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BLACK HOLE MICROSTATES AND THE INFORMATION PARADOX

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DECLARATION

I, *Alessandro Bombini*, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.
I confirm that this thesis has not been previously submitted for the award of a degree by this or any other university.

DETAILS OF COLLABORATION AND PUBLICATIONS:

In this thesis are reported the results of research carried out with my supervisor Prof. Stefano Giusto which was published in [1, 2], with the latter paper written in collaboration with Andrea Galliani, Emanuele Moscato, and Rodolfo Russo. It contains also papers published in collaboration with Elaheh Bakhshaei [3] and Andrea Galliani [4], as well as some unpublished material.

During the graduate program, I have also published other papers that are not reported in this thesis: [5], in collaboration with Stefano Giusto and Rodolfo Russo, [6] with Riccardo Antonelli and Ivano Basile, and [7] with Lorenzo Papini.

CONTENTS

Part I

BLACK HOLES AND THEIR MICROSTATES IN STRING THEORY

1 INTRODUCTION

One of the most intriguing issues that have been puzzling theoretical physicists for almost 50 years now is the so-called *Black Hole information paradox*. Since the seminal papers by Bekenstein [8–10], where he discussed the necessity of associating an entropy to black holes proportional to the area of their event horizon, and by Bardeen, Carter and Hawking [11], where the four law of black hole mechanics where proven, a mysterious relation between black holes and thermodynamics has emerged. It was established that black holes have a temperature, which is inversely proportional to its mass, and an entropy, which is proportional to the square of its mass in four dimensions. But the revolutionary paper by Hawking [12] was needed, both to fix the precise proportionality coefficients, e.g. for a Schwarzschild black hole in four dimensions,

$$T = \frac{\hbar c^3}{8\pi G_N k_B}\frac{1}{M}\,, \quad S = \frac{c^3}{\hbar G_N}\frac{A_H}{4}\,, \tag{1.1}$$

and to highlight their physical meaning, understanding that indeed a black hole behaves as a black-body emitter of particles; in fact, Hawking proved, via a semiclassical computation, that a quantum field on a classical curved background described by a black hole geometry forces the black hole to emit a thermal radiation, whose temperature is indeed the temperature of the black hole, called *Hawking temperature*. What remained unclear - and, in some fashion, it is still not completely clear - was how to compute the entropy of the black hole by counting its microstates via the Boltzmann law [13–15]. Moreover, it was immediately realised that the presence of a black hole undermines the foundation of the quantum theory: indeed, due to its natural tendency to emit a thermal radiation, the black hole seems to be a peculiar thermodynamic object which transforms the pure state of the matter that creates it into a thermal state of pure radiation, after its full evaporation. Since this evolution cannot be described by a unitary operator, it violates one of the principles of quantum mechanics. This is known as *black hole information paradox* [16–19].

A sharp formulation of this information paradox that relies on the contrast between a statistical-mechanical description of the black hole and the (exactly) thermal nature of Hawking radiation, was provided by Page [20], further sharpened adding concepts of quantum information by Mathur [21] and subsequently rephrased by Almheiri, Marolf, Polchinski and Sully [22]. We first start with a brief description of the former; suppose we have an ordinary thermodynamic system, whose Hilbert space we indicate with $\mathcal{H}$, and we want to describe it using microcanonical ensemble. We now find some energy ΔE that is small w.r.t. typical energy scales, but large enough to have a large number of microstates in the interval $(E, E + \Delta E)$. Calling $\mathcal{H}(E)$ the subspace of the Hilbert space spanned by energy eigenstates whose energy eigenvalues are in $(E, E + \Delta E)$, we write $\mathcal{H} = \oplus_E \mathcal{H}(E)$, where the sum runs over energies $E = 0, \Delta E, 2\Delta E, \ldots, N\Delta E, \ldots$. Suppose now the system begins at microcanonical equilibrium at energy E_0 with state contained in $\mathcal{H}(E_0)$, and then cools down emitting thermal radiation, i.e. emitting quanta of radiation in highly mixed states. The original state of the system is pure and,

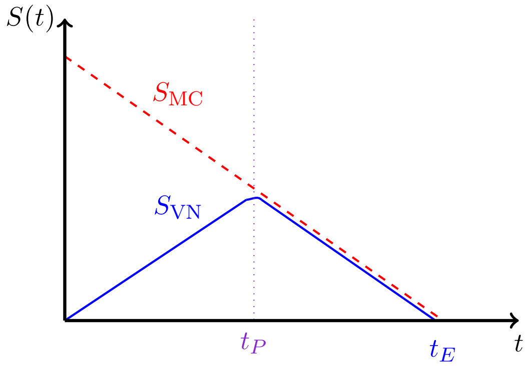


Figure 1.1: The Page curve; the time t_P is the Page time, while t_E is the evaporation time.

since the evolution is unitary, the total state "system+radiation" is pure and contained in $\mathcal{H}(E_0)$; but if the radiation is thermal, each emitted quanta will be in a mixed state and thus entangled with some other system for the total state to be pure; thus each emitted quanta has to be entangled with the system. More concretely, it means that, if the system cools from E_0 to $E < E_0$ with Von Neumann entropy for the radiation equal to S, then also the Von Neumann entropy of the system $S_{\text{VN}}(E_0|E)$ must be equal to S. But this means that if the system has energy $E \in (E(t), E(t) + \Delta E)$ at time t, its state is contained in $\mathcal{H}(E(t))$ and so must have a Von Neumann entropy that is less than $\log \dim \mathcal{H}(E(t))$, that is the microcanonical entropy $S_{\text{MC}}(E)$. In other words,

$$S_{\text{VN}}(E_0|E(t)) \leq S_{\text{MC}}(E(t)), \tag{1.2}$$

with the latter that decreases as the system cools down; it will come a time, called *Page Time*, when the inequality is saturated, and after that the radiation can no longer be exactly thermal, meaning that it has to be entangled with the early time radiation. This process is illustrated in fig. 1.1 and the characteristic time dependence of the entropy is known as *Page curve*.

The Page time for, let us say, the Schwarzschild black hole is approximately half of its total evaporation time when it has approximately radiated away half of its mass. After this Page time, it is not possible for the black hole's radiation to be thermal; instead, it should be maximally entangled with the early-time radiation.

The problem with Hawking radiation, i.e. the core of the information loss in this formulation is that, according to quantum-field theoretic calculations, it is *exactly* thermal, showing no entanglement between early-time and late-time quanta. This is a rude clash between the predictions of QFT (even if carefully refined) and the predictions of black hole statistical mechanics that occurs way long before complete evaporation, when the horizon scale is still macroscopic, and then way before any Planck-size effect may come in to save the day [15, 19, 21].

This paradox can be made even sharper by introducing some quantum information concepts, as first noticed by Mathur [21], and later emphasised by Almheiri, Marolf, Polchiski and Sully [22]. Let us think about an old black hole that has emitted a macroscopic part of its mass through Hawking radiation; we may consider Hawking radiation as a particle-antiparticle pair excitation of the vacuum, where one of the two (let us dub it A) falls back into the black hole, while the other (B) escapes to infinity. In the semiclassical approxima-

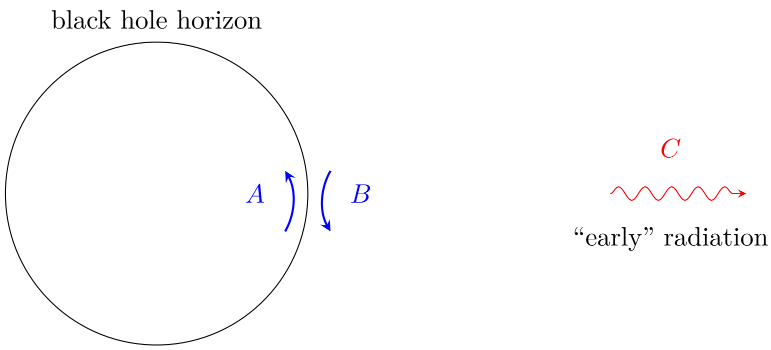


Figure 1.2: A pictorial representation of the Hawking radiation as a particle-antiparticle pair excitation of the vacuum. Hawking radiation can be interpreted as a pair creation; the particle A fall back into the black hole, while its antiparticle B escape to infinity. C are the early times emitted quanta.

tion of Hawking, where the structure of the horizon is the one predicted by general relativity, these two A and B particles are maximally entangled. If we denote with C the early emitted quanta, we see that the previous argument implies that, via the so-called "monogamy of entanglement" [23], A and C (or B and C) cannot be entangled; this means that "late" and "early" radiation are not entangled, and thus cannot form a pure state. This has a dramatic implication for unitarity; it means that the pure state that formed the black hole at the beginning will evolve into a thermal radiation that is entangled only with the hole, that will eventually dissolve, leaving behind quanta of radiation that are in a mixed state but entangled with anything left.

As far as we know, there is no way to avoid these problems in the framework of usual semiclassical general relativity (GR); the impossibility of describing and enumerating all the possible black hole microstates in GR is made sharper due to the existence of uniqueness theorems - or "no-hair theorems" [24–26] - by which a classical black hole can only be described by three classical charges: its mass, its angular momentum and its electric(magnetic) charge. It is then only inside a framework of "quantum gravity" that it is possible to disentangle the paradox[1] and compute the entropy of a black hole by a microstate counting.

One of the biggest achievements of string theory - that is a framework for a "theory of everything" in which gravity is quantized - was the computation of the black hole entropy by microstate counting [33]. In [33], Strominger and Vafa built an extremal black hole as a solution of a low energy limit of type II string theory with a non-vanishing area, and then computed its entropy by counting its stringy microstates, indeed perfectly reproducing the Bekenstein-Hawking entropy in the semiclassical limit, and predicting also the quantum corrections to it. The Strominger and Vafa result was a consequence of the second superstring revolution happened after *Strings '95* and the seminal paper of Polchinski on the Dp-branes [34]; in fact, in order to achieve their results it was crucial both using the established set of dualities among different

1 This point of view is not entirely accepted in the community; there are in fact few other proposal, that we will report briefly here for sake of completeness. We may have unitarity loss [27, 28], remnants (as well as baby universes [29]) and non-local effects [30, 31] (or like the ER=EPR proposal [32]).

string theories as well as engineering a well-defined brane scenario, the so-called D0D4 system in type IIA or D1D5 system in type IIB.

Another consequence of the second superstring revolution was the formulation of the most fruitful conjecture in modern contemporary physics, the so-called *AdS/CFT conjecture* [35–37]. This conjecture states that the Hilbert space of type IIB string theory in $\mathrm{AdS}_5 \times \mathbb{S}^5$ is equal to the Hilbert space of $\mathscr{N} = 4$ Super-Yang-Mills (SYM) living on $\mathbb{R}^{1,4}$, which can be regarded as the boundary of AdS_5. It relies, in its original formulation, on the near-horizon limit of a stack of N D3-branes whose near-horizon geometry is $\mathrm{AdS}_5 \times \mathbb{S}^5$. On one side, we know that the closed string spectrum contains the graviton, so we may take a limit where stringy effect are suppressed reducing us to the supergravity regime, and where the string coupling g_s is small. In this framework, the limit we are looking for is

$$R^2 \sim (g_s N)^{1/2} \alpha' \gg \alpha' \,, \quad g_s \ll 1 \Rightarrow \lambda \equiv g_s N \gg 1 \,, \tag{1.3}$$

where R is the radius of both AdS_5 and $\mathbb{S}^5$ and where λ is the 't Hooft-coupling, it allows to describe the semiclassical limit of a quantum gravity theory. On the other side of the duality, thanks to the open-closed string duality, we can discuss the non-abelian $SU(N)$ theory of the excitations of the open strings attached to the stack of D3-branes, that, in the limit above, is the $\mathscr{N} = 4$ SYM.

This conjecture was generalised (see [38–41] for a review) to different dimensions and context; we will focus on a specific $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence that arises also when studying the Strominger-Vafa black hole [42]. Also, AdS/CFT allows for a useful rephrasing of the black hole information paradox [43–46]; we will discuss its precise formulation later on in the introduction.

Even if Strominger and Vafa counted the stringy microstates of an extremal black hole in the D-brane regime, the question of what these microstates are and what do they look like in the regime where semi-classical supergravity applies remained open. Later, Mathur [47–52] proposed how to realise them in a string-theoretical framework. He introduced the *Fuzzball proposal*, that states that the black hole solution is an effective coarse-grained description that emerges from averaging over horizonless, non-singular microstates with a nontrivial structure whose size could be macroscopically large, differing from the naive black hole solution up to the horizon scale, well above the Planck scale. The geometries of a fuzzball from the five-dimensional point of view can be schematically see as:

- for large radius value, there is the *asymptotically flat regime*; this region looks the same for both the microstate and for the naive geometry (i.e. the black hole) and it is the Minkowski five-dimensional spacetime;
- as r decreases, we encounter a region determined by the global mass and charges of the solution, often dubbed *neck*, in which the functions describing the geometry do not differ between the microstate and the naive geometry, so the two are indistinguishable;
- as we approach the would-be-horizon, we enter in a region called *throat* were the two geometries approaches $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$;
- going further, the microstate geometry starts to differ from the black hole one and depends strongly on the precise microstate we are considering. Since microstates are smooth and horizonless, we don't encounter any coordinate nor curvature singularity, and at $r = 0$ the geometry ends smoothly in a *cap*, whose specific shape is determined by which

microstate we are considering. The black hole and the microstate start to differ just above where the horizon would be [50].

Generic fuzzballs are not always described by semi-classical supergravity solutions, since they can be arbitrarily quantum and arbitrarily strongly curved [53]. All fuzzballs are dual to so-called "heavy" states of the dual CFT, i.e. states whose conformal dimension scales as the central charge c in the large c-limit; this is necessary since in this case the mass of the excitations is big in Planck units, allowing for a non-trivial back-reaction. Up to now, not all the gravity dual for all the heavy states of the dual CFT theory are known. In this thesis we will mainly discuss the largest known family of the so-called *microstate geometries*, dubbed *superstrata*, that are regular and horizonless geometries at the supergravity level, with the same conserved charges as the black hole, whose dual CFT states can be built at the free orbifold point by acting independently on each strand with an element of the global superalgebra.

The fuzzball proposal resolves *de facto* the information paradox: the Hawking radiation is then non-thermal since it is emitted by a precise microstate and unitarity is not lost. The precise mechanism of how it is realised, even in the simpler context of AdS/CFT, is still missing, and one of the goals of this thesis is to make a step further in that direction.

The structure of the thesis is the following: in part i, we review some relevant knowledge; in particular, in chap. 1 we briefly review the Strominger-Vafa microstate counting (sec. 1.1) and the formulation of black hole information paradox in the context of AdS/CFT (sec. 1.2), while in chap. 2 we discuss the two side of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ duality relevant for the fuzzball proposal. In part ii we build new microstate solutions: in chap. 3 we build new three-charge $\frac{1}{8}$-BPS superstrata with both external and internal excitations and that are then described by two free parameters, and we furnish the holographic interpretation in terms of CFT states; in chap. 4 we prove the existence of non-supersymmetric superdescendants and discuss their behaviour. In part iii we discuss the holography of $\frac{1}{4}$- and $\frac{1}{8}$-BPS states, computing four-point functions in the so-called *Heavy-Heavy-Light-Light* (HHLL) limit; in chap. 5 we show explicitly how correlators in microstate geometries show no information-loss, even at the supergravity limit, and how the standard black hole result is related to the microstate one; in chap. 6 we compute HHLL correlators from $\frac{1}{8}$-BPS states, check them using Ward identities that relate them to each other and to correlators involving $\frac{1}{4}$-BPS states, and then we reconstruct all-light (LLLL) correlators involving for the first time $\frac{1}{8}$-BPS operators.

1 THE STROMINGER-VAFA BLACK HOLE

We will now briefly review the Strominger and Vafa argument for the enumeration of stringy microstate of a certain supersymmetric black hole; we begin by constructing the five-dimensional extremal black hole, starting from an M2 system in eleven-dimensional supergravity and then, using compactifications as well as $T-$ and $S-$dualities, we land onto the Strominger-Vafa solution (in the type IIB duality frame), computing its charges and its Bekenstein-Hawking formula; after that, we discuss the counting of microstates.

1.1 *Building the solution*

1.1.1 *The M2 branes and Multi-branes solution in M-theory*

For the notations, we refer to [42]. We recall that the bosonic part of the action of 11-dimensional supergravity is

$$S_{11} \supseteq \frac{1}{2\kappa_{11}} \int \mathrm{d}^{11}x \left[\sqrt{-G_{11}}\left(R_{11} - \frac{1}{48}F_4^2\right) - \frac{1}{6}\, A_3 \wedge \mathrm{d}A_3 \wedge \mathrm{d}A_3\right], \tag{1.4}$$

Where $F_4 \equiv \mathrm{d}A_3$. From here we see that we can have two extended solitonic objects, electo-magnetic dual to each other, that are M2 and M5 branes, that couple respectively electrically and magnetically with A_3.

We want to construct solutions with multiple orthogonal stacks of M2-branes; we start building the single stack of M2 solution as an illustrative example, using the fact that this kind of solution will have a symmetry of $SO(2,1) \times SO(8)$, plus a translation symmetry on the directions of the brane. So it easy to guess an appropriate ansatz

$$\mathrm{d}s_{11}^2 = e^{2f_1(r)}\mathrm{d}x^\mu \mathrm{d}x_\mu + e^{2f_2(r)}\mathrm{d}x^m \mathrm{d}x_m\,, \tag{1.5a}$$

$$A_3 = e^{f_3(r)}\, \mathrm{d}t \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2\,, \tag{1.5b}$$

where we called $x^\mu = (t, x_1, x_2)$ the direction along which the M2 branes are extended while x^m, $m = 3, \ldots, 10$ are the remaining ones and $r^2 \equiv x^m x_m$.

We now want that our stack of M2 branes leaves some subset of the susy unbroken. This implies that the variation of the gravitino must vanish [54]

$$\delta_\epsilon \psi_M = D_M \epsilon + \frac{1}{288}\left(\Gamma_M{}^{NPQR} - 8\Gamma^{PQR}\right) F_{NPQR}\, \epsilon = 0\,, \tag{1.6}$$

where, as usual,

$$D_M \epsilon = \left(\partial_M + \frac{1}{4}\,\omega^{BC}{}_M\, \Gamma_{BC}\right)\epsilon\,. \tag{1.7}$$

Looking at the $M = \mu$ directions we can fix

$$\partial_\mu \epsilon = 0\,, \quad f_3 = 3f_1\,, \quad e_P^{\hat{0}}\, e_Q^{\hat{1}}\, e_R^{\hat{2}}\, \Gamma^{PQR}\epsilon = \epsilon\,, \tag{1.8}$$

where the hat denotes the local Lorentz indices. Looking at the transverse $x^M \supset x^m$ directions on the other hand gives

$$f_1 = -2f_2\,, \tag{1.9}$$

so that, calling $H(r) = e^{-f_3(r)}$,

$$f_1 = -\frac{1}{3}\,\log H\,, \quad f_2 = +\frac{1}{6}\,\log H\,, \quad f_3 = -\log H\,. \tag{1.10}$$

Now the solution is parametrized by the only unknown function H that can be fixed by the equations of motion (EoM) for A_3, that are simply

$$\partial_M \left(\sqrt{-g}\, F^{MNPQ} \right) = 0, \tag{1.11}$$

since

$$F_4 = -f_3'\, e^{f_3(r)}\, \mathrm{d}t \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}r \quad \Rightarrow \quad F_4 \wedge F_4 = 0\,. \tag{1.12}$$

The equation (1.11) reduces to

$$\partial_m \partial^m\, e^{-f_3} = 0 \quad \Rightarrow \partial_m \partial^m H = 0\,, \tag{1.13}$$

whose solution is simply

$$H(r) = 1 + \frac{k}{r^6}\,, \tag{1.14}$$

where 1 and k are integration constants. So finally we have

$$\mathrm{d}s^2_{11} = H^{-2/3}\, \mathrm{d}x^\mu \mathrm{d}x_\mu + H^{1/3}\, \mathrm{d}x^m \mathrm{d}x_m\,, \tag{1.15a}$$

$$A_3 = H^{-1}\, \mathrm{d}t \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2\,. \tag{1.15b}$$

THE BPS NATURE OF THE M2 BRANES: It is easy to see that the charges associated to time translation and gauge symmetry, i.e. mass M and charge q, are related - in the appropriate unit system - as $M = q$. This is the signal that M2 branes are *BPS* objects. We can see it by looking at the susy algebra

$$V_2^{-1}\{Q_\alpha, Q_\beta\} = (C\Gamma^{\hat{0}})_{\alpha\beta}\, M + (C\Gamma^{\hat{1}\hat{2}})_{\alpha\beta}\, q\,, \tag{1.16}$$

where V_2 is the spatial volume of M2 and C is the charge conjugation matrix. We can now notice that, taking the trace and using eq. (1.8), we have

$$(1 - \Gamma^{\hat{0}\hat{1}\hat{2}})\epsilon = 0 \Rightarrow \eta^{\alpha\beta}\{Q_\alpha, Q_\beta\} = 0, \tag{1.17}$$

i.e. the BPS bound is saturated. It is then a $\frac{1}{2}$−BPS solution.

Notice that, in the near-horizon limit, i.e. when $1 \ll k/r^6$, we can write the eight-dimensional part of the metric as

$$H^{1/3}(\mathrm{d}r^2 + r^2 d\Omega_7^2) = k\frac{\mathrm{d}r^2}{r^2} + k\, \mathrm{d}\Omega_7^2\,, \tag{1.18}$$

so that

$$\mathrm{d}s^2_{11} = \left(-f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2 \mathrm{d}\Omega_2^2 \right) + k\, \mathrm{d}\Omega_7^2\,, \quad f^{-1}(r) = \frac{k}{r^2}\,, \tag{1.19}$$

i.e. the metric is asymptotically $\mathrm{AdS}_4 \times S^7$. This is not relevant to us, but it is important in AdS_4 holography.

MULTI-M2 SOLUTIONS: We can now search for a solution with multiple orthogonal stacks of M2 branes. The M2⊥M2 case has $SO(2) \times SO(2) \times SO(6)$ plus five translations symmetry; then a natural ansatz is

$$\mathrm{d}s^2_{11} = -e^{2f_1(r)}\mathrm{d}t^2 + e^{2f_2(r)}(\mathrm{d}x_1^2 + \mathrm{d}x_2^2) + e^{2f_3(r)}(\mathrm{d}x_3^2 + \mathrm{d}x_4^2) + e^{2f_4(r)}\mathrm{d}x_i \mathrm{d}x^i\,, \tag{1.20a}$$

$$A_3 = e^{f_5(r)}\, \mathrm{d}t \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 + e^{f_6(r)}\, \mathrm{d}t \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4\,. \tag{1.20b}$$

M-theory $\xrightarrow{R_{10}}$	IIA $\xrightarrow{T_{567}}$	IIB
M2 (8,9)	D2 (8,9)	D5 (5,6,7,8,9)
M2 (6,7)	D2 (6,7)	D1 (5)

Table 1.1: The compactification and $T-$duality procedure from M-theory to Type IIB.

Now, imposing again $\delta_\epsilon \psi_M = 0$ we fix all the f_i in terms of two harmonic functions H_1, H_2

$$H_i = 1 + \frac{k_i}{r^4}\,, \tag{1.21}$$

that satisfies the equations of motion for the system. The solution, which is $\frac{1}{4}$-BPS, is

$$\mathrm{d}s^2_{11} = (H_1 H_2)^{1/3}\left[-\frac{\mathrm{d}t^2}{H_1 H_2} + \frac{\mathrm{d}x_1^2 + \mathrm{d}x_2^2}{H_1} + \frac{\mathrm{d}x_3^2 + \mathrm{d}x_4^2}{H_2} + \mathrm{d}x_i \mathrm{d}x^i\right], \tag{1.22a}$$

$$A_3 = H_1^{-1}\,\mathrm{d}t \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 + H_2^{-1}\,\mathrm{d}t \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4\,. \tag{1.22b}$$

We can easily generalise to the M2⊥M2⊥M2 case, that will be useful for us. Adding another orthogonal stack of M2 branes to the previous case and following the same steps we get

$$\begin{aligned}\mathrm{d}s^2_{11} = (H_1 H_2 H_3)^{1/3}\Bigg[&-\frac{\mathrm{d}t^2}{H_1 H_2 H_3} \\ &+ \frac{\mathrm{d}x_1^2 + \mathrm{d}x_2^2}{H_1} + \frac{\mathrm{d}x_3^2 + \mathrm{d}x_4^2}{H_2} + \frac{\mathrm{d}x_5^2 + \mathrm{d}x_6^2}{H_3} + \mathrm{d}x_i \mathrm{d}x^i\Bigg],\end{aligned} \tag{1.23a}$$

$$A_3 = H_1^{-1}\,\mathrm{d}t \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 + H_2^{-1}\,\mathrm{d}t \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 + H_2^{-1}\,\mathrm{d}t \wedge \mathrm{d}x^5 \wedge \mathrm{d}x^6\,. \tag{1.23b}$$

1.1.2 *The IIB Black Hole from the M2⊥M2⊥M2 by dimensional reduction and dualities*

THE 6D BLACK STRING FROM M2⊥M2: We recall that, compactifying along the 11th dimension the M-theory solution we obtain a type IIA solution. We recall also that $T-$duality switches between IIA and IIB string theories. So now we will do the following transformations on M2⊥M2:

1. compactify along the 11th dimension;
2. perform a T_5 duality, followed by a T_6 duality, followed by a T_7 duality;

in order to obtain a type IIB solution, as showed in tab. 1.1.

The first step, i.e. the compactification on $\mathbb{S}^1$, simply amounts to the replacement of r^{-4} with r^{-3} in the harmonic functions, plus a suitable redefinition of the integration constants; also we need to rearrange the metric as

$$\begin{aligned}\mathrm{d}s^2_{11} &= e^{-2\phi/3}\,\mathrm{d}s^2_{10} + e^{4\phi/3}(\mathrm{d}x^{10} + C_\mu \mathrm{d}x^\mu)^2\,,\\ A_3 &= B_2 \wedge \mathrm{d}x^{10} + C_3\,,\end{aligned} \tag{1.24}$$

M-theory $\xrightarrow{R_{10}}$	IIA $\xrightarrow{T_{567}}$	IIB
M2 (8,9)	D2 (8,9)	D5 (5,6,7,8,9)
M2 (6,7)	D2 (6,7)	D1 (5)
M2 (5, 10)	NS1 (5)	P (5)

Table 1.2: The compactification and $T-$duality procedure from M-theory to Type IIB for the M2⊥M2⊥M2 case.

where $C_1 = C_\mu \mathrm{d}x^\mu$, C_3 and B_2 are the RR and Kalb-Ramond fields of type IIA. Then, performing the three $T-$dualities[2] $T_{567} = T_5T_6T_7$ and finally compactifying along $\mathbb{T}^4$ - with x^5 identified as living on a large S^1 - we get

$$\begin{aligned} \mathrm{d}s^2_{10} &= -\frac{1}{\sqrt{Z_1Z_2}}(\mathrm{d}t^2 - \mathrm{d}x_5^2) + \sqrt{Z_1Z_2}\,\mathrm{d}s_4^2 + \sqrt{\frac{Z_1}{Z_2}}\,\mathrm{d}\tilde{s}_4^2\,, \\ C^{(2)}_{05} &= -\frac{1}{2}(Z_1^{-1} - 1)\,, \quad F^{(3)}_{ijk} = \partial_{[i}C^{(2)}_{jk]} = \varepsilon_{ijkl}\partial_l Z_2\,, \\ e^\phi &= \sqrt{\frac{Z_1}{Z_2}}\,, \quad Z_1 \equiv 1 + \frac{Q_1}{r^2}\,, \quad Z_2 = 1 + \frac{Q_5}{r^2}\,, \end{aligned} \tag{1.25}$$

where $F^{(3)} = dC^{(2)}$, and where $\mathrm{d}\tilde{s}^2_4$ is the metric of the compact $\mathbb{T}^4$. It is easy to see that the spacetime symmetries are

$$\mathcal{M}_{\mathrm{D1D5}} = SO(1,1) \times SO(4)_E \times SO(4)_I\,, \tag{1.26}$$

where the subscript refers to the large $\mathbb{R}^4$ and to the compact $\mathbb{T}^4$ part of the metric. In fact the first are external directions, while the $SO(4)_I$ refers to the internal directions of the $\mathbb{T}^4$ that, when compactified, get broken. We can read the unbroken susy by starting from M2 theory and perform the transformations or studying directly the Killing spinor equation. In any case Type IIB is chiral and we have

$$\Gamma^{056789}\epsilon_L = \epsilon_R\,, \quad \Gamma^{05}\epsilon_R = \epsilon_L\,. \tag{1.27}$$

The first is related to the D5 brane, while the second to the D1 brane. The solution of those constraints is

$$\Gamma^{6789}\epsilon_L = \epsilon_L\,, \quad \epsilon_R = \Gamma^{05}\epsilon_L\,, \tag{1.28}$$

which means that, since $\epsilon_{L,R}$ has 16 d.o.f., the first equations kills 8 of them while the second fixes completely the ϵ_R. This means that we have $16 + 16 \to 8$, i.e. a $\frac{1}{4}-$BPS state. In the near-horizon limit 8 of those reemerge as unbroken. In fact it is not difficult to see that in the near-horizon limit the metric approaches $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$. This feature will play a major role in this thesis, and its relation with an holographic dual theory will be discussed deeply chap. 2.

1.1.3 *The Strominger-Vafa Black Hole from M2⊥M2⊥M2*

We now do the same procedure with the M2⊥M2⊥M2 case. In this case, from the IIB perspective, we have added a quantized momentum (in form of gravitational wave) to the $\mathbb{S}^1$ direction, obtaining a solution that is described by three charges Q_1, Q_5, Q_P, representing the number of D1, D5 and units of momentum along $\mathbb{S}^1$, respectively.

Explicitly, the solution is a generalisation of the D1D5 case above, and it is usually denoted by D1D5P solution,

2 For the notation on dualities, we refer to app. A.1.

$$
\begin{aligned}
\mathrm{d}s_{10}^2 &= -\frac{1}{\sqrt{Z_1 Z_2}}\left(\mathrm{d}u\mathrm{d}v - \frac{\mathcal{F}}{2}\,\mathrm{d}u^2\right) + \sqrt{Z_1 Z_2}\,\mathrm{d}s_4^2 + \sqrt{\frac{Z_1}{Z_2}}\,\mathrm{d}\tilde{s}_{\mathbb{T}^4}^2\,, \\
C_{05}^{(2)} &= -\frac{1}{2}(Z_1^{-1} - 1)\,, \quad F_{ijk}^{(3)} = \varepsilon_{ijkl}\partial_l Z_2\,, \quad e^{\phi} = \sqrt{\frac{Z_1}{Z_2}}\,, \\
Z_1 &\equiv 1 + \frac{Q_1}{r^2}\,, \quad Z_2 = 1 + \frac{Q_5}{r^2}\,, \quad \mathcal{F} = -\frac{2Q_P}{r^2}\,,
\end{aligned}
\tag{1.29}
$$

where we have introduced the null coordinates $u = t - x^5$, $v = t + x^5$.

The symmetries of the D1D5P system are less than the D1D5 case since the addition of left-moving momenta along x^5 reduces the spacetime symmetries to be

$$
\mathcal{M}_{\text{D1D5P}} = SO(4)_E \times SO(4)_I\,. \tag{1.30}
$$

We have also an additional Killing spinor equation

$$
\Gamma^{05}\epsilon_{L,R} = \epsilon_{L,R}\,, \tag{1.31}
$$

that fixes the D1D5P solution to be $\frac{1}{8}-$BPS.

We can reduce it to 5 dimensions defining two scalars χ, ψ and a $1-$form A_μ and writing the metric as

$$
\mathrm{d}s_{10}^2 = e^{2\chi}\mathrm{d}x_a\mathrm{d}x^a + e^{2\psi}(\mathrm{d}x^5 + A_\mu\mathrm{d}x^\mu) + e^{-\frac{8\chi+2\psi+\phi}{3}}\,\mathrm{d}s_5^2\,, \tag{1.32}
$$

where the exponential factor in front of $\mathrm{d}s_5^2$ comes from the requirement that $\mathrm{d}s_5^2$ must be the five dimensional Einstein metric. Explicitly, it is

$$
\begin{aligned}
\mathrm{d}s_5^2 &= -f^{-2/3}\mathrm{d}t^2 + f^{1/3}(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_3^2)\,, \\
f(r) &= Z_1 Z_5 Z_P\,, \quad Z_i = 1 + \frac{Q_i}{r^2}\,, \quad Q_i = Q_1, Q_5, Q_P\,,
\end{aligned}
\tag{1.33}
$$

which is the Strominger-Vafa black hole, a generalisation of the Reissner-Nördstrom black hole in five dimensions, that is a $\frac{1}{8}$-BPS solution of Type IIB supergravity.

CONSERVED CHARGES AND ENTROPY: While we have have added the "1" as an integration constant for the Z_i in order to have a well-defined asymptotically-flat limit, the Q_i are instead related to the number of D1 and D5 branes, as well as the number of units of momentum along the $\mathbb{S}^1$ circle; in fact it is easy to see that the Strominger-Vafa black hole is a type IIB system with n_1 D1 branes and n_5 D5 branes on $\mathcal{M}^{1,4} \times \mathbb{S}^1 \times \mathbb{T}^4$, with n_P units of momentum along the $\mathbb{S}^1$; the D1 branes wrap the $\mathbb{S}^1$, while the D5 branes wrap the $\mathbb{S}^1 \times \mathbb{T}^4$.

We can now compute the charges of these objects via

$$
Q_{\text{D1}} = \int_{\mathbb{S}^3\times\mathbb{T}^4} *\mathrm{d}C_2\,, \quad Q_{\text{D5}} = \int_{\mathbb{S}^3} \mathrm{d}C_2\,, \tag{1.34}
$$

where we have used that $\mathcal{M}^{1,4} = \mathbb{R}_t \times \mathbb{R}^4 = \mathbb{R}^{1,1} \times \mathbb{S}^3$, so that, at fixed time, $\partial\mathbb{R}^4 = \mathbb{S}^3$; recalling that a BPS bound is involved here, we can also compute the mass (and then check the BPS-ness of the system) by the standard formula

$$
M = \int \mathrm{d}^4x\, T_{00} = -\frac{1}{16\pi G_5}\,\frac{2}{3}\int \mathrm{d}^4x\,\partial^2 h_{00}\,, \tag{1.35}
$$

and using that $M = M_1 + M_5 + M_P$, we get the BPS conditions

$$Q_i = \frac{16\pi G_5}{2(2\pi^2)}\, M_i\,, \tag{1.36}$$

and then, recalling that for a Dp-brane $M_p = \tau_p V_p$, with τ_p brane tension and V_p the volume of the manifold wrapped by the Dp-branes, we get

$$Q_1 = \frac{(2\pi)^4 g_s(\alpha')^3}{V_{\mathbb{T}^4}}\, n_1\,, \quad Q_5 = g_s\alpha'\, n_5\,, \quad Q_P = \frac{(2\pi)^4 g_s^2(\alpha')^4}{V_{\mathbb{T}^4} R_{\mathbb{S}^1}}\, n_P\,. \tag{1.37}$$

We can then compute the Bekenstein-Hawking entropy directly from the five dimensional metric, obtaining

$$S_{\text{BH}} = \frac{\mathcal{A}_5}{4G_5} = 2\pi\sqrt{n_1 n_5 n_P}\,. \tag{1.38}$$

1.1.4 *A small detour: the "original" Strominger-Vafa Black hole*

Up to now, we have discussed the Strominger-Vafa black hole as a type IIB system of the D1D5P kind, coming from the compactification of an M2⊥M2⊥M2 system in 11-dimensional supergravity via dualities; in the original paper, Strominger and Vafa build a $\frac{1}{8}$-BPS solution in type IIA (a D0D4 system) compactified on[3] $\mathbb{S}^1\times$ K3. We will now briefly review it, mainly for historical reason. The bosonic part of the IIA action compactified on $\mathbb{S}^1\times$ K3 is

$$S_{\text{IIA}} = \frac{1}{16\pi}\int \mathrm{d}^5x\sqrt{-g_5}\left[e^{-2\phi}\left(R_5 + 4(\nabla\phi)^2 - \frac{1}{4}\,\widetilde{H}_2^2\right) - \frac{1}{4}\,F_2^2\right], \tag{1.39}$$

where ϕ is the dilaton, F_2 is a RR 2-form strength related to D0 and D4 charges, and $\widetilde{H}_2$ is the dimensionally-reduced Kalb-Ramond field strength. Imposing spherical symmetry, we can solve it as

$$\begin{aligned}
\mathrm{d}s_5 &= -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r}{f(r)} + r^2\mathrm{d}\Omega_3^2\,,\\
f(r) &= 1 - \frac{r_0^2}{r^2}\,, \quad r_0 \equiv \left(\frac{8Q_H Q_F^2}{\pi^2}\right)^{1/6}\,,
\end{aligned} \tag{1.40}$$

and where Q_H and Q_F are the (electric) charges with respect with $\widetilde{H}_2$ and F_2, respectively. It is then easy to compute the entropy via the Bekenstein-Hawking formula as

$$S_{\text{BH}} = 2\pi\sqrt{\frac{Q_H Q_F^2}{2}}\,. \tag{1.41}$$

1.2 *The Strominger-Vafa microstate counting*

The original microstate counting in the Strominger-Vafa black hole relies on the fact that their D0D4 system is dual to a D1D5 system compactified on $\mathbb{S}^1\times$ K3; under that $T-$duality, we have that D1-D3-D5-P in type IIB maps into D0-D2-D4-F1 in type IIA. The BPS states of the D-brane that they consider carry the charges Q_F and Q_H for which the corresponding extremal black hole solutions were found; now it is time to count the degeneracy of the black hole system by counting the number of all the BPS bound states. The key point is that

3 As noted by A. Sen (see [33], footnote 9), the Strominger and Vafa result can be equivalent computed from toroidal compactification of type IIB on $\mathbb{T}^4$; the only difference is the replacement of K3 with $\mathbb{T}^4$ in the symmetric orbifold. In this thesis we will then mainly use $\mathbb{T}^4$ as a compact space.

> *In the limit where* K3 *is small with respect to* $\mathbb{S}^1$*, the number of microstates of the Strominger-Vafa black hole is the number of independent ways in which the D1 branes can move inside the* K3.

This can be computed noticing that in this limit we get a $(1+1)-$dimensional supersymmetric sigma model whose target space is the symmetric product of $\frac{1}{2}Q_F^2+1$ copies of K3 and so, to count states that preserves only $\frac{1}{4}$ of the supercharges, that are states that are killed by the right-moving supercharges so $\bar{L}_0 = 0$, we have to compute the Cardy formula [55]

$$d(n,c) \sim \exp\left[2\pi\sqrt{\frac{nc}{6}}\right] , \tag{1.42}$$

that is valid for $n \gg c$. The crucial feature of the Strominger and Vafa computation is that the central charge c for the D1D5 system is determined solely by the dimension of the moduli space; in particular, we have

$$n = Q_H\,, \quad c = 6\left(\frac{1}{2}\,Q_F^2+1\right). \tag{1.43}$$

This gives

$$S_{\text{micro}} = \log d(n,c) \sim 2\pi\sqrt{Q_H\left(\frac{1}{2}\,Q_F^2+1\right)}\,, \tag{1.44}$$

that, in the right D-brane limit $Q_H \gg Q_F^2 \gg 1$ (that is also the limit where the Cardy formula applies), reproduces the Bekenstein-Hawking formula to leading order.

Reviewed the historical roots of the Strominger-Vafa computations, we reperform it for the D1D5P system (i.e. for the $\frac{1}{8}$-BPS system by computing explicitly the degeneracy of the $\frac{1}{4}$-BPS dual F1P system without using the Cardy formula, and then generalising it). We will explicitly use the fact that in the F1P frame we have a fundamental string carrying momentum, and then the microstate are related to the possible excitation of the string, for which it is possible to compute the partition function and then, taking the log, the free energy. Finally, by Legendre transforming it, we can recover the entropy.

1.3 *The microstate counting for $\frac{1}{8}$-BPS states*

We will now compute the degeneracies of the D1D5 system in an alternative way, by dualising it to the F1P frame and then enumerating them [51,52]. We can perform a chain of dualities from F1P to D1D5 (and back) as

$$\begin{pmatrix}\text{F1}\\ \text{P}\end{pmatrix} \xrightarrow{S} \begin{pmatrix}\text{D1}\\ \text{P}\end{pmatrix} \xrightarrow{T_{6789}} \begin{pmatrix}\text{D5}\\ \text{P}\end{pmatrix} \xrightarrow{S} \begin{pmatrix}\text{NS5}\\ \text{P}\end{pmatrix} \xrightarrow{T_{x^5}} \begin{pmatrix}\text{NS5}\\ \text{F1}\end{pmatrix} \xrightarrow{T_1} \begin{pmatrix}\text{NS5}\\ \text{F1}\end{pmatrix} \xrightarrow{S} \begin{pmatrix}\text{D5}\\ \text{D1}\end{pmatrix}. \tag{1.45}$$

In the F1P system we have a fundamental string that winds n_w times the $\mathbb{S}^1$, and that have n_P units of momentum. The excitations of the fundamental string come from the bosonic X^M and fermionic ψ^M modes, whose operator modes create states with momentum and energy given by

$$|p_k| = e_k = \frac{2\pi k}{L} = \frac{k}{R_{\mathbb{S}^1} n_w}\,, \tag{1.46}$$

where $L = 2\pi R_{\mathbb{S}^1} n_w$ is the effective total length of the string. We can then compute both bosonic and fermionic partition functions

$$\mathcal{Z}_k^B = \sum_{n=0}^{\infty} e^{-\beta e_k n} = \frac{1}{1 - e^{-\beta e_k}} \,, \quad \mathcal{Z}_k^F = 1 + e^{-\beta e_k} \,, \tag{1.47}$$

and then the total partition function is

$$\mathcal{Z} = \left(\prod_{k=1}^{\infty} \mathcal{Z}_k^B \mathcal{Z}_k^F \right)^8 . \tag{1.48}$$

We can compute it by approximate the sum with an integral, that is possible in the large n_w limit,

$$\begin{aligned} \sum_{k=1}^{\infty} \log \mathcal{Z}_k^B &\simeq \int_0^{\infty} \mathrm{d}k \, \log \left(1 - e^{-\frac{\beta}{R_{\mathbb{S}^1}} \frac{k}{n_w}} \right) = \frac{\pi^2}{6} \frac{R_{\mathbb{S}^1}}{\beta} \, n_w \,, \\ \sum_{k=1}^{\infty} \log \mathcal{Z}_k^F &\simeq \int_0^{\infty} \mathrm{d}k \, \log \left(1 + e^{-\frac{\beta}{R_{\mathbb{S}^1}} \frac{k}{n_w}} \right) = \frac{\pi^2}{12} \frac{R_{\mathbb{S}^1}}{\beta} \, n_w \,, \end{aligned} \tag{1.49}$$

so that

$$\log \mathcal{Z} = 8 \sum_{k=1}^{\infty} \left(\log \mathcal{Z}_k^B + \log \mathcal{Z}_k^F \right) = 6 \, \frac{\pi^2}{6} \frac{R_{\mathbb{S}^1}}{\beta} \, n_w \equiv c \, \frac{\pi^2}{6} \frac{R_{\mathbb{S}^1}}{\beta} \, n_w \,, \tag{1.50}$$

where c is the central charge. The energy of the system, given by the total momentum, is

$$E = \frac{n_P}{R_{\mathbb{S}^1}} = -\partial_\beta \log \mathcal{Z} = c \, \frac{\pi^2}{6} \frac{R_{\mathbb{S}^1}}{\beta} \, n_w \,. \tag{1.51}$$

This allows us to find

$$\beta = R_{\mathbb{S}^1} \sqrt{\frac{\pi^2}{6} \frac{R_{\mathbb{S}^1}}{\beta} \, c \, n_w} \,. \tag{1.52}$$

Now, we can compute the entropy by Legendre-transform the free energy

$$S_{\mathrm{F1P}} = \log \mathcal{Z} + \beta E = 2\pi \sqrt{\frac{c}{6} \, n_w n_P} \,. \tag{1.53}$$

To recover the entropy for the D1D5 system we need to perform the chain of dualities back, that amounts to send (n_P, n_w) to (n_1, n_5) and c to $c = 12$.

For the D1D5P system, we have to consider the D1 branes as an effective D1 brane with winding $n_1 n_5$, so that we can simply recover the result from the previous formula as

$$S_{\mathrm{D1D5P}} = 2\pi \sqrt{n_1 n_5 n_P} \,, \tag{1.54}$$

that reproduces the Bekenstein-Hawking entropy of the D1D5P black hole (1.38) in the large n_1, n_5, n_P limit.

2 BLACK HOLE INFORMATION PARADOX IN ADS/CFT

2.1 *Information loss without evaporation*

The information loss paradox becomes very sharp in AdS space [43,56]. This is because, having as a new length scale the AdS radius ℓ, we may build "small", i.e. $M < \ell$, and "big", i.e. $M > \ell$, AdS-black holes; while the former share the same decaying properties as their analogues in flat spacetime, the latter *do not evaporate*. In fact, a big black hole is equivalent to a black hole in a box[4] that arrives in thermal equilibrium with its atmosphere and with the wall of the box, ceasing to emit radiation. The key point is that correlations between field operator at large time separations computed on (any) black hole spacetime fall off exponentially

$$C(t) \equiv \langle \bar{O}(0)O(t)\rangle_{\text{BH}} = \text{Tr}\,[\rho\,\bar{O}(0)O(t)] \sim e^{-\gamma\beta t}\,, \tag{1.55}$$

where ρ is the density matrix of the black hole, O is some operator, $\beta = 1/T$ and γ some dimensionless constant whose value is irrelevant. We will explicitly present an example in sec. 1.2.2.

But, if statistical mechanics applies to black holes, the system is an ordinary quantum system with discrete energy spectrum $H|n\rangle = E_n|k\rangle$, so the Poincaré recurrence theorem applies and thus *it cannot be compatible with exponentially-decaying correlation functions*. To be more precise

$$C(t) = \text{Tr}\left[\frac{e^{-\beta\mathcal{H}}}{\mathcal{Z}}\,\bar{O}(0)O(t)\right] = \sum_{n,m}\frac{e^{-\beta E_n}}{\mathcal{Z}}\,|\langle n|O|m\rangle|^2 e^{-i(E_n-E_m)t}\,, \tag{1.56}$$

where $O(t)$ here is in Heisenberg picture; so that, time-averaging its square on a very long time $\mathcal{T}$ we get [45,58]

$$\begin{aligned}\frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} \mathrm{d}t\,|C(t)|^2 = \sum_{n,m;n',m'}&\frac{e^{-\beta(E_n+E_{n'})}}{\mathcal{Z}}\,|\langle n|O|m\rangle|^2|\langle n'|O|m'\rangle|^2\cdot\\ &\cdot\left[\frac{1}{\mathcal{T}}\int_0^{\mathcal{T}}\mathrm{d}t\,e^{i(E_n-E_m+E_{m'}-E_{n'})}\right].\end{aligned} \tag{1.57}$$

The quantity in the square brackets is a representation of a delta-function, thus approaches 1 as $\mathcal{T}\to\infty$ if $(E_n - E_m + E_{m'} - E_{n'}) = 0$, while approaches zero otherwise; if we assume that the energy spectrum is densely spaced near the energy that dominates the canonical ensemble, with typical spacing of order e^{-S} times the temperature, we have that it can be non-zero only if $E_n - E_m = E_{n'} - E_{m'} = 0$ or $E_n - E_{n'} = E_m - E_{m'} = 0$. Thus the average is finite in the limit $\mathcal{T}\to\infty$, implying that $C(t)$ cannot decrease monotonically to zero for large time separations.

We can estimate the typical size of $C(\infty)$ by employing the *eigenstate thermalization hypothesis* [59,60], that states

$$\langle n'|O|m'\rangle = O(E_n)\delta_{nm} + e^{-\frac{1}{2}S\left[\frac{E_n+E_m}{2}\right]}\,f(E_n,E_m)\,R_{ij}\,, \tag{1.58}$$

where $O(E)$ and $f(E,E')$ are real smooth functions while R_{ij} is a complex matrix; we then have

$$C(t) = C_D(t) + C_R(t) = \langle O_D^2\rangle_\rho + \sum_{n\neq m}\frac{e^{-\beta E_n}}{\mathcal{Z}}\,|\langle n|R|m\rangle|^2 e^{-i(E_n-E_m)t}\,.$$

4 Or to impose reflecting boundary conditions at a distance that is circa 3/2 of the black hole radius [57].

(1.59)

If we want a decaying correlation function, we should choose an observable whose diagonal part is vanishing. In any case, the sums run over e^S states, and $\mathcal{Z} \sim e^{S-\beta E_0}$, where E_0 is the expected ensemble energy. For this sum to be convergent at small times, we expect $|\langle n|R|m\rangle|^2 \sim e^{-2S}$. For large times instead, the factors $e^{-i(E_n-E_m)t}$ can be effectively treated as random so that $C_R(t)$ is a sum of e^{2S} terms of magnitude e^{-2S} and a random phase. The theory of random walks predicts that

$$C_R(t) \sim e^{-2S} \cdot \sqrt{e^{2S}} = e^{-S} \,. \tag{1.60}$$

So, if black hole statistical mechanics is true, the correlators computed on black hole geometry initially decay exponentially, but only until it reaches a (rather small) value of $\sim e^{-S}$, when it start oscillating and it will occasionally fluctuate back to large values due to the (quantum version of the) Poincaré recurrence theorem. Of course all of this is in sharp contradiction with the semiclassical computation that predicts a permanent decay up to zero.

Notice that this formulation of the information paradox afflicts even all the black holes that do not evaporate, such as the BPS or extremal black holes, whose temperature is exactly zero. In fact all the extremal holes have an AdS_2 factor in their near horizon geometry (for the Strominger-Vafa black hole we have seen that there is also an AdS_3 factor), and we can consider the asymptotically flat region as a thermal bath in communication with the AdS throat, in order to not affect the statement of the paradox[5]. We have then a formulation of the paradox even for extremal black holes, that can be treated more easily; in fact, it can be shown that, on extremal black hole geometries, correlators decay *polynomially* to zero for large time separations, again in contrast with what we expect from a unitary theory. We will prove this assertion for a particular extremal black hole in sec. 1.2.2.

One may wonder why AdS/CFT does not solve *de facto* the information paradox: we have actually shows that the boundary CFT, being unitary, does not show decay, thus solving the paradox (at least in this formulation). Even if this is the case, we have no control on the semiclassical computations on the gravity side, and no comprehension whatsoever on the origin their mismatch with the quantum computations on the CFT side; also, as shown by [45], Maldacena's argument in [43] for the existence of subleading saddles in the path integral that can give rise to long-time corrections of order e^{-S}, that is sometimes misunderstood as sufficient for solving the paradox, it is actually not enough for this goal [18].

We will show in part iii how it is possible to address this topic in the context of the *Fuzzball proposal*, already at the supergravity level.

2.2 *The decay of the 2-point function on the massless BTZ geometry*

As an illustrative example on how to holographically compute a 2-point function on a black hole geometry that is also analogous to what we will do in part iii, we will show the case of a scalar perturbation on a massless BTZ black hole. It is well known that gravity in three spacetime dimension in somewhat special, since it is non-dynamical and hence, purely topological[6], it does not have a

5 We thank E. Martinec, G. Bossard and M. Guica for a private discussion on this topic during a RER B trip from Ecolé Normale to Ecolé Polytechnique.

6 in $D = 2+1$ the Riemann and the Ricci have the same degrees of freedom, so imposing Einstein equations fixes completely the Riemann tensor and then the Weyl tensor.

Newtonian limit and it can be quantised as a double copy of a Chern-Simons Theory [61–63]. But, most importantly for us, three-dimensional solutions can be uplifted to ten-dimensional superstring theory [42], and, in particular, as shown in sec. 1.1, the solutions of D1D5 system are asymptotically $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$.

Three-dimensional gravity admits black hole solutions only in anti-de sitter spacetime. These solutions are called BTZ black holes, due to the name of Banados, Teitelboim and Zanelli [64, 65]; these are three-dimensional equivalent of AdS_4-Kerr-Newman black holes. The generic BTZ metric is

$$\begin{aligned} ds^2_{\mathrm{BTZ}} &= -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\left(\mathrm{d}\phi - A(r)\mathrm{d}t\right)^2 , \\ f(r) &= \frac{(r^2-r_-^2)(r^2-r_+^2)}{\ell^2 r^2} , \quad A(r) = \frac{r_+ r_-}{\ell r^2} , \end{aligned} \tag{1.61}$$

where ℓ is the AdS radius; it has two horizons, located at $r = r_\pm$. This black hole has two conserved charges, the mass M and the angular momentum J, that are

$$M = \frac{r_+^2 + r_-^2}{\ell^2} , \quad J = \frac{2r_+ r_-}{\ell} . \tag{1.62}$$

There exists an extremal limit when $M = J$, that is when the two horizons merge, i.e. $r_+ = r_-$, as usual in higher dimensions. More interestingly, and peculiar to the three-dimensional case, the limit $M \to 0$ does *not* reduce the BTZ to empty AdS_3; instead, we have a degenerate black hole solution called *massless* BTZ black hole [65, 66], whose horizon is the point $r = 0$. On the contrary, to recover the standard AdS_3, we need to have $M = -1$ and $J = 0$, so we actually have a gap between $M = -1$ empty solution and $M = 0$ massless BTZ black hole; then, AdS_3 is said to have a mass gap [62].

These black hole solutions can be uplifted to be solutions of type IIB supergravity as $\mathrm{BTZ} \times \mathbb{S}^3 \times \mathbb{T}^4$; We now focus on the simplest, BPS, massless case $M = J = 0$ BTZ case, and compute holographically a 2-point function on that geometry. We know that, in the AdS/CFT correspondence, black hole are dual to coherent sum of heavy states - i.e. states generated by the action of operator whose conformal dimension scale as the central charge in the central charge limit - that are states with a non-trivial back-reaction on the geometry; thus, via operator/state correspondence in the CFT side, we may regard a 2-point function of a Light operator - i.e. an operator whose conformal dimension does not scale with the central charge - on a non-trivial geometry as a 4-point function computed with the two light operators and the two heavy operators that are dual to the fields that generate the non-trivial geometry, i.e.

$$\langle \bar{O}_L(0) O_L(z,\bar{z}) \rangle_{\mathrm{BTZ}} \sim \langle O_{\mathrm{BTZ}}(\infty) O_L(1) O_L(z,\bar{z}) O_{\mathrm{BTZ}}(0) \rangle , \tag{1.63}$$

where O_{BTZ} can be schematically represented as the operator defining the state whose density matrix can be written as[7] [18]

$$\rho_{\mathrm{BTZ}} = \bigotimes_{\omega,\kappa} \left(\sum_n e^{-\beta\omega n} P_{abs}(\omega,\kappa) |n\rangle\langle n| \right) . \tag{1.65}$$

7 This is the form of the density matrix that reproduces the correct energy flux result in a band of late-time outgoing modes with width $\mathrm{d}\omega$ [67, 68]

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\omega \mathrm{d}\omega}{2\pi} \frac{P_{abs}(\omega,\kappa)}{e^{\beta\omega} - 1} , \tag{1.64}$$

where $\beta = 1/T$, κ is the Fourier mode conjugated of ϕ as ω is the Fourier mode conjugated with t, and P_{abs} is the absorption probability or *grey-body factor*.

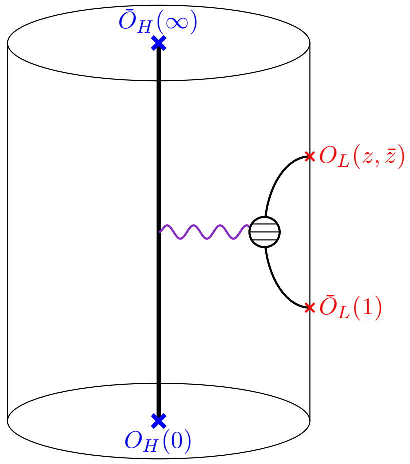


Figure 1.3: A pictorial representation of the 4-point function; the Heavy operators, inserted at $z = 0$ (correspond to $t = -\infty$) and $z = \infty$ (corresponding to $t = +\infty$), due to the state/operator correspondence, are the in- and out- states $\langle H|$ and $|H\rangle$ of the conformal scattering process. They can be regarded as sourcing a non-trivial background supergravity solution in the bulk, on which the supergravity field dual to the light operator moves.

This is pictorially represented in fig. 1.3.

The massless BTZ geometry can be written as

$$\mathrm{d}s^2_{\mathrm{BTZ}} = -r^2\mathrm{d}\tau^2 + r^2\mathrm{d}\sigma^2 + \frac{\mathrm{d}r^2}{r^2}\,, \quad \tau = t/\ell\,, \sigma = \phi\,,\ \sigma \sim \sigma + 2\pi\,. \tag{1.66}$$

We now employ the standard holographic procedure to compute the the Heavy-Heavy-Light-Light (HHLL) correlator. Since the massless BTZ black hole can be uplifted to be a solution of the D1D5 system in type IIB supergravity, that is a system with an $\mathrm{AdS}_3/\mathrm{CFT}_2$ duality, we may look for a Light operator in the CFT side and find its supergravity dual; for sake of simplicity here, let us assume that an operator of that sort exists and that have a supergravity dual that is a minimally coupled massless scalar on AdS_3[8], i.e. it satisfies

$$\Box\phi(\tau,\sigma,r) = 0 \Rightarrow \frac{1}{r}\partial_r(r^3\partial_r\phi) + \frac{1}{r^2}(\partial_\tau^2 - \partial_\sigma^2)\phi = 0\,. \tag{1.67}$$

We have to solve this PDE by imposing that the leading order in the large-r expansion behave as a delta-function of the variables on the plane, so that we can read the 2-point function from the subleading term; since holographically $m = \Delta(\Delta-2) = 0$ implies $\Delta = 2$, it will behave as[9]

$$\phi(\tau,\sigma,r) \sim \delta^{(2)}(\tau,\sigma) + r^{-2}\,\langle\bar{O}_L(0)O_L(\tau,\sigma)\rangle_{\mathrm{BTZ}}\,. \tag{1.70}$$

8 We will prove this statement later on, but for now the detail of the proof are irrelevant.

9 In generic $D = d+1$ dimensions, a field dual to an operator with conformal weight Δ behaves as

$$\phi(z,\vec{x}) \sim z^{d-\Delta}\phi_0(\vec{x}) + z^{\Delta}\phi_1(\vec{x})\,, \quad z \to 0\,, \tag{1.68}$$

where we have employed the Poincaré coordinates, for which $z \sim r^{-1}$ for $r \to \infty$. ϕ_0 is usually called the *non-normalisable* solution, while ϕ_1 is called the *normalisable* solution. If we impose the non-normalisable solution to be a delta-function, the normalisable solution becomes the 4-point function; thus we have, in two dimensions

$$\phi(\tau,\sigma,r) \sim r^{(h+\bar{h})-2}\delta^{(2)}(\tau,\sigma) + r^{-(h+\bar{h})}\langle H|O_L(0)\bar{O}_L(\tau,\sigma)|H\rangle\,, \quad r\to\infty\,. \tag{1.69}$$

In order to solve eq. (1.67), we Fourier transform it

$$\Phi(\tau,\sigma,r) = \int \frac{\mathrm{d}\omega}{2\pi} \sum_{\kappa\in\mathbb{Z}} e^{i\omega\tau}\, e^{i\kappa\sigma}\, g(\omega,\kappa)\, f_{\kappa,\omega}(r)\,, \tag{1.71}$$

thus reducing the PDE to an ODE

$$\frac{1}{r}\partial_r(r^3\partial_r f) + \frac{\omega^2-\kappa^2}{r^2}\, f = 0\,. \tag{1.72}$$

This can be recast in a *Modified Bessel equation* via the definition

$$f(r) = \frac{\sqrt{\kappa^2-\omega^2}}{r}\,\psi(r)\,, \quad x = \frac{\sqrt{\kappa^2-\omega^2}}{r}\,, \tag{1.73}$$

becoming then the (Modified) Bessel equation

$$x^2\psi'' + x\psi' - (x^2+1)\psi = 0\,, \tag{1.74}$$

whose solutions are

$$\psi(x) = c_1 I_1(x) + c_2 K_1(x)\,. \tag{1.75}$$

Going back to the r coordinate and f function, we read

$$f(r) = \frac{\sqrt{\kappa^2-\omega^2}}{r}\left[c_1 I_1\left(\frac{\sqrt{\kappa^2-\omega^2}}{r}\right) + c_2 K_1\left(\frac{\sqrt{\kappa^2-\omega^2}}{r}\right)\right]. \tag{1.76}$$

This geometry has an horizon located at $r = 0$, that is a fuchsian regular double singularity of the ODE. In standard black hole physics [17, 69], especially in the computations of *quasi-normal modes*, it is customary to introduce purely-ingoing radiation at the horizon as a boundary condition; In order to set the right boundary conditions (at $r = 0$ and at $r = \infty$) it is useful to recall that (see [70, 71]):

$$r\to 0\,,\; x\to\infty\;: \quad \begin{cases} I_1(x) \simeq \frac{e^x}{\sqrt{2\pi\, x}} + \cdots\,, \\ K_1(x) \simeq \sqrt{\frac{\pi}{2x}}\, e^{-x} + \cdots\,, \end{cases} \tag{1.77a}$$

$$r\to \infty\,,\; x\to 0\;: \quad \begin{cases} I_1(x) \simeq \frac{x}{2} + \cdots\,, \\ K_1(x) \simeq \frac{1}{x} + \frac{x}{2}\log\frac{x}{2} - (\psi(1)+\psi(2))\,\frac{x}{4} + \cdots\,. \end{cases} \tag{1.77b}$$

It is clear that we have to set $c_1 = 0$ in order to have a purely-ingoing wave at $r = 0$; At $r = \infty$ instead, we have

$$f(r) \simeq 1 - (1-2\gamma_E)\,\frac{\kappa^2-\omega^2}{4r^2} + \frac{\kappa^2-\omega^2}{2}\log\left[\frac{\kappa^2-\omega^2}{2r^2}\right] + \cdots\,. \tag{1.78}$$

This gives that, to have a delta-function at $r = \infty$, we simply impose

$$g(\omega,\kappa) = 1\,, \tag{1.79}$$

since the Fourier-transform of a constant is a delta-function; we can then read from the subleading r^{-2} term the 2-point function as

$$b_1(\tau,\sigma) = -\int \frac{\mathrm{d}\omega}{2\pi}\sum_{\kappa} e^{i(\omega\tau+\kappa\sigma)}\left\{(1-2\gamma_E)\,\frac{\kappa^2-\omega^2}{4} - \frac{\kappa^2-\omega^2}{2}\log\left[\frac{\kappa^2-\omega^2}{2}\right]\right\}.$$

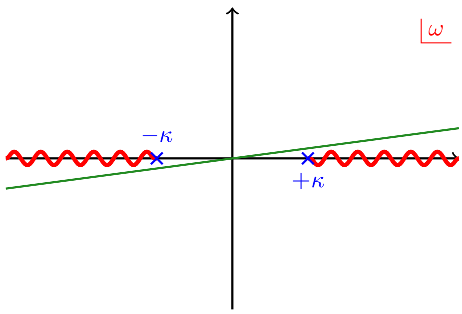


Figure 1.4: The ω-plane for the massless BTZ black hole. In red we show the branch cuts, emerging from the points $\pm\kappa \in \mathbb{Z}$. In green we reported the contour of integration for the Feynman propagator.

(1.80)

The first piece is a pure contact term. We should also notice that we can rewrite $\kappa^2 - \omega^2 \to i(\partial_\tau^2 - \partial_\sigma^2)$ so

$$b_1(\tau,\sigma) = -\frac{i}{2}(\partial_\tau^2 - \partial_\sigma^2)\int \frac{\mathrm{d}\omega}{2\pi}\sum_\kappa e^{i(\omega\tau+\kappa\sigma)}\left\{\frac{1}{2}(1-2\gamma_E) - \log\left[\frac{\kappa^2-\omega^2}{2}\right]\right\}. \tag{1.81}$$

The first term is a contact term, and we can drop it. The second term is indeed very interesting since it has no poles but a branch cut; in the complex ω plane, where we compute it by analytic continuation, there is a branch cut along the Real axis, at $(-\infty, -|\kappa|)$ and $(|\kappa|, +\infty)$. This is somehow expected, since it is known that the massless BTZ black hole does not have any quasi-normal modes [69] (that are naturally associated to poles with non-vanishing imaginary part), since there is no-mass scale and then it is impossible to have $\omega \sim M^{-1}$; but, being a black hole, it still have a non-trivial structure in the phase space, and it is represented here by the presence of two branch-cuts in the ω-plane.

We can now try to compute explicitly b_1. We recall that[10]

$$\int \mathrm{d}\omega\, e^{i\omega\tau}\log|\omega| = -\frac{\pi}{|\tau|} - 2\pi\gamma_E\delta(\tau)\,, \tag{1.83}$$

so that, neglecting contact terms we have to compute

$$b_1(\tau,\sigma) = +\frac{i}{2}(\partial_\tau^2 - \partial_\sigma^2)\int \frac{\mathrm{d}\omega}{2\pi}\sum_\kappa e^{i(\omega\tau+\kappa\sigma)}\log\left[\kappa^2-\omega^2\right]\,. \tag{1.84}$$

Since the prescription is the Feynman prescription, for $\tau > 0$ only the branch cut on the negative real axis matter[11]. To easily perform the computation, we split it three cases: $\kappa = 0, \kappa > 1$ and $\kappa < 1$.

10 We can obtain the Fourier transform of the log by recalling that the Fourier transform of the Principal Value of $|x|^{-1}$ is

$$\mathfrak{F}_T\left[\mathcal{P}\frac{1}{|x|}\right] = \int \mathrm{d}x\, e^{i\omega x}\,\frac{1}{|x|} = -2\gamma_E - 2\log|\omega|\,. \tag{1.82}$$

11 for $\tau < 0$ is the one on the positive real axis, but the result is evidently symmetric and we do not need to compute it twice

CASE $\kappa > 1$: here we have

$$
\begin{aligned}
I &= + \int \frac{\mathrm{d}\omega}{2\pi} \sum_{\kappa=1}^{\infty} e^{i(\omega\tau+\kappa\sigma)} \log\left[\kappa^2-\omega^2\right] \\
&= \sum_{\kappa=1}^{\infty} e^{i\kappa\sigma} \left[\int \frac{\mathrm{d}\omega}{2\pi}\, e^{i\omega\tau} \log(\kappa-\omega) + \int \frac{\mathrm{d}\omega}{2\pi}\, e^{i\omega\tau} \log(\kappa+\omega)\right] \\
&\simeq \sum_{\kappa=1}^{\infty} e^{i\kappa\sigma} \int \frac{\mathrm{d}\omega}{2\pi}\, e^{i\omega\tau} \log(\kappa-\omega) \\
&= -\sum_{\kappa=1}^{\infty} e^{i\kappa(\tau+\sigma)} \int \frac{\mathrm{d}\tilde{\omega}}{2\pi} e^{-i\tilde{\omega}\tau} \log\tilde{\omega} \qquad \text{where } (\tilde{\omega} = \kappa-\omega) \\
&= -\sum_{\kappa=1}^{\infty} e^{i\kappa(\tau+\sigma)} \frac{1}{2\pi}\left[-\frac{\pi}{|\tau|} + 2\pi\gamma_E\delta(\tau)\right] \\
&\simeq -\frac{1}{2\tau}\frac{1}{1-e^{i(\sigma+\tau)}} \,,
\end{aligned}
\tag{1.85}
$$

where we have neglected contact terms and dropped the $\log(\omega+\kappa)$ whose branch cut lies outside the region of integration.

CASE $\kappa < 1$: here we simply shift the $\kappa \in (-\infty,-1)$ to $\kappa \in (1,\infty)$ and recast it as the integral done above:

$$
\begin{aligned}
II &= + \int \frac{\mathrm{d}\omega}{2\pi} \sum_{\kappa=-\infty}^{-1} e^{i(\omega\tau+\kappa\sigma)} \log\left[\kappa^2-\omega^2\right] \\
&= + \int \frac{\mathrm{d}\omega}{2\pi} \sum_{\kappa=1}^{\infty} e^{i(\omega\tau-\kappa\sigma)} \log\left[\kappa^2-\omega^2\right] \\
&\simeq -\frac{1}{2\tau}\frac{1}{1-e^{i(\tau-\sigma)}} \,,
\end{aligned}
\tag{1.86}
$$

where, again, we have neglected contact terms and dropped the $\log(\omega+\kappa)$ whose branch cut is outside the region of integration.

CASE $\kappa = 0$: here we obtain the result as the limit with $\kappa \to 0$ of the above result in order to do not count twice the contribute in $\kappa = 0$:

$$
\begin{aligned}
III &= \lim_{\varepsilon\to 0} \int \frac{\mathrm{d}\omega}{2\pi} \sum_{\kappa} e^{i\omega\tau} \log\left[\varepsilon-\omega\right] \\
&\simeq +\frac{1}{2\tau} \,,
\end{aligned}
\tag{1.87}
$$

where we neglected again contact terms.

THE TOTAL $b_1(\tau,\sigma)$: So adding all together we get

$$
\langle \mathcal{O}(0)\, \bar{\mathcal{O}}(\tau,\sigma)\rangle_{\mathrm{BTZ}} - (\partial_\tau^2 - \partial_\sigma^2)\left[\frac{1}{2i\tau}\left(\frac{1}{1-e^{i(\sigma+\tau)}} + \frac{1}{1-e^{i(\sigma-\tau)}} - 1\right)\right] . \tag{1.88}
$$

It is evident from here that, for $\tau \to \infty$, the correlator decays polynomially in time; this is a minor difference with the massive case, where it decays exponentially - due to the presence of non vanishing quasi-normal modes [72].

This is traceable back to the presence of a point-size horizon, instead of a macroscopic one. Also, notice that, if we define $w = \sigma - \tau, \bar{w} = \sigma + \tau$ we may rewrite eq. (1.88) as eq. (2.14) of [72]

$$\langle \mathcal{O}(0)\, \bar{\mathcal{O}}(w, \bar{w}) \rangle_{\mathrm{BTZ}} = \frac{1}{4(w - \bar{w})^2} \left[\frac{1}{\sin^2 \frac{w}{2}} + \frac{1}{\sin^2 \frac{\bar{w}}{2}} - \frac{4 \sin \frac{w - \bar{w}}{2}}{(w - \bar{w}) \sin \frac{w}{2} \sin \frac{\bar{w}}{2}} \right]. \tag{1.89}$$

2 THE FUZZBALL PROPOSAL

As already emerged in the discussion of the Strominger-Vafa black hole in sec. 1.1, one of the most relevant frameworks for the study of black hole physics in string theory is the D1D5 system [42, 73] that, on the gravity side, it is a type IIB supergravity system with n_1 D1 branes and n_5 D5 branes in a geometry that is asymptotically $\mathbb{R}^{(1,4)} \times \mathbb{S}^1 \times \mathbb{T}^4$; the D5 branes wrap the $\mathbb{S}^1 \times \mathbb{T}^4$, while the D1 wrap the common $\mathbb{S}^1$. The system has a so-called *decoupling region* where the geometry is asymptotically $\mathrm{AdS}_3 \times \mathbb{S}^3$ and, by virtue of the AdS/CFT correspondence, it is dual to a superconformal field theory, often dubbed as D1D5 CFT. This is a $\mathscr{N} = (4,4)$ SCFT with supercurrents $(G_n^{\pm\pm}, \widetilde{G}_n^{\pm\pm})$ and with an affine $SO(4)_R \simeq SU(2)_L \times SU(2)_R$ R-symmetry algebra $(J_n^a, \widetilde{J}_n^a)$, that corresponds in the gravity side to the rotation of the $\mathbb{S}^3$; there exists a special locus in the D1D5 moduli space where the theory can be described as a two dimensional non-linear sigma model whose target space is $(\mathbb{T}^4)^N/S_N$, where S_N is the permutation group of order $N = n_1 n_5$. It is important to recall that the states of an orbifold theory split into different twist sectors that can be described as a collection of effective strings - or "strands" - of different winding number, with the constraint that the total winding must be equal to N. As explained in sec. 1.1, one of the most successful achievements of String Theory was the computation of the number of string microstates of a D1D5 Black Hole and its matching with its Bekenstein-Hawking entropy [33]. Motivated by this, the Fuzzball program aim is to explicitly construct those microstates.

1 THE D1D5 CFT

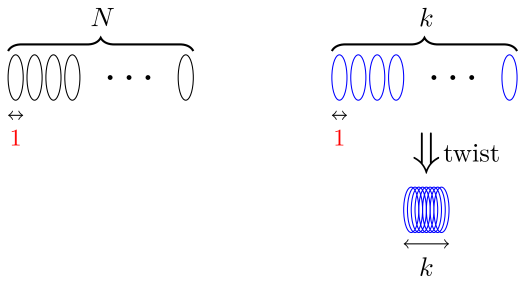
1.1 The Free orbifold point SCFT

The theory we are interested in is often dubbed D1D5 CFT [42, 73–75] that is the dual theory of the type IIB system whose asymptotics is $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$ we encountered in chap. 1, and it is a two dimensional superconformal field theory with $\mathscr{N} = (4,4)$ supercharges and a $SO(4)_R \simeq SU(2)_L \times SU(2)_R$ R-symmetry group which is holographically identified with the rotations of the $\mathbb{S}^3$; there is also a global $SO(4)_I \simeq SU(2)_1 \times SU(2)_2$ "custodial" symmetry group associated to the rotation of the compact $\mathbb{T}^4$ whose spinorial representations are also useful to label the fields in the theory. The D1D5 CFT at a special point of its moduli space, called free orbifold point[1], can be described as a non-linear sigma model with target space $(\mathbb{T}^4)^N/S_N$, where S_N is the permutation group with N element and where $N = n_1 n_5$, with n_1 is the number of D1 branes and n_5 is the number of D5 brane in the supergravity construction; its central charge is

$$c = 6 n_1 n_5 = 6N\,. \tag{2.1}$$

It is then useful to visualize the CFT states by representing the N copies, labelled by an integer index (r), as N strings, on which four bosons and four

1 For the discussion of the SCFT at the free orbifold point, we will follow the notation of [3, 74, 76–78].



(a) (b)

Figure 2.1: (a) The CFT at the orbifold point can be thought of as made of N copies, each of which contains 4 free bosons and 4 free fermions. Each circle in the figure corresponds to a single copy. (b) A twist field intertwines k copies into a single strand of length k.

fermions live; labelling with $\alpha, \dot\alpha = \pm$ the spinorial indexes of the R-symmetry group, with $A, \dot A = 1, 2$ the spinorial indexes of the $SO(4)_I$ group, they are

$$\left(X^{A\dot A}_{(r)}(z,\bar z)\,,\, \psi^{\alpha\dot A}_{(r)}(z)\,,\, \widetilde\psi^{\dot\alpha\dot A}_{(r)}(\bar z)\right). \tag{2.2}$$

The D1D5 CFT contains also twist operators that glue together k copies of the free field into a single strand on length k. This implies that a generic state in the CFT consists in a product of N_{k_i} strands with length k_i, such that the total winding is N, i.e.

$$\sum_i \sum_{k_i} k_i N_{k_i} = N\,. \tag{2.3}$$

The first thing to set up is the picture of the target space: having the peculiar orbifold $(\mathbb{T}^4)^N/S_N$, we can see it as $n_1 n_5 = N$ copies of a $\mathbb{T}^4$, modded out by the identification S_N; on any copy we have a string, or *strand*, wrapped on it.

To recap, this SCFT is a $(1+1)$-dimensional free theory with field content made of

$$\left(X^{\dot A A}_{(r)}(\tau,\sigma),\, \psi^{\alpha\dot A}_{(r)}(\tau+\sigma),\, \tilde\psi^{\dot\alpha\dot A}_{(r)}(\tau-\sigma)\right) \tag{2.4}$$

where $r = 1, \ldots, n_1 n_5$ is the strand index, (τ, σ) the $1+1$ dimension, and the index notation is

$$\begin{aligned} SU(2)_L:&\quad (\alpha,\beta), &\qquad SU(2)_R:&\quad (\dot\alpha,\dot\beta),\\ SU(2)_1:&\quad (A,B), &\qquad SU(2)_2:&\quad (\dot A,\dot B), \end{aligned} \tag{2.5}$$

where we recall that

$$SO(4)_R \simeq SU(2)_L \times SU(2)_R\,, \quad SO(4)_I \simeq SU(2)_1 \times SU(2)_2\,. \tag{2.6}$$

We can wick rotate to euclidean signature via $\tau \to -i\tau_E$ and then pass from the cylinder, where $\sigma \sim \sigma + 2\pi$, to the plane via

$$z = e^{\tau_E + i\sigma}\,, \quad \bar z = e^{\tau_E - i\sigma}\,. \tag{2.7}$$

From the CFT point of view will be more useful to write the elementary bosonic fields as ∂X ($\bar\partial X$) for their holomorphic (anti-holomorphic) properties, as

$$\left(\partial X^{\dot A A}_{(r)}(z),\, \bar\partial X^{\dot A A}_{(r)}(\bar z),\, \psi^{\alpha\dot A}_{(r)}(z),\, \tilde\psi^{\dot\alpha\dot A}_{(r)}(\bar z)\right)\,. \tag{2.8}$$

We will start our discussion describing the *untwisted sector* of the SCFT, i.e. where all the N strands are completely independent, and is actually realised as fig. 2.1(a), with N strands of length 1, i.e. singly winded. The *twisted sector*, on the contrary, has multiple strands glued together.

The CFT is described fully by the Operator Product Expansion (OPE) of the fundamental fields

$$\psi^{1\dot{A}}_{(r)}(z)\,\psi^{2\dot{B}}_{(s)}(w) = \frac{\varepsilon^{\dot{A}\dot{B}}\,\delta_{rs}}{z-w} + \cdots\,, \tag{2.9a}$$

$$\tilde{\psi}^{\dot{1}\dot{A}}_{(r)}(\bar{z})\,\tilde{\psi}^{\dot{2}\dot{B}}_{(s)}(\bar{w}) = \frac{\varepsilon^{\dot{A}\dot{B}}\,\delta_{rs}}{\bar{z}-\bar{w}} + \cdots\,, \tag{2.9b}$$

$$\partial X^{A\dot{A}}_{(r)}(z)\,\partial X^{B\dot{B}}_{(s)}(w) = \frac{\varepsilon^{AB}\,\varepsilon^{\dot{A}\dot{B}}\,\delta_{rs}}{(z-w)^2} + \cdots\,, \tag{2.9c}$$

$$\bar{\partial} X^{A\dot{A}}_{(r)}(\bar{z})\,\bar{\partial} X^{B\dot{B}}_{(s)}(\bar{w}) = \frac{\varepsilon^{AB}\,\varepsilon^{\dot{A}\dot{B}}\,\delta_{rs}}{(\bar{z}-\bar{w})^2} + \cdots\,, \tag{2.9d}$$

where $\varepsilon_{12} = \varepsilon_{\dot{1}\dot{2}} = -\varepsilon^{12} = -\varepsilon^{\dot{1}\dot{2}} = +1$, and where the $\cdots$ represents the regular part of the OPE. These are standard normalisation of the OPE for the fundamental bosonic and fermionic fields (see, for example, [79]). From here we can build the currents operator, i.e. the stress energy tensor, the supercurrents and the local $SU(2)_{L,R}$ R-symmetry currents. In order to do so we have to sum over all strands. The difference in single winded vs multi-winded case will be how the sum rearrange; we will see that in the multi-winded case we can diagonalize the sector.

1.1.1 *The Currents of the SCFT and their OPE*

The way to define the currents is the following: for the stress energy tensor, the usual sum of bosonic and fermionic contribution

$$\begin{aligned} T(z) &= \sum_{r=0}^{N} T_{(r)}(z)\,, \\ T_{(r)}(z) &\equiv \frac{1}{2}\,\varepsilon_{AB}\,\varepsilon_{\dot{A}\dot{B}}\,:\partial X^{A\dot{A}}_{(r)}(z)\,\partial X^{B\dot{B}}_{(r)}(z): + \frac{1}{2}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\dot{A}\dot{B}}\,:\psi^{\alpha\dot{A}}_{(r)}(z)\,\partial\psi^{\beta\dot{B}}_{(r)}(z):\,. \end{aligned} \tag{2.10}$$

The left-moving R-symmetry currents on the other side are, on each strand,

$$J^{+}_{(r)} \equiv +\frac{1}{2}\,\varepsilon_{\dot{A}\dot{B}}\,:\psi^{1\dot{A}}_{(r)}\psi^{1\dot{B}}_{(r)}:\,, \tag{2.11a}$$

$$J^{-}_{(r)} \equiv -\frac{1}{2}\,\varepsilon_{\dot{A}\dot{B}}\,:\psi^{2\dot{A}}_{(r)}\psi^{2\dot{B}}_{(r)}:\,, \tag{2.11b}$$

$$J^{3}_{(r)} \equiv -\frac{1}{2}\left(\varepsilon_{\dot{A}\dot{B}}\,:\psi^{1\dot{A}}_{(r)}\psi^{1\dot{B}}_{(r)}: -1\right), \tag{2.11c}$$

where we will label them as $J^a(z)$, $a = +, -, 3$. For any of them there is of course the Right moving counter part. The last currents are the supercurrents

$$\begin{aligned} G^{\alpha}_{A}(z) &\equiv \sum_{r=1}^{N} :\partial X_{A\dot{A}\,(r)}\,\psi^{\alpha A}_{(r)}:\,, \\ \widetilde{G}^{\alpha}_{A}(z) &\equiv \sum_{r=1}^{N} :\bar{\partial} X_{A\dot{A}\,(r)}\,\widetilde{\psi}^{\dot{\alpha}A}_{(r)}:\,, \end{aligned} \tag{2.12}$$

Now, using the OPE rules (2.9a) we obtain the OPE between the currents:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + 2\frac{T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots\,, \tag{2.13a}$$

$$T(z)J^a(w) = \frac{J^a}{(z-2)^2} + \frac{\partial J^a}{z-w} + \cdots\,, \tag{2.13b}$$

$$T(z)G^{\alpha A} = \frac{3}{2}\frac{G^{\alpha A}}{(z-w)^2} + \frac{\partial G^{\alpha A}}{z-w} + \cdots\,, \tag{2.13c}$$

$$J^a(z)J^b(w) = \frac{c}{12}\frac{\delta^{ab}}{(z-w)^2} + i\varepsilon^{ab}{}_c\frac{J^c(w)}{z-w} + \cdots\,, \tag{2.13d}$$

$$J^a(z)G^{\alpha A}(w) = \frac{1}{2}(\sigma^{*a})^{\alpha}{}_{\beta}\frac{G^{\beta A}(w)}{z-w} + \cdots\,, \tag{2.13e}$$

$$\begin{aligned} G^{\alpha A}(z)G^{\beta B}(w) = &-\frac{c}{3}\frac{\varepsilon^{AB}\varepsilon^{\alpha\beta}}{(z-w)^3} \\ &+ \varepsilon^{AB}\varepsilon^{\beta\gamma}(\sigma^{*a})^{\alpha}{}_{\gamma}\left[\frac{2J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w}\right] \\ &- \varepsilon^{AB}\varepsilon^{\alpha\beta}\frac{T(w)}{z-w} + \cdots\,. \end{aligned} \tag{2.13f}$$

where we have used the OPE between the currents and the Fields:

$$T(z)\partial X^{A\dot{A}}(w) = \frac{\partial X^{A\dot{A}}(w)}{(z-w)^2} + \frac{\partial^2 X^{A\dot{A}}}{(z-w)} + \cdots\,, \tag{2.14a}$$

$$T(z)\psi^{\alpha\dot{A}}(w) = \frac{1}{2}\frac{\psi^{\alpha\dot{A}}(w)}{(z-w)^2} + \frac{\partial\psi^{\alpha\dot{A}}(w)}{z-w} + \cdots\,, \tag{2.14b}$$

$$J^a(z)\partial X^{A\dot{A}}(w) = \cdots\,, \tag{2.14c}$$

$$J^a(z)\psi^{\alpha\dot{A}}(w) = \frac{1}{2}(\sigma^{*a})^{\alpha}{}_{\beta}\frac{\psi^{\beta\dot{A}}(w)}{z-w} + \cdots\,, \tag{2.14d}$$

$$G^{\alpha A}(z)\partial X^{B\dot{B}}(w) = \varepsilon^{AB}\left(\frac{\psi^{\alpha\dot{B}}(w)}{(z-w)^2} + \frac{\partial\psi^{\alpha\dot{B}}(w)}{z-w}\right) + \cdots\,, \tag{2.14e}$$

$$G^{\alpha A}(z)\psi^{\beta\dot{A}}(w) = \varepsilon^{\alpha\beta}\frac{\partial X^{A\dot{A}}(w)}{z-w} + \cdots\,. \tag{2.14f}$$

1.1.2 *Mode Expansion and the Global Supergroup $SU(1,1|2)$*

From here we can read the Global Symmetry group of our free orbifold SCFT and how the fundamental fields transform under its action. We define the mode expansion of an operator according to its weight as

$$\mathcal{O}_m = \oint \frac{\mathrm{d}z}{2\pi i}\,\mathcal{O}(z)\,z^{\Delta+m-1} \iff \mathcal{O}(z) = \sum_m \mathcal{O}_m\,z^{-(\Delta+m)}\,, \tag{2.15}$$

where the weight may be read off from the OPE of the Operator with $T(z)$. All the subtleties in the definitions of the mode (i.e. NS vs R sector; single winded vs multi-winded) lie in the definition of the sum over m. We will postpone the discussion of this fact to the following subsections. We find the algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}\,, \tag{2.16a}$$

$$[L_m, J^a_n] = -nJ^a_{m+n}\,, \tag{2.16b}$$

$$[L_m, G^{\alpha A}_n] = \left(\frac{m}{2}-n\right)G^{\alpha A}_{m+n}\,, \tag{2.16c}$$

$$[J^a_m, J^b_n] = i\varepsilon^{ab}{}_c J^c_{m+n} + \frac{c}{12}\,m\,\delta^{ab}\delta_{m+n,0}\,, \tag{2.16d}$$

$$[J^a_m, G^{\alpha A}_n] = \frac{1}{2}(\sigma^{*a})^\alpha{}_\beta G^{\beta A}_{m+n}\,, \tag{2.16e}$$

$$\begin{aligned}\{G^{\alpha A}_m, G^{\beta B}_n\} = &-\frac{c}{6}\left(m^2-\frac{1}{4}\right)\varepsilon^{AB}\varepsilon^{\alpha\beta}\delta_{m+n,0} \\ &+ (m-n)\varepsilon^{AB}\varepsilon^{\beta\gamma}(\sigma^{*a})^\alpha{}_\gamma J^a_{m+n} - \varepsilon^{AB}\varepsilon^{\alpha\beta}L_{m+n}\,.\end{aligned} \tag{2.16f}$$

As usual for a CFT_2 algebra, this algebra has a global well defined subalgebra that is anomaly-free. The generators of this subalgebra are

$$\left\{L_0, L_\pm, J^a_0, G^{\alpha A}_{\pm\frac{1}{2}}\right\}\,, \tag{2.17}$$

and the resulting subalgebra spans the algebra of the supergroup $SU(1,1|2)_{L,R}$. Its cartan subalgebra is $\{L_0, J^3_0\}$ and so we can classify the states by their eigenvalues (h, m). Also, $\jmath$ will be the eigenvalue of the Casimir $J^2 \equiv J^a J_a$.

1.1.3 *The Short Multiplets of* $SU(1,1|2)$

In the context of AdS/CFT correspondence will be useful the *short multiplets* of $SU(1,1|2)$. To build them we define the states $|\phi\rangle$ that satisfy

$$\text{Chiral}: \quad G^{+A}_{-\frac{1}{2}}|\phi\rangle = 0 \Rightarrow h = \jmath\,. \tag{2.18}$$

and dub them as *chiral* states. We recall that a *Virasoro primary* $|\chi\rangle$ is killed by all the positive Virasoro generators, i.e.

$$\text{Primary}: \quad L_n|\chi\rangle = 0\,, \quad \forall n > 0. \tag{2.19}$$

Finally, a *global primary* is defined by

$$\text{Global Primary}: \quad L_{+1}|\psi\rangle = G^{\alpha A}_{+\frac{1}{2}}|\psi\rangle = 0\,. \tag{2.20}$$

If a state is both Primary and Chiral, is called Chiral Primary; its correspondent operator - by operator-state correspondence of CFT - is called *Chiral Primary Operator* (CPO).

These CPOs are of the most importance: they are the analogue of the highest weight state for $SU(2)$, their energies and their 2- and 3-point functions are protected as one moves in the moduli space and their supergravity dual is well defined. Also they saturate the relation $h \geq m$, i.e. they have $h = m$. So CPO are also the highest weight states of the $SU(2)_{L,R}$ multiplet with $h = \jmath = m$.

Another important feature is that supergravity fields can be identified as the anomaly-free subalgebra descendants of CPO, i.e. CPO on which we act with only $L_{-1}, J^-_0, G^{-A}_{-\frac{1}{2}}$.

	state	h	$\jmath$	m
CP	$\vert c\rangle$	h	h	h
P	$G^{-A}_{-\frac{1}{2}}\vert c\rangle$	$h+\frac{1}{2}$	$h-\frac{1}{2}$	$h-\frac{1}{2}$
S	$(G^{-1}_{-\frac{1}{2}}G^{-2}_{-\frac{1}{2}}+\frac{1}{2h}J_0^-\,L_{-1})\vert c\rangle$	$h+1$	$h-1$	$h-1$

Table 2.1: Here we show the basic structure of a short multiplet. The CP stands for Chiral Primary state, the P for Virasoro Primary and S stands for $SL(2,\mathbb{R})$ primary. Further explanation in the text.

Now we describe how to obtain the short multiplet: in every strand there is a minimal weight CPO; we can apply to it at most two different fermionic modes and obtain 3 more chiral primaries, i.e.

$$|0\rangle\,,\quad \psi^{+\dot{A}}_{-\frac{1}{2}}|0\rangle\,,\quad \psi^{+\dot{1}}_{-\frac{1}{2}}\psi^{+\dot{2}}_{-\frac{1}{2}}|0\rangle\,. \tag{2.21}$$

Each of these CPO will give rise to a distinct short multiplet. We can build from any of them - called a representative of the class $|c\rangle$ - with three different operators, obtaining four different states[2]:

$$|c\rangle\,,\quad G^{-1}_{-\frac{1}{2}}|c\rangle\,,\quad G^{-2}_{-\frac{1}{2}}|c\rangle\,,\quad \left(G^{-1}_{-\frac{1}{2}}G^{-2}_{-\frac{1}{2}}+\frac{1}{2h}J_0^-\,L_{-1}\right)|c\rangle\,. \tag{2.23}$$

Finally, we can act as many times as we want on them with L_{-1}, obtaining the so-called Virasoro descendants of the CPO. Finally, we can span all the $SU(2)$ multiplet acting with J_0^-. The results is summarized in tab. 2.1.

1.2 *Untwisted Sector* $(k=1)$

1.2.1 *Monodromy Conditions and Mode expansion*

Having only single winded strands means that the monodromy conditions we have to impose are $\sigma\to\sigma+2\pi$ on the cylinder or $z\to e^{2\pi i}\,z$ on the plane. This gives for the Bosons

$$\begin{aligned}
&\text{Cylinder}: \quad X^{A\dot{A}}_{(r)}(\tau_E,\sigma+2\pi)=X^{A\dot{A}}_{(r)}(\tau_E,\sigma),\\
&\text{Plane}: \quad \begin{cases}\partial X^{A\dot{A}}_{(r)}(e^{+2\pi i}z)=\partial X^{A\dot{A}}_{(r)}(z),\\ \bar\partial X^{A\dot{A}}_{(r)}(e^{-2\pi i}\bar z)=\bar\partial X^{A\dot{A}}_{(r)}(\bar z).\end{cases}
\end{aligned} \tag{2.24}$$

The Fermions on the other hand are allowed to have either Ramond (R) or Neveu-Schwarz (NS) boundary conditions that are reported in tab. 2.2.

It corresponds, on the Cylinder, as

$$\begin{aligned}
\text{R}\ \ &:\ \psi^{\alpha A}_{(r)}(\tau_E,\sigma+2\pi)=+\psi^{\alpha A}_{(r)}(\tau_E,\sigma)\,,\\
\text{NS}\ &:\ \psi^{\alpha A}_{(r)}(\tau_E,\sigma+2\pi)=-\psi^{\alpha A}_{(r)}(\tau_E,\sigma)\,;
\end{aligned} \tag{2.25}$$

---

2 The last odd operators come from the fact that

$$\begin{aligned}
L_1J_0^+(G^{-1}_{-\frac{1}{2}}G^{-2}_{-\frac{1}{2}}|c\rangle)&=L_1(G^{+1}_{-\frac{1}{2}}G^{-2}_{-\frac{1}{2}}+G^{-1}_{-\frac{1}{2}}J_0^+G^{-2}_{-\frac{1}{2}})|c\rangle\\
&=-L_1L_{-1}|c\rangle\\
&=-2h|c\rangle\,,
\end{aligned} \tag{2.22}$$

that means that *part* of two applications of the supercurrents is equivalent to applying $J_0^-L_{-1}$.

	Cylinder	Plane
R	Periodic	Antiperiodic
NS	Antiperiodic	Periodic

Table 2.2: Cylinder vs plane conditions for R and NS boundary conditions.

while, on the Plane,

$$\begin{aligned} \mathrm{R} \;\; &: \;\; \psi^{\alpha A}_{(r)}(e^{+2\pi i} z) = -\psi^{\alpha A}_{(r)}(z)\,, \\ \mathrm{NS}\, &: \;\; \psi^{\alpha A}_{(r)}(e^{-2\pi i} \bar{z}) = +\psi^{\alpha A}_{(r)}(\bar{z})\,. \end{aligned} \tag{2.26}$$

We recall again that Cylinder and Plane has opposite definitions for the periodicity of R and NS sectors.

We can then Laurent-expand the fundamental Bosons

$$\partial X^{A\dot{A}}_{(r)}(z) = \sum_{n\in\mathbb{Z}} \alpha^{A\dot{A}}_{(r)n} z^{-n-1}\,, \quad \bar{\partial} X^{A\dot{A}}_{(r)}(\bar{z}) = \sum_{n\in\mathbb{Z}} \tilde{\alpha}^{A\dot{A}}_{(r)n} \bar{z}^{-n-1}\,, \tag{2.27}$$

and the fundamental Fermions

$$\begin{aligned} \mathrm{R} \;\; &: \;\; \psi^{\alpha A}_{(r)}(z) = \sum_{n\in\mathbb{Z}} \psi^{\alpha \dot{A}}_{(r)n} z^{-n-\frac{1}{2}}\,, \qquad \tilde{\psi}^{\dot{\alpha} A}_{(r)}(z) = \sum_{n\in\mathbb{Z}} \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)n} \bar{z}^{-n-\frac{1}{2}}\,, \\ \mathrm{NS}\, &: \;\; \psi^{\alpha A}_{(r)}(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \psi^{\alpha \dot{A}}_{(r)n} z^{-n-\frac{1}{2}}\,, \quad \tilde{\psi}^{\dot{\alpha} A}_{(r)}(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)n} \bar{z}^{-n-\frac{1}{2}}\,. \end{aligned} \tag{2.28}$$

The OPE (2.9a) gives the algebra for the modes, both Bosonic

$$\left[\alpha^{A\dot{A}}_{(r)n}\,, \alpha^{B\dot{B}}_{(s)m}\right] = \varepsilon^{AB}\varepsilon^{\dot{A}\dot{B}}\, n\, \delta_{n+m,0}\delta_{rs}, \; \left[\tilde{\alpha}^{A\dot{A}}_{(r)n}\,, \tilde{\alpha}^{B\dot{B}}_{(s)m}\right] = \varepsilon^{AB}\varepsilon^{\dot{A}\dot{B}}\, n\, \delta_{n+m,0}\delta_{rs}, \tag{2.29}$$

and Fermionic (in both sectors they are the same)

$$\left\{\psi^{1\dot{A}}_{(r)n}\,, \psi^{2\dot{B}}_{(r)m}\right\} = \varepsilon^{\dot{A}\dot{B}}\delta_{n+m,0}\delta_{rs}\,, \quad \left\{\tilde{\psi}^{1\dot{A}}_{(r)n}\,, \tilde{\psi}^{2\dot{B}}_{(r)m}\right\} = \varepsilon^{\dot{A}\dot{B}}\delta_{n+m,0}\delta_{rs}\,. \tag{2.30}$$

1.2.2 *Vacuum states*

We now build the vacuum states. For the Bosonic sector we define the vacuum $|0\rangle_{(r)}$ as the state annihilated by all the positive modes of the bosons[3]:

$$\alpha^{A\dot{A}}_{(r)n}|0\rangle_{(r)} = \tilde{\alpha}^{A\dot{A}}_{(r)n}|0\rangle_{(r)} = 0\,, \quad \forall n \geq 0\,,\, \forall A, \dot{A}\,. \tag{2.32}$$

Moving to the Fermions, for the **NS sector** everything work out more or less to be the same as the bosonic one; The vacuum $|0\rangle_{(r)\mathrm{NS}}$ is defined as

$$\psi^{\alpha\dot{A}}_{(r)n}|0\rangle_{(r)\mathrm{NS}} = 0\,, \quad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)n}|0\rangle_{(r)\mathrm{NS}} = 0\,, \quad \forall n > 0\,,\, \forall \alpha, \dot{\alpha}, \dot{A}\,. \tag{2.33}$$

In the **R sector**, on the other hand, works differently since fermions do have zero modes in their expansion, and half of them annihilate the vacuum while the other half does not. So we have a vacuum $|0\rangle_{(r)\mathrm{R}}$ defined as

$$\psi^{\alpha\dot{A}}_{(r)n}|0\rangle_{(r)\mathrm{R}} = 0\,, \quad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)n}|0\rangle_{(r)\mathrm{R}} = 0\,, \quad \forall n > 0\,,\, \forall \alpha, \dot{\alpha}, \dot{A}\,, \tag{2.34}$$

3 Here the strands are all independent and then we assume that

$$_{(r)}\langle 0|0\rangle_{(s)} = \delta_{rs}\,. \tag{2.31}$$

but we have also another vacuum $|{+}{+}\rangle_{(r)\mathrm{R}}$ such that

$$\psi^{1\dot{A}}_{(r)0}\,|{+}{+}\rangle_{(r)\mathrm{R}} = 0\,, \quad \tilde{\psi}^{\dot{1}\dot{A}}_{(r)0}\,|{+}{+}\rangle_{(r)\mathrm{R}} = 0\,. \tag{2.35}$$

We can act on them with zero modes that does not annihilate those vacua; this means that we have a family of non-degenerate vacua generated by applying to $|{+}{+}\rangle_{(r)\mathrm{R}}$ the operators $\psi^{2\dot{A}}_{(r)0}\,, \psi^{\dot{2}\dot{A}}_{(r)0}$. This means that, in the untwisted sector, we have 16 non-equivalent vacua per each strand, that are labelled by their R-symmetry charge under Left and Right $SU(2)$ groups and/or their quantum number under the "custodial" $SU(2)_1 \times SU(2)_2$; they are then[4]

$$|\alpha\dot{\alpha}\rangle_1\,, \quad |\alpha\dot{A}\rangle_1\,, \quad |\dot{A}\dot{\alpha}\rangle_1\,, \quad |\dot{A}\dot{B}\rangle_1\,. \tag{2.36}$$

1.3 *Twisted Sector* $(k > 1)$

1.3.1 *Monodromy conditions and Mode Expansion:*

Here we describe the twisted sector; to have a pictorial image in mind, recall fig. 2.1(b). Here we sew togheter k strands of length 1 in a single strand of length k; this gives a non trivial monodromy to the fields. In the most general case we can have M strands of length k_i such that $\sum_{i=1}^{M} k_i = N$.

In this case, going around the center one time brings the field to the adjacent strand glued together, similarly at what happens in a Riemann sheet:

$$\partial X^{A\dot{A}}_{(r)}(e^{+2\pi i}z) = \partial X^{A\dot{A}}_{(r+1)}(z)\,, \quad \bar{\partial} X^{A\dot{A}}_{(r)}(e^{-2\pi i}\bar{z}) = \bar{\partial} X^{A\dot{A}}_{(r+1)}(\bar{z}), \tag{2.37}$$

We can anyway diagonalize the system rearranging the fields in a linear combination of fields on different copies, i.e. redefining the base of the strands, from $r = 1, \ldots, k$ to $\rho = 0, \ldots, k-1$ via

$$\partial X^{1\dot{1}}_{\rho}(z) = \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{-2\pi i \frac{r\rho}{k}}\,\partial X^{1\dot{1}}_{(r)}(z),\ \partial X^{2\dot{2}}_{\rho}(z) = \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{+2\pi i \frac{r\rho}{k}}\,\partial X^{2\dot{2}}_{(r)}(z), \tag{2.38a}$$

$$\partial X^{1\dot{2}}_{\rho}(z) = \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{+2\pi i \frac{r\rho}{k}}\,\partial X^{1\dot{2}}_{(r)}(z),\ \partial X^{2\dot{1}}_{\rho}(z) = \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{-2\pi i \frac{r\rho}{k}}\,\partial X^{2\dot{1}}_{(r)}(z), \tag{2.38b}$$

$$\bar{\partial} X^{1\dot{1}}_{\rho}(\bar{z}) = \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{+2\pi i \frac{r\rho}{k}}\,\bar{\partial} X^{1\dot{1}}_{(r)}(\bar{z}),\ \bar{\partial} X^{2\dot{2}}_{\rho}(z) = \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{-2\pi i \frac{r\rho}{k}}\,\bar{\partial} X^{2\dot{2}}_{(r)}(\bar{z}), \tag{2.38c}$$

$$\bar{\partial} X^{1\dot{2}}_{\rho}(\bar{z}) = \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{-2\pi i \frac{r\rho}{k}}\,\bar{\partial} X^{1\dot{2}}_{(r)}(\bar{z}),\ \bar{\partial} X^{2\dot{1}}_{\rho}(z) = \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{+2\pi i \frac{r\rho}{k}}\,\bar{\partial} X^{2\dot{1}}_{(r)}(\bar{z}). \tag{2.38d}$$

4 Sometimes it is useful to denote the vacuum state with its left and right R-charges, so we may split the four states $|\dot{A}\dot{B}\rangle_1$ in the irreducible representations of singlet and triplet under the custodial symmetry as $|00\rangle_1 = \varepsilon^{\dot{A}\dot{B}}|\dot{A}\dot{B}\rangle_1$ and $|00\rangle_1^{(\dot{A}\dot{B})}$.

Now we have the diagonalized monodromy relations

$$\partial X^{1\dot{1}}_{\rho}(e^{+2\pi i}z) = e^{+2\pi i\frac{\rho}{k}}\,\partial X^{1\dot{1}}_{\rho}(z), \quad \partial X^{2\dot{2}}_{\rho}(e^{+2\pi i}z) = e^{-2\pi i\frac{\rho}{k}}\,\partial X^{2\dot{2}}_{\rho}(z), \tag{2.39a}$$

$$\partial X^{1\dot{2}}_{\rho}(e^{+2\pi i}z) = e^{-2\pi i\frac{\rho}{k}}\,\partial X^{1\dot{2}}_{\rho}(z), \quad \partial X^{2\dot{1}}_{\rho}(e^{+2\pi i}z) = e^{+2\pi i\frac{\rho}{k}}\,\partial X^{2\dot{1}}_{\rho}(z), \tag{2.39b}$$

$$\bar{\partial} X^{1\dot{1}}_{\rho}(e^{-2\pi i}\bar{z}) = e^{-2\pi i\frac{\rho}{k}}\,\bar{\partial} X^{1\dot{1}}_{\rho}(\bar{z}), \quad \partial X^{2\dot{2}}_{\rho}(e^{-2\pi i}\bar{z}) = e^{+2\pi i\frac{\rho}{k}}\,\partial X^{2\dot{2}}_{\rho}(\bar{z}), \tag{2.39c}$$

$$\bar{\partial} X^{1\dot{2}}_{\rho}(e^{-2\pi i}\bar{z}) = e^{+2\pi i\frac{\rho}{k}}\,\bar{\partial} X^{1\dot{2}}_{\rho}(\bar{z}), \quad \bar{\partial} X^{2\dot{1}}_{\rho}(e^{-2\pi i}\bar{z}) = e^{-2\pi i\frac{\rho}{k}}\,\bar{\partial} X^{2\dot{1}}_{\rho}(\bar{z}), \tag{2.39d}$$

The mode expansions are obtained from the untwisted case with the appropriate substitution $n \to n \pm \frac{\rho}{k}$.

For the fermions we have to distinguish again between R and NS sector. For the **R sector** we have again

$$\psi^{\alpha\dot{A}}_{(r)}(e^{2\pi i}z) = \psi^{\alpha\dot{A}}_{(r+1)}(z)\,, \quad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)}(e^{-2\pi i}\bar{z}) = \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r+1)}(\bar{z})\,. \tag{2.40}$$

We can again diagonalize with a change of basis in analogy with the bosonic case. In the diagonalized frame if we go around the origin k times we get

$$\psi^{\alpha\dot{A}}_{\rho}(e^{2\pi ik}z) = (-1)^k\,\psi^{\alpha\dot{A}}_{\rho}(z)\,, \quad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{\rho}(e^{-2\pi ik}\bar{z}) = (-1)^k\,\tilde{\psi}^{\dot{\alpha}\dot{A}}_{\rho}(\bar{z})\,. \tag{2.41}$$

For the **NS sector** we have the same monodromies on the (r) basis, but with the opposite identification at the end

$$\psi^{\alpha\dot{A}}_{(k+1)} = (-1)^{k+1}\psi^{\alpha\dot{A}}_{(1)}\,, \quad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(k+1)} = (-1)^{k+1}\tilde{\psi}^{\dot{\alpha}\dot{A}}_{(1)}\,, \tag{2.42}$$

so we need to change the diagonalization procedure with

$$(r) \to \ell\,, \quad \ell = -\frac{k-1}{2}\,, -\frac{k-1}{2}+1, \cdots, \frac{k-1}{2}\,. \tag{2.43}$$

At the end of the procedure going around the origin k times will give

$$\psi^{\alpha\dot{A}}_{\ell}(e^{2\pi ik}z) = (-1)^{k+1}\,\psi^{\alpha\dot{A}}_{\ell}(z)\,, \quad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{\ell}(e^{-2\pi ik}\bar{z}) = (-1)^{k+1}\,\tilde{\psi}^{\dot{\alpha}\dot{A}}_{\ell}(\bar{z})\,. \tag{2.44}$$

1.3.2 *Vacuum States*

The vacua of the twisted sector are very close to the one of the untwisted sector, having in mind that we have different monodromy conditions discussed above. Again, the **bosonic vacuum** is

$$\alpha^{A\dot{A}}_{(\rho)n}|0\rangle_k = \tilde{\alpha}^{A\dot{A}}_{(\rho)n}|0\rangle_k = 0\,, \quad \forall n \geq 0\,,\ \forall A, \dot{A}\,. \tag{2.45}$$

In the **NS sector**, we have only $|0\rangle_k$ with again

$$\psi^{\alpha\dot{A}}_{(\ell)n}|0\rangle_{k,\,\mathrm{NS}} = 0\,, \quad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(\ell)n}|0\rangle_{k,\,\mathrm{NS}} = 0\,, \quad \forall n > 0\,,\ \forall \alpha, \dot{\alpha}, \dot{A}\,. \tag{2.46}$$

In the **R sector** instead, we have again, as in the untwisted sector, 16 vacua

$$|\alpha\dot{\alpha}\rangle_k\,, \quad |\alpha\dot{A}\rangle_k\,, \quad |\dot{A}\dot{\alpha}\rangle_k\,, \quad |\dot{A}\dot{B}\rangle_k\,, \tag{2.47}$$

such that

$$\psi^{\alpha\dot{A}}_{(\rho)n}|0\rangle_{k,\,\mathrm{R}} = 0\,, \quad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(\rho)n}|0\rangle_{k,\,\mathrm{R}} = 0\,, \quad \forall n > 0\,,\ \forall \alpha, \dot{\alpha}, \dot{A}\,. \tag{2.48}$$

while[5]

$$\psi^{1\dot{A}}_{(\rho)0}\,|{+}{+}\rangle_{k,\,\mathrm{R}}=0\,,\quad \tilde{\psi}^{\dot{1}\dot{A}}_{(\rho)0}\,|{+}{+}\rangle_{k,\,\mathrm{R}}=0\,. \tag{2.50}$$

The other states - i.e. the one with the minus - are obtained from $|{+}{+}\rangle_{k,\,\mathrm{R}}$ acting with $J^-,\tilde{J}^-$ defined on a length k strand.

1.4 *Bosonization*

As usual for a $(1+1)$-dimensional CFT, we can bosonize the fermion fields [75, 77–79] following the rule

$$\begin{aligned}\psi^{+\dot{1}}_{(r)}&=i:e^{iH_{(r)}(z)}:\,,\quad \psi^{-\dot{2}}_{(r)}=i:e^{-iH_{(r)}(z)}:\,,\\ \psi^{+\dot{2}}_{(r)}&=i:e^{iK_{(r)}(z)}:\,,\quad \psi^{-\dot{1}}_{(r)}=i:e^{-iK_{(r)}(z)}:\,,\end{aligned} \tag{2.51}$$

with the following OPE

$$\begin{aligned}H_{(r)}(z)H_{(s)}(w)&=-\delta_{rs}\log(z-w)+\cdots\,,\\ K_{(r)}(z)K_{(s)}(w)&=-\delta_{rs}\log(z-w)+\cdots\,.\end{aligned} \tag{2.52}$$

Following [79] we have

$$:e^{i\alpha H_{(r)}(z)}::e^{i\beta H_{(r)}(w)}:=(z-w)^{-\alpha\beta}:e^{i\left(\alpha H_{(r)}(z)+\beta H_{(r)}(w)\right)}:\,. \tag{2.53}$$

1.4.1 *Spectral Flow*

The bosonization is useful since it can be used to define an operator that maps the fermions' NS vacuum to the R vacuum: the *Spectral Flow*. On a length 1 strand it is defined as

$$|{+}{+}\rangle_{(r)}=\lim_{z\to0}e^{\frac{i}{2}\left(H_{(r)}(z)+K_{(r)}(z)+\tilde{H}_{(r)}(\bar z)+\tilde{K}_{(r)}(\bar z)\right)}|0\rangle_{(r),\mathrm{NS}}\,. \tag{2.54}$$

The generalization to N single winded strands is straightforward:

$$\bigotimes_{r=1}^{N}|{+}{+}\rangle_{(r)}=\bigotimes_{r=1}^{N}\left[\lim_{z\to0}e^{\frac{i}{2}\left(H_{(r)}(z)+K_{(r)}(z)+\tilde{H}_{(r)}(\bar z)+\tilde{K}_{(r)}(\bar z)\right)}|0\rangle_{(r),\mathrm{NS}}\right]\,. \tag{2.55}$$

The spectral flow will play an important role in the following, when it will turn out to be useful to reconstruct all-light 4-point functions from the Heavy-Heavy-Light-Light ones.

From the definitions of currents in terms of the elemetantary fields, explained in sec. 2.1.1.1, we may see how the spectral flow acts on them, and then see how it changes the elementary charges of the state; in fact it turns out that [4, 74, 78]

$$L_n\mapsto L_n+J^3_m+\frac{1}{4}\,\delta_{m,0}\,,\ J^3_m\mapsto J^3_m-\frac{1}{2}\,\delta_{m,0}\,,\ J^m_m\mapsto J^{\pm}_{m\mp1}\,,\ G^{\pm,A}_m\mapsto G^{\pm,A}_{m\pm\frac{1}{2}}\,. \tag{2.56}$$

In general, the spectral flow in a CFT is a local transformation on operators that leaves the $\mathscr{N}=(4,4)$ algebra invariant. It is defined as a finite transformation generated by the $J^3(z)$, $\tilde{J}^3(\bar z)$ with angles given by

$$\eta(z)=i\alpha\log z\,,\quad \bar\eta(\bar z)=i\bar\alpha\log\bar z\,, \tag{2.57}$$

5 Of course we have also

$$\psi^{\alpha\dot{A}}_{(\rho)n}\,|{+}{+}\rangle_{k,\,\mathrm{R}}=0\,,\quad \tilde{\psi}^{\dot\alpha\dot{A}}_{(\rho)n}\,|{+}{+}\rangle_{k,\,\mathrm{R}}=0\,,\quad \forall n>0\,,\ \forall\alpha,\dot\alpha,\dot{A}\,. \tag{2.49}$$

where α, $\bar{\alpha}$ are the units of spectral flow. So the generalization of eq. (2.56) is

$$\begin{aligned} L_n &\mapsto L_n - \alpha J^3_m + \frac{c\alpha^2}{24}\,\delta_{m,0}\,, \\ J^3_m &\mapsto J^3_m - \frac{c\alpha}{12}\,\delta_{m,0}\,, \quad J^m_m \mapsto J^{\pm}_{m\pm\alpha}\,, \\ G^{\pm,A}_m &\mapsto G^{\pm,A}_{m\pm\frac{\alpha}{2}}\,; \end{aligned} \tag{2.58}$$

this means that the charges of the state change as

$$h \mapsto h + \alpha m + \frac{c\alpha^2}{24}\,, \quad m \mapsto m + \frac{c\alpha}{12}\,. \tag{2.59}$$

To recover the result (2.56), we have used $c = 6$ and imposed $\alpha = -1$. The $\alpha = -1$ case is of special interest, since in this case fermions transforms as $\psi^{\pm\dot{A}}(z) \mapsto z^{\pm\frac{1}{2}}\psi^{\pm\dot{A}}(z)$ changing the periodicity conditions, sending states from NS to R (and viceversa). It is important to notice then that, taking four chiral primaries in the NS untwisted sector

$$|0\rangle\,, \quad \psi^{+\dot{A}}_{-\frac{1}{2}}|0\rangle\,, \quad \psi^{+\dot{1}}_{-\frac{1}{2}}\psi^{+\dot{2}}_{-\frac{1}{2}}|0\rangle\,, \tag{2.60}$$

using the spectral flow rules, we can see that the four states above goes into four R states with $h = 1/4$ with and $SU(2)_L$ charge; joining then left and right sector in all possible way we map 16 chiral primary states in the NS sector into the 16 R ground states

$$|\alpha\dot{\alpha}\rangle\,, \quad |\alpha\dot{A}\rangle\,, \quad |\dot{A}\dot{\alpha}\rangle\,, \quad |\dot{A}\dot{B}\rangle\,. \tag{2.61}$$

One of the most relevant examples of these mapping is that the NS vacuum is mapped into the maximally spinning R ground state, i.e.

$$|0\rangle \mapsto |{+}{+}\rangle\,. \tag{2.62}$$

1.5 *Useful operators*

We now list some families of composite operators that will be relevant in the discussion; later on, for each of these operators that are also CPO, we will discuss their holographic dual in the supergravity side.

1.5.1 *Twist fields*

We will discuss here the *twist operators*, i.e. operators that acts on a tensor product of k strand to give a single strand with a length that is the sum of the lengths of the original strands, sewing them together. We start defining the *fundamental twist* operators σ^X_k, $\overline{\sigma}^X_k$ in the bosonic sectors as

$$\lim_{z,\bar{z}\to 0} \sigma^X_k(z)\overline{\sigma}^X_k(\bar{z}) \left[\otimes^k_{r=1}|0\rangle_{(r)}\right] = |0\rangle_k\,, \tag{2.63}$$

with conformal dimension and spin

$$(h, m) = \left(\frac{k^2-1}{6k}\,, 0\right). \tag{2.64}$$

In the NS fermionic sector instead we define the fundamental twist field $\Sigma_k(z,\bar{z})$ as

$$\lim_{z,\bar{z}\to 0} \Sigma_k(z,\bar{z}) \left[\otimes^k_{r=1}|0\rangle_{(r)}\right] = |0\rangle_k\,, \tag{2.65}$$

so that its conformal dimension and spin are

$$(h, m) = \left(\frac{1}{12}\left(k - \frac{1}{k}\right), 0\right). \tag{2.66}$$

In the R sector, since we have multiple vacua that carries R-charges, we need to have a multiplet of twists that are R-charged, $\Sigma_k^{\alpha_1\dot{\alpha}_2}(z,\bar{z})$, and the highest weight of the multiplet has

$$(h, m) = \left(\frac{(k-1)(2k-1)}{6k}, \frac{k-1}{2}\right), \tag{2.67}$$

where the indexes $\alpha_1\dot{\alpha}_2$ transform in a $\left(\frac{k-1}{2}, \frac{k-1}{2}\right)$ representation of $SU(2)_L \times SU(2)_R$.

1.5.2 *other CPOs*

Another relevant family of operators with $\Delta = h + \bar{h} = 1$ that will be relevant in the following are

$$O_{\text{Fer}}^{\alpha\dot{\alpha}}(z,\bar{z}) = \sum_{r=1}^{N} O_{(r)}^{\alpha\dot{\alpha}}(z,\bar{z}) = \sum_{r=1}^{N} \frac{-i\varepsilon_{\dot{A}\dot{B}}}{\sqrt{2N}}\, \psi_{(r)}^{\alpha\dot{A}}(z)\widetilde{\psi}_{(r)}^{\dot{\alpha}\dot{A}}(\bar{z}), \tag{2.68}$$

that is CPO and has $(h, \bar{h}) = \left(\frac{1}{2}, \frac{1}{2}\right)$. It is worth noticing that its action on the highest R vacua is

$$\lim_{z\to 0} O_{(r)}^{\mp\mp}(z,\bar{z})\, |\pm\pm\rangle_{(r)} = |00\rangle_{(r)}\,. \tag{2.69}$$

These operators are often referred as "fermionic operators" since they are made with two fermions, to distinguish them to the "bosonic operators" that we are going to describe. These have $(h, \bar{h}) = (1, 1)$ and are made with two bosons per strand as

$$O_{\text{Bos}}^{AB}(z,\bar{z}) = \sum_{r=1}^{N} \frac{\varepsilon_{\dot{A}\dot{B}}}{\sqrt{2N}}\, \partial X_{(r)}^{A\dot{A}}(z)\bar{\partial} X_{(r)}^{B\dot{B}}(\bar{z}). \tag{2.70}$$

These bosonic operators are superdescendants of the fermionic ones; in fact, since [2, 78]

$$\begin{aligned}
&\oint_{w\sim z} \frac{\mathrm{d}w}{2\pi i}\, \sqrt{w}\, G_1^1(w)\psi^{2\dot{C}}(z) = \sqrt{z}\, \partial X_{1\dot{E}}(z)\varepsilon^{\dot{E}\dot{C}}\,,\\
&\oint_{w\sim z} \frac{\mathrm{d}w}{2\pi i}\, \sqrt{w}\, G_\alpha^A(w)\partial X^{B\dot{B}}(z) = \delta_A^B \partial_z\left(\sqrt{z}\, \psi^{\alpha\dot{B}}(z)\right),
\end{aligned} \tag{2.71}$$

so

$$\partial X_{A\dot{A}}(z)\bar{\partial} X_{B\dot{B}}(\bar{z}) = -\oint_{w\sim z} \frac{\mathrm{d}w}{2\pi i} \oint_{\bar{w}\sim\bar{z}} \frac{\mathrm{d}\bar{w}}{2\pi i}\, |w|\, G_A^2(w)\widetilde{G}_B^{\dot{2}}(\bar{w})\psi^1{}_{\dot{A}}(z)\widetilde{\psi}^{\dot{1}}{}_{\dot{B}}(\bar{z}), \tag{2.72}$$

that relates the two operators. We have raised and lowered the capital latin letter indexes with $\varepsilon_{\dot{A}\dot{B}}$. We may now construct operators with different *flavours*; we have chosen in eq. (2.68) the anti-symmetric combination over the custodial-symmetry indexes - i.e. we picked the singlet combination. We may select a different configuration, e.g.

$$O^{\alpha\dot{\alpha}}(z,\bar{z}) = -\frac{i}{\sqrt{N}} \sum_{r=1}^{N} \psi^{\alpha\dot{1}}(z)\widetilde{\psi}^{\dot{\alpha}\dot{1}}(\bar{z}). \tag{2.73}$$

We also know that we have a custodial $SU(2)_1 \times SU(2)_2$ symmetry, so we may build a light operator that is a scalar under the Kac-Moody $SU(2)_L \times SU(2)_R$ and has a decomposable representation under the custodial symmetry, as

$$O^{\dot{A}\dot{B}}(z,\bar{z}) = -\frac{i}{\sqrt{2N}} \sum_{r=1}^{N} \varepsilon_{\alpha\dot{\alpha}} \psi^{\alpha\dot{A}}(z) \widetilde{\psi}^{\dot{\alpha}\dot{B}}(\bar{z}); \tag{2.74}$$

similarly, we may have a different "flavour" by building operators involving the twist fields described above, as

$$O^{\alpha\dot{\alpha}}(z,\bar{z}) = \sum_{r>s} \sqrt{\frac{2}{N(N-1)}}\, \Sigma^{\alpha\dot{\alpha}}_{(rs)} \,, \tag{2.75}$$

where $\Sigma^{\alpha\dot{\alpha}}_{(rs)}$ is the R-charged twist operator of dimension $k = k_r + k_s$ that glues together the r-th and s-th strands [80]. These operators are of different "flavour" both from the custodial-symmetric point of view as well as from the point of view of the dual supergravity side, since these operators are dual to different multiplets in the gravity side [81, 82].

All of the above are examples of *Light* operators [2, 4, 77, 78, 83], that are defined on one single strand, and on the orbifold can be described by a symmetrized sum of the schematic form

$$O_L(z,\bar{z}) = \sum_{r=1}^{N} \mathbb{1} \otimes \cdots \otimes O_{(r)}(z,\bar{z}) \otimes \cdots \otimes \mathbb{1} \,, \tag{2.76}$$

so that their conformal dimension $\Delta = h + \bar{h}$ is the same as the one of the operator defined on the single strand, and thus remains small w.r.t. N. In the following paragraph we will encounter examples of *Heavy* states, that on the contrary are products of non-trivial operator on any single strand of the schematic form

$$O_H = \sum \bigotimes_r O_{(r)} \,, \tag{2.77}$$

and thus their conformal dimension scales as N, i.e. is of the same order of the central charge $c = 6N$ of the theory.

1.5.3 $\frac{1}{4}$- *and* $\frac{1}{8}$-*BPS Heavy States*

We have already see that, in the R sector, we have 16 different vacuum states for each strand, both in the untwisted and twisted sector, labelled by their left and right R-charges and custodial-symmetry charges; in order to build a state of the ordifold we need to put together these vacuum states, such that the sum of them, weighted with their winding length, equals N. For example, the vacuum of the NS sector is trivially

$$|0\rangle_{\mathrm{NS}} = [|0\rangle_1]^N \,. \tag{2.78}$$

Spectrally flowing to R sector, since we know that $|0\rangle_1 \mapsto |{+}{+}\rangle_1$, we get

$$[|{+}{+}\rangle_1]^N \,. \tag{2.79}$$

The two-charge microstates of the D1D5 black hole are supergravity solutions dual to states of the CFT that preserves $\frac{1}{4}$ of the total supersymmetries and are constructed with the ground states in the R sector, found in each twisted sector, and combining them in a coherent state in the orbifold theory. The most general two-charge microstate is obtained by taking the tensor product of $N_k^{(m)}$ copies of the vacuum state $|m\rangle_k$, where m is a generic label for any possible R vacuum state, with the constraint that the total winding number has to be N; thus

$$|\psi_{N_k^{(m)}}\rangle = \prod_{k,m} (|m\rangle_k)^{N_k^{(m)}} \quad , \quad \sum_m \sum_k k N_k^{(m)} = N \,, \tag{2.80}$$

where $\{N_k^{(m)}\}$ is a partition of N. The convention for the normalization is such that

$$\langle \psi_{N_k^{(m)}} | \psi_{N_{k'}^{(m')}} \rangle = \mathcal{N}\left(N_k^{(m)}\right) \delta_{N_k^{(m)}, N_{k'}^{(m')}} \,, \tag{2.81}$$

where $\mathcal{N}(N_k^{(m)})$ is the number of ways the strand configuration determined by the partition $\{N_k^{(m)}\}$ can be obtained starting from the R vacuum state (2.79). One relevant example that we will encounter later on is[6]

$$[|{+}{+}\rangle_1]^{N^{(++)}} [|00\rangle_k]^{N_k^{(00)}} \quad , \quad N^{(++)} + k N_k^{(00)} = N \,, \tag{2.82}$$

where $|00\rangle_k = \varepsilon^{\dot{A}\dot{B}} |\dot{A}\dot{B}\rangle_k$.

Usually, typical states of the black hole have multiple strands with different winding and R-charges; we may then generalise this state by summing over all possible twisted sectors [2]

$$[|{+}{+}\rangle_1]^{N^{(++)}} \prod_k [|00\rangle_k]^{N_k^{(00)}} \quad , \quad N^{(++)} + \sum_k k N_k^{(00)} = N \,. \tag{2.83}$$

We may now want to build CFT states dual to the microstates of the three-charge black hole; these are $\frac{1}{8}$-BPS states; one way to obtain them from the $\frac{1}{4}$-BPS states is via the action of a rigid symmetry transformation, i.e. by acting on a generator of the superconformal algebra upon the whole state. The states obtained this way are called *superdescendants* [3, 84–87]. An example of a superdescendant is

$$[|{+}{+}\rangle_1]^{N^{(++)}} \prod_k \left[(J_{-1}^+)^k |00\rangle_k\right]^{N_k^{(00)}} \quad , \quad N^{(++)} + \sum_k k N_k^{(00)} = N \,, \tag{2.84}$$

since $J_{-1}^+ |{+}{+}\rangle_1 = 0$.

Another way to build $\frac{1}{8}$-BPS heavy states is by acting independently on each strand with an element of the global superalgebra generated by L_0, $L_{\pm 1}$, J_0^a, $G_0^{\pm\pm}$; these states are dubbed *superstrata* [3, 76, 88–90]. A prototypical example of a superstrata is given by [90]

6 Here we recall that each R vacuum state has $h = \bar{h} = 1/4$, while the value of the R-charge is $(\jmath, \bar{\jmath}) = (\pm\frac{1}{2}, \pm\frac{1}{2})$ for $|{\pm}{\pm}\rangle_1$ and $(\jmath, \bar{\jmath}) = (0, 0)$ for $|00\rangle_1$.

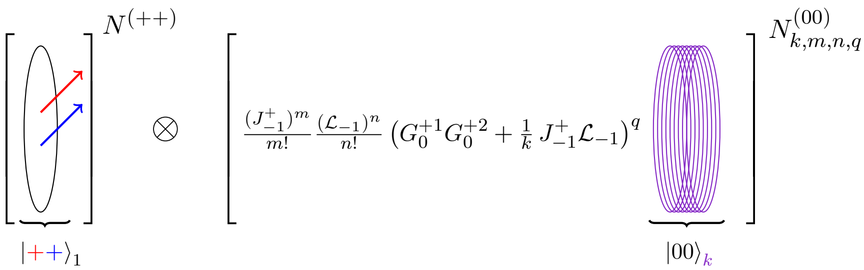


Figure 2.2: A pictorial representation of the state (2.85). We show the untwisted strands with left- and right-R-charges $|{+}{+}\rangle_1$ and the twisted uncharged $|00\rangle_k$ ones.

$$
\begin{aligned}
&[|{+}{+}\rangle_1]^{N^{(++)}}\prod_k[|k,m,n,q\rangle]^{N^{(00)}_{k,m,n,q}}\,,\\
&|k,m,n,q\rangle=\frac{(J^+_{-1})^m}{m!}\frac{(\mathcal{L}_{-1})^n}{n!}\left(G_0^{+1}G_0^{+2}+\frac{1}{k}J^+_{-1}\mathcal{L}_{-1}\right)^q|00\rangle_k\,,\\
&N^{(++)}+\sum_k kN^{(00)}_{k,m,n,q}=N\,,
\end{aligned}
\tag{2.85}
$$

where $\mathcal{L}_n\equiv L_n-J^3_n$, $m\le k-2q$ and $q=0,1$, while $n=0,1,2,\dots$ can be any non-negative integer. A pictorial representation of this state is reported in fig. 2.2.

1.6 *Correlators: basic definition and examples*

As always happens in CFT, the 2- and 3-point functions are fixed by conformal invariance to be [79, 91–94]

$$
\langle O_1(z_1,\bar z_1)O_2(z_2,\bar z_2)\rangle=\frac{\delta_{h_1,h_2}}{z_{12}^{h_1}}\frac{\delta_{\bar h_1,\bar h_2}}{\bar z_{12}^{\bar h_1}}\,,\tag{2.86a}
$$

$$
\langle O_1(z_1,\bar z_1)O_2(z_2,\bar z_2)O_3(z_3,\bar z_3)\rangle=\frac{C_{123}}{z_{12}^{h_{12;3}}z_{13}^{h_{13;2}}z_{23}^{h_{23;1}}}\frac{\bar C_{123}}{\bar z_{12}^{\bar h_{12;3}}\bar z_{13}^{\bar h_{13;2}}\bar z_{23}^{\bar h_{23;1}}}\,,\tag{2.86b}
$$

where we have defined $h_{ij;k}\equiv h_i+h_j-h_k$ and $z_{ij}=z_i-z_j$, where we have exploited the fact that for the CFT_2 the symmetry group is factorised as $\text{Vir}\otimes\overline{\text{Vir}}$, and where we have normalised the operators such that the Zamolodchikov metric is flat[7], $g_{ij}=\delta_{ij}$. The C_{ijk} are the 3-point structure constants, and together with the conformal dimensions forms the so-called *OPE data* of the CFT. From here it is clear that the first non-trivial n-point function that contains information about the dynamics of the theory is the 4-point function

$$
\begin{aligned}
\mathcal{C}(z_1,z_2,z_3,z_4)&=\langle O_1(z_1,\bar z_1)\bar O_1(z_2,\bar z_2)O_2(z_3,\bar z_3)\bar O_2(z_4,\bar z_4)\rangle\\
&=\frac{1}{z_{12}^{2h_1}z_{34}^{2h_2}}\frac{1}{\bar z_{12}^{2\bar h_1}\bar z_{34}^{2\bar h_2}}\mathcal{G}(z,\bar z)\,,
\end{aligned}
\tag{2.87}
$$

7 This choice can always be made locally in moduli space, since it is equivalent to choosing the Riemann normal coordinates on the moduli space manifold.

where $\mathcal{G}$ is a function of the conformal cross ratios

$$z = \frac{z_{14}z_{23}}{z_{13}z_{24}}, \quad \bar{z} = \frac{\bar{z}_{14}\bar{z}_{23}}{\bar{z}_{13}\bar{z}_{24}}, \tag{2.88}$$

and where we have already written down the form of the 4-point function that will be more relevant for us, i.e. with operators that are pairwise equal (up to have opposite R-charges)[8]. The two conformal cross ratios are usually repacked into [94]

$$u = z\bar{z}, \quad v = (1-z)(1-\bar{z}). \tag{2.90}$$

The standard notation is to call *Heavy-Heavy-Light-Light* (HHLL) a correlator that involves two-heavy and two-light operators, while is usual to call *Light-Light-Light-Light* (LLLL) a correlator with four light operators.

Since we will mainly compute HHLL and LLLL operators that might have an insertion of generators of the superalgebra, it is wise to look at the possible Ward identities that may play a role in the following; in fact, the superalgebra is preserved in all points of the moduli space, especially at the supergravity point. This means that the Ward identities have to be satisfied from the 4-point functions computed holographically, and thus they constitute a nice sanity check for holographic correlators.

1.6.1 *An example*

We want now to compute an HHLL 4-point function at the free orbifold point; in order to furnish an example to the reader, we will compute the 4-point functions involving the Heavy state (2.83) and the two light operators (2.68) and (2.70); since we have seen that they are related by the action of a generator of the superalgebra, we will derive the Ward identity that relates the two correlators and we will check that it is indeed satisfied. We have then to compute

$$\langle O_H(z_1,\bar{z}_1)\bar{O}_H(z_2,\bar{z}_2)O_L(z_3,\bar{z}_3)\bar{O}_L(z_4,\bar{z}_4)\rangle = \frac{1}{z_{12}^{2h_H}z_{34}^{2h_L}}\frac{1}{\bar{z}_{12}^{2\bar{h}_H}\bar{z}_{34}^{2\bar{h}_L}}\mathcal{G}(z,\bar{z}), \tag{2.91}$$

In order to easily isolate $\mathcal{G}$ from the correlators one can take the gauge $z_2 \to \infty$, $z_1 = 0$ and $z_3 = 1$, which implies $z = z_4$:

$$\langle \bar{O}_H|O_L(1)\bar{O}_L(z,\bar{z})|O_H\rangle \equiv \mathcal{C}(z,\bar{z}) = \frac{1}{(1-z)^{2h_L}}\frac{1}{(1-\bar{z})^{2\bar{h}_L}}\mathcal{G}(z,\bar{z}). \tag{2.92}$$

With the above choice of light and heavy operators the correlator at the free orbifold point depends only on the strand structure and not on the particular quantum numbers of the R ground state considered; this simply because the elementary bosonic and fermionic fields commute. A standard way to calculate this correlator is to diagonalize the boundary conditions - as in (2.38) - and then to take the linear combination of the contributions of each strand. We

---

8 The generic 4-point function with four different operators can be written as [95]

$$\langle O_1(z_1,\bar{z}_1)\bar{O}_2(z_2,\bar{z}_2)O_3(z_3,\bar{z}_3)O_4(z_4,\bar{z}_4)\rangle = \left(\frac{|z_{24}|}{|z_{14}|}\right)^{\Delta_{12}^-}\left(\frac{|z_{14}|}{|z_{13}|}\right)^{\Delta_{34}^-}\frac{\mathcal{G}(z,\bar{z})}{|z_{12}|^{\Delta_{12}^+}|z_{34}|^{\Delta_{34}^+}}, \tag{2.89}$$

where $\Delta_{ij}^{\pm} = \Delta_i \pm \Delta_j$ and $\Delta = h + \bar{h}$.

start with (2.70) and notice that we can use (2.38) and rewrite the k terms belonging to a single strand in the operators as a sum over ρ

$$\sum_{r=1}^{k} \partial X^{A\dot{B}}_{(r)}(z)\bar{\partial} X^{A\dot{C}}_{(r)}(\bar{z}) = \sum_{\rho=0}^{k-1} \partial X^{A\dot{B}}_{\rho}(z)\bar{\partial} X^{A\dot{C}}_{\rho}(\bar{z}) \,. \tag{2.93}$$

Then by the commutation relations in the twisted sector

$$\left[\alpha^{A\dot{A}}_{\rho_1,n}, \alpha^{B\dot{B}}_{\rho_2,m}\right] = \epsilon^{AB}\epsilon^{\dot{A}\dot{B}}\, n\, \delta_{n+m,0}\, \delta_{\rho_1,\rho_2} \,, \tag{2.94}$$

we can easily calculate the 2-point correlator on strand of length k

$$ {}_k\langle 0|\partial X^{1\dot{1}}_{\rho}(z_1)\, \partial X^{2\dot{2}}_{\rho}(z_2)|0\rangle_k = \frac{1}{(z_1-z_2)^2}\left(\frac{z_1}{z_2}\right)^{-\frac{\rho}{k}} \left\{1-\frac{\rho}{k}\left(1-\frac{z_1}{z_2}\right)\right\}, \tag{2.95}$$

with similar formulae holding for the antiholomorphic sector. Thus we have

$$\begin{aligned} \mathcal{C}^{\rm Bos}_k(z,\bar{z}) &= \frac{1}{(1-z)^2(1-\bar{z})^2}\sum_{\rho=0}^{k-1}|z|^{\frac{2\rho}{k}}\left|1-\frac{\rho}{k}\left(1-\frac{1}{z}\right)\right|^2 \\ &= \partial\bar{\partial}\left[\frac{1-z\bar{z}}{(1-z)(1-\bar{z})\left(1-(z\bar{z})^{\frac{1}{k}}\right)}\right] . \end{aligned} \tag{2.96}$$

This then gives for the bosonic HHLL correlator

$$\mathcal{C}^{\rm Bos} = \frac{1}{N}\sum_{k=1}^{N} N_k \mathcal{C}^{\rm Bos}_k = \frac{1}{N}\sum_{k=1}^{N} N_k \partial\bar{\partial}\left[\frac{1-z\bar{z}}{(1-z)(1-\bar{z})\left(1-(z\bar{z})^{\frac{1}{k}}\right)}\right] , \tag{2.97}$$

Following a similar approach, it is straightforward to calculate the contribution of a strand of length k to the correlator with the fermionic light operators (2.68)

$$\mathcal{C}^{\rm Fer}_{k\,(\jmath\,\bar{\jmath})} = \frac{1}{|z|}\frac{|z|^{\frac{2}{k}}-|z|^2}{(1-z)(1-\bar{z})\left(1-|z|^{\frac{2}{k}}\right)} + f_{(\jmath,\bar{\jmath})}(z,\bar{z}), \tag{2.98}$$

where $f_{k\,(\jmath,\bar{\jmath})}$ is the $\rho = 0$ contribution which depends on the $SU(2)_L \times SU(2)_R$ quantum numbers as

$$\begin{aligned} f_{(\jmath,\bar{\jmath})} &= \frac{z^{\jmath}\bar{z}^{\bar{\jmath}}}{(1-z)(1-\bar{z})}\,, \quad \text{with } \jmath\,, \bar{\jmath} = \pm\frac{1}{2}\,, \\ f_{(0,0)} &= \frac{1}{2|z|(1-z)(1-\bar{z})}\left(1+|z|^2+|1-z|^2\right) . \end{aligned} \tag{2.99}$$

The generic correlator with fermionic light operators is thus

$$\mathcal{C}^{\rm Fer} = \frac{1}{N}\sum_{k=1}^{N}\sum_{s=1}^{8} N^{(s)}_k \mathcal{C}^{\rm Fer}_{k\,(s)}\,, \tag{2.100}$$

where s runs over the 8 different RR ground states (4 with $\jmath, \bar{\jmath} = \pm 1/2$ and 4 with $\jmath, \bar{\jmath} = 0$), $N^{(s)}_k$ is the number of strands of length k in the state s, which has to satisfy the constraint $\sum_{s,k} k N^{(s)}_k = N$.

We have already seen in eq. (2.72) that the action of the fermionic generator of the superalgebra relates the Fermionic and the Bosonic operator; we then deform the contour of integration so that it goes around all the other insertions in the correlator (2.92). This explains why in (2.72) we inserted an extra factor of $\sqrt{w}$ which makes the integration of the supercurrents around the R heavy states at $z = 0, \infty$ well defined. Since we are focusing on the case where O_H are R ground states, the contributions from $w \sim 0$ and $w \sim \infty$ vanish and so the only non trivial terms come from $w \sim z$ and $\bar{w} \sim \bar{z}$, which can be computed using (2.72). In summary we obtain the relation

$$\langle \bar{O}_H | O_{\mathrm{Bos}}(1) \bar{O}_{\mathrm{Bos}}(z, \bar{z}) | O_H \rangle = \partial \bar{\partial} \Big[|z| \langle \bar{O}_H | O_{\mathrm{Fer}}(1) \bar{O}_{\mathrm{Fer}}(z, \bar{z}) | O_H \rangle \Big] . \quad (2.101)$$

This is clearly satisfied by the orbifold point results (2.97) and (2.100), but since this relation uses only the superconformal algebra, we remark that it holds at a generic point of the CFT moduli space, especially at the supergravity point.

2 TYPE IIB SUPERGRAVITY ON $\mathbb{T}^4$

On the supergravity side of the duality, the D1D5 system is a type IIB system on $\mathcal{M}^{1,4} \times \mathbb{S}^1 \times \mathbb{T}^4$ with n_5 D5 branes wrapping $\mathbb{S}^1 \times \mathbb{T}^4$ and n_1 D1 branes wrapping the common $\mathbb{S}^1$ [42, 73]. There will be open strings connecting the branes in all possible way, i.e. $1 \cdot 1$ strings connecting D1 branes, $5 \cdot 5$ strings connecting D5 branes as well as $1 \cdot 5$ connecting D1 with D5 branes. The space that is wrapped by the branes is compact and then the brane system moves like a particle in the non-compact space $\mathcal{M}^{1,4}$, as it can be seen in fig. 2.3. Now, calling V_4 the volume of $\mathbb{T}^4$ and R the radius of the $\mathbb{S}^1$, we see that the various regimes of the D1D5 system are controlled by the set of parameters

$$(n_1,\, n_5,\, V_4,\, R,\, g_s,\, \alpha')\,. \tag{2.102}$$

We will mainly work in the regime where

$$V_4 \sim O\left((\alpha')^2\right), \quad R^2 \gg \alpha'\,, \tag{2.103}$$

so that we may consider the $\mathbb{T}^4$ as small w.r.t. $\mathbb{S}^1$. From the closed string perspective, in order to have a well-behaved supergravity regime we need to have a curvature that is small w.r.t. the string scale; we also need a small string coupling so that we do not need to consider quantum corrections. These requirements will give an ordering that is

$$n_1,\, n_5 \gg g_s^{-1} \gg 1\,. \tag{2.104}$$

Going to the IR fixed point of the D-brane system we decouple the interaction among the branes; this limit is a low-energy limit and it is known as the *decoupling limit* and it is extremely relevant for the AdS/CFT correspondence. As we will see later on, in this limit, the supergravity solutions - that, as explained, is a trustworthy regime - becomes asymptotically $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$ [42].

On the open string sector, within the effective field theory description of their interaction with the Dp-branes that defined the boundary conditions for the string, we have that the d.o.f. of the open string splits into parallel to the brane and perpendicular: the parallel ones have Neumann boundary conditions and give rise to an $U(n)$ gauge field (where n is the number of the branes); the perpendicular ones instead have Dirichelet boundary conditions and give rise to adjoint scalars of the $p+1$-dimensional theory, that describe the transverse oscillation of the Dp-branes. When all the Dp-branes are coincident, the gauge theory is said to be in the *Higgs branch*, since the gauge group $U(n)$ is unbroken, while when (some of) the Dp-branes are separated, so that $U(n)$ breaks down to $U(q) \times U(n-q)$, the gauge theory is said to be in the *Coulomb branch*. The coupling constant g_{YM} for the gauge theory for the Dp-brane is

$$g_{\mathrm{YM}}^2 = (2\pi)^{p-2} (\alpha')^{\frac{p-3}{2}} g_s\,. \tag{2.105}$$

The key point is that the Higgs branch of the D1D5 system flows in the IR to the (1+1)-dimensional $\mathcal{N} = (4,4)$ SCFT described in sec. 2.1, where $c = 6N = 6 n_1 n_5$.

In this framework it is easy to discuss the fuzzball proposal. In particular, we will focus on the study of *microstate geometries*, i.e. of smooth, horizonless solutions of supergravity. These will be (some of the) microstates of the corresponding black hole solution, that will be then regarded as a *naive* solution, with a horizon and a singularity.

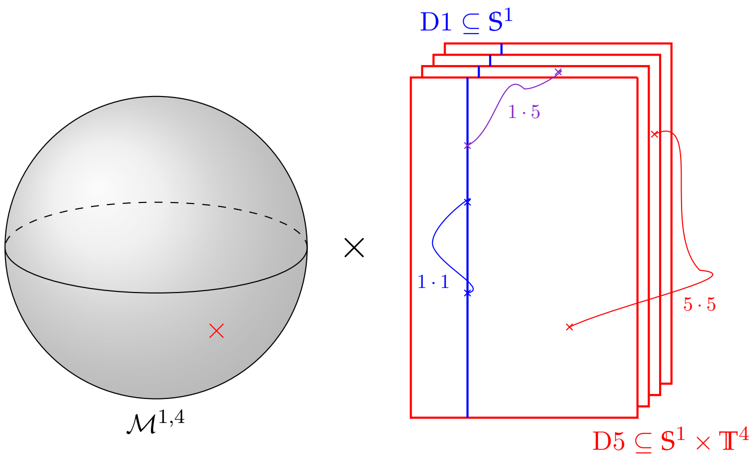


Figure 2.3: A pictorial representation of the D1D5 system. We represent on the left the non-compact $\mathcal{M}^{1,4}$ manifold where the brane system moves like a particle, represented by the small red cross. On the right we show the compact space $\mathbb{S}^1 \times \mathbb{T}^4$ on which the branes live: the D1 are wrapping only the common (blue) $\mathbb{S}^1$, while the D5 are wrapping $\mathbb{S}^1 \times \mathbb{T}^4$. There are also open strings that stretch among the branes; the $1 \cdot 1$ strings have both ends on the D1, while the $5 \cdot 5$ ones have both ends on the D5. Finally, there are the $1 \cdot 5$ strings that have one end on the D1 while the other are on the D5.

2.1 *The equations of motion of type IIB supergravity*

The bosonic content of type IIB supergravity consists in a graviton g_{MN}, a dilaton ϕ, an NSNS 2-form B_2, and a set of RR forms C_0, C_2, C_4 [54,96]. Their field strength are defined as

$$H_3 = \mathrm{d}B_2\,, \quad F_1 = \mathrm{d}C_0\,, \quad F_3 = \mathrm{d}C_2 - H_3\,C_0\,, \quad F_5 = \mathrm{d}C_4 - H_3 \wedge C_2\,, \tag{2.106}$$

so that the following Bianchi identities are satisfied:

$$\mathrm{d}H_3 = 0\,, \quad \mathrm{d}F_1 = 0\,, \quad \mathrm{d}F_3 = H_3 \wedge F_1 \quad \mathrm{d}F_5 = H_3 \wedge F_3\,. \tag{2.107}$$

The EoM they have to satisfy are[9]

$$4\mathrm{d} * \mathrm{d}\phi - 4\mathrm{d}\phi \wedge *\mathrm{d}\phi + *R - \frac{1}{2}\,H_3 \wedge *H_3 = 0\,, \tag{2.108a}$$
$$\mathrm{d} * (e^{-2\phi}H_3) - F_1 \wedge *F_3 - F_3 \wedge F_5 = 0\,, \tag{2.108b}$$
$$\mathrm{d} * F_1 + H_3 \wedge *F_3 = 0\,, \tag{2.108c}$$
$$\mathrm{d} * F_3 + H_3 \wedge F_5 = 0\,, \tag{2.108d}$$
$$F_5 - *F_5 = 0\,, \tag{2.108e}$$
$$\begin{aligned} &e^{-2\phi}\left(R_{MN} + 2\nabla_M\nabla_N\phi - \frac{1}{4}\,H_{MPQ}H_N{}^{PQ}\right) \\ &\quad + \frac{1}{4}\,g_{MN}\left(F_P F^P + \frac{1}{3!}\,F_{PQR}F^{PQR}\right) \\ &- \frac{1}{2}F_M F_N - \frac{1}{2}\frac{1}{2!}F_{MPQ}F_N{}^{PQ} - \frac{1}{4}\frac{1}{4!}\,F_{MPQRS}F_N{}^{PQRS} = 0\,, \end{aligned} \tag{2.108f}$$

9 For the notation of type IIB equations and duality rules, we refer to [3,97].

where the last one are the Einstein equations. They can be obtained from the action in the string frame

$$S_{\mathrm{IIB}} = \int \Bigg[e^{-2\phi} \left(\sqrt{-g}\, R + 4 * \mathrm{d}\phi \wedge \mathrm{d}\phi - \frac{1}{2} * H_3 \wedge H_3 \right) - \frac{1}{2} * F_1 \wedge F_1 \\ - \frac{1}{2} * F_3 \wedge F_3 - \frac{1}{4} * F_5 \wedge F_5 + \frac{1}{2} H_3 \wedge F_3 \wedge C_4 \Bigg] , \tag{2.109}$$

adding by hand the self-duality constraint for F_5[10]. The self-duality of RR fields are expressed in the polyform language as [102]

$$F = *\lambda(F) , \tag{2.110}$$

where on a p-form λ acts as $\lambda(f_p) = (-1)^{\frac{p(p-1)}{2}} f_p$, and where we have defined the RR-flux polyform as

$$F = F_1 + F_3 + F_5 + F_7 + F_9 . \tag{2.111}$$

2.1.1 *The BPS conditions*

In the following we will review some solutions to these set of equations that are especially relevant for the fuzzball proposal, and in part ii we will build some now solutions to these equations. For all the solutions that preserves a minimal amount of supersymmetry, namely all the $\frac{1}{4}$- and $\frac{1}{8}$-BPS solutions, there exists a killing spinor ϵ that satisfies the Killing spinor equation $\mathfrak{D}_M \epsilon = 0$, where $\mathfrak{D}_M$ is the generalised gauge-covariant derivative [54]. With that Killing spinor it is possible to build a vector field $V^M = \bar{\epsilon}\Gamma^M \epsilon$ where Γ^M are the 10-dimensional gamma matrices; this vector is a killing vector for the metric $\mathcal{L}_V g_{MN} = 0$, so it can be used to generate a coordinate u with its integral flow, and the metric will be u-independent by construction. Since we also preserve a subset of supersymmetries, we may employ the remaining BPS conditions to solve the equation of motions. In particular, in [102] it was shown that the minimal set of equations that one has to solve are BPS constraints dubbed with the existence of a null Killing vector whose integral flow generates the u coordinate, the self-duality of RR fields and the vv component of the Einstein equations. Since their results can be applied for all the $\frac{1}{4}$- and $\frac{1}{8}$-BPS solutions, we do not have to solve both equations of motion and BPS in those cases. It will then turns out that, to find $\frac{1}{4}$- and $\frac{1}{8}$-BPS solutions, we need to solve only

$$F_5 - *F_5 = 0 , \tag{2.112a}$$

$$\mathrm{d} * F_3 + H_3 \wedge F_5 = 0 , \tag{2.112b}$$

$$e^{-2\phi} \left(R_{vv} + 2\nabla_v \nabla_v \phi - \frac{1}{4} H_{vPQ} {H_v}^{PQ} \right) \\ + \frac{1}{4} g_{vv} \left(F_P F^P + \frac{1}{3!} F_{PQR} F^{PQR} \right) \\ - \frac{1}{2} F_v F_v - \frac{1}{2}\frac{1}{2!} F_{vPQ} {F_v}^{PQ} - \frac{1}{4}\frac{1}{4!} F_{vPQRS} {F_v}^{PQRS} = 0 . \tag{2.112c}$$

2.2 *General structure of D1D5P solutions*

Over the years, a set of two- and three-charge geometries of the type IIB system described above were built [48, 49, 76, 84–89, 102–113]; if we ask for

10 We will not discuss the Pasti - Sorokin - Tonin formalism [98, 99] for type IIB [100, 101].

the system to be invariant under rotation of the compact manifold $\mathbb{T}^4$, the solution takes the factorized form $\mathbb{R}^{1,1} \times \mathcal{B}_4 \times \mathbb{T}^4$

$$\begin{aligned} \mathrm{d}s_{10}^2 &= \sqrt{\frac{Z_1 Z_2}{\mathcal{P}}}\,\mathrm{d}s_6^2 + \sqrt{\frac{Z_1}{Z_2}}\,\mathrm{d}s_{\mathbb{T}^4}^2 , \\ e^{2\phi} &= \frac{Z_1^2}{\mathcal{P}} , \quad C_0 = \frac{Z_4}{Z_1} , \quad B_2 = \bar{B}_2 , \quad C_2 = \bar{C}_2 , \\ C_4 &= \bar{C}_4 + \frac{Z_4}{Z_2}\,\mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3 \wedge \mathrm{d}z^4 , \end{aligned} \tag{2.113}$$

where everything is z_i-independent, the forms with an over-bar have legs only in the six-dimensional non-compact space and where we have explicitly written down the directions on the $\mathbb{T}^4$. These geometries are $\frac{1}{4}$- or $\frac{1}{8}$-BPS and have a null Killing vector $\frac{\partial}{\partial u}$, inherited by the existence of a Killing spinor. Restricting to v-independent base $\mathcal{B}_4 = \mathbb{R}^4$, the ansatz for the objects appearing there is

$$\begin{aligned} \mathrm{d}s_6^2 &= -\frac{2}{\sqrt{\mathcal{P}}}(\mathrm{d}v + \beta)\left[\mathrm{d}u + \omega + \frac{\mathcal{F}}{2}(\mathrm{d}v + \beta)\right] + \sqrt{\mathcal{P}}\,\mathrm{d}s_4^2 , \\ \mathrm{d}s_4^2 &= \Sigma\left(\frac{\mathrm{d}r^2}{r^2 + a^2} + \mathrm{d}\theta^2\right) + (r^2 + a^2)\sin^2\theta\,\mathrm{d}\phi^2 + r^2\cos^2\theta\,\mathrm{d}\psi^2 , \\ \Sigma &= r^2 + a^2\cos^2\theta , \quad \mathcal{P} = Z_1 Z_2 - Z_4^2 , \quad u = \frac{t - y}{\sqrt{2}} , \quad v = \frac{t + y}{\sqrt{2}} \\ \bar{B}_2 &= -\frac{Z_4}{\mathcal{P}}(\mathrm{d}u + \omega) \wedge (\mathrm{d}v + \beta) + a_4 \wedge (\mathrm{d}v + \beta) + \delta_2 , \\ \bar{C}_2 &= -\frac{Z_2}{\mathcal{P}}(\mathrm{d}u + \omega) \wedge (\mathrm{d}v + \beta) + a_1 \wedge (\mathrm{d}v + \beta) + \gamma_2 , \\ \bar{C}_4 &= -\frac{Z_4}{\mathcal{P}}\gamma_2 \wedge (\mathrm{d}u + \omega) \wedge (\mathrm{d}v + \beta) + x_3 \wedge (\mathrm{d}v + \beta) , \end{aligned} \tag{2.114}$$

where the geometry has two charge and it is $\frac{1}{4}$-BPS if $\mathcal{F} = 0$, and it has three charge and it is $\frac{1}{8}$-BPS otherwise. We also define useful objects that are gauge invariant under the remaining gauge freedom $B_2 \to B_2 + \mathrm{d}\lambda_1$, where λ_1 is an $u\,, v$-independent 1-form and has legs only on the base space $\mathbb{R}^4$ [1,3]:

$$\Theta_1 \equiv \mathcal{D}a_1 + \dot{\gamma}_2 , \quad \Theta_2 \equiv \mathcal{D}a_2 + \dot{\gamma}_1 , \quad \Theta_4 \equiv \mathcal{D}a_4 + \dot{\delta}_2 , \tag{2.115}$$

where $\dot{f} = \partial_v f$ and where

$$\mathcal{D} \equiv \mathrm{d}_4 - \beta \wedge \partial_v . \tag{2.116}$$

In order for this ansatz to be a solution of the type IIB equations of motion, we have to impose the following "layers" of equations, following the notation of [3,76,88,89,112]: the first layer is

$$\begin{aligned} *_4\mathcal{D}\dot{Z}_1 &= \mathcal{D}\Theta_2 , \quad \mathcal{D} *_4 \mathcal{D}Z_1 = -\Theta_2 \wedge d\beta , \quad \Theta_2 = *_4\Theta_2 , \\ *_4\mathcal{D}\dot{Z}_2 &= \mathcal{D}\Theta_1 , \quad \mathcal{D} *_4 \mathcal{D}Z_2 = -\Theta_1 \wedge d\beta , \quad \Theta_1 = *_4\Theta_1 , \\ *_4\mathcal{D}\dot{Z}_4 &= \mathcal{D}\Theta_4 , \quad \mathcal{D} *_4 \mathcal{D}Z_4 = -\Theta_4 \wedge d\beta , \quad \Theta_4 = *_4\Theta_4 , \end{aligned} \tag{2.117}$$

where the $*_4$ is the Hodge relative to the base-space metric, plus the fact that $\dot{\beta} = 0$ and

$$\mathrm{d}\beta = + *_4 \mathrm{d}\beta , \tag{2.118}$$

while the second layer is

$$\begin{aligned}
&\mathcal{D}\omega + *_4\mathcal{D}\omega_4 + \mathcal{F}\,\mathrm{d}\beta = Z_1\Theta_1 + Z_2\Theta_2 - 2Z_4\Theta_4\,,\\
&*_4\mathcal{D}*_4\left(\dot{\omega} - \frac{1}{2}\mathcal{D}\mathcal{F}\right) = \partial_v^2(Z_1Z_2 - Z_4^2) - [\dot{Z}_1\dot{Z}_2 - (\dot{Z}_4)^2]\\
&\qquad\qquad - \frac{1}{2}*_4(\Theta_1\wedge\Theta_2 - \Theta_4\wedge\Theta_4).
\end{aligned} \tag{2.119}$$

One may study, as in [102,112], geometries whose base is possibly v-dependent, having then an almost-hyperkähler structure. Here we will only discuss geometries with $\dot{\beta} = 0$ and consequently a v-independent base.

THE STRUCTURE OF THE D1D5P GEOMETRIES: We are now in the position to describe the generic structure of the D1D5P solution, that have three charges as seen from infinity, corresponding to the three charges of the Strominger-Vafa black hole, i.e. Q_1, Q_5 and Q_P, but are controlled by some additional parameters whose holographic interpretation is clear and that will be furnished in the following[11]; as briefly introduced in sec. 1, we have that these geometries can be described in various regions:

- An asymptotically flat region, for $r \gg \sqrt{Q_i}$, where the naive black hole solution and the microstate geometry are indistinguishable;
- a decoupling region $r \sim \sqrt{Q_i}$, where both the naive black hole solution and the microstate geometry approaches an $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$ region. Going further down, the throat becomes $\mathrm{AdS}_2 \times \mathbb{S}^1 \times \mathbb{S}^3 \times \mathbb{T}^4$, as for the BTZ black hole;
- a "cap" region, where the microstate geometry ends smoothly; it is the end of the spacetime, that has no causal-disconnecting horizon nor a curvature singularity, since the solution is geodesically complete.

A pictorial representation of the spacetime is reported in fig. 2.4.

2.3 *Holographic dictionary*

Now, following the lead of [75, 78, 80, 108, 110], we establish the dictionary between solutions of the D1D5(P) type IIB system of the form (2.113) and Heavy states of the D1D5 CFT of sec. 2.1. We use the holographic renormalization to extract conformal data from the gravity solution, in order to find

11 Notice that we can build solutions of this type IIB system which are Asymptotically flat or asymptotically AdS; the asymptotic structure is encoded in the large r expansion of Z_1 and Z_2: if they decays as r^{-2}, the geometry is asymptotically AdS, while if they go to 1 the geometry is Asymptotically Flat.

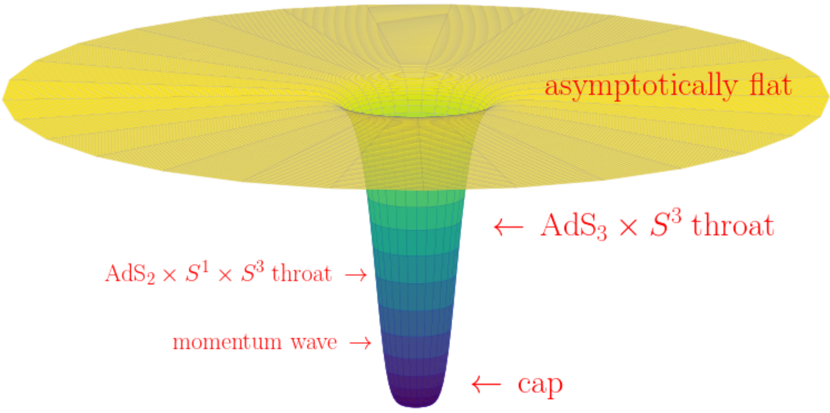


Figure 2.4: A pictorial representation of a microstate geometry; falling from the top, we begin our journey by encountering the Asymptotically Flat region, that we represented in yellow shaded by light blue lines; then the colours shade off and we find the AdS_3 throat region and the $\mathrm{AdS}_2 \times \mathbb{S}^1$ region that are common to both the naive black hole geometry and the microstate geometry. Going further down, we end up in the "cap" for the microstate, that is the region where the spacetime geodesically-ends smoothly.

a precise holographic match between geometry and states. Expanding to the boundary of AdS and following the notations of [75][12]

$$
\begin{aligned}
Z_1 &\simeq \frac{Q_1}{r^2}\left(1+\frac{f^1_{1i}}{r}\,Y^i_1\right)+O(r^{-4})\,, & Z_2 &\simeq \frac{Q_5}{r^2}\left(1+\frac{f^5_{1i}}{r}\,Y^i_1\right)+O(r^{-4})\,,\\
Z_4 &\simeq \frac{\sqrt{Q_1Q_5}}{r^3}\,\mathcal{A}_{1i}Y^i_1+O(r^{-4}), & \mathcal{F} &\simeq -\frac{2Q_p}{r^2}+O(r^{-3}),\\
\beta &\simeq -\frac{\sqrt{2Q_1Q_5}}{r^2}\,a_{\alpha-}Y^{\alpha-}_1+O(r^{-3}), & \omega &\simeq -\frac{\sqrt{2Q_1Q_5}}{r^2}\,a_{\alpha+}Y^{\alpha+}_1+O(r^{-3})\,,
\end{aligned}
\tag{2.122}
$$

where the coordinates are in a gauge where $f^1_{1i}+f^5_{1i}=0$, we can read

$$
\begin{aligned}
\langle H|J^\alpha|H\rangle = c_J\,a_{\alpha+}\,,\quad \langle H|\tilde{J}^\alpha|H\rangle = c_{\tilde{J}}\,a_{\alpha-}\,,\quad \langle H|L_0-\tilde{L}_0|H\rangle = n_p\,,\\
\langle H|O^{(0,0)}_{(2)i}|H\rangle = c_{O^{(0,0)}}\,\frac{f^1_{1i}-f^5_{1i}}{2},\quad \langle H|O^{(1,1)}_{(1)1i}|H\rangle = c_{O^{(1,1)}}\mathcal{A}_{1i}\,,
\end{aligned}
\tag{2.123}
$$

where the coefficients $c_J, c_{\tilde{J}}, c_{O^{(0,0)}}, c_{O^{(1,1)}}$ are state-independent and are fixed by consistency as

$$
c_J = -c_{\tilde{J}} = \frac{NR}{\sqrt{Q_1Q_5}}\,,\quad c_{O^{(0,0)}} = \frac{N^{3/2}R}{\sqrt{Q_1Q_5}}\,,\quad c_{O^{(1,1)}} = \frac{\sqrt{2}NR}{\sqrt{Q_1Q_5}}\,. \tag{2.124}
$$

12 Recall the definitions of the Spherical Harmonics:

$$
Y^i_1 = 2\,\frac{x^i}{r}\,,\quad Y^{\alpha+}_1 = \eta^\alpha_{ij}\,\frac{\mathrm{d}x^i\,x^j}{r^2}\,,\quad Y^{\alpha-}_1 = \bar{\eta}^\alpha_{ij}\,\frac{\mathrm{d}x^i\,x^j}{r^2}\,, \tag{2.120}
$$

where $\eta^\alpha_{ij} = \delta_{\alpha i}\delta_{4j}-\delta_{\alpha j}\delta_{4i}+\varepsilon_{\alpha ij4}$ and $\bar{\eta}^\alpha_{ij} = \delta_{\alpha i}\delta_{4j}-\delta_{\alpha j}\delta_{4i}-\varepsilon_{\alpha ij4}$ with $\alpha = 1,2,3$ are the 't Hooft symbols. Some useful examples are

$$
Y^{3+}_1 = \sin^2\theta\,\mathrm{d}\phi+\cos^2\theta\,\mathrm{d}\psi\,,\quad Y^{3-}_1 = \sin^2\theta\,\mathrm{d}\phi-\cos^2\theta\,\mathrm{d}\psi\,. \tag{2.121}
$$

The nomenclature for the operators follow from [108, 110] and it is

$$\Sigma_2^{++} = O^{(0,0)}_{(2)1} + iO^{(0,0)}_{(2)2}\,, \quad \Sigma_2^{--} = O^{(0,0)}_{(2)1} - iO^{(0,0)}_{(2)2} = +(\Sigma_2^{++})^\dagger\,, \tag{2.125a}$$

$$\Sigma_2^{+-} = O^{(0,0)}_{(2)3} + iO^{(0,0)}_{(2)4}\,, \quad \Sigma_2^{-+} = -\left(O^{(0,0)}_{(2)3} - iO^{(0,0)}_{(2)4}\right) = -(\Sigma_2^{+-})^\dagger\,, \tag{2.125b}$$

$$O^{++} = O^{(1,1)}_{(1)11} + iO^{(1,1)}_{(1)12}\,, \quad O^{--} = O^{(1,1)}_{(1)11} - iO^{(1,1)}_{(1)12} = +(O^{++})^\dagger\,, \tag{2.125c}$$

$$O^{+-} = O^{(1,1)}_{(1)13} + iO^{(1,1)}_{(1)14}\,, \quad O^{-+} = -\left(O^{(1,1)}_{(1)13} - iO^{(1,1)}_{(1)14}\right) = -(O^{+-})^\dagger\,. \tag{2.125d}$$

2.3.1 *The 20 Moduli of the theory*

We have discussed so far type IIB supergravity compactified on $\mathbb{T}^4$; it is known [42] that it has 25 scalar moduli from the six-dimensional point of view, corresponding to 10 components of the deformation of metric on the torus, h_{ij}, 6 component of the perturbation Kalb-Ramond field on the torus, b_{ij} and similarly 6 component of the perturbation of the Ramond-Ramond 2-form on the torus, c_{ij}. The remaining 3 scalars are the perturbation of the dilaton ϕ, the perturbation of the Ramond-Ramond scalar C_0 and the perturbation of the Ramond-Ramond 4-form with four legs on the $\mathbb{T}^4$ $C_{z^1z^2z^3z^4}$. These scalars parametrise the coset $SO(5,5)/(SO(5)\times SO(5))$. But in the near-horizon limit there exists an attractor mechanism that freezes five out of the 25 scalars [73, 114]:

$$\begin{aligned}
v_{\mathbb{T}^4} B_{ij} G^{ik} G^{jl} &= \frac{1}{2}\, B_{ij}\varepsilon^{ijkl}\,,\\
v_{\mathbb{T}^4} C_0 &= C_{z^1z^2z^3z^4} - \frac{1}{8}\,\varepsilon^{ijkl} B_{ij} C_{kl}\,,\\
\frac{n_1}{n_5} &= v_{\mathbb{T}^4} + \frac{1}{8}\,\varepsilon^{ijkl} B_{ij} B_{kl}\,,
\end{aligned} \tag{2.126}$$

where $v_{\mathbb{T}^4}$ is the volume of $\mathbb{T}^4$. We can classify these perturbations by their mass and charges under the symmetry groups, as in tab. 2.3. They are minimally coupled massless scalar around $\mathrm{AdS}_3\times\mathbb{S}^3\times\mathbb{T}^4$, and it seems that these perturbations are minimally coupled massless scalar even on every D1D5P background geometry. The proof of this assertion is rather non-trivial and very involved; we prove in app. A.5 that the traceless, non-diagonal part of h_{ij} indeed satisfies $\Box_6 h_{ij} = 0$, where $\Box_6$ is the Laplacian of the six-dimensional metric in the Einstein frame of a generic D1D5P solution. To get all the other, one may use a set of S- and T-dualities among different background geometries as well as different perturbations. Another way of proving it is to compactify the theory to six dimensions with all the moduli turned on and see that all the moduli are related by the Cremmer-Julia group; since one of them is a minimally-coupled massless scalar, all of them are minimally-coupled massless scalars. Unfortunately, due to its cumbersomeness, a careful proof of this statement is still lacking. In any case, using T- and S-dualities, we will show in app. A.5 that some of the moduli - the ones relevant for the present thesis - are indeed minimally coupled massless scalars.

Sugra	CFT	$SU(2)_1 \times SU(2)_2$	$SU(2)_L \times SU(2)_R$	dof
$h_{ij} - \frac{1}{4}\,\delta_{ij}h^k{}_k$	$\partial X^{(i}\bar\partial X^{j)} - \frac{1}{4}\,\partial X^i\bar\partial X_i$	$(\mathbf{3},\mathbf{3})$	$(\mathbf{1},\mathbf{1})$	9
c_{ij}	$\partial X^{[i}\bar\partial X^{j]}$	$(\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3})$	$(\mathbf{1},\mathbf{1})$	6
b^+_{ij}	$\mathcal{T}^1$	$(\mathbf{3},\mathbf{1})$	$(\mathbf{1},\mathbf{1})$	3
$v_{\mathbb{T}^4}$	$\partial X^i\bar\partial X_i$	$(\mathbf{1},\mathbf{1})$	$(\mathbf{1},\mathbf{1})$	1
Ξ	$\mathcal{T}^0$	$(\mathbf{1},\mathbf{1})$	$(\mathbf{1},\mathbf{1})$	1

Table 2.3: The 20 moduli of type IIB on $\mathbb{T}^4$. We report here the moduli on the gravity side and their holographic CFT dual operator, their representations under the symmetry groups and their number of degrees of freedom. Also, Ξ is the scalar combination of C_0 and C_{ijkl} that is not freezed by the attractor mechanism.

The CFT operators dual to such perturbations are bosonic operators, and come from the decomposition of the reducible operators $\partial X^i\bar\partial X^j$ and $\mathcal{T}^{AB} = G^{-A}\tilde G^{-B}\Sigma^{++}$. In fact, they can be rewritten as

$$\begin{aligned}\partial X^i\bar\partial X^j &= \left(\partial X^{(i}\bar\partial X^{j)} - \frac{1}{4}\,\delta_{kl}\partial X^k\bar\partial X^l\,\delta^{ij}\right)\\ &\qquad + \left(\partial X^{[i}\bar\partial X^{j]}\right) + \left(\frac{1}{4}\,\delta_{kl}\partial X^k\bar\partial X^l\right)\delta^{ij},\\ \mathcal{T}^{AB} &= \mathcal{T}^{[AB]} + \varepsilon^{AB}\varepsilon_{CD}\mathcal{T}^{CD} \equiv \mathcal{T}^1 + \mathcal{T}^0\,,\end{aligned} \tag{2.127}$$

where we use the Pauli matrices $\sigma^i_{A\dot B}$ to pass from the $SO(4)$ indexes to $SU(2)_L \times SU(2)_R$ ones.

The identification between supergravity fields and CFT operators is straightforward by looking at their representation under the symmetry groups. Notice that the bosonic operator (2.70) is one of the marginal operators discussed above. Also, both (2.68) and (2.75) are related by the action of the supercurrents to the moduli of the CFT, respectively to $\partial X^i\bar\partial X^j$ and $\mathcal{T}^{ij}$.

2.4 *All the $\frac{1}{4}$-BPS states*

As we have already pointed out in sec. 1.1, the D1D5 system without momentum, i.e. the system that admits two-charge, $\frac{1}{4}$-BPS geometries, is dual to an F1P system. In that frame, it is possible to build *all* the possible solutions, since there all the supergravity solutions are parametrized by a curve $g_A(v)$ in $\mathcal{B}_4 \times \mathbb{T}^4$, describing the profile of the fundamental string; applying then the set of dualities to bring them back to D1D5 frame, we have all the solutions [108, 110]. This procedure will led to a definitions of the Z_I, β and ω in terms of those profiles:

$$
\begin{aligned}
Z_1 &= \frac{Q_5}{L}\int_0^L \frac{|\dot g_i(v')|^2+|\dot g_5(v')|^2+|\dot g^{\alpha-}(v')|^2}{|x_i-g_i(v')|^2}\,\mathrm{d}v'\,,\\
Z_2 &= \frac{Q_5}{L}\int_0^L \frac{\mathrm{d}v'}{|x_i-g_i(v')|^2}\,,\\
Z_4 &= -\frac{Q_5}{L}\int_0^L \frac{\dot g_5(v')}{|x_i-g_i(v')|^2}\,\mathrm{d}v'\,,\quad Z_5 = -\frac{Q_5}{L}\int_0^L \frac{\dot g^{\alpha-}(v')\omega_{\alpha_-}}{|x_i-g_i(v')|^2}\,\mathrm{d}v'\,,\\
\mathrm{d}\gamma_2 &= *_4\mathrm{d}Z_2\,,\quad \mathrm{d}\gamma_1 = *_4\mathrm{d}Z_1\,,\quad \mathrm{d}\delta_4 = *_4\mathrm{d}Z_4\,,\quad \mathrm{d}\delta_5 = *_4\mathrm{d}Z_5\,,\\
A &= -\frac{Q_5}{L}\int_0^L \frac{\dot g_j(v')\,\mathrm{d}x^j}{|x_i-g_i(v')|^2}\,\mathrm{d}v'\,,\quad \mathrm{d}B = -*_4\,\mathrm{d}A\,,\quad \mathrm{d}s_4^2 = \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\,,\\
\beta &= \frac{-A+B}{\sqrt{2}}\,,\quad \omega = \frac{-A-B}{\sqrt{2}}\,,\quad \mathcal{F}=0\,,\quad a_I=0\,,\quad x_3=0\,,
\end{aligned}
\tag{2.128}
$$

where $L = 2\pi Q_5/R$ and where ω_{α_-} is a basis of anti-self dual two forms on the compact $\mathbb{T}^4$:

$$
\begin{aligned}
\omega^{\alpha-} &= (\omega^{1-}\,,\omega^{2-}\,,\omega^{3-})\,,\\
\omega^{1-} &= \left(\mathrm{d}z^1\wedge\mathrm{d}z^2-\mathrm{d}z^3\wedge\mathrm{d}z^4\right)\,,\\
\omega^{2-} &= \left(\mathrm{d}z^1\wedge\mathrm{d}z^3+\mathrm{d}z^2\wedge\mathrm{d}z^4\right)\,,\\
\omega^{3-} &= \left(\mathrm{d}z^1\wedge\mathrm{d}z^4-\mathrm{d}z^2\wedge\mathrm{d}z^3\right)\,,
\end{aligned}
\tag{2.129}
$$

and where we have parametrized the flat base space $\mathcal{B}_4 = \mathbb{R}^4$ via coordinates x_i that are defined such that

$$
x_1+ix_2 = \widetilde{r}e^{i\phi}\sin\widetilde{\theta}\,,\quad x_3+ix_4 = \widetilde{r}e^{i\psi}\cos\widetilde{\theta}\,,
\tag{2.130}
$$

where $\widetilde{r}^2\sin^2\widetilde{\theta} = (r^2+a^2)\sin^2\theta$, $\widetilde{r}^2\cos^2\widetilde{\theta} = r^2\cos^2\theta$, i.e.

$$
\widetilde{r}^2 = r^2+a^2\sin^2\theta\,,\quad \cos^2\widetilde{\theta} = \frac{r^2\cos^2\theta}{r^2+a^2\sin^2\theta}\,,
\tag{2.131}
$$

so that the flat $\mathbb{R}^4 \simeq \mathbb{R}\times\mathbb{S}^3$ reads

$$
\mathrm{d}s_4^2 = \Sigma\left(\frac{\mathrm{d}r^2}{r^2+a^2}+\mathrm{d}\theta^2\right)+(r^2+a^2)\sin^2\theta\,\mathrm{d}\phi^2+r^2\cos^2\theta\,\mathrm{d}\psi^2\,,
\tag{2.132}
$$

so that we have written the $\mathbb{S}^3$ metric with a Hopf fibration.

Here we have allowed also excitations on the internal manifold $\mathbb{T}^4$, so that the metric is[13] [14]

$$\mathrm{d}s_{10}^2 = \sqrt{\alpha}\,\mathrm{d}s_6^2 + \frac{\sqrt{\widetilde{\mathcal{P}}}}{Z_2}\,\mathrm{d}s_{\mathbb{T}^4}^2\,, \tag{2.134a}$$

$$\mathrm{d}s_6^2 = -\frac{2}{\sqrt{\mathbb{P}}}\,(\mathrm{d}v+\beta)\,(\mathrm{d}u+\omega) + \sqrt{\mathbb{P}}\;\mathrm{d}s_4^2\,, \tag{2.134b}$$

$$\mathcal{P} = Z_1 Z_2 - Z_4^2\,, \quad \widetilde{\mathcal{P}} = Z_1 Z_2 - Z_5^2\,, \quad \mathbb{P} = Z_1 Z_2 - Z_4^2 - Z_5^2\,, \tag{2.134c}$$

$$d\hat{v} = \mathrm{d}v + \beta\,, \quad d\hat{u} = \mathrm{d}u + \omega\,, \quad v = \frac{t+y}{\sqrt{2}}\,, \quad u = \frac{t-y}{\sqrt{2}}\,, \tag{2.134d}$$

$$e^{2\phi} = \alpha\,\frac{\widetilde{\mathcal{P}}}{Z_2^2}\,, \quad \omega_5 = -\ast_{\mathbb{T}^4}\omega_5\,, \quad \alpha = \frac{\widetilde{\mathcal{P}}}{\mathbb{P}}\,, \tag{2.134e}$$

$$B_2 = -\frac{Z_4}{\mathbb{P}}\,d\hat{u}\wedge d\hat{v} + \delta_2 - \frac{Z_5}{Z_2}\,\omega_5\,, \tag{2.134f}$$

$$C_0 = \frac{Z_2 Z_4}{\widetilde{\mathcal{P}}}\,, \tag{2.134g}$$

$$C_2 = -\frac{Z_2}{\mathbb{P}}\,d\hat{u}\wedge d\hat{v} + \gamma_2\,, \tag{2.134h}$$

$$C_4 = \frac{Z_4}{Z_2}\,\mathrm{vol}_{\mathbb{T}^4} - \frac{Z_4}{\mathbb{P}}\,\gamma_2\wedge d\hat{u}\wedge d\hat{v} + \left(\delta_5 - \frac{Z_5}{Z_2}\,\gamma_2\right)\wedge\omega_5\,, \tag{2.134i}$$

where here the ω_5 is any constant two-form that is anti-self dual on the $\mathbb{T}^4$.

2.4.1 *A geometry with $Z_4 \neq 0$ and $Z_5 = 0$*

We want to construct, as an example, a geometry defined by the profile that is circular on two directions of the $\mathbb{R}^4$, has no oscillations in the other two, but oscillates in one direction along the torus, i.e.

$$\begin{aligned} g_1(v') &= a\cos\left(\frac{2\pi v'}{L}\right), \quad g_2(v') = a\sin\left(\frac{2\pi v'}{L}\right), \\ g_5(v') &= -\frac{b}{k}\cos\left(\frac{2\pi k v'}{L}\right). \end{aligned} \tag{2.135}$$

Using the change of coordinates (2.130) we get[15]

$$\begin{aligned} Z_1 &= 1 + \frac{R^2}{Q_5}\,\frac{a^2+\frac{b^2}{2}}{r^2+a^2\cos^2\theta} + \frac{R^2 b^2}{2Q_5}\,\frac{a^{2k}\sin^{2k}\theta\,\cos 2k\phi}{(r^2+a^2)^k(r^2+a^2\cos^2\theta)}\,, \\ Z_2 &= 1 + \frac{Q_5}{r^2+a^2\cos^2\theta}\,, \\ Z_4 &= R\,b\,a^k\,\frac{\sin^k\theta\,\cos k\phi}{(r^2+a^2)^{\frac{k}{2}}(r^2+a^2\cos^2\theta)}\,. \end{aligned} \tag{2.137}$$

13 In the decoupling limit, the dictionary between our notation and the notation of [110] is

$$f_5 = H = Z_1\,,\ K = Z_1\,,\ \mathcal{A}^{\alpha-} = -Z_5\,,\ \mathcal{A} = -Z_4\,,\ \tilde{f}_1 = \frac{\mathbb{P}}{Z_2}\,,\ f_1 = \frac{\widetilde{\mathcal{P}}}{Z_2}\,. \tag{2.133}$$

14 Notice that $d\hat{u}$ and $d\hat{v}$ are *not* 1-forms, since $\mathrm{d}d\hat{u} \neq 0$, $\mathrm{d}d\hat{v} \neq 0$.

15 Recalling that

$$L = 2\pi\frac{Q_5}{R}. \tag{2.136}$$

In order to compute it we have to perform some change of coordinates. In fact[16]

$$\begin{aligned}
\text{den} &\equiv (x_1 - g_1(v'))^2 + (x_2 - g_2(v'))^2 + x_3^2 + x_4^2 \\
&= |(x_1 - g_1(v')) + i(x_2 - g_2(v'))|^2 + |x_3 + ix_4|^2 \\
&= |(x_1 + ix_2) - ae^{iw}|^2 + |x_3 + ix_4|^2 \\
&= |\tilde{r}\sin\tilde{\theta}\, e^{i\phi} - ae^{iw}|^2 + |\tilde{r}\cos\tilde{\theta}e^{i\psi}|^2 \\
&= \tilde{r}^2 + a^2 - a\tilde{r}\sin\tilde{\theta}\left(e^{+i(\phi-w)} + e^{-i(\phi-w)}\right) \\
&= r^2 + a^2 + a^2\sin^2\theta - a\sqrt{r^2+a^2}\,\sin\theta\left(e^{+i(\phi-w)} + e^{-i(\phi-w)}\right) \\
&\equiv A - B\left(e^{+i(\phi-w)} + e^{-i(\phi-w)}\right).
\end{aligned} \tag{2.139}$$

where we have used the change of coordinates $w = \frac{2\pi v'}{L}$ and (2.130). We now introduce the complex coordinate

$$z \equiv e^{i(\phi-w)} \;\Rightarrow\; i\frac{\mathrm{d}z}{z} = \mathrm{d}w = \frac{L}{2\pi}\mathrm{d}v'. \tag{2.140}$$

Also we notice that

$$\dot{g}_5(v') = -b\frac{2\pi}{L}\cos w = -\frac{\pi\, b}{L}\left[e^{-ik\phi}z^k + e^{ik\phi}\bar{z}^k\right]. \tag{2.141}$$

The poles in the denominator are

$$z_\pm = \frac{1}{2B}\left[A \pm \sqrt{A^2 - 4B^2}\right] \tag{2.142}$$

where

$$\Delta = \sqrt{A^2 - 4B^2} = r^2 + a^2\cos^2\theta, \quad z_- = \frac{a\sin\theta}{\sqrt{r^2+a^2}} \tag{2.143}$$

For example, the integral for Z_4 becomes a complex integral

$$\begin{aligned}
Z_4 &= -\frac{Q_5}{L}\frac{\pi\, b}{L}\frac{L}{2\pi}\oint_C \frac{\mathrm{d}z}{z}\, i\, \frac{e^{-ik\phi}z^k + e^{ik\phi}\bar{z}^k}{A - B(z + 1/z)} \\
&= \frac{\pi\, b\, Q_5}{L}\left\{\operatorname*{Res}_{\Omega}\left[\frac{e^{-ik\phi}z^k}{Bz^2 - Az + B}\right] + \operatorname*{Res}_{\tilde{\Omega}}\left[\frac{e^{ik\phi}\bar{z}^k}{Bz^2 - Az + B}\right]\right\} \\
&= R\, b\, a^k \frac{\sin^k\theta\,\cos k\phi}{(r^2+a^2)^{\frac{k}{2}}(r^2 + a^2\cos^2\theta)},
\end{aligned} \tag{2.144}$$

where C is the circle with $|z|^2 = 1$ and $\Omega : \partial\Omega = C$ is the interior of the unitary circle. Notice that the z and $\bar{z}$ terms add up with complex conjugation since, for the $\bar{z}$ case, the contour moves in the opposite direction and that is the same as integrating over the region $\mathbb{C}\setminus\Omega \equiv \overline{\Omega}$. In that region there is only the z_+ pole, that gives a contribute that is exactly the complex conjugate of the z one. With similar computations we can obtain the other two Z_1, Z_2 functions, as in eqs. (2.137).

This state is dual to the heavy state (2.83) with one single mode discussed in sec. 2.1

$$\left[|{+}{+}\rangle_1\right]^{N^{(++)}}\left[|00\rangle_k\right]^{N_k^{(00)}}, \quad N^{(++)} + kN_k^{(00)} = N. \tag{2.145}$$

16 Generically, we have

$$\text{den} = A - \frac{2B}{a}(g_1\cos\phi + g_2\sin\phi) + (g_1^2 + g_2^2 - a^2). \tag{2.138}$$

We may generalise it to include multiple modes [2], i.e.

$$\left[|{+}{+}\rangle_1\right]^{N^{(++)}} \prod_k \left[|00\rangle_k\right]^{N_k^{(00)}} , \quad N^{(++)} + \sum kN_k^{(00)} = N\,. \tag{2.146}$$

The geometry dual to such state is defined by [2]

$$\begin{aligned} Z_1 = \frac{R^2}{Q_5\,\Sigma}\Bigg[a_0^2 + \sum_{k,k'} \frac{b_k b_{k'}}{2}\, \frac{a^{k+k'}}{(r^2+a^2)^{\frac{k+k'}{2}}}\, \sin^{k+k'}\theta \cos((k+k')\phi) \\ + \sum_{k>k'} b_k b_{k'}\, \frac{a^{k-k'}}{(r^2+a^2)^{\frac{k-k'}{2}}}\, \sin^{k-k'}\theta\cos((k-k')\phi)\Bigg]\,, \end{aligned} \tag{2.147a}$$

$$Z_2 = \frac{Q_5}{\Sigma}\,, \quad Z_4 = \frac{R}{\Sigma}\sum_k b_k \frac{a^k}{(r^2+a^2)^{\frac{k}{2}}}\, \sin^k\theta\cos(k\phi)\,, \tag{2.147b}$$

$$\beta = \frac{R\,a^2}{\sqrt{2}\,\Sigma}\,(\sin^2\theta \mathrm{d}\phi - \cos^2\theta\mathrm{d}\psi)\,, \quad \omega = \frac{R\,a^2}{\sqrt{2}\,\Sigma}\,(\sin^2\theta\mathrm{d}\phi + \cos^2\theta\mathrm{d}\psi)\,. \tag{2.147c}$$

For generic values of b_k the geometry is complicated, but it can be shown to be regular and without horizon for any values of the parameters, as far as the regularity constraint

$$a^2 + \sum_k \frac{b_k}{2} = a_0^2 \equiv \frac{Q_1 Q_5}{R^2}\,, \tag{2.148}$$

is satisfied. Using the holographic dictionary, looking at the large r-expansion of the solution, it is possible to read that

$$N_1^{(++)} = N\,\frac{a^2}{a_0^2}\,, \quad kN_k^{(0)} = N\,\frac{b_k^2}{2a_0^2} \quad \text{with} \quad a_0^2 \equiv \frac{Q_1Q_5}{R^2}\,. \tag{2.149}$$

It is trivial to notice that, sending $b \to 0$, we recover pure $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$ as expected since, on the CFT side, the state becomes the vacuum $\left[|{+}{+}\rangle_1\right]^N$.

Sometimes it is useful to rewrite the metric as

$$\mathrm{d}s_6^2 = V^{-2}\mathrm{d}s_3^2 + G_{ab}(\mathrm{d}\theta^a + A^a)(\mathrm{d}\theta^b + A^b)\,, \quad V^2 \equiv \frac{\det G_{ab}}{\det G_{\mathbb{S}^3}}\,. \tag{2.150}$$

For the single-mode solution, the AdS_3 part can be written as

$$\frac{g_{tt}^{(k)}}{\sqrt{Q_1Q_5}} = -\frac{r^2+a^2}{R^2 a_0^4}\left(a^2 + \frac{b_k^2}{2}\frac{r^2}{\Sigma}F_k\right)\,, \tag{2.151a}$$

$$\frac{g_{yy}^{(k)}}{\sqrt{Q_1Q_5}} = \frac{r^2}{R^2 a_0^4}\left(a^2 + \frac{b_k^2}{2}\frac{r^2+a^2}{\Sigma}F_k\right)\,, \tag{2.151b}$$

$$\frac{g_{rr}^{(k)}}{\sqrt{Q_1Q_5}} = \frac{1}{a_0^4(r^2+a^2)}\left(a^2 + \frac{b_k^2}{2}\frac{r^2}{\Sigma}F_k\right)\left(a^2 + \frac{b_k^2}{2}\frac{r^2+a^2}{\Sigma}F_k\right)\,, \tag{2.151c}$$

where

$$F_k \equiv 1 - \left(\frac{a^2\sin^2\theta}{r^2+a^2}\right)^k\,. \tag{2.152}$$

For the single-mode solution, i.e. for $k = 1$, the geometry is *separable* and in the limit $a \to 0$, the metric approaches the massless BTZ metric, while for $b \to 0$ it approaches the vacuum $\mathrm{AdS}_3 \times \mathbb{S}^3$.

2.4.2 *Another non-trivial example:* $\otimes_k |{+}{+}\rangle_k$ *geometry*

We now to build a geometry that contains only $|\pm\pm\rangle$ vacuum states with different length; following [115], we use as a profile function

$$g \equiv g_1 + i g_2 = a \exp\left[i\left(w + \frac{b}{a}\sin w\right)\right], \quad w = \frac{2\pi v}{L}, \tag{2.153}$$

that it is dual to the state

$$|\tilde{H}\rangle = \cdots A\,|{+}{+}\rangle_1\, B\,|{+}{+}\rangle_2\, B^2\,|{+}{+}\rangle_3 \cdots, \tag{2.154}$$

as we can easily see by expanding the profile function in power series of b. With this choice we have

$$\begin{aligned} \text{den} &= |(x_1 + i x_2) - g|^2 + |x_3 + i x_4|^2 \\ &= r^2 + a^2(1+\sin^2\theta) - a\sqrt{r^2+a^2}\cos\theta\left(e^{i\left(\phi - w - \frac{b}{a}\sin w\right)} + \text{c.c.}\right) \\ &= A - B\left(e^{i\left(\phi - w - \frac{b}{a}\sin w\right)} + \text{c.c.}\right). \end{aligned} \tag{2.155}$$

We will compute the Z_i only up to $O(b^2)$, that is the power relevant for holographic purposes. For the computations that follow, we have to use the result of the integral

$$I_{n,k} = \int_0^{2\pi} \mathrm{d}w\, \frac{e^{ikw}}{(A - 2B\cos w)^n}, \tag{2.156}$$

where we have defined, as above,

$$A = r^2 + a^2(1+\sin^2\theta), \quad B = a\sqrt{r^2+a^2}\,\sin\theta. \tag{2.157}$$

Introducing now the complex coordinate

$$z = e^{iw} \Rightarrow i\,\frac{\mathrm{d}z}{z} = \mathrm{d}w, \tag{2.158}$$

we have to compute[17]

$$\begin{aligned} I_{n,k} &= \frac{(-1)^n}{B^n} i \oint \mathrm{d}z\, \frac{z^{n+k-1}}{(z-z_+)^n (z-z_-)^n} \\ &= \frac{2\pi(-1)^{n+1}}{B^n} \lim_{z\to z_-} \frac{1}{(n-1)!}\,\partial_z^{n-1}\left[\frac{z^{n+k-1}}{(z-z_+)^n}\right]. \end{aligned} \tag{2.159}$$

We report here[18] the first three examples,

$$\begin{aligned} I_{n=1,k} &= -\frac{2\pi}{\Sigma}\,\Delta_k, \\ I_{n=2,k} &= +2\pi\,\frac{(k-1)\Sigma + 2(r^2+a^2)}{\Sigma^3}\,\Delta_k, \\ I_{n=3,k} &= 2\pi\frac{\Delta_k}{\Sigma^5}\left[\frac{(k-1)(k-2)}{2}\Sigma^2 + 3(k-2)(r^2+a^2)\Sigma + 6(r^2+a^2)^2\right], \end{aligned} \tag{2.160}$$

where we have defined

$$\Sigma = r^2 + a^2\cos^2\theta, \quad \Delta_k = \frac{a^k \sin^k\theta}{(r^2+a^2)^{\frac{k}{2}}}. \tag{2.161}$$

---

17 We have defined the two roots of the denominator
$$z_\pm = \frac{A \pm \sqrt{A^2 - 4B^2}}{B}.$$

18 Please notice that for $k \leq -n$ we should include the contribution from the $z = 0$ pole.

THE REGULARITY CONDITION: We see that the regularity condition, since

$$|\dot{g}_i|^2 = |\dot{g}|^2 = a^2 \left(1 + \frac{b}{a} \cos w\right)^2 , \tag{2.162}$$

so that

$$\begin{aligned} Q_1 &= \frac{2\pi Q_5}{L^2} \int_0^{2\pi} \mathrm{d}w \, |\dot{g}|^2 \\ &= \frac{2\pi Q_5}{L^2} \int_0^{2\pi} \mathrm{d}w \, [a^2 + 2ab\cos w + b^2 \cos^2 w] \\ &= \frac{4\pi^2 Q_5}{L^2} \left(a^2 + \frac{b^2}{2}\right) , \end{aligned} \tag{2.163}$$

and, recalling that $L = 2\pi Q_5/R$,

$$\frac{Q_1 Q_5}{R^2} = a^2 + \frac{b^2}{2} . \tag{2.164}$$

THE Z_2: We now expand

$$Z_2 = Z_2^{(0)} + \frac{b}{\sqrt{2}\, a} Z_2^{(1)} + \frac{b^2}{2a^2} Z_2^{(2)} + O(b^3) , \tag{2.165}$$

Explicitly we have, using that $A = r^2 + a^2(1 + \sin^2\theta)$, $B = a\sqrt{r^2 + a^2} \cos\theta$,

$$\begin{aligned} Z_2^{(0)} &= +\frac{Q_5}{\Sigma} , \\ Z_2^{(1)} &= -\frac{Q_5}{\Sigma} \frac{\sqrt{2}\, a \sin\theta \cos\phi}{\sqrt{r^2 + a^2}} , \\ Z_2^{(2)} &= +\frac{Q_5}{\Sigma} \frac{2a^2 \sin^2\theta \cos 2\phi}{(r^2 + a^2)} . \end{aligned} \tag{2.166}$$

The full Z_2 at order b^2 is then

$$Z_2 = \frac{Q_5}{\Sigma} \left[1 - \frac{b}{\sqrt{2}\, a} \frac{\sqrt{2}\, a \sin\theta \cos\phi}{\sqrt{r^2 + a^2}} + \frac{b^2}{2a^2} \frac{2a^2 \sin^2\theta \cos 2\phi}{(r^2 + a^2)}\right] . \tag{2.167}$$

THE Z_1: Here we have to compute the numerator, that is

$$|\dot{g}_i|^2 = |\dot{g}|^2 = a^2 \left(1 + \frac{b}{a} \cos w\right)^2 , \tag{2.168}$$

so that, expanding again

$$Z_1 = Z_1^{(0)} + \frac{b}{\sqrt{2}\, a} Z_1^{(1)} + \frac{b^2}{2a^2} Z_1^{(2)} + O(b^3) , \tag{2.169}$$

we get

$$\begin{aligned} Z_1^{(0)} &= +\frac{\frac{a^2 R^2}{Q_5}}{\Sigma} , \\ Z_1^{(1)} &= +\frac{\frac{a^2 R^2}{Q_5}}{\Sigma} \frac{\sqrt{2}\, a \sin\theta \cos\phi}{\sqrt{r^2 + a^2}} , \\ Z_1^{(2)} &= -\frac{\frac{a^2 R^2}{Q_5}}{\Sigma} \left[\frac{a^2 \sin^2\theta \cos 2\phi}{(r^2 + a^2)} - 1\right] . \end{aligned} \tag{2.170}$$

The full Z_1 is then

$$Z_1 = \frac{\frac{a^2 R^2}{Q_5}}{\Sigma}\left[1+\frac{b}{\sqrt{2}\,a}\,\frac{\sqrt{2}\,a\sin\theta\,\cos\phi}{\sqrt{r^2+a^2}}-\frac{b^2}{2a^2}\left(\frac{a^2\sin^2\theta\,\cos 2\phi}{(r^2+a^2)}-1\right)\right]. \tag{2.171}$$

THE ω, β AND Z_4: This is the easy part, since it is trivial to notice that

$$Z_4 = 0\,,\quad \beta=\beta_0+O(b^3),\quad \omega=\omega_0+O(b^3), \tag{2.172}$$

where β_0 and ω_0 are the ones of the $[|{+}{+}\rangle_1]^N$ geometry, i.e.

$$\beta = \frac{R\,a^2}{\sqrt{2}\Sigma}\left(\sin^2\theta\,\mathrm{d}\phi-\cos^2\theta\,\mathrm{d}\psi\right),\quad \omega=\frac{R\,a^2}{\sqrt{2}\Sigma}\left(\sin^2\theta\,\mathrm{d}\phi+\cos^2\theta\,\mathrm{d}\psi\right). \tag{2.173}$$

THE GEOMETRY: To recap we have

$$\begin{aligned}
Z_1 &= \frac{\frac{a^2 R^2}{Q_5}}{\Sigma}\left[1+\frac{b}{\sqrt{2}\,a}\,\frac{\sqrt{2}\,a\sin\theta\,\cos\phi}{\sqrt{r^2+a^2}}-\frac{b^2}{2a^2}\left(\frac{a^2\sin^2\theta\,\cos 2\phi}{(r^2+a^2)}-1\right)\right],\\
Z_2 &= \frac{Q_5}{\Sigma}\left[1-\frac{b}{\sqrt{2}\,a}\,\frac{\sqrt{2}\,a\,\sin\theta\,\cos\phi}{\sqrt{r^2+a^2}}+\frac{b^2}{2a^2}\,\frac{2a^2\sin^2\theta\,\cos 2\phi}{(r^2+a^2)}\right],\\
\beta &= \frac{R\,a^2}{\sqrt{2}\Sigma}\left(\sin^2\theta\,\mathrm{d}\phi-\cos^2\theta\,\mathrm{d}\psi\right),\quad \omega=\frac{R\,a^2}{\sqrt{2}\Sigma}\left(\sin^2\theta\,\mathrm{d}\phi+\cos^2\theta\,\mathrm{d}\psi\right).
\end{aligned} \tag{2.174}$$

This means that we have the same geometry as the state of the previous example, i.e.

$$\begin{aligned}
|\tilde{H}\rangle &= \cdots A\,|{+}{+}\rangle_1\,B\,|{+}{+}\rangle_2\,B^2\,|{+}{+}\rangle_3\cdots \leftrightarrow g_1+ig_2=a\exp\left[i\left(w+\frac{b}{a}\,\sin w\right)\right],\\
|H\rangle &= A\,|{+}{+}\rangle_1\,B|00\rangle_1 \qquad\qquad \leftrightarrow g_1+ig_2=a'e^{iw}\,,\ g_5=-\frac{b'}{\sqrt{2}}\,e^{iw}\,,
\end{aligned} \tag{2.175}$$

where F is the profile on one direction of the Torus, have the same 6D geometry in the Einstein frame

$$\mathrm{d}s^2_{\tilde{H}} = \mathrm{d}s^2_H + O(b^3)\,. \tag{2.176}$$

This simply comes from the fact that the two $\mathcal{P}$ are the same at this order:

$$\begin{aligned}
\mathcal{P}_{|\tilde{H}\rangle'} &\equiv Z_1 Z_2 = \frac{a^2R^2}{\Sigma^2}+\frac{b^2}{2}\,\frac{R^2}{\Sigma(r^2+a^2)}+O(b^3)\,,\\
\mathcal{P}_{|H\rangle} &\equiv Z_1Z_2-Z_4=\frac{(a')^2R^2}{\Sigma^2}+\frac{(b')^2}{2}\,\frac{R^2}{\Sigma(r^2+(a')^2)}+O((b')^3)\,,
\end{aligned} \tag{2.177}$$

where the identification that follows is

$$a=a'\,,\quad b=b'\,. \tag{2.178}$$

Careful holographic computations will give the relation between A, B with a, b so that

$$|\tilde{H}\rangle = \cdots\left[\left(a-\frac{b^2}{4a}\right)|{+}{+}\rangle_1\right](b\,|{+}{+}\rangle_2)\cdots\,. \tag{2.179}$$

2.5 *Some known $\frac{1}{8}$-BPS superstrata*

We now report two relevant examples of $\frac{1}{8}$-BPS solutions that will be useful in chap. 6 and that will be generalised in chap. 3.

2.5.1 *External excitations*

The system we have discussed in eq. (2.113) admits a set of nice solutions [76, 88–90]; one of the most relevant for us will be the so called $|10n\rangle$ superstratum, that is a geometry dual to the state reported in eq. (2.85) with $q = 0$. The explicit solution of the two layers (2.117, 2.119) for the $(k, m, n) = (1, 0, n)$ geometry was found in [88, 89]

$$\mathrm{d}s^2_{10} = \sqrt{\frac{Z_1 Z_2}{\mathcal{P}}}\,\mathrm{d}s^2_6 + \sqrt{\frac{Z_1}{Z_2}}\,\mathrm{d}s^2_{\mathbb{T}^4}\,, \tag{2.180a}$$

$$\mathrm{d}s^2_6 = -\frac{2}{\sqrt{\mathcal{P}}}(\mathrm{d}v+\beta)\left[\mathrm{d}u + \omega + \frac{\mathcal{F}}{2}(\mathrm{d}v+\beta)\right] + \sqrt{\mathcal{P}}\,\mathrm{d}s^2_4\,, \tag{2.180b}$$

$$\hat{v}_{1,0,n} = \frac{\sqrt{2}}{R}\,nv + \phi\,, \quad \hat{v}_{2,0,2n} = \frac{\sqrt{2}}{R}\,2nv + 2\phi\,, \tag{2.180c}$$

$$\Delta_{1,0,n} = \frac{a\,r^n}{(r^2+a^2)^{\frac{n+1}{2}}}\,\sin\theta\,, \quad \Delta_{2,0,2n} = \frac{a^2\,r^{2n}}{(r^2+a^2)^{n+1}}\,\sin^2\theta \tag{2.180d}$$

$$Z_1 = \frac{Q_1}{\Sigma} + \frac{R^2}{2Q_5}\,b^2\,\frac{\Delta_{2,0,2n}}{\Sigma}\cos\hat{v}_{2,0,2n} \qquad Z_2 = \frac{Q_2}{\Sigma}\,, \tag{2.180e}$$

$$Z_4 = bR\,\frac{\Delta_{1,0,n}}{\Sigma}\cos\hat{v}_{1,0,n} \tag{2.180f}$$

$$\omega = \frac{a^2 R}{\sqrt{2}\,\Sigma}\left(\sin^2\theta\,\mathrm{d}\phi + \cos^2\theta\,\mathrm{d}\psi\right) + \frac{b^2}{a^2}\frac{a^2 R}{\sqrt{2}\,\Sigma}\left[1 - \frac{r^{2n}}{(r^2+a^2)^n}\right]\sin^2\theta\,\mathrm{d}\phi\,, \tag{2.180g}$$

$$\mathcal{F} = -\frac{b^2}{a^2}\left[1 - \frac{r^{2n}}{(r^2+a^2)^n}\right]\,, \quad \beta = \frac{a^2 R}{\sqrt{2}\,\Sigma}\left(\sin^2\theta\,\mathrm{d}\phi - \cos^2\theta\,\mathrm{d}\psi\right)\,, \tag{2.180h}$$

so that, calling $F_n \equiv \left[1 - \frac{r^{2n}}{(r^2+a^2)^n}\right]$,

$$\omega = \frac{a^2 R}{\sqrt{2}\,\Sigma}\left[\left(1 + \frac{b^2}{a^2}F_n\right)\sin^2\theta\,\mathrm{d}\phi + \cos^2\theta\,\mathrm{d}\psi\right]\,. \tag{2.181}$$

Notice that this is a three-charge geometry, due to the fact that we have a non-vanishing $\mathcal{F}$, controlled by a non-vanishing n; sending $n \to 0$ will thus reduce it to a 2-charge geometry, as expected. This metric can be rewritten, via the splitting $x^M = (x^\mu, \theta^a)$, with $M = 0, \ldots, 5$, $\mu = 0, 1, 2$ and $a = 3, 4, 5$, as

$$\mathrm{d}s^2_6 = V^{-2} g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + G_{ab}(\mathrm{d}\theta^a + A^a_\mu \mathrm{d}x^\mu)(\mathrm{d}\theta^b + A^b_\nu \mathrm{d}x^\nu), \tag{2.182}$$

where V is fixed requiring that $\sqrt{-\det G_{MN}} = \sqrt{-\det g_{\mu\nu}}\sqrt{\det q_{ab}}$ with q_{ab} the round $\mathbb{S}^3$ metric, for all values of b. The three-dimensional non-compact metric is

$$\begin{aligned}\mathrm{d}s^2_3 = &-\left[r^2\left(1 - \frac{b^2}{2a_0^2}\,F_n\right) + \frac{a^4}{a_0^2}\right]\mathrm{d}\tau + r^2\left(1 + \frac{b^2}{2a_0^2}\,F_n\right)\mathrm{d}\sigma^2 \\ &+ \frac{r^2 + \frac{a^2}{a_0^2}\left(a^2 + \frac{b^2}{2}\,F_n\right)}{(r^2+a_0^2)^2}\,\mathrm{d}r^2\,,\end{aligned} \tag{2.183}$$

where we have defined the adimensional coordinates

$$\tau = \frac{t}{R}\,, \quad \sigma = \frac{y}{R}\,. \tag{2.184}$$

This system has only *external* excitations, and thus this solution is invariant under rotation of the compact manifold $\mathbb{T}^4$; now we see how it is possible, using it as as seed, to use T- and S-dualities to obtain a new solution that has *internal* excitations, i.e. excitations on the compact manifold, so that it is *not* invariant under rotations of the $\mathbb{T}^4$.

2.5.2 *Internal excitations via dualities*

The family of three-charge geometries found in the previous paragraph has the schematic form

$$\begin{aligned}
\mathrm{d}s^2_{10} &= \sqrt{\frac{Z_1 Z_2}{\mathcal{P}}}\,\mathrm{d}s^2_6 + \sqrt{\frac{Z_1}{Z_2}}\,\mathrm{d}s^2_{\mathbb{T}^4}, \\
e^{2\phi} &= \frac{Z_1^2}{\mathcal{P}}\,, \quad C_0 = \frac{Z_4}{Z_1}\,, \\
B_2 &= \bar{B}_2\,, \quad C_2 = \bar{C}_2\,, \\
C_4 &= \bar{C}_4 + \frac{Z_4}{Z_2}\,\mathrm{d}z^1\wedge\mathrm{d}z^2\wedge\mathrm{d}z^3\wedge\mathrm{d}z^4\,,
\end{aligned} \tag{2.185}$$

where barred objects are forms with legs only along the non-compact six-dimensional space.

If we want now a geometry that has only “internal” excitations, i.e. it is not invariant under rotation of the compact manifold, we can simply perform the following chain of dualities:

$$\begin{pmatrix}\mathrm{D1}\\ \mathrm{D5}\end{pmatrix} \xrightarrow{S} \begin{pmatrix}\mathrm{F1}\\ \mathrm{NS5}\end{pmatrix} \xrightarrow{T_{z^1 z^2}} \begin{pmatrix}\mathrm{F1}\\ \mathrm{NS5}\end{pmatrix} \xrightarrow{S} \begin{pmatrix}\mathrm{D1}\\ \mathrm{D5}\end{pmatrix}; \tag{2.186}$$

if performed in order, we get

$$\begin{aligned}
\mathrm{d}s^2_{10} &= \mathrm{d}s^2_6 + \frac{\sqrt{\mathcal{P}}}{Z_2}\,\mathrm{d}s^2_{T^4}\,, \quad e^{2\Phi} = \frac{\mathcal{P}}{Z_2^2}\,, \\
B_2 &= -\frac{Z_5}{Z_2}\,\omega_5\,, \quad \omega_5 = \mathrm{d}z^1\wedge\mathrm{d}z^2 - \mathrm{d}z^3\wedge\mathrm{d}z^4\,, \\
C_0 &= 0\,, \quad C_2 = \bar{C}_2\,, \\
C_4 &= \left[\left(\delta_5 - \frac{Z_5}{Z_2}\,\gamma_2\right) + \left(a_5 - \frac{Z_5}{Z_2}\,a_1\right)\wedge(\mathrm{d}v+\beta)\right]\wedge\omega_5\,,
\end{aligned} \tag{2.187}$$

where the two layers equations are inherited by duality; this means that the objects defining the ansatz here satisfy the previous layers equations; We have then changed the name of Z_4 to Z_5 (and similarly δ_4 with δ_5 and a_4 with a_5), in order to distinguish them later.

Part II

BUILDING SUPERSTRATA

3

THREE-CHARGE SUPERSTRATA WITH INTERNAL EXCITATIONS

In chap. 2 we have discussed some two-charge $\frac{1}{4}$-BPS solutions that have internal and external excitations. Up to now, no explicit three-charge $\frac{1}{8}$-BPS solution having both internal and external excitation has been found. The goal of this chapter is then to find these three-charge solutions that have both excitations and to furnish their holographic interpretation.

In order to do so, we need to generalise the results of [102], in which the supergravity fields are invariant under transformations of the $\mathbb{T}^4$. In this chapter we will this report the results we found in [3], and We will show that the general ansatz we define satisfies the type IIB supergravity equations once we impose a system of partial differential equations that arrange in two *layers* for the objects appearing in the geometry; we want to stress that this set of two layers, if solved in order, constitutes a *linear* system, and it is a generalisation of the system of equations for geometries with only external excitation [76, 87–89, 102, 112].

After having established the system of equations, i.e. the two layers, we build explicitly two asymptotically $\mathrm{AdS}_3 \times \mathbb{S}^3$ solutions for them; one is a three-charge superdescendant, obtained by the solution generating technique of [87], starting from a known two-charge D1D5 solution [110] as a seed; the second one is a superstratum solution with only one Fourier mode for the internal excitation and one Fourier mode for the external excitation; this is the first non-trivial three-charge smooth horizonless solution with an internal excitation. We close the chapter with a brief discussion on the extension of our solutions to Asymptotically Flat geometries.

1 THREE CHARGE SUPERSTRATA WITH INTERNAL EXCITATIONS: THE ANSATZ

We want now to find the most general solution of D1D5P system that has both Z_4 and Z_5 turned on, thus generalising eq. (2.134). This implies that the warp factor is now $\mathcal{P} \mapsto \mathbb{P} = Z_1 Z_2 - Z_4^2 - Z_5^2$. We will also assume again that $\dot{\beta} = 0$, in order to the $\mathrm{d}s_4^2$ base to be $v-$independent.

Motivated from the discussion of chap. i, and from the 2-charges geometry of [110], we formulate the following ansatz[1]:

$$
\begin{aligned}
\mathrm{d}s_{10}^2 &= \sqrt{\alpha}\,\mathrm{d}s_6^2 + \frac{\sqrt{\widetilde{\mathcal{P}}}}{Z_2}\,\mathrm{d}s_{\mathbb{T}^4}^2\,,\\
\mathrm{d}s_6^2 &= -\frac{2}{\sqrt{\mathbb{P}}}\,(\mathrm{d}v+\beta)\left[\mathrm{d}u+\omega+\frac{\mathcal{F}}{2}(\mathrm{d}v+\beta)\right]+\sqrt{\mathbb{P}}\,\mathrm{d}s_4^2\,,\\
\mathrm{d}s_4^2 &= \Sigma\left(\frac{\mathrm{d}r^2}{r^2+a^2}+\mathrm{d}\theta^2\right)+(r^2+a^2)\sin^2\theta\,\mathrm{d}\phi^2+r^2\cos^2\theta\,\mathrm{d}\psi^2\,,\\
\mathcal{P} &= Z_1Z_2-Z_4^2\,,\quad \widetilde{\mathcal{P}} = Z_1Z_2-Z_5^2\,,\quad \mathbb{P}=Z_1Z_2-Z_4^2-Z_5^2\,,\\
d\hat{v} &= \mathrm{d}v+\beta\,,\quad d\hat{u}=\mathrm{d}u+\omega\,,\quad v=\frac{t+y}{\sqrt{2}}\,,\quad u=\frac{t-y}{\sqrt{2}}\,,\\
e^{2\phi} &= \alpha\,\frac{\widetilde{\mathcal{P}}}{Z_2^2}\,,\quad \omega_5=-\ast_{\mathbb{T}^4}\omega_5\,,\quad \alpha=\frac{\widetilde{\mathcal{P}}}{\mathbb{P}}\,,\\
B_2 &= -\frac{Z_4}{\mathbb{P}}\,d\hat{u}\wedge d\hat{v}+a_4\wedge d\hat{v}+\delta_2-\frac{Z_5}{Z_2}\,\omega_5\,,\\
C_0 &= \frac{Z_2Z_4}{\widetilde{\mathcal{P}}}\,,\\
C_2 &= -\frac{Z_2}{\mathbb{P}}\,d\hat{u}\wedge d\hat{v}+a_1\wedge d\hat{v}+\gamma_2\,,\\
C_4 &= \frac{Z_4}{Z_2}\,\mathrm{vol}_4-\frac{Z_4}{\mathbb{P}}\,\gamma_2\wedge d\hat{u}\wedge d\hat{v}+x_3\wedge d\hat{v}\\
&\quad+\left[\left(a_5-\frac{Z_5}{Z_2}\,a_1\right)\wedge d\hat{v}+\left(\delta_5-\frac{Z_5}{Z_2}\,\gamma_2\right)\right]\wedge\omega_5\,,
\end{aligned}
\tag{3.1}
$$

where here the ω_5 is any constant two-form that is anti-self dual on the $\mathbb{T}^4$. It is easy to see that if $Z_5 \to 0$ we recover (2.113), while if $Z_4 \to 0$ we recover (2.187). This ansatz is $\frac{1}{8}-$BPS and three-charge, and this is evident in the geometry by the fact that our geometry has a Killing vector $\frac{\partial}{\partial u}$, so we will always assume that everything is $u-$independent.

1.1 *Rewriting the type IIB Equations of Motion*

In analogy of the discussion on superstrata on chap. 2, we want to find the "layers" that our ansatz has to satisfy in order to be a IIB supergravity solution.

We have already seen in sec. 2 that the bosonic content of type IIB supergravity consists in a graviton g_{MN}, a dilaton ϕ, an NSNS 2-form B_2, and a set of RR forms C_0, C_2, C_4.

In [102], it was shown that the minimal set of equations that one has to solve are BPS constraints dubbed with the existence of a null Killing vector whose integral flow generates the u-coordinate, the self-duality of RR fields and the vv component of the Einstein Equations. We see that their results can be applied here, so we do not have to solve both equations of motion and BPS equations, since it was shown there that one implies the other. So that we will have to solve only one of the two. Our discussion will then focus on solving the gauge equations of motion and the $vv-$component of the Einstein Equations, and that, plus the result of [102], will imply that our ansatz is a BPS solution of type IIB supergravity.

1 We recall that $d\hat{u}$ and $d\hat{v}$ are *not* 1-forms, since $\mathrm{d}d\hat{v} \neq 0$, $\mathrm{d}d\hat{v} \neq 0$.

We will show that to find the complete set of layers we have to study only these equations[2]

$$F_5 - *F_5 = 0\,, \tag{3.2a}$$
$$\mathrm{d} * F_3 + H_3 \wedge F_5 = 0\,, \tag{3.2b}$$
$$\begin{aligned} e^{-2\phi}\left(R_{vv} + 2\nabla_v\nabla_v\phi - \frac{1}{4}\,H_{vPQ}H_v{}^{PQ}\right)& \\ +\frac{1}{4}\,g_{vv}\left(F_PF^P + \frac{1}{3!}\,F_{PQR}F^{PQR}\right)& \\ -\frac{1}{2}F_vF_v - \frac{1}{2}\frac{1}{2!}F_{vPQ}F_v{}^{PQ} - \frac{1}{4}\frac{1}{4!}\,F_{vPQRS}F_v{}^{PQRS} &= 0\,, \end{aligned} \tag{3.2c}$$

and that the other are solved imposing the layers. We will briefly describe how the two layers

$$*_4\mathcal{D}\dot{Z}_1 = \mathcal{D}\Theta_2\,, \quad \mathcal{D}*_4\mathcal{D}Z_1 = -\Theta_2\wedge\mathcal{D}\beta\,, \quad \Theta_1 = *_4\Theta_1\,, \tag{3.3a}$$
$$*_4\mathcal{D}\dot{Z}_2 = \mathcal{D}\Theta_1\,, \quad \mathcal{D}*_4\mathcal{D}Z_2 = -\Theta_1\wedge\mathcal{D}\beta\,, \quad \Theta_2 = *_4\Theta_2\,, \tag{3.3b}$$
$$*_4\mathcal{D}\dot{Z}_4 = \mathcal{D}\Theta_4\,, \quad \mathcal{D}*_4\mathcal{D}Z_4 = -\Theta_4\wedge\mathcal{D}\beta\,, \quad \Theta_4 = *_4\Theta_4\,, \tag{3.3c}$$
$$*_4\mathcal{D}\dot{Z}_5 = \mathcal{D}\Theta_5\,, \quad \mathcal{D}*_4\mathcal{D}Z_5 = -\Theta_5\wedge\mathcal{D}\beta\,, \quad \Theta_5 = *_4\Theta_5\,, \tag{3.3d}$$

and

$$\mathcal{D}\omega + *_4\mathcal{D}\omega + \mathcal{F}\,\mathrm{d}\beta = Z_1\Theta_1 + Z_2\Theta_2 - 2Z_4\Theta_4 - 2Z_5\Theta_5\,, \tag{3.4a}$$
$$\begin{aligned} *_4\mathcal{D}*_4\left(\dot\omega - \frac{1}{2}\,\mathcal{D}\mathcal{F}\right) &= \partial_v^2(Z_1Z_2 - Z_4^2 - Z_5^2) - [\dot Z_1\dot Z_2 - (\dot Z_4)^2 - (\dot Z_5)^2] \\ &\quad - \frac{1}{2}*_4(\Theta_1\wedge\Theta_2 - \Theta_4\wedge\Theta_4 - \Theta_5\wedge\Theta_5)\,, \end{aligned} \tag{3.4b}$$

emerge from the system (2.112).

1.1.1 *The Field Strengths*

The first thing to notice is that everything is $\mathbb{T}^4$-independent, i.e. $\partial_{z^i} = 0\,, \forall i$. By the fact that the solution should be BPS, it is also $u-$independent and then $\partial_u = 0$. Then the ten-dimensional differential operator $\mathrm{d} = \mathrm{d}x^M\wedge\partial_M$ can be split as[3]

$$\mathrm{d} = \mathrm{d}_4 + \mathrm{d}v\wedge\partial_v = \mathcal{D} + (\mathrm{d}v+\beta)\wedge\partial_v \equiv \mathcal{D} + d\hat{v}\wedge\partial_v\,. \tag{3.6}$$

It will be useful to introduce the following gauge invariant objects

$$\begin{aligned} &\Theta_1 \equiv \mathcal{D}a_1 + \dot\gamma_2\,, && \Theta_4 \equiv \mathcal{D}a_4 + \dot\delta_2\,, && \Theta_5 \equiv \mathcal{D}a_5 + \dot\delta_5\,, \\ &\Xi_1 = \mathcal{D}\gamma_2 - a_1\wedge\mathcal{D}\beta\,, && \Xi_4 = \mathcal{D}\delta_2 - a_4\wedge\mathcal{D}\beta\,, && \Xi_5 = \mathcal{D}\delta_5 - a_5\wedge\mathcal{D}\beta\,, \end{aligned} \tag{3.7}$$

2 Notice that the solution of these set of equations will be $\frac{1}{8}$-BPS, due to the results of [102] that are still valid here.

3 Notice that, on a generic form f_p, we have

$$\mathcal{D}^2 f_p = -\mathcal{D}\beta\wedge\dot f_p\,. \tag{3.5}$$

so that we can compute the field strengths via the split[4]

$$H_3 = \mathcal{H}_3^{(3)} + \mathcal{H}_3^{(1)} \wedge \omega_5 \,, \tag{3.9a}$$
$$F_1 = \mathfrak{F}_1^{(1)} \,, \tag{3.9b}$$
$$F_3 = \mathfrak{F}_3^{(3)} + \mathfrak{F}_3^{(1)} \wedge \omega_5 \,, \tag{3.9c}$$
$$F_5 = \mathfrak{F}_5^{(5)} + \mathfrak{F}_5^{(3)} \wedge \omega_5 + \mathfrak{F}_5^{(1)} \wedge \mathrm{vol}_{\mathbb{T}^4} \,, \tag{3.9d}$$

where $\mathrm{vol}_{\mathbb{T}^4}$ is the volume form of the compact $\mathbb{T}^4$. The field strengths are

$$\mathcal{H}_3^{(1)} = -\mathcal{D}\left(\frac{Z_5}{Z_2}\right) - \partial_v \left(\frac{Z_5}{Z_2}\right) d\hat{v} \,, \tag{3.10a}$$
$$\begin{aligned}\mathcal{H}_3^{(3)} &= -\mathcal{D}\left(\frac{Z_4}{\mathbb{P}}\right) \wedge d\hat{u} \wedge d\hat{v} + \frac{Z_4}{\mathbb{P}}\,\mathcal{D}\beta \wedge d\hat{u} \\ &\quad + \left[\Theta_4 - \frac{Z_4}{\mathbb{P}}\,\mathcal{D}\omega\right] \wedge d\hat{v} + \Xi_4 \,,\end{aligned} \tag{3.10b}$$

and

$$\mathfrak{F}_1^{(1)} = \mathcal{D}\left(\frac{Z_2 Z_4}{\widetilde{\mathcal{P}}}\right) + \partial_v \left(\frac{Z_2 Z_4}{\widetilde{\mathcal{P}}}\right) d\hat{v} \,, \tag{3.11}$$

and

$$\mathfrak{F}_3^{(1)} = \frac{Z_2 Z_4}{\widetilde{\mathcal{P}}}\,\mathcal{D}\left(\frac{Z_5}{Z_2}\right) + \frac{Z_2 Z_4}{\widetilde{\mathcal{P}}}\,\partial_v \left(\frac{Z_5}{Z_2}\right) d\hat{v} \tag{3.12a}$$
$$\begin{aligned}\mathfrak{F}_3^{(3)} &= -\left[\mathcal{D}\left(\frac{Z_2}{\mathbb{P}}\right) - \frac{Z_2 Z_4}{\widetilde{\mathcal{P}}}\,\mathcal{D}\left(\frac{Z_4}{\mathbb{P}}\right)\right] \wedge d\hat{u} \wedge d\hat{v} + \frac{Z_2}{\widetilde{\mathcal{P}}}\,\mathcal{D}\beta \wedge d\hat{u} \\ &\quad + \left[\left(\Theta_1 - \frac{Z_2 Z_4}{\widetilde{\mathcal{P}}}\,\Theta_4\right) - \frac{Z_2}{\widetilde{\mathcal{P}}}\,\mathcal{D}\omega\right] \wedge d\hat{v} \\ &\quad + \left[\Xi_1 - \frac{Z_2 Z_4}{\widetilde{\mathcal{P}}}\,\Xi_4\right] ,\end{aligned} \tag{3.12b}$$

where we have used that

$$\frac{Z_2}{\mathbb{P}} - \frac{Z_2 Z_4}{\widetilde{\mathcal{P}}}\,\frac{Z_4}{\mathbb{P}} = \frac{Z_2}{\widetilde{\mathcal{P}}} \,. \tag{3.13}$$

In the end, we have

$$\mathfrak{F}_5^{(1)} = \mathcal{D}\left(\frac{Z_4}{Z_2}\right) + \partial_v \left(\frac{Z_4}{Z_2}\right) d\hat{v} \tag{3.14a}$$
$$\begin{aligned}\mathfrak{F}_5^{(3)} &= -\frac{Z_2}{\mathbb{P}}\,\mathcal{D}\left(\frac{Z_5}{Z_2}\right) \wedge d\hat{u} \wedge d\hat{v} + \left[\Theta_5 - \frac{Z_5}{Z_2}\,\Theta_1\right] \wedge d\hat{v} \\ &\quad + \left[\Xi_5 - \frac{Z_5}{Z_2}\,\Xi_1\right]\end{aligned} \tag{3.14b}$$
$$\begin{aligned}\mathfrak{F}_5^{(5)} &= -\left[\frac{Z_4}{\mathbb{P}}\,\Xi_1 - \frac{Z_2}{\mathbb{P}}\,\Xi_4\right] \wedge d\hat{u} \wedge d\hat{v} \\ &\quad + \left[\mathcal{D}x_3 - \Theta_4 \wedge \gamma_2 + a_1 \wedge \Xi_4\right] \wedge d\hat{v} \\ &\quad + \left[x_3 \wedge \mathcal{D}\beta + \Xi_4 \wedge \gamma_2\right] .\end{aligned} \tag{3.14c}$$

4 Please notice that, since $\mathrm{d}(d\hat{u}) = \mathcal{D}\omega + d\hat{v} \wedge \dot{\omega}$ and $\mathrm{d}(d\hat{v}) = \mathcal{D}\beta + d\hat{v} \wedge \dot{\beta}$,

$$\mathrm{d}(d\hat{u} \wedge d\hat{v}) = \mathcal{D}\omega \wedge d\hat{v} - \mathcal{D}\beta \wedge d\hat{u} - \dot{\beta} \wedge d\hat{u} \wedge d\hat{v} = \mathcal{D}\omega \wedge d\hat{v} - \mathcal{D}\beta \wedge d\hat{u} \,, \tag{3.8}$$

since we had assumed assume $\dot{\beta} = 0$. Also, since $\dot{\beta} = 0$, $\mathcal{D}\beta = \mathrm{d}\beta$.

Now notice that $x_3 \wedge \mathcal{D}\beta + \Xi_4 \wedge \gamma_2 = 0$ since it is a 3-form wedge a 2-form in a 4-dimensional space. We recall that in eq. (2.17) of [76] they define

$$\Omega_4 \equiv \mathcal{D}x_3 - \Theta_4 \wedge \gamma_2 + a_1 \wedge \Xi_4 \,, \tag{3.15}$$

so we recover this combination as expected:

$$\mathfrak{F}_5^{(5)} = +\frac{Z_2}{\mathbb{P}} \left[\Xi_4 - \frac{Z_4}{Z_2}\,\Xi_1\right] \wedge d\hat{u} \wedge d\hat{v} + \Omega_4 \wedge d\hat{v} \,. \tag{3.16}$$

We can also use that $\alpha = \frac{\widetilde{\mathcal{P}}}{\mathbb{P}}$ so

$$\mathfrak{F}_5^{(5)} = \alpha\, \frac{Z_2}{\widetilde{\mathcal{P}}} \left[\Xi_4 - \frac{Z_4}{Z_2}\,\Xi_1\right] \wedge d\hat{u} \wedge d\hat{v} + \Omega_4 \wedge d\hat{v} \,. \tag{3.17}$$

1.1.2 *The eq.* (2.112a)

We now study eq. (2.112a). We employ our split (3.8) and see that

$$*F_5 = \frac{1}{\alpha}\,\frac{\widetilde{\mathcal{P}}}{Z_2^2}\, *_6\, \mathfrak{F}_5^{(5)} \wedge \mathrm{vol}_{\mathbb{T}^4} - *_6\mathfrak{F}_5^{(3)} \wedge \omega_5 + \alpha\,\frac{Z_2^2}{\widetilde{\mathcal{P}}}\, *_6\, \mathfrak{F}_5^{(1)} \,. \tag{3.18}$$

So the type IIB supergravity equation (2.112a) becomes

$$I^{(5)} \equiv \mathfrak{F}_5^{(5)} - \alpha\,\frac{Z_2^2}{\widetilde{\mathcal{P}}}\, *_6\, \mathfrak{F}_5^{(1)} = 0 \,, \tag{3.19a}$$

$$I^{(3)} \equiv \mathfrak{F}_5^{(3)} + *_6\mathfrak{F}_5^{(3)} = 0 \,, \tag{3.19b}$$

$$I^{(1)} \equiv \mathfrak{F}_5^{(1)} - \frac{1}{\alpha}\,\frac{\widetilde{\mathcal{P}}}{Z_2^2}\, *_6\, \mathfrak{F}_5^{(5)} = 0 \,. \tag{3.19c}$$

So then we get, using that $\alpha\,\frac{Z_2^2}{\widetilde{\mathcal{P}}}\,\mathbb{P} = Z_2^2$,

$$\begin{aligned} I^{(1)} = &\left[\mathcal{D}\left(\frac{Z_4}{Z_2}\right) + \frac{1}{Z_2}\left(*_4\Xi_4 + \frac{Z_4}{Z_2}\, *_4\, \Xi_1\right)\right] \\ &+ \left[\partial_v\left(\frac{Z_4}{Z_2}\right) - \frac{1}{Z_2^2}\, *_4\, \Omega_4\right] d\hat{v} \,, \end{aligned} \tag{3.20}$$

that gives two equations

$$\mathcal{D}\left(\frac{Z_4}{Z_2}\right) = -\frac{1}{Z_2}\left(*_4\Xi_4 - \frac{Z_4}{Z_2}\, *_4\, \Xi_1\right) \,, \tag{3.21a}$$

$$Z_2^2\, \partial_v \left(\frac{Z_4}{Z_2}\right) = *_4\Omega_4 \,. \tag{3.21b}$$

These two equations are like eqs. (3.45) and (3.47) of [102]. Notice also that $I^{(1)}$ is the dual of $I^{(5)}$, so the only new equation of motion w.r.t. the case with only external excitation is $I^{(3)}$ that is, broadly speaking, “self-dual":

$$\begin{aligned} I^{(3)} = &\frac{1}{\mathbb{P}}\left[Z_2\mathcal{D}\left(\frac{Z_5}{Z_2}\right) + *_4\Xi_5 - \frac{Z_5}{Z_2}\, *_4\, \Xi_1\right] \wedge d\hat{u} \wedge d\hat{v} \\ &+ \left[\left(\Theta_5 - \frac{Z_5}{Z_2}\,\Theta_1\right) - *_4\left(\Theta_5 - \frac{Z_5}{Z_2}\,\Theta_1\right)\right] \wedge d\hat{v} \\ &- \left[Z_2\, *_4\, \mathcal{D}\left(\frac{Z_5}{Z_2}\right) - \left(\Xi_5 - \frac{Z_5}{Z_2}\,\Xi_1\right)\right] \,. \end{aligned} \tag{3.22}$$

This gives[5]

$$\Theta_5 = *_4\Theta_5\,, \quad \Theta_1 = *_4\Theta_1\,, \tag{3.23}$$

and

$$Z_2\mathcal{D}\left(\frac{Z_5}{Z_2}\right) = -\,*_4\,\Xi_5 + \frac{Z_5}{Z_2}\,*_4\,\Xi_1\,. \tag{3.24}$$

Notice that, since $\mathcal{D}\left(\frac{Z_5}{Z_2}\right) = \frac{1}{Z_2}\,\mathcal{D}Z_5 - \frac{1}{Z_2^2}\,\mathcal{D}Z_2$, we can rewrite the equations as

$$*_4\mathcal{D}Z_2 = \Xi_1\,, \quad *_4\mathcal{D}Z_4 = \Xi_4\,, \quad *_4\mathcal{D}Z_5 = \Xi_5\,, \tag{3.25}$$

Where we have used that $*_d *_d \alpha_p = (-1)^{p(d-p)} s *_d \alpha_p$, where s is the signature value.

So, to recap, we have

$$*_4\mathcal{D}Z_2 = \Xi_1\,, \quad *_4\mathcal{D}Z_4 = \Xi_4\,, \quad *_4\mathcal{D}Z_5 = \Xi_5\,, \tag{3.26a}$$

$$\Theta_1 = *_4\Theta_1\,, \quad \Theta_4 = *_4\Theta_4\,, \quad \Theta_5 = *_4\Theta_5\,, \tag{3.26b}$$

$$\Omega_4 = Z_2^2\,*_4\,\partial_v\left(\frac{Z_4}{Z_2}\right)\,, \qquad *_4\mathcal{D}\beta = \mathcal{D}\beta\,. \tag{3.26c}$$

Now, using that $\mathcal{D}^2 f_p = -\mathcal{D}\beta \wedge \dot{f}_p$, we can rewrite (3.26a) as (3.3).

1.1.3 *The eq.* (2.112b)

We now see how the IIB sugra equation for the F_3

$$\mathrm{d} * F_3 + H_3 \wedge F_5 = 0\,, \tag{3.27}$$

translates in our notation. First, we employ again the splitting (3.8) to have

$$*F_3 = -\alpha\,*_6\,\mathfrak{F}_3^{(1)} \wedge \omega_5 + \frac{\widetilde{\mathcal{P}}}{Z_2^2}\,*_6\,\mathfrak{F}_3^{(3)} \wedge \mathrm{vol}_{\mathbb{T}^4}\,, \tag{3.28}$$

and get the following set of equations

$$\mathrm{d}\left[\frac{\widetilde{\mathcal{P}}}{Z_2^2}\,*_6\,\mathfrak{F}_3^{(3)}\right] + \mathcal{H}_3^{(3)} \wedge \mathfrak{F}_5^{(1)} - 2\mathcal{H}_3^{(1)} \wedge \mathfrak{F}_5^{(3)} = 0\,, \tag{3.29a}$$

$$\mathrm{d}\left[-\alpha\,*_6\,\mathfrak{F}_3^{(1)}\right] + \mathcal{H}_3^{(3)} \wedge \mathfrak{F}_5^{(3)} + \mathcal{H}_3^{(1)} \wedge \mathfrak{F}_5^{(5)} = 0\,, \tag{3.29b}$$

where we have used the fact that $\omega_5 \wedge \omega_5 = -2\mathrm{vol}_{\mathbb{T}^4}$. We will now focus on the first one, and notice that

$$\begin{aligned}
\mathcal{H}_3^{(1)} \wedge \mathfrak{F}_5^{(3)} &= \mathcal{H}_3^{(1)} \wedge \left[\mathrm{d}C_4^{(2)} - \mathcal{H}_3^{(1)} \wedge C_2^{(2)}\right] = \mathcal{H}_3^{(1)} \wedge \mathrm{d}C_4^{(2)} \\
&= \mathrm{d}\left[B_2^{(0)}\mathrm{d}C_4^{(2)}\right]\,, \\
\mathcal{H}_3^{(3)} \wedge \mathfrak{F}_5^{(1)} &= \mathcal{H}_3^{(3)} \wedge \mathrm{d}C_4^{(0)} \\
&= -\mathrm{d}\left[\mathcal{H}_3^{(3)}\,C_4^{(0)}\right]\,,
\end{aligned} \tag{3.30}$$

so that

$$\mathrm{d}\left[\frac{\widetilde{\mathcal{P}}}{Z_2^2}\,*_6\,\mathfrak{F}_3^{(3)} - \mathcal{H}_3^{(3)}\,C_4^{(0)} - 2B_2^{(0)}\mathrm{d}C_4^{(2)}\right] = 0\,. \tag{3.31}$$

5 Actually, this imposes the self-duality of $\left(\Theta_5 - \frac{Z_5}{Z_2}\,\Theta_1\right)$, but imposing also the gauge equation for F_3 we have to solve independently the self-duality condition on all the Θ's.

To extract the solution we then write it as

$$\frac{\widetilde{\mathcal{P}}}{Z_2^2} *_6 \mathfrak{F}_3^{(3)} - \mathcal{H}_3^{(3)}\, C_4^{(0)} - 2B_2^{(0)} \mathrm{d}C_4^{(2)} = -\mathrm{d}\widetilde{C}_2^{(2)} - \mathrm{d}\widetilde{B}_2^{(2)}\,, \tag{3.32}$$

where, in strict analogy with computations in the literature [76,87–89,102,112],

$$\widetilde{C}_2^{(2)} = -\frac{Z_1}{\mathbb{P}}\, d\hat{u}\wedge d\hat{v} + a_2 \wedge d\hat{v} + \gamma_1\,, \quad \widetilde{B}_2^{(2)} = -\frac{1}{Z_2}\frac{Z_5^2}{\mathbb{P}}\, d\hat{u}\wedge d\hat{v}\,, \tag{3.33}$$

we obtain, from the $d\hat{u}$ component of the equation, that

$$*_4 \mathcal{D}\beta = \mathcal{D}\beta\,, \tag{3.34}$$

and, from the $d\hat{v}$ component of the equation, that

$$\mathcal{D}\omega + *_4\mathcal{D}\omega + \mathcal{F}\,\mathrm{d}\beta = Z_1\Theta_1 + Z_2\Theta_2 - 2Z_4\Theta_4 - 2Z_5\Theta_5\,, \tag{3.35}$$

where we have used $\Theta^I = *_4\Theta^I$.

1.1.4 *The eq.* (2.112c)

To solve the last equation, eq. (2.112c), we need to split $x^M = (x^\mu, z^i) = (x^{u_i}, x^a, z^i)$ where $x^{u_i} = (u, v)$ and

$$g_{MN}\mathrm{d}x^M \mathrm{d}x^N = \sqrt{\alpha}\, g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + X\delta_{ij}\mathrm{d}z^i \mathrm{d}z^j\,, \tag{3.36}$$

where $X = \frac{\sqrt{\widetilde{\mathcal{P}}}}{Z_2}$ and $\alpha = \frac{\widetilde{\mathcal{P}}}{\mathbb{P}}$, and

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = G_{u_i u_j}(\mathrm{d}x^{u_i} + A^{u_i})(\mathrm{d}x^{u_j} + A^{u_j}) + \sqrt{\mathbb{P}}\, q_{ab}\mathrm{d}x^a \mathrm{d}x^b\,, \tag{3.37}$$

and where

$$A^{u_i} = A^{u_i}_a \mathrm{d}x^a\,, \quad A^u = \omega\,, \quad A^v = \beta\,, \quad G_{u_i u_j} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{\mathbb{P}}} \\ -\frac{1}{\sqrt{\mathbb{P}}} & -\frac{\mathcal{F}}{\sqrt{\mathbb{P}}} \end{pmatrix}, \tag{3.38}$$

so that we can write [6]

$$\begin{aligned} g_{\mu\nu} &= \begin{pmatrix} G_{u_i u_j} & G_{u_i u_j} A^{u_i}_a \\ A^{u_i}_a G_{u_i u_j} & \sqrt{\mathbb{P}}\, q_{ab} + G_{u_i u_j} A^{u_i}_a A^{u_j}_b \end{pmatrix}, \\ g^{\mu\nu} &= \begin{pmatrix} G^{u_i u_j} + \frac{1}{\sqrt{\mathbb{P}}} q^{ab} A^{u_i}_a A^{u_j}_b & -\frac{1}{\sqrt{\mathbb{P}}} q^{ab} A^{u_i}_b \\ -\frac{1}{\sqrt{\mathbb{P}}} A^{u_i}_a q^{ab} & \frac{1}{\sqrt{\mathbb{P}}} q^{ab} \end{pmatrix}. \end{aligned} \tag{3.40}$$

Notice that we can inherit the results of [116]; in particular, we can use their eq. (3.30). We can thus define the sechsbein as

$$e^+ = \frac{1}{\sqrt{\mathbb{P}}}(\mathrm{d}v + \beta), \quad e^- = \frac{1}{\sqrt{\mathbb{P}}}\left[\mathrm{d}u + \omega + \frac{\mathcal{F}}{2}(\mathrm{d}v + \beta)\right], \quad e^a = \mathbb{P}^{1/4}\,\tilde{e}^a\,, \tag{3.41}$$

6 Notice that

$$G_{u_i u_j} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{\mathbb{P}}} \\ -\frac{1}{\sqrt{\mathbb{P}}} & -\frac{\mathcal{F}}{\sqrt{\mathbb{P}}} \end{pmatrix}, \quad G^{u_i u_j} = \begin{pmatrix} \sqrt{\mathbb{P}}\,\mathcal{F} & -\sqrt{\mathbb{P}} \\ -\sqrt{\mathbb{P}} & 0 \end{pmatrix}. \tag{3.39}$$

so that

$$\eta_{ab} e^a e^b = 2\eta_{+-} e^+ e^- + \sqrt{\mathbb{P}}\, \delta_{ab} \tilde{e}^a \tilde{e}^b = \mathrm{d}s_6^2\,. \tag{3.42}$$

With this kind of metric we can use eq. (3.30) of [116][7]

$$\begin{aligned} R_{vv} = {*_4}\mathcal{D}\, {*_4}\, L + \frac{1}{2}\frac{1}{\mathbb{P}}\left(\mathcal{D}\omega + \frac{1}{2}\mathcal{F}\mathrm{d}\beta\right)^2 \\ - \frac{1}{2}\sqrt{\mathbb{P}}\, q^{ab}\partial_v^2\left(\sqrt{\mathbb{P}}\, q_{ab}\right) - \frac{1}{4}\partial_v\left(\sqrt{\mathbb{P}}\, q^{ab}\right)\partial_v\left(\sqrt{\mathbb{P}}\, q_{ab}\right), \end{aligned} \tag{3.43}$$

where

$$L \equiv \dot{\omega} - \frac{1}{2}\mathcal{D}\mathcal{F}\,. \tag{3.44}$$

With carefulness and using eqs. (3.3) and (3.35) intensively, one can extract the last equation to be

$$\begin{aligned} {*_4}\mathcal{D}\, {*_4}\left(\dot{\omega} - \frac{1}{2}\mathcal{D}\mathcal{F}\right) = \partial_v^2(Z_1 Z_2 - Z_4^2 - Z_5^2) - [\dot{Z}_1\dot{Z}_2 - (\dot{Z}_4)^2 - (\dot{Z}_5)^2] \\ - \frac{1}{2}{*_4}\left(\Theta_1\wedge\Theta_2 - \Theta_4\wedge\Theta_4 - \Theta_5\wedge\Theta_5\right). \end{aligned}$$

2 THE DUAL CFT DESCRIPTION

2.1 *Two-Charge States*

As we have seen repeatedly in chap. 2, the black hole microstates are dual to heavy states in the Ramond sector of the CFT. A typical heavy state will be a product of N_i strands with length k_i, and we will describe them strand by strand. In the R sector we can act on each strand on the vacuum with the fermionic zero-modes $\psi_0^{-\dot{A}}$, $\widetilde{\psi}_0^{-\dot{B}}$ to build 2^4 states and, for concreteness, we pick from the R vacuum states the one with $\jmath_L = \jmath_R = +\frac{1}{2}$, i.e. $|{+}{+}\rangle_k$. Half of these states are fermionic, and do not have a clear holographic dual geometry; the other half are bosonic, and we will focus on those. Out of these eight states, we have the subset of those with zero angular momentum $|00\rangle_k^{(\dot{A}\dot{B})}$. We can extract a combination of those states that is invariant under transformation of the $\mathbb{T}^4$, i.e. $|00\rangle_k = \varepsilon_{\dot{A}\dot{B}}|00\rangle_k^{(\dot{A}\dot{B})}$, while the others are non-invariant under the same transformations; to build our heavy states, we will pick without loss of generality the state $|00\rangle_k^{\dot{1}\dot{1}}$. Here we use the notation commonly used in the literature; we want to stress the fact that all the states written above have zero angular momentum.

To be explicit, we will study in the next section the geometry dual to the two-charge Heavy state

$$\begin{aligned} |H\rangle &= \prod \left[|{+}{+}\rangle_1\right]^{N^{(++)}} \left[|00\rangle_{k_1}\right]^{N^b_{k_1}} \left[|00\rangle_{k_2}^{(\dot{1}\dot{1})}\right]^{N^c_{k_2}}\,, \\ N &= N_1^{(++)} + \sum_{k_1} k_1 N^b_{k_1} + \sum_{k_2} k_2 N^c_{k_2}\,. \end{aligned} \tag{3.45}$$

2.1.1 *The profile functions and their holographic interpretation*

We now want to briefly describe the map between the two-charge states and their dual geometry, as seen in chap. 2. As explained in detail in [75,76,89,110],

7 Notice that in [116] the notation for u and v is exchanged and also they use the mostly plus signature of the metric.

we can construct two-charge solutions of type IIB supergravity in the D1D5 frame by dualities to the F1P frame and assigning a F1P profile and the going back with the proper chain of dualities; this will led to a definitions of the Z_I, β and ω in terms of those profiles:

$$
\begin{aligned}
Z_1 &= \frac{Q_5}{L}\int_0^L \frac{|\dot g_i(v')|^2+|\dot g_5(v')|^2+|\dot g^{\alpha_-}(v')|^2}{|x_i-g_i(v')|^2}\,\mathrm{d}v'\,,\\
Z_2 &= \frac{Q_5}{L}\int_0^L \frac{\mathrm{d}v'}{|x_i-g_i(v')|^2}\,,\\
Z_4 &= -\frac{Q_5}{L}\int_0^L \frac{\dot g_5(v')}{|x_i-g_i(v')|^2}\,\mathrm{d}v'\,,\quad Z_5 = -\frac{Q_5}{L}\int_0^L \frac{\dot g^{\alpha_-}(v')\omega_{\alpha_-}}{|x_i-g_i(v')|^2}\,\mathrm{d}v'\,,\\
\mathrm{d}\gamma_2 &= *_4\mathrm{d}Z_2\,,\quad \mathrm{d}\gamma_1 = *_4\mathrm{d}Z_1\,,\quad \mathrm{d}\delta_4 = *_4\mathrm{d}Z_4\,,\quad \mathrm{d}\delta_5 = *_4\mathrm{d}Z_5\,,\\
A &= -\frac{Q_5}{L}\int_0^L \frac{\dot g_j(v')\,\mathrm{d}x^j}{|x_i-g_i(v')|^2}\,\mathrm{d}v'\,,\quad \mathrm{d}B = -*_4\mathrm{d}A\,,\quad \mathrm{d}s_4^2 = \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\,,\\
\beta &= \frac{-A+B}{\sqrt{2}}\,,\quad \omega = \frac{-A-B}{\sqrt{2}}\,,\quad \mathcal{F}=0\,,\quad a_I=0\,,\quad x_3=0\,,
\end{aligned}
\tag{3.46}
$$

where we recall that $L = 2\pi Q_5/R$ and where ω_{α_-} is the usual basis of anti-self dual two forms on the compact $\mathbb{T}^4$.

The profile we need to construct the geometry dual to the heavy state (3.45) is

$$
\begin{aligned}
g_1+ig_2 &= a\,e^{\frac{2\pi i v'}{L}}\,,\quad g_3+ig_4=0,\\
g_5 &= -\frac{b}{k_1}\sin\left(\frac{2\pi k_1 v'}{L}\right),\quad g^{\alpha_-} = -\frac{c}{k_2}\sin\left(\frac{2\pi k_2 v'}{L}\right).
\end{aligned}
\tag{3.47}
$$

The holographic dictionary then relates [75, 76, 89, 110]

$$
\frac{N^{(++)}}{N} = \frac{a^2}{a_0^2}\,,\quad \frac{k_1 N^b_{k_1}}{N} = \frac{b^2}{2a_0^2}\,,\quad \frac{k_2 N^c_{k_2}}{N} = \frac{c^2}{2a_0^2}\,,
\tag{3.48}
$$

that translates into the regularity condition

$$
a^2+\frac{b^2}{2}+\frac{c^2}{2} = a_0^2 \equiv \frac{Q_1Q_5}{R^2}\,.
\tag{3.49}
$$

Here R is the $\mathbb{S}^1$ radius and Q_1 and Q_5 are the D1 and D5 supergravity charges; they are related to the number n_1 and n_5 via

$$
Q_1 = \frac{(2\pi)^4 g_s(\alpha')^3}{V_{\mathbb{T}^4}}\,n_1\,,\quad Q_5 = g_s\alpha'\,n_5\,,
\tag{3.50}
$$

where g_s is the string coupling constant and where $V_{\mathbb{T}^4}$ is the volume of the $\mathbb{T}^4$.

2.2 *Three charge states*

Starting from the two-charge D1D5 geometry dual to the state in (3.45) we can build a three-charge D1D5P geometry that is dual to a superdescendant of the heavy state (3.45); as explained in [76, 87, 89] we can act with the symmetries of the CFT on each strand to generate new solutions. Since

$$
(J^+_{-1})^{m_1}|00\rangle_{k_1} = 0\,,\quad \forall m_1 > k_1\,,
\tag{3.51}
$$

we can act with a global transformation $e^{\chi(J^+_{-1}-J^-_{+1})}$ on the two-charge state, where we follow the notation of [87]; picking a precise choice for χ, i.e. $\chi = \pi/2$, the resulting state is obtained as a product of states on which we have acted with the maximum number of J^+_{-1}:

$$|\widetilde{H}\rangle = \prod \left[|{+}{+}\rangle_1\right]^{N^{(++)}} \left[(J^+_{-1})^{k_1}|00\rangle_{k_1}\right]^{N^b_{k_1}} \left[(J^+_{-1})^{k_2}\,|00\rangle^{(\mathrm{ii})}_{k_2}\right]^{N^c_{k_2}} . \tag{3.52}$$

To construct its dual geometry we can start from the dual geometry of (3.45) and act, on the supergravity side already in the R sector, with the coordinate transformation

$$\theta \to \frac{\pi}{2} - \theta\,, \quad \phi \to -\psi + \frac{\sqrt{2}\,v}{R}\,, \quad \psi \to -\phi + \frac{\sqrt{2}\,v}{R}\,. \tag{3.53}$$

We will see in the next section what are the dual geometries of these heavy states $|H\rangle$, $|\widetilde{H}\rangle$ and we will check that those geometries satisfies the layer equations (3.3, 3.4), furnishing a non-trivial check.

We will also build new non-superdescendant geometries that are dual to more complicated heavy states, by means of the action of the generators of the algebra; in particular we can act with $(L_{-1} - J^3_{-1})^n$ and with $(J^+_{-1})^m$, as in [89], giving

$$\begin{aligned}\left|H_{(k_1,m_1,n_1),(k_2,m_2,n_2)}\right\rangle = \prod \left[|{+}{+}\rangle_1\right]^{N^{(++)}} \cdot \\ \cdot \left[(L_{-1} - J^3_{-1})^{n_1}(J^+_{-1})^{m_1}|00\rangle_{k_1}\right]^{N^b_{k_1,m_1,n_1}} \cdot \\ \cdot \left[(L_{-1} - J^3_{-1})^{n_2}(J^+_{-1})^{m_2}\,|00\rangle^{(\mathrm{ii})}_{k_2}\right]^{N^c_{k_2,m_2,n_2}} .\end{aligned} \tag{3.54}$$

As explained in chap. 1, they are called *s*uperstrata, in order to distinguish them from the rigidly generated superdescendants [76,89].

We now want to remark that those states will have a dual geometry that will not be invariant under rotation of the compact space $\mathbb{T}^4$; this is easy to see from the CFT point of view because these states have explicit indexes of the torus symmetry group $SO(4)_I$, so are not invariant under $SO(4)_I$ transformations.

3 THREE-CHARGE SUPERSTRATA WITH INTERNAL EXCITATIONS: SOLUTIONS

3.1 *The Superdescendant check*

We now start from the two-charge geometry with both Z_4 and Z_5 excitations; as described in the previous section, the geometry we want to consider is holographically dual to the heavy state (3.45). The dual geometry is then a two-charge geometry [110] described by the general ansatz (3.1) with

$$
\begin{aligned}
Z_1 &= \frac{R^2}{Q_5}\frac{a^2+\frac{b^2}{2}+\frac{c^2}{2}}{\Sigma} + \frac{R^2 a^{2k_1}}{2Q_5}\frac{b^2\sin^{2k_1}\theta\cos 2k_1\phi}{\Sigma\,(r^2+a^2)^{k_1}} + \frac{R^2 a^{2k_2}}{2Q_5}\frac{c^2\sin^{2k_2}\theta\cos 2k_2\phi}{\Sigma\,(r^2+a^2)^{k_2}},\\
Z_2 &= \frac{Q_5}{\Sigma}\,, \quad Z_4 = R\,b\,a^{k_1}\frac{\sin^{k_1}\theta\,\cos k_1\phi}{\Sigma\,(r^2+a^2)^{\frac{k_1}{2}}}\,, \quad Z_5 = R\,c\,a^{k_1}\frac{\sin^{k_2}\theta\,\cos k_2\phi}{\Sigma\,(r^2+a^2)^{\frac{k_2}{2}}}\,,\\
a_1 &= 0\,, \quad a_4 = 0\,, \quad a_5 = 0\,, \quad x_3 = 0\,,\\
\gamma_2 &= -Q_5\frac{r^2+a^2}{\Sigma}\cos^2\theta\,\mathrm{d}\phi\wedge\mathrm{d}\psi\,,\\
\delta_4 &= -\frac{R\,b\,a^{k_1}\sin^{k_1}\theta}{(r^2+a^2)^{\frac{k_1}{2}}}\left(\frac{r^2+a^2}{\Sigma}\cos^2\theta\,\cos k_1\phi\,\mathrm{d}\phi\wedge\mathrm{d}\psi + \sin k_1\phi\,\cot\theta\,\mathrm{d}\theta\wedge\mathrm{d}\psi\right),\\
\delta_5 &= -\frac{R\,c\,a^{k_2}\sin^{k_2}\theta}{(r^2+a^2)^{\frac{k_2}{2}}}\left(\frac{r^2+a^2}{\Sigma}\cos^2\theta\,\cos k_2\phi\,\mathrm{d}\phi\wedge\mathrm{d}\psi + \sin k_2\phi\,\cot\theta\,\mathrm{d}\theta\wedge\mathrm{d}\psi\right).
\end{aligned}
\tag{3.55}
$$

We will now to generate a three-charge solution that is its superdescendant; to do so we will use the generating-solution technique of [87]; on the supergravity side the transformation we have to employ is, already in the R Sector,

$$
\theta \to \frac{\pi}{2}-\theta\,, \quad \phi \to -\psi + \frac{\sqrt{2}\,v}{R}\,, \quad \psi \to -\phi + \frac{\sqrt{2}\,v}{R}\,. \tag{3.56}
$$

This rigidly generated solution is a three-charge, $\frac{1}{8}$−BPS geometry that solves the two layers (3.3, 3.4). We remark that this is a highly non-trivial check for the layers.

The explicit solution is defined in terms of

$$
\begin{aligned}
Z_1 &= \frac{R^2}{2Q_5}\frac{1}{\Sigma}\Bigg[\left(2a^2+b^2+c^2\right) + \frac{a^{2k_1}\cos^{2k_1}\theta}{(r^2+a^2)^{k_1}}\,b^2\cos\left(2k_1\hat{v}\right)\\
&\qquad\qquad + \frac{a^{2k_2}\cos^{2k_2}\theta}{(r^2+a^2)^{k_2}}\,c^2\cos\left(2k_2\hat{v}\right)\Bigg],\\
Z_2 &= \frac{Q_5}{\Sigma}\,,\\
Z_4 &= \frac{R}{\Sigma}\,b\,\frac{a^{k_1}\cos^{k_1}\theta}{(r^2+a^2)^{\frac{k_1}{2}}}\cos\left(k_1\hat{v}\right), \quad Z_5 = \frac{R}{\Sigma}\,c\,\frac{a^{k_2}\cos^{k_2}\theta}{(r^2+a^2)^{\frac{k_2}{2}}}\cos\left(k_2\hat{v}\right),\\
\mathcal{F} &= -\frac{1}{r^2+a^2\sin^2\theta}\left[b^2\left(1-\frac{a^{2k_1}\cos^{2k_1}\theta}{(r^2+a^2)^{k_1}}\right) + c^2\left(1-\frac{a^{2k_2}\cos^{2k_2}\theta}{(r^2+a^2)^{k_2}}\right)\right],
\end{aligned}
\tag{3.57}
$$

and

$$
\begin{aligned}
\omega &= \omega_0 + \omega_1\,, \quad \beta = \beta_0\,,\\
\beta_0 &= \frac{R\,a^2}{\sqrt{2}\,\Sigma}\left(\sin^2\theta\,\mathrm{d}\phi - \cos^2\theta\,\mathrm{d}\psi\right)\,, \quad \omega_0 = \frac{R\,a^2}{\sqrt{2}\,\Sigma}\left(\sin^2\theta\,\mathrm{d}\phi + \cos^2\theta\,\mathrm{d}\psi\right),\\
\omega_1 &= -\frac{R}{\sqrt{2}\,\Sigma}\,\mathcal{F}\left[(r^2+a^2)\sin^2\theta\,\mathrm{d}\phi + r^2\cos^2\theta\,\mathrm{d}\psi\right],\\
\gamma_2 &= \frac{Q_5}{\Sigma}\,(r^2+a^2)\sin^2\theta\,, \quad a_1 = 0\,,\\
a_4 &= \sqrt{2}\,b\,\frac{a^{k_1}\cos^{k_1}\theta}{(r^2+a^2)^{\frac{k_1}{2}}}\left[\tan\theta\,\sin(k_1\hat{v})\mathrm{d}\psi + \cos(k_1\hat{v})\mathrm{d}\psi\right],\\
\delta_4 &= \frac{R}{2\Sigma}\,b\,\frac{a^{k_1}\cos^{k_1}\theta}{(r^2+a^2)^{\frac{k_1}{2}}}\,\tan\theta\left[-2(r^2+a^2)\sin(k_1\hat{v})\mathrm{d}\theta\wedge\mathrm{d}\phi\right.\\
&\quad \left.+2a^2\cos^2\theta\sin(k_1\hat{v})\mathrm{d}\theta\wedge\mathrm{d}\psi + (r^2+a^2)\sin 2\theta\,\cos(k_1\hat{v})\mathrm{d}\phi\wedge\mathrm{d}\psi\right],\\
a_5 &= \sqrt{2}\,c\,\frac{a^{k_2}\cos^{k_2}\theta}{(r^2+a^2)^{\frac{k_2}{2}}}\left[\tan\theta\,\sin(k_2\hat{v})\mathrm{d}\psi + \cos(k_2\hat{v})\mathrm{d}\psi\right],\\
\delta_5 &= \frac{R}{2\Sigma}\,c\,\frac{a^{k_2}\cos^{k_2}\theta}{(r^2+a^2)^{\frac{k_2}{2}}}\,\tan\theta\left[-2(r^2+a^2)\sin(k_2\hat{v})\mathrm{d}\theta\wedge\mathrm{d}\phi\right.\\
&\quad \left.+2a^2\cos^2\theta\sin(k_2\hat{v})\mathrm{d}\theta\wedge\mathrm{d}\psi + (r^2+a^2)\sin 2\theta\,\cos(k_2\hat{v})\mathrm{d}\phi\wedge\mathrm{d}\psi\right].
\end{aligned}
\tag{3.58}
$$

and where we have defined for convenience

$$
\hat{v} = \frac{\sqrt{2}\,v}{R} - \psi\,. \tag{3.59}
$$

We want to stress that the presence of a non-vanishing $\mathcal{F}$ with the proper large r asymptotic means that we have a non-vanishing P charge [75]:

$$
\mathcal{F} = -\frac{2(b^2+c^2)}{2r^2} + O(r^{-3}) \equiv -\frac{2Q_P}{r^2} + O(r^{-3}). \tag{3.60}
$$

3.2 *The superstratum ansatz:* $(k,m,n) = (1,0,n_1)$, $(1,0,n_2)$

We want now to build a superstratum solution, that is not a superdescendant of a two-charge state. In order to do so, we follow the strategy outlined in sec. 2.4 of [76]: we will start with a well-defined ansatz for the Z_I functions that satisfy the first layer (3.3); then we use this ansatz to derive the sources of the second layer (3.4) in order to solve them for ω and $\mathcal{F}$. This gives a set of *linear* partial differential equations for the ω and $\mathcal{F}$ functions.

We start from the simple case of two Fourier modes, one for Z_4 and one for Z_5, described, in the notation of [89], by the triplets $(k,m,n) = (1,0,n_1)$, $(1,0,n_2)$, respectively. We thus define

$$
\begin{aligned}
\Delta_{k,m,n} &= \left(\frac{a}{\sqrt{r^2+a^2}}\right)^k \left(\frac{r}{\sqrt{r^2+a^2}}\right)^n \sin^{k-m}\theta\cos^m\theta\,,\\
\hat{v}_{k,m,n} &= (m+n)\frac{\sqrt{2}\,v}{R} + (k-m)\phi - m\psi\,,\\
\vartheta_{k,m,n} &= -\sqrt{2}\,\Delta_{k,m,n}\left[\left((m+n)r\sin\theta + n\left(\frac{m}{k}-1\right)\frac{\Sigma}{r\sin\theta}\right)\Omega^{(1)}\sin\hat{v}_{k,m,n}\right.\\
&\qquad \left.+\left(m\left(\frac{n}{k}+1\right)\Omega^{(2)} + n\left(\frac{m}{k}-1\right)\Omega^{(3)}\right)\cos\hat{v}_{k,m,n}\right],
\end{aligned}
$$

(3.61)

and the usual basis of self-dual two forms on the base space ds_4^2

$$\begin{aligned}
\Omega^{(1)} &= \frac{dr \wedge d\theta}{(r^2+a^2)\cos\theta} + \frac{r\sin\theta}{\Sigma}\, d\phi \wedge d\psi\,,\\
\Omega^{(2)} &= \frac{r}{r^2+a^2}\, dr \wedge d\psi + \tan\theta\, d\theta \wedge d\phi\,,\\
\Omega^{(3)} &= \frac{dr \wedge d\phi}{r} - \cot\theta\, d\theta \wedge d\phi\,.
\end{aligned} \tag{3.62}$$

We will then build a geometry whose holographic dual is the heavy state (3.54) with $(k,m,n) = (1,0,n_1),(1,0,n_2)$, respectively.

The supergravity ansatz is then[8]

$$\begin{aligned}
Z_1 &= \frac{Q_1}{\Sigma} + \frac{R^2}{2Q_5}\, b^2\, \frac{\Delta_{2,0,2n_1}}{\Sigma}\, \cos\hat{v}_{2,0,2n_1} + \frac{R^2}{2Q_5}\, c^2\, \frac{\Delta_{2,0,2n_2}}{\Sigma}\, \cos\hat{v}_{2,0,2n_2}\,,\\
Z_2 &= \frac{Q_5}{\Sigma}\,, \quad Z_4 = R\, b\, \frac{\Delta_{1,0,n_1}}{\Sigma}\, \cos\hat{v}_{1,0,n_1}\,, \quad Z_5 = R\, c\, \frac{\Delta_{1,0,n_2}}{\Sigma}\, \cos\hat{v}_{1,0,n_2}\,.
\end{aligned} \tag{3.64}$$

where the definition of Z_1 is inspired by the superdescendant case and is defined in a way that assures the *coiffuring* [117–119]. This coiffuring is a procedure that consists in adjusting the Fourier coefficient of (Z_1, Θ_2) in terms of those of (Z_4, Θ_4) and (Z_5, Θ_5), in order to have a smooth geometry at the end of the computations. In fact, since the first layer of equations are a set of decoupled second-order partial differential equations for the couples (Z_I, Θ^I), a priori we have no relation whatsoever among the Z_1, Z_2, Z_4, Z_5 functions. But only an appropriate choice of the Z_1 in terms of the Z_4, Z_5 will lead to a well-defined, smooth solution $(\omega, \mathcal{F})$ for the second layer of equations.

Based on previous result for geometries with a single internal mode for Z_4 [117–119], where the correct Z_1 is such that $Z_1 Z_2 - Z_4^2$ was $v-$independent, we have coiffured our geometry as the following: after having chosen the two Fourier modes for Z_4 and Z_5, i.e. having chosen the dual CFT state (3.54) described by $\left|H_{(1,0,n_1),(1,0,n_2)}\right\rangle$, we pick Z_1 such that $\mathbb{P}$ is $v-$independent. This will also imply that in the second layer we will not have any $v-$dependent sources, allowing us to construct a $v-$independent ω and $\mathcal{F}$ that we will denote as ω^{RMS} and $\mathcal{F}^{\mathrm{RMS}}$.

Please notice that in this thesis we will work with only one Fourier mode for Z_4 and Z_5; one may wonder to generalise it allowing many different Fourier modes for both of them and then have a more involved coiffuring; for sake of simplicity we will restrain to do that here.

Then we define

$$\begin{aligned}
&\Theta_1 = 0\,, \quad \Theta_2 = \frac{R}{2Q_5}\, b\, \vartheta_{2,0,2n_1} + \frac{R}{2Q_5}\, c\, \vartheta_{2,0,2n_2}\,,\\
&\Theta_4 = b\, \vartheta_{1,0,n_1}\,, \quad \Theta_5 = c\, \vartheta_{1,0,n_2}\,.
\end{aligned} \tag{3.65}$$

By construction this set solves the first layer (3.3), as wanted. We are then left to solve the second layer (3.4); in order to do so we need a good ansatz

8 We will use the regularity condition for this case, that is

$$\frac{Q_1 Q_5}{R^2} = a^2 + \frac{b^2}{2} + \frac{c^2}{2}\,. \tag{3.63}$$

Later on we will explain how this condition emerges.

for ω and $\mathcal{F}$. We are searching now for asymptotically AdS solutions, then we define

$$\omega^{\rm AdS} = \omega_0 + \omega^{\rm RMS}(r,\theta)\,, \quad \mathcal{F} = \mathcal{F}^{\rm RMS}(r,\theta)\,, \tag{3.66}$$

where ω_0 is the one of the original two-charge solution[9], and where RMS stands for the non-oscillating part. An important fact is that the oscillating part in asymptotically AdS geometry turns out to decouple from the RMS part, so we do not need to include it, and we avoid its analysis.

The Second Layer gives an equation of motion of the form

$$\begin{aligned} \mathrm{d}\omega^{\rm RMS} + *_4\mathrm{d}\omega^{\rm RMS} + \mathcal{F}^{\rm RMS}\mathrm{d}\beta_0 &= J^{(1)}_{n_1,n_2}\,, \\ \widehat{\mathcal{L}}\mathcal{F}^{\rm RMS} &= J^{(2)}_{n_1,n_2}\,, \end{aligned} \tag{3.68}$$

where β_0 is the one of the two-charge state, and where

$$\widehat{\mathcal{L}}F \equiv \frac{1}{r\,\Sigma}\partial_r\left(r(r^2+a^2)\partial_r F\right) + \frac{1}{\Sigma\,\sin\theta\,\cos\theta}\partial_\theta\left(\sin\theta\,\cos\theta\,\partial_\theta F\right)\,, \tag{3.69}$$

is the scalar Laplacian on the base space, i.e. $\widehat{\mathcal{L}}F = -*_4\mathcal{D}*_4\mathcal{D}F$. We can see that the sources have a linear form in the b, c parameters and also that, in the first equation, there is no $\Omega^{(1)}$ direction, and then $\omega_r = 0 = \omega_\theta$; in particular

$$\begin{aligned} J^{(1)}_{n_1,n_2} &= \sqrt{2}\,R\left(b^2\,\frac{\Delta_{2,0,2n_1}}{\Sigma}\,n_1 + c^2\,\frac{\Delta_{2,0,2n_2}}{\Sigma}\,n_2\right)\Omega^{(3)}\,, \\ J^{(2)}_{n_1,n_2} &= \frac{4}{r^2+a^2}\,\frac{1}{\Sigma\,\cos\theta}\left(b^2\,\Delta_{2,2,2n_1-2}\,n_1^2 + c^2\,\Delta_{2,2,2n_2-2}\,n_2^2\right)\,. \end{aligned} \tag{3.70}$$

We have then to solve the system

$$\begin{aligned} \mathrm{d}\omega^{\rm RMS} + *_4\mathrm{d}\omega^{\rm RMS} + \mathcal{F}^{\rm RMS}\mathrm{d}\beta_0 &= \sqrt{2}\,R\left(b^2\,\frac{\Delta_{2,0,2n_1}}{\Sigma}\,n_1 + c^2\,\frac{\Delta_{2,0,2n_2}}{\Sigma}\,n_2\right)\Omega^{(3)}\,, \\ \widehat{\mathcal{L}}\mathcal{F}^{\rm RMS} &= \frac{4}{r^2+a^2}\,\frac{1}{\Sigma\,\cos\theta}\left(b^2\,\Delta_{2,2,2n_1-2}\,n_1^2 + c^2\,\Delta_{2,2,2n_2-2}\,n_2^2\right)\,. \end{aligned} \tag{3.71}$$

Those equations are linear, and are of the same form of the eqs. (4.7, 4.8) of [89]; then superimposing a linear combination of the form

$$\omega^{\rm AdS} = \omega^{\rm AdS}_{1,0,n_1} + \omega^{\rm AdS}_{1,0,n_2}\,, \quad \mathcal{F}^{\rm AdS} = \mathcal{F}^{\rm AdS}_{1,0,n_1} + \mathcal{F}^{\rm AdS}_{1,0,n_2}\,, \tag{3.72}$$

we simply have two identical set of equations that are *exactly* the one appearing in [89]

$$\begin{aligned} \mathrm{d}\omega^{\rm RMS}_{1,0,n_i} + *_4\mathrm{d}\omega^{\rm RMS}_{1,0,n_i} + \mathcal{F}^{\rm RMS}_{1,0,n_i}\mathrm{d}\beta_0 &= \sqrt{2}\,R\,b_i^2\,\frac{\Delta_{2,0,2n_i}}{\Sigma}\,n_i\,\Omega^{(3)}\,, \\ \widehat{\mathcal{L}}\mathcal{F}^{\rm RMS}_{1,0,n_i} &= \frac{4\,n_i^2}{r^2+a^2}\,b_i^2\,\frac{\Delta_{2,2,2n_i-2}}{\Sigma\,\cos\theta}\,, \quad b_i = (b,c), \end{aligned} \tag{3.73}$$

where they are also solved.

This behaviour is quite natural: having imposed only one mode for Z_4 and one for Z_5, after the right linear coiffuring for Z_1, the linearity of the layers

9 In detail, we have

$$\beta_0 = \frac{R\,a^2}{\sqrt{2}\,\Sigma}\left(\sin^2\theta\,\mathrm{d}\phi - \cos^2\theta\,\mathrm{d}\psi\right)\,, \quad \omega_0 = \frac{R\,a^2}{\sqrt{2}\,\Sigma}\left(\sin^2\theta\,\mathrm{d}\phi + \cos^2\theta\,\mathrm{d}\psi\right)\,. \tag{3.67}$$

equations - and the fact that $\Theta_1 = 0$ - imposes that we can solve separately the equations; to be more explicit let us define

$$Z_1 = Z_1^{(0)} + Z_1^{(b)} + Z_1^{(c)}\,, \quad Z_2 = Z_2^{(0)}\,, \quad \Theta_2 = \Theta_2^{(b)} + \Theta_2^{(b)}\,, \tag{3.74}$$

such that the first layer is separately solved by the couples $\left(Z_1^{(b)}, \Theta_2^{(b)}\right)$ and $\left(Z_1^{(c)}, \Theta_2^{(c)}\right)$, as is our case. Then, defining also

$$\omega^{\text{AdS}} = \omega^{\text{AdS}}_{(b)} + \omega^{\text{AdS}}_{(c)}\,, \quad \mathcal{F}^{\text{AdS}} = \mathcal{F}^{\text{AdS}}_{(b)} + \mathcal{F}^{\text{AdS}}_{(c)}\,, \tag{3.75}$$

and using the layer equations for the ω_0, the second layer decouples into

$$\mathcal{D}\omega_{(i)} + *_4\mathcal{D}\omega_{(i)} + \mathcal{F}_{(i)}\,\mathrm{d}\beta = Z_1^{(i)}\Theta_1^{(0)} + Z_2^{(0)}\Theta_2^{(i)} - 2Z_{(i)}\Theta_{(i)}\,, \tag{3.76a}$$

$$\begin{aligned} *_4\mathcal{D}*_4\left(\dot\omega_{(i)} - \frac{1}{2}\,\mathcal{D}\mathcal{F}_{(i)}\right) &= \partial_v^2(Z_1^{(i)}Z_2^{(0)} - Z_{(i)}) - [\dot Z_1^{(i)}\dot Z_2^{(0)} - (\dot Z_{(i)})^2] \\ &\quad - \frac{1}{2}*_4\left(\Theta_1^{(0)}\wedge\Theta_2^{(i)} - \Theta_{(i)}\wedge\Theta_{(i)}\right)\,, \end{aligned} \tag{3.76b}$$

$$Z_1^{(i)} = \left(Z_1^{(b)}, Z_1^{(c)}\right)\,, \quad Z_{(i)} = (Z_4, Z_5)\,, \tag{3.76c}$$

$$\Theta_2^{(i)} = \left(\Theta_2^{(b)}, \Theta_2^{(c)}\right)\,, \quad \Theta_{(i)} = (\Theta_4, \Theta_5)\,. \tag{3.76d}$$

We can then read the solution directly from [89]; for the $(k,m,n) = (1,0,n_1)$, $(1,0,n_2)$ case we then have

$$\mathcal{F}^{\text{RMS}} = -\frac{b^2}{a^2}\left(1 - \frac{r^{2n_1}}{(r^2+a^2)^{n_1}}\right) - \frac{c^2}{a^2}\left(1 - \frac{r^{2n_2}}{(r^2+a^2)^{n_2}}\right)\,, \tag{3.77a}$$

$$\omega^{\text{RMS}} = \frac{R}{\sqrt{2}\Sigma}\left[b^2\left(1 - \frac{r^{2n_1}}{(r^2+a^2)^{n_1}}\right) + c^2\left(1 - \frac{r^{2n_2}}{(r^2+a^2)^{n_2}}\right)\right]\sin^2\theta\,\mathrm{d}\phi. \tag{3.77b}$$

3.3 *The superstratum ansatz: (k,m,n) generic*

In the light of the discussion of the previous section, it is now easy to generalise the previous solution to generic (k_1,m_1,n_1) and (k_2,m_2,n_2); with the right ansatz for the Z_I

$$\begin{aligned} Z_1 &= \frac{Q_1}{\Sigma} + \frac{R^2}{2Q_5}\,b^2\,\frac{\Delta_{2k_1,2m_1,2n_1}}{\Sigma}\,\cos\hat v_{2k_1,2m_1,2n_1} \\ &\qquad + \frac{R^2}{2Q_5}\,c^2\,\frac{\Delta_{2k_2,2m_2,2n_2}}{\Sigma}\,\cos\hat v_{2k_2,2m_2,2n_2}\,, \\ Z_2 &= \frac{Q_5}{\Sigma}\,, \\ Z_4 &= R\,b\,\frac{\Delta_{k_1,m_1,n_1}}{\Sigma}\,\cos\hat v_{k_1,m_1,n_1}\,, \quad Z_5 = R\,c\,\frac{\Delta_{k_2,m_2,n_2}}{\Sigma}\,\cos\hat v_{k_2,m_2,n_2}\,, \end{aligned} \tag{3.78}$$

and for the Θ^I

$$\begin{aligned} &\Theta_1 = 0\,, \quad \Theta_2 = \frac{R}{2Q_5}\,b\,\vartheta_{2k_1,2m_1,2n_1} + \frac{R}{2Q_5}\,c\,\vartheta_{2k_2,2m_2,2n_2}\,, \\ &\Theta_4 = b\,\vartheta_{k_1,m_1,n_1}\,, \quad \Theta_5 = c\,\vartheta_{k_2,m_2,n_2}\,, \end{aligned} \tag{3.79}$$

splitting also the ω and the $\mathcal{F}$ again as

$$\begin{aligned} \omega^{\mathrm{AdS}} &= \omega_0 + \omega^{\mathrm{RMS}}(r,\theta)\,, & \mathcal{F} &= \mathcal{F}^{\mathrm{RMS}}(r,\theta)\,, \\ \omega^{\mathrm{RMS}} &= \omega^{\mathrm{RMS}}_{k_1,m_1,n_1} + \omega^{\mathrm{RMS}}_{k_2,m_2,n_2}\,, & \mathcal{F}^{\mathrm{RMS}} &= \mathcal{F}^{\mathrm{RMS}}_{k_1,m_1,n_1} + \mathcal{F}^{\mathrm{RMS}}_{k_2,m_2,n_2}\,, \end{aligned} \tag{3.80}$$

we get the two identical systems

$$\mathrm{d}\omega^{\mathrm{RMS}}_{k,m,n} + *_4\mathrm{d}\omega^{\mathrm{RMS}}_{k,m,n} + \mathcal{F}^{\mathrm{RMS}}_{k,m,n}\mathrm{d}\beta_0 = \sqrt{2}\,R\,b_i^2\,\frac{\Delta_{2k,2m,2n}}{\Sigma}\left(\frac{m(k+n)}{k}\,\Omega^{(2)} - \frac{n(k-m)}{k}\,\Omega^{(3)}\right), \tag{3.81a}$$

$$\widehat{\mathcal{L}}\mathcal{F}^{\mathrm{RMS}}_{k,m,n} = \frac{4b_i^2}{r^2+a^2}\,\frac{1}{\Sigma\,\cos^2\theta}\left[\left(\frac{m(k+n)}{k}\right)^2\Delta_{2k,2m,2n} + \left(\frac{n(k-m)}{k}\right)^2\Delta_{2k,2m+2,2n-2}\right], \tag{3.81b}$$

where $b_i = (b,c)$. This system coincides exactly with eqs. (4.7, 4.8) of [89], so we can inherit the solution from there and, in the following, we will briefly review how those solutions are built: we split

$$\omega^{\mathrm{RMS}}_{k,m,n} = \mu_{k,m,n}(\mathrm{d}\psi + \mathrm{d}\phi) + \zeta_{k,m,n}(\mathrm{d}\psi - \mathrm{d}\phi), \tag{3.82}$$

and then, defining

$$\hat{\mu}_{k,m,n} = \mu_{k,m,n} + \frac{R}{4\sqrt{2}}\,\frac{r^2 + a^2\sin^2\theta}{\Sigma}\,\mathcal{F}_{k,m,n} + \frac{R}{4\sqrt{2}}\,b_i^2\,\frac{\Delta_{2k,2m,2n}}{\Sigma}\,, \tag{3.83}$$

we recast[10] the system (3.81) in a system regarding only the two scalar functions $\mathcal{F}_{k,m,n}$ and $\hat{\mu}_{k,m,n}$:

$$\widehat{\mathcal{L}}\hat{\mu}_{k,m,n} = \frac{R\,b_i^2}{4\sqrt{2}\,(r^2+a^2)}\,\frac{1}{\Sigma\,\cos^2\theta}\left(\frac{(k-m)^2(k+n)^2}{k^2}\,\Delta_{2k,2m+2,2n} + \frac{n^2m^2}{k^2}\,\Delta_{2k,2m,2n-2}\right), \tag{3.84a}$$

$$\widehat{\mathcal{L}}\mathcal{F}_{k,m,n} = \frac{4b_i^2}{r^2+a^2}\,\frac{1}{\Sigma\,\cos^2\theta}\left[\left(\frac{m(k+n)}{k}\right)^2\Delta_{2k,2m,2n} + \left(\frac{n(k-m)}{k}\right)^2\Delta_{2k,2m+2,2n-2}\right], \tag{3.84b}$$

while $\zeta_{k,m,n}$ is determined after having determined $\hat{\mu}_{k,m,n}$ by putting (3.82) into eq. (3.81b), as explained in [89]. Since its expression is quite cumbersome and, in the end, not relevant in what follows, we restrain to write it down explicitly.

To solve eqs. (3.84) we need a function $F_{2k,2m,2n}$ such that

$$\widehat{\mathcal{L}}F_{2k,2m,2n} = \frac{\Delta_{2k,2m,2n}}{(r^2+a^2)\Sigma\,\cos^2\theta}\,, \tag{3.85}$$

10 The equation for ω is a system of first-order partial differential equation for the components; we can rearrange it eliminating the unwanted components. This procedure recasts the equation for the wanted degrees of fredom as a second-order partial differential equation, the one we show.

that is

$$F_{2k,2m,2n} = -\sum_{j_1,j_2,j_3=0}^{j_1+j_2+j_3\le k+n-1} \binom{j_1+j_2+j_3}{j_1,j_2,j_3} \times \\ \times \frac{\binom{k+n-1-(j_1+j_2+j_3)}{k-m-j_1,\, m-j_2-1,\, n-j_3}^2}{\binom{k+n-1}{k-m,\, m-1,\, n}^2} \times \\ \times \frac{\Delta_{2(k-1-j_1-j_2),2(m-j_2-1),2(n-j_3)}}{4(k+n)^2(r^2+a^2)}\,, \tag{3.86}$$

where

$$\binom{j_1+j_2+j_3}{j_1,j_2,j_3} \equiv \frac{(j_1+j_2+j_3)!}{j_1!j_2!j_3!}\,. \tag{3.87}$$

Having that, the solution is

$$\mathcal{F}_{k,m,n} = 4b_i^2\left[\frac{m^2(k+n)^2}{k^2}\,F_{2k,2m,2n} + \frac{n^2(k-m)^2}{k^2}\,F_{2k,2m+2,2n-2}\right], \tag{3.88a}$$

$$\mu_{k,m,n} = \frac{R\,b_i^2}{\sqrt{2}}\Big[\frac{(k-m)^2(k+n)^2}{k^2}\,F_{2k,2m+2,2n} + \frac{m^2n^2}{k^2}F_{2k,2m,2n-2} \\ - \frac{r^2+a^2\sin^2\theta}{4\Sigma}\,\frac{1}{b_i^2}\mathcal{F}_{k,m,n} - \frac{\Delta_{2k,2m,2n}}{4\Sigma} + \frac{x^{(i)}_{k,m,n}}{4\Sigma}\Big], \tag{3.88b}$$

where $x^{(i)}_{k,m,n}$ are a set of numbers. Since $\hat{\mu}$ satisfies a generalised Poisson equation $\widehat{\mathcal{L}}\hat{\mu} = J$, we always have the freedom to add a solution of the homogeneus equation $\widehat{\mathcal{L}}G = 0$; this is the role of the piece multiplied by the constant $x^{(i)}_{k,m,n}$; it will be fixed by requiring the regularity of the solution at $r=0, \theta=0$ and $r=0, \theta=\pi/2$. Notice that, by linearity, we have to impose the regularity on the two separate solutions $\left(\mathcal{F}^{(b)}_{k,m,n}, \mu^{(b)}_{k,m,n}\right)$ and $\left(\mathcal{F}^{(c)}_{k,m,n}, \mu^{(c)}_{k,m,n}\right)$, and this will separately fix the two constants $x^{(i)}_{k,m,n}$. This means that we can read the regularity condition from [89]

$$x^{(i)}_{k,m,n} = \left[\binom{k}{m}\binom{k+n-1}{n}\right]^{-1}, \tag{3.89}$$

following from requiring regularity at $r=0, \theta=0$, and

$$\frac{Q_1Q_5}{R^2} = a^2 + x^{(b)}_{k,m,n}\,\frac{b^2}{2} + x^{(c)}_{k,m,n}\,\frac{c^2}{2}\,, \tag{3.90}$$

following from requiring regularity at $r=0, \theta=\pi/2$. In the case $(k,m,n) = (1,0,n_1)$, $(1,0,n_2)$ the two $x^{(i)}_{k,m,n}$ are equals to one, so that

$$x^{(b)}_{1,0,n} = 1\,, \quad x^{(c)}_{1,0,n} = 1\,, \tag{3.91}$$

so the regularity there reads

$$\frac{Q_1Q_5}{R^2} = a^2 + \frac{b^2}{2} + \frac{c^2}{2}. \tag{3.92}$$

Since the most troublesome points in the spacetime are the ones discussed previously, where we have shown the regularity of the solution, we have no reason to expect problem elsewhere.

3.4 *The absence of CTCs*

Another possible issue that could affect the microstates we have built is the existence of *Closed Timelike Curves* (CTCs); if these geometries have CTCs they have to be regarded as unphysical. Proving that all the members of the family of solutions we have found are free of CTCs is particularly involved; we will then focus on the family $(1,0,n_1)$, $(1,0,n_2)$ of sec. 3.3.2 and we will show explicitly that it is indeed regular and free of CTCs. To do that, we have to rewrite the Einstein frame metric in the t, y coordinate as

$$\begin{aligned} \mathrm{d}s_6^2 = {} & G_{tt}\mathrm{d}t^2 + G_{yy}(\mathrm{d}y + A)^2 + G_{\theta\theta}\mathrm{d}\theta^2 + G_{rr}\mathrm{d}r^2 \\ & + G_{\phi\phi}(\mathrm{d}\phi + B_t\mathrm{d}t + B_y\mathrm{d}y)^2 + G_{\psi\psi}(\mathrm{d}\psi + C_t\mathrm{d}t + C_y\mathrm{d}y)^2 . \end{aligned} \tag{3.93}$$

It is indeed easy to show that, thanks to eq. (3.77), we can write

$$\begin{aligned} \omega_{\text{RMS}} &= \frac{R\,a^2}{\sqrt{2}\,\Sigma}\,F\,\sin^2\theta\,\mathrm{d}\phi\,, \quad \mathcal{F}_{\text{RMS}} = -F\,, \\ F &\equiv \frac{b^2}{a^2}\left(1 - \frac{r^{2n_1}}{(r^2+a^2)^{n_1}}\right) + \frac{c^2}{a^2}\left(1 - \frac{r^{2n_2}}{(r^2+a^2)^{n_2}}\right) > 0\,, \end{aligned} \tag{3.94}$$

so that all the angular terms of the metric can be written as

$$\begin{aligned} G_{yy} &= \frac{(2+F)\Lambda\Sigma\sqrt{2a^2+b^2+c^2}}{\sqrt{2}R\left(a^4(F+2)\cos^2\theta + \Lambda^2 r^2\left(2a^2+b^2+c^2\right)\right)}\,r^2\,, \\ G_{\theta\theta} &= R\sqrt{a^2+\frac{b^2}{2}+\frac{c^2}{2}}\,\Lambda\,, \\ G_{\phi\phi} &= \frac{R\sin^2\theta\left(\Lambda^2\left(a^2+r^2\right)\left(2a^2+b^2+c^2\right) - a^4(2+F)\sin^2\theta\right)}{\sqrt{2}\Lambda\Sigma\sqrt{2a^2+b^2+c^2}}\,, \\ G_{\psi\psi} &= \frac{R\cos^2\theta\left(a^4(2+F)\cos^2\theta + \Lambda^2 r^2\left(2a^2+b^2+c^2\right)\right)}{\sqrt{2}\Lambda\Sigma\sqrt{2a^2+b^2+c^2}}\,, \end{aligned} \tag{3.95}$$

where

$$\Lambda = \frac{\sqrt{\mathbb{P}}\,\Sigma}{R\sqrt{a^2+\frac{b^2}{2}+\frac{c^2}{2}}}\,. \tag{3.96}$$

Due to their cumbersomeness and their futility for what follows, we do not show here explicitly the form of all the other coefficients that appear in eq. (3.93).

It is quite straightforward to see that all the angular terms reported in eq. (3.95) are positive and that near $r=0$ they behave as

$$\begin{aligned} G_{yy} &\simeq \frac{\sqrt{2a^2+b^2+c^2}}{\sqrt{2}\,R}\,\frac{r^2}{a^2}\,, \\ G_{\theta\theta} &\simeq R\sqrt{a^2+\frac{b^2}{2}+\frac{c^2}{2}}\,, \\ G_{\phi\phi} &\simeq \frac{R\sqrt{2a^2+b^2+c^2}}{\sqrt{2}}\,\sin^2\theta\,, \\ G_{\psi\psi} &\simeq \frac{R\sqrt{2a^2+b^2+c^2}}{\sqrt{2}}\,\cos^2\theta\,, \end{aligned} \tag{3.97}$$

since $\Lambda \to 1$ and $F \to a^{-2}(b^2+c^2)$ for $r \to 0$. We have then shown that this set of solutions has no CTCs. We want also to stress the fact that, since the $\mathbb{S}^1$ shrinks smoothly at $r=0$, the spacetime is geodesically complete and no possible extension in the $r<0$ region is allowed.

3.5 *A detour: Asymptotically Flat geometries*

One may now ask if it is possible to extend this construction to Asymptotically Flat geometries, rather then Asymptotically Anti-de Sitter ones. In order to do that we need to "add back" the 1, i.e. that we need to perform the shift [76, 87–89, 102]

$$Z_1 \to 1 + Z_1\,, \quad Z_2 \to 1 + Z_2\,, \quad Z_4 \to Z_4\,, \quad Z_5 \to Z_5\,. \tag{3.98}$$

This will give a more involved problem, as pointed out - and then solved - in [89]. But we can see that having both Z_4 and Z_5 with a single mode adds no other difficulties with their analysis. In fact we can easily see that, since $\Theta_1 = 0$ and $\partial_v Z_2 = 0$, the only difference w.r.t. the asymptotically AdS case is that the sources in the second layer equations (3.4) acquires a new term. In fact, defined

$$\begin{aligned}
J_1^{\text{AdS}} &= Z_1\Theta_1 + Z_2\Theta_2 - 2Z_4\Theta_4 - 2Z_5\Theta_5\,,\\
J_2^{\text{AdS}} &= \partial_v^2(Z_1Z_2 - Z_4^2 - Z_5^2) - [\dot{Z}_1\dot{Z}_2 - (\dot{Z}_4)^2 - (\dot{Z}_5)^2]\\
&\quad - \frac{1}{2} *_4 (\Theta_1 \wedge \Theta_2 - \Theta_4 \wedge \Theta_4 - \Theta_5 \wedge \Theta_5)\,,
\end{aligned}$$

we simply have, after the shift (3.98) that

$$J_1^{\text{AF}} = J_1^{\text{AdS}} + \Theta_2\,, \quad J_2^{\text{AF}} = J_2^{\text{AdS}} + \partial_v^2 Z_1\,. \tag{3.99}$$

Now we cannot decouple anymore the $v-$dependent modes, as it was in [89]; but, by linearity of the equation and of the sources, our problem simplifies drastically, leaving us with (twice) the same problem of [89]. We can then again follow their steps and build an Asymptotically Flat superstratum solution. Since this analysis will not add anything to our discussion, and since it is very cumbersome, we will avoid performing that in detail here.

4 RECAP OF THE RESULTS

In this chapter, working in the framework of type IIB string theory on a compact $\mathbb{T}^4$, we have presented the ansatz (3.1) for the most general D1D5P BPS geometry, allowing excitations also in the internal $\mathbb{T}^4$ and we have then shown under which conditions this ansatz is a $\frac{1}{8}-$BPS solution of type IIB supergravity. We have thus built a superdescendant D1D5P geometry from a D1D5 geometry with the generating solution technique of [87] and we furnished a non-trivial check for those equations by proving that this geometry solves our system (3.3, 3.4).

We have also shown how it is possible to construct new asymptotically AdS D1D5P superstratum solutions with both internal and external excitations, inheriting some known results in literature and extending them; we have explicitly built superstrata adding only one mode for the external excitation Z_4 and one mode for the internal excitation Z_5; we have explicitly written down the solution (3.77) for the $(k, m, n) = (1, 0, n_1)$, $(1, 0, n_2)$ case, but we have also built implicitly the generic (k_1, m_1, n_1), (k_2, m_2, n_2) case. We have finally discussed how it is possible to extend these results to the Asymptotically Flat case, that could be useful for the Fuzzball proposal.

One may wonder if this class constitutes the full set of possible microstates; actually, even if it is fairly general, it does not contain all the possible heavy states: one example of microstate that does not fall in this class is the one recently constructed in [90], obtained by acting also with the fermionic generators of the superconformal algebra. But it is also straightforward to notice that we can easily build superstrata that contain excitation of the form discussed in [90], since the steps we made proceed the same way in this case. In fact, repeating the step here reported along the lines of [90], where they need a $\Theta_4 \neq 0$ but $Z_4 = 0$, is fairly straightforward; we simply need to perform the ansatz

$$\begin{aligned} Z_4 &= 0\,, \quad \Theta_4 = \widehat{b}\,\widehat{\vartheta}_{k_1,m_1,n_1}\,, \\ Z_5 &= 0\,, \quad \Theta_5 = \widehat{c}\,\widehat{\vartheta}_{k_2,m_2,n_2}\,, \end{aligned} \tag{3.100}$$

where $\widehat{\vartheta}_{k,m,n}$ is defined in eq. (5.12) of [90].

By the form of the ansatz, the first layer is again solved, while the second layer gives a set of two linearly independent systems of PDEs, that are of the form of [90]. Due to this, it is trivial to solve them by again reproducing exactly the computation reported therein, giving us the most general $\frac{1}{8}$-BPS superstrata with both external and internal excitations, which is dual to the heavy state

$$\begin{aligned} &|{+}{+}\rangle_1^{N^{(++)}} \cdot \\ &\cdot \left[\frac{\left(J^+_{-1}\right)^{m_1}}{m_1!} \frac{\left(L_{-1} - J^3_{-1}\right)^{n_1}}{n_1!} \left(G^{+1}_0 G^{+2}_0 + \frac{1}{k} J^3_{-1} \left(L_{-1} - J^3_{-1}\right)\right)^{q_1} |00\rangle_{k_1} \right]^{N^{(b)}} \\ &\cdot \left[\frac{\left(J^+_{-1}\right)^{m_2}}{m_2!} \frac{\left(L_{-1} - J^3_{-1}\right)^{n_2}}{n_2!} \left(G^{+1}_0 G^{+2}_0 + \frac{1}{k} J^3_{-1} \left(L_{-1} - J^3_{-1}\right)\right)^{q_2} |00\rangle^{(1\dot{1})}_{k_2} \right]^{N^{(c)}} . \end{aligned} \tag{3.101}$$

4 NON-EXTREMAL SUPERDESCENDANTS

We have shown, both in chap. 2 and in chap. 3, that generating the geometries of supersymmetric superdescendants is rather simply. Indeed, the CFT chiral algebra is represented on the gravity side by diffeomorphisms that do not vanish at the AdS boundary and thus the geometries of superdescendants are obtained by applying such diffeomorphisms to the RR ground state geometries, as in [87]. This produces asymptotically AdS solutions, but we are also interested in black hole microstates that are asymptotically flat, i.e. that at large distances the spacetime approaches $\mathbb{R}^{4,1} \times \mathbb{S}^1 \times \mathbb{T}^4$. Even for supersymmetric superdescendants this extension requires solving a non-trivial problem, since the asymptotically flat geometry is not diffeomorphic to the seed $\frac{1}{4}$-BPS solution, with the non-trivial deformation of the geometry occurring in the neck region that joins AdS and asymptotic infinity. This problem was first discussed in [49], where it was solved using a double approximation: first they considered the limit in which the microstate can be described as a perturbation around the background of a simpler state, and then the linear equations for the perturbation are solved approximately using a matching procedure between the AdS and the flat regions. This technique was further generalised in [84, 85, 115], while an exact construction was given, for two different classes of states, in [86] and [87]. The key point is that the existence of an exact solution is again ultimately a consequence of the linearity of the BPS equations.

The purpose of this chapter is to investigate how much of this structure extends to non-supersymmetric microstates. We know only very few non-extremal microstate geometries; the first example, and also the only one with a known CFT dual, was constructed in [120] by generalising to the non-BPS case the technique of [105]. The full holographic interpretation of this solution was found in [121] and it involves states obtained by spectrally-flowing in both left-moving and right-moving sectors to some simple $\frac{1}{4}$-BPS state. The existence of non-supersymmetric supergravity solutions that carry no global charges was conjectured in [117], where a construction of these solutions based on neutral oscillating supertubes was performed.

Powerful techniques to construct exact fully non-linear non-supersymmetric solutions have been developed over the past years [122–125], but the relation between these gravity solutions and the states of the CFT is unclear yet. In fact, the issue of which states of the orbifold theory should survive in the spectrum at the gravity point is less understood for non-supersymmetric states than for supersymmetric ones. The existence of a non-BPS analogue of the graviton gas made of states that are not descendants of RR ground states is far from obvious, since the linear properties of the supergravity equations that allowed the construction of the supergraviton gas in the supersymmetric setting is not guaranteed to exist when supersymmetry is broken. On the other hand the chiral algebra guarantees the existence of superdescendants on the whole moduli space, and crucially at supergravity point, even when the states contain both left- and right-moving generators and, hence, break supersymmetry completely. The geometry of these states in the decoupling limit is thus obtained by the action of a large diffeomorphism on a $\frac{1}{4}$-BPS solution, similarly at what we have done in chap. 3.

The non-trivial task is the extension from the asymptotically AdS geometry so constructed to an asymptotically flat one, which could be interpreted as a black hole microstate of the Strominger-Vafa black hole. In this chapter, we will not attempt here to perform this task at the full non-linear level, and we will work in the regime in which the microstate is merely a linear perturbation around a supersymmetric background. This is enough for us to show that the simplification that allowed for a simple solution of the problem for supersymmetric states does not happen when the perturbation breaks supersymmetry. We reduce the problem to the solution of a partial differential equation, which we will able to solve approximately using the same matching technique of [49].

Our main goal is to prove the existence of a solution that interpolates between the geometry in the decoupling limit and a well-behaved asymptotically-flat solution at large distances. The existence of such a solution is non-trivial since non-extremality has drastic effects on the large-distance behaviour of the geometry. It was shown in [49] that when the perturbation carries more energy than its charge, it will be non-normalisable, i.e. it belongs to the continuum spectrum of excitations around the extremal background. This behaviour is expected for non-extremal perturbations, and it does not signal a pathology of the solution[1]. A similar conclusion was reached more recently in [121], which also considered non-extremal states obtained by applying left- and right-moving transformations to $\frac{1}{4}$-BPS geometries: it was concluded that the only normalisable solutions are given by *extremal* perturbations around the non-extremal background of [120]. Here we consider genuinely non-extremal perturbations and we find that they indeed fall-off very slowly, as $r^{-3/2}$ at large distances. This is the expected asymptotic behaviour for non-extremal states [49]; moreover, we verify that the perturbation does not alter the global charges of the background. Hence we conclude that the perturbative non-extremal solutions we find can be consistently identified with non-supersymmetric microstates of the asymptotically-flat black hole.

1 THE CFT STATES

As we have seen in chap. 2, the RR ground states can be described as a collection of strands that are characterised by the winding number k and the left and right R-charges $(\jmath, \bar\jmath)$; as usual, we will denote the state of each strand by $|\jmath\,\bar\jmath\rangle_k$ and the full D1D5 state containing N_i strands of type $|\jmath_i\,\bar\jmath_i\rangle_{k_i}$ as $\prod_i(|\jmath_i\,\bar\jmath_i\rangle_{k_i})^{N_i}$, plus the constraint that the total winding number must be N, i.e. $\sum_i k_i N_i = N$. The CFT has a spectral flow symmetry which maps R and NS sectors; performing one unit of spectral flow on the left- and the right-sector of the CFT maps the state $(|{+}{+}\rangle_1)^N$ into the SL(2,$\mathbb{C}$)-invariant vacuum. This fact allows to deduce easily the gravity dual geometry for the state $(|{+}{+}\rangle_1)^N$.

Each D1D5 ground state admits a dual description in terms of an asymptotically $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$ geometry [48]; in this chapter, for all the states considered the compact space will just play a spectator role, and we will only focus

1 We thank S. Mathur for discussion on this point.

on the dimensionally reduced six-dimensional theory. The geometry dual to the SL(2,$\mathbb{C}$)-invariant vacuum is simply the global $\mathrm{AdS}_3 \times \mathbb{S}^3$:

$$\mathrm{d}s_6^2 = \sqrt{Q_1 Q_5}\left(\mathrm{d}s^2_{\mathrm{AdS}_3} + \mathrm{d}s^2_{\mathbb{S}^3}\right) , \tag{4.1a}$$

$$\mathrm{d}s^2_{\mathrm{AdS}_3} = \frac{\mathrm{d}r^2}{r^2+a^2} - \frac{r^2+a^2}{Q_1Q_5}\,\mathrm{d}t^2 + \frac{r^2}{Q_1Q_5}\,\mathrm{d}y^2 , \tag{4.1b}$$

$$\mathrm{d}s^2_{\mathbb{S}^3} = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\hat\phi^2 + \cos^2\theta\,\mathrm{d}\hat\psi^2 , \tag{4.1c}$$

$$F_3 = 2Q_5\left(-\mathrm{vol}_{\mathrm{AdS}_3} + \mathrm{vol}_{\mathbb{S}^3}\right) , \quad e^{2\Phi} = \frac{Q_1}{Q_5} , \tag{4.1d}$$

$$\mathrm{vol}_{\mathrm{AdS}_3} = \frac{r}{Q_1Q_5}\,\mathrm{d}r\wedge\mathrm{d}t\wedge\mathrm{d}y , \quad \mathrm{vol}_{\mathbb{S}^3} = \sin\theta\cos\theta\,\mathrm{d}\theta\wedge\mathrm{d}\hat\phi\wedge\mathrm{d}\hat\psi , \tag{4.1e}$$

where $\mathrm{d}s_6^2$ is the Einstein metric in six dimensions, F is the RR 3-form field strength and Φ the dilaton. Q_1 and Q_5 are the supergravity D1 and D5 charges

$$Q_1 = \frac{(2\pi)^4 n_1 g_s (\alpha')^3}{V_4} , \quad Q_5 = g_s n_5 \alpha' , \tag{4.2}$$

where g_s is the string coupling and V_4 is the volume of the $\mathbb{T}^4$. The parameter a is linked to the D-brane charges and the $\mathbb{S}^1$ radius R by

$$a = \frac{\sqrt{Q_1Q_5}}{R} . \tag{4.3}$$

In the following we will slightly simplify our equations by taking

$$Q_1 = Q_5 = Q . \tag{4.4}$$

Spectral flow acts geometrically on the gravity side via the change of coordinates

$$\hat\phi = \phi - \frac{t}{R} , \quad \hat\psi = \psi - \frac{y}{R} . \tag{4.5}$$

Note that this is a diffeomorphism that acts non trivially at the AdS_3 boundary, and hence it changes the state. Thus the geometry dual to the state $(|{+}{+}\rangle_1)^N$ is (4.1) with the above coordinate redefinition (4.5).

As we has intensively shown in the previous chapters, one can construct more generic states by adding strands with different winding numbers and/or different $R-$charges; for example, one can consider the state $|00\rangle_k$, with winding k and $\jmath = \bar\jmath = 0$. The RR ground state $(|{+}{+}\rangle_1)^{N_1}(|00\rangle_k)^{N_2}$ with $N_1 + kN_2 = N$[2] sources a non-trivial geometry when both N_1 and N_2 are of order N. The full geometry is given for example in eq. (3.11) of [87]. The limit of interest here is when the strands of type $|00\rangle_k$ are much fewer then the ones of type $|{+}{+}\rangle_1$, i.e. $N_2 \ll N_1$. In this regime the appropriate gravitational description of the state is as a perturbation around the $\mathrm{AdS}_3 \times \mathbb{S}^3$ background (4.1), which solves the supergravity equations at linear order. After flowing to the NS sector this perturbation is an (anti-)chiral primary of dimension and charge $h = \bar h = -\jmath = -\bar\jmath = k/2$ [49]. The linearised perturbation is thus controlled by a scalar w, which is identified with the RR 0-form, and by the 2-form B-field B_2, and is given by

$$w = \mathcal{B}\,Y , \quad B_2 = \frac{Q}{k}\left(Y *_{\mathrm{AdS}_3} \mathrm{d}\mathcal{B} - \mathcal{B} *_{\mathbb{S}^3} \mathrm{d}Y\right) , \tag{4.6a}$$

$$\mathcal{B} = \frac{b\,R}{Q}\left(\frac{a}{\sqrt{r^2+a^2}}\right)^k e^{-ik\frac{t}{R}} , \quad Y = Y^{\ell,\ell}_{-\ell,-\ell} = \sin^k\theta\, e^{-ik\hat\phi} , \quad k = 2\ell , \tag{4.6b}$$

2 Or, more precisely, a coherent superposition [107] of states of this form.

where $*_{\mathrm{AdS}_3}$ and $*_{\mathbb{S}^3}$ are the Hodge duals with respect to the AdS_3 and $\mathbb{S}^3$ metrics defined in (4.1b) and (4.1c). $Y^{\ell,\ell}_{\jmath,\bar\jmath}$ denotes the $\mathbb{S}^3$ spherical harmonics of order $k = 2\ell$:

$$\Box_{\mathbb{S}^3} Y^{\ell,\ell}_{\jmath,\bar\jmath} = -k(k+2) Y^{\ell,\ell}_{\jmath,\bar\jmath}\,; \tag{4.7}$$

analogously, $\mathcal{B}$ is an eigenfunction of the AdS_3 Laplacian:

$$\Box_{\mathrm{AdS}_3} \mathcal{B} = k(k-2)\mathcal{B}\,. \tag{4.8}$$

The parameter b controls the number of strands of type $|00\rangle_k$ ($b^2 \sim N_2/N$) and the above perturbation solves the supergravity equations at first order in b.

While the holographic description of RR ground states is well-understood [48, 108, 110], as described in chap. 2, the analysis of excited states, and in particular of non-supersymmetric states which carry excitations on both the left- and right-sectors of the CFT, is widely incomplete. An effective way to approach the problem is to use the CFT algebra. As we have widely discussed, simple class of excited states, which are guaranteed to exist at any point of the CFT moduli space, is formed by descendants obtained by acting on RR ground states with an arbitrary string of generators of the superalgebra. In this chapter we focus on the R-charge currents and consider the states

$$[|{+}{+}\rangle_1]^{N_1} [(J^+_{-1})^m (\tilde J^+_{-1})^{\bar m}\, |00\rangle_k]^{N_2}\,. \tag{4.9}$$

When both m and $\bar m$ are non-vanishing, these states are non-supersymmetric. As explained above, in the limit $N_2 \ll N_1$ the states are described by a linearised perturbation around $\mathrm{AdS}_3 \times \mathbb{S}^3$, which is easily built starting from (4.6). Indeed, when one flows from the R sector to the NS sector, J^+_{-1}, $\tilde J^+_{-1}$ $\mapsto J^+_0$, $\tilde J^+_0$, which rotate the perturbation (4.6) while leaving the $\mathrm{AdS}_3 \times \mathbb{S}^3$ background invariant. Thus the perturbation dual to the state (4.9) for $N_2 \ll N_1$ is of the form of eq. (4.6), with the same $\mathcal{B}$ but a rotated spherical harmonic Y, i.e. we have

$$Y = Y^{\ell,\ell}_{-\ell+m,-\ell+\bar m}\,. \tag{4.10}$$

This construction provides a systematic way to generate the geometries dual to descendant states in the decoupling limit, valid when the $\mathbb{S}^1$ radius is large w.r.t. the charges, i.e. $R \gg \sqrt{Q}$, and in the inner region of the spacetime, in which $r \ll \sqrt{Q}$. In this region the geometries have $\mathrm{AdS}_3 \times \mathbb{S}^3$ asymptotics. If the states represent microstates of asymptotically flat black holes, they should admit a description outside this inner region, in which they have to smoothly join to the $\mathbb{R}^{4,1} \times \mathbb{S}^1$ flat spacetime at large distances. The construction of this asymptotically flat extension for non-supersymmetric states will be the focus of the remainder of this chapter. We will focus on two subclasses of states: *non-extremal states* with $m = \bar m = 1$ and *near-extremal states* with $m = k \gg 1$, $\bar m = 1$.

2 ASYMPTOTICALLY FLAT ANSATZ

The non-supersymmetric solutions we are looking to solve are the equations of motion of type IIB supergravity of sec. 2.2.1, linearised around a supersymmetric background, which is the asymptotically flat extension of the $\mathrm{AdS}_3 \times \mathbb{S}^3$ solution (4.1). The fields that make up the background are the metric, the RR 3-form field strength $F_3 = \mathrm{d}C_2$, the dilaton Φ and the volume of $\mathbb{T}^4$; these two scalars trivialise if one takes $Q_1 = Q_5$. The background, which represents the first example [126, 127] of an asymptotically flat solution dual to a D1D5 state, can be conveniently written as

$$\mathrm{d}s_6^2 = -\frac{2}{Z}(\mathrm{d}u+\omega)(\mathrm{d}v+\beta) + Z\,\mathrm{d}s_4^2\,, \tag{4.11a}$$

$$C_2 = -\frac{1}{Z}(\mathrm{d}u+\omega)\wedge(\mathrm{d}v+\beta) + \gamma\,, \tag{4.11b}$$

where

$$\mathrm{d}s_4^2 = \Sigma\left(\frac{\mathrm{d}r^2}{r^2+a^2} + \mathrm{d}\theta^2\right) + (r^2+a^2)\sin^2\theta\,\mathrm{d}\phi^2 + r^2\cos^2\theta\,\mathrm{d}\psi^2\,, \tag{4.12a}$$

$$\beta = \frac{R\,a^2}{\sqrt{2}\,\Sigma}(\sin^2\theta\,\mathrm{d}\phi - \cos^2\theta\,\mathrm{d}\psi)\,,\quad \omega = \frac{R\,a^2}{\sqrt{2}\,\Sigma}(\sin^2\theta\,\mathrm{d}\phi + \cos^2\theta\,\mathrm{d}\psi)\,, \tag{4.12b}$$

$$Z = 1 + \frac{Q}{\Sigma}\,,\quad \gamma = -Q\,\frac{r^2+a^2}{\Sigma}\,\cos^2\theta\,\mathrm{d}\phi\wedge\mathrm{d}\psi\,,\quad \Sigma \equiv r^2 + a^2\cos^2\theta\,, \tag{4.12c}$$

and where, as usual, the light-cone coordinates u and v are related with time t and the $\mathbb{S}^1$ coordinate y by

$$u = \frac{t-y}{\sqrt{2}}\,,\quad v = \frac{t+y}{\sqrt{2}}\,; \tag{4.13}$$

the Euclidean four-dimensional base-space metric ds_4^2 is just flat $\mathbb{R}^4$ written in a convenient system of coordinates, which are related to the usual cartesian coordinates x_i by

$$x_1 + ix_2 = \sqrt{r^2+a^2}\,\sin\theta\,e^{i\phi}\,,\quad x_3 + ix_4 = r\cos\theta\,e^{i\psi}\,. \tag{4.14}$$

Note that the following relations, which are ultimately a consequence of supersymmetry, are satisfied:

$$\mathrm{d}\beta = *_4\mathrm{d}\beta\,,\quad \mathrm{d}\omega = -*_4\,\mathrm{d}\omega\,,\quad *_4\mathrm{d}Z = \mathrm{d}\gamma\,, \tag{4.15}$$

where $*_4$ is the Hodge dual with respect to $\mathrm{d}s_4^2$. The length scale a is defined in (4.3). It is easy to verify that the asymptotically flat geometry (4.11,4.12) reduces to the $\mathrm{AdS}_3\times\mathbb{S}^3$ solution (4.1) in the decoupling limit $r, a \ll \sqrt{Q}$, in which one can simply neglect the 1 in the function Z. Note also that for $Q_1 = Q_5$ the 3-form F_3 is anti-self-dual in the full asymptotically flat geometry, i.e.

$$F_3 + *_6F_3 = 0\,, \tag{4.16}$$

where $*_6$ denotes the Hodge dual with respect to the 6D Einstein metric ds_6^2.

The perturbation that add few strands of the type $(J_{-1}^+)^m(\tilde{J}_{-1}^+)^{\bar{m}}\,|00\rangle_k$ excites the B-field B_2, the RR 0-form χ_1 and the component of the RR 4-form along $\mathbb{T}^4$, χ_2[3]: again a slight simplification happens for $Q_1 = Q_5$, where $\chi_1 = \chi_2 \equiv w$. In the decoupling limit the form of the perturbation is given by (4.6) and (4.10).

We also find that the task of extending the perturbation to the asymptotically flat region is simplified by using as an ansatz

$$w = \frac{Z_4}{Z}\,,\; B = -\frac{Z_4}{Z^2}\,(\mathrm{d}u+\omega)\wedge(\mathrm{d}v+\beta) + a_4\wedge(\mathrm{d}v+\omega) + b_4\wedge(\mathrm{d}u+\beta) + \delta_2\,.$$

3 Here we use the notation of [83] for the type IIB supergravity reduced on $\mathbb{T}^4$.

(4.17)

Here Z, β and ω are the same 0- and 1-forms that appear in the background (4.11), while the 0-form Z_4, the 1-forms a_4, b_4 and the 2-form δ_2 are the unknowns that define the perturbation. All these forms have legs only along the 4D Euclidean base space $\mathrm{d}s_4^2$, but they might depend also on u and v. It was found in [102] that general supersymmetric solutions have the form (4.17) with $b_4 = 0$, if one specialises their results to $Q_1 = Q_5$. Moreover, supersymmetry implies that nothing can depend on u. It is clear that any 0-form w and any 2-form B_2 can be written as in (4.17) for some choice of Z_4, a_4, b_4, δ_2; having chosen the uv component of B_2 to be controlled by the same function Z_4 that appears in w has partially restricted the 2-form gauge invariance $B \to B + \mathrm{d}\lambda$. The remaining gauge freedom left is the one where λ is a u and v-dependent 1-form with only legs on $\mathbb{R}^4$ that acts on our unknowns as

$$a_4 \to a_4 - \partial_v \lambda\,, \quad b_4 \to b_4 - \partial_u \lambda\,, \quad \delta_2 \to \delta_2 + \mathscr{D}\lambda\,, \tag{4.18}$$

where we introduce the generalised non-supersymmetric covariant differential

$$\mathscr{D} \equiv \mathrm{d}_4 - \beta \wedge \partial_v - \omega \wedge \partial_u = \mathcal{D} - \omega \wedge \partial_u\,, \tag{4.19}$$

where d_4 is the exterior differential with respect to the $\mathbb{R}^4$ coordinates; thus, the combinations of a_4, b_4, δ_2 that are left invariant by this residual gauge freedom are

$$\begin{aligned}&\mathcal{A} \equiv \partial_u a_4 - \partial_v b_4\,, \quad \Theta_4 \equiv \mathscr{D} a_4 + \partial_v \delta_2\,, \quad \tilde{\Theta}_4 \equiv \mathscr{D} b_4 + \partial_u \delta_2\,,\\ &\Xi \equiv \mathscr{D}\delta_2 - a_4 \wedge \mathrm{d}\beta - b_4 \wedge \mathrm{d}\omega\,,\end{aligned} \tag{4.20}$$

and it will be convenient to express the equations of motion in terms of these gauge-invariant quantities, as customary for the supersymmetric case. These quantities satisfy the Bianchi identities

$$\partial_u \Theta_4 - \partial_v \tilde{\Theta}_4 = \mathscr{D}\mathcal{A}\,, \tag{4.21a}$$

$$\mathscr{D}\Theta_4 - \partial_v \Xi + \mathcal{A} \wedge \mathrm{d}\omega = 0\,, \quad \mathscr{D}\tilde{\Theta}_4 - \partial_u \Xi - \mathcal{A} \wedge \mathrm{d}\beta = 0\,, \tag{4.21b}$$

$$\mathscr{D}\,\Xi = -\Theta_4 \wedge \mathrm{d}\beta - \tilde{\Theta}_4 \wedge \mathrm{d}\omega\,. \tag{4.21c}$$

We remark here a key difference w.r.t. the supersymmetric case; in supersymmetric solutions, for which $b_4 = 0$ and ∂_u is an isometry, $\mathcal{A}$ and $\tilde{\Theta}_4$ are trivial. In that case the parametrisation (4.17) was particularly useful as it simplifies the problem of finding the asymptotically flat linearised solution given the one in the decoupling limit [102]. The crucial point in that discussion is that it turns out that when supersymmetry is preserved the supergravity equations for Z_4, a_4 and δ_2 do not depend on Z: then, all one has to do to construct the asymptotically flat solution is to keep the same Z_4, a_4 and δ_2 of the inner region solution and simply *add back the 1* in the function Z that appears in [102]. One may hope that a similar simplification also happens for non-extremal solutions: we will see that life is not quite as easy, since the equations of motion couple crucially $\mathcal{A}$ with Z and hence deforming Z, as it is required by asymptotic flatness, necessarily induces deformations of all the objects Z_4, a_4, b_4, δ_2 that control the (w, B_2) fields. Nevertheless we find that using the parametrisation (4.17) helps in simplifying the equations and ultimately reduces the whole problem to a single partial differential equation for a scalar function.

3 LINEARISED SUPERGRAVITY EQUATIONS

Our goal is to construct a solution for the linearised equations of motion around the background (4.11, 4.12); the solution contains the fields w and B_2, parametrised as in (4.17), and must reduce to the near-horizon solution described in sec. 3.2 in the inner region.

The non-trivial equations of motion for (w, B_2) at linear order are

$$\mathrm{d}(*_6 H_3 + 2wF_3) = 0\,, \tag{4.22a}$$
$$\mathrm{d} *_6 \mathrm{d}w + F_3 \wedge H_3 = 0\,, \tag{4.22b}$$

where $H_3 = \mathrm{d}B_2$ is the NSNS 3-form field strength and the Hodge dual $*_6$ and the 3-form $F_3 = \mathrm{d}C_2$ refer to the background (4.11,4.12). The first equation can be partially integrated to

$$*H_3 - H_3 + 2wF_3 = 0\,, \tag{4.23}$$

after taking into account the anti-self-duality of F_3 (4.16). With the ansatz (4.17), eq. (4.23) is indeed equivalent to

$$*_4 \mathscr{D} Z_4 = \ \Xi - Z^2 *_4 \mathcal{A}\,, \tag{4.24a}$$
$$\Theta_4 = *_4\Theta_4\,, \quad \tilde{\Theta}_4 = - *_4 \tilde{\Theta}_4\,. \tag{4.24b}$$

The scalar eq. (4.22b) adds one more differential constraint which, after using (4.24a), can be shown to reduce to

$$*_4 \mathscr{D} *_4 \mathcal{A} = 2\,\partial_u \partial_v Z_4\,. \tag{4.25}$$

One can check that the near-horizon solution (4.6,4.10), when rewritten in the form (4.17), indeed satisfies eqs. (4.24a), (4.24b) and (4.25). When one considers the same equations in the asymptotically-flat background, one has to send $Z \to Z + 1$: then the Z-dependent term in eq. (4.24a) is modified, and this induces a non-trivial change of all other fields. We have already underlined that this complication is a feature of the non-supersymmetric solutions, for which $\mathcal{A} \neq 0$.

Eqs. (4.24a), (4.24b), (4.25) seem to form a complicated set of coupled partial differential equations; one can however simplify the problem drastically by reducing this set to a single equation for the 1-form $\mathcal{A}$. This is done as follows: joining eqs. (4.21a) and (4.24b) gives

$$\partial_u \Theta_4 + \partial_v \tilde{\Theta}_4 = *_4 \mathscr{D} \mathcal{A}\,. \tag{4.26}$$

From (4.21b) and the identity above one derives

$$2\,\partial_u \partial_v \Xi = \mathscr{D} *_4 \mathscr{D} \mathcal{A} + \partial_u \mathcal{A} \wedge \mathrm{d}\omega - \partial_v \mathcal{A} \wedge \mathrm{d}\beta\,. \tag{4.27}$$

Applying $\mathscr{D}$ to (4.25) and using (4.24a), one obtains

$$\begin{aligned}\mathscr{D} *_4 \mathscr{D} *_4 \mathcal{A} &= - 2\,\partial_u \partial_v (*_4 \Xi + Z^2 \mathcal{A}) \\ &= - *_4 \mathscr{D} *_4 \mathscr{D} \mathcal{A} - *_4(\partial_u \mathcal{A} \wedge \mathrm{d}\omega - \partial_v \mathcal{A} \wedge \mathrm{d}\beta) - 2Z^2 \partial_u \partial_v \mathcal{A}\,,\end{aligned} \tag{4.28}$$

where in the last step we have used (4.27). If one defines the Laplacian associated with the covariant differential $\mathscr{D}$:

$$\nabla^2 \equiv -(\mathscr{D} *_4 \mathscr{D} *_4 + *_4 \mathscr{D} *_4 \mathscr{D}) + *_4(\partial_v \mathcal{A} \wedge \mathrm{d}\beta - \partial_u \mathcal{A} \wedge \mathrm{d}\omega)\,, \tag{4.29}$$

one can prove that ∇^2 has a simple action on forms

$$\nabla^2 = \mathscr{D}^i \mathscr{D}_i \,, \tag{4.30}$$

where indices are contracted using the flat metric ds_4^2. Then eq. (4.28) reduces to

$$\nabla^2 \mathcal{A} = 2Z^2 \partial_u \partial_v \mathcal{A} \,, \tag{4.31}$$

which is a set of decoupled PDEs for each component of the 1-form $\mathcal{A}$. These are the main dynamical equations one needs to solve to build the linearised solution. All the other gauge-invariant quantities Z_4, Θ_4, $\tilde{\Theta}_4$, Ξ can be reconstructed from the 1-form $\mathcal{A}$ thanks to eqs. (4.25, 4.21a, 4.26, 4.27). Note that in the examples we consider in this chapter the perturbation has a simple exponential dependence on u and v, hence inverting u and v derivatives is trivial.

In summary, we need to solve eq. (4.31) plus the constraint that $\mathcal{A}$ agrees with the decoupling limit result in the inner region and vanishes sufficiently fast at large distances.

4 A NON-EXTREMAL SOLUTION

Here we start solving the equations of motion for the state

$$[|{+}{+}\rangle_1]^{N_1} [J^+_{-1} \tilde{J}^+_{-1} \, |00\rangle_k]^{N_2}$$

in the $N_2 \ll N_1$ regime: this is a maximally non-extremal perturbation of the background (4.11, 4.12), where one adds energy, through the action of the currents J^+_{-1} and $\tilde{J}^+_{-1}$, without adding any net momentum along the $\mathbb{S}^1$.

In the inner region, the perturbation is given by (4.6) with

$$Y = Y^{\ell,\ell}_{-\ell+1,-\ell+1} = e^{-i(k-2)\phi} \, (k \, \cos^2\theta - 1) \sin^{k-2}\theta \,, \quad \ell = \frac{k}{2} \,. \tag{4.32}$$

The inner region solution can be rewritten in the form (4.17), and thus one can read off the near-horizon values of the gauge-invariant quantities Z_4, $\mathcal{A}$, Θ_4, $\tilde{\Theta}_4$, Ξ that parametrise the perturbation; in particular we find that

$$\mathcal{A}_{\text{n.h.}} = e^{-i \frac{\sqrt{2}\,(u+v)}{R} - i(k-1)\phi} \, f_{\text{n.h.}}(r,\theta) \, (\mathrm{d}x_1 + i \mathrm{d}x_2) \,, \tag{4.33}$$

with

$$f_{\text{n.h.}}(r,\theta) = \frac{2}{R} \, \frac{b \, a^k}{(r^2+a^2)^{\frac{k+1}{2}}} \, \sin^{k-1}\theta \,. \tag{4.34}$$

As explained, all other gauge-invariant quantities easily follow from $\mathcal{A}$; for example

$$Z_{4,\,\text{n.h.}} = R \, e^{-i \frac{\sqrt{2}\,(u+v)}{R} - i(k-2)\phi} \, \frac{b \, a^k}{(r^2+a^2)^{\frac{k}{2}}} \, \sin^{k-2}\theta \, \frac{k \, \cos^2\theta - 1}{r^2 + a^2\cos^2\theta} \,. \tag{4.35}$$

A natural ansatz for the asymptotically flat extension of $\mathcal{A}$ is

$$\mathcal{A} = e^{-i \frac{\sqrt{2}\,(u+v)}{R} - i(k-1)\phi} \, f_{\text{n.h.}}(r,\theta) \, f(r,\theta) \, (\mathrm{d}x_1 + i \mathrm{d}x_2) \,, \tag{4.36}$$

where $f(r,\theta)$ is an unknown function such that $f(r,\theta) \to 1$ for $r, a \ll \sqrt{Q}$ and $\frac{f}{r^{k+1}} \to 0$ for $r \to \infty$. Also, $f(r,\theta)$ is determined by a PDE which comes from (4.31)

$$\begin{aligned}(r^2+a^2)\partial_r^2 f + ((1-2k)r^2+a^2)\frac{\partial_r f}{r} + \partial_\theta^2 f - 2\,\frac{1-2k\cos^2\theta}{\sin 2\theta}\partial_\theta f \\ + \frac{4}{R^2}\left[(r^2+a^2\cos^2\theta)+2Q\right] f = 0\,.\end{aligned} \tag{4.37}$$

Note that the term in the second line is negligible for $r, a \ll \sqrt{Q}$ (which implies $Q \ll R^2$), and hence $f=1$ constitutes a solution in the inner region, as expected. Even if eq. (4.37) is separable, due to the nature of its fuchsian singularities, we were unable to find an exact analytic solution. To provide evidence for the existence of a solution with the appropriate boundary conditions, we resort to a matched asymptotic expansion, as was done in [49] and [128]. This expansion is applicable to the regime in which one has two widely separated scales a and $\sqrt{Q}$ such that $a \ll \sqrt{Q}$: one can then solve the equation separately in the inner region $r \ll \sqrt{Q}$ and in the outer region $r \gg a$, and then require that the two solutions match in the overlapping region $a \ll r \ll \sqrt{Q}$. We will perform the matching at leading order. We already know the inner region solution, that is $f=1$, so we just have to solve eq. (4.37) in the outer region.

4.1 *The solution in the outer region $r \gg a$*

When a is negligible w.r.t. r, the equation for F_3 simplifies to

$$r^2\partial_r^2 f + (1-2k)r\partial_r f + \frac{4}{R^2}(r^2+2Q)f + \partial_\theta^2 f - 2\,\frac{1-2k\cos^2\theta}{\sin 2\theta}\partial_\theta f = 0\,. \tag{4.38}$$

This equation is now separable, and we set $f(r,\theta) = f_1(r)f_2(\theta)$; moreover, since it has to match to a constant for small r, we need to have a constant $f_2(\theta)$. One can also check that a constant is the only solution of the angular equation that does not have unphysical singularities for some values of θ. The radial equation is thus a *Bessel equation*, whose general solution is

$$f(r,\theta) = r^k\left[c_1\, J_\alpha\left(\frac{2r}{R}\right) + c_2\, Y_\alpha\left(\frac{2r}{R}\right)\right] \quad \text{with} \quad \alpha = \sqrt{k^2 - \frac{8Q}{R^2}}\,, \tag{4.39}$$

where c_1 and c_2 are constants. Substituting this result in (4.36), we see that the asymptotic behaviour of the 1-form $\mathcal{A}$ for $r \gg R$ is

$$\mathcal{A} \sim \frac{1}{r^{3/2}} e^{-i\frac{\sqrt{2}\,(u+v)}{R} - i(k-1)\phi} \sin^{k-1}\theta \Big[\tilde{c}_1 \cos\left(\frac{2r}{R}\right) + \tilde{c}_2 \sin\left(\frac{2r}{R}\right)\Big](\mathrm{d}x_1 + i\mathrm{d}x_2)\,, \tag{4.40}$$

with

$$\begin{aligned}\tilde{c}_1 &= c_1 \cos\left(\frac{2\alpha+1}{4}\pi\right) - c_2 \sin\left(\frac{2\alpha+1}{4}\pi\right)\,,\\ \tilde{c}_2 &= c_2 \cos\left(\frac{2\alpha+1}{4}\pi\right) + c_1 \sin\left(\frac{2\alpha+1}{4}\pi\right)\,.\end{aligned} \tag{4.41}$$

We find the same fall-off for the scalar Z_4: $Z_4 \sim r^{-3/2}$. This is a slower fall-off than the one exhibited by the extremal solutions, and agrees with the one estimated in Section 3.3 of [49] for non-extremal perturbations[4]. Notice

4 The same decay was found for the time-dependent non-supersymmetric solutions of [117]. We thank D. Turton for pointing this out.

that this is a general and unavoidable feature of non-extremal perturbations, since all non-trivial solutions of eq. (4.31) have this fall-off; the only way to a obtain a faster asymptotic decay is to have $\mathcal{A} = 0$, which implies that the perturbation is u- and/or v-independent, i.e. it is extremal. We will show that, despite this slow fall-off, the global charges of the solution are not altered by the perturabation, and thus the solution can be consistently identified with a microstate of a D1D5P black hole.

4.2 *The matching region $a \ll r \ll \sqrt{Q}$*

Consistency with the near-horizon solution requires that in the limit $r \ll \sqrt{Q}$ (and $a \ll \sqrt{Q}$) the function (4.39) tends to 1, for some choice of the constants c_i. This is actually guaranteed a priori, since the asymptotic analysis has not imposed any constraint on the integration constants c_i, and since the equation for f has the solution $f = 1$ in the inner region. Indeed one finds in the small r limit

$$f(r,\theta) \simeq r^k \left[\frac{c_1}{k!}\left(\frac{r}{R}\right)^k - \frac{c_2\,(k-1)!}{\pi}\left(\frac{r}{R}\right)^{-k}\right], \tag{4.42}$$

where we have approximated $\alpha \simeq k$ since $Q \ll R^2$ for $a \ll \sqrt{Q}$. Thus the two solutions matches at leading order if $c_2 = -\pi\, R^{-k}/(k-1)!$.

4.3 *Asymptotic charges*

Up to now we have shown the existence of a solution which interpolates between the near-horizon and the asymptotic regions. We have found that the fields of the perturbation fall off at large distances like $r^{-3/2}$; this is a very slow decay: in five non-compact dimensions a field strength carrying a global charge vanishes like r^{-3}, and we expect our perturbation to decay faster, so as to leave the global charges of the solution invariant. We show here that this unusually slow decay is not a problem, since the non-trivial angular dependence of the perturbation guarantees that it does not contribute to the global charges.

Since the perturbation excites the NSNS B-field, it could carry a global F1 and NS5 charge, proportional to

$$Q_{\text{F1}} \propto \int_{S^3} *_6 H\,, \quad Q_{\text{NS5}} \propto \int_{S^3} H\,, \tag{4.43}$$

where the integral is over a 3-sphere with infinite radius in the four non-compact spatial directions. It follows from eq. (4.23) and from the fact that the wF_3 term is negligible at large r, that

$$\int_{S^3} H = \int_{S^3} *_6 H = \int_{S^3} \Xi\,. \tag{4.44}$$

The large r limit of the 3-form Ξ can be computed from the asymptotic expression for $\mathcal{A}$ in (4.40) via eq. (4.27), where one can discard the last two terms at large distances

$$2\,\partial_u\partial_v\,\Xi \approx \mathscr{D} *_4 \mathscr{D}\mathcal{A}\,. \tag{4.45}$$

One thus finds that

$$\int_{S^3} \Xi \sim \lim_{r\to\infty} r^{1/2}\left[\tilde{c}_1 \sin\left(\frac{2r}{R}\right) - \tilde{c}_2 \cos\left(\frac{2r}{R}\right)\right] \times$$
$$\times \int d\theta\, d\phi\, d\psi\, e^{-i\frac{\sqrt{2}\,(u+v)}{R} - i(k-2)\phi}\, \sin^{k-1}\theta \cos\theta\left[(k+2)\cos 2\theta + 3(k-2)\right].$$

(4.46)

Based only on the r-dependence of Ξ one would conclude that the charge carried by the perturbation is not only non-vanishing, but divergent. However the integral over the angular variables vanishes for any $k > 0$. In fact, when $k \neq 2$ the oscillating factor $e^{-i(k-2)\phi}$ kills the ϕ-integral, while when $k = 2$ it is the θ integral that vanishes, since $\int_0^{\pi/2} \mathrm{d}\theta \sin 4\theta = 0$. Note that the state with $k = 2$ is special because it does not depend on either ϕ or ψ: this is a consequence of the fact that the strand $J^+_{-1}\tilde{J}^+_{-1}\,|00\rangle_2$ carries the same angular momenta as $(|++\rangle_1)^2$.

5 A NEAR-EXTREMAL SOLUTION

Here we want to consider a non-supersymmetric state where the departure from extremality could be made arbitrarily small. In order to do so, we could start from the supersymmetric D1D5P state $[|++\rangle_1]^{N_1}[(J^+_{-1})^k|00\rangle_k]^{N_2}$. Note that, as explained in chap. 2, k is the maximum number of times the charge J^+_{-1} can act on the ground state $|00\rangle_k$, since $(J^+_{-1})^{k+1}|00\rangle_k = 0$. We can thus break supersymmetry by acting once with the right-moving current $\tilde{J}^+_{-1}$ to have

$$[|++\rangle_1]^{N_1}[(J^+_{-1})^k\tilde{J}^+_{-1}|00\rangle_k]^{N_2}. \tag{4.47}$$

In the limit of large k one would expect this to be a small perturbation of the supersymmetric state; we will thus work in a large k expansion and keep the first non-trivial order in $1/k$. As usual we will also assume $N_2 \ll N_1$, so we can linearise the supergravity equations around the background (4.11,4.12).

The solution in the inner region is given by (4.6) with

$$Y = Y^{\ell,\ell}_{\ell,-\ell+1} = e^{i(k-1)\psi+i\phi}\cos^{k-1}\theta\,\sin\theta \quad (\ell = k/2)\,, \tag{4.48}$$

where we neglect the spherical harmonic normalisation factor. We can extract from this solution the near-horizon values of the functions that appear in the ansatz (4.17). For example:

$$Z_{4,\text{n.h.}} = R\,b\,e^{-i\frac{\sqrt{2}}{R}(u+kv)+i(k-1)\psi+i\phi}\,\frac{\Delta_{k,k-1}}{\Sigma}\,, \tag{4.49a}$$

$$\mathcal{A}_{\text{n.h.}} = -2\frac{b}{R}\,e^{-i\frac{\sqrt{2}}{R}(u+kv)+i(k-1)\psi}\,\frac{\Delta_{k,k-1}}{\sqrt{r^2+a^2}\sin\theta}\,(\mathrm{d}x_1 + i\mathrm{d}x_2)\,, \tag{4.49b}$$

where we define

$$\Delta_{k,m} \equiv \left(\frac{a}{\sqrt{r^2+a^2}}\right)^k \sin^{k-m}\theta\cos^m\theta\,. \tag{4.50}$$

This is an exact solution of the equations of motion in the inner region for any k.

We expect that the problem of extending the solution outside of the inner region simplifies in the regime of large k, since the state becomes approximately extremal. We thus look for a solution of the equations of motion (4.24), (4.25) with an expansion in $1/k$, and only keep the first non-trivial order

$$Z_4 = Z_{4,0} + k^{-1}Z_{4,1} + \mathcal{O}(k^{-2})\,, \quad \mathcal{A} = \mathcal{A}_0 + k^{-1}\mathcal{A}_1 + \mathcal{O}(k^{-2})\,. \tag{4.51}$$

In defining the large k expansion, we keep the exact k-dependence of exponents; this means that we do not expand the oscillating factor $\exp[-i\frac{\sqrt{2}}{R}(u+$

$kv) + i(k-1)\psi + i\phi]$ nor $\Delta_{k,k-1}$, and only expand the k-dependent coefficients that multiply the various functions. According to this definition, $Z_4, \mathcal{A}, \tilde{\Theta}_4$ begins at order k^0, while the leading term of Θ_4 is of order k^1. Moreover, when v, r, θ- and ψ-derivatives act on our solution, they increase the k-order by one, as a consequence of the k-dependence of $\exp[-i\frac{\sqrt{2}}{R}(u+kv) + i(k-1)\psi + i\phi]$ and $\Delta_{k,k-1}$, while u- and ϕ-derivatives do not change the order in k: schematically $\mathscr{D}, \partial_v \sim k^1$, $\partial_u \sim k^0$.

One can now see how the equations of motion simplify at large k. As explained in sec. 4.3, it is convenient to derive $\mathcal{A}$ using eq. (4.31). The remaining gauge invariant quantities follow from $\mathcal{A}$ without the need to integrate any further differential equation. The leading contribution to the l.h.s. of (4.31) is of order k^2, as it is $\nabla^2 \sim k^2$, while the r.h.s. starts at order k. Thus at leading order one should require

$$\nabla^2 \mathcal{A}_0 = \mathcal{O}(k)\,. \tag{4.52}$$

Since Z has disappeared from the equation above, the solution for $\mathcal{A}$ at leading order in k coincides with the near-horizon solution, even outside the inner region

$$\mathcal{A}_0 = \mathcal{A}_{\text{n.h.}}\,. \tag{4.53}$$

At the next order in $1/k$, the l.h.s. of (4.31) has two contributions: the leading order contribution to $k^{-1}\nabla^2\mathcal{A}_1$, and the order k contribution to $\nabla^2\mathcal{A}_{\text{n.h.}}$, which is given by

$$\nabla^2 \mathcal{A}_{\text{n.h.}} = 2\,\frac{Q^2}{\Sigma^2}\,\partial_u\partial_v\mathcal{A}_{\text{n.h.}}\,, \tag{4.54}$$

as a consequence of the near-horizon equations of motion. On the r.h.s. one can approximate $\mathcal{A}$ with $\mathcal{A}_{\text{n.h.}}$, up to corrections of $\mathcal{O}(k^0)$. Hence the first non-extremal correction to our solution is determined by

$$k^{-1}\nabla^2\mathcal{A}_1 = 2\left(1 + 2\frac{Q}{\Sigma}\right)\partial_u\partial_v\mathcal{A}_{\text{n.h.}} + \mathcal{O}(k^0)\,. \tag{4.55}$$

Given the form of $\mathcal{A}_{\text{n.h.}}$ (4.49b), one can look for a solution $\mathcal{A}_1$ of the form

$$\mathcal{A}_1 = 2\frac{b}{R}\,e^{-i\frac{\sqrt{2}}{R}(u+kv)+i(k-1)\psi}\,G(r,\theta)\,(\mathrm{d}x_1 + i\mathrm{d}x_2)\,. \tag{4.56}$$

Then eq. (4.55) implies

$$\widehat{\mathcal{L}}^{(k,k)}G = \frac{4k^2}{R^2}\left(1 + 2\frac{Q}{\Sigma}\right)\frac{\Delta_{k,k-1}}{\sqrt{r^2+a^2}\sin\theta} + \mathcal{O}(k)\,, \tag{4.57}$$

where $\widehat{\mathcal{L}}^{(k,k)}$ is the covariant Laplacian that was defined in [76]

$$\begin{aligned}\widehat{\mathcal{L}}^{(k,k)}G \equiv{}& \frac{1}{r\Sigma}\partial_r\big(r(r^2+a^2)\partial_r G\big)\\ &+ \frac{1}{\Sigma\sin\theta\cos\theta}\partial_\theta(\sin\theta\cos\theta\,\partial_\theta G) - k^2\frac{r^2+a^2\sin^2\theta}{(r^2+a^2)\Sigma\cos^2\theta}G\\ \approx{}& \frac{r^2+a^2}{\Sigma}\partial_r^2 G + \frac{1}{\Sigma}\partial_\theta^2 G - k^2\frac{r^2+a^2\sin^2\theta}{(r^2+a^2)\Sigma\cos^2\theta}G\,,\end{aligned} \tag{4.58}$$

where in the second line we have kept only the terms of order k^2, according to our working assumption that r and θ derivatives of G give terms of order k.

We have thus reduced our problem to the solution of a Poisson equation for the deformed Laplacian $\widehat{\mathcal{L}}^{(k,k)}G$. Equations of this type usually appears in the construction of extremal superstrata [76], but the source term in (4.57) is different from the one that appears in [76]. Though we do not exclude that a variation of the techniques of [76] could be useful to find an exact solution of (4.57), we have not been able to find one. Thus we resort to a matching technique to show that (4.57) admits a solution that is well behaved at large distances and is negligible with respect to $\mathcal{A}_{\text{n.h.}}$ in the inner region.

5.1 *A matching near-extremal solution*

We now assume as usual that $a \ll \sqrt{Q}$ and look for a solution in the outer region $r \gg a$, where the l.h.s. of (4.57) approximates to

$$\begin{aligned}\widehat{\mathcal{L}}^{(k,k)}G \approx \partial_r^2 G + \frac{1}{r^2}\partial_\theta^2 G - \frac{k^2}{r^2\cos^2\theta}G \\ + \frac{a^2}{r^2}\left[\sin^2\theta\,\partial_r^2 G - \frac{\cos^2\theta}{r^2}\,\partial_\theta^2 G + \frac{2k^2}{r^2}G\right],\end{aligned} \tag{4.59}$$

and where the r.h.s. approximates to

$$\text{r.h.s.} \approx \frac{4k^2}{R^2}\left(1+\frac{2Q}{r^2}\right)\frac{a^k}{r^{k+1}}\cos^{k-1}\theta\,. \tag{4.60}$$

Now we look for a factorised solution of the form

$$G(r,\theta) \approx g(\theta)\left(1+\frac{2Q}{r^2}\right)\frac{a^k}{r^{k+n}}\cos^{k-1}\theta\,, \tag{4.61}$$

where n is a number that we assume to be much smaller than k, i.e. $n \ll k$, and that will be determined shortly. At leading order in $1/k$, one can approximates $\partial_r^2 G \approx k^2/r^2\,G$ and $\partial_\theta^2 G \sim k^2\tan^2\theta\,G$, so that, when one substitutes (4.61) into (4.59), one immediately sees that the leading term in a/r vanishes for any choice of n up to terms of $\mathcal{O}(k)$

$$\partial_r^2 G + \frac{1}{r^2}\partial_\theta^2 G - \frac{k^2}{r^2\cos^2\theta}G = \frac{k^2}{r^2}\left[1+\tan^2\theta - \frac{1}{\cos^2\theta}\right]G + \mathcal{O}(k) = \mathcal{O}(k)\,. \tag{4.62}$$

Thus only the term proportional to a^2/r^2 survives in (4.59), and, in order to match the source (4.60), one needs $n = -3$. The equation for G then becomes

$$\frac{k^2\,a^2}{r^4}[\sin^2\theta - \sin^2\theta + 2]G = \frac{4k^2}{R^2}\left(1+\frac{2Q}{r^2}\right)\frac{a^k}{r^{k+1}}\cos^{k-1}\theta + \mathcal{O}(k)\,, \tag{4.63}$$

which is satisfied by taking

$$g(\theta) = \frac{2}{Q^2}\,. \tag{4.64}$$

Then the solution for $\mathcal{A}_1$ in the outer region is

$$\mathcal{A}_1 \approx 4\frac{b}{Q^2 R}\,e^{-i\frac{\sqrt{2}}{R}(u+kv)+i(k-1)\psi}\left(1+\frac{2Q}{r^2}\right)\frac{a^k}{r^{k-3}}\cos^{k-1}\theta\,(\mathrm{d}x_1 + i\mathrm{d}x_2)\,. \tag{4.65}$$

Consistency requires that in the matching region $a \ll r \ll \sqrt{Q}$ we need $\mathcal{A}_1$ to be suppressed w.r.t. $\mathcal{A}_{\text{n.h.}}$; this is evidently so, since

$$\frac{|\mathcal{A}_1|}{|\mathcal{A}_{\text{n.h.}}|} \sim \frac{r^4}{Q^2} \ll 1 \quad \text{for} \quad a \ll r \ll \sqrt{Q}\,. \tag{4.66}$$

The remaining fields in the outer region can be reconstructed from $\mathcal{A}_1$, e.g.

$$Z_{4,1} \approx -2\frac{R\,b}{Q^2}\, e^{-i\frac{\sqrt{2}}{R}(u+kv)+i(k-1)\psi+i\phi}\left(1+\frac{2Q}{r^2}\right)\frac{a^k}{r^{k-2}}\cos^{k-1}\theta\sin\theta\,. \tag{4.67}$$

One can also see that in the near-extremal regime $k \gg 1$ the fields of the perturbation fall-off very fast at large distances, e.g. $Z_4 \sim 1/r^{k-2}$. This is to be contrasted with the much slower $r^{-3/2}$ fall-off seen for the non-extremal solution. This indicates that the $k \to \infty$ and $r \to \infty$ limits *do not commute*: our near-extremal expansion is valid up to a distance r that grows with k, while for larger distances one should recover the large r behaviour of non-extremal solutions.

6 SUMMARY AND OUTLOOK

In this chapter, we have constructed linearised solutions of the type IIB equations of motion that are dual to non-extremal states of the D1D5P system that are obtained by acting with left- and right-moving algebra generators on a $\frac{1}{4}$−BPS state. We have employed an ansatz inspired by the supersymmetric geometries. However, we have shown that already at the linearised level the solution of the equations is significantly more involved for non-extremal configurations. The main complication arises from the fact that the warp factor Z does not decouple from the equations for the perturbation: thus the problem of extending the solution outside of the inner region requires to solve a non-trivial differential problem. The problem is sourced by the 1-form $\mathcal{A}$, which couples to Z through the last term of eq. (4.24a). It is evident from eq. (4.25) that $\mathcal{A}$ does not vanish exactly when the perturbation is non-extremal and depends on both the light-cone coordinates u and v.

From the technical point of view, the main result of this chapter is the reduction of the differential equations to a single equation (4.31) of the Poisson type for $\mathcal{A}$. The full perturbation can be reconstructed from $\mathcal{A}$ without the need to solve any further differential equation. Even if we have not been able to fully solve the $\mathcal{A}$ equation exactly, we have shown that it admits a matching solution that interpolates between the inner region result and an asymptotically decaying solution. Despite the unusually slow fall-off of the solution at large distances, the global charges of the solution are the ones of the D1D5P black hole, supporting the identification of our solutions with black hole microstates. We have also developed an approximation scheme that allows to expand near-extremal solutions around a supersymmetric background.

Part III

TESTING THE PARADOX

5

NO INFORMATION LOSS ON 2-CHARGE MICROSTATE GEOMETRIES

In this chapter we use the AdS/CFT duality as a tool to study a relatively simple set of heavy operators O_H in D1D5 CFT which are the Ramond-Ramond (RR) ground states. As explained in chap. 2, this ensemble is not dual to a macroscopic black hole at the level of two derivative gravity[1], but it provides a good testing ground as we know the details of the gravitational solutions dual to these states [48, 103, 110]. As explained in sec. 2, it is possible to test the dictionary between the RR ground states on the CFT side and the corresponding bulk description in terms of smooth horizonless solutions of type IIB supergravity [75, 107, 108, 110, 130]: the basic idea is to exploit the AdS/CFT map between protected CFT operators O_L, introduced in sec. 2.1, and the supergravity modes in the bulk, described in sec. 2.2.1, and then compare the 3-point CFT correlators $\langle O_H O_H O_L\rangle$ with the holographic results obtained from the dual microstate geometries. Here the supergravity operators are indicated with a subscript L because they are *light*, meaning that their conformal dimension is fixed in the large central charge limit $c = 6N \to \infty$. This class of 3-point correlators is protected [131] and so it is possible to match the results obtained in the weakly curved gravitational regime and those derived at a different point in the D1D5 SCFT moduli space, where the boundary theory can be described as non-linear sigma model whose target space is the orbifold.

While focusing on non-renormalised quantities is usually useful to established a dictionary between BPS states in different descriptions, this kind of observables is not well suited to study interesting gravitational features of the black hole microstates. Then it is important to extend the analysis to non-protected quantities involving heavy operators. Usually two dynamical quantities of this type have been under detailed scrutiny: the entanglement entropy of a region in a non-trivial state [132–134] and the HHLL 4-point function with two heavy and two light operators

$$\langle O_H(z_1, \bar{z}_1)\bar{O}_H(z_2, \bar{z}_2)O_L(z_3, \bar{z}_3)\bar{O}_L(z_4, \bar{z}_4)\rangle \ . \tag{5.1}$$

In this chapter we study this second observable focusing on the large central charge limit $c \gg 1$. When the D1D5 superconformal field theory is at the free orbifold point in its moduli space, it is possible to calculate the correlator (5.1) exactly by using standard techniques and to study the statistical properties of the result when the heavy operator is chosen from an ensemble of RR ground states [72, 135], as we have already shown in sec. 2.1.6. In order to extract some detailed information on the dual gravitational theory, it is of course important also to deform the CFT away from the free orbifold point, and a possible way for doing this is to insert perturbatively operators corresponding to the interesting superconformal deformations[2]. Here we focus on

1 See [129] for a nice discussion of this system.
2 see [136] and reference therein for a recent discussion of this approach

the opposite limit and discuss how to calculate (5.1) holographically directly in the strongly interacting regime where the CFT is well approximated by type IIB supergravity.

Notice that it is not straightforward to use the standard Witten diagrams technology to compute the correlators above, since here the heavy states correspond to multi-particle operators with a large conformal dimension and are not dual to a single supergravity mode. We overcome this issue by exploiting the known smooth geometries dual to the heavy states; then we use the standard AdS/CFT dictionary to compute the HHLL correlators by studying the quadratic fluctuations of the supergravity field dual to the light operators in the asymptotycally AdS geometry dual to the the heavy operators. This technique was developed in [77, 83] in several concrete examples in the AdS_3/CFT_2 context. In particular, in [83] the authors considered a complicated heavy operator made out of two types of supergravity modes, while they considered (2.68) as a light operator. This case provides the first explicit example of a dynamical HHLL correlator, where the result in the CFT strong coupling region is different from the one valid at the orbifold point. However, the quadratic equations around the asymptotically AdS geometry were explicitly solved resorting to a particular approximation where the two constituents forming the heavy multi-particle state were not on the same footing: the modes carrying a non-trivial R-charge are much more numerous than the modes with no R-charge. In such limit, the HHLL correlators could be recast in terms of the standard D-functions that appear also in the evaluation of the standard Witten diagrams.

In this chapter we generalise the above analysis of [83] in several directions. First, we consider the bosonic light operator (2.68) studied in [72, 135] which is a superdescendant of the chiral primary operator (2.68) mentioned above, as proved in sec. 2.1.6. This implies that the HHLL correlators derived in this chapter satisfy a Ward identity linking them to the correlators computed in [83]; as a consistency check, when we specify our new supergravity results to the heavy state considered in [83], we show that the Ward identity is satisfied. On the gravity side, the derivation of the HHLL correlators is drastically simplified with respect to [83] because the gravity perturbation dual to the light operator is described by the scalar Laplace equation in six dimension, as proven in app. A.5, while for the case of the CPO one had to deal with a coupled system of a scalar and a 3-form. This simplified setup allows to consider more general heavy operators that are formed by many different types of supergravity modes. In one approach we still maintain the approximation where the heavy state constituents include a large number $N_1^{(++)}$ of R-charge carrying modes, that we denote by $|++\rangle_1$, and much smaller numbers $N_k^{(0)}$ of different modes with no R-charge, denoted by $|00\rangle_k$, with k any positive integer. These states form an ensemble, whose generic elements we represent schematically as

$$(|{+}{+}\rangle_1)^{N_1^{(++)}} \prod_k (|00\rangle_k)^{N_k^{(0)}}\,, \quad N_1^{(++)} + \sum_k k N_k^{(0)} = \frac{c}{6}\,. \tag{5.2}$$

Of course these states have a large R-charge $J \sim N_1^{(++)}$, but their ensemble has interesting statistical properties [72, 137] and an entropy that scales like $\sqrt{c/6 - J}$. One of the results of this chapter is an explicit expression for the correlator (5.1) with these heavy states, at the supergravity point of the CFT moduli space. In an alternative approach we focus on a RR ground state that was considered also in [83] and is made out of only the $|{+}{+}\rangle_1$ and $|00\rangle_1$ modes. However we keep the ratio $N_k^{(0)}/N_1^{(++)}$ of the two constituents arbitrary and derive an expression for the HHLL in terms of a Fourier series. While we do not perform the transformation to configuration space in general, we show explicitly that, when it is possible to compare them, the results obtained in the two approaches agree.

In summary the main results of this chapter are:

1. the holographic computation of the correlator of the two bosonic operators in (2.70) in a generic state of the ensemble (5.2) in the limit $N_k^{(0)} \ll N_1^{(++)}$;
2. the check that the bosonic correlator computed here is related via a supersymmetric Ward identity to the fermionic correlator of [83];
3. the holographic computation of the same correlator in a state with $N_k^{(0)} = 0$ for $k \geq 2$, exactly in the ratio $N_1^{(0)}/N_1^{(++)}$.

One of our main motivations for doing these computations is to confront the correlators computed in pure states with those computed in a black hole background. As we mentioned above, the ensemble of BPS two-charge states is not described by a regular black hole in classical supergravity, but by the singular geometry obtained by taking massless limit of the BTZ black hole, as in sec. 1.2.2. This geometry shares some properties with black holes: in particular, as we have shown in sec. 1.2.2, correlators computed in this background vanish at large Lorentzian time, albeit only polynomially. As reviewed in sec. 1.2 the late-time decay of correlators is one of the manifestations of the information loss problem. By contrast correlators in pure states should not decay. It is quite easy to see that this is the case for correlators computed at the orbifold point in a generic D1D5 state [72, 135]. The orbifold point CFT, however, has some special features that distinguish it from the point where a weakly coupled gravitational description is applicable: in particular, at free-orbifold point there exists an infinite series of conserved (bosonic) currents, of which only the Virasoro and the R-currents survive at a generic point. The presence of these currents certainly changes qualitatively the late-time behaviour of the correlators.

In some cases, like the ones considered in [77], even just the R-current is sufficient to completely constrain the form of the correlator, and prevent the vanishing at large Lorentzian times. A mechanism based on the R-current, even if it applies uniformly on the moduli space, can reasonably be argued to be non-generic [138]. The correlator we consider in this chapter, where the light operators are the non-chiral primaries in (2.70), is not constrained by the R-symmetry. This is confirmed by the fact that only the conformal block of the identity[3] contributes to the correlator in the lightcone OPE limit. We can thus use the exact strong coupling result obtained in sec. 5.1.3 to analyse the late-time behaviour of this correlator, and even in this more generic case we find that *it does not decay*. Note that this conclusion applies to a correlator computed in supergravity, and hence at leading order in the $1/N$ expansion. Since all large N Virasoro blocks[4] vanish at late times [140], the only mechanism by which we can explain our findings is that even our non-protected correlator receives contributions from an infinite series of Virasoro primaries[5].

3 As explained in sec. 5.2, it is convenient to use the Virasoro blocks defined with respect to the “reduced" Virasoro generators, given by the full Virasoro blocks minus their R-current Sugawara contribution.

4 For a derivation of Virasoro blocks in the limit of large central charge from AdS_3 gravity see [5, 139].

5 The contribution of these primaries should be relevant also at finite values of the central charge, as each individual Virasoro block is still expected to decay at late times [141].

These primaries cannot be single-particle operators: in fact, such operators are either dual to protected supergravity modes, in which case their contribution appears already in the orbifold-point result, or to string modes, which acquire large anomalous dimensions and decouple when one moves towards the supergravity regime. So the Virasoro primaries that contribute to our correlator at strong coupling must be multi-particle operators.

1 BOSONIC CORRELATORS AT STRONG COUPLING

The aim of this section is to study the HHLL correlators on the bulk side by using the supergravity approximation of type IIB string theory on $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$ reviewed in chap. 2. The case where the light operators are the chiral primaries (2.68) was discussed in [83], so here we consider the correlators with the bosonic light operators of dimension two (2.70). While in the orbifold CFT description of sec. 2.1.6 it was easy to keep the RR ground states completely generic, in the bulk analysis we will find it convenient to focus on a subsector of these heavy states. First we focus on the states that are invariant under the $SU(2)$ acting on the coordinates of $\mathbb{T}^4$, which ensures that the dual solutions are invariant under rotations of the four stringy-sized compact directions. Then, as explained above, we focus on the case where the RR ground states are made of a large number $N_1^{(++)}$ of strands of the type $|{++}\rangle_1$, while the remaining strands have arbitrary winding $k \geq 1$ but are in the unique RR state that is a scalar of $SU(2)_L \times SU(2)_R$; we denote strands of this type as $|00\rangle_k$ and their numbers as $N_k^{(0)}$. These states form the ensemble that was introduced in (5.2). On the bulk side the restriction to this subset of states simplifies the six-dimensional metric (5.3). At some point of our analysis we will also assume that the numbers of $|00\rangle_k$ strands are smaller than the number of $|{++}\rangle_1$ strands ($N_k^{(0)} \ll N_1^{(++)}$): this will allow the perturbative approach in b_k discussed in sec. 5.1.2.

The heavy operators O_H are described in the gravity regime by six dimensional geometries that asymptotically are $\mathrm{AdS}_3 \times \mathbb{S}^3$ and are everywhere regular and horizonless. Operators that are Ramond ground states both in the left and in the right sector are dual to D1D5 geometries with no momentum charge. The six-dimensional Einstein metric dual to RR ground states that are invariant under rotations in the four compact dimensions is (see chap. 2)

$$\mathrm{d}s_6^2 = -\frac{2}{\sqrt{\mathcal{P}}}(\mathrm{d}v+\beta)(\mathrm{d}u+\omega) + \sqrt{\mathcal{P}}\,\mathrm{d}s_4^2\,, \tag{5.3}$$

where

$$\mathcal{P} \equiv Z_1 Z_2 - Z_4^2\,. \tag{5.4}$$

We use lightcone coordinates

$$u \equiv \frac{t-y}{\sqrt{2}}\,, \quad v \equiv \frac{t+y}{\sqrt{2}}\,, \tag{5.5}$$

with t time and y the coordinate along $\mathbb{S}^1$, and denote by ds_4^2 the flat metric on $\mathbb{R}^4$. Z_1, Z_2, Z_4 are harmonic scalar functions on $\mathbb{R}^4$ and β, ω are 1-forms with self-dual and anti-self-dual 2-form field strengths. Apart from the metric, all other fields of type IIB supergravity are non-trivial in the solution, but their expressions will not be relevant for the correlator we compute here.

The form of the supergravity data Z_1, Z_2, Z_4, β and ω depends on the RR ground state and is usually rather complicated. As mentioned above, we focus on the family of D1D5 states described in (5.2). Their dual gravity solutions depend on some continuous parameters: a, whose square is proportional to $N_1^{(++)}$, and b_k, whose square is proportional to $kN_k^{(0)}$ [75]:

$$N_1^{(++)} = N\,\frac{a^2}{a_0^2}\,, \quad kN_k^{(0)} = N\,\frac{b_k^2}{2a_0^2} \quad \text{with} \quad a_0^2 \equiv \frac{Q_1 Q_5}{R^2}\,. \tag{5.6}$$

Here R is the radius of the $\mathbb{S}^1$ and Q_1, Q_5 are the supergravity D1 and D5 charges. The condition that the total number of strands be N implies the constraint

$$a^2 + \sum_k \frac{b_k^2}{2} = a_0^2\,, \tag{5.7}$$

which turns out to be also the regularity condition for the metric. The metrics are more easily written in spheroidal coordinates in which the flat $\mathbb{R}^4$ metric is

$$\mathrm{d}s_4^2 = \Sigma\left(\frac{\mathrm{d}r^2}{r^2+a^2} + \mathrm{d}\theta^2\right) + (r^2+a^2)\sin^2\theta\,\mathrm{d}\phi^2 + r^2\cos^2\theta\,\mathrm{d}\psi^2\,. \tag{5.8}$$

The remaining data encoding the metric are the one reported in eq. (2.147) of sec. 2.2.4.1.

1.1 *The perturbation*

To compute the HHLL correlator one should consider the wave equation for a perturbation in the background (5.3). The bosonic light operator $O_L = O_{\mathrm{Bos}}$ is described by a minimally coupled scalar in the six-dimensional Einstein metric ds_6^2. We show in Appendix A.5 that such scalars arise by dimensional reduction from traceless perturbations of the metric on $\mathbb{T}^4$, and thus have the right quantum numbers to be dual to the CFT operator[6] $\partial X^{(i}\bar\partial X^{j)}$, with $i, j = 1, \ldots, 4$.

Following the line of [77, 83], the gravity computation of the correlator requires solving the wave equation

$$\Box_6 B = 0\,, \tag{5.9}$$

where $\Box_6$ is the scalar Laplace operator w.r.t. $\mathrm{d}s_6^2$, i.e.

$$\Box_6 \cdot \equiv \frac{1}{\sqrt{g_6}}\,\partial_M\left(\sqrt{g_6}\,g_6^{MN}\partial_N \cdot\right)\,, \tag{5.10}$$

dubbed with the boundary condition

$$B(t,y,r) \sim \delta(t,y) + \frac{b(t,y)}{r^2}\,, \tag{5.11}$$

for large r. Since the background metric is regular everywhere, one should also require that B have no singularities at any finite value of r. As the operator O_L is an R-charge singlet, only the projection of B on the trivial scalar spherical harmonic on $\mathbb{S}^3$ contributes to the correlator. The 4-point function computed on the Euclidean plane is thus encoded in the function $b(t,y)$ via

$$\langle O_H(0)\bar O_H(\infty)O_L(1,1)\bar O_L(z,\bar z)\rangle = \frac{1}{|1-z|^4}\,\mathcal{G}^{\mathrm{Bos}}(z,\bar z) = (z\bar z)^{-1}\,b(z,\bar z)\,,$$

6 Actually, the right dual supergravity field dual to operator (2.70) is the self-dual part of Kalb-Ramond field on the $\mathbb{T}^4$, b_{ij}. But, as explained in sec. 2.2.3.1, it is dual to the torus metric perturbation.

$$(5.12)$$

where

$$z = e^{i\,\frac{t+y}{R}} = e^{\frac{t_e+iy}{R}}\,, \quad \bar{z} = e^{i\,\frac{t-y}{R}} = e^{\frac{t_e-iy}{R}}\,, \tag{5.13}$$

with $t_e \equiv it$ the Euclidean time. The factor $(z\bar{z})^{-1}$ on the right hand side of (5.12) comes from the transformation of the primary field

$$\bar{O}_L(z,\bar{z}) = (z\bar{z})^{-1}\,\bar{O}_L(t,y)\,, \tag{5.14}$$

from the cylinder to the plane coordinates.

The Laplacian operator in (5.10) is easily derived if one writes the six-dimensional metric as [75, 133, 142]:

$$ds_6^2 = V^{-2} g_{\mu\nu} dx^\mu dx^\nu + G_{\alpha\beta}(d\theta^\alpha + A^\alpha_\mu dx^\mu)(d\theta^\beta + A^\beta_\nu dx^\nu)\,, \tag{5.15}$$

where

$$V^2 \equiv \frac{\det G}{(Q_1 Q_5)^{3/2}\sin^2\theta\cos^2\theta}\,. \tag{5.16}$$

We have thus split the six-dimensional coordinates in the AdS_3 coordinates $x^\mu, x^\nu, \ldots \equiv (r,t,y)$ and the $\mathbb{S}^3$ coordinates $\theta^\alpha, \theta^\beta, \ldots \equiv (\theta,\phi,\psi)$. The definition of $g_{\mu\nu}$, $G_{\alpha\beta}$, A^α_μ depends of course on the choice of coordinates; the coordinates are fixed at the boundary by the requirement that the metric looks like $AdS_3 \times \mathbb{S}^3$ asymptotically, but one is free to redefine the coordinates in the space-time interior. We will stick to the coordinates defined in (5.8).

If one takes the solution in (2.147) and sets $b_k = 0$ for any k, one finds that $g_{\mu\nu}$ becomes the metric of global AdS_3

$$\begin{aligned} g_{\mu\nu} dx^\mu dx^\nu\Big|_{b_k=0} &= \sqrt{Q_1 Q_5}\left[\frac{dr^2}{r^2+a_0^2} - \frac{r^2+a_0^2}{Q_1Q_5}\,dt^2 + \frac{r^2}{Q_1Q_5}\,dy^2\right] \\ &\equiv \sqrt{Q_1Q_5}\, ds^2_{AdS_3}\,, \end{aligned} \tag{5.17}$$

and $G_{\alpha\beta}$ the metric of the round $\mathbb{S}^3$. When, like in this case, the metric $g_{\mu\nu}$ does not depend on the coordinates of $\mathbb{S}^3$, the six-dimensional Laplace equation (5.9) admits an $\mathbb{S}^3$-independent solution which satisfies the simpler equation

$$\Box_3 B = 0\,, \tag{5.18}$$

with $\Box_3$ the Laplacian of $g_{\mu\nu}$:

$$\Box_3 \cdot \equiv \frac{1}{\sqrt{g}}\,\partial_\mu\left(\sqrt{g}\,g^{\mu\nu}\partial_\nu \cdot\right)\,. \tag{5.19}$$

In general, the six-dimensional metric does not factorise and $g_{\mu\nu}$ and $G_{\alpha\beta}$ depend on both AdS_3 and $\mathbb{S}^3$ coordinates. In this situation solving the six-dimensional equation (5.9) exactly seems hard. When this happens we resort to an approximation scheme that was used already in [83]: we solve the wave equation perturbatively in b_k, keeping only the first non-trivial order $\mathcal{O}(b_k^2)$. In the following we will apply this perturbative method to compute the correlator for generic b_k. In the particular example in which b_1 is the only non-vanishing mode, we will be able to do perform the computation exactly in b_1 since in that case the metric factorise.

1.2 *Perturbative computation for generic b_k's*

As anticipated, we consider here a generic state in the ensemble (5.2) and compute the correlator in the limit $N_k^{(0)} \ll N_1^{(++)}$, keeping the first non-trivial term in an expansion in b_k/a_0. This contribution already depends on the CFT moduli and hence it contains non-trivial information on the dynamics. We perform the b_k-expansion keeping Q_1, Q_5 and R (and thus a_0) fixed: on the CFT side this means we are keeping the central charge fix and we are not changing the size of the circle on which the CFT is defined. At zeroth order in b_k the metric is $\mathrm{AdS}_3 \times \mathbb{S}^3$, and we will expand the terms of order b_k^2 in the basis of spherical harmonics of this unperturbed $\mathbb{S}^3$. We thus write the solution of (5.9) as

$$B = B_0 + B_1 + \mathcal{O}(b_k^4)\,, \tag{5.20}$$

where B_1 quadratic in b_k. The terms of order zero and two of the equation give

$$\Box_0 B_0 = 0\,, \quad \Box_0 B_1 = -\Box_1 B_0\,, \tag{5.21}$$

where $\Box_0$ is the Laplacian of global AdS_3

$$\Box_0 \cdot \equiv \frac{1}{r}\partial_r(r(r^2+a_0^2)\partial_r \cdot) - \frac{a_0^2\,R^2}{r^2+a_0^2}\,\partial_t^2\cdot + \frac{a_0^2\,R^2}{r^2}\,\partial_y^2\cdot\,, \tag{5.22}$$

and $\Box_1$ is the order b_k^2 contribution to the Laplacian $\Box_3$ defined in (5.18). The order zero equation in (5.21), together with the asymptotic boundary condition (5.11) and the regularity condition, implies that B_0 is the usual bulk-to-boundary propagator of dimension $\Delta = 2$ in global AdS_3:

$$\begin{aligned} B_0(r,t,y) &= K_2^{\mathrm{Glob}}(r,t,y|t'=0,y'=0) \\ &= \left[\frac{1}{2}\,\frac{a_0}{\sqrt{r^2+a_0^2}\,\cos(t/R) - r\,\cos(y/R)}\right]^2 . \end{aligned} \tag{5.23}$$

The second-order part in (5.21) gives an equation for B_1. If the metric $g_{\mu\nu}$ is a non-trivial function on $\mathbb{S}^3$, the B_1 that solves this equation has components along non-trivial $\mathbb{S}^3$ spherical harmonics, which we should project away for the purpose of extracting the correlator. In particular all terms in the solution (2.147) that are proportional to $b_k b_{k'}$ for $k \neq k'$ depend non-trivially on ϕ as $\cos((k-k')\phi)$ and source non-trivial spherical harmonics in B_1 and thus they do not contribute to the correlator at quadratic order in b_k. We can then simplify the computation by focusing on a single k-mode at a time. The metric $g_{\mu\nu}$ derived from the solution where a single b_k is non-vanishing is reported in (2.151) of sec. 2.2.4.1.

We see that, unless $k=1$, even for a single mode $g_{\mu\nu}$ depends non-trivially on the $\mathbb{S}^3$ coordinate θ and this is not separable. To compute B_1, one should expand the Laplacian of $g_{\mu\nu}^{(k)}$ up to order b_k^2

$$\Box^{(k)} = \Box_0 + b_k^2\,\Box_1^{(k)} + \mathcal{O}(b_k^4), \tag{5.24}$$

and project on the trivial spherical harmonic. One then finds

$$\begin{aligned} \langle J_k\rangle &\equiv -\langle \Box_1^{(k)} B_0\rangle \\ &= -\frac{r}{(r^2+a_0^2)}\partial_r B_0 + \frac{a_0^2\,R^2}{(r^2+a_0^2)^2}\,\partial_t^2 B_0 + \frac{R^2}{2a_0^2}\,S_k\,(\partial_t^2 B_0 - \partial_y^2 B_0)\,, \end{aligned} \tag{5.25}$$

where

$$S_k \equiv \sum_{p=2}^{k} \left(\frac{a_0^2}{r^2+a_0^2}\right)^p \langle \sin^{2p-2}\theta \rangle = \sum_{p=2}^{k} \frac{1}{p} \left(\frac{a_0^2}{r^2+a_0^2}\right)^p , \tag{5.26}$$

where the bracket $\langle\cdot\rangle$ denotes the average on $\mathbb{S}^3$. In deriving (5.25) we have also used that $\Box_0 B_0 = 0$. The second equation in (5.21) is then easily integrated using the AdS_3 bulk-to-bulk propagator $G_2^{\text{Glob}}(\boldsymbol{r}'|r,t,y)$, and summing over all the modes:

$$B_1(r,t,y) = -i\sum_k b_k^2 \int d^3\boldsymbol{r}' \sqrt{-g_{\text{AdS}_3}}\, G_2^{\text{Glob}}(\boldsymbol{r}'|r,t,y)\, \langle J_k(\boldsymbol{r}')\rangle\,, \tag{5.27}$$

where $\boldsymbol{r}' \equiv \{r',t',y'\}$ is a point in AdS_3 and g_{AdS_3} the metric of global AdS_3.

According to (5.12), the correlator is then determined by the large r limit of B_1, which follows from the asymptotic limit of $G_2^{\text{Glob}}(\boldsymbol{r}'|r,t,y)$, that is

$$G_2^{\text{Glob}}(\boldsymbol{r}'|r,t,y) \to \frac{a_0^2}{2\pi r^2} K_2^{\text{Glob}}(\boldsymbol{r}'|t,y). \tag{5.28}$$

Moving from the Lorentzian cylinder to the Euclidean plane, one finds that the order b_k^2 contribution to the 4-point function is

$$\langle O_H(0)\bar{O}_H(\infty)O_L(1,1)\bar{O}_L(z,\bar{z})\rangle|_{b_k^2} = -\sum_k \frac{b_k^2}{2\pi}\int d^3\boldsymbol{w}\,\sqrt{\bar{g}}\,K_2(\boldsymbol{w}|z,\bar{z})\,\langle J_k(\boldsymbol{w})\rangle\,, \tag{5.29}$$

where $\bar{g}$ is the metric of Euclidean AdS_3 and $K_2(\boldsymbol{w}|z,\bar{z})$ the usual bulk-to-boundary propagator in the Poincaré coordinates $\boldsymbol{w}$. The integral in (5.29), with the source $\langle J_k\rangle$ given in (5.25), can be expressed in terms of D-functions using standard methods; we summarise the various steps in app. A.2. Including also the free contribution at $b_k = 0$, the final result for the strong coupling limit of the bosonic correlator up to order b_k^2 can be written in the suggestive form

$$\mathcal{C}^{\text{Bos}}_{\mathcal{O}(b^2)}(z,\bar{z}) = \partial\bar{\partial}\left[\frac{1}{|1-z|^2} - \sum_k \frac{b_k^2}{a_0^2}\left(\frac{1}{2}\frac{1}{|1-z|^2} - \sum_{p=1}^{k}\frac{|z|^2\hat{D}_{pp22}}{\pi\, p}\right)\right]. \tag{5.30}$$

Comparing this result with the Ward identity (2.101) linking bosonic and fermionic correlators, one is lead to the following natural guess for the correlator with fermionic light operators

$$\mathcal{C}^{\text{Fer}}_{\mathcal{O}(b^2)}(z,\bar{z}) = \frac{1}{|z|}\left[\frac{1}{|1-z|^2} + \frac{b_1^2}{a_0^2}\frac{N}{2} - \sum_k \frac{b_k^2}{a_0^2}\left(\frac{1}{2}\frac{1}{|1-z|^2} - \sum_{p=1}^{k}\frac{|z|^2\hat{D}_{pp22}}{\pi\, p}\right)\right], \tag{5.31}$$

Where the term of order N is the disconnected contribution to the correlator, which cannot be predicted by the Ward identity since it is annihilated by the operator $\partial\bar{\partial}(|z|\cdot)$.

Specialising (5.31) to the heavy state considered in [83], which has $b_1 = b \neq 0$ and $b_k = 0$ for $k > 1$, one can check that the above result is in perfect agreement with eq. (3.58) of [83], using to eq. (D.12a) of the same paper.

This proves that the Ward identity is satisfied for this particular heavy state, and provides a quite non-trivial check of our computations. One can also check that the bosonic correlator (5.30) has the expected symmetry under the exchange of the points z_3 and z_4; this transformation permutes O_L with $\bar{O}_L$ and, according to the definition (2.70), amounts to exchange the $\mathbb{T}^4$ index $A = 1$ with $A = 2$; since the heavy operators we consider are invariant under transformations of the compact space $\mathbb{T}^4$, the correlator should be left invariant. From the definition of the conformal cross ratio z one sees that the transformation $z_3 \to z_4$ is equivalent to $z \to 1/z$ and thus one should have that

$$\mathcal{G}^{\text{Bos}}(z,\bar{z}) = \mathcal{G}^{\text{Bos}}(z^{-1},\bar{z}^{-1})\,. \tag{5.32}$$

That the result (5.30) has this property follows from the symmetry of the $\hat{D}$-functions

$$\hat{D}_{pp22}(z^{-1},\bar{z}^{-1}) = |z|^4 \hat{D}_{pp22}(z,\bar{z})\,. \tag{5.33}$$

1.3 *Exact computation for $b_k = b\,\delta_{k,1}$*

As we have seen, the solution in which only the mode $b_1 \equiv b$ is non-vanishing is particularly simple; in fact it is easy to see from (2.151) and (2.152) that $F_1 = \Sigma/(r^2+a^2)$ and thus the three-dimensional metric $g_{\mu\nu}$ is θ-independent. One can thus look for an exact solution of the three-dimensional Laplace equation (5.18):

$$\frac{r^2+a^2}{r(r^2+a^4/a_0^2)}\,\partial_r[r(r^2+a^2)\partial_r B] - \frac{a_0^2}{r^2+a^4/a_0^2}\,\partial_\tau^2 B + \frac{a_0^2}{r^2}\,\partial_\sigma^2 B = 0\,, \tag{5.34}$$

where we have defined

$$\tau \equiv \frac{t}{R}\,, \quad \sigma \equiv \frac{y}{R}\,. \tag{5.35}$$

The solution of (5.34) that is regular at $r = 0$ and that has the asymptotic behaviour (5.11) for large r is

$$\begin{aligned} B = \frac{1}{(2\pi)^2}\sum_{\ell\in\mathbb{Z}}\int \mathrm{d}\omega\, e^{i\omega\tau+i\ell\sigma}\, g(\omega,\ell)\left(\frac{r}{\sqrt{r^2+a^2}}\right)^{|\ell|}\cdot \\ \cdot\, {}_2F_1\left(\frac{|\ell|+\gamma}{2},\frac{|\ell|-\gamma}{2},1+|\ell|;\frac{r^2}{r^2+a^2}\right)\,, \end{aligned} \tag{5.36}$$

where we need to impose

$$g(\omega,\ell) = \frac{\Gamma\left(1+\frac{|\ell|+\gamma}{2}\right)\Gamma\left(1+\frac{|\ell|-\gamma}{2}\right)}{\Gamma(1+|\ell|)}\,, \tag{5.37}$$

and where we have defined

$$\gamma \equiv \frac{\sqrt{a_0^2\,\omega^2-\frac{1}{2}\,b^2\,\ell^2}}{a}\,. \tag{5.38}$$

The function $b(t,y)$ defined in (5.11) is then extracted from the large r limit of B:

$$\begin{aligned} b(\tau,\sigma) = \frac{a^2}{a_0^2}\sum_{\ell\in\mathbb{Z}}\int\frac{\mathrm{d}\omega}{(2\pi)^2}\,e^{i\omega\tau+i\ell\sigma}\cdot \\ \cdot\left\{-\frac{|\ell|}{2}+\frac{\ell^2-\gamma^2}{4}\left[H\left(\frac{|\ell|+\gamma}{2}\right)+H\left(\frac{|\ell|-\gamma}{2}\right)-1\right]\right\}, \end{aligned}$$

(5.39)

where $H(z)$ is the harmonic number, which is related to the digamma function $\psi(z)$ as

$$H(z) = \psi(z+1) + \gamma_E = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+z}\right) . \tag{5.40}$$

Discarding contact terms proportional to $\delta(\tau)$ and/or $\delta(\sigma)$ and their derivatives, and using the identity

$$\ell^2 - \gamma^2 = \frac{a_0^2}{a^2}\,(\ell^2 - \omega^2)\,, \tag{5.41}$$

one can write

$$b(\tau,\sigma) = \frac{\partial_\tau^2 - \partial_\sigma^2}{4}\, b_F(\tau,\sigma)\,, \tag{5.42}$$

where

$$b_F(\tau,\sigma) = \sum_{\ell\in\mathbb{Z}} \int \frac{\mathrm{d}\omega}{(2\pi)^2}\, e^{i\omega\tau + i\ell\sigma} \sum_{n=1}^{\infty} \left(\frac{2}{\gamma - |\ell| - 2n} - \frac{2}{\gamma + |\ell| + 2n}\right) . \tag{5.43}$$

The ω-integral is performed along Feynman's contour, as in sec. 1.2.2; assuming $\tau > 0$ the contour has to be closed on the upper half plane, so we pick the poles on the negative real axis:

$$\omega_n = -\frac{a}{a_0}\sqrt{(|\ell| + 2n)^2 + \frac{b^2\ell^2}{2a^2}}\,. \tag{5.44}$$

The correlator on the plane is found by transforming from the (τ,σ) coordinates to the $(z = e^{i(\tau+\sigma)}, \bar z = e^{i(\tau-\sigma)})$ coordinates and using (5.12). Dropping an irrelevant overall normalisation one finds

$$\mathcal{C}^{\rm Bos}(z,\bar z) = \partial\bar\partial\left(|z|\,\mathcal{C}^{\rm Fer}(z,\bar z)\right)\,, \tag{5.45}$$

with $\mathcal{C}^{\rm Fer}(z,\bar z) = \mathcal{C}^{\rm Fer}(\tau,\sigma)/|z|$, where the factor $1/|z|$ follows from the transformation of the operator in z, and

$$\mathcal{C}^{\rm Fer}(\tau,\sigma) = \frac{a}{a_0}\sum_{\ell\in\mathbb{Z}} e^{i\ell\sigma} \sum_{n=1}^{\infty} \frac{\exp\left[-i\frac{a}{a_0}\sqrt{(|\ell|+2n)^2 + \frac{b^2\ell^2}{2a^2}}\,\tau\right]}{\sqrt{\ell + \frac{b^2}{2a^2}\frac{\ell^2}{(|\ell|+2n)^2}}}\,. \tag{5.46}$$

In our computation the fermionic correlator $\mathcal{C}^{\rm Fer}(\tau,\sigma)$ is determined only up to terms that are annihilated by the derivatives in (5.42). We have chosen these ambiguous terms such that $\mathcal{C}^{\rm Fer}(\tau,\sigma)$ agrees[7] up to terms of order $O(b^2)$ with the correlator computed in [83]. In order to verify that the $O(b^2)$ expansion of the $\mathcal{C}^{\rm Bos}(z,\bar z)$ and $\mathcal{C}^{\rm Fer}(z,\bar z)$ above agrees with the result obtained via the perturbative method in (5.30) and (5.31) one can start by expanding each term of the series for small b at fix a_0 up to order b^2

$$\begin{aligned}\mathcal{C}^{\rm Fer}(\tau,\sigma) \sim \sum_{\ell\in\mathbb{Z}} e^{i\ell\sigma} \sum_{n=1}^{\infty} e^{i(|\ell|+2n)\tau}\,\cdot \\ \cdot\left[\ell + \frac{b^2}{2a_0^2}\left(-\frac{1}{2} - \frac{\ell^2}{2(|\ell|+2n)^2} + \frac{2i\tau(|\ell|+n)n}{|\ell|+2n}\right)\right].\end{aligned} \tag{5.47}$$

7 Note that in (5.45) we have not included the disconnected contribution to the correlator; this contribution can be computed in the free orbifold theory and is given by the $O(N)$ term in (5.31) at all values of b^2/a_0^2.

The terms in the round parenthesis can be written as ratios of polynomials in the combinations ℓ and $|\ell| + 2n$ that appear in the exponentials. It is thus possible to reduce the sums over ℓ and n in terms of derivative or integrals with respect to τ and σ of the geometric series. In particular, the presence in the denominator of a factor of $(|\ell| + 2n)^2$ implies that we have to integrate twice with respect to τ. It is easy to see that the first integration yields logarithms and the second one dilogarithms, producing exactly the terms proportional to Li_2 in the $\hat{D}$ function present in (5.31). It is possible to check that also all other terms of (5.31) are reproduced by performing the sums for the remaining terms in (5.47).

1.3.1 *Other flavours*

We have seen in sec. 2.2.4.2 that the $\frac{1}{4}$-BPS heavy state

$$|\tilde{H}\rangle = \cdots \left[\left(a - \frac{b^2}{4a}\right) |{+}{+}\rangle_1\right] (b\,|{+}{+}\rangle_2) \cdots \tag{5.48}$$

is dual to a ten-dimensional supergravity solution whose six-dimensional metric in Einstein frame is the same as the one for the state (5.2) if $b_k = b\delta_{k,1}$, up to $O(b^2)$; this immediately means that

$$\langle \tilde{H}|O_B(0)\bar{O}_B(z,\bar{z})|\tilde{H}\rangle = \langle H|O_B(0)\bar{O}_B(z,\bar{z})|H\rangle + O(b^4)\,, \tag{5.49}$$

where $\langle H|O_B(0)\bar{O}_B(z,\bar{z})|H\rangle$ is eq. (5.45). This result will be useful in chap. 6.

2 CFT INTERPRETATION OF THE BULK CORRELATOR

A natural way to make contact of the supergravity results (5.46) and (5.47) with the CFT interpretation is to study the OPE limits. For instance, the leading terms of the $z, \bar{z} \to 1$ limit, corresponding to the OPE where the two light operators are close, do not receive any contributions from the $\hat{D}_{pp22}$ with $p > 1$. By using the definition of appendix A.2, it is straightforward to check that, in this OPE limit, the singular terms obtained from the round parenthesis in (5.30) and (5.31) are

$$\frac{1}{2}\frac{1}{|1-z|^2} - \sum_{p=1}^{k} \frac{|z|^2 \hat{D}_{pp22}}{\pi\, p} \sim \frac{1}{4(1-z)} + \frac{1}{4(1-\bar{z})}\,, \tag{5.50}$$

and so do not contribute to the bosonic correlator (5.30). The two singular terms above capture the contributions to the fermionic correlator of the $SU(2)_R$ and $SU(2)_L$ currents. After having substituted the result (5.50) in (5.31), we can extract the contribution due to the exchange of the $SU(2)_L$ current by looking at the term proportional to $1/(1-\bar{z})$, that is

$$\mathcal{C}^{\mathrm{Fer}}_{\mathcal{O}(b^2)} \sim \frac{1}{1-\bar{z}}\left[\frac{1}{2} - \frac{1}{4}\sum_k \frac{b_k^2}{a_0^2}\right] = \frac{a^2}{2a_0^2}\,\frac{1}{1-\bar{z}}\,, \tag{5.51}$$

where in the last line we used (5.7). This provides a check of the relative normalisation between the free contribution and the terms proportional to b_k^2; at order $1/(1-\bar{z})$ the two combine to produce a result proportional to a^2 which is related to the number of strands with $\jmath = 1/2$. This is the only type of strands in the state considered in sec. 5.1 that can contribute to the exchange of the $SU(2)_L$ currents. In particular, the OPE (5.51) is saturated by the exchange of J^3 and, since the correlator factorises into two protected 3-point functions $\langle O_H \bar{O}_H J^3\rangle$ $\langle J^3 O_L \bar{O}_L\rangle$, it is easy to check the overall normalisation just by using the free theory result for the 3-point building blocks.

It is possible to extend further the result above and focus on the leading term in the $(1-\bar{z})$ expansion, but keep all corrections in $(1-z)$. In Lorentzian signature this corresponds to a lightcone OPE where $y \to t$. Also in this case, only the terms proportional to $\hat{D}_{1122}$ are relevant and we obtain

$$\mathcal{C}^{\text{Bos}}_{\mathcal{O}(b^2)} \sim \frac{1}{|1-z|^4}\left\{1-\sum_k \frac{b_k^2}{a_0^2}\left[1+\frac{1}{2}\frac{1+z}{1-z}\ln z\right]\right\}. \tag{5.52}$$

It is interesting to compare this result with the contribution of the (holomorphic) Virasoro block of the identity[8], but this has to be done with great care. While the heavy operators, being RR ground state, have conformal weight $h_H = \bar{h}_H = c/24$, it is convenient to factor out the contribution of the *Sugawara* part of the stress tensor that is due to the $SU(2)_L \times SU(2)_R$ R-currents. The main reason for doing this is that it is possible to take linear combinations of a Virasoro descendant and an affine descendant constructed with the Sugawara stress-tensor to construct a Virasoro primary, i.e. a state annihilated by L_n for $n > 0$. So, if we try to interpret the correlators (5.30) and (5.31) in terms of the full Virasoro blocks, primaries such as the ones mentioned above would appear as new dynamical contributions, while they should not, since their contributions is completely fixed by the symmetries of the theory. Then it is more convenient to analyse the bulk results above in terms of the Virasoro blocks generated by $L^{[0]} = L - L^{\text{Sug}}$ times the blocks generated by the R-symmetry currents. This approach is particularly apt for the bosonic correlator (5.30), since it is not constrained by the R-symmetry at all. By indicating with a superscript [0] all quantities after factorising out the Sugawara contributions, we have $h_L^{[0]} = \bar{h}_L^{[0]} = 1$ and

$$h_H^{[0]} = \bar{h}_H^{[0]} = \frac{N}{4} - \frac{\langle J^2\rangle}{N} = \frac{N}{4}\left[1-\left(\frac{N_1^{(++)}}{N}\right)^2\right], \tag{5.53}$$

where J^2 is the Casimir operator of the $SU(2)_L$ algebra and, in our case, it is sensitive just to the strands with $\jmath, \bar{\jmath} \neq 0$. Thus we should compare (5.52) with the contribution of the HHLL identity Virasoro block with the $h_H^{[0]}$ and $h_L^{[0]}$ above, and $c \sim 6N$[9]. By using the results of [140], we have that the leading term in $(1-\bar{z})$ expansion of the leading N contribution of such Virasoro block reads

$$\begin{aligned}\mathcal{C}^{\text{Bos}}_{\text{Id}} &\sim \frac{1}{(1-\bar{z})^2}\left[z^{\alpha-1}\left(\frac{\alpha}{1-z^{\alpha}}\right)^2\right] \\ &\sim \frac{1}{|1-z|^4}\left\{1-\sum_k \frac{b_k^2}{a_0^2}\left[1+\frac{1}{2}\frac{1+z}{1-z}\ln z\right]\right\},\end{aligned} \tag{5.54}$$

where in the second step we have used

$$\alpha = \sqrt{1-\frac{24 h_H^{[0]}}{c}} = \frac{N_1^{(++)}}{N} = \frac{a^2}{a_0^2} = 1 - \sum_k \frac{b_k^2}{2a_0^2}\,, \tag{5.55}$$

8 Since it is known that the Virasoro block saturates the leading contribution in the lightcone ope limit.

9 Since subtracting the Sugawara sector does not change the leading N contribution of the D1D5 CFT

and took the approximation $b_k^2 \ll a_0^2$ up to the order b_k^2/a_0^2. This clearly shows that the lightcone OPE (5.52) of the strong coupling correlator (5.30) is entirely saturated by the $L^{[0]}$ Virasoro descendants of the identity (5.54), at least in the $\mathcal{O}(b^2)$ approximation. Of course the full correlator away from the lightcone limit receives contributions from other $L^{[0]}$ Virasoro blocks. By expanding (5.30) for $z \to 1$ and $\bar{z} \to 1$ and comparing with the same expansion of the identity Virasoro block, one may see that the first primaries beyond the identity that appear in the OPE have conformal dimension $h = \bar{h} = 2$. As we argued at the beginning of the chapter, these primaries should be multi-particle operators.

In the case of the heavy state discussed in sec. 5.1.3, it is possible to show that lightcone OPE reproduces the $L^{[0]}$ identity Virasoro block even at finite values of b. Consider first the fermionic correlator in (5.46). The lightcone OPE is captured by the modes with $\ell \gg n$, so we can approximate each term in the series (5.46) by expanding the square roots and by neglecting all terms proportional to $1/\ell$; then, when z^α is not too close to 1, the leading contribution in the $\bar{z} \to 1$ limit is captured by

$$\mathcal{C}^{\rm Fer}(\tau,\sigma) \sim \frac{a^2}{a_0^2} \sum_{\ell=0}^{\infty} e^{i\ell(\sigma-\tau)} \sum_{n=1}^{\infty} e^{-2\,i\,\frac{a^2}{a_0^2} n\tau} = \alpha\, \frac{1}{1-\bar{z}}\, \frac{1}{1-|z|^{2\alpha}}\,. \tag{5.56}$$

By inserting this approximation in (5.45) we have

$$\mathcal{C}^{\rm Bos}(z,\bar{z}) \sim \partial\bar{\partial} \left(\frac{1}{1-\bar{z}}\, \frac{\alpha}{1-|z|^{2\alpha}} \right) \sim \frac{1}{(1-\bar{z})^2}\, z^{\alpha-1} \left(\frac{\alpha}{1-z^\alpha} \right)^2 , \tag{5.57}$$

where we focused on the leading contribution in the limit $\bar{z} \to 1$. As mentioned above, this result agrees with (5.54) even at finite values of b_1.

3 LATE TIME BEHAVIOUR OF THE EXACT CORRELATOR

As we have discussed, for finite b we were not able to sum the series in (5.46). However it is still possible to extract useful information already from (5.46), and in particular one can analyse the behaviour of the correlator for large values of the Lorentzian time τ, in order to contrast it with the black hole result (1.88) discussed in sec. 1.2.2. For large τ we have shown that the correlator vanishes like

$$\mathcal{G}^{\rm Bos}_{\rm BTZ}(\tau,\sigma) \sim \frac{1}{\tau^2}\,. \tag{5.58}$$

This large-time decay is a signal of information loss (see sec. 1.2). As discussed in sec. 1.2.2, the decay in (5.58) is polynomial rather than exponential, because the naive geometry of the massless BTZ black hole is a degenerate zero-temperature limit of a regular finite-temperature black hole.

Let us now consider the correlator in the pure heavy state characterised by $b_k = b\delta_{k,1}$, the one studied in sec. 5.1.3. The result of the previous section implies that, for generic values of $\sigma = \sigma_0$, the correlator given in (5.46) has the same singularities at $\tau_k = \sigma_0 + 2\pi k$ as the vacuum correlator. In fact, in this regime, the leading contribution to the sum arises from the modes with $\ell \gg n$ and so, close to τ_k, the fermionic and bosonic correlators are well

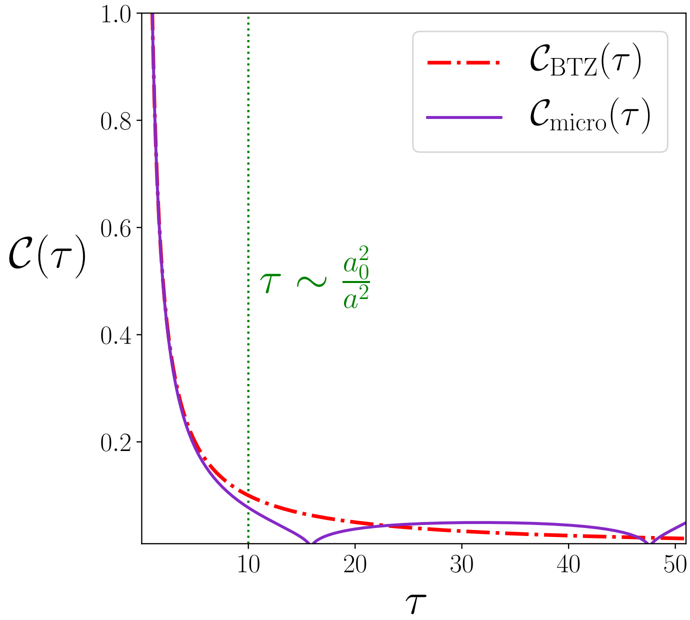


Figure 5.1: A pictorial representation of the HHLL correlator computed in both the naive massless BTZ geometry (in dash-dotted red) and in the pure microstate geometry (in violet). Up to a certain time $\tau \sim a_0^2/a^2$, the two correlators present the same decaying behaviour; after that, the two starts to differ: the BTZ one maintains its decaying behaviour, while the microstate one starts oscillating, as imposed by unitarity.

approximated by (5.56) and (5.57). Then, as expected for a pure state, we have that $\mathcal{G}^{\text{Bos}}_{b_1}$ or $\mathcal{G}^{\text{Fer}}_{b_1}$ tend to a finite value when $\tau \to \tau_k$ for every k, i.e.

$$\begin{aligned} \mathcal{G}^{\text{Fer}}_{b_1} &\sim \alpha \frac{1-e^{2i\sigma_0}}{1-e^{2i\alpha\sigma_0}e^{2\pi i\alpha k}} \,, \\ \mathcal{G}^{\text{Bos}}_{b_1} &\sim \alpha^2 e^{2i\sigma_0(\alpha-1)} e^{2\pi i\alpha k} \left(\frac{1-e^{2i\sigma_0}}{1-e^{2i\alpha\sigma_0}e^{2\pi i\alpha k}} \right)^2 \,. \end{aligned} \tag{5.59}$$

This is in contrast with what happens in the case of the naive geometry, where $\mathcal{G}^{\text{Bos}}_{\text{BTZ}}$ goes to zero at late times.

Since the geometries (2.147), dual to the pure states (5.2), reduce to the naive D1D5 geometry (1.66) in the limit $a \to 0$, it is important to ask *if* the non-unitary correlator (1.88) emerges as the $a \to 0$ limit of the pure state correlator (5.45,5.46). When $a \ll b$, one can distinguish two contributions to the series in (5.46):

$$\frac{a_0}{a}|\ell| \gg 2n : \quad \mathcal{C}^{\text{Fer}} \sim \frac{a^2}{a_0^2} \sum_{\ell,n} \left(1 + \frac{2n}{|\ell|}\right) e^{i(\ell\sigma - |\ell|\tau)} \,; \tag{5.60a}$$

$$\frac{a_0}{a}|\ell| \ll 2n : \quad \mathcal{C}^{\text{Fer}} \sim \frac{a}{a_0} \sum_{\ell,n} e^{i\ell\sigma} e^{-i\frac{a}{a_0}2n\tau} \,, \tag{5.60b}$$

where we have used that $\frac{a}{a_0} \sim \frac{\sqrt{2}a}{b}$. The terms in the first line of the equation above give the sum of a function of $\sigma + \tau$ and a function of $\sigma - \tau$, and thus do not contribute to the bosonic correlator. Then we keep only the second type of contributions, which give

$$\begin{aligned}\mathcal{C}^{\text{Fer}}(\sigma,\tau) &\sim \frac{a}{a_0}\sum_{\ell\in\mathbb{Z}} e^{i\ell\sigma}\sum_{n=\frac{a_0}{2a}|\ell|}^{\infty} e^{-i\frac{a}{a_0}2n\tau}+\dots \\ &= \frac{a}{a_0}\frac{1}{1-e^{-2i\frac{a}{a_0}\tau}}\left[\frac{1}{1-e^{i(\sigma-\tau)}}+\frac{1}{1-e^{-i(\sigma+\tau)}}-1\right]+\dots,\end{aligned} \tag{5.61}$$

where the dots are the terms that do not contribute to $\mathcal{C}^{\text{Bos}}$.

The key point is that, no matter how small a/a_0 is, as far as a is non-zero the correlator in (5.61) and the bosonic correlator derived from it have an oscillating non-vanishing behaviour for large enough τ, as was found in (5.59) for finite a. However, if one observes the correlators at times $\tau \ll a_0/a$, one can approximate (5.61) as

$$\mathcal{C}^{\text{Fer}}(\sigma,\tau) \sim \frac{1}{2i\tau}\left[\frac{1}{1-e^{i(\sigma-\tau)}}+\frac{1}{1-e^{-i(\sigma+\tau)}}-1\right]+\dots, \tag{5.62}$$

and one obtains precisely the naive correlator result given in (1.88). We conclude that

> *the HHLL correlator (5.46), which was computed on a pure two-charge state (5.2), approximates the one computed on the massless BTZ black hole geometry (1.88) in the limit $a \ll a_0$ and for times τ shorter than a_0/a.*

A pictorial representation of this behaviour is reported in fig. 5.1.

4 SUMMARY AND OUTLOOK

In this chapter we used the supergravity approximation of type IIB string theory to derive, using AdS_3/CFT_2 hologrpahy, the strong coupling expression for the HHLL correlators (2.92), where the two light operators are the bosonic states in (2.70) while the heavy operators belong to the ensemble of RR ground states in (5.2). As reviewed in sec. 2.1.6, at the orbifold point in the moduli space, it is easy to compute these correlators in full generality. This was exploited in [72,135] to extract interesting properties of the correlators for *generic* RR ground states. In order to study the problem in a regime where weakly coupled AdS gravity is a valid approximation, one needs to deform the orbifold description and then move to a region where the CFT is strongly coupled. Here we bypassed this challenging task by working with the supergravity description, and to make the computation feasible we restricted to the regime $N_k^{(0)} \ll N_1^{(++)}$, where the states are close to the RR ground state with maximal R-charge. For a particular family of states, the one with $N_k^{(0)} = 0$ for $k \geq 2$, we were able to compute the correlator at strong coupling for all values of the R-charge, even if only implicitly in the form of a Fourier series, including the limit in which the R-charge becomes vanishingly small. To make contact between the gravity results (5.30), (5.31) and (5.45), (5.46) and the CFT, we looked at different OPE limits for the correlator. In the lightcone OPE limit the only contributions to the bosonic correlator come from the Virasoro descendants of the identity, as expected[10] for generic correlators in a CFT where the stress tensor is the only conserved current. In the usual Euclidean OPE, however, other primaries other than identity could contribute,

10 see for instance the discussion in app. A of [138]

and the first ones appears at dimension $h = \bar{h} = 2$ for the bosonic correlator. Summing over these primaries crucially changes the qualitative late time behaviour of the correlator: while each individual classical Virasoro conformal block vanishes at late times, we verified in sec. 5.3 that our correlator has an oscillatory behaviour for arbitrarily large time, as expected in a unitary theory without information loss. Note that, crucially, this result holds also for states that are far from the maximally spinning ground state, for which the correlator is dynamical and not fixed by the symmetries.

6

AdS$_3$ FOUR-POINT FUNCTIONS FROM $\frac{1}{8}$-BPS STATES

As we have briefly discussed in sec. 1, the study of correlators has been one of the key ingredients in AdS/CFT correspondence [36, 143–146] and, more recently, also in the context of black hole physics [48,74,147]. We have seen that in AdS/CFT context, black hole solutions are regarded as supergravity duals to a statistical ensemble of heavy states of the dual field theory (as in chap. 5), and the study of correlation functions provides a powerful tool to shed light on black hole physics and its puzzles. Among all the dynamical quantities one may focus on, four-point functions are especially relevant because of their nature as probes for the black hole and its microscopic structure.

In the D1D5 setup, the four-point function we focused on, often dubbed as Heavy-Heavy-Light-Light (HHLL) correlator, is of the form

$$\langle O_H(z_1,\bar z_1)\bar O_H(z_2,\bar z_2)O_L(z_3,\bar z_3)\bar O_L(z_4,\bar z_4)\rangle\,, \tag{6.1}$$

where we recall that the heavy operators O_H have conformal dimensions scaling with the central charge c, while the light ones O_L have dimensions of order unity. From the gravity point of view the heavy states we will focus on are described by smooth, horizonless solutions of type IIB on $\mathbb{T}^4$, while the light probes are dual to some perturbations around this heavy background. When the D1D5 CFT is at the free point it is possible to calculate the correlator (6.1) with standard technique, while in the opposite limit we have to use holographic methods.

The class of correlators introduced above has recently been studied in [77,83] and we have seen an example in chap. 5 for two-charge microstates, corresponding to $\frac{1}{4}$-BPS RR ground states, whose statistical ensemble is not dual to a macroscopic black hole at the level of classical gravity, but it provides a good testing ground as we know in detail all the gravitational solutions dual to these states [108,110]. We want now to take a step further and study HHLL correlators for a large class of three-charge microstates recently found [88–90][1], and whose thermodynamic description is actually a black hole with a macroscopic entropy already at the level of (semi)classical gravity. These microstates break another half of the supersymmetries of the two-charge seed solution and are therefore $\frac{1}{8}$-BPS states and they are schematically written in the RR sector as

$$|{+}{+}\rangle_1^{N_1}\prod_{k,m,n,q}\Bigg[\frac{\left(J^+_{-1}\right)^m}{m!}\frac{\left(L_{-1}-J^3_{-1}\right)^n}{n!}\cdot \\ \cdot\left(G_0^{+1}G_0^{+2}+\frac{1}{k}J^3_{-1}\left(L_{-1}-J^3_{-1}\right)\right)^q|00\rangle_k\Bigg]^{N_{k,m,n,q}}\,, \tag{6.2}$$

1 For other three-charge geometries, see [84, 86, 142, 148–150].

where L_n, J^a_n, $G^{\alpha A}_{\frac{n}{2}}$ are the generators of the Kac-Moody superalgebra of the CFT. The numbers N_1, $N_{k,m,n}$ (or $N_{k,m,n,q}$) control the number of each *strand* in the D1D5 orbifold picture, and on the gravity side corresponds to some parameters a and b whose strength controls the depth of the throat of the microstates; notice also that $q = 0, 1$ only, while $m \leq k$ and $n \in \mathbb{N}$. The infinite throat limit corresponds to take the limit $a^2/b^2 \to 0$. Even though exact results in the parameters a and b have been found in chap. 5 for a class of two-charge geometries, for the three-charge state considered here, we restrict to the regime where we take the ratio b^2/a^2 to be small, since one of the main goal of this chapter is to reconstruct full LLLL correlators from the HHLL one.

As in chap. 5, it is not straightforward to use Witten diagrams to calculate the correlators (6.1) since the heavy states correspond to multi-particle operators with a large conformal dimension that are not dual to a single supergravity mode. We again bypass these difficulties by seeing the four-point function as a two-point function in a non-trivial background state. In the gravity picture, this boils down to solve a wave equation obtained by perturbing the fields dual to the light operators, around the known smooth geometries, dual to heavy states. The results obtained pass a set of non-trivial consistency checks and they can be related to each other, as well as to the two-charge results obtained in [2, 83], by a set of Ward Identities (WI) encoding, in form of differential operators, the action of all the generators of the global part of the superconformal algebra on the two-charge states.

Moreover in [151] the authos put forward a conjecture to reconstruct the all-light (LLLL) correlators from the corresponding HHLL version. In fact, a class of LLLL four-point functions has been constructed in [151] starting from the HHLL correlator computed in the two-charge geometries of [83]. In analogy to that work, we will also go towards the extraction of the LLLL version of our HHLL four-point function. The LLLL correlators can be thus interpreted in terms of a sum of Witten diagrams, each of them reflects the exchange of fields in AdS in different channels. The prescription of [151] allows to extract the s-channel of the LLLL correlator straightforwardly from the HHLL one. As it will be more clear in the following, in some cases it will be possible to reconstruct all the other channels in order to get the entire LLLL four-point function. Fundamental tools in this approach are the Ward Identities, relating our three-charge correlators to the two-charge ones whose LLLL version is known. Moreover, the Mellin formalism [152–154] constitutes a natural language in which holographic correlators can be interpreted, and it will turn out to be fundamental to analyse the dynamical properties of our results, besides providing another non-trivial consistency check of our results.

1 CFT PICTURE

In this section we use the D1D5 CFT at the free orbifold point to describe the correlators under analysis, introducing the heavy and the light operators we will use, following the notation of sec. 2.1. At the orbifold point, the CFT target space is $(\mathbb{T}^4)^N/S_N$ and the theory can be formulated in terms of N groups of free bosonic and fermionic fields

$$\Big(\partial X^{A\dot{A}}_{(r)}(z),\, \psi^{\alpha\dot{A}}_{(r)}(z)\Big), \quad \Big(\bar{\partial} X^{A\dot{A}}_{(r)}(\bar{z}),\, \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)}(\bar{z})\Big), \tag{6.3}$$

where $(A, \dot{A})$ is a pair of $SU(2)$ indices forming a vector in the CFT target space, while $(\alpha, \dot{\alpha})$ are indices of $SU(2)_L \times SU(2)_R$ which is part of the R-symmetry group and where $r = 1, \ldots N$ is a flavour index running on the various copies of the target space on which the symmetric group S_N acts. The algebra of the theory in all points of the moduli space is given by an affine algebra generated by three conserved currents

$$\Big(T(z),\, J^a(z),\, G^{\alpha A}(z)\Big), \quad \Big(\bar{T}(\bar{z}),\, \bar{J}^a(\bar{z}),\, \bar{G}^{\alpha A}(\bar{z})\Big), \tag{6.4}$$

where $T(z)$ is the stress energy tensor generating the conformal transformations, $J^a(z)$ is the $SU(2)_L$ R-symmetry current and $G^{\alpha A}(z)$ is the supercurrent.

As mentioned in the introduction, we will study four-point functions in the HHLL limit. In order to identify and define these four operators we recall that the spectrum of the theory usually decomposes in two sectors, given by the two different periodicity of the fermionic fields under rotation: the Neveu-Schwarz (NS) and the Ramond (RR) sector. On the CFT side the operation of going to one of these sectors to the other one is implemented via a spectral flow transformation (briefly reviewed in sec. 2.1) that has a particular action on the operators of the theory (see sec. 6.4 for details).

The object we will focus on is then

$$\langle O_H(z_1,\bar{z}_1)\bar{O}_H(z_2,\bar{z}_2)O_L(z_3,\bar{z}_4)\bar{O}_L(z_4,\bar{z}_4)\rangle = \frac{1}{z_{12}^{2h_H} z_{34}^{2h_L}} \frac{1}{\bar{z}_{12}^{2\bar{h}_H} \bar{z}_{34}^{2\bar{h}_L}}\, \mathcal{G}(z,\bar{z})\,, \tag{6.5}$$

where $\mathcal{G}$ is the usual function of the conformal cross ratios (A.32). In order to easily find the function $\mathcal{G}$ one can perform a conformal transformation fixing three of the four points, let us say $z_2 \to \infty$, $z_1 = 0$ and $z_3 = 1$, which further implies $z = z_4$:

$$\langle \bar{O}_H | O_L(1)\bar{O}_L(z,\bar{z})|O_H\rangle \equiv \mathcal{C}(z,\bar{z}) = \frac{1}{(1-z)^{2h_L}} \frac{1}{(1-\bar{z})^{2\bar{h}_L}}\, \mathcal{G}(z,\bar{z}). \tag{6.6}$$

In what follows, we are going to define the operators we will consider in computing the correlators defined above. We will focus on supersymmetric ground states as heavy states, whose dual gravity solutions are known [3, 75, 76, 87–89] and, as light operators, we will consider a class of operators dual to a family of perturbation of the fields that are minimally coupled massless scalars on the gravity side.

The full state in the orbifold theory is a tensor product of ground states for the cyclic twists in the symmetric group conjugacy class, having $N(s)$ copies of k-cycle ground states of the polarisation state s. The class of state then takes the form

$$\psi_{\{N_k^s\}} \equiv \prod_{k,s} \left(|s\rangle_k\right)^{N_k^s} \,. \tag{6.7}$$

These are usually called two-charge states and their gravity dual are known and completely classified [108, 110].

Another class of heavy states, which are the ones we will focus on, are three-charges, $\frac{1}{8}$-BPS state and are given by

$$\psi_{\{N_1, N_{k,m,n}\}} \equiv |{+}{+}\rangle_1^{N_1} \prod_{k,m,n,q} \Bigg[\frac{\left(J_{-1}^{+}\right)^m}{m!} \frac{\left(L_{-1} - J_{-1}^3\right)^n}{n!} \cdot \\ \cdot \left(G_0^{+1}G_0^{+2} + \frac{1}{k} J_{-1}^3 \left(L_{-1} - J_{-1}^3\right)\right)^q |00\rangle_k \Bigg]^{N_{k,m,n,q}} .$$

(6.8)

The integer numbers $\{N_1, N_{k,m,n,q}\}$ specify the number of strands with particular quantum numbers and must satisfy

$$N_1 + \sum_{k,m,n,q} k N_{k,m,n,q} = N \,. \tag{6.9}$$

In order to have a dual classical supergravity solution for these states we need to take a coherent superposition of them. In particular, the heavy states we are interested in with a gravity dual are given by

$$|k,m,n,q\rangle \equiv \sum_{N_1, N_{k,m,n,q}} A_1^{N_1} \left(B_{k,m,n,q}\right)^{N_{k,m,n,q}} \psi_{\{N_1, N_{k,m,n,q}\}} \,, \tag{6.10}$$

where the sum is restricted to $\{N_1, N_{k,m,n,q}\}$ satisfying (6.9), that gives the condition

$$|A_1|^2 + \binom{k}{m}\binom{n+k-1}{n} |B_{k,m,n,q}|^2 = N \,. \tag{6.11}$$

It has been proposed in [88–90] that the states (6.10) are the holographic dual of a class of single-mode supergravity solution whose explicit form can still be found in the same work. We will describe these dual solutions in sec. 6.2.

In particular, we will focus on three classes of heavy states[2]

$$O_H \to |O_H\rangle \equiv \{|1,0,n\rangle,\, |m,m,0\rangle,\, |2,0,0,1\rangle\} \,, \tag{6.12}$$

as defined in (6.10) with the condition

$$|A|^2 + |B|^2 = N \,, \tag{6.13}$$

where we have defined $A \equiv A_1$ and $B \equiv \{B_{1,0,n},\, B_{m,m,0},\, B_{2,0,0,1}\}$.

For what concerns the light operators, we will work again with (2.70), i.e.

$$O_L \to O_{\rm Bos} = \sum_{r=1}^{N} \frac{\epsilon_{\dot A \dot B}}{\sqrt{2N}} \partial X^{1\dot A}_{(r)} \bar\partial X^{1\dot B}_{(r)}, \quad \bar O_L \to \bar O_{\rm Bos} = \sum_{r=1}^{N} \frac{\epsilon_{\dot A \dot B}}{\sqrt{2N}} \partial X^{2\dot A}_{(r)} \bar\partial X^{2\dot B}_{(r)}. \tag{6.14}$$

With this choice of light and heavy operators (in the case $q = 0$), the correlator at the orbifold point depends only on the strand structure, but not on the particular quantum numbers of the RR ground state considered. A standard way to calculate this correlator is to diagonalize the boundary conditions and then to take the linear combination of the contributions of each strand as done in [2]. We write here the results for first two cases, where the heavy states are in the untwisted sector and the strand structure is trivial (6.14)

$$\mathcal{C}_{(1,0,n)} = \mathcal{C}_{(m,m,0)} = \frac{1}{|1-z|^4} \,. \tag{6.15}$$

The computation for $\mathcal{C}_{(2,0,0,1)}$ instead is more involved and, since it is not relevant for the aim of the present chapter, we avoid to report it here. In all the cases we work out, we will see that at the strong coupling point the correlators differ from the ones computed at the free orbifold point.

2 To easen the notation we denote $|k,m,n,q=0\rangle = |k,m,n\rangle$, and analogously for all the objects with k,m,n,q indexes.

2 GRAVITY PICTURE AND HOLOGRAPHIC CORRELATORS IN $\frac{1}{8}$-BPS STATE

We now recall the k, m, n, q geometries built in [3, 88–90] and briefly described in sec. 2.2.1; those are type IIB supergravity solutions described by four scalar functions Z_1, Z_2, Z_4 and $\mathcal{F}$, three 2-forms Θ_1, Θ_2 and Θ_4, and a 1-form ω, by which we can describe all the type IIB fields:

$$\mathrm{d}s^2_{10} = \sqrt{\frac{Z_1 Z_2}{\mathcal{P}}}\,\mathrm{d}s^2_6 + \sqrt{\frac{Z_1}{Z_2}}\,\mathrm{d}s_{\mathbb{T}^4}\,, \tag{6.16a}$$

$$\mathrm{d}s^2_6 = -\frac{2}{\sqrt{\mathcal{P}}}(\mathrm{d}v+\beta)\left[\mathrm{d}u+\omega+\frac{\mathcal{F}}{2}(\mathrm{d}v+\beta)\right]+\sqrt{\mathcal{P}}\,\mathrm{d}s^2_4\,, \tag{6.16b}$$

$$\mathrm{d}s^2_4 = \Sigma\left(\frac{\mathrm{d}r^2}{r^2+a^2}+\mathrm{d}\theta^2\right)+(r^2+a^2)\sin^2\theta\,\mathrm{d}\phi^2+r^2\cos^2\theta\,\mathrm{d}\psi^2\,, \tag{6.16c}$$

$$\Sigma = r^2+a^2\cos^2\theta\,,\quad \mathcal{P}=Z_1Z_2-Z_4^2\,,\quad u=\frac{t-y}{\sqrt{2}}\,,\quad v=\frac{t+y}{\sqrt{2}} \tag{6.16d}$$

$$\beta = \frac{R\,a^2}{\sqrt{2}\,\Sigma}\left(\sin^2\theta\,\mathrm{d}\phi-\cos^2\theta\,\mathrm{d}\psi\right)\,, \tag{6.16e}$$

where we are focusing only on the metric, since it is the only field that will be relevant for us. The objects defining this ansatz have to satisfy the two layers of differential equations (2.117) and (2.119), in order to have a solution of the type IIB equations of motion.

The k, m, n family of solutions built in [88, 89] is described by

$$\begin{aligned}
Z_1 &= \frac{Q_1}{\Sigma}+\frac{R^2}{Q_5}\,\frac{b^2}{2}\,\frac{\Delta_{2k,2m,2n}}{\Sigma}\,\cos\hat{v}_{2k,2m,2n}\,,\\
Z_2 &= \frac{Q_5}{\Sigma}\,,\qquad Z_4 = R\,b\,\frac{\Delta_{k,m,n}}{\Sigma}\,\cos\hat{v}_{k,m,n}\,,\\
\Theta_1 &= 0\,,\quad \Theta_2=\frac{R}{Q_5}\,\frac{b^2}{2}\,\vartheta_{2k,2m,2n}\,,\quad \Theta_4 = b\,\vartheta_{k,m,n}\,,
\end{aligned} \tag{6.17}$$

where we have defined

$$\Delta_{k,m,n} = \left(\frac{a}{\sqrt{r^2+a^2}}\right)^k\left(\frac{r}{\sqrt{r^2+a^2}}\right)^n\cos^m\theta\,\sin^{k-m}\theta\,, \tag{6.18a}$$

$$\hat{v}_{k,m,n} = (m+n)\frac{\sqrt{2}\,v}{R}+(k-m)\phi-m\psi\,, \tag{6.18b}$$

while $\vartheta_{k,m,n}$ is defined in eq. (3.20) of [89]. We do not report here its precise form, since it will not be useful nor relevant. In order to have non-singular geometries we need to impose a regularity condition

$$a^2+x_{k,m,n}\,\frac{b^2}{2}=\frac{Q_1Q_5}{R^2}\equiv a_0^2\,,\qquad x^{-1}_{k,m,n}=\binom{k}{m}\binom{k+n-1}{n}. \tag{6.19}$$

The relation between a, b in the gravity side with the A, B of eq. (6.11) in the CFT side is [75]

$$|A| = R\sqrt{\frac{N}{Q_1Q_5}}\,a\,,\qquad |B| = R\sqrt{\frac{N}{2Q_1Q_5}}\,x_{k,m,n}\,b\,. \tag{6.20}$$

The missing object $\mathcal{F}$ and ω can be computed via the second layer of equations. A close form for them is known only for particular choices of the three parameters [3, 88–90]; we will use here only two of those choices: the $(k,m,n) = (1,0,n)$ and the $(k,m,n) = (m,m,0)$, the second of which we will discuss in detail in the following.

We prove in app. A.5 that, in all the three-charge geometries (6.16), the supergravity field dual to our operator (6.14) is a minimally coupled massless scalar field in six dimensions with Y_0^{00} harmonic[3] on the $\mathbb{S}^3$, i.e.

$$\Box_6 \left(B(\tau,\sigma,r)Y_0^{00}(\theta,\phi,\psi)\right) = 0\,, \tag{6.21}$$

where $\Box_6$ is the scalar Laplacian of the ds_6^2 metric, i.e.

$$\Box_6 \cdot = \frac{1}{\sqrt{g_6}}\,\partial_M(\sqrt{g_6}\,g_6^{MN}\partial_N \cdot), \tag{6.22}$$

and where $\tau = t/R$, $\sigma = y/R$. We then resort to standard holographic methods to extract the Heavy-Light four-point function, as done in chap. 5; In fact, we can compute it by solving the equation of motion for the dual supergravity field with the appropriate boundary conditions

$$B(\tau,\sigma,r) \sim \delta(\tau,\sigma) + \frac{b(\tau,\sigma)}{r^2}\,, \tag{6.23}$$

for $r \to \infty$, plus regularity at $r = 0$. From here we will read the correlator as

$$\langle O_H(0)\bar{O}_H(\infty)O_L(1)\bar{O}_L(z,\bar{z})\rangle = |z|^{-2}\,b(z,\bar{z})\,, \tag{6.24}$$

where we have mapped the cylinder to the plane via $z = e^{\tau_E + i\sigma}$, $\bar{z} = e^{\tau_E - i\sigma}$, with $\tau_E = i\tau$.

Since we were not able to solve the equation, we will resort to a perturbative solution of the equation, expanding it in powers of $\frac{b^2}{2a_0^2}$, that gives a separable equation of motion, since all the (k,m,n) geometries approaches the vacuum $AdS_3 \times \mathbb{S}^3$ solution when $b \to 0$. It will be then useful to rewrite the metric in the following form

$$ds_6^2 = V^{-2}g_{\mu\nu}dx^\mu dx^\nu + G_{ab}(d\theta^a + A^a)(d\theta^b + A^b), \qquad V^2 = \frac{\det G_{ab}}{\sin^2\theta\cos^2\theta}\,, \tag{6.25}$$

where $A^a = A^a{}_\mu dx^\mu$ can be seen as a 1-form on AdS_3. We have split the six-dimensional coordinates as $x^M = (x^\mu, \theta^a)$ where $x^\mu = \{\tau,\sigma,r\}$ and $\theta^a = \{\theta,\phi,\psi\}$. Schematically, we will have then

$$\Box_6\left[\left(B_0 + \frac{b^2}{2a_0^2}\,B_1\right)Y_0^{00}\right] \simeq \left[\Box_0 B_0 + \frac{b^2}{2a_0^2}\left(\Box_0 B_1 + \frac{b^2}{2a_0^2}\,\Box_1 B_0\right)\right]Y_0^{00} + \mathcal{O}\left(\frac{b^4}{4a_0^4}\right) + \text{higher harmonics}, \tag{6.26}$$

where $\Box_0 \cdot = \frac{1}{\sqrt{g_3}}\,\partial_M\left(\sqrt{g_3}g_3^{MN}\partial_N \cdot\right)$ is the scalar Laplacian of AdS_3 in global coordinates, and where the higher harmonics are integrated out; this equation can be solved order by order; at the zeroth order, the solution that respects the correct boundary solutions is the AdS_3 Bulk-to-Boundary propagator:

$$B_0(\tau,\sigma,r) = K_2^{\text{glob}}(\tau,\sigma,r|\tau' = 0,\sigma' = 0) = \left[\frac{1}{2}\,\frac{a_0}{\sqrt{r^2 + a_0^2}\,\cos\tau - r\cos\sigma}\right]^2.$$

3 For the harmonic functions on $\mathbb{S}^3$ we use the notation of [83].

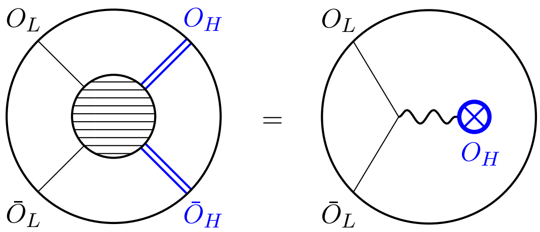


Figure 6.1: A pictorial representation of the method to compute the HHLL 4-point function discussed in the main text, seen as a 2-point function of the light operators on a non-trivial background sourced by the heavy operators. In the left-hand side, the dual supergravity fields of the single-trace light operators are represented by black straight lines in the bulk; on the contrary the heavy operators, being multi-trace, do not have a representation in terms of single-mode supergravity fields, and then their supergravity duals in the bulk are therefore pictorially represented by a blue double-line. In the right-hand side instead, we represent the fact that heavy states source a non-trivial geometry acting like a background field, represented by a crossed blue circle.

(6.27)

The second order can be computed via the Green-function method

$$B_1(\tau,\sigma,r) = -i\int d^3r'\sqrt{-\bar{g}_3}\,G(\boldsymbol{r}|\boldsymbol{r}')J_s(\boldsymbol{r}')\,, \quad J_s \equiv -\langle\Box_1 B_0\rangle\,, \tag{6.28}$$

since at the b^0 order the metric reduces to $AdS_3 \times \mathbb{S}^3$. Now, from the large r-limit of B_1, that is deduced by the large r-limit of the Bulk-to-Bulk propagator[4] as

$$G_2(\boldsymbol{r}'|r,\tau,\sigma) \to \frac{a_0^2}{2\pi r^2}K_2(\boldsymbol{r}'|\tau,\sigma); \tag{6.29}$$

we thus get, mapping onto the plane,

$$\langle O_H(0)\bar{O}_H(\infty)O_L(1)\bar{O}_L(z,\bar{z})\rangle|_{b^2} = -\frac{b^2}{2\pi}\int d^3w'\sqrt{-\bar{g}_3}\,K_2(\boldsymbol{w}'|z,\bar{z})J_s(\boldsymbol{w}')\,. \tag{6.30}$$

This method for computing the 4-point function allows us to avoid using the Witten Diagram technology, that is still not properly defined for AdS_3 [151], since only the cubic coupling have been worked out [155]. To have an intuitive picture in mind, we report in fig. 6.1 a graphic representation of how this method allows us to compute the Heavy-Light 4-point function.

4 Here we explicitly use that $\Delta_L = 2$.

2.1 *HHLL 4-point functions in the* $(k, m, n) = (1, 0, n)$ *geometry*

We start with a case where $q = 0$. The explicit solution of the two layers (3.3, 3.4) for the $(k, m, n) = (1, 0, n)$ geometry was found in [88, 89] and reviewed in sec. 2.2.5

$$\mathrm{d}s_6^2 = -\frac{2}{\sqrt{\mathcal{P}}}(\mathrm{d}v+\beta)\left[\mathrm{d}u+\omega+\frac{\mathcal{F}}{2}(\mathrm{d}v+\beta)\right]+\sqrt{\mathcal{P}}\,\mathrm{d}s_4^2\,, \tag{6.31a}$$

$$\hat{v}_{1,0,n} = \frac{\sqrt{2}}{R}\,nv+\phi\,,\quad \hat{v}_{2,0,2n} = \frac{\sqrt{2}}{R}\,2nv+2\phi\,, \tag{6.31b}$$

$$\Delta_{1,0,n} = \frac{a\,r^n}{(r^2+a^2)^{\frac{n+1}{2}}}\,\sin\theta\,,\quad \Delta_{2,0,2n} = \frac{a^2\,r^{2n}}{(r^2+a^2)^{n+1}}\,\sin^2\theta \tag{6.31c}$$

$$Z_1 = \frac{Q_1}{\Sigma}+\frac{R^2}{2Q_5}\,b^2\,\frac{\Delta_{2,0,2n}}{\Sigma}\cos\hat{v}_{2,0,2n}\quad Z_2 = \frac{Q_2}{\Sigma}\,, \tag{6.31d}$$

$$Z_4 = bR\,\frac{\Delta_{1,0,n}}{\Sigma}\cos\hat{v}_{1,0,n} \tag{6.31e}$$

$$\begin{aligned}\omega &= \frac{a^2R}{\sqrt{2}\,\Sigma}\left(\sin^2\theta\,\mathrm{d}\phi+\cos^2\theta\,\mathrm{d}\psi\right)\\ &\qquad +\frac{b^2}{a^2}\,\frac{a^2R}{\sqrt{2}\,\Sigma}\left[1-\frac{r^{2n}}{(r^2+a^2)^n}\right]\sin^2\theta\,\mathrm{d}\phi\,,\end{aligned} \tag{6.31f}$$

$$\mathcal{F} = -\frac{b^2}{a^2}\left[1-\frac{r^{2n}}{(r^2+a^2)^n}\right]\,,\quad \beta = \frac{a^2R}{\sqrt{2}\,\Sigma}\left(\sin^2\theta\,\mathrm{d}\phi-\cos^2\theta\,\mathrm{d}\psi\right)\,, \tag{6.31g}$$

so that, calling

$$F_n \equiv \left[1-\frac{r^{2n}}{(r^2+a^2)^n}\right]\,, \tag{6.32}$$

we have

$$\omega = \frac{a^2R}{\sqrt{2}\,\Sigma}\left[\left(1+\frac{b^2}{a^2}F_n\right)\sin^2\theta\,\mathrm{d}\phi+\cos^2\theta\,\mathrm{d}\psi\right]\,. \tag{6.33}$$

Notice that this is a 3-charge geometry, due to the fact that we have a non-vanishing $\mathcal{F}$, controlled by a non-vanishing n; sending $n \to 0$ will reduce it to a 2-charge geometry, as expected. This metric can be rewritten, via the splitting of the coordinates $x^M = (x^\mu, \theta^a)$, with $M = 0, \ldots, 5$, $\mu = 0, 1, 2$ and $a = 3, 4, 5$, as

$$\mathrm{d}s_6^2 = V^{-2}g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + G_{ab}(\mathrm{d}\theta^a + A^a_\mu\mathrm{d}x^\mu)(\mathrm{d}\theta^b + A^b_\nu\mathrm{d}x^\nu), \tag{6.34}$$

where V is again fixed requiring that $\sqrt{-\det G_{MN}} = \sqrt{-\det g_{\mu\nu}}\sqrt{\det q_{ab}}$ with q_{ab} the round $\mathbb{S}^3$ metric, for all values of b. The three-dimensional non-compact metric is

$$\begin{aligned}\mathrm{d}s_3^2 = &-\left[r^2\left(1-\frac{b^2}{2a_0^2}\,F_n\right)+\frac{a^4}{a_0^2}\right]\mathrm{d}\tau+r^2\left(1+\frac{b^2}{2a_0^2}\,F_n\right)\mathrm{d}\sigma^2\\ &+\frac{r^2+\frac{a^2}{a_0^2}\left(a^2+\frac{b^2}{2}\,F_n\right)}{(r^2+a_0^2)^2}\,\mathrm{d}r^2\,,\end{aligned} \tag{6.35}$$

where we recall that we have defined

$$\tau = \frac{t}{R}\,,\quad \sigma = \frac{y}{R}\,. \tag{6.36}$$

We then try to solve the scalar equation perturbatively, giving us the source

$$
\begin{aligned}
J_s = {} & \frac{R^2}{2(r^2+a_0^2)^2}\left[(r^2+2a_0^2) - \frac{r^n}{(r^2+a_0^2)^{n-1}}\right]\partial_t^2 B_0 \\
& - \frac{R^2}{r^2+a_0^2}\left[1 - \frac{r^n}{(r^2+a_0^2)^n}\right]\partial_t\partial_y B_0 + \frac{1}{2}\left(\frac{1}{r^2} - \frac{r^n}{(r^2+a_0^2)^{n-1}}\right)R^2\partial_y^2 B_0 \\
& + \frac{1}{r}\,\partial_r\left(r\partial_r B_0\right) .
\end{aligned}
\tag{6.37}
$$

In the case $n=0$ we recover the result of the 2-charge case

$$
J_s^{(n=0)} = \frac{a_0^2 R^2}{2(r^2+a_0^2)^2}\,\partial_t^2 B_0 + \frac{a_0^2 R^2}{2r^2(r^2+a_0^2)}\,\partial_y^2 B_0 + \frac{1}{r}\,\partial_r\left(r\partial_r B_0\right) , \tag{6.38}
$$

as wanted. After some simple algebraic manipulations, we can rewrite the source as[5]

$$
J_s = J_s^{(n=0)} + J_s^{(n>0)} , \tag{6.39a}
$$

$$
J_s^{(n=0)} = \frac{a_0^2 R^2}{2(r^2+a_0^2)^2}\,\partial_t^2 B_0 + \frac{a_0^2 R^2}{2r^2(r^2+a_0^2)}\,\partial_y^2 B_0 + \frac{1}{r}\,\partial_r\left(r\partial_r B_0\right) , \tag{6.39b}
$$

$$
J_s^{(n>0)} = \frac{R^2}{a_0^2}\,B_+^{(0)}B_-^{(0)}\left[1 - \frac{r^{2n}}{(r^2+a_0^2)^n}\right]\partial_u^2 B_0 . \tag{6.39c}
$$

The integration of the $J_s^{(n=0)}$ source piece is exactly the same as the one 5 (once we specialise there $k=1$), giving

$$
\begin{aligned}
B^{(n=0)}(z,\bar z) &= \frac{b^2}{\pi a_0^2}\left[-\frac{1}{2}\hat D_{2222} + \frac{1}{|1-z|^4}\,\left(2(1+|z|^2)\hat D_{3311} - \pi\right)\right] \\
&= \frac{b^2}{\pi a_0^2}\,\partial\bar\partial\left[-\frac{\pi}{2}\,\frac{1}{|1-z|^2} + |z|^2\hat D_{1122}\right] ,
\end{aligned}
\tag{6.40}
$$

while the $n>0$ term can be integrated noticing that

$$
J_s^{(n>0)} = -\frac{R^2}{2a_0^2}\sum_{p=1}^{n}\binom{n}{k}(-1)^p(B_+B_-)^{p+1}\partial_u^2 B_0 , \tag{6.41}
$$

so that, using the fact that $\partial_{u'} = -\partial_u$ by the dependence upon $\tau_E' - \tau_E$, $\sigma' - \sigma$ of the integrand, we can integrate by parts each p-term as

$$
\begin{aligned}
|z|^2 I_p(z,\bar z) &= \partial_u^2 \int d^3 r_e'\sqrt{-g}\,B_0(\boldsymbol{r}_e'|0,0)B_+^{p+1}(\boldsymbol{r}')B_-^{p+1}(\boldsymbol{r}')B_0(\boldsymbol{r}_e'|t_e,y) \\
&= 4(\bar z^2\,\bar\partial^2 + \bar z\,\bar\partial)\left(|z|^2\hat D_{(p+1)(p+1)22}\right) ,
\end{aligned}
\tag{6.42}
$$

5 Notice that, since

$$
B_\pm = \frac{a_0}{\sqrt{r^2+a_0^2}}\,e^{\pm i\tau} ,
$$

we have

$$
\frac{R^2}{r^2+a_0^2} = \frac{R^2}{a_0^2}(B_-^{(1)}B_+^{(1)}) .
$$

where we used the notation of app. A.2, A.3, and that $\partial_u^2 = -(\bar{z}\,\bar{\partial})^2 = -\bar{z}^2\,\bar{\partial}^2 - \bar{z}\,\bar{\partial}$. Summing all the p terms, we thus get the final result

$$\mathcal{C}^{\mathcal{O}(b^2)}_{(1,0,n)} = \mathcal{C}_{n=0} + \mathcal{C}_{n>0}\,, \tag{6.43a}$$

$$\mathcal{C}_{n=0} = +\frac{b^2}{\pi a_0^2}\left[\frac{1}{|1-z|^4}\,\left(2(1+|z|^2)\hat{D}_{3311} - \pi\right) - \frac{1}{2}\,\hat{D}_{2222}\right], \tag{6.43b}$$

$$\mathcal{C}_{n>0} = -\frac{b^2}{\pi a_0^2}\sum_{p=1}^{n}\binom{n}{k}(-1)^p\frac{1}{z}(\bar{z}\,\bar{\partial}^2 + \bar{\partial})\left(|z|^2\hat{D}_{(p+1)(p+1)22}\right). \tag{6.43c}$$

Adding the free part, we have the first 4-point function involving $\frac{1}{8}$-BPS operators in AdS$_3$:

$$\boxed{\begin{aligned}\mathcal{C}_{(1,0,n)} = \frac{1}{|1-z|^4} + \frac{b^2}{\pi a_0^2}\Bigg[&\frac{1}{|1-z|^4}\,\left(2(1+|z|^2)\hat{D}_{3311} - \pi\right) - \frac{1}{2}\,\hat{D}_{2222}\\ &-\sum_{p=1}^{n}\binom{n}{k}(-1)^p\frac{1}{z}(\bar{z}\,\bar{\partial}^2 + \bar{\partial})\left(|z|^2\hat{D}_{(p+1)(p+1)22}\right)\Bigg].\end{aligned}} \tag{6.44}$$

2.2 *HHLL 4-point functions in the $(k,m,n) = (m,m,0)$ geometry*

We want now to discuss the bosonic 4-point function in a different three-charge geometry, the $k = m$, $n = 0$, $q = 0$ one; this is a superdescendant of a two charge geometry and was built in [3, 87–89]:

$$ds_6^2 = -\frac{2}{\sqrt{\mathcal{P}}}(dv+\beta)\left[du + \omega + \frac{1}{2}\,\mathcal{F}(dv+\beta)\right] + \sqrt{\mathcal{P}}\,ds_4^2\,, \tag{6.45a}$$

$$ds_4^2 = \Sigma\left(\frac{dr^2}{r^2+a^2} + d\theta^2\right) + (r^2+a^2)\sin^2\theta\,d\phi^2 + r^2\cos^2\theta\,d\psi^2\,, \tag{6.45b}$$

$$Z_1 = \frac{R^2}{2Q_5}\,\frac{1}{\Sigma}\left[(2a^2+b^2) + \frac{a^{2m}\cos^{2m}\theta}{(r^2+a^2)^m}\,b^2\cos(2m\hat{v})\right], \quad Z_2 = \frac{Q_5}{\Sigma}\,, \tag{6.45c}$$

$$Z_4 = \frac{R}{\Sigma}\,b\,\frac{a^m\cos^m\theta}{(r^2+a^2)^{\frac{m}{2}}}\,\cos(m\hat{v})\,, \qquad \Sigma = r^2 + a^2\cos^2\theta\,, \tag{6.45d}$$

$$\mathcal{F} = -\frac{b^2}{r^2+a^2\sin^2\theta}\left(1 - \frac{a^{2m}\cos^{2m}\theta}{(r^2+a^2)^m}\right), \qquad \mathcal{P} = Z_1 Z_2 - Z_4^2\,, \tag{6.45e}$$

$$\omega = \omega_0 - \frac{R}{\sqrt{2}\,\Sigma}\,\mathcal{F}\left[(r^2+a^2)\sin^2\theta\,d\phi + r^2\cos^2\theta\,d\psi\right], \tag{6.45f}$$

$$\beta_0 = \frac{R\,a^2}{\sqrt{2}\,\Sigma}(\sin^2\theta\,d\phi - \cos^2\theta\,d\psi), \quad \omega_0 = \frac{R\,a^2}{\sqrt{2}\,\Sigma}(\sin^2\theta\,d\phi + \cos^2\theta\,d\psi), \tag{6.45g}$$

where $\hat{v} = \frac{\sqrt{2}\,v}{R} - \psi$. As before, it may be useful to rewrite this geometry as

$$ds_6^2 = V^{-2}g_{\mu\nu}dx^\mu dx^\nu + G_{ab}(d\theta^a + A^a{}_\mu dx^\mu)(d\theta^b + A^b{}_\nu dx^\nu). \tag{6.46}$$

For sake of simplicity, we report here only the g_3 for the case $k = m = 1$, that is

$$\frac{\mathrm{d}s_3^2}{\sqrt{Q_1 Q_5}} = -\left(1 + \frac{r^2 - b^2}{a_0^2} - \frac{b^4}{4a_0^4}\right)\mathrm{d}\tau^2 + \frac{r^2}{a_0^2}\,\mathrm{d}\sigma^2 + \frac{r^2 + a_0^2\left(1 - \frac{b^2}{2a_0^2}\right)}{\left(r^2 + a_0^2 - \frac{b^2}{2}\right)^2}\,\mathrm{d}r^2\,, \tag{6.47}$$

since the $m > 1$ differs from this only via higher scalar harmonics term, that gives rise only to terms that are integrated out in the extraction of the solution.

In order to compute the four-point function we will again perform the same procedure; this will give us the source, once we have integrated on the appropriate harmonic, that is

$$J_s = -\frac{b^2}{2a_0^2}\left[2a_0^2\,\frac{r}{r^2 + a_0^2}\,\partial_r B_0 - 2\frac{a_0^4}{(r^2 + a_0^2)^2}\,\partial_\tau^2 B_0\right], \tag{6.48}$$

for all m. This is due to the fact that m sources higher harmonics that are projected out. This source can be easily integrated and, using the notation of app. A.2, we get

$$\mathcal{C}^{\mathcal{O}(b^2)}_{(m,m,0)} = \frac{b^2}{\pi a_0^2}\left[\frac{I_1 + I_2}{2} - I_3\right], \tag{6.49}$$

that is, explicitly,

$$\mathcal{C}^{\mathcal{O}(b^2)}_{(m,m,0)} = \frac{b^2}{\pi a_0^2}\left[\frac{1}{|1-z|^4}\left(2(1+|z|^2)\hat{D}_{3311} - \pi\right) - \frac{1}{2}\,\hat{D}_{2222}\right]. \tag{6.50}$$

This means that the full correlator is

$$\mathcal{C}_{(m,m,0)} = \frac{1}{|1-z|^4} + \frac{b^2}{\pi a_0^2}\left[\frac{1}{|1-z|^4}\left(2(1+|z|^2)\hat{D}_{3311} - \pi\right) - \frac{1}{2}\,\hat{D}_{2222}\right]. \tag{6.51}$$

2.3 *HHLL 4-point functions in the* $(k, m, n, q) = (2, 0, 0, 1)$ *geometry*

Here we study an example of a geometry with $q = 1$. The geometry with $k = 2, m = n = 0, q = 1$, built in [90] has the usual form (6.16) with

$$\mathrm{d}s_6^2 = -\frac{2}{\sqrt{\mathcal{P}}}(\mathrm{d}v + \beta)\left[\mathrm{d}u + \omega + \frac{\mathcal{F}}{2}(\mathrm{d}v + \beta)\right] + \sqrt{\mathcal{P}}\,\mathrm{d}s_4^2\,, \tag{6.52a}$$

$$\mathrm{d}s_4^2 = \Sigma\left(\frac{\mathrm{d}r^2}{r^2 + a^2} + \mathrm{d}\theta^2\right) + (r^2 + a^2)\sin^2\theta\,\mathrm{d}\phi^2 + r^2\cos^2\theta\,\mathrm{d}\psi^2\,, \tag{6.52b}$$

$$Z_1 = \frac{Q_1}{\Sigma}\,,\quad Z_2 = \frac{Q_5}{\Sigma}\,,\quad Z_4 = 0\,,\quad \mathcal{P} = Z_1 Z_2\,, \tag{6.52c}$$

$$\mathcal{F} = -\frac{b^2\left(3r^4 + 8r^2a^2 + 5a^4 - a^2\left(r^2 + 2a^2\right)\sin^2\theta\right)}{3\left(r^2 + a^2\right)^3}\,, \tag{6.52d}$$

$$\begin{aligned}\omega = {}& \frac{R\,a^2}{\sqrt{2}\,\Sigma}\left(\sin^2\theta\,\mathrm{d}\phi + \cos^2\theta\,\mathrm{d}\psi\right) \\ & + \frac{1}{6}\,\frac{R\,b^2}{\sqrt{2}\,\Sigma}\,\frac{(3r^4 + 8r^2a^2 + 6a^4)\sin^2\theta\,\mathrm{d}\phi + r^2(3r^2 + 4a^2)\cos^2\theta\,\mathrm{d}\psi}{(r^2 + a^2)^2}\,,\end{aligned} \tag{6.52e}$$

and with the regularity condition

$$a^2 + \frac{b^2}{2} = \frac{Q_1 Q_2}{R^2} \equiv a_0^2\,. \tag{6.53}$$

This peculiar metric is separable at all order in b. Performing the same small b procedure, we get a source that is

$$J_s = \frac{b^2}{2a_0}\Bigg[-2a_0^2\,\frac{r}{r^2+a_0^2}\,\partial_r B_0 + \frac{1}{3}\left(2\left(\frac{a_0^2}{r^2+a_0^2}\right)^3 + 9\left(\frac{a_0^2}{r^2+a_0^2}\right)^2\right)\partial_\tau^2 B_0 \\ - \left(\left(\frac{a_0^2}{r^2+a_0^2}\right)^3 + \left(\frac{a_0^2}{r^2+a_0^2}\right)^2\right)\partial_\tau\partial_\sigma B_0 + \frac{1}{3}\left(\frac{a_0^2}{r^2+a_0^2}\right)^3\partial_\sigma^2 B_0\Bigg]. \tag{6.54}$$

This can be easily integrated noticing that

$$(B_+B_-)^2 = \left(\frac{a_0^2}{r^2+a_0^2}\right)^2, \quad (B_+B_-)^3 = \left(\frac{a_0^2}{r^2+a_0^2}\right)^3, \tag{6.55}$$

and that

$$-2a_0^2\,\frac{r}{r^2+a_0^2}\,\partial_r B_0\,, \tag{6.56}$$

is the term we already encountered in eq. (6.48), so that, using the results of app. A.2 and mapping from the cylinder to the plane and the free part, we get

$$\mathcal{C}_{(2,0,0,1)}(z,\bar z) = \frac{1}{|1-z|^4} - \frac{b^2}{2\pi a_0}\frac{1}{|z|^2}\Bigg[|z|^2\hat D_{2222} \\ - \frac{1}{3}(z\partial+\bar z\bar\partial)^2\left(2|z|^2\hat D_{3322} + 9|z|^2\hat D_{2222}\right) \\ + (z\partial+\bar z\bar\partial)(z\partial-\bar z\bar\partial)\left(|z|^2\hat D_{3322} + |z|^2\hat D_{2222}\right) \\ - \frac{1}{3}(z\partial-\bar z\bar\partial)^2\left(|z|^2\hat D_{3322}\right)\Bigg]. \tag{6.57}$$

With this correlator, we have computed all the four-point functions involving $\frac{1}{8}$-BPS operators whose dual geometries are explicitly known that we have described in sec. 6.1. Using the properties of the D-functions reported in app. A.3, it is straightforward to notice that all the 4-point function results (6.44, 6.51, 6.57) posses the symmetry under the exchange of the two bosonic operators $z_3 \leftrightarrow z_4$, i.e. $z \to z^{-1}$, as required. This is a first check on the validity of the computations.

3 From HHLL to LLLL correlators

Here in this section our goal is to follow the arguments of [151] to extract the s-channel of a correlator containing only single-trace operators from the correlators we computed in the previous sections. Here we focus on 4-point functions of the type

$$\langle O_1(z_1,\bar z_1)O_2(z_2,\bar z_2)O_3(z_3,\bar z_3)O_4(z_4,\bar z_4)\rangle = \frac{1}{|z_{12}|^{2\Delta_1}|z_{34}|^{2\Delta_3}}\,\mathcal{G}(z,\bar z), \tag{6.58}$$

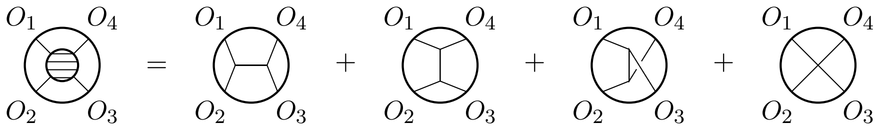


Figure 6.2: Pictorial representation of all the channels that enters in the computation of a generic four-point function. Recall that each diagram correspond to a channel contribution to the correlator $\mathcal{C} = \langle O_1(z_1,\bar{z}_1)O_2(z_2,\bar{z}_2)O_3(z_3,\bar{z}_3)O_4(z_4,\bar{z}_4)\rangle$.

where we assume that the conformal dimensions satisfy $\Delta_1 = \Delta_2$ and $\Delta_3 = \Delta_4$ and they does not scale with the central charge. From the general structure of the correlator we can write it as a sum over channels[6]

$$\mathcal{G} = \mathcal{G}_s + \mathcal{G}_t + \mathcal{G}_u + \mathcal{G}_{\text{cont}}\,, \tag{6.59}$$

where the s, t, u contributions take into account the single-trace operators exchanged in each of those channel, while $\mathcal{G}_{\text{cont}}$ encode the contribution of the possible contact terms. On the complex plane, the three s, t, u channels are $z \to 0$, $z \to 1$, $z \to \infty$, respectively. Each of these terms has a bulk picture in terms of Witten diagram. In particular we have that the s, t, u contributions come from three different Witten diagrams arising from 3-point vertexes in the bulk and the contact terms come from the 4-point vertexes. A pictorial representation of eq. (6.59) is reported in fig. 6.2.

In the case of HHLL 4-point functions we bypassed the Witten diagram machinery and we computed them as a 2-point function of the light operators in a non-trivial background, sourced by the heavy operators. The possibility to extract the all-light (LLLL) four-point functions from the HHLL relies in the fact that the heavy multi-trace operators involved in the HHLL correlators are made of the single-trace constituents involved in the LLLL one, where the number of these constituents is controlled by the free parameter that we called b in the previous sections.

Since the D1D5 has two sector, as explained in sec. 6.1, we can use the spectral flow to go from the NS to the R sector, and vice versa. In particular, the HHLL correlators described above are computed in the R sector, and would be interesting to see how they flow into the NS sector. This interest arise in the fact that the spectral flow, that acts on the generator as (see sec. 2.1)

$$L_n \mapsto L_n + J^3_m + \frac{1}{4}\,\delta_{m,0}\,,\;\; J^3_m \mapsto J^3_m - \frac{1}{2}\,\delta_{m,0}\,,\;\; J^m_m \mapsto J^{\pm}_{m\mp 1}\,,\;\; G^{\pm,A}_m \mapsto G^{\pm,A}_{m\pm\frac{1}{2}}\,, \tag{6.60}$$

on the R-vacua acts as

$$|{+}{+}\rangle_1 \mapsto |0\rangle_1\,, \quad |00\rangle_1 \mapsto O^{--}|0\rangle_1\,, \tag{6.61}$$

where O^{--}, is the fermionic operator[7] [2,75,77,83], that is one of the element of the family (2.68)

$$O^{\alpha\dot{\alpha}}(z,\bar{z}) = \sum_{r=1}^{N} \frac{-i\varepsilon_{\dot{A}\dot{B}}}{\sqrt{2N}}\,\psi^{\alpha\dot{A}}_{(r)}\tilde{\psi}^{\dot{\alpha}\dot{B}}_{(r)}\,. \tag{6.62}$$

---

6 Here we use the usual notation for the s, t and u channels used when computations with Witten diagrams are performed.

7 The nomenclature comes from the fact that it is a product of two fermions of the two different chiral sectors of the theory, i.e. ψ and $\tilde{\psi}$.

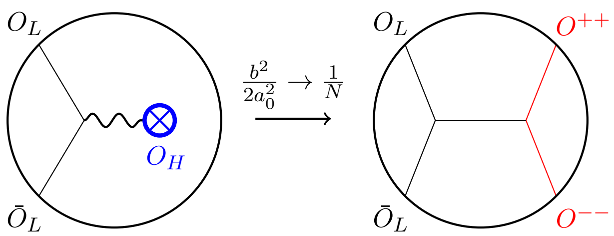


Figure 6.3: The spectral flow from the Ramond sector to the Neveu-Schwarz sector lightens the heavy operator, since the $|{+}{+}\rangle_1$ states flow into the NS-vacua; since we are keeping $\frac{b^2}{2a_0^2}$ small, it is like having $N^{++} \ll N$. Taking then the limit $\frac{b^2}{2a_0^2} \to \frac{1}{N}$ while in the NS sector, we obtain a LLLL 4-point function s-channel out of the spectrally-flowed HHLL one.

This last point is crucial, since it means that the heavy state (6.8) flows in the NS sector to

$$|k, m, n\rangle \mapsto [|0\rangle_1]^{N_1} \prod_{k,m,n} \left[\frac{(L_{-1})^n}{n!} \frac{(J_0^+)^m}{m!} O^{--} |0\rangle_1 \right]^{N_{k,m,n}} . \tag{6.63}$$

This explicitly means that we can now play with the number of operators insertion by controlling $N_{k,m,n}/N$, i.e. $\frac{b^2}{2a_0^2}$; in the NS the $b^2 \ll 2a_0^2$ limit is the "lightening" limit $N^{k,m,n} \ll N$; sending

$$\frac{b^2}{2a_0^2} = \frac{N_{k,m,n}}{N} \to \frac{1}{N} , \tag{6.64}$$

we reduce the HHLL to a LLLL in the s-channel [151]. One may wonder why this limit does not reproduce the full LLLL correlator, but only the s-channel contribution; this is due to the fact that, in the HHLL computation in the R-sector, no single-trace operator is exchanged in the channels where one heavy and one light operator fuse together, i.e. in the t- or u-channels.

Since this statement is true for all values of b, this survives the lightening limit. This implies that the correlator we extract via this lightening ansatz of [151], lacks of the t- and u-channel contribution, and the contact terms[8]. We report a pictorial representation of that in fig. 6.3. In [151] it is also shown how, at least for a certain subset of all the possible D1D5 4-point functions, it is possible to fix unambiguously all the missing terms, effectively reconstructing the full LLLL 4-point functions. In particular, they focus on 4-point functions where the four operators are all the possible combinations of the $O^{\alpha\dot{\alpha}}$ operators of eq. (6.62).

We start from the multi-particle heavy states that create the background in the R sector. As explained in eq. (6.63), these heavy operators flow in the NS in:

$$|1, 0, n\rangle \to \left(L_{-1}^n O^{--}\right)^{N_b} , \tag{6.65a}$$

$$|m, m, 0\rangle \to \left((J_0^+)^m O^{--}\right)^{N_b} , \tag{6.65b}$$

with $N_b = Nb^2/(2a_0^2)$ in the parameter controlling the number of single trace operator inside the heavy state.

8 since even in contact terms heavy and light operator fuse, so are not taken in account.

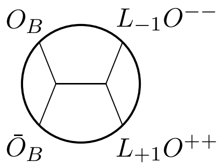
Let us start analysing the LLLL 4-point function (6.58), coming from the states in (6.65a)

$$O_1 = \left(L_{-1}^n O^{--}\right), \quad O_2 = \left(L_{+1}^n O^{++}\right), \quad O_3 = O_{\text{Bos}}, \quad O_4 = \bar{O}_{\text{Bos}}\,. \tag{6.66}$$

Following [151] we have that the s-channel of the this correlator is given by the correlator (6.44) flowed in the NS sector and setting $N_b = 1$:

$$\mathcal{G}_s^{(1,0,n)}(z,\bar z) = \frac{|1-z|^4}{\pi N}\Bigg[\frac{1}{|1-z|^4}\,\left(4(1+|z|^2)\hat D_{3311} - 2\pi\right) - \hat D_{2222} \\ - 2\sum_{p=1}^{n}\binom{n}{k}(-1)^p\frac{1}{z}(\bar z\,\bar\partial^2 + \bar\partial)\left(|z|^2\hat D_{(p+1)(p+1)22}\right)\Bigg], \tag{6.67}$$

i.e., for $n = 1$ case,



$$= \frac{1}{\pi N}\Bigg[\frac{4(1+|z|^2)\hat D_{3311} - 2\pi}{|1-z|^4} - \hat D_{2222} \\ + 2\,\frac{(\bar z\bar\partial^2 + \bar\partial)}{z}\left(|z|^2\hat D_{2222}\right)\Bigg]. \tag{6.68}$$

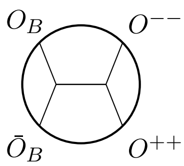
Notice that taking the case $n = 0$ we obtain the s-channel for the LLLL correlators with

$$O_1 = O^{--}\,, \quad O_2 = O^{++}\,, \quad O_3 = O_{\text{Bos}}, \quad O_4 = \bar O_{\text{Bos}}\,. \tag{6.69}$$

This contribution could be directly found by performing this analysis starting from the HHLL correlator computed in [2], and applying the prescription described above. The results agree and they are given by

$$\mathcal{G}_s(z,\bar z) = \frac{|1-z|^4}{\pi N}\left[\frac{1}{|1-z|^4}\,\left(4(1+|z|^2)\hat D_{3311} - 2\pi\right) - \hat D_{2222}\right], \tag{6.70}$$

that is



$$= \frac{1}{\pi N}\left[\frac{1}{|1-z|^4}\,\left(4(1+|z|^2)\hat D_{3311} - 2\pi\right) - \hat D_{2222}\right]. \tag{6.71}$$

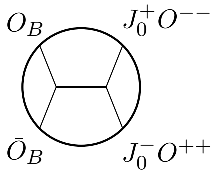
With similar analysis we can extract the s-channel for the LLLL correlator (6.58) with

$$O_1 = \left((J_0^+)^m O^{--}\right),\; O_2 = \left((J_0^-)^m O^{++}\right),\; O_3 = O_{\text{Bos}},\; O_4 = \bar O_{\text{Bos}}, \tag{6.72}$$

coming from the heavy states (6.65b). The result for the s-channel reads

$$\mathcal{G}_s^{(m,m,0)}(z,\bar z) = \frac{|1-z|^4}{\pi N}\left[\frac{1}{|1-z|^4}\,\left(4(1+|z|^2)\hat D_{3311} - 2\pi\right) - \hat D_{2222}\right], \tag{6.73}$$

that is



$$= \frac{1}{\pi N}\left[\frac{1}{|1-z|^4}\left(4(1+|z|^2)\hat{D}_{3311}-2\pi\right)-\hat{D}_{2222}\right]. \tag{6.74}$$

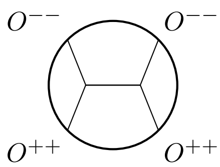
For sake of completeness, we report here also a result already obtained in [83, 151], where it was computed the HHLL 4-point function with four fermionic operators by which it is possible to extract the LLLL s-channel contribution

$$\left[\langle O^{--}(0)O^{++}(\infty)O^{++}(1)O^{--}(z,\bar{z})\rangle\right]_{\text{s-channel}} \equiv \frac{1}{|1-z|^2}\,\mathcal{G}_s^{\text{Fer}}(z,\bar{z}), \tag{6.75}$$

that is

$$\mathcal{G}_s^{\text{Fer}} = \frac{1}{\pi\, N}\left[2\,|z|^2|1-z|^2\hat{D}_{1122}-\pi\right], \tag{6.76}$$

i.e.



$$= \frac{1}{\pi N}\left(2|z|^2\hat{D}_{1122}-\frac{\pi}{|1-z|^2}\right). \tag{6.77}$$

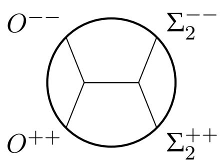
In the next section we will explain how, using Ward identities, we will be able to reconstruct the full LLLL correlators out of the one written here.

3.1 *LLLL correlators with other flavours*

We now start from the state (2.179) and the result of sec. 5.1.3.1. We recall that the state (2.75) constitutes an example of a "fermionic" operator with different "flavour" w.r.t. the operator (2.68); if now we use the lightening ansatz and the supersymmetric WI we have that

$$\langle \Sigma_2^{--}(0)\Sigma_2^{++}(\infty)O^{++}(1)O^{--}(z,\bar{z})\rangle\Big|_{\text{s}} = \langle O^{--}(0)O^{++}(\infty)O^{++}(1)O^{--}(z,\bar{z})\rangle\Big|_{\text{s}}. \tag{6.78}$$

i.e.



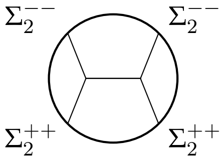
$$= \frac{1}{\pi N}\left(2|z|^2\hat{D}_{1122}-\frac{\pi}{|1-z|^2}\right). \tag{6.79}$$

Moreover, since we have seen in sec. 2.2.3.1 that the moduli of the D1D5 CFT are all minimally coupled massless scalar; since also the modulus $\mathcal{T}^{[ij]} \equiv \mathcal{T}^1$ is the superdescendant of $\Sigma_2^{\pm\pm}$, if we compute the $\langle 1,0,0|\mathcal{T}^1(1)\bar{\mathcal{T}}^1(z,\bar{z})|1,0,0\rangle$ correlator, it will be the same as $\langle 1,0,0|O_{\text{Bos}}(1)\bar{O}_{\text{Bos}}(z,\bar{z})|1,0,0\rangle$; also, we can use the same Ward Identity, since the state is again $\frac{1}{4}$-BPS and thus annihilated by the action of the supercharges. This means that

$$\langle \Sigma_2^{--}(0)\Sigma_2^{++}(\infty)\Sigma_2^{++}(1)\Sigma_2^{--}(z,\bar{z})\rangle\Big|_{\text{s}} = \langle O^{--}(0)O^{++}(\infty)O^{++}(1)O^{--}(z,\bar{z})\rangle\Big|_{\text{s}}.$$

(6.80)

i.e.



$$= \frac{1}{\pi N}\left(2|z|^2\hat{D}_{1122} - \frac{\pi}{|1-z|^2}\right). \tag{6.81}$$

4 WARD IDENTITIES FOR FOUR-POINT FUNCTIONS

As explained in sec. 2.1, the D1D5 CFT has a $\mathcal{N} = (4,4)$ superconformal algebra with a chiral $SU(2)_{L,R}$ Kac-Moody symmetry, with an $SU(2)_1 \times SU(2)_2$ custodial global symmetry.

The main purpose of this section is to find the Ward Identities (WI) for the generators of the global subalgebra L_{-1}, J_0^a, G_0^{+A} that appeared in the correlators computed in the previous section. Concretely, one wants to find differential operators encoding the effects of the insertion of the algebra generators, that acts on the correlators without any insertion, namely the four-point function in (6.70).

The WI referring to the spectrally-flowed state (6.65a) can be found by considering

$$\langle\left(L_{-1}^n O^{--}(z_1,\bar z_1)\right)\left(L_{+1}^n O^{++}(z_2,\bar z_2)\right) O_{\text{Bos}}(z_3,\bar z_3)\bar O_{\text{Bos}}(z_4,\bar z_4)\rangle, \tag{6.82}$$

and finding the differential representation of the operators $L_{\pm1}^n$ on the operators. In order to do that we focus on the case $n=1$ for simplicity. We have

$$\begin{aligned}
&\langle\left(L_{-1} O^{--}(z_1,\bar z_1)\right)\left(L_{+1} O^{++}(z_2,\bar z_2)\right) O_{\text{Bos}}(z_3,\bar z_3)\bar O_{\text{Bos}}(z_4,\bar z_4)\rangle\\
&= \oint_{z_1}\frac{dw_1}{2\pi i}\oint_{z_2}\frac{dw_2}{2\pi i}\, w_2^2\cdot\\
&\quad\cdot\langle T(w_1)O^{--}(z_1,\bar z_1)T(w_2)O^{++}(z_2,\bar z_2)O_{\text{Bos}}(z_3,\bar z_3)\bar O_{\text{Bos}}(z_4,\bar z_4)\rangle\\
&= \left[\left(z_2^2\partial_{z_2} + 2h_1 z_2\right)\partial_{z_1}\right]\langle O^{--}(z_1,\bar z_1)O^{++}(z_2,\bar z_2)O_{\text{Bos}}(z_3,\bar z_3)\bar O_{\text{Bos}}(z_4,\bar z_4)\rangle,
\end{aligned} \tag{6.83}$$

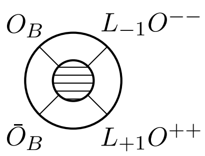
where we used the action of the stress energy tensor on primaries (see sec. 2.1). Using the definition of the cross ratios (A.32) it is straightforward to write the derivative as

$$\frac{\partial}{\partial z_i} = \frac{\partial z}{\partial z_i}\frac{\partial}{\partial z} + \frac{\partial \bar z}{\partial z_i}\frac{\partial}{\partial \bar z}, \tag{6.84}$$

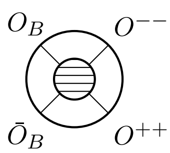
such that in the $(z,\bar z)$ coordinates the WI is realised as

$$\mathcal{G}^{(1,0,1)}(z,\bar z) = \left[(1-z)^2\partial(z\partial) + 1\right]\mathcal{G}(z,\bar z), \tag{6.85}$$

where $\mathcal{G}^{(1,0,1)}(z,\bar z)$ and $\mathcal{G}(z,\bar z)$ are defined to be in the full LLLL correlators. Diagrammatically, we have



$$= |1-z|^4\left[(1-z)^2\partial(z\partial) + 1\right]\left(|1-z|^4 \right.$$



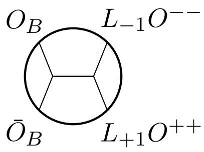
$$\left.\right).$$

(6.86)

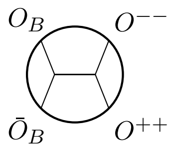
It can be checked from the explicit expressions (6.67) and (6.70) that the WI works also in the s-channel and thus we can write

$$\mathcal{G}_s^{(1,0,1)}(z,\bar{z}) = \left[(1-z)^2\partial(z\partial)+1\right]\mathcal{G}_s(z,\bar{z}). \tag{6.87}$$

Diagrammatically it is represented by



$$= |1-z|^4\ \left[(1-z)^2\partial(z\partial)+1\right]\left(|1-z|^4\right.$$



$$\left.\right). \tag{6.88}$$

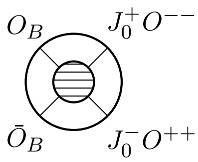
Similarly we can write the WI for spectrally-flowed state (6.65b) for the simple case $m=1$

$$\langle\left(J_0^+O^{--}(z_1,\bar{z}_1)\right)\left(J_0^-O^{++}(z_2,\bar{z}_2)\right)O_{\rm Bos}(z_3,\bar{z}_3)\bar{O}_{\rm Bos}(z_4,\bar{z}_4)\rangle. \tag{6.89}$$

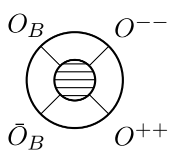
This can be done in the same way of the previous case by two insertion of the currents and by using the OPE relations (see eqs. 2.1) to get the result

$$\mathcal{G}^{(1,1,0)}(z,\bar{z}) = \mathcal{G}(z,\bar{z}), \tag{6.90}$$

i.e., diagrammatically,



$$=$$



$$. \tag{6.91}$$

It is straightforward to check, using the explicit expression in (6.73) and (6.67), that also in this case the WI works also in the s-channel and we have

$$\mathcal{G}_s^{(1,1,0)}(z,\bar{z}) = \mathcal{G}_s(z,\bar{z}). \tag{6.92}$$

We remark that these computations then furnish a highly non-trivial check to our results.

The last WI we want to recall, that will be useful in the next section, is the one coming from supersymmetry. This has been found in chap. 5 (see eq. 5.30) and it reads

$$\mathcal{G}(z,\bar{z}) = |1-z|^4\,\partial\bar{\partial}\left[\frac{\mathcal{G}^{\rm Fer}(z,\bar{z})}{|1-z|^2}\right], \tag{6.93}$$

with $\mathcal{G}^{\rm Fer}(z,\bar{z})$ defined as

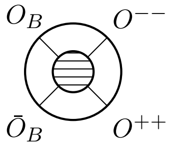
$$\langle O^{--}(z_1,\bar{z}_1)O^{++}(z_2,\bar{z}_2)O^{++}(z_3,\bar{z}_3)O^{--}(z_4,\bar{z}_4)\rangle = \frac{1}{|z_{12}|^2|z_{34}|^2}\,\mathcal{G}^{\rm Fer}(z,\bar{z}), \tag{6.94}$$

and the explicit expression has been found in eq. (3.10) of [151] to be

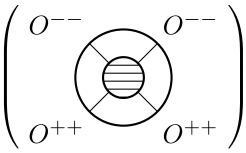
$$\mathcal{G}^{\rm Fer}(z,\bar{z}) = \left(1-\frac{1}{N}\right)(1+|1-z|^2)+\frac{2}{\pi\,N}\,|z|^2|1-z|^2(\hat{D}_{1122}+\hat{D}_{1212}+\hat{D}_{2112}),$$

(6.95)

while the $s-$channel piece is the one reported in eq. (6.76). Again, diagrammatically it is



$= \partial\bar{\partial}$



. (6.96)

Also in this case the WI are valid in the $s-$channel and then we have

$$\mathcal{G}_s(z,\bar{z}) = |1-z|^4 \partial\bar{\partial}\left[\frac{\mathcal{G}_s^{\mathrm{Fer}}(z,\bar{z})}{|1-z|^2}\right]. \tag{6.97}$$

4.1 *Reconstructing the full LLLL correlators using the Ward identities*

So far we have used the validity of WI in the s-channel to check our computations. Now we will instead use it to construct the full LLLL correlators involving the $\frac{1}{8}$-BPS operators. It is easy to see that, via eq. (6.96), using eq. (6.95),

$$\mathcal{G}(z,\bar{z}) = 1 + \frac{1}{\pi N}\left[\left(4(1+|z|^2)\hat{D}_{3311} - 2\pi\right) + |1-z|^4 \hat{D}_{2222}\right]. \tag{6.98}$$

From here, using eq. (6.90), we can build

$$\mathcal{G}^{(1,1,0)}(z,\bar{z}) = 1 + \frac{1}{\pi N}\left[\left(4(1+|z|^2)\hat{D}_{3311} - 2\pi\right) + |1-z|^4 \hat{D}_{2222}\right], \tag{6.99}$$

and, using eq. (6.85),

$$\begin{aligned}\mathcal{G}^{(1,0,1)}(z,\bar{z}) = 1 + \frac{1}{\pi N}\Big[&\left(4(1+|z|^2)\hat{D}_{3311} - 2\pi\right) + 5|1-z|^4 \hat{D}_{2222} \\ &+ 2|1-z|^4 \frac{(\bar{z}\bar{\partial}^2 + \bar{\partial})}{z}\left(|z|^2 \hat{D}_{2222}\right) - 8|1-z|^4 \hat{D}_{3322}\Big].\end{aligned} \tag{6.100}$$

These last two equations constitute the first example of all-light AdS_3 4-point functions at strong coupling involving $\frac{1}{8}$-BPS operators.

5 DISCUSSION

In this chapter we have holographically computed the 4-point functions (6.44, 6.51, 6.57) in the HHLL limit from a large family of $\frac{1}{8}$-BPS states whose dual supergravity solution is explicitly known. These 4-point functions involve two heavy states (6.8) and two Bosonic light states (2.70). In order to perform these computations we used a standard holographic technique: we identified the supergravity field dual to the light state, and we consider it as a perturbation around the type IIB supergravity solution dual to the heavy state, that can be regarded as acting as a source for the equation of motion of the perturbation [2, 77, 83] (as in fig. 6.1).

This was possible to be achieved in the D1D5 SCFT, due to the existence of a precise dictionary that was established between heavy states and type IIB solutions (see chap. 2). We were also able to perform these computations without using the Witten diagram technology, that is still not properly defined in AdS$_3$ [151, 156], since, up to now, only the cubic coupling are known [155].

With respect to the analysis put forward in 5, we have computed the HHLL correlators on three-charge microstates of a black hole with non-degenerate horizon, in the sense that the ensemble of such states is described by a black hole with a finite horizon in the (semi)classical limit of supergravity; our analysis was performed in the small-b limit. In the regime under scrutiny, the correlators show a behaviour that is compatible with unitarity, as expected for pure states, as it was found for two-charge 5. It would then be very interesting to extend the results of sec. 5 in the three-charge geometries analysed here, that was done in the infinite throat limit, i.e. $a \ll b$ in [157], to see if the general mechanism for information conservation described in sec. 5 is extended to an ensemble dual to a regular black hole.

A conjecture on how to extract LLLL four-point functions from the HHLL one was put forward in [151], where the authors suggest that, under a proper lightening limit, the HHLL correlator reduces to a single-channel contribution of the LLLL, the s-channel one. The reason is quite simple: in the HHLL case, no single-trace operators are exchanged in the cross channels where one heavy and one light operator fuses, i.e. in the t- and the u-channels; this implies that the HHLL correlator contains only contributions from the channel where the two heavy operators fuses. Crucial to this lightening limit was the fact that the theory contains two sectors, a Ramond and a Neveu-Schwarz sector, that are connected by a spectral flow. We have shown in sec. 6.3 how spectral-flowing the one-parameter family of heavy states from the R sector to the NS sector and then taking an appropriate limit on the free parameter, it was possible to extract the aforementioned s-channel contribution of the LLLL correlator.

This conjecture is further supported by looking at the correlator in Mellin space (see app. A.4); in fact, from the Mellin transform of the 4-point functions we can read all the single-trace operators that are exchanged simply by looking at their poles [152]. The location of the poles describes the twist of the single trace, while the residue at the pole is related to the 3-point function at the vertexes. It is easy to see that the s-channel contribution deduced from the HHLL correlator (6.76) reads in Mellin space

$$M_s^{\mathrm{Fer}}(s,t,u) = -2\left(1-\frac{t}{2}\right)^2 \frac{1}{s}\,, \tag{6.101}$$

showing then only the pole in s. Since this correlator is of the form

$$\langle O(z_1)\bar{O}(z_2)\bar{O}(z_3)O(z_4)\rangle, \tag{6.102}$$

one may expect that also a u-channel pole should arise, due to the evident symmetry under $z_1 \leftrightarrow z_3$. Indeed, the fully reconstructed one (6.95) in Mellin space reads

$$M^{\mathrm{Fer}}(s,t,u) = \left(1-\frac{t}{2}\right) - 2\left(1-\frac{t}{2}\right)^2\left(\frac{1}{s}+\frac{1}{u}\right), \tag{6.103}$$

showing clearly the u-channel pole as well as all the correct symmetries, i.e. $s \leftrightarrow u$. The only poles arising are then $s = 0$ and $u = 0$. As already pointed out, from here we read that the exchanged single-traces have zero twist and correspond therefore to conserved currents [83].

We have further checked our results by using Ward identities that relate the correlators. This was possible since in the R sector the heavy states are generated by acting with certain generators of the global part of the superconformal Affine algebra. This constitutes a non-trivial check on our computations, but it furnishes a way to reconstruct the full LLLL correlators (6.99, 6.100), using the result of [151], eq. (6.95), as a seed.

We can then compute the Mellin transform of both the s-channel result (6.73) for the $\langle (J_0^+ O^{--})(J_0^- O^{++}) O_B \bar{O}_B \rangle$ correlator, that reads

$$M_s^{(1,1,0)}(s,t,u) = +(s-2) - \frac{(t-3)^2}{s} - \frac{(u-3)^2}{s}\,, \tag{6.104}$$

as well as the full LLLL result (6.99);

$$M^{(1,1,0)}(s,t,u) = -(s-2) - \frac{(t-3)^2}{s} - \frac{(u-3)^2}{s}\,. \tag{6.105}$$

These result have the right symmetry manifestly, that is $z_3 \leftrightarrow z_4$, i.e. $t \leftrightarrow u$. Also, we see that, again, only twist-zero single-trace operators are exchanged, namely the identity and the affine currents. From the latter we also learn that there is no exchange of single-trace operators in the t- and u-channels, meaning that the only difference among the s-channel part and the full LLLL is a term containing no poles, that can be interpreted as a contact term [156].

7

CONCLUSIONS

1 SUMMARY OF RESULTS

In this thesis, we have reported the results of research conducted during the three years of the graduate programme at the University of Padua. We devoted the part ii to the construction of a new set of microstate geometries, while we dedicated part iii to their applications in the context of holography.

The main result of chap. 4 is the explicit construction of the asymptotically flat type IIB supergravity solutions dual to two non-extremal superdescendants of the D1D5 CFT, corresponding to the states

$$[|{+}{+}\rangle_1]^{N_1}[J^+_{-1}\tilde{J}^+_{-1}\,|00\rangle_k]^{N_2}\,,\quad [|{+}{+}\rangle_1]^{N_1}[(J^+_{-1})^k\tilde{J}^+_{-1}\,|00\rangle_k]^{N_2}\,,\tag{7.1}$$

in the perturbative limit, where $N_1 \gg N_2$. While the first one is genuinely non-extremal, since we act on the $|00\rangle_k$ strand with Left- and Right-moving generators the same amount of times, the second is near-extremal in the limit $k \gg 1$, that is the limit we employed to construct the solution. Unfortunately, due to the difficulties arising in non-supersymmetric settings, we were not able to find any analytic solution valid both in the throat region and in the asymptotically flat region. However, we were able to solve the type IIB equations of motion approximately by a matching procedure, showing that indeed a solution exists.

In chap. 3 instead, we focused on three-charge, $\frac{1}{8}$-BPS solutions of type IIB supergravity equations on $\mathbb{T}^4$ that have both internal and external excitations, meaning that these solutions are not invariant under rotations of the compact $\mathbb{T}^4$; these solutions enlarge the family of superstrata discussed in chap. 2, they are dual to the state

$$[|{+}{+}\rangle_1]^{N^{++}}\left[(L_{-1}-J^3_{-1})^{n_1}(J^+_{-1})^{m_1}|00\rangle_{k_1}\right]^{N^b}\left[(L_{-1}-J^3_{-1})^{n_2}(J^+_{-1})^{m_2}\,|00\rangle^{(\dot{1}\dot{1})}_{k_2}\right]^{N^c},\tag{7.2}$$

where we recall that $|00\rangle_k$ and $|00\rangle^{(\dot{1}\dot{1})}_k$ are respectively singlet and triplet w.r.t. $\mathbb{T}^4$ rotations. Since we have to satisfy the regularity condition $N^{++} + k_1 N^b + k_2 N^c = N$, these solutions are described by two free parameters, the ratios N^b/N and N^c/N. All the geometries we have explicitly built are smooth, horizonless and geodesically complete and have the same conserved charges as the naive black hole geometry.

In chap. 5 we computed Heavy-Heavy-Light-Light (HHLL) 4-point functions; we use the CFT state $[|{+}{+}\rangle_1]^{N^{++}}[|00\rangle_k]^{N^{00}_k}$ as heavy operator, which is dual to a two-charge, $\frac{1}{4}$-BPS type IIB supergravity solution, and we use the bosonic operator (2.70) as light operator, which is a superdescendant of the fermionic operator (2.68). The correlator is extracted via a standard holographic procedure by solving the semi-classical equation of motion for the supergravity field dual to the light operator, which deforms the background geometry sourced by the heavy states. In the $N^{00}_k \ll N$ limit, we were able to compute the correlator for all k. We further checked the result by virtue of the supersymmetric Ward identities that relate this correlator with the one involving the fermionic operator (2.68). For the $k = 1$ case, we were able to solve the equation of motion exactly in the N_1^{00}/N limit and, extracting the correlator, we found that it does not show any decay in Lorentzian time, as expected from the unitarity of the CFT. Indeed, the correlator computed on the microstate shows the same decaying behaviour of the correlator computed on the naive black hole geometry only up to a certain time, which is proportional to N/N_k^{00}; after that, the correlator computed on the pure microstate starts to grow again and then oscillate, thus preserving unitarity.

Finally, in chap. 6 we extended the computations of HHLL 4-point functions by employing heavy states of the form

$$|{+}{+}\rangle_1^{N_1}\left[\frac{\left(J_{-1}^{+}\right)^m}{m!}\frac{\left(L_{-1}-J_{-1}^3\right)^n}{n!}\left(G_0^{+1}G_0^{+2}+\frac{1}{k}\,J_{-1}^3\left(L_{-1}-J_{-1}^3\right)\right)^q|00\rangle_k\right]^{N_{k,m,n,q}}, \tag{7.3}$$

that are dual to three-charge, $\frac{1}{8}$-BPS geometries. We put forward the computations explicitly for three cases: $(k,m,n,q)=(1,0,n,0)$, $(m,m,0,0)$ and $(2,0,0,1)$. After extracting the three corresponding HHLL correlators, we were able to employ a set of Ward identities to relate all of them to each other as well as to the correlator computed in chap. 5, furnishing a non-trivial check of our results. We also made a step further: using the procedure discussed in [151] as well as the Ward identities, we were able to reconstruct the LLLL correlators out of the HHLL ones, furnishing the first example of LLLL correlators on AdS_3 involving $\frac{1}{8}$-BPS operators; it was achieved by spectrally flowing from the Ramond sector to the Neveu-Schwarz sector, since the vacuum state $|{+}{+}\rangle_1$ flows into the NS vacuum $|0\rangle$, and then setting the ratio $N_{k,m,n,q}/N \to 1/N$, thus "lightening" the heavy state to the state $O^{--}|0\rangle$. This procedure gave us the s-channel contribution to the LLLL 4-point function. In [151], exploiting the symmetries of the correlator and requiring the right flat space limit of the Mellin transform, the LLLL involving only $\frac{1}{4}$-BPS fermionic operators of the form (2.68) were reconstructed. We extended such construction by employing the Ward Identities to extract other LLLL correlators of $\frac{1}{8}$-BPS operators, reported in eq. (6.99, 6.100). We further motivated the results by looking at their structure in Mellin space, finding the expected poles at the expected locations.

2 FUTURE DIRECTIONS

One may wonder what are the possible future developments that rely on the results presented in this thesis. On the line of what was presented in part ii, it would be nice to push further the analysis of non-extremal microstate geometries by extending the construction to the non-linear level, thus trying to find a non-supersymmetric analogue of the superstrata discussed in chap. 2; due to the absence of supersymmetry, we have a more involved framework, that was not worked out yet. If in the case of $\frac{1}{8}$-BPS geometries, as explained extensively in chap. 2, 3, the non-linear type IIB equations of motion reduces to a set of two linear systems of partial differential equations, this would not probably hold in a non-supersymmetric setting, leaving us to a set of non-linear partial differential equations.

It could be possible to try to generalise the results of [158,159] to the two-parameter family of geometries built in chap. 3, that has a non-trivial cap structure dictated by the two free parameters. One may also try to compute holographically quantum information-theoretical quantities such as entanglement entropy or complexity on these geometries, extending some of the results of [133,160].

Another interesting direction to follow could be the generalisation of the discussion about unitarity restoration described in chap. 5 to more general states, possibly refining the nice results of [157], where it was argued that HHLL correlators computed on $(1, 0, n)$ geometries with $n > 0$ do not decay in Lorentzian time. It will be nice to elucidate how a similar result holds once we expand the 4-point function in conformal blocks, since each of them decays exponentially in the $(1, 0, n)$ framework; we expect that the unitarity restoration appears in the infinite sum over the primaries, but working out the detail would be a crucial step in the resolution of black hole information paradox.

One may also think to expand the results of chap. 6 by computing more general LLLL correlators from the HHLL ones, along the line of [80, 82], by changing either the Light or the Heavy states employed.

APPENDICES

A

NOTATION AND USEFUL RESULTS

1 NOTATION AND DUALITY RULES

1.1 *Duality rules*

For the notation on dualities, we follow [97,110]; for the S-duality we need to define

$$\tau = C_0 + ie^{-\phi} \Rightarrow \tau \to -\frac{1}{\tau}\,. \tag{A.1}$$

Since

$$\tau \to -\frac{1}{\tau} = -\frac{1}{C_0 + ie^{-\phi}} = -\frac{C_0 - ie^{-\phi}}{C_0^2 + e^{-2\phi}}\,, \tag{A.2}$$

we then have

$$\begin{aligned} &\mathrm{d}s_E^2 \to \mathrm{d}s_E^2\,, \quad e^{-\phi} \to \frac{e^{-\phi}}{C_0^2 + e^{-2\phi}} \quad C_0 \to -\frac{C_0}{C_0^2 + e^{-2\phi}}\,, \\ &B_2 \to C_2\,, \qquad C_2 \to -B_2\,, \qquad\qquad C_4 \to C_4 - B_2 \wedge C_2\,, \end{aligned} \tag{A.3}$$

where $\mathrm{d}s_E^2$ is the metric in the Einstein frame. For the string frame metric we thus $\mathrm{d}s_S^2 \to |\tau|\,\mathrm{d}s_S^2$.

If we arrange the solution as

$$\begin{aligned} \mathrm{d}s^2 &= G_{yy}(\mathrm{d}y + A_\mu \mathrm{d}x^\mu)^2 + \mathrm{d}\hat{s}_9^2\,, \\ B_2 &= B_{\mu y}\mathrm{d}x^\mu \wedge (\mathrm{d}y + A_\mu \mathrm{d}x^\mu) + \hat{B}_2\,, \\ C_p &= C_{p-1}^y \wedge (\mathrm{d}y + A_\mu \mathrm{d}x^\mu) + \hat{C}_p\,, \end{aligned} \tag{A.4}$$

where the hatted quantities are object on the nine-dimensional part of the metric, and C_{p-1}^y is the $(p-1)$-form obtained factorising out the $(\mathrm{d}y + A_\mu \mathrm{d}x^\mu)$ leg of C_p, the action of the T-duality along the y direction is

$$\begin{aligned} \mathrm{d}\tilde{s}^2 &= G_{yy}^{-1}(\mathrm{d}y - B_{\mu y}\mathrm{d}x^\mu)^2 + \mathrm{d}\hat{s}_9^2\,, \quad e^{2\tilde{\phi}} = \frac{e^{2\phi}}{G_{yy}}\,, \\ \tilde{B}_2 &= -A_\mu \mathrm{d}x^\mu \wedge \mathrm{d}y + \hat{B}_2\,, \\ \tilde{C}_{p-1} &= \hat{C}_{p-2} \wedge (\mathrm{d}y - B_{\mu y}\mathrm{d}x^\mu) + C_{p-1}^y\,, \end{aligned} \tag{A.5}$$

where C_{p-1}^y is the previous $(p-1)-$form obtained factorizing out the $(\mathrm{d}y + A_\mu \mathrm{d}x^\mu)$ leg of C_p, while $\hat{C}_{p-2}$ is the part of C_{p-2} with no $(\mathrm{d}y + A_\mu \mathrm{d}x^\mu)$ legs.

1.2 *Hodge star operator on warped geometries*

Our notation for the Hodge operator is the following: on a $p-$form, in d dimensions,

$$*_d\alpha^{(p)} = \frac{\sqrt{|g|}}{p!(d-p)!}\,\mathrm{d}x^{M_1} \wedge \cdots \mathrm{d}x^{M_{d-p}}\,\varepsilon_{M_1\cdots M_{d-p}N_1\cdots N_p}\,g_{(d)}^{N_1N_1'}\cdots g_{(d)}^{N_pN_p'}\,\alpha^{(p)}_{N_1'\cdots N_p'}\,.$$

(A.6)

We will repeatedly use that our most general geometry of chap. 3 is factorized as $\mathbb{R}^{(1,1)} \times \mathcal{B} \times \mathbb{T}^4$; we will refer to $*_4$ and $*_{\mathbb{T}^4}$ as performed w.r.t. the flat metrics

$$\begin{aligned} \mathrm{d}s_4^2 &= \Sigma\left(\frac{\mathrm{d}r^2}{r^2+a^2}+\mathrm{d}\theta^2\right)+(r^2+a^2)\sin^2\theta\,\mathrm{d}\phi^2+r^2\cos^2\theta\,\mathrm{d}\psi^2\,, \\ \mathrm{d}s_{\mathbb{T}^4}^2 &= \mathrm{d}z_1^2+\mathrm{d}z_2^2+\mathrm{d}z_3^2+\mathrm{d}z_4^2\,. \end{aligned} \tag{A.7}$$

Now we employ the diagonal split between the six dimensional metric in the Einstein frame and the flat torus metric, as

$$g_{MN}\mathrm{d}x^M\mathrm{d}x^N = \sqrt{\alpha}\,g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \frac{\sqrt{\widetilde{\mathcal{P}}}}{Z_2}\,\delta_{ij}\mathrm{d}z^i\mathrm{d}z^j\,, \tag{A.8}$$

where we have split $x^M=(x^\mu,z^i)$, so that

$$\sqrt{|g_{10}|} = (\sqrt{\alpha})^3\left(\frac{\sqrt{\widetilde{\mathcal{P}}}}{Z_2}\right)^2\sqrt{|g_6|}\,. \tag{A.9}$$

We have then that, called $\lambda^{(p)}$ a $p-$form (with $p\leq 6$) with legs only along x^μ directions,

$$\begin{aligned} *_{10}\left[\lambda^{(p)}\wedge\widehat{\mathrm{vol}}_{\mathbb{T}^4}\right] &= \left(\sqrt{\alpha}\right)^{3-p}\left(\frac{Z_2}{\sqrt{\widetilde{\mathcal{P}}}}\right)^2 *_6\lambda^{(p)}\,, \\ *_{10}\left[\lambda^{(p)}\wedge\omega_5\right] &= \left(\sqrt{\alpha}\right)^{3-p} *_6\alpha^{(p)}\wedge *_{\mathbb{T}^4}\omega_5 = -\left(\sqrt{\alpha}\right)^{3-p} *_6\lambda^{(p)}\wedge\omega_5\,, \\ *_{10}\left[\lambda^{(p)}\right] &= \left(\sqrt{\alpha}\right)^{3-p}\left(\frac{\sqrt{\widetilde{\mathcal{P}}}}{Z_2}\right)^2 *_6\lambda^{(p)}\wedge\widehat{\mathrm{vol}}_{\mathbb{T}^4}\,. \end{aligned} \tag{A.10}$$

We can go further and split the $*_6$ into $*_4$ and the null directions. We split

$$\begin{aligned} \mathrm{d}s_6^2 &= -\frac{2}{\sqrt{\mathbb{P}}}\,d\hat{v}\left[d\hat{u}+\frac{\mathcal{F}}{2}\,d\hat{v}\right]+\sqrt{\mathbb{P}}\,\mathrm{d}s_4^2\,, \\ &= -\frac{2}{\sqrt{\mathbb{P}}}\,d\hat{u}d\hat{v}-\frac{\mathcal{F}}{\sqrt{\mathbb{P}}}\,d\hat{v}^2+\sqrt{\mathbb{P}}\,q_{ab}\mathrm{d}x^a\mathrm{d}x^b\,, \end{aligned} \tag{A.11}$$

where we have split $x^\mu=(\hat{u},\hat{v},x^a)$, so that

$$\sqrt{|g_6|} = \sqrt{\mathbb{P}}\,\sqrt{|q|}\,. \tag{A.12}$$

We have then that, called $\lambda^{(p)}$ a $p-$form (with $p\leq 4$) with legs only along x^a directions[1],

$$\begin{aligned} *_6\left[\lambda^{(p)}\wedge d\hat{u}\wedge d\hat{v}\right] &= -\left(\sqrt{\mathbb{P}}\right)^{3-p} *_4\lambda^{(p)}\,, \\ *_6\left[\lambda^{(p)}\wedge d\hat{u}\right] &= (-1)^{p+1}\left(\sqrt{\mathbb{P}}\right)^{2-p} *_4\lambda^{(p)}\wedge(d\hat{u}+\mathcal{F}\,d\hat{v})\,, \\ *_6\left[\lambda^{(p)}\wedge d\hat{v}\right] &= (-1)^{p}\left(\sqrt{\mathbb{P}}\right)^{2-p} *_4\lambda^{(p)}\wedge d\hat{v}\,, \\ *_6\left[\lambda^{(p)}\right] &= +\left(\sqrt{\mathbb{P}}\right)^{1-p} *_4\lambda^{(p)}\wedge d\hat{u}\wedge d\hat{v}\,. \end{aligned} \tag{A.14}$$

1 Notice that

$$\left[-\frac{1}{\mathbb{P}}\begin{pmatrix}\mathcal{F} & 1\\ 1 & 0\end{pmatrix}\right]^{-1} = \mathbb{P}\begin{pmatrix}0 & -1\\ -1 & \mathcal{F}\end{pmatrix} \tag{A.13}$$

2 BULK INTEGRALS

In this appendix we describe how to compute Bulk integrals appearing in the formulae for the correlators. In order to do so, we need to introduce the Bulk-to-Boundary propagator of conformal dimension Δ

$$\begin{aligned} B_0(\boldsymbol{r}'|\tau_E\,,\sigma) &\equiv K_\Delta(\boldsymbol{r}'|\tau_E\,,\sigma) \\ &= \left[\frac{1}{2}\,\frac{a_0}{\sqrt{r^2+a_0^2}\,\cos(\tau_E'-\tau_E)-r\,\sin(\sigma'-\sigma)}\right]^\Delta ; \end{aligned} \tag{A.15}$$

in this notation, we read $B_0(\boldsymbol{r}) \equiv B_0(\boldsymbol{r}'|0,0)$. We also have that

$$B_\pm \equiv \frac{a_0}{\sqrt{r^2+a_0^2}}\,e^{\pm\tau_E}\,, \tag{A.16}$$

are the bulk-to-boundary propagators with conformal dimension $\Delta = 1$ evaluated at the points $z = \infty$ and $z = 0$

$$B_0(\boldsymbol{r}'|\tau_E,\sigma) = |z|^2 K_2(\boldsymbol{w}|z,\bar{z}), \tag{A.17a}$$

$$B_+(\boldsymbol{r}') = \lim_{z_2\to\infty} |z_2|^2 K_1(\boldsymbol{w}|z_2,\bar{z}_2) \equiv K_1(\boldsymbol{w}|\infty) = w_0\,, \tag{A.17b}$$

$$B_-(\boldsymbol{r}') = K_1(\boldsymbol{w}|0)\,, \tag{A.17c}$$

where $\boldsymbol{w} = \{w_0, w, \bar{w}\}$ are the AdS_3 Poincaré patch coordinates. We will also need to notice that

$$-2\,a_0^2\frac{r}{r^2+a_0^2}\,\partial_r B_0 = (B_+\partial_\mu B_+ B_-\partial_\mu B_+)\partial^\mu B_0\,. \tag{A.18}$$

The bulk integrals that appear in the main part are

$$I_1 \equiv \int d^3\boldsymbol{r}_e'\sqrt{g}\,B_0(\boldsymbol{r}_e'|\tau_E,\sigma)\,\partial'^\mu B_0(\boldsymbol{r}_e'|0,0)\,B_-(\boldsymbol{r}_e')\,\partial_\mu' B_+(\boldsymbol{r}_e')\,, \tag{A.19a}$$

$$I_2 \equiv \int d^3\boldsymbol{r}_e'\sqrt{g}\,B_0(\boldsymbol{r}_e'|\tau_E,\sigma)\,\partial'^\mu B_0(\boldsymbol{r}_e'|0,0)\,B_+(\boldsymbol{r}_e')\,\partial_\mu' B_-(\boldsymbol{r}_e')\,, \tag{A.19b}$$

$$I_3 \equiv \int d^3\boldsymbol{r}_e'\sqrt{g}\,B_0(\boldsymbol{r}_e'|\tau_E,\sigma)\,\partial_{\tau_E'}^2 B_0(\boldsymbol{r}_e'|0,0)\,\frac{a_0^4}{(r'^2+a_0^2)^2}\,, \tag{A.19c}$$

$$I^{(p)} \equiv \int d^3\boldsymbol{r}_e'\sqrt{g}\,B_0(\boldsymbol{r}_e'|\tau_E,\sigma)\,B_0(\boldsymbol{r}_e'|0,0)\,\left(\frac{a_0^2}{r'^2+a_0^2}\right)^p\,, \tag{A.19d}$$

These integrals can be written in terms of the same D-functions $D_{\Delta_1\Delta_2\Delta_3\Delta_4}$, defined in eq. (A.28), that usually appear in the computations of Witten diagrams.

The first integral can be computed as in [83] by writing it in Poincaré coordinates as

$$\begin{aligned} |z|^{-2}I_1 &= \int d^3\boldsymbol{w}\,w_0^{-1}\left(\frac{w_0}{w_0^2+|w-z|^2}\right)^2 \cdot \\ &\qquad \cdot\left[\frac{2w_0}{(w_0^2+|w-1|^2)^2}-\frac{4w_0^3}{(w_0^2+|w-1|^2)^3}\right]\frac{w_0}{w_0^2+|z|^2} \\ &= 2\hat{D}_{1122}-4\hat{D}_{1232}\,. \end{aligned} \tag{A.20}$$

Similarly, we get

$$|z|^{-2}I_2 = 2\hat{D}_{2222}-2\hat{D}_{1122}+4\hat{D}_{1232}\,. \tag{A.21}$$

The computation of I_3 instead is done using that we can integrate by parts exchanging $\partial^2_{\tau'_E}$ with $-\partial^2_{\tau_E}$, since the bulk-to-boundary depends only on the difference $\tau'_E - \tau_E$:

$$\begin{aligned} I_3 = \partial_{\tau_E} \frac{I_1 - I_2}{2} &= (z\partial + \bar{z}\bar{\partial}) \left(|z|^2 (2\hat{D}_{1122} - 4\hat{D}_{1232} - \hat{D}_{2222}) \right) \\ &= \frac{2|z|^2}{|1-z|^4} \left(2(1+|z|^2)\hat{D}_{3311} - \pi \right) , \end{aligned} \tag{A.22}$$

where the last identity follows from a computation that uses properties of the $\hat{D}$-functions reported in app. A.3.

The last integral is similarly computed as

$$I^{(p)} = |z|^2 \hat{D}_{pp22} \, . \tag{A.23}$$

Now, using again that we can integrate by parts exchanging $\partial^2_{\tau'_E}$ with $-\partial^2_{\tau_E}$, since everything depends only on the difference $\tau'_E - \tau_E$, we can compute

$$\tilde{I}^{(p)} \equiv \int d^3\boldsymbol{r}'_e \sqrt{\tilde{g}}\, B_0(\boldsymbol{r}'_e|\tau_E, y)\, \partial_i\partial_j B_0(\boldsymbol{r}'_e|0,0) \left(\frac{a_0^2}{r'^2 + a_0^2} \right)^p , \tag{A.24}$$

with i,j that can be either τ or σ, giving

$$\tilde{I}^{(p)} = \partial_i\partial_j \left(|z|^2 \hat{D}_{pp22} \right) . \tag{A.25}$$

Substituting (A.19d) for the integrals, transforming to the Euclidean plane and adding the trivial contribution $1/|1-z|^4$ from $b_k = 0$, one finds the correlator

$$\begin{aligned} \frac{1}{|1-z|^4} \mathcal{G}^{\mathrm{Bos}}(z,\bar{z}) = \frac{1}{|1-z|^4} + \sum_k \frac{b_k^2}{\pi a_0^2} \Bigg[& \frac{1}{|1-z|^4} \left(2(1+|z|^2)\hat{D}_{3311} - \pi \right) \\ & - \frac{1}{2}\hat{D}_{2222} + \sum_{p=2}^{k} \frac{1}{p}\, \partial\bar{\partial}(|z|^2 \hat{D}_{pp22}) \Bigg] . \end{aligned} \tag{A.26}$$

The first line can be rewritten in a more suggestive form by making use of the identity

$$\frac{1}{|1-z|^4} \left(2(1+|z|^2)\hat{D}_{3311} - \pi \right) - \frac{1}{2}\hat{D}_{2222} = \partial\bar{\partial} \left[-\frac{\pi}{2} \frac{1}{|1-z|^2} + |z|^2 \hat{D}_{1122} \right] , \tag{A.27}$$

that can be verified explicitly. Substituting this identity in (A.26) one arrives at (5.30).

3 D-FUNCTION TECHNOLOGY

The D-functions are defined as[2]

$$D_{\Delta_1\Delta_2\Delta_3\Delta_4}(z_i) = \int d^{d+1}w \,\sqrt{g} \prod_{i=1}^{4} K_{\Delta_i}(w, \vec{z}_i) \, , \tag{A.28}$$

2 We follow the notation of [4, 152, 161, 162]

where the AdS_{d+1} metric in the Euclidean Poincaré coordinates is[3]

$$d\bar{s}^2 = \frac{1}{w_0^2}\left(dw_0^2 + \sum_{i=1}^{d} dw_i^2\right). \tag{A.29}$$

and the boundary-to-bulk propagator for a scalar field propagating in Euclidean AdS_{d+1} is

$$K_\Delta(w, \vec{z}) = \left[\frac{w_0}{w_0^2 + (\vec{w} - \vec{z})^2}\right]^\Delta = \frac{1}{\Gamma(\Delta)} \int_0^\infty dt\, w_0^\Delta\, t^{\Delta-1} e^{-t(w_0^2 + (\vec{w}-\vec{z})^2)}, \tag{A.30}$$

where Δ is the conformal dimension of the dual operator. By using the representation of the propagator in terms of Schwinger parameters given in (A.30) we can perform the integration over the interaction point $(w_0, \vec{w})$, obtaining

$$D_{\Delta_1\Delta_2\Delta_3\Delta_4}(z_i) = \Gamma\left(\frac{\hat{\Delta} - d}{2}\right) \int_0^\infty \prod_i \left[dt_i \frac{t_i^{\Delta_i - 1}}{\Gamma(\Delta_i)}\right] \frac{\pi^{d/2}}{2T^{\frac{\hat{\Delta}}{2}}} e^{-\sum_{i,j=1}^4 |z_{ij}|^2 \frac{t_i t_j}{2T}}, \tag{A.31}$$

where $T = \sum_i t_i$, $\hat{\Delta} = \sum_i \Delta_i$ and $z_{ij} = z_i - z_j$.

We also recall the definition for the conformal cross ratios

$$u = (1-z)(1-\bar{z}) = \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2}, \quad v = z\bar{z} = \frac{z_{14}^2 z_{23}^2}{z_{13}^2 z_{24}^2}. \tag{A.32}$$

The $\hat{D}$-functions that appear in the bulk computation of the correlators are thus defined as

$$\hat{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(z, \bar{z}) = \lim_{z_2 \to \infty} |z_2|^{2\Delta_2} D_{\Delta_1\Delta_2\Delta_3\Delta_4}(z_1 = 0, z_2, z_3 = 1, z_4 = z). \tag{A.33}$$

Once written in terms of Schwinger parameter, one can see that D_{1111} is proportional to the massless box-integral in four dimensions with external massive state [161, 162], whose result can be written in term of logarithms and dilogarithms

$$D_{1111} = \frac{\pi}{2|z_{13}|^2 |z_{24}|^2 (z - \bar{z})} \left[2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + \ln(z\bar{z}) \ln\frac{1-z}{1-\bar{z}}\right], \tag{A.34}$$

The result in (A.34) is proportional to the Bloch-Wigner dilogarithm $D(z, \bar{z})$ [2, 4, 83]

$$D_{1111} = \frac{2\pi i}{|z_{13}|^2 |z_{24}|^2 (z - \bar{z})} D(z, \bar{z}), \tag{A.35}$$

where

$$\begin{aligned} D(z, \bar{z}) &= \mathrm{Im}[\mathrm{Li}_2(z)] + \mathrm{Arg}[\ln(1-z)] \ln|z| \\ &= \frac{1}{2i}\left[\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2}\ln(z\bar{z}) \ln\frac{1-z}{1-\bar{z}}\right]. \end{aligned} \tag{A.36}$$

3 We recall that the change of coordinates that brings global AdS_3 into this coordinate set is

$$w_0 = \frac{a_0}{\sqrt{r^2 + a_0^2}} e^{i\tau}, \quad w = \frac{r}{\sqrt{r^2 + a_0^2}} e^{i(\tau+\sigma)}, \quad \bar{w} = \frac{r}{\sqrt{r^2 + a_0^2}} e^{i(\tau-\sigma)}.$$

Moreover we report here the following useful identity

$$D(z,\bar z) = -D\left(\frac{1}{z},\frac{1}{\bar z}\right) = -D\left(1-z,1-\bar z\right), \tag{A.37}$$

which implies

$$D(z,\bar z) = D\left(1-\frac{1}{z},1-\frac{1}{\bar z}\right) = D\left(\frac{1}{1-z},\frac{1}{1-\bar z}\right) = -D\left(\frac{-z}{1-z},\frac{-\bar z}{1-\bar z}\right). \tag{A.38}$$

Other useful relations are [151]

$$\hat D_{\Delta_2\Delta_1\Delta_3\Delta_4}\left(\frac{1}{z},\frac{1}{\bar z}\right) = |z|^{2\Delta_4}\hat D_{\Delta_1\Delta_2\Delta_3\Delta_4}(z,\bar z), \tag{A.39a}$$

$$\hat D_{\Delta_3\Delta_2\Delta_1\Delta_4}\left(1-z,1-\bar z\right) = \hat D_{\Delta_1\Delta_2\Delta_3\Delta_4}(z,\bar z), \tag{A.39b}$$

$$\hat D_{\Delta_2\Delta_1\Delta_4\Delta_3}\left(z,\bar z\right) = |z|^{\Delta_1-\Delta_2-\Delta_3+\Delta_4}\hat D_{\Delta_1\Delta_2\Delta_3\Delta_4}(z,\bar z). \tag{A.39c}$$

Using in order (A.39b), (A.39c), (A.39b) we get the relation

$$\hat D_{\Delta_1\Delta_2\Delta_3\Delta_4}(z,\bar z) = |1-z|^{\Delta_2-\Delta_3-\Delta_4+\Delta_1}\hat D_{\Delta_4\Delta_3\Delta_2\Delta_1}(z,\bar z). \tag{A.40}$$

In the literature, it is customary to introduce the $\bar D$-functions, which depend only on the cross ratios [152, 161, 162]; we will briefly review how they are defined in our notation, where they read

$$\bar D_{\Delta_1\Delta_2\Delta_3\Delta_4}(z,\bar z) = \frac{2\prod_{i=1}^4\Gamma(\Delta_i)}{\pi^{d/2}\Gamma\left(\frac{\hat\Delta-d}{2}\right)}\frac{|z_{13}|^{\hat\Delta-2\Delta_4}|z_{24}|^{2\Delta_2}}{|z_{14}|^{\hat\Delta-2\Delta_1-2\Delta_4}|z_{34}|^{\hat\Delta-2\Delta_3-2\Delta_4}}D_{\Delta_1\Delta_2\Delta_3\Delta_4}(z_i). \tag{A.41}$$

The relation between $\hat D$ and $\bar D$-functions is thus

$$\hat D_{\Delta_1\Delta_2\Delta_3\Delta_4}(z,\bar z) = \frac{\pi\Gamma\left(\frac{\hat\Delta-2}{2}\right)}{2\prod_{i=1}^4\Gamma(\Delta_i)}|z|^{\hat\Delta-2\Delta_1-2\Delta_4}|1-z|^{\hat\Delta-2\Delta_3-2\Delta_4}\bar D_{\Delta_1\Delta_2\Delta_3\Delta_4}(z,\bar z). \tag{A.42}$$

In the results for the correlators we encountered in the main text appears the derivative of the $\hat D$-functions with respect z or $\bar z$. In order to handle with $\hat D$-functions, it is useful to write these contributions in terms of $\hat D$-functions without derivatives. In order to find an useful expression we rewrite the generic $\hat D$-functions in terms of $\bar D$-functions using (A.42) and (A.41), and expressing them as function of u and v. We thus have

$$\hat D_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v) = \frac{\pi\Gamma\left(\frac{\hat\Delta-2}{2}\right)}{2\prod_{i=1}^4\Gamma(\Delta_i)}v^{\frac{\hat\Delta-2\Delta_1-2\Delta_4}{2}}u^{\frac{\hat\Delta-2\Delta_3-2\Delta_4}{2}}\bar D_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v). \tag{A.43}$$

The object to be computed is therefore

$$\frac{\partial}{\partial z}\hat D_{\Delta_1\Delta_2\Delta_3\Delta_4}(z,\bar z) = \left(\frac{\partial v}{\partial z}\frac{\partial}{\partial v}+\frac{\partial u}{\partial z}\frac{\partial}{\partial u}\right)\hat D_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v). \tag{A.44}$$

Rewriting now the $\hat D_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v)$ in terms of $\bar D_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v)$ and using (A.43) and the fact that $\frac{\partial v}{\partial z}=\bar z$, $\frac{\partial u}{\partial z}=-(1-\bar z)$, we end up with an expression containing $\bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v)$-functions and its derivatives w.r.t. u and v. We can use the result of [161–163]

$$\frac{\partial}{\partial u}\bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v) = -\bar{D}_{\Delta_1+1\Delta_2+1\Delta_3\Delta_4}(u,v), \tag{A.45}$$

$$\frac{\partial}{\partial v}\bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v) = -\bar{D}_{\Delta_1\Delta_2+1\Delta_3+1\Delta_4}(u,v). \tag{A.46}$$

Reconstructing now the $\hat{D}$-functions and coming back to $z, \bar{z}$, we get

$$\begin{aligned}\frac{\partial}{\partial z}\hat{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(z,\bar{z}) = &\left(\frac{\hat{\Delta}-2\Delta_1-2\Delta_4}{2z(1-z)}+\frac{\Delta_4-\Delta_2}{1-z}\right)\hat{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(z,\bar{z})\\ &+\frac{2\Delta_2\Delta_3}{(2-\hat{\Delta})z}\hat{D}_{\Delta_1\Delta_2+1\Delta_3+1\Delta_4}(z,\bar{z})\\ &-\frac{2\Delta_1\Delta_2}{(2-\hat{\Delta})(1-z)}\hat{D}_{\Delta_1+1\Delta_2+1\Delta_3\Delta_4}(z,\bar{z}).\end{aligned} \tag{A.47}$$

With very similar procedure it is straightforward to find

$$\begin{aligned}\frac{\partial}{\partial \bar{z}}\hat{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(z,\bar{z}) = &\left(\frac{\hat{\Delta}-2\Delta_1-2\Delta_4}{2\bar{z}(1-\bar{z})}+\frac{\Delta_4-\Delta_2}{1-\bar{z}}\right)\hat{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(z,\bar{z})\\ &+\frac{2\Delta_2\Delta_3}{(2-\hat{\Delta})\bar{z}}\hat{D}_{\Delta_1\Delta_2+1\Delta_3+1\Delta_4}(z,\bar{z})\\ &-\frac{2\Delta_1\Delta_2}{(2-\hat{\Delta})(1-\bar{z})}\hat{D}_{\Delta_1+1\Delta_2+1\Delta_3\Delta_4}(z,\bar{z}).\end{aligned} \tag{A.48}$$

The above results are enough to find the generic term containing higher derivatives of $\hat{D}$-functions and to trade it as a sum of $\hat{D}$-functions.

Acting with ∂_z on the relations (A.39c), (A.40), and using them, we can easily find non-trivial relations among D-functions of different $\hat{\Delta}$, as

$$\begin{aligned}\hat{D}_{2442}(z,\bar{z}) &= \frac{4}{9}\left(\hat{D}_{2332}(z,\bar{z})+|z|^2\hat{D}_{3333}(z,\bar{z})\right),\\ \hat{D}_{4422}(z,\bar{z}) &= \frac{4}{9}\left(\hat{D}_{3322}(z,\bar{z})+|1-z|^2\hat{D}_{3333}(z,\bar{z})\right).\end{aligned} \tag{A.49}$$

4 MELLIN REPRESENTATION

Following [152], we define the Mellin amplitudes for four-point function

$$\mathcal{C}_{\Delta_1\Delta_2\Delta_3\Delta_4}(z_i) = \langle O_1(z_1)O_2(z_2)O_3(z_3)O_4(z_4)\rangle, \tag{A.50}$$

as

$$\mathcal{C}_{\Delta_1\Delta_2\Delta_3\Delta_4}(z_i) = \prod_{i<j}^{4}\int_{c-i\infty}^{c+i\infty}\frac{\mathrm{d}s_{ij}}{2\pi i}M(s_{ij})\,\Gamma(s_{ij})(z_{ij}^2)^{-s_{ij}}, \tag{A.51}$$

where we need to impose the following constraint

$$\sum_{j=1,j\neq i}^{4} s_{ij} = \Delta_i\,. \tag{A.52}$$

The variables s_{ij} can be defined introducing four Lorentzian vectors p_i, satisfying

$$\sum_{i=1}^{4} p_i = 0\,, \qquad p_i^2 = -\Delta_i\,, \tag{A.53a}$$

$$s_{ij} = p_i\cdot p_j = \frac{1}{2}(\Delta_i+\Delta_j+(p_i+p_j)^2). \tag{A.53b}$$

In order to compute the Mellin transform of D-functions let us recall the definition (A.31)

$$D_{\Delta_1\Delta_2\Delta_3\Delta_4}(z_i) = \Gamma\left(\frac{\hat{\Delta}-d}{2}\right)\int_0^\infty \prod_i \left[\mathrm{d}t_i\,\frac{t_i^{\Delta_i-1}}{\Gamma(\Delta_i)}\right]\frac{\pi^{d/2}}{2T^{\frac{\hat{\Delta}}{2}}}\,e^{-\sum_{i,j=1}^4 z_{ij}^2\frac{t_it_j}{2T}}\,, \tag{A.54}$$

where we have defined $T=\sum_i t_i$, $\hat{\Delta}=\sum_i \Delta_i$. Then performing the following change of variables

$$t_i \longrightarrow T^{\frac{1}{2}}\,t_i\,, \tag{A.55}$$

the D-function becomes

$$D_{\Delta_1\Delta_2\Delta_3\Delta_4}(z_i) = \pi^{d/2}\Gamma\left(\frac{\hat{\Delta}-d}{2}\right)\int_0^\infty \prod_i \left[\mathrm{d}t_i\,\frac{t_i^{\Delta_i-1}}{\Gamma(\Delta_i)}\right]e^{-\sum_{i,j=1}^4 z_{ij}^2 t_it_j}\,. \tag{A.56}$$

Now, by rescaling the coordinates

$$t_1 = \frac{|z_{23}|}{|z_{12}||z_{13}|}\,\hat{t}_1,\quad t_2 = \frac{|z_{13}|}{|z_{12}||z_{23}|}\,\hat{t}_2, \tag{A.57a}$$

$$t_3 = \frac{|z_{12}|}{|z_{13}||z_{23}|}\,\hat{t}_3,\quad t_4 = \frac{|z_{12}||z_{23}|}{|z_{24}|^2|z_{13}|}\,\hat{t}_4, \tag{A.57b}$$

and then using the Mellin representation of the exponential

$$\exp[-z_{ij}^2] = \int_{c-i\infty}^{c+i\infty}\frac{\mathrm{d}s_{ij}}{2\pi i}\,\Gamma(s_{ij})(z_{ij}^2)^{-s_{ij}}, \tag{A.58}$$

and finally performing the Gaussian integrals over $\hat{t}_i$, we get

$$D_{\Delta_1\Delta_2\Delta_3\Delta_4}(z_i) = \frac{\pi^{d/2}\Gamma\left(\frac{\hat{\Delta}-d}{2}\right)}{\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)\Gamma(\Delta_4)}\left[\prod_{i<j}^4\int_{c-i\infty}^{c+i\infty}\frac{\mathrm{d}s_{ij}}{2\pi i}\,\Gamma(s_{ij})(z_{ij}^2)^{-s_{ij}}\right], \tag{A.59}$$

that means that the Mellin transform of the $D_{\Delta_1\Delta_2\Delta_3\Delta_4}(z_i)$ is indeed a constant function

$$M(s_{ij}) = \frac{\pi^{d/2}\Gamma\left(\frac{\hat{\Delta}-d}{2}\right)}{\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)\Gamma(\Delta_4)}\,, \tag{A.60}$$

in agreement with [152].

Since in the correlators we computed usually appears more generic terms, let us discuss terms of the form $\mathcal{C}_{\Delta_1\Delta_2\Delta_3\Delta_4}(z_i)$ as

$$\mathcal{G}^{(p,q)}_{\Lambda_1\Lambda_2\Lambda_3\Lambda_4}(u,v) = u^pv^q\hat{D}_{\Delta_1+\Lambda_1\Delta_2+\Lambda_1\Delta_3+\Lambda_3\Delta_4+\Lambda_4}(u,v), \tag{A.61}$$

where $p,q,\Lambda_i\in\mathbb{Z}$ and u, v are the usual cross ratios defined in (A.32). We define the following quantities[4]

$$s=-(p_1+p_2)^2=-(p_3+p_4)^2,\quad t=-(p_1+p_4)^2=-(p_2+p_3)^2\,, \tag{A.62a}$$

4 We also have the variable $u=-(p_2+p_4)^2$. Notice that, because of the conservation of momentum, we have the relation between the three Mandelstam variables $s+t+u=\Delta_1+\Delta_2+\Delta_3+\Delta_4$.

and writing the variables s_{ij} in (A.53b) in terms of the above Mandelstam variables we get

$$\begin{aligned}\mathcal{C}_{\Delta_1\Delta_2\Delta_3\Delta_4}(z_i) = {} & \frac{z_{23}^{\Delta_1-\Delta_2-\Delta_3+\Delta_4} z_{24}^{\Delta_1-\Delta_2+\Delta_3-\Delta_4}}{z_{12}^{2\Delta_1} z_{34}^{\Delta_1-\Delta_2+\Delta_3+\Delta_4}} \cdot \\ & \cdot \int \frac{\mathrm{d}s\,\mathrm{d}t}{(2\pi i)^2} u^{\frac{s}{2}+\frac{\Delta_1-\Delta_2}{2}} v^{\frac{t}{2}-\frac{\Delta_1+\Delta_4}{2}} M(s,t)\bar{\Gamma}(s,t)\,,\end{aligned} \tag{A.63a}$$

where we have defined the block of Γ-function as

$$\begin{aligned}\bar{\Gamma}(s,t) \equiv {} & \Gamma\left(-\frac{s}{2}+\frac{\Delta_1+\Delta_2}{2}\right)\Gamma\left(-\frac{s}{2}+\frac{\Delta_3+\Delta_4}{2}\right)\Gamma\left(-\frac{t}{2}+\frac{\Delta_1+\Delta_4}{2}\right) \\ & \times\Gamma\left(-\frac{t}{2}+\frac{\Delta_2+\Delta_3}{2}\right)\Gamma\left(\frac{s+t}{2}-\frac{\Delta_1+\Delta_3}{2}\right)\Gamma\left(\frac{s+t}{2}-\frac{\Delta_2+\Delta_4}{2}\right).\end{aligned} \tag{A.64a}$$

We focus on the case $d=2$, $\Delta_1=\Delta_2$, $\Delta_3=\Delta_4$ even if the generalisation can be performed for generic Δ_i. The generic term appear inside the correlators as

$$\mathcal{C}_{\Delta_1\Delta_1\Delta_3\Delta_3}(z_i) = \frac{1}{z_{12}^{2\Delta_1} z_{34}^{2\Delta_3}} \left[\mathcal{G}^{(p,q)}_{\Lambda_1\Lambda_2\Lambda_3\Lambda_4}(u,v) + \cdots\right]. \tag{A.65}$$

The logic to find the Mellin representation of this term inside the correlator is to put it in the same form of (A.63) and considering the function $M(s,t)$ as the unknown.

Concretely, we first notice that we are dealing with $\hat{D}$-function and so we have to rewrite it as a non-hatted D-function using (A.41) and (A.42). Then we use the Mellin representation of the D-function (A.59). The factor $u^p v^q$ in front of the D-function and the other factors coming from passing from $\hat{D}$ to D-function will shift the power of the terms $(z_{ij})^{-s_{ij}}$. Then we can extract the function by performing a change of variable in order to have the correct power as in (A.63) and the same block of Γ-functions.

The Mellin transform of the generic terms then reads

$$\begin{aligned}M^{(p,q)}_{\Lambda_1\Lambda_2\Lambda_3\Lambda_4}(s,t) = {} & \pi\mathcal{N}(\Lambda_i,\Delta_i)\left(-\frac{s}{2}+\Delta_1\right)_a\left(-\frac{s}{2}+\Delta_3\right)_b \\ & \times\left(-\frac{t}{2}+\frac{\Delta_1+\Delta_3}{2}\right)_c\left(-\frac{t}{2}+\frac{\Delta_1+\Delta_3}{2}\right)_d \\ & \times\left(\frac{s+t}{2}-\frac{\Delta_1+\Delta_3}{2}\right)_e\left(\frac{s+t}{2}-\frac{\Delta_1+\Delta_3}{2}\right)_f\,,\end{aligned} \tag{A.66a}$$

with

$$\begin{aligned} & a = p-\Delta_1, \quad b = p+\Delta_1-2\Delta_3+\frac{\Lambda_1+\Lambda_2-\Lambda_3-\Lambda_4}{2} \\ & c = q, \qquad d = q-\frac{\Lambda_1-\Lambda_2-\Lambda_3+\Lambda_4}{2} \\ & e = -p-q+\Delta_3+\Lambda_4, \quad f = -p-q+\Delta_3+\frac{\Lambda_1-\Lambda_2+\Lambda_3+\Lambda_4}{2}\,,\end{aligned} \tag{A.67a}$$

and with the normalisation factor given by

$$\mathcal{N}(\Lambda_i,\Delta_i) = \frac{\Gamma\left(-1+\Delta_1+\Delta_3+\frac{\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4}{2}\right)}{\Gamma(\Delta_1+\Lambda_1)\,\Gamma(\Delta_2+\Lambda_2)\,\Gamma(\Delta_3+\Lambda_3)\,\Gamma(\Delta_4+\Lambda_4)}\,, \tag{A.68}$$

and where the Pochammer symbol is defined as $(x)_a = \frac{\Gamma(x+a)}{\Gamma(a)}$.

5 WAVE EQUATION ON SUPERSTRATA

The goal of this appendix is to compute the linearized equation of motion of a traceless perturbation h_{ij} of the torus metric, i.e.

$$\begin{aligned}
\mathrm{d}s_{10}^2 &= \sqrt{\alpha}\,\mathrm{d}s_6^2 + X(\delta_{ij}+h_{ij})\mathrm{d}z^i\mathrm{d}z^j\,,\\
\mathrm{d}s_6^2 &= -\frac{2}{\sqrt{\mathcal{P}}}(\mathrm{d}v+\beta)\left[\mathrm{d}u+\omega+\frac{\mathcal{F}}{2}(\mathrm{d}v+\beta)\right]+\sqrt{\mathcal{P}}\,\mathrm{d}s_4^2\,,\\
X &= \sqrt{\frac{Z_1}{Z_2}}\,,\quad e^{2\bar{\phi}}=\frac{Z_1^2}{\mathcal{P}}\,,\quad \mathcal{P}=Z_1Z_2-Z_4^2\,,\quad \alpha=\frac{Z_1Z_2}{\mathcal{P}}\,,
\end{aligned} \tag{A.69}$$

while the gauge fields are

$$\begin{aligned}
H_3 = -\mathcal{D}\left(\frac{Z_4}{\mathcal{P}}\right)\wedge d\hat{u}\wedge d\hat{v} + \frac{Z_4}{\mathcal{P}}\,\mathcal{D}\beta\wedge d\hat{u}\\
+\left[\Theta_4-\frac{Z_4}{\mathcal{P}}\,\mathcal{D}\omega\right]\wedge d\hat{v}+\Xi_4\,,
\end{aligned} \tag{A.70a}$$

$$F_1 = \mathcal{D}\left(\frac{Z_4}{Z_1}\right)+\partial_v\left(\frac{Z_4}{Z_1}\right)d\hat{v}\,, \tag{A.70b}$$

$$\begin{aligned}
F_3 = -\left[\mathcal{D}\left(\frac{Z_2}{\mathcal{P}}\right)-\frac{Z_4}{Z_1}\,\mathcal{D}\left(\frac{Z_4}{\mathcal{P}}\right)\right]\wedge d\hat{u}\wedge d\hat{v}+\frac{1}{Z_1}\,\mathcal{D}\beta\wedge d\hat{u}\\
+\left[\left(\Theta_1-\frac{Z_4}{Z_1}\,\Theta_4\right)-\frac{1}{Z_1}\,\mathcal{D}\omega\right]\wedge d\hat{v}+\left[\Xi_1-\frac{Z_4}{Z_1}\,\Xi_4\right],
\end{aligned} \tag{A.70c}$$

$$\begin{aligned}
F_5 = \left[\mathcal{D}\left(\frac{Z_4}{Z_2}\right)+\partial_v\left(\frac{Z_4}{Z_2}\right)d\hat{v}\right]\wedge dz^1\wedge dz^2\wedge dz^3\wedge dz^4\\
-\left[\frac{Z_4}{\mathcal{P}}\,\Xi_1-\frac{Z_2}{\mathcal{P}}\,Z_2\right]\wedge d\hat{u}\wedge d\hat{v}+\Omega_4\wedge d\hat{v}\,,
\end{aligned} \tag{A.70d}$$

where we recall that $d\hat{u}=\mathrm{d}u+\omega$, $d\hat{v}=\mathrm{d}v+\beta$ and that

$$\mathcal{D}=\mathrm{d}_4-\beta\wedge\partial_v\,, \tag{A.71}$$

and where the object appearing here are defined as

$$\begin{aligned}
\Omega_4 &= \mathcal{D}x_3-\Theta_4\wedge\gamma_2+a_1\wedge\Xi_4\,,\\
\Theta_1 &= \mathcal{D}a_1+\dot{\gamma}_2\,, \qquad \Theta_4=\mathcal{D}a_4+\dot{\delta}_2\,,\\
\Xi_1 &= \mathcal{D}\gamma_2-a_1\wedge\mathcal{D}\beta\,,\quad \Xi_4=\mathcal{D}\delta_2-a_4\wedge\mathcal{D}\beta\,,
\end{aligned} \tag{A.72}$$

where the dot represent a derivative w.r.t. v, and have to satisfy the following differential equations

$$*_4\mathcal{D}Z_1=\Xi_1\,,\quad *_4\mathcal{D}Z_4=\Xi_4\,,\quad \Omega_4=Z_2^2 *_4\partial_v\left(\frac{Z_4}{Z_2}\right)\,, \tag{A.73a}$$

and

$$*_4\mathcal{D}\dot{Z}_1=\mathcal{D}\Theta_2\,,\quad \mathcal{D}*_4\mathcal{D}Z_1=-\Theta_2\wedge\mathcal{D}\beta\,,\quad \Theta_1=*_4\Theta_1\,, \tag{A.74a}$$

$$*_4\mathcal{D}\dot{Z}_2=\mathcal{D}\Theta_1\,,\quad \mathcal{D}*_4\mathcal{D}Z_2=-\Theta_1\wedge\mathcal{D}\beta\,,\quad \Theta_2=*_4\Theta_2\,, \tag{A.74b}$$

$$*_4\mathcal{D}\dot{Z}_4=\mathcal{D}\Theta_4\,,\quad \mathcal{D}*_4\mathcal{D}Z_4=-\Theta_4\wedge\mathcal{D}\beta\,,\quad \Theta_4=*_4\Theta_4\,, \tag{A.74c}$$

and

$$\mathcal{D}\omega+*_4\mathcal{D}\omega+\mathcal{F}\,\mathrm{d}\beta=Z_1\Theta_1+Z_2\Theta_2-2Z_4\Theta_4\,, \tag{A.75a}$$

$$\begin{aligned}
*_4\mathcal{D}*_4\left(\dot{\omega}-\frac{1}{2}\,\mathcal{D}\mathcal{F}\right) = \partial_v^2(Z_1Z_2-Z_4^2)-[\dot{Z}_1\dot{Z}_2-(\dot{Z}_4)^2]\\
-\frac{1}{2}*_4(\Theta_1\wedge\Theta_2-\Theta_4\wedge\Theta_4)\,.
\end{aligned} \tag{A.75b}$$

The relevant equation of type IIB we are interested in the Einstein equation

$$\begin{aligned} & e^{-2\phi}\left(R_{MN}+2\nabla_M\nabla_N\phi-\frac{1}{4}\,H_{MPQ}H_N{}^{PQ}\right) \\ & \quad +\frac{1}{4}\,g_{MN}\left(F_PF^P+\frac{1}{3!}\,F_{PQR}F^{PQR}\right) \\ & \quad -\frac{1}{2}F_MF_N-\frac{1}{2}\frac{1}{2!}F_{MPQ}F_N{}^{PQ}-\frac{1}{4}\frac{1}{4!}\,F_{MPQRS}F_N{}^{PQRS}=0\,. \end{aligned} \tag{A.76}$$

5.1 *Zehnbein and Spin connection*

We may rewrite the 10-dimensional metric for the $\frac{1}{8}$-BPS geometries

$$g_{MN}\mathrm{d}x^M\mathrm{d}x^N=\sqrt{\alpha}\,g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu+X\delta_{ij}\mathrm{d}z^i\mathrm{d}z^j\,, \tag{A.77}$$

where we split $x^M=(x^\mu,z^i)=(x^{u_i},x^a,z^i)$ with $x^{u_i}=(u,v)$, where $X=\sqrt{\frac{Z_1}{Z_2}}$ and $\alpha=\frac{Z_1Z_2}{\mathcal{P}}$, and

$$g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu=G_{u_iu_j}(\mathrm{d}x^{u_i}+A^{u_i})(\mathrm{d}x^{u_j}+A^{u_j})+\sqrt{\mathcal{P}}\,q_{ab}\mathrm{d}x^a\mathrm{d}x^b\,, \tag{A.78}$$

and where

$$A^{u_i}=A^{u_i}_a\mathrm{d}x^a\,,\quad A^u=\omega\,,\quad A^v=\beta\,,\quad G_{u_iu_j}=\begin{pmatrix}0 & -\frac{1}{\sqrt{\mathcal{P}}}\\ -\frac{1}{\sqrt{\mathcal{P}}} & -\frac{\mathcal{F}}{2\sqrt{\mathcal{P}}}\end{pmatrix}, \tag{A.79}$$

so that[5]

$$\begin{aligned} g_{\mu\nu}&=\begin{pmatrix}G_{u_iu_j} & G_{u_iu_j}A^{u_i}_a\\ A^{u_i}_aG_{u_iu_j} & \sqrt{\mathcal{P}}\,q_{ab}+G_{u_iu_j}A^{u_i}_a\,A^{u_j}_b\end{pmatrix},\\ g^{\mu\nu}&=\begin{pmatrix}G^{u_iu_j}+\frac{1}{\sqrt{\mathcal{P}}}\,q^{ab}A^{u_i}_aA^{u_j}_b & -\frac{1}{\sqrt{\mathcal{P}}}\,q^{ab}A^{u_i}_b\\ -\frac{1}{\sqrt{\mathcal{P}}}\,A^{u_i}_aq^{ab} & \frac{1}{\sqrt{\mathcal{P}}}\,q^{ab}\end{pmatrix}. \end{aligned} \tag{A.81}$$

We can now introduce the 10-dimensional vielbein, i.e. the zehnbein, e^A as

$$g_{MN}\mathrm{d}x^M\mathrm{d}x^N=\eta_{AB}e^Ae^B\,, \tag{A.82}$$

splitting them in a set of sechsbein e^α and vierbein e^i; We can define the sechsbein as

$$e^+=\frac{1}{\sqrt{\mathcal{P}}}(\mathrm{d}v+\beta),\quad e^-=\frac{1}{\sqrt{\mathcal{P}}}\left[\mathrm{d}u+\frac{\mathcal{F}}{2}\,\mathrm{d}v+\left(\omega+\frac{\mathcal{F}}{2}\,\beta\right)\right],\quad e^I=\mathcal{P}^{1/4}\,\tilde{e}^I\,, \tag{A.83}$$

where we split $\mu=u,v,a$ and $\alpha=+,-,I$, so

$$\eta_{\alpha\beta}e^\alpha e^\beta=2\eta_{+-}e^+e^-+\sqrt{\mathcal{P}}\,\delta_{IJ}\tilde{e}^I\tilde{e}^J=\mathrm{d}s_6^2\,. \tag{A.84}$$

This allows us to define the ten dimensional metric as

$$\mathrm{d}s_{10}^2=\sqrt{\alpha}\left[2\eta_{+-}e^+e^-+\sqrt{\mathcal{P}}\,\delta_{IJ}\tilde{e}^I\tilde{e}^J\right]+X\delta_{ij}e^ie^j\,. \tag{A.85}$$

5 Notice that

$$G_{u_iu_j}=\begin{pmatrix}0 & -\frac{1}{\sqrt{\mathcal{P}}}\\ -\frac{1}{\sqrt{\mathcal{P}}} & -\frac{\mathcal{F}}{2\sqrt{\mathcal{P}}}\end{pmatrix},\quad G^{u_iu_j}=\begin{pmatrix}\frac{1}{2}\sqrt{\mathcal{P}}\mathcal{F} & -\sqrt{\mathcal{P}}\\ -\sqrt{\mathcal{P}} & 0\end{pmatrix}. \tag{A.80}$$

With this definitions, using that

$$e_A^M = g^{MN}\eta_{AB}e_M^A\,, \tag{A.86}$$

we may compute

$$\begin{aligned}
e_+ &= \left(1+\frac{\mathcal{F}}{2\sqrt{\mathcal{P}}}\,\beta\wedge *_4\beta\right)\partial_v\,,\\
e_- &= \left[1-\frac{1}{\mathcal{P}}\,\beta\wedge *_4(\omega-\beta)\right]\partial_v - \left[\frac{\mathcal{F}}{2}-\frac{1}{\mathcal{P}}\,\omega\wedge *_4(\omega-\beta)\right]\partial_u\,,\\
e_I &= \delta_{IJ}\left[-q^{cd}\omega_c\tilde{e}_d^J\partial_v - q^{cd}\beta_c\tilde{e}_d^J\partial_u + q^{cd}\tilde{e}_c^J\partial_d\right] = -\tilde{e}_I^a\omega_a\partial_v - \tilde{e}_I^a\beta_a\partial_u + \tilde{e}_I^a\partial_a\,,
\end{aligned} \tag{A.87}$$

where the $*_4$ is the Hodge star operator on the 4-dimensional flat manifold whose metric is $\delta_{IJ}\tilde{e}^I\tilde{e}^J$.

We may also refer to $g^S_{\mu\nu}$ as the string-frame metric in 6D, and with $g^E_{\mu\nu}$ as the Einstein-frame metric, so that

$$\mathrm{d}s_S^2 = \sqrt{\alpha}\,\mathrm{d}s_E^2 = \sqrt{\alpha}\left\{-\frac{2}{\sqrt{\mathcal{P}}}(\mathrm{d}v+\beta)\left[\mathrm{d}u+\omega+\frac{\mathcal{F}}{2}(\mathrm{d}v+\beta)\right]+\sqrt{\mathcal{P}}\,\mathrm{d}s_4^2\right\}. \tag{A.88}$$

5.2 *The linearized Einstein equations*

5.2.1 *The linearized Ricci*

We want to compute the linearized Ricci, via

$$\delta R_{ij} = \bar{\nabla}_A\delta\Gamma^A_{ij} - \bar{\nabla}_i\delta\Gamma^A_{Aj}\,, \tag{A.89}$$

where we refer to the background objects with an overbar, and where

$$\delta\Gamma^A_{MN} = \frac{1}{2}\,\bar{g}^{AB}\left(\bar{\nabla}_M h_{NB} + \bar{\nabla}_N h_{MB} - \bar{\nabla}_B h_{MN}\right), \tag{A.90}$$

so that the only non-vanishing perturbations are[6]

$$\begin{aligned}
\delta\Gamma^\mu_{ij} &= -\frac{1}{2}\,\bar{g}^{\mu\nu}\partial_\nu\left(Xh_{ij}\right)\\
\delta\Gamma^i_{\mu j} &= +\frac{1}{2}\,X^{-1}\,\delta^{ik}\left[\partial_\mu\left(Xh_{jk}\right) - X(\partial_\mu\log X)h_{jk}\right]\,.
\end{aligned} \tag{A.92}$$

We then get in the string frame

$$\delta R_{ij} = -\frac{1}{2}\left[\bar{\Box}_S\left(Xh_{ij}\right) + X\bar{g}_S^{\mu\nu}(\partial_\mu\log X)(\partial_\nu\log X)h_{ij}\right]. \tag{A.93}$$

We may express it in Einstein frame using the general formula that under Weyl transformations of the metric $\tilde{g}_{AB} = e^{2\varphi}g_{AB}$,

$$\Box_{\tilde{g}} = e^{-2\varphi}\Box_g + (D-2)e^{-2\varphi}g^{AB}(\partial_A\varphi)\partial_B\,. \tag{A.94}$$

6 The only non-vanishing background Christoffels are, other that the six-dimensional string-frame $\bar{\Gamma}^\mu_{\nu\rho}$ ones, whose explicit form is not relevant for now,

$$\begin{aligned}
\bar{\Gamma}^i_{\mu j} &= +\frac{1}{2}(\partial_\mu\log X)\delta^i{}_j\\
\bar{\Gamma}^\mu_{ij} &= -\frac{1}{2}(\partial^\mu X)\delta_{ij}\,.
\end{aligned} \tag{A.91}$$

gives

$$\Box_S h_{ij} = \alpha^{-1/2}\,\Box_E h_{ij} + \alpha^{-1/2}\, g_E^{ab}(\partial_a \log\alpha)\partial_b h_{ij}\,. \tag{A.95}$$

So we get

$$\begin{aligned}\delta R_{ij} = -\frac{1}{2}\,\alpha^{-1/2}\Big[\bar{\Box}_E\,(X h_{ij}) + \bar{g}_E^{\mu\nu}(\partial_\mu \log\alpha)(\partial_\nu \log(X h_{ij}))\\ + X\bar{g}_E^{\mu\nu}(\partial_\mu \log X)(\partial_\nu \log X) h_{ij}\Big]\,.\end{aligned} \tag{A.96}$$

It is also easy to see that $\delta R_{\mu\nu} = 0 = \delta R_{\mu j}$, so that the variation of the Ricci is only along the torus.

5.2.2 *The scalar variation*

We now compute

$$\begin{aligned}\delta\left(\nabla_M\nabla_N\bar{\phi}\right) &= -\delta\Gamma^A_{MN}\,\partial_A\bar{\phi} = -\delta\Gamma^\mu_{MN}\,\partial_\mu\bar{\phi} = -\delta^i_M\delta^j_N\delta\Gamma^\mu_{ij}\,\partial_\mu\bar{\phi}\\ &= +\frac{1}{2}\,\delta^i_M\delta^j_N\,\bar{g}_S^{\mu\nu}\partial_\mu\left(X h_{ij}\right)\,\partial_\nu\bar{\phi}\\ &= +\frac{1}{2}\,\delta^i_M\delta^j_N\,\bar{g}_S^{\mu\nu}(\partial_\mu \log X)\partial_\nu\left(X h_{ij}\right)\,.\end{aligned} \tag{A.97}$$

so that the variation is again only on the torus.

5.2.3 *The linearized Einstein equation: part I*

If we now sum the two together as they appear in the variation of the Einstein equation, i.e. as

$$\delta R_{MN} + 2\delta\left(\nabla_M\nabla_N\bar{\phi}\right), \tag{A.98}$$

using the fact that

$$\bar{\phi} = \log\frac{Z_1}{\sqrt{\mathcal{P}}} = \log\left(\sqrt{\frac{Z_1}{Z_2}}\sqrt{\frac{Z_1 Z_2}{\mathcal{P}}}\right) = \log(X\sqrt{\alpha}) = \log X + \frac{1}{2}\log\alpha\,, \tag{A.99}$$

so

$$\bar{g}_S^{\mu\nu}(\partial_\mu\bar{\phi})\partial_\nu\left(X h_{ij}\right) = \alpha^{-1/2}\bar{g}_E^{\mu\nu}\left(\frac{1}{2}\partial_\mu\log\alpha + \partial_\mu\log X\right)\partial_\nu\left(X h_{ij}\right), \tag{A.100}$$

and then, after some manipulation, we get, for the linearized Einstein equations (A.76)

$$\begin{aligned}0 = \bar{\Box}_E h_{ij} - \frac{1}{4!}\,X^2\alpha^{3/2}\,h^{k\ell}\bar{F}_{ikPQR}\bar{F}_{j\ell}{}^{PQR}\\ + h_{ij}\left[X^{-1}\bar{\Box}_E X - \bar{g}_E^{\mu\nu}(\partial_\mu\log X)(\partial_\nu\log X) - \frac{1}{2}\,X^2\alpha^{3/2}\left(\bar{F}_1^2 + \frac{1}{3!}\,\bar{F}_3^2\right)\right],\end{aligned} \tag{A.101}$$

where we have used that $\bar{F}_1$, $\bar{F}_3$ and $\bar{H}_3$ have no legs on $\mathbb{T}^4$, so the terms F_iF_j, $F_{iPQ}F_j{}^{PQ}$ and $H_{iPQ}H_j{}^{PQ}$ are all zero, while

$$\delta\left(F_{MPQRS}F_N{}^{PQRS}\right) = 2\delta^i_M\delta^j_N h^{k\ell}F_{ikPQR}F_{j\ell}{}^{PQR}\,. \tag{A.102}$$

5.2.4 *The* $\bar{F}_1$ *term*

We now compute, using the formulae of app. A.1.2

$$\begin{aligned}\bar{F}_1^2 &= *_{10}(\bar{F}_1 \wedge *_{10}\bar{F}_1) \\ &= \alpha X^2 *_{10}\left[\mathcal{D}\left(\frac{Z_4}{Z_1}\right) \wedge *_4\mathcal{D}\left(\frac{Z_4}{Z_1}\right) \wedge d\hat{u} \wedge d\hat{v} \wedge \mathrm{vol}_{\mathbb{T}^4}\right] \\ &= \alpha^{-1/2}\mathcal{P}^{-1/2} *_4\left[\mathcal{D}\left(\frac{Z_4}{Z_1}\right) \wedge *_4\mathcal{D}\left(\frac{Z_4}{Z_1}\right)\right] .\end{aligned} \tag{A.103}$$

since

$$\begin{aligned}*_{10}\bar{F}_1 &= \alpha X^2 *_E \bar{F}_1 \wedge \mathrm{vol}_{\mathbb{T}^4} \\ &= \alpha X^2\left[*_4\mathcal{D}\left(\frac{Z_4}{Z_1}\right) \wedge d\hat{u} \wedge d\hat{v} - \mathcal{P} *_4 \partial_v\left(\frac{Z_4}{Z_1}\right) \wedge d\hat{v}\right] \wedge \mathrm{vol}_{\mathbb{T}^4} .\end{aligned} \tag{A.104}$$

5.2.5 *The* $\bar{F}_3$ *term*

Here we have, again using the formulae of app. A.1.2 that

$$\begin{aligned}*_{10}\bar{F}_3 &= X^2 *_E \bar{F}_3 \wedge \mathrm{vol}_{\mathbb{T}^4} \\ &= \Bigg\{\mathcal{P} *_4\left[\frac{Z_1}{Z_2}\mathcal{D}\left(\frac{Z_2}{\mathcal{P}}\right) - \frac{Z_4}{Z_2}\mathcal{D}\left(\frac{Z_4}{\mathcal{P}}\right)\right] - \frac{1}{Z_1} *_4 \mathcal{D}\beta \wedge d\hat{u} \\ &\quad + \left[\left(\frac{Z_1}{Z_2} *_4 \Theta_1 - \frac{Z_4}{Z_2} *_4 \Theta_4\right) - \frac{1}{Z_2}\left(*_4\mathcal{D}\omega + \mathcal{F} *_4 \mathcal{D}\beta\right)\right] \wedge d\hat{v} \\ &\quad + \frac{1}{\mathcal{P}}\left[\frac{Z_1}{Z_2} *_4 \Xi_1 - \frac{Z_4}{Z_2} *_4 \Xi_4\right] \wedge d\hat{u} \wedge d\hat{v}\Bigg\} \wedge \mathrm{vol}_{\mathbb{T}^4} ,\end{aligned} \tag{A.105}$$

so that, using the self-duality conditions for the Θ_I and $*_4\mathcal{D}\beta = d\beta$, we get

$$\begin{aligned}\bar{F}_3^2 &= *_{10}\left(\bar{F}_3 \wedge *_{10}\bar{F}_3\right) \\ &= *_{10}\Big\{-\mathcal{P}X\left[\mathcal{D}\left(\frac{Z_2}{\mathcal{P}}\right) - \frac{Z_4}{Z_1}\mathcal{D}\left(\frac{Z_4}{\mathcal{P}}\right)\right] \wedge *_4\left[\mathcal{D}\left(\frac{Z_2}{\mathcal{P}}\right) - \frac{Z_4}{Z_1}\mathcal{D}\left(\frac{Z_4}{\mathcal{P}}\right)\right] \wedge d\hat{u} \wedge d\hat{v} \\ &\quad + \frac{1}{Z_1 Z_2}d\beta \wedge \left[\mathcal{D}\omega + *_4\mathcal{D}\omega + \mathcal{F}d\beta - (2Z_1\Theta_1 - 2Z_4\Theta_4)\right] \wedge d\hat{u} \wedge d\hat{v} \\ &\quad + \frac{X}{\mathcal{P}}\left[\Xi_1 - \frac{Z_4}{Z_1}\Xi_4\right] \wedge *_4\left[\Xi_1 - \frac{Z_4}{Z_1}\Xi_4\right] \wedge d\hat{u} \wedge d\hat{v}\Big\} \wedge \mathrm{vol}_{\mathbb{T}^4} .\end{aligned} \tag{A.106}$$

Please notice that, using eq. (A.73, A.74) we can reconstruct eq. (A.75) as

$$\begin{aligned}\bar{F}_3^2 &= *_{10}\left(\bar{F}_3 \wedge *_{10}\bar{F}_3\right) \\ &= *_{10}\Big\{-\mathcal{P}X\left[\mathcal{D}\left(\frac{Z_2}{\mathcal{P}}\right) - \frac{Z_4}{Z_1}\mathcal{D}\left(\frac{Z_4}{\mathcal{P}}\right)\right] \wedge *_4\left[\mathcal{D}\left(\frac{Z_2}{\mathcal{P}}\right) - \frac{Z_4}{Z_1}\mathcal{D}\left(\frac{Z_4}{\mathcal{P}}\right)\right] \wedge d\hat{u} \wedge d\hat{v} \\ &\quad + \frac{1}{Z_1 Z_2}d\beta \wedge \left[\mathcal{D}\omega + *_4\mathcal{D}\omega + \mathcal{F}d\beta - (Z_1\Theta_1 + Z_2\Theta_2 - 2Z_4\Theta_4)\right] \wedge d\hat{u} \wedge d\hat{v} \\ &\quad + \frac{1}{Z_1 Z_2}\left[Z_1\mathcal{D} *_4 \mathcal{D}Z_2 - Z_2\mathcal{D} *_4 \mathcal{D}Z_1\right] \wedge d\hat{u} \wedge d\hat{v} \\ &\quad + \frac{X}{\mathcal{P}}\left[\mathcal{D}Z_1 - \frac{Z_4}{Z_1}\mathcal{D}Z_4\right] \wedge *_4\left[\mathcal{D}Z_1 - \frac{Z_4}{Z_1}\mathcal{D}Z_1\right] \wedge d\hat{u} \wedge d\hat{v}\Big\} \wedge \mathrm{vol}_{\mathbb{T}^4} \\ &= *_{10}\Big\{-\mathcal{P}X\left[\mathcal{D}\left(\frac{Z_2}{\mathcal{P}}\right) - \frac{Z_4}{Z_1}\mathcal{D}\left(\frac{Z_4}{\mathcal{P}}\right)\right] \wedge *_4\left[\mathcal{D}\left(\frac{Z_2}{\mathcal{P}}\right) - \frac{Z_4}{Z_1}\mathcal{D}\left(\frac{Z_4}{\mathcal{P}}\right)\right] \wedge d\hat{u} \wedge d\hat{v} \\ &\quad + \frac{1}{Z_1 Z_2}\left[Z_1\mathcal{D} *_4 \mathcal{D}Z_2 - Z_2\mathcal{D} *_4 \mathcal{D}Z_1\right] \wedge d\hat{u} \wedge d\hat{v} \\ &\quad + \frac{X}{\mathcal{P}}\left[\mathcal{D}Z_1 - \frac{Z_4}{Z_1}\mathcal{D}Z_4\right] \wedge *_4\left[\mathcal{D}Z_1 - \frac{Z_4}{Z_1}\mathcal{D}Z_1\right] \wedge d\hat{u} \wedge d\hat{v}\Big\} \wedge \mathrm{vol}_{\mathbb{T}^4} .\end{aligned}$$

(A.107)

where we have also used that $*_4 *_4 \lambda_1 = -\lambda_1$, for every-one form λ_1. Using again the formulae of app. A.1.2

$$\begin{aligned}\bar{F}_3^2 = X^{-2}\alpha^{-3/2}\mathcal{P}^{-1/2} *_4 \Bigg\{ &\frac{X}{\mathcal{P}}\left[\mathcal{D}Z_1 - \frac{Z_4}{Z_1}\mathcal{D}Z_4\right] \wedge *_4 \left[\mathcal{D}Z_1 - \frac{Z_4}{Z_1}\mathcal{D}Z_1\right] \\ &+ \frac{1}{Z_1 Z_2}\left[Z_1 \mathcal{D} *_4 \mathcal{D}Z_2 - Z_2 \mathcal{D} *_4 \mathcal{D}Z_1\right] \\ &- \mathcal{P}X\left[\mathcal{D}\left(\frac{Z_2}{\mathcal{P}}\right) - \frac{Z_4}{Z_1}\mathcal{D}\left(\frac{Z_4}{\mathcal{P}}\right)\right] \wedge *_4 \left[\mathcal{D}\left(\frac{Z_2}{\mathcal{P}}\right) - \frac{Z_4}{Z_1}\mathcal{D}\left(\frac{Z_4}{\mathcal{P}}\right)\right]\Bigg\}.\end{aligned} \tag{A.108}$$

5.2.6 *The $\bar{F}_5$ term*

Using the notation of chap. 3, we see that the only relevant part of the $\bar{F}_5$ for us is the one with legs on the torus, i.e.

$$\bar{F}_5 \supseteq \left[\mathcal{D}\left(\frac{Z_4}{Z_2}\right) + \partial_v\left(\frac{Z_4}{Z_2}\right)\right] \wedge \mathrm{vol}_{\mathbb{T}^4} \equiv \mathfrak{F}_5^{(1)} \wedge \mathrm{vol}_{\mathbb{T}^4}\,, \tag{A.109}$$

so that $F_{ijk\ell\mu}dx^\mu = \mathfrak{F}_5^{(1)}$ and then

$$\begin{aligned}F_{ikPQR}F_{j\ell}{}^{PQR} &= \varepsilon_{ikmn}\varepsilon_{j\ell}{}^{mn}\mathfrak{F}_\mu\mathfrak{F}^\mu \\ &= 4\alpha^{-1/2}\mathcal{P}^{-1/2}(\delta_{ij}\delta_{k\ell} - \delta_{ik}\delta_{j\ell}) *_4 \left[\mathcal{D}\left(\frac{Z_4}{Z_2}\right) \wedge *_4\mathcal{D}\left(\frac{Z_4}{Z_2}\right)\right],\end{aligned} \tag{A.110}$$

so that, since $h^{ij}\delta_{ij} = 0$,

$$h^{k\ell}F_{ikPQR}F_{j\ell}{}^{PQR} = 4\alpha^{-1/2}\mathcal{P}^{-1/2} *_4 \left[\mathcal{D}\left(\frac{Z_4}{Z_2}\right) \wedge *_4\mathcal{D}\left(\frac{Z_4}{Z_2}\right)\right] h_{ij}\,. \tag{A.111}$$

5.2.7 *The linearized Einstein equation: part II*

We now restart from eq. (A.101), and we plug in it eqs. (A.103, A.108, A.111) using that

$$\begin{aligned}\bar{\Box}_E X &= \mathcal{P}^{-1/2} *_4 (\mathcal{D} *_4 \mathcal{D}X) \\ &= \frac{1}{2}\mathcal{P}^{-1/2} *_4 \left[\frac{X}{Z_1 Z_2}(Z_2 \mathcal{D} *_4 \mathcal{D}Z_1 - Z_1 \mathcal{D} *_4 \mathcal{D}Z_2)\right. \\ &\qquad \left.+ \frac{1}{2}\mathcal{D}\left(\frac{X}{Z_1}\right) \wedge *_4\mathcal{D}Z_1 - \frac{1}{2}\mathcal{D}\left(\frac{X}{Z_2}\right) \wedge *_4\mathcal{D}Z_2\right].\end{aligned} \tag{A.112}$$

After careful computations, one gets

$$\bar{\Box}_E h_{ij} = 0\,. \tag{A.113}$$

5.3 *A short derivation of the linearized equation using dualities*

There is a very fast way to prove eq. (A.113): we simply use that the geometries (2.113) and (2.187) are related by the action of dualities, as explained in sec. 2.2.5. Now, it is easy to see that if we turn on a linear perturbation $c_2 = c\,\widetilde{\omega}_2$, where e.g. $\widetilde{\omega}_2 = \mathrm{d}z^1 \wedge \mathrm{d}z^3$, is a 2-form on the flat Torus on the geometry (2.187), so that

$$\widetilde{\omega}_2 \wedge \omega_2 = 0\,, \quad \omega_2 = \mathrm{d}z^1 \wedge \mathrm{d}z^2 - \mathrm{d}z^3 \wedge \mathrm{d}z^4\,, \tag{A.114}$$

the only type IIB equation (2.108) that is modified is

$$\mathrm{d} * F_3 + H_3 \wedge F_5 = 0 \Rightarrow \mathrm{d} *_6 \mathrm{d}c = 0\,, \tag{A.115}$$

while the others are trivially satisfied due to eq. (A.114). This meas that this kind of perturbation is a minimally-coupled massless scalar on the geometry (2.187). If we now perform the dualities that relates (2.187) to (2.113), i.e. $ST_{z^2}T_{z^1}S$, we obtain

$$\begin{aligned}
&\mathrm{d}s^2_{10} = \sqrt{\frac{Z_1 Z_2}{\mathcal{P}}}\,\mathrm{d}s^2_6 + \sqrt{\frac{Z_1}{Z_2}}\left(\mathrm{d}s^2_{\mathbb{T}^4} - c\,\mathrm{d}z^1\mathrm{d}z^2\right),\\
&e^{2\phi} = \frac{Z_1^2}{\mathcal{P}}\,, \quad B_2 = \bar{B}_2\,,\\
&C_0 = \frac{Z_4}{Z_1}\,, \quad C_2 = \bar{C}_2\,, \quad C_4 = \bar{C}_4 + \frac{Z_4}{Z_2}\,\mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3 \wedge \mathrm{d}z^4\,.
\end{aligned} \tag{A.116}$$

If we have chosen a different $\widetilde{\omega}_2$, let us say $\widetilde{\omega}_2 = \mathrm{d}z^1 \wedge \mathrm{d}z^2 + \mathrm{d}z^3 \wedge \mathrm{d}z^4$, we have instead turned on a modulus for the B_2:

$$\begin{aligned}
&\mathrm{d}s^2_{10} = \sqrt{\frac{Z_1 Z_2}{\mathcal{P}}}\,\mathrm{d}s^2_6 + \sqrt{\frac{Z_1}{Z_2}}\,\mathrm{d}s^2_{\mathbb{T}^4}, \quad e^{2\phi} = \frac{Z_1^2}{\mathcal{P}}\,,\\
&B_2 = \bar{B}_2 + \frac{Z_4}{Z_2}\,c\,\widetilde{\omega}_2\,, \quad \widetilde{\omega}_2 = \mathrm{d}z^1 \wedge \mathrm{d}z^2 + \mathrm{d}z^3 \wedge \mathrm{d}z^4\,,\\
&C_0 = \frac{Z_4}{Z_1}\,, \quad C_2 = \bar{C}_2 + c\,\widetilde{\omega}_2\,,\\
&C_4 = \bar{C}_4 + \frac{Z_4}{Z_2}\,\mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3 \wedge \mathrm{d}z^4 + \frac{Z_4}{Z_2}\,\bar{C}_2 \wedge c\,\widetilde{\omega}_2\,.
\end{aligned} \tag{A.117}$$

This is a strong indication (but not a proof) that all the 20 moduli are minimally coupled massless scalar; one simply need to find the appropriate frame, in which the equation of motion is trivial, and then find the appropriate set of dualities to move to the chosen background geometry.

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