# Overall equilibrium in the coupling of peridynamics and classical continuum mechanics 

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#### Abstract

Coupling peridynamics based computational tools with those using classical continuum mechanics can be very beneficial, because it can provide a means to generate a computational method that combines the efficiency of classical continuum mechanics with the capability to simulate crack propagation, typical of peridynamics. This paper presents an overlooked issue in this type of coupled computational methods: the lack of overall equilibrium. This can be the case even if the coupling strategy satisfies the usual numerical tests involving rigid body motions as well as uniform and linear strain distributions. We focus our investigation on the lack of overall equilibrium in an approach to couple peridynamics and classical continuum mechanics recently proposed by the authors. In our examples, the magnitude of the out-of-balance forces is a fraction of a per cent of the applied forces, but it cannot be assumed to be a numerical round-off error. We show analytically and numerically that the main reason for the existence of out-of-balance forces is a lack of balance between the local and nonlocal tractions at the coupling interface. This usually results from the presence of high-order derivatives


[^0]of displacements in the coupling zone.
Keywords: Peridynamics, classical continuum mechanics, CCM-PD coupling, overall static equilibrium, out-of-balance forces

## 1. Introduction

The unavoidable presence of small or large cracks in many aeronautical and aerospace structures still represents a major challenge for engineers who want to simulate a full structural life cycle [1, Chapter 16], [2-5]. Even though classical continuum mechan- structures, which can be affected by different levels of damage in their various parts or components, many scientists have tried to equip CCM based numerical methods, in particular the finite element method (FEM), with the capability to simulate crack formation and propagation. The most popular approaches that have been developed in the last years are: extended finite element method [8, 9], element erosion [10, 11], phase field model [12, 13], interface elements with a cohesive zone model [14-16], and the partition of unity finite element method [17, 18]. Even if these different strategies have all been used so far, they all present some drawbacks [7, Chapter 10]. The issues related to the use of the extended finite element method to model propagating cracks are manifold. First, in dynamic brittle fracture problems, one may need to significantly modify the input fracture energy in the numerical method in order to match the values of the crack propagation speeds obtained from experimental investigations. Second, this
method requires crack path tracking, phenomenological damage models, extra damage the crack propagation speeds obtained from experimental investigations. Second, this
method requires crack path tracking, phenomenological damage models, extra damage 25 ics (CCM) based numerical methods are extensively used for the simulation of different structural problems, their application for damage prediction introduces some challenges arising from the presence of spatial derivatives of displacements in the governing equations, which are undefined when the displacement fields are discontinuous [6], [7, Chapter 1]. Since cracks are, in fact, discontinuities in the domain where the problem is defined, they do not satisfy the basic underlying continuum hypothesis of classical continuum mechanics. In order to achieve an accurate description of large and complex criteria regarding the angle of propagation and the stress state around the crack tip, and branching criteria, which are not reliable in practice. Third, the implementation of this
method may introduce some computational burdens related to the need to subdivide the cut elements to perform the numerical integration process. The inherent complexity and computational cost of this strategy prevent its application for problems involving

In recent years, innovative computational methods based on peridynamics have been proposed and implemented in order to solve complex problems involving damage initiation and crack propagation. The peridynamic (PD) theory is a nonlocal reformulation of CCM based on integro-differential equations, which introduces a concept of damage for a material point, allowing to predict the evolution of cracks, including their nucleation, their propagation direction, and the points where they start and stop, with-
out having to define any criteria for triggering, bifurcation, and deviation phenomena. The PD theory was proposed in the year 2000. The original bond-based version of the theory was presented in [6] and then extended in the year 2007 to its final form called state-based PD theory in [29]. Related nonlocal models were proposed previously by other authors [30-32]. Despite the effectiveness of PD models in solving problems concerning crack propagation [33-46], PD models are computationally more expensive than CCM models due to their nonlocal nature. The computational expense issue is even more evident when implicit time integration is considered, since the number of nonzero elements in the PD tangent stiffness matrix is typically much bigger than that in the corresponding CCM model solved with the FEM [7, Chapter 14]. Therefore, the considerable computational cost of PD models hinders their application in large-scale, geometrically complex simulations [47]. Furthermore, PD numerical implementations may be affected by some additional difficulties related to the definition o of nonlocal boundary conditions [6]. In nonlocal theories the boundaries are fuzzy, so that prescribed displacement or load conditions have to be imposed in finite volumetric regions rather than on boundary surfaces [48-50]. Most of the time, such extension of classical boundary conditions is not clearly defined [7, Chapter 14]. Hence, it would be convenient to couple PD and CCM models in order to take advantage of the benefits of both models while avoiding their aforementioned drawbacks.

In CCM-PD coupling, usually small areas of a domain, which might be affected by the presence of discontinuities, are described with a PD model, whereas the remaining parts of the domain are represented through a more efficient CCM model. In particular, it is common practice to couple PD models based on the meshfree discretization of [44] with CCM models discretized using the FEM. Even though PD models can be also discretized with the FEM [51], in this work FEM is used only to denote discretization of CCM models. Coupling FEM meshes with PD grids (or, more generally, coupling local and nonlocal models) is not as simple as sharing nodes between meshes, as is frequently done in FEM codes when different types of elements are connected to each

Local-to-nonlocal coupling has led to a great research effort (much of it concerning the
coupling of CCM and PD models) resulting in the development of a variety of techniques, including the optimization-based [53-55], partitioned [56, 57], Arlequin [58], morphing [59-62], quasi-nonlocal [63], blending [64, 65], splice [66], variable hori- zon [66], and partial stress [66] methods, among others; a recent review of these methods can be found in [67]. In the context of the PD theory, the first paper to deal with this type of coupling was [68], in which bond-based PD grids and FEM meshes were coupled by embedding PD nodes within FEM elements. Other early works on the coupling of FEM meshes and PD grids can be found in [69-72]. CCM-PD coupling is still an posed in the above listed papers are affected by some kind of arbitrariness or spurious effects that need to be overcome. In the present work, we are interested in the coupling technique proposed in a series of papers [73-77], which can be seen as an application of the splice method. For the first time, we address the problem of the overall equilibrium in CCM-PD coupled models. This paper studies the origin of out-of-balance forces and discusses possible ways to reduce them.

The analysis presented in this paper shows that the absence of overall equilibrium in a CCM-PD coupled system results from the lack of balance between the local and nonlocal tractions at the coupling interface. The concept of lack of force reciprocity in CCM-PD coupled models and how it leads to failure of Newton's third law between two given objects was discussed in [64]. However, a thorough investigation of this effect and its manifestation in global structural equilibrium in CCM-PD coupled systems has not been presented. The closest studies in this regard from the literature concern patch-test consistency and the so-called "ghost" forces; these are non-physical forces that arise in the transition between local and nonlocal regions whenever a coupling method does not pass a patch test [67]. Unfortunately, such studies generally provide only a qualitative assessment of whether a CCM-PD coupled model passes or not a patch test and often limit the analysis to a simple constant strain solution (i.e., a linear patch test). Our work, in contrast, presents a detailed analysis of the balance between local and nonlocal tractions at coupling interfaces along with a practical quantitative way to assess the resulting out-of-balance through computation of the reaction forces.

The contents of this paper are organized as follows. In Section 2, a short summary of the CCM-PD coupling strategy presented in $[75,76]$ and exploited in the present work is provided. In Section 3, the overall static equilibrium issue is presented, disrespectively. Section 6 closes the paper with some remarks and proposals for future research.

## 2. CCM-PD coupling strategy

The proposed coupling approach is based on the idea presented in [75], where the coupled stiffness matrix is defined and used to solve linear static bond-based PD problems, and then extended to the solution of dynamic problems in [76]. A further extension of this coupling approach to state-based PD models is presented in [77]. In order to better introduce the main features of the proposed CCM-PD coupling strategy, it is necessary to provide a brief outline of the PD theory. More details can be found in [6], where the bond-based version of the PD theory is presented, and in [29], where the more general state-based PD theory is introduced.

In a domain $\mathcal{B} \subset \mathbb{R}^{n}$ with $n$ the spatial dimension, described with a PD model, each material point $\mathbf{x} \in \mathcal{B}$ interacts with all the other material points located within a finite neighbourhood, $\mathcal{H}_{\mathbf{x}}$, of that material point. The state-based PD equation of motion for any material point $\mathbf{x} \in \mathcal{B}$ at time $t \geqslant 0$ is given by [29]:

$$
\begin{equation*}
\rho(\mathbf{x}) \ddot{\mathbf{u}}(\mathbf{x}, t)=\int_{\mathcal{H}_{\mathbf{x}}}\left\{\underline{\mathbf{T}}[\mathbf{x}, t]\left\langle\mathbf{x}^{\prime}-\mathbf{x}\right\rangle-\underline{\mathbf{T}}\left[\mathbf{x}^{\prime}, t\right]\left\langle\mathbf{x}-\mathbf{x}^{\prime}\right\rangle\right\} d \mathbf{x}^{\prime}+\mathbf{b}(\mathbf{x}, t), \tag{1}
\end{equation*}
$$

where $\rho$ is the mass density, $\ddot{\mathbf{u}}$ is the second derivative in time of the displacement field $\mathbf{u}$, $\underline{T}[\mathbf{x}, t]\left\langle\mathbf{x}^{\prime}-\mathbf{x}\right\rangle$ is the force state defined at the material point $\mathbf{x}$ at time $t$ mapping the bond $\mathbf{x}^{\prime}-\mathbf{x}$ to force per unit volume squared, and $\mathbf{b}$ is a prescribed body force density
field. The neighbourhood, $\mathcal{H}_{\mathbf{x}}$, is defined by:

$$
\begin{equation*}
\mathcal{H}_{\mathbf{x}}:=\left\{\mathbf{x}^{\prime} \in \mathcal{B}:\left\|\mathbf{x}^{\prime}-\mathbf{x}\right\| \leqslant \delta\right\} \tag{2}
\end{equation*}
$$

$$
\begin{equation*}
\underline{\mathbf{T}}[\mathbf{x}, t]\langle\boldsymbol{\xi}\rangle=\frac{1}{2} \mathbf{f}(\boldsymbol{\eta}, \boldsymbol{\xi}) \tag{5}
\end{equation*}
$$

where $\mathbf{f}$ is the pairwise force function in the bond-based PD theory [6]. A linear isotropic bond-based PD model, provided a pairwise equilibrated reference configuration, is given by [6]:

$$
\begin{equation*}
\mathbf{f}(\boldsymbol{\eta}, \boldsymbol{\xi})=\lambda(\|\xi\|) \xi \otimes \xi \boldsymbol{\eta} \tag{6}
\end{equation*}
$$

where $\lambda$ is a micromodulus function.

In the proposed CCM-PD coupling technique, the domain $\mathcal{B}$ is partially described with a CCM model discretized using the FEM. The remaining part of the domain is described with a PD model discretized with a meshfree method based on [44]. The two parts of the domain have to be coupled in a way that ensures an adequate transfer of force between the two regions. Figure 1 illustrates the CCM-PD coupled model in a one-dimensional PD bonds to all nodes, FEM or PD nodes, within its neighbourhood. In a similar way, all PD nodes, the neighbourhood of which contains FEM nodes, are nonlocally interacting through PD bonds with those FEM nodes.


Figure 1: Illustration of the CCM-PD coupled model in a one-dimensional system. Blue diamonds are FEM nodes and green circles are PD nodes. Blue thick straight lines represent FEM elements and green thin curved lines represent PD bonds. Adapted from [75].

The proposed CCM-PD coupling technique assumes that internal forces acting on a node are of the same nature as the node itself: only internal forces evaluated using the FEM approach act on FEM nodes, whereas only internal forces computed through the PD formulation are applied on PD nodes. A coupling zone can be defined where forces are exchanged between the CCM and PD parts of the domain. In the example presented in Fig. 1, the coupling zone is composed of the FEM nodes 3 and 4; the PD nodes 5 and 6 ; the PD bonds $3-5,4-5$, and $4-6$; and the FEM element $d$. The coupling method assumes that the internal force exerted by the FEM element $d$ acts only on the FEM node 4, whereas the internal forces exerted by the PD bonds $3-5$ and $4-5$ as well as $4-6$ act only on the PD nodes 5 and 6 , respectively. Consequently, the assembly
of the global stiffness matrix is performed by making sure that equilibrium equations equations of PD nodes include only terms derived from the PD formulation.

The case of Fig. 1 produces the following system of equations:

$$
\left[\begin{array}{cccccccccc}
l & -l & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \vdots  \tag{7}\\
-l & 2 l & -l & 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\
0 & -l & 2 l & -l & 0 & 0 & 0 & 0 & 0 & \vdots \\
0 & 0 & -l & 2 l & -l & 0 & 0 & 0 & 0 & \vdots \\
0 & 0 & -\frac{1}{4} p & -p & \frac{5}{2} p & -p & -\frac{1}{4} p & 0 & 0 & \vdots \\
0 & 0 & 0 & -\frac{1}{4} p & -p & \frac{5}{2} p & -p & -\frac{1}{4} p & 0 & \vdots \\
0 & 0 & 0 & 0 & -\frac{1}{4} p & -p & \frac{5}{2} p & -p & -\frac{1}{4} p & \vdots \\
0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \ddots & \vdots & \vdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \vdots
\end{array}\right]\left[\begin{array}{c}
u_{1} \\
u_{2} \\
u_{3} \\
u_{4} \\
u_{5} \\
\vdots \\
\vdots \\
\vdots \\
u_{N}
\end{array}\right]=\left[\begin{array}{c}
F_{1} \\
F_{2} \\
F_{3} \\
F_{4} \\
F_{5} \\
\vdots \\
\vdots \\
F_{N}
\end{array}\right],
$$

where $l:=E A / \Delta x[52], p:=c A^{2} \Delta x[78], N$ is the total number of nodes (including FEM and PD nodes), $\left\{u_{i}\right\}_{i=1, \ldots, N}$ are the nodal displacements, $\left\{F_{i}\right\}_{i=1, \ldots, N}$ are the external nodal forces, $E$ is Young's modulus, $c$ is the micromodulus constant, and the cross-sectional area $A$ is assumed to be $A=1$; the same assumption applies hereafter. To obtain (7), we assumed a CCM model given by (8) and a PD model given by (15) with a micromodulus function $c(|\xi|)=c /|\xi|$. The meshfree PD discretization employed in (7) uses a partial-volume correction [79], which applies a factor of $\frac{1}{2}$ to the contribution of second-nearest neighbors. The solution of a single equation satisfies node equilibrium. The overall equilibrium of the whole structure, however, requires the sum of the external nodal forces to be equal to zero.

In the numerical examples of Sections 4.1 and 4.2, a nodal displacement vector will be input into the system and the relevant external nodal forces (i.e., 'reactions', since the displacements are imposed) will be computed according to a system of the type of (7) in the one-dimensional case and a corresponding system in the two-dimensional case. In some cases of CCM-PD coupled systems, the force vector will have a non-zero resultant, i.e., $\sum_{i=1}^{N} F_{i} \neq 0$, and therefore overall equilibrium will not be satisfied, even coupling problems carried out by imposing rigid body motions as well as uniform and linear strain distributions [75-77].

## 3. Theoretical background: Out-of-balance analysis in CCM-PD coupling

Traditionally, the consistency between CCM and PD models is studied through the analysis of the corresponding governing equations, as reported in Appendix A. However, this analysis does not reveal the responsible for the existence of out-of-balance forces in CCM-PD coupled systems. For this reason, we present instead an analysis of the force balance in one-dimensional and two-dimensional CCM-PD coupled models in Section 3.1 and Section 3.2, respectively. For a detailed analysis of the consistency between CCM and PD models, please refer to Appendix A. 1 for the one-dimensional case and Appendix A. 2 for the two-dimensional case.

### 3.1. One-dimensional case

### 3.1.1. CCM model



Figure 2: One-dimensional domain $\mathcal{B}=(0, L)$.

Assume a one-dimensional domain $\mathcal{B}=(0, L)$ as in Fig. 2 and consider the CCM static equation

$$
\begin{equation*}
-E \frac{d^{2} u}{d x^{2}}(x)=b(x) \quad x \in(0, L) \tag{8}
\end{equation*}
$$

where $E$ is Young's modulus. Integrating the equation over the domain $\mathcal{B}$, we obtain

$$
\begin{equation*}
-\int_{0}^{L} E \frac{d^{2} u}{d x^{2}}(x) d x=\int_{0}^{L} b(x) d x \tag{9}
\end{equation*}
$$

Performing the integration on the left-hand side, we have

$$
\begin{equation*}
-E \frac{d u}{d x}(L)+E \frac{d u}{d x}(0)=\int_{0}^{L} b(x) d x \tag{10}
\end{equation*}
$$

In this case, the stress at $x_{0}$ is given by:

$$
\begin{equation*}
\sigma\left(x_{0}\right)=E \frac{d u}{d x}\left(x_{0}\right) \tag{11}
\end{equation*}
$$

and the corresponding local traction is defined by:

$$
\begin{equation*}
\mathrm{t}\left(x_{0}, n\right):=\sigma\left(x_{0}\right) n \tag{12}
\end{equation*}
$$

where $n= \pm 1$ represents a normal in one dimension. Note that

$$
\begin{equation*}
\mathfrak{t}\left(x_{0},-n\right)=-\mathrm{t}\left(x_{0}, n\right) \tag{13}
\end{equation*}
$$

We then obtain from (10) the force balance equation

$$
\begin{equation*}
\mathrm{t}(L,+1)+\mathrm{t}(0,-1)+\int_{0}^{L} b(x) d x=0 \tag{14}
\end{equation*}
$$

where $t(L,+1)$ and $t(0,-1)$ are boundary local tractions. The boundary local tractions balance the external forces.

### 3.1.2. PD model



Figure 3: One-dimensional domain $\mathcal{B}=(0, L)$ with nonlocal boundary $[-\delta, 0] \cup[L, L+\delta]$.

Assume a one-dimensional domain $\mathcal{B}=(0, L)$ as in Fig. 3, where $L \geqslant \delta>0$, with nonlocal boundary $[-\delta, 0] \cup[L, L+\delta]$ and consider the bond-based PD static equation

$$
\begin{equation*}
-\int_{\mathcal{H}_{x}} c\left(\left|x^{\prime}-x\right|\right)\left(u\left(x^{\prime}\right)-u(x)\right) d x^{\prime}=b(x) \quad x \in(0, L) \tag{15}
\end{equation*}
$$

where $c(|\xi|)$ is a micromodulus function and $\mathcal{H}_{x}=[x-\delta, x+\delta]$. Note the relation $c(|\xi|)=\lambda(|\xi|)|\xi|^{2}$ in one dimension with $\lambda(|\xi|)$ from (6). Introducing the characteristic function

$$
\chi_{\delta}(|\xi|):=\left\{\begin{array}{cc}
1 & |\xi| \leqslant \delta  \tag{16}\\
0 & \text { else }
\end{array}\right.
$$

we can extend the domain of integration in (15) to the union of the domain $\mathcal{B}$ and its nonlocal boundary (see Fig. 3):

$$
\begin{equation*}
-\int_{-\delta}^{L+\delta} \chi_{\delta}\left(\left|x^{\prime}-x\right|\right) c\left(\left|x^{\prime}-x\right|\right)\left(u\left(x^{\prime}\right)-u(x)\right) d x^{\prime}=b(x) \quad x \in(0, L) \tag{17}
\end{equation*}
$$

We now integrate the equation over the domain $\mathcal{B}$ :

$$
\begin{equation*}
-\int_{0}^{L} \int_{-\delta}^{L+\delta} \chi_{\delta}\left(\left|x^{\prime}-x\right|\right) c\left(\left|x^{\prime}-x\right|\right)\left(u\left(x^{\prime}\right)-u(x)\right) d x^{\prime} d x=\int_{0}^{L} b(x) d x \tag{18}
\end{equation*}
$$

Due to the antisymmetry of the integrand on the left-hand side of (18),

$$
\begin{equation*}
-\int_{0}^{L} \int_{0}^{L} \chi_{\delta}\left(\left|x^{\prime}-x\right|\right) c\left(\left|x^{\prime}-x\right|\right)\left(u\left(x^{\prime}\right)-u(x)\right) d x^{\prime} d x=0 \tag{19}
\end{equation*}
$$

meaning that internal forces are balanced in a bounded PD body. Consequently, we obtain

$$
\begin{align*}
& -\int_{0}^{L} \int_{-\delta}^{0} \chi_{\delta}\left(\left|x^{\prime}-x\right|\right) c\left(\left|x^{\prime}-x\right|\right)\left(u\left(x^{\prime}\right)-u(x)\right) d x^{\prime} d x \\
& -\int_{0}^{L} \int_{L}^{L+\delta} \chi_{\delta}\left(\left|x^{\prime}-x\right|\right) c\left(\left|x^{\prime}-x\right|\right)\left(u\left(x^{\prime}\right)-u(x)\right) d x^{\prime} d x=\int_{0}^{L} b(x) d x \tag{20}
\end{align*}
$$

which can now be written without the characteristic function (recall $L \geqslant \delta$ ) as

$$
\begin{align*}
& -\int_{0}^{\delta} \int_{x-\delta}^{0} c\left(\left|x^{\prime}-x\right|\right)\left(u\left(x^{\prime}\right)-u(x)\right) d x^{\prime} d x \\
& -\int_{L-\delta}^{L} \int_{L}^{x+\delta} c\left(\left|x^{\prime}-x\right|\right)\left(u\left(x^{\prime}\right)-u(x)\right) d x^{\prime} d x=\int_{0}^{L} b(x) d x \tag{21}
\end{align*}
$$

Following the concept of areal force density introduced in [6], we define the nonlocal traction at $x_{0}$ with normal $n= \pm 1$ in one dimension by:

$$
\tau\left(x_{0}, n\right):= \begin{cases}\int_{x_{0}-\delta}^{x_{0}} \int_{x_{0}}^{x+\delta} c\left(\left|x^{\prime}-x\right|\right)\left(u\left(x^{\prime}\right)-u(x)\right) d x^{\prime} d x & n=+1,  \tag{22}\\ \int_{x_{0}}^{x_{0}+\delta} \int_{x-\delta}^{x_{0}} c\left(\left|x^{\prime}-x\right|\right)\left(u\left(x^{\prime}\right)-u(x)\right) d x^{\prime} d x & n=-1 .\end{cases}
$$

Note that (see Remark B. 1 in Appendix B.1)

$$
\begin{equation*}
\tau\left(x_{0},-n\right)=-\tau\left(x_{0}, n\right) \tag{23}
\end{equation*}
$$

We then obtain from (21) the force balance equation

$$
\begin{equation*}
\tau(L,+1)+\tau(0,-1)+\int_{0}^{L} b(x) d x=0 \tag{24}
\end{equation*}
$$

where $\tau(L,+1)$ and $\tau(0,-1)$ are boundary nonlocal tractions. The boundary nonlocal tractions balance the external forces.

### 3.1.3. CCM-PD coupled model



Figure 4: Decomposition of a one-dimensional domain $\mathcal{B}=(0, L)$ into a PD subdomain $\mathcal{B}_{\mathrm{PD}}=\left(x_{I L}, x_{I R}\right)$ embedded into a CCM subdomain $\mathcal{B}_{\mathrm{CCM}}=\mathcal{B}_{\mathrm{CCM}}^{\mathrm{L}} \cup \mathcal{B}_{\mathrm{CCM}}^{\mathrm{R}}=\left(0, x_{I L}\right) \cup\left(x_{I R}, L\right)$. The transition between the PD and CCM subdomains occurs at the interfaces $x_{I L}$ and $x_{I R}$.

Assume a one-dimensional domain $\mathcal{B}=(0, L)$ and consider a CCM-PD coupled configuration where a PD subdomain is embedded into a CCM subdomain, as illustrated in Fig. 4. This configuration enables the use of classical local boundary conditions. Consider two interfaces, $x_{I L}$ and $x_{I R}$, such that $0<x_{I L}<x_{I R}<L$. Assume points $x \in$ $\left(x_{I L}, x_{I R}\right)$ are described by the PD model (15), whereas points $x \in\left(0, x_{I L}\right) \cup\left(x_{I R}, L\right)$ are described by the CCM model (8). We assume the length of the PD subdomain is at least $\delta$, so that $x_{I R}-x_{I L} \geqslant \delta$. We further assume the CCM subdomain is large enough, so that $x_{I L}-\delta \geqslant 0$ and $x_{I R}+\delta \leqslant L$. The corresponding coupled system of equations can be written as:

$$
\begin{align*}
-E \frac{d^{2} u}{d x^{2}}(x)=b(x) & x \in\left(0, x_{I L}\right),  \tag{25a}\\
-\int_{\mathcal{H}_{x}} c\left(\left|x^{\prime}-x\right|\right)\left(u\left(x^{\prime}\right)-u(x)\right) d x^{\prime}=b(x) & x \in\left(x_{I L}, x_{I R}\right),  \tag{25b}\\
-E \frac{d^{2} u}{d x^{2}}(x)=b(x) & x \in\left(x_{I R}, L\right) . \tag{25c}
\end{align*}
$$

Integrating the equations over their respective subdomains, we obtain

$$
\left.\begin{array}{rl}
-\int_{0}^{x_{I L}} E \frac{d^{2} u}{d x^{2}}(x) d x & =\int_{0}^{x_{I L}} b(x) d x \\
-\int_{x_{I L}}^{x_{I R}} \int_{\mathcal{H}_{x}} c\left(\left|x^{\prime}-x\right|\right)\left(u\left(x^{\prime}\right)-u(x)\right) d x^{\prime} d x & =\int_{x_{I L}}^{x_{I R}} b(x) d x \\
& -\int_{x_{I R}}^{L} E \frac{d^{2} u}{d x^{2}}(x) d x \tag{26c}
\end{array}\right)=\int_{x_{I R}}^{L} b(x) d x .
$$

Adding the equations in (26), we get

$$
\begin{align*}
-\int_{0}^{x_{I L}} E \frac{d^{2} u}{d x^{2}}(x) d x & -\int_{x_{I L}}^{x_{I R}} \int_{\mathcal{H}_{x}} c\left(\left|x^{\prime}-x\right|\right)\left(u\left(x^{\prime}\right)-u(x)\right) d x^{\prime} d x \\
& -\int_{x_{I R}}^{L} E \frac{d^{2} u}{d x^{2}}(x) d x=\int_{0}^{L} b(x) d x \tag{27}
\end{align*}
$$

Performing the integration in the first and third terms on the left-hand side and using the characteristic function (16) for the second term on the left-hand side, we have

$$
\begin{align*}
-E \frac{d u}{d x}\left(x_{I L}\right)+E \frac{d u}{d x}(0) & -\int_{x_{I L}}^{x_{I R}} \int_{x_{I L}-\delta}^{x_{I R}+\delta} \chi_{\delta}\left(\left|x^{\prime}-x\right|\right) c\left(\left|x^{\prime}-x\right|\right)\left(u\left(x^{\prime}\right)-u(x)\right) d x^{\prime} d x \\
& -E \frac{d u}{d x}(L)+E \frac{d u}{d x}\left(x_{I R}\right)=\int_{0}^{L} b(x) d x \tag{28}
\end{align*}
$$

Similar to (19),

$$
\begin{equation*}
-\int_{x_{I L}}^{x_{I R}} \int_{x_{I L}}^{x_{I R}} \chi_{\delta}\left(\left|x^{\prime}-x\right|\right) c\left(\left|x^{\prime}-x\right|\right)\left(u\left(x^{\prime}\right)-u(x)\right) d x^{\prime} d x=0 \tag{29}
\end{equation*}
$$

Therefore, we obtain

$$
\begin{align*}
-E \frac{d u}{d x}\left(x_{I L}\right)+E \frac{d u}{d x}(0) & -\int_{x_{I L}}^{x_{I R}} \int_{x_{I L}-\delta}^{x_{I L}} \chi_{\delta}\left(\left|x^{\prime}-x\right|\right) c\left(\left|x^{\prime}-x\right|\right)\left(u\left(x^{\prime}\right)-u(x)\right) d x^{\prime} d x \\
& -\int_{x_{I L}}^{x_{I R}} \int_{x_{I R}}^{x_{I R}+\delta} \chi_{\delta}\left(\left|x^{\prime}-x\right|\right) c\left(\left|x^{\prime}-x\right|\right)\left(u\left(x^{\prime}\right)-u(x)\right) d x^{\prime} d x \\
& -E \frac{d u}{d x}(L)+E \frac{d u}{d x}\left(x_{I R}\right)=\int_{0}^{L} b(x) d x \tag{30}
\end{align*}
$$

Removing the characteristic function (recall $x_{I R}-x_{I L} \geqslant \delta$ ), we have

$$
\begin{align*}
-E \frac{d u}{d x}\left(x_{I L}\right)+E \frac{d u}{d x}(0) & -\int_{x_{I L}}^{x_{I L}+\delta} \int_{x-\delta}^{x_{I L}} c\left(\left|x^{\prime}-x\right|\right)\left(u\left(x^{\prime}\right)-u(x)\right) d x^{\prime} d x \\
& -\int_{x_{I R}-\delta}^{x_{I R}} \int_{x_{I R}}^{x+\delta} c\left(\left|x^{\prime}-x\right|\right)\left(u\left(x^{\prime}\right)-u(x)\right) d x^{\prime} d x \\
& -E \frac{d u}{d x}(L)+E \frac{d u}{d x}\left(x_{I R}\right)=\int_{0}^{L} b(x) d x \tag{31}
\end{align*}
$$

Using the definitions for the local and nonlocal tractions in (12) and (22), respectively, we can express this equation as follows:
$\mathrm{t}(L,+1)+\mathrm{t}(0,-1)+\int_{0}^{L} b(x) d x=-\left\{\left[\mathrm{t}\left(x_{I L},+1\right)+\tau\left(x_{I L},-1\right)\right]+\left[\tau\left(x_{I R},+1\right)+\mathrm{t}\left(x_{I R},-1\right)\right]\right\}$.

The net force, $\mathcal{F}$, applied on the domain $\mathcal{B}$ is given by (cf. (14)):

$$
\begin{equation*}
\mathcal{F}=\mathrm{t}(L,+1)+\mathrm{t}(0,-1)+\int_{0}^{L} b(x) d x \tag{33}
\end{equation*}
$$

We then conclude that overall equilibrium, i.e., $\mathcal{F}=0$, requires the balance between the local and nonlocal tractions at the interfaces (see (32)):

$$
\begin{align*}
& \tau\left(x_{I L},-1\right)=-\mathrm{t}\left(x_{I L},+1\right)  \tag{34a}\\
& \tau\left(x_{I R},+1\right)=-\mathrm{t}\left(x_{I R},-1\right) \tag{34b}
\end{align*}
$$

In Section 3.1.4, we discuss the convergence of the nonlocal traction to the local traction and provide conditions under which (34) is satisfied.

### 3.1.4. Convergence of the nonlocal traction to the local traction in one dimension

This section presents for brevity only the main results derived from the analysis of the convergence of the nonlocal traction to the local traction in one-dimensional CCM-PD coupled models. For a comprehensive derivation, please refer to Appendix B.1.

Consider the nonlocal traction at $x_{0} \in \mathcal{B}$ in the bulk of the body with normal $n=+1$, i.e., $\tau\left(x_{0},+1\right)$ in (22) and assume a micromodulus function of the form $c(|\xi|)=c /|\xi|^{\alpha}$ with $c$ a constant and $\alpha<2$. Assuming a smooth deformation and performing some Taylor expansions, we obtain

$$
\begin{equation*}
\tau\left(x_{0},+1\right)=\mathrm{t}\left(x_{0},+1\right)+\frac{1}{12} \frac{3-\alpha}{5-\alpha} E \delta^{2} \frac{d^{3} u}{d x^{3}}\left(x_{0}\right)+\ldots, \tag{35}
\end{equation*}
$$

where the relation $c=\frac{3-\alpha}{\delta^{3-\alpha}} E$ has been employed (cf. (A.12)), $\mathrm{t}\left(x_{0},+1\right)$ is the local traction at $x_{0}$ with normal $n=+1(c f .(12))$, and the dots indicate higher-order derivative terms. In the limit as $\delta \rightarrow 0$, we get

$$
\begin{equation*}
\tau\left(x_{0},+1\right)=\mathrm{t}\left(x_{0},+1\right)+\mathcal{O}\left(\delta^{2}\right) \tag{36}
\end{equation*}
$$ this result reveals that, even though the discrepancy between the PD and CCM models depends upon fourth-order and higher derivatives of displacements (see Appendix A.1), the discrepancy between the nonlocal and local tractions depends upon third-order and higher derivatives of displacements. Nevertheless, the leading terms in both the model and traction discrepancies are both of order $\boldsymbol{O}\left(\delta^{2}\right)$ (see Appendix A.1).

The result in (35) implies that, if the deformation around the interfaces $x_{I L}$ and $x_{I R}$ in Fig. 4 is smooth and third-order and higher derivatives of displacements are negligible, we have (recall (13) and (23))

$$
\begin{align*}
& \tau\left(x_{I L},-1\right)=-\tau\left(x_{I L},+1\right)=-\mathrm{t}\left(x_{I L},+1\right)  \tag{37a}\\
& \tau\left(x_{I R},+1\right)=\mathrm{t}\left(x_{I R},+1\right)=-\mathrm{t}\left(x_{I R},-1\right) \tag{37b}
\end{align*}
$$

so that (34) is satisfied and overall equilibrium is attained. However, whenever thirdorder or higher derivatives of displacements are not negligible around either of the interfaces, lack of overall equilibrium is in general expected. In this case, the net out-ofbalance force is given by (cf. (33) and (32)):

$$
\begin{align*}
\mathcal{F} & =-\left\{\left[\mathrm{t}\left(x_{I L},+1\right)+\tau\left(x_{I L},-1\right)\right]+\left[\tau\left(x_{I R},+1\right)+\mathrm{t}\left(x_{I R},-1\right)\right]\right\} \\
& =\frac{1}{12} \frac{3-\alpha}{5-\alpha} E \delta^{2}\left(\frac{d^{3} u}{d x^{3}}\left(x_{I L}\right)-\frac{d^{3} u}{d x^{3}}\left(x_{I R}\right)\right)+\ldots, \tag{38}
\end{align*}
$$

where we employed (35) in combination with (13) and (23) in the last equality. In the limit as $\delta \rightarrow 0$, the net out-of-balance force vanishes at a rate of $\mathcal{O}\left(\delta^{2}\right)$.

### 3.2. Two-dimensional case

### 3.2.1. CCM model

Assume a two-dimensional domain $\mathcal{B}$ with boundary $\partial \mathcal{B}$ as in Fig. 5 and consider the CCM static equation

$$
\begin{equation*}
-\nabla \cdot \sigma(\mathbf{x})=\mathbf{b}(\mathbf{x}) \quad \mathbf{x} \in \mathcal{B} \tag{39}
\end{equation*}
$$

where $\sigma$ is a Piola-Kirchhoff stress tensor field. Integrating the equation over the domain $\mathcal{B}$, we obtain

$$
\begin{equation*}
-\int_{\mathcal{B}} \nabla \cdot \sigma(\mathbf{x}) d \mathbf{x}=\int_{\mathcal{B}} \mathbf{b}(\mathbf{x}) d \mathbf{x} \tag{40}
\end{equation*}
$$



Figure 5: Two-dimensional domain $\mathcal{B}$ with boundary $\partial \mathcal{B}$.

Using Gauss's theorem for the left-hand side, we have

$$
\begin{equation*}
-\int_{\partial \mathcal{B}} \sigma(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) d l=\int_{\mathcal{B}} \mathbf{b}(\mathbf{x}) d \mathbf{x} \tag{41}
\end{equation*}
$$

where $\mathbf{n}(\mathbf{x})$ is the outward unit normal to the boundary $\partial \mathcal{B}$ at $\mathbf{x} \in \partial \mathcal{B}$ and the integral over $\partial \mathcal{B}$ is a line integral. The local traction is defined by:

$$
\begin{equation*}
\mathfrak{t}(\mathbf{x}, \mathbf{n}):=\sigma(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \tag{42}
\end{equation*}
$$

Note that

$$
\begin{equation*}
\mathbf{t}(\mathbf{x},-\mathbf{n})=-\mathbf{t}(\mathbf{x}, \mathbf{n}) \tag{43}
\end{equation*}
$$

We then obtain from (41) the force balance equation

$$
\begin{equation*}
\int_{\partial \mathcal{B}} \mathbf{t}(\mathbf{x}, \mathbf{n}) d l+\int_{\mathcal{B}} \mathbf{b}(\mathbf{x}) d \mathbf{x}=\mathbf{0} \tag{44}
\end{equation*}
$$

where the boundary local tractions balance the external forces.

### 3.2.2. PD model

Assume a two-dimensional domain $\mathcal{B}$ with a nonlocal boundary layer as in Fig. 6 and consider the PD static equation

$$
\begin{equation*}
-\int_{\mathcal{H}_{\mathbf{x}}} \boldsymbol{f}\left(\mathbf{x}^{\prime}, \mathbf{x}\right) d \mathbf{x}^{\prime}=\mathbf{b}(\mathbf{x}) \quad \mathbf{x} \in \mathcal{B} \tag{45}
\end{equation*}
$$

where (cf. (1))

$$
\begin{equation*}
\boldsymbol{f}\left(\mathbf{x}^{\prime}, \mathbf{x}\right):=\underline{\mathbf{T}}[\mathbf{x}, t]\left\langle\mathbf{x}^{\prime}-\mathbf{x}\right\rangle-\underline{\mathbf{T}}\left[\mathbf{x}^{\prime}, t\right]\left\langle\mathbf{x}-\mathbf{x}^{\prime}\right\rangle \tag{46}
\end{equation*}
$$



Figure 6: Two-dimensional domain $\mathcal{B}$ with nonlocal boundary layer (in gray).
and $\mathcal{H}_{\mathbf{x}}$ is the neighbourhood of $\mathbf{x}$. Note that the following antisymmetric property holds:

$$
\begin{equation*}
f\left(\mathbf{x}, \mathbf{x}^{\prime}\right)=-f\left(\mathbf{x}^{\prime}, \mathbf{x}\right) \tag{47}
\end{equation*}
$$

Introducing the characteristic function

$$
\chi_{\delta}(\|\xi\|):=\left\{\begin{array}{cc}
1 & \|\xi\| \leqslant \delta  \tag{48}\\
0 & \text { else }
\end{array}\right.
$$

we can extend the domain of integration in (45) to the union of the domain $\mathcal{B}$ and its nonlocal boundary, which we denote together by $\overline{\overline{\mathcal{B}}}$ :

$$
\begin{equation*}
-\int_{\overline{\bar{B}}} \chi_{\delta}\left(\left\|\mathbf{x}^{\prime}-\mathbf{x}\right\|\right) \boldsymbol{f}\left(\mathbf{x}^{\prime}, \mathbf{x}\right) d \mathbf{x}^{\prime}=\mathbf{b}(\mathbf{x}) \quad \mathbf{x} \in \mathcal{B} \tag{49}
\end{equation*}
$$

We now integrate the equation over the domain $\mathcal{B}$ :

$$
\begin{equation*}
-\int_{\mathcal{B}} \int_{\overline{\bar{B}}} \chi_{\delta}\left(\left\|\mathbf{x}^{\prime}-\mathbf{x}\right\|\right) \boldsymbol{f}\left(\mathbf{x}^{\prime}, \mathbf{x}\right) d \mathbf{x}^{\prime} d \mathbf{x}=\int_{\mathcal{B}} \mathbf{b}(\mathbf{x}) d \mathbf{x} \tag{50}
\end{equation*}
$$

Due to the antisymmetric property (47), we have

$$
\begin{equation*}
-\int_{\mathcal{B}} \int_{\mathcal{B}} \chi_{\delta}\left(\left\|\mathbf{x}^{\prime}-\mathbf{x}\right\|\right) \boldsymbol{f}\left(\mathbf{x}^{\prime}, \mathbf{x}\right) d \mathbf{x}^{\prime} d \mathbf{x}=\mathbf{0} \tag{51}
\end{equation*}
$$

meaning that internal forces are balanced in a bounded PD body. Consequently, we obtain

$$
\begin{equation*}
-\int_{\mathcal{B}} \int_{\overline{\overline{\mathcal{B}}} \backslash \mathcal{B}} \chi_{\delta}\left(\left\|\mathbf{x}^{\prime}-\mathbf{x}\right\|\right) \boldsymbol{f}\left(\mathbf{x}^{\prime}, \mathbf{x}\right) d \mathbf{x}^{\prime} d \mathbf{x}=\int_{\mathcal{B}} \mathbf{b}(\mathbf{x}) d \mathbf{x} . \tag{52}
\end{equation*}
$$

We now have the force balance equation

$$
\begin{equation*}
\int_{\mathcal{B}} \int_{\overline{\overline{\mathcal{B}}} \backslash \mathcal{B}} \chi_{\delta}\left(\left\|\mathbf{x}^{\prime}-\mathbf{x}\right\|\right) \boldsymbol{f}\left(\mathbf{x}^{\prime}, \mathbf{x}\right) d \mathbf{x}^{\prime} d \mathbf{x}+\int_{\mathcal{B}} \mathbf{b}(\mathbf{x}) d \mathbf{x}=\mathbf{0} . \tag{53}
\end{equation*}
$$

Assume there exists a function $\boldsymbol{\tau}(\mathbf{x}, \mathbf{n})$ satisfying

$$
\begin{equation*}
\int_{\partial \mathcal{B}} \tau(\mathbf{x}, \mathbf{n}) d l=\int_{\mathcal{B}} \int_{\overline{\bar{B}} \backslash \mathcal{B}} \chi_{\delta}\left(\left\|\mathbf{x}^{\prime}-\mathbf{x}\right\|\right) f\left(\mathbf{x}^{\prime}, \mathbf{x}\right) d \mathbf{x}^{\prime} d \mathbf{x}, \tag{54}
\end{equation*}
$$

where $\mathbf{n}=\mathbf{n}(\mathbf{x})$ is the outward unit normal to the boundary $\partial \mathcal{B}$ at $\mathbf{x} \in \partial \mathcal{B}$. In this case, we refer to $\boldsymbol{\tau}(\mathbf{x}, \mathbf{n})$ as the nonlocal traction at $\mathbf{x}$ with normal $\mathbf{n}(\mathbf{x})$, and we can express (53) as

$$
\begin{equation*}
\int_{\partial \mathcal{B}} \tau(\mathbf{x}, \mathbf{n}) d l+\int_{\mathcal{B}} \mathbf{b}(\mathbf{x}) d \mathbf{x}=\mathbf{0} \tag{55}
\end{equation*}
$$

where the boundary nonlocal tractions balance the external forces.

### 3.2.3. CCM-PD coupled model



Figure 7: Decomposition of a two-dimensional domain $\mathcal{B}$ with boundary $\partial \mathcal{B}$ into a PD subdomain $\mathcal{B}_{\mathrm{PD}}$ embedded into a CCM subdomain $\mathcal{B}_{\mathrm{CCM}}$. The interface between the PD and CCM subdomains is denoted by $\Gamma$.

Assume a two-dimensional domain $\mathcal{B}$ with boundary $\partial \mathcal{B}$ and consider a CCM-PD coupled configuration where a PD subdomain, $\mathcal{B}_{\mathrm{PD}}$, is embedded into a CCM subdomain, $\mathcal{B}_{\mathrm{CCM}}$, as illustrated in Fig. 7, such that $\overline{\mathcal{B}}=\overline{\mathcal{B}_{\mathrm{PD}}} \cup \overline{\mathcal{B}_{\mathrm{CCM}}}, \mathcal{B}_{\mathrm{PD}} \cap \mathcal{B}_{\mathrm{CCM}}=\emptyset$, and $\overline{\mathcal{B}_{\mathrm{PD}}} \cap \overline{\mathcal{B}_{\mathrm{CCM}}}=\partial \mathcal{B}_{\mathrm{PD}}=: \Gamma$. This configuration enables the use of classical local boundary conditions. We assume the CCM subdomain is large enough, so that $\overline{\overline{\mathcal{B}_{\mathrm{PD}}}} \backslash \mathcal{B}_{\mathrm{PD}} \subset \overline{\mathcal{B}_{\mathrm{CCM}}}$. The corresponding coupled system of equations can be written as:

$$
\begin{align*}
-\nabla \cdot \sigma(\mathbf{x})=\mathbf{b}(\mathbf{x}) & \mathbf{x} \in \mathcal{B}_{\mathrm{CCM}}  \tag{56a}\\
-\int_{\mathcal{H}_{\mathbf{x}}} \boldsymbol{f}\left(\mathbf{x}^{\prime}, \mathbf{x}\right) d \mathbf{x}^{\prime}=\mathbf{b}(\mathbf{x}) & \mathbf{x} \in \mathcal{B}_{\mathrm{PD}} \tag{56b}
\end{align*}
$$

Integrating the equations over their respective subdomains, we obtain

$$
\begin{align*}
-\int_{\mathcal{B}_{\mathrm{CCM}}} \nabla \cdot \sigma(\mathbf{x}) d \mathbf{x} & =\int_{\mathcal{B}_{\mathrm{CCM}}} \mathbf{b}(\mathbf{x}) d \mathbf{x}  \tag{57a}\\
-\int_{\mathcal{B}_{\mathrm{PD}}} \int_{\mathcal{H}_{\mathbf{x}}} f\left(\mathbf{x}^{\prime}, \mathbf{x}\right) d \mathbf{x}^{\prime} d \mathbf{x} & =\int_{\mathcal{B}_{\mathrm{PD}}} \mathbf{b}(\mathbf{x}) d \mathbf{x} \tag{57b}
\end{align*}
$$

Adding the equations in (57), we get

$$
\begin{equation*}
-\int_{\mathcal{B}_{\mathrm{CCM}}} \nabla \cdot \sigma(\mathbf{x}) d \mathbf{x}-\int_{\mathcal{B}_{\mathrm{PD}}} \int_{\mathcal{H}_{\mathbf{x}}} f\left(\mathbf{x}^{\prime}, \mathbf{x}\right) d \mathbf{x}^{\prime} d \mathbf{x}=\int_{\mathcal{B}} \mathbf{b}(\mathbf{x}) d \mathbf{x} \tag{58}
\end{equation*}
$$

Using Gauss's theorem for the first term on the left-hand side and the characteristic function (48) for the second term on the left-hand side, we have

$$
\begin{equation*}
-\int_{\partial \mathcal{B}_{\mathrm{CCM}}} \sigma(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) d l-\int_{\mathcal{B}_{\mathrm{PD}}} \int_{\overline{\overline{\mathcal{B}_{\mathrm{PD}}}}} \chi_{\delta}\left(\left\|\mathbf{x}^{\prime}-\mathbf{x}\right\|\right) \boldsymbol{f}\left(\mathbf{x}^{\prime}, \mathbf{x}\right) d \mathbf{x}^{\prime} d \mathbf{x}=\int_{\mathcal{B}} \mathbf{b}(\mathbf{x}) d \mathbf{x} \tag{59}
\end{equation*}
$$

Note that $\partial \mathcal{B}_{\mathrm{CCM}}=\partial \mathcal{B} \cup \Gamma$ (see Fig. 7). In addition, similar to (51),

$$
\begin{equation*}
-\int_{\mathcal{B}_{\mathrm{PD}}} \int_{\mathcal{B}_{\mathrm{PD}}} \chi_{\delta}\left(\left\|\mathbf{x}^{\prime}-\mathbf{x}\right\|\right) \boldsymbol{f}\left(\mathbf{x}^{\prime}, \mathbf{x}\right) d \mathbf{x}^{\prime} d \mathbf{x}=\mathbf{0} \tag{60}
\end{equation*}
$$

Therefore, we obtain
$-\int_{\partial \mathcal{B}} \sigma(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) d l-\int_{\Gamma} \sigma(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) d l-\int_{\mathcal{B}_{\mathrm{PD}}} \int_{\overline{\overline{\mathcal{B}_{\mathrm{PD}}}} \backslash \mathcal{B}_{\mathrm{PD}}} \chi_{\delta}\left(\left\|\mathbf{x}^{\prime}-\mathbf{x}\right\|\right) f\left(\mathbf{x}^{\prime}, \mathbf{x}\right) d \mathbf{x}^{\prime} d \mathbf{x}=\int_{\mathcal{B}} \mathbf{b}(\mathbf{x}) d \mathbf{x}$.

Using the definition for the local traction in (42), we can express this equation as follows:

$$
\begin{equation*}
\int_{\partial \mathcal{B}} \mathbf{t}(\mathbf{x}, \mathbf{n}) d l+\int_{\mathcal{B}} \mathbf{b}(\mathbf{x}) d \mathbf{x}=-\left[\int_{\Gamma} \mathbf{t}(\mathbf{x}, \mathbf{n}) d l+\int_{\mathcal{B}_{\mathrm{PD}}} \int_{\mathcal{B}_{\mathrm{CCM}}} \chi_{\delta}\left(\left\|\mathbf{x}^{\prime}-\mathbf{x}\right\|\right) \boldsymbol{f}\left(\mathbf{x}^{\prime}, \mathbf{x}\right) d \mathbf{x}^{\prime} d \mathbf{x}\right], \tag{62}
\end{equation*}
$$

where we used the assumption $\overline{\overline{\mathcal{B}_{\mathrm{PD}}}} \backslash \mathcal{B}_{\mathrm{PD}} \subset \overline{\mathcal{B}_{\mathrm{CCM}}}$ to rewrite the inner domain of integration of the second term inside the square brackets on the right-hand side. We recall that the normal $\mathbf{n}$ on $\Gamma$ in the first term inside the square brackets on the righthand side points outwards relative to $\mathcal{B}_{\mathrm{CCM}}$. The net force, $\boldsymbol{F}$, applied on the domain $\mathcal{B}$ is given by (cf. (44)):

$$
\begin{equation*}
\mathcal{F}=\int_{\partial \mathcal{B}} \mathbf{t}(\mathbf{x}, \mathbf{n}) d l+\int_{\mathcal{B}} \mathbf{b}(\mathbf{x}) d \mathbf{x} . \tag{63}
\end{equation*}
$$

We then conclude that overall equilibrium, i.e., $\boldsymbol{F}=\mathbf{0}$, requires (see (62))

$$
\begin{equation*}
\int_{\mathcal{B}_{\mathrm{PD}}} \int_{\mathcal{B}_{\mathrm{CCM}}} \chi_{\delta}\left(\left\|\mathbf{x}^{\prime}-\mathbf{x}\right\|\right) \boldsymbol{f}\left(\mathbf{x}^{\prime}, \mathbf{x}\right) d \mathbf{x}^{\prime} d \mathbf{x}=-\int_{\Gamma} \mathbf{t}(\mathbf{x}, \mathbf{n}) d l . \tag{64}
\end{equation*}
$$

Similar to (54), assume there exists a nonlocal traction $\boldsymbol{\tau}(\mathbf{x},-\mathbf{n})$ satisfying

$$
\begin{equation*}
\int_{\Gamma} \boldsymbol{\tau}(\mathbf{x},-\mathbf{n}) d l=\int_{\mathcal{B}_{\mathrm{PD}}} \int_{\mathcal{B}_{\mathrm{CCM}}} \chi_{\delta}\left(\left\|\mathbf{x}^{\prime}-\mathbf{x}\right\|\right) \boldsymbol{f}\left(\mathbf{x}^{\prime}, \mathbf{x}\right) d \mathbf{x}^{\prime} d \mathbf{x} \tag{65}
\end{equation*}
$$

where, in this case, $-\mathbf{n}$ on $\Gamma$ points outwards relative to $\mathcal{B}_{\mathrm{PD}}$. Then, we can express (64) as

$$
\begin{equation*}
\int_{\Gamma} \boldsymbol{\tau}(\mathbf{x},-\mathbf{n}) d l=-\int_{\Gamma} \mathbf{t}(\mathbf{x}, \mathbf{n}) d l \tag{66}
\end{equation*}
$$

i.e., overall equilibrium requires the balance between the local and nonlocal tractions at the interface $\Gamma$. In particular, overall equilibrium is attained if the following (stronger) condition is satisfied:

$$
\begin{equation*}
\boldsymbol{\tau}(\mathbf{x},-\mathbf{n})=-\mathbf{t}(\mathbf{x}, \mathbf{n}) \quad \mathbf{x} \in \Gamma \tag{67}
\end{equation*}
$$

In Section 3.2.4, we discuss the convergence of the nonlocal traction to the local traction and provide conditions under which (67) is satisfied.

### 3.2.4. Convergence of the nonlocal traction to the local traction in two dimensions

This section presents for brevity only the main results derived from the analysis of the convergence of the nonlocal traction to the local traction in two-dimensional CCM-PD coupled models. For a comprehensive derivation, please refer to Appendix B.2.

We consider the simplified case of two non-overlapping subdomains $\Omega_{A}$ and $\Omega_{B}$ with a straight interface $\Gamma$ connecting them, i.e., $\Omega_{A} \cap \Omega_{B}=\emptyset$ and $\overline{\Omega_{A}} \cap \overline{\Omega_{B}}=\Gamma$. We assume the normal $\mathbf{n}$ to the interface $\Gamma$ points outwards relative to $\Omega_{A}$. In this case, following the concept of areal force density introduced in [6], we define the nonlocal traction at $\mathbf{x}_{0} \in \Gamma$ with normal $\mathbf{n}$ by:

$$
\begin{equation*}
\tau\left(\mathbf{x}_{0}, \mathbf{n}\right):=\int_{\mathcal{L}} \int_{\Omega_{B}} \chi_{\delta}\left(\left\|\mathbf{x}^{\prime}-\mathbf{x}\right\|\right) \boldsymbol{f}\left(\mathbf{x}^{\prime}, \mathbf{x}\right) d \mathbf{x}^{\prime} d \ell \tag{68}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{L}:=\left\{\mathbf{x} \in \Omega_{A}: \mathbf{x}=\mathbf{x}_{0}-s \mathbf{n}, 0 \leqslant s \leqslant \delta\right\} . \tag{69}
\end{equation*}
$$

Given the linear isotropic bond-based PD model (6), we can express (68) as (recall (46) and (5)):

$$
\begin{equation*}
\tau\left(\mathbf{x}_{0}, \mathbf{n}\right)=\int_{\mathcal{L}} \int_{\Omega_{B}} \chi_{\delta}\left(\left\|\mathbf{x}^{\prime}-\mathbf{x}\right\|\right) \lambda\left(\left\|\mathbf{x}^{\prime}-\mathbf{x}\right\|\right)\left(\mathbf{x}^{\prime}-\mathbf{x}\right) \otimes\left(\mathbf{x}^{\prime}-\mathbf{x}\right)\left(\mathbf{u}\left(\mathbf{x}^{\prime}\right)-\mathbf{u}(\mathbf{x})\right) d \mathbf{x}^{\prime} d \ell \tag{70}
\end{equation*}
$$

or, in component form,

$$
\begin{equation*}
\tau_{i}\left(\mathbf{x}_{0}, \mathbf{n}\right)=\int_{\mathcal{L}} \int_{\Omega_{B}} \chi_{\delta}(\|\xi\|) \lambda(\|\xi\|) \xi_{i} \xi_{j}\left(u_{j}(\mathbf{x}+\xi)-u_{j}(\mathbf{x})\right) d \mathbf{x}^{\prime} d \ell, \quad i=1,2 \tag{71}
\end{equation*}
$$

where we used the notation $\boldsymbol{\xi}=\mathbf{x}^{\prime}-\mathbf{x}$ for brevity and repeated indices imply a summation by 1 and 2 . We consider below two cases, the first one given by a horizontal interface $\Gamma$ with normal $\mathbf{n}=\mathbf{e}_{2}$ (see Fig. B.26a) and the second one given by a vertical interface $\Gamma$ with normal $\mathbf{n}=\mathbf{e}_{1}$ (see Fig. B.26b); the normals $\left\{\mathbf{e}_{1}, \mathbf{e}_{2}\right\}$ correspond to the standard Cartesian orthonormal basis. In both cases, the normal points outwards relative to $\Omega_{A}$. We assume the point $\mathbf{x}_{0}$ is in the bulk of the body. We further assume a micromodulus function of the form $\lambda(\|\xi\|)=c /\|\xi\|^{\alpha}$ with $c$ a constant and $\alpha<6$.

Horizontal Interface. Assuming a smooth deformation and performing some Taylor expansions, we obtain (see (B.30))

$$
\begin{align*}
& \tau_{1}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)=\mathrm{t}_{1}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)+\frac{2 E}{5 \pi} \frac{6-\alpha}{7-\alpha} \delta\left(\frac{3}{2} \frac{\partial^{2} u_{1}}{\partial x^{2}}\left(\mathbf{x}_{0}\right)+\frac{\partial^{2} u_{2}}{\partial x \partial y}\left(\mathbf{x}_{0}\right)\right)+\ldots,  \tag{72a}\\
& \tau_{2}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)=\mathrm{t}_{2}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)+\frac{2 E}{5 \pi} \frac{6-\alpha}{7-\alpha} \delta\left(\frac{\partial^{2} u_{1}}{\partial x \partial y}\left(\mathbf{x}_{0}\right)+\frac{\partial^{2} u_{2}}{\partial x^{2}}\left(\mathbf{x}_{0}\right)\right)+\ldots, \tag{72b}
\end{align*}
$$

where the relation $c=\frac{3(6-\alpha) E}{\pi \delta^{6-\alpha}}$ has been employed (cf. (A.31)), $\mathrm{t}_{1}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)$ and $\mathrm{t}_{2}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)$ are the $x$ - and $y$-components, respectively, of the local traction (cf. (42)) evaluated at
${ }_{340} \quad \mathbf{x}_{0} \in \Gamma$ in classical plane stress (cf. (A.18)) for $v=1 / 3$, and the dots indicate higherorder derivative terms. In the limit as $\delta \rightarrow 0$, we get

$$
\begin{equation*}
\boldsymbol{\tau}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)=\mathbf{t}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)+\mathcal{O}(\delta), \tag{73}
\end{equation*}
$$

i.e., the nonlocal traction converges to the local traction at a rate of $\mathcal{O}(\delta)$. Equation (72) implies that, if the deformation around the interface $\Gamma$ is smooth and second-order and higher derivatives of displacements are negligible, (67) is satisfied (note $\mathbf{t}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)=$ $-\mathbf{t}\left(\mathbf{x}_{0},-\mathbf{e}_{2}\right)$ by (43)) and overall equilibrium is attained. However, whenever secondorder or higher derivatives of displacements are not negligible around the interface $\Gamma$, lack of overall equilibrium is in general expected. In this case, the components of the
net out-of-balance force are given by (see (62), (63), and (65)):

$$
\begin{align*}
& \mathcal{F}_{1} \approx-\int_{\Gamma} \frac{2 E}{5 \pi} \frac{6-\alpha}{7-\alpha} \delta\left(\frac{3}{2} \frac{\partial^{2} u_{1}}{\partial x^{2}}\left(x, y_{0}\right)+\frac{\partial^{2} u_{2}}{\partial x \partial y}\left(x, y_{0}\right)\right) d x+\ldots  \tag{74a}\\
& \mathcal{F}_{2} \approx-\int_{\Gamma} \frac{2 E}{5 \pi} \frac{6-\alpha}{7-\alpha} \delta\left(\frac{\partial^{2} u_{1}}{\partial x \partial y}\left(x, y_{0}\right)+\frac{\partial^{2} u_{2}}{\partial x^{2}}\left(x, y_{0}\right)\right) d x+\ldots \tag{74b}
\end{align*}
$$

where all points along $\Gamma$ have $y$-coordinate $y_{0}$. Note that approximations and not equalities appear in (74). The reason for that is the fact that (72) holds for a point $\mathbf{x}_{0}$ in the bulk of the body; this assumption may not hold for all the points in $\Gamma$, which may then introduce a surface effect [80]. Nevertheless, this effect, if present, would normally vanish in the limit as $\delta \rightarrow 0$; in this limit, the net out-of-balance force thus vanishes at a rate of $\mathcal{O}(\delta)$.

Vertical Interface. The treatment of the case with a vertical interface is identical to that of the horizontal interface, except that the limits of integration change. Assuming a smooth deformation and performing some Taylor expansions, we obtain (see (B.34))

$$
\begin{align*}
& \tau_{1}\left(\mathbf{x}_{0}, \mathbf{e}_{1}\right)=\mathrm{t}_{1}\left(\mathbf{x}_{0}, \mathbf{e}_{1}\right)+\frac{2 E}{5 \pi} \frac{6-\alpha}{7-\alpha} \delta\left(\frac{\partial^{2} u_{1}}{\partial y^{2}}\left(\mathbf{x}_{0}\right)+\frac{\partial^{2} u_{2}}{\partial x \partial y}\left(\mathbf{x}_{0}\right)\right)+\ldots,  \tag{75a}\\
& \tau_{2}\left(\mathbf{x}_{0}, \mathbf{e}_{1}\right)=\mathrm{t}_{2}\left(\mathbf{x}_{0}, \mathbf{e}_{1}\right)+\frac{2 E}{5 \pi} \frac{6-\alpha}{7-\alpha} \delta\left(\frac{\partial^{2} u_{1}}{\partial x \partial y}\left(\mathbf{x}_{0}\right)+\frac{3}{2} \frac{\partial^{2} u_{2}}{\partial y^{2}}\left(\mathbf{x}_{0}\right)\right)+\ldots, \tag{75b}
\end{align*}
$$

where the relation $c=\frac{3(6-\alpha) E}{\pi \delta^{6-\alpha}}$ has been employed (cf. (A.31)), $\mathrm{t}_{1}\left(\mathbf{x}_{0}, \mathbf{e}_{1}\right)$ and $\mathrm{t}_{2}\left(\mathbf{x}_{0}, \mathbf{e}_{1}\right)$ are the $x$ - and $y$-components, respectively, of the local traction (cf. (42)) evaluated at $\mathbf{x}_{0} \in \Gamma$ in classical plane stress ( $c f$. (A.18)) for $v=1 / 3$, and the dots indicate higherorder derivative terms. In the limit as $\delta \rightarrow 0$, we get

$$
\begin{equation*}
\boldsymbol{\tau}\left(\mathbf{x}_{0}, \mathbf{e}_{1}\right)=\mathbf{t}\left(\mathbf{x}_{0}, \mathbf{e}_{1}\right)+\mathcal{O}(\delta), \tag{76}
\end{equation*}
$$

i.e., the nonlocal traction converges to the local traction at a rate of $\mathcal{O}(\delta)$. Equation (75) implies that, if the deformation around the interface $\Gamma$ is smooth and second-order and higher derivatives of displacements are negligible, (67) is satisfied (note $\mathbf{t}\left(\mathbf{x}_{0}, \mathbf{e}_{1}\right)=$ $-\mathbf{t}\left(\mathbf{x}_{0},-\mathbf{e}_{1}\right)$ by (43)) and overall equilibrium is attained. However, whenever secondorder or higher derivatives of displacements are not negligible around the interface $\Gamma$, lack of overall equilibrium is in general expected. In this case, the components of the
net out-of-balance force are given by (see (62), (63), and (65)):

$$
\begin{align*}
& \mathcal{F}_{1} \approx-\int_{\Gamma} \frac{2 E}{5 \pi} \frac{6-\alpha}{7-\alpha} \delta\left(\frac{\partial^{2} u_{1}}{\partial y^{2}}\left(x_{0}, y\right)+\frac{\partial^{2} u_{2}}{\partial x \partial y}\left(x_{0}, y\right)\right) d y+\ldots  \tag{77a}\\
& \mathcal{F}_{2} \approx-\int_{\Gamma} \frac{2 E}{5 \pi} \frac{6-\alpha}{7-\alpha} \delta\left(\frac{\partial^{2} u_{1}}{\partial x \partial y}\left(x_{0}, y\right)+\frac{3}{2} \frac{\partial^{2} u_{2}}{\partial y^{2}}\left(x_{0}, y\right)\right) d y+\ldots \tag{77b}
\end{align*}
$$

where all points along $\Gamma$ have $x$-coordinate $x_{0}$. Note that, similar to (74), approximations and not equalities appear in (77). In the limit as $\delta \rightarrow 0$, the net out-of-balance force vanishes at a rate of $\mathcal{O}(\delta)$.

### 4.1. One-dimensional case

In this section, an equilibrium check is carried out on one-dimensional cases adopting the CCM-PD coupling strategy described in Section 2. We consider a bar discretized with $N=31$ nodes uniformly distributed with $\Delta x=1$ as shown in Fig. 8. The PD portion of the domain is composed of nodes with coordinates in the interval [10 . . 20], while the remaining part of the domain is modelled with a CCM model discretized using two-node bar FEM elements with linear shape functions. The values of the main problem parameters are $L=30$ (bar length), $E=1$ (Young's modulus), and $A=1$ (bar cross-sectional area) in consistent units. We assume a CCM model given by (8)


Figure 8: CCM-PD coupled model for the one-dimensional case. Blue diamonds are FEM nodes, green circles are PD nodes, and blue solid lines represent FEM elements.


Figure 9: Imposed displacement field along the bar length.
and a PD model given by (15) with a micromodulus function $c(|\xi|)=c /|\xi|$. The PD horizon is taken as $\delta=3$ (i.e., $m=\delta / \Delta x=3$ ) and the micromodulus constant $c$ has been evaluated through the following relation (see (A.12) with $\alpha=1$ ):

$$
\begin{equation*}
c=\frac{2 E}{\delta^{2}} \tag{78}
\end{equation*}
$$

The PD portion of the domain employs a meshfree discretization with a partial-volume correction [79]. action forces are computed using a system of the type of (7). The imposed displacement field is a piecewise polynomial function composed of three curves: two linear functions connected by a cubic function as shown in Fig. 9. The cubic function has been selected to ensure $C^{1}$ continuity of the displacement field along the bar length. The three curves used for the imposed displacement field are described in Table 1; the value of the coefficient $a$ is set to $a=0.0001$.

Table 1: Piecewise displacement field for the one-dimensional case.

| Displacement type | Displacement field equation | Domain |
| :--- | :---: | :---: |
| Curve 1: linear | $u(x)=a x$ | $x \in\left(0, X_{1}\right)$ |
| Curve 2: cubic | $u(x)=\frac{a}{3 X_{1}^{2}} x^{3}+\frac{2 a}{3} X_{1}$ | $x \in\left(X_{1}, X_{2}\right)$ |
| Curve 3: linear | $u(x)=a\left(\frac{X_{2}}{X_{1}}\right)^{2}\left(x-X_{2}\right)+\frac{a}{3 X_{1}^{2}}\left(X_{2}^{3}+2 X_{1}^{3}\right)$ | $x \in\left(X_{2}, L\right)$ |

The following part of this section presents five different cases of displacement distributions imposed on the bar. For all the cases, we keep fixed the location of the PD portion of the domain and all the problem parameters, while only changing the position of the cubic displacement curve along the bar length. The resulting relative out-of-balance error is evaluated through the following quantity:

$$
\begin{equation*}
e_{r}:=\frac{\left|\sum_{i=1}^{N} F_{i}\right|}{\sum_{i=1}^{N}\left|F_{i}\right|} \tag{79}
\end{equation*}
$$

where $F_{i}$ is the reaction force generated at node $i$ after the imposition of the displacement field. In the case of overall equilibrium of the whole structure, the sum of the reaction forces is equal to zero (see Section 2).

Table 2 lists the results in terms of relative out-of-balance error for the five different cases investigated (see Fig. 10). In the first three cases, i.e., configurations (a), (b), and (c) in Fig. 10, the cubic displacement curve is located away from the two coupling zones. In the configurations (a) and (c) the cubic displacement curve is placed within the CCM portion of the domain, whereas in the configuration (b) the cubic displacement curve is located in the PD region. As shown in Table 2, none of these cases exhibit out-of-balance, since the magnitude of the resulting relative out-of-balance errors is on the order of machine precision. In the last two cases, i.e., configurations (d) and (e) in Fig. 10, the cubic displacement curve is located over the left and right coupling zones, respectively. In these cases, the resulting relative out-of-balance errors are about twelve

Table 2: Relative out-of-balance errors for the configurations described in Fig. 10.

| Coupled model | $e_{r}$ |
| :---: | :---: |
| Case (a) | $9.04 \times 10^{-16}$ |
| Case (b) | $1.59 \times 10^{-16}$ |
| Case (c) | $3.02 \times 10^{-16}$ |
| Case (d) | $1.98 \times 10^{-03}$ |
| Case (e) | $5.78 \times 10^{-04}$ |

orders of magnitude larger than the ones computed for the first three cases (see Table 2). These results confirm what was found in Section 3.1: if displacements across either of the interfaces between the PD and CCM portions of the domain are characterized by cubic or higher-order polynomial distributions, lack of overall equilibrium is experienced (cf. (38)).

We now present the outputs obtained by performing an $m$ - and a $\delta$-convergence study [78]. In the $m$-convergence study we keep $\delta$ fixed and increase the value of $m$, whereas in the $\delta$-convergence study we keep the value of $m$ fixed and decrease $\delta$; in both cases, the value of $\Delta x$ decreases resulting in an increase in the total number of nodes, $N$. Both ${ }_{415}$ studies consider the configuration (e) in Fig. 10, where the cubic displacement curve is located over the right coupling zone. The resulting relative out-of-balance errors are listed in Table 3, where it is evident that the increase in $m$ has no clear effect on the out-of-balance level of the CCM-PD coupled model (see cases (f) and (g)). On the contrary, when a $\delta$-convergence study is performed, the out-of-balance level decreases with the horizon value (see cases (h) and (i)). To verify the $\delta$-dependence of the out-of-balance, Table 3 also lists the sum of the reaction forces, $\sum_{i=1}^{N} F_{i}$, scaled by $\delta^{2}$. The results confirm the analysis presented in Section 3.1, where the leading term of the net out-of-balance force, $\mathcal{F}$, depends on $\delta^{2}$ (see (38)).

In the last part of this section, we present a quantitative comparison between the nu-


Figure 10: Imposed displacement fields on the CCM-PD coupled model with a cubic displacement curve placed in different locations along the bar. The cubic displacement curve, represented by magenta lines, is located in (a) the left CCM region, (b) the central PD part, (c) the right CCM region, (d) the left coupling zone, and (e) the right coupling zone. Long dashed gray vertical lines indicate the interfaces between the PD and CCM portions of the domain, while short dashed-dotted red vertical lines define the coupling zones of the model. The values of the parameters $X_{1}$ and $X_{2}$ defining the curves in Table 1 are indicated for each case. For clarity reasons, the vertical axis scale changes from plot to plot.

Table 3: Relative out-of-balance errors and scaled sums of reaction forces for the $m$ - and $\delta$-convergence studies.

| Coupled model | $e_{r}$ | $\left(\sum_{i=1}^{N} F_{i}\right) / \delta^{2}$ |
| :---: | :---: | :---: |
| Case (e), $\delta=3, m=3$ | $5.78 \times 10^{-04}$ | $-2.56 \times 10^{-08}$ |
| Case (f), $\delta=3, m=6$ | $6.33 \times 10^{-04}$ | $-2.80 \times 10^{-08}$ |
| Case (g), $\delta=3, m=12$ | $6.46 \times 10^{-04}$ | $-2.86 \times 10^{-08}$ |
| Case (h), $\delta=1.5, m=3$ | $1.45 \times 10^{-04}$ | $-2.56 \times 10^{-08}$ |
| Case (i), $\delta=0.5, m=3$ | $1.61 \times 10^{-05}$ | $-2.56 \times 10^{-08}$ |

merically computed sum of the reaction forces, $\sum_{i=1}^{N} F_{i}$, and the analytically calculated net out-of-balance force, $\mathcal{F}$, using (38). We consider the cases (e), (f), and (g) listed in Table 3. Additionally, we numerically compute the nonlocal and local tractions at the corresponding interface, $x_{I R}=20+\frac{\Delta x}{2}$, and report their sum. The nonlocal traction is computed by $\tau^{\text {num }}\left(x_{I R},+1\right)$ in (B.15) and the local traction is computed by (cf. (7))

$$
\begin{equation*}
\mathrm{t}^{\mathrm{num}}\left(x_{I R},-1\right):=-\frac{E}{\Delta x}\left(u_{x_{I R}}^{\mathrm{FEM}}-u_{x_{I R}}^{\mathrm{PD}}\right), \tag{80}
\end{equation*}
$$

where $u_{x_{I R}}^{\mathrm{FEM}}$ and $u_{x_{I R}}^{\mathrm{PD}}$ are the displacements of the FEM node and PD node, respectively, closest to the interface $x_{I R}$. The results are presented in Table 4. Various observations are drawn from these results. First, the sum of the reaction forces has the same magnitude as, but opposite sign to the sum of the nonlocal and local tractions:

$$
\begin{equation*}
\sum_{i=1}^{N} F_{i}=-\left[\tau^{\mathrm{num}}\left(x_{I R},+1\right)+\mathrm{t}^{\mathrm{num}}\left(x_{I R},-1\right)\right] \tag{81}
\end{equation*}
$$

which confirms the force balance equation (32); note that, in this case, the corresponding nonlocal and local tractions at $x_{I L}$ are equal in magnitude because the displacement field around that interface is linear ( $c f$. (35)). Second, the sum of the reaction forces provides a suitable approximation to the net out-of-balance force:

$$
\begin{equation*}
\mathcal{F} \approx \sum_{i=1}^{N} F_{i}, \tag{82}
\end{equation*}
$$

and the numerical values approach the analytical ones as $m$ increases. Third, as shown in Appendix B.1.1, the numerical nonlocal traction, $\tau^{\text {num }}\left(x_{I R},+1\right)$, accurately reproduces the analytical nonlocal traction, $\tau\left(x_{I R},+1\right)$, for linear, quadratic, and cubic displacement fields (see Table B.12). Consequently, the discrepancy between the sum of the reaction forces and the net out-of-balance force in Table 4 originates from a numerical error in the approximation of the local traction. To show this, consider the numerical local traction in (80), and note that $u_{x_{I R}}^{\mathrm{FEM}}=u\left(x_{I R}+\frac{\Delta x}{2}\right)$ and $u_{x_{I R}}^{\mathrm{PD}}=u\left(x_{I R}-\frac{\Delta x}{2}\right)$. Performing Taylor expansions (recall the displacement field is cubic around $x_{I R}$ ), we obtain

$$
\begin{align*}
\mathrm{t}^{\mathrm{num}}\left(x_{I R},-1\right) & =-\frac{E}{\Delta x}\left(u\left(x_{I R}+\frac{\Delta x}{2}\right)-u\left(x_{I R}-\frac{\Delta x}{2}\right)\right) \\
& =-E\left(\frac{d u}{d x}\left(x_{I R}\right)+\frac{1}{4!} \frac{d^{3} u}{d x^{3}}\left(x_{I R}\right)(\Delta x)^{2}\right) \\
& =-E \frac{d u}{d x}\left(x_{I R}\right)+\mathcal{O}\left((\Delta x)^{2}\right)=\mathrm{t}\left(x_{I R},-1\right)+\mathcal{O}\left((\Delta x)^{2}\right) \tag{83}
\end{align*}
$$

The numerical local traction, $\mathrm{t}^{\text {num }}\left(x_{I R},-1\right)$, is thus an accurate estimator of the analytical local traction, $\mathrm{t}\left(x_{I R},-1\right)$, for constant, linear, and quadratic displacement fields, while it is an order $\mathcal{O}\left((\Delta x)^{2}\right)$ approximation of the analytical local traction for cubic or higher-order polynomial displacement fields. The error in this approximation vanishes in the limit as $\Delta x \rightarrow 0$, which coincides with the limit of $m \rightarrow \infty$ in the $m$-convergence study, explaining why the sum of the reaction forces approaches the net out-of-balance force in this limit (see Table 4).

Remark 1. The observation concerning the numerical error in the approximation of the local traction provides an explanation of why the scaled sums of the reaction forces in Table 3 possess a fixed value for $m=3$, regardless of the value of $\delta$, while varying when changing $m$. To explain this, consider a scaled sum of the numerical nonlocal and local tractions. Using the fact that, for the cases considered in Table 3, the numerical nonlocal traction accurately estimates the analytical nonlocal traction, and employing (35), (83), and (13), we have (recall the displacement field is cubic around $x_{I R}$,

$$
\alpha=1, \text { and } \delta=m \Delta x)
$$

$$
\begin{align*}
& \frac{1}{\delta^{2}}\left(\tau^{\mathrm{num}}\left(x_{I R},+1\right)+\mathrm{t}^{\mathrm{num}}\left(x_{I R},-1\right)\right)= \frac{1}{\delta^{2}}( \\
&\left(\tau\left(x_{I R},+1\right)+\mathrm{t}^{\mathrm{num}}\left(x_{I R},-1\right)\right) \\
&= \frac{1}{\delta^{2}}\left(\mathrm{t}\left(x_{I R},+1\right)+\frac{1}{4!} E \delta^{2} \frac{d^{3} u}{d x^{3}}\left(x_{I R}\right)\right. \\
&\left.+\mathrm{t}\left(x_{I R},-1\right)-\frac{1}{4!} E(\Delta x)^{2} \frac{d^{3} u}{d x^{3}}\left(x_{I R}\right)\right)  \tag{84}\\
&= \frac{1}{4!}\left(1-\frac{1}{m^{2}}\right) E \frac{d^{3} u}{d x^{3}}\left(x_{I R}\right) .
\end{align*}
$$

This expression is independent of $\delta$ for a fixed value of $m$, and it increases in magnitude with increasing $m$. Using (84) for the cases in Table 3 gives values with the same magnitude as, but opposite sign to the ones reported for the scaled sums of the reaction forces in that table.

Table 4: Comparison between sums of reaction forces, net out-of-balance forces, and sums of numerical nonlocal and local tractions.

| Coupled model | $\sum_{i=1}^{N} F_{i}$ | $\mathcal{F}$ | $\tau^{\mathrm{num}}\left(x_{I R},+1\right)+\mathrm{t}^{\mathrm{num}}\left(x_{I R},-1\right)$ |
| :---: | :---: | :---: | :---: |
| Case (e), $\delta=3, m=3$ | $-2.31 \times 10^{-07}$ | $-2.60 \times 10^{-07}$ | $2.31 \times 10^{-07}$ |
| Case (f), $\delta=3, m=6$ | $-2.52 \times 10^{-07}$ | $-2.60 \times 10^{-07}$ | $2.52 \times 10^{-07}$ |
| Case (g), $\delta=3, m=12$ | $-2.58 \times 10^{-07}$ | $-2.60 \times 10^{-07}$ | $2.58 \times 10^{-07}$ |

Remark 2. The analytical expression for the net out-of-balance force in (38) implies that imposing a cubic displacement field along the whole bar results in $\mathcal{F}=0$, because the contributions of the nonlocal tractions at the interfaces cancel each other (note that the third derivative of the displacement, in this case, is constant). For this reason, the numerical results in this section were based on cases where a cubic displacement field occurs at most at one of the two interfaces. A similar reasoning is employed in Section 4.2 below in the choice of the imposed displacement fields for the two-dimensional case.


Figure 11: CCM-PD coupled model for the two-dimensional case. Green circles are PD nodes and blue (empty) squares are FEM elements. The dashed gray lines represent the interface between the PD and CCM regions, while the portion of the domain bounded by the dashed-dotted red lines is the coupling zone. For clarity reasons, in the figure, $\Delta x=\Delta y=1$ and $m=\delta / \Delta x=2$.

### 4.2. Two-dimensional case

In this section, equilibrium checks are carried out on two-dimensional plane stress cases adopting the CCM-PD coupling strategy described in Section 2. We consider a twodimensional rectangular plate with an internal PD region as shown in Fig. 11. The PD portion of the domain is a square of edge length $L_{P D x}=L_{P D y}=10$, and its centre has coordinates $(13,19)$. The remaining part of the domain, the CCM region, is discretized using four-node square plane stress FEM elements for which the element stiffness matrix has been evaluated with exact integration [81]. The discretization of the domain employs a uniform grid with $\Delta x=\Delta y=0.25$, where $\Delta x$ and $\Delta y$ are the grid spacings in the $x$ - and $y$-directions, respectively. The values of the main problem parameters are $L_{x}=24$ and $L_{y}=34$ (plate dimensions), $E=1$ (Young's modulus), $v=\frac{1}{3}$ (Poisson's ratio), and $h=1$ (plate thickness) in consistent units. We assume a CCM model given by the classical linear elasticity plane stress isotropic model (cf. (A.20)) and a PD model given by a linear bond-based isotropic model (cf. (A.16)) with a micromodulus function $\lambda(\|\xi\|)=\frac{c}{\|\xi\|^{3}}$. The PD horizon is taken as $\delta=0.75$ (i.e., $m=\delta / \Delta x=3$ ) and the micromodulus constant $c$ has been evaluated through the
following relation (see (A.31) with $\alpha=3$ ):

$$
\begin{equation*}
c=\frac{9 E}{\pi \delta^{3}} \gamma \tag{85}
\end{equation*}
$$

which corresponds to a plane stress condition, where $\gamma$ is a correction factor (see Remark B. 2 in Appendix B.2.1). The PD portion of the domain employs a meshfree discretization with a partial-volume correction [82].

In all the cases considered in this section, a displacement is imposed on all the nodes of the plate in such a way as to examine either a single straight interface between the PD and CCM portions of the domain (Case I in Section 4.2.1 and Case II in Section 4.2.2) or a single interface corner (Case III in Section 4.2.3). For all the cases, we keep fixed the location of the PD portion of the domain and all the problem parameters, while only changing the characteristics of the displacement distributions within the plate. The resulting relative out-of-balance error is evaluated both in the $x$ - and $y$-directions through the following quantities:

$$
\begin{align*}
& e_{r_{x}}:=\frac{\left|\sum_{i=1}^{N} F_{1 i}\right|}{\sum_{i=1}^{N}\left|F_{1 i}\right|},  \tag{86a}\\
& e_{r_{y}}:=\frac{\left|\sum_{i=1}^{N} F_{2 i}\right|}{\sum_{i=1}^{N}\left|F_{2 i}\right|}, \tag{86b}
\end{align*}
$$

where $N$ is the total number of nodes and $F_{1 i}$ and $F_{2 i}$ are the $x$ - and $y$-components, respectively, of the reaction force generated at node $i$ after the imposition of the displacement field. In the case of overall equilibrium of the whole structure, the sum of the reaction forces for each component is equal to zero (see Section 2).

### 4.2.1. Case I: Bilinear displacement over a straight interface

In this section, an equilibrium check is carried out by imposing a piecewise displacement field composed of a bilinear function connected to a constant function. The imposed displacement field is described by the set of equations in Table 5, where $u_{1}$ and $u_{2}$ are the $x$ - and $y$-components, respectively, of the displacement field $\mathbf{u}=\left(u_{1}, u_{2}\right)$, and the value of the coefficient $d$ is set to $d=0.5$. The bilinear portion of the displacement component $u_{2}$ is shown in Fig. 12, where the two subdomains $B_{1}$ and $B_{2}$ are defined as

Table 5: Piecewise displacement field for the two-dimensional Case I.

| Displacement type | Displacement field equation | Domain |
| :---: | :---: | :---: |
| Constant | $u_{1}(x, y)=0$ | $(x, y) \in \mathcal{B}$ |
| Bilinear | $u_{2}(x, y)=d \frac{y-Y_{1}}{Y_{B}-Y_{1}}$ | $(x, y) \in B_{1}$ |
| Constant | $u_{2}(x, y)=d\left(1-\frac{y-Y_{B}}{Y_{2}-Y_{B}}\right)$ | $(x, y) \in B_{2}$ |
|  | $u_{2}(x, y)=0$ | $(x, y) \in \mathcal{B} \backslash\left(B_{1} \cup B_{2}\right)$ |

follows:

$$
\begin{align*}
& B_{1}:=\left\{(x, y) \in \mathcal{B}: x \in\left(X_{1}, X_{2}\right) \wedge y \in\left(Y_{1}, Y_{B}\right)\right\},  \tag{87a}\\
& B_{2}:=\left\{(x, y) \in \mathcal{B}: x \in\left(X_{1}, X_{2}\right) \wedge y \in\left(Y_{B}, Y_{2}\right)\right\}, \tag{87b}
\end{align*}
$$

where $X_{1}, X_{2}, Y_{1}, Y_{2}$, and $Y_{B}$ are the bounds of the two subdomains, as shown in Fig. 12c. The bilinear displacement portion is located over the lower horizontal interface (see Fig. 12a and Fig. 12b). The values of the bounds of the subdomains $B_{1}$ and $B_{2}$ are set to $X_{1}=9.25, X_{2}=13.75, Y_{1}=12.75, Y_{2}=17.25$, and $Y_{B}=\left(Y_{1}+Y_{2}\right) / 2=15$. Table 6 lists the results in terms of relative out-of-balance error along the $x$ - and $y$ directions. In this case, the force equilibrium is verified along both the $x$ - and $y$ directions. We performed a similar study by imposing instead the bilinear distribution described in Table 5 on the displacement component $u_{1}$ over a vertical interface. Also, in this case, no appreciable out-of-balance error was found. These results confirm what was found in Section 3.2: if displacements across a straight (horizontal or vertical) interface are characterized by linear or constant distributions, overall equilibrium is attained (cf. (74) and (77)).


Figure 12: Imposed displacement field on the plate for Case I: (a) top view, (b) 3D view, and (c) characteristic parameters of the bilinear displacement portion. The square part of the domain bounded by thick straight white lines represents the PD region, while the remaining part of the domain is the CCM region.

Table 6: Relative out-of-balance error along the $x$ - and $y$-directions for Case I in Fig. 12.

| Coupled model | $e_{r_{x}}$ | $e_{r_{y}}$ |
| :---: | :---: | :---: |
| Case I | $3.28 \times 10^{-16}$ | $5.64 \times 10^{-17}$ |

Table 7: Piecewise displacement field for the two-dimensional Case II.

| Displacement type | Displacement field equation | Domain |
| :---: | :---: | :---: |
| Constant | $u_{1}(x, y)=0$ | $(x, y) \in \mathcal{B}$ |
| Quadratic | $u_{2}(x, y)=\frac{-\left(x-X_{Q}\right)^{2}-\left(y-Y_{Q}\right)^{2}+R^{2}}{q^{2}}$ | $(x, y) \in Q$ |
| Constant | $u_{2}(x, y)=0$ | $(x, y) \in \mathcal{B} \backslash Q$ |

### 4.2.2. Case II: Quadratic displacement over a straight interface

In this section, an equilibrium check is carried out by imposing a piecewise displace- ment field composed of a quadratic function connected to a constant function. The imposed displacement field is described by the set of equations in Table 7, where the value of the coefficient $q$ is set to $q=15$. The quadratic portion of the displacement component $u_{2}$ is shown in Fig. 13, and it is applied to a circular subdomain, $Q$, defined as follows:

$$
\begin{equation*}
Q:=\left\{(x, y) \in \mathcal{B}:\left(x-X_{Q}\right)^{2}+\left(y-Y_{Q}\right)^{2} \leqslant R^{2}\right\} \tag{88}
\end{equation*}
$$

where $X_{Q}$ and $Y_{Q}$ are the $x$ - and $y$-coordinates, respectively, of the centre of the subdomain and $R$ indicates its radius (see Fig. 13c). The quadratic displacement portion is located over the lower horizontal interface (see Fig. 13a and Fig. 13b). The centre of $Q$ has coordinates $X_{Q}=11.5$ and $Y_{Q}=15$, and its radius is set to $R=2.25$. Table 8 lists the results obtained in terms of relative out-of-balance error along the $x$ - and $y$-directions. In this case, the force equilibrium is verified only along the $x$-direction. This result is consistent with the net out-of-balance force in (74); specifically, in this case, $\mathcal{F}_{1}$ is expected to vanish, while $\mathcal{F}_{2}$ is expected to be non-zero due to the contribution of the second derivative of the displacement component $u_{2}$ with respect to $x$. We performed a similar study by imposing instead the quadratic distribution described in Table 7 on the displacement component $u_{1}$ over a vertical interface. In this case, the


Figure 13: Imposed displacement field on the plate for Case II: (a) top view, (b) 3D view, and (c) characteristic parameters of the quadratic displacement portion. The square part of the domain bounded by thick straight white lines represents the PD region, while the remaining part of the domain is the CCM region.
force equilibrium is verified only along the $y$-direction. This result is consistent with the net out-of-balance force in (77); specifically, in this case, $\mathcal{F}_{2}$ is expected to vanish, while $\mathcal{F}_{1}$ is expected to be non-zero due to the contribution of the second derivative of the displacement component $u_{1}$ with respect to $y$. These results confirm what was found in Section 3.2: if displacements across a straight (horizontal or vertical) interface are characterized by quadratic or higher-order polynomial distributions, lack of overall equilibrium is experienced ( $c f$. (74) and (77)).

In the remaining part of this section, the outputs obtained by performing a $\delta$-convergence study are presented. We consider the case where the quadratic displacement distribution described in Table 7, which is applied to the displacement component $u_{2}$, is located

Table 8: Relative out-of-balance error along the $x$ - and $y$-directions for Case II in Fig. 13.

| Coupled model | $e_{r_{x}}$ | $e_{r_{y}}$ |
| :---: | :---: | :---: |
| Case II | $1.40 \times 10^{-16}$ | $5.00 \times 10^{-04}$ |

over the lower horizontal interface (see Fig. 13). As demonstrated in Appendix B.2.1, large values of $m$ are required to obtain accurate computations of nonlocal tractions (see Table B.14). For this reason, we perform the $\delta$-convergence study using a larger value of $m$, chosen as $m=8$; this value has been selected as a compromise between computational cost and numerical accuracy in two-dimensional simulations. The resulting relative out-of-balance errors are listed in Table 9, where, as expected, the force equilibrium is verified only along the $x$-direction. The results for $e_{r_{y}}$ demonstrate that the out-of-balance level decreases with the horizon. To verify the $\delta$-dependence of the out-of-balance, Table 9 also lists the sum of the $y$-component of the reaction forces, ${ }_{525} \quad \sum_{i=1}^{N} F_{2 i}$, scaled by $\delta$. The results do not exactly give a linear dependence on $\delta$, which is the theoretically predicted behavior in (74). The potential reasons for this discrepancy are twofold. First, it is possible that the value of $m$ is not large enough to provide the required numerical accuracy (cf. Table B.14). Second, the configuration presented in Fig. 13 cannot satisfy one of the hypotheses on which the analytical derivations leading to (74) rely, i.e., the assumption that, for each PD node, the entire neighbourhood on the CCM side is subjected to a uniform, non-piecewise displacement field.

The reason for the choice of the piecewise displacement fields in Fig. 12 and Fig. 13 was to consider a displacement variation around a single straight (horizontal or vertical) interface, for consistency with the analysis presented in Section 3.2.4. In particular, that choice was aimed at isolating the effect of corners, i.e., non-straight interfaces; this effect is investigated in Section 4.2.3 below.

### 4.2.3. Case III: Bilinear displacement over an interface corner

The theoretical results for the net out-of-balance forces presented in Section 3.2.4 hold for a straight (horizontal or vertical) interface and may not hold for an interface of ar-

Table 9: Relative out-of-balance errors and scaled sums of reaction forces for the $\delta$-convergence study.

| Coupled model | $e_{r_{x}}$ | $e_{r_{y}}$ | $\left(\sum_{i=1}^{N} F_{2 i}\right) / \delta$ |
| :---: | :---: | :---: | :---: |
| Case II, $\delta=0.75, m=8$ | $1.03 \times 10^{-16}$ | $1.86 \times 10^{-03}$ | $-8.08 \times 10^{-04}$ |
| Case II, $\delta=0.375, m=8$ | $5.23 \times 10^{-17}$ | $7.23 \times 10^{-04}$ | $-7.05 \times 10^{-04}$ |
| Case II, $\delta=0.1875, m=8$ | $1.05 \times 10^{-15}$ | $1.60 \times 10^{-04}$ | $-3.33 \times 10^{-04}$ |

Table 10: Relative out-of-balance error along the $x$ - and $y$-directions for Case III in Fig. 14.

| Coupled model | $e_{r_{x}}$ | $e_{r_{y}}$ |
| :---: | :---: | :---: |
| Case III | $4.31 \times 10^{-03}$ | $3.09 \times 10^{-16}$ |

bitrary shape (e.g., a corner). In this section, we consider a non-straight interface. An equilibrium check is carried out by imposing the displacement field described in Table 5 in Section 4.2.1, with the bilinear displacement portion located over the lower right interface corner of the CCM-PD coupled model. In this case, the values of the bounds of the subdomains $B_{1}$ and $B_{2}$ are set to $X_{1}=14.75, X_{2}=19.25, Y_{1}=12.75, Y_{2}=17.25$, and $Y_{B}=15$. Fig 14 shows the configuration under investigation. Table 10 lists the results obtained in terms of relative out-of-balance error along the $x$ - and $y$-directions. In this case, the resulting relative out-of-balance error $e_{r_{x}}$ is not negligible. This result demonstrates that, in contrast to the results reported in Section 4.2.1, if displacements across a non-straight interface are characterized by linear distributions, lack of overall equilibrium may be experienced.

The results of this section suggest that, for a two-dimensional CCM-PD coupled model, the overall static equilibrium is affected not only by the location of the coupling interface but also by its shape. Consequently, a future extension of the out-of-balance analysis in CCM-PD coupled models could be focused on controlling the relative out-of-balance error by optimizing the shape of the interface between the PD and CCM portions of the domain. In Section 5, we study the effect of the location of the coupling interface in the context of crack propagation problems.


Figure 14: Imposed displacement field on the plate for Case III (top view). The square part of the domain bounded by thick straight white lines represents the PD region, while the remaining part of the domain is the CCM region.

## 5. Simulation of crack propagation using the CCM-PD coupled model

In this section, we present quasi-static crack propagation problems. Initially, the entire domain with an initial crack is discretized with the FEM; then, a PD region is introduced, which adaptively follows an advancing crack [76, 77]. We show that the position of the coupling interface of the CCM-PD coupled model affects the values of the out-of-balance forces. The main idea is that, in crack propagation problems, high spatial strains normally appear in the region near the crack tip. Therefore, the coupling interface of the CCM-PD coupled model should not be too close to the crack tip. The CCM region is linear in terms of material response and deformation, and it is described by a classical linear elasticity model given by the plane stress isotropic model (cf. (A.20)) in two dimensions and the Navier equation in three dimensions. The PD region is described by the linearized state-based PD model from [83, 84]. The values given to the parameters of the problems are associated to the usual units.

### 5.1. Two-dimensional case: three-point bending test

In Fig. 15, we present the geometric parameters and boundary conditions of a threepoint bending test carried out in this section. The domain is discretized using a uniform grid with $\Delta x=\Delta y=0.05[\mathrm{~m}]$, resulting in a total of 64,561 nodes. The material


Figure 15: Geometric parameters and boundary conditions of the three-point bending test. $G_{0}=500\left[\mathrm{~J} / \mathrm{m}^{2}\right]$ (fracture energy) [85]. The CCM region is discretized using four-node FEM elements with bilinear shape functions and four integration points. For the PD portion of the domain, we employ the same PD discretization used in Section 4.2. The horizon is taken as $\delta=0.15[\mathrm{~m}]$ (i.e., $m=\delta / \Delta x=3$ ), and the micromodulus function and influence function described in [77] are used. A downward vertical displacement of $u_{y}=0.001[\mathrm{~m}]$ is imposed on the central point of the top edge of the plate. The imposed displacement is divided into 1000 steps. A crack at the bottom, the initial length of which is $1[\mathrm{~m}]$, propagates in the vertical direction as the imposed vertical displacement increases. Using the algorithm in [85], we solve the structural problem and compute the three vertical reaction forces of the system: $F_{y_{A}}, F_{y_{B}}$, and $F_{y_{C}}$, the first two at the supports $A$ and $B$, and the third one at $C$ where the vertical displacement is imposed (see Fig. 15). The relative out-of-balance error is given by:

$$
\begin{equation*}
e_{r}:=\frac{\left|F_{y_{A}}+F_{y_{B}}+F_{y_{C}}\right|}{\left|F_{y_{A}}\right|+\left|F_{y_{B}}\right|+\left|F_{y_{C}}\right|} . \tag{89}
\end{equation*}
$$

The second-order derivatives of the displacement field, $\frac{\partial^{2} u_{1}}{\partial x^{2}}, \frac{\partial^{2} u_{1}}{\partial y^{2}}, \frac{\partial^{2} u_{1}}{\partial x \partial y}, \frac{\partial^{2} u_{2}}{\partial x^{2}}, \frac{\partial^{2} u_{2}}{\partial y^{2}}$, and $\frac{\partial^{2} u_{2}}{\partial x \partial y}$, are calculated using the PD differential operators [86, 87]. An indicator for


Figure 16: Distribution of $D^{2}(\mathbf{u})$ based on the CCM model for the three-point bending test in Fig. 15 with an applied vertical displacement of $u_{y}=1 \times 10^{-06}[\mathrm{~m}]$. The colour plot is displayed in logarithmic scale.
the distribution of the overall second-order derivatives is defined as:

$$
\begin{equation*}
D^{2}(\mathbf{u}):=\left|\frac{\partial^{2} u_{1}}{\partial x^{2}}\right|+\left|\frac{\partial^{2} u_{1}}{\partial y^{2}}\right|+\left|\frac{\partial^{2} u_{1}}{\partial x \partial y}\right|+\left|\frac{\partial^{2} u_{2}}{\partial x^{2}}\right|+\left|\frac{\partial^{2} u_{2}}{\partial y^{2}}\right|+\left|\frac{\partial^{2} u_{2}}{\partial x \partial y}\right| . \tag{90}
\end{equation*}
$$

Given the configuration in Fig. 15, the distribution of $D^{2}(\mathbf{u})$ for the CCM model with a displacement of $u_{y}=1 \times 10^{-06}[\mathrm{~m}]$ is shown in Fig. 16. It is obvious that the values of $D^{2}(\mathbf{u})$ around the crack tip as well as around the point $C$ where the displacement is imposed and around the supports $A$ and $B$ are greater than in other zones, and this feature is preserved during the crack propagation. When the crack propagates, we adopt two switching schemes to convert FEM nodes to PD nodes [76, 77]:

Switching scheme 1: FEM nodes within one horizon radius from PD nodes with broken bonds are transformed into PD nodes, as shown in Fig. 17a.

Switching scheme 2: FEM nodes within a distance of twice the horizon radius from PD nodes with broken bonds are transformed into PD nodes, as shown in Fig. 17b.

In order to ensure that the solutions of the two switching schemes are comparable, we perform the simulation using the switching scheme 1 , and then post-process the solution using both switching schemes to study the behaviour of the relative out-of-balance error.

Figure 18 shows the distribution of $D^{2}(\mathbf{u})$ around the crack tip for different load step numbers (step $=200,400,600,800,1000$ ). The relative out-of-balance error computed with (89) is plotted in Fig. 19. We observe that the relative out-of-balance error is larger


Figure 17: Schemes for switching nodes around the crack tip. Blue diamonds are FEM nodes and green circles are PD nodes. The black line represents the crack.
when the interface between the CCM and PD regions falls into an area with larger values of $D^{2}(\mathbf{u})$ (switching scheme 1) compared to the case where that interface falls into an area with smaller values of $D^{2}(\mathbf{u})$ (switching scheme 2).

### 5.2. Three-dimensional case: Brokenshire torsion experiment

This section is only intended to demonstrate that the study presented in Section 5.1 can be extended to a three-dimensional case. We consider a crack propagation problem given by the Brokenshire torsion experiment, which is comprehensively described in [88]. The geometric parameters of the prismatic specimen and the boundary conditions are presented in Fig. 20. The CCM region is discretized using eight-node FEM elements with trilinear shape functions and eight integration points. The initial FEM mesh used in the simulation is shown in Fig. 21. In the central part of the specimen a uniform hexahedral mesh with mesh size $\Delta x=\Delta y=\Delta z=0.0025[\mathrm{~m}]$ is adopted, whereas the remaining parts of the domain are discretized using non-uniform hexahedral meshes to reduce the computational cost of the simulation. The FEM mesh has a total of 173,082 nodes and 161,824 elements. The material parameters are: $E=35$ [GPa] (Young's modulus), $v=0.2$ (Poisson's ratio), and $G_{0}=80\left[\mathrm{~J} / \mathrm{m}^{2}\right.$ ] (fracture energy) [77]. For the PD portion of the domain, the standard meshfree PD discretization presented in [44] is employed. The horizon is taken as $\delta=0.0075$ [m] (i.e., $m=\delta / \Delta x=3$ ), and the


Figure 18: Distribution of $D^{2}(\mathbf{u})$ around the crack tip for different load step numbers, based on the CCM-PD coupled model with the switching scheme 1, for the three-point bending test in Fig. 15. The colour plot is displayed in logarithmic scale. The black solid line is the crack. The inner dotted piecewise linear red curve represents the interface between the CCM and PD regions generated by the switching scheme 1 . The outer dashed-dotted piecewise linear red curve represents the corresponding interface generated by the switching scheme 2 , which is used only for post-processing putpuses.


Figure 19: Relative out-of-balance error in the CCM-PD coupled model for different load step numbers for the three-point bending test in Fig. 15 with the two switching schemes.
micromodulus function and influence function described in [77] are used. A downward vertical displacement of $u_{z}=0.001[\mathrm{~m}]$ is divided into 7000 steps and applied as shown in Fig. 20. As the imposed vertical displacement increases, a non-planar crack propagates in the notched prismatic specimen. Using the algorithm in [85], we solve the fracture problem and compute the four vertical reaction forces of the system: $F_{z_{A}}, F_{z_{B}}$, $F_{z_{C}}$, and $F_{z_{D}}$, the first one at $A$ where the vertical displacement $u_{z}$ is imposed, and the other three at the supports B, C, and D (see Fig. 20). The relative out-of-balance error is given by:

$$
\begin{equation*}
e_{r}:=\frac{\left|F_{z_{A}}+F_{z_{B}}+F_{z_{C}}+F_{z_{D}}\right|}{\left|F_{z_{A}}\right|+\left|F_{z_{B}}\right|+\left|F_{z_{C}}\right|+\left|F_{z_{D}}\right|} \tag{91}
\end{equation*}
$$

Similar to Section 5.1, when the non-planar crack propagates, we employ a switching scheme to convert FEM nodes to PD nodes and adaptively follow the advancing crack [76, 77]. Following the same procedure adopted in Section 5.1, we perform the simulation using the switching scheme 1 , and then post-process the solution using both switching schemes, i.e., the switching scheme 1 and the switching scheme 2 , to study the behaviour of the relative out-of-balance error.


Figure 20: Geometric parameters and boundary conditions of the Brokenshire torsion experiment. Adapted from [77].


Figure 21: Initial FEM mesh used for the Brokenshire torsion experiment.

Figure 22 shows the shape of the propagating crack for different load step numbers $($ step $=1000,2000,3000,4000,5000,6000,7000)$. The corresponding relative out-ofbalance error computed with (91) is plotted in Fig. 23. This latter result is consistent with the output of the study carried out in Section 5.1 and plotted in Fig. 19, since the switching scheme 2 again demonstrates to generally perform better than the switching scheme 1 in terms of relative out-of-balance error. As in the two-dimensional case in Section 5.1, we observe that the relative out-of-balance error is affected by the location of the coupling interface of the CCM-PD coupled model, since its magnitude is generally smaller when the interface between the CCM and PD regions is further from the crack tip (switching scheme 2), i.e., further from the area with larger values of highorder derivatives of displacements, compared to the case where the interface is closer to the crack tip (switching scheme 1).


Figure 22: Shapes of the non-planar crack for different load step numbers, based on the CCM-PD coupled model with the switching scheme 1, for the Brokenshire torsion experiment in Fig. 20. The colours indicate damage [44].


Figure 23: Relative out-of-balance error in the CCM-PD coupled model for the different load step numbers shown in Fig. 22 for the Brokenshire torsion experiment in Fig. 20 with the two switching schemes.

## 6. Conclusions

This work concerned the coupling of peridynamics and classical continuum mechanics, focusing on an error given by the lack of overall equilibrium in static problems. This coupling error has been overlooked in the literature. We provided a theoretical analysis describing the reason for the appearance of this spurious effect, and we supported the analysis with numerical simulations. While this paper considered a particular strategy to couple peridynamics and classical continuum mechanics, proposed by the authors [75, 76], this issue most probably affects other coupling approaches. We observed that a lack of overall equilibrium may occur even if the coupling method satisfies the usual numerical tests for static problems, given by rigid body motions as well as uniform and linear strain distributions. The theoretical analysis and the supporting numerical simulations allow us to conclude:

- The out-of-balance forces are related to the order of the derivatives of displacements in the coupling zone.
- It is easy to evaluate the magnitude of the out-of-balance error by computing the reaction forces.
- In the numerical examples investigated in this paper, the relative out-of-balance error is a fraction of a per cent and reduces as $\delta \rightarrow 0$.
- It is usually possible to reduce the out-of-balance error by moving the coupling interface away from regions of high gradients of displacements.
- The two-dimensional numerical examples suggest that the shape of the coupling interface may have a significant impact on the overall out-of-balance error: corners in the coupling interface can introduce additional out-of-balance contributions.

The impacts of these findings on coupled simulations are twofold. First, the tolerance used in an implicit solution of a coupled computational problem should be carefully chosen: if the tolerance is smaller than the out-of-balance forces, then the computation will not converge. Second, the proper location and shape of the coupling interface in a computational problem can be defined by using an adaptive approach to convert FEM
nodes into peridynamic nodes. The use of adaptivity, focused on controlling the out-of-balance error, can reduce the computational effort considerably with respect to that required by a fully peridynamic simulation and will pave the way to future applications of the coupling of peridynamics and classical continuum mechanics to the solution of many practical problems.

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## Appendices

## A. Consistency between linear bond-based PD and CCM models

## A.1. One-dimensional case

5 Assume a domain $\mathcal{B} \subset \mathbb{R}$ and let a one-dimensional linear bond-based PD model be given by (cf. (6)):

$$
\begin{equation*}
\rho(x) \ddot{u}(x, t)=\int_{\mathcal{H}_{x}} c\left(\left|x^{\prime}-x\right|\right)\left(u\left(x^{\prime}, t\right)-u(x, t)\right) d x^{\prime}+b(x, t), \tag{A.1}
\end{equation*}
$$

where $\rho$ is the mass density, $\ddot{u}$ is the second derivative in time of the displacement field $u$, $c(|\xi|)$ is a micromodulus function with $\xi=x^{\prime}-x$ a PD bond, $\mathcal{H}_{x}$ is the neighbourhood of the material point $x$, and $b$ is a prescribed body force density field. The relation $c(|\xi|)=\lambda(|\xi|)|\xi|^{2}(c f .(6))$ holds in one dimension. For points in the bulk of the body,
the neighbourhood is $\mathcal{H}_{x}=[x-\delta, x+\delta]$ and we can use the change of variable $\xi=x^{\prime}-x$ to express (A.1) as

$$
\begin{equation*}
\rho(x) \ddot{u}(x, t)=\int_{-\delta}^{\delta} c(|\xi|)(u(x+\xi, t)-u(x, t)) d \xi+b(x, t) \tag{A.2}
\end{equation*}
$$

To establish a connection between (A.2) and the corresponding CCM model, given by the classical wave equation

$$
\begin{equation*}
\rho(x) \ddot{u}(x, t)=E \frac{\partial^{2} u}{\partial x^{2}}(x, t)+b(x, t) \tag{A.3}
\end{equation*}
$$

with $E$ Young's modulus, we assume the displacement field is smooth and perform a Taylor expansion of $u(x+\xi, t)$ about $x$ :
$u(x+\xi, t)=u(x, t)+\frac{\partial u}{\partial x}(x, t) \xi+\frac{1}{2} \frac{\partial^{2} u}{\partial x^{2}}(x, t) \xi^{2}+\frac{1}{3!} \frac{\partial^{3} u}{\partial x^{3}}(x, t) \xi^{3}+\frac{1}{4!} \frac{\partial^{4} u}{\partial x^{4}}(x, t) \xi^{4}+\ldots$.

Substituting (A.4) in (A.2), we obtain
$\rho(x) \ddot{u}(x, t)=\int_{-\delta}^{\delta} c(|\xi|)\left(\frac{\partial u}{\partial x}(x, t) \xi+\frac{1}{2} \frac{\partial^{2} u}{\partial x^{2}}(x, t) \xi^{2}+\frac{1}{3!} \frac{\partial^{3} u}{\partial x^{3}}(x, t) \xi^{3}+\frac{1}{4!} \frac{\partial^{4} u}{\partial x^{4}}(x, t) \xi^{4}+\ldots\right) d \xi$
$+b(x, t)$.

We observe that terms with an odd power of $\xi$ vanish due to their antisymmetry and the symmetry of the integration domain. Then, we have
$\rho(x) \ddot{u}(x, t)=\left[\frac{1}{2} \int_{-\delta}^{\delta} c(|\xi|) \xi^{2} d \xi\right] \frac{\partial^{2} u}{\partial x^{2}}(x, t)+\left[\frac{1}{4!} \int_{-\delta}^{\delta} c(|\xi|) \xi^{4} d \xi\right] \frac{\partial^{4} u}{\partial x^{4}}(x, t)+\ldots+b(x, t)$.

Assuming fourth-order and higher derivatives of displacements are negligible, we obtain

$$
\begin{equation*}
\rho(x) \ddot{u}(x, t)=\left[\frac{1}{2} \int_{-\delta}^{\delta} c(|\xi|) \xi^{2} d \xi\right] \frac{\partial^{2} u}{\partial x^{2}}(x, t)+b(x, t) \tag{A.7}
\end{equation*}
$$

This allows us to relate the micromodulus function $c(|\xi|)$ in (A.2) to Young's modulus $E$ in (A.3):

$$
\begin{equation*}
\frac{1}{2} \int_{-\delta}^{\delta} c(|\xi|) \xi^{2} d \xi=E \tag{A.8}
\end{equation*}
$$

so that the PD model (A.2) reduces to the CCM model (A.3). between the PD and CCM models, in Section A.1.1 below we assume a particular form for the micromodulus function.

## A.1.1. Model discrepancy between PD and CCM models in one dimension

Assume a micromodulus function of the following form:

$$
\begin{equation*}
c(|\xi|)=\frac{c}{|\xi|^{\alpha}} \tag{A.9}
\end{equation*}
$$

with $c$ a constant and $\alpha<3$ (see below). Then, we can compute the following integrals appearing in (A.6):

$$
\begin{align*}
\frac{1}{2} \int_{-\delta}^{\delta} c(|\xi|) \xi^{2} d \xi & =\int_{0}^{\delta} c \xi^{2-\alpha} d \xi=\frac{\delta^{3-\alpha}}{3-\alpha} c  \tag{A.10}\\
\frac{1}{4!} \int_{-\delta}^{\delta} c(|\xi|) \xi^{4} d \xi & =\frac{2}{4!} \int_{0}^{\delta} c \xi^{4-\alpha} d \xi=\frac{2}{4!} \frac{\delta^{5-\alpha}}{5-\alpha} c \tag{A.11}
\end{align*}
$$

Equating (A.8) and (A.10), we obtain

$$
\begin{equation*}
c=\frac{(3-\alpha) E}{\delta^{3-\alpha}} \tag{A.12}
\end{equation*}
$$

Note that the case $\alpha=1$ recovers the micromodulus definition reported in [78] for a one-dimensional bar with unit cross-sectional area. Using (A.10)-(A.12) in (A.6), we get

$$
\begin{equation*}
\rho(x) \ddot{u}(x, t)=E\left[\frac{\partial^{2} u}{\partial x^{2}}(x, t)+\frac{1}{12}\left(\frac{3-\alpha}{5-\alpha}\right) \delta^{2} \frac{\partial^{4} u}{\partial x^{4}}(x, t)+\ldots\right]+b(x, t) . \tag{A.13}
\end{equation*}
$$

In the limit as $\delta \rightarrow 0$, we have

$$
\begin{equation*}
\rho(x) \ddot{u}(x, t)=E \frac{\partial^{2} u}{\partial x^{2}}(x, t)+O\left(\delta^{2}\right)+b(x, t), \tag{A.14}
\end{equation*}
$$

so that the PD model (A.2) converges to the CCM model (A.3) at a rate of $O\left(\delta^{2}\right)$. The leading term in the model discrepancy is of order $O\left(\delta^{2}\right)$.

## A.2. Two-dimensional case

Assume a domain $\mathcal{B} \subset \mathbb{R}^{2}$ and let a two-dimensional linear bond-based PD model be given by (cf. (6)):

$$
\rho(\mathbf{x}) \ddot{\mathbf{u}}(\mathbf{x}, t)=\int_{\mathcal{H}_{\mathbf{x}}} \lambda\left(\left\|\mathbf{x}^{\prime}-\mathbf{x}\right\|\right)\left(\mathbf{x}^{\prime}-\mathbf{x}\right) \otimes\left(\mathbf{x}^{\prime}-\mathbf{x}\right)\left(\mathbf{u}\left(\mathbf{x}^{\prime}, t\right)-\mathbf{u}(\mathbf{x}, t)\right) d \mathbf{x}^{\prime}+\mathbf{b}(\mathbf{x}, t)
$$

where $\rho$ is the mass density, $\ddot{\mathbf{u}}$ is the second derivative in time of the displacement field $\mathbf{u}$, $\lambda(\|\xi\|)$ is a micromodulus function with $\xi=\mathbf{x}^{\prime}-\mathbf{x}$ a PD bond, $\mathcal{H}_{\mathbf{x}}$ is the neighbourhood of the material point $\mathbf{x}$, and $\mathbf{b}$ is a prescribed body force density field. For points in the bulk of the body, we can use the change of variable $\boldsymbol{\xi}=\mathbf{x}^{\prime}-\mathbf{x}$ to express (A.15) as

$$
\begin{equation*}
\rho(\mathbf{x}) \ddot{\mathbf{u}}(\mathbf{x}, t)=\int_{\mathcal{H}} \lambda(\|\xi\|) \xi \otimes \xi(\mathbf{u}(\mathbf{x}+\xi, t)-\mathbf{u}(\mathbf{x}, t)) d \xi+\mathbf{b}(\mathbf{x}, t) \tag{A.16}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{H}:=\left\{\xi \in \mathbb{R}^{2}:\|\xi\| \leqslant \delta\right\} . \tag{A.17}
\end{equation*}
$$

We would like to establish a connection between the PD model (A.16) and the twodimensional classical linear elasticity plane stress model given by: ${ }^{2}$

$$
\begin{align*}
& \rho(\mathbf{x}) \ddot{u}_{1}(\mathbf{x}, t)=\frac{9 E}{8}\left[\frac{\partial^{2} u_{1}}{\partial x^{2}}(\mathbf{x}, t)+\frac{2}{3} \frac{\partial^{2} u_{2}}{\partial x \partial y}(\mathbf{x}, t)+\frac{1}{3} \frac{\partial^{2} u_{1}}{\partial y^{2}}(\mathbf{x}, t)\right]+b_{1}(\mathbf{x}, t),  \tag{A.20a}\\
& \rho(\mathbf{x}) \ddot{u}_{2}(\mathbf{x}, t)=\frac{9 E}{8}\left[\frac{\partial^{2} u_{2}}{\partial y^{2}}(\mathbf{x}, t)+\frac{2}{3} \frac{\partial^{2} u_{1}}{\partial x \partial y}(\mathbf{x}, t)+\frac{1}{3} \frac{\partial^{2} u_{2}}{\partial x^{2}}(\mathbf{x}, t)\right]+b_{2}(\mathbf{x}, t), \tag{A.20b}
\end{align*}
$$

[^1]where $E$ is Young's modulus, $v$ is Poisson's ratio, $\sigma_{i j}$ are the components of the stress tensor, and $\varepsilon_{i j}$ are the components of the infinitesimal strain tensor, $\varepsilon_{i j}=\frac{1}{2}\left(\frac{\partial u_{i}}{\partial x_{j}}+\frac{\partial u_{j}}{\partial x_{i}}\right)$; both $\sigma$ and $\varepsilon$ are symmetric tensors. The equation of motion is given, in component form, by
\[

$$
\begin{equation*}
\rho(\mathbf{x}) \ddot{u}_{i}(\mathbf{x}, t)=\frac{\partial \sigma_{i j}}{\partial x_{j}}(\mathbf{x}, t)+b_{i}(\mathbf{x}, t), \quad i=1,2 \tag{A.19}
\end{equation*}
$$

\]

where repeated indices imply summation by 1 and 2 .
where $E$ is Young's modulus and we assumed a Poisson's ratio of $v=1 / 3[89,90]$. For this purpose, we assume the displacement field is smooth and perform a Taylor expansion of $\mathbf{u}(\mathbf{x}+\boldsymbol{\xi}, t)$ about $\mathbf{x}$ :

$$
\begin{align*}
u_{j}(\mathbf{x}+\xi, t)= & u_{j}(\mathbf{x}, t)+\frac{\partial u_{j}}{\partial x_{k}}(\mathbf{x}, t) \xi_{k}+\frac{1}{2} \frac{\partial^{2} u_{j}}{\partial x_{k} \partial x_{l}}(\mathbf{x}, t) \xi_{k} \xi_{l}+\frac{1}{3!} \frac{\partial^{3} u_{j}}{\partial x_{k} \partial x_{l} \partial x_{m}}(\mathbf{x}, t) \xi_{k} \xi_{l} \xi_{m} \\
& +\frac{1}{4!} \frac{\partial^{4} u_{j}}{\partial x_{k} \partial x_{l} \partial x_{m} \partial x_{n}}(\mathbf{x}, t) \xi_{k} \xi_{l} \xi_{m} \xi_{n}+\ldots, \quad j=1,2 \tag{A.21}
\end{align*}
$$

where repeated indices imply a summation by 1 and 2. Employing (A.21) for the $i$ th component of (A.16), we obtain

$$
\begin{align*}
\rho(\mathbf{x}) \ddot{u}_{i}(\mathbf{x}, t)= & \int_{\mathcal{H}} \lambda(\|\xi\|) \xi_{i} \xi_{j}\left(u_{j}(\mathbf{x}+\xi, t)-u_{j}(\mathbf{x}, t)\right) d \xi+b_{i}(\mathbf{x}, t) \\
= & \int_{\mathcal{H}} \lambda(\|\xi\|) \xi_{i} \xi_{j}\left(\frac{\partial u_{j}}{\partial x_{k}}(\mathbf{x}, t) \xi_{k}+\frac{1}{2} \frac{\partial^{2} u_{j}}{\partial x_{k} \partial x_{l}}(\mathbf{x}, t) \xi_{k} \xi_{l}+\frac{1}{3!} \frac{\partial^{3} u_{j}}{\partial x_{k} \partial x_{l} \partial x_{m}}(\mathbf{x}, t) \xi_{k} \xi_{l} \xi_{m}\right. \\
& \left.+\frac{1}{4!} \frac{\partial^{4} u_{j}}{\partial x_{k} \partial x_{l} \partial x_{m} \partial x_{n}}(\mathbf{x}, t) \xi_{k} \xi_{l} \xi_{m} \xi_{n}+\ldots\right) d \xi+b_{i}(\mathbf{x}, t) . \tag{A.22}
\end{align*}
$$

We observe that terms with an odd number of components of $\boldsymbol{\xi}$ vanish due to their antisymmetry and the symmetry of the integration domain. Then, we have

$$
\begin{align*}
\rho(\mathbf{x}) \ddot{u}_{i}(\mathbf{x}, t)= & {\left[\frac{1}{2} \int_{\mathcal{H}} \lambda(\|\xi\|) \xi_{i} \xi_{j} \xi_{k} \xi_{l} d \xi\right] \frac{\partial^{2} u_{j}}{\partial x_{k} \partial x_{l}}(\mathbf{x}, t)+\left[\frac{1}{4!} \int_{\mathcal{H}} \lambda(\|\xi\|) \xi_{i} \xi_{j} \xi_{k} \xi_{l} \xi_{m} \xi_{n} d \xi\right] \frac{\partial^{4} u_{j}}{\partial x_{k} \partial x_{l} \partial x_{m} \partial x_{n}}(\mathbf{x}, t) } \\
& +\ldots+b_{i}(\mathbf{x}, t) \tag{A.23}
\end{align*}
$$

Assuming fourth-order and higher derivatives of displacements are negligible, we obtain

$$
\begin{equation*}
\rho(\mathbf{x}) \ddot{u}_{i}(\mathbf{x}, t)=\left[\frac{1}{2} \int_{\mathcal{H}} \lambda(\|\xi\|) \xi_{i} \xi_{j} \xi_{k} \xi_{l} d \xi\right] \frac{\partial^{2} u_{j}}{\partial x_{k} \partial x_{l}}(\mathbf{x}, t)+b_{i}(\mathbf{x}, t) . \tag{A.24}
\end{equation*}
$$

Employing polar coordinates, $\xi_{1}=r \cos (\theta)$ and $\xi_{2}=r \sin (\theta)$, we can compute the
following integrals:

$$
\begin{align*}
\frac{1}{2} \int_{\mathcal{H}} \lambda(\|\xi\|) \xi_{1}^{4} d \xi & =\frac{1}{2} \int_{0}^{2 \pi} \int_{0}^{\delta} \lambda(r)(r \cos (\theta))^{4} r d r d \theta \\
& =\left(\frac{1}{2} \int_{0}^{\delta} \lambda(r) r^{5} d r\right) \int_{0}^{2 \pi} \cos ^{4}(\theta) d \theta=\Lambda  \tag{A.25a}\\
\frac{1}{2} \int_{\mathcal{H}} \lambda(\|\xi\|) \xi_{1}^{2} \xi_{2}^{2} d \xi & =\frac{1}{2} \int_{0}^{2 \pi} \int_{0}^{\delta} \lambda(r)(r \cos (\theta))^{2}(r \sin (\theta))^{2} r d r d \theta \\
& =\left(\frac{1}{2} \int_{0}^{\delta} \lambda(r) r^{5} d r\right) \int_{0}^{2 \pi} \cos ^{2}(\theta) \sin ^{2}(\theta) d \theta=\frac{\Lambda}{3}  \tag{A.25b}\\
\frac{1}{2} \int_{\mathcal{H}} \lambda(\|\xi\|) \xi_{2}^{4} d \xi & =\frac{1}{2} \int_{0}^{2 \pi} \int_{0}^{\delta} \lambda(r)(r \sin (\theta))^{4} r d r d \theta \\
& =\left(\frac{1}{2} \int_{0}^{\delta} \lambda(r) r^{5} d r\right) \int_{0}^{2 \pi} \sin ^{4}(\theta) d \theta=\Lambda  \tag{A.25c}\\
\frac{1}{2} \int_{\mathcal{H}} \lambda(\|\xi\|) \xi_{1}^{3} \xi_{2} d \xi & =\frac{1}{2} \int_{\mathcal{H}} \lambda(\|\xi\|) \xi_{1} \xi_{2}^{3} d \xi=0, \tag{A.25d}
\end{align*}
$$

where

$$
\begin{equation*}
\Lambda:=\frac{3 \pi}{4}\left(\frac{1}{2} \int_{0}^{\delta} \lambda(r) r^{5} d r\right) \tag{A.26}
\end{equation*}
$$

Substituting (A.25) in (A.24), we get

$$
\begin{align*}
\rho(\mathbf{x}) \ddot{u}_{1}(\mathbf{x}, t)= & {\left[\frac{1}{2} \int_{\mathcal{H}} \lambda(\|\xi\|) \xi_{1} \xi_{j} \xi_{k} \xi_{l} d \xi\right] \frac{\partial^{2} u_{j}}{\partial x_{k} \partial x_{l}}(\mathbf{x}, t)+b_{1}(\mathbf{x}, t) } \\
= & {\left[\frac{1}{2} \int_{\mathcal{H}} \lambda(\|\xi\|) \xi_{1}^{4} d \xi\right] \frac{\partial^{2} u_{1}}{\partial x^{2}}(\mathbf{x}, t)+\left[\frac{1}{2} \int_{\mathcal{H}} \lambda(\|\xi\|) \xi_{1}^{2} \xi_{2}^{2} d \xi\right] \frac{\partial^{2} u_{1}}{\partial y^{2}}(\mathbf{x}, t) } \\
& +2\left[\frac{1}{2} \int_{\mathcal{H}} \lambda(\|\xi\|) \xi_{1}^{2} \xi_{2}^{2} d \xi\right] \frac{\partial^{2} u_{2}}{\partial x \partial y}(\mathbf{x}, t)+b_{1}(\mathbf{x}, t) \\
= & \Lambda\left[\frac{\partial^{2} u_{1}}{\partial x^{2}}(\mathbf{x}, t)+\frac{2}{3} \frac{\partial^{2} u_{2}}{\partial x \partial y}(\mathbf{x}, t)+\frac{1}{3} \frac{\partial^{2} u_{1}}{\partial y^{2}}(\mathbf{x}, t)\right]+b_{1}(\mathbf{x}, t),  \tag{A.27a}\\
\rho(\mathbf{x}) \ddot{u}_{2}(\mathbf{x}, t)= & {\left[\frac{1}{2} \int_{\mathcal{H}} \lambda(\|\xi\|) \xi_{2} \xi_{j} \xi_{k} \xi_{l} d \xi\right] \frac{\partial^{2} u_{j}}{\partial x_{k} \partial x_{l}}(\mathbf{x}, t)+b_{2}(\mathbf{x}, t) } \\
= & {\left[\frac{1}{2} \int_{\mathcal{H}} \lambda(\|\xi\|) \xi_{2}^{4} d \xi\right] \frac{\partial^{2} u_{2}}{\partial y^{2}}(\mathbf{x}, t)+\left[\frac{1}{2} \int_{\mathcal{H}} \lambda(\|\xi\|) \xi_{1}^{2} \xi_{2}^{2} d \xi\right] \frac{\partial^{2} u_{2}}{\partial x^{2}}(\mathbf{x}, t) } \\
& +2\left[\frac{1}{2} \int_{\mathcal{H}} \lambda(\|\xi\|) \xi_{1}^{2} \xi_{2}^{2} d \xi\right] \frac{\partial^{2} u_{1}}{\partial x \partial y}(\mathbf{x}, t)+b_{2}(\mathbf{x}, t) \\
= & \Lambda\left[\frac{\partial^{2} u_{2}}{\partial y^{2}}(\mathbf{x}, t)+\frac{2}{3} \frac{\partial^{2} u_{1}}{\partial x \partial y}(\mathbf{x}, t)+\frac{1}{3} \frac{\partial^{2} u_{2}}{\partial x^{2}}(\mathbf{x}, t)\right]+b_{2}(\mathbf{x}, t) . \tag{A.27b}
\end{align*}
$$

Equating (A.27a) and (A.20a) or (A.27b) and (A.20b), we obtain

$$
\begin{equation*}
\Lambda=\frac{9 E}{8} . \tag{A.28}
\end{equation*}
$$ and CCM models possess the same static solution for problems with constant, linear, quadratic, or cubic solutions. To characterize the model discrepancy between the PD and CCM models, in Section A. 2.1 below we assume a particular form for the micromodulus function.

## A.2.1. Model discrepancy between PD and CCM models in two dimensions

745 Assume a micromodulus function of the following form:

$$
\begin{equation*}
\lambda(\|\xi\|)=\frac{c}{\|\xi\|^{\alpha}} \tag{A.29}
\end{equation*}
$$

with $c$ a constant and $\alpha<6$ (see below). Then, we can compute $\Lambda$ in (A.26):

$$
\begin{equation*}
\Lambda=\frac{3 \pi}{4}\left(\frac{1}{2} \int_{0}^{\delta} c r^{5-\alpha} d r\right)=c \frac{3 \pi}{8} \frac{\delta^{6-\alpha}}{6-\alpha} \tag{A.30}
\end{equation*}
$$

By equating (A.30) and (A.28), we get

$$
\begin{equation*}
c=\frac{3(6-\alpha) E}{\pi \delta^{6-\alpha}} . \tag{A.31}
\end{equation*}
$$

Note that the case $\alpha=3$ recovers the micromodulus definition reported in [89] for a plane stress structure with unit thickness. Employing polar coordinates as in (A.25), we can express the coefficients of the fourth-order derivatives in (A.23) as

$$
\begin{align*}
\frac{1}{4!} \int_{\mathcal{H}} \lambda(\|\xi\|) \xi_{i} \xi_{j} \xi_{k} \xi_{l} \xi_{m} \xi_{n} d \xi & =\frac{1}{4!} \int_{0}^{2 \pi} \int_{0}^{\delta} \lambda(r)(r \cos (\theta))^{a}(r \sin (\theta))^{6-a} r d r d \theta \\
& =\frac{1}{4!}\left(\int_{0}^{\delta} \lambda(r) r^{7} d r\right) \int_{0}^{2 \pi}(\cos (\theta))^{a}(\sin (\theta))^{6-a} d \theta \tag{A.32}
\end{align*}
$$

where $a$ is the number of 1 s in $\{i, j, k, l, m, n\}$. We now have ( $c f$. (A.29) and (A.31)),

$$
\begin{equation*}
\int_{0}^{\delta} \lambda(r) r^{7} d r=\int_{0}^{\delta} c r^{7-\alpha} d r=c \frac{\delta^{8-\alpha}}{8-\alpha}=\frac{3}{\pi} \frac{6-\alpha}{8-\alpha} E \delta^{2} \tag{A.33}
\end{equation*}
$$

In the limit as $\delta \rightarrow 0$, (A.23) gives ( $c f$. (A.27) and (A.28))

$$
\begin{align*}
& \rho(\mathbf{x}) \ddot{u}_{1}(\mathbf{x}, t)=\frac{9 E}{8}\left[\frac{\partial^{2} u_{1}}{\partial x^{2}}(\mathbf{x}, t)+\frac{2}{3} \frac{\partial^{2} u_{2}}{\partial x \partial y}(\mathbf{x}, t)+\frac{1}{3} \frac{\partial^{2} u_{1}}{\partial y^{2}}(\mathbf{x}, t)\right]+O\left(\delta^{2}\right)+b_{1}(\mathbf{x}, t), \\
& \rho(\mathbf{x}) \ddot{u}_{2}(\mathbf{x}, t)=\frac{9 E}{8}\left[\frac{\partial^{2} u_{2}}{\partial y^{2}}(\mathbf{x}, t)+\frac{2}{3} \frac{\partial^{2} u_{1}}{\partial x \partial y}(\mathbf{x}, t)+\frac{1}{3} \frac{\partial^{2} u_{2}}{\partial x^{2}}(\mathbf{x}, t)\right]+O\left(\delta^{2}\right)+b_{2}(\mathbf{x}, t), \tag{A.34a}
\end{align*}
$$

so that the PD model (A.16) converges to the CCM model (A.20) at a rate of $O\left(\delta^{2}\right)$. The leading term in the model discrepancy is of order $O\left(\delta^{2}\right)$, similar to the result obtained in Section A.1.1.

## B. Convergence of the nonlocal traction to the local traction

## B.1. One-dimensional case

Assume a domain $\mathcal{B} \subset \mathbb{R}$ and consider the nonlocal traction at $x_{0} \in \mathcal{B}$ in the bulk of the body with normal $n=+1$ (cf. (22)):

$$
\begin{equation*}
\tau\left(x_{0},+1\right)=\int_{x_{0}-\delta}^{x_{0}} \int_{x_{0}}^{x+\delta} c\left(\left|x^{\prime}-x\right|\right)\left(u\left(x^{\prime}\right)-u(x)\right) d x^{\prime} d x . \tag{B.1}
\end{equation*}
$$

Assuming a smooth deformation, we begin by employing a first Taylor expansion of $u\left(x^{\prime}\right)$ about $x$ (cf. (A.4)):

$$
\begin{align*}
\tau\left(x_{0},+1\right) & =\int_{x_{0}-\delta}^{x_{0}} \int_{x_{0}}^{x+\delta} c\left(\left|x^{\prime}-x\right|\right)\left(\frac{d u}{d x}(x)\left(x^{\prime}-x\right)+\frac{1}{2} \frac{d^{2} u}{d x^{2}}(x)\left(x^{\prime}-x\right)^{2}+\frac{1}{3!} \frac{d^{3} u}{d x^{3}}(x)\left(x^{\prime}-x\right)^{3}+\ldots\right) d x^{\prime} d x \\
& =\int_{x_{0}-\delta}^{x_{0}} \int_{x_{0}-x}^{\delta} c(|\xi|)\left(\frac{d u}{d x}(x) \xi+\frac{1}{2} \frac{d^{2} u}{d x^{2}}(x) \xi^{2}+\frac{1}{3!} \frac{d^{3} u}{d x^{3}}(x) \xi^{3}+\ldots\right) d \xi d x \tag{B.2}
\end{align*}
$$

where we used the change of variable $\xi=x^{\prime}-x$ in the last equality. Note that due to the limits of integration, $x^{\prime}>x$ and thus $\xi>0$. Assume the micromodulus function (A.9)
with $\alpha<2$ (see below). Then,

$$
\begin{align*}
\tau\left(x_{0},+1\right)= & \int_{x_{0}-\delta}^{x_{0}} \int_{x_{0}-x}^{\delta} \frac{c}{|\xi|^{\alpha}}\left(\frac{d u}{d x}(x) \xi+\frac{1}{2} \frac{d^{2} u}{d x^{2}}(x) \xi^{2}+\frac{1}{3!} \frac{d^{3} u}{d x^{3}}(x) \xi^{3}+\ldots\right) d \xi d x \\
= & c \int_{x_{0}-\delta}^{x_{0}} \int_{x_{0}-x}^{\delta}\left(\frac{d u}{d x}(x) \xi^{1-\alpha}+\frac{1}{2} \frac{d^{2} u}{d x^{2}}(x) \xi^{2-\alpha}+\frac{1}{3!} \frac{d^{3} u}{d x^{3}}(x) \xi^{3-\alpha}+\ldots\right) d \xi d x \\
= & c \int_{x_{0}-\delta}^{x_{0}}\left(\frac{d u}{d x}(x) \frac{1}{2-\alpha}\left[\delta^{2-\alpha}-\left(x_{0}-x\right)^{2-\alpha}\right]+\frac{1}{2} \frac{d^{2} u}{d x^{2}}(x) \frac{1}{3-\alpha}\left[\delta^{3-\alpha}-\left(x_{0}-x\right)^{3-\alpha}\right]\right. \\
& \left.\quad+\frac{1}{3!} \frac{d^{3} u}{d x^{3}}(x) \frac{1}{4-\alpha}\left[\delta^{4-\alpha}-\left(x_{0}-x\right)^{4-\alpha}\right]+\ldots\right) d x . \tag{B.3}
\end{align*}
$$

We now perform a second Taylor expansion, this time for the derivatives evaluated at $x$ about $x_{0}$. Explicitly writing terms up to third derivatives, we have

$$
\begin{align*}
\tau\left(x_{0},+1\right)=c \int_{x_{0}-\delta}^{x_{0}}( & \frac{1}{2-\alpha}\left\{\frac{d u}{d x}\left(x_{0}\right)+\frac{d^{2} u}{d x^{2}}\left(x_{0}\right)\left(x-x_{0}\right)+\frac{1}{2} \frac{d^{3} u}{d x^{3}}\left(x_{0}\right)\left(x-x_{0}\right)^{2}+\ldots\right\}\left[\delta^{2-\alpha}-\left(x_{0}-x\right)^{2-\alpha}\right] \\
& +\frac{1}{2} \frac{1}{3-\alpha}\left\{\frac{d^{2} u}{d x^{2}}\left(x_{0}\right)+\frac{d^{3} u}{d x^{3}}\left(x_{0}\right)\left(x-x_{0}\right)+\ldots\right\}\left[\delta^{3-\alpha}-\left(x_{0}-x\right)^{3-\alpha}\right] \\
& \left.+\frac{1}{3!} \frac{1}{4-\alpha}\left\{\frac{d^{3} u}{d x^{3}}\left(x_{0}\right)+\ldots\right\}\left[\delta^{4-\alpha}-\left(x_{0}-x\right)^{4-\alpha}\right]\right) d x . \tag{B.4}
\end{align*}
$$

Collecting the contributions to each derivative, we have

$$
\begin{align*}
& \tau\left(x_{0},+1\right)=c\left(\left\{\int_{x_{0}-\delta}^{x_{0}} \frac{1}{2-\alpha}\left[\delta^{2-\alpha}-\left(x_{0}-x\right)^{2-\alpha}\right] d x\right\} \frac{d u}{d x}\left(x_{0}\right)\right. \\
&+\left\{\int_{x_{0}-\delta}^{x_{0}} \frac{1}{2-\alpha}\left(x-x_{0}\right)\left[\delta^{2-\alpha}-\left(x_{0}-x\right)^{2-\alpha}\right]+\frac{1}{2} \frac{1}{3-\alpha}\left[\delta^{3-\alpha}-\left(x_{0}-x\right)^{3-\alpha}\right] d x\right\} \frac{d^{2} u}{d x^{2}}\left(x_{0}\right) \\
&+\left\{\int_{x_{0}-\delta}^{x_{0}} \frac{1}{2-\alpha} \frac{\left(x-x_{0}\right)^{2}}{2}\left[\delta^{2-\alpha}-\left(x_{0}-x\right)^{2-\alpha}\right]+\frac{1}{2} \frac{1}{3-\alpha}\left(x-x_{0}\right)\left[\delta^{3-\alpha}-\left(x_{0}-x\right)^{3-\alpha}\right]\right. \\
&\left.\left.+\frac{1}{3!} \frac{1}{4-\alpha}\left[\delta^{4-\alpha}-\left(x_{0}-x\right)^{4-\alpha}\right] d x\right\} \frac{d^{3} u}{d x^{3}}\left(x_{0}\right)+\ldots\right) \tag{B.5}
\end{align*}
$$

We compute the following integral (for $n \geqslant 0$ and $m>1$ ):

$$
\begin{align*}
& \int_{x_{0}-\delta}^{x_{0}} \frac{1}{(m-1)!} \frac{1}{m-\alpha} \frac{\left(x-x_{0}\right)^{n}}{n!}\left[\delta^{m-\alpha}-\left(x_{0}-x\right)^{m-\alpha}\right] d x \\
& =\left.\frac{1}{(m-1)!} \frac{1}{m-\alpha} \frac{1}{n!}(-1)^{n}\left[-\delta^{m-\alpha} \frac{1}{n+1}\left(x_{0}-x\right)^{n+1}+\frac{1}{n+m+1-\alpha}\left(x_{0}-x\right)^{n+m+1-\alpha}\right]\right|_{x_{0}-\delta} ^{x_{0}} \\
& =\frac{1}{(m-1)!} \frac{(-1)^{n}}{(n+1)!} \frac{\delta^{n+m+1-\alpha}}{n+m+1-\alpha} \tag{B.6}
\end{align*}
$$

Using (B.6) to compute the integrals in (B.5), we obtain

$$
\begin{equation*}
\tau\left(x_{0},+1\right)=c\left(\frac{\delta^{3-\alpha}}{3-\alpha} \frac{d u}{d x}\left(x_{0}\right)+\frac{1}{12} \frac{\delta^{5-\alpha}}{5-\alpha} \frac{d^{3} u}{d x^{3}}\left(x_{0}\right)+\ldots\right) \tag{B.7}
\end{equation*}
$$

Employing the relation in (A.12), we finally obtain

$$
\begin{align*}
\tau\left(x_{0},+1\right) & =E \frac{d u}{d x}\left(x_{0}\right)+\frac{1}{12} \frac{3-\alpha}{5-\alpha} E \delta^{2} \frac{d^{3} u}{d x^{3}}\left(x_{0}\right)+\ldots \\
& =\mathrm{t}\left(x_{0},+1\right)+\frac{1}{12} \frac{3-\alpha}{5-\alpha} E \delta^{2} \frac{d^{3} u}{d x^{3}}\left(x_{0}\right)+\ldots \tag{B.8}
\end{align*}
$$

where $\mathrm{t}\left(x_{0},+1\right)$ is the local traction at $x_{0}$ with normal $n=+1(c f .(12)$ and (11)). In the limit as $\delta \rightarrow 0$, we have

$$
\begin{equation*}
\tau\left(x_{0},+1\right)=\mathrm{t}\left(x_{0},+1\right)+\mathcal{O}\left(\delta^{2}\right) \tag{B.9}
\end{equation*}
$$

i.e., the nonlocal traction converges to the local traction at a rate of $\mathcal{O}\left(\delta^{2}\right)$.


Figure B.24: Domain of integration (shaded region) for the one-dimensional nonlocal traction in (B.10).
Remark B.1. Consider the nonlocal traction at $x_{0}$ with normal $n=-1$ (cf. (22)):

$$
\begin{equation*}
\tau\left(x_{0},-1\right)=\int_{x_{0}}^{x_{0}+\delta} \int_{x-\delta}^{x_{0}} c\left(\left|x^{\prime}-x\right|\right)\left(u\left(x^{\prime}\right)-u(x)\right) d x^{\prime} d x . \tag{B.10}
\end{equation*}
$$

The two-dimensional region of integration is illustrated in Fig. B.24. Changing the order of integration, we have

$$
\begin{align*}
\tau\left(x_{0},-1\right) & =\int_{x_{0}-\delta}^{x_{0}} \int_{x_{0}}^{x^{\prime}+\delta} c\left(\left|x^{\prime}-x\right|\right)\left(u\left(x^{\prime}\right)-u(x)\right) d x d x^{\prime} \\
& =-\int_{x_{0}-\delta}^{x_{0}} \int_{x_{0}}^{x^{\prime}+\delta} c\left(\left|x-x^{\prime}\right|\right)\left(u(x)-u\left(x^{\prime}\right)\right) d x d x^{\prime} \\
& =-\int_{x_{0}-\delta}^{x_{0}} \int_{x_{0}}^{\hat{x}+\delta} c\left(\left|\hat{x}^{\prime}-\hat{x}\right|\right)\left(u\left(\hat{x}^{\prime}\right)-u(\hat{x})\right) d \hat{x}^{\prime} d \hat{x}=-\tau\left(x_{0},+1\right) \tag{B.11}
\end{align*}
$$

where we used the change of variables $\hat{x}=x^{\prime}$ and $\hat{x}^{\prime}=x$ as well as (22) in the second to last and last equalities, respectively. Using (B.8) and (13), we have

$$
\begin{align*}
\tau\left(x_{0},-1\right) & =-\tau\left(x_{0},+1\right)=-\mathfrak{t}\left(x_{0},+1\right)-\frac{1}{12} \frac{3-\alpha}{5-\alpha} E \delta^{2} \frac{d^{3} u}{d x^{3}}\left(x_{0}\right)+\ldots \\
& =\mathfrak{t}\left(x_{0},-1\right)-\frac{1}{12} \frac{3-\alpha}{5-\alpha} E \delta^{2} \frac{d^{3} u}{d x^{3}}\left(x_{0}\right)+\ldots \tag{B.12}
\end{align*}
$$

## B.1.1. Numerical examples for the nonlocal traction in one dimension



Figure B.25: Interface between the PD and CCM regions for the nonlocal traction computation in a onedimensional CCM-PD coupled model. Blue diamonds are FEM nodes and green circles are PD nodes. The dashed gray vertical line indicates the interface at $x_{0}$. A uniform discretization with grid spacing $\Delta x=1$ is employed, and the PD horizon is taken as $\delta=3$.

We present some numerical examples to confirm the result in (B.8). To put these examples within the context of a CCM-PD coupled model, we consider a one-dimensional system with an interface at $x_{0}$ in the bulk of the body between a PD region and a CCM region. We assume points $x<x_{0}$ belong to the PD region, whereas points $x>x_{0}$ correspond to the CCM region. To numerically compute the nonlocal traction in (B.1),
we employ a uniform discretization with grid spacing $\Delta x$ and define a set of $N_{\mathrm{PD}}$ PD nodes with positions given by

$$
\begin{equation*}
S_{\mathrm{PD}}=\left\{x_{0}-\delta+\frac{\Delta x}{2}, \ldots, x_{0}-\frac{\Delta x}{2}\right\} \tag{B.13}
\end{equation*}
$$

and a set of $N_{\text {FEM }}$ FEM nodes with positions given by

$$
\begin{equation*}
S_{\mathrm{FEM}}=\left\{x_{0}+\frac{\Delta x}{2}, \ldots, x_{0}+\delta-\frac{\Delta x}{2}\right\} . \tag{B.14}
\end{equation*}
$$

To relate the examples in this section to the numerical results in Section 4.1, we choose $\delta=3$ and $\Delta x=1$ (i.e., $m=\delta / \Delta x=3$ ), and we consider the case of an interface at $x_{0}=20.5$ (i.e., $x_{0}=20+\frac{\Delta x}{2}$ ), which corresponds to the configuration (e) in Fig. 10; an illustration is presented in Fig. B.25. We denote by $x_{i}^{\mathrm{PD}} \in S_{\mathrm{PD}}, i=1, \ldots, N_{\mathrm{PD}}$, and $x_{j}^{\mathrm{FEM}} \in \mathcal{S}_{\mathrm{FEM}}, j=1, \ldots, N_{\mathrm{FEM}}$, the reference positions of the PD and FEM nodes, respectively. Define the displacements of the PD and FEM nodes, respectively, by $u_{i}^{\mathrm{PD}}:=u\left(x_{i}^{\mathrm{PD}}\right), i=1, \ldots, N_{\mathrm{PD}}$, and $u_{j}^{\mathrm{FEM}}:=u\left(x_{j}^{\mathrm{FEM}}\right), j=1, \ldots, N_{\mathrm{FEM}}$. Then, we can compute the nonlocal traction in (B.1) by
$\tau^{\mathrm{num}}\left(x_{0},+1\right):=\sum_{i=1}^{N_{\mathrm{PD}}} \sum_{j=1}^{N_{\mathrm{FEM}}} \chi_{\delta}\left(\left|x_{j}^{\mathrm{CE}}-x_{i}^{\mathrm{PD}}\right|\right) c\left(\left|x_{j}^{\mathrm{CE}}-x_{i}^{\mathrm{PD}}\right|\right)\left(u_{j}^{\mathrm{FEM}}-u_{i}^{\mathrm{PD}}\right) \Delta x_{j}^{(i)} \Delta x$,
where $\chi_{\delta}$ is the characteristic function in (16) and a partial-volume correction [79] is used for $m$ th neighbors, so that $\Delta x_{j}^{(i)}=\frac{1}{2} \Delta x$ if $\left|x_{j}^{\mathrm{CE}}-x_{i}^{\mathrm{PD}}\right|=\delta$ and $\Delta x_{j}^{(i)}=\Delta x$ otherwise. We consider the micromodulus function (A.9) with $\alpha=1$.

We compare the numerical computation of the nonlocal traction given by (B.15) with the analytical calculation using (B.8) for the case of linear, quadratic, and cubic displacement fields, described in Table B.11; the values of the coefficients are $a=0.0001$ and $X_{1}=17$. As a comparison, we analytically calculate the local traction using (12) with (11). The results are reported in Table B.12. In addition to reporting the values for the nonlocal and local tractions, we present the error of the nonlocal traction computation given by the absolute value of the difference between the numerical and analytical values. We observe that the values of the numerical nonlocal traction obtained by (B.15), which is a discretization of (B.1), accurately match the values given by the analytical nonlocal traction in (B.8) for all the cases, linear, quadratic, and cubic

Table B.11: Displacement fields for the nonlocal traction computation in one dimension.

| Displacement type | Displacement field equation |
| :---: | :---: |
| Linear | $u(x)=a x$ |
| Quadratic | $u(x)=\frac{a}{2 X_{1}} x^{2}$ |
| Cubic | $u(x)=\frac{a}{3 X_{1}^{2}} x^{3}$ |

Table B.12: Comparison between numerical and analytical tractions in one dimension.

| Displacement type | Nonlocal traction |  |  | Local traction |
| :---: | :---: | :---: | :---: | :---: |
|  | $\tau^{\mathrm{num}}\left(x_{0},+1\right)$ | $\tau\left(x_{0},+1\right)$ | $\left\|\tau^{\mathrm{num}}\left(x_{0},+1\right)-\tau\left(x_{0},+1\right)\right\|$ | $\mathrm{t}\left(x_{0},+1\right)$ |
| Linear | $1.00 \times 10^{-04}$ | $1.00 \times 10^{-04}$ | $9.49 \times 10^{-20}$ | $1.00 \times 10^{-04}$ |
| Quadratic | $1.21 \times 10^{-04}$ | $1.21 \times 10^{-04}$ | $4.07 \times 10^{-20}$ | $1.21 \times 10^{-04}$ |
| Cubic | $1.46 \times 10^{-04}$ | $1.46 \times 10^{-04}$ | $2.71 \times 10^{-20}$ | $1.45 \times 10^{-04}$ |

displacements. Moreover, for the linear and quadratic displacements, the values of the nonlocal and local tractions coincide, as predicted by (B.8). These two observations confirm the result in (B.8).

## B.2. Two-dimensional case

Assume a domain $\mathcal{B} \subset \mathbb{R}^{2}$ and consider two non-overlapping subdomains $\Omega_{A}$ and $\Omega_{B}$ with a straight interface $\Gamma$ connecting them (see, e.g., Fig. B.26), i.e., $\Omega_{A} \cap \Omega_{B}=\emptyset$ and $\overline{\Omega_{A}} \cap \overline{\Omega_{B}}=\Gamma$. We assume the normal $\mathbf{n}$ to the interface $\Gamma$ points outwards relative to $\Omega_{A}$. Given the linear isotropic bond-based PD model (6), consider the nonlocal traction at $\mathbf{x}_{0}=\left(x_{0}, y_{0}\right) \in \Gamma$ in the bulk of the body with normal $\mathbf{n}(c f .(70)):$

$$
\tau\left(\mathbf{x}_{0}, \mathbf{n}\right)=\int_{\mathcal{L}} \int_{\Omega_{B}} \chi_{\delta}\left(\left\|\mathbf{x}^{\prime}-\mathbf{x}\right\|\right) \lambda\left(\left\|\mathbf{x}^{\prime}-\mathbf{x}\right\|\right)\left(\mathbf{x}^{\prime}-\mathbf{x}\right) \otimes\left(\mathbf{x}^{\prime}-\mathbf{x}\right)\left(\mathbf{u}\left(\mathbf{x}^{\prime}\right)-\mathbf{u}(\mathbf{x})\right) d \mathbf{x}^{\prime} d \ell, \text { (B.16) }
$$

where $\mathcal{L}$ is defined in (69). In component form, we have

$$
\tau_{i}\left(\mathbf{x}_{0}, \mathbf{n}\right)=\int_{\mathcal{L}} \int_{\Omega_{B}} \chi_{\delta}(\|\xi\|) \lambda(\|\xi\|) \xi_{i} \xi_{j}\left(u_{j}(\mathbf{x}+\boldsymbol{\xi})-u_{j}(\mathbf{x})\right) d \mathbf{x}^{\prime} d \ell, \quad i=1,2, \text { (B.17) }
$$



Figure B.26: Illustration of two adjacent subdomains $\Omega_{\mathrm{A}}$ and $\Omega_{\mathrm{B}}$ separated by a straight interface $\Gamma$ for the calculation of the nonlocal traction.
where the notation $\boldsymbol{\xi}=\mathbf{x}^{\prime}-\mathbf{x}$ is used for brevity and repeated indices imply a summation by 1 and 2. Assuming a smooth deformation, we begin by employing a first Taylor expansion of $u_{j}(\mathbf{x}+\boldsymbol{\xi})$ about $\mathbf{x}(c f$. (A.21)) for the $i$ th component of (B.17):
$\tau_{i}\left(\mathbf{x}_{0}, \mathbf{n}\right)=\int_{\mathcal{L}} \int_{\Omega_{B}} \chi_{\delta}(\|\xi\|) \lambda(\|\xi\|) \xi_{i} \xi_{j}\left(\frac{\partial u_{j}}{\partial x_{k}}(\mathbf{x}) \xi_{k}+\frac{1}{2} \frac{\partial^{2} u_{j}}{\partial x_{k} \partial x_{l}}(\mathbf{x}) \xi_{k} \xi_{l}+\ldots\right) d \mathbf{x}^{\prime} d \ell$.

We consider below two cases, the first one given by a horizontal interface $\Gamma$ with normal $\mathbf{n}=\mathbf{e}_{2}$ (see Fig. B.26a) and the second one given by a vertical interface $\Gamma$ with normal $\mathbf{n}=\mathbf{e}_{1}$ (see Fig. B.26b); the normals $\left\{\mathbf{e}_{1}, \mathbf{e}_{2}\right\}$ correspond to the standard Cartesian orthonormal basis. We assume the micromodulus function (A.29).

Horizontal Interface. For the case of a horizontal interface with normal $\mathbf{n}=\mathbf{e}_{2}$, we can compute (B.18) using polar coordinates, $\xi_{1}=r \sin (\theta)$ and $\xi_{2}=r \cos (\theta)$, as follows
(see Fig. B.26a for the limits of integration):

$$
\begin{align*}
\tau_{i}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)= & \int_{y_{0}-\delta}^{y_{0}} \int_{\mathcal{H}_{\mathbf{x}} \cap \Omega_{B}} \lambda(\|\xi\|) \xi_{i} \xi_{j}\left(\frac{\partial u_{j}}{\partial x_{k}}\left(x_{0}, y\right) \xi_{k}+\frac{1}{2} \frac{\partial^{2} u_{j}}{\partial x_{k} \partial x_{l}}\left(x_{0}, y\right) \xi_{k} \xi_{l}+\ldots\right) d \mathbf{x}^{\prime} d y \\
= & \int_{y_{0}-\delta}^{y_{0}}\left[\int_{\mathcal{H}_{\mathbf{x}} \cap \Omega_{B}} \lambda(\|\xi\|) \xi_{i} \xi_{j} \xi_{k} d \mathbf{x}^{\prime}\right] \frac{\partial u_{j}}{\partial x_{k}}\left(x_{0}, y\right) d y \\
& +\frac{1}{2} \int_{y_{0}-\delta}^{y_{0}}\left[\int_{\mathcal{H}_{\mathbf{x}} \cap \Omega_{B}} \lambda(\|\xi\|) \xi_{i} \xi_{j} \xi_{k} \xi_{l} d \mathbf{x}^{\prime}\right] \frac{\partial^{2} u_{j}}{\partial x_{k} \partial x_{l}}\left(x_{0}, y\right) d y+\ldots \\
= & \int_{y_{0}-\delta}^{y_{0}}\left[\int_{y_{0}-y}^{\delta} \int_{-\cos ^{-1}\left(\frac{y_{0}-y}{r}\right)}^{\cos ^{-1}\left(\frac{y_{0}-y}{r}\right)} \lambda(r)(r \cos (\theta))^{a_{1}}(r \sin (\theta))^{3-a_{1}} r d \theta d r\right] \frac{\partial u_{j}}{\partial x_{k}}\left(x_{0}, y\right) d y \\
& +\frac{1}{2} \int_{y_{0}-\delta}^{y_{0}}\left[\int_{y_{0}-y}^{\delta} \int_{-\cos ^{-1}\left(\frac{y_{0}-y}{r}\right)}^{\cos ^{-1}\left(\frac{y_{0}-y}{r}\right)} \lambda(r)(r \cos (\theta))^{a_{2}}(r \sin (\theta))^{4-a_{2}} r d \theta d r\right] \frac{\partial^{2} u_{j}}{\partial x_{k} \partial x_{l}}\left(x_{0}, y\right) d y+\ldots, \tag{B.19}
\end{align*}
$$

where $a_{1}$ is the number of 2 s in $\{i, j, k\}$ in the coefficients of the first-order derivatives and $a_{2}$ is the number of 2 s in $\{i, j, k, l\}$ in the coefficients of the second-order derivatives. Employing the change of variable $s=y_{0}-y$, we obtain

$$
\begin{align*}
\tau_{i}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)= & \int_{0}^{\delta}\left[\int_{s}^{\delta} \lambda(r) r^{4} \int_{-\cos ^{-1}\left(\frac{s}{r}\right)}^{\cos ^{-1}\left(\frac{s}{r}\right)}(\cos (\theta))^{a_{1}}(\sin (\theta))^{3-a_{1}} d \theta d r\right] \frac{\partial u_{j}}{\partial x_{k}}\left(x_{0}, y_{0}-s\right) d s \\
& +\frac{1}{2} \int_{0}^{\delta}\left[\int_{s}^{\delta} \lambda(r) r^{5} \int_{-\cos ^{-1}\left(\frac{s}{r}\right)}^{\cos ^{-1}\left(\frac{s}{r}\right)}(\cos (\theta))^{a_{2}}(\sin (\theta))^{4-a_{2}} d \theta d r\right] \frac{\partial^{2} u_{j}}{\partial x_{k} \partial x_{l}}\left(x_{0}, y_{0}-s\right) d s+\ldots \tag{B.20}
\end{align*}
$$

Changing the order of integration between $r$ and $s$ according to Fig. B.27, and then


Figure B.27: Domain of integration (shaded region) in the variables $s$ and $r$ in (B.20) and corresponding limits for the change in the order of integration.
using the change of variable $\kappa=s / r$, we obtain

$$
\begin{align*}
\tau_{i}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)= & \int_{0}^{\delta} \int_{0}^{r} \lambda(r) r^{4} \int_{-\cos ^{-1}\left(\frac{s}{r}\right)}^{\cos ^{-1}\left(\frac{s}{r}\right)}(\cos (\theta))^{a_{1}}(\sin (\theta))^{3-a_{1}} d \theta \frac{\partial u_{j}}{\partial x_{k}}\left(x_{0}, y_{0}-s\right) d s d r \\
& +\frac{1}{2} \int_{0}^{\delta} \int_{0}^{r} \lambda(r) r^{5} \int_{-\cos ^{-1}\left(\frac{s}{r}\right)}^{\cos ^{-1}\left(\frac{s}{r}\right)}(\cos (\theta))^{a_{2}}(\sin (\theta))^{4-a_{2}} d \theta \frac{\partial^{2} u_{j}}{\partial x_{k} \partial x_{l}}\left(x_{0}, y_{0}-s\right) d s d r+\ldots \\
= & \int_{0}^{\delta} \int_{0}^{1} \lambda(r) r^{4} \int_{-\cos ^{-1}(\kappa)}^{\cos ^{-1}(\kappa)}(\cos (\theta))^{a_{1}}(\sin (\theta))^{3-a_{1}} d \theta \frac{\partial u_{j}}{\partial x_{k}}\left(x_{0}, y_{0}-r \kappa\right) r d \kappa d r \\
& +\frac{1}{2} \int_{0}^{\delta} \int_{0}^{1} \lambda(r) r^{5} \int_{-\cos ^{-1}(\kappa)}^{\cos ^{-1}(\kappa)}(\cos (\theta))^{a_{2}}(\sin (\theta))^{4-a_{2}} d \theta \frac{\partial^{2} u_{j}}{\partial x_{k} \partial x_{l}}\left(x_{0}, y_{0}-r \kappa\right) r d \kappa d r+\ldots \tag{B.21}
\end{align*}
$$

Using a second Taylor expansion for each term:

$$
\begin{align*}
\frac{\partial u_{j}}{\partial x_{k}}\left(x_{0}, y_{0}-r \kappa\right) & =\frac{\partial u_{j}}{\partial x_{k}}\left(x_{0}, y_{0}\right)-\frac{\partial^{2} u_{j}}{\partial y \partial x_{k}}\left(x_{0}, y_{0}\right) r \kappa+\ldots,  \tag{B.22a}\\
\frac{\partial^{2} u_{j}}{\partial x_{k} \partial x_{l}}\left(x_{0}, y_{0}-r \kappa\right) & =\frac{\partial^{2} u_{j}}{\partial x_{k} \partial x_{l}}\left(x_{0}, y_{0}\right)+\ldots, \tag{B.22b}
\end{align*}
$$

we obtain

$$
\begin{align*}
\tau_{i}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)= & \int_{0}^{\delta} \int_{0}^{1} \lambda(r) r^{4} \int_{-\cos ^{-1}(\kappa)}^{\cos ^{-1}(\kappa)}(\cos (\theta))^{a_{1}}(\sin (\theta))^{3-a_{1}} d \theta\left[\frac{\partial u_{j}}{\partial x_{k}}\left(x_{0}, y_{0}\right)-\frac{\partial^{2} u_{j}}{\partial y \partial x_{k}}\left(x_{0}, y_{0}\right) r \kappa+\ldots\right] r d \kappa d r \\
& +\frac{1}{2} \int_{0}^{\delta} \int_{0}^{1} \lambda(r) r^{5} \int_{-\cos ^{-1}(\kappa)}^{\cos ^{-1}(\kappa)}(\cos (\theta))^{a_{2}}(\sin (\theta))^{4-a_{2}} d \theta\left[\frac{\partial^{2} u_{j}}{\partial x_{k} \partial x_{l}}\left(x_{0}, y_{0}\right)+\ldots\right] r d \kappa d r+\ldots \\
= & \left(\int_{0}^{\delta} \lambda(r) r^{5} d r\right)\left[\int_{0}^{1} \int_{-\cos ^{-1}(\kappa)}^{\cos ^{-1}(\kappa)}(\cos (\theta))^{a_{1}}(\sin (\theta))^{3-a_{1}} d \theta d \kappa\right] \frac{\partial u_{j}}{\partial x_{k}}\left(x_{0}, y_{0}\right) \\
& -\left(\int_{0}^{\delta} \lambda(r) r^{6} d r\right)\left[\int_{0}^{1} \kappa \int_{-\cos ^{-1}(\kappa)}^{\cos ^{-1}(\kappa)}(\cos (\theta))^{a_{1}}(\sin (\theta))^{3-a_{1}} d \theta d \kappa\right] \frac{\partial^{2} u_{j}}{\partial y \partial x_{k}}\left(x_{0}, y_{0}\right) \\
& +\frac{1}{2}\left(\int_{0}^{\delta} \lambda(r) r^{6} d r\right)\left[\int_{0}^{1} \int_{-\cos ^{-1}(\kappa)}^{\cos ^{-1}(\kappa)}(\cos (\theta))^{a_{2}}(\sin (\theta))^{4-a_{2}} d \theta d \kappa\right] \frac{\partial^{2} u_{j}}{\partial x_{k} \partial x_{l}}\left(x_{0}, y_{0}\right)+\ldots \tag{B.23}
\end{align*}
$$

We now have (recall (A.29) and (A.31)):

$$
\begin{align*}
& \int_{0}^{\delta} \lambda(r) r^{5} d r=c \int_{0}^{\delta} r^{5-\alpha} d r=\frac{3(6-\alpha) E}{\pi \delta^{6-\alpha}} \frac{\delta^{6-\alpha}}{6-\alpha}=\frac{3 E}{\pi}  \tag{B.24}\\
& \int_{0}^{\delta} \lambda(r) r^{6} d r=c \int_{0}^{\delta} r^{6-\alpha} d r=\frac{3(6-\alpha) E}{\pi \delta^{6-\alpha}} \frac{\delta^{7-\alpha}}{7-\alpha}=\frac{3 E}{\pi} \frac{6-\alpha}{7-\alpha} \delta \tag{B.25}
\end{align*}
$$

Consequently, we can write (B.23) as (recall $\mathbf{x}_{0}=\left(x_{0}, y_{0}\right)$ )

$$
\begin{align*}
\tau_{i}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)= & \frac{3 E}{\pi}\left[\int_{0}^{1} \int_{-\cos ^{-1}(\kappa)}^{\cos ^{-1}(\kappa)}(\cos (\theta))^{a_{1}}(\sin (\theta))^{3-a_{1}} d \theta d \kappa\right] \frac{\partial u_{j}}{\partial x_{k}}\left(\mathbf{x}_{0}\right) \\
& -\frac{3 E}{\pi} \frac{6-\alpha}{7-\alpha} \delta\left[\int_{0}^{1} \kappa \int_{-\cos ^{-1}(\kappa)}^{\cos ^{-1}(\kappa)}(\cos (\theta))^{a_{1}}(\sin (\theta))^{3-a_{1}} d \theta d \kappa\right] \frac{\partial^{2} u_{j}}{\partial y \partial x_{k}}\left(\mathbf{x}_{0}\right) \\
& +\frac{1}{2} \frac{3 E}{\pi} \frac{6-\alpha}{7-\alpha} \delta\left[\int_{0}^{1} \int_{-\cos ^{-1}(\kappa)}^{\cos ^{-1}(\kappa)}(\cos (\theta))^{a_{2}}(\sin (\theta))^{4-a_{2}} d \theta d \kappa\right] \frac{\partial^{2} u_{j}}{\partial x_{k} \partial x_{l}}\left(\mathbf{x}_{0}\right)+\ldots . \tag{B.26}
\end{align*}
$$

Computing with Mathematica [91], we have

$$
\begin{gather*}
\int_{0}^{1} \int_{-\cos ^{-1}(\kappa)}^{\cos ^{-1}(\kappa)}(\cos (\theta))^{a_{1}}(\sin (\theta))^{3-a_{1}} d \theta d \kappa= \begin{cases}\frac{3 \pi}{8} & a_{1}=3 \\
0 & a_{1}=2 \\
\frac{\pi}{8} & a_{1}=1 \\
0 & a_{1}=0\end{cases}  \tag{B.27a}\\
\int_{0}^{1} \kappa \int_{-\cos ^{-1}(\kappa)}^{\cos ^{-1}(\kappa)}(\cos (\theta))^{a_{1}}(\sin (\theta))^{3-a_{1}} d \theta d \kappa=\left\{\begin{array}{cl}
\frac{8}{15} & a_{1}=3 \\
0 & a_{1}=2 \\
\frac{2}{15} & a_{1}=1 \\
0 & a_{1}=0
\end{array}\right.  \tag{B.27b}\\
\int_{0} \int_{-\cos ^{-1}(\kappa)}(\cos (\theta))^{a_{2}(\sin (\theta))^{4-a_{2}} d \theta d \kappa=\left\{\begin{array}{cl}
\frac{16}{15} & a_{2}=4 \\
\frac{4}{15} & a_{2}=2
\end{array}\right.} \begin{array}{l}
a_{2}=3 \\
0 \\
\cos ^{-1}(\kappa)
\end{array}  \tag{B.27c}\\
a_{2}=1 \\
\frac{2}{5} \\
a_{2}=0
\end{gather*}
$$

Employing (B.27) to compute the coefficients in (B.26) and collecting the contributions from each derivative term, we finally obtain

$$
\begin{align*}
& \tau_{1}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)=\frac{9 E}{8}\left[\frac{1}{3}\left(\frac{\partial u_{1}}{\partial y}\left(\mathbf{x}_{0}\right)+\frac{\partial u_{2}}{\partial x}\left(\mathbf{x}_{0}\right)\right)+\frac{16}{45 \pi} \frac{6-\alpha}{7-\alpha} \delta\left(\frac{3}{2} \frac{\partial^{2} u_{1}}{\partial x^{2}}\left(\mathbf{x}_{0}\right)+\frac{\partial^{2} u_{2}}{\partial x \partial y}\left(\mathbf{x}_{0}\right)\right)+\ldots\right], \\
& (\text { B. } 28 \mathrm{a}) \\
& \tau_{2}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)=\frac{9 E}{8}\left[\frac{1}{3} \frac{\partial u_{1}}{\partial x}\left(\mathbf{x}_{0}\right)+\frac{\partial u_{2}}{\partial y}\left(\mathbf{x}_{0}\right)+\frac{16}{45 \pi} \frac{6-\alpha}{7-\alpha} \delta\left(\frac{\partial^{2} u_{1}}{\partial x \partial y}\left(\mathbf{x}_{0}\right)+\frac{\partial^{2} u_{2}}{\partial x^{2}}\left(\mathbf{x}_{0}\right)\right)+\ldots\right], \tag{B.28b}
\end{align*}
$$

where the dots indicate higher-order derivative terms.
Consider a classical linear elasticity plane stress isotropic model (see (A.18)). Given a Young's modulus $E$ and a Poisson's ratio $v=1 / 3[89,90]$, the components of the
stress tensor are given by:

$$
\begin{align*}
\sigma_{11} & =\frac{E}{1-v^{2}}\left[\varepsilon_{11}+v \varepsilon_{22}\right]=\frac{9 E}{8}\left[\frac{\partial u_{1}}{\partial x}+\frac{1}{3} \frac{\partial u_{2}}{\partial y}\right]  \tag{B.29a}\\
\sigma_{22} & =\frac{E}{1-v^{2}}\left[v \varepsilon_{11}+\varepsilon_{22}\right]=\frac{9 E}{8}\left[\frac{1}{3} \frac{\partial u_{1}}{\partial x}+\frac{\partial u_{2}}{\partial y}\right]  \tag{B.29b}\\
\sigma_{12} & =\frac{E}{1-v^{2}}(1-v) \varepsilon_{12}=\frac{9 E}{8}\left[\frac{1}{3}\left(\frac{\partial u_{1}}{\partial y}+\frac{\partial u_{2}}{\partial x}\right)\right] \tag{B.29c}
\end{align*}
$$

We can then express (B.28a) and (B.28b), respectively, as (recall (42))

$$
\begin{align*}
\tau_{1}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right) & =\sigma_{12}\left(\mathbf{x}_{0}\right)+\frac{2 E}{5 \pi} \frac{6-\alpha}{7-\alpha} \delta\left(\frac{3}{2} \frac{\partial^{2} u_{1}}{\partial x^{2}}\left(\mathbf{x}_{0}\right)+\frac{\partial^{2} u_{2}}{\partial x \partial y}\left(\mathbf{x}_{0}\right)\right)+\ldots \\
& =\mathrm{t}_{1}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)+\frac{2 E}{5 \pi} \frac{6-\alpha}{7-\alpha} \delta\left(\frac{3}{2} \frac{\partial^{2} u_{1}}{\partial x^{2}}\left(\mathbf{x}_{0}\right)+\frac{\partial^{2} u_{2}}{\partial x \partial y}\left(\mathbf{x}_{0}\right)\right)+\ldots  \tag{B.30a}\\
\tau_{2}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right) & =\sigma_{22}\left(\mathbf{x}_{0}\right)+\frac{2 E}{5 \pi} \frac{6-\alpha}{7-\alpha} \delta\left(\frac{\partial^{2} u_{1}}{\partial x \partial y}\left(\mathbf{x}_{0}\right)+\frac{\partial^{2} u_{2}}{\partial x^{2}}\left(\mathbf{x}_{0}\right)\right)+\ldots \\
& =\mathrm{t}_{2}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)+\frac{2 E}{5 \pi} \frac{6-\alpha}{7-\alpha} \delta\left(\frac{\partial^{2} u_{1}}{\partial x \partial y}\left(\mathbf{x}_{0}\right)+\frac{\partial^{2} u_{2}}{\partial x^{2}}\left(\mathbf{x}_{0}\right)\right)+\ldots \tag{B.30b}
\end{align*}
$$

In the limit as $\delta \rightarrow 0$, we have

$$
\begin{equation*}
\boldsymbol{\tau}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)=\mathbf{t}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)+\mathcal{O}(\delta) \tag{B.31}
\end{equation*}
$$

i.e., the nonlocal traction converges to the local traction at a rate of $\mathcal{O}(\delta)$.

Vertical Interface. The treatment of the case with a vertical interface is identical to that of the horizontal interface, except that the limits of integration change. For the case of a vertical interface with normal $\mathbf{n}=\mathbf{e}_{1}$, we can compute (B.18) using polar coordinates, $\xi_{1}=r \cos (\theta)$ and $\xi_{2}=r \sin (\theta)$, as follows (see Fig. B.26b for the limits of
integration):

$$
\begin{align*}
\tau_{i}\left(\mathbf{x}_{0}, \mathbf{e}_{1}\right)= & \int_{x_{0}-\delta}^{x_{0}} \int_{\mathcal{H}_{\mathbf{x}} \cap \Omega_{B}} \lambda(\|\xi\|) \xi_{i} \xi_{j}\left(\frac{\partial u_{j}}{\partial x_{k}}\left(x, y_{0}\right) \xi_{k}+\frac{1}{2} \frac{\partial^{2} u_{j}}{\partial x_{k} \partial x_{l}}\left(x, y_{0}\right) \xi_{k} \xi_{l}+\ldots\right) d \mathbf{x}^{\prime} d x \\
= & \int_{x_{0}-\delta}^{x_{0}}\left[\int_{\mathcal{H}_{\mathbf{x}} \cap \Omega_{B}} \lambda(\|\xi\|) \xi_{i} \xi_{j} \xi_{k} d \mathbf{x}^{\prime}\right] \frac{\partial u_{j}}{\partial x_{k}}\left(x, y_{0}\right) d x \\
& +\frac{1}{2} \int_{x_{0}-\delta}^{x_{0}}\left[\int_{\mathcal{H}_{\mathbf{x}} \cap \Omega_{B}} \lambda(\|\xi\|) \xi_{i} \xi_{j} \xi_{k} \xi_{l} d \mathbf{x}^{\prime}\right] \frac{\partial^{2} u_{j}}{\partial x_{k} \partial x_{l}}\left(x, y_{0}\right) d x+\ldots \\
= & \int_{x_{0}-\delta}^{x_{0}}\left[\int_{x_{0}-x}^{\delta} \int_{-\cos ^{-1}\left(\frac{x_{0}-x}{r}\right)}^{\cos ^{-1}\left(\frac{x_{0}-x}{r}\right)} \lambda(r)(r \cos (\theta))^{a_{1}}(r \sin (\theta))^{3-a_{1}} r d \theta d r\right] \frac{\partial u_{j}}{\partial x_{k}}\left(x, y_{0}\right) d x \\
& +\frac{1}{2} \int_{x_{0}-\delta}^{x_{0}}\left[\int_{x_{0}-x}^{\delta} \int_{-\cos ^{-1}\left(\frac{x_{0}-x}{r}\right)}^{\cos ^{-1}\left(\frac{x_{0}-x}{r}\right)} \lambda(r)(r \cos (\theta))^{a_{2}}(r \sin (\theta))^{4-a_{2}} r d \theta d r\right] \frac{\partial^{2} u_{j}}{\partial x_{k} \partial x_{l}}\left(x, y_{0}\right) d x+\ldots, \tag{B.32}
\end{align*}
$$

where, in this case, $a_{1}$ is the number of 1 s in $\{i, j, k\}$ in the coefficients of the first-order derivatives and $a_{2}$ is the number of 1 s in $\{i, j, k, l\}$ in the coefficients of the second-order derivatives. Employing a similar procedure to the one used from (B.19) to (B.28), we obtain

$$
\begin{align*}
& \tau_{1}\left(\mathbf{x}_{0}, \mathbf{e}_{1}\right)=\frac{9 E}{8}\left[\frac{\partial u_{1}}{\partial x}\left(\mathbf{x}_{0}\right)+\frac{1}{3} \frac{\partial u_{2}}{\partial y}\left(\mathbf{x}_{0}\right)+\frac{16}{45 \pi} \frac{6-\alpha}{7-\alpha} \delta\left(\frac{\partial^{2} u_{1}}{\partial y^{2}}\left(\mathbf{x}_{0}\right)+\frac{\partial^{2} u_{2}}{\partial x \partial y}\left(\mathbf{x}_{0}\right)\right)+\ldots\right],  \tag{B.33a}\\
& \tau_{2}\left(\mathbf{x}_{0}, \mathbf{e}_{1}\right)=\frac{9 E}{8}\left[\frac{1}{3}\left(\frac{\partial u_{1}}{\partial y}\left(\mathbf{x}_{0}\right)+\frac{\partial u_{2}}{\partial x}\left(\mathbf{x}_{0}\right)\right)+\frac{16}{45 \pi} \frac{6-\alpha}{7-\alpha} \delta\left(\frac{\partial^{2} u_{1}}{\partial x \partial y}\left(\mathbf{x}_{0}\right)+\frac{3}{2} \frac{\partial^{2} u_{2}}{\partial y^{2}}\left(\mathbf{x}_{0}\right)\right)+\ldots\right], \tag{B.33b}
\end{align*}
$$

where the dots indicate higher-order derivative terms. Employing (B.29), we can express (B.33a) and (B.33b), respectively, as (recall (42))

$$
\begin{align*}
\tau_{1}\left(\mathbf{x}_{0}, \mathbf{e}_{1}\right) & =\sigma_{11}\left(\mathbf{x}_{0}\right)+\frac{2 E}{5 \pi} \frac{6-\alpha}{7-\alpha} \delta\left(\frac{\partial^{2} u_{1}}{\partial y^{2}}\left(\mathbf{x}_{0}\right)+\frac{\partial^{2} u_{2}}{\partial x \partial y}\left(\mathbf{x}_{0}\right)\right)+\ldots \\
& =\mathrm{t}_{1}\left(\mathbf{x}_{0}, \mathbf{e}_{1}\right)+\frac{2 E}{5 \pi} \frac{6-\alpha}{7-\alpha} \delta\left(\frac{\partial^{2} u_{1}}{\partial y^{2}}\left(\mathbf{x}_{0}\right)+\frac{\partial^{2} u_{2}}{\partial x \partial y}\left(\mathbf{x}_{0}\right)\right)+\ldots  \tag{B.34a}\\
\tau_{2}\left(\mathbf{x}_{0}, \mathbf{e}_{1}\right) & =\sigma_{12}\left(\mathbf{x}_{0}\right)+\frac{2 E}{5 \pi} \frac{6-\alpha}{7-\alpha} \delta\left(\frac{\partial^{2} u_{1}}{\partial x \partial y}\left(\mathbf{x}_{0}\right)+\frac{3}{2} \frac{\partial^{2} u_{2}}{\partial y^{2}}\left(\mathbf{x}_{0}\right)\right)+\ldots \\
& =\mathrm{t}_{2}\left(\mathbf{x}_{0}, \mathbf{e}_{1}\right)+\frac{2 E}{5 \pi} \frac{6-\alpha}{7-\alpha} \delta\left(\frac{\partial^{2} u_{1}}{\partial x \partial y}\left(\mathbf{x}_{0}\right)+\frac{3}{2} \frac{\partial^{2} u_{2}}{\partial y^{2}}\left(\mathbf{x}_{0}\right)\right)+\ldots \tag{B.34b}
\end{align*}
$$



Figure B.28: Interface $\Gamma$ between the PD and CCM regions for the nonlocal traction computation in a twodimensional CCM-PD coupled model. Blue diamonds are FEM nodes and green circles are PD nodes; only PD nodes located along the line $\mathcal{L}(c f .(69))$ are indicated. The dashed gray line indicates the interface $\Gamma$. The point $\mathbf{x}_{0}$ where $\mathcal{L}$ intersects $\Gamma$ is the point where the nonlocal traction is computed. A uniform discretization with grid spacing $\Delta x=\Delta y=0.25$ is employed, and the PD horizon is taken as $\delta=0.75$. For illustration, the red dotted curve represents the part of the boundary of the neigborhood of the PD node closest to $\Gamma$ located in the CCM region; for clarity, the dotted black lines represent the radius of that neighborhood.

In the limit as $\delta \rightarrow 0$, we have

$$
\begin{equation*}
\boldsymbol{\tau}\left(\mathbf{x}_{0}, \mathbf{e}_{1}\right)=\mathbf{t}\left(\mathbf{x}_{0}, \mathbf{e}_{1}\right)+\mathcal{O}(\delta), \tag{B.35}
\end{equation*}
$$

i.e., the nonlocal traction converges to the local traction at a rate of $\mathcal{O}(\delta)$.

## B.2.1. Numerical examples for the nonlocal traction in two dimensions

We present some numerical examples to confirm the results in (B.30) and (B.34). To put these examples within the context of a CCM-PD coupled model, we consider a twodimensional system with an interface $\Gamma$ between a PD region, given by $\Omega_{A}$, and a CCM region, given by $\Omega_{B}$ (see Fig. B.26). We consider the case of a horizontal interface $\Gamma$ with normal $\mathbf{n}=\mathbf{e}_{2}$ (see Fig. B.26a) and the nonlocal traction at $\mathbf{x}_{0}=\left(x_{0}, y_{0}\right) \in \Gamma$ in the bulk of the body. To numerically compute the nonlocal traction in (B.16), we employ a uniform discretization with grid spacing $\Delta x=\Delta y$ and define a set of $N_{\text {PD }}$ PD nodes with positions given by

$$
\begin{equation*}
\mathcal{S}_{\mathrm{PD}}=\left\{\left(x_{0}, y_{0}-\delta+\frac{\Delta y}{2}\right), \ldots,\left(x_{0}, y_{0}-\frac{\Delta y}{2}\right)\right\} \tag{B.36}
\end{equation*}
$$

and a set of $N_{\text {FEM }}$ FEM nodes with positions given by

$$
\begin{equation*}
S_{\mathrm{FEM}}=\left\{x_{0}-\delta, \ldots, x_{0}+\delta\right\} \times\left\{y_{0}+\frac{\Delta y}{2}, \ldots, y_{0}+\delta-\frac{\Delta y}{2}\right\}, \tag{B.37}
\end{equation*}
$$

which is built as a Cartesian product. To relate the examples in this section to the numerical results in Section 4.2, we choose $\delta=0.75$ and $\Delta x=0.25$ (i.e., $m=\delta / \Delta x=$ 3 ), and we consider the case of an interface vertically located at $y_{0}=24.125$ (i.e., $y_{0}=$ $24+\frac{\Delta y}{2}$ ); an illustration is presented in Fig. B.28a. We compute the nonlocal traction at $\mathbf{x}_{0}=\left(x_{0}, y_{0}\right)$ with $x_{0}=11.5$. We denote by $\mathbf{x}_{i}^{\mathrm{PD}} \in \mathcal{S}_{\mathrm{PD}}, i=1, \ldots, N_{\mathrm{PD}}$, and $\mathbf{x}_{j}^{\mathrm{FEM}} \in$ $S_{\mathrm{FEM}}, j=1, \ldots, N_{\mathrm{FEM}}$, the reference positions of the PD and FEM nodes, respectively. Define the displacements of the PD and FEM nodes, respectively, by $\mathbf{u}_{i}^{\mathrm{PD}}:=\mathbf{u}\left(\mathbf{x}_{i}^{\mathrm{PD}}\right)$, $Y_{B}=15$. For the quadratic displacement field, the values of the coefficients are $q=15$, $X_{Q}=11.5, Y_{Q}=23$, and $R=2.25$. As a comparison, we analytically calculate the local traction using (42) with (B.29). The results are reported in Table B.14. In addition to reporting the values for the $x$ - and $y$-components of the nonlocal and local tractions, we present the error of the nonlocal traction computation, for each component, given by the absolute value of the difference between the numerical and analytical values. To study the improvement in accuracy gained by using an increased value of $m$, we also re-
port the calculations for $m=8$ and the same $\delta$. Both values of $m$ reported in Table B. 14 are used in Section 4.2. We observe that the values of the numerical nonlocal traction obtained by (B.38), which is a discretization of (B.16), approximately recover the values given by the analytical nonlocal traction in (B.30) for both cases, linear and quadratic displacements. For the linear displacement, the values of the analytical nonlocal and local tractions coincide, as expected from (B.30). We note that the $x$-component of the tractions is zero, while the $y$-component of the tractions is non-zero. We performed a similar study by imposing instead the linear and quadratic distributions described in Table B. 13 on the displacement component $u_{1}$ over a vertical interface (see Fig. B.28b). In this case, similar results were obtained, where instead the $x$-component of the tractions is non-zero, while the $y$-component of the tractions is zero. These findings confirm the results in (B.30) and (B.34).
${ }_{855}$ Remark B.2. In the numerical studies in Section 4.2, we investigate the overall equilibrium in two-dimensional CCM-PD coupled systems, which requires the balance between the local and nonlocal tractions at the coupling interface. The results in (B.30) and (B.34) imply that, for linear deformations, the nonlocal and local tractions should be balanced at the coupling interface. However, the results in Table B. 14 show that the numerical computation of the nonlocal traction only approximately recovers the analytical nonlocal traction. To allow a numerical verification of the force equilibrium for linear deformations in Section 4.2, we introduce a correction factor given by

$$
\begin{equation*}
\gamma:=\frac{\tau_{2}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)}{\tau_{2}^{\text {num }}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)}, \tag{B.39}
\end{equation*}
$$

which is computed with the values reported in Table B. 14 for the linear displacement case. For $m=3, \gamma=0.9784710341$, whereas for $m=8, \gamma=0.9974762599$.

Table B.13: Displacement fields for the nonlocal traction computation in two dimensions.

| Displacement type | Displacement field equation |
| :---: | :---: |
| Linear | $u_{1}(x, y)=0$ |
| $u_{2}(x, y)=d \frac{y-Y_{1}}{Y_{B}-Y_{1}}$ |  |
| Quadratic | $u_{1}(x, y)=0$ |
|  | $u_{2}(x, y)=\frac{-\left(x-X_{Q}\right)^{2}-\left(y-Y_{Q}\right)^{2}+R^{2}}{q^{2}}$ |

Table B.14: Comparison between numerical and analytical tractions in two dimensions.

| Displacement type | Nonlocal traction ( $x$-component) |  |  | Local traction ( $x$-component) |
| :---: | :---: | :---: | :---: | :---: |
|  | $\tau_{1}^{\mathrm{num}}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)$ | $\tau_{1}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)$ | $\left\|\tau_{1}^{\text {num }}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)-\tau_{1}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)\right\|$ | $\mathrm{t}_{1}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)$ |
| Linear, $\delta=0.75, m=3$ | $-5.83 \times 10^{-19}$ | 0.00 | $5.83 \times 10^{-19}$ | 0.00 |
| Linear, $\delta=0.75, m=8$ | $-4.86 \times 10^{-18}$ | 0.00 | $4.86 \times 10^{-18}$ | 0.00 |
| Quadratic, $\delta=0.75, m=3$ | $2.62 \times 10^{-19}$ | 0.00 | $2.62 \times 10^{-19}$ | 0.00 |
| Quadratic, $\delta=0.75, m=8$ | $-4.40 \times 10^{-17}$ | 0.00 | $4.40 \times 10^{-17}$ | 0.00 |
| Displacement type |  | Nonlocal ( $y$-comp | raction nent) | Local traction ( $y$-component) |
|  | $\tau_{2}^{\text {num }}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)$ | $\tau_{2}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)$ | $\left\|\tau_{2}^{\text {num }}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)-\tau_{2}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)\right\|$ | $\mathrm{t}_{2}\left(\mathbf{x}_{0}, \mathbf{e}_{2}\right)$ |
| Linear, $\delta=0.75, m=3$ | $2.56 \times 10^{-01}$ | $2.50 \times 10^{-01}$ | $5.50 \times 10^{-03}$ | $2.50 \times 10^{-01}$ |
| Linear, $\delta=0.75, m=8$ | $2.51 \times 10^{-01}$ | $2.50 \times 10^{-01}$ | $6.33 \times 10^{-04}$ | $2.50 \times 10^{-01}$ |
| Quadratic, $\delta=0.75, m=3$ | $-1.22 \times 10^{-02}$ | $-1.19 \times 10^{-02}$ | $2.69 \times 10^{-04}$ | $-1.12 \times 10^{-02}$ |
| Quadratic, $\delta=0.75, m=8$ | $-1.19 \times 10^{-02}$ | $-1.19 \times 10^{-02}$ | $3.18 \times 10^{-05}$ | $-1.12 \times 10^{-02}$ |

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[^1]:    ${ }^{2}$ In classical linear elasticity, the stress-strain relation for isotropic materials under plane stress is given by:

    $$
    \left[\begin{array}{c}
    \sigma_{11}  \tag{A.18}\\
    \sigma_{22} \\
    \sigma_{12}
    \end{array}\right]=\frac{E}{1-v^{2}}\left[\begin{array}{ccc}
    1 & v & 0 \\
    v & 1 & 0 \\
    0 & 0 & 1-v
    \end{array}\right]\left[\begin{array}{l}
    \varepsilon_{11} \\
    \varepsilon_{22} \\
    \varepsilon_{12}
    \end{array}\right]
    $$

