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CICLO XXXII

Minimal approximations for cotorsion pairs generated by modules of projective dimension at most one over commutative rings

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Abstract

In this thesis we study cotorsion pairs $(\mathcal{A}, \mathcal{B})$ generated by classes of R-modules of projective dimension at most one. We are interested in when these cotorsion pairs provide covers or envelopes over commutative rings. More precisely, we investigate Enochs' Conjecture in this setting. That is, for a class $\mathcal{A}$ contained in the class of modules of projective dimension at most one, denoted $\mathcal{P}_1(R)$, we investigate the question of whether $\mathcal{A}$ is covering necessarily implies that $\mathcal{A}$ is closed under direct limits. Additionally, under certain restrictions we characterise the rings which satisfy this property. To this end, there were two cases to consider: when the cotorsion pair is of finite type and when it is not of finite type.

For the case that the cotorsion pair $(\mathcal{P}_1(R), \mathcal{P}_1(R)^{\perp})$ is not (necessarily) of finite type, we show that over a semihereditary ring R, if $\mathcal{P}_1(R)$ is covering it must be closed under direct limits. This gives an example of a cotorsion pair not of finite type which satisfies Enochs' Conjecture.

The next part of the thesis is dedicated toward cotorsion pairs of finite type, specifically the 1-tilting cotorsion pairs over commutative rings. We rely heavily on work of Hrbek who characterises these cotorsion pairs over commutative rings, as well as work of Positselski and Bazzoni-Positselski in their work on contramodules.

We consider the case of a 1-tilting cotorsion pair $(\mathcal{A}, \mathcal{T})$ over a commutative ring with an associated Gabriel topology $\mathcal{G}$, and begin by investigating when $\mathcal{T}$ is an enveloping class. We find that if $\mathcal{T}$ is enveloping, then the associated Gabriel topology must arise from a perfect localisation. That is, $\mathcal{G}$ must arise from a flat ring epimorphism denoted $R \to R_{\mathcal{G}}$. Furthermore, if $\mathcal{G}$ arises from a perfect localisation, $\mathcal{T}$ is enveloping in Mod-R if and only if $\operatorname{p.dim} R_{\mathcal{G}} \leq 1$ and R/J is a perfect ring for every ideal $J \in \mathcal{G}$ if and only if $\operatorname{p.dim} R_{\mathcal{G}} \leq 1$ and the topological ring $\operatorname{End}(R_{\mathcal{G}}/R)$ is pro-perfect. Next, we consider the case that $\mathcal{A}$ is a covering class, and we prove that $\mathcal{A}$ is covering in Mod-R if and only if $\operatorname{p.dim} R_{\mathcal{G}} \leq 1$ and both the localisation $R_{\mathcal{G}}$ is a perfect ring and R/J is a perfect ring for every $J \in \mathcal{G}$.

Additionally, we study general cotorsion pairs, as well as conditions for an approximation to be a minimal approximation. Moreover, we consider a hereditary cotorsion pair and show that if it provides covers it must provide envelopes.

Riassunto

In questa tesi studiamo le coppie di cotorsione $(\mathcal{A}, \mathcal{B})$ generate da classi di R-moduli di dimensione proiettiva al più uno. Siamo interessati nel caso in cui queste coppie di cotorsione ammettano ricoprimenti o inviluppi su anelli commutativi. Più precisamente, indaghiamo la congettura di Enochs per $\mathcal{A}$. Cioè, per $\mathcal{A}$ contenuta nella classe $\mathcal{P}_1(R)$, che denota la classe di R-moduli di dimensione proiettiva al più uno, cerchiamo di capire se per $\mathcal{A}$ una classe ricoprente allora necessariamente implica che $\mathcal{A}$ è chiusa per limiti diretti. In più, con certe restrizioni, descriviamo gli anelli che soddisfano questa proprietà. Ci sono due casi da considerare: il caso di coppia di cotorsione di tipo finito e il caso non di tipo finito.

Quando la coppia di cotorsione non è (necessariamente) di tipo finito, dimostriamo che per un anello commutativo semiereditario R, se $\mathcal{P}_1(R)$ è una classe ricoprente, deve essere chiusa per limiti diretti. Questo ci da un esempio di una coppia di cotorsione che non è di tipo finito che soddisfa la congettura di Enochs.

Successivamente, analizziamo le coppie di cotorsione di tipo finito. Specificamente, le coppie di cotorsione 1-tilting su anelli commutativi. A questo scopo sono indispensabili il lavoro di Hrbek, che caratterizza tali coppie di cotorsione su anelli commutativi, e il lavoro di Positselski e Bazzoni-Positselski nel loro lavoro sui contramoduli.

Consideriamo il caso di una coppia di cotorsione 1-tilting $(\mathcal{A}, \mathcal{T})$ su un anello commutativo con una topologia di Gabriel associata $\mathcal{G}$, e studiamo quando $(\mathcal{A}, \mathcal{T})$ ammette inviluppi. Troviamo che se $\mathcal{T}$ ammette inviluppi, $\mathcal{G}$ è una topologia di Gabriel perfetta. Cioè, $\mathcal{G}$ viene da un epimorfismo piatto di anelli $R \to R_{\mathcal{G}}$ dove $R_{\mathcal{G}}$ è la localizzazione di R rispetto a $\mathcal{G}$. Inoltre, se $\mathcal{G}$ è una topologia di Gabriel perfetta, $\mathcal{T}$ ammette inviluppi se e solo se $\operatorname{p.dim} R_{\mathcal{G}} \leq 1$ e R/J è un anello perfetto per tutti gli ideali $J \in \mathcal{G}$ se e solo se $\operatorname{p.dim} R_{\mathcal{G}} \leq 1$ e l'anello topologico $\operatorname{End}(R_{\mathcal{G}}/R)$ è pro-perfetto. Poi consideriamo il caso in cui $\mathcal{A}$ è ricoprente. Dimostriamo che $\mathcal{A}$ è ricoprente in Mod-R se e solo se $\operatorname{p.dim} R_{\mathcal{G}} \leq 1$ e $R_{\mathcal{G}}$ è un anello perfetto e R/J è perfetto per ogni $J \in \mathcal{G}$.

In aggiunta, studiamo coppie di cotorsione in generale e studiamo condizioni sufficienti affinchè una approssimazione sia minimale. Inoltre, consideriamo una coppia di cotorsione ereditaria e dimostriamo che se ammette ricoprimenti deve ammettere inviluppi.

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Introduction

The classification problem for classes of modules over arbitrary rings is in general very difficult, perhaps even hopeless. Nonetheless, approximation theory was developed as a tool to approximate arbitrary modules by modules in classes where the classification is more manageable. Left and right approximations by an arbitrary class were studied in the case of modules over finite dimensional modules by work of Auslander, Reiten, and Smalø and independently by Enochs and Xu for modules over arbitrary rings using the terminology of preenvelopes and precovers.

An important problem in approximation theory is when minimal approximations, that is covers or envelopes, over certain classes exist. In other words, for a certain class $\mathcal{C}$, the aim is to characterise the rings over which every module has a minimal approximation provided by $\mathcal{C}$ and furthermore to characterise the class $\mathcal{C}$ itself. The most famous positive result of when minimal approximations exist is the construction of an injective envelope for every module [20]. Instead, Bass proved in [8] that projective covers rarely exist. In his paper, Bass introduced and characterised the class of perfect rings which are exactly the rings over which every module admits a projective cover. These results motivated the the study of minimal approximations for an arbitrary class $\mathcal{C}$.

Among the many characterisations of perfect rings, the most important from the homological point of view is the closure under direct limits of the class of projective modules. In fact, a famous theorem of Enochs says that for a well-behaved class $\mathcal{C}$ in Mod-R (that is $\mathcal{C}$ is closed under direct summands and isomorphisms), if $\mathcal{C}$ is closed under direct limits, then any module that has a $\mathcal{C}$-precover has a $\mathcal{C}$-cover [22]. The converse problem, that is if $\mathcal{C}$ is a covering class then it is closed under direct limits, is still an open problem which is known as Enochs' Conjecture.

Interestingly, one has a similar conclusion under slightly stronger assumptions for the dual notion of $\mathcal{C}$-envelopes. That is, if $\mathcal{C}$ is a class in Mod-R that is closed under direct limits and extensions, then if a module M has a $\mathcal{C}^{\perp_1}$-preenvelope which is a monomorphism with cokernel in $\mathcal{C}$, then M has a $\mathcal{C}^{\perp_1}$-envelope. A class $\mathcal{C}$ of modules is called covering, respectively enveloping, if every module admits a $\mathcal{C}$-cover, respectively a $\mathcal{C}$-envelope.

Approximations and minimal approximations are strongly linked with another notion in homological algebra: cotorsion pairs. Cotorsion pairs were introduced by Salce in the 1970s as an analogue to the notion of a torsion pair [37]. That is, cotorsion pairs are pairs of classes in the category of R-modules which are mutually Ext^1_R-orthogonal. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ provides a natural way to look at a well behaved type of approximation: a special $\mathcal{B}$-preenvelope, which is a monomorphism with cokernel in $\mathcal{A}$ in addition to being a $\mathcal{B}$-preenvelope, or dually a special $\mathcal{A}$-precover, which is an epimorphism with kernel in $\mathcal{B}$ in addition to being an $\mathcal{A}$-precover. Note that the stronger assumptions on the class $\mathcal{C}$ in the previous paragraph turn out to be that the $\mathcal{C}$-preenvelope is a special $\mathcal{C}$-preenvelope.

In a cotorsion pair $(\mathcal{A}, \mathcal{B})$, Salce demonstrated that there is a symmetry between special $\mathcal{A}$-precovers and special $\mathcal{B}$-preenvelopes. More precisely, a cotorsion pair $(\mathcal{A}, \mathcal{B})$ provides special $\mathcal{A}$-precovers if and only if it provides special $\mathcal{B}$-preenvelopes. This observation lead to the notion of complete cotorsion pairs, that is cotorsion pairs which provide approximations. Moreover, by a theorem of Eklof and Trlifaj, complete cotorsion pairs are abundant. More precisely, in [21] Eklof and Trlifaj proved that that any cotorsion pair generated by a set is complete, that is for a set $\mathcal{S}$ of R-modules, the cotorsion pair $({}^{\perp_1}(\mathcal{S}^{\perp_1}), \mathcal{S}^{\perp_1})$ is complete. Thus many well-known cotorsion pairs were shown to be complete by demonstrating that they can be generated by a set.

A particularly important accomplishment of Eklof and Trlifaj's result is the role it played in the famous flat cover conjecture, which was asserted by Enochs in 1981 and states that every R-module has a flat cover. This conjecture could be seen to be influenced by the positive result for the existence of an injective envelope for every R-module. Finally, the Flat cover Conjecture was resolved in 2001 by Bican-El Bashir-Enochs [18] who proved that the flat cotorsion pair $(\mathcal{F}, \mathcal{C})$ is generated by a set and thus is complete, and therefore $\mathcal{F}$ is covering since $\mathcal{F}$ is closed under direct limits.

Another important type of cotorsion pair is a hereditary cotorsion pair, which is a cotorsion pair $(\mathcal{A}, \mathcal{B})$ where $\mathcal{A}$ is resolving, that is closed under kernels of epimorphisms, or equivalently $\mathcal{B}$ is coresolving, that is closed under cokernels of monomorphisms. Most of the cotorsion pairs we work with in this thesis are hereditary, to name a few $(\mathcal{P}_n(R), \mathcal{P}_n(R)^{\perp})$ where $\mathcal{P}_n(R)$ is the class of modules of projective dimension less than or equal to n, the injective cotorsion pair $(\text{Mod-}R, \mathcal{I})$ and the flat cotorsion pair $(\mathcal{F}, \mathcal{C})$.

As mentioned before, by results of Bass Enochs' Conjecture holds for the class of projective modules. Morover, Bass gives both a ring-theoretic characterisation of the rings for which the class $\mathcal{P}_0(R)$ is closed under direct limits, as well as homological characterisations in the category of R-modules. In this thesis we are interested in developing a similar characterisation for the class of modules of projective dimension less than or equal to one, denoted

$\mathcal{P}_1(R)$. Before we discuss the subject of this thesis in more detail, it is important to take into consideration some significant advancements made towards Enochs' Conjecture in recent years.

In 2017, Angeleri Hügel-Šaroch-Trlifaj in [5] proved that Enochs' Conjecture holds for a large class of cotorsion pairs. Explicity, they proved that for a cotorsion pair $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{B}$ is closed under direct limits, $\mathcal{A}$ is covering if and only if it is closed under direct limits. To prove this, Angeleri Hügel-Šaroch-Trlifaj used methods developed in Šaroch's paper [41], which uses sophisticated set-theoretical methods in homological algebra. In particular, this result tells us that Enochs' Conjecture holds for a large collection of cotorsion pairs (which are not necessarily hereditary) and include the 1-tilting cotorsion pairs, which we will introduce below.

In this thesis we will avoid using the results of Angeleri Hügel-Šaroch-Trlifaj discussed in the previous paragraph, rather, we choose to take an algebraic approach. We will show that one can come to the same conclusion under our assumptions without needing to use these deep results. With this in mind, we introduce the main topic of this thesis.

The original goal of our research was to investigate Enochs' Conjecture in the specific case of the class of modules of projective dimension less than or equal to one in Mod-R (denoted $\mathcal{P}_1(R)$), and to investigate the properties of the (commutative) rings which satisfy these conditions. To be precise, we wanted to investigate the following question over a ring R.

Question 0.0.1. *Let R be an associative ring. If $\mathcal{P}_1(R)$ is a covering class, is $\mathcal{P}_1(R)$ necessarily closed under direct limits?*

Question 0.0.1 in general is very difficult, so we restrict our investigation to when R is a commutative ring. To this end, there were two natural directions of research in consideration of some practical issues when working with certain cotorsion pairs, which we will outline after introducing 1-tilting cotorsion pairs.

In this thesis we are interested in infinitely generated 1-tilting modules, where a module T in Mod-R is 1-tilting if and only if $\mathrm{Gen}(T) = T^{\perp}$, where $\mathrm{Gen}(T)$ is the class of epimorphic images of direct sums of T [19]. Moreover, we consider the cotorsion pairs generated by 1-tilting modules, known as 1-tilting cotorsion pairs and the 1-tilting class $\mathcal{T} = T^{\perp}$. These cotorsion pairs are not only complete and hereditary, but also have another important property. First recall that a class $\mathcal{C}$ is of finite type if there exists a set $\mathcal{S}$ of modules each with a projective resolution of finitely generated modules (i.e. $\mathcal{S} \subset$ mod-R) such that $\mathcal{C} = \mathcal{S}^{\perp_\infty}$. We say a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is of finite type if $\mathcal{B}$ is of finite type. Even though the 1-tilting modules are not finitely generated, they are well behaved in this sense. That is, in [10] Bazzoni-Herbera proved that 1-tilting classes are of finite type.

In the context of Question 0.0.1, the cotorsion pair $(\mathcal{P}_1(R), \mathcal{P}_1(R)^{\perp})$ is of finite type if and only if it is a 1-tilting cotorsion pair [11]. Furthermore, in the study of cotorsion pairs where the right hand class is closed under direct sums (as in the case of 1-tilting cotorsion pairs), one can use a sort of T-nilpotency result of Enochs-Xu which turns out to be a powerful tool when working with minimal approximations.

We briefly clarify this concept of T-nilpotency. Recall that if the ring R is perfect, the Jacobson radical $J(R)$ of R must be T-nilpotent. A theorem of Enochs-Xu states that if $(\mathcal{A}, \mathcal{B})$ has the property that $\mathcal{B}$ is closed under direct sums, a direct sum of minimal approximations remains a minimal approximation. Moreover, if one has a countable direct sum of minimal approximations with homomorphisms $(f_1, f_2, \dots, f_i, \dots)$ where $f_i \colon X_i \to X_{i+1}$ between each term of the minimal approximations that satisfy some hypothesis, then for each $x \in X_1$, there exists an integer n such that $f_n f_{n-1} \cdots f_1(x) = 0$, see Theorem 1.2.4. Hence, this can be considered a type of T-nilpotency which is a sort of generalisation of the notion of a T-nilpotent ideal of the Jacobson radical when the ring R is perfect.

Thus, in the investigation of the cotorsion pair $(\mathcal{P}_1(R), \mathcal{P}_1(R)^{\perp})$, we begin by studying the case when this cotorsion pair is not of finite type, and in particular to study Enochs' Conjecture in this context. Up until now, there are no examples of a positive answer to Question 0.0.1 for when $(\mathcal{P}_1(R), \mathcal{P}_1(R)^{\perp})$ is not of finite type (as far as we are aware). Moreover, this investigation opens other questions about the class $\varinjlim \mathcal{P}_1(R)$. The second direction of research was to investigate cotorsion pairs $(\mathcal{A}, \mathcal{B})$ where $\mathcal{A}$ is contained in $\mathcal{P}_1(R)$ and $(\mathcal{A}, \mathcal{B})$ is of finite type, and in particular a 1-tilting cotorsion pair and investigate the rings over which these cotorsion pairs admit minimal approximations. Thus, this second direction of research opened natural questions about when 1-tilting classes are enveloping.

For the first direction when $(\mathcal{P}_1(R), \mathcal{P}_1(R)^{\perp})$ is not of finite type, we show that over a semihereditary commutative ring, if $\mathcal{P}_1(R)$ is covering then $\mathcal{P}_1(R)$ is closed under direct limits, hence R is hereditary (Theorem 3.2.18). This provides an example of when the cotorsion pair $(\mathcal{P}_1(R), \mathcal{P}_1(R)^{\perp})$ is not of finite type and Enochs' Conjecture holds.

In the investigation of when $\mathcal{P}_1(R)$ is covering, the class $\varinjlim \mathcal{P}_1(R)$ plays an important role, even though it is not always well understood. Unlike the case of the projective modules where $\varinjlim \mathcal{P}_0(R) = \mathcal{F}_0(R)$, it is not necessarily true that the direct limit closure $\varinjlim \mathcal{P}_1(R)$ coincides with the modules of weak dimension less than or equal to one, $\mathcal{F}_1(R)$, though one inclusion always holds. For certain nice rings, such as commutative domains, the two classes $\varinjlim \mathcal{P}_1(R)$ and $\mathcal{F}_1(R)$ coincide. This motivated us to give a characterisation of the class $\varinjlim \mathcal{P}_1(R)$ when R has a classical ring of quotients, which is an extension of a result from [11]. We prove that $\varinjlim \mathcal{P}_1(R)$ is exactly the intersection of $\mathcal{F}_1(R)$ and the left Tor_1^R-orthogonal of the minimal cotilting class of Q-Mod, $\mathcal{C}(Q) := \mathcal{P}_1(\text{mod-}Q)^\intercal$ (see Proposition 3.1.8) where $\mathcal{P}_1(\text{mod-}Q)$ denotes the finitely presented Q-modules of projective dimension less than or equal to one.

In our study of 1-tilting cotorsion pairs over commutative rings, that is, in the second direction of our research, we used extensively the bijective correspondence between 1-tilting cotorsion pairs and faithful finitely generated Gabriel topologies as demonstrated by Hrbek in [30]. Classically it is known more generally that right Gabriel topologies on a ring R are in bijective correspondence with hereditary torsion pairs in Mod-R. In [30], over commutative rings Hrbek extended this bijective correspondence between faithful finitely generated Gabriel topologies (and faithful hereditary torsion pairs of finite type) to 1-tilting cotorsion pairs, amongst others. In this characterisation he associates to a 1-tilting class $\mathcal{T}$ the collection of ideals which "divide" $\mathcal{T}$, that is $\{J \mid JT = T, \forall T \in \mathcal{T}\}$, and in the converse direction he associates to a faithful finitely generated Gabriel topology $\mathcal{G}$ the 1-tilting class of $\mathcal{G}$-divisible modules. This was extended to a correspondence between silting classes and finitely generated Gabriel topologies in [6].

Classically, it is known that with a Gabriel topology of right ideals of a ring R one can describe a ring of quotients of R with respect to $\mathcal{G}$, denoted $R_\mathcal{G}$, which has the property that for any $J \in \mathcal{G}$, a homomorphism $J \to R_\mathcal{G}$ can be extended uniquely to map $R \to R_\mathcal{G}$. This can be considered a sort of generalisation of a localisation of a commutative ring with respect to a multiplicative subset, where the Gabriel topology is made up of the principal ideals $\{sR \mid s \in S\}$. In fact, in the case of a localisation of a ring with respect to a multiplicative subset, there are many other nice properties of the ring of quotients map, in particular that it is a flat ring epimorphism and $R_\mathcal{G}$ is $\mathcal{G}$-divisible.

There are many advantageous properties of perfect localisations. Therefore, when we work with a faithful finitely generated Gabriel topology over a commutative ring, or equivalently a 1-tilting cotorsion pair and assume that this cotorsion pair admits covers or envelopes, the first thing we do is to deduce that $\mathcal{G}$ arises from a perfect localisation and that $\operatorname{p.dim} R_\mathcal{G} \leq 1$ as in Proposition 6.1.6 and Lemma 7.1.3. This is because when $\mathcal{G}$ arises from a perfect localisation and $\operatorname{p.dim} R_\mathcal{G} \leq 1$, we find that the 1-tilting class arises from the flat injective ring epimorphism $R \to R_\mathcal{G}$, so by work of Angeleri Hügel-Sánchez in [4], $R_\mathcal{G} \oplus R_\mathcal{G}/R$ is a 1-tilting module associated to $(\mathcal{A}, \mathcal{D}_\mathcal{G})$. When we work in this setting, we have a much larger range of theories to use, in particular the work of Bazzoni-Positselski and Positselski, some of which we outline now.

There are already some classification results of cotorsion pairs which satisfy Enochs' Conjecture for a particularly well behaved type of 1-tilting cotorsion pair—those which arise from an injective homological ring epimorphism $u\colon R \to U$ in the sense of [4]. Explicitly, in [14] Bazzoni-Positselski showed the following that in a case that includes all commutative rings (we will state it here for commutative rings). Consider an injective homological ring epimorphism $u\colon R \to U$ of commutative rings with $\operatorname{p.dim} U \leq 1$ where $K := U/R$, $\mathcal{T} = \operatorname{Gen}(U) = K^{\perp}$, and the topological ring $\mathfrak{R} := \operatorname{End}_R(K)$. First in [14, Proposition 13.3], $\operatorname{Add}(K)$ is closed under direct limits if and only if the discrete quotient rings of $\mathfrak{R}$ are perfect rings. We prove that it follows that the 1-tilting class $\mathcal{T}$ is enveloping in Theorem 6.3.3. We also prove that a 1-tilting class $\mathcal{T}$ is enveloping if and only if the discrete quotient rings of $\mathfrak{R}$ are perfect and additionally $\operatorname{p.dim} U \leq 1$ in Theorem 6.3.4. In Theorem 6.3.5, we start from a more general setting with respect to [14]. That is, we begin with a general 1-tilting class and show that it must arise from a flat injective ring epimorphism and thus the above results hold. Moreover we use Gabriel topologies to state these results.

Furthermore for the covering side, in [14, Theorem 13.5] Bazzoni-Positselski proved that $\operatorname{Add}(U \oplus K)$ is closed under direct limits if and only if both the ring U and the discrete quotient rings are perfect if and only if the left hand class of the 1-tilting cotorsion pair $(\mathcal{A}, \operatorname{Gen}(U))$ is covering. In this thesis we start from a slightly more general starting point of a 1-tilting cotorsion pair $(\mathcal{A}, \mathcal{T})$ with associated Gabriel topology $\mathcal{G}$ and show that $\mathcal{A}$ is covering if and only if $\operatorname{p.dim} R_{\mathcal{G}} \leq 1$, $R_{\mathcal{G}}$ and the R/J are perfect rings for each $J \in \mathcal{G}$ in Theorem 7.3.16. A consequence of this result is that $\mathcal{G}$ must arise from a perfect localisation. Our point of difference is that in [14] Bazzoni-Positselski use deep results like the tilting-cotilting correspondence and the notion of a contramodule over a topological ring, whereas in our case we state and prove results using simpler algebraic methods which are significantly inspired by the original proofs.

We briefly summarise some of the results mentioned in the previous two paragraphs as well as some implications. Consider the situation of 1-tilting cotorsion pair $(\mathcal{A}, \mathcal{T})$ over a commutative ring. Then the ring R is characterised as follows.

$$\mathcal{T} \text{ is enveloping} \Leftrightarrow \begin{cases} \operatorname{p.dim} R_{\mathcal{G}} \leq 1 \\ R/J \text{ is a perfect ring for each } J \in \mathcal{G} \\ \mathcal{G} \text{ is a perfect Gabriel topology} \end{cases}$$

$$\mathcal{A} \text{ is covering} \Leftrightarrow \begin{cases} \operatorname{p.dim} R_{\mathcal{G}} \leq 1 \\ R_{\mathcal{G}} \text{ is a perfect ring} \\ R/J \text{ is a perfect ring for each } J \in \mathcal{G} \end{cases}$$

It is important to note that if $\operatorname{p.dim} R_{\mathcal{G}} \leq 1$ and $R_{\mathcal{G}}$ is a perfect ring, then it follows that $\mathcal{G}$ is a perfect Gabriel topology. Thus, the characterisation of the ring implies that $\mathcal{A}$ is covering if and only if $\mathcal{T}$ is enveloping *and* the ring of quotients $R_{\mathcal{G}}$ is a perfect ring. Moreover, in both if $\mathcal{T}$ is enveloping or $\mathcal{A}$ is covering $R_{\mathcal{G}} \oplus R_{\mathcal{G}}/R$ is a 1-tilting module associated to the cotorsion pair $(\mathcal{A}, \mathcal{T})$.

In [15, Theorem 1.2], Bazzoni and Positselski state that for a (not necessarily injective) ring epimorphism $u\colon R \to U$ such that $\operatorname{Tor}_1^R(U, U) = 0$ and $K := U/u(R)$, there is an equivalence of additive categories between the u-h-divisible u-comodules and the u-torsion-free u-contramodules via the adjoint functors $\big((- \otimes_R K), \operatorname{Hom}_R(K, -)\big)$. In fact, when $u\colon R \to U$ is a flat injective ring epimorphism of commutative rings as in our case, this becomes an equivalence between the u-h-divisible $\mathcal{G}$-torsion modules and the $\mathcal{G}$-torsion-free u-contramodules.

The advantage of this Matlis category equivalence is that in particular when $\operatorname{p.dim} U \leq 1$, the category of u-contramodules is an abelian category with a projective generator. This Matlis category equivalence is crucial for the reverse direction of the characterisation of the rings for which the 1-tilting cotorsion pair $(\mathcal{A}, \mathcal{T})$ admits covers or envelopes, that is in particular for the result in Proposition 7.3.13. More explicitly, the Matlis category equivalence is required when we begin with the assumption that all the $\mathcal{G}$-torsion factor rings of R (that is the R/J for $J \in \mathcal{G}$) are perfect rings to show that K is Σ-pure-split, or equivalently $\operatorname{Add}(K)$ is closed under direct limits, see Proposition 7.3.14.

For obvious reasons, another focus of my thesis is cotorsion pairs in a more general setting, specifically hereditary cotorsion pairs and cotorsion pairs which admit covers or envelopes.

In the study of injective envelopes, to show that a homomorphism from a module M to an injective module E is an injective envelope, it is sufficient to show that it is a monomorphism and additionally the image of M is essential in E, which in some sense is an intrinsic property of the module E. Analogously, to show that a homomorphism from a projective module P to a module M is a projective cover, it is sufficient to show that the homomorphism is an epimorphism and additionally its kernel is superfluous in P. Thus the property of being essential or superfluous is a sufficient condition for the existence of an injective envelope or projective cover, respectively.

In this thesis we include a generalisation of essential and superfluous submodules to general cotorsion pairs $(\mathcal{A}, \mathcal{B})$ in Proposition 2.1.5 and Proposition 2.2.5. Unfortunately, these do not provide a sufficient condition for the existence of an envelope or cover unless $\mathcal{B}$ is closed under epimorphic images. Although this does not include all the cases that we would like, we found it interesting to include. However, if a minimal approximation does exist, then Proposition 2.1.5 and Proposition 2.2.5 provide some sufficient conditions to show that an approximation is a minimal approximation.

If a cotorsion pair $(\mathcal{A}, \mathcal{B})$ admits both covers and envelopes, then it is called perfect. Examples of perfect cotorsion pairs include the cotorsion pairs such that $\mathcal{A}$ is closed under direct limits by the work of Enochs and Xu, Theorems 1.2.5 and 1.2.12. Furthermore, assuming that Enochs' Conjecture holds and $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair which provides covers, then $\mathcal{B}$ must be enveloping as $\mathcal{A}$ is closed under direct limits. Additionally by the work of Angeleri Hügel-Šaroch-Trlifaj, [5] if $\mathcal{A}$ is covering and $\mathcal{B}$ is closed under direct limits then $\mathcal{B}$ is enveloping. Therefore there are many examples of cotorsion pairs for which providing covers is a sufficient condition to be a perfect cotorsion pair.

However, the converse does not hold, and there are many examples of cotorsion pairs which admit envelopes and not covers: for example the projective cotorsion pair $(\mathcal{P}_0(R), \text{Mod-}R)$ when R is not a right perfect ring. Thus we became interested in the implication that any cotorsion pair that provides covers must provide envelopes. We found this to be true using simple algebraic methods for the case of a hereditary cotorsion pair.

Specifically we found that if $\mathcal{A}$ is covering, every module in $\mathcal{A}$ must have a $\mathcal{B}$-envelope in Proposition 2.3.1. This proposition holds for all cotorsion pairs, and moreover demonstrates that one can extract the $\mathcal{B}$-envelope of $A \in \mathcal{A}$ from the injective envelope of A. The next step requires the assumption that the cotorsion pair is hereditary. We show that for a hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ if every module in $\mathcal{A}$ has a $\mathcal{B}$-envelope, then $\mathcal{B}$ is enveloping in Lemma 2.3.5. This lemma was taken from the dual result presented in [14, Lemma 8.3].

We will now outline the structure of this thesis as it will be presented.

In Chapter 1 we begin with some preliminaries and basic definitions. In particular we cover some basics about minimal approximations and cotorsion pairs, characterisations of perfect rings and some sufficient conditions for a ring to be perfect. Next we introduce ring epimorphisms, 1-tilting and 1-cotilting modules, silting classes, torsion theories and linear topologies. In particular we present what we will need for Gabriel topologies, and Hrbek's characterisation of 1-tilting classes over commutative rings. The work on Gabriel topologies will be extended in Chapter 4.

Next in Chapter 2, for a cotorsion pair $(\mathcal{A}, \mathcal{B})$ we introduce $\mathcal{A}$-essential and $\mathcal{B}$-essential submodules, and dually $\mathcal{A}$-superfluous and $\mathcal{B}$-superfluous submodules. We give some sufficient theorems using these notions for an approximation to be a minimal approximation. Finally in Section 2.3, we show that if a hereditary cotorsion pair admits covers, it admits envelopes in Theorem 2.3.6. Moreover, for any cotorsion pair, if a module $A \in \mathcal{A}$ has a $\mathcal{B}$-envelope, this envelope can be extracted from the injective envelope of A by Proposition 2.3.1.

Our first results in the investigation of Question 0.0.1 are in Chapter 3, where we state our results relating to when the cotorsion pair $(\mathcal{P}_1(R), \mathcal{P}_1(R)^\perp)$ is not of finite type. In Section 3.1 we prove that if an associative ring R has a classical ring of quotients Q, then we can describe the direct limit closure of $\mathcal{P}_1(R)$ as $\varinjlim \mathcal{P}_1(R) = \mathcal{F}_1(R) \cap {}^{\intercal}\mathcal{C}(Q)$ where ${}^{\intercal}\mathcal{C}(Q)$ represents the left Tor_1^R-orthogonal of $\mathcal{C}(Q)$ in Mod-R, as stated in Proposition 3.1.8. Next in Section 3.2 we show that if R is a commutative semihereditary ring, then if $\mathcal{P}_1(R)$ is covering, then $\mathcal{P}_1(R)$ is closed under direct limits in Theorem 3.2.18.

Next we prepare for the results for cotorsion pairs of finite type, in particular the 1-tilting cotorsion pairs over a commutative ring. In Chapter 4 and Chapter 5 we introduce the necessary background results for the results in the final Chapter 6 and Chapter 7.

Chapter 4, Section 4.1 is dedicated to Gabriel topologies, and in particular when the Gabriel topology is faithful with a basis of finitely generated ideals. Additionally we needed to generalise some standard results true for localisations of rings with respect to a multiplicative subset to perfect localisations with respect to Gabriel topologies. Next in Section 4.2, we introduce $\mathcal{H}$-h-local rings for a linear topology $\mathcal{H}$ over a commutative ring, which is a generalisation of results in [13]. The $\mathcal{H}$-h-local rings provide a way to describe the $\mathcal{H}$-discrete modules as direct sums of their localisations as seen in Proposition 4.2.6.

Chapter 5 is divided into five sections. In Section 5.1 we introduce topological rings as well as completions $\Lambda_{\mathcal{H}}(M)$ for a module M. In Section 5.2 we provide a theorem for when the $\mathcal{H}$-topology and the projective limit topology coincide on the completion $\Lambda_{\mathcal{H}}(M)$ in Theorem 5.2.1, which is our generalisation of [34, Theorem 2.3]. In Section 5.3 we discuss u-contramodules where $u\colon R \to U$ is a flat injective ring epimorphism of commutative rings. We cover the case when there is the additional condition that $\mathrm{p.dim}\, U \leq 1$ in Subsection 5.3.2. In Section 5.4 we continue with a flat injective ring epimorphism of commutative rings $u\colon R \to U$ (with no assumptions of the projective dimension of U) and state the category equivalence between $\mathcal{G}$-torsion u-h-divisible modules and $\mathcal{G}$-torsion-free u-contramodules as in [15]. Finally in Section 5.5, as before we let $u\colon R \to U$ be a flat injective ring epimorphism of commutative rings and show that the $\mathcal{G}$-torsion-free modules satisfy the equivalent conditions of Theorem 5.2.1.

In Chapter 6 we look at when $\mathcal{T}$ was an enveloping class in the 1-tilting cotorsion pair $(\mathcal{A}, \mathcal{T})$ over a commutative ring. We find that if $\mathcal{T}$ is enveloping, then the associated Gabriel topology must arise from a perfect localisation in Proposition 6.1.6. Furthermore, if $\mathcal{G}$ arises from a perfect localisation $u\colon R \to U$, $\mathcal{T}$ is enveloping in Mod-R if and only if $\mathrm{p.dim}\, U \leq 1$ and R/J is perfect for every ideal J in the associated Gabriel topology of $\mathcal{T}$ if and only if $\mathrm{p.dim}\, U \leq 1$ and the topological ring $\mathrm{End} K$ is pro-perfect where $K := R_{\mathcal{G}}/R$. We also consider the generalisation to the case that a silting class is enveloping in Theorem 6.4.3. This chapter is the main content of the paper [12].

Finally in Chapter 7, we consider the 1-tilting cotorsion pair $(\mathcal{A}, \mathcal{T})$ such that $\mathcal{A}$ is a covering class. Although it is already known in this case that if

$\mathcal{A}$ is covering then $\mathcal{T}$ is enveloping by Theorem 2.3.6, we prove directly that the rings R/J are perfect for every J in the associated Gabriel topology of $\mathcal{T}$. Specifically, we prove that if $\mathcal{A}$ is covering then the associated Gabriel topology must arise from a perfect localisation. Also, $\mathcal{A}$ is covering in Mod-R if and only if $\operatorname{p.dim}_R R_{\mathcal{G}} \leq 1$ and both the localisation $R_{\mathcal{G}}$ is a perfect ring and R/J is a perfect ring for each ideal J in the associated Gabriel topology $\mathcal{G}$ of $\mathcal{T}$. Rings with the latter two properties, that is both the localisation $R_{\mathcal{G}}$ is a perfect ring and R/J is a perfect ring for each ideal $J \in \mathcal{G}$, will be called $\mathcal{G}$-almost perfect rings.

Contents

Chapter 1

Preliminaries

1.1 Preliminaries

In this section we will recall some definitions and some notation.

All rings will be associative with a unit, Mod-R (R-Mod) the category of right (left) R-modules over the ring R, and mod-R the full subcategory of Mod-R which is composed of all the modules which have a projective resolution consisting of only finitely generated projective modules.

For a commutative ring R, we let $\operatorname{Spec} R$ denote the collection of all the prime ideals of R and $\operatorname{mSpec} R$ denote the collection of all the maximal ideals of R.

For a right R-module M and a right ideal I of R, we let $M[I]$ denote the submodule of M of elements which are annihilated by the ideal I. That is, $M[I] := \{x \in M \mid xI = 0\}$.

Let $\mathcal{C}$ be a class of right R-modules. The right Ext^1_R-orthogonal and right $\operatorname{Ext}^\infty_R$-orthogonal classes of $\mathcal{C}$ are defined as follows.

$$\mathcal{C}^{\perp_1} = \{M \in \text{Mod-}R \mid \operatorname{Ext}^1_R(C, M) = 0 \text{ for all } C \in \mathcal{C}\}$$

$$\mathcal{C}^{\perp} = \{M \in \text{Mod-}R \mid \operatorname{Ext}^i_R(C, M) = 0 \text{ for all } C \in \mathcal{C}, \text{ for all } i \geq 1\}$$

The left Ext-orthogonal classes ${}^{\perp_1}\mathcal{C}$ and ${}^{\perp}\mathcal{C}$ are defined symmetrically. We note that this notation changes depending on the exposition, for example in [29] which we cite extensively, $^{\perp}$ and $^{\perp_\infty}$ are used where instead here we use $^{\perp_1}$ and $^{\perp}$ respectively.

For $\mathcal{C}$ a class in Mod-R, the right Tor^R_1-orthogonal and right $\operatorname{Tor}^R_\infty$-orthogonal classes are classes in R-Mod defined as follows.

$$\mathcal{C}^{\intercal_1} = \{M \in R\text{-Mod} \mid \operatorname{Tor}^R_1(C, M) = 0, \text{ for all } C \in \mathcal{C}\},$$

$$\mathcal{C}^{\intercal} = \{M \in R\text{-Mod} \mid \operatorname{Tor}_i^R(C, M) = 0 \text{ for all } C \in \mathcal{C}, \text{ for all } i \geq 1\}$$

The left Tor_1^R-orthogonal and left $\operatorname{Tor}_\infty^R$-orthogonal classes are classes in Mod-R which are defined symmetrically for a class $\mathcal{C}$ in R-Mod.

If the class $\mathcal{C}$ has only one element, say $\mathcal{C} = \{X\}$, we write $X^{\perp_1}$ instead of $\{X\}^{\perp_1}$, and similarly for the other Ext-orthogonal and Tor-orthogonal classes.

We denote by $\mathcal{P}_n(R)$ ($\mathcal{F}_n(R)$, $\mathcal{I}_n(R)$) the class of right R-modules of projective (flat, injective) dimension at most n, or simply $\mathcal{P}_n$ ($\mathcal{F}_n$, $\mathcal{I}_n$) when the ring is clear from the context. We let $\mathcal{P}_n(\text{mod-}R)$ denote the intersection of mod-R and $\mathcal{P}_n$. The projective dimension (weak or flat dimension, injective dimension) of a right R-module M is denoted $\operatorname{p.dim} M_R$ ($\operatorname{w.dim} M_R$, $\operatorname{inj.dim} M_R$).

Given a ring R, the *right big finitistic dimension*, $\operatorname{F.dim} R$, is the supremum of the projective dimension of right R-modules with finite projective dimension. The *big weak finitistic dimension*, $\operatorname{F.w.dim} R$ is the supremum of the flat dimension of right R-modules with finite flat dimension, or equivalently the supremum of the flat dimension of left R-modules with finite flat dimension.

The *right little finitistic dimension*, f.dim R, is the supremum of the projective dimension of right R-modules in mod-R with finite projective dimension.

For an R-module C, we let $\operatorname{Add}(C)$ denote the class of R-modules which are direct summands of direct sums of copies of C, and $\operatorname{Gen}(C)$ the class of R-modules which are homomorphic images of direct sums of copies of C. Dually, we let $\operatorname{Prod}(C)$ denote the class of R-modules which are direct summands of a direct product of copies of C, and $\operatorname{Cogen}(C)$ the class of R-modules which are submodules of direct products of copies of C.

Consider the following short exact sequence (of right R-modules) where A will be considered a submodule of B.

$$0 \to A \to B \to B/A \to 0 \tag{1.1}$$

Recall that A is a *pure submodule* of B, or $A \subseteq_* B$, if for each finitely presented module F, the functor $\operatorname{Hom}_R(F, -)$ is exact when applied to (1.1). Equivalently, for every left R-module M, $(- \otimes_R M)$ is exact when applied to (1.1). The embedding $A \hookrightarrow B$ is called a *pure embedding*, the epimorphism $B \twoheadrightarrow B/A$ a *pure epimorphism* and the short exact sequence (1.1) a *pure exact sequence*.

Modules that are injective with respect to pure embeddings are called *pure-injective*. That is, a module M is pure-injective if it has the property that $\operatorname{Hom}_R(-, M)$ is exact when applied to a pure exact sequence.

Example 1.1.1. Let $(M_i, f_{ji} \mid i, j \in I)$ be a direct system of modules and consider the direct limit $\varinjlim_I M_i$. The following canonical presentation of $\varinjlim_I M_i$ is an example of a pure exact sequence by [29, Corollary 2.9].

$$0 \to K \to \bigoplus_{i \in I} M_i \to \varinjlim_I M_i \to 0$$

A module X is called Σ-*pure-split* if every pure embedding $A \subseteq_* B$ with $B \in \operatorname{Add}(X)$ splits.

For an R-module M, let the following be a projective resolution of M.

$$\cdots \to P_{i+1} \stackrel{f_{i+1}}{\to} P_i \stackrel{f_i}{\to} P_{i-1} \to \cdots \stackrel{f_1}{\to} P_0 \stackrel{f_0}{\to} M \to 0$$

Then for $i > 0$, the *i-th syzygy* of M is the module $\operatorname{Im} f_i = \operatorname{Ker} f_{i-1}$. The class $\Omega_i(M)$ denotes the class of all the i-th syzygies of M.
Next we do the same thing for an injective resolution of M to construct a dual class of objects. Let the following be an injective resolution of M.

$$0 \to M \stackrel{g_0}{\to} I^0 \stackrel{g_1}{\to} I^1 \to \cdots \to I^{i-1} \stackrel{g_i}{\to} I^i \stackrel{g_{i+1}}{\to} I^{i+1} \to \cdots$$

Then for $i > 0$, the *i-th cosyzygy* of M is the module $\operatorname{Im} g_i = \operatorname{Coker} g_{i-1}$. The class $\Omega^i(M)$ denotes the class of all the i-th cosyzygies of M.
For a class $\mathcal{C}$ of modules, we denote by $\Omega_i(\mathcal{C})$ the union $\bigcup_{M \in \mathcal{C}} \Omega_i(M)$, and analogously by $\Omega^i(\mathcal{C})$ the union $\bigcup_{M \in \mathcal{C}} \Omega^i(M)$ for all $i > 0$.

1.1.1 Homological formulae

The following facts will be useful. Let F_R be a right R-module ${}_R G_S$ be an R-S-bimodule such that $\operatorname{Tor}_1^R(F, G) = 0$. Then, for every right S-module M_S there is the following injective map of abelian groups.

$$\operatorname{Ext}^1_R(F, \operatorname{Hom}_S(G, M)) \hookrightarrow \operatorname{Ext}^1_S(F \otimes_R G, M)) \tag{1.2}$$

To see this, apply $(- \otimes_R G)$ to the following projective presentation of F.

$$0 \to L \to P \to F \to 0$$

$$0 \to L \otimes_R G \to P \otimes_R G \to F \otimes_R G \to 0$$

Then we apply $\operatorname{Hom}_R(-, \operatorname{Hom}_S(G, M))$ to the above short exact sequence to find the top row of (1.3). The bottom row with the two left-hand isomorphisms of (1.3) follow by the tensor-Hom adjunction, and these induce a homomorphism α such that the diagram commutes.

$$\begin{array}{ccccc}
\operatorname{Hom}_R(P, \operatorname{Hom}_S(G, M)) & \longrightarrow & \operatorname{Hom}_R(L, \operatorname{Hom}_S(G, M)) & \longrightarrow & \\
\downarrow \cong & & \downarrow \cong & & \\
\operatorname{Hom}_S(P \otimes_R G, M) & \longrightarrow & \operatorname{Hom}_S(L \otimes_R G, M) & \xrightarrow{\partial} &
\end{array} \tag{1.3}$$

$$\begin{array}{cccc} \longrightarrow & \mathrm{Ext}^1_R(F,\mathrm{Hom}_S(G,M)) & \longrightarrow & 0 \\ & \Big\downarrow \alpha & & \Big\downarrow \\ \xrightarrow{\partial} & \mathrm{Ext}^1_S(F\otimes_R G, M) & \longrightarrow & \mathrm{Coker}\,\partial \end{array}$$

By the four-lemma, α is the desired monomorphism as the rightmost vertical homomorphism is a monomorphism and the two left-most homomorphisms are isomorphisms.

Let $s\colon R \to S$ be a ring homomorphism. Suppose $\mathrm{Tor}_i^R(M,S) = 0$ for $M \in \text{Mod-}R$ for all $1 \leq i \leq n$ and N_S is a right S-module (and also a right R-module via the restriction of scalars functor). Then the following holds for all i such that $1 \leq i \leq n$ (see for example [36, Lemma 4.2].

$$\mathrm{Ext}^i_R(M_R, N_R) \cong \mathrm{Ext}^i_S(M_R \otimes_R S, N_S) \tag{1.4}$$

Moreover, if M is as above and N is a left S-module, then the following holds.

$$\mathrm{Tor}^R_i(M_R,{}_R N) \cong \mathrm{Tor}^S_i(M_R \otimes_R S,{}_S N) \tag{1.5}$$

1.2 Envelopes and covers

For this section, $\mathcal{C}$ will be a class of right R-modules closed under isomorphisms and direct summands. We will begin by defining envelopes as well as giving some properties of envelopes and enveloping classes before moving onto the dual notion of covers.

Many of the results in this section are taken from Xu's book [42], which generalises work based on Enochs' paper [22] where he works mainly in the setting where $\mathcal{C}$ is the class of injective modules or flat modules. For this reason, many results are attributed to Enochs-Xu rather than just Enochs.

Definition 1.2.1. A *$\mathcal{C}$-preenvelope* of M is a homomorphism $\varepsilon\colon M \to C$ where $C \in \mathcal{C}$ with the property that for every homomorphism $g\colon M \to C'$ with $C' \in \mathcal{C}$, there exists $g'\colon C \to C'$ such that $g'\varepsilon = g$.

$$\begin{array}{ccc} M & \xrightarrow{\varepsilon} & C \\ & {\scriptstyle g}\searrow & \Big\downarrow \exists g' \\ & & C' \end{array}$$

A *$\mathcal{C}$-envelope* of M is a $\mathcal{C}$-preenvelope with the property that for every $g\colon C \to C$ such that $g\varepsilon = \varepsilon$, g is an isomorphism.

$$\begin{array}{ccc} M & \xrightarrow{\varepsilon} & C \\ & {\scriptstyle \varepsilon}\searrow & \cong\Big\downarrow g \\ & & C \end{array}$$

A $\mathcal{C}$-preenvelope $\mu\colon M \to C$ of M is called a *special $\mathcal{C}$-preenvelope* if μ is a monomorphism and $\operatorname{Coker}\mu \in {}^{\perp_1}\mathcal{C}$. We point out that for any module M, any sequence of the following form with $C \in \mathcal{C}$ and $\operatorname{Coker}\mu \in {}^{\perp_1}\mathcal{C}$ is a special $\mathcal{C}$-preenvelope.

$$0 \to M \xrightarrow{\mu} C \to \operatorname{Coker}\mu \to 0$$

The existence of a $\mathcal{C}$-envelope or a (special) $\mathcal{C}$-preenvelope of a module depends on the class $\mathcal{C}$, the module M, and the ring R. When every R-module has a $\mathcal{C}$-envelope ($\mathcal{C}$-preenvelope, special $\mathcal{C}$-preenvelope), the class $\mathcal{C}$ is called *enveloping* (respectively, *preenveloping*, *special preenveloping*). If M does have a $\mathcal{C}$-envelope, one can describe the relationship between the $\mathcal{C}$-preenvelopes and $\mathcal{C}$-envelopes of a module M which was shown by Xu as follows.

Proposition 1.2.2. *[42, Proposition 1.2.2] Let $\mathcal{C}$ be a class of modules and assume that a module N admits a $\mathcal{C}$-envelope. If $\mu\colon N \to C$ is a $\mathcal{C}$-preenvelope of N, then $C = C' \oplus H$ for some submodules C' and H such that the composition $N \to C \to C'$ is a $\mathcal{C}$-envelope of N.*

Corollary 1.2.3. *[42, Corollary 1.2.3] Suppose M has a $\mathcal{C}$-envelope. Let $\mu\colon M \to C$ be a $\mathcal{C}$-preenvelope. Then μ is an envelope if and only if there is no direct sum decomposition $C = C_1 \oplus K$ with $K \neq 0$ and $\operatorname{Im}\mu \leq C_1$.*

We will often consider $\mathcal{C}$-envelopes where $\mathcal{C}$ is a class closed under direct sums and therefore we will make use of the following result. In fact, the theorem is strongly connected with the notion of T-nilpotency of an ideal of a ring (see Section 1.4).

Theorem 1.2.4. *[42, Theorem 1.4.4, Theorem 1.4.6]*

(i) Let $\mathcal{C}$ be a class closed under countable direct sums. Assume that for every $n \geq 1$, $\mu_n\colon M_n \to C_n$ are $\mathcal{C}$-envelopes of M_n and that $\bigoplus_n M_n$ admits a $\mathcal{C}$-envelope. Then $\bigoplus \mu_n\colon \bigoplus_n M_n \to \bigoplus_n C_n$ is a $\mathcal{C}$-envelope of $\bigoplus_n M_n$.

(ii) Assume that $\bigoplus \mu_n\colon \bigoplus_n M_n \to \bigoplus_n C_n$ is a $\mathcal{C}$-envelope of $\bigoplus_n M_n$ with $M_n \leq C_n$ and let $f_n\colon C_n \to C_{n+1}$ be a family of homomorphisms such that $f_n(M_n) = 0$. Then, for each $x \in C_1$ there is an integer m such that $f_m f_{m-1} \dots f_1(x) = 0$.

The following is an important result originally from which gives a sufficient condition for a class $\mathcal{C}$ to be enveloping. The crucial steps are found in the paper [22], and the theorem is generally attributed to Enochs and Xu or sometimes just Enochs.

Theorem 1.2.5. *[42, Theorem 2.2.6][29, Theorem 5.27] Assume that $\mathcal{C}$ is a class of modules closed under direct limits and extensions. If a module M admits a special $\mathcal{C}^{\perp_1}$-preenvelope with cokernel in $\mathcal{C}$, then M admits a $\mathcal{C}^{\perp_1}$-envelope.*

Theorem 1.2.5 plays an important part in this thesis as one of our main questions is when certain classes are enveloping.

If the class $\mathcal{C}$ contains the injective modules, then it follows that every $\mathcal{C}$-preenvelope must be a monomorphism as every module can be embedded in an injective module. Moreover, if $\mathcal{C} = \mathcal{I}_0$, then every $\mathcal{I}_0$-preenvelope is special, because the cokernel is trivially in ${}^{\perp}\mathcal{I}_0 = \text{Mod-}R$. We can say even more about injective envelopes, which coincide with the classical notion of an injective hull. Injective hulls were first introduced by Eckmann and Schopf in 1953 and were one of the motivating examples for the study of minimal approximations.

Before stating their theorem, we first recall the notion of an essential submodule. Let N be a submodule of M. Then N is *essential* in M, denoted $N \subseteq_e M$, if for a submodule H of M, $N \cap H = 0$ implies that $H = 0$. Moreover, M is called an *essential extension* of N.

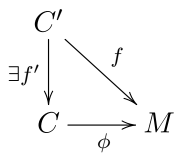
Proposition 1.2.6. *[20][23, Theorem 3.1.14] Every R-module has an injective hull, that is an essential extension to an injective module.*

Therefore by the following well-known lemma and Proposition 1.2.6, one can deduce that every module has an injective envelope. In other words, the following lemma states that the notion of an injective envelope and an injective hull coincide.

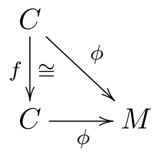
Lemma 1.2.7. *[42, Theorem 1.2.11] An $\mathcal{I}_0$-preenvelope $\varepsilon \colon M \to E$ is an $\mathcal{I}_0$-envelope if and only if M is essential in E.*

We now will discuss the dual concepts of $\mathcal{C}$-precovers and $\mathcal{C}$-covers.

Definition 1.2.8. A $\mathcal{C}$-*precover* of M is a homomorphism $\phi \colon C \to M$ where $C \in \mathcal{C}$ with the property that for every homomorphism $f \colon C' \to M$ where $C' \in \mathcal{C}$, there exists $f' \colon C' \to C$ such that $\phi f' = f$.



A $\mathcal{C}$-*cover* of M is a $\mathcal{C}$-precover with the additional property that for every homomorphism $f\colon C \to C$ such that $\phi f = \phi$, f is an isomorphism.



A $\mathcal{C}$-precover $\phi\colon C \to M$ of M is called a *special $\mathcal{C}$-precover* if ϕ is an epimorphism and $\operatorname{Ker}\phi \in \mathcal{C}^{\perp}$. We point out that for any module M, any sequence of the following form with $C \in \mathcal{C}$ and $\operatorname{Ker}\phi \in \mathcal{C}^{\perp_1}$ is a special $\mathcal{C}$-precover.

$$0 \to \operatorname{Ker}\phi \to C \xrightarrow{\phi} M \to 0$$

Moreover, if the class $\mathcal{C}$ contains R, any $\mathcal{C}$-precover must be surjective.

If every R-module has a $\mathcal{C}$-cover ($\mathcal{C}$-precover, special $\mathcal{C}$-precover), the class $\mathcal{C}$ is called *covering* (respectively, *precovering*, *special precovering*). Like in the case of envelopes and preenvelopes, the existence of covers and precovers depends on the module M, the class $\mathcal{C}$, and the ring R. If a cover does exist, we can describe the relationship between a $\mathcal{C}$-cover and a $\mathcal{C}$-precover of M, which is analagous to Proposition 1.2.2.

Theorem 1.2.9. *[42, Theorem 1.2.7] Suppose $\mathcal{C}$ is a class of modules and M admits a $\mathcal{C}$-cover and $\phi\colon C \to M$ is a $\mathcal{C}$-precover. Then $C = C' \oplus K$ for submodules C', K such that the restriction $\phi{\restriction_{C'}}$ gives rise to a $\mathcal{C}$-cover of M and $K \subseteq \operatorname{Ker}\phi$.*

The following is dual to Corollary 1.2.3.

Corollary 1.2.10. *[42, Corollary 1.2.8] Suppose M admits a $\mathcal{C}$-cover. Then a $\mathcal{C}$-precover $\phi\colon C \to M$ is a $\mathcal{C}$-cover if and only if there is no non-zero direct summand K of C contained in $\operatorname{Ker}\phi$.*

The following two theorems are dual to Theorem 1.2.4 and Theorem 1.2.5.

Theorem 1.2.11. *[42, Theorem 1.4.7, Theorem 1.4.1]*

(i) Suppose for each integer $n \geq 1$, $\phi_n\colon C_n \to M_n$ is a $\mathcal{C}$-cover. Then if $\bigoplus_n \phi_n\colon \bigoplus_n C_n \to \bigoplus_n M_n$ is a $\mathcal{C}$-precover, then it is also a $\mathcal{C}$-cover.

(ii) Assume that $\bigoplus \mu_n\colon \bigoplus_n C_n \to \bigoplus_n M_n$ is a $\mathcal{C}$-cover of $\bigoplus_n M_n$ and let $f_n\colon C_n \to C_{n+1}$ be a family of homomorphisms such that $\operatorname{Im} f_n \subseteq \operatorname{Ker}\phi_{n+1}$. Then, for each $x \in C_1$ there is an integer m such that $f_m f_{m-1} \dots f_1(x) = 0$.

As before, the crucial steps of the following theorem are due to Enochs in [22].

Theorem 1.2.12. *[42, Theorem 2.2.8] [29, Theorem 5.31] Assume that $\mathcal{C}$ is a class of modules closed under direct limits. If a module M admits a $\mathcal{C}$-precover, then M admits a $\mathcal{C}$-cover.*

Furthermore, analogously to the case of injective envelopes and essential submodules, we can describe projective covers more explicitly. Recall that a module N is *superfluous* in M (or $N \ll M$) if for $H \leq M$, $N + H = M$ implies that $H = M$.

Lemma 1.2.13. *[42, Theorem 1.2.12]A projective precover $\phi\colon P \to M$ is a $\mathcal{P}_0$-cover if and only if $\operatorname{Ker}\phi$ is superfluous in P.*

Unlike in the case of injective envelopes, projective covers do not necessarily exist for every module, though projective precovers do. Thus it is then natural to ask over what rings does every module have a projective cover. These rings are called perfect rings and were characterised by Bass in [8], which will be discussed more in Section 1.4.

In 1981, Enochs conjectured that every R-module has a flat cover in [22]. This was proven to be true by Bican-El Bashir-Enochs in [18] in 2001 following work of Trlifaj and Eklof. More explicitly, the Flat Cover Conjecture was shown to be true by showing that $\mathcal{F}_0$ is precovering, so by Theorem 1.2.12 it follows that $\mathcal{F}_0$ is covering.

By Theorem 1.2.12, if $\mathcal{C}$ is a precovering class that is closed under direct limits, it is also a covering class. The converse implication is a major point of interest in approximation theory and is known as Enochs' Conjecture.

Conjecture 1.2.14 (Enochs' Conjecture)**.** *If $\mathcal{C}$ is a covering class then $\mathcal{C}$ is closed under direct limits.*

A well-known positive result of the above conjecture (and perhaps even a motivating factor for its statement) is the case of projective modules as given by Bass's Theorem P (see Theorem 1.4.2). That is, Enochs' Conjecture holds for the class $\mathcal{P}_0$ of projective modules, or to be explicit, $\mathcal{P}_0$ is covering if and only if $\mathcal{P}_0$ is closed under direct limits. We will discuss some more recent positive results in the next section.

1.3 Cotorsion pairs

In this thesis, we consider precovers and preenvelopes for particular classes of modules, that is classes which form a cotorsion pair. Cotorsion pairs were introduced by Salce in [37].

Definition 1.3.1. A pair of classes of modules $(\mathcal{A}, \mathcal{B})$ is a *cotorsion pair* provided that $\mathcal{A} = {}^{\perp_1}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp_1}$.

Examples of cotorsion pairs include $(\mathcal{P}_0, \text{Mod-}R)$, $(\text{Mod-}R, \mathcal{I}_0)$ and $(\mathcal{F}_0, \mathcal{C})$ where $\mathcal{C} := \mathcal{F}_0^{\perp_1}$. For any class $\mathcal{C}$, $\left({}^{\perp_1}(\mathcal{C}^{\perp_1}), \mathcal{C}^{\perp_1}\right)$ is a cotorsion pair and is said to be the cotorsion pair *generated* by $\mathcal{C}$ and $\left({}^{\perp_1}\mathcal{C}, ({}^{\perp_1}\mathcal{C})^{\perp_1}\right)$ is a cotorsion pair and is said to be *cogenerated* by $\mathcal{C}$. For a fixed ring R, the cotorsion pairs of right (or left) R-modules have a partial ordering, where $(\mathcal{A}, \mathcal{B}) \leq (\mathcal{A}', \mathcal{B}')$ if $\mathcal{A} \subseteq \mathcal{A}'$ or equivalently $\mathcal{B}' \subseteq \mathcal{B}$. The lattice of cotorsion pairs has a minimal element $(\mathcal{P}_0, \text{Mod-}R)$ and maximal element $(\text{Mod-}R, \mathcal{I}_0)$. For example, we have the following for any ring R.

$$(\mathcal{P}_0, \text{Mod-}R) \leq (\mathcal{F}_0, \mathcal{C}) \leq (\text{Mod-}R, \mathcal{I}_0)$$

The following is known as the Wakamatsu Lemma which we will state using cotorsion pairs instead of in its full generality. In the case of cotorsion pairs which provide envelopes (covers), it allows us to work always with special preenvelopes (special precovers).

Lemma 1.3.2 (Wakamatsu Lemma)**.** *[29, Lemma 5.13] Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair and M a module. Then the following hold.*

(i) If M has a $\mathcal{B}$-envelope, then it is special.

(ii) If M has an $\mathcal{A}$-cover, then it is special.

Moreover, there is a symmetry between preenvelopes and precovers with respect to a cotorsion pair.

Lemma 1.3.3 (Salce Lemma)**.** *[29, Lemma 5.20][37] Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Then $\mathcal{B}$ is special preenveloping if and only if $\mathcal{A}$ is special precovering.*

Therefore, one calls a cotorsion pair *complete* if either $\mathcal{B}$ is special preenveloping or $\mathcal{A}$ is special precovering. The following important result due to Trlifaj and Eklof shows that complete cotorsion pairs are abundant.

Theorem 1.3.4. *[29, Theorem 6.11] Suppose $\mathcal{S}$ is a set of modules. Then the cotorsion pair generated by $\mathcal{S}$, $\left({}^{\perp_1}(\mathcal{S}^{\perp_1}), \mathcal{S}^{\perp_1}\right)$, is complete.*

There is a sort of dual for cotorsion pairs cogenerated by a particular class which we will mention shortly in Theorem 1.3.5.

In analogy with the terminology introduced by Bass, a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *perfect* if every R-module M admits an $\mathcal{A}$-cover and a $\mathcal{B}$-envelope. This terminology was motivated by perfect rings in the sense of Bass. More precisely, by Theorem 1.4.2, the cotorsion pair $(\mathcal{P}_0(R), \text{Mod-}R)$ is perfect if and only if the ring is right perfect. In addition, by Theorem 1.2.5 and Theorem 1.2.12 a if $(\mathcal{A}, \mathcal{B})$ is a complete cotorsion pair such that $\mathcal{A}$ is closed under direct limits, then $(\mathcal{A}, \mathcal{B})$ is perfect. Another class of examples of perfect cotorsion pairs is given in the following theorem.

Theorem 1.3.5. *[29, Theorem 6.19] Suppose $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair cogenerated by a class $\mathcal{C}$ which is contained in the class of pure-injective modules. Then $(\mathcal{A}, \mathcal{B})$ is a complete cotorsion pair and $\mathcal{A} = \varinjlim \mathcal{A}$, therefore $(\mathcal{A}, \mathcal{B})$ is perfect.*

We note that there is an asymmetry with respect to the existence of $\mathcal{B}$-envelopes and $\mathcal{A}$-covers. That is, given a complete cotorsion pair $(\mathcal{A}, \mathcal{B})$ the existence of $\mathcal{B}$-envelopes doesn't imply the existence of $\mathcal{A}$-covers. An easy example of this is $(\mathcal{P}_0, \text{Mod-}R)$ for a non-right perfect ring.

Another example is the complete cotorsion pair generated by the quotient field Q of a commutative domain R, and denoted $(\mathcal{SF}, \mathcal{WC})$, that is $\mathcal{WC} = Q^{\perp_1}$ and $\mathcal{SF} = {}^{\perp_1} \mathcal{WC}$. A module $M \in \mathcal{SF}$ is called *strongly flat* and a module $M \in \mathcal{WC}$ is called *weakly cotorsion*. In fact, $\mathcal{WC}$ coincides with $(\text{Mod-}Q)^{\perp_1}$, hence it is special preenveloping and by Theorem 1.2.5, every R-module admits a $\mathcal{WC}$-envelope (see [29, Corollary 7.42]). On the other hand, $\mathcal{SF}$-covers don't necessarily exist. In the case of commutative domains, $\mathcal{SF}$-covers were shown to exist for every module if and only if every flat module is strongly flat in [16], i.e., if $\mathcal{SF}$ is closed under direct limits.

For the converse problem, we note that if there exists a complete cotorsion pair such that $\mathcal{A}$ is covering and $\mathcal{B}$ is not enveloping, this cotorsion pair would be a counterexample of Enoch's Conjecture, because $\mathcal{A}$ is covering and not closed under direct limits. In fact, we show in Theorem 2.3.6 that for a particular type of cotorsion pair, a hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ (see Lemma 1.3.7), if $\mathcal{A}$ is covering, then $\mathcal{B}$ is enveloping, that is the cotorsion pair is perfect.

There have been some important advancements made toward Enochs' Conjecture in recent years. The following theorem states that Enochs' Conjecture holds for a large class of cotorsion pairs.

Theorem 1.3.6. *[5, Corollary 5.5] Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair such that $\mathcal{B}$ is closed under direct limits. Then $\mathcal{A}$ is closed under direct limits if and only if $\mathcal{A}$ is covering.*

We note that if $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair such that $\mathcal{B}$ is closed under direct limits, then by [41, Theorem 6.1], $(\mathcal{A}, \mathcal{B})$ is complete. Moreover, in this paper Šaroch gives some examples of non-hereditary cotorsion pairs that satisfy this condition.

A class $\mathcal{C}$ is called *syzygy closed* if $\Omega_i(\mathcal{C}) \subseteq \mathcal{C}$ for all $i > 0$, and dually a class $\mathcal{C}$ is called *cosyzygy closed* if $\Omega^i(\mathcal{C}) \subseteq \mathcal{C}$ for all $i > 0$.
A class $\mathcal{C}$ is a *resolving* class if it is closed under extensions, kernels of epimorphisms and contains the projective modules. In particular, all resolving classes are syzygy closed. Dually, a class $\mathcal{C}$ is *coresolving* if it is closed under

extensions, cokernels of monomorphisms and contains the injective modules. Additionally, coresolving classes are cosyzygy closed.

Resolving classes and coresolving classes are related to another type of cotorsion pair.

Lemma 1.3.7. *[29, Lemma 5.24] Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Then the following are equivalent.*

(i) *$\mathcal{A}$ is resolving.*

(ii) *$\mathcal{B}$ is coresolving.*

(iii) $\mathrm{Ext}^i_R(A, B) = 0$ *for every* $A \in \mathcal{A}$, $B \in \mathcal{B}$ *and* $i > 0$.

If the above conditions hold, then the cotorsion pair $(\mathcal{A}, \mathcal{B})$ is called hereditary.

Thus if a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is hereditary, then $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$, thus there is no need to differentiate between $\perp_1$ and $\perp$.

For the most part, in this thesis all the cotorsion pairs we are interested in will be hereditary, in particular $(\mathcal{P}_1(R), \mathcal{P}_1(R)^{\perp})$.

Theorem 1.3.8. *[29, Theorem 8.10] Let R be a ring. Then the pair $(\mathcal{P}_1(R), \mathcal{P}_1(R)^{\perp})$ is a complete hereditary cotorsion pair. In fact, if $(\mathcal{A}, \mathcal{B})$ is a cotorsion pairs such that $\mathcal{A} \subseteq \mathcal{P}_1(R)$, then $(\mathcal{A}, \mathcal{B})$ is hereditary.*

A hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ is of *finite type* if there is a set $\mathcal{S}$ of modules in mod-R such that $\mathcal{S}^{\perp} = \mathcal{B}$ (recall mod-R denotes the class of modules admitting a projective resolution consisting of finitely generated projective modules). In other words, $(\mathcal{A}, \mathcal{B})$ is of finite type if and only if $\mathcal{B} = (\mathcal{A} \cap \text{mod-}R)^{\perp}$. Dually, a class $\mathcal{C}$ is of *cofinite type* is there is a set $\mathcal{S}$ of modules in $\mathcal{P}_n(\text{mod-}R)$ for some $n \geq 0$ such that $\mathcal{C} = \mathcal{S}^{\intercal}$.

Almost all of the cotorsion pairs we work with in this thesis will be hereditary and moreover often the right-hand class will be closed under direct sums.

1.4 Projective covers and perfect rings

Given Eckmann and Schopf's result on injective envelopes (Proposition 1.2.6), it is natural to ask over what rings does every module have a projective cover. These rings were characterised by Bass in the classical Theorem P (Theorem 1.4.2), both in terms of homological properties of Mod-R and ring theoretic properties of R, and are called *perfect rings*. In this section we define perfect rings and discuss some of their nice properties, as well as giving some sufficient conditions for a ring to be perfect.

Before giving the characterisation of perfect rings, we must introduce a particular type of ideal. One can generalise the notion of a nilpotent ideals, to a T-nilpotent ideal where the T stands for "transfinite." An ideal I of R is said to be *right T-nilpotent* if for every sequence of elements $a_1, a_2, ..., a_i, ...$ in I, there exists an $n > 0$ such that $a_n a_{n-1} \cdots a_1 = 0$. For *left T-nilpotence*, one must have $a_1 a_2 \cdots a_n = 0$.

The property of T-nilpotence of an ideal has interesting consequences.

Lemma 1.4.1. *[1, Lemma 28.3] Let I be a right ideal in a ring R. Then the following are equivalent.*

(i) I is right T-nilpotent.

(ii) $MI \neq M$ for every non-zero right R-module M.

(iii) $MI \ll M$ for every non-zero right R-module M.

(iv) $R^{(\mathbb{N})} I \ll R^{(\mathbb{N})}$.

We now define and characterise the right perfect rings as follows. We let $J(R)$ denote the Jacobson radical of the ring R.

Theorem 1.4.2 (Theorem P, [8])**.** *For a ring R, the following conditions are equivalent.*

(i) R is right perfect (that is, every right R-module has a projective cover).

(ii) Every flat right R-module is projective.

(iii) $J(R)$ is right T-nilpotent and $R/J(R)$ is semisimple.

(iv) The class of projective right R-modules $\mathcal{P}_0$ is closed under direct limits ($\varinjlim \mathcal{P}_0 = \mathcal{P}_0$).

(v) Every decreasing chain of left principal ideals terminates. That is, for the sequence of principal ideals

$$Ra_1 \supseteq Ra_2 \supseteq \cdots \supseteq Ra_i \supseteq \cdots$$

where $a_i \in R$, there exists an $n \in \mathbb{N}$ such that $Ra_n = Ra_m$ for all $m \geq n$.

Moreover, all idempotents can be lifted modulo $J(R)$.

We also recall that a *semi-perfect ring* is a ring such that all finitely generated modules have a projective cover. Examples of right perfect rings include left artinian rings.
In the case R is a commutative ring, there is the following characterisation of R. First recall that a ring R is *semilocal* if $R/J(R)$ is semisimple. If R has finitely many maximal ideals, then R is semisimple, and if R is commutative the converse also holds. A ring R is *semiartinian* if every non-zero factor of R contains a simple R-module.

Proposition 1.4.3. *Suppose R is a commutative ring. The following statements are equivalent for R.*

(i) *R is perfect (that is, every R-module has a projective cover).*

(ii) $\mathrm{F.dim}\, R = 0$.

(iii) *R is a finite product of local rings, each one with a T-nilpotent maximal ideal.*

(iv) *R is semilocal and semiartinian, i.e., R has only finitely many maximal ideals and every non-zero factor of R contains a simple R-module.*

Additionally, if R is perfect then every element of R is either a unit or a zero-divisor.

Proof. The equivalence of (i), (ii), and (iii) follows from the introduction in [8], where the equivalence of (ii) and (iii) is a famous result of Kaplansky. We will show the equivalence of (ii) and (iv) to (i) explicitly using theorems from Bass' paper [8].

(i) ⇔ (ii). By [8, Theorem 6.3], for any ring R, $\mathrm{F.dim}\, R = 0$ if and only if both R is perfect and every finitely generated proper ideal of R has a non-zero annihilator. Therefore, (ii) implies (i) always holds.

For the converse, suppose R is a perfect commutative ring and take a finitely generated proper ideal I of R. Then by [8, Lemma 2.4], either I is contained in $J(R)$ or I contains a direct summand of R, that is it contains a non-zero non-unit idempotent e.

In the first case, suppose $I \subseteq J(R)$. We know that $J(R)$ is T-nilpotent, so as I is finitely generated, it must also be nilpotent. Therefore there exists a minimal n such that $I^n = 0$. Take as the annihilator I^{n-1}, which is non-zero as n is minimal, so I has a non-zero annihilator.
Instead, if I contains a non-zero nilpotent element e, we have that $I = eI \oplus (1-e)I = eR \oplus (1-e)I$. That is, we have the following diagram where $P_{R/I} \to R/I$ is the extracted projective cover of R/I from the natural quotient homomorphism $R \to R/I$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L \oplus eR & \longrightarrow & P_{R/I} \oplus eR & \longrightarrow & R/I & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & I & \longrightarrow & R & \xrightarrow{\text{nat}} & R/I & \longrightarrow & 0 \end{array}$$

Thus, from the above diagram, one can see that $L = (1-e)I$. Thus $(1-e)I$ must be contained in the Jacobson radical as $(1-e)I$ is superfluous in $(1-e)R$ so also $(1-e)I$ is superfluous in R. Therefore, $(1-e)I$ has a non-zero annihilator of the form $(1-e)I^{n-1}$ and clearly this ideal also annihilates eR.

(i) $\Leftrightarrow$ (iii). This is [32, Theorem 23.24].

(i) $\Rightarrow$ (iv). We use ideas in [1, Theorem 28.4, Remark 28.5]. Assume that R is a perfect commutative ring. Then by Theorem 1.4.2, $R/J(R)$ is semisimple, which happens if and only if R is semilocal. We now show that when R is commutative, R is semiartinian when R is perfect. Fix an ideal I of R and suppose that R/I does not contain a simple module. Then there exists $a \in R$ such that $R/I \supsetneq a(R/I) = (aR + I)/I \neq 0$, as otherwise I would be maximal. As $(aR+I)/I$ is a submodule of R/I, by assumption it also does not contain a simple module. Thus one can construct an infinite chain of principal ideals, which contradicts (v) of Theorem 1.4.2, therefore every non-zero factor of R contains a simple module.

(iv) $\Rightarrow$ (i). We now show the converse. Assume that (iv) holds. We will show the ring is perfect by showing that R satisfies Theorem 1.4.2(iii). It remains to show that $J(R)$ is T-nilpotent. Take a sequence of elements $a_1, a_2, \dots$ in $J(R)$ and suppose that $a_1 a_2 \cdots a_n \neq 0$ for every $n > 0$. Consider the poset of ideals Φ where $I \in \Phi$ if $a_1 a_2 \cdots a_n \notin I$ for every n. The set Φ is non-empty as it contains the zero ideal and it is straightforward to see that every chain in Φ has an upper bound in Φ. Then by Zorn's Lemma, there exists a maximal ideal I with respect to this property, and $R/I \neq 0$. By assumption, each factor module contains a simple submodule denoted $K/I \leq R/I$, where $I \subset K$. By maximality of I, there exists an n for which $a_1 a_2 \cdots a_n \in K \setminus I$, and also $a_1 a_2 \cdots a_{n+1} \in K \setminus I$. Again as K/I is simple, $a_1 a_2 \cdots a_{n+1}$ has an inverse element, that is there exists an r such that $a_1 a_2 \cdots a_n(1 - a_{n+1} r) \in I$. However as $a_{n+1} \in J(R)$, $1 - a_{n+1} r$ is an invertible element in R, thus $a_1 a_2 \cdots a_n \in I$, a contradiction. Therefore, $J(R)$ is T-nilpotent.

To prove the last statement, take a non-zero divisor r of R. Then R/rR is of projective dimension one so is projective by (ii). Therefore rR is a direct summand of R and therefore is isomorphic to eR for some e idempotent. As r is regular, $e = 1$, so r is a unit. $\square$

It was noticed by Bass in [8] that it is sufficient to look at the following nice class of modules to find if the ring is perfect.
If R is a ring and $\{a_1, a_2, \dots, a_n, \dots\}$ is a sequence of elements of R, a *Bass right R-module* is a flat module of the following form.

$$F = \varinjlim (R \xrightarrow{a_1} R \xrightarrow{a_2} R \xrightarrow{a_3} \cdots).$$

That is, F is the direct limit of the direct system obtained by considering the left multiplications by the elements a_i on R. The direct limit presentation of F is given by the following short exact sequence, where the homomorphism can be represented by the matrix below.

$$0 \to \bigoplus_{n \in \mathbb{N}} R \xrightarrow{\phi} \bigoplus_{n \in \mathbb{N}} R \to F \to 0$$

$$\phi = \begin{pmatrix} 1 & 0 & \dots & 0 & \dots \\ -a_1 & 1 & & & \\ 0 & -a_2 & \ddots & & \\ \vdots & & \ddots & 1 & \\ \vdots & & & -a_n & \ddots \\ \vdots & & & & \ddots \end{pmatrix}$$

By the above projective presentation, it is clear that all Bass R-modules have projective dimension at most one. Thus the class of Bass R-modules is contained in $\mathcal{F}_0 \cap \mathcal{P}_1$. The following result is implicitly proved in Bass' paper [8].

Lemma 1.4.4. *[35, Proposition 3.2, Lemma 4.1]*

(i) If all flat right R-modules have projective covers, then all the flat right R-modules are projective, so the ring is right perfect.

(ii) If all Bass right R-modules have projective covers then the ring R is right perfect.

Recall that the *socle* of a module M, denoted $\operatorname{soc}(M)$ is the sum of its simple submodules. The notion of a socle of a module is related to the *Loewy series* of an R-module M, which is constructed by transfinite induction as follows. Set $M_0 = \operatorname{soc}_0 := 0$. For an ordinal α, $\operatorname{soc}_{\alpha+1}(M)$ is defined from $\operatorname{soc}_\alpha(M)$ by $\operatorname{soc}_{\alpha+1}(M)/\operatorname{soc}_\alpha(M) = \operatorname{soc}(M/\operatorname{soc}_\alpha(M))$. For τ a limit ordinal, let $\operatorname{soc}_\beta(M) := \bigcup_{\alpha<\beta} \operatorname{soc}_\alpha(M)$. Thus for every ordinal α, there is the following short exact sequence, where the last term is a semisimple module.

$$0 \to \operatorname{soc}_\alpha(M) \to \operatorname{soc}_{\alpha+1}(M) \to \operatorname{soc}(M/\operatorname{soc}_\alpha(M)) \to 0$$

For every module M, there exists an ordinal α such that $\operatorname{soc}_\alpha(M) = \operatorname{soc}_\beta(M)$ for every $\beta \geq \alpha$. In the case that there exists an α such that $\operatorname{soc}_\alpha(M) = M$, M is called a *Loewy module*. In other words, if M is a Loewy module, the Loewy series is a continuous filtration of M by the semisimple modules of Mod-R. The following gives a condition for when a module is a Loewy module.

Lemma 1.4.5. *[24, Lemma 2.58] A module M is a Loewy module if and only if every non-zero homomorphic image of M has a non-zero socle.*

Thus the following corollary follows from Lemma 1.4.5 and Proposition 1.4.3 (iv), as over a perfect commutative ring, every module has a non-zero socle.

Corollary 1.4.6. *If R is a perfect commutative ring, then every module is a Loewy module.*

It will be useful to observe that the notion of projective covers can be generalised from the category of R-modules to an abelian category. In Section 3 of [35], it was pointed out that the notion of projective covers can be extended to abelian categories.

Let $\mathcal{A}$ be an abelian category with enough projective objects (that is, objects with a lifting property with respect to epimorphisms). Then, an epimorphism $h\colon P \to C$ with P a projective object is a *projective cover* of the object C if for any endomorphism $e\colon P \to P$ of P such that $he = h$ implies that e is an automorphism.

Similarly as in the case of R-modules, one can extend the notion of superfluous submodule to a superfluous subobject. A subobject K of an object Q is *superfluous* if for $H \leq Q$, $H + K = Q$ implies that $H = Q$.

Lemma 1.4.7. *[35, Lemma 3.1] Let $P \in \mathcal{A}$ be a projective object. Then an epimorphism $h\colon P \to C$ in $\mathcal{A}$ is a projective cover if and only if its kernel is superfluous in P.*

To conclude this section, we return to the category of R-modules and mention some more recent generalisations of perfect rings. Let R be a commutative ring with a classical ring of quotients $Q = Q(R) = R[\Sigma^{-1}]$ where Σ denotes the regular elements of R. Then there is a generalisation of perfect rings which was first defined for domains by Bazzoni-Salce, and then for commutative rings by Fuchs-Salce.

Definition 1.4.8. [16][26] A commutative ring R is *almost perfect* if Q is a perfect ring and all the quotient rings R/sR for $s \in \Sigma$ are perfect.

In [13, Definition 7.6], Bazzoni-Positselski defined an *S-almost perfect* ring for a multiplicative subset S of R, which is a ring R such that $R[S^{-1}]$ is perfect and all the quotient rings R/sR for $s \in S$ are perfect. Thus a Σ-almost perfect ring in the sense of Bazzoni-Positselski is equivalent to an almost perfect ring in the sense of Fuchs-Salce. These definitions will be generalised further in Section 1.7.

Moreover, the notion of an almost perfect ring is related to the direct limit closure of $\mathcal{P}_1(R)$, which will be discussed more in Chapter 3.

1.5 1-tilting modules and 1-cotilting modules

We now introduce 1-tilting classes and modules, as well as some properties that we will use. In this section we will also define 1-cotilting classes and silting classes.

1.5.1 1-tilting modules and classes

A right R-module T is 1-*tilting* if the following conditions hold (as defined in [19]).

(T1) $\mathrm{p.dim}\, T \leq 1$.

(T2) $\mathrm{Ext}^i_R(T, T^{(\kappa)}) = 0$ for every cardinal κ and every $i > 0$.

(T3) There exists an exact sequence of the following form where T_0, T_1 are modules in $\mathrm{Add}(T)$.

$$0 \to R \to T_0 \to T_1 \to 0$$

Equivalently, T is 1-tilting if and only if $T^\perp = \mathrm{Gen}(T)$. The cotorsion pair generated by T, $({}^\perp(T^\perp), T^\perp)$, is called a 1-*tilting cotorsion pair* and the torsion class $T^\perp$ is called the 1-*tilting class*. Often we let $\mathcal{T}$ denote the 1-tilting class $T^\perp$. Two 1-tilting modules T and T' are *equivalent* if they define the same 1-tilting class, that is $T^\perp = T'^\perp$ (equivalently, if $\mathrm{Add}(T) = \mathrm{Add}(T')$). If T is a 1-tilting module which generates a 1-tilting class $\mathcal{T}$, then we say that T is a 1-*tilting module associated to* $\mathcal{T}$.

The *kernel* of the cotorsion pair $({}^\perp\mathcal{T}, \mathcal{T})$ is the class $\mathcal{T} \cap {}^\perp\mathcal{T}$, and the 1-tilting cotorsion pairs have the nice property that, $\mathcal{T} \cap {}^\perp\mathcal{T}$ coincides with $\mathrm{Add}(T)$ (see [29, Lemma 13.10]). As the 1-tilting cotorsion pair is generated by a set, by Theorem 1.3.4, the tilting cotorsion pair is complete. Also, it is hereditary as the right-hand class $\mathcal{T} = \mathrm{Gen}(T) = T^\perp$ is clearly closed under epimorphic images, so is a coresolving class. Moreover, by [10], the 1-tilting cotorsion pair $({}^\perp\mathcal{T}, \mathcal{T})$ is of finite type.

The following proposition says that 1-tilting modules behave well with respect to localisations of a commutative ring.

Proposition 1.5.1. *[29, Proposition 13.50] Let R be a commutative ring and T a 1-tilting module, and $\mathcal{T} = T^\perp$ the 1-tilting class. Then the following hold.*

(i) For S a multiplicative subset of R, $T[S^{-1}]$ is a 1-tilting module in $\mathrm{Mod}\text{-}R[S^{-1}]$ with the corresponding tilting class $\mathcal{T}_S = T[S^{-1}]^\perp = \mathcal{T} \cap \mathrm{Mod}\text{-}R[S^{-1}]$.

(ii) Let M be an R-module. Then $M \in \mathcal{T}$ if and only if $M_\mathfrak{m} \in \mathcal{T}_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of R.

The following proposition and theorem relate 1-tilting cotorsion pairs to approximations and minimal approximations. The first gives a sufficient condition for the left-hand side of cotorsion pair to be closed under direct limits, and hence implies that the cotorsion pair be perfect. Recall from Section 1.1 that a module X is called Σ-pure-split if every pure embedding $A \subseteq_* B$ with $B \in \mathrm{Add}(X)$ splits.

Proposition 1.5.2. *[29, Proposition 13.55] Let T be a tilting module with $(\mathcal{A}, \mathcal{T})$ the associated tilting cotorsion pair. Then $\mathcal{A}$ is closed under direct limits if and only if T is Σ-pure-split.*

All 1-tilting classes are torsion classes (see Section 1.7 for definitions), and as mentioned above the 1-tilting cotorsion pair is complete so provides special preenvelopes. In fact, there is a sort of converse.

Theorem 1.5.3. *[29, Theorem 14.4] Let R be a ring and $\mathcal{T}$ be a class of modules. The following conditions are equivalent.*

(i) $\mathcal{T}$ is a 1-tilting torsion class.

(ii) $\mathcal{T}$ is a special preenveloping torsion class.

(iii) $\mathcal{T}$ is a pretorsion class such that R has a special $\mathcal{T}$-preenvelope.

More specifically, by (T3) of the definition of a 1-tilting module we have the following short exact sequence where $T_0, T_1 \in \mathrm{Add}(T)$.

$$\text{(T3)} \quad 0 \to R \xrightarrow{\varepsilon} T_0 \to T_1 \to 0$$

In fact, this short exact sequence is a special $\mathcal{T}$-preenvelope of R, and $T_0 \oplus T_1$ is a 1-tilting module associated to $\mathcal{T}$ by [29, Theorem 13.18 and Remark 13.19].

1.5.2 Silting modules and classes

A 1-tilting class can be generalised in the following way. For a homomorphism $\sigma : P_{-1} \to P_0$ between projective modules in Mod-R, consider the following class of modules.

$$D_\sigma := \{X \in \text{Mod-}R \mid \mathrm{Hom}_R(\sigma, X) \text{ is surjective}\}$$

An R-module T is said to be *silting* if it admits a projective presentation

$$P_{-1} \xrightarrow{\sigma} P_0 \to T \to 0$$

such that $\mathrm{Gen}(T) = D_\sigma$. The class $\mathrm{Gen}(T)$ is called the *silting class*. In the case that σ is a monomorphism, $\mathrm{Gen}(T)$ is a 1-tilting class.

1.5.3 1-cotilting modules and classes

We now move on to the dual notion of tilting, a cotilting class. A right module C is 1-cotilting if the following conditions hold.

(C1) $\mathrm{inj.dim}\, C \leq 1$.

(C2) $\mathrm{Ext}^i_R(C^\kappa, C) = 0$ for every cardinal κ and every $i > 0$.

(C3) There exists an exact sequence of the following form where each C_i is in $\mathrm{Prod}(C)$ and W is an injective generator for Mod-R.

$$0 \to C_1 \to C_0 \to W \to 0$$

Equivalently, C is 1-cotilting if and only if ${}^\perp C = \mathrm{Cogen}(C)$, see [29, Lemma 15.21]. The cotorsion pair cogenerated by C, $({}^\perp C, ({}^\perp C)^\perp)$, is called a *1-cotilting cotorsion pair* and the torsion-free class ${}^\perp C$ is called *1-cotilting class*. Two 1-cotilting modules C, C' are *equivalent* if they define the same 1-cotilting class ${}^\perp C = {}^\perp C'$ (equivalently, if $\mathrm{Prod}(C) = \mathrm{Prod}(C')$).

Unlike the case of 1-tilting modules, 1-cotilting modules are not always of cofinite type. In fact, there is an example of a 1-cotilting module not of cofinite type due to Bazzoni, see [29, Example 15.33].

1.6 Ring epimorphisms

A *ring epimorphism* is a ring homomorphism $R \xrightarrow{u} U$ such that u is an epimorphism in the category of unital rings. That is, for ring maps $v, w \colon U \rightrightarrows V$ where V is a ring, $vu = wu$ implies that $v = w$. This is equivalent to the natural map $U \otimes_R U \to U$ induced by the multiplication in U being an isomorphism, or equivalently that $U \otimes_R (U/u(R)) = 0$ (see [39, Chapter XI.1]).

We note that if R is commutative and $u \colon R \to U$ a ring epimorphism, then also U is commutative by [38, Corollary 1.2]. Two ring epimorphisms $R \xrightarrow{u} U$ and $R \xrightarrow{u'} U'$ are equivalent if there is a ring isomorphism $\sigma \colon U \to U'$ such that $\sigma u = u'$.

A ring homomorphism $u \colon R \to U$ (not necessarily a ring epimorphism) induces two adjoint functors. These are the *extension of scalars* functor $u^* \colon \text{Mod-}R \to \text{Mod-}U$ which maps a $M_R \in \text{Mod-}R$ to the right U-module $u^*(M) = M \otimes_R U$, and the *restriction of scalars* functor $u_* \colon \text{Mod-}U \to \text{Mod-}R$ which for a module $M \in \text{Mod-}U$, $u_*(M_U)$ is the R-module M where the action of R is defined by $m \cdot r := m \cdot u(r)$ for $m \in M$.

The following proposition gives some well-known characterisations of ring epimorphisms.

Proposition 1.6.1. *[39, Proposition XI.1.2] Let $u \colon R \to U$ be a ring homomorphism. Then the following are equivalent.*

(i) u is an epimorphism of rings.

(ii) $U \otimes_R (U/u(R)) = 0$.

(iii) $U \otimes_R U \cong U$ via the natural map $u_1 \otimes_R u_2 \mapsto u_1 u_2$.

(iv) The restriction functor $u_*\colon \text{Mod-}U \to \text{Mod-}R$ *is fully faithful. That is,* $\operatorname{Hom}_R(M_U, N_U) \cong \operatorname{Hom}_U(M_U, N_U)$.

(v) $u^*u_* \to \mathrm{id}_{\text{Mod-}U}$ *is a natural equivalence of functors.*

A ring epimorphism is *homological* if $\operatorname{Tor}_n^R(U_R, {}_RU) = 0$ for all $n > 0$. A ring epimorphism is called *left (right) flat* if u makes U into a flat left (right) R-module. Clearly all left or right flat ring epimorphisms are homological. We will denote the cokernel of u by K and sometimes by U/R or $U/u(R)$.

In [4], Angeleri Hügel-Sánchez proved that there is a connection between injective ring epimorphisms and 1-tilting classes as follows.

Theorem 1.6.2. *[4, Theorem 2.5] Let* $u\colon R \to U$ *be an injective ring epimorphism with the additional property that* $\operatorname{Tor}_1^R(U, U) = 0$. *Then the following are equivalent.*

(i) $\mathrm{p.dim}\, U_R \leq 1$

(ii) $(U/R)^\perp = \operatorname{Gen}(U)$

(iii) $U \oplus U/R$ *is a* 1*-tilting module.*

Theorem 1.6.3. *[4, Theorem 2.10] Let* R *be a ring and* T *be a* 1*-tilting module in* Mod-R. *The following statements are equivalent.*

(i) There is an injective ring epimorphism $u\colon R \to U$ *such that* $\operatorname{Tor}_1^R(U, U) = 0$ *and* $U \oplus U/R$ *is a* 1*-tilting module equivalent to* T.

(ii) There is an exact sequence $0 \to R \xrightarrow{\varepsilon} T_0 \to T_1 \to 0$ *such that* $T_0, T_1 \in \operatorname{Add}(T)$ *and* $\operatorname{Hom}_R(T_1, T_0) = 0$.

Moreover, under these conditions $\varepsilon\colon R \to T_0$ *is a* $T^\perp$*-envelope of* R *and* $u\colon R \to U$ *is a homological ring epimorphism.*

If a 1-tilting module T satisfies the equivalent properties of Theorem 1.6.3, then T or $T^\perp$ is said to *arise from an injective homological ring epimorphism*.

1.7 Gabriel topologies

In this section we introduce torsion pairs and Gabriel topologies as well as proving some results that will be useful to us later on. We will conclude by discussing some advancements that relate Gabriel topologies to 1-tilting classes and silting classes over commutative rings as done in [30] and [6]. The reference for this section, particularly for torsion pairs and Gabriel topologies, is Stenström's book [39, Chapters VI and IX].

We will start by giving definitions in the case of a general ring with unit (not necessarily commutative). Everything will be done with reference to right R-modules (and right Gabriel topologies), but everything can be done for left R-modules.

1.7.1 Torsion classes

A class $\mathcal{C}$ of R-modules is called a *pretorsion class* if it is closed under direct sums and epimorphic images. A pretorsion class is called *hereditary* if it is also closed under submodules. A class $\mathcal{C}$ of R-modules is called a *pretorsion-free class* if it is closed under products and submodules. The hereditary pretorsion classes are in bijective correspondence with left exact preradicals, that is with subfunctors r of the identity functor on Mod-R which are left exact ([39, Corollary VI.1.8]). In other words, a left exact preradical is a left exact functor $r\colon \text{Mod-}R \to \text{Mod-}R$ such that $r(C) \subseteq C$ for every $C \in \text{Mod-}R$, and for a map $f\colon C \to D$ in Mod-R, there is an induced map $r(f)\colon r(C) \to r(D)$ which is the restriction of f to $r(C)$. We note that by [39, Proposition VI.1.7] all left exact preradicals are idempotent, that is $r(r(C)) = r(C)$.

For a left exact preradical, the associated hereditary pretorsion class is the class $\mathcal{E}_r = \{C \in \text{Mod-}R \mid r(C) = C\}$. Conversely, for a hereditary pretorsion class $\mathcal{C}$, the associated left exact preradical assigns to each R-module M the sum of submodules of M which are contained in $\mathcal{C}$.

A class $\mathcal{C}$ of modules is a *torsion class* if it is closed under extensions, direct sums, and epimorphic images, that is it is a pretorsion class with the additional condition of being closed under extensions. A *hereditary* torsion class is a torsion class which is additionally closed under submodules. A radical r is a preradical with the additional property that $r(C/r(C)) = 0$ for every R-module C. The hereditary torsion classes are in bijective correspondence with left exact radicals, [39, Proposition VI.3.1], via the same associations described for hereditary pretorsion classes and left exact preradicals in the previous paragraph.

A *torsion pair* $(\mathcal{E}, \mathcal{F})$ in Mod-R is a pair of classes of modules in Mod-R which are mutually orthogonal with respect to the Hom-functor and maximal with respect to this property. That is, $\mathcal{E} = \{M \mid \text{Hom}_R(M, F) = 0 \text{ for every } F \in \mathcal{F}\}$ and $\mathcal{F} = \{M \mid \text{Hom}_R(X, M) = 0 \text{ for every } X \in \mathcal{E}\}$. The class $\mathcal{E}$ is called a *torsion class* and $\mathcal{F}$ a *torsion-free class*.

Every torsion class forms the left-hand class of a torsion pair in Mod-R, and in this way the collection of torsion classes of R-modules is in bijective correspondence with the collection of torsion pairs in Mod-R. Analogously for the right-hand class, a class $\mathcal{C}$ is a *torsion-free class* if and only if it is closed under extensions, products and submodules, and these classes are in bijective correspondence with the collection of torsion classes.

A torsion pair $(\mathcal{E}, \mathcal{F})$ is called *hereditary* if $\mathcal{E}$ is also closed under submodules, which is equivalent to $\mathcal{F}$ being closed under injective envelopes.

A torsion pair $(\mathcal{E}, \mathcal{F})$ is *generated* by a class $\mathcal{C}$ if $\mathcal{F}$ consists of all the modules F such that $\text{Hom}_R(C, F) = 0$ for every $C \in \mathcal{C}$. A torsion pair $(\mathcal{E}, \mathcal{F})$ is of *finite type* if $\mathcal{F}$ is closed under direct limits.

1.7.2 Linear topologies

Recall that for a right ideal I in R and an element $t \in R$, $(I : t) := \{r \in R \mid tr \in I\}$. A *right linear topology* on R is a collection of ideals of R, denoted $\mathcal{H}$, which satisfy the following properties.

(G1) If $I \in \mathcal{H}$ and $I \subseteq J$, then $J \in \mathcal{H}$.

(G2) If $I, J \in \mathcal{H}$, then $I \cap J \in \mathcal{H}$.

(G3) If $I \in \mathcal{H}$ and $r \in R$ then $(I : r) \in \mathcal{H}$.

The first two conditions just say that $\mathcal{H}$ is a filter of right ideals of R. The right linear topologies on R are in bijective correspondence with hereditary pretorsion classes in R. This bijection associates to a hereditary pretorsion class $\mathcal{C}$ in Mod-R the right linear topology $\{I \mid R/I \in \mathcal{C}\}$, and to each right linear topology $\mathcal{H}$ the pretorsion class $\{M \mid \text{Ann}x \in \mathcal{H} \text{ for all } x \in M\}$, called the class of *$\mathcal{H}$-discrete* modules (see [39, Proposition VI.4.2]). For example, for every $J \in \mathcal{H}$, R/J is $\mathcal{H}$-discrete. Thus, there is the following bijection.

$$\left\{\begin{matrix}\text{right linear topologies}\\ \text{on } R\end{matrix}\right\} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \left\{\begin{matrix}\text{hereditary pretorsion}\\ \text{classes in Mod-}R\end{matrix}\right\}$$

$$\Phi\colon \mathcal{H} \longmapsto \{M \mid \text{Ann}x \in \mathcal{H}, \forall x \in M\}$$

$$\{I \mid R/I \in \mathcal{C}\} \longleftarrow\!\shortmid\, \mathcal{C}\colon \Psi$$

A *basis* of a right linear topology $\mathcal{H}$ is a subset $\mathcal{B}$ of $\mathcal{H}$ such that every ideal in $\mathcal{H}$ contains some ideal in $\mathcal{B}$.

A *(right) Gabriel topology* on R is a right linear topology on R, denoted $\mathcal{G}$, such that the following additional condition holds.

(G4) If J is a right ideal of R and there exists a $I \in \mathcal{G}$ such that $(J : t) \in \mathcal{G}$ for every $t \in I$, then $J \in \mathcal{G}$.

This extra condition gives the associated hereditary pretorsion class the extra property that it is closed under extensions, and therefore is a hereditary torsion class. As with right linear topologies, we have the following equivalence.

$$\left\{\begin{matrix}\text{right Gabriel topologies}\\ \text{on } R\end{matrix}\right\} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \left\{\begin{matrix}\text{hereditary torsion}\\ \text{classes in Mod-}R\end{matrix}\right\}$$

$$\Phi\colon \mathcal{G} \longmapsto \mathcal{E}_{\mathcal{G}} = \{M \mid \text{Ann}x \in \mathcal{G}, \forall x \in M\}$$

$$\{I \leq R \mid R/I \in \mathcal{E}\} \longleftarrow\!\shortmid\, \mathcal{E}\colon \Psi$$

We denote the corresponding torsion pair by $(\mathcal{E}_\mathcal{G}, \mathcal{F}_\mathcal{G})$, which is generated by the cyclic modules R/J where $J \in \mathcal{G}$. The classes $\mathcal{E}_\mathcal{G}$ and $\mathcal{F}_\mathcal{G}$ are referred to as the $\mathcal{G}$-*torsion* and $\mathcal{G}$-*torsion-free* classes, respectively.

For a right R-module M let $t_\mathcal{G}(M)$ denote the associated (left exact) radical, also called the $\mathcal{G}$-torsion submodule of M, or sometimes $t(M)$ when the Gabriel topology is clear from context. Thus for every module M, there is the following unique decomposition.

$$0 \to t_\mathcal{G}(M) \to M \to M/t_\mathcal{G}(M) \to 0$$

1.7.3 Modules of quotients

A Gabriel topology allows us to generalise localisations of commutative rings with respect to a multiplicative subset to non-commutative rings. In the specific case of localisation with respect to a multiplicative subset of a commutative ring, the Gabriel topology would have as a basis collection of principal ideals generated by the elements of the multiplicative subset. We will come back to this example later, and will begin by defining the module of quotients of a module with respect to a Gabriel topology $\mathcal{G}$.
The following subsection uses results from [39, Chapter IX].

The *module of quotients* of the Gabriel topology $\mathcal{G}$ of a right R-module M is the module $M_\mathcal{G}$ defined as follows.

$$M_\mathcal{G} := \varinjlim_{J \in \mathcal{G}} \mathrm{Hom}_R(J, M/t_\mathcal{G}(M))$$

Furthermore, there is the following canonical homomorphism.

$$\psi_M \colon M \cong \mathrm{Hom}_R(R, M) \to M_\mathcal{G}$$

For each R-module M, the homomorphism ψ_M is part of the following exact sequence, where both the kernel and cokernel of the map ψ_M are $\mathcal{G}$-torsion R-modules.

$$0 \to t_\mathcal{G}(M) \to M \overset{\psi_M}{\to} M_\mathcal{G} \to M_\mathcal{G}/\psi_M(M) \to 0 \tag{1.6}$$

By substituting $M = R$, the assignment gives a ring homomorphism $\psi_R \colon R \to R_\mathcal{G}$ and furthermore, for each R-module M the module $M_\mathcal{G}$ is both an R-module and an $R_\mathcal{G}$-module.

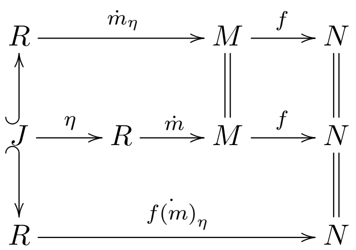
A right R-module is $\mathcal{G}$-*closed* if the following natural homomorphisms are all isomorphisms for each $J \in \mathcal{G}$.

$$M \cong \mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(J, M) \tag{1.7}$$

This amounts to saying that $\mathrm{Hom}_R(R/J, M) = 0$ for every $J \in \mathcal{G}$ (i.e. M is $\mathcal{G}$-*torsion-free*) and $\mathrm{Ext}^1_R(R/J, M) = 0$ for every $J \in \mathcal{G}$ (i.e. M is $\mathcal{G}$-*injective*). Moreover, if M is $\mathcal{G}$-closed then M is isomorphic to its module of quotients $M_{\mathcal{G}}$ via ψ_M. Conversely, every R-module of the form $M_{\mathcal{G}}$ is $\mathcal{G}$-closed. The $\mathcal{G}$-closed modules form a full subcategory of both Mod-R and Mod-$R_{\mathcal{G}}$. The full subcategory of $\mathcal{G}$-closed modules in Mod-R is denoted Mod-$(R, \mathcal{G})$. In fact, we now show that every R-linear morphism of $\mathcal{G}$-closed modules is also a $R_{\mathcal{G}}$-linear, thus the full subcategory of $\mathcal{G}$-closed modules considered as $R_{\mathcal{G}}$-modules is equivalent to Mod-$(R, \mathcal{G})$. Furthermore in Proposition 1.7.1, we see that the canonical homomorphism ψ_M is well behaved with respect to the $\mathcal{G}$-closed modules.

We recall briefly what it means for two elements to be equivalent for a $\mathcal{G}$-closed module. Consider two elements x, y of a $\mathcal{G}$-closed module M and let the map $\dot{x}\colon R \to M$ denote the homomorphism which maps $1 \mapsto x$, and similarly for y. Then, via the isomorphism in (1.7), if there exists an ideal $J \in \mathcal{G}$ such that the restriction of the maps $R \overset{\dot{x}}{\to} M$ and $R \overset{\dot{y}}{\to} M$ to J coincide.

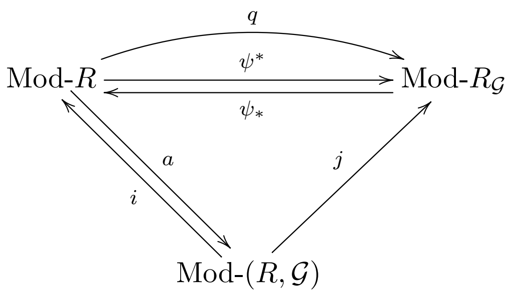
Let M, N be $\mathcal{G}$-closed and consider an R-linear map $f\colon M \to N$, and fix an element $x \in R_{\mathcal{G}}$ which is represented by $\eta\colon J \to R$. We must show that $f(mx) = f(m)x$. For an element $m \in M$, as before let the map $\dot{m}\colon R \to M$ denote the homomorphism which maps $1 \mapsto m$. The element mx is represented by the map $J \overset{\eta}{\to} R \overset{\dot{m}}{\to} M$. Then as M is $\mathcal{G}$-closed, $\mathrm{Hom}_R(J, M) \cong \mathrm{Hom}_R(R, M)$, so each maps $J \to M$ extends uniquely to a map $R \to M$, and similarly for N. These extended morphisms of $\dot{m}\eta$ and $\dot{f(m)}\eta$ are denoted $\dot{m}_\eta$ and $\dot{f(m)}_\eta$ in the following diagram.



Therefore it is clear from the above diagram that $f \circ \dot{m}_\eta$ and $\dot{f(m)}_\eta$ represent the same element in N. Moreover, mx can be represented by the homomorphism $\dot{m}_\eta$ and similarly $f(m)x$ can be represented by the morphism $\dot{f(m)}_\eta$. Thus we have shown that $f(mx) = f(m)x$ as desired.

We have the following diagram of functors. We will use the notation of

Stenström which is explained below.



(i) ψ_* is the restriction of scalars functor which is exact and faithful.

(ii) $\psi^* := (- \otimes_R R_{\mathcal{G}})$ is the extension of scalars functor $M_R \mapsto M_R \otimes_R R_{\mathcal{G}}$.

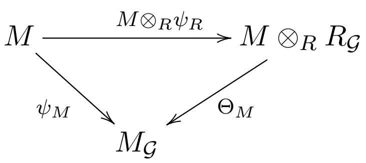
(iii) a maps $M \mapsto M_{\mathcal{G}}$ in Mod-R.

(iv) i is the inclusion functor of the $\mathcal{G}$-closed modules into Mod-R.

(v) j is the inclusion functor of the $\mathcal{G}$-closed modules into Mod-$R_{\mathcal{G}}$ and is full and faithful.

(vi) q maps $M \mapsto M_{\mathcal{G}}$ in Mod-$R_{\mathcal{G}}$ and is left exact.

In general, there is a natural transformation $\Theta\colon \psi^* \to q$ with $\Theta_M \colon M \otimes_R R_{\mathcal{G}} \to M_{\mathcal{G}}$ which is defined as $m \otimes_R \eta \mapsto \psi_M(m) \cdot \eta$. That is, for every M the following triangle commutes.



(1.8)

The Gabriel topologies for which Θ is a natural equivalence form an important class of Gabriel topologies, which we will discuss more in detail in Subsection 1.7.4.

Proposition 1.7.1. *[39, Proposition IX.1.11] The functor a is left adjoint to i. Therefore for a $\mathcal{G}$-closed module N the canonical map* $\mathrm{Hom}_R(\psi_M, N)\colon \mathrm{Hom}_R(M_{\mathcal{G}}, N) \to \mathrm{Hom}_R(M, N)$ *is an isomorphism.*

A left R-module N is called *$\mathcal{G}$-divisible* if $JN = N$ for every $J \in \mathcal{G}$. Equivalently, N is $\mathcal{G}$-divisible if and only if $R/J \otimes_R N = 0$ for each $J \in \mathcal{G}$. We denote the class of $\mathcal{G}$-divisible modules by $\mathcal{D}_{\mathcal{G}}$. It is straightforward to see that $\mathcal{D}_{\mathcal{G}}$ is a torsion class in R-Mod.

A right Gabriel topology is *faithful* if $\mathrm{Hom}_R(R/J, R) = 0$ for every $J \in \mathcal{G}$, or equivalently if R is $\mathcal{G}$-torsion-free, that is the natural map $\psi_R \colon R \to R_{\mathcal{G}}$

is injective. A right Gabriel topology is *finitely generated* if it has a basis consisting of finitely generated right ideals. The torsion pairs associated to the finitely generated Gabriel topologies have an additional property, which we will state after recalling the following definition.

Consider a direct system of submodules of M, denoted $\{M_i, f_{ji} \mid M_i \to M_j, i \leq j\}_{i,j \in I}$ such that the natural maps f_{ji} are all inclusion maps. Then $\varinjlim M = \sum_i M_i$ is called a *direct union* of the $\{M_i, f_{ji} \mid M_i \to M_j, i \leq j\}_{i,j \in I}$. For example, every module M is a direct union of its finitely generated submodules.

Proposition 1.7.2. *[39, Proposition XIII.1.2] The following properties of a right Gabriel topology $\mathcal{G}$ are equivalent.*

(i) $\mathcal{G}$ has a basis of finitely generated ideals.

(ii) Every direct union of $\mathcal{G}$-closed modules is $\mathcal{G}$-closed.

(iii) The $\mathcal{G}$-torsion radical preserves direct limits (that is there is a natural isomorphism $t_{\mathcal{G}}(\varinjlim_i M_i) \cong \varinjlim_i (t_{\mathcal{G}}(M_i))$).

(iv) The $\mathcal{G}$-torsion-free modules are closed under direct limits (that is, the associated torsion pair is of finite type).

Proof. The equivalence of (i), (ii), and (iii) are in [39, Proposition XIII.1.2]. The equivalence of these conditions with (iv) was noted by Hrbek in the discussion before [30, Lemma 2.4]. We will prove the equivalence of (iv) with (iii) here for completeness.

We will denote $t_{\mathcal{G}}(M)$ by $t(M)$. Assume (iii) holds, and take a direct limit $M = \varinjlim_i M_i$ of $\mathcal{G}$-torsion-free modules M_i. Then $t(M) = t(\varinjlim_i M_i) \cong \varinjlim_i t(M_i) = 0$ as each of the M_i is $\mathcal{G}$-torsion-free, so we conclude that also M is $\mathcal{G}$-torsion-free. For the converse, suppose (iv) holds, and consider a direct system of module $\{M_i, f_{ji} \mid M_i \to M_j, i \leq j\}_{i,j \in I}$. Then the image of restriction $f_{ji}\restriction_{t(M_i)}$ is contained in $t(M_j)$, thus there is an induced directed system $\{t(M_i), f_{ji}\restriction_{t(M_i)} \colon t(M_i) \to t(M_j), i \leq j\}_{i,j \in I}$. As the $\mathcal{G}$-torsion modules are closed under direct sums and homomorphic images, $\varinjlim_i t(M_i)$ is also $\mathcal{G}$-torsion. Thus as the direct limit functor is exact, we have the following commuting diagram, where α is the natural restriction of the inclusion map.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varinjlim_i t(M_i) & \longrightarrow & \varinjlim_i M_i & \longrightarrow & \varinjlim_i M_i/t(M_i) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \| & & \downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & t(\varinjlim_i M_i) & \longrightarrow & \varinjlim_i M_i & \longrightarrow & \varinjlim_i M_i/t(\varinjlim_i M_i) & \longrightarrow & 0 \end{array}$$

By the snake lemma, the cokernel of α is isomorphic to the kernel of β. Additionally, the cokernel of α is $\mathcal{G}$-torsion, and by assumption $\varinjlim_i (M_i/t(M_i))$ is $\mathcal{G}$-torsion-free, so the kernel of β is $\mathcal{G}$-torsion-free. Thus $\operatorname{Coker} \alpha \cong \operatorname{Ker} \beta = 0$, so α is an isomorphism as required. □

1.7.4 Perfect localisations

We will often be concerned with a particular type of Gabriel topology that has many useful properties. The following theorem tells us that a flat ring epimorphism gives rise to a Gabriel topology.

Theorem 1.7.3. *[39, Theorem XI.2.1] Suppose $u\colon R \to U$ is a ring homomorphism. Then the following are equivalent.*

(i) u is an epimorphism of rings which makes U into a flat left R-module.

(ii) The family $\mathcal{G}$ of right ideals J such that $u(J)U = U$ is a Gabriel topology, and the natural ring homomorphism $\psi\colon R \to R_{\mathcal{G}}$ is equivalent to $u\colon R \to U$. That is, there is a ring isomorphism $\sigma\colon U \to R_{\mathcal{G}}$ such that $\sigma u\colon R \to R_{\mathcal{G}}$ is the canonical homomorphism $\psi_R\colon R \to R_{\mathcal{G}}$.

This theorem allows us to give the following definition. A left flat ring epimorphism $R \xrightarrow{u} U$ is called a *perfect right localisation* of R, and there is the following associated right Gabriel topology.

$$\mathcal{G}_u = \{J \leq R \mid u(J)U = U\}$$

Note also that a right ideal J of R is in $\mathcal{G}_u$ if and only if $R/J \otimes_R U = 0$. Often we will simply write $\mathcal{G}$ instead of $\mathcal{G}_u$ for the associated Gabriel topology of a ring epimorphism u. Moreover, when we find that a Gabriel topology arises from a perfect localisation, we will often denote $R_{\mathcal{G}}$ by U and $R_{\mathcal{G}}/\psi_R(R)$ by K, thus $\psi_R\colon R \to R_{\mathcal{G}}$ is denoted $u\colon R \to U$.

We note that the adjective "perfect" for a Gabriel topology can be slightly confusing as it is not related in any way to perfect rings. However, we will continue to use this nomenclature as it is already commonly used in the literature.

In Stenström's book the perfect right localisations are characterised in the following way. Recall that $q\colon \text{Mod-}R \to \text{Mod-}R_{\mathcal{G}}$ is the functor that maps each module to its module of quotients $M \mapsto M_{\mathcal{G}}$.

Proposition 1.7.4. *[39, Proposition XI.3.4] Let $\mathcal{G}$ be a right Gabriel topology. Then the following conditions are equivalent.*

(i) $\psi_R\colon R \to R_{\mathcal{G}}$ is a perfect right localisation and $\mathcal{G} = \{J \leq R \mid \psi_R(J)R_{\mathcal{G}} = R_{\mathcal{G}}\}$.

(ii) $R_{\mathcal{G}}$ is $\mathcal{G}$-divisible.

(iii) $\mathcal{G}$ has a basis of finitely generated ideals and q is exact.

(iv) $\operatorname{Ker}(M \to M \otimes_R R_{\mathcal{G}})$ is the $\mathcal{G}$-torsion submodule of M.

(v) $\Theta\colon \psi^* \to q$ is a natural equivalence of functors.

In particular, Proposition XI.3.4 in Stenström's book states that the right Gabriel topology $\mathcal{G}$ associated to a flat ring epimorphism $R \overset{u}{\to} U$ is finitely generated and the $\mathcal{G}$-torsion submodule $t_{\mathcal{G}}(M)$ of a right R-module M is the kernel of the canonical homomorphism $M \to M \otimes_R U$. Additionally it is clear that $K = U/u(R)$ is $\mathcal{G}$-torsion, hence $\mathrm{Hom}_R(K, U) = 0$. If moreover the flat ring epimorphism $R \overset{u}{\to} U$ is injective, then $\mathrm{Tor}_1^R(M, K) \cong t_{\mathcal{G}}(M)$ and $\mathcal{G}$ is faithful.

1.7.5 Gabriel topologies and 1-tilting classes

As mentioned before, much useful and important research has already been done in this direction. Specifically, in [30], Hrbek showed that over commutative rings the faithful finitely generated Gabriel topologies are in bijective correspondence with 1-tilting classes, and that the latter are exactly the classes of $\mathcal{G}$-divisible modules for some faithful finitely generated Gabriel topology $\mathcal{G}$. Before we state this theorem, we recall briefly a definition and a Lemma, which were used to prove Hrbek's correspondence.

Suppose M is a finitely presented right R-module with projective presentation $P_1 \overset{\rho}{\to} P_0 \to M \to 0$ where P_0, P_1 are finitely generated projective modules. Recall that the *transpose* of M, denoted $\mathrm{Tr}(M)$, is the cokernel of the map $\rho^* \colon P_0^* \to P_1^*$ where $(-)^* := \mathrm{Hom}_R(-, R)$. Moreover, we have the following relations from [3, Lemma 2.9] or [30, Lemma 3.3].

Lemma 1.7.5. *Let R be a ring and M a non-zero finitely presented right R-module such that $\mathrm{Hom}_R(M, R) = 0$. Then the following hold.*

(i) $\mathrm{p.dim}_R \mathrm{Tr}\, M = 1$ *and* $\mathrm{Tr}\, M$ *is a finitely presented left R-module.*

(ii) $\mathrm{Hom}_R(M, -)$ *and* $\mathrm{Tor}_1^R(-, \mathrm{Tr}\, M)$ *are isomorphic functors.*

(iii) $\mathrm{Ext}_R^1(\mathrm{Tr}\, M, -)$ *and* $(M \otimes_R -)$ *are isomorphic functors.*

We now state the following theorem that is an indispensable starting point for Chapters 6 and 7 of this thesis.

Theorem 1.7.6. *[30, Theorem 3.16] Let R be a commutative ring. There are bijections between the following collections.*

(i) 1-tilting classes $\mathcal{T}$.

(ii) Faithful finitely generated Gabriel topologies $\mathcal{G}$.

(iii) Faithful hereditary torsion pairs $(\mathcal{E}, \mathcal{F})$ of finite type in $\mathrm{Mod}\text{-}R$.

Moreover, the tilting class $\mathcal{T}$ is the class of $\mathcal{G}$-divisible modules with respect to the Gabriel topology $\mathcal{G}$.

That is, Hrbek shows that to a faithful finitely generated Gabriel topology $\mathcal{G}$, one considers the modules $\mathrm{Tr}(R/J) \in \mathcal{P}_1(\text{mod-}R)$ for each finitely generated $J \in \mathcal{G}$, and associates the 1-tilting class $\mathcal{T} = \left(\bigoplus_{J\in\mathcal{G}; J \text{ f.g.}} \mathrm{Tr}(R/J)\right)^{\perp}$. For the reverse direction, he associates to each 1-tilting class $\mathcal{T}$ the Gabriel topology $\mathcal{G} = \{J \leq R \mid JT = T, \forall T \in \mathcal{T}\}$.

When we refer to the *Gabriel topology associated to a 1-tilting class $\mathcal{T}$* we will always mean the Gabriel topology in the sense of the above theorem. In addition we will often denote $\mathcal{D}_\mathcal{G}$ to be the 1-tilting class associated to $\mathcal{G}$ and $\mathcal{A}$ to be the left Ext-orthogonal class to $\mathcal{D}_\mathcal{G}$ so $(\mathcal{A}, \mathcal{D}_\mathcal{G})$ will denote the 1-tilting cotorsion pair associated to $\mathcal{G}$.

Moreover, in the case of a Gabriel topology that arises from a perfect localisation such that $\mathrm{p.dim}\, R_\mathcal{G} \leq 1$, we can describe the 1-tilting class more explicitly. One can conclude that the 1-tilting class arises from the flat ring epimorphism $\psi_R \colon R \to R_\mathcal{G}$ in the sense of Theorem 1.6.2. This observation is crucial as the 1-tilting module $R_\mathcal{G} \oplus R_\mathcal{G}/R$ is much more convenient to work with than the 1-tilting class $\mathcal{D}_\mathcal{G}$.

Proposition 1.7.7. *[30, Proposition 5.4] Let R be a commutative ring, T a 1-tilting module, and $\mathcal{G}$ the Gabriel topology associated to $\mathcal{D}_\mathcal{G} = T^{\perp}$ in the sense of Hrbek. Then the following are equivalent.*

(i) $\mathcal{G}$ is a perfect Gabriel topology and $\mathrm{p.dim}\, R_\mathcal{G} \leq 1$.

(ii) $\mathrm{Gen}(R_\mathcal{G}) = \mathcal{D}_\mathcal{G}$

If the above equivalent conditions hold, T or the 1-tilting class $\mathcal{D}_\mathcal{G}$ is said to arise from a perfect localisation.

We note that there is yet more confusion with our terminology. That is the 1-tilting class arises from a perfect localisation if and only if the Gabriel topology arises from a perfect localisation *and* $\mathrm{p.dim}\, R_\mathcal{G} \leq 1$. Therefore we often include the statement $\mathrm{p.dim}\, R_\mathcal{G} \leq 1$ for clarity.

In [6] the correspondence between faithful finitely generated Gabriel topologies and 1-tilting classes over commutative rings was extended to finitely generated Gabriel topologies which were shown to be in bijective correspondence with silting classes. Thus in this case the class $\mathcal{D}_\mathcal{G}$ of $\mathcal{G}$-divisible modules coincides with the class $\mathrm{Gen}(T)$ for some silting module T. We state the result formally.

Theorem 1.7.8. *[6, Theorem 4.7] Let R be a commutative ring. There is a one-to-one correspondence between silting classes $\mathcal{T}$ in $\text{Mod-}R$ and Gabriel topologies with a basis of finitely generated ideals over R.*

We conclude this section with a definition which extends S-almost perfect rings.

Definition 1.7.9. Let R be a ring with a right Gabriel topology $\mathcal{G}$. Then R is $\mathcal{G}$-*almost perfect* if $R_{\mathcal{G}}$ is a perfect ring and the quotient rings R/J are perfect for each $J \in \mathcal{G}$.

Chapter 2

Cotorsion pairs and minimal approximations

In this chapter we study classes which form the left-hand and right-hand classes of cotorsion pairs in relation to approximations. In Sections 2.1 and 2.2 we describe some results which extend the relationship between injective envelopes and essential submodules of injective modules to a general cotorsion pair $(\mathcal{A}, \mathcal{B})$. That is we relate $\mathcal{B}$-envelopes to the so called $\mathcal{A}$-essential submodules and $\mathcal{B}$-essential submodules. Analogously, the relationship between projective covers and superfluous submodules of projective modules is extended to a general cotorsion pair $(\mathcal{A}, \mathcal{B})$, that is to $\mathcal{A}$-covers and the so called $\mathcal{A}$-superfluous submodules and $\mathcal{B}$-superfluous submodules. In later chapters we will often refer to Examples 2.1.9 and 2.2.10.

Next in Section 2.3 we consider the relationship between $\mathcal{A}$ being covering and $\mathcal{B}$ being enveloping for a cotorison pair $(\mathcal{A}, \mathcal{B})$. As mentioned in Section 1.3, there are many examples and even simple examples of cotorsion pairs where $\mathcal{B}$ is enveloping and $\mathcal{A}$ is not covering. In this section, we consider the converse: if $\mathcal{A}$ is covering, is $\mathcal{B}$ enveloping. There are already some positive results. If $\mathcal{A}$ is closed under direct limits, then if $\mathcal{A}$ is covering then $\mathcal{B}$ is enveloping by Theorem 1.2.5 and Theorem 1.2.12. Moreover, if $\mathcal{B}$ is closed under direct limits and $\mathcal{A}$ is covering, then $\mathcal{A}$ is closed under direct limits by Theorem 1.3.6 and so $\mathcal{B}$ is enveloping by Theorem 1.2.5. Cotorsion pairs with the assumption that $\mathcal{B}$ is closed under direct limits are not necessarily hereditary cotorsion pairs.

In Section 2.3 we give a purely algebraic proof that for any hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$, if $\mathcal{A}$ is covering then $\mathcal{B}$ is enveloping in Theorem 2.3.6. We note that if there existed a cotorsion pair $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{A}$ is covering and $\mathcal{B}$ is not enveloping, this would be a counterexample to Enochs' Conjecture as $\mathcal{A}$ would not be closed under direct limits.

The proof of Theorem 2.3.6 is divided into two parts. First we show that if $\mathcal{A}$ is covering then every module in $\mathcal{A}$ has a $\mathcal{B}$-envelope. This holds for all cotorsion pairs, that is the hereditary condition is not required. Next we use [14, Lemma 8.3] of Bazzoni-Positselski (where the dual statement was shown) to show that if $(\mathcal{A}, \mathcal{B})$ hereditary and every module in $\mathcal{A}$ has a $\mathcal{B}$-envelope then $\mathcal{B}$ is enveloping.

2.1 $\mathcal{C}$-essential submodules

For a complete cotorsion pair $(\mathcal{A}, \mathcal{B})$, we investigate the properties of $\mathcal{B}$-envelopes of arbitrary R-modules. First of all we state two lemmas.

Lemma 2.1.1. *Let* $0 \to N \overset{\phi}{\to} B \overset{\pi}{\to} A \to 0$ *be an exact sequence. Let* f *be an endomorphism of* B *such that* $\phi = f \circ \phi$. *Then* $f(B) \supseteq \phi(N)$ *and* $\operatorname{Ker} f \cap \phi(N) = 0$.

Lemma 2.1.2. *Let* $0 \to N \overset{\phi}{\to} B \overset{\pi}{\to} A \to 0$ *be an exact sequence. For every endomorphism* f *of* B*, the following are equivalent*

(i) $\phi = f \circ \phi$.

(ii) The restriction of f *to* $\phi(N)$ *is the identity of* $\phi(N)$.

(iii) There is a morphism $g \in \operatorname{Hom}_R(A, B)$ *such that* $f = id_B - g \circ \pi$.

Proof. (i) $\implies$ (ii). This is clear as $f\phi(n) = f(n)$ for every $n \in N$.
(ii) $\implies$ (iii). As $(\mathrm{id}_B - f)\phi = 0$ and π is the cokernel of ϕ, there exists a unique map g such that $g\pi = \mathrm{id}_B - f$, as required.
(iii) $\implies$ (i). If such a g exists, then $f\phi = \phi - g\pi\phi = \phi$. □

Recall that a submodule N of a module X is *essential* in X if for every submodule H of X, $H \cap X = 0$ implies $H = 0$. We define a notion of essential submodule with respect to classes in a cotorsion pair.

Definition 2.1.3. Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Let $B \in \mathcal{B}$ and let N be a submodule of B.

(i) We say that N is *$\mathcal{A}$-essential* in B if for every submodule K of B, $K \cap N = 0$ and $B/(N + K) \in \mathcal{A}$ imply $K = 0$.

(ii) We say that N is *$\mathcal{B}$-essential* in B if for every submodule H of B containing N, $H \in \mathcal{B}$ implies $H = B$.

Remark 2.1.4. *Consider the class* $\mathcal{I}_0$ *of injective right* R*-modules and the complete cotorsion pair* $(\text{Mod-}R, \mathcal{I}_0)$. *Our definition of* Mod-R*-essential submodule coincides with the classical definition of essential submodule. Moreover, if* E *is an injective envelope of a module* M*, and* $M \subseteq X \subseteq E$ *where* X *is also injective, then* $X = E$ *so* M *is* $\mathcal{I}_0$*-essential in the sense of the above definition.*

The next proposition illustrates the relations between the notions of envelopes and the restricted notions of essential submodules.

Proposition 2.1.5. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Assume that*

$$0 \to N \overset{\phi}{\hookrightarrow} B \overset{\pi}{\to} A \to 0 \tag{2.1}$$

is a special $\mathcal{B}$-preenvelope of the R-module N and identify N with $\phi(N)$. Then the following hold.

(i) If (2.1) is a $\mathcal{B}$-envelope of N, then N is $\mathcal{A}$-essential and $\mathcal{B}$-essential in B.

(ii) If N admits a $\mathcal{B}$-envelope and N is $\mathcal{B}$-essential in B, then (2.1) is a $\mathcal{B}$-envelope.

(iii) If N admits a $\mathcal{B}$-envelope and N is $\mathcal{A}$-essential in B, then (2.1) is a $\mathcal{B}$-envelope.

(iv) Assume that $\mathcal{B}$ is closed under epimorphic images. Then (2.1) is a $\mathcal{B}$-envelope if and only if N is $\mathcal{A}$-essential and $\mathcal{B}$-essential in B.

Proof. (i) Assume that (2.1) is a $\mathcal{B}$-envelope of N. We first show that N is $\mathcal{A}$-essential in B. Let $K \leq B$ be such that $K \cap N = 0$ and $\dfrac{B}{(N+K)} \in \mathcal{A}$. Let $\sigma \colon B \to B/K$ and $\nu \colon (N+K)/K \to B/K$ be the canonical projection and inclusion, respectively. There is an isomorphism $h \colon N \to (N+K)/K$ such that $\sigma \circ \phi = \nu \circ h$. Consider the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (N+K)/K & \xrightarrow{\nu} & B/K & \longrightarrow & B/(N+K) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi\circ h^{-1}} & \swarrow{\scriptstyle \psi} & & & & & \\ & & B & & & & & & \end{array}$$

The diagram can be completed by ψ, since $B/(N+K) \in \mathcal{A}$. Consider the endomorphism $\psi \circ \sigma$ of B. We have that $\phi \circ (\psi \circ \sigma) = \phi$, hence, by assumption $\psi \circ \sigma$ is an automorphism of B. In particular, $\mathrm{Ker}(\psi \circ \sigma) = 0$, so $K = 0$.

Now we show that N is $\mathcal{B}$-essential in B. Let $N \leq H \leq B$, $H \in \mathcal{B}$. Then B/H is a quotient of $B/N \cong A$; hence there is a canonical projection $\tau \colon A \to B/H$.

Consider the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \xrightarrow{\mu} & B & \xrightarrow{\rho} & B/H & \longrightarrow & 0 \\ & & & & \uparrow{\scriptstyle g} & \nearrow{\scriptstyle \tau} & & & \\ & & & & A & & & & \end{array}$$

where μ, ρ are the inclusion and the projection maps. The diagram can be completed by g, since $\mathrm{Ext}^1_R(A,H) = 0$. Consider the morphism $f = id_B - g\circ\pi$. Then f is an endomorphism of B satisfying $f\circ\phi = \phi$. By assumption f is an isomorphism, hence $f(B) = B$.

We show $f(B) \leq H$, so that we must have $H = B$. In fact, $\rho\circ f = \rho - \rho\circ g\circ\pi = \rho - \tau\pi = 0$. Hence $f(B) \subseteq \operatorname{Ker}\rho = H$.

(ii) and (iii) By Xu's result quoted in Proposition 1.2.2, it is enough to show that if N is $\mathcal{A}$-essential or $\mathcal{B}$-essential in B, then B doesn't contain any proper direct summand containing N.

Assume that $B = Y \oplus B_1$ with $Y \geq N$; then $Y \in \mathcal{B}$. So, if N is $\mathcal{B}$-essential in B we conclude that $Y = B$.

We also have $B_1 \cap N = 0$ and $A \cong Y/N \oplus B_1$ where $Y/N \cong B/(N \oplus B_1)$. Thus, $B/(N + B_1) \in \mathcal{A}$, since it is isomorphic to a summand of A. Hence, if N is $\mathcal{A}$-essential in B we conclude that $B_1 = 0$.

(iv) The necessary part is proved in (i). We show sufficiency. Let f be an endomorphism of B such that $\phi = f\circ\phi$. We must prove that in our assumption f is an automorphism of B. First we show that f is an epimorphism. By Lemma 2.1.1, $f(B) \supseteq N$ and $f(B)$ is an epimorphic image of B, hence by assumption $f(B) \in \mathcal{B}$. Thus $f(B) = B$, since N is $\mathcal{B}$-essential in B. The morphism $\pi\circ f\colon B \to A$ is an epimorphim, since f and π are surjective and $\operatorname{Ker}\pi\circ f$ is the preimage N under f. Since $\phi = f\circ\phi$ we have $\operatorname{Ker}(\pi\circ f) = \operatorname{Ker} f + N$ and $\operatorname{Ker} f \cap N = 0$. Thus, $\operatorname{Ker} f = 0$, since $B/(\operatorname{Ker} f + N) \cong A$ and N is $\mathcal{A}$-essential in B. □

Remark 2.1.6. *The condition that $\mathcal{B}$ is closed under epimorphic images in (iv) of Proposition 2.1.5 is stronger than one would like. In fact, in the case of the cotorsion pair* (Mod-$R, \mathcal{I}_0$)*, if a preenvelope $0 \to M \xrightarrow{\mu} E$ has the condition that $\mu(M)$ is* Mod-R*-essential in E, then μ is an envelope, even though $\mathcal{I}_0$ is not closed under epimorphic images.*

Proposition 2.1.7. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair and assume that*

$$0 \to N \xrightarrow{\phi} B \xrightarrow{\pi} A \to 0$$

is a special $\mathcal{B}$-preenvelope of N. The following are equivalent.

(i) ϕ is a $\mathcal{B}$-envelope of N.

(ii) For every $g \in \mathrm{Hom}_R(A,B)$, $id_B - g\circ\pi$ is an isomorphism of B.

(iii) $\mathrm{Hom}_R(A,B)\pi$ is a left ideal of $\mathrm{End}(B)$ contained in the Jacobson radical of $\mathrm{End}(B)$.

Proof. (i) $\Rightarrow$ (ii) Let $g \in \mathrm{Hom}_R(A,B)$ and let $f = id_B - g\circ\pi$. f is an endomorphism of B and $f\circ\phi = \phi$, since $\pi\circ\phi = 0$. By assumption f is an isomorphism of B.

(ii) $\Rightarrow$ (iii) It is clear that $\mathrm{Hom}_R(A,B)\pi$ is a left ideal of $\mathrm{End}(B)$. Let $g \in \mathrm{Hom}_R(A,B)$. For every $h \in \mathrm{End}(B)$, $id_B - h \circ g \circ \pi$ is an invertible element of $\mathrm{End}(B)$, by (ii). Hence $\mathrm{Hom}_R(A,B)\pi$ is contained in the Jacobson radical of $\mathrm{End}(B)$.

(iii) $\Rightarrow$ (i) Let $f \in \mathrm{End}(B)$ be such that $f \circ \phi = \phi$. By Lemma 2.1.2 (iii), there exists a morphism $g \in \mathrm{Hom}_R(A,B)$ such that $id_B - f = g \circ \pi$. By assumption, $-g \circ \pi$ belongs to the Jacobson radical of $\mathrm{End}(B)$, hence $f = id_B + g \circ \pi$ is a unit of $\mathrm{End}(B)$.

□

The following was originally shown in [9]. We have modified the proof slightly.

Proposition 2.1.8. *Let $(\mathcal{A},\mathcal{B})$ be a complete cotorsion pair over a ring R. Assume that $0 \to M \overset{\mu}{\to} B$ is a $\mathcal{B}$-envelope of the R-module M. Let α be an automorphism of M and let β be any endomorphism of B such that $\beta\mu = \mu\alpha$. Then β is an automorphism of B.*

Proof. By the Wakamatsu Lemma (Lemma 1.3.2), μ induces an exact sequence

$$0 \to M \overset{\mu}{\to} B \overset{\pi}{\to} A \to 0$$

with $A \in \mathcal{A}$. Since α is an automorphism of M, it is immediate to see $\mathrm{Coker}\,\mu\alpha \cong A \in \mathcal{A}$ and $f\mu\alpha = \mu\alpha$ implies that f is an isomorphism, so the following is a $\mathcal{B}$-envelope of M.

$$0 \to M \overset{\mu\alpha}{\to} B \to A \to 0$$

is a $\mathcal{B}$-envelope of M. Let β be as assumed and consider an endomorphism g of B such that $g\mu\alpha = \mu$. Then $g\beta\mu = \mu$ and thus $g\beta$ is an automorphism of B, since μ is a $\mathcal{B}$-envelope. This implies that β is a monomorphism.
To see that β is an epimorphism, note that $\beta g\mu\alpha = \beta g\beta\mu = \beta\mu = \mu\alpha$, so by the envelope property of $\mu\alpha$, βg is an automorphism, thus β is an epimorphism as required. □

The following is an important application of Proposition 2.1.8 which we will use often.

Example 2.1.9. Let R be a commutative ring and S a multiplicative subset, and fix a cotorsion pair $(\mathcal{A},\mathcal{B})$ in Mod-R. Consider the localisation of R at the multiplicative subset S, denoted $R[S^{-1}]$, and let M be an $R[S^{-1}]$-module with a $\mathcal{B}$-envelope in Mod-R.

$$0 \to M \overset{\mu}{\to} B \to A \to 0 \tag{2.2}$$

As M is an $R[S^{-1}]$-module, multiplication by an element of S is an automorphism of M. Therefore applying Proposition 2.1.8, multiplication by an element of S is also an automorphism of B and so B is an $R[S^{-1}]$-module. One concludes that the short exact sequence (2.2) is a sequence in Mod-$R[S^{-1}]$, as moreover $R \to R[S^{-1}]$ is a ring epimorphism so the embedding Mod-$R[S^{-1}] \to$ Mod-R is fully faithful.

2.2 $\mathcal{C}$-superfluous submodules

We investigate the properties of $\mathcal{A}$-covers when $\mathcal{A}$ is the left component of a cotorsion pair. First of all we need two easy lemmas whose proofs are similar to the proofs of Lemmas 2.1.1 and 2.1.2.

Lemma 2.2.1. *Let* $0 \to B \hookrightarrow A \xrightarrow{\phi} M \to 0$ *be an exact sequence. Let* f *be an endomorphism of* A *such that* $\phi = \phi \circ f$*. Then* $f(A) + B = A$*,* $\operatorname{Ker} f \leq B$ *and* $B \cap f(A) = f(B)$*.*

Lemma 2.2.2. *Let* $0 \to B \overset{\varepsilon}{\hookrightarrow} A \xrightarrow{\phi} M \to 0$ *be an exact sequence. For every endomorphism* f *of* A*, the following are equivalent.*

(i) $\phi = \phi \circ f$*.*

(ii) The restriction of f *to* B *is an endomorphism of* B *and the morphism induced by* f *on* M *is the identity of* M*.*

(iii) There is a morphism $g \in \operatorname{Hom}_R(A, B)$ *such that* $f = id_A - \varepsilon \circ g$*.*

Proof. (i) ⇔ (ii) If $\phi = \phi \circ f$, then $f(B) \subseteq B$, by Lemma 2.2.1, so f induces an endomorphism of B. Consider the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{\varepsilon} & A & \xrightarrow{\phi} & M & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle f_{|B}} & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle h} & & \\ 0 & \longrightarrow & B & \xrightarrow{\varepsilon} & A & \xrightarrow{\phi} & M & \longrightarrow & 0 \end{array}$$

Then it follows that if $\phi = \phi \circ f$, then h is the identity on M.

The converse follows again by the above commutative diagram.

(i) ⇔ (iii) $\phi \circ f = \phi$ if and only if $\phi \circ (id_A - f) = 0$, that is if and only if the image of $id_A - f$ is contained in B, so $\mathrm{id}_A - f$ factors through ε. Equivalently, there is $g \in \operatorname{Hom}_R(A, B)$ such that $id_A - f = \varepsilon \circ g$. □

Recall that a submodule N of a module X is superfluous in X, if for every submodule H of X, $N + H = X$ implies $H = X$.

We define a notion of superfluous submodule with respect to classes in a cotorsion pair.

Definition 2.2.3. Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Let $B \in \mathcal{B}$ be a submodule of a right R-module X.

(i) We say that B is *$\mathcal{B}$-superfluous* in X if for every submodule H of X, $H + B = X$ and $H \cap B \in \mathcal{B}$ imply $H = X$.

(ii) We say that B is *$\mathcal{A}$-superfluous* in X if for every submodule K of B, $X/K \in \mathcal{A}$ implies $K = 0$.

Remark 2.2.4. *Let $\mathcal{P}_0$ be the class of projective right R-modules and consider the complete cotorsion pair $(\mathcal{P}_0, \text{Mod-}R)$. Our definition of* Mod-R-*superfluous submodule coincides with the classical definition of superfluous submodule.*

The next proposition shows how our the notions of $\mathcal{A}$-superfluous and $\mathcal{B}$-superfluous submodules are related to $\mathcal{A}$-covers.

Proposition 2.2.5. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Assume that*

$$0 \to B \hookrightarrow A \xrightarrow{\phi} M \to 0 \tag{2.3}$$

is a special $\mathcal{A}$-precover of the R-module M. The following hold.

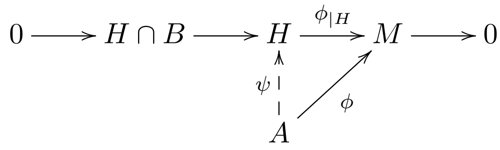
(i) If (2.3) is an $\mathcal{A}$-cover of M, then B is at the same time $\mathcal{B}$-superfluous and $\mathcal{A}$-superfluous in A.

(ii) If M admits an $\mathcal{A}$-cover and B is $\mathcal{B}$-superfluous in A, then (2.3) is an $\mathcal{A}$-cover.

(iii) If M admits an $\mathcal{A}$-cover and B is $\mathcal{A}$-superfluous in A , then (2.3) is an $\mathcal{A}$-cover.

(iv) Assume that $\mathcal{B}$ is closed under epimorphic images. Then (2.3) is an $\mathcal{A}$-cover if and only if B is at the same time $\mathcal{B}$-superfluous and $\mathcal{A}$-superfluous in A.

Proof. (i) Assume that (2.3) is an $\mathcal{A}$-cover of M. We first show that B is $\mathcal{B}$-superfluous in $\mathcal{A}$. Let $H \leq A$ be such that $H + B = A$ and $H \cap B \in \mathcal{B}$. Consider the following diagram where $\phi_{|H}$ is the restriction of ϕ to H.



The diagram can be completed by ψ, since $\text{Ext}^1_R(A, H \cap B) = 0$. Consider the inclusion ν of H into A; then by the above diagram it is clear that $\phi = \phi\nu\psi$. Since (2.3) is an $\mathcal{A}$-cover of M, we conclude that $\nu\psi$ is an automorphism of A, hence $H = A$.

Now we show that B is $\mathcal{A}$-superfluous in A. Let K be a submodule of B such that $A/K \in \mathcal{A}$. Consider the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{\mu} & A & \longrightarrow & A/K & \longrightarrow & 0 \\ & & {\scriptstyle \iota}\downarrow & \swarrow{\scriptstyle g} & & & & & \\ & & B & & & & & & \end{array}$$

where μ, ι are the inclusion maps. The diagram can be completed by g, since $\operatorname{Ext}^1_R(A/K, B) = 0$. Consider the inclusion ε of B into A. Then, by Lemma 2.2.2, $f = id_A - \varepsilon \circ g$ is an endomorphism of A satisfying $\phi \circ f = \phi$. By assumption f is an isomorphism, hence $\operatorname{Ker} f = 0$. Since g extends the inclusion of K into B we infer that $K \subseteq \operatorname{Ker} f$, hence $K = 0$.

(ii) and (iii) By Xu's result quoted in Corollary 1.2.10, it is enough to show that if B is $\mathcal{A}$-superfluous in A or if B is $\mathcal{B}$-superfluous in A, then B doesn't contain any non-zero summand of A.

Assume that $A = Y \oplus A_1$ with $Y \leq B$; then $A/Y \in \mathcal{A}$. So, if B is $\mathcal{A}$-superfluous in A we conclude that $Y = 0$.

We have $B + A_1 = A$ and $B = Y \oplus (B \cap A_1)$. $B \cap A_1 \in \mathcal{B}$, since it is a summand of B. Hence, if B is $\mathcal{B}$-superfluous in A we conclude that $A_1 = A$.

(iv) By (i), we have only to prove that if B is $\mathcal{B}$-superfluous and $\mathcal{A}$-superfluous in A, then (2.3) is an $\mathcal{A}$-cover of M. Let f be an endomorphism of A such that $\phi = \phi \circ f$. We must prove that f is an automorphism of A. First we show that f is an epimorphism. By Lemma 2.2.1, $\operatorname{Ker} f \leq B$, $f(B) = B \cap f(A)$ and $f(A) + B = A$. Consider the sequence $0 \to \operatorname{Ker} f \to B \stackrel{f}{\to} f(B) \to 0$. By assumption $f(B) \in \mathcal{B}$, thus $f(A) = A$, since B is $\mathcal{B}$-superfluous in A. It remains to show that f is a monomorphism. Since f is an epimorphism, we have $A \cong A/\operatorname{Ker} f$ and $\operatorname{Ker} f \subseteq B$. Hence $\operatorname{Ker} f = 0$, since B is $\mathcal{A}$-superfluous in A. □

Remark 2.2.6. *Note that the condition of the closure of the class $\mathcal{B}$ under epimorphic images in Proposition 2.1.5(iv) is the same as in Proposition 2.2.5(iv) and not a condition on the class $\mathcal{A}$ as one would expect.*

Proposition 2.2.7. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair and assume that*

$$0 \to B \stackrel{\varepsilon}{\hookrightarrow} A \stackrel{\phi}{\to} M \to 0$$

is a special $\mathcal{A}$-precover of M. The following are equivalent.

(i) ϕ is an $\mathcal{A}$-cover of M.

(ii) For every $g \in \operatorname{Hom}_R(A, B)$, $id_A - \varepsilon \circ g$ is an automorphism of A

(iii) $\varepsilon \operatorname{Hom}_R(A, B)$ is a right ideal of $\operatorname{End}(A)$ *contained in the Jacobson radical of* $\operatorname{End}(A)$.

Proof. (i) $\Rightarrow$ (ii). Let $g \in \mathrm{Hom}_R(A, B)$ and let $f = id_A - \varepsilon \circ g$. Then $\phi \circ f = \phi - \phi \circ \varepsilon \circ g = \phi$, hence f is an isomorphism of A.

(ii) $\Rightarrow$ (iii). It is clear that $\varepsilon \,\mathrm{Hom}_R(A, B)$ is a right ideal of $\mathrm{End}(A)$. Let $g \in \mathrm{Hom}_R(A, B)$. For every $h \in \mathrm{End}(A)$, $id_A - \varepsilon \circ g \circ h$ is an invertible element of $\mathrm{End}(A)$, by (ii). Hence $\varepsilon \,\mathrm{Hom}_R(A, B)$ is contained in the Jacobson radical of $\mathrm{End}(A)$.

(iii) $\Rightarrow$ (i). Let $f \in \mathrm{End}(A)$ be such that $\phi \circ f = \phi$. By Lemma 2.2.2 (iii), $id_A - f = \varepsilon \circ g$ for a morphism $g \in \mathrm{Hom}_R(A, B)$ and by assumption, $\varepsilon \circ g$ belongs to the Jacobson radical of $\mathrm{End}(A)$. Hence $f = id_A - \varepsilon \circ g$ is a unit of $\mathrm{End}(A)$.

□

Proposition 2.2.8. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair and assume that a module M admits an $\mathcal{A}$-cover, and consider the following special $\mathcal{A}$-precover of M.*

$$0 \to B \overset{\varepsilon}{\hookrightarrow} A \overset{\phi}{\to} M \to 0 \tag{2.4}$$

Then (2.4) is an $\mathcal{A}$-cover of M if and only if for every $g \in \mathrm{Hom}_R(A, B)$, $1_A - \varepsilon \circ g$ is a monomorphism.

Proof. The necessary condition follows by Proposition 2.2.7. Assume that $1_A - \varepsilon \circ g$ is a monomorphism for every $g \in \mathrm{Hom}_R(A, B)$ and that π is not a cover. By Theorem 1.2.9 there is a direct summand $0 \neq X$ of A contained in B. Let $p_X \colon A \to X$ be the projection map and $i \colon X \to B$ the inclusion map. Then $g = i \circ p_X$ is a map in $\mathrm{Hom}_R(A, B)$ and for every $x \in X$ we have $\varepsilon \circ g(x) = x$. Hence $\mathrm{Ker}(1_A - \varepsilon \circ g)$ contains X and $1_A - \varepsilon \circ g$ is not a monomorphism. □

Another useful result is the following, which as in Proposition 2.1.8 is due to [9] but with a slightly modified proof.

Proposition 2.2.9. *Let $(\mathcal{A}, \mathcal{B})$ be a complete cotorsion pair over a ring R. Assume that $A \overset{\phi}{\to} M \to 0$ is an $\mathcal{A}$-cover of the R-module M. Let α be an automorphism of M and let β be an endomorphism of A such that $\phi\beta = \alpha\phi$. Then β is an automorphism of A.*

Proof. By the Wakamatsu Lemma (Lemma 1.3.2) ϕ gives rise to an exact sequence

$$0 \to B \overset{\mu}{\to} A \overset{\phi}{\to} M \to 0$$

with $B \in \mathcal{B}$. Since α is an automorphism of M, it is immediate to see that $\mathrm{Ker}\,\alpha\phi \cong B \in \mathcal{B}$ and $\alpha\phi f = \alpha\phi$ implies f is an isomorphism, so the following is an $\mathcal{A}$-cover of M.

$$0 \to B \to A \overset{\alpha\phi}{\to} M \to 0$$

Let β be as assumed and consider an endomorphism g of A such that $\alpha\phi g = \phi$. Then $\phi\beta g = \phi$ and thus βg is an automorphism of A, since ϕ is an $\mathcal{A}$ cover of M. This implies that β is an epimorphism.
To see that β is a monomorphism, note that $\alpha\phi g\beta = \phi\beta g\beta = \phi\beta = \alpha\phi$, thus by the cover property of $\alpha\phi$, $g\beta$ is an automorphism, thus β is an automorphism as required. □

An analogous example of Example 2.1.9 holds.

Example 2.2.10. Let R be a commutative ring and S a multiplicative subset, and fix a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in Mod-R. Consider the localisation of R at the multiplicative subset S, denoted $R[S^{-1}]$, and let M be an $R[S^{-1}]$-module with a $\mathcal{A}$-cover in Mod-R.

$$0 \to B \to A \xrightarrow{\phi} M \to 0 \tag{2.5}$$

As M is an $R[S^{-1}]$-module, multiplication by element of S is an automorphism of M. Therefore applying Proposition 2.1.8, multiplication by an element of S is also an automorphism of A and so A is an $R[S^{-1}]$-module. One concludes that the short exact sequence (2.5) is a sequence in Mod-$R[S^{-1}]$, as moreover $R \to R[S^{-1}]$ is a ring epimorphism so the embedding Mod-$R[S^{-1}] \to$ Mod-R is fully faithful.

2.3 Covering implies enveloping for hereditary cotorsion pairs

We want to show that for a cotorsion pair $(\mathcal{A}, \mathcal{B})$ if $\mathcal{A}$ is covering then $\mathcal{B}$ is enveloping. We begin by showing the following.

Proposition 2.3.1. *If $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair such that $\mathcal{A}$ is covering, then every $A \in \mathcal{A}$ has a $\mathcal{B}$-envelope.*

Proof part 1. Let $\varepsilon\colon A \to E(A)$ be the injective envelope of A and take $\gamma\colon A' \to E(A)/A$ to be an $\mathcal{A}$-cover of $E(A)/A$, and consider the following pullback of $E(A) \xrightarrow{\pi} E(A)/A \xleftarrow{\gamma} A'$.

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & A & \xrightarrow{\varepsilon} & E(A) & \xrightarrow{\pi} & E(A)/A & \longrightarrow & 0 \\ & & \| & & {\scriptstyle\alpha}\uparrow & & {\scriptstyle\gamma}\uparrow & & \\ 0 & \longrightarrow & A & \xrightarrow{\mu} & Z & \xrightarrow{\rho} & A' & \longrightarrow & 0 \\ & & & & {\scriptstyle\beta}\uparrow & & {\scriptstyle\delta}\uparrow & & \\ & & & & B & = & B & & \\ & & & & \uparrow & & \uparrow & & \\ & & & & 0 & & 0 & & \end{array} \tag{2.6}$$

Then $A' \in \mathcal{A}$ by construction and $Z \in \mathcal{B}$ as $\mathcal{B}$ is closed under extensions, thus μ is a special $\mathcal{B}$-preenvelope of A. Moreover, Z is also in $\mathcal{A}$ as $\mathcal{A}$ is closed under extensions. We claim that μ is a $\mathcal{B}$-envelope of A. Take $f\colon Z \to Z$ such that $f\mu = \mu$. We will show that this f is an isomorphism which will be proved in the following three lemmas. $\square$

We begin by proving a lemma which allows us to use that a module is an essential submodule of its injective envelope.

Lemma 2.3.2. *Consider the following commuting diagram of short exact sequences such that $\mu'(A)$ is essential in M'.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\mu} & M & \xrightarrow{\rho} & N & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle k} & & \\ 0 & \longrightarrow & A & \xrightarrow{\mu'} & M' & \xrightarrow{\rho'} & N' & \longrightarrow & 0 \end{array}$$

Then for every submodule H of M such that $H \cap \mu(A) = 0$, H is contained in the kernel of f.

Proof. Take $H \leq M$ such that $H \cap \mu(A) = 0$, so $\mu(A) + H = \mu(A) \oplus H$. Then we have the following induced diagram, where $\mu_0\colon A \to \mu(A) \oplus H$ is μ with the codomain restricted to $\mu(A) \oplus H \subseteq M$, and $f_{\restriction_{\mu(A)\oplus H}}$, $k_{\restriction}$ are the obvious restriction maps.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\mu_0} & \mu A \oplus H & \xrightarrow{\rho} & \mu A \oplus H/\mu A \cong H & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f_{\restriction_{\mu A \oplus H}}} & & \downarrow{\scriptstyle k_{\restriction_{(\mu A \oplus H)/\mu A}}} & & \\ 0 & \longrightarrow & A & \xrightarrow{\mu'} & M' & \xrightarrow{\rho'} & N' & \longrightarrow & 0 \end{array}$$

We will show that $f(H) = 0$. First we claim that $f(H) \cap \mu'(A) = 0$. Take $x \in f(H) \cap \mu'(A)$. Then $x = f(y) = \mu'(a)$ for $a \in A$ and $y \in H$. So $k\rho(y) =$

$\rho' f(y) = \rho'\mu'(a) = 0$, so $\rho(y) \in \operatorname{Ker} k$. However, the snake lemma induces an isomorphism $\rho_{\restriction \operatorname{Ker} f \cap \mu(A)+H} \mid \operatorname{Ker}(f_{\restriction(\mu(A)+H)}) \to \operatorname{Ker}(k_{\restriction(\mu(A)+H)/\mu(A)})$, thus $y \in \operatorname{Ker} f$, so $x = f(y) = 0$. By assumption, $\mu'(A)$ is essential in M', so $f(H) \cap \mu'(A) = 0$ implies $f(H) = 0$. ☐

Lemma 2.3.3. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair such that $\mathcal{A}$ is covering. Take $f\colon Z \to Z$ such that $f\mu = \mu$ using the notation of Diagram (2.6). Then f is a monomorphism.*

Proof. By Lemma 2.3.2, as $\mu(A) \cap \operatorname{Ker} f = 0$ it follows that $\alpha(\operatorname{Ker} f) = 0$ so $\operatorname{Ker} f \subseteq \operatorname{Ker} \alpha$. Therefore, the maps $f\colon Z \to Z$ and $\alpha\colon Z \to E(A)$ both factor through $p\colon Z \to Z/\operatorname{Ker} f$ so that $f = \bar{f}p$ and $\alpha = \bar{\alpha}p$ where $\bar{f}, \bar{\alpha}$ are as in (2.7). Using that $E(A)$ is an injective module we find that α factors through f as follows.

$$\begin{array}{ccccc} 0 & \longrightarrow & Z/\operatorname{Ker} f & \xrightarrow{\bar{f}} & Z \\ & & \bar{\alpha}\downarrow & \swarrow g & \\ & & E(A) & & \end{array} \tag{2.7}$$

That is, there exists a $g\colon Z \to E(A)$ such that $gf = g\bar{f}p = \bar{\alpha}p = \alpha$. Let k and l be the unique induced maps that make the following short exact sequences commute as follows (where the columns are not exact).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\mu} & Z & \xrightarrow{\rho} & A' & \longrightarrow & 0 \\ & & \| & & \downarrow f & & \downarrow k & & \\ 0 & \longrightarrow & A & \xrightarrow{\mu} & Z & \xrightarrow{\rho} & A' & \longrightarrow & 0 \\ & & \| & & \downarrow g & & \downarrow l & & \\ 0 & \longrightarrow & A & \xrightarrow{\varepsilon} & E(A) & \xrightarrow{\pi} & E(A)/A & \longrightarrow & 0 \end{array} \tag{2.8}$$

Using also the commutativity of (2.6), we have that $\gamma\rho = \pi\alpha = \pi g f = l\rho f = lk\rho$. As ρ is an epimorphism, we conclude that $lk = \gamma$.

We now claim that k is a monomorphism by using the $\mathcal{A}$-cover property of γ.

Consider the map $l\colon A' \to E(A)/A$. By the $\mathcal{A}$-precover property of γ, there exists an $h\colon A' \to A'$ such that $l = \gamma h$. Thus as $\gamma h k = lk = \gamma$, hk is an automorphism, thus k is a monomorphism and therefore also f, which is seen by an application of the snake lemma to the upper two short exact sequences of (2.8) to find that $\operatorname{Ker} f \cong \operatorname{Ker} k$. ☐

Lemma 2.3.4. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair such that $\mathcal{A}$ is covering. Take $f\colon Z \to Z$ such that $f\mu = \mu$ as in (2.6). Then f is an isomorphism.*

Proof. By Lemma 2.3.3 we have that f is a monomorphism. We now use the fact that $Z \in \mathcal{A} \cap \mathcal{B}$, thus f composed with the canonical epimorphism $Z \to Z/f(Z)$ is an $\mathcal{A}$-precover of $Z/f(Z)$. We extract the following $\mathcal{A}$-cover of $Z/f(Z)$ where $\nu = (\operatorname{coker} f)_{\restriction_{Z'}}$.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Z'' & \xrightarrow{f'} & Z' & \xrightarrow{\nu} & Z/f(Z) & \longrightarrow & 0 \\
 & & \big\downarrow \oplus & & \big\downarrow \oplus & & \| & & \\
0 & \longrightarrow & Z & \xrightarrow{f} & Z & \longrightarrow & Z/f(Z) & \longrightarrow & 0
\end{array}
$$

We want to show that $Z' = 0$ which implies that $Z/f(Z) = 0$ and this proves our claim. In fact, we will see that if Z' is non-zero, there exists a non-zero direct summand of A' which is contained in $\operatorname{Ker} \gamma$, which contradicts the $\mathcal{A}$-cover property of $\gamma \colon A' \to E(A)/A$ by Corollary 1.2.10.

Firstly, by the snake lemma applied to the top row of diagram (2.8), $\rho \colon Z \to A'$ induces an isomorphism $\bar{\rho} \colon Z/f(Z) \cong A'/k(A') = \operatorname{Coker} k$. Thus by the precover property of ν, the map $\bar{\rho}^{-1} \operatorname{coker} k \colon A' \to Z/f(Z)$ must factor through ν, as in the bottom row of the following diagram (2.9), i.e. there exists a $j \colon A' \to Z'$ such that $\nu j = \bar{\rho}^{-1} \operatorname{coker} k$. The top square of (2.9) commutes by the construction of ν, that is $(\operatorname{coker} k)(\rho_{\restriction_{Z'}}) = \bar{\rho}\nu$.

$$
\begin{array}{ccc}
Z' & \xrightarrow{\quad \nu \quad} & Z/f(Z) \\
\big\downarrow \rho_{\restriction_{Z'}} & & \| \\
A' & \xrightarrow{\bar{\rho}^{-1} \operatorname{coker} k} & Z/f(Z) \\
\big\downarrow j & & \| \\
Z' & \xrightarrow{\quad \nu \quad} & Z/f(Z)
\end{array}
\tag{2.9}
$$

By the $\mathcal{A}$-cover property of ν, $j\rho_{\restriction_{Z'}}$ is an isomorphism, so $\rho_{\restriction_{Z'}}$ is a monomorphism, and moreover $\rho(Z')$ is a direct summand in A'. It follows that $Z' \cap \mu(A) = 0$ since $\mu(A) = \operatorname{Ker} \rho$ from (2.6). Therefore by Lemma 2.3.2, $Z' \subseteq \operatorname{Ker} \alpha$, and as the restriction map $\rho_{\restriction_{\operatorname{Ker} \alpha}} \colon \operatorname{Ker} \alpha \cong \operatorname{Ker} \gamma$ is an isomorphism by the snake lemma, $\rho(Z') \subseteq \operatorname{Ker} \gamma$. Thus we have shown that $\rho(Z')$ is a direct summand of A' which is contained in $\operatorname{Ker} \gamma$, thus $\rho(Z') = 0$ by the cover property of γ and by Corollary 1.2.10. Therefore also $Z' = 0$ as $\rho_{\restriction_{Z'}}$ is a monomorphism. We conclude that $Z/f(Z) = 0$, so f is an isomorphism. □

Next we would like to prove the following lemma, whose dual result was proved by Bazzoni-Positselski in [14, Lemma 8.3]. Here we will present the dual result. For this proof, one must assume that the cotorsion pair $(\mathcal{A}, \mathcal{B})$ is hereditary.

Lemma 2.3.5. *[14, Dual of Lemma 8.3] Let $(\mathcal{A}, \mathcal{B})$ be a hereditary complete cotorsion pair in* Mod-R. *Suppose every module in $\mathcal{A}$ has a $\mathcal{B}$-envelope. Then $\mathcal{B}$ is enveloping, that is every module has a $\mathcal{B}$-envelope.*

Proof. Fix an R-module M and take $0 \to B \overset{\varepsilon}{\to} A \overset{\pi}{\to} M \to 0$ to be a special $\mathcal{A}$-precover of M. Additionally, by assumption A has a $\mathcal{B}$-envelope which is denoted $0 \to A \overset{\gamma}{\to} B' \overset{\delta}{\to} A' \to 0$. Thus we take the pushout of the maps $B' \overset{\gamma}{\leftarrow} A \overset{\pi}{\to} M$ as follows.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \xrightarrow{\varepsilon} & A & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \mu} & & \\
0 & \longrightarrow & B & \xrightarrow{\alpha} & B' & \xrightarrow{\beta} & Z & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \rho} & & \\
 & & & & A' & = & A' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{2.10}$$

As $\mathcal{B}$ is a coresolving class, $Z \in \mathcal{B}$ by the middle row as $B, B' \in \mathcal{B}$. Therefore, the right vertical column is a $\mathcal{B}$-preenvelope of M. Moreover, $B' \in \mathcal{A}$ from the middle vertical short exact sequence as $\mathcal{A}$ is closed under extensions.

Fix an automorphism f of Z such that $f\mu = \mu$. We will show that this f is an isomorphism to conclude that μ is in fact a $\mathcal{B}$-envelope. Take the pullback of $Z \overset{f}{\to} Z \overset{\beta}{\leftarrow} B'$, and denote it by W as follows. Moreover, consider the maps $B' \overset{\gamma}{\leftarrow} A \overset{\mu\pi}{\to} Z$, which make the following diagram commute. Thus there exists a homomorphism h such that $gh = \gamma$.

$$\begin{array}{ccccc}
A & & & \xrightarrow{\mu\pi} & \\
 & \overset{h}{\searrow} & & & \\
 & & W & \xrightarrow{\beta'} & Z \\
 & {\scriptstyle \gamma} & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
 & & B' & \xrightarrow{\beta} & Z
\end{array} \tag{2.11}$$

Thus we have the following commuting diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \xrightarrow{\varepsilon} & A & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \mu} & & \\
0 & \longrightarrow & B & \xrightarrow{\alpha'} & W & \xrightarrow{\beta'} & Z & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & B & \xrightarrow{\alpha} & B' & \xrightarrow{\beta} & Z & \longrightarrow & 0
\end{array} \tag{2.12}$$

The composition of right column $f\mu$ is μ by assumption and the centre column is γ as previously shown. Moreover, $W \in \mathcal{B}$ as both $B, Z \in \mathcal{B}$.
We will now use the envelope property of γ. As $W \in \mathcal{B}$, there exists a map $k\colon B' \to W$ such that $k\gamma = h$ as follows in the top left square. The lower-left square follows from Diagram 2.11.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\gamma} & B' & \longrightarrow & A' & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle k} & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{h} & W & \longrightarrow & \operatorname{Coker} h & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle g} & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{\gamma} & B' & \longrightarrow & A' & \longrightarrow & 0
\end{array}$$

Therefore as γ is an envelope and $\gamma = gk\gamma$, gk is an automorphism of B'. This means that there exists a map $k'\colon B' \to B'$ such that $gkk' = k'gk = \mathrm{id}_{B'}$. First note that $\gamma = k'gk\gamma = k'\gamma$. Moreover $kk'\alpha = kk'\gamma\varepsilon = k\gamma\varepsilon = h\varepsilon = \alpha'$. Therefore there exists an l such that the following diagram commutes.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \xrightarrow{\alpha} & B' & \xrightarrow{\beta} & Z & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle kk'} & & \downarrow{\scriptstyle l} & & \\
0 & \longrightarrow & B & \xrightarrow{\alpha'} & W & \xrightarrow{\beta'} & Z & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & B & \xrightarrow{\alpha} & B' & \xrightarrow{\beta} & Z & \longrightarrow & 0
\end{array} \tag{2.13}$$

Moreover, the centre column composed is $\mathrm{id}_{B'}$, therefore $fl\beta = \beta gkk' = \beta$, and we conclude that $fl = \mathrm{id}_Z$ as β is surjective.
Finally, we have the following equalities.

$$l\mu\pi = \beta kk'\gamma = \beta' k\gamma = \beta' h = \mu\pi$$

Therefore $l\mu = \mu$. By the above proof, we can conclude that also l is an epimorphism, therefore f must be a monomorphism, so is an automorphism as required. □

We now state the main result of this section.

Theorem 2.3.6. *Let $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair. If $\mathcal{A}$ is covering then $\mathcal{B}$ is enveloping.*

Proof. Suppose $(\mathcal{A}, \mathcal{B})$ is a hereditary cotorsion pair such that $\mathcal{A}$ is covering. Then by Proposition 2.3.1, every module in $\mathcal{A}$ has a $\mathcal{B}$-envelope, and since $(\mathcal{A}, \mathcal{B})$ is hereditary from Lemma 2.3.5 this is sufficient for $\mathcal{B}$ to be enveloping. □

Chapter 3

The class $\mathcal{P}_1(R)$

In this chapter we study the class $\mathcal{P}_1(R)$ over rings which have a classical ring of quotients, with the aim to study Enochs' Conjecture in this setting, that is the question of when $\mathcal{P}_1(R)$ is covering implies that $\mathcal{P}_1(R)$ is closed under direct limits. We were mainly interested in finding an example of a positive result when the cotorsion pair $(\mathcal{P}_1(R), \mathcal{P}_1(R)^{\perp})$ is not of finite type, since a consequence of Theorem 1.3.6 is that Enochs' Conjecture holds for the cotorsion pairs $(\mathcal{P}_1(R), \mathcal{P}_1(R)^{\perp})$ of finite type.

In the investigation of when $\mathcal{P}_1(R)$ is covering, the class $\varinjlim \mathcal{P}_1(R)$ plays an important role, although it is not always well understood. Unlike the case of the projective modules where $\varinjlim \mathcal{P}_0(R) = \mathcal{F}_0(R)$, it is not necessarily true that the direct limit closure $\varinjlim \mathcal{P}_1(R)$ coincides with the modules of weak dimension less than or equal to 1, $\mathcal{F}_1(R)$. However the inclusion $\varinjlim \mathcal{P}_1(R) \subseteq \mathcal{F}_1(R)$ always holds, although an example of rings where $\varinjlim \mathcal{P}_1(R) \subsetneq \mathcal{F}_1(R)$ can be found in [29, Example 9.12]. For certain nice rings, such as commutative domains, the two classes $\varinjlim \mathcal{P}_1(R)$ and $\mathcal{F}_1(R)$ coincide ([29, Theorem 9.10]).

The aim of Section 3.1 is to give a characterisation of the class $\varinjlim \mathcal{P}_1(R)$ when R has a classical ring of quotients, as a generalisation of a result from [11] without the assumption that $\operatorname{f.dim} Q = 0$. The main result of this section is in Proposition 3.1.8, which states that $\varinjlim \mathcal{P}_1(R)$ is exactly the intersection of $\mathcal{F}_1(R)$ with the left Tor_1^R-orthogonal of the minimal cotilting class of Q-Mod, $\mathcal{C}(Q) := \mathcal{P}_1(\text{mod-}Q)^{\intercal}$.

Next, Section 3.2 is divided into three parts, where the main aim is to show that if $\mathcal{P}_1(R)$ is covering, $\mathcal{P}_1(R)$ must be closed under direct limits for all commutative semihereditary rings. This provides us with an example of a class of rings for which $\mathcal{P}_1(R)$ satisfies Enochs' Conjecture and $(\mathcal{P}_1(R), \mathcal{P}_1(R)^{\perp})$ is not (necessarily) of finite type.

We begin in Section 3.2 by looking at some consequences of $\mathcal{P}_1(R)$ being covering on the classical ring of quotients of R in Subsection 3.2.1. Next we collect some equivalent characterisations of certain commutative rings, specifically semihereditary rings, hereditary rings, and the larger class of rings of weak global dimension less than or equal to one. Additionally we provide some theorems and lemmas which will be required in the final subsection. Finally the aforementioned main result is Theorem 3.2.18 in Subsection 3.2.3.

In this chapter, R will always be a ring such that the regular elements Σ satisfy both the left and right Ore conditions. Then the *classical ring of quotients* of R, denoted $Q = Q(R)$ is the ring $R[\Sigma^{-1}] = [\Sigma^{-1}]R$ which is flat both as a right and left R-module. Additionally, we recall that an ideal I of R is called *regular* if I contains a regular element of R, that is $I \cap \Sigma \neq \emptyset$.

Recall that $\mathcal{P}_1(R)$ denotes the class of right R-modules with projective dimension less than or equal to one. We denote by $\mathcal{B}(R)$ the right orthogonal class $\mathcal{P}_1(R)^{\perp}$.

For a ring R, we will denote by $\mathcal{C}(R)$ the minimal 1-cotilting class of cofinite type, so $\mathcal{C}(R) = \mathcal{P}_1(\text{mod-}R)^{\intercal_1} = \mathcal{P}_1(\text{mod-}R)^{\intercal}$. To see that this is in fact minimal, recall that a 1-cotilting class $\mathcal{C}$ is of cofinite type if there exists a set $\mathcal{S} \subseteq \mathcal{P}_1(\text{mod-}R)$ such that $\mathcal{S}^{\intercal} = \mathcal{C}$, thus the 1-cotilting class $\mathcal{C}(R)$ is contained in every other 1-cotilting class $\mathcal{C}$ as $\mathcal{S}^{\intercal} \supseteq \mathcal{P}_1(\text{mod-}R)^{\intercal}$.

When it is not clear in which module category we are taking the Tor-orthogonal, for clarity we will denote the Tor_1^R-orthogonal class in Mod-R as ${\intercal}^R$.

3.1 The direct limit closure of $\mathcal{P}_1(R)$

The purpose of this section is to describe the class $\varinjlim \mathcal{P}_1(R)$. It is a generalisation of [11, Theorem 6.7 (vi)] which states that when $\operatorname{f.dim} Q = 0$, $\varinjlim \mathcal{P}_1(R) = \mathcal{F}_1(R) \cap {}^{\intercal}Q\text{-Mod}$.

We begin by recalling the following Corollary, which states that one can consider only the finitely presented modules in $\mathcal{P}_1(R)$ to find its direct limit closure.

Theorem 3.1.1. *[29, Corollary 9.8] Let R be a ring. Then $\varinjlim \mathcal{P}_1(R) = \varinjlim \mathcal{P}_1(\text{mod-}R) = {}^{\intercal}(\mathcal{P}_1(\text{mod-}R)^{\intercal}) = {}^{\intercal}\mathcal{C}(R)$ and $\mathcal{P}_1(\text{mod-}R)^{\intercal} = \left(\varinjlim \mathcal{P}_1(R)\right)^{\intercal}$*

We now state some results from [11]. Following the nomenclature of [11], in this chapter $\mathcal{D}$ will denote the class of right R-modules $\{D \mid \mathrm{Ext}_R^1(R/rR, D) = 0, r \in \Sigma\}$ which are called the divisible modules in Mod-R. Similarly, $\mathcal{TF}$ will denote the class of left R-modules $\{N \mid \mathrm{Tor}_1^R(R/rR, N) = 0, r \in \Sigma\}$ which are called the torsion-free modules in R-Mod. The analogous statements hold for the divisible modules in R-Mod ($\mathcal{D} = \{D \mid \mathrm{Ext}_R^1(R/Rr, D) =$

$0, r \in \Sigma\}$) and the torsion-free modules in Mod-R ($\mathcal{TF} = \{N \mid \mathrm{Tor}_1^R(N, R/Rr) = 0, r \in \Sigma\}$).

Remark 3.1.2. The transpose of each of R/rR and R/Rr is the other. Explicitly, $\mathrm{Tr}\, R/rR \cong R/Rr$ and $\mathrm{Tr}\, R/Rr \cong R/rR$ for $r \in \Sigma$ (up to stable equivalence). So by Lemma 1.7.5, $\mathrm{Ext}_R^1(R/rR, M) \cong \mathrm{Ext}_R^1(\mathrm{Tr}\, R/Rr, M) \cong M \otimes_R R/Rr$, and $\mathrm{Tor}_1^R(R/rR, N) \cong \mathrm{Tor}_1^R(\mathrm{Tr}\, R/Rr, N) \cong \mathrm{Hom}_R(R/Rr, N)$, so the definition of $\mathcal{D}$ and $\mathcal{TF}$ coincide with the Σ-divisible and Σ-torsion-free classes defined in Section 1.7 with respect to the left Gabriel topology generated by principal ideals of the form Rr where $r \in \Sigma$.

We will first quote three useful lemmas from [11]. The first one is a well-known fact about Ore localisations which we paraphrase for our convenience. It allows us to state that the torsion-free class defined above coincides with the torsion-free class that arises from the perfect localisation $R \to Q$. From another point of view, it is straightforward to conclude from Remark 3.1.2.

Lemma 3.1.3. *[11, Lemma 5.3] Let R be a ring with classical ring of quotients Q. Then for every torsion-free left R-module ${}_RN$, $\mathrm{Tor}_1^R(Q/R, N) = 0$. Analogously, for every torsion-free right R-module N_R, $\mathrm{Tor}_1^R(N, Q/R) = 0$.*

The following lemma establishes a relationship between $\mathcal{P}_1(R)$ and $\mathcal{P}_1(Q)$.

Lemma 3.1.4. *[11, Lemma 6.2] Let R be a ring with classical ring of quotients Q. Then a right Q-module V is in $\mathcal{P}_1(Q)$ if and only if there exists $M_R \in \mathcal{P}_1(R)$ such that $V = M \otimes_R Q$.*

Finally we have a lemma about projective presentations of finitely presented modules.

Lemma 3.1.5. *[11, Lemma 6.4] Let R be a ring and C a module in $\mathcal{P}_1(\text{mod-}R)$. Then there is a finitely generated projective module P and a short exact sequence of the following form.*

$$0 \to R^m \to R^n \to C \oplus P \to 0$$

We now apply these lemmas to the minimal 1-cotilting class of cofinite type, $\mathcal{C}(R) = \mathcal{P}_1(\text{mod-}R)^{\intercal}$.

The following lemma is a sort of analogue to Lemma 3.1.5 using the results of Lemma 3.1.4 for the finitely presented case.

Lemma 3.1.6. *Let R be a ring with classical ring of quotients Q. Then the following hold.*

(i) If $M \in \mathcal{P}_1(\text{mod-}R)$, then $M \otimes_R Q \in \mathcal{P}_1(\text{mod-}Q)$.

(ii) If $C_Q \in \mathcal{P}_1(\text{mod-}Q)$, then there exists a $P_Q \in \mathcal{P}_0(\text{mod-}Q)$ and $N_R \in \mathcal{P}_1(\text{mod-}R)$ such that $C_Q \oplus P_Q \cong N \otimes_R Q$.

Proof. (i) is straightforward as Q is both left and right flat.
(ii) Take $C_Q \in \mathcal{P}_1(\text{mod-}Q)$. Then by Lemma 3.1.5, there exists a $P_Q \in \mathcal{P}_0(\text{mod-}Q)$ such that there is the following short exact sequence.

$$0 \to Q^m \xrightarrow{\mu} Q^n \to C_Q \oplus P_Q \to 0$$

Arguing as in the proof of Lemma 3.1.4, it follows that there exists a finitely presented N_R in $\mathcal{P}_1(R)$ of the desired form. We will rewrite the proof here for this special case. Fix a basis $(e_1, \ldots, e_m)$ and $(f_1, \ldots, f_n)$ of Q^m and Q^n in Mod-Q respectively. Let A the $n \times m$ matrix with entries in Q which represents right multiplication of the monomorphism μ on each of the basis entries $(e_1, \ldots, e_m)$. As A is a finite matrix (specifically, a finite column matrix), one can take the multiple of the denominators of all the entries in the ith column of A. As this multiple is a unit in Q (and is a regular element of R) one can adjust the corresponding basis elements of Q^m accordingly. Explicitly, if $[a_1 \cdots a_n]^\intercal$ is the ith column of A and $d_i \in \Sigma$ is such that $d_i[a_1 \cdots a_n]^\intercal$ is a column in R^n, then one takes $(e_1 d_1, \ldots, e_m d_m)$ as a basis of Q^m. Therefore μ can be represented by a matrix B with entries in R. The matrix B represents a morphism ε of right free R-modules $R^m \to R^n$, and ε is moreover a monomorphism as $R^m \subseteq Q^m$ and the matrix B commutes with this inclusion map. Therefore $\varepsilon \otimes_R \mathrm{id}_Q = \mu$ so $\operatorname{Coker} \varepsilon \in \mathcal{P}_1(\text{mod-}R)$ is the desired N. □

Lemma 3.1.7. *Let R be a ring with classical ring of quotients Q. Then the following hold.*

(i) $\mathcal{C}(R) \cap Q\text{-Mod} = \mathcal{C}(Q)$.

(ii) *If* $Z \in \mathcal{C}(R)$, *then* $Q \otimes_R Z \in \mathcal{C}(Q)$.

Proof. (i) For modules $M \in \text{Mod-}R$ and $N \in Q\text{-Mod}$, there is the following isomorphism as Q is flat from Equation 1.5.

$$\operatorname{Tor}_1^R(M, N) \cong \operatorname{Tor}_1^Q(M \otimes_R Q, N) \tag{3.1}$$

Suppose $M \in \mathcal{P}_1(\text{mod-}R)$. Then $N \in \mathcal{C}(R) \cap Q\text{-Mod}$ if and only if the left-hand side of (3.1) vanishes. On the other hand, $N \in \mathcal{C}(Q)$ if and only if $N \in \mathcal{P}_1(Q)^\intercal$ which in view of Lemma 3.1.4 amounts to the right-hand side of (3.1) vanishing. Therefore $N \in \mathcal{C}(R) \cap Q\text{-Mod}$ if and only if $N \in \mathcal{C}(Q)$, which proves $\mathcal{C}(R) \cap Q\text{-Mod} = \mathcal{C}(Q)$.

For (ii), we first note that $\mathcal{C}(R)$ is closed under direct limits as Tor commutes with direct limits so a Tor-orthogonal must always be closed under direct limits.

As Q is both left and right flat, one can write Q as a direct limit of finitely generated free right R-modules $\varinjlim_\alpha R^{n_\alpha}$. Fix a $Z \in \mathcal{C}(R)$. Then $Q \otimes_R Z \cong \varinjlim_\alpha Z^{n_\alpha}$ which must be in $\mathcal{C}(R)$ as $\mathcal{C}(R)$ is closed under direct limits. Moreover, $Q \otimes_R Z \in Q\text{-Mod}$, thus since $\mathcal{C}(R) \cap Q\text{-Mod} = \mathcal{C}(Q)$ from (i), $Q \otimes_R Z \in \mathcal{C}(Q)$. □

Proposition 3.1.8. *Let R be a ring with classical ring of quotients Q. Then the direct limit closure of $\mathcal{P}_1(R)$ can be written as follows.*

$$\varinjlim \mathcal{P}_1(R) = \mathcal{F}_1(R) \cap {}^{\intercal_R}\mathcal{C}(Q)$$

In particular, if $\mathrm{f.dim}\, Q = 0$, $\varinjlim \mathcal{P}_1(R) = \mathcal{F}_1(R) \cap {}^{\intercal}Q\text{-Mod}$.

Proof. We will simply write ${}^{\intercal}\mathcal{C}(Q)$ for the left Tor_1^R-orthogonal of $\mathcal{C}(Q)$ in Mod-R instead of ${}^{\intercal_R}\mathcal{C}(Q)$.

First we suppose that $M \in \varinjlim \mathcal{P}_1(R)$ and show that $M \in \mathcal{F}_1(R) \cap {}^{\intercal}\mathcal{C}(Q)$. The inclusion $\varinjlim \mathcal{P}_1(R) \subseteq \mathcal{F}_1(R)$ always holds so it remains to show that $M \in {}^{\intercal}\mathcal{C}(Q)$. By Theorem 3.1.1, if $M \in \varinjlim \mathcal{P}_1(R)$, then $M \in {}^{\intercal}\mathcal{C}(R)$. As $\mathcal{C}(Q) \subseteq \mathcal{C}(R)$ by Lemma 3.1.7(i), it follows that ${}^{\intercal}\mathcal{C}(R) \subseteq {}^{\intercal}\mathcal{C}(Q)$, so $M \in {}^{\intercal}\mathcal{C}(Q)$ as required.

For the converse, fix $M \in \mathcal{F}_1(R) \cap {}^{\intercal}\mathcal{C}(Q)$. We will show $M \in \varinjlim \mathcal{P}_1(R) = {}^{\intercal}\mathcal{C}(R)$.

As $\{R/sR \mid s \in \Sigma\} \subseteq \mathcal{P}_1(\text{mod-}R)$, the class $\mathcal{C}(R)$ is contained in $\{R/sR \mid s \in \Sigma\}^{\intercal}$, so is torsion-free. Hence by Lemma 3.1.3, $\mathrm{Tor}_1^R(Q/R, N) = 0$ for every $N \in \mathcal{C}(R)$.

Therefore for every $N \in \mathcal{C}(R)$ we have the following short exact sequence in R-Mod.

$$0 \to N \to Q \otimes_R N \to Q/R \otimes_R N \to 0 \tag{3.2}$$

We apply $(M \otimes_R -)$ to the short exact sequence (3.2) to get the following short exact sequence (3.3).

$$0 = \mathrm{Tor}_2^R(M, Q/R \otimes_R N) \to \mathrm{Tor}_1^R(M, N) \to \mathrm{Tor}_1^R(M, Q \otimes_R N) = 0 \tag{3.3}$$

The first term vanishes as $M \in \mathcal{F}_1(R)$ and the last term vanishes as $M \in {}^{\intercal}\mathcal{C}(Q)$ and $N \in \mathcal{C}(R)$ implies $Q \otimes_R N \in \mathcal{C}(Q)$ by Lemma 3.1.7(ii). Therefore the centre term vanishes for every $N \in \mathcal{C}(R)$ so we have shown that $M \in {}^{\intercal}\mathcal{C}(R)$.

The final statement follows as if $\mathrm{f.dim}\, Q = 0$, $\mathcal{P}_1(\text{mod-}Q) = \mathcal{P}_0(\text{mod-}Q)$. Therefore $\mathcal{C}(Q) = \mathcal{P}_0(\text{mod-}Q)^{\intercal} = Q\text{-Mod}$. □

We now state some consequences that were already known for the class $\mathcal{P}_1(R)$ when $\mathrm{f.dim}\, Q = 0$.

Proposition 3.1.9. *[11, Corollary 6.8] Let R be a commutative ring such that* $\mathrm{f.dim}\, Q = 0$*. The following are equivalent*

(i) $\varinjlim \mathcal{P}_1(R) = \mathcal{F}_1(R)$.

(ii) $\mathrm{F.w.dim}\, Q = 0$

Proof. First note that for every $M \in R$-Mod and $Z \in Q$-Mod we have $\mathrm{Tor}_1^R(M, Z) \cong \mathrm{Tor}_1^Q(M \otimes_R Q, Z)$ by Equation 1.5.

(i) $\Rightarrow$ (ii) We show that $\mathcal{F}_1(Q) = \mathcal{F}_0(Q)$. Since Q is a flat R-module, every flat Q-module is flat as an R-module. Let $X \in \mathcal{F}_1(Q)$, then viewing X as an R-module, $X \in \mathcal{F}_1(R)$. Thus, by combining the assumption $\varinjlim \mathcal{P}_1(R) = \mathcal{F}_1(R)$ and Proposition 3.1.8, $\mathcal{F}_1 \subseteq {}^{\intercal}Q$-Mod, therefore $X \in \mathcal{F}_0(Q)$.

(ii)$\Rightarrow$(i) Let $M \in \mathcal{F}_1(R)$. If $\operatorname{F.w.dim} Q = 0$, then $M \otimes_R Q$ is a flat Q-module, hence, $\mathcal{F}_1 \subseteq^{\intercal} Q$-Mod and thus, $\varinjlim \mathcal{P}_1 = \mathcal{F}_1$, by Proposition 3.1.8. □

Proposition 3.1.10. *Let R be a commutative ring such that Q is a perfect ring. The following are equivalent.*

(i) $\mathcal{P}_1$ is closed under direct limits.

(ii) $\mathcal{P}_1 = \mathcal{F}_1$.

(iii) R is an almost perfect ring.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows by Proposition 3.1.9.

The equivalence (ii) $\Leftrightarrow$ (iii) is proved in [26] or [13, Proposition 8.5]. □

We note some consequences for certain almost perfect rings. Our results in the next section regard semihereditary rings and valuation domains, in particular in Subsection 3.2.2.

Corollary 3.1.11. *Let R be a commutative ring. Then the following hold.*

(i) If R is an almost perfect semihereditary ring, then R is hereditary.

(ii) If R is an almost perfect valuation domain, it is a discrete valuation domain.

Proof. (i) This follows by the equivalence of (i) and (iii) in Proposition 3.1.10 and Lemma 3.2.6.

(ii) This follows by [16, Theorem 4.4 and Proposition 4.5], as a noetherian valuation domain is a discrete valuation domain. □

We note that there exist commutative (hereditary) rings with non-perfect total quotient ring, therefore it is necessary to impose the condition that the classical ring of quotients is perfect.

Example 3.1.12. In [17, 5.1] it is shown that there is a totally disconnected topological space X whose ring of continuous functions K is Von Neumann regular and hereditary. Moreover, every regular element of K is invertible. Hence K coincides with its own quotient ring and $\mathcal{P}_1 = \mathcal{F}_0 =$ Mod-K, but K is not perfect, since it is not semisimple.

3.2 When $\mathcal{P}_1(R)$ is covering

The purpose of this section is to show that if R is a semihereditary commutative ring, $\mathcal{P}_1(R)$ is covering if and only if $\mathcal{P}_1(R)$ is closed under direct limits. This section will be divided into three subsections. First in Subsection 3.2.1 we collect some facts about when $\mathcal{P}_1(R)$ is a covering class, and in particular state consequences for Mod-Q where Q is the ring of quotients of R. Next we discuss semihereditary rings and state some useful theorems in Subsection 3.2.2. In the final Subsection 3.2.3, we assume that R is semihereditary and $\mathcal{P}_1(R)$ is covering, and we will show that R is hereditary, thus $\mathcal{P}_1(R)$ is closed under direct limits.

3.2.1 Classical rings of quotients when $\mathcal{P}_1(R)$ is covering

We restate the following theorem in a way which will be more convenient to us.

Theorem 3.2.1. *[11, Theorem 7.2] Let R be a ring with an $\aleph_0$-noetherian classical ring of quotients Q such that* $\operatorname{f.dim} Q = 0$. *Then* $\operatorname{F.dim} Q = 0$ *if and only if $(\mathcal{P}_1(R), \mathcal{B})$ is of finite type.*

Moreover, in this case $\mathcal{B} = \mathcal{D}$, the class of divisible modules.

Remark 3.2.2. When R is a commutative domain, the classical ring of quotients Q is a field, so the conditions of Theorem 3.2.1 are satisfied. Therefore $\mathcal{B} = \mathcal{D}$, the class of divisible modules and thus is closed under direct limits and the cotorsion pair $(\mathcal{P}_1(R), \mathcal{B})$ is of finite type. Furthermore by Theorem 1.3.6 $\mathcal{P}_1(R)$ is covering if and only if $\mathcal{P}_1(R)$ is closed under direct limits.

However, the result that $\mathcal{P}_1(R)$ is closed under direct limits when $\mathcal{P}_1(R)$ is covering for when R is a commutative domain can be proved easily without referring to Theorem 1.3.6. For commutative domains, the proof is even easier using classical Matlis equivalence and some theorems about strongly flat covers. We do not include an explicit proof here, as it will follow for arbitrary commutative rings from Theorem 7.3.16 and from the properties of almost perfect domains, see Proposition 3.1.10.

Lemma 3.2.3. *Let R be a commutative ring. Then $\mathcal{B}(R) \cap$ Mod-Q $=$ $\mathcal{P}_1(Q)^{\perp}$.*

Proof. Fix some $P \in \mathcal{P}_1(R)$ and $B \in$ Mod-Q. Then there is the following natural bijection.

$$\operatorname{Ext}^1_R(P, B) \cong \operatorname{Ext}^1_Q(P \otimes_R Q, B)$$

If $B \in \mathcal{B}(R)$, then the left-hand side vanishes, thus $B \in \mathcal{P}_1(Q)^{\perp}$, as every module in $\mathcal{P}_1(Q)$ is of the form $P \otimes_R Q$ for some $P \in \mathcal{P}_1(R)$ by Lemma 3.1.4. Conversely, if $B \in \mathcal{P}_1(Q)^{\perp}$, then the right-hand side vanishes, so we conclude that $B \in \mathcal{B}(R)$ via the isomorphism. ☐

We now find some consequences of when $\mathcal{P}_1(R)$ is covering.

Lemma 3.2.4. *Let R be a commutative ring and let S be a multiplicative subset of R. Then the following hold.*

(i) If $R[S^{-1}]$ has a $\mathcal{P}_1(R)$-cover, then $\operatorname{p.dim}_R R[S^{-1}] \leq 1$.

(ii) If $M \in \mathcal{P}_1(R)$ and $M \otimes_R R[S^{-1}]$ admits a $\mathcal{P}_1(R)$-cover, then $\operatorname{p.dim}_R(M \otimes_R R[S^{-1}]) \leq 1$.

(iii) Suppose $\mathcal{P}_1(R)$ is covering. Let S, T be multiplicative systems of R with $S \subseteq \Sigma$. Then $\operatorname{p.dim}_R \dfrac{R[S^{-1}] \otimes_R R[T^{-1}]}{R[T^{-1}]} \leq 1$.

Proof. (i) Let the following be a $\mathcal{P}_1(R)$-cover of $R[S^{-1}]$.

$$0 \to Y \to A \to R[S^{-1}] \to 0 \tag{3.4}$$

By Example 2.2.10, both A and Y are $R[S^{-1}]$-modules as well. Thus (3.4) is an exact sequence of $R[S^{-1}]$-modules, hence it splits. We conclude that $R[S^{-1}]$ is a direct summand of A as an R-module, hence $\operatorname{p.dim} R[S^{-1}] \leq 1$.

(ii)Suppose $M \in \mathcal{P}_1(R)$ and let the following be a $\mathcal{P}_1(R)$-cover of $M \otimes_R R[S^{-1}]$.

$$0 \to Y \to A \to M \otimes_R R[S^{-1}] \to 0 \tag{3.5}$$

As in (i), we conclude by Example 2.2.10 that the sequence (3.5) is in $\text{Mod-}R[S^{-1}]$. Thus, $\operatorname{Ext}^1_{R[S^{-1}]}(M \otimes_R R[S^{-1}], Y) \cong \operatorname{Ext}^1_R(M, Y) = 0$ since $Y \in \mathcal{P}_1(R)^{\perp}$ and $M \in \mathcal{P}_1(R)$. Therefore $M \otimes_R R[S^{-1}]$ is a summand of A, hence it has projective dimension at most one.

(iii)Suppose $\mathcal{P}_1(R)$ is covering in Mod-R. By (i) $\operatorname{p.dim}_R R[S^{-1}]/R \leq 1$ as $S \subseteq \Sigma$, hence (iii) follows by (ii).

□

Lemma 3.2.5. *Let R be a commutative ring and suppose $\mathcal{P}_1(R)$ is covering in* Mod-R*. Then the following hold.*

(i) $\mathcal{P}_1(R) \cap \text{Mod-}Q = \mathcal{P}_1(Q)$

(ii) $\mathcal{P}_1(Q)$ is covering in Mod-Q*.*

Proof. (i) The inclusion $\mathcal{P}_1(R) \cap \text{Mod-}Q \subseteq \mathcal{P}_1(Q)$ is clear. For the converse, take $M \in \mathcal{P}_1(Q)$ and consider the following $\mathcal{P}_1(R)$-cover of M.

$$0 \to B \to A \xrightarrow{\phi} M \to 0 \tag{3.6}$$

Then by Example 2.2.10, $A, B \in \text{Mod-}Q$. From Lemma 3.2.3, $B \in \mathcal{P}_1(Q)^{\perp}$, so ϕ splits, so M must be in $\mathcal{P}_1(R)$.

(ii) Fix a Q-module M and let its $\mathcal{P}_1(R)$-cover be as in (3.6). Then again by Example 2.2.10, (3.6) is a short exact sequence in Mod-Q, and by (i) and Lemma 3.2.3, $A \in \mathcal{P}_1(Q)$ and $B \in \mathcal{P}_1(Q)^{\perp}$ so ϕ must be a $\mathcal{P}_1(Q)$-precover. To see that is a cover, any endomorphism f of A in Mod-Q is also a homomorphism in Mod-R, therefore ϕ is also a $\mathcal{P}_1(Q)$-cover. □

3.2.2 Properties of semihereditary rings

The following lemma holds symmetrically for the case of a left semihereditary ring.

Lemma 3.2.6. *Suppose R is a right semihereditary ring. Then $\mathcal{P}_1(R)$ is closed under direct limits if and only if R is right hereditary.*

Proof. Recall that R is right hereditary if and only if $\mathcal{P}_1(R)$ coincides with the category of R-modules Mod-R.

Every R-module can be written as a direct limit of finitely presented modules, and as by assumption R is semihereditary, the finitely presented modules are contained in $\mathcal{P}_1(R)$. Thus, every module M is a direct limit of (finitely presented) modules in $\mathcal{P}_1(R)$, so Mod-$R \subseteq \varinjlim \mathcal{P}_1(R)$. Therefore for a commutative semihereditary ring we have the inclusions $\mathcal{P}_1(R) \subseteq \varinjlim \mathcal{P}_1(R) = \text{Mod-}R$.

If R is hereditary, then $\mathcal{P}_1(R) = \varinjlim \mathcal{P}_1(R)$ as $\mathcal{P}_1(R) = \text{Mod-}R$. Conversely, if $\mathcal{P}_1(R) = \varinjlim \mathcal{P}_1(R)$ then $\mathcal{P}_1(R) = \text{Mod-}R$, so the conclusion follows. □

We now state some equivalent characterisations of commutative semihereditary rings. We found it interesting to include the rings of weak global dimension one as the semihereditary rings form a subclass of these rings. We first recall some definitions.

A *valutation domain* is a commutative domain R such that for every $x \in Q(R)$, the quotient field of R, $x \in R$ or $x^{-1} \in R$. Equivalently, all the ideals of R are totally ordered by inclusion.

A *discrete valuation domain* is a local principal ideal domain which is not a field. A discrete valuation has exactly two prime ideals: the zero ideal and its unique maximal ideal. Moreover, a discrete valuation domain is a valuation domain.

Theorem 3.2.7. *[28, Corollary 4.2.6] Let R be a commutative ring. Then* $\text{w.gl.dim}\, R \leq 1$ *if and only if $R_{\mathfrak{p}}$ is a valuation domain for all prime ideals $\mathfrak{p}$ of R.*

Proposition 3.2.8. *[28, Corollary 4.2.19] The following are equivalent for a commutative ring R where $Q(R)$ denotes the classical ring of quotients of R.*

1. *R is semihereditary.*
2. *$Q(R)$ is Von Neumann regular and for every prime ideal $\mathfrak{p}$, $R_{\mathfrak{p}}$ is a valuation domain.*

Moreover, R is a reduced ring (that is it does not contain any nilpotent elements) and $\mathrm{w.gl.dim}\, R \leq 1$.

We will use the following characterisation of hereditary rings.

Theorem 3.2.9. *[28, Corollary 4.2.20],[40, Theorem 1.2] Let R be a commutative ring. Then R is hereditary if and only if $Q(R)$ is hereditary and any ideal of R that is not contained in any minimal prime ideal of R is projective.*

We also have the following proposition from a paper of Vasconcelos.

Proposition 3.2.10. *[40, Proposition 1.1] Let R be a commutative ring with a projective ideal I. If I is not contained in any minimal prime ideal it is finitely generated.*

The following proposition in modelled on Cohen's Theorem which states that if all prime ideals are finitely generated, then all ideals are finitely generated, providing a sufficient condition for a ring to be noetherian, for example see [31, Theorem 8]. In the following we consider only the regular ideals.

Proposition 3.2.11. *If every regular prime ideal is finitely generated, then every regular ideal is finitely generated.*

Proof. Let Θ be the collection of regular ideals which are not finitely generated, $\{J_\alpha\}$ with a partial ordering by inclusion and assume that Θ is not empty. Let Φ be a totally ordered subset of Θ. We first claim that Φ has an upper bound in Θ, which implies that the set Θ has a maximal element by Zorn's Lemma. Set $I := \bigcup_{J_\alpha \in \Phi} J_\alpha$ to be an upper bound of Φ. Clearly I contains a regular element, so it remains to show that I is not finitely generated. Suppose for contradiction that I has a finite set of generators $\{a_1, \dots, a_n\}$. Then there exists a $J_\alpha \in \Phi$ such that $I =< a_1, \dots, a_n >\subseteq J_\alpha \subseteq I$, therefore J_α is finitely generated which is a contradiction.

Thus we can apply Zorn's Lemma, so we fix a maximal element L of Θ. We will show that such a maximal element must be prime, so that Θ must be empty. Fix a maximal element L of Θ, and suppose it is not prime, that is there exist two elements $a, b \in R \setminus L$ such that $ab \in L$. Then both $L + aR$ and $L+bR$ strictly contain L, so they are both finitely generated. Therefore, there exist $x_1, \dots, x_n \in L$ and $y_1, \dots, y_n \in R$ so that the following forms a generating set of $L + aR$.

$$\{x_1 + ay_1, \dots, x_n + ay_n\}$$

Consider the ideal $H := (L : a) = \{r \mid ra \in L\}$. Then $L \subsetneq L + bR \subseteq H$, therefore also H is finitely generated, and so also aH is finitely generated.

We now will show that $L =< x_1, \ldots, x_n > +aH$ so L is finitely generated. Take an element $r \in L \subsetneq L + aR$ which can be written in the following form.

$$r = s_1(x_1 + ay_1) + \cdots + s_n(x_n + ay_n) = \Sigma_i s_i x_i + a\bigl(\Sigma_i s_i y_i\bigr)$$

In fact, $\Sigma_i s_i y_i \in H$ as $a\bigl(\Sigma_i s_i y_i\bigr) = r - \Sigma_i s_i x_i \in L$. Therefore $r \in< x_1, \ldots, x_n > +aH$, so $L \subseteq< x_1, \ldots, x_n > +aH$. The converse inclusion is clear, so $L =< x_1, \ldots, x_n > +aH$ which implies that L is finitely generated as H is, a contradiction. Therefore L is prime, and so by the assumption that every prime ideal is finitely generated, Θ must be empty. □

The following lemma is due to Glaz.

Lemma 3.2.12. *Let R be a semihereditary commutative ring and I an ideal of R. Then I is contained in a minimal prime ideal of R if and only if I is not regular.*

Proof. First suppose that $I \subseteq \mathfrak{p}$ where $\mathfrak{p}$ is a minimal prime ideal of R. By assumption R is semihereditary, therefore $R_\mathfrak{p}$ is a domain. Therefore $\mathfrak{p}_\mathfrak{p} = 0$ as the prime ideals of $R_\mathfrak{p}$ are in bijective correspondence with the prime ideals of R contained in $\mathfrak{p}$. Therefore also $I_\mathfrak{p} = 0$, so for each $a \in I$, there exists a $s \notin \mathfrak{p}$ such that $as = 0$. Therefore for every $a \in I$, $a \notin \Sigma$, that is I is not regular.

For the converse, suppose that I is not regular. This proof follows a technique of Kaplansky, see [31, Theorem 1]. Take an ideal L such that $I \subseteq L$, $L \cap \Sigma = \emptyset$ and L is maximal with respect to this property. Then we claim that L is prime, which we will show by contradiction.

Suppose that there exist $a, b \notin L$ such that $ab \in L$. Then $L + aR$ and $L + bR$ strictly contain L, and so by the maximality of L with respect to $L \cap \Sigma = \emptyset$, there exist $s_1 \in L + aR$ and $s_2 \in L + bR$. Rewrite s_1 as $l_1 + ar_1$ and s_2 as $l_2 + br_2$ and consider the following.

$$s_1 s_2 = (l_1 + ar_1)(l_2 + br_2)$$

Clearly the right-hand side is contained in L as $l_1, l_2 \in L$ and $ab \in L$. Therefore $s_1 s_2 \in L \cap \Sigma$, a contradiction.

It remains to show that L is minimal. Suppose there exists a prime $\mathfrak{p}$ such that $\mathfrak{p} \subseteq L$. Then $\mathfrak{p}Q \subseteq LQ \subsetneq Q$, where the last strict inclusion follows as L is not regular. However, as R is semihereditary, Q is Von Neumann regular by Proposition 3.2.8 and therefore has Krull dimension zero. As $\mathfrak{p}Q \subseteq LQ$ are both primes of Q by the bijective correspondence between the non-regular primes of R and the primes of Q, $\mathfrak{p} = L$, so L must be minimal. □

3.2.3 When $\mathcal{P}_1(R)$ is covering and semihereditary rings

We are now ready to show that if R is a semihereditary commutative ring such that $\mathcal{P}_1(R)$ is covering, then R is hereditary or equivalently $\mathcal{P}_1(R)$ is closed under direct limits.

We first show that if $\mathcal{P}_1(R)$ is covering for a Von Neumann regular commutative ring R, then $\mathcal{P}_1(R)$ is closed under direct limits.

Proposition 3.2.13. *Let R be a Von Neumann regular commutative ring. Then $\mathcal{P}_1(R)$ is covering if and only if R is a hereditary ring.*

Proof. R is semi-hereditary, so it remains to show that every infinitely generated ideal I of R is projective. Let the following be a $\mathcal{P}_1(R)$-cover of R/I.

$$0 \to B \to A \to R/I \to 0 \tag{3.7}$$

The ideal I is the sum of its finitely generated ideals which are all of the form Re, for some idempotent element $e \in R$. For every idempotent element $e \in I$, we have $Ae \subseteq B$, hence by Corollary 1.2.10 $Ae = 0$. We conclude that $AI = 0$. On the other hand, $A = B + xR$ for some element $x \in A$ such that $xR \cap B = xI$. Thus $B \cap xR = 0$, since $AI = 0$ and we infer that the the sequence (3.7) splits, thus p.dim $R/I \leq 1$ and I is projective. □

Note that if R is a Von Neumann regular commutative ring such that $\mathcal{P}_1(R)$ is closed under direct limits, the ring R is hereditary as $\mathcal{F}_0(R) \subseteq \mathcal{P}_1(R)$.

By Proposition 3.2.8, if R is semihereditary then $R_\mathfrak{p}$ is a commutative valuation ring for each $\mathfrak{p} \in \operatorname{Spec} R$. Thus $R_\mathfrak{p}$ is semihereditary (every finitely generated ideal is principal so is free). Statement (i) clearly holds for all localisations at a multiplicative subset.

Lemma 3.2.14. *Let R be a commutative semihereditary ring and $\mathfrak{p}$ a prime ideal of R. Then the following statements hold.*

(i) For every $M \in \mathcal{P}_1(R)$, $M \otimes_R R_\mathfrak{p} \in \mathcal{P}_1(R_\mathfrak{p})$

(ii) For every $N \in \mathcal{B}(R)$, $N \otimes_R R_\mathfrak{p} \in \mathcal{B}(R_\mathfrak{p})$.

(iii) If $\mathcal{P}_1(R)$ is covering then $\mathcal{P}_1(R_\mathfrak{p})$ is covering.

Proof. (i) Clear as $R_\mathfrak{p}$ is flat.
(ii) As $R_\mathfrak{p}$ is a commutative domain, the cotorsion pair $(\mathcal{P}_1(R_\mathfrak{p}), \mathcal{B}(R_\mathfrak{p}))$ is of finite type, and $\mathcal{B}(R_\mathfrak{p})$ coincides with the divisible modules by Theorem 3.2.1. Thus it is sufficient to show that for every $N \in \mathcal{B}(R)$, $\operatorname{Ext}^1_{R_\mathfrak{p}}(R_\mathfrak{p}/aR_\mathfrak{p}, N \otimes_R R_\mathfrak{p}) = 0$ for each $a \in R_\mathfrak{p}$. Without loss of generality, we can assume that $a \in R$, since if $a = x/y$ with $y \notin \mathfrak{p}$, $R_\mathfrak{p}/aR_\mathfrak{p} \cong R_\mathfrak{p}/xR_\mathfrak{p}$. As R is commutative

and $R/aR \in \mathcal{P}_1(\text{mod-}R)$ since R is semihereditary, there is the following isomorphism.

$$\text{Ext}^1_R(R/aR, N)_\mathfrak{p} \cong \text{Ext}^1_{R_\mathfrak{p}}(R_\mathfrak{p}/aR_\mathfrak{p}, N_\mathfrak{p})$$

As $R/aR \in \mathcal{P}_1(R)$, the left-hand side vanishes as required.
(iii) Consider the following $\mathcal{P}_1(R)$-cover of $M \in \text{Mod-}R_\mathfrak{p}$.

$$0 \to B \to A \overset{\phi}{\to} M \to 0 \tag{3.8}$$

Then $A, B \in \text{Mod-}R_\mathfrak{p}$ by Example 2.2.10, and by (i) and (ii), $A \in \mathcal{P}_1(R_\mathfrak{p})$ and $B \in \mathcal{B}(R_\mathfrak{p})$. Therefore, (3.8) is also a $\mathcal{P}_1(R_\mathfrak{p})$-precover of M in $\text{Mod-}R_\mathfrak{p}$. Moreover, it follows that any $R_\mathfrak{p}$-module homomorphism is also an R-module homomorphism, thus (3.8) is a $\mathcal{P}_1(R)$-cover of M.

□

Lemma 3.2.15. *Let R be a commutative semihereditary ring such that $\mathcal{P}_1(R)$ is covering. Then for each prime $\mathfrak{p}$, the ring $R_\mathfrak{p}$ is a discrete valuation domain (or a field) and so is an almost perfect hereditary ring. Therefore,* $\text{Mod-}R_\mathfrak{p} \subseteq \mathcal{P}_1(R)$.

As a consequence, every maximal ideal $\mathfrak{m}$ in R is projective.

Proof. First note that as $R_\mathfrak{p}$ is a valuation domain, it is also semihereditary. By Lemma 3.2.14(iii), $\mathcal{P}_1(R_\mathfrak{p})$ is covering. Therefore by Remark 3.2.2, $\mathcal{P}_1(R_\mathfrak{p})$ is closed under direct limits, so by Proposition 3.1.10, $R_\mathfrak{p}$ is almost perfect. Thus by Corollary 3.1.11, $R_\mathfrak{p}$ is a discrete valuation domain.

To see that $\text{Mod-}R_\mathfrak{p} \subseteq \mathcal{P}_1(R)$, consider an $R_\mathfrak{p}$-module M with the following $\mathcal{P}_1(R)$-cover.

$$0 \to B \to A \overset{\phi}{\to} M \to 0 \tag{3.9}$$

By Example 2.2.10, the short exact sequence (3.9) is also a $\mathcal{P}_1(R_\mathfrak{p})$-cover of M in $\text{Mod-}R_\mathfrak{p}$ by Lemma 3.2.14. We have just shown that $R_\mathfrak{p}$ is hereditary, so $M \in \mathcal{P}_1(R_\mathfrak{p})$, so the sequence must split. We conclude that as $A \in \mathcal{P}_1(R)$, also $M \in \mathcal{P}_1(R)$ for any $R_\mathfrak{p}$-module M.

For the second statement, let $\mathfrak{m}$ be a maximal ideal of R. Once one observes that $R/\mathfrak{m}$ is a $R_\mathfrak{m}$-module, it follows that $\mathfrak{m}$ is projective as $\text{Mod-}R_\mathfrak{m} \subseteq \mathcal{P}_1(R)$ by the first part of this lemma. □

Lemma 3.2.16. *Let R be a commutative semihereditary ring such that $\mathcal{P}_1(R)$ is covering. Then every regular prime ideal is maximal.*

Proof. Take $\mathfrak{p}$ to be a regular prime ideal of R. Then by Lemma 3.2.12, $\mathfrak{p}$ cannot be minimal. Fix a maximal ideal $\mathfrak{m}$ such that $\mathfrak{p} \subseteq \mathfrak{m}$. Then by Lemma 3.2.15 in the localisation $R_\mathfrak{m}$, there are exactly two prime ideals, as $R_\mathfrak{m}$ is a discrete valuation domain by Lemma 3.2.15 and $\mathfrak{m}$ is not minimal by assumption. These are the prime ideals 0 and $\mathfrak{m}_\mathfrak{m}$, which are in bijective correspondence with the prime ideals of R contained in $\mathfrak{m}$. As $\mathfrak{p}$ cannot be minimal, one concludes that $\mathfrak{p} = \mathfrak{m}$, therefore $\mathfrak{p}$ is maximal. □

The following corollary follows easily. It states that all primes not contained in a minimal prime ideal are finitely generated.

Corollary 3.2.17. *Let R be a commutative semihereditary ring such that $\mathcal{P}_1(R)$ is covering. Then every regular prime (hence maximal) ideal is finitely generated.*

Proof. Let R be as in the statement of the corollary. From Lemma 3.2.16 we know that all regular primes are maximal and therefore projective by Lemma 3.2.15. By assumption R is semihereditary, so we can apply Lemma 3.2.12 to conclude that regular primes are not minimal. Thus by Proposition 3.2.10, as the regular primes are not contained in any minimal prime ideals and are projective, they are finitely generated. □

We now can state the main result of this section.

Theorem 3.2.18. *Let R be a commutative semihereditary ring such that $\mathcal{P}_1(R)$ is covering. Then R is hereditary. Therefore $\mathcal{P}_1(R)$ is closed under direct limits.*

Proof. We use Theorem 3.2.9 to show that R must be hereditary. First we show that the the classical ring of quotients, Q, is hereditary. From Lemma 3.2.5 and the assumption that $\mathcal{P}_1(R)$ is covering, we know that $\mathcal{P}_1(Q)$ is covering. Additionally, as R is semihereditary, we know that Q is Von Neumann regular by Proposition 3.2.8. Therefore, Q must be hereditary by Proposition 3.2.13.

Now we show that any ideal not contained in a minimal prime ideal is projective. By Lemma 3.2.12, it is enough to show that any regular ideal is projective, which follows if any regular ideal is finitely generated as R is semihereditary. By Proposition 3.2.11, it is sufficient to show that the regular prime ideals are finitely generated, which follows from Corollary 3.2.17. We conclude that all ideals not contained in a minimal prime ideal are finitely generated, and hence are projective as R is semihereditary.

□

Chapter 4

Gabriel topologies and $\mathcal{H}$-h-local rings

One of the main interests in this thesis are the enveloping and covering properties of 1-tilting cotorsion pairs over commutative rings. As discussed in Section 1.7, these 1-tilting cotorsion pairs over commutative rings are in bijective correspondence with faithful finitely generated Gabriel topologies. Therefore, for Chapter 6 and Chapter 7 we will require some more properties of Gabriel topologies, in particular faithful finitely generated Gabriel topologies, although often these properties are true in more generality. These results are outlined in Section 4.1.

In Section 4.2, we study $\mathcal{H}$-h-local rings with respect to a linear topology $\mathcal{H}$ on a commutative ring R. The $\mathcal{H}$-h-local rings are a generalisation of the S-h-local rings with respect to a multiplicative subset S of R, which were defined and studied in [13, Section 4]. The results and proofs on S-h-local rings can be extended in a straightforward way to a linear topology $\mathcal{H}$. The important property of $\mathcal{H}$-h-local rings is that they can be characterised by the properties of the $\mathcal{H}$-discrete modules (or $\mathcal{H}$-torsion when $\mathcal{H}$ is a Gabriel topology), as seen in Proposition 4.2.6 which is the main result of Section 4.2.

Eventually, we will use these results for the case of a Gabriel topology associated to a 1-tilting class, but we found it interesting to generalise to the case of a linear topology. That is, we only require that the associated hereditary pretorsion class (in this case the $\mathcal{H}$-pretorsion class) is closed under direct sums, submodules and quotients. In other words, the additional property of closure under extensions as in the case of Gabriel topologies is not required for Proposition 4.2.6.

4.1 Some properties of Gabriel topologies

We note that in the following two lemmas, all statements hold in the non-commutative case except for Lemma 4.1.1 (iii). Otherwise, all the Gabriel topologies will be right Gabriel topologies, therefore the associated torsion pair $(\mathcal{E}_\mathcal{G}, \mathcal{F}_\mathcal{G})$ are right R-modules and the $\mathcal{G}$-divisible modules are left R-modules.

We will often refer to the following exact sequence where ψ_R is the ring of quotients homomorphism discussed in Section 1.7. We often will denote $t_\mathcal{G}(M)$ simply by $t(M)$ and when clear from the context, ψ instead of ψ_R.

$$0 \to t_\mathcal{G}(R) \to R \overset{\psi_R}{\to} R_\mathcal{G} \to R_\mathcal{G}/\psi_R(R) \to 0 \tag{4.1}$$

Lemma 4.1.1. *Suppose $\mathcal{G}$ is a right Gabriel topology. Then the following statements hold.*

(i) If M is a $\mathcal{G}$-torsion (right) R-module and D is a $\mathcal{G}$-divisible module then $M \otimes_R D = 0$.

(ii) If N is a $\mathcal{G}$-torsion-free module then the natural map $\mathrm{id}_N \otimes_R \psi_R \colon N \to N \otimes_R R_\mathcal{G}$ is a monomorphism and $N \to N \otimes_R R/t(R)$ is an isomorphism.

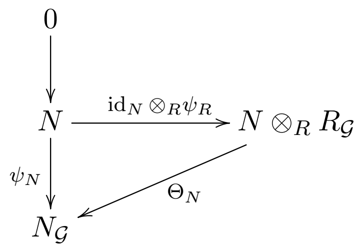
(iii) Suppose R is commutative. If D is both $\mathcal{G}$-divisible and $\mathcal{G}$-torsion-free, then D is a $R_\mathcal{G}$-module and $D \cong D \otimes_R R_\mathcal{G}$ via the natural map $\mathrm{id}_D \otimes_R \psi_R \colon D \otimes_R R \to D \otimes_R R_\mathcal{G}$.

(iv) If X is an R-R-bimodule and is $\mathcal{G}$-torsion, then $M \otimes_R X$ is $\mathcal{G}$-torsion for every $M \in \mathrm{Mod}\text{-}R$.

Proof. (i) This is from [39, Proposition VI.9.1]. Suppose M is a $\mathcal{G}$-torsion module and D is a $\mathcal{G}$-divisible module. Then there is the following surjection.

$$\bigoplus_{\substack{\alpha \in A \\ J_\alpha \in \mathcal{G}}} R/J_\alpha \to M \to 0$$

As $R/J \otimes_R D = 0$ for every $J \in \mathcal{G}$ by definition, the conclusion follows by applying $(- \otimes_R D)$ to the above sequence.
(ii) Consider the following commuting triangle where N is $\mathcal{G}$-torsion-free in Mod-R.



Then ψ_N is a monomorphism and since $\psi_N = \Theta_N \circ (\mathrm{id}_N \otimes_R \psi_R)$, also $\mathrm{id}_N \otimes_R \psi_R$ is a monomorphism. Moreover, we know that $\mathrm{id}_N \otimes_R \psi_R$ factors as follows.

$$N \twoheadrightarrow N \otimes_R R/t(R) \to N \otimes_R R_\mathcal{G}$$

Thus also $N \twoheadrightarrow N \otimes_R R/t(R)$ is a monomorphism, and therefore is an isomorphism.
(iii) Consider the following commuting diagram where the horizontal sequence is exact by (i) as D is $\mathcal{G}$-torsion-free.

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & D & \xrightarrow{\mathrm{id}_D \otimes_R \psi_R} & D \otimes_R R_{\mathcal{G}} & \longrightarrow & D \otimes_R R_{\mathcal{G}}/\psi(R) & \longrightarrow & 0 \\ & & {\scriptstyle \psi_D}\downarrow & \swarrow {\scriptstyle \Theta_D} & & & & & \\ & & D_{\mathcal{G}} & & & & & & \end{array}$$

Additionally, $D \otimes_R R_{\mathcal{G}}/\psi(R) = 0$, since $R_{\mathcal{G}}/\psi(R)$ is $\mathcal{G}$-torsion. Therefore the following map is an isomorphism.

$$\mathrm{id}_D \otimes_R \psi_R \colon D \to D \otimes_R R_{\mathcal{G}}$$

(iv)Fix X a $\mathcal{G}$-torsion R-R-bimodule and $M \in$ Mod-R. Take a free presentation of M, $R^{(\alpha)} \to M \to 0$. Apply $(- \otimes_R X)$ to find the following exact sequence.

$$X^{(\alpha)} \to M \otimes_R X \to 0$$

As $X^{(\alpha)}$ is $\mathcal{G}$-torsion and the $\mathcal{G}$-torsion modules are closed under quotients, also $M \otimes_R X$ is $\mathcal{G}$-torsion. □

Lemma 4.1.2. *Suppose $\mathcal{G}$ is Gabriel topology of right ideals. Then the following hold.*

- *(i) If* $\mathrm{p.dim}\, M_R \leq 1$, *then* $\mathrm{Tor}_1^R(M, R_{\mathcal{G}}) = 0$.
- *(ii) If* $\mathrm{p.dim}\, M_R \leq 1$ *and M is $\mathcal{G}$-torsion-free, then* $\mathrm{Tor}_1^R(M, R_{\mathcal{G}}) = 0 = \mathrm{Tor}_1^R(M, R_{\mathcal{G}}/\psi(R))$.

Proof. (i) By assumption $\mathrm{p.dim}\, M_R \leq 1$, so there is the following projective resolution of M, where P_0, P_1 are projective right R-modules.

$$0 \to P_1 \xrightarrow{\gamma} P_0 \to M \to 0 \tag{4.2}$$

We first note that from the following short exact sequence, $\mathrm{Tor}_1^R(M, R/t(R))$ is $\mathcal{G}$-torsion, as it is contained in the $\mathcal{G}$-torsion module $M \otimes_R t(R)$ (see Lemma 4.1.1(iv)) and is itself a right R-module as $R/t(R)$ is an R-R-bimodule.

$$0 \to \mathrm{Tor}_1^R(M, R/t(R)) \to M \otimes_R t(R) \to M \to M \otimes_R R/t(R) \to 0$$

Next, we note that from the following short exact sequence, $\operatorname{Tor}_1^R(M, R_\mathcal{G})$ is $\mathcal{G}$-torsion-free as it is contained in the $\mathcal{G}$-torsion-free module $P_1 \otimes_R R_\mathcal{G}$.

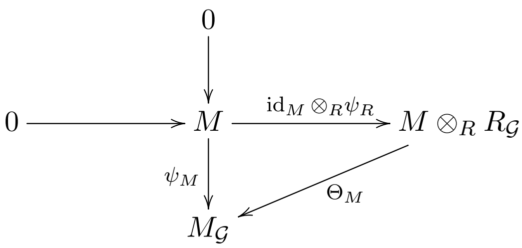
$$0 \to \operatorname{Tor}_1^R(M, R_\mathcal{G}) \to P_1 \otimes_R R_\mathcal{G} \to P_0 \otimes_R R_\mathcal{G} \to M \otimes_R R_\mathcal{G} \to 0 \qquad (4.3)$$

Thus from the following short exact sequence, $\operatorname{Tor}_1^R(M, R/t(R))$ is $\mathcal{G}$-torsion-free as by assumption $\operatorname{Tor}_2^R(M, R_\mathcal{G}/\psi(R)) = 0$. Therefore we conclude that $\operatorname{Tor}_1^R(M, R/t(R)) = 0$ as it is both $\mathcal{G}$-torsion and $\mathcal{G}$-torsion-free.

$$0 \to \operatorname{Tor}_1^R(M, R/t(R)) \to \operatorname{Tor}_1^R(M, R_\mathcal{G}) \to \operatorname{Tor}_1^R(M, R_\mathcal{G}/\psi(R)) \qquad (4.4)$$

Moreover, also $\operatorname{Tor}_1^R(M, R_\mathcal{G}/\psi(R))$ is $\mathcal{G}$-torsion by applying $(-\otimes_R R_\mathcal{G}/\psi(R))$ to the short exact sequence (4.2). Therefore $\operatorname{Tor}_1^R(M, R_\mathcal{G}) = 0$ as it is both $\mathcal{G}$-torsion by (4.4) and $\mathcal{G}$-torsion-free.

(ii) Consider the following commuting triangle where $\mathrm{id}_M \otimes_R \psi_R$ is a monomorphism by Lemma 4.1.1(ii).



By applying the functor $(M \otimes_R -)$ to the short exact sequence (4.1), we have the following exact sequence, as the connecting map $\operatorname{Tor}_1^R(M, R_\mathcal{G}/\psi(R)) \to M \otimes_R R/t(R)$ is zero.

$$0 = \operatorname{Tor}_1^R(M, R_\mathcal{G}) \longrightarrow \operatorname{Tor}_1^R(M, R_\mathcal{G}/\psi(R)) \longrightarrow 0$$

By (i), $\operatorname{Tor}_1^R(M, R_\mathcal{G}) = 0$, and $\mathrm{id}_M \otimes_R \psi_R$ is a monomorphism by thus also $\operatorname{Tor}_1^R(M, R_\mathcal{G}/\psi(R)) = 0$ as these two modules are isomorphic from the above short exact sequence.

□

Lemma 4.1.3. *Consider a right Gabriel topology $\mathcal{G}$. Let M be a $\mathcal{G}$-torsion module and N a $\mathcal{G}$-closed module in* Mod-R. *Then* $\operatorname{Ext}_R^1(M, N) = 0$.

Proof. Let $\mathcal{G}$ be a Gabriel topology of right ideals and $\mathcal{E}$ its associated hereditary torsion class in Mod-R which is generated by the cyclic modules R/J where $J \in \mathcal{G}$. Therefore, for M a $\mathcal{G}$-torsion module, there exists a presentation of M as follows.

$$0 \to H \to \bigoplus_{J_\alpha \in \mathcal{G}} R/J_\alpha \to M \to 0 \qquad (4.5)$$

The module H is $\mathcal{G}$-torsion since $\mathcal{E}$ is a hereditary torsion class. Take a $\mathcal{G}$-closed module N and apply the functor $\mathrm{Hom}_R(-,N)$ to (4.5).

$$0=\mathrm{Hom}_R(H,N)\to \mathrm{Ext}^1_R(M,N)\to \mathrm{Ext}^1_R(\bigoplus R/J_\alpha,N)=0 \qquad (4.6)$$

The first abelian group of the sequence (4.6) vanishes since H is $\mathcal{G}$-torsion and the last abelian group vanishes since direct sums commute with $\mathrm{Ext}^i_R(-,N)$ and $\mathrm{Ext}^1_R(R/J_\alpha,N)=0$ for every $J_\alpha\in\mathcal{G}$. Therefore $\mathrm{Ext}^1_R(M,N)=0$ as desired. □

We note that in the case of a Gabriel topology with a basis of finitely generated ideals, Lemma 4.1.2 can be generalised slightly to include all modules of $\mathrm{w.dim}\, M_R\le 1$. This interests us as the Gabriel topologies associated to silting classes and 1-tilting classes are finitely generated.

Lemma 4.1.4. *Suppose $\mathcal{G}$ is a right Gabriel topology with a basis of finitely generated ideals. Then the following hold.*

(i) If $\mathrm{p.dim}\, M_R\le 1$*, then* $M_R\otimes_R R_\mathcal{G}$ *is* $\mathcal{G}$*-torsion-free.*

(ii) If $\mathrm{w.dim}\, M_R\le 1$*, then* $\mathrm{Tor}^R_1(M,R_\mathcal{G})=0$.

(iii) If $\mathrm{w.dim}\, M_R\le 1$ *and* M *is* $\mathcal{G}$*-torsion-free, then* $\mathrm{Tor}^R_1(M,R_\mathcal{G})=0=\mathrm{Tor}^R_1(M,R_\mathcal{G}/\psi(R))$.

Proof. For (i), first note that for any projective right R-module P_R, $P_R\otimes_R R_\mathcal{G}\underset{\oplus}{\le} R_\mathcal{G}^{(\alpha)}$. By the assumption that $\mathcal{G}$ is finitely generated, by Proposition 1.7.2, we have that arbitrary direct sums of copies of $\mathcal{G}$-closed modules are $\mathcal{G}$-closed, thus we conclude that $P_R\otimes_R R_\mathcal{G}$ is $\mathcal{G}$-closed. Now consider the presentation $0\to P_1\to P_0\to M\to 0$ of M with P_0,P_1 projective. Then $0\to P_1\otimes_R R_\mathcal{G}\to P_0\otimes_R R_\mathcal{G}\to M\otimes_R R_\mathcal{G}\to 0$ is exact as $\mathrm{Tor}^R_1(M,R_\mathcal{G})=0$ by Lemma 4.1.2(i). As the middle term $P_0\otimes_R R_\mathcal{G}$ is $\mathcal{G}$-torsion-free and $P_1\otimes_R R_\mathcal{G}$ is $\mathcal{G}$-closed, it follows that $M\otimes_R R_\mathcal{G}$ is $\mathcal{G}$-torsion-free.

For (ii), it is enough to show that for every flat right R-module F, $F\otimes_R R_\mathcal{G}$ is $\mathcal{G}$-torsion-free, then by an identical argument to that in Lemma 4.1.2, the conclusion follows. Explicitly, for (ii) one finds that $\mathrm{Tor}^R_1(M,R_\mathcal{G})$ is $\mathcal{G}$-torsion-free and at the same time a submodule of the $\mathcal{G}$-torsion module $\mathrm{Tor}^R_1(M,R_\mathcal{G}/\psi(R))$, and similarly for (iii).

We now show that for every flat right R-module F, $F\otimes_R R_\mathcal{G}$ is $\mathcal{G}$-torsion-free. By assumption, $\mathcal{G}$ has a basis of finitely generated ideals. By Proposition 1.7.2, the $\mathcal{G}$-torsion-free modules are closed under direct limits. As F can be seen as a direct limit of projective modules, and the tensor product commutes with direct limits, $F\otimes_R R_\mathcal{G}$ is a direct limit of modules $P_\alpha\otimes_R R_\mathcal{G}$ with P_α projective. Each $P_\alpha\otimes_R R_\mathcal{G}$ is $\mathcal{G}$-torsion-free, thus $F\otimes_R R_\mathcal{G}$ is a direct limit of $\mathcal{G}$-torsion-free modules, thus is itself $\mathcal{G}$-torsion-free by Proposition 1.7.2. □

The following lemma is taken from [39, Exercise IX.1.4], although we state the lemma in a slightly more convenient way for us. We let $E(M)$ denote the injective envelope of M and $E(M)/M$ the cokernel of the natural inclusion map.

Lemma 4.1.5. *Let $\mathcal{G}$ be a right Gabriel topology on R. Then the following are equivalent.*

(i) The functor $q\colon \text{Mod-}R \to \text{Mod-}R_{\mathcal{G}}$ is exact.

(ii) The module $E(M)/M$ is $\mathcal{G}$-closed for every $\mathcal{G}$-closed module M.

(iii) For every $\mathcal{G}$-closed module M and each $J \in \mathcal{G}$, $\operatorname{Ext}^2_R(R/J, M) = 0$.

Proof. We will show (i) $\Longrightarrow$ (ii)$\Leftrightarrow$(iii) $\Longrightarrow$ (i) The equivalence of (ii) and (iii) follows applying $\operatorname{Hom}_R(R/J, -)$ to the injective envelope of M. Therefore we have the following isomorphism for each $J \in \mathcal{G}$.

$$0 \to \operatorname{Ext}^1_R(R/J, E(M)/M) \to \operatorname{Ext}^2_R(R/J, M) \to 0$$

For the equivalence of (i) and (iii), we begin by assuming that q is exact. Fix a $J \in \mathcal{G}$ and take a $\mathcal{G}$-closed R-module M. Let $0 \to M \to E(M) \to E(M)/M \to 0$ be the injective envelope of M in Mod-R. Then since M is essential in $E(M)$, $E(M)$ must be $\mathcal{G}$-torsion-free and thus is $\mathcal{G}$-closed. Thus we have the following commuting diagram, where the exactness of the bottom row follows by our assumption that q is exact.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E(M) & \longrightarrow & E(M)/M & \longrightarrow & 0 \\ & & \cong\big\downarrow \psi_M & & \cong\big\downarrow \psi_{E(M)} & & \big\downarrow \psi_{E(M)/M} & & \\ 0 & \longrightarrow & M_{\mathcal{G}} & \longrightarrow & E(M)_{\mathcal{G}} & \xrightarrow{\pi} & (E(M)/M)_{\mathcal{G}} & \longrightarrow & 0 \end{array}$$

It follows by the snake lemma that $E(M)/M$ is isomorphic to its module of quotients so is $\mathcal{G}$-closed. Therefore, by applying the functor $\operatorname{Hom}_R(R/J, -)$ to the injective envelope of M, we find the following sequence is exact and so the conclusion follows.

$$0 = \operatorname{Ext}^1_R(R/J, E(M)/M) \to \operatorname{Ext}^2_R(R/J, M) \to \operatorname{Ext}^2_R(R/J, E(M)) = 0$$

For the converse, assume that for every $\mathcal{G}$-closed module M and every $J \in \mathcal{G}$, $\operatorname{Ext}^2_R(R/J, M) = 0$. First note that by assumption that the $\mathcal{G}$-closed modules are closed under cokernels of monomorphisms, which follows easily by applying $\operatorname{Hom}_R(R/J, -)$. Now consider q applied to the exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$. Recall that q is left exact, so it remains only to show that the induced map $g_{\mathcal{G}}$ is a surjection. We have the following commuting diagram where the top row is in Mod-R and the bottom row is in Mod-$R_{\mathcal{G}}$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi_L} & & \downarrow{\scriptstyle \psi_M} & & \downarrow{\scriptstyle \psi_N} & & \\ 0 & \longrightarrow & L_{\mathcal{G}} & \xrightarrow{f_{\mathcal{G}}} & M_{\mathcal{G}} & \xrightarrow{g_{\mathcal{G}}} & N_{\mathcal{G}} & & \end{array} \tag{4.7}$$

As $R \to R_{\mathcal{G}}$ is a ring homomorphism and the canonical homomorphisms ψ_L, ψ_M, ψ_N are R-linear, the diagram can be considered an exact sequence in Mod-R (and so also $g_{\mathcal{G}}$). Therefore it is sufficient to show that $g_{\mathcal{G}}$ is a surjection in Mod-R, and it will follow that it is a surjection in Mod-$R_{\mathcal{G}}$. Firstly, the canonical homomorphisms ψ_L and ψ_M induce a map $h\colon N \to \operatorname{Coker} f_{\mathcal{G}}$ and $k\colon \operatorname{Coker} f_{\mathcal{G}} \to N_{\mathcal{G}}$ by the universal property of the cokernel as in the following commuting diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi_L} & & \downarrow{\scriptstyle \psi_M} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & L_{\mathcal{G}} & \xrightarrow{f_{\mathcal{G}}} & M_{\mathcal{G}} & \xrightarrow{\pi} & \operatorname{Coker} f_{\mathcal{G}} & \longrightarrow & 0 \\ & & \| & & \| & & \downarrow{\scriptstyle k} & & \\ 0 & \longrightarrow & L_{\mathcal{G}} & \xrightarrow{f_{\mathcal{G}}} & M_{\mathcal{G}} & \xrightarrow{g_{\mathcal{G}}} & N_{\mathcal{G}} & & \end{array} \tag{4.8}$$

Thus from (4.8) $khg = k\pi\psi_M = g_{\mathcal{G}}\psi_M$. Additionally from (4.7) $\psi_N g = g_{\mathcal{G}}\psi_M$, so by the surjectivity of g, $kh = \psi_N$. We would like to show that $k\colon \operatorname{Coker} f_{\mathcal{G}} N_{\mathcal{G}}$ is an epimorphism, forcing the bottom row of (4.8) to be exact as we want.

By assumption $\operatorname{Coker} f_{\mathcal{G}}$ is $\mathcal{G}$-closed. Additionally from Proposition 1.7.1we have the isomorphism $\psi_N^*\colon \operatorname{Hom}_R(N_{\mathcal{G}}, X) \overset{\cong}{\to} \operatorname{Hom}_R(N, X)$ for any $\mathcal{G}$-closed module X. It follows that below we have the existence and uniqueness of h' and k' such that $h'\psi_N = h$ and $k\psi_N = \psi_N$.

$$\begin{array}{ccc} N & \xrightarrow{\psi_N} & N_{\mathcal{G}} \\ {\scriptstyle h}\downarrow & \swarrow{\scriptstyle h'} & \\ \operatorname{Coker} f_{\mathcal{G}} & & \end{array} \qquad\qquad \begin{array}{ccc} N & \xrightarrow{\psi_N} & N_{\mathcal{G}} \\ {\scriptstyle \psi_N}\downarrow & \swarrow{\scriptstyle k'} & \\ N_{\mathcal{G}} & & \end{array}$$

Since k' is unique and both kh' and $\operatorname{id}_{N_{\mathcal{G}}}$ make the above-right triangle commute, we conclude that $kh' = \operatorname{id}_{N_{\mathcal{G}}}$. Therefore k is surjective, so also $g_{\mathcal{G}}$ must be surjective, as required.

□

Lemma 4.1.6. *Let $\mathcal{G}$ be a right Gabriel topology and $\psi_R\colon R \to R_{\mathcal{G}}$ the natural ring homomorphism. Then ψ_R is a ring epimorphism if and only if $R_{\mathcal{G}} \otimes_R R_{\mathcal{G}}$ is $\mathcal{G}$-torsion-free.*

Proof. If ψ_R is a ring epimorphism, then $R_\mathcal{G} \cong R_\mathcal{G} \otimes_R R_\mathcal{G}$, so $R_\mathcal{G}$ is clearly $\mathcal{G}$-torsion-free. For the converse, by Lemma 4.1.1 (ii), the following is a short exact sequence.

$$0 \to R_\mathcal{G} \overset{\mathrm{id}_{R_\mathcal{G}} \otimes_R \psi_R}{\to} R_\mathcal{G} \otimes_R R_\mathcal{G} \to R_\mathcal{G} \otimes_R R_\mathcal{G}/\psi(R) \to 0$$

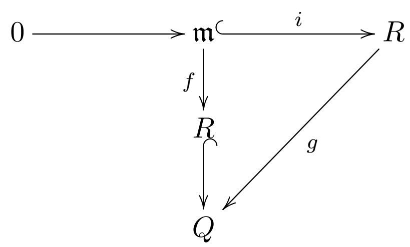
If $R_\mathcal{G} \otimes_R R_\mathcal{G}$ is $\mathcal{G}$-torsion-free, then $R_\mathcal{G} \otimes_R R_\mathcal{G}/\psi(R)$ is $\mathcal{G}$-torsion-free as additionally $R_\mathcal{G}$ is $\mathcal{G}$-closed. Thus $R_\mathcal{G} \otimes_R R_\mathcal{G}/\psi(R) = 0$ as it is both $\mathcal{G}$-torsion and $\mathcal{G}$-torsion-free. □

More generally, if every $R_\mathcal{G}$-module is $\mathcal{G}$-torsion-free as an R-module, then $R_\mathcal{G}$ is a right perfect localisation, see [39, Exercise XI.6].

Next we want to show that if the ring of quotients is flat, the localisation map must be an epimorphism of rings. We note that this doesn't necessarily mean that the Gabriel topology arises from a perfect localisation, as in the following example.

Example 4.1.7. Let R be a valuation domain which has a (non-principal) idempotent maximal ideal $\mathfrak{m}$. Consider the Gabriel topology $\mathcal{G} = \{\mathfrak{m}, R\}$ on R. We claim that $R \to R_\mathcal{G}$ is a flat ring epimorphism but $\mathcal{G}$ is not a perfect localisation.

We first compute $R_\mathcal{G} = \varinjlim \mathrm{Hom}_R(\mathfrak{m}, R)$. Fix a map $f \colon \mathfrak{m} \to R$, and let $\varepsilon \colon R \to Q$ denote the inclusion of R in its field of fractions. Then there exists a $g \colon R \to Q$ which makes the following diagram commute as Q is an injective module.



We want to show that $\mathrm{Im}\, g \subseteq R$, that is that $g(1) \in R$. As R is a valuation domain, for every $z \in Q$, $z \in R$ or $z^{-1} \in R$. Thus we suppose that $g(1) := r/s$ for $r, s \in R$. If $r/s \in R$, we are done. If $s/r \in R$, then $r/s = 1/t$ for some $t \in R$. We will show that this t is necessarily a unit.

For every $x \in \mathfrak{m}$, $g(x) \in R$, so $g(x) = x/t \in R$. Therefore, $\mathfrak{m} \subseteq tR$, but as $\mathfrak{m}$ is maximal not principal, $tR = R$. Therefore, t is a unit, so every map f extends uniquely to a map $h \colon R \to R$. We conclude that $\mathrm{Hom}_R(\mathfrak{m}, R) \cong \mathrm{Hom}_R(R, R)$, so $R_\mathcal{G} \cong R$. Therefore, the map $\psi_R \colon R \to R_\mathcal{G}$ is an isomorphism, so is clearly a flat ring epimorphism. Instead, $R_\mathcal{G}$ is clearly not $\mathcal{G}$-divisible, as $\mathfrak{m}R \neq R$, so $\mathcal{G}$ is not a perfect Gabriel topology.

Proposition 4.1.8. *Let $\mathcal{G}$ be a finitely generated right Gabriel topology over R. Suppose $R_\mathcal{G}$ is flat as a left R-module. Then the localisation map $R \to R_\mathcal{G}$ is an epimorphism of rings.*

Proof. By Lemma 4.1.6, it is enough to show that $R_\mathcal{G} \otimes_R R_\mathcal{G}$ is $\mathcal{G}$-torsion-free. As ${}_R R_\mathcal{G}$ is flat, it is a direct limit of free left R-modules F_α. Thus as the tensor product commutes with direct limits, we have that $R_\mathcal{G} \otimes_R R_\mathcal{G} \cong \varinjlim_\alpha (R_\mathcal{G} \otimes_R F_\alpha)$. Each $R_\mathcal{G} \otimes_R F_\alpha$ is $\mathcal{G}$-torsion-free, thus by the assumption that $\mathcal{G}$ is finitely generated, by Proposition 1.7.2, also the direct limit of $\mathcal{G}$-torsion-free modules is $\mathcal{G}$-torsion-free. Thus $R_\mathcal{G} \otimes_R R_\mathcal{G}$ is $\mathcal{G}$-torsion-free. □

Similarly, by Lemma 4.1.6 and Lemma 4.1.4(i), if $\mathcal{G}$ is a finitely generated Gabriel topology and $\operatorname{p.dim}(R_\mathcal{G})_R \leq 1$, then $\psi_R \colon R \to R_\mathcal{G}$ is a ring epimorphism.

The following lemma will be useful when working with a faithful Gabriel topology over a commutative ring that arises from a perfect localisation.

Lemma 4.1.9. *Let R be a commutative ring, $u \colon R \to U$ a flat injective ring epimorphism, and $\mathcal{G}$ the associated Gabriel topology. Then the annihilators of the elements of U/R form a sub-basis for the Gabriel topology $\mathcal{G}$. That is, for every $J \in \mathcal{G}$ there exist $z_1, z_2, \dots, z_n \in U$ such that*

$$\bigcap_{0 \leq i \leq n} \operatorname{Ann}_R(z_i + R) \subseteq J.$$

Proof. Every ideal of the form $\operatorname{Ann}_R(z + R)$ is an ideal in $\mathcal{G}$ since $K = U/R$ is $\mathcal{G}$-torsion.

Fix an ideal $J \in \mathcal{G}$. Then, $U = JU$, so $1_U = \sum_{0 \leq i \leq n} a_i z_i$ where $a_i \in J$ and $z_i \in U$. We claim that

$$\bigcap_{0 \leq i \leq n} \operatorname{Ann}_R(z_i + R) \subseteq J.$$

Take $b \in \bigcap_{0 \leq i \leq n} \operatorname{Ann}_R(z_i + R)$. Then

$$b = \sum_{0 \leq i \leq n} b a_i z_i \in J$$

since each $b z_i \in R$, hence $b a_i z_i \in J$, and it follows that $b \in J$. □

4.2 $\mathcal{H}$-h-local rings

This section concerns a class of rings which includes the commutative local rings and the h-local rings. We will be looking at $\mathcal{H}$-h-local rings. The main result of this section is that the $\mathcal{H}$-h-local rings can be characterised by the properties of the $\mathcal{H}$-discrete modules, as will be shown in Proposition 4.2.6 which is the main result of this section.

The following observations were made in [13, Section 4] but in the case of a multiplicative subset of R. All the proofs can be extended easily to the case of a linear topology $\mathcal{H}$ on a commutative ring R.

Definition 4.2.1. A ring R is *$\mathcal{H}$-h-local* if for every $J \in \mathcal{H}$, J is contained only in finitely many maximal ideals of R and every prime ideal in $\mathcal{H}$ is contained in only one maximal ideal.

We say that the ring R is *$\mathcal{H}$-h-nil* if every element $J \in \mathcal{H}$ is contained only in finitely many maximal ideals of R and every prime ideal of R in $\mathcal{H}$ is maximal.

It is clear that every $\mathcal{H}$-h-nil ring is $\mathcal{H}$-h-local. We first give a sufficient condition for a ring to be $\mathcal{H}$-h-nil.

Lemma 4.2.2. *Let $\mathcal{H}$ be a linear topology on R. If R/J is perfect for every $J \in \mathcal{H}$, then R is $\mathcal{H}$-h-nil.*

Proof. By Proposition 1.4.3, it follows that there are only finitely many maximal ideals of R/J thus each $J \in \mathcal{H}$ is contained in finitely many maximal ideals of R.

Take a prime $\mathfrak{p} \in \mathcal{H}$. Then $R/\mathfrak{p}$ is a perfect domain, so is a field (by the comment at the end of Proposition 1.4.3), so it follows that $\mathfrak{p}$ must be maximal. □

Before continuing with linear topologies, we will need a well-known fact about localisations and a corollary.

Lemma 4.2.3. *[7, Exercise 3.3] Suppose S, T are multiplicative subsets of a commutative ring R and $\bar{T}$ is the image of T in $R[S^{-1}]$, or in other words the multiplicative subset $\{(t/s)_\sim \in R[S^{-1}] \mid t \in T, s \in S\}$ of $R[S^{-1}]$. Then $R[(ST)^{-1}] \cong (R[S^{-1}])[\bar{T}^{-1}]$.*

Corollary 4.2.4. *Let S be a multiplicative subset of a commutative ring R, $\phi_S \colon R \to R[S^{-1}]$ the localisation map and $\mathfrak{p}, \mathfrak{q}$ where $\mathfrak{p} = \phi_S^{-1}\mathfrak{q}$ be primes of R and $R[S^{-1}]$ respectively, so $\mathfrak{p} \cap S = \emptyset$. Then the localisation of $R[S^{-1}]$ with respect to $\mathfrak{q}$ and the localisation of R with respect to the prime $\mathfrak{p}$ are isomorphic as rings and as R-modules.*

Proof. First fix a prime $\mathfrak{q}$ of $R[S^{-1}]$ and a prime $\mathfrak{p}$ of R as above.

The image of $R \setminus \mathfrak{p}$ can be considered as a multiplicative subset of $R[S^{-1}]$ via the natural ring map ϕ_S. Also, as $\mathfrak{q} = \phi_S(\mathfrak{p})(R[S^{-1}])$, $\phi_S(R \setminus \mathfrak{p})$ is contained in $R[S^{-1}] \setminus \mathfrak{q}$.

Moreover, every element of $R[S^{-1}] \setminus \mathfrak{q}$ can be written as a multiple of $\phi_S(R \setminus \mathfrak{p})$ and a unit of $R[S^{-1}]$, thus the localisations of $R[S^{-1}]$ with respect to $\phi_S(R \setminus \mathfrak{p})$ and with respect to $R[S^{-1}] \setminus \mathfrak{q}$ are the same, so $(R[S^{-1}])[\phi_S(R \setminus \mathfrak{p})^{-1}] = (R[S^{-1}])[(R[S^{-1}] \setminus \mathfrak{q})^{-1}]$. Therefore, setting S to be S and T to be $R \setminus \mathfrak{p}$ in Lemma 4.2.3, we find $ST = S(R \setminus \mathfrak{p}) = R \setminus \mathfrak{p} = T$ as $\mathfrak{p} \cap S = \emptyset$. We conclude that there is the following isomorphisms of rings. As all the ring maps are R-linear, it is also true for R-modules.

$$R_{\mathfrak{p}} = R[(ST)^{-1}] \cong (R[S^{-1}])[\bar{T}^{-1}] = R[S^{-1}]_{\mathfrak{q}}$$

□

The following holds for any linear topology on R of a commutative ring.

Lemma 4.2.5. *Let $\mathcal{H}$ be a linear topology on R such that every prime in $\mathcal{H}$ is contained in only one maximal ideal. Then for $\mathfrak{m} \neq \mathfrak{n}$ maximal ideals of R, the following hold.*

(i) $J(R_{\mathfrak{m}} \otimes_R R_{\mathfrak{n}}) = R_{\mathfrak{m}} \otimes_R R_{\mathfrak{n}}$ for each $J \in \mathcal{H}$.

(ii) For each $\mathcal{H}$-discrete module N, $N \otimes_R R_{\mathfrak{m}} \otimes_R R_{\mathfrak{n}} = 0$.

Proof. Let $\phi \colon R \to R_{\mathfrak{m}} \otimes_R R_{\mathfrak{n}}$ denote the natural ring map. For (i), first fix $J \in \mathcal{H}$. Take $\mathfrak{q}$ a prime ideal in $R_{\mathfrak{m}} \otimes_R R_{\mathfrak{n}}$. Then there is a unique prime $\mathfrak{p}$ of R such that $\mathfrak{p} \subseteq \mathfrak{m} \cap \mathfrak{n}$ and $\mathfrak{q} = \mathfrak{p}(R_{\mathfrak{m}} \otimes_R R_{\mathfrak{n}})$. By assumption, $\mathfrak{p} \notin \mathcal{H}$ as it is a prime contained in two maximal ideals. Therefore, $J \nsubseteq \mathfrak{p}$ so there exists $a \in J$ such that $a \notin \mathfrak{p}$ and we conclude that $JR_{\mathfrak{p}} = R_{\mathfrak{p}}$.

Applying the exact functors $(- \otimes_R R_{\mathfrak{m}})$ and $(- \otimes_R R_{\mathfrak{n}})$ to the inclusion $0 \to J \to R$, we find the following inclusion where $\varepsilon_{\mathfrak{mn}} := \varepsilon \otimes_R \mathrm{id}_{R_{\mathfrak{m}}} \otimes_R \mathrm{id}_{R_{\mathfrak{n}}} = \varepsilon \otimes_R \mathrm{id}_{R_{\mathfrak{m}} \otimes_R R_{\mathfrak{n}}}$.

$$0 \to J \otimes_R R_{\mathfrak{m}} \otimes_R R_{\mathfrak{n}} \cong J(R_{\mathfrak{m}} \otimes_R R_{\mathfrak{n}}) \xrightarrow{\varepsilon_{\mathfrak{mn}}} R_{\mathfrak{m}} \otimes_R R_{\mathfrak{n}}$$

We will show $\varepsilon_{\mathfrak{mn}}$ is an isomorphism of $R_{\mathfrak{m}} \otimes_R R_{\mathfrak{n}}$-modules by showing that for every prime $\mathfrak{q}$ of $R_{\mathfrak{m}} \otimes_R R_{\mathfrak{n}}$, the localisation $\varepsilon_{\mathfrak{mn}}$ is an isomorphism. Fix a prime $\mathfrak{q}$ of $R_{\mathfrak{m}} \otimes_R R_{\mathfrak{n}}$. From Corollary 4.2.4, we know that localisation of R with respect to $\mathfrak{p}$ is the same as localisation of $R_{\mathfrak{m}} \otimes_R R_{\mathfrak{n}}$ with respect to $\mathfrak{q}$, that is $R_{\mathfrak{p}} \cong (R_{\mathfrak{m}} \otimes_R R_{\mathfrak{n}})_{\mathfrak{q}}$ as R-modules. Moreover, as we know that $J(R_{\mathfrak{m}} \otimes_R R_{\mathfrak{n}}) \otimes_R R_{\mathfrak{p}} \cong R_{\mathfrak{m}} \otimes_R R_{\mathfrak{n}} \otimes_R R_{\mathfrak{p}}$ by the argument in the first paragraph, we conclude the first statement of the lemma.

The statement (ii) follows easily as $R/J \otimes_R R_{\mathfrak{m}} \otimes_R R_{\mathfrak{n}} = 0$ for every $J \in \mathcal{H}$, so as every $\mathcal{H}$-discrete module N is an epimorphic image of modules of the form $\bigoplus_\alpha R/J_\alpha$ with $J_\alpha \in \mathcal{H}$.

$$0 = (\bigoplus_\alpha R/J_\alpha) \otimes_R R_{\mathfrak{m}} \otimes_R R_{\mathfrak{n}} \to N \otimes_R R_{\mathfrak{m}} \otimes_R R_{\mathfrak{n}} \to 0$$

□

The following two propositions are the main results of this section, which generalise [13, Proposition 4.3 and Lemma 4.4].

Proposition 4.2.6. *Suppose $\mathcal{H}$ is a linear topology over a commutative ring R. The following are equivalent.*

1. *R is $\mathcal{H}$-h-local.*
2. *$N \cong \bigoplus_{\mathfrak{m}\in \mathrm{mSpec}\, R} N_{\mathfrak{m}}$ for every $\mathcal{H}$-discrete module N.*
3. *$N \cong \bigoplus_{\substack{\mathfrak{m}\in\mathcal{H} \\ \mathfrak{m}\in \mathrm{mSpec}\, R}} N_{\mathfrak{m}}$ for every $\mathcal{H}$-discrete module N.*

Moreover, the above conditions hold when R/J is perfect for every $J \in \mathcal{H}$.

Proof. (i) $\Rightarrow$ (ii). We begin by showing that statement (ii) holds for the cyclic modules R/J with $J \in \mathcal{H}$. By assumption, J is contained in finitely many maximal ideals so $1+J$ is mapped to a non-zero element of $R_{\mathfrak{m}} \otimes_R R/J$ for only finitely many maximal ideals. Thus there is the following natural monomorphism.

$$\Psi_{R/J}\colon R/J \hookrightarrow \bigoplus_{\mathfrak{m}\in \mathrm{mSpec}\, R} (R/J)_{\mathfrak{m}} \quad \subseteq \prod_{\mathfrak{m}\in \mathrm{mSpec}\, R} (R/J)_{\mathfrak{m}}$$

$$r + J \longmapsto \sum_{\mathfrak{m}\in \mathrm{mSpec}\, R} (r+J)_{\mathfrak{m}}$$

We will show that $\Psi_{R/J}$ is surjective by showing that $\Psi_{R/J}(R/J)$ and $(\quad R/J)_{\mathfrak{m}}$ coincide for every localisation at a maximal ideal of R. To $_{\mathfrak{m}\in \mathrm{mSpec}\, R}$ begin, if $\mathfrak{n} \notin \mathcal{H}$ is maximal, then for each $J \in \mathcal{G}$ $(R/J)_{\mathfrak{n}} = 0$ as there exists an $a \in J \setminus \mathfrak{n}$, and it also follows that $\left(\bigoplus_{\mathfrak{m}\in \mathrm{mSpec}\, R} (R/J)_{\mathfrak{m}}\right)_{\mathfrak{n}} = 0$. For a maximal ideal $\mathfrak{n} \in \mathcal{H}$, by Lemma 4.2.5, $(R/J)_{\mathfrak{m}} \otimes_R R_{\mathfrak{n}} = 0$ for $\mathfrak{m} \neq \mathfrak{n}$. So clearly $\left(\bigoplus_{\mathfrak{m}\in \mathrm{mSpec}\, R} (R/J)_{\mathfrak{m}}\right)_{\mathfrak{n}} \cong (R/J)_{\mathfrak{n}} \cong \Psi_{R/J}(R/J)_{\mathfrak{n}}$, so we are done.

For N a general $\mathcal{H}$-discrete module, consider a short exact sequence of the following form where $J_\alpha \in \mathcal{H}$ and all the modules are $\mathcal{H}$-discrete as the class of $\mathcal{H}$-discrete modules is closed under submodules and quotients (that is, it is hereditary pretorsion).

$$0 \to H \to \bigoplus_{\alpha} R/J_\alpha \to N \to 0 \tag{4.9}$$

Consider the following commuting diagram formed by taking the direct sum of all $\bigoplus_{\mathfrak{m}\in \mathrm{mSpec}\, R} (R_{\mathfrak{m}} \otimes_R -)$ applied to (4.9), and ψ_H, ψ_N the natural maps sending each element to its image in the localisations, which can be seen to be well defined (that is, contained in the direct sum) considering the isomorphism for each R/J.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H & \longrightarrow & \bigoplus_\alpha R/J_\alpha & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi_H} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \psi_N} & & \\
0 & \longrightarrow & \bigoplus_{\mathfrak{m}\in \mathrm{mSpec}\, R} H_{\mathfrak{m}} & \longrightarrow & \bigoplus_{\mathfrak{m}\in \mathrm{mSpec}\, R} \left(\bigoplus_{\alpha} R/J_\alpha\right) & \longrightarrow & \bigoplus_{\mathfrak{m}\in \mathrm{mSpec}\, R} N_{\mathfrak{m}} & \longrightarrow & 0
\end{array} \tag{4.10}$$

Thus ψ_N is surjective by the snake lemma applied to (4.10). Additionally, as also H is $\mathcal{H}$-discrete, the same argument says that ψ_H is surjective. Thus ψ_N must be an isomorphism again by the snake lemma applied to (4.10).

(ii) $\Rightarrow$ (iii). We claim that if $J \in \mathcal{H}$, $(R/J)_{\mathfrak{n}} = 0$ for every $\mathfrak{n} \notin \mathcal{H}$ maximal, as there exists an element $a \in J$, $a \notin \mathfrak{n}$, so $JR_{\mathfrak{n}} = R_{\mathfrak{n}}$ as J contains a unit of $R_{\mathfrak{n}}$. Therefore, using that every $\mathcal{H}$-discrete module N is the image of cyclic $\mathcal{H}$-discrete modules, $N_{\mathfrak{n}} = 0$ for every maximal $\mathfrak{n} \notin \mathcal{H}$.

(iii) $\Rightarrow$ (i). If $R/J \cong \bigoplus_{\mathfrak{m}\in \operatorname{mSpec} R; \mathfrak{m}\in\mathcal{H}} (R/J)_{\mathfrak{m}}$, this direct sum must be finite as R/J is cyclic. Moreover, if $(R/J)_{\mathfrak{n}} = 0$ for $\mathfrak{n}$ maximal, $JR_{\mathfrak{n}} = R_{\mathfrak{n}}$ so J must contain a unit of $R_{\mathfrak{n}}$, so $J \nsubseteq \mathfrak{n}$. This shows J is contained in finitely many maximal ideals. To see that every prime $\mathfrak{p}$ of $\mathcal{H}$ must be contained only in one maximal ideal, suppose $\mathfrak{p} \subseteq \mathfrak{m} \cap \mathfrak{n}$ where $\mathfrak{m} \neq \mathfrak{n}$ are maximal and consider $R/\mathfrak{p} \cong \bigoplus_{\mathfrak{m}\in \operatorname{mSpec} R; \mathfrak{m}\in\mathcal{H}} (R/\mathfrak{p})_{\mathfrak{m}}$. Then applying $(R_{\mathfrak{p}} \otimes_R -)$, $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \cong \bigoplus_{\mathfrak{m}\in \operatorname{mSpec} R; \mathfrak{m}\in\mathcal{H}} (R/\mathfrak{p})_{\mathfrak{p}}$ as $\mathfrak{p} \subseteq \mathfrak{m}$, so $R_{\mathfrak{m}} \otimes_R R_{\mathfrak{p}} = R_{\mathfrak{p}}$. This is a contradiction, as $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ cannot contain two direct sum copies of itself, so $\mathfrak{p}$ is contained in exactly one maximal ideal. □

Proposition 4.2.7. *Let R be a $\mathcal{H}$-h-local ring. Let $M(\mathfrak{m})$, $N(\mathfrak{m})$ be two collections of $R_{\mathfrak{m}}$-modules indexed by maximal ideals $\mathfrak{m}$ of R. Suppose the modules $M(\mathfrak{m})$ are $\mathcal{H}$-discrete. Then any morphism $\bigoplus_{\mathfrak{m}} M(\mathfrak{m}) \to \bigoplus_{\mathfrak{m}} N(\mathfrak{m})$ is a direct sum of $R_{\mathfrak{m}}$-module homomorphisms $M(\mathfrak{m}) \to N(\mathfrak{m})$.*

Proof. We will show that $\operatorname{Hom}_R(M(\mathfrak{m}), N(\mathfrak{n})) = 0$ for $\mathfrak{m} \neq \mathfrak{n}$ maximal ideals. As $R \to R_{\mathfrak{n}}$ is a ring epimorphism, note that $N(\mathfrak{n}) \cong \operatorname{Hom}_{R_{\mathfrak{n}}}(R_{\mathfrak{n}}, N(\mathfrak{n})) \cong \operatorname{Hom}_R(R_{\mathfrak{n}}, N(\mathfrak{n}))$. Thus using the tensor-Hom adjunction we have the following.

$$\operatorname{Hom}_R\left(M(\mathfrak{m}), \operatorname{Hom}_R(R_{\mathfrak{n}}, N(\mathfrak{n}))\right) \cong \operatorname{Hom}_R(M(\mathfrak{m}) \otimes_R R_{\mathfrak{n}}, N(\mathfrak{n}))$$

However, as $M(\mathfrak{m})$ is $\mathcal{H}$-discrete and $R/J \otimes_R R_{\mathfrak{m}} \otimes_R R_{\mathfrak{n}} = 0$ for every $J \in \mathcal{H}$ and $\mathfrak{m} \neq \mathfrak{n}$, $M(\mathfrak{m}) \otimes_R R_{\mathfrak{n}} \cong M(\mathfrak{m}) \otimes_R R_{\mathfrak{m}} \otimes_R R_{\mathfrak{n}} = 0$, as required. □

Chapter 5

Topological rings and contramodules

Along with Chapter 4, the purpose of this chapter is to introduce some concepts that we will need for the final Chapter 6 and Chapter 7. The material covered in this chapter is a combination of ideas from [34], [15], and [14], and covers mostly methods using contramodules and topological rings.

In Section 5.1 we begin by defining the completion of an R-module with respect to the right linear topology $\mathcal{H}$ on R, as well as stating some preliminaries. In particular, for when $\mathcal{H}$ is nice enough, we introduce the topological ring $\Lambda_{\mathcal{H}}(R)$. Moreover when $\mathcal{G}$ is a faithful perfect Gabriel topology, there is a ring $\mathfrak{R} := \mathrm{End}_R(K)$ where $K = R_{\mathcal{G}}/R$. The two $\Lambda_{\mathcal{G}}(R)$ and $\mathfrak{R}$, as well as more generally the modules $\Lambda_{\mathcal{G}}M$ and $\mathrm{Hom}_R(K, M \otimes_R K)$ for a $\mathcal{G}$-torsion-free R-module M will be particularily interesting for us later in Section 5.5 and moreover in Chapter 6 and Chapter 7.

The main result of Section 5.2 is an extension of [34, Theorem 2.3] which itself a generalisation of [33, Theorem 6.8]. In this theorem, Positselski considered a commutative ring R and a multiplicative subset S of R and considered the so called S-completion of a module M. He proceeded to give a characterisation of the modules M such that the projective limit topology and the S-topology coincide on the completion $\Lambda_S(M)$ of M. We extend this to to a right linear topology $\mathcal{H}$. Furthermore, we give some equivalent conditions for these topologies to coincide in Theorem 5.2.1 which is an an extension of [34, Theorem 2.3].

In Section 5.3 we discuss u-contramodules with respect to a ring epimorphism of commutative rings $u : R \to U$ and let $K := U/u(R)$. We begin with u a general ring homomorphism before specialising first to the case that u is a flat injective ring epimorphism, and then furthermore with the additional assumption that $\mathrm{p.dim}\, U \leq 1$ in Subsection 5.3.2. Here we introduce some modules, for example $\mathrm{Hom}_R(K, M \otimes_R K)$ and $\Delta_u(M) := \mathrm{Ext}^1_R(K, M)$, which in view of the category equivalence in Theorem 5.4.2, will be very useful. When we use these results in Chapter 6 and Chapter 7, we will be in this setting of a flat injective ring epimorphism $u\colon R \to U$ of commutative rings such that $\mathrm{p.dim}\, U \leq 1$.

The brief Section 5.4 recalls the equivalence of two additive subcategories of Mod-R with respect to a flat injective ring epimorphism of commutative rings $u\colon R \to U$ with associated Gabriel topology $\mathcal{G}$: the u-h-divisible $\mathcal{G}$-torsion modules and the $\mathcal{G}$-torsion-free u-contramodules. This is a special case of [15, Theorem 1.3], where the equivalence is proved for a general ring epimorphism $u\colon R \to U$ such that $\mathrm{Tor}_1^R(U, U) = 0$ (note that U is not necessarily flat).

Finally in Section 5.5 we show that if $u\colon R \to U$ is a flat injective ring epimorphism of commutative rings, the equivalent conditions of Theorem 5.2.1 hold for $\mathcal{G}$-torsion-free modules. This was proved in [34, Lemma 2.4] for the S-topology which is a generalisation of [33, Lemma 6.9]. However, in our more general case we need to use a different approach which is more involved than that for the S-topology.

As mentioned above, for most of this chapter, $u\colon R \to U$ will always denote a ring epimorphism of commutative rings, and for the most part (outside the beginning of Section 5.3) u will denote a flat injective ring epimorphism. Only in Subsection 5.3.2 will there also be the assumption that $\mathrm{p.dim}\, U \leq 1$.

Even though Section 5.1 on topological rings covers only preliminary material, we chose to include it in this chapter rather than in Chapter 1 to make this chapter self-contained.

5.1 Topological rings

A ring R is a *topological ring* if it has a topology such that the ring operations are continuous.

A topological ring R is *right linearly topological* if it has a topology with a basis of neighbourhoods of zero consisting of right ideals of R. The ring R with a right Gabriel topology is an example of a right linearly topological ring.

If R is a right linearly topological ring, then the set of right ideals J in a basis $\mathfrak{B}$ of the topology form a directed set. The $\mathfrak{B}$-*topology* on a right R-module M is the topology where the base of neighbourhoods of 0 are the submodules MJ for $J \in \mathfrak{B}$. Furthermore, for every R-module M, $\{M/MJ \mid J \in \mathfrak{B}\}$ is an inverse system. The *completion* of M with respect to $\mathfrak{B}$ is the module

$$\Lambda_{\mathfrak{B}}(M) := \varprojlim_{J \in \mathfrak{B}} M/MJ.$$

There is a canonical map $\lambda_M \colon M \to \Lambda_{\mathfrak{B}}(M)$ which sends the element $x \in M$ to $(x + MJ)_{J\in\mathfrak{B}}$. Each element in $\Lambda_{\mathfrak{B}}(M)$ is of the form $(x_J + MJ)_{J\in\mathfrak{B}}$ with the relation that for $J \subseteq J'$, $x_J - x_{J'} \in MJ'$. The module M is called *$\mathcal{H}$-separated* if the homomorphism λ_M is injective, which is equivalent to $\bigcap_{J\in\mathfrak{B}} MJ = 0$. The module M is called *$\mathcal{H}$-complete* if the map λ_M is surjective.

The *projective limit topology* on $\Lambda_{\mathfrak{B}}(M)$ is the topology where a subbasis of neighbourhoods of zero is given by the the kernels of the projection maps $\Lambda_{\mathfrak{B}}(M) \to M/MJ$. That is, it is the subspace topology of $\prod_{J\in\mathfrak{B}} M/MJ$, where the topology on $\prod_{J\in\mathfrak{B}} M/MJ$ is the product topology of the discrete topologies on each of the M/MJ. In this case, the R-module $\Lambda_{\mathfrak{B}}(M)$ is both separated and complete with this topology. We will simply write $\Lambda(M)$ when the basis $\mathfrak{B}$ is clear from the context.

The assignment $\Lambda_{\mathfrak{B}}$ is a functor from Mod-R to the full subcategory of complete and separated right R-modules, and the assignment $\{\lambda_M\}_{M\in\text{Mod-}R}$ forms a natural transformation $\mathrm{id}_{\text{Mod-}R} \to \Lambda_{\mathfrak{B}}$. Thus if $f \colon M \to N$ is a homomorphism of R-modules, $\lambda(f)$ maps $(m_J + MJ)_{J\in\mathfrak{B}}$ to $(f(m_J) + NJ)_{J\in\mathfrak{B}}$, and there is the following commuting diagram.

$$\begin{array}{ccc} M & \xrightarrow{\lambda_M} & \Lambda_{\mathfrak{B}}(M) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \Lambda_{\mathfrak{B}}(f)} \\ N & \xrightarrow{\lambda_N} & \Lambda_{\mathfrak{B}}(N) \end{array}$$

If the ideals in $\mathfrak{B}$ are two-sided in R then the module $\Lambda_{\mathfrak{B}}(R)$ is a ring. Furthermore, it is a linearly topological ring with respect to the projective limit topology.

Let $\mathcal{H}$ be a right linear topology with basis $\mathfrak{B}$. Then it follows that the module $\varprojlim_{J\in\mathcal{H}} M/MJ$ is isomorphic to $\Lambda_{\mathfrak{B}}(M)$, thus we sometimes denote the completion by $\Lambda_{\mathcal{H}}(M)$.

Remark 5.1.1. For each $J \in \mathfrak{B}$, let V_J denote the kernel of the projection $p_J \colon \Lambda_{\mathfrak{B}}(M) \to M/MJ$. Then there is always the inclusion $V_J \supseteq \Lambda_{\mathfrak{B}}(M)J$.

The following is [34, Proposition 2.2(a)].

Lemma 5.1.2. *Let $\mathfrak{B}$ be a basis for a right linear topology $\mathcal{H}$ on R and M a right R-module. Then the completion $\Lambda_{\mathfrak{B}}(M)$ is $\mathcal{H}$-separated.*

Proof. By Remark 5.1.1, $\Lambda_{\mathfrak{B}}(M)J \subseteq V_J$ for each $J \in \mathfrak{B}$. Therefore $\bigcap_{J\in\mathfrak{B}} \Lambda_{\mathfrak{B}}(M)J \subseteq \bigcap_{J\in\mathfrak{B}} V_J = 0$, since $\Lambda_{\mathfrak{B}}(M)$ is always separated with respect to the projective limit topology. □

Let R be a linearly topological ring. A right R-module M is *discrete* if for every $x \in M$, the annihilator ideal $\mathrm{Ann}_R(x) = \{r \in R \mid xr = 0\}$ is open in the topology of R. So if $\mathcal{H}$ is a right linear topology on R, then M is $\mathcal{H}$-discrete if and only if M is in the associated pretorsion class of $\mathcal{H}$. The $\mathcal{H}$-discrete modules are naturally $\Lambda_{\mathfrak{B}}(R)$-modules, where the action of $\Lambda_{\mathfrak{B}}(R)$ is defined as follows. Fix $x \in M$ and $\tilde{r} := (r_J + J)_{J \in \mathfrak{B}}$ and let $I \in \mathfrak{B}$ be an ideal of R such that $I \subseteq \operatorname{Ann} x$. Then the action is defined as $x \cdot \tilde{r} := x r_I$. Furthermore, in the case that the topology on R is a Gabriel topology $\mathcal{G}$ on R, then a module is discrete if and only if it is $\mathcal{G}$-torsion.

Definition 5.1.3. A linearly topological ring is *pro-perfect* ([14, 35]) if it is separated, complete, and with a basis of neighbourhoods of zero formed by two-sided ideals such that all of its discrete quotient rings are perfect.

5.1.1 Perfect Gabriel topologies and the ring $\Lambda_{\mathcal{G}} R$

In this subsection we consider the case where the linear topology is a faithful perfect Gabriel topology over a commutative ring R. Therefore $\psi_R \colon R \to R_{\mathcal{G}}$ is a flat injective ring epimorphism of commutative rings, which as usual we will denote by $u \colon R \to U$. Let $\mathfrak{R}$ denote the endomorphism ring $\operatorname{End}_R(K)$ where $K = R_{\mathcal{G}}/R = U/R$. Take a finitely generated submodule F of K, and consider the ideal formed by the elements of $\mathfrak{R}$ which annihilate F. The ideals of this form form a base of neighbourhoods of zero in $\mathfrak{R}$. Note that this is the same as considering $\operatorname{End}_R(K)$ with the subspace topology of the product topology on K^K where the topology on K is the discrete topology. We will consider $\mathfrak{R}$ endowed with this topology, which we will call the *finite topology*.

We will now state the above in terms of a Gabriel topology that arises from a perfect localisation. Let $\mathcal{G}$ be the Gabriel topology associated to the flat ring epimorphism $u \colon R \to U$ of commutative rings. As $K \otimes_R U = 0$, K is $\mathcal{G}$-torsion, or equivalently a discrete module. Thus there is a natural well-defined action of $\Lambda_{\mathcal{G}}(R)$ on K. As well as the natural map $\lambda_R \colon R \to \Lambda_{\mathcal{G}}(R)$, there is also a natural map $\nu \colon R \to \mathfrak{R}$ where each element of R is mapped to the endomorphism of K which is multiplication by that element.

5.2 The $\mathcal{H}$-topology and the projective limit topology

We now return to the more general case of a right linear topology $\mathcal{H}$. We would like to prove a theorem which is an analogue of [34, Theorem 2.3] with a basically identical proof. Instead of considering localisations, we consider right linear topologies on R which we will denote $\mathcal{H}$, and $\mathfrak{B}$ a basis for $\mathcal{H}$.

The definition of an S-pure submodule given in [34] is extended to right linear topologies in the following way. A submodule L of a right R-module M is $\mathcal{H}$-*pure* in M, if $MJ \cap L = LJ$ for every $J \in \mathcal{H}$. We note that this is not the same definition given by Stenström in [39, Exercise IX.23] for Gabriel topologies, and coincides with his definition of a $\mathcal{H}$-copure submodule.

We let $\prod_{\mathfrak{B}} M$ denote $\prod_{J\in\mathfrak{B}} M/MJ$ and V_J the kernel of the natural projection map $p_J \colon \Lambda_{\mathfrak{B}}(M) \to M/MJ$.

Theorem 5.2.1. *Let M be a right R-module and and $\mathfrak{B}$ the basis for a right linear topology $\mathcal{H}$ on R. Consider the following conditions.*

(i) The R-module $\Lambda_{\mathfrak{B}}(M)$ is ($\mathcal{H}$-separated and) $\mathcal{H}$-complete.

(ii) The $\mathcal{H}$-topology and the projective limit topology on $\Lambda_{\mathfrak{B}}(M)$ coincide.

(iii) For every $J \in \mathfrak{B}$, the submodule $V_J \subseteq \Lambda_{\mathfrak{B}}(M)$ coincides with $\Lambda_{\mathfrak{B}}(M)J$.

(iv) For every $J \in \mathfrak{B}$, $\bar{\lambda}_M \colon M/MJ \to \Lambda_{\mathfrak{B}}(M)/\Lambda_{\mathfrak{B}}(M)J$ is an isomorphism.

(v) For every $\mathcal{H}$-separated and $\mathcal{H}$-complete module D and every R-module homomorphism $f \colon M \to D$ there exists a unique R-module homomorphism $f' \colon \Lambda_{\mathfrak{B}}(M) \to D$ such that the following commutes.

$$\begin{array}{ccc} M & \xrightarrow{\lambda_M} & \Lambda_{\mathfrak{B}}(M) \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle f'} & \\ D & & \end{array}$$

(vi) The R-module $\operatorname{Coker} \lambda_M$ *equals* $(\operatorname{Coker} \lambda_M)J$ *for every $J \in \mathfrak{B}$.*

(vii) $\Lambda_{\mathfrak{B}}M$ is $\mathcal{H}$-pure in $\prod_{J\in\mathfrak{B}} M/MJ$ (or $(\prod_{\mathfrak{B}} M)J \cap \Lambda_{\mathfrak{B}}(M) = \Lambda_{\mathfrak{B}}(M)J$ for every $J \in \mathfrak{B}$).

Then (i)-(v) are always equivalent, and we have the implications (vi) $\Rightarrow$ (i)-(v) $\Rightarrow$ (vii). If moreover $\mathcal{H}$ has a basis of finitely generated ideals $\mathfrak{B}$, (i)-(v) and (vii) are equivalent.

If $\mathfrak{B}$ consists only of two-sided ideals, then (i)-(vi) are equivalent.

Thus, for the case that R is commutative and $\mathfrak{B}$ is a basis of finitely generated ideals, all the above conditions are equivalent, and $\operatorname{Coker} \lambda_M$ *is an $\mathcal{H}$-divisible module.*

Proof. We will often omit the $\mathfrak{B}$ in $\Lambda_{\mathfrak{B}}$ and M in λ_M for clarity of exposition.

We first show the equivalence of (i), (ii), and (iii). Recall that the module $\Lambda(M)$ is always $\mathcal{H}$-separated by Lemma 5.1.2.

(ii) $\Rightarrow$ (i). This is clear as the module $\Lambda(M)$ is always complete with respect to the projective limit topology. Similarly, (iii) $\Rightarrow$ (ii) is clear as (iii) implies that the bases of the $\mathcal{H}$-topology and projective topology coincide on $\Lambda(M)$.

(i) ⇒ (iii). Consider the following two morphisms.

$$\lambda_{\Lambda(M)}, \Lambda(\lambda_M) \colon \Lambda(M) \rightrightarrows \Lambda(\Lambda(M))$$

The morphism $\lambda_{\Lambda(M)}$ is an isomorphism by assumption. Take $x \in \Lambda(M)$. Then there exists a $y \in \Lambda(M)$ such that $\lambda_{\Lambda(M)}(y) = \Lambda(\lambda_M)(x)$. Denote x and y by $(x_J + MJ)_{J\in\mathfrak{B}}$ and $(y_J + MJ)_{J\in\mathfrak{B}}$ respectively. Then we can rewrite the elements as $\Lambda(\lambda_M)((x_J + MJ)_{J\in\mathfrak{B}}) = (\lambda_M(x_J) + \Lambda(M)J)_{J\in\mathfrak{B}}$ and $\lambda_{\Lambda(M)}(y) = (y + \Lambda(M)J)_{J\in\mathfrak{B}}$. Hence $\lambda_M(x_J) - y \in \Lambda(M)J$ for each $J \in \mathcal{H}$, so looking at the J-th component, $(x_J - y_J + MJ) = (z + MJ)J$ for some $z \in M$, but $(z + MJ)J = MJ$. So $x_J - y_J \in MJ$ and $x = y$ in $\Lambda(M)$. Thus $x - \lambda(x_J) \in \Lambda(M)J$ for each $J \in \mathfrak{B}$.

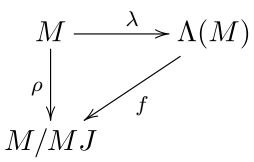
Thus, in particular, any element of $\Lambda(M)J$ can be written in this form, and this element is clearly mapped to 0 via the projection map $p_J \colon \Lambda(M) \to M/MJ$. So $\Lambda(M)J \subseteq V_J$, as required.

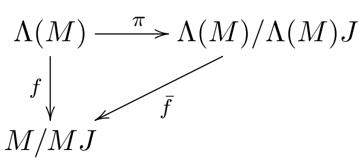
(iii) ⇔ (iv). Consider the following two natural maps, where $\bar\lambda$ is induced by $\lambda\colon M \to \Lambda(M)$ and $\bar p_J$ is induced by $p_J \colon \Lambda(M) \to M/MJ$.

$$M/MJ \overset{\bar\lambda_M}{\to} \Lambda(M)/\Lambda(M)J \overset{\bar p_J}{\to} M/MJ$$

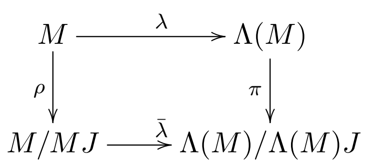
The composition is the identity on M/MJ, so $\bar p_J$ is a monomorphism if and only if $\bar\lambda$ is an epimorphism. Moreover, (iii) holds if and only if $\bar p_J$ is a monomorphism if and only if $\bar\lambda$ is an epimorphism if and only if (iv) holds.

(v) ⇒ (iv). Note that for $J \in \mathfrak{B}$, any R/J-module is $\mathcal{H}$-discrete, so is $\mathcal{H}$-separated and $\mathcal{H}$-complete. So, there exists a unique f such that the left triangle below commutes. The map f induces $\bar f$ since $\Lambda(M)J \subseteq \operatorname{Ker} f$, so the right triangle below also commutes.





Let $\bar\lambda$ be the map induced by λ as in the following commuting diagram. We will show that $\bar f$ and $\bar\lambda$ are mutually inverse.



Then, we have that $\pi\lambda = \bar\lambda\rho$, and so using the above commuting triangles it follows that $\bar f\bar\lambda\rho = \bar f\pi\lambda = f\lambda = \rho$. As ρ is surjective, $\bar f\bar\lambda = \mathrm{id}_{M/MJ}$. We now show that $\bar\lambda\bar f = \mathrm{id}_{\Lambda(M)/\Lambda(M)J}$. By assumption there exists a unique map h which makes the following diagram commute.

$$\begin{array}{ccc} M & \xrightarrow{\lambda} & \Lambda(M) \\ {\scriptstyle \pi\lambda}\downarrow & \swarrow{\scriptstyle h} & \\ \Lambda(M)/\Lambda(M)J & & \end{array}$$

By uniqueness, π is the unique map that fits into the triangle above, that is $\pi\lambda = h\lambda$ implies that $h = \pi$. So,

$$\pi\lambda = \bar{\lambda}\rho = \bar{\lambda}f\lambda = \bar{\lambda}\bar{f}\pi\lambda$$

Therefore $\pi = \bar{\lambda}\bar{f}\pi$, and as π is surjective, $\bar{\lambda}\bar{f} = \mathrm{id}_{\Lambda(M)/\Lambda(M)J}$ as required.
(iv) $\Rightarrow$ (v). Note that for $J \in \mathfrak{B}$, any right R/J-module is $\mathcal{H}$-separated and $\mathcal{H}$-complete. Therefore, for any right R-module D, one can extend any homomorphism $M \to D/DJ$ uniquely to $\Lambda(M) \to D/DJ$ via the isomorphism $M/MJ \cong \Lambda(M)/\Lambda(M)J$ of our assumption. So $\lambda_M^*\colon \mathrm{Hom}_R(M, D/DJ) \overset{\cong}{\to} \mathrm{Hom}_R(\Lambda(M), D/DJ)$.

Now let D a $\mathcal{H}$-separated and $\mathcal{H}$-complete module. Then we have the following isomorphisms via the homomorphism λ_M.

$$\begin{aligned} \mathrm{Hom}_R(M, D) &\cong \varprojlim_{J\in\mathfrak{B}} \mathrm{Hom}_R(M, D/DJ) \\ &\cong \varprojlim_{J\in\mathfrak{B}} \mathrm{Hom}_R(\Lambda(M), D/DJ) \qquad (5.1) \\ &\cong \mathrm{Hom}_R(\Lambda(M), D) \end{aligned}$$

We have shown that the statements $(i) - (v)$ are equivalent. We now show that (vi) $\Rightarrow$ (v) and (iii) $\Rightarrow$ (vi), showing the equivalence of (vi) with (i)-(v).

(vi) $\Rightarrow$ (v). Take a $\mathcal{H}$-complete $\mathcal{H}$-separated module D and a homomorphism $f\colon M \to D$. Then consider $\Lambda(f)$ which makes the following diagram commute. We must show that $\Lambda(f)$ is unique.

$$\begin{array}{ccccc} M & \xrightarrow{\lambda} & \Lambda(M) & \xrightarrow{\rho} & \mathrm{Coker}\,\lambda \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \Lambda(f)} & & \\ D & \underset{\lambda}{\xrightarrow{\cong}} & \Lambda(D) & & \end{array}$$

It is enough to show that if $h\lambda = 0$ for $h\colon \Lambda(M) \to D$, then $h = 0$. Take h such that $h\lambda = 0$. By the universal property of the cokernel, there exists a unique map $h'\colon \mathrm{Coker}\,\lambda \to D$ such that $h'\rho = h$. By assumption $\mathrm{Coker}\,\lambda = (\mathrm{Coker}\,\lambda)J$ for each $J \in \mathfrak{B}$, so $h'(\mathrm{Coker}\,\lambda) = h'((\mathrm{Coker}\,\lambda)J) = (h'(\mathrm{Coker}\,\lambda))J \subseteq DJ$. Therefore, $\mathrm{Im}\,h' \subseteq \bigcap_{J\in\mathfrak{B}} DJ = 0$, as D is $\mathcal{H}$-separated.

(iii) $\Rightarrow$ (vi). We must assume that $\mathfrak{B}$ is a basis of two-sided ideals. Suppose (iii) holds, and for some $J \in \mathfrak{B}$, apply $(- \otimes_R R/J)$ to the following exact sequence.

$$M \xrightarrow{\lambda} \Lambda(M) \to \operatorname{Coker} \lambda \to 0$$

As $\bar{\lambda} = \lambda \otimes_R \mathrm{id}_{R/J}$ is an isomorphism by assumption, we conclude that $\operatorname{Coker} \lambda \otimes_R R/J = 0$, so $\operatorname{Coker} \lambda = (\operatorname{Coker} \lambda)J$ for each $J \in \mathfrak{B}$.

(iii) $\Rightarrow$ (vii), (vii) when $\mathfrak{B}$ is finitely generated $\Rightarrow$(iii). We will revert back to the notation $\prod(M/MJ)$. We will show that for each ideal $J \in \mathfrak{B}$, there are the following inclusions, and that the second inclusion is an equality when J is finitely generated.

$$\Lambda(M)J \subseteq \Lambda(M) \cap (\prod(M/MJ))J \subseteq V_J \tag{5.2}$$

This is sufficient as (iii) holds if and only if all the all the modules in (5.2) coincide (as $\Lambda(M)J = V_J$), and (vii) holds if and only if the first inclusion is an equality.

The first inclusion of (5.2) is clear. For the second, take $x \in \Lambda(M) \cap (\prod(M/MJ))J$. Then clearly $x_J \in MJ$, so $x \in V_J$.

Now we show $\Lambda(M) \cap (\prod(M/MJ))J_0 = V_{J_0}$ when J_0 is finitely generated. Take $x \in V_{J_0}$ and let it be represented by the element $(x_J + MJ)_{J \in \mathfrak{B}}$. Then $x_{J_0} \in MJ_0$, and for all $J' \subseteq J_0$ in $\mathfrak{B}$, $x_{J'} - x_{J_0} \in MJ_0$, so also $x_{J'} \in MJ_0$. Set $y_J := x_{JJ_0}$, and let $y = (y_J + MJ)_{J \in \mathfrak{B}}$. Then $y = x$ as $y_J - x_J = x_{JJ_0} - x_J \in MJ$. Moreover, $y_J \in MJ_0$ for each $J \in \mathfrak{B}$ as $JJ_0 \subseteq J_0$, so $y \in \prod((M/MJ)J_0)$. As J_0 is finitely generated, $\prod((M/MJ)J_0) = \prod(M/MJ)J_0$, thus $x = y \in \Lambda(M) \cap (\prod M)J_0$, as required. ☐

The following lemma states that one can extend the properties of the basis elements as shown in Theorem 5.2.1 to all ideals in $\mathcal{H}$.

Lemma 5.2.2. *Suppose $\mathcal{H}$ is a right linear topology with a basis $\mathfrak{B}$, and consider the completion $\Lambda(M)$ of a module M such that $\bar{\lambda}\colon M/MJ \to \Lambda(M)/\Lambda(M)J$ is an isomorphism for every basis element $J \in \mathfrak{B}$. Then $\bar{\lambda}\colon M/MJ \to \Lambda(M)/\Lambda(M)J$ is an isomorphism for every $J \in \mathcal{H}$.*

Proof. As $\mathcal{H}$ has a basis $\mathfrak{B}$, for every $J \in \mathcal{H}$, there exists an ideal $J_0 \in \mathfrak{B}$ such that $J_0 \subseteq J$. First recall that for every $J \in \mathcal{H}$, there is a monomorphism $\bar{\lambda}_J\colon M/MJ \to \Lambda(M)/\Lambda(M)J$ induced by the morphism $\lambda\colon M \to \Lambda(M)$. That is, if $x \in M$ is such that $\bar{\lambda}(x + MJ) = \lambda(x) + \Lambda(M)J$, then clearly $x \in MJ$. Now consider the following diagram where the horizontal homomorphisms are the natural quotient morphisms, and by assumption $\bar{\lambda}_{J_0}$ is an isomorphism.

$$\begin{array}{ccccc}
M/MJ_0 & \longrightarrow & M/MJ & \longrightarrow & 0 \\
{\scriptstyle \bar{\lambda}_{J_0}}\downarrow{\scriptstyle \cong} & & {\scriptstyle \bar{\lambda}_J}\downarrow & & \\
\Lambda(M)/\Lambda(M)J_0 & \longrightarrow & \Lambda(M)/\Lambda(M)J & \longrightarrow & 0
\end{array}$$

The above diagram shows that $\bar{\lambda}_J$ must be an epimorphism, so is an isomorphism. □

5.3 u-contramodules

We will begin by discussing a general commutative ring epimorphism $u\colon R \to U$ before moving onto flat injective ring epimorphisms. We begin with some definitions.

A module M is *u-h-divisible* if M is an epimorphic image of $U^{(\alpha)}$ for some cardinal α. An R-module M has a unique u-h-divisible submodule denoted $h_u(M)$, which is the image of the map $u^*\colon \operatorname{Hom}_R(U, M) \to \operatorname{Hom}_R(R, M) \cong M$. In nice situations, the u-h-divisible modules are related to the $\mathcal{G}$-divisible modules, which we will discuss later in Subsection 5.3.2.

Definition 5.3.1. Let $u\colon R \to U$ be a ring epimorphism. A *u-contramodule* is an R-module M such that the following holds.

$$\operatorname{Hom}_R(U, M) = 0 = \operatorname{Ext}^1_R(U, M)$$

We let u-**contra** denote the full subcategory of u-contramodules in Mod-R.

Dually, a *u-comodule* is a module M such that the following holds.

$$M \otimes_R U = 0 = \operatorname{Tor}^R_1(M, U)$$

Lemma 5.3.2. *[27, Proposition 1.1] The category of u-contramodules is closed under kernels of morphisms, extensions, infinite products and projective limits in* Mod-R.

The following two lemmas are proved in [34] for the case of the localisation of R at a multiplicative subset. For completeness of this work, we include their proofs in our setting.

Lemma 5.3.3. *[34, Lemma 1.2] Let $u\colon R \to U$ be a ring epimorphism of commutative rings and let M be an R-module.*

(i) If $\operatorname{Hom}_R(U, M) = 0$*, then* $\operatorname{Hom}_R(Z, M) = 0$ *for any u-h-divisible module Z.*

(ii) If M is a u-contramodule, then $\operatorname{Ext}^1_R(Z, M) = 0 = \operatorname{Hom}_R(Z, M)$ *for any U-module Z.*

Proof. For (i), by the u-h-divisibility of Z there exists a map $U^{(\alpha)} \to Z \to 0$. As $\operatorname{Hom}_R(U^{(\alpha)}, M) = 0$, it follows that $\operatorname{Hom}_R(Z, M) = 0$.

For (ii), let $Z \in$ Mod-U. Then there is U-projective presentation of Z, $0 \to H \to U^{(\alpha)} \to Z \to 0$ of U-modules in Mod-R. We apply $\operatorname{Hom}_R(-, M)$ to this projective presentation. As u is a ring epimorphism, we have that $\operatorname{Ext}^1_R(U^{(\alpha)}, M) \cong \operatorname{Ext}^1_U(U^{(\alpha)}, M) = 0$, so $\operatorname{Hom}_R(H, M) \cong \operatorname{Ext}^1_R(Z, M)$. However, by (i) of this lemma, $\operatorname{Hom}_R(H, M) = 0$, so also $\operatorname{Ext}^1_R(Z, M) = 0$. □

Lemma 5.3.4. *[34, Lemma 1.2] Let $u\colon R \to U$ be a ring epimorphism of commutative rings and let M be an R-module.*

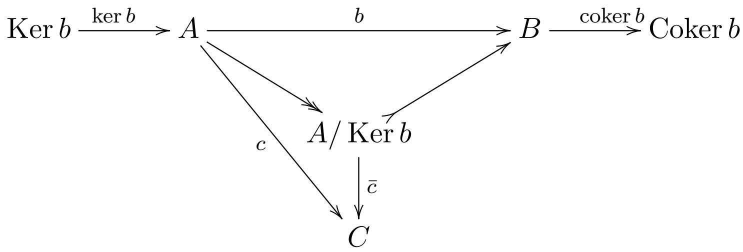
(i) If $M \otimes_R U = 0$, then $M \otimes_R Z = 0$ for any u-h-divisible module Z.

(ii) If M is a u-comodule, then $\mathrm{Tor}_1^R(M, Z) = 0 = M \otimes_R Z$ for any U-module Z.

Proof. The proof is dual-analogous to that of Lemma 5.3.3. □

Lemma 5.3.5. *[34, Lemma 1.10] Let $b\colon A \to B$ and $c\colon A \to C$ be two R-module homomorphisms such that C is a u-contramodule and* $\mathrm{Ker}(b)$ *is a u-h-divisible R-module and* $\mathrm{Coker}(b)$ *is a U-module. Then there exists a unique homomorphism $f\colon B \to C$ such that $c = fb$.*

Proof. First we show the existence of a homomorphism $f\colon B \to C$ such that $c = fb$. $\mathrm{Ker}\, b$ is a u-h-divisible module, so the composition $c \circ \ker b = 0$ by Lemma 5.3.3 (i), hence the map c factors through $\bar{c}\colon A/\mathrm{Ker}\, b \to C$ as in the following diagram.



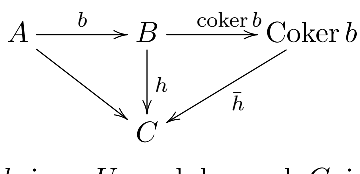
(5.3)

By applying the functor $\mathrm{Hom}_R(-, C)$ to the right short exact sequence above we get the following exact sequence.

$$\mathrm{Hom}_R(B, C) \to \mathrm{Hom}_R(A/\mathrm{Ker}\, b, C) \to \mathrm{Ext}^1_R(\mathrm{Coker}\, b, C)$$

By Lemma 5.3.3 (ii), $\mathrm{Ext}^1_R(\mathrm{Coker}\, b, C) = 0$ as $\mathrm{Coker}\, b$ is a U-module. Thus $\bar{c}\colon A/\mathrm{Ker}\, b \to C$ factors through $A/\mathrm{Ker}\, b \to B$, and we conclude that c factors through b.

Now we show the uniqueness of such a homomorphism. Suppose $h = f - g$ is such that $hb = 0$. Then there exists a homomorphism $\bar{h}\colon \mathrm{Coker}\, b \to C$ such that $\bar{h} \circ \mathrm{coker}\, b = h$.



By assumption, $\mathrm{Coker}\, b$ is a U-module, and C is a u-contramodule, so $\mathrm{Hom}_R(\mathrm{Coker}\, b, C) = 0$ by Lemma 5.3.3 (ii). Thus h must be the zero homomorphism, so $f = g$. □

From now on, $u\colon R \to U$ will always be a flat injective ring epimorphism of commutative rings.

5.3.1 Flat injective ring epimorphisms

Let $0 \to R \overset{u}{\to} U$ be a flat injective ring epimorphism of commutative rings where $U = R_{\mathcal{G}}$, $K = R_{\mathcal{G}}/R$ and $\mathcal{G}$ is the associated Gabriel topology $\{J \leq R \mid JU = U\}$. We will often refer to the following short exact sequence.

$$0 \to R \overset{u}{\to} U \overset{w}{\to} K \to 0 \tag{5.4}$$

In general we will use N to denote a $\mathcal{G}$-torsion-free module, while M to denote any R-module.

In our case, we already assume that U is flat, so $0 = \operatorname{Tor}_1^R(M, U)$ always holds, and moreover, $M \otimes_R U = 0$ if and only if M is $\mathcal{G}$-torsion, as seen by the following exact sequence.

$$0 \to t_{\mathcal{G}}(M) = \operatorname{Tor}_1^R(M, K) \to M \to M \otimes_R U \to M \otimes_R K \to 0$$

Thus the u-comodules coincide with the $\mathcal{G}$-torsion modules.

Every u-h-divisible module is $\mathcal{G}$-divisible, but the converse doesn't necessarily hold. The converse holds exactly when $\operatorname{Gen} U = \mathcal{D}_{\mathcal{G}}$, so the equivalent conditions of Proposition 1.7.7 hold, therefore exactly when $\operatorname{p.dim} U \leq 1$. We will cover what this condition means in more detail in Subsection 5.3.2.

Hence for an R-module M, by applying the contravariant functor $\operatorname{Hom}_R(-, M)$ to the short exact sequence (5.4) we have the following short exact sequences.

$$0 \to \operatorname{Hom}_R(K, M) \to \operatorname{Hom}_R(U, M) \to h_u(M) \to 0 \tag{5.5}$$

$$0 \to h_u(M) \to M \to M/h_u(M) \to 0 \tag{5.6}$$

$$0 \to M/h_u(M) \to \operatorname{Ext}_R^1(K, M) \to \operatorname{Ext}_R^1(U, M) \to 0 \tag{5.7}$$

For an R-module M, we let $\Delta_u(M)$ denote the module $\operatorname{Ext}_R^1(K, M)$ and $\delta_M \colon M \to \Delta_u(M)$ the natural connecting map from the exact sequences (5.6) and (5.7).

For each R-module M, let ν_M be the unit of the adjunction $\big((- \otimes_R K), \operatorname{Hom}_R(K, -)\big)$ evaluated at M.

$$\begin{aligned} \nu_M \colon M &\longrightarrow \operatorname{Hom}_R(K, M \otimes_R K) \\ m &\longmapsto [m^* \colon z + R \to m \otimes_R (z + R)] \qquad z \in U \end{aligned}$$

In the next lemmas we want to show that for a $\mathcal{G}$-torsion-free module N, the modules $\operatorname{Hom}_R(K, N \otimes_R K)$ and $\Delta_u(N)$ are isomorphic in a natural way. More precisely, they are isomorphic via a natural connecting homomorphism μ_N (see (5.9)), and moreover $\delta_N = \mu_N \nu_N$. This will be useful when proving Corollary 5.5.7.

We will sometimes not include the subscript on the homomorphisms δ_N, μ_N and ν_N for clarity of exposition.

Lemma 5.3.6. *Let $u\colon R \to U$ be a flat injective ring epimorphism of commutative rings. Then if N is a $\mathcal{G}$-torsion-free R-module, $\mu_N\colon \operatorname{Hom}_R(K, N\otimes_R K) \overset{\cong}{\to} \Delta_u(N)$ where μ_N is a connecting homomorphism of the long exact sequence in (5.9).*

Proof. We apply the covariant functor $\operatorname{Hom}_R(K, -)$ to the short exact sequence in (5.8), which is exact since N is $\mathcal{G}$-torsion-free.

$$0 \to N \to N \otimes_R U \to N \otimes_R K \to 0 \tag{5.8}$$

$$\begin{aligned}0 = \operatorname{Hom}_R(K, N\otimes_R U) \to \operatorname{Hom}_R(K, N\otimes_R K) \overset{\mu_N}{\to} \operatorname{Ext}^1_R(K, N) \to \\ \to \operatorname{Ext}^1_R(K, N \otimes_R U) = 0\end{aligned} \tag{5.9}$$

The first term vanishes as K is $\mathcal{G}$-torsion and $N \otimes_R U$ is $\mathcal{G}$-torsion-free. The last term vanishes since by the flatness of the ring U, there is an isomorphism $\operatorname{Ext}^1_R(K, N \otimes_R U) \cong \operatorname{Ext}^1_U(K \otimes_R U, N \otimes_R U) = 0$. Thus note that $\operatorname{Hom}_R(K, N \otimes_R K)$ is isomorphic to $\operatorname{Ext}^1_R(K, N) = \Delta_u(N)$ via μ.

Alternatively, one can use Lemma 4.1.3 as K is $\mathcal{G}$-torsion and $N \otimes_R U$ is $\mathcal{G}$-closed. □

Before continuing with the goal of proving that $\delta_N = \mu_N \nu_N$, we state a consequence of the Lemma 5.3.6. We note that in the reference provided, the statement is more general thus requires a more sophisticated proof, whereas here we choose to provide a simpler proof.

Lemma 5.3.7. *[15, Lemma 2.5(a),(b)] Let $u\colon R \to U$ be a flat injective ring epimorphism. Then the following hold.*

(i) $\operatorname{Hom}_R(K, M)$ *is a u-contramodule for every R-module M.*

(ii) $\Delta_u(N)$ *is a u-contramodule for every $\mathcal{G}$-torsion-free R-module N.*

Proof. (i)Now we will show that $\operatorname{Hom}_R(K, M)$ is a u-contramodule. By the tensor-Hom adjunction, we have the following isomorphism.

$$\operatorname{Hom}_R(U, \operatorname{Hom}_R(K, M)) \cong \operatorname{Hom}_R(U \otimes_R K, M) = 0$$

Similarly, to see that $\operatorname{Ext}^1_R(U, \operatorname{Hom}_R(K, M)) = 0$, we use the flatness of U so $\operatorname{Tor}^R_1(U, K) = 0$. Hence there is the following inclusion (see the homological formulas in Section 1.1).

$$0 \to \operatorname{Ext}^1_R(U, \operatorname{Hom}_R(K, M)) \to \operatorname{Ext}^1_R(U \otimes_R K, M) = 0$$

(ii)This follows by Lemma 5.3.6 and (i) of this lemma. □

The second statement in the above lemma can be extended with the additional assumption that $\operatorname{p.dim} U \leq 1$, which is stated in Lemma 5.3.17.

Lemma 5.3.8. *Let $u\colon R \to U$ be a flat injective ring epimorphism of commutative rings. Consider two short exact sequences ζ, ζ' such that B' is $\mathcal{G}$-torsion-free and C' is $\mathcal{G}$-torsion, and fix a morphism $\alpha\colon A \to A'$.*

$$
\begin{array}{lccccccccc}
\zeta\colon & 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \bar\beta} & & \\
\zeta'\colon & 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0
\end{array}
$$

If there exists a $\beta\colon B \to B'$ which makes the above into a commuting diagram, then β is unique.

Proof. Suppose there exist β, β' such that $\beta f = \beta' f = f'\alpha$ and $\bar\beta, \bar\beta'\colon C \to C'$ are the induced homomorphisms by β, β'. Then it follows that $\mathrm{Im}(\beta - \beta') = \mathrm{Im}(\bar\beta - \bar\beta')$, which is a submodule of both B' and C', therefore both $\mathcal{G}$-torsion and $\mathcal{G}$-torsion-free, and therefore is 0. Thus $\beta = \beta'$, as required. □

Lemma 5.3.9. *Let $u\colon R \to U$ be a flat injective ring epimorphism of commutative rings. For N a $\mathcal{G}$-torsion-free module, the following diagram commutes.*

$$
\begin{array}{ccc}
N & \xrightarrow{\delta_N} & \mathrm{Ext}^1_R(K, N) \\
\downarrow{\scriptstyle \nu_N} & \nearrow{\scriptstyle \mu,\ \cong} & \\
\mathrm{Hom}_R(K, N \otimes_R K) & &
\end{array}
$$

Proof. As $\Phi(\nu_N) = \mathrm{id}_{N \otimes_R K}$ and Φ is an isomorphism, it is enough to show that $\Phi(\mu^{-1}\delta_N) = \mathrm{id}_{N \otimes_R K}$, that is that $(\mu^{-1}\delta_N)(x)(k) = n \otimes_R k$ for every $x \in N$ and $k \in K$.

Fix $m \in N$ and $k \in K$. Consider the map $f_x\colon 1_R \mapsto x \in \mathrm{Hom}_R(R, N)$. Then $\delta_N(f_x)$ is the map associated to the pushout of $N \stackrel{f_x}{\leftarrow} R \stackrel{u}{\to} U$ which is shown in the top two rows of short exact sequences of Diagram 5.10. As μ is an isomorphism, for each extension ζ_x of K by N, one can associate a map $\mu^{-1}(\zeta_x) = g_x\colon K \to N \otimes_R K$ such that the bottom two rows of short exact sequences in (5.10) commute and form part of a pullback diagram.

$$
\begin{array}{lccccccccc}
 & 0 & \longrightarrow & R & \longrightarrow & U & \longrightarrow & K & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle f_x} & & \downarrow & & \| & & \\
\zeta_x\colon & 0 & \longrightarrow & N & \longrightarrow & Z_x & \longrightarrow & K & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow{\scriptstyle g_x} & & \\
 & 0 & \longrightarrow & N & \longrightarrow & N \otimes_R U & \longrightarrow & N \otimes_R K & \longrightarrow & 0
\end{array}
\qquad (5.10)
$$

As the larger squares also commute, we apply Lemma 5.3.8 to say that the map $U \to Z_x \to N \otimes_R U$ is exactly the map $z \mapsto x \otimes_R z$, as this map makes the larger left square commute. Thus $g_x\colon z + R \mapsto x \otimes_R (z + R)$. It is now straightforward to see that $(\mu^{-1}\delta_N)(x)(k) = (\mu^{-1})(\zeta_x)(k) = g_x(k) = x \otimes_R k$. □

The next corollary states that ν_N satisfies the assumptions of $b\colon A \to B$ Lemma 5.3.5

Corollary 5.3.10. *Let $u\colon R \to U$ be a flat injective ring epimorphism of commutative rings and N a $\mathcal{G}$-torsion-free module. Then the kernel of $\nu_N\colon N \to \mathrm{Hom}_R(K, N \otimes_R K)$ is u-h-divisible and the cokernel is a U-module.*

Proof. This follows from Lemma 5.3.9 as $\mu\nu_N = \delta_N$ and μ is an isomorphism, so $\mathrm{Ker}\,\nu_N \cong \mathrm{Ker}\,\delta_N = h_u(N)$, a u-h-divisible module and $\mathrm{Coker}\,\nu_N = \mathrm{Coker}\,\delta_N = \mathrm{Ext}^1_R(U, N)$, a U-module as required. □

The following two lemmas will be useful in Chapter 7.

Lemma 5.3.11. *[34, Lemma 1.11] Let $u\colon R \to U$ be a flat injective ring epimorphism with associated Gabriel topology $\mathcal{G}$, and let M by any R-module. Then $M/MJ \cong \Delta_u(M) \otimes_R R/J$ is an isomorphism for every $J \in \mathcal{G}$.*

Proof. Consider the combination of Equations (5.6) and (5.7).

$$0 \to h_u(M) \to M \to M/h_u(M) \to 0 \tag{5.6}$$

$$0 \to M/h_u(M) \to \mathrm{Ext}^1_R(K, M) \to \mathrm{Ext}^1_R(U, M) \to 0 \tag{5.7}$$

Applying $(- \otimes_R R/J)$ to (5.6) we first find that $M \otimes_R R/J \cong M/h_u(M) \otimes_R R/J$ as $h_u(M)$ is $\mathcal{G}$-divisible. Applying $(- \otimes_R R/J)$ to (5.7) we find $M/h_u(M) \otimes_R R/J \cong \mathrm{Ext}^1_R(K, M) \otimes_R R/J$ as $\mathrm{Ext}^1_R(U, M)$ is a U-module, so by Lemma 5.3.4 $\mathrm{Tor}^R_1(\mathrm{Ext}^1_R(U, M), R/J) = 0 = \mathrm{Ext}^1_R(U, M) \otimes_R R/J$. □

Lemma 5.3.12. *Let $u\colon R \to U$ be a flat injective ring epimorphism with associated Gabriel topology $\mathcal{G}$, and let N by a $\mathcal{G}$-torsion-free R-module. Then $\mathrm{Tor}^R_1(N, R/J) \cong \mathrm{Tor}^R_1(\Delta_u(N), R/J)$ is an isomorphism for every $J \in \mathcal{G}$.*

Proof. The R/J are u-comodules, so we apply Lemma 5.3.4(ii). Otherwise, note that $\mathrm{Tor}^R_i(Z, R/J) = 0 = Z \otimes_R R/J$ for any U-module Z and $i > 0$ as U is flat, so $\mathrm{Tor}^R_i(Z, R/J) \cong \mathrm{Tor}^U_i(Z, U \otimes_R R/J) = 0$.

Consider the combination of Equations (5.6) and (5.7).

$$0 \to h_u(N) \to N \to N/h_u(N) \to 0 \tag{5.6}$$

$$0 \to N/h_u(N) \to \mathrm{Ext}^1_R(K, N) \to \mathrm{Ext}^1_R(U, N) \to 0 \tag{5.7}$$

A N is $\mathcal{G}$-torsion-free, also $h_u(N)$ is $\mathcal{G}$-torsion-free and $\mathcal{G}$-divisible, so is a U-module. Thus applying $(- \otimes_R R/J)$ to the above sequences, we use the observation in the first lines of this proof, and find the following isomorphisms.

$$\mathrm{Tor}^R_1(N, R/J) \cong \mathrm{Tor}^R_1(N/h_u(N), R/J) \cong \mathrm{Tor}^R_1(\Delta_u(N), R/J)$$

□

The following lemma and corollary link u-contramodules with completions of a module with respect to the $\mathcal{G}$-topology.

Lemma 5.3.13. *Pos Let $u\colon R \to U$ be a flat injective ring epimorphism with associated Gabriel topology $\mathcal{G}$. Then for every $J \in \mathcal{G}$, every R/J-module M is a u-contramodule.*

Proof. To see that $\mathrm{Hom}_R(U, M) = 0$, take $f\colon U \to M$. Then $f(U) = f(JU) = Jf(U) = 0$ as J annihilates M.

To see $\mathrm{Ext}^1_R(U, M) = 0$, as $\mathrm{Tor}^R_i(R/J, U) = 0$ and $R \to R/J$ is a ring epimorphism, one has the following isomorphism.

$$\mathrm{Ext}^1_R(U, M) \cong \mathrm{Ext}^1_{R/J}(R/J \otimes_R U, M) = 0$$

□

Corollary 5.3.14. *Let $u\colon R \to U$ be a flat injective ring epimorphism of commutative rings with associated Gabriel topology $\mathcal{G}$. Then $\Lambda_{\mathcal{G}}(M)$ is a u-contramodule.*

Proof. This follows immediately by Lemma 5.3.13 and by the closure properties of u-contramodules in Lemma 5.3.2. □

5.3.2 When $\mathrm{p.dim}\, U \leq 1$

Let $u\colon R \to U$ be an flat injective ring epimorphism as before. The additional condition that $\mathrm{p.dim}\, U \leq 1$ has many important consequences for the category u-**contra**, and moreover these consequences will be crucial to the future chapters.

Remark 5.3.15. Suppose $u\colon R \to U$ is a flat injective ring epimorphism of commutative rings and let $\mathcal{G}$ be the associated Gabriel topology. Then $\mathrm{p.dim}\, U \leq 1$ if and only if the u-h-divisible modules and the $\mathcal{G}$-divisible modules coincide by Proposition 1.7.7.

Lemma 5.3.16. *[15, Proposition 3.2] Let $u\colon R \to U$ be a flat injective ring epimorphism such that* $\mathrm{p.dim}\, U \leq 1$. *Then the category of u-contramodules is closed under cokernels, and so is an abelian category.*

Lemma 5.3.17. *[15, Lemma 2.5(c)] Let $u\colon R \to U$ be a flat injective ring epimorphism such that* $\mathrm{p.dim}\, U \leq 1$. *Then the following hold. Then $\Delta_u(M)$ is a u-contramodule for every R-module M.*

Proof. Let the following short exact sequence denote the injective envelope of M.

$$0 \to M \to E(M) \to E(M)/M \to 0$$

We apply $\mathrm{Hom}_R(K, -)$ to the injective envelope of M, and find the exact sequence in (5.11).

$$\mathrm{Hom}_R(K, E(M)) \to \mathrm{Hom}_R(K, E(M)/M) \to \mathrm{Ext}^1_R(K, M) \to 0 \qquad (5.11)$$

Moreover, the first two modules in (5.11) are u-contramodules by Lemma 5.3.7(i). Therefore, $\Delta_u(M) := \mathrm{Ext}^1_R(K, M)$ is a cokernel of u-contramodule, therefore by Proposition 5.3.18, $\Delta_u(M)$ is also a u-contramodule. □

Lemma 5.3.18. *[15, Proposition 3.2] Let $u\colon R \to U$ be a flat injective ring epimorphism such that* $\mathrm{p.dim}\, U \leq 1$*. Then the category of u-contramodules is closed under cokernels, and so is an abelian category.*

Proposition 5.3.19. *[15, Proposition 3.2(b)] Let $u\colon R \to U$ be a flat injective ring epimorphism such that* $\mathrm{p.dim}\, U \leq 1$*. Then $\iota\colon u$-**contra** $\hookrightarrow$ Mod-R is an exact embedding and the functor $\Delta_u = \mathrm{Ext}^1_R(K, -)$ defines a left adjoint to this embedding. In particular, $\Delta_u(R)$ is a projective generator of u-**contra**.*

Sketch of proof. To prove that Δ_u is left adjoint to the inclusion, the crucial observation is Lemma 5.3.5, that is, any R-module homomorphism $B \to C$ where C is a u-contramodule extends uniquely to a homomorphism of u-contramodules $\Delta_u(B) \to C$.

Furthermore the inclusion $\iota\colon u$-**contra** $\to$ Mod-R is exact, $\Delta_u(R)$ is a projective object in u-**contra** and so is a generator since $C \cong \mathrm{Hom}_R(R, C) \cong \mathrm{Hom}_{u\text{-}\mathbf{contra}}(\Delta_u(R), C)$. □

Lemma 5.3.20. *Let $u\colon R \to U$ be a flat injective ring epimorphism of commutative rings such that* $\mathrm{p.dim}\, U \leq 1$*. The coproduct of $\Delta_u(R)$ indexed over a set X is $\Delta_u(R^{(X)})$.*

*The projective modules in u-**contra** are the direct summands of the modules of the form $\Delta(R^{(X)})$ for some set X.*

Proof. By Proposition 5.3.19, $\Delta_u(-)$ is a left adjoint, and thus the lemma follows as left adjoints preserve coproducts. The second statement then follows by the fact that $\Delta_u(R)$ is a projective generator of u-**contra**. That is, for every $C \in u$-**contra**, there exists a u-**contra**-projective presentation of the following form.

$$\coprod_X \Delta_u(R) \to C \to 0 \qquad (5.12)$$

As $\coprod_X \Delta_u(R) \cong \Delta_u(R^{(X)})$, the sequence (5.12) splits.

□

It is straightforward to see that a direct summand of a projective u-contramodule is still a projective u-contramodule.

5.4 The equivalence of categories

The following is a specific case of [15, Theorem 1.2], which we will prove here for completeness. We will associate the Gabriel topology $\mathcal{G}$ to the flat injective ring epimorphism u. We note that in [15] for u a (not necessarily injective nor flat nor commutative) ring epimorphism such that $\mathrm{Tor}_1^R(U,U) = 0$, Bazzoni-Positselski show a more general equivalence.

The *u-torsion-free* modules are modules which are contained in a U-module, or equivalently a module N is u-torsion-free if the natural map $N \to N \otimes_R U$ is a monomorphism.

In [15, Theorem 1.2], it is shown that $\big((- \otimes_R K), \mathrm{Hom}_R(K, -)\big)$ where $K = U/u(R)$ defines an equivalence between the u-h-divisible right u-comodules and the u-torsion-free u-contramodules.

Recall from Subsection 5.3.1, the $\mathcal{G}$-torsion and u-comodules coincide when u is a flat ring epimorphism. Additionally, as when U is flat, there is an associated Gabriel topology and all the U-modules are $\mathcal{G}$-torsion-free, therefore the $\mathcal{G}$-torsion-free modules and the u-torsion-free modules coincide.

Lemma 5.4.1. *[15, Lemma 1.3] Let $u\colon R \to U$ be a flat injective ring epimorphism of commutative rings. Then the following hold.*

(i) For any R-module M, $\mathrm{Hom}_R(K, M)$ is $\mathcal{G}$-torsion-free u-contramodule.

(ii) For any R-module M, $M \otimes_R K$ is u-h-divisible and $\mathcal{G}$-torsion.

Proof. Fix an R-module M.
(i) To see that $\mathrm{Hom}_R(K, M)$ is $\mathcal{G}$-torsion-free, note that by applying $\mathrm{Hom}_R(-, M)$ to u, we have the following inclusion.

$$0 \to \mathrm{Hom}_R(K, M) \to \mathrm{Hom}_R(U, M)$$

As $\mathrm{Hom}_R(U, M)$ is a U-module, it is also $\mathcal{G}$-torsion-free, thus so is $\mathrm{Hom}_R(K, M)$.

That $\mathrm{Hom}_R(K, M)$ is a u-contramodule follows by Lemma 5.3.7.

(ii)$M \otimes_R K$ is $\mathcal{G}$-divisible as K is, that is for each $J \in \mathcal{G}$, $(M \otimes_R K)J = MJ \otimes_R K = M \otimes_R K$. Furthermore $U \otimes_R (M \otimes_R K) \cong 0 \otimes_R M = 0$, so $M \otimes_R K$ is $\mathcal{G}$-torsion. □

Theorem 5.4.2. *[15, Theorem 1.2] Let $u\colon R \to U$ be a flat injective ring epimorphism of commutative rings. Then the restrictions of the adjoint functors $\mathrm{Hom}_R(K, -)$ and $(- \otimes_R K)$ are mutually inverse equivalences between the additive categories of u-h-divisible $\mathcal{G}$-torsion modules and $\mathcal{G}$-torsion-free u-contramodules.*

$$\left\{\begin{array}{c}\mathcal{G}\textit{-torsion-free}\\ u\textit{-contramodules}\end{array}\right\} \underset{\mathrm{Hom}_R(K,-)}{\overset{(-\otimes_R K)}{\rightleftarrows}} \left\{\begin{array}{c}u\textit{-h-divisible and}\\ \mathcal{G}\textit{-torsion modules}\end{array}\right\}$$

Proof. Let M be u-h-divisible and $\mathcal{G}$-torsion. We would like to show that $\mathrm{Hom}_R(K, M) \otimes_R K \to M$ is an isomorphism. We have the following exact sequence as M is u-h-divisible so $M = h_u(M) := u^*(\mathrm{Hom}_R(U, M))$.

$$0 \to \mathrm{Hom}_R(K, M) \to \mathrm{Hom}_R(U, M) \overset{u^*}{\to} M \to 0$$

As M is $\mathcal{G}$-torsion, we have that $\mathrm{Hom}_R(K, M)\otimes_R U \cong \mathrm{Hom}_R(U, M)\otimes_R U \cong \mathrm{Hom}_R(U, M)$ since $\mathrm{Hom}_R(U, M)$ is a U-module. As $\mathrm{Hom}_R(K, M)$ is $\mathcal{G}$-torsion-free, we also have the following exact sequence.

$$0 \to \mathrm{Hom}_R(K, M) \to \mathrm{Hom}_R(K, M) \otimes_R U \to \mathrm{Hom}_R(K, M) \otimes_R K \to 0$$

Thus we have the following commuting diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_R(K, M) & \longrightarrow & \mathrm{Hom}_R(K, M) \otimes_R U & \longrightarrow & \mathrm{Hom}_R(K, M) \otimes_R K & \longrightarrow & 0 \\
 & & \| & & \downarrow{\cong} & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(K, M) & \longrightarrow & \mathrm{Hom}_R(U, M) & \longrightarrow & M & \longrightarrow & 0
\end{array} \tag{5.13}$$

So it follows that the natural map $\mathrm{Hom}_R(K, M) \otimes_R K \to M$, which is the counit $(z + R) \otimes_R f \mapsto f(z + R)$, is an isomorphism by applying the snake lemma to the above diagram. To explicitly see that $\mathrm{Hom}_R(K, M)\otimes_R K \to M$ is the counit in Diagram 5.13 will follow from a dual-analogous proof which we write explicitly in the second part of this proof (when we show that $M \to \mathrm{Hom}_R(K, M \otimes_R K)$ is the unit).

We now consider the case of M a $\mathcal{G}$-torsion-free u-contramodule. Consider

$$0 \to M \overset{u\otimes_R \mathrm{id}_M}{\to} M \otimes_R U \to M \otimes_R K \to 0 \tag{5.14}$$

We first note that $\mathrm{Hom}_R(U, -)$ applied to (5.14) gives us the following isomorphism as M is a u-contramodule.

$$0 = \mathrm{Hom}_R(U, M) \to \mathrm{Hom}_R(U, M\otimes_R U) \overset{\cong}{\to} \mathrm{Hom}_R(U, M\otimes_R K) \to \mathrm{Ext}^1_R(U, M) = 0$$

Next we apply $\mathrm{Hom}_R(-, M \otimes_R K)$ to u.

$$0 \to \mathrm{Hom}_R(K, M\otimes_R K) \to \mathrm{Hom}_R(U, M\otimes_R K) \to M\otimes_R K \to \mathrm{Ext}^1_R(K, M\otimes_R K) = 0$$

Moreover, there is the isomorphism $\mathrm{Hom}_R(u, M\otimes_R U) = u^*\colon\ \mathrm{Hom}_R(U, M\otimes_R U) \overset{\cong}{\to} M\otimes_R U$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{u\otimes_R \mathrm{id}_M} & M\otimes_R U & \xrightarrow{w\otimes_R \mathrm{id}_M} & M\otimes_R K & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\beta} & & \cong\downarrow{\scriptstyle\alpha} & & \Vert & & \\ 0 & \longrightarrow & \mathrm{Hom}_R(K, M\otimes_R K) & \xrightarrow{w^*} & \mathrm{Hom}_R(U, M\otimes_R K) & \xrightarrow{u^*} & M\otimes_R K & \longrightarrow & 0 \end{array} \tag{5.15}$$

We will explicitly show that β is indeed the unit of M. The map α is the composition $\mathrm{Hom}_R(U, w\otimes_R \mathrm{id}_M)\circ\mathrm{Hom}_R(u, M\otimes_R U)^{-1} = (w\otimes_R \mathrm{id}_M)^*(u^*)^{-1}$, so it is clear the right square commutes.

$$\begin{array}{ccc} \mathrm{Hom}_R(U, M\otimes_R U) & \xrightarrow[(-)\circ u]{\cong} & M\otimes_R U \\ {\scriptstyle w\otimes_R \mathrm{id}_M\circ(-)}\downarrow\cong & & \downarrow{\scriptstyle w\otimes_R \mathrm{id}_M} \\ \mathrm{Hom}_R(U, M\otimes_R K) & \xrightarrow{(-)\circ u} & M\otimes_R K \end{array}$$

It follows that the map β is the unit, as when computed explicitly from the above commuting diagram of short exact sequences Diagram 5.15, $\beta(m)$ is the map $m\mapsto [m^*\colon z+R\to m\otimes_R (z+R)]$ for $z\in U$.
Therefore, by the snake lemma applied to Diagram 5.15, we find the exact sequence that we wanted.

□

5.5 The equivalence of $\Lambda_{\mathcal{G}}(N)$ and $\Delta_u(N)$ for $\mathcal{G}$-torsion-free modules

For the rest of this subsection, we will be considering a flat injective ring epimorphism of commutative rings denoted $0\to R\overset{u}{\to} U$, and we will denote by K the cokernel U/R of u.

As before, we denote by $h_u(M)$ the u-h-divisible submodule of M. Additionally, as before we will often drop the subscripts on the homomorphisms ν_M, δ_M and λ_M for clarity of exposition.

Lemma 5.5.1. *Let $u\colon R\to U$ be a flat injective ring epimorphism with associated Gabriel topology $\mathcal{G}$ and suppose N is $\mathcal{G}$-torsion-free. Then $h_u(N)\subseteq \bigcap_{J\in\mathcal{G}} NJ$.*

Proof. As $h_u(N)$ is u-h-divisible, it is $\mathcal{G}$-divisible, so $h_u(N)J = h_u(N)$ for every $J\in\mathcal{G}$, so clearly $h_u(N)\subseteq NJ$ for every $J\in\mathcal{G}$, as required. □

As before, for each R-module M, let ν_M be the unit of the adjunction $\big((-\otimes_R K), \mathrm{Hom}_R(K,-)\big)$ evaluated at M.

Lemma 5.5.2. *Let $u\colon R \to U$ be a flat injective ring epimorphism of commutative rings and $\mathcal{G}$ the associated Gabriel topology. For every $\mathcal{G}$-torsion-free module N, there exists an R-module homomorphism $\tilde{\nu}_N$ which makes the following diagram commute.*

$$\begin{array}{ccc} N & \xrightarrow{\nu_N} & \mathrm{Hom}_R(K, N \otimes_R K) \\ {\scriptstyle \lambda_N}\downarrow & \nearrow{\scriptstyle \tilde{\nu}_N} & \\ \Lambda_{\mathcal{G}}(N) & & \end{array}$$

Proof. Take $\tilde{n} = (n_J + NJ)_{J \in \mathcal{G}} \in \Lambda_{\mathcal{G}}(N)$. We will first define how $\tilde{\nu}((n_J + NJ)_{J \in \mathcal{G}})$ acts on a fixed $z + R \in K$.

Let the annihilator of $z + R \in K$ be denoted as $I_z = \mathrm{Ann}_R(z + R)$.

$$\tilde{\nu}(\tilde{n})\colon K \longrightarrow N \otimes_R K$$

$$z + R \longmapsto n_{I_z} \otimes_R (z + R)$$

First we claim that $\tilde{\nu}(\tilde{n})$ is a well-defined map. Consider $\tilde{n}' = (n'_J + NJ)_{J \in \mathcal{G}} \in \Lambda_{\mathcal{G}}(N)$ such that $\tilde{n} = \tilde{n}'$, so $n_J - n'_J \in NJ$ for each $J \in \mathcal{G}$. Then $n_{I_z} - n'_{I_z} \otimes_R (z + R) = 0$ for each $z + R \in K$, so $\tilde{\nu}(\tilde{n}) = \tilde{\nu}(\tilde{n}')$.

We claim the map $\tilde{\nu}(\tilde{n})$ is R-linear. Note that $\mathrm{Ann}_R(z+R) \subseteq \mathrm{Ann}_R(rz+R)$ for every $r \in R$ so $n_{I_{rz}} - n_{I_z} \in NI_{rz}$. Therefore the two elements on the right represent the same element in $N \otimes_R K$.

$$\big(\tilde{\nu}(\tilde{n})\big)(rz + R) = n_{I_{rz}} \otimes_R (rz + R)$$

$$r\big(\tilde{\nu}(\tilde{n})\big)(z + R) = r(n_{I_z} \otimes_R (z + R)) = n_{I_z} \otimes_R (rz + R)$$

It remains to see that the homomorphism $\tilde{\nu}(\tilde{n})$ respects addition. We first note that for $z, y \in U$, $0 \neq \mathrm{Ann}_R(z) \cap \mathrm{Ann}_R(y) \subseteq \mathrm{Ann}_R(z + y)$, so $n_{I_{z+y}} - n_{I_z \cap I_y} \in NI_{z+y}$.

$$\big(\tilde{\nu}(\tilde{n})\big)(z + y + R) = n_{I_{z+y}} \otimes_R (z + y + R)$$

$$\begin{aligned} \big(\tilde{\nu}(\tilde{n})\big)(z + R) + \big(\tilde{\nu}(\tilde{n})\big)(y + R) &= y_{I_z} \otimes_R (z + R) + n_{I_y} \otimes_R (y + R) \\ &= n_{I_z \cap I_y} \otimes_R (z + R) + n_{I_z \cap I_y} \otimes_R (y + R) \\ &= n_{I_z \cap I_y} \otimes_R (z + y + R) \end{aligned}$$

It is straightforward to see that $\tilde{\nu}$ is an R-linear homomorphism of R-modules. It is clear that the above triangle commutes, $\tilde{\nu}\lambda = \nu$ as $\tilde{\nu}\lambda(n) = \tilde{\nu}((n + NJ)_{J \in \mathcal{G}})\colon z + R \mapsto n \otimes_R (z + R)$, as required. □

The following uses ideas related to Lemma 4.1.9.

Lemma 5.5.3. *Let $u\colon R \to U$ be a flat injective ring epimorphism of commutative rings and $\mathcal{G}$ the associated Gabriel topology. For every $\mathcal{G}$-torsion-free module N, the R-module homomorphism $\tilde{\nu}_N$ is a monomorphism.*

Proof. Take $\tilde{n} \in \Lambda_{\mathcal{G}}(N)$ such that $\tilde{\nu}(\tilde{n}) = 0$. That is, $n_{I_z} \otimes_R (z + R) = 0$ for every $z \in U$ as defined in Lemma 5.5.2. We first note that if $n \otimes_R (z+R) = 0$, then one can rewrite each $n \otimes_R z \in N \otimes_R U$ as an element of $N \otimes_R R$, as seen from the following short exact sequence.

$$0 \longrightarrow N \longrightarrow N \otimes_R U \longrightarrow N \otimes_R K \longrightarrow 0$$
$$n_{I_z} \otimes_R z \longmapsto 0$$

Therefore for each $z \in U$ there exists an $m \in N$ such that $m \otimes_R 1 = n_{I_z} \otimes_R z \in N \otimes_R U$.

Fix a $J \in \mathcal{G}$, and we claim that $n_J \in NJ$. As $JU = U$, one can write $1_U = \sum_{0 \le i \le k} a_i z_i$ where $a_i \in J$ and $z_i \in U$. Recall that by the argument in Lemma 4.1.9, that $\bigcap_{0 \le i \le n} I_{z_i} \subseteq J$ where $I_{z_i} = \mathrm{Ann}(z_i + R)$. Thus it is sufficient to show that $n_{\bigcap I_{z_i}} \in NJ$ as $n_J - n_{\bigcap I_{z_i}} \in NJ$. Rearranging, we find the following.

$$\begin{aligned} n_{\bigcap I_{z_i}} \otimes_R 1 &= n_{\bigcap I_{z_i}} \otimes_R (\sum_i a_i z_i) \\ &= \sum_i a_i (n_{\bigcap I_{z_i}} \otimes_R z_i) \\ &= \sum_i a_i (n_{I_{z_i}} \otimes_R z_i) \\ &= \sum_i a_i (m_i \otimes_R 1) \\ &= (\sum_i a_i m_i) \otimes_R 1 \end{aligned} \tag{5.16}$$

Therefore we have shown that $n_{\bigcap I_{z_i}} = \sum_i a_i m_i \in NJ$, so also $n_J \in NJ$ as desired. □

Proposition 5.5.4. *For every $\mathcal{G}$-torsion-free module N, the homomorphism $\tilde{\nu}_N$ is an isomorphism of R-modules. That is, $\tilde{\nu}_N\colon \Lambda_{\mathcal{G}}(N) \cong \mathrm{Hom}_R(K, N \otimes_R K)$.*

Proof. This follows by applying Lemma 5.3.5 to the unit $\nu_N\colon N \to \mathrm{Hom}_R(K, N \otimes_R K)$, which satisfies the assumption of the lemma. That is, $\mathrm{Ker}\,\nu_N$ is u-h-divisible and $\mathrm{Coker}\,\nu_N$ is a U-module. Moreover, $\Lambda_{\mathcal{G}}(N)$ and $\mathrm{Hom}_R(K, N \otimes_R K)$ are both u-contramodules by Corollary 5.3.14 and Lemma 5.4.1. Thus consider the following two commuting diagrams, with the existence and uniqueness of f and $\mathrm{id}_{\mathrm{Hom}_R(K,N\otimes_R K)}$ respectively in the following diagrams follow by Lemma 5.3.5.

$$\begin{array}{ccc} N & \xrightarrow{\nu_N} & \mathrm{Hom}_R(K, N \otimes_R K) \\ {\scriptstyle\lambda}\downarrow & \swarrow {\scriptstyle f} & \\ \Lambda_{\mathcal{G}}(N) & & \end{array} \qquad \begin{array}{ccc} N & \xrightarrow{\nu_N} & \mathrm{Hom}_R(K, N \otimes_R K) \\ {\scriptstyle\nu_N}\downarrow & \swarrow {\scriptstyle \mathrm{id}_{\mathrm{Hom}_R(K,N\otimes_R K)}} & \\ \mathrm{Hom}_R(K, N \otimes_R K) & & \end{array}$$

We already know that $\tilde{\nu}\lambda = \nu_N$ and that $\tilde{\nu}$ is a monomorphism by Lemma 5.5.3. Thus $\tilde{\nu} f \nu_N = \tilde{\nu}\lambda = \nu_N$, so $\tilde{\nu} f = \mathrm{id}_{\mathrm{Hom}_R(K,N\otimes_R K)}$. Therefore $\tilde{\nu}$ is an isomorphism, as required.

□

Corollary 5.5.5. *Let $u\colon R \to U$ be a flat injective ring epimorphism and let $\mathfrak{R} := \mathrm{End}_R(K)$. Then $\tilde{\nu}_R\colon \Lambda_{\mathcal{G}}(R) \to \mathfrak{R}$ is a ring isomorphism.*

Proof. By Proposition 5.5.4 we know that $\tilde{\nu}_R\colon \Lambda_{\mathcal{G}}(R) \to \mathfrak{R}$ is an isomorphism of R-modules. It remains only to check that this is a homomorphism of rings.

Let $\tilde{r} = (r_J + J)_{J\in\mathcal{G}}$ and $\tilde{s} = (s_J + J)_{J\in\mathcal{G}}$ denote elements of $\Lambda_{\mathcal{G}}(R)$. As before let I_z denote $\mathrm{Ann}_R(z + R)$ and note that $I_z \subseteq I_{tz}$ for each $t \in R$. We let $z' := s_{I_z} z$.

$$\tilde{\nu}(\tilde{r} \cdot \tilde{s})\colon K \to K\colon z + R \mapsto r_{I_z} s_{I_z} z + R$$

$$\tilde{\nu}(\tilde{r})\tilde{\nu}(\tilde{s}) = (K \xrightarrow{\tilde{r}} K)(K \xrightarrow{\tilde{s}} K)\colon z + R \mapsto s_{I_z} z + R \mapsto r_{I_{z'}} s_{I_z} z + R$$

Then as $I_{z'} = I_{s_{I_z} z} \supseteq I_z$, clearly $r_{I_{z'}} - r_{I_z} \in I_{z'}$, so $(r_{I_z} - r_{I_{z'}})s_{I_z} z = 0$. We conclude the endomorphisms $\tilde{\nu}(\tilde{r} \cdot \tilde{s})$ and $\tilde{\nu}(\tilde{r})\tilde{\nu}(\tilde{s})$ are equal. □

Corollary 5.5.6. *If V is an open ideal in the topology of $\mathfrak{R} = \mathrm{End}_R(K)$, then there is $J \in \mathcal{G}$ and a surjective ring homomorphism $R/J \to \mathfrak{R}/V$.*

Proof. By the definition of the topology on $\mathfrak{R}$, if V is an open ideal, then by Corollary 5.5.5, $W = \alpha^{-1}(V)$ is an open ideal in the projective limit topology of $\Lambda_{\mathcal{G}}(R)$. Hence by Remark 5.1.1, there is $J \in \mathcal{G}$ such that $W \supseteq \Lambda_{\mathcal{G}}(R)J$. By Corollary 5.5.5 there is a surjective ring homomorphism $R/J \cong \mathfrak{R}/\mathfrak{R}J \to \mathfrak{R}/V$. □

Corollary 5.5.7. *For every $\mathcal{G}$-torsion-free module N, the projective limit topology and the $\mathcal{G}$-topology coincide in $\Lambda_{\mathcal{G}}(N)$. That is, the equivalent conditions of Theorem 5.2.1 hold. Moreover, $\bigcap_{J\in\mathcal{G}} NJ = h_u(N)$.*

Proof. This is straightforward to see as $\tilde{\nu}\lambda = \nu$ where $\tilde{\nu}$ is an isomorphism, so $\mathrm{Coker}\,\lambda \cong \mathrm{Coker}\,\nu$. We know that $\mathrm{Coker}\,\nu \cong \mathrm{Ext}^1_R(U, N)$ is a U-module so is $\mathcal{G}$-divisible as required.

□

Chapter 6

Enveloping classes and 1-tilting cotorsion pairs over commutative rings

The results of this chapter form the main content of the preprint [12]. We begin with a 1-tilting cotorsion pair over a commutative ring, $(\mathcal{A}, \mathcal{D}_\mathcal{G})$. The aim of this chapter is to characterise the commutative rings for which $\mathcal{D}_\mathcal{G}$ is an enveloping class. Some of the results in this chapter are proved directly or more generally in other parts of this thesis, but we chose to leave it as written in [12].

Section 6.1 is dedicated to showing that if $\mathcal{D}_\mathcal{G}$ is enveloping then $R \to R_\mathcal{G}$ must arise from a perfect localisation, or equivalently $R_\mathcal{G} \in \mathcal{D}_\mathcal{G}$. In particular, the assumption that $\mathcal{D}_\mathcal{G}$ is enveloping and that $(\mathcal{A}, \mathcal{D}_\mathcal{G})$ is of finite type is used to show that the $\mathcal{D}_\mathcal{G}$-envelopes of $\mathcal{G}$-torsion modules must be $\mathcal{G}$-torsion in Lemma 6.1.3. Let T be an associated 1-tilting module of $(\mathcal{A}, \mathcal{D}_\mathcal{G})$. Our results in this first section use the $\mathrm{Add}(T)$ coresolution of R in (T3) (see Section 1.5) and facts about the Gabriel topology $\mathcal{G}$ which were shown in Chapter 4. Proposition 6.1.6 is the main result of this section. We note that Lemma 6.1.2 is a particular case of Lemma 4.1.4, although we choose to leave the proof as it is presented in [12].

Next, in Section 6.2 we continue with the assumption that the class $\mathcal{D}_\mathcal{G}$ is enveloping, with the additional information from Section 6.1 that $\mathcal{G}$ arises from a perfect localisation. Thus this information along with that $\operatorname{p.dim} U \leq 1$ in Proposition 6.2.1 implies that $\mathrm{Gen}(U) = \mathcal{D}_\mathcal{G} = K^\perp$ and $U \oplus K$ is the associated 1-tilting module and $U = R_\mathcal{G}, K = R_\mathcal{G}/R$ by Proposition 1.7.7 so $\mathcal{D}_\mathcal{G}$ arises from a perfect localisation. We prove that all the quotient rings R/J, for $J \in \mathcal{G}$ are perfect rings and so are all the discrete quotient rings of the topological ring $\mathfrak{R} = \mathrm{End}(K)$ (Theorems 6.2.14 and 6.2.15). In the terminology of [35] or [14] this means that $\mathfrak{R}$ is a pro-perfect topological ring. Moreover, Example 6.2.3 provides an instance of when R has a $\mathcal{T}$-envelope, but the class $\mathcal{T}$ is not enveloping. We note additionally that Proposition 6.2.1 partly generalises [2, Theorem 1.1], which was already partly generalised in [4].

In Section 6.3 we show that the converse holds, that is if $\mathfrak{R} = \mathrm{End}(K)$ is a pro-perfect topological ring and the projective dimension of U is at most one, then the class of $\mathcal{G}$-divisible modules is enveloping, as stated in Theorem 6.3.4. Consequently, applying results from [14, Theorem 13.3], we obtain that $\mathrm{Add}(K)$, the class of direct summands of direct sums of copies of K, is closed under direct limits. This result is shown directly in Section 7.3, where we show that if the R/J are perfect rings for each $J \in \mathcal{G}$, K is Σ-pure-split and so $\mathrm{Add}(K)$ must be closed under direct limits. The method we use in Section 7.3 is that used by Positselski in [35, Sections 3].

Since $\mathcal{D}_\mathcal{G}$ coincides with the right Ext-orthogonal of $\mathrm{Add}(K)$, we have an instance of the necessity of the closure under direct limits of a class whose right Ext-orthogonal admits envelopes. Therefore in our situation we prove a converse of the result by Enochs and Xu Theorem 1.2.5 which states that if a class $\mathcal{A}$ of modules is closed under direct limits and extensions and whose right Ext-orthogonal $\mathcal{A}^\perp$ admits special preenvelopes with cokernel in $\mathcal{A}$, then $\mathcal{A}^\perp$ is enveloping.

As a byproduct we obtain that a 1-tilting torsion class over a commutative ring is enveloping if and only if it arises from a flat injective ring epimorphism $u\colon R \to U$ such that $\mathrm{p.dim}\, U \leq 1$ with associated Gabriel topology $\mathcal{G}$ where the factor rings R/J are perfect rings for every $J \in \mathcal{G}$ (Theorem 6.3.5). This provides a partial answer to Problem 1 of [29, Section 13.5] and generalises the result proved in [9] for the case of commutative domains and divisible modules.

In Section 6.4, we consider the case that $u\colon R \to U$ is not a monomorphism. This case is easily reduced to the injective case, since the class of $\mathcal{G}$-divisible modules is annihilated by the kernel I of u, so all the results proved for R apply to the ring R/I and to the cokernel K of u.

As mentioned previously in this thesis, if the assumption states that the Gabriel topology $\mathcal{G}$ arises from a perfect localisation, we denote $R_\mathcal{G}$ by U and $R_\mathcal{G}/R$ by K, and the natural ring homomorphism by $u\colon R \to U$.

6.1 Enveloping 1-tilting classes over commutative rings

For this section, R will always be a commutative ring and $\mathcal{T}$ a 1-tilting class.

By Theorem 1.7.6 there is a faithful finitely generated Gabriel topology $\mathcal{G}$ such that $\mathcal{T}$ is the class of $\mathcal{G}$-divisible modules. We denote again by $(\mathcal{E}_\mathcal{G}, \mathcal{F}_\mathcal{G})$ the associated faithful hereditary torsion pair of finite type. We use $\mathcal{D}_\mathcal{G}$ and

$\mathcal{T} = \mathrm{Gen}(T) = T^{\perp}$ interchangeably to denote the 1-tilting class, and $\mathcal{A}$ to denote the left orthogonal class ${}^{\perp}\mathcal{D}_{\mathcal{G}}$.

Recall that if $\mathcal{T}$ is 1-tilting, $\mathcal{T} \cap {}^{\perp}\mathcal{T} = \mathrm{Add}(T)$ (see Section 1.5). By (T3) of the definition of a 1-tilting module we have the following short exact sequence where $T_0, T_1 \in \mathrm{Add}(T)$.

$$\text{(T3)} \qquad 0 \to R \overset{\varepsilon}{\to} T_0 \to T_1 \to 0 \tag{6.1}$$

In fact, this short exact sequence is a special $\mathcal{D}_{\mathcal{G}}$-preenvelope of R, and $T_0 \oplus T_1$ is a 1-tilting module which generates $\mathcal{T}$.

Furthermore, assuming that R has a $\mathcal{D}_{\mathcal{G}}$-envelope, we can suppose without loss of generality that the sequence (T3) is the $\mathcal{D}_{\mathcal{G}}$-envelope of R, since an envelope is extracted from a special preenvelope by passing to direct summands (Proposition 1.2.2). For the rest of the section we will denote the $\mathcal{D}_{\mathcal{G}}$-envelope of R by ε as in (6.1).

Recall from Section 1.7 that for every $M \in \text{Mod-}R$ there is the commuting diagram (1.8). Since $\mathcal{G}$ is faithful we have the following short exact sequence where ψ_R is a ring homomorphism and $R_{\mathcal{G}}/R$ is $\mathcal{G}$-torsion.

$$0 \to R \overset{\psi_R}{\to} R_{\mathcal{G}} \to R_{\mathcal{G}}/R \to 0 \tag{6.2}$$

We now show two lemmas about the 1-tilting module $T_0 \oplus T_1$ and the class $\mathrm{Add}(T_0 \oplus T_1)$ assuming that R has a $\mathcal{D}_{\mathcal{G}}$-envelope.

Lemma 6.1.1. *Let the following short exact sequence be a $\mathcal{D}_{\mathcal{G}}$-envelope of R.*

$$0 \to R \overset{\varepsilon}{\to} T_0 \to T_1 \to 0$$

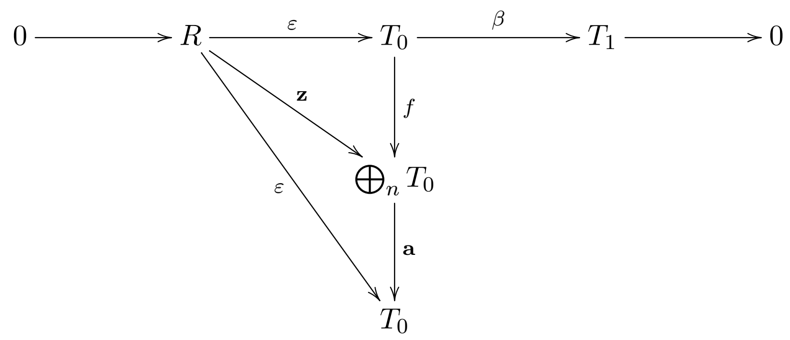
Then T_0 is $\mathcal{G}$-torsion-free and $T_0 \cong T_0 \otimes_R R_{\mathcal{G}}$.

Proof. We will show that for every $J \in \mathcal{G}$, $T_0[J]$, the submodule of elements of T_0 which are annihilated by J, is zero. Set $w := \varepsilon(1_R)$ and fix a $J \in \mathcal{G}$. As $T_0 = JT_0$, $w = \sum_{1 \le i \le n} a_i z_i$ where $a_i \in J$ and $z_i \in T_0$. This sum is finite, so we can define the following maps.

$$\mathbf{z}\colon R \longrightarrow \bigoplus_{1\le i\le n} T_0 \qquad \mathbf{a}\colon \bigoplus_{1\le i\le n} T_0 \longrightarrow T_0$$
$$1_R \longmapsto (z_1, ..., z_n) \qquad (x_1, ..., x_n) \longmapsto \sum_i a_i x_i$$

As $\bigoplus_n T_0$ is also $\mathcal{G}$-divisible, by the preenvelope property of ε there exists a map $f\colon T_0 \to \bigoplus_n T_0$ such that $f\varepsilon = \mathbf{z}$. Also, $\mathbf{az}(1_R) = \sum_{1\le i\le n} a_i z_i = w$, so

$\mathbf{az} = \varepsilon$ and the following diagram commutes.



By the envelope property of ε, $\mathbf{a}f$ is an automorphism of T_0. The restriction of the automorphism $\mathbf{a}f$ to $T_0[J]$ is an automorphism of $T_0[J]$, and factors through the module $\bigoplus_n T_0[J]$. However $\mathbf{a}(\bigoplus_n T_0[J]) = 0$, so $\mathbf{a}f(T_0[J]) = 0$, but $\mathbf{a}f$ restricted to $T_0[J]$ is an automorphism, thus $T_0[J] = 0$.
It follows from (iii) of Lemma 4.1.2 that $T_0 \cong T_0 \otimes_R R_{\mathcal{G}}$ since T_0 is $\mathcal{G}$-divisible. □

As mentioned in the introduction of this chapter, the following is a particular case of Lemma 4.1.4(i).

Lemma 6.1.2. *Suppose R has a $\mathcal{D}_{\mathcal{G}}$-envelope in* Mod-$R$. *Then for every $M \in \mathrm{Add}(T_0 \oplus T_1)$, $M \otimes_R R_{\mathcal{G}}$ is $\mathcal{G}$-torsion-free.*

Proof. From Lemma 6.1.1, $T_0 \cong T_0 \otimes_R R_{\mathcal{G}}$ is $\mathcal{G}$-torsion-free. We first show that $T_1 \otimes_R R_{\mathcal{G}}$ is $\mathcal{G}$-torsion-free. Consider the following short exact sequence obtained by applying $(- \otimes_R R_{\mathcal{G}})$ to the envelope of R, and note that $\mathrm{Tor}_1^R(T_1, R_{\mathcal{G}}) = 0$ by Lemma 4.1.2 (iii).

$$0 \to R_{\mathcal{G}} \to T_0 \otimes_R R_{\mathcal{G}} \to T_1 \otimes_R R_{\mathcal{G}} \to 0$$

As $R_{\mathcal{G}}$ is $\mathcal{G}$-closed and $T_0 \otimes_R R_{\mathcal{G}}$ is $\mathcal{G}$-torsion-free, by applying the covariant functor $\mathrm{Hom}_R(R/J, -)$ to the above sequence for every $J \in \mathcal{G}$, we obtain that $T_1 \otimes_R R_{\mathcal{G}}$ must be $\mathcal{G}$-torsion-free.
It is now straightforward to see that the statement holds for any direct summand of $(T_0 \oplus T_1)^{(\alpha)}$. □

We look at $\mathcal{D}_{\mathcal{G}}$-envelopes of $\mathcal{G}$-torsion modules in Mod-R, and find that they are also $\mathcal{G}$-torsion.

Lemma 6.1.3. *Suppose $\mathcal{D}_{\mathcal{G}}$ is enveloping in* Mod-R *and M is a $\mathcal{G}$-torsion R-module. Then the $\mathcal{D}_{\mathcal{G}}$-envelope of M is $\mathcal{G}$-torsion.*

Proof. To begin with, fix a finitely generated $J \in \mathcal{G}$ with a set $\{a_1, \ldots, a_t\}$ of generators and consider a $\mathcal{D}_{\mathcal{G}}$-envelope $D(J)$ of the cyclic $\mathcal{G}$-torsion module R/J, denoted as follows.

$$0 \to R/J \hookrightarrow D(J) \to A(J) \to 0$$

We will use the T-nilpotency of direct sums of envelopes as in Theorem 1.2.4 (ii). Consider the following countable direct sum of envelopes of R/J which is itself an envelope, by Theorem 1.2.4 (i):

$$0 \to \bigoplus_n (R/J)_n \hookrightarrow \bigoplus_n D(J)_n \to \bigoplus_n A(J)_n \to 0.$$

Choose an element $a \in J$ and for each n set $f_n \colon D(J)_n \to D(J)_{n+1}$ to be the multiplication by a.

Then clearly $(R/J)_n$ vanishes under the action of f_n, hence we can apply Theorem 1.2.4 (ii). For every $d \in D(J)$, there exists an m such that

$$f_m \circ \cdots \circ f_2 \circ f_1(d) = 0 \in D(J)_{(m+1)}.$$

Hence for every $d \in D$ there is an integer m for which $a^m d = 0$.
Fix $d \in D$ and let m_i be the minimal natural number for which $(a_i)^{m_i} d = 0$ and set $m := \sup\{m_i \mid 1 \le i \le t\}$. Then for a large enough integer k we have that $J^k d = 0$ (for example set $k = tm$), and $J^k \in \mathcal{G}$. Thus every element of $D(J)$ is annihilated by an ideal contained in $\mathcal{G}$, therefore $D(J)$ is $\mathcal{G}$-torsion.

Now consider an arbitrary $\mathcal{G}$-torsion module M. Then M has a presentation $\bigoplus_{\alpha\in\Lambda} R/J_\alpha \xrightarrow{p} M \to 0$ for a family $\{J_\alpha\}_{\alpha\in\Lambda}$ of ideals of $\mathcal{G}$. Since $\mathcal{G}$ is of finite type, we may assume that each J_α is finitely generated.

Take the push-out of this map with the $\mathcal{D}_\mathcal{G}$-envelope of $\bigoplus_\alpha R/J_\alpha$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{\alpha\in\Lambda} R/J_\alpha & \longrightarrow & \bigoplus_{\alpha\in\Lambda} D(J_\alpha) & \longrightarrow & \bigoplus_{\alpha\in\Lambda} A(J_\alpha) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p} & & \downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & Z & \longrightarrow & \bigoplus_{\alpha\in\Lambda} A(J_\alpha) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The bottom short exact sequence forms a preenvelope of M. We have shown above that for every α in A, $D(J_\alpha)$ is $\mathcal{G}$-torsion, so also Z is $\mathcal{G}$-torsion. Therefore, as the $\mathcal{D}_\mathcal{G}$-envelope of M must be a direct summand of Z by Proposition 1.2.2, also the $\mathcal{D}_\mathcal{G}$-envelope of M is $\mathcal{G}$-torsion. □

The following is a corollary to Lemma 6.1.2 and Lemma 6.1.3.

Corollary 6.1.4. *Suppose $\mathcal{D}_\mathcal{G}$ is enveloping in* Mod-R *and suppose M is a $\mathcal{G}$-torsion R-module. Then $M \otimes_R R_\mathcal{G}$ is $\mathcal{G}$-divisible.*

Proof. Let the following be a $\mathcal{D}_\mathcal{G}$-envelope of a $\mathcal{G}$-torsion module M, where both D and A are $\mathcal{G}$-torsion by Lemma 6.1.3.

$$0 \to M \to D \to A \to 0$$

The module A is $\mathcal{G}$-divisible and $R_\mathcal{G}/R$ is $\mathcal{G}$-torsion so $A \otimes_R R_\mathcal{G}/R = 0$, hence $A \to A \otimes_R R_\mathcal{G}$ is surjective. In particular, $A \otimes_R R_\mathcal{G}$ is $\mathcal{G}$-torsion. Also as $A \in \mathrm{Add}(T_0 \oplus T_1)$, $A \otimes_R R_\mathcal{G}$ is $\mathcal{G}$-torsion-free by Lemma 6.1.2 (ii). It follows that $A \otimes_R R_\mathcal{G}$ is both $\mathcal{G}$-torsion and $\mathcal{G}$-torsion-free so $A \otimes_R R_\mathcal{G} = 0$. Additionally as $\mathrm{p.dim}\, A \leq 1$, $\mathrm{Tor}_1^R(A, R_\mathcal{G}) = 0$, so the functor $(- \otimes_R R_\mathcal{G})$ applied to the envelope of M reduces to the following isomorphism.

$$0 = \mathrm{Tor}_1^R(A, R_\mathcal{G}) \to M \otimes_R R_\mathcal{G} \overset{\cong}{\to} D \otimes_R R_\mathcal{G} \to A \otimes_R R_\mathcal{G} = 0$$

Hence as $D \otimes_R R_\mathcal{G}$ is $\mathcal{G}$-divisible, also $M \otimes_R R_\mathcal{G}$ is $\mathcal{G}$-divisible, as required. □

Proposition 6.1.5. *Suppose $\mathcal{D}_\mathcal{G}$ is enveloping in* Mod-R. *Then $R_\mathcal{G}$ is $\mathcal{G}$-divisible.*

Proof. We will show that for each $J \in \mathcal{G}$, $R/J \otimes_R R_\mathcal{G} = 0$. Fix a $J \in \mathcal{G}$. By Corollary 6.1.4, $R/J \otimes_R R_\mathcal{G}$ is $\mathcal{G}$-divisible, Thus we have $R/J \otimes_R (R/J \otimes_R R_\mathcal{G}) = 0$. However

$$0 = R/J \otimes_R (R/J \otimes_R R_\mathcal{G}) \cong (R/J \otimes_R R/J) \otimes_R R_\mathcal{G} \cong R/J \otimes_R R_\mathcal{G},$$

since $R \to R/J$ is a ring epimorphism, thus $R_\mathcal{G}$ is $\mathcal{G}$-divisible. □

Using the characterisation of a perfect localisation of [39, Chapter XI.3, Proposition 3.4], we can state the main result of this section. Note that it remains to show that $\mathrm{p.dim}\, R_\mathcal{G} \leq 1$ to say the equivalent conditions of Proposition 1.7.7 hold.

Proposition 6.1.6. *Assume that $\mathcal{T}$ is a* 1*-tilting class over a commutative ring R such that the class $\mathcal{T}$ is enveloping. Then the associated Gabriel topology $\mathcal{G}$ of $\mathcal{T}$ arises from a perfect localisation.*

Proof. By Proposition 6.1.5, $R_\mathcal{G}$ is $\mathcal{G}$-divisible, hence the Gabriel topology $\mathcal{G}$ is perfect by Proposition 1.7.4. Hence $\psi\colon R \to R_\mathcal{G}$ is flat ring epimorphism and moreover it is injective. □

6.2 When a $\mathcal{G}$-divisible class is enveloping

For this section, R will always be a commutative ring. Fix a flat injective ring epimorphism u and an exact sequence

$$0 \to R \overset{u}{\to} U \to K \to 0.$$

Denote by $\mathcal{G}$ the corresponding Gabriel topology.

The aim of this section is to show that if $\mathcal{D}_\mathcal{G}$ is enveloping then for each $J \in \mathcal{G}$ the ring R/J is perfect. It will follow from Corollary 5.5.5 that also $\mathfrak{R}$ is pro-perfect.

We begin by showing that for a local ring R the rings R/J are perfect, before extending the result to all commutative rings by showing that all $\mathcal{G}$-torsion modules (specifically the R/J for $J \in \mathcal{G}$) are isomorphic to the direct sum of their localisations.

In Lemma 6.1.1, it was shown that if $\varepsilon\colon R \to D$ is a $\mathcal{D}_\mathcal{G}$-envelope of R in Mod-R, then D must be $\mathcal{G}$-torsion-free. Furthermore, if $\mathcal{G}$ arises from a perfect localisation $u\colon R \to U$ and R has a $\mathcal{D}_\mathcal{G}$ envelope, then the following proposition allows us to work in the setting that $\mathcal{D}_\mathcal{G} = \text{Gen}(U)$, thus $(\mathcal{A}, \mathcal{D}_\mathcal{G})$ is the 1-tilting cotorsion pair associated to the 1-tilting module $U \oplus K$ (see Proposition 1.7.7).

Proposition 6.2.1. *Let $u\colon R \to U$ be a (non-trivial) flat injective ring epimorphism and suppose R has a $\mathcal{D}_\mathcal{G}$-envelope. Then* $\text{p.dim}_R U \leq 1$.

Proof. Let

$$0 \to R \overset{\varepsilon}{\to} D \to D/R \to 0 \tag{6.3}$$

denote the $\mathcal{D}_\mathcal{G}$-envelope of R. First we claim that D is a U-module by showing that D is $\mathcal{G}$-closed, or that $D \cong U \otimes_R D$. Consider the following exact sequence.

$$0 \to \text{Tor}_1^R(D, K) \to D \to D \otimes_R U \to D \otimes_R K \to 0$$

Therefore we must show that $\text{Tor}_1^R(D, K) = 0 = D \otimes_R K$. As D is $\mathcal{G}$-divisible and K is $\mathcal{G}$-torsion it follows that $D \otimes_R K = 0$. By Lemma 6.1.1 D is $\mathcal{G}$-torsion-free, hence $D \cong D \otimes_R U$ and D is a U-module. The cotorsion pair $(\mathcal{A}, \mathcal{D}_\mathcal{G})$ is complete, which implies R-module D/R is in $\mathcal{A}$, so $\text{p.dim}_R D/R \leq 1$. From the short exact sequence (6.3) it follows that also $\text{p.dim}_R D \leq 1$. Consider the following short exact sequence of U-modules

$$0 \to U \to D \otimes_R U \cong D \to D/R \otimes_R U \to 0$$

We now claim that $D/R \otimes_R U$ is U-projective. Indeed, take any $Z \in U$-Mod and note that $Z \in \mathcal{D}_\mathcal{G}$. Then $0 = \text{Ext}_R^1(D/R, Z) \cong \text{Ext}_U^1(D/R \otimes_R U, Z)$. Therefore the short exact sequence above splits in Mod-U and so U is a direct summand of D also as an R-module, and the conclusion follows. □

Combined with Proposition 1.7.7, this provides a generalisation of [2, Theorem 1.1]. More precisely, it shows that conditions (1),(4), and (6) in that theorem hold also in this more general context. The equivalence of (1),(2), and (3) was already shown in [4].

Corollary 6.2.2. *Let $u\colon R \to U$ be a (non-trivial) flat injective ring epimorphism and suppose R has a $\mathcal{D}_\mathcal{G}$-envelope. Then*

$$0 \to R \overset{u}{\to} U \to K \to 0$$

is a $\mathcal{D}_\mathcal{G}$-envelope of R.

Proof. By Proposition 6.2.1 $\mathrm{p.dim}\, U \leq 1$, so by Proposition 1.7.7, $U \oplus K$ is a 1-tilting module such that $(U \oplus K)^{\perp} = \mathcal{D}_{\mathcal{G}}$. Thus $K \in \mathcal{A}$ and so u is a $\mathcal{D}_{\mathcal{G}}$-preenvelope. To see that u is an envelope, note that $\mathrm{Hom}_R(K, U) = 0$, so by Lemma 2.1.2, if $u = fu$, then $f = \mathrm{id}_U$ is an automorphism of U, thus u is a $\mathcal{D}_{\mathcal{G}}$-envelope as required. □

Before continuing, we give an example of a ring R and 1-tilting cotorsion class $\mathcal{T}$ where R has a $\mathcal{T}$-envelope, but $\mathcal{T}$ is not enveloping. This result uses our characterisation of the rings over which a 1-tilting class $\mathcal{T}$ is enveloping in Theorem 6.3.4.

Example 6.2.3. Let R be a valuation domain with valuation v and valuation group $\Gamma(R) = \mathbb{R}$, and an idempotent maximal ideal $\mathfrak{m} =< r_n \in R \mid v(r_n) = 1/n, n \in \mathbb{Z}^{>0} >$ (see [25, Section II.3] for details on valuation rings). Then as $\mathfrak{m}$ is countably generated, it follows that the field of quotients Q of R is also countably generated and therefore of projective dimension at most one. Thus $Q \oplus Q/R$ is a 1-tilting module with 1-tilting cotorsion pair $(\mathcal{A}, \mathcal{T})$, and the Gabriel topology is made up of the principal ideals generated by the non-zero elements of R. Moreover, the following is a $\mathcal{T}$-envelope of R.

$$0 \to R \to Q \to Q/R \to 0$$

However, we claim that $\mathcal{T}$ is not enveloping, using results stated at later in this chapter. If $\mathcal{T}$ is enveloping, then in particular R is an almost perfect ring by Theorem 6.3.4 and as Q is a field, so perfect. By Corollary 3.1.11, almost perfect valuation domains are discrete valuation domains, but by assumption R is not discrete as $\mathfrak{m}$ is countably generated, a contradiction.

We now begin by showing that when R is a commutative local ring, if $\mathcal{D}_{\mathcal{G}}$ is enveloping in Mod-R then for each $J \in \mathcal{G}$, R/J is a perfect ring. We will use the ring isomorphism $\tilde{\nu}_R \colon \Lambda(R) \cong \mathfrak{R}$ of Corollary 5.5.5.

Lemma 6.2.4. *Let R be a commutative local ring and $u \colon R \to U$ a flat injective ring epimorphism and let K denote U/R. Then K is indecomposable.*

Proof. It is enough to show that every idempotent of $\mathfrak{R} := \mathrm{End}_R(K)$ is either the zero homomorphism or the identity on K. Let $\mathfrak{m}$ denote the maximal ideal of R. Take a non-zero idempotent $e \in \mathrm{End}_R(K)$. Then there is an associated element $\tilde{\nu}_R^{-1}(e) = \tilde{r} := (r_J + J)_{J \in \mathcal{G}} \in \Lambda(R)$ via the ring isomorphism $\tilde{\nu}_R \colon \Lambda(R) \cong \mathfrak{R}$ of Corollary 5.5.5. Clearly $\tilde{r}$ is also non-zero and an idempotent in $\Lambda(R)$. We will show this element is the identity in $\Lambda(R)$.

As $\tilde{r}$ is non-zero, there exists a $J_0 \in \mathcal{G}$ such that $r_{J_0} \notin J_0$. Also, $\tilde{r} \cdot \tilde{r} - \tilde{r} = 0$, hence

$$r_{J_0} r_{J_0} - r_{J_0} = r_{J_0}(r_{J_0} - 1_R) \in J_0.$$

We claim that r_{J_0} is a unit in R. Suppose not, then $r_{J_0} \in \mathfrak{m}$, hence $r_{J_0} - 1_R$ is a unit, which implies that $r_{J_0} \in J_0$, a contradiction.

Consider some other $J \in \mathcal{G}$ such that $J \neq R$. $r_{J\cap J_0} - r_{J_0} \in J_0$, hence $r_{J\cap J_0} \notin J_0$. Therefore, by a similar argument as above, $r_{J\cap J_0}$ is a unit in R. As $r_{J\cap J_0} - r_J \in J$ and $r_{J\cap J_0}$ is a unit, $r_J \notin J$. Therefore by a similar argument as above r_J is a unit in R for each $J \in \mathcal{G}$ and we conclude that $\tilde{r}$ is a unit in $\Lambda(R)$.

Finally, as $r_J(r_J - 1_R) \in J$ for every J, and $\tilde{r} := (r_J + J)_{J\in\mathcal{G}}$ is a unit, it follows that $r_J - 1_R \in J$ for each J, implying that $\tilde{r}$ is the identity in $\Lambda(R)$. □

Proposition 6.2.5. *Let R be a commutative local ring and consider the 1-tilting cotorsion pair $(\mathcal{A}, \mathcal{D}_\mathcal{G})$ induced by the flat injective ring epimorphism $u\colon R \to U$. If $\mathcal{D}_\mathcal{G}$ is enveloping in* Mod-R*, then R/J is a perfect ring for each $J \in \mathcal{G}$.*

Proof. Let $\mathfrak{m}$ denote the maximal ideal of R. As R is local, to show that R/J is perfect it is enough to show that for every sequence of elements $\{a_1, a_2, \dots, a_i, \dots\}$ with $a_i \in \mathfrak{m} \setminus J$, there exists an $m > 0$ such that the product $a_1a_2\cdots a_m \in J$ (that is $\mathfrak{m}/J$ is T-nilpotent) by Proposition 1.4.3.

Fix a $J \in \mathcal{G}$ and take $\{a_1, a_2, \dots, a_i, \dots\}$ as above. Consider the following preenvelope of R/a_iR.

$$0 \to R/a_iR \hookrightarrow U/a_iR \to K \to 0$$

As R is local, by Lemma 6.2.4, K is indecomposable, and as R/a_iR is not $\mathcal{G}$-divisible this is an envelope of R/a_iR.

We will use the T-nilpotency of direct sums of envelopes from Theorem 1.2.4. Consider the following countable direct sum of envelopes of R/a_iR which is itself an envelope by Theorem 1.2.4 (i).

$$0 \to \bigoplus_{i>0} R/a_iR \hookrightarrow \bigoplus_{i>0} U/a_iR \to \bigoplus_{i>0} K \to 0$$

For each $i > 0$, we define a homomorphism $f_i\colon U/a_iR \to U/a_{i+1}R$ between the direct summands to be the multiplication by the element a_{i+1}. Then clearly $R/a_iR \subseteq U/a_iR$ vanishes under the action of $f_i = \dot{a}_{i+1}$, hence we can apply Theorem 1.2.4 (ii) to the homomorphisms $\{f_i\}_{i>0}$. So, for every $z + a_1R \in U/a_1R$, there exists an $n > 0$ such that

$$f_n \cdots f_2f_1(z + a_1R) = 0 \in U/a_{n+1}R,$$

which can be rewritten as

$$a_{n+1} \cdots a_3a_2(z) \in a_{n+1}R.$$

By Lemma 4.1.9, there exist $z_1, z_2, \ldots, z_n \in U$ such that

$$\bigcap_{0 \leq j \leq n} \mathrm{Ann}_R(z_j + R) \subseteq J.$$

Let $\Omega = \{z_1, z_2, \ldots, z_n\}$. For each z_j, there exists an n_j such that $a_{n_j+1} \cdots a_3 a_2$ annihilates z_j. That is,

$$a_{n_j+1} \cdots a_3 a_2(z_j) \in a_{n_j+1} R \subseteq R.$$

We now choose an integer m such that $a_m \cdots a_3 a_2$ annihilates all the z_j for $a \leq j \leq n$. Set $m = max\{n_j \mid j = 1, 2 \ldots, n\}$. Then this m satisfies the following, which finishes the proof.

$$a_m a_{m-1} \cdots a_3 a_2 \in \bigcap_{0 \leq j \leq n} \mathrm{Ann}_R(z_j + R) \subseteq J$$

□

Now we extend the result to general commutative rings. Our assumption is that the Gabriel topology $\mathcal{G}$ is arises from a perfect localisation $u \colon R \to U$ and that the associated 1-tilting class $\mathcal{D}_{\mathcal{G}}$ is enveloping in Mod-R.

Notation 6.2.6. There is a preenvelope of the following form induced by the map u.

$$0 \to R/\mathfrak{m} \to U/\mathfrak{m} \to K \to 0$$

Let the following sequence denote an envelope of $R/\mathfrak{m}$.

$$0 \to R/\mathfrak{m} \to D(\mathfrak{m}) \to X(\mathfrak{m}) \to 0$$

By Proposition 1.2.2, $D(\mathfrak{m})$ and $X(\mathfrak{m})$ are direct summands of $U/\mathfrak{m}$ and $K = U/R$ respectively. For convenience we will consider $R/\mathfrak{m}$ as a submodule of $D(\mathfrak{m})$ and $X(\mathfrak{m})$ as a submodule of K.

Remark 6.2.7.

(i) Note that for every maximal ideal $\mathfrak{m}$ of R, $R/\mathfrak{m}$ is $\mathcal{G}$-divisible if and only if, for every $J \in \mathcal{G}$, $J + \mathfrak{m} = R$ if and only if $J \not\subseteq \mathfrak{m}$ if and only if $\mathfrak{m} \notin \mathcal{G}$. Therefore, we will only consider the envelopes of $R/\mathfrak{m}$ where $\mathfrak{m} \in \mathcal{G}$. The modules $D(\mathfrak{m})$ and $X(\mathfrak{m})$ will always refer to the components of the envelope of some $R/\mathfrak{m}$ where $\mathfrak{m} \in \mathcal{G}$. Additionally, as $R/\mathfrak{m}$ is also an $R_\mathfrak{m}$-module, it follows by Example 2.2.10 that $D(\mathfrak{m})$ and $X(\mathfrak{m})$ are also $R_\mathfrak{m}$-modules.

(ii) For every $J \in \mathcal{G}$, $(R/J)_\mathfrak{m} = 0$ if and only if $J \not\subseteq \mathfrak{m}$.

(iii) If M is a $\mathcal{G}$-torsion R-module, then $M_\mathfrak{m} = 0$ for every $\mathfrak{m} \notin \mathcal{G}$ which follows by (ii).

The following lemma allows us to use Proposition 6.2.5 to say that if $D_\mathcal{G}$ is enveloping in R, all localisations $R_\mathfrak{m}/J_\mathfrak{m}$ are perfect rings where $\mathfrak{m}$ is a maximal ideal in $\mathcal{G}$ and $J \in \mathcal{G}$.

Lemma 6.2.8. *Let R be a commutative ring and consider the 1-tilting cotorsion pair $(\mathcal{A}, \mathcal{D}_\mathcal{G})$ induced from the flat injective ring epimorphism $u\colon R \to U$. Fix a maximal ideal $\mathfrak{m}$ of R and let $u_\mathfrak{m}\colon R_\mathfrak{m} \to U_\mathfrak{m}$ be the corresponding flat injective ring epimorphism in* Mod-$R_\mathfrak{m}$. *Then the following hold.*

(i) $K_\mathfrak{m} = 0$ *if and only if* $\mathfrak{m} \notin \mathcal{G}$.

(ii) The induced Gabriel topology of $u_\mathfrak{m}$ denoted

$$\mathcal{G}(\mathfrak{m}) = \{L \leq R_\mathfrak{m} \mid LU_\mathfrak{m} = U_\mathfrak{m}\}$$

contains the localisations $\mathcal{G}_\mathfrak{m} = \{J_\mathfrak{m} \mid J \in \mathcal{G}\}$.

(iii) Suppose $\operatorname{p.dim} U \leq 1$. *Then $(\mathcal{A}_\mathfrak{m}, (\mathcal{D}_\mathcal{G})_\mathfrak{m})$ is the 1-tilting cotorsion pair associated to the flat injective ring epimorphism $u_\mathfrak{m}\colon R_\mathfrak{m} \to U_\mathfrak{m}$. That is, $(\mathcal{D}_\mathcal{G})_\mathfrak{m} = \mathcal{D}_{\mathcal{G}(\mathfrak{m})}$ and $\mathcal{A}_\mathfrak{m} = {}^\perp\mathcal{D}_{\mathcal{G}(\mathfrak{m})}$.*

(iv) If $\mathcal{D}_\mathcal{G}$ is enveloping in Mod-R, *then $\mathcal{D}_{\mathcal{G}(\mathfrak{m})}$ is enveloping in* Mod-$R_\mathfrak{m}$.

Proof. (i) Since K is $\mathcal{G}$-torsion, $\mathfrak{m} \notin \mathcal{G}$ implies $K_\mathfrak{m} = 0$ follows by Remark 6.2.7 (iii). For the converse, suppose $K_\mathfrak{m} = 0$. If $\mathfrak{m} \in \mathcal{G}$ then $R_\mathfrak{m} \cong U_\mathfrak{m} = \mathfrak{m}_\mathfrak{m} U_\mathfrak{m} \cong \mathfrak{m}_\mathfrak{m} R_\mathfrak{m}$, a contradiction.

Note that if $\mathfrak{m} \notin \mathcal{G}$ the rest of the lemma follows trivially.

(ii) Take $J_\mathfrak{m} \in \mathcal{G}_\mathfrak{m}$. Then $R_\mathfrak{m}/J_\mathfrak{m} \otimes_R U_\mathfrak{m} \cong (R/J \otimes_R U) \otimes_R R_\mathfrak{m} = 0$, so $J_\mathfrak{m} \in \mathcal{G}(\mathfrak{m})$.

(iii) That $(\mathcal{A}_\mathfrak{m}, (\mathcal{D}_\mathcal{G})_\mathfrak{m})$ is the 1-tilting cotorsion pair associated to the 1-tilting module $(U \oplus K)_\mathfrak{m}$ is Proposition 1.5.1, therefore $\operatorname{Gen}(U_\mathfrak{m}) = (\mathcal{D}_\mathcal{G})_\mathfrak{m}$ in Mod-$R_\mathfrak{m}$. As $u_\mathfrak{m}\colon R_\mathfrak{m} \to U_\mathfrak{m}$ is a flat injective ring epimorphism and $\operatorname{p.dim}_{R_\mathfrak{m}} U_\mathfrak{m} \leq 1$ the 1-tilting classes $\operatorname{Gen}(U_\mathfrak{m})$ and $\mathcal{D}_{\mathcal{G}(\mathfrak{m})}$ coincide in Mod-$R_\mathfrak{m}$ by Proposition 1.7.7. Thus $(\mathcal{D}_\mathcal{G})_\mathfrak{m} = \mathcal{D}_{\mathcal{G}(\mathfrak{m})}$ and it follows that $\mathcal{A}_\mathfrak{m} = {}^\perp\mathcal{D}_{\mathcal{G}(\mathfrak{m})}$.

(iv) Assume that $\mathcal{D}_\mathcal{G}$ is enveloping in Mod-R and take some $M \in$ Mod-$R_\mathfrak{m}$ with the following $\mathcal{D}_\mathcal{G}$-envelope.

$$0 \to M \to D \to X \to 0$$

We claim that M has a $\mathcal{D}_{\mathcal{G}(\mathfrak{m})}$-envelope in Mod-$R_\mathfrak{m}$. Since $M \in$ Mod-$R_\mathfrak{m}$, D and X are $R_\mathfrak{m}$-modules by Proposition 2.1.8. By Proposition 6.2.1 $\operatorname{p.dim} U \leq 1$. By (iii), $(\mathcal{D}_\mathcal{G})_\mathfrak{m} = \mathcal{D}_{\mathcal{G}(\mathfrak{m})}$ so $D \in \mathcal{D}_{\mathcal{G}(\mathfrak{m})}$

and $X \in {}^{\perp}\mathcal{D}_{\mathcal{G}(\mathfrak{m})}$. Since $R \to R_{\mathfrak{m}}$ is a ring epimorphism, any direct summand of D which contains M in Mod-$R_{\mathfrak{m}}$ would also be a direct summand in Mod-R. Thus we conclude that $0 \to M \to D \to X \to 0$ is a $\mathcal{D}_{\mathcal{G}(\mathfrak{m})}$-envelope of M in Mod-$R_{\mathfrak{m}}$.

□

By the above lemma, if $\mathcal{D}_{\mathcal{G}}$ is enveloping in Mod-R, then $\mathcal{D}_{\mathcal{G}(\mathfrak{m})}$ is enveloping in Mod-$R_{\mathfrak{m}}$. Next we show that, under our enveloping assumption, all $\mathcal{G}$-torsion modules are isomorphic to the direct sums of their localisations at maximal ideals.

The proof of the following lemma uses an almost identical argument to the proof of Lemma 6.1.3.

Lemma 6.2.9. *Let $u\colon R \to U$ be a flat injective ring epimorphism, $\mathcal{G}$ the associated Gabriel topology and suppose that $\mathcal{D}_{\mathcal{G}}$ is enveloping. Let $D(\mathfrak{m})$ and $X(\mathfrak{m})$ be as in Notation 6.2.6 and fix a maximal ideal $\mathfrak{m} \in \mathcal{G}$. For every element $d \in D(\mathfrak{m})$ and every element $a \in \mathfrak{m}$, there is a natural number $n > 0$ such that $a^n d = 0$. Moreover, for every element $x \in X(\mathfrak{m})$ and every element $a \in \mathfrak{m}$, there is a natural number $n > 0$ such that $a^n x = 0$.*

Proof. We will use the T-nilpotency of direct sums of envelopes as in Theorem 1.2.4 (ii). Consider the following countable direct sum of envelopes of $R/\mathfrak{m}$ which is itself an envelope by Theorem 1.2.4 (i).

$$0 \to \bigoplus_{0<i}(R/\mathfrak{m})_{(i)} \to \bigoplus_{0<i} D(\mathfrak{m})_{(i)} \to \bigoplus_{0<i} X(\mathfrak{m})_{(i)} \to 0$$

For a fixed element $a \in \mathfrak{m}$, we choose the homomorphisms $f_i\colon D(\mathfrak{m})_{(i)} \to D(\mathfrak{m})_{(i+1)}$ between the direct summands to all be multiplication by a. Then clearly $R/\mathfrak{m} \subseteq D(\mathfrak{m})$ vanishes under the action of $f_i = \dot{a}$, hence we can apply Xu's Theorem: for every $d \in D(\mathfrak{m})$, there exists an n such that

$$f_n \cdots f_2 f_1(d) = 0 \in D(\mathfrak{m})_{(n+1)}.$$

Since each f_i acts as multiplication by a, for every $d \in D$ there is an integer n for which $a^n d = 0$, as required.

It is straightforward to see that $X(\mathfrak{m})$ has the same property as $X(\mathfrak{m})$ is an epimorphic image of $D(\mathfrak{m})$. □

Lemma 6.2.10. *Let $u\colon R \to U$ be a flat injective ring epimorphism and suppose $\mathcal{D}_{\mathcal{G}}$ is enveloping. Let $\mathfrak{m} \in \mathcal{G}$ and let $X(\mathfrak{m})$ be as in Notation 6.2.6. The support of $X(\mathfrak{m})$ is exactly $\{\mathfrak{m}\}$, and each $X(\mathfrak{m}) \cong X(\mathfrak{m})_{\mathfrak{m}}$ is $K_{\mathfrak{m}}$.*

Proof. We claim that $X(\mathfrak{m})$ is non-zero. Otherwise, $X(\mathfrak{m}) = 0$ would imply that $R/\mathfrak{m}$ is $\mathcal{G}$-divisible, so $R/\mathfrak{m} = \mathfrak{m}(R/\mathfrak{m}) = 0$, a contradiction.

Consider a maximal ideal $\mathfrak{n} \neq \mathfrak{m}$. Take an element $a \in \mathfrak{m} \setminus \mathfrak{n}$. Then for any $x \in X(\mathfrak{m})$, $a^n x = 0$ for some $n > 0$, by Lemma 6.2.9 and since a is an invertible element in $R_{\mathfrak{n}}$, x is zero in the localisation with respect to $\mathfrak{n}$. This holds for any element $x \in X(\mathfrak{m})$, hence $X(\mathfrak{m})_{\mathfrak{n}} = 0$.

It follows that since $X(\mathfrak{m})$ is non-zero, $X(\mathfrak{m})_{\mathfrak{m}} \neq 0$. As mentioned in Remark 6.2.7, $X(\mathfrak{m})$ is an $R_{\mathfrak{m}}$-module and since $X(\mathfrak{m})$ is a direct summand of K, $X(\mathfrak{m})$ is a direct summand of $K_{\mathfrak{m}}$ which is indecomposable, by Lemma 6.2.4. Therefore $X(\mathfrak{m})$ is non-zero and is isomorphic to $K_{\mathfrak{m}}$. □

Lemma 6.2.11. *Let $u\colon R \to U$ be a flat injective ring epimorphism and suppose $\mathcal{D}_{\mathcal{G}}$ is enveloping. Then the sum of the submodules $X(\mathfrak{m})$ in K is a direct sum.*

$$\sum_{\mathfrak{m}\in\mathcal{G}} X(\mathfrak{m}) = \bigoplus_{\mathfrak{m}\in\mathcal{G}} X(\mathfrak{m})$$

Proof. Recall that $X(\mathfrak{m})$ is non-zero only for $\mathfrak{m} \in \mathcal{G}$ by Remark 6.2.7. Consider an element

$$x \in X(\mathfrak{m}) \cap \sum_{\substack{\mathfrak{n}\neq\mathfrak{m}\\ \mathfrak{n}\in\mathcal{G}}} X(\mathfrak{n}).$$

We will show that this element must be zero. By Lemma 6.2.10, since $x \in X(\mathfrak{m})$, x is zero in the localisation with respect to all maximal ideals $\mathfrak{n} \neq \mathfrak{m}$. But x can also be written as a finite sum of elements $x_i \in X(\mathfrak{n}_i)$, each of which is zero in the localisation with respect to $\mathfrak{m}$, by Lemma 6.2.10. Therefore, $(x)_{\mathfrak{n}} = 0$ for all maximal ideals $\mathfrak{n}$, hence $x = 0$. □

Proposition 6.2.12. *Let $u\colon R \to U$ be a flat injective ring epimorphism and suppose $\mathcal{D}_{\mathcal{G}}$ is enveloping. The module K can be written as a direct sum of its localisations $K_{\mathfrak{m}}$, as follows.*

$$K \cong \bigoplus_{\mathfrak{m}\in\mathcal{G}} K_{\mathfrak{m}} = \bigoplus_{\mathfrak{m}\in\operatorname{mSpec} R} K_{\mathfrak{m}}$$

Proof. From Lemma 6.2.11, we have the following inclusion.

$$\bigoplus_{\mathfrak{m}\in\mathcal{G}} X(\mathfrak{m}) \leq K$$

To see that this is an equality we show that these two modules have the same localisation with respect to every $\mathfrak{m}$ maximal in R. Recall that by Lemma 6.2.8(i) if $\mathfrak{n}$ is maximal, then $K_{\mathfrak{n}} = 0$ if and only if $\mathfrak{n} \notin \mathcal{G}$ and by Lemma 6.2.10, $\operatorname{Supp}(X(\mathfrak{m})) = \{\mathfrak{m}\}$. Using these lemmas, it follows that for $\mathfrak{n} \notin \mathcal{G}$, $K_{\mathfrak{n}} = 0 = (\bigoplus_{\mathfrak{m}\in\mathcal{G}} X(\mathfrak{m}))_{\mathfrak{n}}$. Similarly, if $\mathfrak{m} \in \mathcal{G}$, then $K_{\mathfrak{m}} = X(\mathfrak{m})_{\mathfrak{m}}$. Hence we have shown the following.

$$\bigoplus_{\mathfrak{m}\in\mathcal{G}} X(\mathfrak{m}) = K$$

Since $K_{\mathfrak{m}} = X(\mathfrak{m})_{\mathfrak{m}}$, it only remains to see that $X(\mathfrak{m}) \cong X(\mathfrak{m})_{\mathfrak{m}}$, which follows from Remark 6.2.7. □

Corollary 6.2.13. *Let $u\colon R \to U$ be a flat injective ring epimorphism and suppose $\mathcal{D}_{\mathcal{G}}$ is enveloping. Then for every $\mathcal{G}$-torsion module M, the following isomorphism holds.*

$$M \cong \bigoplus_{\mathfrak{m}\in\mathcal{G}} M_{\mathfrak{m}} = \bigoplus_{\mathfrak{m}\in \operatorname{mSpec} R} M_{\mathfrak{m}}$$

Furthermore, it follows that for every $J \in \mathcal{G}$, J is contained only in finitely many maximal ideals of R.

Proof. For the first isomorphism, recall that if an R-module M is $\mathcal{G}$-torsion, then $M \cong \operatorname{Tor}_1^R(M, K)$. Also, note that in this case, $M_{\mathfrak{m}} \cong \operatorname{Tor}_1^R(M, K)_{\mathfrak{m}} \cong \operatorname{Tor}_1^{R_{\mathfrak{m}}}(M_{\mathfrak{m}}, K_{\mathfrak{m}}) \cong \operatorname{Tor}_1^R(M, K_{\mathfrak{m}})$. Hence we have the following isomorphisms.

$$M \cong \operatorname{Tor}_1^R(M, K) \cong \operatorname{Tor}_1^R(M, \bigoplus_{\mathfrak{m}\in\mathcal{G}} K_{\mathfrak{m}}) \cong \bigoplus_{\mathfrak{m}\in\mathcal{G}} \operatorname{Tor}_1^R(M, K_{\mathfrak{m}}) \cong \bigoplus_{\mathfrak{m}\in\mathcal{G}} M_{\mathfrak{m}}$$

The fact that

$$\bigoplus_{\mathfrak{m}\in\mathcal{G}} M_{\mathfrak{m}} = \bigoplus_{\mathfrak{m}\in \operatorname{mSpec} R} M_{\mathfrak{m}}$$

follows from Remark 6.2.7 (iii).

For the final statement of the proposition, one only has to replace M with the $\mathcal{G}$-torsion module R/J where $J \in \mathcal{G}$. Hence as R/J is cyclic, it cannot be isomorphic to an infinite direct sum. Therefore, $(R/J)_{\mathfrak{m}}$ is non-zero only for finitely many maximal ideals and the conclusion follows. □

We are now in the position to show the main results of this section.

Theorem 6.2.14. *Let $u\colon R \to U$ be a flat injective ring epimorphism and suppose $\mathcal{D}_{\mathcal{G}}$ is enveloping. Then R/J is a perfect ring for every $J \in \mathcal{G}$.*

Proof. By Corollary 6.2.13, every R/J is a finite product of local rings $R_{\mathfrak{m}}/J_{\mathfrak{m}}$. Additionally as $(\mathcal{D}_{\mathcal{G}})_{\mathfrak{m}}$ is enveloping in Mod-$R_{\mathfrak{m}}$ by Lemma 6.2.8 each $R_{\mathfrak{m}}/J_{\mathfrak{m}}$ is a perfect ring by Proposition 6.2.5. Therefore, by Proposition 1.4.3, R/J itself is perfect. □

Theorem 6.2.15. *Let $u\colon R \to U$ be a flat injective ring epimorphism and suppose $\mathcal{D}_{\mathcal{G}}$ is enveloping in* Mod-R*. Then the topological ring $\mathfrak{R} = \operatorname{End}(K)$ is pro-perfect.*

Proof. Recall that the topology of $\mathfrak{R}$ is given by the annihilators of finitely generated submodules of K, so that $\mathfrak{R} = \operatorname{End}_R(K)$ is separated and complete in its topology. Let V be an open ideal in the topology of $\mathfrak{R}$. By Corollary 5.5.6 there is $J \in \mathcal{G}$ and a surjective ring homomorphism $R/J \to \mathfrak{R}/V$. By Theorem 6.2.14 R/J is a perfect ring and thus so are the quotient rings $\mathfrak{R}/V$. □

6.3 $\mathcal{D}_\mathcal{G}$ is enveloping if and only if $\mathfrak{R}$ is pro-perfect

Suppose that $u\colon R \to U$ is a commutative flat injective ring epimorphism where $\mathrm{p.dim}_R U \leq 1$ and denote $K = U/R$. In this section we show that if the endomorphism ring $\mathfrak{R} = \mathrm{End}_R(K)$ is pro-perfect, then $\mathcal{D}_\mathcal{G}$ is enveloping in Mod-R. So combining with the results in the Section 6.2 we obtain that $\mathcal{D}_\mathcal{G}$ is enveloping if and only if $\mathrm{p.dim}\, U \leq 1$ and $\mathfrak{R}$ is pro-perfect.

Recall that if $\mathrm{p.dim}\, U \leq 1$, $(\mathcal{A}, \mathcal{D}_\mathcal{G})$ denotes the 1-tilting cotorsion pair associated to the 1-tilting module $U \oplus K$. The following theorem of Positselski is vital for this section.

Theorem 6.3.1. *([14, Theorem 13.3]) Suppose R is a commutative ring and $u\colon R \to U$ a flat injective ring epimorphism with* $\mathrm{p.dim}_R U \leq 1$*. Then the topological ring* $\mathfrak{R} = \mathrm{End}(K)$ *is pro-perfect if and only if* $\varinjlim \mathrm{Add}(K) = \mathrm{Add}(K)$*.*

A second crucial result that we will use is Theorem 1.2.5. It states that $\mathcal{C}$ is a class of modules closed under direct limits and extensions, then if a M admits a special $\mathcal{C}^{\perp_1}$-preenvelope with cokernel in $\mathcal{C}$, then M admits a $\mathcal{C}^{\perp_1}$-envelope.

We now show that if $\mathfrak{R}$ is pro-perfect, then $\mathrm{Add}(K)$ does in fact satisfy the conditions of Theorem 1.2.5. From Theorem 6.3.1 $\mathrm{Add}(K)$ is closed under direct limits. Moreover, $\mathrm{Add}(K)$ is closed under extensions as any short exact sequence $0 \to L \to M \to N \to 0$ with $L, N \in \mathrm{Add}(K)$ splits.

As the cotorsion pair $(\mathcal{A}, \mathcal{D}_\mathcal{G})$ is complete, every R-module M has an injective $\mathcal{D}_\mathcal{G}$-preenvelope, and as $\mathcal{D}_\mathcal{G} = K^\perp = (\mathrm{Add}(K))^\perp$, M has a $(\mathrm{Add}(K))^\perp$-preenvelope. It remains to be seen that every M has a special preenvelope ν such that $\mathrm{Coker}\, \nu \in \mathrm{Add}(K)$, which we will now show.

Lemma 6.3.2. *Suppose $u\colon R \to U$ is a flat injective ring epimorphism where* $\mathrm{p.dim}_R U \leq 1$*. Let $(\mathcal{A}, \mathcal{D}_\mathcal{G})$ be the 1-tilting cotorsion pair associated to the 1-tilting module $U \oplus K$. Then every module has a special $\mathcal{D}_\mathcal{G}$-preenvelope ν such that* $\mathrm{Coker}\, \nu \in \mathrm{Add}(K)$*.*

Proof. For every cardinal α the short exact sequence $0 \to R^{(\alpha)} \to U^{(\alpha)} \to K^{(\alpha)} \to 0$ is a $\mathcal{D}_\mathcal{G}$-preenvelope and is of the desired form. Take an R-module M and consider the canonical surjection $R^{(\alpha)} \xrightarrow{p} M \to 0$. Consider the following pushout Z of $M \leftarrow R^{(\alpha)} \rightarrow U^{(\alpha)}$.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \ker p & = & \ker p & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & R^{(\alpha)} & \longrightarrow & U^{(\alpha)} & \longrightarrow & K^{(\alpha)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p} & & \downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & Z & \longrightarrow & K^{(\alpha)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The module Z is in $\mathrm{Gen}(U) = \mathcal{D}_{\mathcal{G}}$, and so the bottom row of the above diagram is a $\mathcal{D}_{\mathcal{G}}$-preenvelope of M of the desired form.

□

The following theorem follows easily from the above discussion.

Theorem 6.3.3. *Suppose $u\colon R \to U$ is a flat injective ring epimorphism with $\mathrm{p.dim}_R U \leq 1$. If the topological ring $\mathfrak{R}$ is pro-perfect, then $\mathcal{D}_{\mathcal{G}}$ is enveloping in* Mod-R.

Proof. From Theorem 6.3.1 and Lemma 6.3.2, $\mathrm{Add}(K)$ does satisfy the conditions of Theorem 1.2.5. Thus the conclusion follows, since $\mathcal{D}_{\mathcal{G}} = \mathrm{Add}(K)^{\perp}$. □

Finally combining the above theorem with the results in Section 6.1 and Section 6.2 we obtain the two main results of this chapter.

Theorem 6.3.4. *Suppose $u\colon R \to U$ is a commutative flat injective ring epimorphism, $\mathcal{G}$ the associated Gabriel topology and $\mathfrak{R}$ the topological ring* $\mathrm{End}_R(K)$. *The following are equivalent.*

(i) $\mathcal{D}_{\mathcal{G}}$ *is enveloping.*

(ii) $\mathrm{p.dim}\, U \leq 1$ *and R/J is a perfect ring for every $J \in \mathcal{G}$.*

(iii) $\mathrm{p.dim}\, U \leq 1$ *and $\mathfrak{R}$ is pro-perfect.*

In particular, if $\mathcal{D}_{\mathcal{G}}$ is enveloping then the class $\mathrm{Add}(K)$ *is closed under direct limits.*

Proof. (i)⇒(ii) Follows by Proposition 6.2.1 and Theorem 6.2.14.

(ii)⇒(iii) Follows from Corollary 5.5.5.

(iii)⇒(i) Follows from Theorem 6.3.3. □

Theorem 6.3.5. *Assume that T is a 1-tilting module over a commutative ring R such that the class $T^{\perp}$ is enveloping, and let $\mathcal{G}$ be the associated Gabriel topology of $\mathcal{T}$. Then we have the following equivalence.*

$$\mathcal{T} \text{ is enveloping} \Leftrightarrow \begin{cases} \mathrm{p.dim}\, R_{\mathcal{G}} \leq 1 \\ R/J \text{ is a perfect ring for each } J \in \mathcal{G} \\ \mathcal{G} \text{ is a perfect Gabriel topology} \end{cases}$$

That is, there is a flat injective ring epimorphism $u\colon R \to U$ such that $U \oplus U/R$ is equivalent to T.

Proof. ($\Rightarrow$) By Proposition 6.1.6, the Gabriel topology $\mathcal{G}$ associated to $T^{\perp}$ arises from a perfect localisation. Moreover, $\psi\colon R \to R_{\mathcal{G}}$ is injective so by setting $U = R_{\mathcal{G}}$ we can apply Theorem 6.3.4 to conclude.

($\Leftarrow$)This follows by Theorem 6.3.4. □

6.4 The case of a non-injective flat ring epimorphism

Now we extend the results of the previous section to the case of a non-injective flat ring epimorphism $u\colon R \to U$ with $K = \operatorname{Coker} u$.

As before, the Gabriel topology $\mathcal{G}_u = \{J \leq R \mid JU = U\}$ associated to u is finitely generated and the class

$$\mathcal{D}_{\mathcal{G}_u} = \{M \in \text{Mod-}R \mid JM = M \text{ for every } J \in \mathcal{G}_u\}$$

of $\mathcal{G}_u$-divisible modules is a torsion class. Moreover, by [6] it is a silting class, that is there is a silting module T such that $\mathrm{Gen}(T) = \mathcal{D}_{\mathcal{G}_u}$.

The ideal I will denote the kernel of u and $\overline{R}$ the ring R/I so that there is a flat injective ring epimorphism $\overline{u}\colon \overline{R} \to U$.

To $\overline{u}$, one can associate the Gabriel topology $\mathcal{G}_{\overline{u}} = \{L/I \leq \overline{R} \mid LU = U, I \subseteq L\}$ on $\overline{R}$ and the following class of $\overline{R}$-modules.

$$\mathcal{D}_{\mathcal{G}_{\overline{u}}} = \{M \in \text{Mod-}\overline{R} \mid (L/I)M = M, \text{ for every } L/I \in \mathcal{G}_{\overline{u}}\}$$

That is, we have that if $J \in \mathcal{G}_u$, then $J + I/I \in \mathcal{G}_{\overline{u}}$, and conversely if $L/I \in \mathcal{G}_{\overline{u}}$, $L \in \mathcal{G}_u$.

We first note the following.

Lemma 6.4.1. *Every module in $\mathcal{D}_{\mathcal{G}_u}$ is annihilated by I, thus $\mathcal{D}_{\mathcal{G}_u} = \mathcal{D}_{\mathcal{G}_{\overline{u}}}$.*

Proof. Note that $\operatorname{Ker} u = I$ is the $\mathcal{G}_u$-torsion submodule of R. Hence for every $b \in I$ there is $J \in \mathcal{G}_u$ such that $bJ = 0$. Let $M \in \mathcal{D}_{\mathcal{G}_u}$, then $bM = bJM = 0$, thus $IM = 0$. We conclude that $\mathcal{D}_{\mathcal{G}_u}$ can be considered a class in Mod-$\overline{R}$ and coincides with $\mathcal{D}_{\mathcal{G}_{\overline{u}}}$. □

Proposition 6.4.2. *The class* $\mathcal{D}_{\mathcal{G}_u}$ *is enveloping in* Mod-R *if and only if* $\mathcal{D}_{\mathcal{G}_{\overline{u}}}$ *is enveloping in* Mod-$\overline{R}$.

Proof. Assume that $\mathcal{D}_{\mathcal{G}_u}$ is enveloping in Mod-R and let $\overline{M} \in$ Mod-$\overline{R}$. Consider a $\mathcal{D}_{\mathcal{G}_u}$-envelope $\overline{\psi}\colon \overline{M} \to D$ in Mod-R. Since $R \to R/I$ is a ring epimorphism and D is annihilated by I by Lemma 6.4.1, it is immediate to conclude that $\overline{\psi}$ is also a $\mathcal{D}_{\mathcal{G}_{\overline{u}}}$-envelope of $\overline{M}$.

Conversely, assume that $\mathcal{D}_{\mathcal{G}_{\overline{u}}}$ is enveloping in Mod-$\overline{R}$. Take $M \in$ Mod-R and let $\overline{\psi}\colon M/IM \to D$ be a $\mathcal{D}_{\mathcal{G}_{\overline{u}}}$-envelope of M/IM in Mod-$\overline{R}$. Let $\pi\colon M \to M/IM$ be the canonical projection. We claim that $\psi = \overline{\psi}\pi$ is a $\mathcal{D}_{\mathcal{G}_u}$-envelope of M in Mod-R. Indeed, if $f\colon D \to D$ satisfies $f\psi = \psi$, then $f\overline{\psi}\pi = \overline{\psi}\pi$. As π is a surjection, $f\overline{\psi} = \overline{\psi}$ and so f is an automorphism of D. □

Note that $\mathrm{End}_R(K)$ coincides with $\mathrm{End}_{\overline{R}}(K)$ both as a ring and as a topological ring. It will be still denoted by $\mathfrak{R}$. Thus if $\mathcal{D}_{\mathcal{G}_u}$ is enveloping in Mod-R we can apply the results of the previous sections to the ring $\overline{R}$, in particular Theorem 6.3.4.

Theorem 6.4.3. *Let* $u\colon R \to U$ *be a commutative flat ring epimorphism with kernel* I. *Let* $\mathcal{G}_u$ *be the associated Gabriel topology and* $\mathfrak{R}$ *the topological ring* $\mathrm{End}_R(K)$. *The following are equivalent.*

(i) $\mathcal{D}_{\mathcal{G}_u}$ *is enveloping.*

(ii) $\mathrm{p.dim}_{\overline{R}} U \leq 1$ *and* R/L *is a perfect ring for every* $L \in \mathcal{G}$ *such that* $L \supseteq I$.

(iii) $\mathrm{p.dim}_{\overline{R}} U \leq 1$ *and* $\mathfrak{R}$ *is pro-perfect.*

In particular, $U \oplus K$ *is a* 1*-tilting module over the ring* $\overline{R}$ *and since* $\mathrm{Gen}(U)$ *is contained in* Mod-$\overline{R}$, $\mathcal{D}_{\mathcal{G}_u} = \mathrm{Gen}(U)$.

As already noted, results from [6] imply that $\mathrm{Gen}(U)$ is a silting class in Mod-R. Since we have that $U \oplus K$ is a 1-tilting module in Mod-$\overline{R}$ inducing the silting class $\mathrm{Gen}(U)$, it is natural to ask the following question.

Question 6.4.4. *Is* $U \oplus K$ *a silting module in* Mod-R*?*

Chapter 7

Covering classes and 1-tilting cotorsion pairs over commutative rings

Suppose $(\mathcal{A}, \mathcal{D}_\mathcal{G})$ is a 1-tilting cotorsion pair over a commutative ring with associated Gabriel topology $\mathcal{G}$. The aim of this chapter is to characterise the commutative rings R over which the class $\mathcal{A}$ of the cotorsion pair $(\mathcal{A}, \mathcal{D}_\mathcal{G})$ is covering. We find that $\mathcal{A}$ is covering if and only if both $\operatorname{p.dim} R_\mathcal{G} \leq 1$ and R is $\mathcal{G}$-almost perfect, that is $R_\mathcal{G}$ and R/J for each $J \in \mathcal{G}$ are perfect rings. Moreover in these cases the Gabriel topology $\mathcal{G}$ is perfect, that is $R \to R_\mathcal{G}$ is a perfect localisation and $\mathcal{G} = \{J \leq R \mid JR_\mathcal{G} = R_\mathcal{G}\}$, and $R_\mathcal{G} \oplus R_\mathcal{G}/R$ is a 1-tilting module associated to $(\mathcal{A}, \mathcal{D}_\mathcal{G})$.

We note that one can conclude that if $\mathcal{A}$ is covering, the R/J are perfect rings for every $J \in \mathcal{G}$ by applying Theorem 2.3.6, that is if $\mathcal{A}$ is covering then $\mathcal{D}_\mathcal{G}$ is enveloping. However, we found it interesting to show it directly, as one uses different techniques to study covers and envelopes. Additionally, in Section 7.3 we show explicitly that if the R/J are perfect rings, K is Σ-pure-split. Thus by Example 1.1.1, $\operatorname{Add}(K)$ is closed under direct limits, and so this gives an explicit proof of the implication (ii)⇒(i) in Theorem 6.3.4.

To find the closure under direct limits of $\mathcal{A}$ in $(\mathcal{A}, \mathcal{D}_\mathcal{G})$, we could apply Theorem 1.3.6, but we chose to use purely algebraic methods to get moreover a ring theoretic characterisation of R. Another possibility is to apply [14, Theorem 13.5] once we prove that the Gabriel topology $\mathcal{G}$ associated to the 1-tilting pair arises from a perfect localisation and additionally $\operatorname{p.dim} R_\mathcal{G} \leq 1$. This will be shown in Lemma 7.1.3 and Proposition 7.1.4.

In Section 7.1 we first begin by showing that if $\mathcal{A}$ is covering, then the Gabriel topology $\mathcal{G}$ is perfect. We use some properties of covers of R/J-modules where $J \in \mathcal{G}$, using in particular the fact that $(\mathcal{A}, \mathcal{D}_\mathcal{G})$ is of finite type, so $\mathcal{D}_\mathcal{G}$ is closed under direct sums. Moreover in this section we show in Proposition 7.1.4 that if $\mathcal{A}$ is covering then $\operatorname{p.dim} R_{\mathcal{G}} \leq 1$, so $R_{\mathcal{G}} \oplus R_{\mathcal{G}}/R$ is a 1-tilting module associated to $(\mathcal{A}, \mathcal{D}_{\mathcal{G}})$ and $\operatorname{Gen}(R_{\mathcal{G}}) = \mathcal{D}_{\mathcal{G}}$.

Next in Section 7.2, we continue with the assumption that $\mathcal{A}$ is covering and show that R must be $\mathcal{G}$-almost perfect. First we show that every $R_{\mathcal{G}}$-module has a projective cover, thus is perfect. For R/J, we show that every Bass R/J-module N has a projective cover. To do so we must use the free presentation of N, and find a short exact sequence that corresponds to this free presentation from which we can extract an $\mathcal{A}$-cover. Next we must pass to the category of u-contramodules where we show that the sequence becomes a projective cover in the category of u-contramodules. Finally, one can show that the original free presentation must split, thus N is projective in Mod-R/J.

Finally, Section 7.3 is divided into three subsections. The aim is to show that if we begin with a 1-tilting cotorsion pair $(\mathcal{A}, \mathcal{D}_{\mathcal{G}})$ over a $\mathcal{G}$-almost perfect commutative ring R, and $\operatorname{p.dim} R_{\mathcal{G}} \leq 1$, then $\mathcal{A}$ must be covering. We begin Section 7.3 by showing that $\mathcal{G}$ must arise from a perfect localisation, thus we can work in the case that $U \oplus K$ is the associated 1-tilting module of $(\mathcal{A}, \mathcal{D}_{\mathcal{G}})$. The main result is Lemma 7.3.2 and only requires that $\operatorname{f.dim} R_{\mathcal{G}} = 0$.

In Subsection 7.3.1 we prove some initial results about the pure short exact sequence (7.11), and that by Lemma 7.3.3 and Lemma 7.3.5, it only remains to show that K is Σ-pure-split. Next in Subsection 7.3.2, we prove that K is Σ-pure-split when R is local by using the equivalence of $\mathcal{G}$-torsion $\mathcal{G}$-divisible modules and $\mathcal{G}$-torsion-free u-contramodules via the functors $\big((- \otimes_R K), \operatorname{Hom}_R(K, -)\big)$ as in Section 5.4. The results of Subsection 7.3.2 are from [35, Sections 3] which in our context can be proved in a much more straightforward way. Finally, we extend these results on local rings to the global case. We use that if the R/J are perfect, the ring R is $\mathcal{G}$-h-local as shown in Proposition 4.2.6. Additionally we summarise the main result of this chapter in Subsection 7.3.3, and in particular Theorem 7.3.16.

As mentioned previously in this thesis, if the assumption states that the Gabriel topology $\mathcal{G}$ arises from a perfect localisation, we denote $R_{\mathcal{G}}$ by U and $R_{\mathcal{G}}/R$ by K, and the natural ring homomorphism by $u\colon R \to U$.

7.1 $\mathcal{G}$ rises from a perfect localisation and $R_{\mathcal{G}} \oplus R_{\mathcal{G}}/R$ is 1-tilting

The next two sections are devoted to the study of the following situation.

Setting 7.1.1. Let R be a commutative ring and let $(\mathcal{A}, \mathcal{D}_{\mathcal{G}})$ be a 1-tilting cotorsion pair with associated Gabriel topology $\mathcal{G}$. We suppose moreover that $\mathcal{A}$ is covering.

We begin by describing covers of modules annihilated by some $J \in \mathcal{G}$.

Lemma 7.1.2. *Suppose R is commutative and let $(\mathcal{A}, \mathcal{D}_\mathcal{G})$ be a 1-tilting cotorsion pair with associated Gabriel topology $\mathcal{G}$. Consider an R-module M such that $MJ = 0$ for some $J \in \mathcal{G}$ and let the following be an $\mathcal{A}$-cover of M.*

$$0 \to B \to A \xrightarrow{\phi} M \to 0$$

Then both A and B are $\mathcal{G}$-torsion.

Proof. We will use the T-nilpotency of direct sums of covers as in Theorem 1.2.11 (ii). Suppose M has the property that $MJ = 0$ for some $J \in \mathcal{G}$, and let the sequence above be an $\mathcal{A}$-cover of M. Consider the following countable direct sum of covers of M which is itself a cover, by Theorem 1.2.11 (ii).

$$0 \to \bigoplus_n B_n \to \bigoplus_n A_n \xrightarrow{\bigoplus \phi_n} \bigoplus_n M_n \to 0.$$

Choose an element $x \in J$ and for each n set $f_n \colon A_n \to A_{n+1}$ to be the multiplication by x.

Then clearly $\phi(f_n(A_n)) = 0$ for every $n > 0$, hence we can apply Theorem 1.2.11 (ii). For every $a \in A$, there exists an m such that

$$f_m \circ \cdots \circ f_2 \circ f_1(a) = 0 \in A_{m+1}.$$

Hence for every $a \in A$ there is an integer m for which $x^m a = 0$.

Fix $a \in A$ and let m_i be the minimal natural number for which $(x_i)^{m_i} a = 0$ and set $m := \sup\{m_i \mid 1 \leq i \leq t\}$. Then for a large enough integer k we have that $J^k a = 0$ (for example set $k = tm$), and $J^k \in \mathcal{G}$. Thus every element of A is annihilated by an ideal contained in $\mathcal{G}$, therefore A is $\mathcal{G}$-torsion. Since the associated torsion pair of the Gabriel topology is hereditary, also $B \subseteq A$ is $\mathcal{G}$-torsion. □

Next we show that $\mathcal{G}$ must arise from a perfect localisation using an exercise from Stenström, Lemma 4.1.5.

Lemma 7.1.3. *Suppose R is commutative and let $(\mathcal{A}, \mathcal{D}_\mathcal{G})$ be a 1-tilting cotorsion pair with associated Gabriel topology $\mathcal{G}$. Suppose $\mathcal{A}$ is covering. Then $\mathcal{G}$ is a perfect Gabriel topology.*

Proof. From Proposition 1.7.4, $R_\mathcal{G}$ arises from a perfect localisation if and only if both the functor q is exact and $\mathcal{G}$ has a basis of finitely generated ideals. The associated Gabriel topology, $\mathcal{G}$ of a 1-tilting class has a basis of finitely generated ideals by Hrbek's characterisation in Theorem 1.7.6, so it remains only to show that q is exact.
We will show that $\mathrm{Ext}^2_R(R/J, M) = 0$ for every $\mathcal{G}$-closed R-module M and every $J \in \mathcal{G}$, and then apply Lemma 4.1.5 to conclude that q is exact.

Let M be any $\mathcal{G}$-closed R-module and $J \in \mathcal{G}$, and consider the following $\mathcal{A}$-cover of R/J.

$$0 \to B_J \to A_J \to R/J \to 0$$

By Lemma 7.1.2, A_J and B_J are $\mathcal{G}$-torsion. We apply the contravariant functor $\operatorname{Hom}_R(-, M)$ to the above cover, and find the following exact sequence.

$$0 = \operatorname{Ext}^1_R(B_J, M) \to \operatorname{Ext}^2_R(R/J, M) \to \operatorname{Ext}^2_R(A_J, M) = 0$$

The first module $\operatorname{Ext}^1_R(B_J, M)$ vanishes by Lemma 4.1.3 since B_J is $\mathcal{G}$-torsion and M is $\mathcal{G}$-closed. The last module $\operatorname{Ext}^2_R(A_J, M)$ vanishes since $\operatorname{p.dim} A_J \leq 1$. Therefore $\operatorname{Ext}^2_R(R/J, M) = 0$ for every M $\mathcal{G}$-closed and every $J \in \mathcal{G}$, as required. □

The above lemma allows us to use the equivalent conditions of Proposition 1.7.4. In particular, we have that $\psi_R \colon R \to R_\mathcal{G}$ is a flat injective ring epimorphism and that $R_\mathcal{G}$ is $\mathcal{G}$-divisible, so $R_\mathcal{G} \in \mathcal{D}_\mathcal{G}$. It remains to see that if $\mathcal{A}$ is covering in $(\mathcal{A}, \mathcal{D}_\mathcal{G})$ then $R_\mathcal{G} \oplus R_\mathcal{G}/R$ is the associated 1-tilting module, that is the equivalent conditions of Proposition 1.7.7. This amounts to showing that $\operatorname{p.dim} R_\mathcal{G} \leq 1$.

Proposition 7.1.4. *Suppose R is commutative and let $(\mathcal{A}, \mathcal{D}_\mathcal{G})$ be a 1-tilting cotorsion pair with associated Gabriel topology $\mathcal{G}$. Suppose $\mathcal{A}$ is covering. Then the module $R_\mathcal{G} \oplus R_\mathcal{G}/R$ is a 1-tilting module associated to the cotorsion pair $(\mathcal{A}, \mathcal{D}_\mathcal{G})$ and moreover $\operatorname{Gen}(R_\mathcal{G}) = \mathcal{D}_\mathcal{G}$.*

Proof. We know that $\mathcal{G}$ is perfect, so that $R_\mathcal{G}$ is $\mathcal{G}$-divisible by Lemma 7.1.3. By Proposition 1.7.7 it only remains to show that $\operatorname{p.dim} R_\mathcal{G} \leq 1$, that is we will show that $R_\mathcal{G} \in \mathcal{A}$. Let the following be an $\mathcal{A}$-cover of $R_\mathcal{G}$.

$$0 \to D \to A \xrightarrow{\phi} R_\mathcal{G} \to 0 \tag{7.1}$$

Note that A is $\mathcal{G}$-divisible since both $R_\mathcal{G}$ and D are $\mathcal{G}$-divisible. We will first show that A must be $\mathcal{G}$-torsion-free, and therefore an $R_\mathcal{G}$-module. Fix a finitely generated $J \in \mathcal{G}$ with generators $x_1, \ldots, x_n$. We will show that $A[J] = 0$, that is the only element of A annihilated by J is 0. Since $R_\mathcal{G}$ is divisible, one can write $1_R = 1_{R_\mathcal{G}} = \sum x_i \eta_i$ for some $x_i \in J$ and $\eta_i \in R_\mathcal{G}$. Let $\mathbf{x}$ and $\mathbf{s}$ be the following homomorphisms.

$$\mathbf{x} \colon R_\mathcal{G} \longrightarrow \bigoplus_{1 \leq i \leq n} R_\mathcal{G} \qquad 1_{R_\mathcal{G}} \longmapsto (\eta_1, \ldots, \eta_n)$$

$$\mathbf{s} \colon \bigoplus_{1 \leq i \leq n} R_\mathcal{G} \longrightarrow R_\mathcal{G} \qquad (\nu_1, \ldots, \nu_n) \longmapsto \textstyle\sum_i x_i \nu_i$$

$$\mathbf{s}_A \colon \bigoplus_{1 \leq i \leq n} A \longrightarrow A \qquad (a_1, \ldots, a_n) \longmapsto \textstyle\sum_i x_i a_i$$

By the definition of $\mathbf{s}$ and $\mathbf{s}_A$, the lower square of (7.2) commutes. Clearly ϕ^n is a precover of $R_{\mathcal{G}}^n$ as $D^n \in \mathcal{D}_{\mathcal{G}}$ and $A^n \in \mathcal{A}$ (it is in fact a cover). Therefore, there exists a map f such that the upper square of (7.2) commutes.

$$\begin{array}{ccccc}
A & \xrightarrow{\phi} & R_{\mathcal{G}} & \longrightarrow & 0 \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathbf{x}} & & \\
A^n & \xrightarrow{\oplus\phi} & {R_{\mathcal{G}}}^n & \longrightarrow & 0 \\
\downarrow{\scriptstyle \mathbf{s}_A} & & \downarrow{\scriptstyle \mathbf{s}} & & \\
A & \xrightarrow{\phi} & R_{\mathcal{G}} & \longrightarrow & 0
\end{array} \tag{7.2}$$

The map $\mathbf{sx}$ is the identity on $R_{\mathcal{G}}$, so we have that $\phi\mathbf{s}_A f = \phi$ and by the $\mathcal{A}$-cover property of ϕ, $\mathbf{s}_A f$ is an automorphism of A. Consider an element $a \in A[J]$ and let $f(a) = (f_1(a), \dots, f_n(a)) \in A^n$. Then $\mathbf{s}_A(f(a)) = \sum x_i f_i(a) = \sum f_i(x_i a) = 0$ as $x_i \in J$, and by the injectivity of $\mathbf{s}_A f$, $a = 0$.

We have shown that A, D are both $\mathcal{G}$-torsion-free and $\mathcal{G}$-divisible, so by Lemma 4.1.1 they are $R_{\mathcal{G}}$-modules. Then the sequence (7.1) is a sequence in Mod-$R_{\mathcal{G}}$ as $R \to R_{\mathcal{G}}$ is a ring epimorphism and Mod-$R_{\mathcal{G}} \to$ Mod-R is fully faithful. Thus (7.1) splits so $R_{\mathcal{G}} \in \mathcal{A}$ and $\operatorname{p.dim} R_{\mathcal{G}} \leq 1$ as required. □

7.2 When $\mathcal{A}$ is covering, R is $\mathcal{G}$-almost perfect

In this section we continue with the situation of Setting 7.1.1, that is we suppose that $(\mathcal{A}, \mathcal{D}_{\mathcal{G}})$ is a 1-tilting cotorion pair such that $\mathcal{A}$ is covering. We now additionally know that $U \oplus K$ is a 1-tilting module associated to $(\mathcal{A}, \mathcal{D}_{\mathcal{G}})$ by Proposition 7.1.4.

In Proposition 7.2.1 we prove that $R_{\mathcal{G}}$ is a perfect ring and in Proposition 7.2.4 that the R/J are perfect for every $J \in \mathcal{G}$. The main result is stated in Theorem 7.2.5.

By Proposition 7.1.4 if $(\mathcal{A}, \mathcal{D}_{\mathcal{G}})$ is a 1-tilting pair such that $\mathcal{A}$ is covering then the associated tilting module arises from a flat injective ring epimorphism $u\colon R \to U$ and $U \oplus K$ is a 1-tilting module for $(\mathcal{A}, \mathcal{D}_{\mathcal{G}})$, thus $\mathcal{D}_{\mathcal{G}} = \operatorname{Gen}(U)$.

Proposition 7.2.1. *Suppose R is commutative and let $(\mathcal{A}, \mathcal{D}_{\mathcal{G}})$ be a 1-tilting cotorsion pair with associated Gabriel topology $\mathcal{G}$. Suppose $\mathcal{A}$ is covering. Then $R_{\mathcal{G}}$ is a perfect ring.*

Proof. We will show that every $R_{\mathcal{G}}$-module has a projective cover in Mod-$R_{\mathcal{G}}$ (see Theorem 1.4.2). Consider some $M \in$ Mod-$R_{\mathcal{G}}$ with the following short exact sequence in Mod-$R_{\mathcal{G}}$.

$$0 \to L \to R_{\mathcal{G}}^{(\alpha)} \xrightarrow{\phi} M \to 0 \tag{7.3}$$

Then this sequence is also a short exact sequence of R-modules with $R_{\mathcal{G}}^{(\alpha)} \in \mathcal{A}$ by Proposition 7.1.4 and $L \in \mathcal{D}_{\mathcal{G}}$ since all $R_{\mathcal{G}}$-modules are $\mathcal{G}$-divisible, thus it is an $\mathcal{A}$-precover of M_R. By the assumption that $\mathcal{A}$ is covering, one can extract the $\mathcal{A}$-cover (7.4) from the above short exact sequence (7.3), so that L' and P are direct summands of L and $R_{\mathcal{G}}^{(\alpha)}$ respectively as R-modules. As $R \to R_{\mathcal{G}}$ is a ring epimorphism, the direct summand of a $R_{\mathcal{G}}$-module must be also an $R_{\mathcal{G}}$-module.

$$0 \to L' \to P \overset{\phi'}{\to} M \to 0 \tag{7.4}$$

We now have a $\mathcal{P}_0(R_{\mathcal{G}})$-precover of M in Mod-$R_{\mathcal{G}}$ as above, which is also an $\mathcal{A}$-cover when considered in Mod-R. It remains to see that it is a $\mathcal{P}_0(R_{\mathcal{G}})$-cover, that is that L' is superfluous in P. Consider $H \leq P$ an $R_{\mathcal{G}}$-submodule of P such that $H + L' = P$. Then $H \cap L'$ is an $R_{\mathcal{G}}$-module, hence $\mathcal{G}$-divisible. Therefore, by the $\mathcal{A}$-cover property of ϕ', H is $\mathcal{D}_{\mathcal{G}}$-superfluous in P, thus by Proposition 2.2.5(i) $H = P$. It follows that L' is superfluous in P so ϕ' is a $\mathcal{P}_0(R_{\mathcal{G}})$-cover of M, as required.

Another way to see that (7.4) is a $\mathcal{P}_0(R_{\mathcal{G}})$-cover, is to use that every $R_{\mathcal{G}}$-homomorphism f such that $\phi' f = \phi'$ is also an R-homomorphism, and therefore is an automorphism by the $\mathcal{A}$-cover property of ϕ'. □

We will now show that R/J is perfect for each $J \in \mathcal{G}$ by showing that every Bass R/J-module has a $\mathcal{P}_0(R/J)$-cover, that is using Lemma 1.4.4.

Take $a_1, a_2, \dots, a_i, \dots$ a sequence of elements of R and let N be the Bass module with presentation as in the sequence (7.5), where $(e_i)_{i \in \mathbb{N}}$ and $(f_i)_{i \in \mathbb{N}}$ are basis of the domain and codomain of ϕ respectively.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_{\mathbb{N}} R/J & \xrightarrow{\phi} & \bigoplus_{\mathbb{N}} R/J & \longrightarrow & N & \longrightarrow & 0 \\ & & e_i & \longrightarrow & f_i - a_i f_{i+1} & & & & \end{array} \tag{7.5}$$

As the elements $a_1, a_2, \dots, a_i, \dots$ are in R, we can also define a Bass R-module, which is a lift of N. That is, we consider the following Bass R-module.

$$0 \to \bigoplus_{\mathbb{N}} R \overset{\phi}{\to} \bigoplus_{\mathbb{N}} R \to F \to 0 \tag{7.6}$$

It is clear that applying $(- \otimes_R R/J)$ to (7.6) will give us (7.5), thus $F \otimes_R R/J = N$, where F is flat.

We will make use of results in Section 5.3 and Section 5.4.

Lemma 7.2.2. *Suppose $\mathcal{A}$ is covering and F is a Bass R-module. Then $\operatorname{Hom}_R(K, F \otimes_R K)$ has a projective cover in the category of u-contramodules.*

Proof. Apply the functor $(- \otimes_R K)$ to (7.6).

$$0 \to \bigoplus_{\mathbb{N}} K \overset{\phi}{\to} \bigoplus_{\mathbb{N}} K \to F \otimes_R K \to 0$$

The above is an $\mathcal{A}$-precover by Proposition 7.1.4. As by assumption $\mathcal{A}$ is covering, one can extract the $\mathcal{A}$-cover from the above sequence, which we will denote as follows.

$$0 \to D_1 \overset{\phi\restriction_{D_1}}{\to} D_0 \overset{\pi}{\to} F \otimes_R K \to 0 \tag{7.7}$$

Now we apply $\mathrm{Hom}_R(K, -)$ to the above sequence, and we claim that it is a projective cover in the category of u-contramodules.

$$0 \to \mathrm{Hom}_R(K, D_1) \to \mathrm{Hom}_R(K, D_0) \overset{\rho}{\to} \mathrm{Hom}_R(K, F \otimes_R K) \to 0 \tag{7.8}$$

Firstly, $\mathrm{Hom}_R(K, D_1)$ and $\mathrm{Hom}_R(K, D_0)$ are direct summands of modules of the form $\mu_{R^{(\alpha)}} : \mathrm{Hom}_R(K, K^{(\alpha)}) \cong \Delta_u(R^{(\alpha)})$ (see Lemma 5.3.6), thus are projective objects in the category u-**contra**. We will show that $\rho := \mathrm{Hom}_R(K, \pi)$ is a projective cover in u-**contra**. Take $f\colon \mathrm{Hom}_R(K, D_0) \to \mathrm{Hom}_R(K, D_0)$ such that $\rho f = \rho$. By Theorem 5.4.2, the adjoint functors $\big((- \otimes_R K), \mathrm{Hom}_R(K, -)\big)$ form equivalences between the subcategories of $\mathcal{G}$-torsion $\mathcal{G}$-divisible modules $\mathcal{E}_\mathcal{G} \cap \mathcal{D}_\mathcal{G}$ and $\mathcal{G}$-torsion-free u-contramodules $\mathcal{F}_\mathcal{G} \cap u$-**contra**. Thus in particular the functor $\mathrm{Hom}_R(K, -)$ restricted to the subcategories $\mathcal{E}_\mathcal{G} \cap \mathcal{D}_\mathcal{G} \to \mathcal{F}_\mathcal{G} \cap u$-**contra** is full so there exists a $g\colon D_0 \to D_0$ such that $\mathrm{Hom}_R(K, g) = f$. Thus as $\pi g = \pi$ implies that g is an automorphism, we conclude that also f is an automorphism, as required. □

Lemma 7.2.3. *Suppose R is a ring with unit and take a right ideal I of R and take a monomorphism $f\colon A \to B \in \mathrm{Mod}\text{-}R$. Suppose $f(A)$ is superfluous in B. Then $\bar{f}(A/AJ)$ is superfluous in B/BJ, where $\bar{f}$ is the naturally induced map from f.*

Proof. Take $H/BJ \subseteq B/BJ$ such that $H/BJ + \bar{f}(A/AJ) = B/BJ$. Note that $\bar{f}(A/AJ) = (f(A)+BJ)/BJ$. Then it follows that $H + f(A) + BJ = B$. As $BJ \subseteq H$ and $f(A)$ is superfluous in B, we find $H = B$, as required. □

Proposition 7.2.4. *Suppose $(\mathcal{A}, \mathcal{D}_\mathcal{G})$ is a 1-tilting cotorsion pair over a commutative ring where $\mathcal{A}$ is covering. If F is a Bass R-module, then $F \otimes_R R/J$ has a $\mathcal{P}_0(R/J)$-cover.*

Proof. By Lemma 7.2.2, there is a projective cover of $\mathrm{Hom}_R(K, F \otimes_R K)$ in the category of u-contramodules. Moreover, as F is flat it is $\mathcal{G}$-torsion-free, so $\mathrm{Hom}_R(K, F \otimes_R K) \otimes_R R/J \cong F/FJ$ by Lemma 5.3.11, and similarly $\mathrm{Hom}_R(K, K^{(\alpha)}) \otimes_R R/J \cong (R/J)^{(\alpha)}$.

$$\begin{aligned} \mathrm{Hom}_R(K, D_0) \otimes_R R/J &\underset{\oplus}{\leq} \mathrm{Hom}_R(K, K^{(\alpha)}) \otimes_R R/J \\ &\cong (R/J)^{(\alpha)} \end{aligned} \tag{7.9}$$

Moreover we have the following short exact sequence. The first isomorphism is by Lemma 5.3.12 and Lemma 5.3.6, the latter of which states $\mathrm{Hom}_R(K, F \otimes_R K) \cong \Delta_u(F)$. Finally we use that F is flat.

$$\mathrm{Tor}_1^R(\mathrm{Hom}_R(K, F \otimes_R K), R/J) \cong \mathrm{Tor}_1^R(F, R/J) = 0$$

It will be enough to show the following sequence (7.10), which is $(- \otimes_R R/J)$ applied to (7.8), is a projective cover in Mod-R/J.

$$0 \to \mathrm{Hom}_R(K, D_1) \otimes_R R/J \to \mathrm{Hom}_R(K, D_0) \otimes_R R/J \overset{\rho \otimes_R \mathrm{id}_{R/J}}{\to} F/FJ \to 0 \tag{7.10}$$

By Lemma 7.2.3, $\mathrm{Hom}_R(K, D_1) \otimes_R R/J \ll \mathrm{Hom}_R(K, D_0) \otimes_R R/J$, so (7.10) is a $\mathcal{P}_0(R/J)$-cover of F/FJ. □

Theorem 7.2.5. *Suppose R is a commutative ring and $(\mathcal{A}, \mathcal{D}_\mathcal{G})$ a 1-tilting cotorsion pair. Then if $\mathcal{A}$ is covering, $R \to R_\mathcal{G}$ is a perfect localisation, $\mathrm{p.dim}\, R_\mathcal{G} \leq 1$, and R is $\mathcal{G}$-almost perfect.*

Proof. That $R \to R_\mathcal{G}$ is a perfect localisation and $\mathrm{p.dim}\, R_\mathcal{G} \leq 1$ are by Lemma 7.1.3, Proposition 7.1.4. That R is $\mathcal{G}$-almost perfect is by Proposition 7.2.1 and Proposition 7.2.4. □

7.3 When R is a $\mathcal{G}$-almost perfect ring

Let R be a commutative ring. We suppose that $(\mathcal{A}, \mathcal{D}_\mathcal{G})$ is a 1-tilting cotorsion pair and R is $\mathcal{G}$-almost perfect (that is $R_\mathcal{G}$ is a perfect ring and R/J is a perfect ring for every $J \in \mathcal{G}$) and additionally $\mathrm{p.dim}\, R_\mathcal{G} \leq 1$. The purpose of this section is to show that under these assumptions $\mathcal{A}$ is covering. We will do this by first showing that $\mathcal{G}$ arises from a perfect localisation. Next it is sufficient to show that we will show that $U \oplus K$ is Σ-pure-split, as then $\mathcal{A}$ is closed under direct limits using Proposition 1.5.2. To show that $U \oplus K$ is Σ-pure-split, the problem naturally splits into two parts: showing that each of U and K are Σ-pure-split.

Setting 7.3.1. Let R be a commutative ring. We suppose that $(\mathcal{A}, \mathcal{D}_\mathcal{G})$ is a 1-tilting cotorsion pair and R is $\mathcal{G}$-almost perfect ($U = R_\mathcal{G}$ is perfect and R/J is perfect for every $J \in \mathcal{G}$) and additionally $\mathrm{p.dim}\, U \leq 1$.

We first prove a lemma. Recall that $\mathcal{G}$ is faithful and finitely generated, so Lemma 1.7.5 is satisfied for the modules R/J for the finitely generated $J \in \mathcal{G}$.

Lemma 7.3.2. *Let R be a commutative ring. Suppose $(\mathcal{A}, \mathcal{D}_\mathcal{G})$ is a 1-tilting cotorsion pair, $\mathcal{G}$ the associated Gabriel topology and $\mathrm{f.dim}\, R_\mathcal{G} = 0$. Then $\mathcal{G}$ arises from a perfect localisation, or equivalently $R_\mathcal{G}$ is $\mathcal{G}$-divisible.*

Proof. We use the relations in Lemma 1.7.5 to show that $R_{\mathcal{G}} \otimes_R R/J$ for each finitely generated $J \in \mathcal{G}$. That is, for a finitely generated $J \in \mathcal{G}$, $R_{\mathcal{G}} \otimes_R R/J \cong \operatorname{Ext}^1_R(\operatorname{Tr} R/J, R_{\mathcal{G}})$, and as $\operatorname{p.dim} \operatorname{Tr} R/J \leq 1$, by Lemma 4.1.2(i) $\operatorname{Tor}^R_1(\operatorname{Tr} R/J, R_{\mathcal{G}}) = 0$. Thus applying $(R_{\mathcal{G}} \otimes_R -)$ to the projective resolution of $\operatorname{Tr} R/J$, we find the following.

$$0 \to R_{\mathcal{G}} \to R^n_{\mathcal{G}} \to R_{\mathcal{G}} \otimes_R \operatorname{Tr} R/J \to 0$$

By assumption $\operatorname{f.dim} R_{\mathcal{G}} = 0$ so $R_{\mathcal{G}} \otimes_R \operatorname{Tr} R/J$ is $R_{\mathcal{G}}$-projective. Next consider the following isomorphism.

$$\operatorname{Ext}^1_R(\operatorname{Tr} R/J, R_{\mathcal{G}}) \cong \operatorname{Ext}^1_{R_{\mathcal{G}}}(R_{\mathcal{G}} \otimes_R \operatorname{Tr} R/J, R_{\mathcal{G}}) = 0$$

The last module vanishes as $R_{\mathcal{G}} \otimes_R \operatorname{Tr} R/J$ is $R_{\mathcal{G}}$-projective, so $R/J \otimes_R R_{\mathcal{G}} \cong \operatorname{Ext}^1_R(\operatorname{Tr} R/J, R_{\mathcal{G}}) = 0$ for each $J \in \mathcal{G}$ so $R_{\mathcal{G}}$ is $\mathcal{G}$-divisible. □

Our assumption states that $R_{\mathcal{G}}$ is perfect, so by Proposition 1.4.3, $\operatorname{F.dim} R_{\mathcal{G}} = 0$, so we can use Lemma 7.3.2. Thus $\mathcal{G}$ is a perfect Gabriel topology and we can work in the case that the 1-tilting cotorsion pair $(\mathcal{A}, \mathcal{D}_{\mathcal{G}})$ arises from a flat injective ring epimorphism $u \colon R \to U$ and $\operatorname{Gen}(U) = \mathcal{D}_{\mathcal{G}}$ as in Proposition 1.7.7.

7.3.1 $U \oplus K$ is Σ-pure spilt

Take $X \in \text{Mod-}R$ such that the following is a pure exact sequence and $T \in \operatorname{Add}(U \oplus K)$.

$$0 \to X \to T \to Y \to 0 \tag{7.11}$$

It follows that $X, Y \in \mathcal{D}_{\mathcal{G}}$ as $T \in \mathcal{D}_{\mathcal{G}}$ and the sequence (7.11) is pure exact, so the sequence vanishes when one applies $(R/J \otimes_R -)$.

Lemma 7.3.3. *Let $\mathcal{G}$ be a faithful finitely generated perfect Gabriel topology over R such that $\operatorname{p.dim} U \leq 1$ and R/J is a perfect ring for each $J \in \mathcal{G}$. Suppose furthermore that U is a perfect ring. Then the following sequence (which is $(- \otimes_R U)$ applied to (7.11)) splits.*

$$0 \to X \otimes_R U \to T \otimes_R U \to Y \otimes_R U \to 0 \tag{7.12}$$

Furthermore, (7.12) is a short exact sequence of projective modules in Mod-U.

Proof. The sequence (7.12) is pure exact as it is a tensor product of a pure exact sequence and a flat module. Moreover, as $T \in \operatorname{Add}(U \oplus K)$, it is straightforward to see that $T \otimes_R U \in \operatorname{Add}(U)$, thus is U-projective.

Since the sequence (7.12) is pure and $T \otimes_R U$ is projective as a U-module, also $Y \otimes_R U$ is also U-flat and therefore U-projective as U is perfect. So the sequence splits both in Mod-U and hence in Mod-R. Also note that this implies that $X \otimes_R U$ is flat in Mod-R. □

Lemma 7.3.4. *Let $\mathcal{G}$ be a faithful finitely generated perfect Gabriel topology over R such that* $\mathrm{p.dim}\, U \leq 1$ *and R/J is a perfect ring for each $J \in \mathcal{G}$. Suppose D is $\mathcal{G}$-divisible module. Then its $\mathcal{G}$-torsion module $t(D)$ is also $\mathcal{G}$-divisible.*

Proof. Take the $\mathcal{G}$-torsion decomposition of D as follows.

$$0 \to t(D) \to D \to D/t(D) \to 0$$

Apply $\mathrm{Hom}_R(K, -)$ to the above sequence. We have that $0 = \mathrm{Hom}_R(K, D/t(D))$ as $D/t(D)$ is $\mathcal{G}$-torsion-free and $\mathrm{Ext}^1_R(K, D) = 0$ as D is $\mathcal{G}$-divisible and $\mathcal{D}_\mathcal{G} = K^{\perp}$.

$$0 = \mathrm{Hom}_R(K, D/t(D)) \to \mathrm{Ext}^1_R(K, t(D)) \to \mathrm{Ext}^1_R(K, D) = 0$$

□

Lemma 7.3.5. *Let $\mathcal{G}$ be a faithful finitely generated perfect Gabriel topology over a commutative ring such that U is a perfect ring and* $\mathrm{p.dim}\, U \leq 1$*. Let X, T, Y be as in (7.11). Then the following sequence is pure exact.*

$$0 \to t(X) \to t(T) \to t(Y) \to 0$$

Proof. We claim diagram (7.13) has exact rows and exact columns. This is because the bottom row is exact as U is flat, and by the snake lemma and the fact that $X \otimes_R K = 0$ as X is $\mathcal{G}$-divisible, forces the top row to be exact.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & t(X) & \longrightarrow & t(T) & \longrightarrow & t(Y) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & T & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X \otimes_R U & \longrightarrow & T \otimes_R U & \longrightarrow & Y \otimes_R U & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{7.13}$$

We apply $(- \otimes_R N)$ for some $N \in \text{Mod-}R$ to the diagram above, and it is enough to show that in the first row the connection map $\delta\colon \mathrm{Tor}^R_1(t(Y), N) \to t(X) \otimes_R N$ is zero. As we have shown $X \otimes_R U$ is flat in Lemma 7.3.3,

$\mathrm{Tor}_1^R(X \otimes_R U, N) = 0$. We want to show that $\delta = 0$.

$$\begin{array}{ccc} & & \mathrm{Tor}_1^R(X \otimes_R U, N) = 0 \\ & & \downarrow \\ \mathrm{Tor}_1^R(t(Y), N) & \xrightarrow{\delta} & t(X) \otimes_R N \\ \downarrow & & \downarrow \varepsilon \\ \mathrm{Tor}_1^R(Y, N) & \xrightarrow{0} & X \otimes_R N \end{array}$$

So $\varepsilon\delta = 0$ and as ε is a monomorphism, $\delta = 0$ as required. $\square$

Corollary 7.3.6. *Let $\mathcal{G}$ be a faithful finitely generated perfect Gabriel topology over R such that* $\mathrm{p.dim}\, U \leq 1$ *and R/J is a perfect ring for each $J \in \mathcal{G}$ and U is a perfect ring. Suppose that K is Σ-pure split. Then $U \oplus K$ is Σ-pure split, that is every pure embedding as in (7.11) splits.*

Proof. Consider a pure exact sequence as in (7.11). Then the pure exact sequence forms the middle row in Diagram (7.13). By assumption, the top row splits, so $t(Y) \in \mathrm{Add}(K)$. Moreover by Lemma 7.3.3, the bottom row splits so $Y \otimes_R U \in \mathrm{Add}(U)$.

$$0 \to t(Y) \to Y \to Y \otimes_R U \to 0$$

Thus by the above short exact sequence, $Y \in \mathrm{Add}(U \oplus K)$ as $t(Y), Y \otimes_R U$ are in $\mathrm{Add}(U \oplus K)$ and $\mathrm{Add}(U \oplus K)$ is closed under extensions. Thus (7.11) splits as $X \in \mathcal{D}_{\mathcal{G}}$. $\square$

We will let (7.14) denote a pure exact sequence with $T_t \in \mathrm{Add}(K)$. That is, it denotes the torsion part of the pure exact sequence (7.11).

$$0 \to X_t \to T_t \to Y_t \to 0 \tag{7.14}$$

Thus from Corollary 7.3.6, it remains only to see that (7.14) splits, that is if K is Σ-pure-split. In this case, (7.14) is a sequence of $\mathcal{G}$-torsion $\mathcal{G}$-divisible modules. Hence we can use the results of Section 5.4, that is the equivalence between the categories of $\mathcal{G}$-torsion $\mathcal{G}$-divisible modules and the $\mathcal{G}$-torsion-free u-contramodules via the adjoint functors $\big((- \otimes_R K), \mathrm{Hom}_R(K, -)\big)$. As we will see, once in u-**contra** we can show that $\mathrm{Hom}_R(K, Y_t)$ is a projective u-contramodule and moreover the sequence splits in the category of $\mathcal{G}$-torsion-free u-contramodules, and thus also the original sequence (7.14) in the category of $\mathcal{G}$-torsion $\mathcal{G}$-divisible modules.

Moreover, we will use that for a $\mathcal{G}$-torsion-free module N, (in particular a free module $R^{(\beta)}$), there is an isomorphism $\tilde{\nu}_N : \Delta_u(N) \cong \Lambda_{\mathcal{G}}(N)$ as stated in Proposition 5.5.4, and these are modules are u-contramodules by Lemma 5.3.7. Also we use regularly Lemma 5.3.11, that is $M/MJ \cong \Delta_u(M) \otimes_R R/J$ for any R-module M and every $J \in \mathcal{G}$. Finally, we also recall that with the assumption $\mathrm{p.dim}\, U \leq 1$, u-**contra** is an abelian category by Proposition 5.3.18 and the direct summands of modules of the form $\Delta_u(R^{(\beta)})$ for some cardinal β are the projective modules in u-**contra** as stated in Lemma 5.3.20.

Before working in the local case, we want to prove a lemma about a $\mathcal{G}$-torsion-free u-contramodule M such that $M \otimes_R K \in \mathcal{F}_1(R)$. In particular, $\mathrm{Hom}_R(K, Y_t)$ with Y_t from the sequence (7.14) satisfies this condition as $X_t, Y_t \in \mathcal{F}_1(R)$ since $T_t \in \mathcal{F}_1(R)$ by the pure-exactness of the sequence (7.14), and moreover $Y_t \cong \mathrm{Hom}_R(K, Y_t) \otimes_R K$ by Theorem 5.4.2.

Lemma 7.3.7. *Let $\mathcal{G}$ be a faithful finitely generated perfect Gabriel topology over R such that $\mathrm{p.dim}\, U \leq 1$ and R/J is a perfect ring for each $J \in \mathcal{G}$. Suppose M is a $\mathcal{G}$-torsion-free u-contramodule such that $M \otimes_R K \in \mathcal{F}_1(R)$ and N is a $\mathcal{G}$-torsion module. Then $\mathrm{Tor}_1^R(M, N) = 0$.*

Proof. Fix an M and N as above. We apply $(- \otimes_R N)$ to the following short exact sequence.

$$0 \to M \to M \otimes_R U \to M \otimes_R K \to 0$$

$$0 = \mathrm{Tor}_2^R(M \otimes_R K, N) \to \mathrm{Tor}_1^R(M, N) \to \mathrm{Tor}_1^R(M \otimes_R U, N)$$

Thus as U is flat, we have that $\mathrm{Tor}_1^R(M \otimes_R U, N) \cong \mathrm{Tor}_1^U(M \otimes_R U, U \otimes_R N)$ which is zero as in the case that N is $\mathcal{G}$-torsion, $U \otimes_R N = 0$. □

7.3.2 When R is local and R/J is a perfect ring for each $J \in \mathcal{G}$

In this subsection we will assume that R is local with maximal ideal $\mathfrak{m}$.
The module M will denote $\mathrm{Hom}_R(K, Y_t)$ where Y_t is the module in (7.14), although as mentioned before everything can be generalised to a $\mathcal{G}$-torsion-free u-contramodule M such that $M \otimes_R K \in \mathcal{F}_1(R)$.

Remark 7.3.8. By assumption R is local, so R/J is a perfect local ring for each $J \in \mathcal{G}$. Therefore the maximal ideal of each R/J is T-nilpotent, so by Lemma 1.4.1 it follows that $N\mathfrak{m} \ll N$ for every R/J-module N. Moreover by Proposition 1.4.3 (iv), every R/J-module has a non-zero socle.

Lemma 7.3.9. *Suppose R is a local commutative ring and $\mathcal{G}$ a faithful finitely generated perfect Gabriel topology over R such that $\mathrm{p.dim}\, U \leq 1$ and R/J is a perfect ring for each $J \in \mathcal{G}$. Then every non-zero R-module M is either in $\mathcal{D}_\mathcal{G}$ or $M\mathfrak{m} \neq M$.*

Proof. Suppose M is not in $\mathcal{D}_\mathcal{G}$. Then there exists a $J \in \mathcal{G}$ such that $M/MJ \neq 0$. By Remark 7.3.8 we have the following strict inclusion.

$$(M/MJ)\mathfrak{m} = (M\mathfrak{m})/(MJ) \subsetneq M/MJ$$

So it follows that $M\mathfrak{m} \subsetneq M$, as required. □

We will show the following using the method of Positselski in [35, Lemma 3.3 and Proposition 3.4], although in a much simpler setting. That is, we will show that $M = \operatorname{Hom}_R(K, Y)$ is projective in u-**contra**. Consider the following short exact sequence of the natural inclusion of $M\mathfrak{m}$ in M.

$$0 \to M\mathfrak{m} \to M \overset{p_M}{\to} M/M\mathfrak{m} \to 0$$

As $M/M\mathfrak{m}$ is an $R/\mathfrak{m}$ module and $R/\mathfrak{m}$ is simple, there exists a β such that $(R/\mathfrak{m})^{(\beta)} \cong M/M\mathfrak{m}$. Thus we let $p\colon \Delta_u(R^{(\beta)}) \to (R/\mathfrak{m})^{(\beta)}$ be the natural projection map where $(R/\mathfrak{m})^{(\beta)}$ is associated with $\Delta_u(R^{(\beta)})/\Delta_u(R^{(\beta)})\mathfrak{m}$.

$$\Delta_u(R^{(\beta)}) \overset{p}{\to} \Delta_u(R^{(\beta)})/\Delta_u(R^{(\beta)})\mathfrak{m} \cong (R/\mathfrak{m})^{(\beta)} \to 0$$

Proposition 7.3.10. *Suppose R is a local commutative ring and $\mathcal{G}$ a faithful finitely generated perfect Gabriel topology over R such that* $\operatorname{p.dim} U \leq 1$ *and R/J is a perfect ring for each $J \in \mathcal{G}$. Then there is a map f that makes the following diagram commute.*

$$\begin{array}{ccccccccc} & & & & \Delta_u(R^{(\beta)}) & \xrightarrow{p} & (R/\mathfrak{m})^{(\beta)} & \longrightarrow & 0 \\ & & & & \downarrow f & & \downarrow \cong & & \\ 0 & \longrightarrow & M\mathfrak{m} & \longrightarrow & M & \xrightarrow{p_M} & M/M\mathfrak{m} & \longrightarrow & 0 \end{array} \tag{7.15}$$

Furthermore, f is an epimorphism.

Proof. The fact that f exists follows from the fact that all the modules in the above diagram are u-contramodules and $\Delta_u(R^{(\beta)})$ is a projective u-contramodule.

To see that f is an epimorphism, note that as $\Delta_u(R^{(\beta)}) \to M/M\mathfrak{m}$ is an epimorphism, it follows that $f + M\mathfrak{m} = M$. By the following computation and Lemma 7.3.9, it follows that $M/\operatorname{Im} f$ is $\mathcal{G}$-divisible.

$$(M/\operatorname{Im} f)\mathfrak{m} = (M\mathfrak{m} + \operatorname{Im} f)/\operatorname{Im} f = M/\operatorname{Im} f$$

However, f is a map of u-contramodules, so also $\operatorname{Coker} f = M/\operatorname{Im} f$ is a u-contramodule, thus $M/\operatorname{Im} f$ contains no non-zero $\mathcal{G}$-divisible submodule. We conclude that $M/\operatorname{Im} f = 0$, so f is an epimorphism as required. □

As R/J is a perfect ring for each $J \in \mathcal{G}$, as mentioned in Remark 7.3.8, every R/J-module contains a non-zero socle. As R and thus also R/J are local, the only simple module to consider is $R/\mathfrak{m}$. We will use a minor modification of the Loewy series of R/J. We construct it as follows.

Let $J_0 = J$ so that $N_0 = J_0/J = 0$. Next, for each ordinal σ, let $N_\sigma/N_{\sigma-1} = J_\sigma/J_{\sigma-1}$ be the image of $0 \to R/\mathfrak{m} \to R/J_{\sigma-1}$ in $R/J_{\sigma-1}$ for some choice of map $R/\mathfrak{m} \to R/J_{\sigma-1}$. Thus we have the following.

$$0 \to R/\mathfrak{m} \cong J_\sigma/J_{\sigma-1} \to R/J_{\sigma-1} \to R/J_\sigma \to 0$$

Rearranging, we find the following short exact sequence.

$$0 \to J_{\sigma-1}/J \to J_\sigma/J \to J_\sigma/J_{\sigma-1} \cong R/\mathfrak{m} \to 0$$

For a limit ordinal τ, define $M_\tau = J_\tau/J$ to be the union $\bigcup_{\sigma<\tau} M_\sigma = \bigcup_{\sigma<\tau} J_\sigma/J$.

Thus one can write $R/J = \bigcup_\alpha J_\alpha/J$, that is as a direct limit of modules in its Loewy series.

Proposition 7.3.11. *Suppose R is a local commutative ring and $\mathcal{G}$ a faithful finitely generated perfect Gabriel topology over R such that $\operatorname{p.dim} U \leq 1$ and R/J is a perfect ring for each $J \in \mathcal{G}$. Then we have the following isomorphism where M is a $\mathcal{G}$-torsion-free u-contramodule and f is as in (7.15).*

$$\mathrm{id}_{R/J} \otimes_R f \colon (R/J)^{(\beta)} \cong M/MJ$$

Proof. For each ordinal σ, we have the following commuting diagram as $\operatorname{Tor}_1^R(R/\mathfrak{m}, \Delta_u(R^{(\beta)})) \cong \operatorname{Tor}_1^R(R/\mathfrak{m}, M) = 0$ by Lemma 5.3.12 and Lemma 7.3.7, and f is an epimorphism by Proposition 7.3.10.

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & J_{\sigma-1}/J \otimes_R \Delta_u(R^{(\beta)}) & \longrightarrow & J_\sigma/J \otimes_R \Delta_u(R^{(\beta)}) & \longrightarrow & R/\mathfrak{m} \otimes_R \Delta_u(R^{(\beta)}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}_{J_{\sigma-1}/J} \otimes_R f} & & \downarrow{\scriptstyle \mathrm{id}_{J_\sigma/J} \otimes_R f} & & {\scriptstyle \cong}\downarrow{\scriptstyle \mathrm{id}_{R/\mathfrak{m}} \otimes_R f} & & \\
0 & \longrightarrow & J_{\sigma-1}/J \otimes_R M & \longrightarrow & J_\sigma/J \otimes_R M & \longrightarrow & R/\mathfrak{m} \otimes_R M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We will show the isomorphism by transfinite induction on σ. It is clear in the base case of $\sigma = 0$. If $\mathrm{id}_{J_{\sigma-1}/J} \otimes_R f$ is an isomorphism, then by the five-lemma, as the two outer vertical morphisms of the above diagram are isomorphisms, also $\mathrm{id}_{J_\sigma/J} \otimes_R f$ is an isomorphism.

For τ a limit ordinal, we have that the tensor product commutes with direct limits and that $\mathrm{id}_{J_\sigma/J} \otimes_R f$ is an isomorphism for every $\sigma < \tau$. Thus these maps induce an isomorphism $\mathrm{id}_{J_\tau/J} \otimes_R f$.

$$\begin{aligned}
\big(\bigcup_{\sigma<\tau} J_\sigma/J\big) \otimes_R \Delta_u(R^{(\beta)}) &= \bigcup_{\sigma<\tau} \big(J_\sigma/J \otimes_R \Delta_u(R^{(\beta)})\big) \\
&\cong \bigcup_{\sigma<\tau} (J_\sigma/J \otimes_R M) \\
&= \big(\bigcup_{\sigma<\tau} J_\sigma/J\big) \otimes_R M
\end{aligned}$$

As $R/J = \bigcup_\alpha J_\alpha/J$, we have shown that $R/J^{(\beta)} \cong R/J \otimes_R \Delta_u(R^{(\beta)}) \cong M/MJ$, as required. □

The following proposition uses work done in Section 5.2 and Section 5.5. That is, we finally use the isomorphism $\Delta_u(N) \cong \Lambda_{\mathcal{G}}(N)$ for a $\mathcal{G}$-torsion-free module N and that $\Lambda_{\mathcal{G}}(N)$ is $\mathcal{G}$-separated.

Proposition 7.3.12. *Suppose R is a local commutative ring and $\mathcal{G}$ a faithful finitely generated perfect Gabriel topology over R such that* $\operatorname{p.dim} U \leq 1$ *and R/J is a perfect ring for each $J \in \mathcal{G}$. Then the morphism $f\colon \Delta_u(R^{(\beta)}) \to M$ as defined in (7.15) is an isomorphism.*

Proof. Note that the kernel of $f\colon \Delta_u(R^{(\beta)}) \to M$ is contained in every $\Delta_u(R^{(\beta)})J$, thus $\operatorname{Ker} f \subseteq \bigcap_{J\in\mathcal{G}} \Delta_u(R^{(\beta)})J$. However, as $R^{(\beta)}$ is $\mathcal{G}$-torsion-free by Corollary 5.5.7 $\Delta_u(R^{(\beta)}) \cong \Lambda_{\mathcal{G}}(R^{(\beta)})$, which is already $\mathcal{G}$-separated by Lemma 5.1.2, so $\bigcap_{J\in\mathcal{G}} \Delta_u(R^{(\beta)})J = \bigcap_{J\in\mathcal{G}} \Lambda_{\mathcal{G}}(R^{(\beta)})J$ vanishes. □

Proposition 7.3.13. *Let $\mathcal{G}$ be a faithful finitely generated perfect Gabriel topology over a commutative ring such that* $\operatorname{p.dim} R_{\mathcal{G}} \leq 1$*. Suppose R is local and R/J is perfect for each $J \in \mathcal{G}$. Consider the pure exact sequence with $T_t \in \operatorname{Add}(K)$.*

$$0 \to X_t \to T_t \to Y_t \to 0 \tag{7.14}$$

Then the sequence splits. In other words, K is Σ-pure-split.

Proof. In Proposition 7.3.12, we have shown that $M = \operatorname{Hom}_R(K, Y_t)$ is a projective u-contramodule, therefore the following sequence (which is $\operatorname{Hom}_R(K, -)$ applied to (7.14)) splits.

$$0 \to \operatorname{Hom}_R(K, X_t) \to \operatorname{Hom}_R(K, T_t) \to \operatorname{Hom}_R(K, Y_t) \to 0$$

Applying $(- \otimes_R K)$, we recover the original short exact sequence up to isomorphism, which also splits. □

We have shown that for R a local (commutative) ring, if R/J is perfect for every $J \in \mathcal{G}$, a pure submodule of a module $T_t \in \operatorname{Add}(K)$ splits. We will now extend this to the global case in the following final subsection.

7.3.3 Final results

We recall that since R/J is perfect for each $J \in \mathcal{G}$, by Lemma 4.2.2 the ring R is $\mathcal{G}$-h-nil. In particular, the equivalent statements of Proposition 4.2.6 holds. That is, we use in particular that for every $\mathcal{G}$-torsion module M, $M \cong \bigoplus_{\mathfrak{m} \max} M_{\mathfrak{m}}$, where $\mathfrak{m}$ runs over all the maximal ideals of R.

Proposition 7.3.14. *Let $\mathcal{G}$ be a faithful finitely generated perfect Gabriel topology over a commutative ring such that* $\operatorname{p.dim} R_{\mathcal{G}} \leq 1$*. Suppose that the R/J are perfect rings for each $J \in \mathcal{G}$. Then K is Σ-pure-split.*

Proof. Take $0 \to X_t \to T_t \xrightarrow{\rho} Y_t \to 0$ a pure exact sequence. By Proposition 4.2.6, $T_t = \bigoplus_{\mathfrak{m}}(T_t)_{\mathfrak{m}}$ and $Y_t = \bigoplus_{\mathfrak{m}}(Y_t)_{\mathfrak{m}}$. Additionally by Proposition 4.2.7, the morphism ρ is a direct sum of surjective maps $(T_t)_{\mathfrak{m}} \to (Y_t)_{\mathfrak{m}}$ and also is a pure epimorphism (it is $(- \otimes_R R_{\mathfrak{m}})$ applied to a pure epimorphism). By Proposition 7.3.13, each $(Y_t)_{\mathfrak{m}}$ is in $\mathrm{Add}(K)_{\mathfrak{m}}$, thus also $Y_t \in \mathrm{Add}(K)$. Thus ρ is a split epimorphism as $X_t \in \mathcal{D}_{\mathcal{G}}$. □

Theorem 7.3.15. *Suppose $(\mathcal{A}, \mathcal{D}_{\mathcal{G}})$ is a 1-tilting cotorsion pair over a commutative ring R such that R is $\mathcal{G}$-almost perfect and* $\mathrm{p.dim}\, R_{\mathcal{G}} \leq 1$. *Then $R_{\mathcal{G}}$ is $\mathcal{G}$-divisible and $R_{\mathcal{G}} \oplus R_{\mathcal{G}}/R$ is Σ-pure-split, so $\mathcal{A}$ is closed under direct limits.*

Proof. Lemma 7.3.2 states that $R_{\mathcal{G}}$ is $\mathcal{G}$-divisible. That $R_{\mathcal{G}} \oplus R_{\mathcal{G}}/R$ is Σ-pure-split is a combination of Corollary 7.3.6 and Proposition 7.3.14. Finally by Proposition 1.5.2 we conclude that $\mathcal{A}$ is closed under direct limits. □

Finally combining the above theorem with the results in Section 7.3 and Section 7.2 we obtain the main result of this chapter.

Theorem 7.3.16. *Suppose $(\mathcal{A}, \mathcal{D}_{\mathcal{G}})$ is a 1-tilting cotorsion pair, $\mathcal{G}$ the associated Gabriel topology and $\mathfrak{R}$ the topological ring* $\mathrm{End}_R(K)$. *The following are equivalent.*

(i) $\mathcal{A}$ is covering.

(ii) $\mathcal{A}$ is closed under direct limits.

(iii) $\mathrm{p.dim}\, R_{\mathcal{G}} \leq 1$ *and R is $\mathcal{G}$-almost perfect.*

Moreover, if these equivalent conditions hold $R \to R_{\mathcal{G}}$ is a perfect localisation and $\mathfrak{R}$ is pro-perfect.

Proof. The implications (ii) ⇒ (i) is Theorem 1.2.12. (iii) ⇒ (ii) is using Theorem 7.3.15. Finally, (i) ⇒ (iii) is Theorem 7.2.5.

The final statement holds by Lemma 7.1.3 and that $\mathfrak{R}$ is a separated compete topological ring and $\mathfrak{R}/\mathfrak{R}J \cong R/J$ by Lemma 5.3.6 and Lemma 5.3.11. □

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