due to Theorem 4.3.23, this means that the NUPBR condition holds for S if and only if the NUPBR condition holds for $\bar{S}$. For a related result on the finiteness of the mean-variance tradeoff process under a change of numéraire, see also Lemma 4.5 of Delbaen & Shirakawa (1996).

Summing up, Theorems 4.4.3, 4.4.5 and 4.4.7 together show that all the weak no-arbitrage conditions considered in Section 4.3 are stable with respect to a change of numéraire. As we already argued above, the robustness with respect to the choice of the numéraire should be regarded from an economic point of view as a fundamental property of a no-arbitrage condition.

Let us close this Section by comparing the stability properties of the weak no-arbitrage conditions discussed so far with the stability properties (or the lack thereof) of the classical *No Arbitrage (NA)* and *No Free Lunch with Vanishing Risk (NFLVR)* conditions. In Delbaen & Schachermayer (1995c), the authors study the impact of a change of numéraire on the classical NA and NFLVR conditions, showing that the NA condition (and hence, recalling part *(a)* of Proposition 4.3.28, also the NFLVR condition) is not necessarily stable with respect to a change of numéraire. More precisely (see Delbaen & Schachermayer (1995c), Theorem 4.4), assuming that the NFLVR condition holds for S, the NA condition (and, hence, in view of part *(a)* of Proposition 4.3.28 and Theorem 4.4.7, also the NFLVR condition) holds for $\bar{S}$ if and only if there exists a probability measure Q on $(\Omega, \mathcal{F})$ with $Q \sim P$ such that S is a local Q-martingale *and* V is a uniformly integrable Q-martingale, i.e. an ELMM Q for S such that the numéraire process is a uniformly integrable Q-martingale. It follows that the NA condition is not necessarily preserved by a change of numéraire. For a simple counterexample, we refer the interested reader to Example 4.1 in Delbaen & Schachermayer (1995c). In a nutshell, the reason why the NA condition is not stable under a change of numéraire is due to the definition of admissible strategy (see Definition 4.2.2). Indeed, it could be that there is arbitrage with respect to $\bar{S} = \left(\frac{S}{V}, \frac{1}{V}\right)$ but the NA condition still holds for S, since the strategy which realizes the arbitrage with respect to $\bar{S}$ is not admissible for S because it fails to be lower bounded by some constant. From an economic point of view, the fact that the NA and NFLVR conditions are affected by the choice of the reference numéraire represents a significant drawback of the classical no-arbitrage theory based on NA and NFLVR.

4.4.2 Absolutely continuous changes of measure

This Section continues the study of the robustness properties of the weak no-arbitrage conditions discussed in Section 4.3. More specifically, in the present Section we shall be concerned with the stability of the NUIP/NIAO/NUPBR conditions under an absolutely continuous change of the reference probability measure. Recall that we started in Section 4.2 by considering an abstract financial market model formulated with respect to a fixed probability measure P on $(\Omega, \mathcal{F})$. Let us now suppose that one of the NUIP/NIAO/NUPBR conditions holds with respect to P and consider another probability measure Q on $(\Omega, \mathcal{F})$ with $Q \ll P$. We can then ask the following question. Does the NUIP/NIAO/NUPBR condition still holds with respect to Q? What about the classical NA and NFLVR conditions? In this Section, we shall provide an answer to these questions.

Let Q be a given probability measure on $(\Omega, \mathcal{F})$ with $Q \ll P$ and denote by $Z^Q = \left(Z_t^Q\right)_{0 \leq t \leq T}$ its density process with respect to P, i.e. $Z_t^Q = E\left[\frac{dQ}{dP} | \mathcal{F}_t\right]$ for all $t \in [0,T]$. Recall that Z^Q is a uniformly integrable P-martingale, which also satisfies $Q\left(Z_t^Q > 0 \text{ and } Z_{t-}^Q > 0 \text{ for all } t \in [0,T]\right) = 1$ (see e.g. Revuz & Yor (1999), Proposition VIII.1.2). Note that the P-martingale Z^Q is not necessarily continuous. Recall also that S is supposed to be an $\mathbb{R}^d$-valued continuous semimartingale on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Hence, due to Theorem II.2 of Protter (2005), the process S is also a semimartingale with respect to the probability measure Q. Furthermore, due to the Girsanov-Lenglart Theorem for absolutely continuous changes of measure (see Protter (2005), Theorem III.41), we can compute the canonical decomposition of S with respect to Q, as shown in the following Lemma.

Lemma 4.4.9. *Let Q be a probability measure on $(\Omega, \mathcal{F})$ with $Q \ll P$. Then the density process Z^Q of Q with respect to P can be represented as follows:*

$$Z^Q = Z_0^Q + \theta \cdot M + N \tag{4.16}$$

for some $\mathbb{R}^d$-valued predictable process $\theta = (\theta_t)_{0\leq t\leq T}$ in $L^2_{loc}(M)$ and for some real-valued local martingale $N = (N_t)_{0\leq t\leq T}$ strongly orthogonal to M with $N_0 = 0$. Moreover, the $\mathbb{R}^d$-valued continuous semimartingale S admits the canonical decomposition $S = S_0 + A^Q + M^Q$ with respect to Q, where A^Q in an $\mathbb{R}^d$-valued continuous predictable process of finite variation with $A_0^Q = 0$ and M^Q is an $\mathbb{R}^d$-valued continuous local martingale with $M_0^Q = 0$, explicitly given as follows:

$$A^Q = A + \int \frac{1}{Z_-^Q} d\langle M, M\rangle\theta \qquad \text{and} \qquad M^Q = M - \int \frac{1}{Z_-^Q} d\langle M, M\rangle\theta \tag{4.17}$$

where A and M denote the finite variation and the local martingale part, respectively, in the canonical decomposition of S with respect to P.

Proof. Equation (4.16) represents the Galtchouk-Kunita-Watanabe decomposition of the (uniformly integrable) martingale Z^Q with respect to the continuous local martingale M, see Ansel & Stricker (1993). Then, Theorem III.41 of Protter (2005) implies that the process $M^Q = \left(M_t^Q\right)_{0\leq t\leq T}$ defined by $M_t^Q := M_t - \int_0^t \frac{1}{Z_{u-}^Q} d\langle M, Z^Q\rangle_u$ is a local Q-martingale. Note that, since M is continuous, the predictable quadratic variation $\langle M, Z^Q\rangle$ always exists. Hence, we get the following canonical decomposition of S with respect to Q:

$$S = S_0 + \left(A + \int \frac{1}{Z_-^Q} d\langle M, Z^Q\rangle\right) + \left(M - \int \frac{1}{Z_-^Q} d\langle M, Z^Q\rangle\right)$$

The expressions in (4.17) then follow from (4.16), using also the fact that $\langle M, N\rangle \equiv 0$ due to the continuity of M and to the strong orthogonality of M and N. □

By relying on the above Lemma, we can easily prove the stability of the NUIP condition with respect to an absolutely continuous change of the reference probability measure.

Theorem 4.4.10. *Let Q be a probability measure on $(\Omega, \mathcal{F})$ with $Q \ll P$ and suppose that the NUIP condition holds with respect to P. Then the NUIP condition holds with respect to Q as well.*

Proof. Suppose that the NUIP condition holds with respect to P. Then, due to Theorem 4.3.2, there exists an $\mathbb{R}^d$-valued predictable process $\lambda = (\lambda_t)_{0\leq t\leq T}$ such that $dA = d\langle M, M\rangle\lambda$. Hence, if Q is a probability measure on $(\Omega, \mathcal{F})$ with $Q \ll P$, Lemma 4.4.9 implies the following:

$$A^Q = A + \int \frac{1}{Z_-^Q}\langle M, M\rangle\theta = \int d\langle M, M\rangle\left(\lambda + \frac{\theta}{Z_-^Q}\right) = \int d\langle M^Q, M^Q\rangle\left(\lambda + \frac{\theta}{Z_-^Q}\right) \tag{4.18}$$

where we have also used the fact that $\langle M, M\rangle = \langle M^Q, M^Q\rangle$. This shows that $dA^Q = d\langle M^Q, M^Q\rangle\lambda^Q$, where the $\mathbb{R}^d$-valued predictable process $\lambda^Q = \left(\lambda_t^Q\right)_{0\leq t\leq T}$ is given by $\lambda^Q := \lambda + \frac{\theta}{Z_-^Q}$. Theorem 4.3.2 then implies that the NUIP condition holds with respect to Q as well. □

In order to study the stability of the NIAO and NUPBR conditions under an absolutely continuous change of measure, let us first compute the mean-variance tradeoff process $\widehat{K}^Q = \big(\widehat{K}_t^Q\big)_{0\leq t\leq T}$ of S with respect to the probability measure Q.

Lemma 4.4.11. *Let Q be a probability measure on $(\Omega, \mathcal{F})$ with $Q \ll P$. Then the mean-variance tradeoff process $\widehat{K}^Q = \big(\widehat{K}_t^Q\big)_{0\leq t\leq T}$ of S with respect to the probability measure Q is explicitly given as follows, for all $t \in [0, T]$:*

$$\widehat{K}_t^Q := \sum_{i,j=1}^{d} \int_0^t \left(\lambda_u^Q\right)^i \left(\lambda_u^Q\right)^j d\Big\langle \left(M^Q\right)^i, \left(M^Q\right)^j \Big\rangle_u = \int \left(\lambda + \frac{\theta}{Z_-^Q}\right)' d\langle M, M\rangle \left(\lambda + \frac{\theta}{Z_-^Q}\right) \quad (4.19)$$

Furthermore, the process $\widehat{K}^Q$ satisfies the following inequality, for all $t \in [0, T]$:

$$\widehat{K}_t^Q \leq \widehat{K}_t + \int_0^t \frac{1}{\left(Z_{u-}^Q\right)^2} \theta_u' d\langle M, M\rangle_u \theta_u \quad (4.20)$$

Proof. Equation (4.19) follows directly from equation (4.18), using the fact that $\langle M,M\rangle = \langle M^Q,M^Q\rangle$. Equation (4.20) is then a simple consequence of the Cauchy-Schwarz inequality together with (4.4). □

We can now easily prove the following Theorem, which shows the stability of the NIAO and NUPBR conditions with respect to an absolutely continuous change of the reference probability measure.

Theorem 4.4.12. *Let Q be a probability measure on $(\Omega, \mathcal{F})$ with $Q \ll P$. If the NIAO condition holds with respect to P then it holds with respect to Q as well. In addition, if the NUPBR condition holds with respect to P then it holds with respect to Q as well.*

Proof. Suppose first that the NIAO condition holds with respect to P. As pointed out in Section 4.3.2, the NIAO condition implies the NUIP condition and Theorem 4.4.10 shows that the NUIP condition holds with respect to Q. In particular, there exists an $\mathbb{R}^d$-valued predictable process $\lambda^Q = \big(\lambda_t^Q\big)_{0\leq t\leq T}$ such that $dA^Q = d\langle M^Q, M^Q\rangle\lambda^Q$. Hence, in view of Theorem 4.3.10, to show that the NIAO condition holds with respect to Q it suffices to show that the mean-variance tradeoff process $\widehat{K}^Q$ defined in (4.19) does not jump to infinity Q-a.s. Since the NIAO condition holds under P, Theorem 4.3.2 implies that $\alpha := \inf\big\{t > 0 : \widehat{K}_t^{t+h} = \infty, \forall h \in (0, T-t]\big\} = \infty$ P-a.s. and also Q-a.s., since $Q \ll P$. Note that the second term in the right hand side of (4.20) is continuous and P-a.s. (and, hence, also Q-a.s.) finite, because $\theta \in L^2_{loc}(M)$ and the process $1/Z_-^Q$ is locally bounded, being left-continuous. Due to the inequality (4.20), this implies the following:

$$\alpha^Q := \inf\left\{t > 0 : \widehat{K}_{t+h}^Q - \widehat{K}_t^Q = \infty, \forall h \in (0, T-t]\right\} = \alpha = \infty \qquad Q\text{-a.s.}$$

Due to Theorem 4.3.10, we can conclude that the NIAO condition holds with respect to Q. Suppose now that the NUPBR condition holds with respect to P. In view of Theorem 4.3.23, since the NUPBR condition is stronger than the NUIP/NIAO conditions and since we already know that the latter hold under Q, to show that NUPBR holds with respect to Q it suffices to show that $\widehat{K}_T^Q < \infty$ Q-a.s. Since NUPBR holds under P, Theorem 4.3.23 implies that $\widehat{K}_T < \infty$ P-a.s. and also Q-a.s., since $Q \ll P$. Thus, equation (4.20) gives $\widehat{K}_T^Q < \infty$ Q-a.s. Due to Theorem 4.3.23, we can conclude that the NUPBR condition holds with respect to Q. □

Summing up, Theorems 4.4.10 and 4.4.12 show that the weak no-arbitrage conditions considered in Section 4.3 are all stable with respect to an absolutely continuous change of the reference probability measure. However, this stability property is not enjoyed by the classical *No Arbitrage (NA)* and *No Free Lunch with Vanishing Risk (NFLVR)* conditions, as we are going to show in the next Example, which is based on Delbaen & Schachermayer (1995a).

Example 4.4.13. Let $W = (W_t)_{0 \leq t \leq T}$ be a standard Brownian motion starting from $W_0 = 1$ and define the stopping time $\tau := \inf\{t \in [0,T] : W_t = 0\} \wedge T$. We define the discounted price process S of a single risky asset as the stopped process $S := W^\tau$ and we assume that the filtration $\mathbb{F}$ is the P-augmented natural filtration of S, with $\mathcal{F} = \mathcal{F}_T$. Clearly, the process S is a martingale. Hence, recalling Proposition 4.3.28, the NA and NFLVR no-arbitrage conditions hold with respect to P. Let us also define a probability measure Q on $(\Omega, \mathcal{F})$ by letting $\frac{dQ}{dP} := S_T = W_{T\wedge\tau} = W_\tau$. Clearly, we have $Q \ll P$ but $P \ll Q$ does not hold, meaning that the probability measures Q and P are not equivalent. Observe that the process S represents also the density process of Q with respect to P. In fact, for all $t \in [0,T]$:

$$E\left[\frac{dQ}{dP}\Big|\mathcal{F}_t\right] = E\left[S_T|\mathcal{F}_t\right] = E\left[W_\tau|\mathcal{F}_t\right] = W_{t\wedge\tau} = S_t$$

By the Girsanov-Lenglart Theorem for absolutely continuous changes of measure (see Protter (2005), Theorem III.41), the process $N = (N_t)_{0 \leq t \leq T}$ defined by $N := S - \int \frac{1}{S} d\langle S, S\rangle$ is a continuous local Q-martingale. Note that $\langle N, N\rangle_t = \langle S, S\rangle_t = t \wedge \tau$ for all $t \in [0,T]$. Furthermore, since $Q(S_T = 0) = E\left[\mathbf{1}_{\{S_T=0\}} S_T\right] = 0$, so that $\tau = T$ Q-a.s., we have $\langle N, N\rangle_t = t$ Q-a.s. for all $t \in [0,T]$. Lévy's characterization of Brownian motion (see e.g. Protter (2005), Theorem II.39) implies then that N is a Q-Brownian motion starting at $N_0 = 1$. Let us denote by $\mathbb{G}$ the Q-augmented natural filtration of N (or, equivalently, of S). It can be easily seen that the filtration $\mathbb{G}$ coincides with $\mathbb{F}$ augmented by the subsets of $\{S_T = 0\}$. We now show that S does not satisfy the NA condition on the filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, Q)$, with $\mathcal{G} = \mathcal{G}_T$. Indeed, suppose on the contrary that S satisfies the NA condition on $(\Omega, \mathcal{G}, \mathbb{G}, Q)$. Then, due to Theorem 4.4.12 together with Proposition 4.3.28, there exists a probability measure Q' on $(\Omega, \mathcal{G})$ with $Q' \sim Q$ such that S is a local Q'-martingale. Let us denote by $Z^{Q'}$ the density process of Q' with respect to Q. The $(Q, \mathbb{G})$-Brownian motion N enjoys the representation property, since $\mathbb{G}$ is its Q-augmented natural filtration. Hence, due to Corollary 2 to Theorem IV.43 of Protter (2005), we have the representation $Z^{Q'} = \mathcal{E}(-\theta \cdot N)$, for some $\mathbb{G}$-predictable process $\theta = (\theta_t)_{0 \leq t \leq T}$ such that $\int_0^T \theta_t^2 dt < \infty$ Q-a.s. Let us compute the stochastic differential of the product $Z^{Q'} S$:

$$d\big(Z_t^{Q'} S_t\big) = Z_t^{Q'} dS_t + S_t dZ_t^{Q'} + d\langle Z^{Q'}, S\rangle_t = Z_t^{Q'} dN_t + \frac{Z_t^{Q'}}{S_t} dt - Z_t^{Q'} S_t \theta_t dN_t - Z_t^{Q'} \theta_t dt$$

Since $Z^{Q'}S$ is a local Q-martingale and $Z^{Q'} > 0$ Q-a.s., this implies that $\theta_t = \frac{1}{S_t}$ Q-a.s. for all $t \in [0,T]$, thus yielding $Z^{Q'} = \mathcal{E}\left(-\frac{1}{S} \cdot N\right)$. Equivalently, the process $Z^{Q'} = \left(Z_t^{Q'}\right)_{0 \leq t \leq T}$ is the unique solution to the following SDE:

$$dZ_t^{Q'} = -Z_t^{Q'} \frac{1}{S_t} dN_t \qquad Z_0^{Q'} = 1 \tag{4.21}$$

On the other hand, Itô's formula gives the following:

$$d\frac{1}{S_t} = -\frac{1}{S_t^2} dS_t + \frac{1}{S_t^3} d\langle S \rangle_t = -\frac{1}{S_t^2} dN_t - \frac{1}{S_t^3} dt + \frac{1}{S_t^3} dt = -\frac{1}{S_t^2} dN_t \qquad \frac{1}{S_0} = 1 \tag{4.22}$$

Equations (4.21)-(4.22) show that the processes $Z^{Q'} = \left(Z_t^{Q'}\right)_{0 \leq t \leq T}$ and $1/S = \left(1/S_t\right)_{0 \leq t \leq T}$ solve the same SDE with the same initial condition and, hence, we must have $Z_t^{Q'} = \frac{1}{S_t}$ Q-a.s. for all $t \in [0,T]$. At this point, observe that the process $S = (S_t)_{0 \leq t \leq T}$, which satisfies $dS_t = dN_t + \frac{1}{S_t} dt$ with respect to the $(Q, \mathbb{G})$-Brownian motion N, is a Bessel process of dimension three (see Revuz & Yor (1999), Section XI.1). It is well-known that the reciprocal of a 3-dimensional Bessel process is a *strict* local martingale, i.e. a local martingale which is not a true martingale (see e.g. Revuz & Yor (1999), Exercise XI.1.16). This implies that the process $Z^{Q'}$ cannot be a true martingale, thus contradicting the existence of the measure Q'. Due to part *(c)* of Proposition 4.3.28, this implies that the NFLVR condition does not hold on $(\Omega, \mathcal{G}, \mathbb{G}, Q)$. On the other hand, observe that the process $Z^{Q'}$ is a martingale deflator for S on $(\Omega, \mathcal{G}, \mathbb{G}, Q)$, so that, due to Theorem 4.3.23, the NUPBR condition holds. Hence, part *(a)* of Proposition 4.3.28 implies that the NA condition fails on $(\Omega, \mathcal{G}, \mathbb{G}, Q)$. So, there exists a $\mathbb{G}$-predictable S-integrable (with respect to $(Q, \mathbb{G})$) process $H = (H_t)_{0 \leq t \leq T}$ with $G(H) \geq -a$ Q-a.s., for some $a \in \mathbb{R}_+$, and such that $G_T(H) \geq 0$ Q-a.s. and $Q\big(G_T(H) > 0\big) > 0$. As pointed out on page 360 of Delbaen & Schachermayer (1995a), there also exists an $\mathbb{F}$-predictable process $K = (K_t)_{0 \leq t \leq T}$ which is Q-indistinguishable from H, so that $K \cdot S = H \cdot S$, where both stochastic integrals are considered with respect to $(Q, \mathbb{G})$. Since S and K are both $\mathbb{F}$-predictable, Theorem 7 of Jacod (1980) shows that the stochastic integral $K \cdot S$ viewed with respect to $(Q, \mathbb{G})$ is the same as the stochastic integral viewed with respect to $(Q, \mathbb{F})$. This means that the $\mathbb{F}$-predictable process $K = (K_t)_{0 \leq t \leq T}$ is integrable with respect to S (with respect to $(Q, \mathbb{F})$) and satisfies $G(K) \geq -a$ Q-a.s. and $G_T(K) \geq 0$ Q-a.s. and $Q\big(G_T(K) > 0\big) > 0$. Hence, we can conclude that the NA condition fails on $(\Omega, \mathcal{F}, \mathbb{F}, Q)$, thus finishing the Example.

Remark 4.4.14 (**On equivalent changes of measure**). We close this Section by pointing out that, if we consider an *equivalent* change of measure, rather than an absolutely continuous change of measure, then the NA and NFLVR conditions are also preserved. Indeed, suppose that the NA condition holds with respect to P and let Q be a probability measure on $(\Omega, \mathcal{F})$ with $Q \sim P$. Arguing by contradiction, suppose that there exists an $\mathbb{R}^d$-valued predictable S-integrable (with respect to Q) process $H = (H_t)_{0 \leq t \leq T}$ with $G(H) \geq -a$ Q-a.s., for some $a \in \mathbb{R}_+$, and such that $G_T(H) \geq 0$ Q-a.s. and $Q\big(G_T(H) > 0\big) > 0$. Then, since $Q \sim P$, Theorem IV.25 of Protter (2005) shows that the process H is also S-integrable with respect to P and the stochastic integral $H \cdot S$ viewed with respect to P coincides with the stochastic integral $H \cdot S$ viewed with respect to Q. Since $Q \sim P$, we also have $G(H) \geq -a$ P-a.s. and $G_T(H) \geq 0$ P-a.s. and

$P\big(G_T(H) > 0\big) > 0$. Clearly, this contradicts the assumption that the NA condition holds with respect to P, thus showing that an equivalent change of measure does not affect the validity of the NA condition. Due to part *(a)* of Proposition 4.3.28 together with Theorem 4.4.12, the same holds true for the NFLVR condition.

4.4.3 Changes of the reference filtration

In this Section we shall be concerned with the issue of the robustness of the weak no-arbitrage conditions analyzed in Section 4.3 with respect to changes in the reference filtration. Intuitively, this amounts to study the impact on the weak no-arbitrage conditions of restrictions/enlargements of the information set available to market participants. This study is not only of theoretical interest but has also important implications from an economic point of view. Indeed, it allows to show in a rigorous way the relation between the possibility of making arbitrages (in a suitable sense) and the quality of the information available to the investors. Furthermore, these issues are also connected to the analysis of economically relevant situations like *insider trading* and *partial information*. In Section 4.4.3 we shall consider the case where the information set is restricted, while Section 4.4.3 will be devoted to the analysis of the case where the information set is expanded.

Restriction of the filtration

We continue to work within the general framework described in Section 4.2 and we let S be an $\mathbb{R}^d$-valued continuous semimartingale on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with canonical decomposition $S = S_0 + A + M$. Furthermore, we let $\mathbb{E} = (\mathscr{E}_t)_{0 \leq t \leq T}$ be a filtration satisfying the usual conditions on the probability space $(\Omega, \mathcal{F}, P)$ and such that $\mathbb{E} \subseteq \mathbb{F}$, meaning that $\mathscr{E}_t \subseteq \mathcal{F}_t$ for all $t \in [0, T]$. We also assume that the $\mathbb{F}$-semimartingale S is $\mathbb{E}$-adapted. The basic question we shall answer in this Section can be formulated in the following terms. Suppose that one of the NUIP/NIAO/NUPBR no-arbitrage conditions holds on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Does the same no-arbitrage condition hold with respect to $(\Omega, \mathcal{F}, \mathbb{E}, P)$ as well? In other words: does the restriction of the information set affect the no-arbitrage condition? Our intuition suggests that, if there is no-arbitrage (in a suitable sense) in the original full-information financial market (i.e. with respect to $\mathbb{F}$) and we restrict the information set available to market participants, then arbitrage should not be possible in the restricted information financial market (i.e. with respect to $\mathbb{E}$) as well. As will be shown below, this conjecture is indeed valid and can be rigorously justified.

As a preliminary, recall that S is a continuous (and, hence, special) $\mathbb{F}$-semimartingale, with canonical decomposition $S = S_0 + A + M$, where A is an $\mathbb{R}^d$-valued continuous $\mathbb{F}$-predictable process of finite variation with $A_0 = 0$ and M is an $\mathbb{R}^d$-valued continuous $\mathbb{F}$-local martingale with $M_0 = 0$. Since $\mathbb{E} \subseteq \mathbb{F}$ and S is supposed to be $\mathbb{E}$-adapted, Stricker's Theorem (see Protter (2005), Theorem II.4) implies that S is also a special $\mathbb{E}$-semimartingale and we denote by $S = S_0 + \widetilde{A} + \widetilde{M}$ its canonical decomposition with respect to $\mathbb{E}$, where $\widetilde{A}$ is an $\mathbb{R}^d$-valued continuous $\mathbb{E}$-predictable process of finite variation with $\widetilde{A}_0 = 0$ and $\widetilde{M}$ is an $\mathbb{R}^d$-valued continuous $\mathbb{E}$-local martingale with $\widetilde{M}_0 = 0$. Furthermore, we denote by $L(S, \mathbb{F})$ the set of all $\mathbb{R}^d$-valued $\mathbb{F}$-predictable S-integrable (with respect to the filtration $\mathbb{F}$) processes and, analogously, we denote by $L(S,\mathbb{E})$ the set of all $\mathbb{R}^d$-valued $\mathbb{E}$-predictable S-integrable (with respect to the filtration $\mathbb{E}$) processes.

At this point, note that it is tempting to answer the questions formulated at the beginning of this Section by relying on the following reasoning. Suppose the one of the NUIP/NIAO/NUPBR conditions holds with respect to the filtration $\mathbb{F}$ and let $H=(H_t)_{0\leq t\leq T}$ be an element of $L(S,\mathbb{E})$ which realizes an arbitrage (in the sense of UIP/IAO/UPBR, respectively) in the filtration $\mathbb{E}$. Then, since $\mathbb{E}\subseteq\mathbb{F}$, the process H is also $\mathbb{F}$-predictable and, hence, can be viewed as a trading strategy in the filtration $\mathbb{F}$ (in plain words, nothing should prevent a more informed trader from neglecting part of the information available to her). Hence, one is led to conjecture that $L(S,\mathbb{E})\subseteq L(S,\mathbb{F})$, so that the strategy H, now viewed with respect to the filtration $\mathbb{F}$, yields an arbitrage (in the sense of UIP/IAO/UPBR, respectively) also in the filtration $\mathbb{F}$, thus contradicting the assumption that NUIP/NIAO/NUPBR holds with respect to the filtration $\mathbb{F}$. Hence, one would conclude that the NUIP/NIAO/NUPBR conditions are stable with respect to a restriction of the reference filtration. The flaw in this line of reasoning is that the inclusion $L(S,\mathbb{E})\subseteq L(S,\mathbb{F})$ is not true in general, as shown by an explicit counterexample in Chou et al. (1980) (see also Jeulin (1980), Theorem 3.23). The inclusion holds true if we restrict our attention to locally bounded integrands, as shown in Theorem 9.19 of Jacod (1979) and in Theorem IV.33 of Protter (2005). However, as pointed out in Remark 4.2.1, we have chosen to work with the most general class of integrands, which includes non-locally bounded processes. Hence, the above reasoning fails and special care must be taken.

The following Theorem studies the impact of a restriction of the reference filtration on the *No Unbounded Increasing Profit (NUIP)* condition. Furthermore, it gives a characterization of the finite variation part $\widetilde{A}$ in the canonical decomposition of S with respect to the filtration $\mathbb{E}$.

Theorem 4.4.15. *Let $\mathbb{E}$ be a filtration with $\mathbb{E}\subseteq\mathbb{F}$ and suppose that S is $\mathbb{E}$-adapted. If the NUIP condition holds with respect to $\mathbb{F}$ then it holds with respect to $\mathbb{E}$ as well. Furthermore, the finite variation part $\widetilde{A}$ in the canonical decomposition of S with respect to $\mathbb{E}$ can be represented as follows, for all $t\in[0,T]$:*

$$\widetilde{A}^i_t=\int_0^t d\langle\widetilde{M}^i,\widetilde{M}\rangle_u\,{}^{(p,\mathbb{E})}\lambda_u \qquad i=1,\dots,d$$

where $\widetilde{M}$ is the local martingale part in the canonical decomposition of S with respect to $\mathbb{E}$ and ${}^{(p,\mathbb{E})}\lambda$ denotes the $\mathbb{E}$-predictable projection of the process λ, the latter being as in Theorem 4.3.2.

Proof. Note first that we have $\langle M,M\rangle=\langle S,S\rangle=\langle\widetilde{M},\widetilde{M}\rangle$ and $\langle S,S\rangle$ does not depend on the choice of the reference filtration. Indeed, due to the continuity of S, we have $\langle S,S\rangle=[S,S]$ and the latter does not depend on the reference filtration, being the limit (in probability) of the pathwise quadratic variation of S (compare also Jacod (1979), part *(b)* of Theorem 9.19) For every $n\in\mathbb{N}$, let us define the $\mathbb{E}$-stopping time τ_n as follows:

$$\tau_n:=\inf\left\{t\in[0,T]:\|S_t-S_0\|\geq n\text{ or }\sum_{i,j=1}^d\langle S^i,S^j\rangle_t\geq n\right\}\wedge T$$

Clearly, since both S and $\langle S, S\rangle$ are P-a.s. finite valued, we have $\tau_n \nearrow T$ P-a.s. as $n \to \infty$. For every $n \in \mathbb{N}$, the stopped process $\langle M, M\rangle^{\tau_n} = \langle M^{\tau_n}, M^{\tau_n}\rangle$ is bounded and, hence, the $\mathbb{F}$-local martingale M^{τ_n} is an $\mathbb{R}^d$-valued uniformly integrable $\mathbb{F}$-martingale. Similarly, the same holds true for the stopped process $\widetilde{M}^{\tau_n}$, for every $n \in \mathbb{N}$, with respect to the filtration $\mathbb{E}$. Note that, for every $n \in \mathbb{N}$, we have $A^{\tau_n} = S^{\tau_n} - S_0 - M^{\tau_n} \leq n - M^{\tau_n}$ and, hence, $A_t^{\tau_n} \in L^1(P)$ for all $t \in [0, T]$. Similarly, the same holds true for the stopped process $\widetilde{A}^{\tau_n}$, for every $n \in \mathbb{N}$, with respect to the filtration $\mathbb{E}$. So, for any $\mathbb{E}$-stopping time τ and for every $n \in \mathbb{N}$:

$$E\left[A_\tau^{\tau_n}\right] = E\left[S_\tau^{\tau_n} - S_0\right] - E\left[M_\tau^{\tau_n}\right] = E\left[S_\tau^{\tau_n} - S_0\right] = E\left[S_\tau^{\tau_n} - S_0\right] - E\big[\widetilde{M}_\tau^{\tau_n}\big] = E\big[\widetilde{A}_\tau^{\tau_n}\big]$$

where the second and the third equalities follow from the optional sampling theorem (see e.g. Protter (2005), Theorem I.16) together with the uniform integrability of M^{τ_n} and $\widetilde{M}^{\tau_n}$, respectively. Due to Theorem 9.22 of Jacod (1979), we can conclude that $\widetilde{A}^{\tau_n}$ is the dual predictable projection of A^{τ_n} with respect to $\mathbb{E}$, i.e. $\widetilde{A}^{\tau_n} = (A^{\tau_n})^{(p,\mathbb{E})}$, for every $n \in \mathbb{N}$. Furthermore, the following holds:

$$\widetilde{A}^{\tau_n} = (A^{\tau_n})^{(p,\mathbb{E})} = \left(\mathbf{1}_{[\![0,\tau_n]\!]} \cdot A\right)^{(p,\mathbb{E})} = \mathbf{1}_{[\![0,\tau_n]\!]} \cdot A^{(p,\mathbb{E})} = \left(A^{(p,\mathbb{E})}\right)^{\tau_n}$$

where the third equality follows from 9.23 of Jacod (1979), since τ_n is an $\mathbb{E}$-stopping time for all $n \in \mathbb{N}$. Since $\tau_n \nearrow T$ P-a.s. as $n \to \infty$, this shows that $\widetilde{A} = A^{(p,\mathbb{E})}$ (see also Jacod (1979), Proposition 9.24). Suppose now that the NUIP condition holds with respect to $\mathbb{F}$. Then, due to Theorem 4.3.2, there exists an $\mathbb{R}^d$-valued $\mathbb{F}$-predictable process $\lambda = (\lambda_t)_{0\leq t\leq T}$ such that $dA = d\langle M, M\rangle\lambda$. We can thus write the following (compare also with Kohlmann et al. (2007), Lemma 2.2):

$$\widetilde{A} = A^{(p,\mathbb{E})} = \left(\int d\langle M, M\rangle\lambda\right)^{(p,\mathbb{E})} = \left(\int d\langle \widetilde{M}, \widetilde{M}\rangle\lambda\right)^{(p,\mathbb{E})} = \int d\langle \widetilde{M}, \widetilde{M}\rangle\, {}^{(p,\mathbb{E})}\lambda$$

where the last equality follows from 9.23 of Jacod (1979) and ${}^{(p,\mathbb{E})}\lambda$ denotes the predictable projection of λ with respect to the filtration $\mathbb{E}$. Due to Theorem 4.3.2, this shows that the NUIP condition holds with respect to the filtration $\mathbb{E}$ as well. □

Theorem 4.4.15 shows that, if the continuous $\mathbb{E}$-adapted $\mathbb{F}$-semimartingale S satisfies the NUIP condition with respect to the filtration $\mathbb{F}$, then it satisfies the NUIP condition with respect to the filtration $\mathbb{E}$ as well. Note that the proof of Theorem 4.4.15 makes use of the continuity of the semimartingale S. Our next goal consists in showing that the same result holds true also in the case where S is a general (possibly discontinuous and non-locally bounded) semimartingale, provided we make the following natural assumption concerning the structure of the filtrations $\mathbb{E}$ and $\mathbb{F}$.

Assumption 4.4.16. *Every $\mathbb{E}$-semimartingale is also an $\mathbb{F}$-semimartingale.*

Remark 4.4.17. In the theory of *enlargement of filtrations*, Assumption 4.4.16 is commonly known as the (H')-*hypothesis*, see e.g. Chapter II of Jeulin (1980), where the interested reader can also find necessary and sufficient conditions for its validity (simpler results can also be found in Jacod (1979), Proposition 9.32). Observe that Assumption 4.4.16 automatically holds if one assumes the stronger (H)-*hypothesis*, i.e. if every $\mathbb{E}$-local martingale is also an $\mathbb{F}$-local martingale, see e.g. Proposition 9.28 of Jacod (1979). However, there are situations where Assumption 4.4.16 holds but the (H)-hypothesis fails, see for instance part 2 of Remark 9.37 in Jacod (1979).

As soon as Assumption 4.4.16 holds, we are able to prove the following technical result, where we also allow the process S to be a general (possibly discontinuous and non-locally bounded) semimartingale.

Proposition 4.4.18. *Let $\mathbb{E}$ be a filtration with $\mathbb{E} \subseteq \mathbb{F}$ and assume that the $\mathbb{R}^d$-valued (general) $\mathbb{F}$-semimartingale S is $\mathbb{E}$-adapted. Suppose that Assumption 4.4.16 holds. Then we have $L(S,\mathbb{E}) \subseteq L(S,\mathbb{F})$.*

Proof. The claim can be proved by relying on arguments similar to those used in the proofs of Theorem 7 of Jacod (1980) and Theorem III.6.19 of Jacod & Shiryaev (2003). Suppose that $H \in L(S,\mathbb{E})$. Clearly, since $\mathbb{E} \subseteq \mathbb{F}$, the process H is $\mathbb{F}$-predictable, being $\mathbb{E}$-predictable. Let $Y := H \cdot S$, where the stochastic integral is viewed with respect to $\mathbb{E}$, and define the set $D := \{|\Delta S| > 1\} \cup \{|\Delta Y| > 1\} \subset \Omega \times [0,T]$. The optional set D is discrete, meaning that for all $(\omega,t) \in \Omega \times [0,T]$, the set $\{s \in [0,t] : (\omega,s) \in D\}$ is finite. Therefore, we can define the processes $\widetilde{S}^D = \big(\widetilde{S}^D_t\big)_{0 \leq t \leq T}$ and $\widetilde{Y}^D = \big(\widetilde{Y}^D_t\big)_{0 \leq t \leq T}$ as follows, for $t \in [0,T]$:

$$\widetilde{S}^D_t := S_0 + \sum_{s \leq t} \Delta S_s \mathbf{1}_D(s) \qquad \widetilde{Y}^D_t := \sum_{s \leq t} \Delta Y_s \mathbf{1}_D(s)$$

Since D is discrete we obviously have $H \in L\big(\widetilde{S}^D,\mathbb{E}\big)$ and $H \cdot \widetilde{S}^D = \widetilde{Y}^D$. Clearly, the stochastic integral $H \cdot \widetilde{S}^D$ is a Stieltjes integral and, as such, does not depend on the choice of the reference filtration. Hence, we also have $H \in L\big(\widetilde{S}^D,\mathbb{F}\big)$. Let us define $S^D := S - \widetilde{S}^D$. Then, to show that $H \in L(S,\mathbb{F})$, it suffices to show that $H \in L\left(S^D,\mathbb{F}\right)$. In fact, due to part *(d)* of Theorem III.6.19 of Jacod & Shiryaev (2003), if we have $H \in L\big(\widetilde{S}^D,\mathbb{F}\big) \cap L\left(S^D,\mathbb{F}\right)$, then we also have $H \in L\big(\widetilde{S}^D + S^D,\mathbb{F}\big) = L(S,\mathbb{F})$. To show that $H \in L\left(S^D,\mathbb{F}\right)$, let us define the following processes, for every $n \in \mathbb{N}$:

$$F_n := \mathbf{1}_{\{\|H\| \leq n\}} \qquad H(n) := H \mathbf{1}_{F_n} \qquad Y^D(n) := \mathbf{1}_{F_n} \cdot Y^D = H(n) \cdot S^D$$

where $Y^D := Y - \widetilde{Y}^D$ and the stochastic integrals are viewed with respect to $\mathbb{E}$. Note that the process Y^D is a special $\mathbb{E}$-semimartingale, since it has bounded jumps. Hence, due to Assumption 4.4.16, it is also a special $\mathbb{F}$-semimartingale, thus implying that the stochastic integral $Y^D(n) = \mathbf{1}_{F_n} \cdot Y^D$ is also well-defined with respect to $\mathbb{F}$. On the other hand, note that we have $H(n) \in L\left(S^D,\mathbb{F}\right) \cap L\left(S^D,\mathbb{E}\right)$, for every $n \in \mathbb{N}$, since $H(n)$ is a bounded $\mathbb{E}$-predictable (and, hence, also $\mathbb{F}$-predictable) process. Furthermore, due to part *(c)* of Theorem 9.19 of Jacod (1979), there exists a common version (i.e. with respect to both $\mathbb{E}$ and $\mathbb{F}$) of the stochastic integral $H(n) \cdot S^D$, for every $n \in \mathbb{N}$. This implies that we have, for every $n \in \mathbb{N}$:

$$H(n) \cdot S^D = \mathbf{1}_{F_n} \cdot Y^D \tag{4.23}$$

where both stochastic integrals are viewed with respect to $\mathbb{F}$. Let us now denote by $Y^D = N^D + C^D$ the canonical decomposition of Y^D with respect to $\mathbb{F}$, where N^D is an $\mathbb{F}$-locally square-integrable $\mathbb{F}$-local martingale with $N_0^D = 0$ and C^D is an $\mathbb{F}$-predictable process of finite variation with $C_0^D = 0$. Analogously, let us denote by $S^D = M^D + A^D$ the canonical decomposition of S^D with respect to $\mathbb{F}$, where M^D is an $\mathbb{R}^d$-valued $\mathbb{F}$-locally square-integrable $\mathbb{F}$-local martingale with $M_0^D = 0$ and A^D is an $\mathbb{R}^d$-valued $\mathbb{F}$-predictable process of finite variation with $A_0^D = 0$. Now, (4.23), together with Proposition 2 of Jacod (1980), implies that, for every $n \in \mathbb{N}$:

$$H(n) \cdot M^D = \mathbf{1}_{F_n} \cdot N^D \qquad H(n) \cdot A^D = \mathbf{1}_{F_n} \cdot C^D$$

As in Section 4.2, let B be a real-valued $\mathbb{F}$-predictable increasing process of $\mathbb{F}$-locally integrable variation such that $A^D = \int a dB$ and $\langle M^D, M^D \rangle = \int c dB$, where a and c are suitable $\mathbb{F}$-predictable processes taking values in $\mathbb{R}^d$ and $\mathbb{R}^{d \times d}$, respectively. Then, for every $n \in \mathbb{N}$ and $t \in [0, T]$:

$$\begin{aligned}\left(\left(\sum_{i,j=1}^{d} H^i c^{ij} H^j\right) \mathbf{1}_{F_n} \cdot B\right)_t &= \left(\left(\sum_{i,j=1}^{d} H^i(n) c^{ij} H^j(n)\right) \cdot B\right)_t = \left\langle H(n) \cdot M^D, H(n) \cdot M^D \right\rangle_t \\ &= \left(\mathbf{1}_{F_n} \cdot \langle N^D, N^D \rangle\right)_t < \infty \qquad P\text{-a.s.}\end{aligned}$$

Letting $n \to \infty$, we get (for $t = T$):

$$\sum_{i,j=1}^{d} \int_0^T H_t^i c_t^{ij} H_t^j dB_t = \langle N^D, N^D \rangle_T < \infty \qquad P\text{-a.s.}$$

This shows that $H \in L^2_{loc}\left(M^D, \mathbb{F}\right)$. Analogously, for every $n \in \mathbb{N}$ and $t \in [0, T]$:

$$\left(\left(\mathbf{1}_{F_n} \left| \sum_{i=1}^{d} H^i a^i \right|\right) \cdot B\right)_t = \left(\left| \sum_{i=1}^{d} H^i(n) a^i \right| \cdot B\right)_t = \mathrm{Var}\left(H(n) \cdot A^D\right)_t = \left(\mathbf{1}_{F_n} \cdot \mathrm{Var}\left(C^D\right)\right)_t < \infty \ P\text{-a.s.}$$

where Var denotes the total variation. Letting $n \to \infty$, we get (for $t = T$):

$$\int_0^T \left| \sum_{i=1}^{d} H_t^i a_t^i \right| dB_t = \mathrm{Var}\left(C^D\right)_t < \infty \qquad P\text{-a.s.}$$

This shows that $H \in L^0\left(A^D, \mathbb{F}\right)$. Summing up, we have proved the following:

$$H \in L^2_{loc}\left(M^D, \mathbb{F}\right) \cap L^0\left(A^D, \mathbb{F}\right) = L\left(S^D, \mathbb{F}\right)$$

thus completing the proof of the Proposition. □

Remark 4.4.19.

1. As can be seen by inspecting the proof of Proposition 4.4.18, we can actually replace Assumption 4.4.16 with the weaker assumption that every $\mathbb{E}$-semimartingale X such that $X = H \cdot S$, for some $H \in L(S, \mathbb{E})$, is also an $\mathbb{F}$-semimartingale.

2. Suppose that the stronger (H)-hypothesis (see Remark 4.4.17) holds. Then the proof of Proposition 4.4.18 can be substantially simplified. Indeed, let $H \in L(S,\mathbb{E})$. According to Definition III.6.17 of Jacod & Shiryaev (2003), there exists a decomposition $S = S_0+B+N$ of S with respect to $\mathbb{E}$, with B an $\mathbb{R}^d$-valued $\mathbb{E}$-adapted process of finite variation with $B_0 = 0$ and N an $\mathbb{R}^d$-valued $\mathbb{E}$-locally square-integrable $\mathbb{E}$-local martingale with $N_0 = 0$, such that $H \in L^2_{loc}(N,\mathbb{E}) \cap L^0(B,\mathbb{E})$. Since $\mathbb{E} \subseteq \mathbb{F}$ and every $\mathbb{E}$-local martingale is also an $\mathbb{F}$-local martingale, due to the (H)-hypothesis, the decomposition $S = S_0 + B + N$ can also be viewed with respect to the larger filtration $\mathbb{F}$. Clearly, the process H is $\mathbb{F}$-predictable, being $\mathbb{E}$-predictable, and, hence, we have $H \in L^2_{loc}(N,\mathbb{F}) \cap L^0(B,\mathbb{F})$. Definition III.6.17 of Jacod & Shiryaev (2003) then implies that H is S-integrable with respect to the filtration $\mathbb{F}$ as well, thus showing that $L(S,\mathbb{E}) \subseteq L(S,\mathbb{F})$.

We can now state the following Corollary, the proof of which relies on Proposition 4.4.18. In particular, note that the result of Corollary 4.4.20 holds true also for financial market models based on general (possibly discontinuous and non-locally bounded) semimartingales.

Corollary 4.4.20. *Let $\mathbb{E}$ be a filtration with $\mathbb{E} \subseteq \mathbb{F}$ and assume that the $\mathbb{R}^d$-valued (general) $\mathbb{F}$-semimartingale S is $\mathbb{E}$-adapted. Suppose that Assumption 4.4.16 holds. Then the following hold:*

(a) if the NUIP condition holds with respect to $\mathbb{F}$ then it holds with respect to $\mathbb{E}$ as well;

(b) if the NIAO condition holds with respect to $\mathbb{F}$ then it holds with respect to $\mathbb{E}$ as well;

(c) if the NUPBR condition holds with respect to $\mathbb{F}$ then it holds with respect to $\mathbb{E}$ as well;

(d) if the NA condition holds with respect to $\mathbb{F}$ then it holds with respect to $\mathbb{E}$ as well;

(e) if the NFLVR condition holds with respect to $\mathbb{F}$ then it holds with respect to $\mathbb{E}$ as well.

Proof. Part *(a)* directly follows from Definition 4.3.1 and Proposition 4.4.18, since the only property in Definition 4.3.1 that depends on the choice of the reference filtration is related to the S-integrability of the strategy generating the unbounded increasing profit. To show part *(b)*, suppose that the NIAO condition holds with respect to $\mathbb{F}$ and let $H = (H_t)_{0 \leq t \leq T}$ generate an immediate arbitrage opportunity with respect to $\mathbb{E}$. Due to Definition 4.3.6, this means that H is an element of $L(S,\mathbb{E})$ such that $H = H\mathbf{1}_{]\!]\tau,T]\!]}$ and $(H \cdot S)_t > 0$ P-a.s. on $\{\tau < T\}$ for all $t \in (\tau, T]$, where τ is an $\mathbb{E}$-stopping time such that $P(\tau < T) > 0$. Since $\mathbb{E} \subseteq \mathbb{F}$, we have that τ is also an $\mathbb{F}$-stopping time and, due to Proposition 4.4.18, we also have $H \in L(S,\mathbb{F})$. This implies that H generates an immediate arbitrage opportunity also with respect to $\mathbb{F}$, thus contradicting the assumption that the NIAO condition holds with respect to $\mathbb{F}$. To show part *(c)*, note that, due to Proposition 4.4.18:

$$\Big\{1+G_T(H) : H \in L(S,\mathbb{E}), G(H) \geq -1 \ P\text{-a.s.}\Big\} \subseteq \Big\{1+G_T(H) : H \in L(S,\mathbb{F}), G(H) \geq -1 \ P\text{-a.s.}\Big\}$$

Hence, the set on the right hand side cannot be bounded in probability if the set on the left hand side is not bounded in probability. Similarly, part *(d)* directly follows from the following relation, which is due to Proposition 4.4.18, for any $a > 0$:

$$\Big\{G_T(H) : H \in L(S,\mathbb{E}), G(H) \geq -a \ P\text{-a.s.}\Big\} \subseteq \Big\{G_T(H) : H \in L(S,\mathbb{F}), G(H) \geq -a \ P\text{-a.s.}\Big\}$$

Finally, part *(e)* is a direct consequence of *(c)*-*(d)* together with part *(a)* of Proposition 4.3.28. □

Corollary 4.4.20 shows that, if Assumption 4.4.16 holds, then all the no-arbitrage conditions considered in Section 4.3 are preserved under a restriction of the available information. In particular, Corollary 4.4.20 shows the stability not only of the weak NUIP/NIAO/NUPBR conditions but also of the classical NA and NFLVR conditions. Of course, the result of Corollary 4.4.20 is very natural from an economic point of view, since it amounts to saying that, if it is not possible to construct an arbitrage by using all the available information, then it should also not be possible to construct an arbitrage by using only a subset of the available information.

Enlargement of the filtration

In this Section we shall be concerned with the analysis of the impact on the weak no-arbitrage conditions discussed in Section 4.3 of an enlargement of the reference filtration. We continue to work within the general setting described in Section 4.2 and we let S be an $\mathbb{R}^d$-valued continuous semimartingale with canonical decomposition $S = S_0 + A + M$ on the probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Furthermore, we let $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T}$ be a filtration satisfying the usual conditions with $\mathbb{F} \subseteq \mathbb{G}$, meaning that $\mathcal{F}_t \subseteq \mathcal{G}_t$ for all $t \in [0, T]$. Intuitively, the basic question we shall try to answer in the present Section can be formulated as follows. Suppose that one of the NUIP/NIAO/NUPBR no-arbitrage conditions holds with respect to the filtration $\mathbb{F}$. Does the same no-arbitrage condition hold with respect to the larger filtration $\mathbb{G}$ as well? In other words, assuming that there are no arbitrage opportunities (in a suitable sense) in the original financial market (i.e. with respect to $\mathbb{F}$), is it possible that an enlargement of the information set available to the market participants introduces (suitable types of) arbitrage opportunities?

If compared to the results obtained in Section 4.4.3, the results of the present Section are slightly less general, since they depend on how the original filtration $\mathbb{F}$ is enlarged. Indeed, many fundamental properties, like the semimartingale property, the martingale property, the canonical decomposition, the integrability with respect to a semimartingale, are not necessarily preserved under an enlargement of the reference filtration. Hence, we shall distinguish different situations, depending on the hypotheses we make on the enlarged filtration $\mathbb{G}$. Let us first study a rather easy case, where the semimartingale S admits the same canonical decomposition with respect to both $\mathbb{F}$ and $\mathbb{G}$. As a preliminary, let us state the following simple Lemma.

Lemma 4.4.21. *Let $S = S_0 + A + M$ be the canonical decomposition of the semimartingale S with respect to the filtration $\mathbb{F}$ and let $\mathbb{G}$ be a filtration with $\mathbb{F} \subseteq \mathbb{G}$. If the $\mathbb{F}$-local martingale M is also a $\mathbb{G}$-local martingale, then the canonical decomposition of S with respect to the filtration $\mathbb{G}$ is also given by $S = S_0 + A + M$.*

Proof. Let $S = S_0 + A + M$ be the canonical decomposition of S with respect to $\mathbb{F}$, where A is an $\mathbb{R}^d$-valued $\mathbb{F}$-predictable process of finite variation with $A_0 = 0$ and M is an $\mathbb{R}^d$-valued $\mathbb{F}$-local martingale with $M_0 = 0$. By assumption M is also a $\mathbb{G}$-local martingale. Furthermore, since $\mathbb{F} \subseteq \mathbb{G}$, the process A is $\mathbb{G}$-predictable, being $\mathbb{F}$-predictable. Hence, $S - S_0 = A + M$ gives a decomposition of $S - S_0$ into the sum of the $\mathbb{R}^d$-valued $\mathbb{G}$-predictable process of finite variation

A with $A_0 = 0$ and the $\mathbb{R}^d$-valued $\mathbb{G}$-local martingale M with $M_0 = 0$. Due to Theorem III.34 of Protter (2005), such decomposition is unique and it coincides with the canonical decomposition of S with respect to the filtration $\mathbb{G}$. □

Remark 4.4.22 (On the (H)-hypothesis). Suppose that the (H)-hypothesis holds, meaning that every $\mathbb{F}$-local martingale is also a $\mathbb{G}$-local martingale. In this case, the assumptions of Lemma 4.4.21 are obviously satisfied (compare also with Jacod (1979), part *(a)* of Proposition 9.28). Conversely, it can be easily shown that if the $\mathbb{F}$-local martingale part M in the canonical decomposition of S (with respect to $\mathbb{F}$) is also a $\mathbb{G}$-local martingale and has the representation property with respect to the filtration $\mathbb{F}$, then the (H)-hypothesis holds, see e.g. Theorem 9.30 of Jacod (1979). Equivalent characterizations of the (H)-hypothesis are well-known in the literature: see for instance Proposition 5.9.1.1 of Jeanblanc et al. (2009) or Proposition 3.2 of Fontana (2010b).

By relying on Lemma 4.4.21, we can immediately prove the following Corollary.

Corollary 4.4.23. *Let $S = S_0 + A + M$ be the canonical decomposition of the semimartingale S with respect to the filtration $\mathbb{F}$ and let $\mathbb{G}$ be a filtration with $\mathbb{F} \subseteq \mathbb{G}$. Assume that the $\mathbb{F}$-local martingale M is also a $\mathbb{G}$-local martingale. Then the following hold:*

(a) if the NUIP condition holds with respect to $\mathbb{F}$ then it holds with respect to $\mathbb{G}$ as well;

(b) if the NIAO condition holds with respect to $\mathbb{F}$ then it holds with respect to $\mathbb{G}$ as well;

(c) if the NUPBR condition holds with respect to $\mathbb{F}$ then it holds with respect to $\mathbb{G}$ as well.

Proof. Suppose first that S satisfies the NUIP condition with respect to $\mathbb{F}$. Due to Theorem 4.3.2, this means that there exists an $\mathbb{R}^d$-valued $\mathbb{F}$-predictable process $\lambda = (\lambda_t)_{0 \leq t \leq T}$ such that $a = c\lambda$ holds $P \otimes B$-a.e. Clearly, the process λ is also $\mathbb{G}$-predictable and, due to Lemma 4.4.21, the semimartingale S admits the same canonical decomposition with respect to both $\mathbb{F}$ and $\mathbb{G}$. Theorem 4.3.2 then implies that the NUIP condition holds with respect to $\mathbb{G}$ as well. To show parts *(b)* and *(c)*, recall that, due to Theorems 4.3.10 and 4.3.23 and since we already know that the NUIP condition holds with respect to $\mathbb{G}$, the validity of the NIAO/NUPBR conditions is characterized in terms of the mean-variance tradeoff process $\widehat{K} = \left(\widehat{K}_t\right)_{0 \leq t \leq T}$. Since the latter depends only on the canonical decomposition of S, which is the same with respect to both $\mathbb{F}$ and $\mathbb{G}$, we can conclude that if the NIAO/NUPBR conditions hold in the filtration $\mathbb{F}$ then they also hold in the filtration $\mathbb{G}$. □

Corollary 4.4.23 shows that, in the special situation considered in Lemma 4.4.21, all the weak no-arbitrage conditions discussed in Section 4.3 are stable with respect to an enlargement of the reference filtration. It is worth pointing out that the fact that the $\mathbb{F}$-local martingale part M in the canonical decomposition of S (with respect to $\mathbb{F}$) is also a $\mathbb{G}$-local martingale does not suffice to ensure that the classical NA and NFLVR no-arbitrage conditions are stable with respect to an enlargement of the filtration. See also Coculescu et al. (2008) and Section 3.1 of Fontana (2010b) for a related discussion on the NFLVR condition with respect to an enlarged filtration $\mathbb{G}$.

Progressive enlargement of the reference filtration

Let us now consider the interesting situation which arises when the filtration $\mathbb{F}$ is *progressively enlarged* with respect to a random time τ. This type of enlargement of filtration has been studied by several authors, starting already from the seventies, and represents one of the classical topics in the theory of *enlargement of filtrations*, see Section VI.3 of Protter (2005) or Section 5.9 of Jeanblanc et al. (2009) for an overview of the main results and Chapter IV of Jeulin (1980) for a more detailed treatment. This type of enlargement of filtration comes up very naturally in the context of credit risk modeling, where the random time τ represents the random occurrence of a default event, see for instance Chapter 7 of Jeanblanc et al. (2009).

In order to study the stability of the weak no-arbitrage conditions discussed in Section 4.3 with respect to a progressively enlarged filtration, we first need some preliminaries. Let us start from a given random time τ, i.e. a random variable taking values in $[0,\infty]$. Let us define the filtration $\mathbb{G}^0 = (\mathcal{G}^0_t)_{0\leq t\leq T}$ by $\mathcal{G}^0_t := \mathcal{F}_t \vee \sigma\{\tau\wedge t\}$, for all $t\in[0,T]$, and define the enlarged filtration $\mathbb{G} = (\mathcal{G}_t)_{0\leq t\leq T}$ by $\mathcal{G}_t := \mathcal{G}^0_{t+}$, for all $t\in[0,T]$. It is well-known that the filtration $\mathbb{G}$ is the smallest filtration satisfying the usual conditions which makes τ a $\mathbb{G}$-stopping time. Let us also denote by $Z^\tau = \left(Z^\tau_t\right)_{0\leq t\leq T}$ the $\mathbb{F}$-optional projection of the process $\mathbf{1}_{[\![0,\tau[\![}$, i.e. a cadlag version of the process $\left(P\left(\tau>t|\mathcal{F}_t\right)\right)_{0\leq t\leq T}$, and let α^τ be the dual $\mathbb{F}$-optional projection[5] of the increasing process $\mathbf{1}_{[\![\tau,\infty[\![}$. Then, the process $\mu^\tau := Z^\tau + \alpha^\tau$ is an $\mathbb{F}$-martingale, see e.g. Lemma 9.51 of Jacod (1979). Furthermore, it can also be shown that μ^τ is a *BMO* $\mathbb{F}$-martingale. Recall also that, for any $\mathbb{F}$-local martingale $M=(M_t)_{0\leq t\leq T}$, the process $m=(m_t)_{0\leq t\leq T}$ defined as:

$$m_t := M_{\tau\wedge t} - \int_0^{\tau\wedge t}\frac{1}{Z^\tau_{u-}}d\langle M,\mu^\tau\rangle_u \qquad t\in[0,T] \tag{4.24}$$

is a $\mathbb{G}$-local martingale (see e.g. Jeulin (1980), Proposition 4.16). We want to remark that, as shown in Theorem IV.13 of Protter (2005), the set $\left\{t\in[0,T] : Z^\tau_{t-}=0\right\}$ is contained in the set $(\tau,\infty]$, so that the right hand side of (4.24) is indeed well-defined.

We are now in a position to study the impact of a progressive enlargement of the filtration $\mathbb{F}$ with respect to a random time τ on the weak no-arbitrage conditions studied in Section 4.3. In the following Proposition, we let τ be a general random time and we restrict our attention to what happens on the stochastic interval $[\![0,\tau]\!]$. Note that we do not make any assumption concerning the validity of the (H)-hypothesis, meaning that the following Proposition holds true even when not all $\mathbb{F}$-local martingales are necessarily $\mathbb{G}$-local martingales (to this effect, see also Barbarin (2008), Section 5).

Proposition 4.4.24. *Let τ be a random time and let $\mathbb{G}$ be the progressive enlargement of the filtration $\mathbb{F}$ with respect to τ. Then the following hold:*

(a) if S satisfies the NUIP condition with respect to $\mathbb{F}$, then the stopped process S^τ satisfies the NUIP condition with respect to $\mathbb{G}$ as well;

[5] Note that, as can be deduced from the proof of Lemma 9.51 of Jacod (1979), if the random time τ avoids $\mathbb{F}$-stopping times, in the sense that $P(\tau=\rho)=0$ for any $\mathbb{F}$-stopping time ρ, the process α^τ coincides with the dual $\mathbb{F}$-predictable projection of the process $\mathbf{1}_{[\![\tau,\infty[\![}$.

(b) if S satisfies the NIAO condition with respect to $\mathbb{F}$, then the stopped process S^τ satisfies the NIAO condition with respect to $\mathbb{G}$ as well;

(c) if S satisfies the NUPBR condition with respect to $\mathbb{F}$, then the stopped process S^τ satisfies the NUPBR condition with respect to $\mathbb{G}$ as well.

Proof. Suppose first that S satisfies the NUIP condition with respect to the filtration $\mathbb{F}$. Then, due to Theorem 4.3.2, there exists an $\mathbb{R}^d$-valued $\mathbb{F}$-predictable process $\lambda = (\lambda_t)_{0\leq t\leq T}$ such that $dA = d\langle M, M\rangle\lambda$, where A and M denote the finite variation and the $\mathbb{F}$-local martingale part, respectively, in the canonical decomposition of S with respect to $\mathbb{F}$. Equation (4.24) gives then the following decomposition of the stopped process S^τ, for all $t \in [0, T]$:

$$S_t^\tau = S_0 + m_t + \int_0^{\tau\wedge t} d\langle M, M\rangle_u \lambda_u + \int_0^{\tau\wedge t} \frac{1}{Z_{u-}^\tau} d\langle M, \mu^\tau\rangle_u \tag{4.25}$$

where the $\mathbb{G}$-local martingale m is as in (4.24). Since the $\mathbb{F}$-local martingale M is continuous, we have the following Galtchouk-Kunita-Watanabe decomposition of the $\mathbb{F}$-martingale μ^τ with respect to M, see Ansel & Stricker (1993):

$$\mu^\tau = \mu_0^\tau + \psi \cdot M + N \tag{4.26}$$

for some $\mathbb{R}^d$-valued $\mathbb{F}$-predictable process $\psi = (\psi_t)_{0\leq t\leq T} \in L^2_{loc}(M, \mathbb{F})$ and for some $\mathbb{F}$-local martingale $N = (N_t)_{0\leq t\leq T}$ strongly orthogonal to M with $N_0 = 0$. In particular, strong orthogonality and continuity of M imply that $\langle M, N\rangle \equiv 0$. Hence, we can rewrite (4.25) as follows, for all $t \in [0, T]$:

$$S_t^\tau = S_0 + m_t + \int_0^{\tau\wedge t} d\langle M, M\rangle_u \left(\lambda_u + \frac{\psi_u}{Z_{u-}^\tau}\right) = S_0 + m_t + \int_0^{\tau\wedge t} d\langle m, m\rangle_u \left(\lambda_u + \frac{\psi_u}{Z_{u-}^\tau}\right) \tag{4.27}$$

Due to Theorem 4.3.2, this shows that S^τ satisfies the NUIP condition with respect to the filtration $\mathbb{G}$. In order to prove parts *(b)* and *(c)*, note that, due to equation (4.27), the mean-variance tradeoff process of S with respect to the filtration $\mathbb{G}$ satisfies the following inequality (compare also with Lemma 4.4.11), for all $t \in [0, T]$:

$$\widehat{K}^{\mathbb{G}}_{\tau\wedge t} = \int_0^{\tau\wedge t} \left(\lambda_u + \frac{\psi_u}{Z_{u-}^\tau}\right)' d\langle m, m\rangle_u \left(\lambda_u + \frac{\psi_u}{Z_{u-}^\tau}\right) \leq \widehat{K}_{\tau\wedge t} + \int_0^{\tau\wedge t} \frac{1}{\left(Z_{u-}^\tau\right)^2} \psi_u' d\langle M, M\rangle_u \psi_u \tag{4.28}$$

where $\widehat{K}$ is as in (4.4). Note that the process $1/Z_-^\tau$ is $\mathbb{F}$-predictable and locally bounded, being $\mathbb{F}$-adapted and left-continuous. Since $\psi \in L^2_{loc}(M, \mathbb{F})$, this implies that $\int_0^T \frac{1}{(Z_{t-}^\tau)^2} \psi_t' d\langle m, m\rangle_t \psi_t < \infty$ P-a.s. Inequality (4.28) shows that, if the process $\widehat{K}^{\mathbb{G}}$ jumps to infinity, then the process $\widehat{K}$ jumps to infinity as well. Due to Theorem 4.3.10 and given *(a)*, this implies that, if S satisfies the NIAO condition with respect to $\mathbb{F}$, then S^τ satisfies the NIAO condition with respect to $\mathbb{G}$. Similarly, inequality (4.28) (for $t = T$) implies that, if $\widehat{K}_T < \infty$ P-a.s., then also $\widehat{K}^{\mathbb{G}}_{\tau\wedge T} < \infty$ P-a.s. Due to Theorem 4.3.23, this shows that, if S satisfies the NUPBR condition with respect to $\mathbb{F}$, then S^τ satisfies the NUPBR condition with respect to $\mathbb{G}$. □

Proposition 4.4.24 shows that, in the case where the enlarged filtration $\mathbb{G}$ is obtained as the progressive enlargement of $\mathbb{F}$ with respect to a random time τ, the weak no-arbitrage conditions considered in Section 4.3 are stable with respect to the expansion of the information set. Observe that the result of Proposition 4.4.24 is general in the sense that it does not need any hypothesis on the random time τ. In the context of credit risk modeling, where the random time τ models the random occurrence of a default event, the result of Proposition 4.4.24 can be rephrased as follows. Suppose that the filtration $\mathbb{F}$ represents the information set characterizing the default-free financial market while the enlarged filtration $\mathbb{G}$ represents the full market information, i.e. the information associated to the default-free financial market enriched with the knowledge of whether the default event has occurred. Then, Proposition 4.4.24 says that if one of the NUIP/NIAO/NUPBR no-arbitrage conditions holds in the default-free financial market (i.e. with respect to $\mathbb{F}$), then the same no-arbitrage condition holds with respect to the defaultable market (i.e. with respect to $\mathbb{G}$) as well (at least up to the occurrence of the default event). Observe that the proof of Proposition 4.4.24 exploits the fact that the validity of the weak no-arbitrage conditions studied in Section 4.3 can be directly checked by looking at the characteristics of the process S. As pointed out in Section 4.3.3, this is not possible for the classical NA and NFLVR conditions and, hence, one needs to introduce further assumptions in order to ensure the stability of the NA and NFLVR conditions under a progressive enlargement of the reference filtration.

Remark 4.4.25 (**Extension to discontinuous semimartingales**). We want to point out that Proposition 4.4.24 can be extended to the case where the semimartingale S is only assumed to be $\mathbb{F}$-locally square-integrable, in the sense of Definition II.2.27 of Jacod & Shiryaev (2003), and not necessarily continuous. In this case, the predictable quadratic variation $\langle M, M\rangle$ is still well-defined. Observe that the proof of Proposition 4.4.24 makes use of the continuity of S only in the Galtchouk-Kunita-Watanabe decomposition of the $\mathbb{F}$-martingale μ^τ. However, as shown in Ansel & Stricker (1993), we can obtain an analogous Galtchouk-Kunita-Watanabe decomposition even if S is only assumed to be $\mathbb{F}$-locally square-integrable. This is due to the fact that the $\mathbb{F}$-martingale μ^τ is in *BMO* and, as such, also square-integrable (see e.g. Protter (2005), Section IV.4).

Our next goal consists in extending the analysis of Proposition 4.4.24 to see what happens after the random time τ. To this effect, we need to introduce some further hypotheses on τ. In particular, in view of credit risk applications, let us consider the case where the random time τ is an *initial time*, in the sense of the following Definition.

Definition 4.4.26. *A random time τ is said to be an* initial time *if there exists a measure η on $\mathcal{B}(\mathbb{R}_+)$ such that $Q_t(\omega, du) \ll \eta(du)$ holds P-a.s. for all $t \in [0,T]$, where $Q_t(\omega, du)$ denotes the regular $\mathcal{F}_t$-conditional distribution of τ.*

As pointed out in Jeanblanc & Le Cam (2009), Definition 4.4.26 is equivalent to the existence of a family of positive $\mathbb{F}$-adapted processes $(\beta_t^u)_{0\leq t\leq T}$, $u \in \mathbb{R}_+$, such that $P(\tau > u|\mathcal{F}_t) = \int_u^\infty \beta_t^s \eta(ds)$, for any $t \in [0,T]$, see also El Karoui et al. (2010). Furthermore, for every $u \geq 0$, the process $\beta^u = (\beta_t^u)_{0\leq t\leq T}$ is an $\mathbb{F}$-martingale and satisfies the following:

$$\rho^\tau := \inf\left\{t \in [0,T] : \beta_{t-}^\tau = 0 \text{ or } \beta_t^\tau = 0\right\} = \infty \qquad P\text{-a.s.}$$

so that the processes β^τ and β^τ_- are P-a.s. strictly positive. For any $\mathbb{F}$-local martingale $M = (M_t)_{0\leq t\leq T}$, Theorem 3.1 of Jeanblanc & Le Cam (2009) shows that the process $m = (m_t)_{0\leq t\leq T}$ defined as:

$$m_t := M_t - \int_0^{\tau\wedge t} \frac{1}{Z^\tau_{u-}} d\langle M, \mu^\tau\rangle_u - \int_{\tau\wedge t}^t \frac{1}{\beta^\theta_{u-}} d\langle M, \beta^\theta\rangle_u \Big|_{\theta=\tau} \qquad t\in[0,T] \tag{4.29}$$

is a $\mathbb{G}$-local martingale. We are now in a position to state the following version of Proposition 4.4.24 in the case where the random time τ is an initial time.

Proposition 4.4.27. *Let τ be an initial time and let $\mathbb{G}$ be the progressive enlargement of the filtration $\mathbb{F}$ with respect to τ. Then the following hold:*

(a) if the NUIP condition holds with respect to $\mathbb{F}$ then it holds with respect to $\mathbb{G}$ as well;

(b) if the NIAO condition holds with respect to $\mathbb{F}$ then it holds with respect to $\mathbb{G}$ as well;

(c) if the NUPBR condition holds with respect to $\mathbb{F}$ then it holds with respect to $\mathbb{G}$ as well.

Proof. The proof is based on arguments similar to those used in the proof of Proposition 4.4.24. Indeed, suppose that the NUIP condition holds with respect to $\mathbb{F}$. Then, due to Theorem 4.3.2, there exists an $\mathbb{R}^d$-valued $\mathbb{F}$-predictable process $\lambda = (\lambda_t)_{0\leq t\leq T}$ such that $dA = d\langle M, M\rangle\lambda$, where A and M denote the finite variation and the $\mathbb{F}$-local martingale part, respectively, in the canonical decomposition of S with respect to $\mathbb{F}$. Equation (4.29) gives then the following, for all $t\in[0,T]$:

$$S_t = S_0 + m_t + \int_0^t d\langle M, M\rangle_u \lambda_u + \int_0^{\tau\wedge t} \frac{1}{Z^\tau_{u-}} d\langle M, \mu^\tau\rangle_u + \int_{\tau\wedge t}^t \frac{1}{\beta^\theta_{u-}} d\langle M, \beta^\theta\rangle_u \Big|_{\theta=\tau} \tag{4.30}$$

Since $\beta^\theta = \left(\beta^\theta_t\right)_{0\leq t\leq T}$ is an $\mathbb{F}$-martingale, for every $\theta\geq 0$, and the $\mathbb{F}$-local martingale M is continuous, we can write the following Galtchouk-Kunita-Watanabe decomposition of β^θ with respect to M, see Ansel & Stricker (1993):

$$\beta^\theta = \beta^\theta_0 + \varphi(\theta)\cdot M + L^\theta$$

for some $\mathbb{R}^d$-valued $\mathbb{F}$-predictable process $\varphi(\theta) = \left(\varphi_t(\theta)\right)_{0\leq t\leq T}$ such that $\varphi(\theta)\in L^2_{loc}(M,\mathbb{F})$, for some $\mathbb{F}$-local martingale $L^\theta = \left(L^\theta_t\right)_{0\leq t\leq T}$ strongly orthogonal to M with $L^\theta_0 = 0$, for every $\theta\geq 0$. Hence, using also the Galtchouk-Kunita-Watanabe decomposition (4.26), we can write as follows, for all $t\in[0,T]$:

$$S_t = S_0 + m_t + \int_0^t d\langle M, M\rangle_u \left(\lambda_u + \mathbf{1}_{[\![0,\tau]\!]}\frac{\psi_u}{Z^\tau_{u-}} + \mathbf{1}_{]\!]\tau,T]\!]}\frac{\varphi_u(\theta)}{\beta^\theta_{u-}}\right)\Big|_{\theta=\tau} = S_0 + m_t + \int_0^t d\langle m, m\rangle_u \lambda^{\mathbb{G}}_u$$

where the $\mathbb{R}^d$-valued $\mathbb{G}$-predictable process $\lambda^{\mathbb{G}} = \left(\lambda^{\mathbb{G}}_t\right)_{0\leq t\leq T}$ is defined as follows:

$$\lambda^{\mathbb{G}} := \lambda + \mathbf{1}_{[\![0,\tau]\!]}\frac{\psi}{Z^\tau_-} + \mathbf{1}_{]\!]\tau,T]\!]}\frac{\varphi(\theta)}{\beta^\theta_-}\Big|_{\theta=\tau}$$

We have thus obtained the canonical decomposition of S with respect to $\mathbb{G}$ and, due to Theorem 4.3.2, we can conclude that the NUIP condition holds with respect to the enlarged filtration $\mathbb{G}$, thus proving part *(a)*. In order to prove parts *(b)* and *(c)*, observe that the mean-variance tradeoff process $\widehat{K}^{\mathbb{G}}$ of S with respect to $\mathbb{G}$ satisfies the following inequality, for all $t \in [0,T]$:

$$\begin{aligned}\widehat{K}^{\mathbb{G}}_t &:= \sum_{i,j=1}^{d} \int_0^t \left(\lambda^{\mathbb{G}}_u\right)^i \left(\lambda^{\mathbb{G}}_u\right)^j d\langle m^i, m^j\rangle_u \\ &\leq \widehat{K}_t + \int_0^{\tau\wedge t} \frac{1}{\left(Z^{\tau}_{u-}\right)^2} \psi'_u d\langle M, M\rangle_u \psi_u + \int_{\tau\wedge t}^{t} \frac{1}{\left(\beta^{\theta}_{u-}\right)^2} \varphi_u\left(\theta\right)' d\langle M, M\rangle_u \varphi_u\left(\theta\right)\Big|_{\theta=\tau}\end{aligned}$$

Then, by relying on the same arguments used in the last part of the proof of Proposition 4.4.24, we can easily prove parts *(b)* and *(c)*. □

In the context of credit risk modeling[6], Proposition 4.4.27 shows that, if one of the NUIP/NIAO/NUPBR no-arbitrage conditions holds with respect to the default-free financial market (i.e. with respect to $\mathbb{F}$), then the same condition holds with respect to the defaultable financial market (i.e. with respect to $\mathbb{G}$) as well, also after the random default time τ.

Remark 4.4.28 (Extension to discontinuous semimartingales). We want to point out that, similarly as in Remark 4.4.25, Proposition 4.4.27 can be extended to the more general situation where S is only assumed to be an $\mathbb{R}^d$-valued $\mathbb{F}$-locally square-integrable semimartingale, in the sense of Definition II.2.27 of Jacod & Shiryaev (2003). Indeed, in this case the predictable quadratic variation $\langle M, M\rangle$ is still well-defined. Furthermore, due to equation (4.30) together with the Kunita-Watanabe inequality (see e.g. Protter (2005), Theorem II.25), the finite variation part in the canonical decomposition of S with respect to $\mathbb{G}$ is absolutely continuous with respect to $\langle m, m\rangle = \langle M, M\rangle$. The validity of the NUIP condition with respect to the filtration $\mathbb{G}$ then follows from Theorem 4.3.2. In order to extend parts *(b)* and *(c)* of Proposition 4.4.27 to the case where S is not necessarily continuous, we need some further assumptions on the family of $\mathbb{F}$-martingales β^u, $u \in \mathbb{R}_+$. In particular, suppose that β^u, is an $\mathbb{F}$-locally square-integrable $\mathbb{F}$-martingale, for every $u \in \mathbb{R}_+$. Then, due to Ansel & Stricker (1993), there exists a Galtchouk-Kunita-Watanabe decomposition of β^u with respect to the $\mathbb{F}$-locally square-integrable $\mathbb{F}$-local martingale M. Since the remaining part of the proof of Proposition 4.4.27 does not rely on the continuity of S, we can then prove parts *(b)* and *(c)* also in the case where S is a general $\mathbb{F}$-locally square-integrable semimartingale.

Remark 4.4.29 (Progressive enlargements with respect to honest times). Let us now briefly consider the case where the filtration $\mathbb{G}$ is obtained as the progressive enlargement of $\mathbb{F}$ with respect to a random time τ, where the latter is supposed to be an *honest time*. This means that for every

[6] As pointed out in Jeanblanc & Le Cam (2009), initial times are well-suited to the modeling of credit risk. For instance, typical *intensity-based* (or *reduced-form*) models for default risk assume that the random default time τ is given by the first jump time of a Poisson process with stochastic intensity (i.e. a *doubly stochastic Poisson process*), see e.g. Chapter 7 of Jeanblanc et al. (2009). It can be easily shown that such a random time is a particular example of an initial time, see Section 5 of Jeanblanc & Le Cam (2009).

$t \geq 0$ there exists an $\mathcal{F}_t$-measurable random variable $\xi(t)$ such that $\tau = \xi(t)$ P-a.s. on the set $\{\tau < t\}$. Honest times have been extensively studied in the theory of enlargement of filtrations, see e.g. Chapter V of Jeulin (1980). In particular, Theorem 5.10 of Jeulin (1980) (compare also with Protter (2005), Theorem VI.18), shows that any $\mathbb{F}$-local martingale $M = (M_t)_{0 \leq t \leq T}$ with $M_0 = 0$ can be decomposed as follows:

$$M_t = m_t + \int_0^{\tau \wedge t} \frac{1}{Z^{\tau}_{u-}} d\langle M, \mu^{\tau}\rangle_u - \int_{\tau}^{\tau \wedge t} \frac{1}{1 - Z^{\tau}_{u-}} d\langle M, \mu^{\tau}\rangle_u \tag{4.31}$$

where the process $m = (m_t)_{0 \leq t \leq T}$ is a $\mathbb{G}$-local martingale and the $\mathbb{F}$-martingale μ^{τ} is defined as before (4.24). Observe that $\langle M, \mu^{\tau}\rangle = \langle m, \mu^{\tau}\rangle$ and, due to the Kunita-Watanabe inequality, the finite variation part in equation (4.31) is absolutely continuous with respect to $\langle m, m\rangle = \langle M, M\rangle$. Due to Theorem 4.3.2, this implies that, if the NUIP condition holds with respect to $\mathbb{F}$, then it holds with respect to $\mathbb{G}$ as well. However, stronger no-arbitrage conditions such as the NIAO and the NUPBR conditions (and, hence, also the classical NA and NFLVR conditions) are not necessarily preserved by a progressive enlargement of the filtration with respect to an honest time[7]. Indeed, an honest time can be equivalently characterized as "the end of an optional set" (see Jeulin (1980), Proposition 5.1, and Protter (2005), Theorem VI.16) and, therefore, an honest time can represent the time at which the discounted price process S of the risky assets crosses for the last time a given threshold or achieves its maximum value, see for instance Imkeller (2002). Therefore, from an economic point of view, the knowledge of an honest time allows one to construct very strong arbitrage strategies, which can also lead to immediate arbitrage opportunities, as pointed out in Imkeller (2002) and Zwierz (2007).

Initial enlargement of the reference filtration

Let us close this Section by briefly considering a different but related situation, where the filtration $\mathbb{G}$ is no longer supposed to be obtained as the progressive enlargement of $\mathbb{F}$ with respect to a random time τ but as the *initial enlargement* of $\mathbb{F}$ with respect to a random variable ξ. More formally, let us define the filtration $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T}$ as follows, for all $t \in [0, T]$:

$$\mathcal{G}_t := \bigcap_{s > t} \big(\mathcal{F}_s \vee \sigma\{\xi\}\big)$$

Similarly as in Definition 4.4.26, we make the following Assumption, first introduced by Jacod (1985).

[7] The decomposition (4.31) looks rather similar to the decomposition (4.29). Hence, one may wonder what could go wrong if one were to apply the arguments used in the proof of Proposition 4.4.27 to the case where the filtration $\mathbb{G}$ is obtained as the progressive enlargement of $\mathbb{F}$ with respect to an honest time τ. Technically, the problem is that in general the process $\frac{1}{1-Z^{\tau}_{-}}$ may fail to be locally bounded on the stochastic interval $\rrbracket \tau, T \rrbracket$. Indeed, suppose for instance that there exists a predictable set $\Gamma \subset \Omega \times [0, T]$ such that $\tau(\omega) = \sup\{t \in [0, T] : (\omega, t) \in \Gamma\}$. Then the Theorem on page 299 of Azéma (1972) gives that $Z^{\tau}_{\tau-} = 1$ P-a.s on the set $\{\tau > 0\}$, thus implying that the process $\frac{1}{1-Z^{\tau}_{-}}$ is not locally bounded on the stochastic interval $\rrbracket \tau, T \rrbracket$.

Assumption 4.4.30. *There exists a positive σ-finite measure η on $\mathcal{B}(\mathbb{R}_+)$ such that $Q_t(\omega, du) \ll \eta(du)$ holds P-a.s. for all $t \in [0,T]$, where $Q_t(\omega, du)$ denotes the regular $\mathcal{F}_t$-conditional distribution of the random variable ξ.*

Due to Lemma 1.8 of Jacod (1985), there exists a family of positive $\mathbb{F}$-martingales $q^u = (q^u_t)_{0 \leq t \leq T}$, $u \in \mathbb{R}_+$, such that, for all $t \in [0,T]$, the measure $\eta(du)\, q^u_t(\omega)$ is a version of $Q_t(\omega, du)$. Due to Theorem 1.1 of Jacod (1985), the (H')-hypothesis holds between the filtrations $\mathbb{F}$ and $\mathbb{G}$, meaning that any $\mathbb{F}$-semimartingale is also a $\mathbb{G}$-semimartingale. Furthermore, due to Theorem 2.5 of Jacod (1985), any $\mathbb{F}$-locally square-integrable $\mathbb{F}$-local martingale $M = (M_t)_{0 \leq t \leq T}$ with $M_0 = 0$ admits the following $\mathbb{G}$-canonical decomposition:

$$M_t = m_t + \int_0^t \frac{1}{q^\theta_{u-}} d\langle q^\theta, M\rangle_u \Big|_{\theta=\xi} \qquad t \in [0,T]$$

where $m = (m_t)_{0 \leq t \leq T}$ is a $\mathbb{G}$-local martingale. Note that Corollary 1.11 of Jacod (1985) ensures that the process q^ξ_- is P-a.s. strictly positive. Thus, the same arguments used in the proofs of Propositions 4.4.24 and 4.4.27 allow to easily show the following result.

Proposition 4.4.31. *Let $\mathbb{G}$ be the initial enlargement of the filtration $\mathbb{F}$ with respect to the random variable ξ and suppose that Assumption 4.4.30 holds. Then the following hold:*

(a) if the NUIP condition holds with respect to $\mathbb{F}$ then it holds with respect to $\mathbb{G}$ as well;

(b) if the NIAO condition holds with respect to $\mathbb{F}$ then it holds with respect to $\mathbb{G}$ as well;

(c) if the NUPBR condition holds with respect to $\mathbb{F}$ then it holds with respect to $\mathbb{G}$ as well.

In the case where S is continuous, some related results on the preservation of the structure condition under an initial enlargement of the reference filtration can also be found in Section 2 of Campi (2005). Similarly as in Remark 4.4.28, the above Proposition can be extended to the case where S is a general (not necessarily continuous) $\mathbb{F}$-locally square-integrable semimartingale.

Summing up, the results of the present Section show that the NUIP no-arbitrage condition is stable with respect to an enlargement of the reference filtration under minimal conditions, regardless of the nature of the information added to the original filtration $\mathbb{F}$. The NIAO/NUPBR no-arbitrage conditions also enjoy good stability properties with respect to an enlargement of the filtration, at least under (not very restrictive) technical assumptions. However, in the present degree of generality, one cannot prove general results on the preservation of the classical NA and NFLVR no-arbitrage conditions under an enlargement of the filtration, the reason being that one cannot check the validity of these no-arbitrage conditions by looking only at the characteristics of the discounted price process S. Of course, this represents a limitation of the classical no-arbitrage theory based on the NA and NFLVR conditions.

4.5 General characterizations of hedgeable contingent claims

So far, we have been concerned with general characterizations of weak no-arbitrage conditions and with the study of their stability properties with respect to changes in the structure of the underlying financial market model. This last Section deals with a more "practical" issue, namely the problem of characterizing the set of contingent claims which can be perfectly replicated in the financial market. Under the traditional *No Free Lunch with Vanishing Risk (NFLVR)* condition, general results on the attainability of contingent claims have been obtained in the classical works Jacka (1992), Ansel & Stricker (1994) and Delbaen & Schachermayer (1994),(1995c),(1998b). The novelty of the present Section is represented by the fact that we do not assume that the NFLVR condition holds, but only that the weaker *No Unbounded Profit with Bounded Risk (NUPBR)* condition holds. This allows us to extend the scope of the analysis of hedging problems towards interesting financial market models which do not satisfy the NFLVR condition, as in the context of *Stochastic Portfolio Theory* and *Benchmark Approach*, see Fernholz & Karatzas (2009) and Chapters 12-13 of Platen & Heath (2006), respectively.

As in Section 4.2, let us suppose that the discounted price process $S = (S_t)_{0\leq t\leq T}$ of d risky assets is a continuous semimartingale on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. For simplicity, we assume that the initial σ-field $\mathcal{F}_0$ is trivial and we let $\mathcal{F} = \mathcal{F}_T$. As we have seen in Section 4.3.3, as soon as the NUPBR condition holds, the process $\widehat{Z} = \mathcal{E}\left(-\lambda \cdot M\right)$ is well-defined as a martingale deflator, in the sense of Definition 4.3.11, where the $\mathbb{R}^d$-valued predictable process $\lambda = (\lambda_t)_{0\leq t\leq T}$ is as in Theorem 4.3.2. Furthermore, the process $1/\widehat{Z}$ satisfies the following SDE (compare also with Remark 4.3.16):

$$d\frac{1}{\widehat{Z}_t} = -\frac{1}{\widehat{Z}_t^2}d\widehat{Z}_t + \frac{1}{\widehat{Z}_t^3}d\langle\widehat{Z}\rangle_t = \frac{1}{\widehat{Z}_t}\lambda_t dM_t + \frac{1}{\widehat{Z}_t}\lambda'_t d\langle M, M\rangle_t \lambda_t = \frac{\lambda_t}{\widehat{Z}_t}dS_t$$

The process $\widehat{Z}$ is continuous and, hence, also predictable and locally bounded and Theorem 4.3.23 shows that $\lambda \in L^2_{loc}\left(M\right)$ if the NUPBR condition holds. This implies that, as soon as the NUPBR condition holds, we have $\lambda/\widehat{Z} \in L\left(S\right)$. We have thus shown that the martingale deflator $\widehat{Z}$ can be represented as the reciprocal of the wealth process $V = (V_t)_{0\leq t\leq T}$ corresponding to a self-financing trading strategy $H^V = \left(H_t^V\right)_{0\leq t\leq T}$, namely[8]:

$$1/\widehat{Z} = V = 1 + H^V \cdot S \qquad \text{where } H_t^V = \lambda_t/\widehat{Z}_t,\ t \in [0, T]$$

Furthermore, since $\widehat{Z} > 0$ P-a.s., the strategy H^V is 1-admissible, i.e. $H^V \in \mathcal{A}_1$. In particular, due to Definition 4.4.1, the process $V = (V_t)_{0\leq t\leq T}$ can be viewed as a numéraire for S. In the following we shall often use $V = 1/\widehat{Z}$ as a numéraire for S and we denote by $\bar{S} = \left(\frac{S}{V}, \frac{1}{V}\right)$ the price process expressed in terms of the numéraire V, as in Section 4.4.1. At this point, observe the

[8]It is well-known that, under the NUPBR condition, the wealth process $V = (V_t)_{0\leq t\leq T}$ generated by the strategy $H^V = \lambda/\widehat{Z}$ corresponds to the so-called *growth-optimal portfolio*, see for instance Christensen & Larsen (2007), Karatzas & Kardaras (2007), Hulley & Schweizer (2010) and, in the special case of an Itô-process model, Fontana & Runggaldier (2011). Formally, the growth-optimal portfolio is defined as the element $V = 1 + H \cdot S$ with $V > 0$ P-a.s. such that $E\left[\log\left(V'_T/V_T\right)\right] \leq 0$ for all $V' = 1 + H' \cdot S$ with $H' \in \mathcal{A}_1$.

following fundamental fact. Since $\widehat{Z}$ is a martingale deflator, the process $\bar{S} = \left(\frac{S}{V}, \frac{1}{V}\right) = \left(\widehat{Z}S, \widehat{Z}\right)$ is a local P-martingale. In particular, this implies that the probability measure P is (trivially) an *Equivalent Local Martingale Measure* for $\bar{S}$. Hence, due to part *(c)* of Proposition 4.3.28, the V-discounted price process $\bar{S}$ satisfies the NFLVR condition. In the following, this key property will be used as follows. Suppose that the original price process S satisfies the NUPBR condition. As a first step, apply a change of numéraire with respect to $V = 1/\widehat{Z}$. The V-discounted price process $\bar{S}$ then satisfies the NFLVR condition and, hence, we can apply the classical results of Delbaen & Schachermayer (1994),(1995c) with respect to the price process $\bar{S} = \left(\frac{S}{V}, \frac{1}{V}\right)$. Finally, we go back to the original price process S.

4.5.1 Maximal elements

In order to make this approach work, we first need some preliminaries. Let us denote by $\mathcal{D}$ the set of all martingale deflators for S, in the sense of Definition 4.3.11. Due to Theorem 4.3.23, as soon as the NUPBR condition holds, the set $\mathcal{D}$ is non-empty. Furthermore, in view of Proposition 4.3.24, we have:

$$\mathcal{D} = \left\{\widehat{Z}\,\mathcal{E}(N) : N \text{ local martingale, with } N_0 = 0, N \perp M, \Delta N > -1 \; P\text{-a.s.}\right\} \tag{4.32}$$

where $\perp$ denotes strong orthogonality, in the sense of Definition I.4.11 of Jacod & Shiryaev (2003). Furthermore, since $\bar{S} = \left(\frac{S}{V}, \frac{1}{V}\right)$ satisfies the NFLVR condition, the set of all *Equivalent Local Martingale Measures (ELMMs)* for $\bar{S}$ is non-empty (recall part *(c)* of Proposition 4.3.28). Let us denote by $\mathcal{M}^e\left(P, \bar{S}\right)$ the set of all density processes of ELMMs:

$$\mathcal{M}^e\left(P, \bar{S}\right) := \left\{Z^Q = \left(Z_t^Q\right)_{0\leq t\leq T} : Z_t^Q = \frac{dQ|_{\mathcal{F}_t}}{dP|_{\mathcal{F}_t}} \text{ for all } t \in [0,T], Q \text{ is an ELMM for } \bar{S}\right\}$$

With a slight abuse of notation, we shall sometimes simply write $Q \in \mathcal{M}^e\left(P, \bar{S}\right)$ with the meaning that the density process Z^Q of Q with respect to P belongs to $\mathcal{M}^e\left(P, \bar{S}\right)$. Since $\bar{S}$ is already a local P-martingale, we have $P \in \mathcal{M}^e\left(P, \bar{S}\right)$. Note however that the set $\mathcal{M}^e\left(P, \bar{S}\right)$ may contain infinitely many elements other than P. The following Lemma shows the general structure of the set $\mathcal{M}^e\left(P, \bar{S}\right)$.

Lemma 4.5.1. *Suppose that the NUPBR condition holds and let the $\mathbb{R}^{d+1}$-valued process $\bar{S} = \left(\bar{S}_t\right)_{0\leq t\leq T}$ be defined as above. Then the following hold:*

$$\mathcal{M}^e\left(P, \bar{S}\right) = \left\{\mathcal{E}(N) : N \text{ local martingale, with } N_0 = 0, N \perp M, \Delta N > -1 \; P\text{-a.s. and } E\left[\mathcal{E}(N)_T\right] = 1\right\}$$

where M denotes the local martingale part in the canonical decomposition of S.

Proof. As soon as the NUPBR condition holds, the process $\bar{S}$ is a local martingale. More precisely, due to equations (4.45) and (4.43) with $H = \lambda/\widehat{Z}$, for all $t \in [0,T]$:

$$\begin{aligned} \bar{S}_t^i &= S_0^i + \int_0^t \frac{1}{V_u}\left(e^i - \frac{\bar{S}_u^i \lambda_u}{\widehat{Z}_u}\right) dM_u = S_0^i + \int_0^t \left(\widehat{Z}_u e^i - \bar{S}_u^i \lambda_u\right) dM_u \qquad i = 1, \ldots, d \\ \bar{S}_t^{d+1} &= 1 - \int_0^t \frac{1}{V_u^2}\frac{\lambda_u}{\widehat{Z}_u} dM_u = 1 - \int_0^t \widehat{Z}_u \lambda_u dM_u \end{aligned} \tag{4.33}$$

Analogously as in Definition 4.3.11, a martingale deflator for $\bar{S}$ is a P-a.s. strictly positive local martingale $\bar{Z} = \left(\bar{Z}_t\right)_{0\leq t\leq T}$ with $\bar{Z}_0 = 1$ such that the product $\bar{Z}\bar{S}^i$ is a local martingale for all $i = 1,\dots,d+1$. It can be easily seen (see the first part of the proof of Proposition 4.3.24) that every martingale deflator $\bar{Z} = \left(\bar{Z}_t\right)_{0\leq t\leq T}$ for $\bar{S}$ can be represented as $\bar{Z} = \mathcal{E}(N)$, for some local martingale $N = (N_t)_{0\leq t\leq T}$ strongly orthogonal to $\bar{S}$ with $N_0 = 0$ and $\Delta N > -1$ P-a.s. Furthermore, a martingale deflator $\bar{Z}$ for $\bar{S}$ is the density process of an ELMM for $\bar{S}$ if and only if $E\left[\bar{Z}_T\right] = 1$. Hence, to prove the Lemma it remains to show that N is strongly orthogonal to $\bar{S}$ if and only if it is strongly orthogonal to M. If N is strongly orthogonal to M, it readily follows from (4.33), together with the integration by parts formula, that N is also strongly orthogonal to $\bar{S}$. Conversely, suppose that N is strongly orthogonal to $\bar{S}$, so that $\langle\bar{S}^i,N\rangle\equiv 0$ for all $i=1,\dots,d+1$. Then, for $i=1,\dots,d$, using the second equation in (4.33) and recalling that $\bar{S}^i=\widehat{Z}S^i$:

$$0=\langle\bar{S}^i,N\rangle=\int\left(\widehat{Z}e^i-\bar{S}^i\lambda\right)d\langle M,N\rangle=\int\widehat{Z}d\langle M^i,N\rangle+\int S^i d\langle\bar{S}^{d+1},N\rangle=\int\widehat{Z}d\langle M^i,N\rangle$$

Since $\widehat{Z}>0$ P-a.s., this implies that $\langle M^i,N\rangle\equiv 0$ for all $i=1,\dots,d$, meaning that N is strongly orthogonal to M. □

Let us now introduce the following Definition, in the spirit of Delbaen & Schachermayer (1994),(1995c). For any $a>0$, the sets $\mathcal{A}_a$ and $\bar{\mathcal{A}}_a$ are defined as in Sections 4.2 and 4.4.1, respectively.

Definition 4.5.2. *For any $a>0$, let the sets $\mathcal{K}_a$ and $\bar{\mathcal{K}}_a$ be defined as follows:*

$$\mathcal{K}_a:=\left\{(H\cdot S)_T : H\in\mathcal{A}_a\right\}\qquad\bar{\mathcal{K}}_a:=\left\{(\bar{H}\cdot\bar{S})_T:\bar{H}\in\bar{\mathcal{A}}_a\right\}$$

For any $a>0$, we say that an element $f\in\mathcal{K}_a$ is maximal *in $\mathcal{K}_a$ if the properties $g\geq f$ P-a.s. and $g\in\mathcal{K}_a$ imply that $g=f$ P-a.s. Analogously, for any $a>0$, we say that an element $\bar{f}\in\bar{\mathcal{K}}_a$ is* maximal *in $\bar{K}_a$ if the properties $\bar{g}\geq\bar{f}$ P-a.s. and $\bar{g}\in\bar{K}_a$ imply that $\bar{g}=\bar{f}$ P-a.s.*

We have then the following simple Lemma, which refines the result of Lemma 4.4.2. As in Lemma 4.4.2, we let $\bar{H}^V:=(0,\dots,0,1)'\in\mathbb{R}^{d+1}$.

Lemma 4.5.3. *Let V be a numéraire for S, with $V=1+H^V\cdot S$, and let $a>0$. If $H\in\mathcal{A}_a$, then there exists an element $\bar{H}\in\bar{\mathcal{A}}_a$ such that $\frac{1}{V}(H\cdot S)=\left(\bar{H}-a\bar{H}^V\right)\cdot\bar{S}$. In addition, if $(H\cdot S)_T$ is maximal in $\mathcal{K}_a$, then $\left(\bar{H}\cdot\bar{S}\right)_T$ is maximal in $\bar{\mathcal{K}}_a$. Conversely, if $\bar{H}\in\bar{\mathcal{A}}_a$, then there exists an element $H\in\mathcal{A}_a$ such that $V\left(\bar{H}\cdot\bar{S}\right)=\left(H-aH^V\right)\cdot S$. In addition, if $\left(\bar{H}\cdot\bar{S}\right)_T$ is maximal in $\bar{\mathcal{K}}_a$, then $(H\cdot S)_T$ is maximal in $\mathcal{K}_a$.*

Proof. Let $H\in\mathcal{A}_a$ and suppose that $(H\cdot S)_T$ is maximal in $\mathcal{K}_a$. The existence of an element $\bar{H}\in\bar{\mathcal{A}}_a$ such that $\frac{1}{V}(H\cdot S)=\left(\bar{H}-a\bar{H}^V\right)\cdot\bar{S}$ follows from Lemma 4.4.2. It remains to show the maximality of $\left(\bar{H}\cdot\bar{S}\right)_T$ in $\bar{\mathcal{K}}_a$. So, let $\bar{K}\in\bar{\mathcal{A}}_a$ and suppose that $\left(\bar{K}\cdot\bar{S}\right)_T\geq\left(\bar{H}\cdot\bar{S}\right)_T$ P-a.s. Then:

$$\frac{1}{V_T}(H\cdot S)_T=\left(\left(\bar{H}-a\bar{H}^V\right)\cdot\bar{S}\right)_T\leq\left(\left(\bar{K}-a\bar{H}^V\right)\cdot\bar{S}\right)_T\qquad P\text{-a.s.}\tag{4.34}$$

Due to Lemma 4.4.2, there exists an element $K \in \mathcal{A}_a$ such that $\left(\bar{K} - a\bar{H}^V\right) \cdot \bar{S} = \frac{1}{V}\left(K \cdot S\right)$. Then, the inequality in (4.34) implies that $(H \cdot S)_T \leq (K \cdot S)_T$ P-a.s. Since $(H \cdot S)_T$ is assumed to be maximal in $\mathcal{K}_a$, we have $(H \cdot S)_T = (K \cdot S)_T$ P-a.s., which in turn implies $\left(\bar{H} \cdot \bar{S}\right)_T = \left(\bar{K} \cdot \bar{S}\right)_T$ P-a.s., thus showing the maximality of $\left(\bar{H} \cdot \bar{S}\right)_T$ in $\bar{\mathcal{K}}_a$. The converse statement can be proved in a similar way. □

The following Proposition provides several equivalent characterizations of maximal elements. Its proof is based on a change of numéraire with respect to $V = 1/\widehat{Z}$, on Corollary 4.6 of Delbaen & Schachermayer (1995c) (applied to the V-discounted price process $\bar{S}$) and on Lemma 4.5.3.

Proposition 4.5.4. *Suppose that the NUPBR condition holds and let $H \in \mathcal{A}_a$, for some $a > 0$. Let $\bar{H}$ be an element in $\bar{\mathcal{A}}_a$ such that $\frac{1}{V}\left(H \cdot S\right) = \left(\bar{H} - a\bar{H}^V\right) \cdot \bar{S}$. Then the following are equivalent:*

(i) $(H \cdot S)_T$ *is maximal in* $\mathcal{K}_a$*;*

(ii) $\left(\bar{H} \cdot \bar{S}\right)_T$ *is maximal in* $\bar{\mathcal{K}}_a$*;*

(iii) there exists an element $Q \in \mathcal{M}^e\left(P, \bar{S}\right)$ *such that* $E^Q\left[\left(\bar{H} \cdot \bar{S}\right)_T\right] = 0$*;*

(iv) there exists an element $Q \in \mathcal{M}^e\left(P, \bar{S}\right)$ *such that* $\bar{H} \cdot \bar{S}$ *is a* Q*-martingale;*

(v) there exists an element $Z = (Z_t)_{0 \leq t \leq T} \in \mathcal{D}$ *such that* $E\big[Z_T\big(a + (H \cdot S)_T\big)\big] = a$*;*

(vi) there exists an element $Z = (Z_t)_{0 \leq t \leq T} \in \mathcal{D}$ *such that* $Z\left(a + H \cdot S\right)$ *is a* P*-martingale.*

Proof. The implication *(i)* ⇒ *(ii)* follows from Lemma 4.5.3. Since the $\mathbb{R}^{d+1}$-valued continuous semimartingale $\bar{S}$ satisfies the NFLVR condition, the equivalence *(ii)* ⇔ *(iii)* ⇔ *(iv)* follows from Corollary 4.6 of Delbaen & Schachermayer (1995c). We now show that *(iii)* ⇒ *(v)*. Let $Q \in \mathcal{M}^e\left(P, \bar{S}\right)$ and suppose that $E^Q\left[\left(\bar{H} \cdot \bar{S}\right)_T\right] = 0$. Then, due to Lemma 4.5.1, this means that there exists a local martingale $N = (N_t)_{0 \leq t \leq T}$ strongly orthogonal to M with $N_0 = 0$, $\Delta N > -1$ P-a.s. and $E\left[\mathcal{E}\left(N\right)_T\right] = 1$ such that:

$$0 = E^Q\left[\left(\bar{H} \cdot \bar{S}\right)_T\right] = E\left[\frac{dQ}{dP}\left(\bar{H} \cdot \bar{S}\right)_T\right] = E\left[\mathcal{E}\left(N\right)_T\left(\bar{H} \cdot \bar{S}\right)_T\right] \tag{4.35}$$

Now observe that, by the definition of $\bar{H}$ in the statement of the Proposition:

$$\bar{H} \cdot \bar{S} = \frac{1}{V}\left(H \cdot S\right) + a\left(\bar{H}^V \cdot \bar{S}\right) = \frac{1}{V}\left(H \cdot S\right) + a\left(\frac{1}{V} - 1\right) \tag{4.36}$$

By combining (4.35) and (4.36) we get:

$$0 = E\left[\frac{\mathcal{E}\left(N\right)_T}{V_T}\left(H \cdot S\right)_T\right] + aE\left[\frac{\mathcal{E}\left(N\right)_T}{V_T}\right] - a = E\left[\widehat{Z}_T\,\mathcal{E}\left(N\right)_T\left(a + \left(H \cdot S\right)_T\right)\right] - a$$

Due to Proposition 4.3.24, the product $\widehat{Z}\,\mathcal{E}\left(N\right)$ is a martingale deflator for S, thus showing *(v)*. To prove that *(v)* ⇒ *(vi)*, note that, due to Proposition 4.3.13, the product $Z\left(a + H \cdot S\right)$ is a non-negative local P-martingale for every $H \in \mathcal{A}_a$, and, due to Fatou's Lemma, it is also a supermartingale. So, the product $Z\left(a + H \cdot S\right)$ is a P-martingale if and only if $E\big[Z_T\big(a + (H \cdot S)_T\big)\big] =$

$E\left[Z_0\big(a+(H\cdot S)_0\big)\right]=a$. It remains to show that *(vi)* $\Rightarrow$ *(i)*. Suppose that Z is an element of $\mathcal{D}$ such that $Z\,(a+H\cdot S)$ is a P-martingale and let $K\in\mathcal{A}_a$ with $(K\cdot S)_T\geq(H\cdot S)_T$ P-a.s. Then:

$$a=E\big[Z_0\big(a+(H\cdot S)_0\big)\big]=E\big[Z_T\big(a+(H\cdot S)_T\big)\big]\leq E\big[Z_T\big(a+(K\cdot S)_T\big)\big]\leq a$$

where the last inequality follows from the fact that the product $Z\,(a+K\cdot S)$ is a supermartingale, being a non-negative local martingale (see Lemma 4.3.13). This shows that $E\left[Z_T\,(H\cdot S)_T\right]=E\left[Z_T\,(K\cdot S)_T\right]$, thus implying $(K\cdot S)_T=(H\cdot S)_T$ P-a.s. Due to Definition 4.5.2, this shows that $(H\cdot S)_T$ is maximal in $\mathcal{K}_a$. □

4.5.2 Attainable contingent claims and market completeness

We are now in a position to start dealing with the main theme of this Section, namely the characterization of those contingent claims which can be perfectly replicated by trading in the financial market according to a self-financing trading strategy. More precisely, let us give the following Definition.

Definition 4.5.5. *A* contingent claim *is a non-negative $\mathcal{F}$-measurable random variable. A contingent claim f is said to be* attainable (or hedgeable) for S *if there exists a pair $(x,H)\in\mathbb{R}_+\times\mathcal{A}_x$ such that $f=x+(H\cdot S)_T$ holds P-a.s. and $(H\cdot S)_T$ is maximal in $\mathcal{K}_x$. Analogously, we say that f is* attainable (or hedgeable) for $\bar{S}$ *if there exists a pair $\big(\bar{x},\bar{H}\big)\in\mathbb{R}_+\times\bar{\mathcal{A}}_{\bar{x}}$ such that $f=\bar{x}+\big(\bar{H}\cdot\bar{S}\big)_T$ holds P-a.s. and $\big(\bar{H}\cdot\bar{S}\big)_T$ is maximal in $\bar{\mathcal{K}}_{\bar{x}}$.*

As pointed out in Delbaen & Schachermayer (1995c), there is a good reason to require the use of maximal elements in Definition 4.5.5. In fact, let f be a given contingent claim and suppose that there exists a pair $(x,H)\in\mathbb{R}_+\times\mathcal{A}_x$ such that $f=x+(H\cdot S)_T$ holds P-a.s. but $(H\cdot S)_T$ is not maximal in $\mathcal{K}_x$. Then, there exists an element $K\in\mathcal{A}_x$ such that $x+(K\cdot S)_T\geq f$ P-a.s., with strict inequality on some set with non-zero probability. Hence, an investor who starts from the initial endowment x will obtain a better final payoff by trading according to the strategy $K\in\mathcal{A}_x$. Clearly, in this case the contingent claim f cannot be the result of an optimal hedging policy. The following Theorem gives necessary and sufficient conditions for a given contingent claim f to be attainable. Again, the proof is based on a change of numéraire with respect to $V=1/\widehat{Z}$ combined with the classical results of Delbaen & Schachermayer (1995c), here applied to the V-discounted price process $\bar{S}$.

Theorem 4.5.6. *Suppose that the NUPBR condition holds. Let f be a contingent claim and define $\bar{f}:=f\widehat{Z}_T$. Then the following are equivalent:*

(i) f is attainable for S, i.e. there exists a pair $(x,H)\in\mathbb{R}_+\times\mathcal{A}_x$ such that $f=x+(H\cdot S)_T$ holds P-a.s. and $(H\cdot S)_T$ is maximal in K_x;

(ii) $\bar{f}$ is attainable for $\bar{S}$, i.e. there exists a pair $\big(\bar{x},\bar{H}\big)\in\mathbb{R}_+\times\bar{\mathcal{A}}_{\bar{x}}$ such that $\bar{f}=\bar{x}+\big(\bar{H}\cdot\bar{S}\big)_T$ holds P-a.s. and $\big(\bar{H}\cdot\bar{S}\big)_T$ is maximal in $\bar{\mathcal{K}}_{\bar{x}}$;

(iii) there exists an element $Q \in \mathcal{M}^e\left(P, \bar{S}\right)$ such that:

$$E^Q\left[\,\bar{f}\,\right] = \sup\Big\{E^R\left[\,\bar{f}\,\right] : R \in \mathcal{M}^e\left(P, \bar{S}\right)\Big\} < \infty$$

(iv) there exists an element $Z^* = (Z^*_t)_{0\leq t\leq T} \in \mathcal{D}$ such that:

$$E\left[Z^*_T f\right] = \sup\Big\{E\left[Z_T f\right] : Z \in \mathcal{D}\Big\} < \infty$$

Furthermore, parts (i) and (ii) are related as follows:

$$x = \bar{x} \qquad \text{and} \qquad \bar{H}\cdot\bar{S} = \frac{1}{V}\Big(\left(H - xH^V\right)\cdot S\Big) \tag{4.37}$$

Proof. Since the $\mathbb{R}^{d+1}$-valued continuous semimartingale $\bar{S}$ satisfies the NFLVR condition and $\widehat{Z}_T > 0$ P-a.s., the equivalence *(ii)* $\Leftrightarrow$ *(iii)* follows from Theorem 4.11 of Delbaen & Schachermayer (1995c). Let us now show that *(ii)* $\Rightarrow$ *(i)*. Suppose that there exists a pair $\left(\bar{x}, \bar{H}\right) \in \mathbb{R}_+ \times \bar{\mathcal{A}}_{\bar{x}}$ such that $\bar{f} = \bar{x} + \left(\bar{H}\cdot\bar{S}\right)_T$ holds P-a.s. and $\left(\bar{H}\cdot\bar{S}\right)_T$ is maximal in $\bar{\mathcal{K}}_{\bar{x}}$. Let H be an element of $\mathcal{A}_{\bar{x}}$ such that $\bar{H}\cdot\bar{S} = \frac{1}{V}\big((H - \bar{x}H^V)\cdot S\big)$, as in Lemma 4.5.3. Then:

$$\begin{aligned} f = \bar{f}V_T &= \Big(\bar{x} + \left(\bar{H}\cdot\bar{S}\right)_T\Big)V_T = \bar{x}V_T + \Big(\left(H - \bar{x}H^V\right)\cdot S\Big)_T = \bar{x}V_T + (H\cdot S)_T - \bar{x}\left(V_T - 1\right) \\ &= \bar{x} + (H\cdot S)_T \end{aligned} \tag{4.38}$$

Furthermore, Lemma 4.5.3 implies that $(H\cdot S)_T$ is maximal in $\mathcal{K}_{\bar{x}}$, thus showing that the contingent claim f is attainable for S, in the sense of Definition 4.5.5. Let us now show that *(i)* $\Rightarrow$ *(iv)*. If f is attainable for S, there exists a pair $(x, H) \in \mathbb{R}_+ \times \mathcal{A}_x$ such that $f = x + (H\cdot S)_T$ holds P-a.s. and $(H\cdot S)_T$ is maximal in $\mathcal{K}_x$. Due to Proposition 4.5.4, this implies that there exists an element $Z^* = (Z^*_t)_{0\leq t\leq T} \in \mathcal{D}$ such that $E\left[Z^*_T\big(x + (H\cdot S)_T\big)\right] = x$. Thus, for any $Z = (Z_t)_{0\leq t\leq T} \in \mathcal{D}$:

$$E\left[Z^*_T f\right] = E\left[Z^*_T\big(x + (H\cdot S)_T\big)\right] = x \geq E\left[Z_T\big(x + (H\cdot S)_T\big)\right] = E\left[Z_T f\right]$$

where the inequality follows since the product $Z\left(x + H\cdot S\right)$ is a non-negative local martingale (see Lemma 4.3.13) and, due to Fatou's Lemma, also a supermartingale. This shows *(iv)*.

Finally, let us prove that *(iv)* implies *(ii)*. Due to Lemma 4.5.1 and (4.32), it can be easily seen that the following holds:

$$\sup\Big\{E^R\left[\,\bar{f}\,\right] : R \in \mathcal{M}^e\left(P, \bar{S}\right)\Big\} \leq \sup\Big\{E\left[Z_T f\right] : Z \in \mathcal{D}\Big\} = E\left[Z^*_T f\right] < \infty \tag{4.39}$$

Conversely, let $N = (N_t)_{0\leq t\leq T}$ be an arbitrary local martingale strongly orthogonal to M with $N_0 = 0$ and $\Delta N > -1$ P-a.s. and let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence for $\mathcal{E}\left(N\right)$. Then, for every $n \in \mathbb{N}$, the stopped process $\mathcal{E}\left(N\right)^{\tau_n} = \mathcal{E}\left(N^{\tau_n}\right)$ is a martingale and, since strong orthogonality is preserved under stopping (see Jacod & Shiryaev (2003), Lemma I.4.13), we have $\mathcal{E}\left(N^{\tau_n}\right) \in \mathcal{M}^e\left(P, \bar{S}\right)$. Thus, due to Fatou's Lemma:

$$\begin{aligned} E\left[\widehat{Z}_T\,\mathcal{E}\left(N\right)_T f\right] &= E\left[\lim_{n\to\infty}\widehat{Z}_T\,\mathcal{E}\left(N\right)_{\tau_n\wedge T} f\right] \leq \liminf_{n\to\infty} E\left[\mathcal{E}\left(N^{\tau_n}\right)_T \bar{f}\,\right] \\ &\leq \sup\Big\{E^R\left[\,\bar{f}\,\right] : R \in \mathcal{M}^e\left(P, \bar{S}\right)\Big\} \end{aligned} \tag{4.40}$$

By combining (4.39) and (4.40) and recalling Proposition 4.3.24, we have thus shown the following equality:

$$\sup\left\{E^R\left[\bar{f}\right] : R \in \mathcal{M}^e\left(P, \bar{S}\right)\right\} = \sup\bigl\{E\left[Z_T f\right] : Z \in \mathcal{D}\bigr\} = E\left[Z_T^* f\right] < \infty$$

Since $\bar{S}$ is continuous and satisfies NFLVR, Corollary 3.5 of Delbaen & Schachermayer (1995c) gives the existence of an element $\bar{H} \in \bar{\mathcal{A}}$ such that $\bar{f} \leq \bar{x} + \left(\bar{H} \cdot \bar{S}\right)_T$ holds P-a.s., where $\bar{x} := \sup\left\{E^R\left[\bar{f}\right] :\ R \in \mathcal{M}^e\left(P, \bar{S}\right)\right\}$, and such that $\left(\bar{H} \cdot \bar{S}\right)_T$ is maximal in $\bigcup_{a>0} \bar{\mathcal{K}}_a$. Furthermore, due to Proposition 3.5 of Delbaen & Schachermayer (1994), we have $\bar{H} \in \bar{\mathcal{A}}_{\bar{x}}$. Observe now that, due to Proposition 4.3.24, we can write $Z^* = \widehat{Z}\,\mathcal{E}\left(N^*\right)$ for some local martingale $N^* = (N_t^*)_{0 \leq t \leq T}$ strongly orthogonal to M with $N_0^* = 0$ and $\Delta N^* > -1$ P-a.s. Furthermore, the product $\mathcal{E}\left(N^*\right)\left(\bar{H} \cdot \bar{S}\right)$ is a local martingale. In fact, due to the integration by parts formula:

$$\mathcal{E}\left(N^*\right)\left(\bar{H} \cdot \bar{S}\right) = \left((\bar{H} \cdot \bar{S})\mathcal{E}\left(N^*\right)_-\right) \cdot N^* + \left(\mathcal{E}\left(N^*\right)_- \bar{H}\right) \cdot \bar{S} + \left(\mathcal{E}\left(N^*\right)_- \bar{H}\right) \cdot \left[\bar{S}, N^*\right] \quad (4.41)$$

Since N^* and $\bar{S}$ are strongly orthogonal local martingales (see the proof of Lemma 4.5.1), this shows the local martingale property of $\mathcal{E}\left(N^*\right)\left(\bar{H} \cdot \bar{S}\right)$. In particular, since $\bar{H} \in \bar{\mathcal{A}}_{\bar{x}}$, Fatou's Lemma implies that $\mathcal{E}\left(N^*\right)\left(\bar{x} + \bar{H} \cdot \bar{S}\right)$ is also a supermartingale. Hence:

$$\bar{x} = E\left[Z_T^* f\right] = E\left[\mathcal{E}\left(N^*\right)_T \bar{f}\,\right] \leq E\left[\mathcal{E}\left(N^*\right)_T \left(\bar{x} + (\bar{H} \cdot \bar{S})_T\right)\right] \leq \bar{x}$$

We have thus shown that $E\left[\mathcal{E}\left(N^*\right)_T \bar{f}\,\right] = E\left[\mathcal{E}\left(N^*\right)_T \left(\bar{x} + (\bar{H} \cdot \bar{S})_T\right)\right]$, which in turn implies that $\bar{f} = \bar{x} + \left(\bar{H} \cdot \bar{S}\right)_T$ holds P-a.s. Recalling that $\left(\bar{H} \cdot \bar{S}\right)_T$ is maximal in $\bar{\mathcal{K}}_{\bar{x}}$, we have thus proved *(ii)*. The last assertion of the Theorem follows by elementary computations as in (4.38). □

In particular, the most interesting result of Theorem 4.5.6 is given by the equivalence between parts *(i)* and *(iv)*. This equivalence result represents an extension of the classical results of Jacka (1992), Ansel & Stricker (1994) and Delbaen & Schachermayer (1995c) on the attainability of contingent claims to the more general situation where, instead of the NFLVR condition, only the weaker NUPBR condition is supposed to hold. Furthermore, Theorem 4.5.6 highlights the fact that even in the absence of a well-defined ELMM the situation is not hopeless, since we can easily work with martingale deflators. In the special case of an Itô process model, the implication *(iv)* ⇒ *(i)* has been recently shown in Chapter 2 of Ruf (2011b), albeit with a different (and, to our mind, more involved) proof.

Remark 4.5.7 (Comparison with the results of Stricker & Yan (1998)). The equivalence *(i)* ⇔ *(iv)* in Theorem 4.5.6 bears a close similarity with the result of Theorem 3.2 of Stricker & Yan (1998). Indeed, working under the same set of assumptions as in the present Section, in Stricker & Yan (1998) the authors show the equivalence between the condition in part *(iv)* of our Theorem 4.5.6 and the *replicability* of the contingent claim f, in the sense that there exists a pair $(x, H) \in \mathbb{R}_+ \times L\left(S\right)$ such that $f = x + (H \cdot S)_T$ holds P-a.s. and the product $Z\left(x + H \cdot S\right)$ is a martingale for some $Z \in \mathcal{D}$. Our Proposition 4.5.4 shows that this definition of *replicability* is equivalent to our Definition 4.5.5. However, it seems to us more natural from an economic point of view to define the concept of attainability as in our Definition 4.5.5, which does not involve the abstract concept of a martingale deflator and only uses the financially sound concept of *maximal element*. Furthermore, the proof given in Stricker & Yan (1998) is entirely different from ours, since it relies on a generalization of the *optional decomposition theorem*, while our approach is based on a rather simple change of numéraire technique combined with the classical results of Delbaen & Schachermayer (1995c).

In the particular case where the contingent claim f is such that $\bar{f} := f\widehat{Z}_T$ is bounded we can also say something more, as shown in the next Proposition. As a preliminary, let us state the following simple Lemma.

Lemma 4.5.8. *Let $L = (L_t)_{0\leq t\leq T}$ be a local martingale and let $M^1 = (M^1_t)_{0\leq t\leq T}$ and $M^2 = (M^2_t)_{0\leq t\leq T}$ be two martingales such that $M^1_t \leq L_t \leq M^2_t$ P-a.s. for all $t \in [0,T]$. Then $L = (L_t)_{0\leq t\leq T}$ is a martingale.*

Proof. Note first that the difference $L - M^1$ is a non-negative local martingale, because it is difference of a local martingale and a martingale and $L_t \geq M^1_t$ P-a.s. for all $t \in [0,T]$. Fatou's Lemma implies then that $L - M^1$ is also a supermartingale. Since M^1 is a martingale, this shows that L is a supermartingale. Analogously, the difference $M^2 - L$ is a non-negative supermartingale, again due to Fatou's Lemma together with the fact that $M^2 - L$ is a non-negative local martingale, because it is the difference of a martingale and a local martingale and $M^2_t \geq L_t$ P-a.s. for all $t \in [0,T]$. However, $M^2 - L$ is also the difference of a martingale and a supermartingale and, hence, it is a submartingale. We have thus shown that $M^2 - L$ is both a supermartingale and a submartingale and, hence, it is a martingale. In particular, this implies that $L = M^2 - (M^2 - L)$ is also a martingale, being the difference of two martingales. □

Proposition 4.5.9. *Suppose that the NUPBR condition holds and let f be a contingent claim such that $\bar{f} := f\widehat{Z}_T$ is bounded. Then the following are equivalent:*

- *(i) f is attainable for S, i.e. there exists a pair $(x,H) \in \mathbb{R}_+ \times \mathcal{A}_x$ such that $f = x + (H\cdot S)_T$ holds P-a.s. and $(H\cdot S)_T$ is maximal in K_x, and the product $\widehat{Z}\,(x + H\cdot S)$ is a martingale;*
- *(ii) $\bar{f}$ is attainable for $\bar{S}$;*
- *(iii) $E^R\left[\,\bar{f}\,\right]$ is constant as a function of $R \in \mathcal{M}^e\left(P,\bar{S}\right)$.*

Proof. We first show that *(iii) ⇒ (ii)*. Suppose that $E^R\left[\,\bar{f}\,\right]$ is constant as a function of $R \in \mathcal{M}^e\left(P,\bar{S}\right)$. This obviously implies that the condition in part *(iii)* of Theorem 4.5.6 is satisfied and, hence, due to the equivalence between parts *(ii)* and *(iii)* of Theorem 4.5.6, we can conclude that $\bar{f}$ is attainable for $\bar{S}$. Let us now show that *(ii) ⇒ (i)*. Suppose that $\bar{f}$ is attainable for $\bar{S}$, meaning that there exists a pair $\left(x,\bar{H}\right) \in \mathbb{R}_+ \times \bar{\mathcal{A}}_x$ such that $\bar{f} = x + \left(\bar{H}\cdot\bar{S}\right)_T$ holds P-a.s. and $\left(\bar{H}\cdot\bar{S}\right)_T$ is maximal in $\bar{\mathcal{K}}_x$. Due to the equivalence between parts *(i)* and *(ii)* of Theorem 4.5.6, we have $f = x + (H\cdot S)_T$ P-a.s., where $H \in \mathcal{A}_x$ satisfies (4.37). Recall that, due to Lemma 4.3.13, the product $\widehat{Z}\,(x + H\cdot S)$ is a local martingale and, being non-negative, also a supermartingale. Recall also that, since $P \in \mathcal{M}^e\left(P,\bar{S}\right)$, the process $\bar{H}\cdot\bar{S}$ is a local P-martingale and let $(\tau_n)_{n\in\mathbb{N}}$

be a localizing sequence for it. Due to Proposition 4.5.4, there is an element $Q \in \mathcal{M}^e\left(P, \bar{S}\right)$ such that $\bar{H} \cdot \bar{S}$ is a Q-martingale. Thus, for every $n \in \mathbb{N}$:

$$\left(\bar{H} \cdot \bar{S}\right)_{\tau_n \wedge T} = E^Q\left[\left(\bar{H} \cdot \bar{S}\right)_T |\mathcal{F}_{\tau_n \wedge T}\right] = E^Q\left[\bar{f}\,|\mathcal{F}_{\tau_n \wedge T}\right] - x$$

Since by assumption $\bar{f}$ is bounded, this shows that $\left(\bar{H} \cdot \bar{S}\right)_{\tau_n \wedge T}$ is bounded, for every $n \in \mathbb{N}$. Hence, due to the supermartingale property of $\widehat{Z}\left(x + H \cdot S\right)$ and to the dominated convergence theorem:

$$\begin{aligned} x \geq E\left[\widehat{Z}_T\big(x + (H \cdot S)_T\big)\right] = E\left[\,\bar{f}\,\right] = E\left[x + \left(\bar{H} \cdot \bar{S}\right)_T\right] = x + E\left[\lim_{n\to\infty}\left(\bar{H} \cdot \bar{S}\right)_{\tau_n \wedge T}\right] \\ = x + \lim_{n\to\infty} E\left[\left(\bar{H} \cdot \bar{S}\right)_{\tau_n \wedge T}\right] = x \end{aligned}$$

Clearly, this shows the martingale property of $\widehat{Z}\left(x + H \cdot S\right)$, thus proving part *(i)* of the Corollary. Finally, let us show that *(i)* $\Rightarrow$ *(iii)*. Suppose that f is attainable for S, meaning that there exists a pair $(x, H) \in \mathbb{R}_+ \times \mathcal{A}_x$ such that $f = x + (H \cdot S)_T$ holds P-a.s. and $(H \cdot S)_T$ is maximal in $\mathcal{K}_x$, and suppose that $\widehat{Z}\left(x + H \cdot S\right)$ is a martingale. Due to the equivalence between parts *(i)* and *(ii)* in Theorem 4.5.6, this implies that $\bar{f}$ is attainable for $\bar{S}$. Furthermore, due to equation (4.37), we have that:

$$\bar{H} \cdot \bar{S} = \frac{1}{V}\Big(\big(H - xH^V\big) \cdot S\Big) = \widehat{Z}\big(H \cdot S - x\left(V - 1\right)\big) = \widehat{Z}\left(x + H \cdot S\right) - x$$

thus showing that $\bar{H} \cdot \bar{S}$ is a martingale. The martingale property implies the following, for all $t \in [0, T]$:

$$\left(\bar{H} \cdot \bar{S}\right)_t = E\left[\left(\bar{H} \cdot \bar{S}\right)_T |\mathcal{F}_t\right] = E\left[\bar{f}\,|\mathcal{F}_t\right] - x$$

Since $\bar{f}$ is bounded, this implies that there exists a positive constant K such that $\bar{H} \cdot \bar{S} \leq K$ P-a.s. Let now $Q \in \mathcal{M}^e\left(P, \bar{S}\right)$, so that, due to Lemma 4.5.1, there exists a local martingale $N = (N_t)_{0 \leq t \leq T}$ strongly orthogonal to M with $N_0 = 0$, $\Delta N > -1$ P-a.s. and $E\left[\mathcal{E}\left(N\right)_T\right] = 1$ such that $\frac{dQ}{dP} = \mathcal{E}\left(N\right)_T$. As shown in (4.41), the product $\mathcal{E}\left(N\right)\left(\bar{H} \cdot \bar{S}\right)$ is a local martingale. Furthermore, it is bounded from below by the martingale $-x\mathcal{E}\left(N\right)$, since $\bar{H} \in \bar{\mathcal{A}}_x$, and from above by the martingale $K\mathcal{E}\left(N\right)$. Lemma 4.5.8 implies then that $\mathcal{E}\left(N\right)\left(\bar{H} \cdot \bar{S}\right)$ is a martingale[9], so that:

$$E\left[\,\bar{f}\,\right] = x + E\left[\left(\bar{H} \cdot \bar{S}\right)_T\right] = x = x + E\left[\mathcal{E}\left(N\right)_T\left(\bar{H} \cdot \bar{S}\right)_T\right] = x + E^Q\left[\left(\bar{H} \cdot \bar{S}\right)_T\right] = E^Q\left[\,\bar{f}\,\right]$$

Since $Q \in \mathcal{M}^e\left(P, \bar{S}\right)$ is arbitrary, this shows that $E^R\left[\bar{f}\,\right]$ is constant as a function of $R \in \mathcal{M}^e\left(P, \bar{S}\right)$. □

Remark 4.5.10. A probability measure Q on $(\Omega, \mathcal{F})$ is said to be an *Absolutely Continuous Local Martingale Measure (ACLMM)* for $\bar{S}$ if $Q \ll P$ and $\bar{S}$ is a local Q-martingale. Let us denote by $\mathcal{M}\left(P, \bar{S}\right)$ the set of all ACLMMs for $\bar{S}$. Clearly, we have $\mathcal{M}^e\left(P, \bar{S}\right) \subseteq \mathcal{M}\left(P, \bar{S}\right)$. It can

[9] More succinctly, we can also argue as follows. We have shown that there exists a positive constant K such that $\left|\left(\bar{H} \cdot \bar{S}\right)_t\right| \leq K$ P-a.s. for all $t \in [0, T]$. For any $Q \in \mathcal{M}^e\left(P, \bar{S}\right)$, the stochastic integral $\bar{H} \cdot \bar{S}$ is a local Q-martingale and, being bounded, also a Q-martingale. Hence, for any $Q \in \mathcal{M}^e\left(P, \bar{S}\right)$, we have $E\left[\,\bar{f}\,\right] = x + E\left[\left(\bar{H} \cdot \bar{S}\right)_T\right] = x = x + E^Q\left[\left(\bar{H} \cdot \bar{S}\right)_T\right] = E^Q\left[\,\bar{f}\,\right]$. Compare also with Lemma 5.1 of Delbaen & Schachermayer (1994).

be shown that condition *(iii)* of Proposition 4.5.9 is equivalent to the condition that $E^R\left[\bar{f}\right]$ is constant as a function of $R \in \mathcal{M}\left(P, \bar{S}\right)$. As pointed out in the proof of Theorem 5.2 of Delbaen & Schachermayer (1994), this is due to the fact that the set $\mathcal{M}^e\left(P, \bar{S}\right)$ is dense in $L^1(P)$ in $\mathcal{M}\left(P, \bar{S}\right)$ and, hence, $E^R\left[\bar{f}\right]$ is constant on $\mathcal{M}^e\left(P, \bar{S}\right)$ if and only if it is constant in $\mathcal{M}\left(P, \bar{S}\right)$.

Remark 4.5.11 (Connections with the benchmark approach). Let us briefly comment on an interesting implication of Proposition 4.5.9 in the context of the Benchmark Approach (see Platen & Heath (2006) for a detailed account). As we have seen, the martingale deflator $\widehat{Z}$ is the reciprocal of the wealth process V generated by the self-financing 1-admissible trading strategy $H^V = \lambda/\widehat{Z}$. More precisely, it can be shown that V coincides with the so-called *growth-optimal portfolio (GOP)*, see for instance Christensen & Larsen (2007), Karatzas & Kardaras (2007), Hulley & Schweizer (2010) and, in the special case of an Itô-process model, Fontana & Runggaldier (2011). A self-financing trading strategy $H \in \mathcal{A}_a$, for some $a \in \mathbb{R}_+$, is said to be *fair* if the GOP-discounted (i.e. V-discounted) value of the corresponding wealth process $a + H \cdot S$ is a martingale. Since $V = 1/\widehat{Z}$, we can equivalently say that a trading strategy $H \in \mathcal{A}_a$ is fair if the product $\widehat{Z}\,(a + H \cdot S)$ is a martingale. In the context of the benchmark approach, the valuation of a given contingent claim f is performed by taking the expectation of its GOP-discounted value $f/V_T = f\widehat{Z}_T$ under the original probability measure P, thus giving rise to the so-called *real-world pricing formula*, see Section 10.4 of Platen & Heath (2006), Section 5 of Platen (2009) and Section 5 of Fontana & Runggaldier (2011). However, the valuation via the real-world pricing formula is fully justified only for those contingent claims which can be perfectly replicated by means of a fair trading strategy, see for instance Corollary 5.1 of Platen (2009). However, at the present time, we are not aware of any result in the literature providing a characterization of those contingent claims which can be attained by fair strategies. Hence, the result of our Proposition 4.5.9 can be of interest, since it provides a general characterization of those contingent claims f (such that $f\widehat{Z}_T$ is bounded) which can be attained by fair strategies in the context of financial market models based on continuous semimartingales.

Let us close this Section by studying the notion of market completeness, which intuitively means that all contingent claims can be perfectly replicated by trading in the market. More precisely, let us give the following Definition.

Definition 4.5.12. *The financial market is said to be* complete *if every contingent claim f such that $E\big[\widehat{Z}_T f\big] < \infty$ is attainable for S, in the sense of Definition 4.5.5.*

The following Theorem represents an extension of the so-called *second fundamental theorem of asset pricing* to financial market models based on continuous semimartingales which do not necessarily satisfy the classical NFLVR condition. Theorem 4.5.13 is essentially a corollary to Theorem 4.5.6. However, due to its importance, we prefer to state it as a theorem.

Theorem 4.5.13. *Suppose that the NUPBR condition holds. Then the financial market is complete, in the sense of Definition 4.5.12, if and only if there is an unique martingale deflator, i.e. $\mathcal{D} = \big\{\widehat{Z}\big\}$.*

Proof. Due to the equivalence between parts *(i)* and *(iv)* of Theorem 4.5.6, the financial market is obviously complete if $\mathcal{D} = \big\{\widehat{Z}\big\}$. Conversely, suppose that the financial market is complete.

Let ξ be a bounded $\mathcal{F}$-measurable random variable and let ξ^+, ξ^- denote the positive and the negative part of ξ, respectively. Let us define $f^\pm := \xi^\pm/\widehat{Z}_T$, so that we have $f^\pm \geq 0$ P-a.s. and $E\big[\widehat{Z}_T f^\pm\big] = E\left[\xi^\pm\right] < \infty$. Since the financial market is complete, $f^\pm$ is attainable for S and, due to the equivalence between parts *(i)* and *(ii)* of Theorem 4.5.6, $\bar{f}^\pm := \xi^\pm$ is attainable for $\bar{S}$. This implies that there exists a pair $\left(x^\pm, \bar{H}^\pm\right) \in \mathbb{R}_+ \times \bar{\mathcal{A}}_{x^\pm}$ such that $\xi^\pm = x^\pm + \left(\bar{H}^\pm \cdot \bar{S}\right)_T$ holds P-a.s. Furthermore, since $\xi^\pm$ is bounded, (the proof of) Proposition 4.5.9 shows that $\bar{H}^\pm \cdot \bar{S}$ is a martingale for all $Q \in \mathcal{M}^e\left(P, \bar{S}\right)$. In particular, $\bar{H}^\pm \cdot \bar{S}$ is a P-martingale. Let $x := x^+ - x^-$ and $\bar{H} := \bar{H}^+ - \bar{H}^-$. Since $L\left(\bar{S}\right)$ is linear (see e.g. Protter (2005), Theorem IV.16), we have $\bar{H} \in L\left(\bar{S}\right)$. Hence, we can write $\xi = x + \left(\bar{H} \cdot \bar{S}\right)_T$ and $\bar{H} \cdot \bar{S}$ is a P-martingale. Let now $N = (N_t)_{0 \leq t \leq T}$ be a bounded martingale strongly orthogonal to M with $N_0 = 0$. By applying the previous arguments to $\xi := N_T$, we get the representation $N = x + \bar{H} \cdot \bar{S}$, for some $\bar{H} \in L\left(\bar{S}\right)$ such that $\bar{H} \cdot \bar{S}$ is a P-martingale. In particular, this implies that N has a continuous version. Thus:

$$\langle N, N \rangle = \bar{H} \cdot \langle \bar{S}, N \rangle = \left(\bar{H}\bar{\lambda}\right) \cdot \langle M, N \rangle = 0$$

where the second equality follows from (4.33), for a suitable $\mathbb{R}^{(d+1)\times d}$-valued process $\bar{\lambda}$, and the last is due to the strong orthogonality of M and N. Due to Corollary 1 to Theorem II.27 of Protter (2005), this implies that N is trivial, meaning that $N_t = N_0$ P-a.s. for all $t \in [0, T]$. In turn, due to Corollary III.4.27 of Jacod & Shiryaev (2003), this implies that all local martingales strongly orthogonal to M are trivial. Due to Proposition 4.3.24, we can conclude that $\widehat{Z}$ is the unique martingale deflator, i.e. $\mathcal{D} = \{\widehat{Z}\}$. □

In the special case of an Itô-process-based financial market model, related results have been recently shown in Fontana & Runggaldier (2011) and Ruf (2011b). See also Corollary 2.1 of Stricker & Yan (1998), where a similar result is obtained by relying on a generalization of the optional decomposition theorem.

Remark 4.5.14 (Connections with the benchmark approach). Recall that, as we have pointed out in Remark 4.5.11, the capability of replicating a given contingent claim by following a *fair* trading strategy plays a crucial role in the context of the benchmark approach. Recall that a strategy $H \in \mathcal{A}_a$ is said to be *fair* if the product $\widehat{Z}\,(a + H \cdot S)$ is a P-martingale. We can now prove the following fact: if the financial market is complete, in the sense of Definition 4.5.12, then every contingent claim f with $E\big[\widehat{Z}_T f\big] < \infty$ can be attained by a fair strategy. In fact, let f be a contingent claim such that $E\big[\widehat{Z}_T f\big] < \infty$ and suppose that the financial market is complete, so that there exists a couple $(x, H) \in \mathbb{R}_+ \times \mathcal{A}_x$ such that $f = x + (H \cdot S)_T$ holds P-a.s. and $(H \cdot S)_T$ is maximal in $\mathcal{K}_x$. Theorem 4.5.13 gives that $\mathcal{D} = \{\widehat{Z}\}$ and, therefore, the equivalence between parts *(i)* and *(vi)* of Proposition 4.5.4 implies that the product $\widehat{Z}\,(x + H \cdot S)$ is a P-martingale, thus showing that the hedging strategy H is fair.

The following Proposition can be seen as a generalization of Theorem 5.4 of Delbaen & Schachermayer (1994) to financial market models which do not satisfy the classical NFLVR condition and gives a sufficient condition for the financial market to be complete, in the sense of Definition 4.5.12. Recall that, as in Remark 4.5.10, the set $\mathcal{M}\left(P, \bar{S}\right)$ is the set of the density processes of all ACLMMs for $\bar{S}$.

Proposition 4.5.15. *Suppose that the NUPBR condition holds. If we have $\mathcal{M}\left(P,\bar{S}\right)=\mathcal{M}^e\left(P,\bar{S}\right)$, then the financial market is complete.*

Proof. Recall that, as soon as the NUPBR condition holds, the $\mathbb{R}^{d+1}$-valued continuous semimartingale $\bar{S}$ satisfies the NFLVR condition and, since $\bar{S}$ is a local P-martingale, we have $P\in\mathcal{M}^e\left(P,\bar{S}\right)$. Theorem 5.4 of Delbaen & Schachermayer (1994) implies then that $\mathcal{M}^e\left(P,\bar{S}\right)=\mathcal{M}\left(P,\bar{S}\right)=\{1\}$, meaning that P is the only ELMM for $\bar{S}$. Due to Lemma 4.5.1, this means that there exists no non-trivial local martingale $N=(N_t)_{0\leq t\leq T}$ strongly orthogonal to M with $N_0=0$ and such that $\Delta N>-1$ P-a.s. and $E\left[\mathcal{E}\left(N\right)_T\right]=1$. Arguing by contradiction, suppose now that the set $\mathcal{D}$ contains more than one element. Due to Proposition 4.3.24, this implies that there exists a non-trivial local martingale $N=(N_t)_{0\leq t\leq T}$ strongly orthogonal to M with $N_0=0$ and such that $\Delta N>-1$ P-a.s. Let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence for $\mathcal{E}\left(N\right)$, so that $E\left[\mathcal{E}\left(N\right)_{\tau_n\wedge T}\right]=1$, for all $n\in\mathbb{N}$. Since strong orthogonality is preserved by stopping (see Jacod & Shiryaev (2003), Lemma I.4.13), the fact that $\mathcal{E}\left(N^{\tau_n}\right)\in\mathcal{M}^e\left(P,\bar{S}\right)=\{1\}$ implies that N^{τ_n} is trivial, for every $n\in\mathbb{N}$. Since $\tau_n\nearrow T$ P-a.s. as $n\to\infty$, this implies that N is trivial as well, thus contradicting the assumption that $\mathcal{D}$ contains more than one element. We have thus shown that $\mathcal{D}=\left\{\widehat{Z}\right\}$. Due to Theorem 4.5.13, this implies that the financial market is complete. □

4.6 Conclusions and further developments

In the present Chapter, we have studied no-arbitrage conditions which are weaker than the classical *No Free Lunch with Vanishing Risk (NFLVR)* criterion, namely: the *No Unbounded Increasing Profit (NUIP)* condition, the *No Immediate Arbitrage Opportunity (NIAO)* condition and the *No Unbounded Profit with Bounded Risk (NUPBR)* condition. In the context of general financial market models based on continuous semimartingales, we have shown that:

$$\text{NFLVR}\;\Rightarrow\;\text{NUPBR}\;\Rightarrow\;\text{NIAO}\;\Rightarrow\;\text{NUIP}\tag{4.42}$$

Furthermore, we have shown by means of explicit counterexamples that the converse implications fail, meaning that each of the NFLVR/NUPBR/NIAO/NUIP no-arbitrage conditions is strictly stronger than the following one in (4.42), and we have compared the above no-arbitrage conditions which other related notions which have appeared in the literature. A crucial aspect of the weak NUIP/NIAO/NUPBR conditions, not shared by the classical NFLVR condition, is represented by the fact that their validity can be directly checked by looking at the properties of the characteristics of the discounted price process of the risky assets. By relying on this key observation, we have shown that the NUIP/NIAO/NUPBR conditions are robust with respect to changes of numéraire, absolutely continuous changes of the reference probability measure and restrictions/enlargements of the reference filtration. In contrast, the classical NFLVR condition does not generally possess these stability properties. Extending the classical results on the replication of contingent claims to financial market models which do not necessarily satisfy the NFLVR condition, we have also provided a general characterization of attainable contingent claims by relying on the concept of martingale deflator, which can be regarded as a weaker counterpart to the classical notion of density process of an *Equivalent Local Martingale Measure*.

Among the possible further developments of the present Chapter, the extension to general (possibly discontinuous and non-locally bounded) semimartingales appears particularly interesting. As we have pointed out throughout this Chapter, not all of our results rely on the continuity of the underlying discounted price process S and, hence, we have already given in the text some hints at possible generalizations to discontinuous semimartingales. However, the results of Section 4.5 on the attainability of contingent claims rely on the continuity of the process S. In fact, if S is a general (possibly discontinuous and non-locally bounded) semimartingale, the nice and natural relations linking the candidate density process $\widehat{Z} = \mathcal{E}\left(-\lambda \cdot M\right)$ of the minimal martingale measure (see Remark 4.3.15) with the existence of a numéraire V for S such that the V-discounted price process $\left(\frac{S}{V}, \frac{1}{V}\right)$ is a local martingale under the original probability measure P may fail, see also Becherer (2001), Christensen & Larsen (2007) and Karatzas & Kardaras (2007). Hence, we expect that the extension of the results of Section 4.5 to general semimartingales requires a more substantial effort.

Finally, we think that it would also be of interest to apply the rather abstract results of the present Chapter to specific financial market models. As an example, one could try to apply the results on the preservation (or, symmetrically, on the failure) of the weak NUIP/NIAO/NUPBR conditions with respect to restrictions/enlargements of the filtration to financial market models in which a structure with multiple filtrations naturally arises. Typical instances of such models include credit risk models and models representing incomplete/partial information situations.

4.7 Appendix

Proof of Theorem 4.4.3

The proof is based on some lengthy but rather simple computations, using integration by parts and Itô's formula. Suppose first that the NUIP condition holds for S. Then, due to Theorem 4.3.2, there exists an $\mathbb{R}^d$-valued predictable process $\lambda = (\lambda_t)_{0 \leq t \leq T}$ such that $dA = d\langle M, M\rangle\lambda$, where A and M denote the finite variation and the local martingale parts, respectively, in the canonical decomposition $S = S_0 + A + M$ of the semimartingale S. Let us start by computing the stochastic differential of $\frac{1}{V}$ (recall that $V > 0$ P-a.s., due to Definition 4.4.1):

$$\begin{aligned} d\frac{1}{V_t} &= -\frac{1}{V_t^2}dV_t + \frac{1}{V_t^3}d\langle V\rangle_t = -\frac{1}{V_t^2}H_t dS_t + \frac{1}{V_t^3}H_t' d\langle M, M\rangle_t H_t \\ &= -\frac{1}{V_t^2}H_t dM_t + \frac{1}{V_t^2}H_t' d\langle M, M\rangle_t \left(\frac{H_t}{V_t} - \lambda_t\right) \end{aligned} \tag{4.43}$$

with quadratic variation:

$$d\left\langle \frac{1}{V}\right\rangle_t = \frac{1}{V_t^4}H_t' d\langle M, M\rangle_t H_t \tag{4.44}$$

Let us now compute the stochastic differential of the ratio S^i/V, for $i = 1, \ldots, d$:

$$\begin{aligned} d\frac{S^i_t}{V_t} &= \frac{1}{V_t}dS^i_t + S^i_t d\frac{1}{V_t} + d\Big\langle \frac{1}{V}, S^i \Big\rangle_t \\ &= \frac{1}{V_t}dS^i_t - S^i_t \frac{1}{V_t^2} H_t dM_t + \frac{S^i_t}{V_t^2} H'_t d\langle M, M\rangle_t \left(\frac{H_t}{V_t} - \lambda_t\right) - \frac{1}{V_t^2} H'_t d\langle M, M^i\rangle_t \\ &= \frac{1}{V_t}dM^i_t + \frac{1}{V_t} d\langle M^i, M\rangle_t \lambda_t - S^i_t \frac{1}{V_t^2} H_t dM_t + \frac{S^i_t}{V_t^2} H'_t d\langle M, M\rangle_t \left(\frac{H_t}{V_t} - \lambda_t\right) - \frac{1}{V_t^2} H'_t d\langle M, M^i\rangle_t \\ &= \frac{1}{V_t}\left(e^i - \frac{S^i_t H_t}{V_t}\right) dM_t + \frac{1}{V_t}\left(e^i - \frac{S^i_t H_t}{V_t}\right)' d\langle M, M\rangle_t \left(\lambda_t - \frac{H_t}{V_t}\right) \end{aligned} \tag{4.45}$$

where e^i denotes the vector in $\mathbb{R}^d$ with the i-th component equal to 1 and the remaining components equal to 0. Let us denote by $\bar{S} = S_0 + \bar{A} + \bar{M}$ the canonical decomposition of the $\mathbb{R}^{d+1}$-valued continuous semimartingale $\bar{S} = \left(\frac{S}{V}, \frac{1}{V}\right)$. Equations (4.43)-(4.45) imply the following, for all $i = 1, \ldots, d$ and $t \in [0, T]$:

$$\begin{aligned} \langle \bar{M}^i, \bar{M}^{d+1}\rangle_t &= -\int_0^t \frac{1}{V_u^3}\left(e^i - \bar{S}^i_u H_u\right)' d\langle M, M\rangle_u H_u \\ &= -\int_0^t \frac{1}{V_u^3} d\langle M^i, M\rangle_u H_u + \int_0^t \frac{\bar{S}^i_u}{V_u^3} H'_u d\langle M, M\rangle_u H_u \end{aligned} \tag{4.46}$$

and similarly, for all $i, j = 1, \ldots, d$ and $t \in [0, T]$:

$$\begin{aligned} \langle \bar{M}^i, \bar{M}^j\rangle_t &= \int_0^t \frac{1}{V_u^2}\left(e^i - \bar{S}^i_u H_u\right)' d\langle M, M\rangle_u \left(e^j - \bar{S}^j_u H_u\right) \\ &= \int_0^t \frac{1}{V_u^2}\left(e^i - \bar{S}^i_u H_u\right)' d\langle M, M^j\rangle_u - \int_0^t \frac{1}{V_u^2}\left(e^i - \bar{S}^i_u H_u\right)' d\langle M, M\rangle_u H_u \bar{S}^j_u \end{aligned} \tag{4.47}$$

The finite variation part $\bar{A}^{d+1}$ in the canonical decomposition (4.43) of the process $\bar{S}^{d+1} = 1/V$ can be rewritten as follows, for all $t \in [0, T]$:

$$\begin{aligned} \bar{A}^{d+1}_t &= \int_0^t \frac{1}{V_u^2} H'_u d\langle M, M\rangle_u \left(\frac{H_u}{V_u} - \lambda_u\right) \\ &= \int_0^t V_u d\langle \bar{M}^{d+1}, \bar{M}^{d+1}\rangle_u + \sum_{j=1}^d \int_0^t V_u \lambda^j_u d\langle \bar{M}^{d+1}, \bar{M}^j\rangle_u - \sum_{j=1}^d \int_0^t \frac{\bar{S}^j_u}{V_u^2} \lambda^j_u H'_u d\langle M, M\rangle_u H_u \\ &= \int_0^t V_u \Big(1 - \sum_{j=1}^d S^j_u \lambda^j_u\Big) d\langle \bar{M}^{d+1}, \bar{M}^{d+1}\rangle_u + \sum_{j=1}^d \int_0^t V_u \lambda^j_u d\langle \bar{M}^{d+1}, \bar{M}^j\rangle_u \end{aligned}$$

where the first equality follows from (4.43) and the second and third equalities from (4.44) and (4.46). Similarly, for all $i = 1, \ldots, d$, the finite variation part $\bar{A}^i$ in the canonical decomposition

(4.45) of the process $\bar{S}^i = S^i/V$ can be rewritten as follows, for all $t \in [0,T]$:

$$\begin{aligned}
\bar{A}^i_t &= \int_0^t \frac{1}{V_u}\left(e^i - \bar{S}^i_u H_u\right)' d\langle M,M\rangle_u \left(\lambda_u - \frac{H_u}{V_u}\right) \\
&= \sum_{j=1}^d \int_0^t V_u \lambda^j_u d\langle \bar{M}^i, \bar{M}^j\rangle_u + \sum_{j=1}^d \int_0^t \frac{\lambda^j_u}{V_u}\left(e^i - \bar{S}^i_u H_u\right)' d\langle M,M\rangle_u H_u \bar{S}^j_u \\
&\quad - \int_0^t \frac{1}{V_u^2}\left(e^i - \bar{S}^i_u H_u\right)' d\langle M,M\rangle_u H_u \\
&= \sum_{j=1}^d \int_0^t V_u \lambda^j_u d\langle \bar{M}^i, \bar{M}^j\rangle_u - \int_0^t \frac{1}{V_u^2}\left(e^i - \bar{S}^i_u H_u\right)' d\langle M,M\rangle_u H_u \left(1 - \sum_{j=1}^d S^j_u \lambda^j_u\right) \\
&= \sum_{j=1}^d \int_0^t V_u \lambda^j_u d\langle \bar{M}^i, \bar{M}^j\rangle_u + \int_0^t V_u \left(1 - \sum_{j=1}^d S^j_u \lambda^j_u\right) d\langle \bar{M}^i, \bar{M}^{d+1}\rangle_u
\end{aligned}$$

where the first equality follows from (4.45) and the subsequent equalities from (4.47) and (4.46). We have thus shown the following, for all $i = 1,\ldots,d+1$ and $t \in [0,T]$:

$$\bar{A}^i_t = \sum_{j=1}^{d+1} \int_0^t \bar{\lambda}^j_u d\big\langle \bar{M}^i, \bar{M}^j\big\rangle_u$$

where $\bar{\lambda}^j_t := V_t \lambda^j_t$, for all $j = 1,\ldots,d$, and $\bar{\lambda}^{d+1}_t := V_t\left(1 - \sum_{k=1}^d S^k_t \lambda^k_t\right)$, for all $t \in [0,T]$. Theorem 4.3.2 implies then that the NUIP condition holds for $\bar{S}$ as well, since $d\bar{A} \ll d\langle \bar{M}, \bar{M}\rangle$.

Conversely, suppose that $\bar{S}$ satisfies the NUIP condition. Recall that, as pointed out before Lemma 4.4.2, if the process $V = 1 + H \cdot S$ is a numéraire for S, then the process $1/V$ is a numéraire for $\bar{S}$ and we have the trivial identity $(S,1) = \bar{S}/\frac{1}{V}$. Hence, the converse implication can be proved similarly as in the first part of the Theorem, interchanging the roles of S and $\bar{S}$ and of V and $1/V$. However, for the sake of completeness, we prefer to give full details. Recall first that, due to Theorem 4.3.2, if $\bar{S}$ satisfies the NUIP condition, there exists an $\mathbb{R}^{d+1}$-valued predictable process $\bar{\lambda} = \left(\bar{\lambda}_t\right)_{0 \leq t \leq T}$ such that $d\bar{A}_t = d\langle \bar{M}, \bar{M}\rangle_t \bar{\lambda}_t$, where $\bar{A}$ and $\bar{M}$ denote the finite variation and the local martingale part, respectively, in the canonical decomposition of $\bar{S}$. Observe now the following:

$$dV_t = d\frac{1}{\bar{S}^{d+1}_t} = -\frac{1}{\left(\bar{S}^{d+1}_t\right)^2} d\bar{S}^{d+1}_t + \frac{1}{\left(\bar{S}^{d+1}_t\right)^3} d\langle \bar{M}^{d+1}, \bar{M}^{d+1}\rangle_t$$

Then, for all $i = 1,\ldots,d$:

$$\begin{aligned}
dS^i_t &= d\left(V_t \bar{S}^i_t\right) = d\frac{\bar{S}^i_t}{\bar{S}^{d+1}_t} = \bar{S}^i_t d\frac{1}{\bar{S}^{d+1}_t} + \frac{1}{\bar{S}^{d+1}_t} d\bar{S}^i_t + d\Big\langle \bar{S}^i, \frac{1}{\bar{S}^{d+1}}\Big\rangle_t \\
&= -\frac{\bar{S}^i_t}{\left(\bar{S}^{d+1}_t\right)^2} d\bar{S}^{d+1}_t + \frac{\bar{S}^i_t}{\left(\bar{S}^{d+1}_t\right)^3} d\langle \bar{M}^{d+1}, \bar{M}^{d+1}\rangle_t + \frac{1}{\bar{S}^{d+1}_t} d\bar{S}^i_t - \frac{1}{\left(\bar{S}^{d+1}_t\right)^2} d\langle \bar{M}^i, \bar{M}^{d+1}\rangle_t \\
&= -\frac{\bar{S}^i_t}{\left(\bar{S}^{d+1}_t\right)^2} d\bar{M}^{d+1}_t - \frac{\bar{S}^i_t}{\left(\bar{S}^{d+1}_t\right)^2} d\langle \bar{M}^{d+1}, \bar{M}\rangle_t \bar{\lambda}_t + \frac{\bar{S}^i_t}{\left(\bar{S}^{d+1}_t\right)^3} d\langle \bar{M}^{d+1}, \bar{M}^{d+1}\rangle_t + \frac{1}{\bar{S}^{d+1}_t} d\bar{M}^i_t \\
&\quad + \frac{1}{\bar{S}^{d+1}_t} d\langle \bar{M}^i, \bar{M}\rangle_t \bar{\lambda}_t - \frac{1}{\left(\bar{S}^{d+1}_t\right)^2} d\langle \bar{M}^i, \bar{M}^{d+1}\rangle_t
\end{aligned}$$

This shows that, for all $i = 1, \dots, d$, the finite variation part A^i in the canonical decomposition of S^i is given by the following expression, for all $t \in [0, T]$:

$$\begin{aligned} A_t^i = &\int_0^t \frac{1}{\bar{S}_u^{d+1}} d\langle \bar{M}^i, \bar{M} \rangle_u \bar{\lambda}_u - \int_0^t \frac{\bar{S}_u^i}{\left(\bar{S}_u^{d+1}\right)^2} d\langle \bar{M}^{d+1}, \bar{M} \rangle_u \bar{\lambda}_u - \int_0^t \frac{1}{\left(\bar{S}_u^{d+1}\right)^2} d\langle \bar{M}^i, \bar{M}^{d+1} \rangle_u \\ &+ \int_0^t \frac{\bar{S}_u^i}{\left(\bar{S}_u^{d+1}\right)^3} d\langle \bar{M}^{d+1}, \bar{M}^{d+1} \rangle_u \end{aligned} \tag{4.48}$$

and the predictable quadratic variation between the martingale parts M^i, M^j in the canonical decompositions of S^i, S^j is given by the following expression, for all $i, j = 1, \dots, d$ and $t \in [0, T]$:

$$\begin{aligned} \langle M^i, M^j \rangle_t = &\int_0^t \frac{\bar{S}_u^i}{\left(\bar{S}_u^{d+1}\right)^2} \frac{\bar{S}_u^j}{\left(\bar{S}_u^{d+1}\right)^2} d\langle \bar{M}^{d+1}, \bar{M}^{d+1} \rangle_u + \int_0^t \frac{1}{\left(\bar{S}_u^{d+1}\right)^2} d\langle \bar{M}^i, \bar{M}^j \rangle_u \\ &- \int_0^t \frac{\bar{S}_u^i}{\left(\bar{S}_u^{d+1}\right)^3} d\langle \bar{M}^{d+1}, \bar{M}^j \rangle_u - \int_0^t \frac{\bar{S}_u^j}{\left(\bar{S}_u^{d+1}\right)^3} d\langle \bar{M}^{d+1}, \bar{M}^i \rangle_u \end{aligned} \tag{4.49}$$

Hence, identifying the common terms between (4.48) and (4.49), we can write (4.48) as follows, for all $i = 1, \dots, d$ and $t \in [0, T]$:

$$\begin{aligned} A_t^i = &\sum_{j=1}^d \int_0^t \bar{S}_u^{d+1} \bar{\lambda}_u^j d\langle M^i, M^j \rangle_u + \int_0^t \frac{\bar{\lambda}_u^{d+1}}{\bar{S}_u^{d+1}} d\langle \bar{M}^i, \bar{M}^{d+1} \rangle_u - \int_0^t \frac{\bar{S}_u^i \bar{\lambda}_u^{d+1}}{\left(\bar{S}_u^{d+1}\right)^2} d\langle \bar{M}^{d+1}, \bar{M}^{d+1} \rangle_u \\ &- \int_0^t \frac{1}{\left(\bar{S}_u^{d+1}\right)^2} d\langle \bar{M}^i, \bar{M}^{d+1} \rangle_u + \int_0^t \frac{\bar{S}_u^i}{\left(\bar{S}_u^{d+1}\right)^3} d\langle \bar{M}^{d+1}, \bar{M}^{d+1} \rangle_u \\ &- \sum_{j=1}^d \int_0^t \frac{\bar{S}_u^i \bar{S}_u^j \bar{\lambda}_u^j}{\left(\bar{S}_u^{d+1}\right)^3} d\langle \bar{M}^{d+1}, \bar{M}^{d+1} \rangle_u + \sum_{j=1}^d \int_0^t \frac{\bar{S}_u^j \bar{\lambda}_u^j}{\left(\bar{S}_u^{d+1}\right)^2} d\langle \bar{M}^{d+1}, \bar{M}^i \rangle_u \\ = &\sum_{j=1}^d \int_0^t \bar{S}_u^{d+1} \bar{\lambda}_u^j d\langle M^i, M^j \rangle_u - \int_0^t \frac{1}{\left(\bar{S}_u^{d+1}\right)^2} \left(1 - \sum_{j=1}^d \bar{S}_u^j \bar{\lambda}_u^j - \bar{S}_u^{d+1} \bar{\lambda}_u^{d+1}\right) d\langle \bar{M}^i, \bar{M}^{d+1} \rangle_u \\ &+ \int_0^t \frac{\bar{S}_u^i}{\left(\bar{S}_u^{d+1}\right)^3} \left(1 - \sum_{j=1}^d \bar{S}_u^j \bar{\lambda}_u^j - \bar{S}_u^{d+1} \bar{\lambda}_u^{d+1}\right) d\langle \bar{M}^{d+1}, \bar{M}^{d+1} \rangle_u \end{aligned} \tag{4.50}$$

Recall now that, due to equations (4.43) and (4.45):

$$\bar{M}_t^{d+1} = -\int_0^t \left(\bar{S}_u^{d+1}\right)^2 H_u dM_u \qquad \text{and} \qquad \bar{M}_t^i = \int_0^t \bar{S}_u^{d+1} \left(e^i - \bar{S}_u^i H_u\right) dM_u \quad i = 1, \dots, d$$

Hence, we can rewrite (4.50) as follows:

$$
\begin{aligned}
A_t^i &= \sum_{j=1}^{d}\int_0^t \bar{S}_u^{d+1}\bar{\lambda}_u^j d\langle M^i,M^j\rangle_u + \int_0^t \bar{S}_u^{d+1}\Big(1-\sum_{j=1}^{d+1}\bar{S}_u^j\bar{\lambda}_u^j\Big)\left(e^i-\bar{S}_u^iH_u\right)' d\langle M,M\rangle_u H_u \\
&\quad + \int_0^t \bar{S}_u^i\bar{S}_u^{d+1}\Big(1-\sum_{j=1}^{d+1}\bar{S}_u^j\bar{\lambda}_u^j\Big)H_u' d\langle M,M\rangle_u H_u \\
&= \sum_{j=1}^{d}\int_0^t \bar{S}_u^{d+1}\bar{\lambda}_u^j d\langle M^i,M^j\rangle_u + \int_0^t \bar{S}_u^{d+1}\Big(1-\sum_{j=1}^{d+1}\bar{S}_u^j\bar{\lambda}_u^j\Big)d\langle M^i,M\rangle_u H_u
\end{aligned}
$$

We have thus shown the following, for all $i=1,\dots,d$ and $t\in[0,T]$:

$$
A_t^i = \sum_{j=1}^{d}\int_0^t \lambda_u^j d\langle M^i,M^j\rangle_u \qquad \text{where} \qquad \lambda_t^j := \frac{\bar{\lambda}_t^j}{V_t} + \left(1-\sum_{k=1}^{d+1}\frac{S_t^k\bar{\lambda}_t^k}{V_t}\right)\frac{H_t^j}{V_t}, \quad j=1,\dots,d
$$

We have thus shown that $dA \ll d\langle M,M\rangle$. Due to Theorem 4.3.2, this implies that S satisfies the NUIP condition.

Proof of Corollary 4.4.4

The claim can be shown by direct computations, using equations (4.44), (4.46) and (4.47):

$$
\begin{aligned}
\bar{K}_t &:= \int_0^t \left(\bar{\lambda}_u^{d+1}\right)^2 d\langle \bar{M}^{d+1},\bar{M}^{d+1}\rangle_u + \sum_{i,j=1}^{d}\int_0^t \bar{\lambda}_u^i\bar{\lambda}_u^j d\langle \bar{M}^i,\bar{M}^j\rangle_u + 2\sum_{i=1}^{d}\int_0^t \bar{\lambda}_u^i\bar{\lambda}_u^{d+1} d\langle \bar{M}^i,\bar{M}^{d+1}\rangle_u \\
&= \int_0^t \frac{1}{V_u^2}\Big(1-\sum_{k=1}^{d}S_u^k\lambda_u^k\Big)^2 H_u' d\langle M,M\rangle_u H_u + \sum_{i,j=1}^{d}\int_0^t \lambda_u^i\lambda_u^j\left(e^i-\frac{S_u^i}{V_u}H_u\right)' d\langle M,M\rangle_u\left(e^j-\frac{S_u^j}{V_u}H_u\right) \\
&\quad - 2\sum_{i=1}^{d}\int_0^t \frac{1}{V_u}\Big(1-\sum_{k=1}^{d}S_u^k\lambda_u^k\Big)\lambda_u^i\left(e^i-\frac{S_u^i}{V_u}H_u\right)' d\langle M,M\rangle_u H_u \\
&= \int_0^t \left(\frac{1}{V_u^2}\Big(1-\sum_{k=1}^{d}S_u^k\lambda_u^k\Big)^2 + \sum_{i,j=1}^{d}\lambda_u^i\lambda_u^j\frac{S_u^i}{V_u}\frac{S_u^j}{V_u} + 2\sum_{i=1}^{d}\Big(1-\sum_{k=1}^{d}S_u^k\lambda_u^k\Big)\frac{\lambda_u^i}{V_u}\frac{S_u^i}{V_u}\right)H_u' d\langle M,M\rangle_u H_u \\
&\quad - 2\sum_{i,j=1}^{d}\int_0^t \lambda_u^i\lambda_u^j\frac{S_u^i}{V_u}H_u' d\langle M,M^j\rangle_u - 2\sum_{i=1}^{d}\int_0^t\Big(1-\sum_{k=1}^{d}S_u^k\lambda_u^k\Big)\frac{\lambda_u^i}{V_u}d\langle M^i,M\rangle_u H_u \\
&\quad + \sum_{i,j=1}^{d}\int_0^t \lambda_u^i\lambda_u^j d\langle M^i,M^j\rangle_u \\
&= \int_0^t \frac{1}{V_u^2}H_u' d\langle M,M\rangle_u H_u - \sum_{i=1}^{d}\int_0^t \frac{1}{V_u}\lambda_u^i d\langle M^i,M\rangle_u H_u - \sum_{j=1}^{d}\int_0^t \frac{1}{V_u}H_u' d\langle M,M^j\rangle_u\lambda_u^j + \widehat{K}_t \\
&= \int_0^t \left(\frac{H_u}{V_u}-\lambda_u\right)' d\langle M,M\rangle_u\left(\frac{H_u}{V_u}-\lambda_u\right)
\end{aligned}
$$

Bibliography

AHLIP, R. & RUTKOWSKI, M. (2009), Forward start options under stochastic volatility and stochastic interest rates, *International Journal of Theoretical and Applied Finance*, 12(2): 209–225.

ANSEL, J.P. & STRICKER, C. (1993), Décomposition de Kunita-Watanabe, *Séminaire de Probabilités*, XXVII: 30–32.

ANSEL, J.P. & STRICKER, C. (1994), Couverture des actifs contingents et prix maximum, *Annales de l'Institut Henri Poincaré*, 30(2): 303–315.

AZÉMA, J. (1972), Quelques applications de le théorie générale des processus, *Inventiones Mathematicae*, 18: 293–336.

BARBARIN, J. (2008), Local risk-minimization for payment processes. Application to insurance contracts with surrender option or to default sensitive contingent claims, working paper, Université Catholique de Louvain.

BECHERER, D. (2001), The numeraire portfolio for unbounded semimartingales, *Finance and Stochastics*, 5: 327-341.

BECHERER, D. (2003), Rational hedging and valuation of integrated risks under constant absolute risk aversion, *Insurance: Mathematics and Economics*, 33: 1–28.

BECHERER, D. (2006), Bounded solutions to backward SDEs with jumps for utility optimization and indifference hedging, *Annals of Applied Probability*, 16: 2027–2054.

BECHERER, D. (2010), Utility indifference valuation, in: Cont, R. (ed.), *Encyclopedia of Quantitative Finance*, 1854–1869, Wiley, Chichester.

BICHTELER, K. (2002), *Stochastic Integration with Jumps*, Cambridge University Press, Cambridge.

BIELECKI, T.R., JEANBLANC, M. & RUTKOWSKI, M. (2004), Hedging of defaultable claims, in: Carmona, R. et al. (eds.), *Paris-Princeton Lectures on Mathematical Finance 2003*, 1–132, Springer, Berlin - Heidelberg.

BIELECKI, T.R. & RUTKOWSKI, M. (2002), *Credit Risk: Modeling, Valuation and Hedging*, Springer, Berlin - Heidelberg - New York.

BJÖRK, T., KABANOV, Y. & RUNGGALDIER, W.J. (1997), Bond market structure in the presence of marked point processes, *Mathematical Finance*, 7(2): 211–223.

BLEI, S. & ENGELBERT, H.J. (2009), On exponential local martingales associated with strong Markov continuous local martingales, *Stochastic Processes and their Applications*, 119: 2859–2880.

BRUTI-LIBERATI, N., NIKITOPOULOS-SKLIBOSIOS, C. & PLATEN, E. (2010), Real-world jump-diffusion term structure models, *Quantitative Finance*, 10(1): 23–37.

CAMPI, L. (2005), Some results on quadratic hedging with insider trading, *Stochastics*, 77(4): 327–348.

CAMPI, L., POLBENNIKOV, S. & SBUELZ, A. (2009), Systematic equity-based credit risk: A CEV model with jump to default, *Journal of Economic Dynamics and Control*, 33: 93–108.

CARR, P. & LINETSKY, V. (2006), A jump to default extended CEV model: An application of Bessel processes, *Finance and Stochastics*, 10: 303–330.

CARR, P. & MADAN, D.B. (1999), Option valuation using the Fast Fourier Transform, *Journal of Computational Finance*, 2(4): 61–73.

CARR, P. & MADAN, D.B. (2010), Local volatility enhanced by a jump to default, *SIAM Journal of Financial Mathematics*, 1: 2–15.

CARR, P. & SCHOUTENS, W. (2008), Hedging under the Heston model with jump-to-default, *International Journal of Theoretical and Applied Finance*, 11(4): 403–414.

CARR, P. & WU, L. (2010), Stock options and credit default swaps: A joint framework for valuation and estimation, *Journal of Financial Econometrics*, 8(4): 409–449.

CASSESE, G. (2005), A note on asset bubbles in continuous-time, *International Journal of Theoretical and Applied Finance*, 8(4): 523–536.

CHAMBERLAIN, G. & ROTHSCHILD, M. (1983), Arbitrage, factor structure, and mean-variance analysis on large asset markets, *Econometrica*, 51: 1281–1304.

CHERIDITO, P., FILIPOVIĆ, D. & KIMMEL, R.L. (2007), Market price of risk specifications for affine models: Theory and evidence, *Journal of Financial Economics*, 83: 123–170.

CHERIDITO, P., FILIPOVIĆ, D. & KIMMEL, R.L. (2010), A note on the Dai-Singleton canonical representation of affine term structure models, *Mathematical Finance*, 20(3): 509–519.

CHERIDITO, P., FILIPOVIĆ, D. & YOR, M. (2005), Equivalent and absolutely continuous measure changes for jump-diffusion processes, *Annals of Applied Probability*, 15(3): 1713–1732.

CHERIDITO, P. & WUGALTER, A. (2011), Pricing and hedging in affine models with possibility of default, working paper, Princeton University.

CHOU, C.S., MEYER, P.A. & STRICKER, C. (1980), Sur les intégrales stochastiques de processus prévisibles non bornés, *Séminaire de Probabilités*, XIV: 128–139.

CHOULLI, T. & STRICKER, C. (1996), Deux applications de la décomposition de Galtchouk-Kunita-Watanabe, *Séminaire de Probabilités*, XXX: 12–23.

CHRISTENSEN, M.M. & LARSEN, K. (2007), No arbitrage and the growth optimal portfolio, *Stochastic Analysis and Applications*, 25: 255–280.

COCHRANE, J.H. (2005), *Asset Pricing*, revised edition, Princeton University Press, Princeton.

COCULESCU, D., JEANBLANC, M. & NIKEGHBALI, A. (2008), Default times, no arbitrage conditions and change of probability measures, preprint.

COCULESCU, D. & NIKEGHBALI, A. (2010), Hazard processes and martingale hazard processes, to appear in: *Mathematical Finance*.

COURTAULT, J.M., DELBAEN, F., KABANOV, Y. & STRICKER, C. (2004), On the law of one price, *Finance and Stochastics*, 8: 525–530.

COX, A.M.G. & HOBSON, D.G. (2005), Local martingales, bubbles and options prices, *Finance and Stochastics*, 9: 477–492.

CZICHOWSKY, C. & SCHWEIZER, M. (2010), Convex duality in mean-variance hedging under convex trading constraints, NCCR FINRISK working paper No. 667, ETH Zürich.

CZICHOWSKY, C. & SCHWEIZER, M. (2011), Cone-constrained continuous-time Markowitz problems, NCCR FINRISK working paper No. 683, ETH Zürich.

DAI, Q. & SINGLETON, K.J. (2000), Specification analysis of affine term structure models, *Journal of Finance*, LV(5): 1943–1978.

DAVIS, M.H.A. (1997), Option pricing in incomplete markets, in: Dempster, M.A.H. & Pliska, S.R. (eds.): *Mathematics of Derivative Securities*, 227–254, Cambridge University Press.

DELBAEN, F. & SCHACHERMAYER, W. (1994), A general version of the fundamental theorem of asset pricing, *Mathematische Annalen*, 300: 463–520.

DELBAEN, F. & SCHACHERMAYER, W. (1995a), Arbitrage possibilities in Bessel processes and their relations to local martingales, *Probability Theory and Related Fields*, 102: 357–366.

DELBAEN, F. & SCHACHERMAYER, W. (1995b), The existence of absolutely continuous local martingale measures, *Annals of Applied Probability*, 5(4): 926–945.

DELBAEN, F. & SCHACHERMAYER, W. (1995c), The no-arbitrage property under a change of numéraire, *Stochastics*, 53(3-4): 213–226.

DELBAEN, F. & SCHACHERMAYER, W. (1998a), A simple counterexample to several problems in the theory of asset pricing, *Mathematical Finance*, 8: 1–12.

DELBAEN, F. & SCHACHERMAYER, W. (1998b), The fundamental theorem of asset pricing for unbounded stochastic processes, *Mathematische Annalen*, 312: 215–250.

DELBAEN, F. & SCHACHERMAYER, W. (2006), *The Mathematics of Arbitrage*, Springer, Berlin - Heidelberg - New York.

DELBAEN, F. & SHIRAKAWA, H. (1996), A note on the no-arbitrage condition for international financial markets, *Asia-Pacific Financial Markets*, 3(3): 239–251.

DRIESSEN, J. (2005), Is default event risk priced in corporate bonds?, *Review of Financial Studies*, 18(1): 165–195.

DUFFEE, G.R. (1999), Estimating the price of default risk, *Review of Financial Studies*, 12(1): 197–226.

DUFFEE, G.R. (2002), Term premia and interest rate forecasts in affine models, *Journal of Finance*, LVII(1): 405–443.

DUFFIE, D., ECKNER, A., HOREL, G. & SAITA, L. (2009), Frailty correlated default, *Journal of Finance*, LXIV(5): 2089–2123.

DUFFIE, D. & KAN, R. (1996), A yield-factor model of interest rates, *Mathematical Finance*, 6(4): 379–406.

DUFFIE, D., PAN, J. & SINGLETON, K.J. (2000), Transform analysis and asset pricing for affine jump-diffusions, *Econometrica*, 68(6): 1343–1376.

DUFFIE, D. & SINGLETON, K.J. (1999), Modeling term structures of defaultable bonds, *Review of Financial Studies*, 12(4): 687–720.

EL KAROUI, N., JEANBLANC, M. & JIAO, Y. (2010), What happens after a default: The conditional density approach, *Stochastic Processes and Their Applications*, 120: 1011–1032.

EL KAROUI, N. & MARTELLINI, L. (2001), A theoretical inspection of the market price for default risk, working paper, University of Southern California.

ETHIER, S.N. & KURTZ, T.G. (1986), *Markov Processes - Characterization and Convergence*, Wiley, New York.

FERNHOLZ, R. & KARATZAS, I. (2009), Stochastic portfolio theory: an overview, in: Bensoussan, A. & Zhang, Q. (eds.): *Mathematical Modeling and Numerical Methods in Finance*, Handbook of Numerical Analysis, vol. XV, 89–167, North-Holland, Oxford.

FILIPOVIĆ, D. (2009), *Term-Structure Models: A Graduate Course*, Springer, Berlin - Heidelberg.

FÖLLMER, H. & SCHWEIZER, M. (1991), Hedging of contingent claims under incomplete information, in: Davis, M.H.A. & Elliott, R.J. (eds.), *Applied Stochastic Analysis*, Stochastic monographs, vol. 5, 389–414, Gordon and Breach, London - New York.

FONTANA, C. (2010a), Credit risk and incomplete information: A filtering framework for pricing and risk management, to appear in: Perna, C. & Sibillo, M. (eds.), *Mathematical and Statistical Methods for Actuarial Sciences and Finance*, Springer.

FONTANA, C. (2010b), *Mean-Variance Problems with Applications to Credit Risk Models*, MAS Finance thesis, ETH Zürich and University of Zürich.

FONTANA, C. & RUNGGALDIER, W.J. (2010), Credit risk and incomplete information: Filtering and EM parameter estimation, *International Journal of Theoretical and Applied Finance*, 13(5): 683–715.

FONTANA, C. & RUNGGALDIER, W.J. (2011), Diffusion-based models for financial markets without martingale measures, preprint, University of Padova.

FONTANA, C. & SCHWEIZER, M. (2011), The mathematics and financial economics of mean-variance portfolio optimisation, preprint, ETH Zürich.

GALESSO, G. & RUNGGALDIER, W.J. (2010), Pricing without equivalent martingale measures under complete and incomplete observation, in: Chiarella, C. & Novikov, A. (eds.), *Contemporary Quantitative Finance: Essays in Honour of Eckhard Platen*, 99–121, Springer, Berlin - Heidelberg.

GATHERAL, J. (2006), *The Volatility Surface: A Practitioner's Guide*, Wiley, Hoboken (NJ).

GIESECKE, K. & GOLDBERG, L.R. (2008), The market price of credit risk: The impact of asymmetric information, working paper, Stanford University.

GOMBANI, A., JASCHKE, S. & RUNGGALDIER, W.J. (2007), Consistent price systems for subfiltrations, *ESAIM: Probability and Statistics*, 11: 35–39.

HANSEN, L.P. & RICHARD, S.F. (1987), The role of conditioning information in deducing testable restrictions implied by dynamic asset pricing models, *Econometrica*, 55: 587–613.

HENDERSON, V. & HOBSON, D. (2009), Utility indifference pricing: An overview, in: Carmona, R. (ed.), *Indifference Pricing: Theory and Applications*, 44–74, Princeton University Press, Princeton.

HESTON, S.L. (1993), A closed-form solution for options with stochastic volatility, with applications to bond and currency options, *Review of Financial Studies*, 6: 327–343.

HESTON, S.L., LOEWENSTEIN, M. & WILLARD, G.A. (2007), Options and bubbles, *Review of Financial Studies*, 20(2): 359–390.

HODGES, S.D. & NEUBERGER, A. (1989), Optimal replication of contingent claims under transaction costs, *Review of Futures Markets*, 8: 222–239.

HOU, C. & KARATZAS, I. (2004), Least-squares approximation of random variables by stochastic integrals, in: Kunita, H. et al. (eds.), *Stochastic Analysis and Related Topics in Kyoto*, Advanced Studies in Pure Mathematics, vol. 41: 141–166, Mathematical Society of Japan, Tokyo.

HUANG, C. & LITZENBERGER, R.H. (1988), *Foundations for Financial Economics*, North-Holland, New York.

HULLEY, H. (2010), The economic plausibility of strict local martingales in financial modelling, in: Chiarella, C. & Novikov, A. (eds.): *Contemporary Quantitative Finance: Essays in Honour of Eckhard Platen*, 53–75, Springer, Berlin - Heidelberg.

HULLEY, H. & SCHWEIZER, M. (2010), M^6 - on minimal market models and minimal martingale measures, in: Chiarella, C. & Novikov, A. (eds.): *Contemporary Quantitative Finance: Essays in Honour of Eckhard Platen*, 35–51, Springer, Berlin - Heidelberg.

IMKELLER, P. (2002), Random times at which insiders can have free lunches, *Stochastics*, 74(1-2): 465–487.

INGERSOLL, J.E. (1987), *Theory of Financial Decision Making*, Rowman & Littlefield, Totowa.

JACKA, S.D. (1992), A martingale representation result and an application to incomplete financial markets, *Mathematical Finance*, 2(4): 239–250.

JACOD, J. (1979), *Calcul Stochastique et Problèmes de Martingales*, Lecture Notes in Mathematics, Vol. 714, Springer, Berlin - Heidelberg.

JACOD, J. (1980), Intégrales stochastiques par rapport à une semi-martingale vectorielle et changements de filtration, *Séminaire de Probabilités*, XIV: 161–172.

JACOD, J. (1985), Grossissement initial, hypothèse (H'), et théorème de Girsanov, in: Jeulin, T. & Yor, M. (eds): *Grossissement de Filtrations: Exemples et Applications*, Lecture Notes in Mathematics, Vol. 1118, 15–35, Springer, Berlin - Heidelberg.

JACOD, J. & SHIRYAEV, A.N. (2003), *Limit Theorems for Stochastic Processes*, second edition, Springer, Berlin - Heidelberg - New York.

JARROW, R.A., LANDO, D. & YU, F. (2005), Default risk and diversification: Theory and empirical implications, *Mathematical Finance*, 15(1): 1–26.

JARROW, R.A. & MADAN, D.B. (1999), Hedging contingent claims on semimartingales, *Finance and Stochastics*, 3: 111–134.

JARROW, R. & PROTTER, P. (2005), Large traders, hidden arbitrage, and complete markets, *Journal of Banking and Finance*, 29: 2803–2820.

JARROW, R.A., PROTTER, P. & SHIMBO, K. (2007), Asset price bubbles in complete markets, in: Fu, M.C., Jarrow, R.A., Yen, J.-Y.J. & Elliott, R.J. (eds.), *Advances in Mathematical Finance*, 97–122, Birkhäuser, Boston.

JARROW, R.A., PROTTER, P. & SHIMBO, K. (2010), Asset price bubbles in incomplete markets, *Mathematical Finance*, 20(2): 145–185.

JEANBLANC, M. & LE CAM, Y. (2009), Progressive enlargement of filtrations with initial times, *Stochastic Processes and Their Applications*, 119: 2523–2543.

JEANBLANC, M., YOR, M. & CHESNEY, M. (2009), *Mathematical Methods for Financial Markets*, Springer, London.

JEULIN, T. (1980), *Semi-Martingales et Grossissement d'une Filtration*, Lecture Notes in Mathematics, Vol. 833, Springer, Berlin - Heidelberg - New York.

KABANOV, Y. (1997), On the FTAP of Kreps-Delbaen-Schachermayer, in: Kabanov, Y., Rozovskii, B.L. & Shiryaev, A.N. (eds.), *Statistics and Control of Stochastic Processes: The Liptser Festschrift*, 191–203, World Scientific, Singapore.

KABANOV, Y. & STRICKER, C. (2005), Remarks on the true no-arbitrage property, *Séminaire de Probabilités*, XXXVIII: 186–194.

KALLSEN, J. (2004), σ-localization and σ-martingales, *Theory of Probability and its Applications*, 48: 152–163.

KALLSEN, J. & MUHLE-KARBE, J. (2010), Exponentially affine martingales, affine measure changes and exponential moments of affine processes, *Stochastic Processes and Their Applications*, 120: 163–181.

KALLSEN, J. & SHIRYAEV, A.N. (2002), The cumulant process and Esscher's change of measure, *Finance and Stochastics*, 6: 397–428.

KARATZAS, I. & KARDARAS, K. (2007), The numeraire portfolio in semimartingale financial models, *Finance and Stochastics*, 11: 447–493.

KARATZAS, I. & SHREVE, S.E. (1991), *Brownian Motion and Stochastic Calculus*, second edition, Springer, New York.

KARATZAS, I. & SHREVE, S.E. (1998), *Methods of Mathematical Finance*, Springer, New York.

KARDARAS, C. (2010a), Finitely additive probabilities and the fundamental theorem of asset pricing, in: Chiarella, C. & Novikov, A. (eds.): *Contemporary Quantitative Finance: Essays in Honour of Eckhard Platen*, 19–34, Springer, Berlin - Heidelberg.

KARDARAS, C. (2010b), Market viability via absence of arbitrage of the first kind, to appear in: *Finance and Stochastics*.

KOHLMANN, M., XIONG, D. & YE, Z. (2007), Change of filtrations and mean-variance hedging, *Stochastics*, 79(6): 539–562.

KRAMKOV, D. & SCHACHERMAYER, W. (1999), The asymptotic elasticity of utility functions and optimal investment in incomplete markets, *Annals of Applied Probability*, 9(3): 904–950.

KUSUOKA, S. (1999), A remark on default risk models, *Advances in Mathematical Economics*, 1: 69–82.

LANDO, D. (1998), On Cox processes and credit risky securities, *Review of Derivatives Research*, 2: 99–120.

LANDO, D. (2004), *Credit Risk Modeling - Theory and Applications*, Princeton University Press, Princeton (NJ).

LEIPPOLD, M., TROJANI, F. & VANINI, P. (2004), A geometric approach to multiperiod mean variance optimization of assets and liabilities, *Journal of Economic Dynamics and Control*, 28: 1079–1113.

LEITNER, D. (2000), Mean-variance efficiency and intertemporal price for risk, *Discussion Paper CoFE 35, University of Konstanz*.

LEPINGLE, D. & MÉMIN, J. (1978), Sur l'intégrabilité uniforme des martingales exponentielles, *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 42(3): 175–203.

LEVENTAL, S. & SKOROHOD, A.S. (1995), A necessary and sufficient condition for absence of arbitrage with tame portfolios, *Annals of Applied Probability*, 5(4): 906–925.

LI, D. & NG, W.L. (2000), Optimal dynamic portfolio selection: Multiperiod mean-variance formulation, *Mathematical Finance*, 10: 387–406.

LOEWENSTEIN, M. & WILLARD, G.A. (2000), Local martingales, arbitrage, and viability, *Economic Theory*, 16: 135–161.

LONDONO, J.A. (2004), State tameness: a new approach for credit constraints, *Electronic Communications in Probability*, 9: 1–13.

LUENBERGER, D.G. (1998), *Investment Science*, Oxford University Press, New York.

LUENBERGER, D.G. (2001), Projection pricing, *Journal of Optimization Theory and Applications*, 109: 1–25.

MACLEAN, L.C., THORP, E.O. & ZIEMBA, W.T. (2010), Long-term capital growth: the good and bad properties of the Kelly and fractional Kelly capital growth criteria, *Quantitative Finance*, 10(7): 681–687.

MARKOWITZ, H. (1952), Portfolio selection, *Journal of Finance*, 7: 77–91.

MARKOWITZ, H. (1987), *Mean-variance Analysis in Portfolio Choice and Capital Markets*, Basil Blackwell, Oxford New York.

MAYERHOFER, E., MUHLE-KARBE, J. & SMIRNOV, A.G. (2011), A characterization of the martingale property of exponentially affine processes, *Stochastic Processes and their Applications*, 121: 568–582.

MCNEIL, A.J., FREY, R. & EMBRECHTS, P. (2005), *Quantitative Risk Management - Concepts, Techniques and Tools*, Princeton University Press, Princeton (NJ).

MÉMIN, J. (1980), Espaces des semimartingales et changement de probabilitè, *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 52: 9-39.

MERCURIO, F. (2001), Claim pricing and hedging under market incompleteness and "mean-variance" preferences, *European Journal of Operations Research*, 133: 635–652.

MERTON, R.C. (1974), On the pricing of corporate debt: On the risk structure of interest rates, *Journal of Finance*, 29(2): 449–470.

MIJATOVIĆ, A. & URUSOV, M. (2010a), On the martingale property of certain local martingales, to appear in: *Probability Theory and Related Fields*.

MIJATOVIĆ, A. & URUSOV, M. (2010b), Deterministic criteria for the absence of arbitrage in one-dimensional diffusion models, to appear in: *Finance and Stochastics*.

MØLLER, T. (2001), On transformations of actuarial valuation principles, *Insurance: Mathematics and Economics*, 28: 281–303.

NAJNUDEL, J. & NIKEGHBALI, A. (2011), A new kind of augmentation of filtrations, *ESAIM: Probability and Statistics - Supplement: In Honor of Marc Yor*, 15: 39–57.

PAOLELLA, M.S. (2007), *Intermediate Probability: A Computational Approach*, Wiley, Chichester.

PHAM, H. (2000), On quadratic hedging in continuous time, *Mathematical Methods of Operations Research*, 51: 315–339.

PLATEN, E. (2002), Arbitrage in continuous complete markets, *Advances in Applied Probability*, 34: 540–558.

PLATEN, E. (2006), A benchmark approach to finance, *Mathematical Finance*, 16(1): 131–151.

PLATEN, E. (2009), A benchmark approach to investing and pricing, research paper No. 253, Quantitative Finance Research Centre, University of Technology Sydney.

PLATEN, E. & HEATH, D. (2006), *A Benchmark Approach to Quantitative Finance*, Springer, Berlin - Heidelberg.

PLATEN, E. & RUNGGALDIER, W.J. (2005), A benchmark approach to filtering in finance, *Asia-Pacific Financial Markets*, 11: 79–105.

PLATEN, E. & RUNGGALDIER, W.J. (2007), A benchmark approach to portfolio optimization under partial information, *Asia-Pacific Financial Markets*, 14: 25–43.

PROTTER, P. (2005), *Stochastic Integration and Differential Equations*, second edition, version 2.1, Springer, Berlin - Heidelberg - New York.

PROTTER, P. & SHIMBO, K. (2008), No arbitrage and general semimartingales, in: Ethier, S.N., Feng, J. & Stockbridge, R.H. (eds.), *Markov Processes and Related Topics: A Festschrift for Thomas G. Kurtz*, 267–283, Institute of mathematical statistics, Beachwood (OH).

REVUZ, D. & YOR, M. (1999), *Continuous Martingales and Brownian Motion*, third edition, Springer, Berlin - Heidelberg.

RHEINLÄNDER, T. (1999), *Optimal Martingale Measures and their Applications in Mathematical Finance*, PhD thesis, Technische Universität Berlin.

RHEINLÄNDER, T. & SCHWEIZER, M. (1997), On L^2-projections on a space of stochastic integrals, *Annals of Probability*, 25(4): 1810–1831.

RUF, J. (2011a), Hedging under arbitrage, working paper, Columbia University.

RUF, J. (2011b), Optimal trading strategies under arbitrage, PhD Thesis, Columbia University.

RUNGGALDIER, W.J. (2003), Jump-diffusion models, in: Rachev, S.T. (ed.), *Handbook of Heavy Tailed Distributions in Finance*, Elsevier/North Holland, Amsterdam.

SCHÖNBUCHER, P.J. (2003a), *Credit Derivatives Pricing Models - Models, Pricing and Implementation*, Wiley, Chichester.

SCHÖNBUCHER, P.J. (2003b), Information-driven default contagion, working paper, Department of Mathematics, ETH Zürich.

SCHWEIZER, M. (1992), Martingale densities for general asset prices, *Journal of Mathematical Economics*, 21: 363–378.

SCHWEIZER, M. (1995), On the minimal martingale measure and the Föllmer-Schweizer decomposition, *Stochastic Analysis and Applications*, 13: 573–599.

SCHWEIZER, M. (1996), Approximation pricing and the variance-optimal martingale measure, *Annals of Probability*, 24: 206–236.

SCHWEIZER, M. (1997), From actuarial to financial valuation principles, *Proceedings of the 7th AFIR Colloquium and the 28th ASTIN Colloquium, Cairns (1997), Joint Day Volume*, 261–282.

SCHWEIZER, M. (1999), Risky options simplified, *International Journal of Theoretical and Applied Finance*, 2: 59–82.

SCHWEIZER, M. (2001a), From actuarial to financial valuation principles, *Insurance: Mathematics and Economics*, 28: 31–47.

SCHWEIZER, M. (2001b), A guided tour through quadratic hedging approaches, in: Jouini, E. et al. (eds.), *Option Pricing, Interest Rates and Risk Management*, 538–574, Cambridge University Press, Cambridge.

SCHWEIZER, M. (2010), Mean-variance hedging, in: Cont, R. (ed.), *Encyclopedia of Quantitative Finance*, 1177–1181, Wiley, Chichester.

STEINBACH, M.C. (2001), Markowitz revisited: Mean-variance models in financial portfolio analysis, *SIAM Review*, 43: 31–85.

STRASSER, E. (2005), Characterization of arbitrage-free markets, *Annals of Applied Probability*, 15(1): 116–124.

STRICKER, C. & YAN, J.-A. (1998), Some remarks on the optional decomposition theorem, *Séminaire de Probabilités*, XXXII: 56–66.

SUN, W.G. & WANG, F.W. (2005), Claim hedging in an incomplete market, *Journal of Systems Science and Complexity*, 18: 193–201.

SUN, W.G. & WANG, F.W. (2006), The mean-variance investment problem in a constrained financial market, *Journal of Mathematical Economics*, 42: 885–895.

TAKAOKA, K. (2010), On the condition of no unbounded profit with bounded risk, working paper, Hitotsubashi University.

WONG, B. & HEYDE, C.C. (2006), On changes of measure in stochastic volatility models, *Journal of Applied Mathematics and Stochastic Analysis*, 1-13.

XIA, J. & YAN, J.A. (2006), Markowitz's portfolio optimization in an incomplete market, *Mathematical Finance*, 16: 203–216.

ZHOU, X.Y. (2003), Markowitz's world in continuous time, and beyond, in: Yao, D.D. et al. (eds.), *Stochastic Modeling and Optimization*, 279–310, Springer, New York.

ZWIERZ, J. (2007), On existence of local martingale measures for insiders who can stop at honest times, *Bulletin of the Polish Academy of Sciences: Mathematics*, 55(2): 183–192.