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BOUNDEDNESS AND COMPACTNESS
OF MATRIX OPERATORS
IN WEIGHTED SPACES OF SEQUENCES
AND THEIR APPLICATIONS

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To my family

Abstract

The present Thesis is dedicated to the investigation of necessary and sufficient conditions for which a weighted Hardy type inequality holds in weighted spaces of sequences and on the cone of non-negative monotone sequences, and their applications. We prove a new discrete Hardy type inequality involving a kernel which has a more general form than those known in the literature.

This Thesis consists of four chapters.

In Chapter 1, we shortly describe the development and current status of the theory of Hardy type inequalities. Moreover, Chapter 1 includes the statement and motivation of the problems and the main results. In Chapter 1, we also present some well-known auxiliary facts and necessary notation on Hardy type inequalities in weighted spaces of sequences and on the cone of non-negative monotone sequences.

In Chapter 2, we study the problems of boundedness and compactness of matrix operators in weighted spaces of sequences. We introduce a general class of matrices, and introduce their properties. Moreover, Chapter 2 contains examples of matrices from the introduced classes and here we show that such classes of matrices include well-known classical operators such as the operator of multiple summation, Hölder's operator, Cesàro operator and others. We establish necessary and sufficient conditions for the boundedness and compactness of the matrix operators in weighted spaces of sequences, where the corresponding matrices belong to such classes. Such classes of matrices are wider than those which have been previously studied in the theory of discrete Hardy type inequalities. Moreover, some related results are also proved.

In Chapter 3, we investigate a Hardy type inequality restricted to the cone of non-negative and non-increasing sequences under weaker conditions than those studied before in the literature. We obtain new results, which generalize the known results concerning this subject.

Chapter 4 is devoted to the application of the main results. Here we apply the main results of Chapter 2 in order to obtain criteria on boundedness and compactness of composition of matrix operators in weighted spaces of sequences. By using the results of Chapter 2 we obtain necessary and sufficient conditions for which three-weighted Hardy type inequalities hold. Moreover, in Chapter 4, by exploiting the main results of Chapter 2 and 3 we obtain two-sided estimates for summable matrices in weighted spaces of sequences and on the cone of non-negative and non-increasing sequences.

Sunto

Questa tesi è dedicata allo studio di condizioni necessarie e sufficienti per cui valga una disuguaglianza di tipo Hardy con peso in uno spazio pesato di successioni e nel cono delle successioni monotone non-negative, e alle corrispondenti applicazioni.

Proviamo una nuova disuguaglianza discreta di tipo Hardy con nucleo di forma più generale di quelli noti in letteratura.

La tesi consiste di quattro capitoli.

Nel Capitolo 1 descriviamo brevemente lo sviluppo e lo stato attuale della teoria delle disuguaglianze di tipo Hardy. Inoltre il Capitolo 1 contiene l'enunciato e la motivazione dei problemi e dei principali risultati. Nel Capitolo 1 presentiamo anche alcuni fatti ausiliari ben noti e la notazione necessaria per le disuguaglianze di tipo Hardy negli spazi pesati di successioni e nel cono delle successioni monotone non-negative.

Nel Capitolo 2 studiamo il problema della limitatezza e compattezza degli operatori matriciali negli spazi pesati di successioni. Introduciamo una classe generale di matrici e le loro proprietà. Inoltre il Capitolo 2 contiene esempi di matrici delle classi introdotte e qui mostriamo che tali classi di matrici contengono ben noti operatori classici come l'operatore di sommazione multipla, l'operatore di Hölder, l'operatore di Cesàro ed altri. Stabiliamo condizioni necessarie e sufficienti per la limitatezza e la compattezza di operatori matriciali in spazi pesati di successioni, nel caso in cui le corrispondenti matrici appartengano a tali classi. Tali classi di matrici sono più grandi di quelle che sono state studiate in precedenza nella teoria delle disuguaglianza discrete di

tipo Hardy. Inoltre, si dimostrano anche dei risultati ad esse relativi.

Nel Capitolo 3, studiamo una disuguaglianza di tipo Hardy ristretta al cono delle successioni non-negative e non crescenti in condizioni più deboli di quelle studiate prima nella letteratura. Otteniamo dei nuovi risultati che generalizzano i risultati noti su questo argomento.

Il Capitolo 4 è dedicato alle applicazioni dei risultati principali. Qui applichiamo i risultati principali del Capitolo 2 al fine di ottenere criteri di limitatezza e compattezza per la composizione di operatori matriciali in spazi pesati di successioni. Utilizzando i risultati del Capitolo 2 otteniamo condizioni necessarie e sufficienti affinchè valgano disuguaglianze di tipo Hardy a tre pesi. Inoltre nel Capitolo 4, sfruttando i risultati dei Capitoli 2 e 3 otteniamo stime bilatere per matrici sommabili in spazi pesati di successioni e nel cono delle successioni non negative e non crescenti.

Acknowledgements

Contents

Chapter 1

Introduction

1.1 Preliminaries.

One of the main problems in the theory of matrices is to find necessary and sufficient conditions for the elements of a matrix so that the corresponding matrix operator maps continuously one normed space of sequences into another normed space of sequences. Thus it is very important to find the norm of a matrix operator, or at least, an upper or lower bound for the norm. However, in several spaces, which are very important both theoretically and in the applications, such problems have not been solved yet in full generality for operators corresponding to arbitrary matrices. Therefore, in such spaces researchers have considered some specific classes of matrix operators and have established criteria of boundedness and compactness for operators of such classes.

For a summary of results on matrix operators acting in 11 spaces of sequences and on their norms, we refer to [1]. However, as pointed out in [1], general criteria for the action of a matrix operator from l_p to l_q with $p > 1$, $q > 1$ and for the corresponding norms are not known yet. Such operators have their own self interest and they are also a discrete analogue of integral operators, which play a very important role in functional analysis (see [2], [3]).

In the second half of last century researchers singled out a class of integral

operators, which is called the class of Hardy type operators, which is related to the work [4] of G.H. Hardy (1925). Hardy has established the boundedness of the operator H in $L_p(0, \infty)$ for $1 < p < \infty$ defined by

$$(Hf)(x) = \frac{1}{x} \int\limits_0^x f(s)ds \quad \forall f \in L_p(0, \infty),$$

and has proved that $\|H\|_{p \to p} = \frac{p}{p-1}$.

However, in several applications in function theory, harmonic analysis and differential equations, one needs to consider weighed forms of Hardy operators. Namely, one needs to consider non-negative weights $u(x)$ and $v(x)$ in Lebesgue spaces and operators K_0 of the form $(\mathcal{K}_0 f)(x) = u(x) \int\limits_0^x v(s)f(s)ds$. The problem was not easy. Only in 1969 the Italian mathematicians G. Talenti [5] and G. Tomaselli [6] have established, independently of each other, criteria of boundedness of the operator $\mathcal{K}_0$ in $L_p(0, \infty)$. During the next 11 years B. Muckenhoupt [7], J.S. Bradley [8], V.M. Kokilashvili [9], V.G. Maz'ya [10] have obtained criteria of boundedness for the operator $\mathcal{K}_0$ from $L_p(0, \infty)$ to $L_q(0, \infty)$ with $1 \leq p, q \leq \infty$. The initial results of G. Talenti, G. Tomaselli, B. Muckenhoupt gave a new impetus in the analysis of weighted embedding theorems and spectral problems for singular differential operators. Thus, for example, M. Otelbaev and his school have obtained important results in the 1970s concerning such topics (see e.g. [11], [12], [13]).

The next step was a study of the operator

$$(\mathcal{K}f)(x) = \int\limits_0^x K(x, s)f(s)ds$$

with non-negative kernel $K(\cdot, \cdot)$. Such type of operators are called Hardy type operators. However, even in the space $L_2(0, \infty)$ finding a criterion of boundedness for such general form operators in terms of the kernel $K(\cdot, \cdot)$ is very difficult and is still an unsolved problem. Therefore, many researchers have identified several classes of kernels, which satisfy some specific conditions and have proved boundedness criteria for the corresponding integral operators from $L_p(0, \infty)$ to $L_q(0, \infty)$, $1 < p, q < \infty$.

The first impulse in this direction was given by the works of F. Martin-Reyes and E. Sawyer [14], and V.D. Stepanov [15]-[18]. They have obtained criteria of boundedness of the Riemann-Liouville fractional integration operator from $L_{p,v}$ to $L_{q,u}$, $1 < p, q < \infty$, which has several applications in various fields of science. In [19] V.D. Stepanov has investigated the operator $\mathcal{K}$ with kernel $K(x,s) = k(x-s)$, where $k(\cdot)$ is not decreasing and for which there exists $d \geq 1$ such that $k(x+s) \leq d(k(x) + k(s))$, $x, s \in (0, \infty)$. In 1989-1990 R. Oinarov in [20] and independently the American mathematicians S. Bloom and R. Kerman in [21] in 1991 have studied the operator $\mathcal{K}$, when its kernel satisfies the following condition

$$\frac{1}{d}(K(x,t) + K(t,s)) \leq K(x,s) \leq d(K(x,t) + K(t,s)),$$

$x \geq t \geq s > 0$, $d \geq 1$. One of the important feature of this class of operators is that it includes almost all known operators of fractional integration. Nowadays, this condition imposed on the kernel $K(\cdot, \cdot)$ of the operator $\mathcal{K}$ is called the "Oinarov condition" in the mathematical literature. An operator $\mathcal{K}$ with Oinarov condition has been investigated by many authors (see e.g. [2], [22]). Necessary and sufficient conditions for the boundedness and compactness of operator $\mathcal{K}$ for a more general classes of kernels have been obtained by R. Oinarov in [23].

In the twenties of the last century G.H. Hardy considered the discrete analogue of the operator H in the form $(H^d f)_i = \frac{1}{i} \sum_{j=1}^{i} f_j$ and proved the boundedness of H^d in the space of sequences l_p and a formula for the norm $\|H^d\|_{p \to p} = \frac{p}{p-1}$, $1 < p < \infty$. As in the continuous case, this result of Hardy had various applications in many problems. The discrete analogue $(K_0^d f)_i = u_i \sum_{j=1}^{i} v_j f_j$ of the operator K_0 has been studied by many authors and the main final results have been obtained in [24]-[29] only in 1987-1994. Such delay of decades is related with the discrete changes of sequences $\{f_j\}$ and $\{(K_0^d f)_i\}$, which do not enable to transfer methods of the continuous case based on the continuity of the function $(K_0 f)(\cdot)$. The results which were obtained for the operator K_0^d have been successfully applied by mathematicians of different

countries. For example in Kazakhstan M. Otelbaev [13], E.S. Smailov [30]-[32], A. Stikharnyi [33], R. Oinarov, A. Stikharnyi [34] and other authors have proved several applications in various problems of analysis.

An attempt to investigate more general matrix operators of Hardy type $(Af)_i = u_i \sum_{j=1}^{i} a_{i,j} v_j f_j$, $a_{i,j} \geq 0$ has been done by K.F Andersen and H.P. Heinig [24], who have proved sufficient conditions for the boundedness of the operator A in the space l_p under some conditions on the matrix $(a_{i,j})$.

In recent years M.L. Goldman [35] has introduced the method of discretization for solving various problems in the embedding theory and in the theory of integral operators, where the estimate of matrix operators plays a main role.

Thus, not only the theory of matrix operators has an important significance, but also different and versatile applications.

Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and let $u = \{u_i\}_{i=1}^{\infty}$, $v = \{v_i\}_{i=1}^{\infty}$ be sequences of positive real numbers. Let $l_{p,v}$ be the space of sequences $f = \{f_i\}_{i=1}^{\infty}$ of real numbers with the following norm

$$\|f\|_{p,v} := \left(\sum_{i=1}^{\infty} |v_i f_i|^p \right)^{\frac{1}{p}}, \qquad 1 < p < \infty.$$

In Chapter 2, we consider the problems of boundedness and compactness from the weighted $l_{p,v}$ space into the weighted $l_{q,u}$ space of the matrix operators of Hardy type

$$\left(A^+ f \right)_i := \sum_{j=1}^{i} a_{i,j} f_j, \qquad i \geq 1, \tag{1.1}$$

$$\left(A^- f \right)_j := \sum_{i=j}^{\infty} a_{i,j} f_i, \qquad j \geq 1. \tag{1.2}$$

The boundedness of such operators is equivalent to the validity of the following Hardy type inequality

$$\|A^{\pm} f\|_{q,u} \leq C \|f\|_{p,v} \qquad \forall f \in l_{p,v}, \tag{1.3}$$

where C is a positive finite constant independent of f and $(a_{i,j})$ is a triangular matrix with entries $a_{i,j} \geq 0$ for $i \geq j \geq 1$ and $a_{i,j} = 0$ for $i < j$.

For $a_{i,j} = 1$, $i \geq j \geq 1$, the operators (1.1), (1.2) coincide with the discrete Hardy operators of the forms $\left(A_0^+ f\right)_i := \sum_{j=1}^{i} f_j$, $\left(A_0^- f\right)_j := \sum_{i=j}^{\infty} f_i$, respectively. References about generalizations of the original forms of the discrete and continuous Hardy inequalities can be found in different books, see e.g., [2, 3, 36].

R. Oinarov, S.Kh. Shalgynbaeva [37] and R. Oinarov, C.A. Okpoti, L-E. Persson [38] have proved criteria of boundedness and compactness for the operators A^+ and A^- from $l_{p,v}$ to $l_{q,u}$, when the entries of the matrix $(a_{i,j})$ satisfy a discrete analogue of the "Oinarov condition".

R. Oinarov, L-E. Persson, A.M. Temirkhanova [39] and A.M. Temirkhanova [40] have obtained necessary and sufficient conditions for the boundedness of operators A^+ and A^- from the weighted $l_{p,v}$ space into the weighted $l_{q,u}$ space for a slightly more general classes of matrix operators.

For more information, we refer to the PhD dissertations of A.M. Temirkhanova [41] and C.A. Okpoti [42].

Moreover, C.A. Okpoti in his PhD thesis [42, Chapter 4] has pointed out the following open questions.

Open question 1. Find necessary and sufficient conditions for (1.3) to hold for all non-negative sequences $\{f_i\}_{i=1}^{\infty}$ for as general kernels as possible.

Open question 2. Find necessary and sufficient conditions for (1.3) to hold on the cone of non-increasing sequences for as general kernels as possible.

The main part of the present PhD thesis is dedicated to the above mentioned open problems. In the present PhD thesis we consider a discrete Hardy type inequality involving a kernel which has a more general form than those known in the literature.

In Chapter 2 we have obtained the following new results.

– Firstly, we have introduced the classes of matrices $\mathcal{O}_n^+$ and $\mathcal{O}_n^-$ for $n \geq 0$, which cover much wider classes of matrix operators than those studied before. Such classes of matrices include well-known matrices of analysis such as summable matrices, and in particular, matrices satisfying a discrete analogue of the "Oinarov condition" and the condition considered in [39], [40].

Moreover, we have introduced properties of such classes, which allowed us to to consider the problem of boundedness and compactness of composition of matrix operators.

– We have shown that the classes of matrices $\mathcal{O}_n^+$ and $\mathcal{O}_n^-$ for $n \geq 0$ include well-known classical operators such as the operator of multiple summation, Hölder's operator, Cesàro operator and others.

– We have obtained necessary and sufficient condition for the boundedness and compactness of the matrix operators A^+ and A^- from the weighted $l_{p,v}$ space into the weighted $l_{q,u}$ space when the corresponding matrices $(a_{i,j})$ belong to the class $\mathcal{O}_n^+ \cup \mathcal{O}_n^-$, $n \geq 0$ for $1 < p \leq q < \infty$.

– We have obtained boundedness criteria of the matrix operators A^+ and A^- from the weighted $l_{p,v}$ space into the weighted $l_{q,u}$ space when the corresponding matrices $(a_{i,j})$ belong to the class $\mathcal{O}_1^-$ for $1 < q < p < \infty$.

Chapter 3 is devoted to the second open problem, which is pointed out in [42]. Actually, we consider an inequality of the following form

$$\left(\sum_{i=1}^{\infty} u_i^q \left(\sum_{j=1}^{i} a_{i,j} f_j \right)^q \right)^{\frac{1}{q}} \leq C \left(\sum_{i=1}^{\infty} v_i^p f_i^p \right)^{\frac{1}{p}} \tag{1.4}$$

on the cone of non-negative and non-increasing sequences $f = \{f_i\}_{i=1}^{\infty}$ of $l_{p,v}$, where C is a positive constant independent of f and $(a_{i,j})$ is a triangular matrix with entries $a_{i,j} \geq 0$ for $i \geq j \geq 1$ and $a_{i,j} = 0$ for $i < j$.

If $a_{i,j} = \frac{1}{i}$ for $i \geq j \geq 1$ and $a_{i,j} = 0$ for $i < j$, then we obtain a discrete Hardy inequality of the form

$$\left(\sum_{i=1}^{\infty} u_i^q \left(\frac{1}{i} \sum_{j=1}^{i} f_j \right)^q \right)^{\frac{1}{q}} \leq C \left(\sum_{i=1}^{\infty} v_i^p f_i^p \right)^{\frac{1}{p}} \tag{1.5}$$

for all non-negative and non-increasing sequences $f \in l_{p,v}$.

The Hardy type inequalities restricted to the cones of monotone functions and sequences have been actively studied in the last two decades. This problem has some applications in the investigation of boundedness of operators in Lorentz spaces and in the embedding theory in Lorentz spaces. For a history of Hardy type inequalities on the cones of monotone functions and sequences

and for references to related results we refer to the book of A. Kufner, L. Maligranda and L.-E. Persson [3, Chapter 10], and to the PhD thesis of O. Popova [43].

Note that the impulse to study Hardy inequality on the cone of monotone functions was given by the work of E. Sawyer [44], which allows to reduce an inequality restricted to the cone of monotone functions to a corresponding inequality for some positive linear operators on the cone of positive functions. Now this result of E. Sawyer is known as the Sawyer duality principle. Some generalizations of Sawyer duality principle and a corresponding result for the discrete case were studied by many authors under some conditions on weights (see [3, Chapter 10]).

In 1998 R. Oinarov and S.Kh. Shalgynbaeva [45] obtained an analogue of the Sawyer duality principle for the discrete case for $1 < p, q < \infty$. Moreover, S.Kh. Shalgynbaeva in her PhD thesis [46] obtained criteria for the validity of inequality (1.5) for some other values of the parameters p and q. Also inequality (1.4) was studied in [46] under some conditions on the entries of the matrix $(a_{i,j})$.

In 2006 G. Bennett and K.-G. Grosse-Erdmann [47] obtained a complete characterization of the weights for which the Hardy inequality (1.5) holds on the cone of monotone sequences of different nature of the conditions of S.Kh. Shalgynbaeva.

In [48] S.Kh. Shalgynbaeva has obtained necessary and sufficient conditions for the validity of (1.4) on the cone of monotone sequences for $1 < p \leq q < \infty$, when the entries of the matrix $(a_{i,j})$ satisfy a discrete analogue of the "Oinarov condition".

Chapter 3 contains the following new results.

– Necessary and sufficient conditions for the validity of inequality (1.4) on the cone of non-negative and non-increasing sequences $f \geq 0$ when the corresponding matrix $(a_{i,j})$ belongs to the class $\mathcal{O}_n^+ \cup \mathcal{O}_n^-$, $n \geq 0$ for $1 < p \leq q < \infty$.

– Two-sided estimates for matrix operators on the cone of non-negative and non-increasing sequences $f \geq 0$ for the case $1 < q < p < \infty$, when the corresponding matrices $(a_{i,j})$ belong to the classes $\mathcal{O}_1^+$ and $\mathcal{O}_1^-$.

Chapter 4 is devoted to the applications of the main results. Here we apply the results of Chapter 2 and 3 for composition of matrix operators, for three-weighted inequalities of Hardy type, and for summable matrices.

In Chapter 4 we have obtained the following new results.

–We have proved both boundedness and compactness results for composition of matrix operators in weighted spaces of sequences when the corresponding matrices $(a_{i,j})$ belong to the classes $\mathcal{O}_n^+ \cup \mathcal{O}_n^-$, $n \geq 0$ for the case $1 < p \leq q < \infty$.

–Necessary and sufficient conditions for which three-weighted inequalities of Hardy type hold when the corresponding matrices belong to the class $\mathcal{O}_n^-$, $n \geq 0$ in the case $1 < p \leq q < \infty$.

–Necessary and sufficient conditions for the validity of three-weighted inequalities of Hardy type when the corresponding matrices belong to the class $\mathcal{O}_1^-$ in the case $1 < q < p < \infty$.

–Two-sided estimates for summable matrices in weighted spaces of sequences and on the cone of non-negative and non-increasing sequences.

1.2   The history and development of weighted Hardy type inequalities.

Since the present Thesis deals with some generalizations of weighted Hardy type inequalities, in this section we focus our interest on the history and references results on the weighted Hardy type inequalities.

The theory of Hardy type inequalities is a wonderful subject with a proud history and a great future. The study of Hardy type inequalities in weighted Lebesgue spaces began in 1915 with the work of G.H. Hardy. In 1925 G.H. Hardy proved the following result [4]:

Let $p > 1$ and $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers, such that the series $\sum\limits_{n=1}^{\infty} a_n^p$ converges. Then the well-known *discrete Hardy inequality*

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} a_k \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p \qquad (1.6)$$

holds.

The continuous Hardy inequality reads: if $p > 1$ and f is a non-negative p-integrable function on $(0, \infty)$, then f is integrable over the interval $(0, x)$ for all $x > 0$ and

$$\int_0^{\infty} \left(\frac{1}{x} \int_0^x f(t)dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^{\infty} f^p(x)dx. \qquad (1.7)$$

It should be noted that the constant $\left(\dfrac{p}{p-1} \right)^p$ in both inequalities (1.6) and (1.7) is sharp in the sense that it can not be replaced by any smaller number.

The inequalities (1.6) and (1.7) imply the following information, respectively.

$$\text{If} \quad \sum_{n=1}^{\infty} a_n^p < \infty, \quad \text{then} \quad \sum_{n=1}^{\infty} h_n^p(a) < \infty, \qquad (1.8)$$

where $a = \{a_n\}$ with $a_n \geq 0$ and $h(a) = \{h_n(a)\}$, $h_n(a) := \frac{1}{n} \sum\limits_{k=1}^{n} a_k < \infty$ is

the *discrete Hardy operator*. Similarly, in the continuous case, we have the following.

$$\text{If} \quad \int_0^\infty f^p(x)dx < \infty, \quad \text{then} \quad \int_0^\infty (Hf(x))^p\, dx < \infty, \qquad (1.9)$$

where $f(x) \geq 0$ and $Hf(x) := \frac{1}{x}\int_0^x f(t)dt$ is the *continuous Hardy operator*.

Note that (1.8) and (1.9) are called the weak forms of (1.6) and (1.7), respectively.

The inequalities (1.6) and (1.7) imply that the Hardy operators h and H map the spaces l_p into l_p and L_p into L_p ($p > 1$), respectively, and that their norms are equal to $p' = \frac{p}{p-1}$. Here, as usual, the spaces l_p and L_p are the Lebesgue spaces of all sequences $a = \{a_n\}_{n=1}^\infty$ of real numbers and all (equivalence classes modulo equality almost everywhere of) measurable functions $f = f(x)$ on $(0, \infty)$, respectively, with the following norms

$$\|a\|_{l_p} := \left(\sum_{n=1}^\infty |a_n|^p\right)^{\frac{1}{p}} \quad \text{and} \quad \|f\|_{L_p} := \left(\int_0^\infty |f(x)|^p\right)^{\frac{1}{p}}.$$

Hardy's original aim was to find a new, more elementary proof of Hilbert's double series theorem and he showed that in fact it follows from (1.6). Simple proofs of inequality (1.6) and its generalizations were obtained by Hardy (1925), Elloit (1926,1929), Copson (1927, 1928), Kaluza-Szegö (1927), Hadry-Littlewood (1927), Broadbent (1928), Grandjot (1928), Knopp (1928, 1929, 1930) and Mulholland (1932).

Many books and articles have been devoted to the investigation and generalization of Hardy inequalities ever since. The first book on the Hardy inequality was the book of G.H. Hardy, J.E. Littlewood and G. Pólya *Inequalities* [49] of 1934. The first book has been fully devoted to the Hardy inequality, and has been published in 1990 by B. Opic, A. Kufner [50]. We also mention here the book of A. Kufner and L.-E. Persson *Weighted Inequalities of Hardy Type* [2], which is devoted to basic overview of the subject of weighted Hardy type

inequalities in weighted Lebesgue spaces. A description of the most important steps in the development of the Hardy inequalities has been published by A. Kufner, L. Maligranda and L.-E. Persson [3].

In 1928 G.H. Hardy [51] proved the first "weighted" modification of inequality (1.7) for some integral operators, namely he proved the inequality

$$\int\limits_{o}^{\infty} \left(\frac{1}{x} \int\limits_{0}^{x} f(t)dt \right)^p x^{\varepsilon}dx \leq \left(\frac{p}{p-\varepsilon-1} \right)^p \int\limits_{0}^{\infty} f^p(x)x^{\varepsilon}dx. \qquad (1.10)$$

for all measurable non-negative functions f and for $p > 1$, $\varepsilon < p-1$. Here the constant $\left(\dfrac{p}{p-\varepsilon-1} \right)^p$ is the best possible.

Some generalizations of the Hardy inequalities (1.6) and (1.7) have been studied by Higaki (1935), Takahashi (1935), Chow (1939), Beesack (1961), Petersen (1962), Levinson (1964) and others.

During the last decades inequalities (1.6) and (1.7) have been developed in the following forms

$$\left(\sum_{n=1}^{\infty} u_n \left| \sum_{k=1}^{n} a_k \right|^q \right)^{\frac{1}{q}} \leq C \left(\sum_{n=1}^{\infty} |a_n|^p v_n \right)^{\frac{1}{p}}, \qquad (1.11)$$

$$\left(\int\limits_{a}^{b} \left| \int\limits_{a}^{x} f(t)dt \right|^q u(x)dx \right)^{\frac{1}{q}} \leq C \left(\int\limits_{a}^{b} |f(x)|^p v(x)dx \right)^{\frac{1}{p}}, \qquad (1.12)$$

respectively, which are called weighted Hardy inequalities.

In 1930 G.H. Hardy and J.E. Littlewood [52], and G.A. Bliss [53] studied inequality (1.12) with different parameters p and q in the case $1 < p < q < \infty$. They considered the interval $(0, \infty)$ and the weight functions $v(x) \equiv 1$, $u(x) = x^{r-q}$ with $r = \frac{q-p}{p}$ and derived inequality (1.12). Moreover, G.A. Bliss found the best constant in this case.

The systematic investigation of (1.12) started in the fifties-sixties for the case $p = q$ in the papers of P.R. Beesack [54], [55], J. Kadlec and A. Kufner [56], V.R. Portnov [57], V.N. Sedov [58], F.A. Sysoeva [59], G. Talenti [5], G. Tomaselli [6]. Note that G. Talenti and G. Tomaselli obtained the following

necessary and sufficient condition for the validity of inequality (1.12) in the case $p = q$

$$\sup_{t>0} \left(\int_t^\infty u(x)dx \right)^{\frac{1}{p}} \left(\int_0^t v^{1-p'}(y)dy \right)^{\frac{1}{p'}} < \infty, \quad p' = \frac{p}{p-1}.$$

Nowadays this condition is called the Muckenhoupt condition in honour of B. Muckenhoupt, who presented a very nice proof of this result in [7] even in a more general form with $1 \le p = q < \infty$ and for some Borel measures $d\mu(x)$, $d\nu(y)$ instead of $u(x)dx$ and $v(y)dy$.

The study of the case of different parameters p and q has been started by J.S. Bradley in [8]. He consider inequality (1.12) with $(a,\ b) = (0,\ \infty)$ and proved that the condition

$$\sup_{t>0} \left(\int_t^\infty u(x)dx \right)^{\frac{1}{q}} \left(\int_0^t v^{1-p'}(y)dy \right)^{\frac{1}{p'}} < \infty$$

is necessary for (1.12) to hold for all $1 \le p, q \le \infty$ and that it is also sufficient for $1 \le p \le q < \infty$.

From the 60's of the last century weighted Hardy inequalities have been intensively studied by many authors. Let us mention here some of them: V.M. Kokilashvili [9], V.G. Maz'ya [10], K.F. Andersen and B. Muckenhoupt [60], L.-E. Persson and V.D. Stepanov [61], R.K. Juberg [62], V.D. Stepanov [63], G. Bennett [28], V.M. Manakov [64], V.G. Maz'ya and A.L. Rozin (see [10]), G. Sinnamon [65], [66], G. Sinnamon and V.D. Stepanov [67] and others. Moreover, for more information we refer to the books [49], [50], [2], [68], [69] and to the PhD dissertations of M. Nassyrova [70], D.V. Prokhorov [71], and A. Wedestig [72].

Almost all references concern the continuous case (1.12), and surprisingly little has been done for the discrete case.

The first result related to inequality (1.11) belongs to K.F. Andersen and H.P. Heinig ([24], Theorem 4.1). In 1983 they proved that if $1 \le p \le q < \infty$

and

$$\sup_{n \in N} \left(\sum_{k=n}^{\infty} u_k \right)^{\frac{1}{q}} \left(\sum_{k=1}^{n} v_k^{1-p'} \right)^{\frac{1}{p'}} < \infty,$$

then the inequality (1.11) holds.

In 1985 H.P. Heinig [25] obtained a sufficient condition for the validity of inequality (1.11). Namely, he proved that if $1 \le q < p < \infty$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ and

$$B := \left(\sum_{n=-\infty}^{\infty} \left(\sum_{k=n}^{\infty} u_k \right)^{\frac{r}{q}} \left(\sum_{k=-\infty}^{n} v_k^{1-p'} \right)^{\frac{r}{q'}} v_n^{1-p'} \right)^{\frac{1}{r}} < \infty,$$

then inequality (1.11) holds with $C \le q^{\frac{1}{q}} (p')^{\frac{1}{q'}} B$.

In 1987-1991 G. Bennett [26], [27], [28] gave a full characterization of the weighted inequality (1.11), except for the case $0 < q < 1 < p < \infty$. The remaining case was obtained by M.S. Braverman and V.D. Stepanov [29] in 1992.

Now we state the following important result mainly taken from the Bennett's paper [28] (see also [3]):

Theorem 1.1. *(i) If $1 < p \le q < \infty$, then inequality (1.11) holds if and only if either*

$$A_1 := \sup_{n \in N} \left(\sum_{k=n}^{\infty} u_k \right)^{\frac{1}{q}} \left(\sum_{k=1}^{n} v_k^{1-p'} \right)^{\frac{1}{p'}} < \infty$$

or

$$A_2 := \sup_{n \in N} \left(\sum_{k=1}^{n} v_k^{1-p'} \right)^{-\frac{1}{p}} \left(\sum_{k=1}^{n} u_k \left(\sum_{m=1}^{k} v_m^{1-p'} \right)^q \right)^{\frac{1}{q}} < \infty$$

or

$$A_3 := \sup_{n \in N} \left(\sum_{k=n}^{\infty} u_k \right)^{-\frac{1}{q'}} \left(\sum_{k=n}^{\infty} v_k^{1-p'} \left(\sum_{m=k}^{\infty} u_m \right)^{p'} \right)^{\frac{1}{p'}} < \infty.$$

(ii) If $0 < p \le 1$, $p \le q < \infty$, then inequality (1.11) holds if and only if

$$A_4 := \sup_{n \in N} \left(\sum_{k=n}^{\infty} u_k \right)^{\frac{1}{q}} v_n^{-\frac{1}{p}} < \infty.$$

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(iii) If $1 < p < \infty$, $0 < q < p$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, then inequality (1.11) holds if and only if

$$A_5 := \sum_{n=1}^{\infty} \left(u_n \left(\sum_{k=n}^{\infty} u_k \right)^{\frac{r}{p}} \left(\sum_{k=1}^{n} v_k^{1-p'} \right)^{\frac{r}{p'}} \right) < \infty.$$

(iv) If $q < p = 1$, then inequality (1.11) holds if and only if

$$A_6 := \sum_{n=1}^{\infty} \left(u_n \left(\sum_{k=n}^{\infty} u_k \right)^{\frac{q}{1-q}} \max_{1 \leq k \leq n} v_k^{\frac{q}{q-1}} \right) < \infty.$$

(v) If $0 < q < 1 < p < \infty$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, then inequality (1.11) holds if and only if

$$A_7 := \left(\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} u_k \right)^{\frac{r}{q}} \left(\sum_{k=1}^{n} v_k^{1-p'} \right)^{\frac{r}{q'}} v_n^{1-p'} \right)^{\frac{1}{r}} < \infty.$$

C.A. Okpoti [42] in his PhD thesis has proved that for the case $1 < p \leq q < \infty$ there are infinite many conditions characterizing (1.11).

More general Hardy type operators $(\mathcal{K}f)(x) = \int_0^x K(x,s)f(s)ds$ with non-negative kernel $K(\cdot,\cdot)$ have been studied by many authors including F. Martin-Reyes and E. Sawyer [14], V.D. Stepanov [15]-[19], R. Oinarov [20], [23], S. Bloom and R. Kerman [21], L.-E. Persson, V.D. Stepanov [61], D.V. Prokhorov [71].

It is now natural to study the following general discrete Hardy type operators

$$\left(A^+ f \right)_i := \sum_{j=1}^{i} a_{i,j} f_j, \qquad i \geq 1, \tag{1.13}$$

$$\left(A^- f \right)_j := \sum_{i=j}^{\infty} a_{i,j} f_i, \qquad j \geq 1, \tag{1.14}$$

where $(a_{i,j})$ is a triangular matrix with entries $a_{i,j} \geq 0$ for $i \geq j \geq 1$ and $a_{i,j} = 0$ for $i < j$.

The boundedness of such operators from the weighted $l_{p,v}$ space into the weighted $l_{q,u}$ space is equivalent to finding necessary and sufficient conditions

on the weight sequences $\{u_i\}_{i=1}^{\infty}$ and $\{v_i\}_{i=1}^{\infty}$ such that the following general discrete Hardy type inequality

$$\left(\sum_{i=1}^{\infty} u_i^q \left|(A^{\pm}f)_i\right|^q\right)^{\frac{1}{q}} \leq C \left(\sum_{i=1}^{\infty} v_i^p |f_i|^p\right)^{\frac{1}{p}} \tag{1.15}$$

holds for all $f \in l_{p,v}$.

When one of the parameters p or q is equal to 1 or ∞, necessary and sufficient conditions for the validity of (1.15) with the exact value of the best constant $C > 0$ have been obtained in [1]. In case $1 < p, q < \infty$ inequalities as (1.15) have not been established yet for arbitrary matrices $(a_{i,j})$. Instead inequality (1.15) has been established with certain restrictions on the matrix $(a_{i,j})$.

The first result in this direction has been obtained by K.F. Andersen and H.P. Heinig [24], who proved a sufficient condition for general discrete Hardy type inequality (1.15) to hold for the case $1 \leq p \leq q < \infty$ with special non-negative kernels $(a_{i,j})$ that was assumed to be non-increasing in i and non-decreasing in j.

In [73] C.A. Okpoti, L-E. Persson, A. Wedestig have studied inequality (1.15) for the case $a_{i,j} = \alpha_i \beta_j$, $i \geq j \geq 1$, where $\{\alpha_i\}_{i=1}^{\infty}$ and $\{\beta_j\}_{j=1}^{\infty}$ are positive sequences. Moreover, in [74] they have obtained a sufficient condition for which inequality (1.15) holds for a general kernel.

Now we introduce the following notation.

Notation. If M and K are real valued functionals of sequences, then we understand that the symbol $M \ll K$ means that there exists $c > 0$ such that $M \leq cK$, where c is a constant which may depend only on parameters such as p, q, r_n and h_n. If $M \ll K \ll M$, then we write $M \approx K$.

R. Oinarov, S.Kh. Shalgynbaeva [37] and R. Oinarov, C.A.Okpoti, L-E. Persson [38] have obtained necessary and sufficient conditions for the validity of (1.15) for $1 < p, q < \infty$ under the assumption that there exists $d \geq 1$ such that

$$a_{i,j} \approx \frac{a_{i,k}}{c_k} c_j + \frac{a_{k,j}}{b_k} b_i, \quad i \geq k \geq j \geq 1 \tag{1.16}$$

holds, where $c = \{c_i\}_{i=1}^{\infty}$, $b = \{b_i\}_{i=1}^{\infty}$ are sequences of positive numbers.

However, inequality (1.15) is equivalent to the following inequality

$$\|\widetilde{A}^{\pm} f\|_{q,\widetilde{u}} \le C\|f\|_{p,\widetilde{v}},$$

where $\widetilde{u}_i = u_i b_i^q$, $\widetilde{v}_i = v_i c_i^{-p}$, $i \ge 1$ and the entries $\widetilde{a}_{i,j} = \frac{a_{i,j}}{b_i c_j}$ of matrix $(\widetilde{a}_{i,j})$ of operator $\widetilde{A}$ satisfy the following condition

$$\widetilde{a}_{i,j} \approx \widetilde{a}_{i,k} + \widetilde{a}_{k,j} \quad i \ge k \ge j \ge 1,$$

which is equivalent to the condition (1.16).

We now list some of the most relevant results of R. Oinarov, S.Kh. Shalgynbaeva [37] and R. Oinarov, C.A.Okpoti, L-E. Persson [38].

Theorem 1.2. *[37] Let $1 < p \le q < \infty$. Let the entries of the matrix $(a_{i,j})$ satisfy condition (1.16). Then inequality (1.15) for the operator (1.13) holds if and only if $M = \max\{M_1, M_2\} < \infty$, where*

$$M_1 = \sup_{k \ge 1} \frac{1}{c_k} \left(\sum_{j=1}^{k} c_j^{p'} v_j^{-p'} \right)^{\frac{1}{p'}} \left(\sum_{i=k}^{\infty} a_{i,k}^{q} u_i^{q} \right)^{\frac{1}{q}},$$

$$M_2 = \sup_{k \ge 1} \frac{1}{b_k} \left(\sum_{j=1}^{k} a_{k,j}^{p'} v_j^{-p'} \right)^{\frac{1}{p'}} \left(\sum_{i=k}^{\infty} b_i^{q} u_i^{q} \right)^{\frac{1}{q}}.$$

Moreover, $M \approx C$, where C is the best constant in (1.15).

Theorem 1.3. *[37] Let $1 < p \le q < \infty$. Let the entries of the matrix $(a_{i,j})$ satisfy condition (1.16). Then inequality (1.15) for the operator (1.14) holds if and only if $M^* = \max\{M_1^*, M_2^*\} < \infty$, where*

$$M_1^* = \sup_{k \ge 1} \frac{1}{c_k} \left(\sum_{j=1}^{k} c_j^{q} u_j^{q} \right)^{\frac{1}{q}} \left(\sum_{i=k}^{\infty} a_{i,k}^{p'} v_i^{-p'} \right)^{\frac{1}{p'}},$$

$$M_2^* = \sup_{k \ge 1} \frac{1}{b_k} \left(\sum_{j=1}^{k} a_{k,j}^{q} u_j^{q} \right)^{\frac{1}{q}} \left(\sum_{i=k}^{\infty} b_i^{-p'} v_i^{-p'} \right)^{\frac{1}{p'}}.$$

Moreover, $M^* \approx C$, where C is the best constant in (1.15).

Theorem 1.4. *[38] Let $1 < q < p < \infty$. Let the entries of the matrix $(a_{i,j})$ satisfy condition (1.16). Then inequality (1.15) for the operator (1.13) holds if and only if $B := \max\{B_1, B_2\} < \infty$, where*

$$B_1 := \left(\sum_{k=1}^{\infty} v_k^{-p'} \left(\sum_{i=k}^{\infty} a_{i,k}^q u_i^q \right)^{\frac{p}{p-q}} \left(\frac{1}{c_k^{p'}} \sum_{i=1}^{k} c_i^{p'} v_i^{-p'} \right)^{\frac{p(q-1)}{p-q}} \right)^{\frac{p-q}{pq}},$$

$$B_2 := \left(\sum_{k=1}^{\infty} u_k^q \left(\sum_{i=1}^{k} a_{i,k}^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} \left(\frac{1}{b_k^q} \sum_{i=k}^{\infty} b_i^q u_i^q \right)^{\frac{q}{p-q}} \right)^{\frac{p-q}{pq}}.$$

Moreover, $B \approx C$, where C is the best constant in (1.15).

Theorem 1.5. *[38] Let $1 < q < p < \infty$. Let the entries of the matrix $(a_{i,j})$ satisfy condition (1.16). Then inequality (1.15) for the operator (1.14) holds if and only if $B^* := \max\{B_1^*, B_2^*\} < \infty$, where*

$$B_1^* := \left(\sum_{k=1}^{\infty} u_k^q \left(\sum_{i=k}^{\infty} a_{i,k}^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} \left(\frac{1}{c_k^q} \sum_{i=1}^{k} c_i^q u_i^q \right)^{\frac{q}{p-q}} \right)^{\frac{p-q}{pq}},$$

$$B_2^* := \left(\sum_{k=1}^{\infty} v_k^{-p'} \left(\sum_{i=1}^{k} a_{i,k}^q u_i^q \right)^{\frac{p}{p-q}} \left(\frac{1}{b_k^{p'}} \sum_{i=k}^{\infty} b_i^{p'} v_i^{-p'} \right)^{\frac{p(q-1)}{p-q}} \right)^{\frac{p-q}{pq}}.$$

Moreover, $B^* \approx C$, where C is the best constant in (1.15).

Now we introduce the following definition.

Definition 1.6. *A sequence $\{\alpha_i\}_{i=1}^{\infty}$ is called almost non-decreasing (non-increasing), if there exists $c > 0$ such that $c\alpha_i \geq \alpha_k$ ($\alpha_k \leq c\alpha_j$) for all $i \geq k \geq j \geq 1$.*

R. Oinarov, L-E. Persson, A.M. Temirkhanova [39] and A.M. Temirkhanova [40] have studied estimate (1.15) for $1 < p, q < \infty$ under the assumption that there exist $d \geq 1$ and a sequence of positive numbers $\{\omega_k\}_{k=1}^{\infty}$, and a non-negative matrix $(b_{i,j})$, where $b_{i,j}$ is almost non-decreasing in i and almost non-increasing in j, such that the inequalities

$$\frac{1}{d}(b_{i,k}\omega_j + a_{k,j}) \leq a_{i,j} \leq d(b_{i,k}\omega_j + a_{k,j}) \tag{1.17}$$

hold for all $i \geq k \geq j \geq 1$.

We now state the main results of [39] and [40].

Theorem 1.7. *[39] Let $1 < p \leq q < \infty$. Let the entries of the matrix $(a_{i,j})$ of the operator (1.13) satisfy assumption (1.17). Then inequality (1.15) for the operator (1.13) holds if and only if $\mathbb{G} = \max\{\mathbb{G}_1, \ \mathbb{G}_2\} < \infty$, where*

$$\mathbb{G}_1 = \sup_{n \geq 1} \left(\sum_{i=n}^{\infty} b_{i,n}^q u_i^q \right)^{\frac{1}{q}} \left(\sum_{j=1}^{n} \omega_j^{p'} v_j^{-p'} \right)^{\frac{1}{p'}}, \ \mathbb{G}_2 = \sup_{n \geq 1} \left(\sum_{i=n}^{\infty} u_i^q \right)^{\frac{1}{q}} \left(\sum_{j=1}^{n} a_{n,j}^{p'} v_j^{-p'} \right)^{\frac{1}{p'}}.$$

Moreover, $\mathbb{G} \approx C$, where C is the best constant in (1.15).

Theorem 1.8. *[39] Let $1 < p \leq q < \infty$. Let the entries of the matrix $(a_{i,j})$ of the operator (1.14) satisfy assumption (1.17). Then inequality (1.15) for the operator (1.14) holds if and only if $\mathbb{G}^* = \max\{\mathbb{G}_1^*, \mathbb{G}_2^*\} < \infty$, where*

$$\mathbb{G}_1^* = \sup_{k \geq 1} \left(\sum_{i=k}^{\infty} b_{i,k}^{p'} v_i^{-p'} \right)^{\frac{1}{p'}} \left(\sum_{j=1}^{k} \omega_j^q u_j^q \right)^{\frac{1}{q}}, \ \mathbb{G}_2^* = \sup_{k \geq 1} \left(\sum_{i=k}^{\infty} v_i^{-p'} \right)^{\frac{1}{p'}} \left(\sum_{j=1}^{k} a_{k,j}^q u_j^q \right)^{\frac{1}{q}}.$$

Moreover, $\mathbb{G}^* \approx C$, where C is the best constant in (1.15).

Theorem 1.9. *[40] Let $1 < q < p < \infty$. Let the entries of the matrix $(a_{i,j})$ of the operator (1.13) satisfy assumption (1.17). Then inequality (1.15) for the operator (1.13) holds if and only if $\mathcal{G} = \max\{\mathcal{G}_1, \ \mathcal{G}_2\} < \infty$, where*

$$\mathcal{G}_1 = \left(\sum_{k=1}^{\infty} \left(\sum_{j=k}^{\infty} b_{j,k}^q u_j^q \right)^{\frac{p}{p-q}} \left(\sum_{i=1}^{k} \omega_i^{p'} v_i^{-p'} \right)^{\frac{p(q-1)}{p-q}} \omega_k^{p'} v_k^{-p'} \right)^{\frac{p-q}{pq}},$$

$$\mathcal{G}_2 = \left(\sum_{k=1}^{\infty} \left(\sum_{j=k}^{\infty} u_j^q \right)^{\frac{q}{p-q}} \left(\sum_{i=1}^{k} a_{k,i}^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} u_k^q \right)^{\frac{p-q}{pq}}.$$

Moreover, $\mathcal{G} \approx C$, where C is the best constant in (1.15).

Theorem 1.10. *[40] Let $1 < q < p < \infty$. Let the entries of the matrix $(a_{i,j})$ of the operator (1.14) satisfy assumption (1.17). Then inequality (1.15) for the operator (1.14) holds if and only if $\mathcal{G}^* = \max\{\mathcal{G}_1^*, \ \mathcal{G}_2^*\} < \infty$, where*

$$\mathcal{G}_1^* = \left(\sum_{k=1}^{\infty} \left(\sum_{j=k}^{\infty} b_{j,k}^{p'} v_j^{-p'} \right)^{\frac{q(p-1)}{p-q}} \left(\sum_{i=1}^{k} \omega_i^q u_i^q \right)^{\frac{q}{p-q}} \omega_k^q u_k^q \right)^{\frac{p-q}{pq}},$$

$$\mathcal{G}_2^* = \left(\sum_{k=1}^{\infty} \left(\sum_{j=k}^{\infty} v_j^{-p'} \right)^{\frac{p(q-1)}{p-q}} \left(\sum_{i=1}^{k} a_{k,i}^q u_i^q \right)^{\frac{p}{p-q}} v_k^{-p'} \right)^{\frac{p-q}{pq}}.$$

Moreover, $\mathcal{G}^* \approx C$, where C is the best constant in (1.15).

1.3 Weighted Hardy type inequalities on the cones of monotone functions and sequences.

The properties of the cone of monotone sequences of real numbers, of the cone of monotone functions, and several related extremum problems have an important significance in functional analysis, in the problems of the mathematical economics, of the theory of probability and statistics. The approximation numbers of operators, quantitative characteristics of the best approximations of functions, moment sequences of function are monotone sequences of numbers, which carry certain information. Many qualitative properties of this type can be expressed by functional relations of monotone sequences.

It is known that the properties of a class of functions or of a class of sequences of numbers can be obtained from the functional relations of their non-increasing rearrangements, which are monotone functions and monotone sequences, respectively. Therefore, the problem of establishing the various functional relationships on the cone of monotone sequences of numbers is an actual direction of mathematical analysis.

Hardy type inequalities on the cone of monotone functions and sequences have some applications in the investigation of boundedness of operators in Lorentz spaces and in the embedding theory in Lorentz spaces. In 1951 G. Lorentz [75] first introduced the Lorentz spaces $\Lambda^p(u)$, $0 < p < \infty$.

$$\Lambda^p(u) := \left\{ f: \ \|f^*\|_{p,u} = \left(\int\limits_0^\infty (f^*(t))^p \, u(t) dt \right)^{\frac{1}{p}} < \infty \right\}.$$

Here f^* is the equimeasurable decreasing rearrangement of $|f|$ defined by

$$f^*(t) := \inf\{y > 0: \ \lambda_f(y) \le t\},$$

where λ_f is the distribution function:

$$\lambda_f(y) := meas\{x \in X: \ |f(x)| > y\}.$$

A weight characterization of classical operators in Lorentz spaces led to the necessity to study operators defined on the cone of decreasing functions.

Now we consider the Hardy-Littlewood maximal function Mf, which is defined by the following formula

$$(Mf)(x) := \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(z)|dz, \quad x \in \mathbb{R}^n,$$

where Q ia a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes and $|Q|$ is its n-dimensional Lebesgue measure. It is known that

$$(Mf)^*(t) \approx \frac{1}{t} \int_0^t f^*(s)ds, \quad t > 0$$

(for the proofs and historical notes concerning this estimate, see e.g. [76], [77]). Thus, the characterization of weight functions u and v, for which the mapping

$$M: \quad \Lambda^p(v) \to \Lambda^q(u), \quad 1 < p, q < \infty$$

is bounded between Lorentz spaces, is equivalent to the characterization of weight functions u and v, for which the following Hardy inequality

$$\left(\int_0^\infty \left(\frac{1}{t} \int_0^t f(s)ds \right)^q u(t)dt \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty f^p(t)v(t)dt \right)^{\frac{1}{p}} \tag{1.18}$$

holds for all decreasing functions $f \geq 0$.

Hardy type inequalities on the cone of monotone functions and sequences have been intensively studied during the last two decades. In 1990 M. Ariño and B. Muckenhoupt [78] obtained a necessary and sufficient condition for the validity of (1.18) on the cone of non-negative and non-increasing functions f in the case $1 \leq p = q < \infty$ and $u(t) = v(t)$. The result is the following.

Let $1 \leq p < \infty$. Then the inequality

$$\left(\int_0^\infty \left(\frac{1}{t} \int_0^t f(s)ds \right)^p v(t)dt \right)^{\frac{1}{p}} \leq C \left(\int_0^\infty f^p(t)v(t)dt \right)^{\frac{1}{p}}$$

*holds for all non-negative and non-increasing functions f if and only if there
is a constant $D > 0$ such that*

$$\int\limits_{t}^{\infty} s^{-p} v(s) ds \leq D t^{-p} \int\limits_{0}^{t} v(s) ds \ \text{ for all } \ t > 0.$$

Previously, such problems were studied by D.W. Boyd [79] in 1967 and by
S.G. Krein, Yu.I. Petunin, E.M. Semenov [80].

E. Sawyer [44] has extended the result of M. Ariño and B. Muckenhoupt to
the case of different weights v and u and $1 < p, q < \infty$. Nowadays this result
of E. Sawyer is known as the Sawyer duality principle.

The Sawyer duality principle. *Let $1 < p < \infty$, g, v be non-negative
measurable functions on $(0, \infty)$ with v locally integrable. Then*

$$\sup_{0 \leq f \downarrow} \frac{\int\limits_{0}^{\infty} f(x) g(x) dx}{\left(\int\limits_{0}^{\infty} f^p(x) v(x) dx \right)^{\frac{1}{p}}}$$

$$\approx \left(\int\limits_{0}^{\infty} \left(\int\limits_{0}^{x} g(t) dt \right)^{p'} \left(\int\limits_{0}^{x} v(t) dt \right)^{-p'} v(x) dx \right)^{\frac{1}{p'}} + \frac{\int\limits_{0}^{\infty} g(x) dx}{\left(\int\limits_{0}^{\infty} v(x) dx \right)^{\frac{1}{p}}}$$

*If $\int\limits_{0}^{\infty} v(x) dx = \infty$, then the second term on the right hand side of the last
equivalence can be omitted.*

Moreover, E. Sawyer [44] has used this duality result to obtain necessary
and sufficient conditions for which (1.18) holds for all non-negative and non-
increasing functions f in the case $1 < p, q < \infty$. This result of E. Sawyer
was extended by V.D. Stepanov [81] to the cases $0 < q < 1 < p < \infty$ and
$0 < p \leq q < \infty, 0 < p < 1$. M.L. Goldman [35], G. Bennett and K.-G. Grosse-
Erdmann [47] have characterized the weights u and v, for which inequality
(1.18) holds for all non-negative and non-increasing functions f in the case
$0 < q < p < 1$. The duality principle for the case $0 < p \leq 1$ has been
proved in [81], [82] and [83]. A simpler proof of the Sawyer duality principle
has been obtained by by V.D. Stepanov [81], M.J. Carro and J. Soria [82].

Some generalizations of Sawer duality formula were proved by D.E. Edmunds, R. Kerman, L. Pick [84] and A. Kamińska, M. Mastylo [85].

Various aspects of weighted inequalities for monotone functions and their applications in the estimation of maximal functions, in the theory of interpolation of operators, in the embedding theory and their relations with the stochastic inequalities have been studied in [21], [81], [86]-[94]. Such intensive investigation of various weighted inequalities on the cone of monotone functions in recent years has been made possible due to the achievements of the study of weighted inequalities in various spaces (see e.g. [7], [8], [9], [10], [16], [17], [20], [50] and many others).

At the same time the investigation and generalization of the discrete Hardy inequality

$$\left(\sum_{i=1}^{\infty} u_i \left(\frac{1}{i} \sum_{j=1}^{i} f_j \right)^q \right)^{\frac{1}{q}} \leq C \left(\sum_{i=1}^{\infty} v_i f_i^p \right)^{\frac{1}{p}} \tag{1.19}$$

on the cone of monotone sequences $f \geq 0$ was developed. Results on weighted Hardy inequalities on the cone of monotone sequences have been obtained by K.F. Andersen, H.P. Heinig [24], H.P. Heinig [25], L. Leindler [95], M.Sh. Braverman, V.D. Stepanov [29], J. Nemeth [96], F.P. Cass, W. Kratz [97], P.D. Johnson, R.N. Mohapatra, David Ross [98], R. Oinarov, S.Kh. Shalgynbaeva [45], G. Bennett, K.-G. Grosse-Erdmann [47], M.L. Goldman [35], [99], [100], S.Kh. Shalgynbaeva [48] and others.

In 1998 R. Oinarov, S.Kh. Shalgynbaeva [45] obtained an analogue of the Sawyer duality principle for the discrete case if $1 < p, q < \infty$. This result of R. Oinarov, S.Kh. Shalgynbaeva allows to reduce a Hardy type inequality on the cone of monotone sequences to a corresponding inequality on the cone of non-negative sequences from $l_{p,v}$. Indeed, we have the following.

Theorem 1.11. *[45] Let $1 < p$, $q < \infty$. Let $(a_{i,j})$ be a triangular matrix with entries $a_{i,j} \geq 0$ for $i \geq j \geq 1$ and $a_{i,j} = 0$ for $i < j$. Let $\mathbb{V}_k = \sum_{i=1}^{k} v_i$ for $k \geq 1$.*

Then inequality

$$\left(\sum_{i=1}^{\infty} u_i \left(\sum_{j=1}^{i} a_{i,j} f_j \right)^q \right)^{\frac{1}{q}} \leq C \left(\sum_{i=1}^{\infty} v_i f_i^p \right)^{\frac{1}{p}} \tag{1.20}$$

on the cone of non-negative and non-increasing sequences $f = \{f_i\}_{i=1}^{\infty}$ *of* $l_{p,v}$ *is equivalent to the following inequality*

$$\left(\sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} \sum_{i=j}^{\infty} a_{i,j} g_i \right)^{p'} \left(\mathbb{V}_k^{-\frac{p'}{p}} - \mathbb{V}_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{1}{p'}} \leq \widetilde{C} \left(\sum_{i=1}^{\infty} g_i^{q'} u_i^{1-q'} \right)^{\frac{1}{q'}} \tag{1.21}$$

for all non-negative sequences $g = \{g_i\}_{i=1}^{\infty}$, *if* $\mathbb{V}_{\infty} = \lim\limits_{k \to \infty} \mathbb{V}_k = \infty$, *and to the inequality*

$$\left(\sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} \sum_{i=j}^{\infty} a_{i,j} g_i \right)^{p'} \left(\mathbb{V}_k^{-\frac{p'}{p}} - \mathbb{V}_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{1}{p'}}$$

$$+ \left(\sum_{j=1}^{\infty} \sum_{i=j}^{\infty} a_{i,j} g_i \right) \left(\sum_{k=1}^{\infty} v_k \right)^{-\frac{1}{p}} \leq \overline{C} \left(\sum_{i=1}^{\infty} g_i^{q'} u_i^{1-q'} \right)^{\frac{1}{q'}} \tag{1.22}$$

for all non-negative sequences $g = \{g_i\}_{i=1}^{\infty}$, *if* $\mathbb{V}_{\infty} < \infty$.

Moreover, $\widetilde{C} \approx C$ *if* $\mathbb{V}_{\infty} = \infty$, *and* $\overline{C} \approx C$ *if* $\mathbb{V}_{\infty} < \infty$, *where* C, $\widetilde{C}$ *and* $\overline{C}$ *are the best constants in (1.20), (1.21), (1.22), respectively.*

Moreover, S.Kh. Shalgynbaeva in her PhD thesis [46] has obtained criteria for the validity of inequality (1.19) for some other values of the parameters p and q.

In 2006 G. Bennett and K.-G. Grosse-Erdmann [47] obtained a complete characterization of the weights for which the Hardy inequality (1.19) holds on the cone of monotone sequences of different nature of the conditions of S.Kh. Shalgynbaeva.

M.L. Goldman in his papers has studied inequalities of the type (1.19) on the cone of monotone sequences and has applied the corresponding results to establish Hardy inequalities on the cone of quasi-monotone sequences, see e.g. [35], [99], [100].

In [48] S.Kh. Shalgynbaeva has obtained necessary and sufficient conditions for the validity of (1.20) on the cone of monotone sequences for $1 < p \leq q < \infty$ under the assumption that there exists $d \geq 1$ such that the inequalities

$$\frac{1}{d}(a_{i,k} + a_{k,j}) \leq a_{i,j} \leq d(a_{i,k} + a_{k,j}), \qquad i \geq k \geq j \geq 1 \tag{1.23}$$

hold. Namely, the following result.

Theorem 1.12. *[48] Let $1 < p \leq q < \infty$. Let the entries of the matrix $(a_{i,j})$ in (1.20) satisfy condition (1.23). Let $\mathbb{V}_k = \sum_{i=1}^{k} v_i$ for $k \geq 1$ and $A_{ik} = \sum_{j=1}^{k} a_{i,j}$ for $i \geq k \geq 1$. Then inequality (1.20) on the cone of non-negative and non-increasing sequences $f \geq 0$ holds if and only if $\mathcal{H}_0 \equiv \max\{\mathcal{H}_1,\ \mathcal{H}_2,\ \mathcal{H}_3\} < \infty$, where*

$$\mathcal{H}_1 = \sup_{s \geq 1} \mathbb{V}_s^{-\frac{1}{p}} \left(\sum_{i=1}^{s} A_{ii}^q u_i \right)^{\frac{1}{q}},$$

$$\mathcal{H}_2 = \sup_{s \geq 1} \left(\sum_{k=1}^{s} k^{p'} \left(\mathbb{V}_k^{-\frac{p'}{p}} - \mathbb{V}_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{1}{p'}} \left(\sum_{i=s}^{\infty} a_{ik}^q u_i \right)^{\frac{1}{q}},$$

$$\mathcal{H}_3 = \sup_{s \geq 1} \left(\sum_{i=1}^{s} A_{si}^{p'} \left(\mathbb{V}_i^{-\frac{p'}{p}} - \mathbb{V}_{i+1}^{\frac{p'}{p}} \right) \right)^{\frac{1}{p'}} \left(\sum_{k=s}^{\infty} u_k \right)^{\frac{1}{q}}.$$

Moreover, $\mathcal{H}_0 \approx C$, where C is the best constant in (1.20).

Nowadays inequalities on the cone of monotone functions and sequences are still being intensively developed. This fact is confirmed by a great number of recent publications [2], [36], [67], [88], [92], [101]-[114] and most recently [100] and [115]. For a history of Hardy type inequalities on the cones of monotone functions and sequences and for references to related results we refer to the book of A. Kufner, L. Maligranda and L.-E. Persson [3, Chapter 10], and to the PhD thesis of O. Popova [43].

Chapter 2

Boundedness and compactness of matrix operators in weighted Lebesgue spaces

2.1 Preliminaries and notation.

In this Chapter we consider the problems of boundedness and compactness from the weighted $l_{p,v}$ space into the weighted $l_{q,u}$ space of the matrix operators

$$\left(A^+ f\right)_i := \sum_{j=1}^{i} a_{i,j} f_j, \qquad i \geq 1, \tag{2.1}$$

$$\left(A^- f\right)_j := \sum_{i=j}^{\infty} a_{i,j} f_i, \qquad j \geq 1. \tag{2.2}$$

The boundedness of such operators is equivalent to the validity of the following Hardy type inequality

$$\|A^\pm f\|_{q,u} \leq C \|f\|_{p,v} \qquad \forall f \in l_{p,v}, \tag{2.3}$$

where C is a positive finite constant independent of f and $(a_{i,j})$ is a triangular matrix with entries $a_{i,j} \geq 0$ for $i \geq j \geq 1$ and $a_{i,j} = 0$ for $i < j$.

Here and further $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $u = \{u_i\}_{i=1}^{\infty}$, $v = \{v_i\}_{i=1}^{\infty}$ are positive sequences of real numbers. $l_{p,v}$ is the space of sequences $f = \{f_i\}_{i=1}^{\infty}$ of real numbers such that

$$\|f\|_{p,v} := \left(\sum_{i=1}^{\infty} |v_i f_i|^p \right)^{\frac{1}{p}} < \infty, \qquad 1 < p < \infty.$$

In this Chapter we consider inequality (2.3) under the following assumption in the case $1 < q < p < \infty$.

Assumption A. There exist $d \geq 1$, a sequence of positive numbers $\{\omega_k\}_{k=1}^{\infty}$, and a non-negative matrix $(b_{i,j})$, whose entries $b_{i,j}$ are almost non-decreasing in i and almost non-increasing in j such that the inequalities

$$\frac{1}{d}(a_{i,k} + b_{k,j}\omega_i) \leq a_{i,j} \leq d(a_{i,k} + b_{k,j}\omega_i) \tag{2.4}$$

hold for all $i \geq k \geq j \geq 1$.

Here and further, a matrix is said to be non-negative provided that all its entries are non-negative.

Let $\alpha > 0$. Let $a_{i,j} = (b_i - d_j)^{\alpha}$ for $i \geq j \geq 1$ and $a_{i,j} = 0$ for $i < j$, where the sequences $\{b_i\}_{i=1}^{\infty}$ and $\{d_i\}_{i=1}^{\infty}$ are such that $b_i \geq d_j$, $i \geq j \geq 1$. If $\{b_i\}_{i=1}^{\infty}$ is a non-decreasing sequence and $\{d_i\}_{i=1}^{\infty}$ is an arbitrary sequence, then the entries of the matrix $(a_{i,j})$ satisfy condition (1.17). Indeed, we have $a_{i,j} \approx (b_i - b_k)^{\alpha} + a_{k,j}$, $i \geq k \geq j \geq 1$. In general, the entries $a_{i,j}$ do not satisfy condition (2.4). If $\{d_i\}_{i=1}^{\infty}$ is a non-decreasing sequence and $\{b_i\}_{i=1}^{\infty}$ is an arbitrary sequence, then the entries $a_{i,j}$ satisfy condition (2.4), but in general, condition (1.17) does not hold for the entries of the matrix $(a_{i,j})$.

Thus, conditions (1.17), (2.4) include condition (1.16) and complement each other.

Moreover, in this Chapter we introduce a general class of matrices. We establish necessary and sufficient conditions for the boundedness and compactness of the operators (2.1) and (2.2) when the corresponding matrices belong to such classes. Such classes of matrices are wider than those which have been previously studied in the theory of discrete Hardy type inequalities.

The content of the Chapter is as follows. In Section 2.2, we introduce the classes of matrices $\mathcal{O}_n^+$ and $\mathcal{O}_n^-$ for $n \geq 0$, which include well-known matrices of analysis such as summable matrices, and in particular, matrices satisfying conditions (1.16), (1.17), (2.4). Moreover, in this section we introduce properties of such classes. Section 2.3 contains examples of matrices from the introduced classes and here we show that such classes of matrices include well-known classical operators such as the operator of multiple summation, Hölder's operator, Cesàro operator. In Section 2.4, we prove the theorems, which give criteria of boundedness of the operators defined by (2.1) and (2.2) in weighted spaces of sequences when the corresponding matrices belong to the classes $\mathcal{O}_n^+$ and $\mathcal{O}_n^-$, $n \geq 0$ in case $1 < p \leq q < \infty$. In Section 2.5, we obtain necessary and sufficient conditions for the compactness of the matrix operators defined by (2.1) and (2.2) when the corresponding matrices belong to one of the classes $\mathcal{O}_n^+$ and $\mathcal{O}_n^-$, $n \geq 0$. Section 2.6 contains the main results for the case $1 < p \leq q < \infty$ and their proofs. Section 2.7 is devoted to the problem of boundedness of the operators defined by (2.1) and (2.2) in weighted spaces of sequences when the entries of the corresponding matrices satisfy Assumption A in case $1 < q < p < \infty$.

For the proof of our main theorem we will need the following well-known results for the discrete weighted Hardy inequality (see [3]) and the criteria of precompactness of sets in l_p (see [116, p. 32]).

Theorem A. *Let $\{\mu_j\}_{j=1}^{\infty}$ be a non-negative sequence of real numbers.*
(i) If $1 < p \leq q < \infty$, then the inequality

$$\left(\sum_{i=1}^{\infty} \left| \sum_{j=1}^{i} \mu_j f_j \right|^q u_i^q \right)^{\frac{1}{q}} \leq C \left(\sum_{i=1}^{\infty} |v_i f_i|^p \right)^{\frac{1}{p}} \tag{2.5}$$

holds for all $f \in l_{p,v}$ if and only if

$$H_0 = \sup_{n \geq 1} \left(\sum_{j=n}^{\infty} u_j^q \right)^{\frac{1}{q}} \left(\sum_{i=1}^{n} \mu_i^{p'} v_i^{-p'} \right)^{\frac{1}{p'}} < \infty.$$

Moreover, $H_0 \approx C$, where C is the best constant in (2.5).

(ii) If $1 < q < p < \infty$, then the inequality (2.5) for all $f \in l_{p,v}$ holds if and only if

$$H_1 = \left(\sum_{k=1}^{\infty} \left(\sum_{i=k}^{\infty} u_i^q \right)^{\frac{p}{p-q}} \left(\sum_{j=1}^{k} \mu_j^{p'} v_j^{-p'} \right)^{\frac{p(q-1)}{p-q}} \mu_k^{p'} v_k^{-p'} \right)^{\frac{p-q}{pq}} < \infty.$$

Moreover, $H_1 \approx C$, where C is the best constant in (2.5).

Theorem B. *Let $\{\mu_j\}_{j=1}^{\infty}$ be a non-negative sequence of real numbers.*

(i) If $1 < p \le q < \infty$, then the inequality

$$\left(\sum_{i=1}^{\infty} \left| \sum_{j=i}^{\infty} \mu_j f_j \right|^q u_i^q \right)^{\frac{1}{q}} \le C \left(\sum_{i=1}^{\infty} |v_i f_i|^p \right)^{\frac{1}{p}} \tag{2.6}$$

holds for all $f \in l_{p,v}$ if and only if

$$H_2 = \sup_{n \ge 1} \left(\sum_{j=1}^{n} u_j^q \right)^{\frac{1}{q}} \left(\sum_{i=n}^{\infty} \mu_i^{p'} v_i^{-p'} \right)^{\frac{1}{p'}} < \infty.$$

Moreover, $H_2 \approx C$, where C is the best constant in (2.6).

(ii) If $1 < q < p < \infty$, then the inequality (2.6) for all $f \in l_{p,v}$ holds if and only if

$$H_3 = \left(\sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} u_i^q \right)^{\frac{p}{p-q}} \left(\sum_{j=k}^{\infty} \mu_j^{p'} v_j^{-p'} \right)^{\frac{p(q-1)}{p-q}} \mu_k^{p'} v_k^{-p'} \right)^{\frac{p-q}{pq}} < \infty.$$

Moreover, $H_3 \approx C$, where C is the best constant in (2.6).

Theorem C. *Let T be a set in l_p, $1 \le p < \infty$. The set T is compact if and only if T is bounded and for all $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that for all $x = \{x_i\}_{i=1}^{\infty} \in T$ the inequality*

$$\sum_{i=N}^{\infty} |x_i|^p < \varepsilon$$

holds.

We also need the following well-known result (see [38]).

Lemma D. *Let $\gamma > 0$. Then there exists $c > 0$ such that*

$$\frac{1}{c} \left(\sum_{k=1}^{j} \beta_k \right)^{\gamma} \le \sum_{k=1}^{j} \beta_k \left(\sum_{i=1}^{k} \beta_i \right)^{\gamma-1} \le c \left(\sum_{k=1}^{j} \beta_k \right)^{\gamma} \quad \forall j \in \mathbb{N} \tag{2.7}$$

for all sequences $\{\beta_k\}_{k=1}^{\infty}$ of positive real numbers.

Moreover, there exists $c_1 \geq 1$ such that

$$\frac{1}{c_1} \left(\sum_{k=j}^{N} \beta_k \right)^{\gamma} \leq \sum_{k=j}^{N} \beta_k \left(\sum_{i=k}^{N} \beta_i \right)^{\gamma - 1} \leq c_1 \left(\sum_{k=j}^{N} \beta_k \right)^{\gamma} \qquad (2.8)$$

for all $j, k \in \{1, 2, ..., N\}$, $N \in \mathbb{N} \cup \{\infty\}$ and for all sequences $\{\beta_k\}_{k=1}^{\infty}$ of positive real numbers such that $\sum\limits_{k=1}^{\infty} \beta_k < \infty$.

2.2   Introduction of classes of matrices and their properties.

For $n \geq 1$, we introduce the classes $\mathcal{O}_n^+$ and $\mathcal{O}_n^-$ of matrices $(a_{i,j})$. We denote by $\left(a_{i,j}^{(n)} \right)$ a generic element $(a_{i,j})$ of $\mathcal{O}_n^+$ or $(a_{i,j})$ of $\mathcal{O}_n^-$.

We now introduce the set M^+ of non-negative matrices $(a_{i,j})$ such that $a_{i,j}$ is non-decreasing in the first index for all $i \geq j \geq 1$. We define the classes $\mathcal{O}_n^+$, $n \geq 0$ by induction. If $n = 0$ the class $\mathcal{O}_0^+$ is defined as the set of matrices of M^+ of the type $a_{i,j}^{(0)} = \alpha_j$, $\forall i \geq j \geq 1$. Next we assume that the classes $\mathcal{O}_\gamma^+$ have already been defined for $\gamma = 0, 1, \ldots, n-1$, $n \geq 1$. By definition a matrix $(a_{i,j}) \equiv (a_{i,j}^{(n)})$ of M^+ belongs to the class $\mathcal{O}_n^+$ if and only if there exist matrices $(a_{i,j}^{(\gamma)}) \in \mathcal{O}_\gamma^+$, $\gamma = 0, 1, \ldots, n-1$ and a number $r_n > 0$ such that

$$a_{i,j}^{(n)} \leq r_n \sum_{\gamma=0}^{n} b_{i,k}^{n,\gamma} a_{k,j}^{(\gamma)} \tag{2.9}$$

for all $i \geq k \geq j \geq 1$, where $b_{i,k}^{n,n} \equiv 1$ and

$$b_{i,k}^{n,\gamma} = \inf_{1 \leq j \leq k} \frac{a_{i,j}^{(n)}}{a_{k,j}^{(\gamma)}}, \quad \gamma = 0, 1, \ldots, n-1. \tag{2.10}$$

From (2.10) it follows that entries of the matrices $(b_{i,k}^{n,\gamma})$ are non-decreasing in the first index and are non-increasing in the second index. Moreover, (2.10) provides the validity of the following inequality

$$a_{i,j}^{(n)} \geq b_{i,k}^{n,\gamma} a_{k,j}^{(\gamma)} \tag{2.11}$$

for all $i \geq k \geq j \geq 1$, $\gamma = 0, 1, \ldots, n$, $n = 0, 1, \ldots$ Then for $(a_{i,j}^{(n)}) \in \mathcal{O}_n^+$ we have

$$a_{i,j}^{(n)} \approx \sum_{\gamma=0}^{n} b_{i,k}^{n,\gamma} a_{k,j}^{(\gamma)}, \qquad n \geq 0 \tag{2.12}$$

for all $i \geq k \geq j \geq 1$.

Remark 2.1. *It is easy to show that if for the matrix $(a_{i,j}^{(n)})$, $n \geq 0$ there exist matrices $(a_{i,j}^{(\gamma)}) \in \mathcal{O}_\gamma^+$, $\gamma = 0, 1, \ldots, n-1$, and matrices $(\widetilde{b}_{i,k}^{n,\gamma})$, $\gamma = 0, 1, \ldots, n$*

such that the equivalence (2.12) is valid for all $i \geq k \geq j \geq 1$, then $(a_{i,j}^{(n)}) \in \mathcal{O}_n^+$ and $\widetilde{b}_{i,k}^{n,\gamma} \approx b_{i,k}^{n,\gamma}$. Hence, we may assume that the matrices $(b_{i,k}^{n,\gamma})$ are arbitrary non-negative matrices which satisfy (2.12).

For the proof of our main results we also need the following inequality. Let $n \geq l \geq \gamma$. Then we have

$$b_{i,k}^{n,\gamma} \geq b_{i,s}^{n,l} \cdot b_{s,k}^{l,\gamma} \qquad \forall i \geq s \geq k \geq 1. \tag{2.13}$$

Indeed, using (2.11), for $i \geq s \geq k \geq 1$, $n \geq l \geq \gamma$ we obtain

$$b_{i,k}^{n,\gamma} = \inf_{1 \leq j \leq k} \frac{a_{i,j}^{(n)}}{a_{k,j}^{(\gamma)}} \geq b_{i,s}^{n,l} \cdot \inf_{1 \leq j \leq k} \frac{a_{s,j}^{(l)}}{a_{k,j}^{(\gamma)}} = b_{i,s}^{n,l} \cdot b_{s,k}^{l,\gamma}.$$

As above, we introduce the classes $\mathcal{O}_m^-$, $m \geq 0$. We now define the set M^- of non-negative matrices $(a_{i,j})$ such that $a_{i,j}$ is non-increasing in the second index for all $i \geq j \geq 1$. By definition a matrix $(a_{i,j}) = (a_{i,j}^{(0)})$ of M^- belongs to the class $\mathcal{O}_0^-$ if and only if it has the form $a_{i,j}^{(0)} = \beta_i$ for all $i \geq j \geq 1$. Let the classes $\mathcal{O}_\gamma^-$, $\gamma = 0, 1, \ldots, m-1$, $m \geq 1$ be defined. A matrix $(a_{i,j}) = (a_{i,j}^{(m)})$ of M^- belongs to the class $\mathcal{O}_m^-$ if and only if there exist matrices $(a_{i,j}^{(\gamma)}) \in \mathcal{O}_\gamma^-$, $\gamma = 0, 1, \ldots, m-1$ and a number $h_m > 0$ such that

$$a_{i,j}^{(m)} \leq h_m \sum_{\gamma=0}^{m} a_{i,k}^{(\gamma)} d_{k,j}^{\gamma,m} \tag{2.14}$$

for all $i \geq k \geq j \geq 1$, where $d_{k,j}^{m,m} \equiv 1$ and

$$d_{k,j}^{\gamma,m} = \inf_{k \leq i \leq \infty} \frac{a_{i,j}^{(m)}}{a_{i,k}^{(\gamma)}}, \quad \gamma = 0, 1, \ldots, m-1. \tag{2.15}$$

From the definition of the matrix $(d_{k,j}^{\gamma,m})$, $\gamma = 0, 1, \ldots, m-1$, $m = 0, 1, \ldots$, it is obvious that the entries of the matrix $(d_{k,j}^{\gamma,m})$ do not decrease in the first index and do not increase in the second index and for $m \geq l \geq \gamma$, $k \geq s \geq j$ satisfy the following inequality

$$d_{k,j}^{\gamma,m} \geq d_{k,s}^{\gamma,l} \cdot d_{s,j}^{l,m}. \tag{2.16}$$

From (2.15) it follows that for all $i \geq k \geq j \geq 1$

$$a_{i,j}^{(m)} \geq a_{i,k}^{(\gamma)} d_{k,j}^{\gamma,m}, \quad \gamma = 0, 1, \ldots, m-1. \tag{2.17}$$

As in (2.12) every class $\mathcal{O}_m^-$, $m \geq 0$ of matrices $(a_{i,j}^{(m)})$ is characterized by the following relation

$$a_{i,j}^{(m)} \approx \sum_{\gamma=0}^{m} a_{i,k}^{(\gamma)} d_{k,j}^{\gamma,m} \tag{2.18}$$

for all $i \geq k \geq j$, where $d_{k,j}^{\gamma,m}$, $\gamma = 0, 1, \ldots, m$ are defined by the formula (2.15).

Remark 2.2. *As mentioned before, we may assume that the matrices $(d_{k,j}^{\gamma,m})$, $\gamma = 0, 1, \ldots, m$, $m \geq 0$ are arbitrary non-negative matrices which satisfy (2.18).*

Remark 2.3. *By the definitions of the classes $\mathcal{O}_n^\pm$, $n \geq 0$ we have $\mathcal{O}_0^\pm \subset \mathcal{O}_1^\pm \subset \cdots \subset \mathcal{O}_n^\pm \subset \cdots$*

In particular, the matrices of the classes $\mathcal{O}_1^+$ and $\mathcal{O}_1^-$ are characterized by the following relations, respectively,

$$a_{i,j}^{(1)} \approx b_{i,k}^{1,0} a_{k,j}^{(0)} + a_{k,j}^{(1)} \qquad \forall i \geq k \geq j \geq 1,$$

$$a_{i,j}^{(1)} \approx a_{i,k}^{(1)} + a_{i,k}^{(0)} d_{k,j}^{0,1} \qquad \forall i \geq k \geq j \geq 1.$$

It is obvious that the class $\mathcal{O}_1^+$ include the matrices, whose entries satisfy conditions (1.16) and (1.17). Also it should be noted that the matrices with conditions (1.16) and (2.4) belong to the class $\mathcal{O}_1^-$. This implies that the classes $\mathcal{O}_n^+$, $n \geq 1$ and $\mathcal{O}_m^-$, $m \geq 1$ are wider than the classes of matrices which have been used in this connection before.

The matrices of the classes $\mathcal{O}_2^+$ and $\mathcal{O}_2^-$ are described by the following relations, respectively,

$$a_{i,j}^{(2)} \approx b_{i,k}^{2,0} a_{k,j}^{(0)} + b_{i,k}^{2,1} a_{k,j}^{(1)} + a_{k,j}^{(2)} \qquad \forall i \geq k \geq j \geq 1,$$

$$a_{i,j}^{(2)} \approx a_{i,k}^{(2)} + a_{i,k}^{(1)} d_{k,j}^{1,2} + a_{i,k}^{(0)} d_{k,j}^{0,2} \qquad \forall i \geq k \geq j \geq 1.$$

A continuous analogue of the classes $\mathcal{O}_n^+$ and $\mathcal{O}_n^-$, $n \geq 0$ has been studied by R. Oinarov in [23].

Definition 2.4. *If there exist a non-negative sequence* $\alpha = \{\alpha_i\}_{i=1}^{\infty}$ *and a matrix* $\widetilde{a}_{i,j} \in \mathcal{O}_n^{\pm}$, $n \geq 0$ *such that* $a_{i,j} \approx \alpha_i \widetilde{a}_{i,j}$, *then we say that* $(a_{i,j}) \in \alpha \mathcal{O}_n^{\pm}$, $n \geq 0$.

Definition 2.5. *If there exist a non-negative sequence* $\beta = \{\beta_j\}_{j=1}^{\infty}$ *and a matrix* $\widetilde{a}_{i,j} \in \mathcal{O}_n^{\pm}$, $n \geq 0$ *such that* $a_{i,j} \approx \widetilde{a}_{i,j} \beta_j$, *then we say that* $(a_{i,j}) \in \mathcal{O}_n^{\pm} \beta$, $n \geq 0$.

Such classes of operators include a lot of well-known operators, which play significant role in analysis. As an example of matrices from the classes $\alpha \mathcal{O}_n^{\pm}$ and $\mathcal{O}_n^{\pm} \beta$, $n \geq 0$ we can take Cesàro matrix, Hölder's matrix and others. For more detailed information, see Section 2.3.

Next, we show properties of the classes of matrices $\mathcal{O}_n^+$ and $\mathcal{O}_n^-$, $n \geq 0$.

We set

$$w_{i,k} = \sum_{j=k}^{i} a_{i,j} \sigma_{j,k}.$$

Then we have the following

Lemma 2.6. *Let* $(a_{i,j}) \in \mathcal{O}_n^+$, $(\sigma_{j,k}) \in \mathcal{O}_m^+$. *Then* $(w_{i,k}) \in \mathcal{O}_{m+n+1}^+$.

Proof of Lemma 2.6. Since $(a_{i,j}) \in \mathcal{O}_n^+$, there exist matrices $(a_{i,j}^{(\gamma)}) \in \mathcal{O}_\gamma^+$, $\gamma = 0, 1, \dots, n-1$, and matrices $(\delta_{i,l}^{n,\gamma})$ such that

$$a_{i,j} \equiv a_{i,j}^{(n)} \approx \sum_{\gamma=0}^{n} \delta_{i,l}^{n,\gamma} a_{l,j}^{(\gamma)}, \qquad n = 0, 1, \dots, \quad \delta_{i,l}^{n,n} \equiv 1$$

for all $i \geq l \geq j \geq 1$.

Since $(\sigma_{j,k}) \in \mathcal{O}_m^+$, there exist matrices $(\sigma_{j,k}^{(\mu)}) \in \mathcal{O}_\mu^+$, $\mu = 0, 1, \dots, m-1$, and matrices $(e_{j,l}^{m,\mu})$ such that

$$\sigma_{j,k} \equiv \sigma_{j,k}^{(m)} \approx \sum_{\mu=0}^{m} e_{j,l}^{m,\mu} \sigma_{l,k}^{(\mu)}, \qquad m = 0, 1, \dots, \quad e_{j,l}^{m,m} \equiv 1$$

for all $j \geq l \geq k \geq 1$.

We set

$$w_{i,k} \equiv w_{i,k}^{n,m} = \sum_{j=k}^{i} a_{i,j}^{(n)} \sigma_{j,k}^{(m)}.$$

First, we consider the case when $m \geq 0$, $n = 0$. In this case $a_{i,j}^{(0)} = \alpha_j$, $\forall i \geq j \geq 1$. For $\forall i \geq l \geq k$ we obtain

$$
\begin{aligned}
w_{i,k}^{o,m} &= \sum_{j=k}^{i} \alpha_j \sigma_{j,k}^{(m)} \approx \sum_{j=k}^{l} \alpha_j \sigma_{j,k}^{(m)} + \sum_{j=l}^{i} \alpha_j \sigma_{j,k}^{(m)} \\
&\approx w_{l,k}^{o,m} + \sum_{\mu=0}^{m} \sigma_{l,k}^{(\mu)} \sum_{j=l}^{i} \alpha_j e_{j,l}^{m,\mu} \\
&= w_{l,k}^{o,m} + \sum_{\mu=0}^{m} \widetilde{e}_{i,l}^{m+1,\mu} \sigma_{l,k}^{(\mu)},
\end{aligned}
$$

where $\widetilde{e}_{i,l}^{m+1,\mu} = \sum_{j=l}^{i} \alpha_j e_{j,l}^{m,\mu}$, $\mu = 0, 1, \ldots, m$. Suppose that $\widetilde{e}_{i,l}^{m+1,m+1} \equiv 1$. Since $\left(\sigma_{l,k}^{(\mu)} \right) \in \mathcal{O}_{\mu}^{+}$ for $\mu = 0, 1, \ldots, m$, by definition we easily see that $w_{i,k}^{o,m} \in \mathcal{O}_{m+1}^{+}$. By induction, we assume that for $n = 0, 1, \ldots, r-1$, $r \geq 1$ $\left(w_{i,k}^{n,m} \right)$ belong to the classes $\mathcal{O}_{n+m+1}^{+}$. For $i \geq l \geq k$ we have

$$
\begin{aligned}
w_{i,k}^{r,m} &= \sum_{j=k}^{i} a_{i,j}^{(r)} \sigma_{j,k}^{(m)} \approx \sum_{j=k}^{l} a_{i,j}^{(r)} \sigma_{j,k}^{(m)} + \sum_{j=l}^{i} a_{i,j}^{(r)} \sigma_{j,k}^{(m)} \\
&\approx \sum_{j=k}^{l} \left(\sum_{\gamma=0}^{r} \delta_{i,l}^{r,\gamma} a_{l,j}^{(\gamma)} \right) \sigma_{j,k}^{(m)} + \sum_{j=l}^{i} a_{i,j}^{(r)} \left(\sum_{\mu=0}^{m} e_{j,l}^{m,\mu} \sigma_{l,k}^{(\mu)} \right) \\
&= \sum_{\gamma=0}^{r} \delta_{i,l}^{r,\gamma} \sum_{j=k}^{l} a_{l,j}^{(\gamma)} \sigma_{j,k}^{(m)} + \sum_{\mu=0}^{m} \sigma_{l,k}^{(\mu)} \sum_{j=l}^{i} a_{i,j}^{(r)} e_{j,l}^{m,\mu} \\
&= \sum_{j=k}^{l} a_{l,j}^{(r)} \sigma_{j,k}^{(m)} + \sum_{\gamma=0}^{r-1} \delta_{i,l}^{r,\gamma} \sum_{j=k}^{l} a_{l,j}^{(\gamma)} \sigma_{j,k}^{(m)} + \sum_{\mu=0}^{m} \sigma_{l,k}^{(\mu)} \sum_{j=l}^{i} a_{i,j}^{(r)} e_{j,l}^{m,\mu} \\
&= w_{l,k}^{r,m} + \sum_{\gamma=0}^{r-1} \delta_{i,l}^{r,\gamma} \widetilde{\sigma}_{l,k}^{(\gamma+m+1)} + \sum_{\mu=0}^{m} \widetilde{e}_{i,l}^{m+1,\mu} \sigma_{l,k}^{(\mu)},
\end{aligned}
$$

where $\widetilde{\sigma}_{l,k}^{(\gamma+m+1)} \equiv \sum_{j=k}^{l} a_{l,j}^{(\gamma)} \sigma_{j,k}^{(m)}$, $\gamma = 0, \ldots, r-1$ and $\widetilde{e}_{i,l}^{m+1,\mu} \equiv \sum_{j=l}^{i} a_{i,j}^{(r)} e_{j,l}^{m,\mu}$ $\mu = 0, \ldots, m$. We denote $\gamma + m + 1$ by μ. Then we have

$$
\begin{aligned}
w_{i,k}^{r,m} &\approx w_{l,k}^{r,m} + \sum_{\mu=m+1}^{r+m} \delta_{i,l}^{r,\mu-m-1} \widetilde{\sigma}_{l,k}^{(\mu)} + \sum_{\mu=0}^{m} \widetilde{e}_{i,l}^{m+1,\mu} \sigma_{l,k}^{(\mu)} \\
&= w_{l,k}^{r,m} + \sum_{\mu=0}^{r+m} \widetilde{\delta}_{i,l}^{r+m,\mu} \widetilde{\sigma}_{l,k}^{(\mu)},
\end{aligned}
$$

where

$$\widetilde{\delta}_{i,l}^{r+m,\mu} = \begin{cases} \widetilde{e}_{i,l}^{m+1,\mu}, & 0 \leq \mu \leq m, \\ \delta_{i,l}^{r,\mu-m-1}, & m+1 \leq \mu \leq r+m, \end{cases}$$

and

$$\widetilde{\sigma}_{l,k}^{(\mu)} = \begin{cases} \sigma_{l,k}^{(\mu)}, & 0 \leq \mu \leq m, \\ \widetilde{\sigma}_{l,k}^{(\mu)}, & m+1 \leq \mu \leq r+m. \end{cases}$$

Since $\widetilde{\sigma}_{l,k}^{(\mu)} \in \mathcal{O}_{\mu}^{+}$, $\mu = 0, 1, \dots, r+m$ we obtain that $w_{i,k}^{r,m} \in \mathcal{O}_{r+m+1}^{+}$. The proof is complete.

Now we set

$$\zeta_{k,j} = \sum_{i=j}^{k} \sigma_{k,i} a_{i,j}.$$

Then we have the following lemma.

Lemma 2.7. *Let* $(a_{i,j}) \in \mathcal{O}_{n}^{-}$, $(\sigma_{k,i}) \in \mathcal{O}_{m}^{-}$. *Then* $(\zeta_{k,j}) \in \mathcal{O}_{m+n+1}^{-}$.

Lemma 2.7 can be proved in the same way as Lemma 2.6.

We define

$$\varpi_{i,k} = \sum_{j=1}^{k} a_{i,j} \sigma_{k,j}, \quad i \geq k,$$

$$\varphi_{k,i} = \sum_{j=1}^{i} \sigma_{k,j} a_{i,j}, \quad k \geq i.$$

Lemma 2.8. *i)* *If* $(a_{i,j}) \in \mathcal{O}_{n}^{+}$, $n \geq 0$ *and* $(\sigma_{k,j})$ *is an arbitrary non-negative lower triangular matrix, then* $(\varpi_{i,k})$ *belongs to the class* $\mathcal{O}_{n}^{+}$.

ii) *If* $(\sigma_{k,j}) \in \mathcal{O}_{m}^{+}$, $m \geq 0$ *and* $(a_{i,j})$ *is an arbitrary non-negative lower triangular matrix, then* $(\varphi_{k,i})$ *belongs to the class* $\mathcal{O}_{m}^{+}$.

iii) *If* $(a_{i,j}) \in \mathcal{O}_{n}^{-}$, $n \geq 0$ *and* $(\sigma_{k,j})$ *is an arbitrary non-negative lower triangular matrix, then* $(\varpi_{i,k})$ *belongs to the class* $\mathcal{O}_{n}^{-}\beta$, *where* $\beta = \{\beta_k\}_{k=1}^{\infty}$ *and* $\beta_k = \sum_{j=1}^{k} \sigma_{k,j}$.

iv) *If* $(\sigma_{k,j}) \in \mathcal{O}_{m}^{-}$, $m \geq 0$ *and* $(a_{i,j})$ *is an arbitrary non-negative lower triangular matrix, then* $(\varphi_{k,i})$ *belongs to the class* $\mathcal{O}_{m}^{-}\alpha$, *where* $\alpha = \{\alpha_i\}_{i=1}^{\infty}$ *and* $\alpha_i = \sum_{j=1}^{i} a_{i,j}$.

Proof of Lemma 2.8

i) Since $(a_{i,j}) \in \mathcal{O}_n^+$, there exist matrices $(a_{i,j}^{(\gamma)}) \in \mathcal{O}_\gamma^+$, $\gamma = 0, 1, ..., n-1$, and matrices $(d_{i,k}^{n,\gamma})$ such that

$$a_{i,j} \equiv a_{i,j}^{(n)} \approx \sum_{\gamma=0}^{n} d_{i,s}^{n,\gamma} a_{s,j}^{(\gamma)}, \qquad d_{i,s}^{n,n} \equiv 1$$

for all $i \geq s \geq j \geq 1$.

We set

$$\varpi_{i,k} \equiv \varpi_{i,k}^{(n)} = \sum_{j=1}^{k} a_{i,j}^{(n)} \sigma_{k,j}.$$

First we consider the case when $n = 0$. In this case $\varpi_{i,k}^{(0)} = \sum_{j=1}^{k} \alpha_j \sigma_{k,j} \equiv \widetilde{\alpha}_k$, which means that $(\varpi_{i,k}^{(0)})$ belongs to the class $\mathcal{O}_0^+$. By induction we assume that for $n = 0, 1, ..., r-1$, $r \geq 1$ $(\varpi_{i,k}^{(n)})$ belong to the classes $\mathcal{O}_n^+$. For $i \geq s \geq k$ we have

$$\varpi_{i,k}^{(r)} \approx \sum_{j=1}^{k} \sum_{\gamma=0}^{r} d_{i,s}^{r,\gamma} a_{s,j}^{(\gamma)} \sigma_{k,j} = \sum_{\gamma=0}^{r} d_{i,s}^{r,\gamma} \sum_{j=1}^{k} a_{s,j}^{(\gamma)} \sigma_{k,j} = \sum_{\gamma=0}^{r} d_{i,s}^{r,\gamma} \varpi_{s,k}^{(\gamma)},$$

which implies that $(\varpi_{i,k}^{(r)})$ belongs to the class $\mathcal{O}_r^+$, $r \geq 0$.

ii) Since $(\sigma_{k,j}) \in \mathcal{O}_m^+$, there exist matrices $(\sigma_{i,j}^{(\gamma)}) \in \mathcal{O}_\gamma^+$, $\gamma = 0, 1, ..., m-1$, and matrices $(e_{k,i}^{m,\gamma})$ such that

$$\sigma_{k,j} \equiv \sigma_{k,j}^{(m)} \approx \sum_{\gamma=0}^{m} e_{k,s}^{m,\gamma} \sigma_{s,j}^{(\gamma)}, \qquad e_{k,s}^{m,m} \equiv 1$$

for all $k \geq s \geq j \geq 1$.

We set

$$\varphi_{k,i} \equiv \varphi_{k,i}^{(m)} = \sum_{j=1}^{i} \sigma_{k,j}^{(m)} a_{i,j}.$$

First we consider the case when $m = 0$. In this case $\varphi_{k,i}^{(0)} = \sum_{j=1}^{i} \beta_j a_{i,j} \equiv \widetilde{\beta}_i$, which means that $(\varphi_{k,i}^{(0)})$ belongs to the class $\mathcal{O}_0^+$. By induction we assume that for $m = 0, 1, ..., r-1$, $r \geq 1$ $(\varphi_{k,i}^{(m)})$ belong to the classes $\mathcal{O}_m^+$. For $k \geq s \geq i$ we obtain

$$\varphi_{k,i}^{(r)} \approx \sum_{j=1}^{i} \sum_{\gamma=0}^{r} e_{k,s}^{r,\gamma} \sigma_{s,j}^{(\gamma)} a_{i,j} = \sum_{\gamma=0}^{r} e_{k,s}^{r,\gamma} \sum_{j=1}^{i} \sigma_{s,j}^{(\gamma)} a_{i,j} = \sum_{\gamma=0}^{r} e_{k,s}^{r,\gamma} \varphi_{s,i}^{(\gamma)}.$$

According to definition we obtain that $(\varphi_{k,i}^{(r)})$ belongs to the class $\mathcal{O}_r^+$, $r \geq 0$.

iii) Since $(a_{i,j}) \in \mathcal{O}_n^-$, there exist matrices $(a_{i,s}^{(\gamma)}) \in \mathcal{O}_\gamma^-$, $\gamma = 0, 1, ..., n-1$, and matrices $(d_{s,j}^{\gamma,n})$ such that

$$a_{i,j} \equiv a_{i,j}^{(n)} \approx \sum_{\gamma=0}^{n} a_{i,s}^{(\gamma)} d_{s,j}^{\gamma,n}, \qquad d_{s,j}^{n,n} \equiv 1$$

for all $i \geq s \geq j \geq 1$.

Let $i \geq s \geq k$. Then

$$\varpi_{i,k} \equiv \varpi_{i,k}^{(n)} = \sum_{j=1}^{k} a_{i,j}^{(n)} \sigma_{k,j} \approx \sum_{j=1}^{k} \sum_{\gamma=0}^{n} a_{i,s}^{(\gamma)} d_{s,j}^{\gamma,n} \sigma_{k,j}$$

$$= \sum_{\gamma=0}^{n} a_{i,s}^{(\gamma)} \sum_{j=1}^{k} d_{s,j}^{\gamma,n} \sigma_{k,j}$$

$$= a_{i,s}^{(n)} \sum_{j=1}^{k} \sigma_{k,j} + \sum_{\gamma=0}^{n-1} a_{i,s}^{(\gamma)} \sum_{j=1}^{k} d_{s,j}^{\gamma,n} \sigma_{k,j}$$

$$= \beta_k \left(a_{i,s}^{(n)} + \sum_{\gamma=0}^{n-1} a_{i,s}^{(\gamma)} D_{s,k}^{\gamma,n} \right) = \beta_k \widetilde{a}_{i,k},$$

where $\beta_k = \sum_{j=1}^{k} \sigma_{k,j}$, $\widetilde{a}_{i,k} = a_{i,s}^{(n)} + \sum_{\gamma=0}^{n-1} a_{i,s}^{(\gamma)} D_{s,k}^{\gamma,n}$ and $D_{s,k}^{\gamma,n} = \frac{1}{\beta_k} \left(\sum_{j=1}^{k} d_{s,j}^{\gamma,n} \sigma_{k,j} \right)$.

By definition of $(\widetilde{a}_{i,k})$ we see that $(\widetilde{a}_{i,k}) \in \mathcal{O}_n^-$, $n \geq 0$. Therefore $(\varpi_{i,k})$ belongs to the class $\mathcal{O}_n^- \beta$, $n \geq 0$.

iv) Since $(\sigma_{k,j}) \in \mathcal{O}_m^-$, there exist matrices $(\sigma_{k,s}^{(\gamma)}) \in \mathcal{O}_\gamma^-$, $\gamma = 0, 1, ..., m-1$, and matrices $(e_{s,j}^{\gamma,m})$ such that

$$\sigma_{k,j} \equiv \sigma_{k,j}^{(m)} \approx \sum_{\gamma=0}^{m} \sigma_{k,s}^{(\gamma)} e_{s,j}^{\gamma,m}, \qquad e_{s,j}^{m,m} \equiv 1$$

for all $k \geq s \geq j \geq 1$.

Let $k \geq s \geq i$. Then

$$\varphi_{k,i} \equiv \varphi_{k,i}^{(m)} = \sum_{j=1}^{i} \sigma_{k,j}^{(m)} a_{i,j} \approx \sum_{j=1}^{i} \sum_{\gamma=0}^{m} \sigma_{k,s}^{(\gamma)} e_{s,j}^{\gamma,m} a_{i,j}$$

$$= \sigma_{k,s}^{(m)} \sum_{j=1}^{i} a_{i,j} + \sum_{\gamma=0}^{m-1} \sigma_{k,s}^{(\gamma)} \sum_{j=1}^{i} e_{s,j}^{\gamma,m} a_{i,j}$$

$$= \alpha_i \left(\sigma_{k,s}^{(m)} + \sum_{\gamma=0}^{m-1} \sigma_{k,s}^{(\gamma)} E_{s,i}^{\gamma,m} \right) = \alpha_i \widetilde{\sigma}_{k,i},$$

where $\alpha_i = \sum\limits_{j=1}^{i} a_{i,j}$, $\widetilde{\sigma}_{k,i} = \sigma_{k,s}^{(m)} + \sum\limits_{\gamma=0}^{m-1} \sigma_{k,s}^{(\gamma)} E_{s,i}^{\gamma,m}$ and $E_{s,i}^{\gamma,m} = \frac{1}{\alpha_i} \left(\sum\limits_{j=1}^{i} e_{s,j}^{\gamma,m} a_{i,j} \right).$

By definition of $(\widetilde{\sigma}_{k,i})$ we see that $(\widetilde{\sigma}_{k,i}) \in \mathcal{O}_m^-$, $m \geq 0$. Since $\varphi_{k,i} \approx \alpha_i \widetilde{\sigma}_{k,i}$, we obtain $(\varphi_{k,i})$ belongs to the class $\mathcal{O}_m^- \alpha$, $m \geq 0$.

Thus the proof of Lemma 2.8 is complete.

We define

$$\psi_{k,j} = \sum_{i=k}^{\infty} \sigma_{i,k} a_{i,j}, \quad k \geq j,$$

$$\rho_{j,k} = \sum_{i=j}^{\infty} a_{i,j} \sigma_{i,k}, \quad j \geq k.$$

Lemma 2.9. *i)   If $(a_{i,j}) \in \mathcal{O}_n^-$, $n \geq 0$ and $(\sigma_{k,j})$ is an arbitrary non-negative lower triangular matrix, then $(\psi_{k,j})$ belongs to the class $\mathcal{O}_n^-$.*

ii)   If $(\sigma_{i,k}) \in \mathcal{O}_m^-$, $m \geq 0$ and $(a_{i,j})$ is an arbitrary non-negative lower triangular matrix, then $(\rho_{j,k})$ belongs to the class $\mathcal{O}_m^-$.

iii)   If $(a_{i,j}) \in \mathcal{O}_n^+$, $n \geq 0$ and $(\sigma_{k,j})$ is an arbitrary non-negative lower triangular matrix, then $(\psi_{k,j})$ belongs to the class $\mathcal{B}\mathcal{O}_n^+$, where $\mathcal{B} = \{\mathcal{B}_k\}_{k=1}^{\infty}$ and $\mathcal{B}_k = \sum\limits_{i=k}^{\infty} \sigma_{i,k}$.

iv)   If $(\sigma_{k,j}) \in \mathcal{O}_m^+$, $m \geq 0$ and $(a_{i,j})$ is an arbitrary non-negative lower triangular matrix, then $(\rho_{j,k})$ belongs to the class $\mathcal{A}\mathcal{O}_m^+$, where $\mathcal{A} = \{\mathcal{A}_j\}_{j=1}^{\infty}$ and $\mathcal{A}_j = \sum\limits_{i=j}^{\infty} a_{i,j}$.

Proof of Lemma 2.9

i) Since $(a_{i,j}) \in \mathcal{O}_n^-$, there exist matrices $(a_{i,j}^{(\gamma)}) \in \mathcal{O}_\gamma^-$, $\gamma = 0, 1, ..., n-1$, and matrices $(d_{s,j}^{\gamma,n})$ such that

$$a_{i,j} \equiv a_{i,j}^{(n)} \approx \sum_{\gamma=0}^{n} a_{i,s}^{(\gamma)} d_{s,j}^{\gamma,n}, \qquad d_{s,j}^{n,n} \equiv 1$$

for all $i \geq s \geq j \geq 1$.

We set

$$\psi_{k,j} \equiv \psi_{k,j}^{(n)} = \sum_{i=k}^{\infty} \sigma_{i,k} a_{i,j}^{(n)}.$$

First we consider the case when $n = 0$. In this case $\psi_{k,j}^{(0)} = \sum\limits_{i=k}^{\infty} \sigma_{i,k} \alpha_i \equiv \widetilde{\alpha}_k$, which means that $(\psi_{k,j}^{(0)})$ belongs to the class $\mathcal{O}_0^-$. By induction we assume that for $n = 0, 1, ..., r-1$, $r \geq 1$ $(\psi_{k,j}^{(n)})$ belong to the classes $\mathcal{O}_n^-$. For $k \geq s \geq j$ we have

$$\psi_{k,j}^{(r)} \approx \sum_{i=k}^{\infty} \sigma_{i,k} \sum_{\gamma=0}^{r} a_{i,s}^{(\gamma)} d_{s,j}^{\gamma,r} = \sum_{\gamma=0}^{r} d_{s,j}^{\gamma,r} \sum_{j=k}^{\infty} \sigma_{i,k} a_{i,s}^{(\gamma)} = \sum_{\gamma=0}^{r} \psi_{k,s}^{(\gamma)} d_{s,j}^{\gamma,r},$$

which means that $(\psi_{k,j}^{(r)})$ belongs to the class $\mathcal{O}_r^-$, $r \geq 0$.

ii) Since $(\sigma_{i,k}) \in \mathcal{O}_m^-$, there exist matrices $(\sigma_{i,j}^{(\gamma)}) \in \mathcal{O}_\gamma^-$, $\gamma = 0, 1, ..., m-1$, and matrices $(e_{s,k}^{\gamma,m})$ such that

$$\sigma_{i,k} \equiv \sigma_{i,k}^{(m)} \approx \sum_{\gamma=0}^{m} \sigma_{i,s}^{(\gamma)} e_{s,k}^{\gamma,m}, \qquad e_{s,k}^{m,m} \equiv 1$$

for all $i \geq s \geq k \geq 1$.

We set

$$\rho_{j,k} \equiv \rho_{j,k}^{(m)} = \sum_{i=j}^{\infty} a_{i,j} \sigma_{i,k}^{(m)}.$$

First we consider the case when $m = 0$. In this case $\rho_{j,k}^{(0)} = \sum\limits_{i=j}^{\infty} a_{i,j} \beta_i \equiv \widetilde{\beta}_j$, which means that $(\rho_{j,k}^{(0)})$ belongs to the class $\mathcal{O}_0^-$. By induction we assume that for $m = 0, 1, ..., r-1$, $r \geq 1$ $(\rho_{j,k}^{(0)})$ belong to the classes $\mathcal{O}_m^-$. For $i \geq j \geq k$ we have

$$\rho_{j,k}^{(r)} \approx \sum_{i=j}^{\infty} a_{i,j} \sum_{\gamma=0}^{r} \sigma_{i,s}^{(\gamma)} e_{s,k}^{\gamma,r} = \sum_{\gamma=0}^{r} e_{s,k}^{\gamma,r} \sum_{i=j}^{\infty} a_{i,j} \sigma_{i,s}^{(\gamma)} = \sum_{\gamma=0}^{r} \rho_{j,s}^{(\gamma)} e_{s,k}^{\gamma,r},$$

which implies that $(\rho_{j,k}^{(r)})$ belongs to the class $\mathcal{O}_r^-$, $r \geq 0$.

iii) Since $(a_{i,j}) \in \mathcal{O}_n^+$, there exist matrices $(a_{s,j}^{(\gamma)}) \in \mathcal{O}_\gamma^+$, $\gamma = 0, 1, ..., n-1$, and matrices $(d_{i,s}^{n,\gamma})$ such that

$$a_{i,j} \equiv a_{i,j}^{(n)} \approx \sum_{\gamma=0}^{n} d_{i,s}^{n,\gamma} a_{s,j}^{(\gamma)}, \qquad d_{i,s}^{n,n} \equiv 1$$

for all $i \geq s \geq j \geq 1$.

Let $k \geq s \geq j$. Then we have

$$
\begin{aligned}
\psi_{k,j} \;\equiv\; \psi_{k,j}^{(n)} &= \sum_{i=k}^{\infty} \sigma_{i,k} a_{i,j}^{(n)} \approx \sum_{i=k}^{\infty} \sigma_{i,k} \sum_{\gamma=0}^{n} d_{i,s}^{n,\gamma} a_{s,j}^{(\gamma)} \\
&= a_{s,j}^{(n)} \sum_{i=k}^{\infty} \sigma_{i,k} + \sum_{\gamma=0}^{n-1} a_{s,j}^{(\gamma)} \sum_{i=k}^{\infty} \sigma_{i,k} d_{i,s}^{n,\gamma} \\
&= \mathcal{B}_k \left(a_{s,j}^{(n)} + \sum_{\gamma=0}^{n-1} \mathcal{D}_{k,s}^{n,\gamma} a_{s,j}^{(\gamma)} \right) = \mathcal{B}_k \widehat{a}_{k,j},
\end{aligned}
$$

where $\mathcal{B}_k = \sum\limits_{i=k}^{\infty} \sigma_{i,k}$, $\widehat{a}_{k,j} = a_{s,j}^{(n)} + \sum\limits_{\gamma=0}^{n-1} \mathcal{D}_{k,s}^{n,\gamma} a_{s,j}^{(\gamma)}$ and $\mathcal{D}_{k,s}^{n,\gamma} = \frac{1}{\mathcal{B}_k} \left(\sum\limits_{i=k}^{\infty} \sigma_{i,k} d_{i,s}^{n,\gamma} \right)$.

By definition of $(\widehat{a}_{k,j})$ we see that $(\widehat{a}_{k,j}) \in \mathcal{O}_n^+$, $n \geq 0$. Therefore $(\psi_{k,j})$ belongs to the class $\mathcal{BO}_n^+$, $n \geq 0$.

iv) Since $(\sigma_{i,k}) \in \mathcal{O}_m^+$, there exist matrices $(\sigma_{s,k}^{(\gamma)}) \in \mathcal{O}_\gamma^+$, $\gamma = 0, 1, ..., m-1$, and matrices $(e_{i,s}^{m,\gamma})$ such that

$$
\sigma_{i,k} \equiv \sigma_{i,k}^{(m)} \approx \sum_{\gamma=0}^{m} e_{i,s}^{m,\gamma} \sigma_{s,k}^{(\gamma)}, \qquad e_{i,s}^{m,m} \equiv 1
$$

for all $i \geq s \geq k \geq 1$.

Let $i \geq j \geq k$. Then we have

$$
\begin{aligned}
\rho_{j,k} \;\equiv\; \rho_{j,k}^{(m)} &= \sum_{i=j}^{\infty} a_{i,j} \sigma_{i,k}^{(m)} \approx \sum_{i=j}^{\infty} a_{i,j} \sum_{\gamma=0}^{m} e_{i,s}^{m,\gamma} \sigma_{s,k}^{(\gamma)} \\
&= \sigma_{s,k}^{(m)} \sum_{i=j}^{\infty} a_{i,j} + \sum_{\gamma=0}^{m-1} \sigma_{s,k}^{(\gamma)} \sum_{i=j}^{\infty} a_{i,j} e_{i,s}^{m,\gamma} \\
&= \mathcal{A}_j \left(\sigma_{s,k}^{(m)} + \sum_{\gamma=0}^{m-1} \mathcal{E}_{j,s}^{m,\gamma} \sigma_{s,k}^{(\gamma)} \right) = \mathcal{A}_j \widehat{\sigma}_{j,k},
\end{aligned}
$$

where $\mathcal{A}_j = \sum\limits_{i=j}^{\infty} a_{i,j}$, $\widehat{\sigma}_{j,k} = \sigma_{s,k}^{(m)} + \sum\limits_{\gamma=0}^{m-1} \mathcal{E}_{j,s}^{m,\gamma} \sigma_{s,k}^{(\gamma)}$ and $\mathcal{E}_{j,s}^{m,\gamma} = \frac{1}{\mathcal{A}_i} \left(\sum\limits_{i=j}^{\infty} a_{i,j} e_{i,s}^{m,\gamma} \right)$.

By definition of $(\widehat{\sigma}_{j,k})$ we see that $(\widehat{\sigma}_{j,k}) \in \mathcal{O}_m^+$, $m \geq 0$. Hence, $(\rho_{j,k})$ belongs to the class $\mathcal{AO}_m^+$, $m \geq 0$.

Thus the proof of Lemma 2.9 is complete.

2.3 Examples of matrices of the classes $\alpha\mathcal{O}_n^{\pm}$ and $\mathcal{O}_n^{\pm}\beta$.

In this section we give examples of matrices of the classes $\alpha\mathcal{O}_n^{\pm}$, $\mathcal{O}_n^{\pm}\beta$, $n \geq 0$, which can be estimated in $l_{p,v}$ and on the cone of monotone sequences by using our main results.

1. We consider an operator of multiple summation

$$(S_n f)_i = \sum_{k_1=1}^{i} \omega_{1,k_1} \sum_{k_2=1}^{k_1} \omega_{2,k_2} \sum_{k_3=1}^{k_2} \omega_{3,k_3} ... \sum_{k_{n-1}=1}^{k_{n-2}} \omega_{n-1,k_{n-1}} \sum_{j=1}^{k_{n-1}} f_j. \tag{2.19}$$

By changing the order of summation in (2.19), we obtain

$$(S_n f)_i = \sum_{j=1}^{i} f_j W_{n-1,1}(i,j), \tag{2.20}$$

where $W_{n-1,1}(i,j) \equiv 1$ for $n = 1$ and

$$W_{n-1,1}(i,j) = \sum_{k_{n-1}=j}^{i} \omega_{n-1,k_{n-1}} \sum_{k_{n-2}=k_{n-1}}^{i} \omega_{n-2,k_{n-2}} ... \sum_{k_1=k_2}^{i} \omega_{1,k_1}$$

for $n \geq 2$.

The boundedness and compactness criteria of such operators have been established in [117].

Assume that $W_{l,m}(i,j) \equiv 1$ for $m > l$ and

$$W_{l,m}(i,j) = \sum_{k_l=j}^{i} \omega_{l,k_l} \sum_{k_{l-1}=k_l}^{i} \omega_{l-1,k_{l-1}} ... \sum_{k_m=k_{m+1}}^{i} \omega_{m,k_m}$$

for $n - 1 \geq l \geq m \geq 1$.

By Lemma 1 of [117] it easily follows that

$$W_{l,m}(i,j) \approx \sum_{r=m}^{l+1} W_{r-1,m}(i,\tau) W_{l,r}(\tau,j) \tag{2.21}$$

for $n - 1 \geq l \geq m \geq 1$ and for all i, τ, j such that $1 \leq j \leq \tau \leq i < \infty$.

For $m = 1$ and $n - 1 \geq l \geq 1$ (2.21) implies that

$$W_{l,1}(i,j) \approx \sum_{r=0}^{l} W_{r,1}(i,\tau) W_{l,r+1}(\tau,j) \tag{2.22}$$

for all $i \geq \tau \geq j \geq 1$.

We define $a^{(r)}(i,j) \equiv W_{r,1}(i,j)$ for $r = 0, 1, ..., l$. Then we have

$$a^{(l)}(i,j) \approx \sum_{r=0}^{l} a^{(r)}(i,\tau)W_{l,r+1}(\tau,j) \tag{2.23}$$

for $i \geq \tau \geq j \geq 1$. If we prove that $(a^{(l)}(i,j)) \in \mathcal{O}_l^-$ for $l = 0, 1, ..., n-1$, then we obtain that $(a^{(n-1)}(i,j)) \equiv (W_{n-1,1}(i,j)) \in \mathcal{O}_{n-1}^-$.

Indeed, if $l = 0$, then $a^{(0)}(i,j) = W_{0,1}(i,j) \equiv 1$, which implies that $(a^{(0)}(i,j)) \in \mathcal{O}_0^-$.

For $l = 1$, by (2.23) and (2.18) we have

$$a^{(1)}(i,j) \approx a^{(1)}(i,\tau) + a^{(1)}(\tau,j)$$

for $i \geq \tau \geq j \geq 1$. Hence, $(a^{(1)}(i,j)) \in \mathcal{O}_1^-$.

Assume that $(a^{(r)}(i,j)) \in \mathcal{O}_r^-$ for $r = 0, 1, ..., l-1$, $n-1 \geq l > 1$. Then taking into account that $W_{l,l+1}(\tau,j) \equiv 1$, by (2.23) and (2.18) we deduce that $(a^{(l)}(i,j)) \in \mathcal{O}_l^-$. Consequently, $(W_{n-1,1}(i,j)) \in \mathcal{O}_{n-1}^-$ for $n \geq 1$.

Next we prove that $(W_{n-1,1}(i,j)) \in \mathcal{O}_{n-1}^+$ for $n \geq 1$.

For $m = 1$ and $l = n - 1$ relation (2.21) implies that

$$W_{n-1,1}(i,j) \approx \sum_{k=0}^{n-1} W_{n-k-1,1}(i,\tau)W_{n-1,n-k}(\tau,j)$$

for all $i \geq \tau \geq j \geq 1$.

We define $a^{(k)}(i,j) \equiv W_{n-1,n-k}(i,j)$ for $k = 0, 1, ..., n-1$. Then we have $W_{n-1,1}(i,j) \equiv a^{(n-1)}(i,j)$ and

$$a^{(n-1)}(i,j) \approx \sum_{k=0}^{n-1} W_{n-k-1,1}(i,\tau)a^{(k)}(\tau,j), \qquad i \geq \tau \geq j.$$

Since $W_{0,1}(i,\tau) \equiv 1$, if $(a^{(k)}(i,j)) \in \mathcal{O}_k^+$ for $k = 0, 1, ...n-2$, then we obtain that $(a^{(n-1)}(i,j)) \in \mathcal{O}_{n-1}^+$.

By assuming that $l = n - 1$ and $m = n - k$, $0 \leq k \leq n - 1$ in (2.21), we obtain

$$\begin{aligned} a^{(k)}(i,j) &\approx \sum_{r=n-k}^{n} W_{r-1,n-k}(i,\tau)W_{n-1,r}(\tau,j) \tag{2.24} \\ &= \sum_{r=0}^{k} W_{n-r-1,n-k}(i,\tau)a^{(r)}(\tau,j) \end{aligned}$$

for all $i \geq \tau \geq j \geq 1$.

If $k = 0$, then $a^{(0)}(i,j) \equiv W_{n-1,n}(i,j) \equiv 1$. Therefore, $(a^{(0)}(i,j)) \in \mathcal{O}_0^+$.

If $k = 1$ by (2.12) and (2.24) we obtain that

$$
\begin{aligned}
a^{(1)}(i,j) &\approx W_{n-1,n-1}(i,\tau)a^{(0)}(\tau,j) + W_{n-2,n-1}(i,\tau)a^{(1)}(\tau,j) \\
&= a^{(1)}(i,\tau) + a^{(1)}(\tau,j),
\end{aligned}
$$

which implies that $(a^{(1)}(i,j)) \in \mathcal{O}_1^+$.

Assume that $(a^{(r)}(i,j)) \in \mathcal{O}_r^+$ for $r = 0,1,...,k-1$, $k > 1$. Since $W_{n-k-1,n-k}(i,\tau) \equiv 1$ for $k = 0,1,...,n-1$ from (2.24) it follows that $(a^{(k)}(i,j)) \in \mathcal{O}_k^+$ for $k = 0,1,...,n-1$. Therefore, $(W_{n-1,1}(i,j)) \in \mathcal{O}_{n-1}^+$ for $n \geq 1$.

Thus, we prove that the matrix $(W_{n-1,1}(i,j))$ belongs to the class $\mathcal{O}_{n-1}^+ \cap \mathcal{O}_{n-1}^-$.

2. By definition, the series $\sum\limits_{j=1}^{\infty} f_j$ is summable with sum s by the Hölder method (H, n) if

$$
\lim_{i \to \infty} H_i^n = s,
$$

where

$$
H_i^0 = s_i = \sum_{j=1}^{i} f_j,
$$

$$
H_i^n = \frac{H_1^{n-1} + ... + H_i^{n-1}}{i}, \quad i = 1,2,...
$$

This summation method was introduced by O. Hölder [118] (see also [119]) as a generalization of the summation method of arithmetic averages.

The expression H_i^n can be written in the following form

$$
H_i^n \equiv (H^n f)_i = \frac{1}{i} \sum_{k_1=1}^{i} \frac{1}{k_1} \sum_{k_2=1}^{k_1} \frac{1}{k_2} \cdots \sum_{k_{n-1}=1}^{k_{n-2}} \frac{1}{k_{n-1}} \sum_{k_n=1}^{k_{n-1}} \sum_{j=1}^{k_n} f_j.
$$

If we consider the operator of multiple summation of order $(n+1)$ by taking $\omega_{l,k_l} = \frac{1}{k_l}$ for $1 \leq l \leq n-1$ and $\omega_{n,k_n} = 1$, then we obtain

$$
(S_{n+1}f)_i = \sum_{k_1=1}^{i} \frac{1}{k_1} \sum_{k_2=1}^{k_1} \frac{1}{k_2} \cdots \sum_{k_{n-1}=1}^{k_{n-2}} \frac{1}{k_{n-1}} \sum_{k_n=1}^{k_{n-1}} \sum_{j=1}^{k_n} f_j = i(H^n f)_i.
$$

According to the previous example, the matrix of the operator of multiple summation of order $(n+1)$ belongs to the class $\mathcal{O}_n^+ \cap \mathcal{O}_n^-$, $n \geq 0$. Therefore, we obtain that the Hölder's matrix of order n belongs to the class $\varphi \mathcal{O}_n^+ \cap \varphi \mathcal{O}_n^-$, $n \geq 0$, where $\varphi = \{\frac{1}{i}\}_{i=1}^{\infty}$.

3. We consider Cesàro summation method, which is a matrix summation method with matrix $(a_{i,j}^k)$, where

$$a_{i,j}^k = \begin{cases} \frac{T_{i-j}^{k-1}}{T_i^k}, & j \leq i, \\ 0, & j > i \end{cases}$$

and $T_i^k = \frac{(k+1)(k+2)...(k+i)}{i!}$.

If $k = 1$, then $a_{i,j}^1 = \frac{1}{i+1}$, which gives a Hardy operator and we already know how to estimate a Hardy operator in weighted spaces of sequences, on the cone of monotone sequences (see e.g. [3]).

If $k \geq 2$ for $j \leq i$, we have

$$a_{i,j}^k = \frac{T_{i-j}^{k-1}}{T_i^k} = \frac{k}{(i+1)(i+2)...(i+k)}(i-j+1)(i-j+2)...(i-j+k-1).$$

We set

$$a_{i,j}^k = \psi_i^k \widetilde{a}_{i,j}^{(k-1)},$$

where $\widetilde{a}_{i,j}^{(k-1)} = (i-j+1)(i-j+2)...(i-j+k-1)$ and $\psi_i^k = \frac{k}{(i+1)(i+2)...(i+k)}$.

Now we prove that the Cesàro matrix $(a_{i,j}^k)$ belongs to the class $\psi^{(k)}\mathcal{O}_{k-1}^+ \cap \psi^{(k)}\mathcal{O}_{k-1}^-$, $k \geq 1$, where $\psi^{(k)} = \{\psi_i^k\}_{i=1}^{\infty}$. For that purpose, we need to prove that $(\widetilde{a}_{i,j}^{(k-1)}) \in \mathcal{O}_{k-1}^+ \cap \mathcal{O}_{k-1}^-$, $k \geq 2$.

We assert that for $\forall l \geq 1$ the following factorization is valid

$$\widetilde{a}_{i,j}^{(l)} = (i-j+1)(i-j+2)...(i-j+l) = \sum_{\gamma=0}^{l} \widetilde{a}_{i,s}^{(\gamma)} d_{s,j}^{\gamma,l} \quad \forall i \geq s \geq j, \quad (2.25)$$

where $d_{s,j}^{l,l} \equiv 1$ for any $l \geq 0$, $d_{s,j}^{-1,l} \equiv 0$ for any $l \geq 0$, and $d_{s,j}^{\gamma,l} = d_{s,j}^{\gamma-1,l-1} + d_{s,j}^{\gamma,l-1}(l-\gamma-1+s-j)$ for $\gamma = 0,...,l-1$, $l \geq 1$.

If $l = 1$, then $\widetilde{a}_{i,j}^{(1)} = (i-j+1) = (i-s+1) + (s-j) = \widetilde{a}_{i,s}^{(1)} + d_{s,j}^{0,1}$, where $d_{s,j}^{0,1} = s-j$ and $\widetilde{a}_{i,s}^{(0)} = 1$.

By induction, we assume that (2.25) holds for $l = 2, ..., r-1$, $r \geq 2$ and we prove that (2.25) holds for $l = r$. We assume that $d_{s,j}^{\gamma,r-1} = 0$ if $\gamma < 0$. For $\forall i \geq s \geq j$ we obtain

$$
\begin{aligned}
\widetilde{a}_{i,j}^{(r)} &= (i-j+1)(i-j+2)...(i-j+r) = \widetilde{a}_{i,j}^{(r-1)}(i-j+r) \\
&= \sum_{\gamma=0}^{r-1} \widetilde{a}_{i,s}^{(\gamma)} d_{s,j}^{\gamma,r-1} \left((i-s+\gamma+1) + (r-\gamma-1) + (s-j) \right) \\
&= \sum_{\gamma=0}^{r-1} \widetilde{a}_{i,s}^{(\gamma)} d_{s,j}^{\gamma,r-1}(i-s+\gamma+1) \\
&\quad + \sum_{\gamma=0}^{r-1} \left(\widetilde{a}_{i,s}^{(\gamma)} d_{s,j}^{\gamma,r-1}(r-\gamma-1) + \widetilde{a}_{i,s}^{(\gamma)} d_{s,j}^{\gamma,r-1}(s-j) \right) \\
&= \sum_{\gamma=0}^{r-1} \widetilde{a}_{i,s}^{(\gamma+1)} d_{s,j}^{\gamma,r-1} + \sum_{\gamma=0}^{r-1} \widetilde{a}_{i,s}^{(\gamma)} d_{s,j}^{\gamma,r-1}(r-\gamma-1+s-j) \\
&= \sum_{\gamma=1}^{r} \widetilde{a}_{i,s}^{(\gamma)} d_{s,j}^{\gamma-1,r-1} + \sum_{\gamma=1}^{r-1} \widetilde{a}_{i,s}^{(\gamma)} d_{s,j}^{\gamma,r-1}(r-\gamma-1+s-j) \\
&\quad + d_{s,j}^{0,r-1}(r-1+s-j) \\
&= \widetilde{a}_{i,s}^{(r)} + \sum_{\gamma=1}^{r-1} \left(\widetilde{a}_{i,s}^{(\gamma)} d_{s,j}^{\gamma-1,r-1} + \widetilde{a}_{i,s}^{(\gamma)} d_{s,j}^{\gamma,r-1}(r-\gamma-1+s-j) \right) \\
&\quad + d_{s,j}^{0,r-1}(r-1+s-j) \\
&= \widetilde{a}_{i,s}^{(r)} + \sum_{\gamma=0}^{r-1} \widetilde{a}_{i,s}^{(\gamma)}(d_{s,j}^{\gamma-1,r-1} + d_{s,j}^{\gamma,r-1}(r-\gamma-1+s-j)) \\
&= \widetilde{a}_{i,s}^{(r)} + \sum_{\gamma=0}^{r-1} \widetilde{a}_{i,s}^{(\gamma)} d_{s,j}^{\gamma,r},
\end{aligned}
$$

where $d_{s,j}^{\gamma,r} = d_{s,j}^{\gamma-1,r-1} + d_{s,j}^{\gamma,r-1}(r-\gamma-1+s-j)$.

By definition of classes $\mathcal{O}_l^-$, $l \geq 1$ we see that the matrix $(\widetilde{a}_{i,j}^{(1)})$ belongs to the class $\mathcal{O}_1^-$. We assume that the matrices $(\widetilde{a}_{i,j}^{(l)})$ belong to the classes $\mathcal{O}_l^-$ for $l = 2, ...r-1$. We see that for $(\widetilde{a}_{i,j}^{(r)})$ equality (2.25) holds. By assumption $(\widetilde{a}_{i,j}^{(l)})$ belong to the classes $\mathcal{O}_l^-$ for $l = 1, 2, ...r-1$. Then equality (2.25) implies that $(\widetilde{a}_{i,j}^{(r)})$ belongs to the class $\mathcal{O}_r^-$.

In the same way one proves that for $(\widetilde{a}_{i,j}^{(l)})$, $l \geq 1$ the following factorization

is valid

$$\widetilde{a}_{i,j}^{(l)} = \sum_{\gamma=0}^{l} e_{i,s}^{l,\gamma} \widetilde{a}_{s,j}^{(\gamma)} \quad \forall i \geq s \geq j, \tag{2.26}$$

where $e_{i,s}^{l,l} \equiv 1$ for any $l \geq 0$, $e_{i,s}^{-1,l} \equiv 1$ for any $l \geq 0$, and $e_{i,s}^{l,\gamma} = e_{i,s}^{l-1,\gamma-1} + e_{i,s}^{l-1,\gamma}(i - s + l - \gamma - 1)$ for $\gamma = 0, ..., l-1$, $l \geq 1$. Moreover, as above we obtain that $(\widetilde{a}_{i,j}^{(l)})$ belongs to the class $\mathcal{O}_l^+$, $l \geq 1$.

Therefore, we see that $(\widetilde{a}_{i,j}^{(k-1)}) \in \mathcal{O}_{k-1}^+ \cap \mathcal{O}_{k-1}^-$, $k \geq 1$. This implies that the Cesàro matrix $(a_{i,j}^k)$ belongs to the class $\psi^{(k)}\mathcal{O}_{k-1}^+ \cap \psi^{(k)}\mathcal{O}_{k-1}^-$, $k \geq 1$.

2.4 Necessary and sufficient conditions for the boundedness of matrix operators in weighted spaces of sequences, the case $1 < p \le q < \infty$.

In this section we give necessary and sufficient conditions for the boundedness of the operators A^+ and A^- from the weighted $l_{p,v}$ space into the $l_{q,u}$ space when the corresponding matrices belong to one of the classes $\mathcal{O}_n^+$ and $\mathcal{O}_n^-$, $n \ge 0$.

We define

$$\left(\mathcal{B}_{p,q}^+\right)_k = \left(\sum_{j=1}^{k} v_j^{-p'} \left(\sum_{i=k}^{\infty} a_{i,j}^q u_i^q\right)^{\frac{p'}{q}}\right)^{\frac{1}{p'}},$$

$$\left(\mathcal{B}_{p,q}^-\right)_k = \left(\sum_{i=k}^{\infty} u_i^q \left(\sum_{j=1}^{k} a_{i,j}^{p'} v_j^{-p'}\right)^{\frac{q}{p'}}\right)^{\frac{1}{q}},$$

$$\left(\mathcal{A}_{p,q}^+\right)_k = \left(\sum_{j=1}^{k} u_j^q \left(\sum_{i=k}^{\infty} a_{i,j}^{p'} v_i^{-p'}\right)^{\frac{q}{p'}}\right)^{\frac{1}{q}},$$

$$\left(\mathcal{A}_{p,q}^-\right)_k = \left(\sum_{i=k}^{\infty} v_i^{-p'} \left(\sum_{j=1}^{k} a_{i,j}^q u_j^q\right)^{\frac{p'}{q}}\right)^{\frac{1}{p'}}.$$

We set $\mathcal{B}^+ = \sup_{k \ge 1} \left(\mathcal{B}_{p,q}^+\right)_k$, $\mathcal{B}^- = \sup_{k \ge 1} \left(\mathcal{B}_{p,q}^-\right)_k$, $\mathcal{A}^+ = \sup_{k \ge 1} \left(\mathcal{A}_{p,q}^+\right)_k$ and $\mathcal{A}^- = \sup_{k \ge 1} \left(\mathcal{A}_{p,q}^-\right)_k$.

Theorem 2.10. *Let $1 < p \le q < \infty$. Let the matrix $(a_{i,j})$ in (2.1) belong to the class $\mathcal{O}_n^+$, $n \ge 0$. Then the estimate (2.3) for the operator defined by (2.1) holds if and only if at least one of the conditions $\mathcal{B}^+ < \infty$ and $\mathcal{B}^- < \infty$ holds. Moreover, $\mathcal{B}^+ \approx \mathcal{B}^- \approx C$, where C is the best constant in (2.3).*

Theorem 2.11. *Let $1 < p \leq q < \infty$. Let the matrix $(a_{i,j})$ in (2.2) belong to the class $\mathcal{O}_m^-$, $m \geq 0$. Then the estimate (2.3) for the operator defined by (2.2) holds if and only if at least one of the conditions $\mathcal{A}^+ < \infty$ and $\mathcal{A}^- < \infty$ holds. Moreover, $\mathcal{A}^+ \approx \mathcal{A}^- \approx C$, where C is the best constant in (2.3).*

Here we present only the proof of Theorem 2.11, since the proof of Theorem 2.10 is very similar.

For the proof of Theorem 2.11 we need the following.

Lemma 2.12. *Let the matrix in (2.2) belong to the class $\mathcal{O}_m^-$, $m \geq 0$. Then for $k \geq 1$ we have the following equivalence*

$$\left(\mathcal{A}_{p,q}^+\right)_k \approx \left(\mathcal{A}_m\right)_k \equiv \max_{0 \leq \gamma \leq m} \left(\mathcal{A}_{\gamma,m}\right)_k \approx \left(\mathcal{A}_{p,q}^-\right)_k, \tag{2.27}$$

where

$$\left(\mathcal{A}_{\gamma,m}\right)_k = \left(\sum_{j=1}^{k} \left(d_{k,j}^{\gamma,m}\right)^q u_j^q\right)^{\frac{1}{q}} \left(\sum_{i=k}^{\infty} \left(a_{i,k}^{(\gamma)}\right)^{p'} v_i^{-p'}\right)^{\frac{1}{p'}}.$$

By (2.27) it follows that

$$\mathcal{A}^+ \approx \mathcal{A}_m = \sup_{k \geq 1} \left(\mathcal{A}_m\right)_k \approx \mathcal{A}^-, \quad \forall m \geq 0. \tag{2.28}$$

Indeed, this equivalence follows from (2.18).

Proof of Theorem 2.11. *Necessity.* Suppose that the matrix of the operator (2.2) belongs to the class $\mathcal{O}_m^-$, $m \geq 0$ and (2.3) holds.

For $k > 1$ we assume that $\widetilde{g} = \{\widetilde{g}_i\}_{i=1}^{\infty}$: $\quad \widetilde{g}_i = \begin{cases} u_i, & 1 \leq i \leq k \\ 0, & i > k. \end{cases}$

It is known that inequality (2.3) holds if and only if the following dual inequality

$$\|A^* g\|_{p',v^{-1}} \leq C \|g\|_{q',u^{-1}}, \qquad g \in l_{q',u^{-1}} \tag{2.29}$$

holds for the conjugate operator A^*, which coincides with the operator defined by (2.1). Moreover, the best constants in (2.3) and (2.29) coincide (see e.g., [2]).

Hence, choosing $g = \widetilde{g}$ in (2.29) and by exploiting (2.17), we obtain

$$Ck^{\frac{1}{q'}} \geq \|A^*\widetilde{g}\|_{p',v^{-1}} \geq \left(\sum_{i=k}^{\infty} \left(\sum_{j=1}^{k} a_{i,j}^{(m)} u_j \right)^{p'} v_i^{-p'} \right)^{\frac{1}{p'}}$$

$$\gg \left(\sum_{i=k}^{\infty} \left(a_{i,k}^{(\gamma)} \right)^{p'} v_i^{-p'} \right)^{\frac{1}{p'}} \left(\sum_{j=1}^{k} d_{k,j}^{\gamma,m} u_j \right), \quad \gamma = 0, 1, \ldots, m.$$

Therefore $\{a_{i,k}^{(\gamma)}\}_{i=1}^{\infty} \in l_{p',v^{-1}}$.

Now for $1 \leq r < M < \infty$, we assume that $\widetilde{f} = \{\widetilde{f}_s\}_{s=1}^{\infty}$, where

$$\widetilde{f}_s = \begin{cases} (a_{s,r}^{(\gamma)})^{p'-1} v_s^{-p'}, & r \leq s \leq M \\ 0, & s < r \quad \text{or} \quad s > M. \end{cases}$$

By choosing $f = \widetilde{f}$ in inequality (2.3) and exploiting (2.17), we find that

$$C \left(\sum_{s=r}^{M} \left(a_{s,r}^{(\gamma)} \right)^{p'} v_s^{-p'} \right)^{\frac{1}{p}} \geq \|A^-\widetilde{f}\|_{q,u} = \left(\sum_{j=1}^{\infty} \left(\sum_{s=j}^{\infty} a_{s,j}^{(m)} \widetilde{f}_s \right)^q u_j^q \right)^{\frac{1}{q}}$$

$$\gg \left(\sum_{j=1}^{r} \left(d_{r,j}^{\gamma,m} \right)^q u_j^q \right)^{\frac{1}{q}} \left(\sum_{s=r}^{M} \left(a_{s,r}^{(\gamma)} \right)^{p'} v_s^{-p'} \right),$$

which implies that

$$C \gg \left(\sum_{j=1}^{r} \left(d_{r,j}^{\gamma,m} \right)^q u_j^q \right)^{\frac{1}{q}} \left(\sum_{s=r}^{M} \left(a_{s,r}^{(\gamma)} \right)^{p'} v_s^{-p'} \right)^{\frac{1}{p'}}. \tag{2.30}$$

Since inequality (2.30) holds for all $\gamma = 0, 1, \ldots, m$ and $r \geq 1$ is arbitrary, passing to the limit as $M \to \infty$ we have

$$\sup_{k \geq 1} (\mathcal{A}_m)_k \ll C. \tag{2.31}$$

Then by Lemma 2.12, we obtain

$$\mathcal{A}^+ \approx \mathcal{A}^- \ll C \tag{2.32}$$

and thus the proof of the necessity is complete.

Sufficiency. Let the matrix $(a_{i,j})$ of the operator (2.2) belong to the class $\mathcal{O}_m^-$, $m \geq 0$. Let $0 \leq f \in l_{p,v}$ and at least one of the conditions $\mathcal{A}^+ < \infty$ and $\mathcal{A}^- < \infty$ hold. Assume that $m = 0$. By the definition of $\mathcal{O}_0^-$, the matrix of

the operator (2.2) has the form $a_{i,j}^{(0)} = \beta_i \ \forall i \geq j \geq 1$. Then the estimate (2.3) coincides with the estimate (2.6) and the operator (2.2) is the matrix operator A_0^-. Hence from Theorem B it follows that

$$\|A_0^- f\|_{q,u} \ll \mathcal{A}_0 \|f\|_{p,v} \qquad \forall f \in l_{p,v}.$$

Based on Lemma 2.12 it follows that the inequality (2.3) holds for $m = 0$ and for the best constant in (2.3) the following estimate is valid

$$C \ll \mathcal{A}^+ \approx \mathcal{A}^-. \tag{2.33}$$

Now we assume that the inequality (2.3) holds for $m = 0, 1, \ldots, n-1$, $n \geq 1$ and for the best constant in (2.3) the estimate (2.33) is valid. We consider the inequality

$$\|A_m^- f\|_{q,u} \ll \mathcal{A}_m \|f\|_{p,v} \qquad \forall f \in l_{p,v}, \tag{2.34}$$

where A_m^- is given by (2.2) with the matrix $(a_{i,j}^{(m)}) \in \mathcal{O}_m^-$.

Now our aim is to show that the inequality (2.34) holds for $m = n$ with the estimate (2.33).

Let $h \equiv h_n$, where h_n is the constant in (2.14) with $m = n$. For all $j \geq 1$ we define the following set:

$$T_j = \{k \in \mathbb{Z} : (h+1)^{-k} \leq \left(A_n^- f\right)_j\},$$

where $\mathbb{Z}$ is the set of integers. We assume that $k_j = \inf T_j$, if $T_j \neq \emptyset$ and $k_j = \infty$, if $T_j = \emptyset$. In order to avoid trivial cases we directly suppose that $(A_n^- f)_1 \neq 0$. Since $a_{i,j}^{(n)}$ is non-increasing in j, we have $k_j \leq k_{j+1}$. If $k_j < \infty$, then

$$(h+1)^{-k_j} \leq \left(A_n^- f\right)_j < (h+1)^{-(k_j-1)}, \qquad j \geq 1. \tag{2.35}$$

Let $m_1 = 0$, $k_1 = k_{m_1+1}$ and $M_1 = \{j \in \mathbb{N} : k_j = k_1 = k_{m_1+1}\}$, where $\mathbb{N}$ is the set of natural numbers. Suppose that m_2 is such that $\sup M_1 = m_2$. Obviously $m_2 > m_1$ and if the set M_1 is bounded from above, then $m_2 < \infty$ and $m_2 = \max M_1$. We now define the numbers $0 = m_1 < m_2 < \cdots < m_s < \infty$,

$s \geq 1$ by induction. To define m_{s+1} we assume that $m_{s+1} = \sup M_s$, where $M_s = \{j \in \mathbb{N} : k_j = k_{m_s+1}\}$.

Let $N_0 = \{s \in \mathbb{N} : m_s < \infty\}$. Further, we assume that $k_{m_s+1} = n_{s+1}$, $s \in N_0$. From the definition of m_s and from (2.35) it follows that, for $s \in N_0$,

$$(h+1)^{-n_{s+1}} \leq \left(A_n^- f\right)_j < (h+1)^{-n_{s+1}+1}, \quad m_s + 1 \leq j \leq m_{s+1} \qquad (2.36)$$

and

$$\mathbb{N} = \bigcup_{s \in N_0} [m_s + 1, m_{s+1}], \quad \text{where } [m_s + 1, m_{s+1}] \cap [m_l + 1, m_{l+1}] = \emptyset, \ s \neq l.$$

Therefore, for $0 \leq f \in l_{p,v}$ the left-hand side of (2.3) has the following form

$$\|A_n^- f\|_{q,u}^q = \sum_{s \in N_0} \sum_{j=m_s+1}^{m_{s+1}} \left(A_n^- f\right)_j^q u_j^q. \qquad (2.37)$$

We assume that $\sum_{j=m_s+1}^{m_{s+1}} = 0$, if $m_s = \infty$.

There are two possible cases: $N_0 = \mathbb{N}$ and $N_0 \neq \mathbb{N}$.

1. If $N_0 = \mathbb{N}$, then we estimate (2.37) in the following way.

Clearly inequalities $n_{s+1} < n_{s+2} < n_{s+3}$ imply that $-n_{s+3} + 1 \leq -n_{s+1} - 1$ for all $s \in \mathbb{N}$. Hence, (2.36), (2.18) imply that

$$
\begin{aligned}
(h+1)^{-n_{s+1}-1} &= (h+1)^{-n_{s+1}} - h(h+1)^{-n_{s+1}-1} \qquad (2.38)\\
&\leq (h+1)^{-n_{s+1}} - h(h+1)^{-n_{s+3}+1} \\
&< \left(A_n^- f\right)_{m_s+1} - h\left(A_n^- f\right)_{m_s+3} \\
&= \sum_{i=m_s+1}^{\infty} a_{i,m_s+1}^{(n)} f_i - h \sum_{i=m_s+3}^{\infty} a_{i,m_s+3}^{(n)} f_i \\
&\leq \sum_{i=m_s+1}^{m_s+3} a_{i,m_s+1}^{(n)} f_i + \sum_{i=m_s+3}^{\infty} [a_{i,m_s+1}^{(n)} - h a_{i,m_s+3}^{(n)}] f_i \\
&\leq \sum_{i=m_s+1}^{m_s+3} a_{i,m_s+1}^{(n)} f_i + \sum_{i=m_s+3}^{\infty} [h \sum_{\gamma=0}^{n} a_{i,m_s+3}^{(\gamma)} d_{m_s+3,m_s+1}^{\gamma,n} - h a_{i,m_s+3}^{(n)}] f_i \\
&= \sum_{i=m_s+1}^{m_s+3} a_{i,m_s+1}^{(n)} f_i + h \sum_{i=m_s+3}^{\infty} \sum_{\gamma=0}^{n-1} a_{i,m_s+3}^{(\gamma)} d_{m_s+3,m_s+1}^{\gamma,n} f_i.
\end{aligned}
$$

Now, by using (2.36) and (2.38), we can estimate (2.37) in the following way.

$$\sum_{s \in N} \sum_{j=m_s+1}^{m_{s+1}} \left(A_n^- f\right)_j^q u_j^q < \sum_{s \in N} \sum_{j=m_s+1}^{m_{s+1}} (h+1)^{(-n_{s+1}+1)q} u_j^q \tag{2.39}$$

$$= (h+1)^{2q} \sum_{s \in N} (h+1)^{(-n_{s+1}-1)q} \sum_{j=m_s+1}^{m_{s+1}} u_j^q$$

$$\ll \sum_{s \in N} \left(\sum_{i=m_s+1}^{m_{s+3}} a_{i,m_{s+1}}^{(n)} f_i + h \sum_{\gamma=0}^{n-1} \sum_{i=m_{s+3}}^{\infty} a_{i,m_{s+3}}^{(\gamma)} d_{m_{s+3},m_{s+1}}^{\gamma,n} f_i \right)^q \sum_{j=m_s+1}^{m_{s+1}} u_j^q$$

$$\ll \sum_{s \in N} \left(\sum_{i=m_s+1}^{m_{s+3}} a_{i,m_{s+1}}^{(n)} f_i \right)^q \sum_{j=m_s+1}^{m_{s+1}} u_j^q$$

$$+ \sum_{\gamma=0}^{n-1} \sum_{s \in N} \left(d_{m_{s+3},m_{s+1}}^{\gamma,n}\right)^q \left(\sum_{i=m_{s+3}}^{\infty} a_{i,m_{s+3}}^{(\gamma)} f_i \right)^q \sum_{j=m_s+1}^{m_{s+1}} u_j^q := I_n + \sum_{\gamma=0}^{n-1} I_\gamma,$$

where

$$I_n = \sum_{s \in N} \left(\sum_{i=m_s+1}^{m_{s+3}} a_{i,m_{s+1}}^{(n)} f_i \right)^q \sum_{j=m_s+1}^{m_{s+1}} u_j^q$$

and

$$I_\gamma = \sum_{s \in N} \left(d_{m_{s+3},m_{s+1}}^{\gamma,n}\right)^q \left(\sum_{i=m_{s+3}}^{\infty} a_{i,m_{s+3}}^{(\gamma)} f_i \right)^q \sum_{j=m_s+1}^{m_{s+1}} u_j^q, \quad 0 \le \gamma \le n-1.$$

To estimate I_n we apply Hölder's and Jensen's inequalities and find that

$$I_n \le \sum_{s \in N} \left(\sum_{i=m_s+1}^{m_{s+3}} (a_{i,m_{s+1}}^{(n)})^{p'} v_i^{-p'} \right)^{\frac{q}{p'}} \sum_{j=m_s+1}^{m_{s+1}} u_j^q \left(\sum_{i=m_s+1}^{m_{s+3}} |f_i v_i|^p \right)^{\frac{q}{p}} \tag{2.40}$$

$$\le \left[\sup_{k \ge 1} \left(\sum_{j=1}^{k} u_j^q \right)^{\frac{1}{q}} \left(\sum_{i=k}^{\infty} (a_{i,k}^{(n)})^{p'} v_i^{-p'} \right)^{\frac{1}{p'}} \right]^q \sum_{s \in N} \left(\sum_{j=m_s+1}^{m_{s+3}} |f_i v_i|^p \right)^{\frac{q}{p}}$$

$$\le \left[\sup_{k \ge 1} \left(\sum_{j=1}^{k} (d_{k,j}^{n,n})^q u_j^q \right)^{\frac{1}{q}} \left(\sum_{i=k}^{\infty} (a_{i,k}^{(n)})^{p'} v_i^{-p'} \right)^{\frac{1}{p'}} \right]^q \left(\sum_{s \in N} \sum_{i=m_s+1}^{m_{s+3}} |f_i v_i|^p \right)^{\frac{q}{p}}$$

$$\ll \mathcal{A}_n^q \|f\|_{p,v}^q.$$

We introduce the sequence $\{\Delta_j\}_{j=1}^{\infty}$ defined by $\Delta_j = \left(d_{m_{s+3},m_{s+1}}^{\gamma,n}\right)^q \sum_{j=m_s+1}^{m_{s+1}} u_j^q$, $j = m_{s+3}$ and $\Delta_j = 0$, $j \ne m_{s+3}$, $s \in N$. Hence, we can rewrite I_γ,

$\gamma = 0, \ldots, n - 1$ in the following form.

$$I_\gamma = \sum_{s \in N} \left(\sum_{i=m_s+3}^{\infty} a_{i,m_s+3}^{(\gamma)} f_i \right)^q \left(d_{m_s+3,m_s+1}^{\gamma,n} \right)^q \sum_{i=m_s+1}^{m_{s+1}} u_i^q \qquad (2.41)$$

$$= \sum_{j=1}^{\infty} \left(\sum_{i=j}^{\infty} a_{i,j}^{(\gamma)} f_i \right)^q \Delta_j.$$

By the assumptions on $a_{i,j}^{(\gamma)}$, $\gamma = 0, \ldots, n-1$, $i \geq j \geq 1$, we have the validity of (2.34). Therefore,

$$I_\gamma \ll \widetilde{\mathcal{A}}_\gamma^q \|f\|_p^q, \quad \gamma = 0, \ldots, n-1, \qquad (2.42)$$

where

$$\widetilde{\mathcal{A}}_\gamma = \max_{0 \leq l \leq \gamma} \sup_{k \geq 1} \left(\sum_{j=1}^{k} \left(d_{k,j}^{l,\gamma} \right)^q \Delta_j \right)^{\frac{1}{q}} \left(\sum_{i=k}^{\infty} \left(a_{i,k}^{(l)} \right)^{p'} v_i^{-p'} \right)^{\frac{1}{p'}}. \qquad (2.43)$$

Using (2.16) and taking into account that $d_{i,j}^{l,n}$ is non-decreasing in i and non-increasing in j, we find that

$$\sum_{j=1}^{k} \left(d_{k,j}^{l,\gamma} \right)^q \Delta_j = \sum_{m_s+3 \leq k} \left(d_{k,m_s+3}^{l,\gamma} \right)^q \left(d_{m_s+3,m_s+1}^{\gamma,n} \right)^q \sum_{j=m_s+1}^{m_{s+1}} u_j^q \qquad (2.44)$$

$$\ll \sum_{m_s+3 \leq k} \sum_{i=m_s+1}^{m_{s+1}} \left(d_{k,i}^{l,n} \right)^q u_i^q \leq \sum_{i=1}^{k} \left(d_{k,i}^{l,n} \right)^q u_i^q.$$

By combining (2.42), (2.43), and (2.44), we obtain that

$$I_\gamma \ll \mathcal{A}_n^q \|f\|_{p,v}^q. \qquad (2.45)$$

Thus, from (2.39), (2.40), and (2.45) it follows that

$$\|A_n^- f\|_{q,u} \ll \mathcal{A}_n \|f\|_{p,v}, \qquad f \geq 0, \qquad (2.46)$$

which means that inequality (2.3) is valid. Hence, by Lemma 2.12, we obtain that

$$C \ll \mathcal{A}_n \approx \mathcal{A}^+ \approx \mathcal{A}^-. \qquad (2.47)$$

2. If $N_0 \neq N$, then $\max N_0 < \infty$ and $N_0 = \{1, 2, \ldots, s_0\}$, $s_0 \geq 1$. Therefore, $m_{s_0} < \infty$ and $m_{s_0+1} = \infty$. We assume that $\sum_{s=k}^{n} = 0$, if $k > n$ and

$\sum\limits_{s=k}^{n} = \sum\limits_{s=1}^{n}$, if $k \leq 0$. We have two possible cases: $n_{s_0+1} < \infty$ and $n_{s_0+1} = \infty$. We consider these cases separately.

1) If $n_{s_0+1} < \infty$, then from (2.37) it follows that

$$
\begin{aligned}
\|A_n^- f\|_{q,u}^q &= \sum_{s \in N_0} \sum_{j=m_s+1}^{m_{s+1}} \left(A_n^- f\right)_j^q u_j^q && (2.48) \\
&= \sum_{s=1}^{s_0} \sum_{j=m_s+1}^{m_{s+1}} \left(A_n^- f\right)_j^q u_j^q \\
&= \sum_{s=1}^{s_0-3} \sum_{j=m_s+1}^{m_{s+1}} \left(A_n^- f\right)_j^q u_j^q + \sum_{s=s_0-2}^{s_0-1} \sum_{j=m_s+1}^{m_{s+1}} \left(A_n^- f\right)_j^q u_j^q \\
&\quad + \sum_{j=m_{s_0}+1}^{\infty} \left(A_n^- f\right)_j^q u_j^q \\
&= J_1 + J_2 + J_3.
\end{aligned}
$$

If $J_1 \neq 0$, then for $s_0 > 3$, we estimate J_1 using (2.38) and the previous proof for the case $N_0 = \mathbb{N}$ as in estimate I_γ. Hence, we get

$$
J_1 \ll \mathcal{A}_n^q \|f\|_{p,v}^q. \tag{2.49}
$$

If $J_2 \neq 0$, then by using (2.36) and applying Hölder's and Jensen's inequalities, we obtain the following estimate

$$
\begin{aligned}
J_2 &= \sum_{s=s_0-2}^{s_0-1} \sum_{j=m_s+1}^{m_{s+1}} \left(A_n^- f\right)_j^q u_j^q && (2.50) \\
&< \sum_{s=s_0-2}^{s_0-1} \sum_{j=m_s+1}^{m_{s+1}} (h+1)^{(-n_{s+1}+1)q} u_j^q \\
&= (h+1)^q \sum_{s=s_0-2}^{s_0-1} (h+1)^{-n_{s+1}q} \sum_{j=m_s+1}^{m_{s+1}} u_j^q \\
&\ll \sum_{s=s_0-2}^{s_0-1} \left(A_n^- f\right)_{m_{s+1}}^q \sum_{j=m_s+1}^{m_{s+1}} u_j^q \\
&= \sum_{s=s_0-2}^{s_0-1} \left(\sum_{i=m_{s+1}}^{\infty} a_{i,m_{s+1}}^{(n)} f_i\right)^q \sum_{j=m_s+1}^{m_{s+1}} u_j^q \\
&\leq \sum_{s=s_0-2}^{s_0-1} \left[\left(\sum_{i=m_{s+1}}^{\infty} (a_{i,m_{s+1}}^{(n)})^{p'} v_i^{-p'}\right)^{\frac{1}{p'}} \left(\sum_{j=m_s+1}^{m_{s+1}} u_j^q\right)^{\frac{1}{q}}\right]^q \left(\sum_{j=m_{s+1}}^{\infty} |v_i f_i|^p\right)^{\frac{q}{p}}
\end{aligned}
$$

$$\leq \left[\sup_{k\geq 1}\left(\sum_{i=k}^{\infty}(a_{i,k}^{(n)})^{p'}v_i^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{j=1}^{k}u_j^q\right)^{\frac{1}{q}}\right]^q\left(\sum_{s=s_0-2}^{s_0-1}\sum_{j=m_s+1}^{\infty}|v_if_i|^p\right)^{\frac{q}{p}}$$

$$\leq 2\mathcal{A}_n^q\|f\|_{p,v}^q \ll \mathcal{A}_n^q\|f\|_{p,v}^q.$$

Using (2.36) and applying Hölder's inequality we estimate J_3 in the following way.

$$J_3 = \sum_{j=m_{s_0}+1}^{\infty}\left(A_n^-f\right)_j^q u_j^q \tag{2.51}$$

$$\leq \sup_{t\geq m_{s_0}+1}\sum_{j=m_{s_0}+1}^{t}\left(A_n^-f\right)_j^q u_j^q$$

$$\leq (h+1)^q\sup_{t\geq m_{s_0}+1}(h+1)^{-n_{s_0+1}q}\sum_{j=m_{s_0}+1}^{t}u_j^q$$

$$\ll \sup_{t\geq m_{s_0}+1}\left(A_n^-f\right)_t^q\sum_{j=m_{s_0}+1}^{t}u_j^q$$

$$= \sup_{t\geq m_{s_0}+1}\left(\sum_{i=t}^{\infty}a_{i,t}^{(n)}f_i\right)^q\sum_{j=m_{s_0}+1}^{t}u_j^q$$

$$\leq \sup_{t\geq m_{s_0}+1}\left[\left(\sum_{i=t}^{\infty}(a_{i,t}^{(n)})^{p'}v_i^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{j=m_{s_0}+1}^{t}u_j^q\right)^{\frac{1}{q}}\right]^q\|f\|_{p,v}^q$$

$$\leq \mathcal{A}_n^q\|f\|_{p,v}^q.$$

By (2.48), (2.49), (2.50), and (2.51) we obtain (2.46) and, consequently (2.47).

2) If $n_{s_0+1}=\infty$, which means that $k_{m_{s_0+1}}=\infty$, then by the definition of m_{s_0+1} we have $k_j=\infty$ and $T_j=\emptyset$, if $j\geq m_{s_0}+1$, i.e., $(A_n^-f)_j=0$, if $j\geq m_{s_0}+1$. By the assumption that $(A_n^-f)_1\neq 0$ it follows that $s_0>1$. Therefore, $m_2<\infty$ and $s_0\geq 2$. Thus by (2.37) we have

$$\|A_n^-f\|_{q,u}^q = \sum_{s\in N_0}\sum_{j=m_s+1}^{m_{s+1}}\left(A_n^-f\right)_j^q u_j^q \tag{2.52}$$

$$= \sum_{s=1}^{s_0-3}\sum_{j=m_s+1}^{m_{s+1}}\left(A_n^-f\right)_j^q u_j^q + \sum_{s=s_0-2}^{s_0-1}\sum_{j=m_s+1}^{m_{s+1}}\left(A_n^-f\right)_j^q u_j^q$$

$$= J_1' + J_2'.$$

By estimating J_1' and J_2' as J_1 and J_2, respectively, from (2.52) we obtain (2.46) and, consequently (2.47). Therefore, we see that inequality (2.34) holds for $m = n$ and the estimate (2.33) is valid. This means that inequality (2.34) holds for all $m \geq 0$ with the estimate (2.33), which together with (2.32) gives $C \approx \mathcal{A}_n$. Thus the proof is complete.

2.5 Compactness criteria of matrix operators in weighted Lebesgue spaces.

This section is devoted to the compactness criteria of the matrix operators A^+ and A^- from the weighted $l_{p,v}$ space into the $l_{q,u}$ space when the corresponding matrices belong to one of the classes $\mathcal{O}_n^+$ and $\mathcal{O}_n^-$ $n \geq 0$.

Theorem 2.13. *Let* $1 < p \leq q < \infty$. *Let the matrix* $(a_{i,j})$ *of (2.1) belong to the class* $\mathcal{O}_n^+$, $n \geq 0$. *Then the operator defined by (2.1) is compact from* $l_{p,v}$ *into* $l_{q,u}$ *if and only if at least one of the following conditions holds*

$$\lim_{k \to \infty} \left(\mathcal{B}_{p,q}^+ \right)_k = 0, \tag{2.53}$$

$$\lim_{k \to \infty} \left(\mathcal{B}_{p,q}^- \right)_k = 0. \tag{2.54}$$

Theorem 2.14. *Let* $1 < p \leq q < \infty$. *Let the matrix* $(a_{i,j})$ *of (2.2) belong to the class* $\mathcal{O}_m^-$, $m \geq 0$. *Then the operator defined by (2.2) is compact from* $l_{p,v}$ *into* $l_{q,u}$ *if and only if at least one of the following conditions holds*

$$\lim_{k \to \infty} \left(\mathcal{A}_{p,q}^+ \right)_k = 0, \tag{2.55}$$

$$\lim_{k \to \infty} \left(\mathcal{A}_{p,q}^- \right)_k = 0. \tag{2.56}$$

Now we give the proof of compactness for the class $\mathcal{O}_n^+$, $n \geq 0$.

Proof of Theorem 2.13. For the proof of Theorem 2.13, we need the following equivalence

$$\left(\mathcal{B}_{p,q}^+ \right)_k \approx \left(\mathcal{B}_n \right)_k \equiv \max_{0 \leq \gamma \leq n} \left(\mathcal{B}_{\gamma,n} \right)_k \approx \left(\mathcal{B}_{p,q}^- \right)_k, \tag{2.57}$$

where

$$\left(\mathcal{B}_{\gamma,n} \right)_k = \left(\sum_{i=k}^{\infty} \left(b_{i,k}^{n,\gamma} \right)^q u_i^q \right)^{\frac{1}{q}} \left(\sum_{j=1}^{k} \left(a_{k,j}^{(\gamma)} \right)^{p'} v_j^{-p'} \right)^{\frac{1}{p'}}.$$

The equivalence directly follows from (2.12).

Necessity. Suppose that the matrix of operator (2.1) belongs to the class $\mathcal{O}_n^+$, $n \geq 0$. Let the operator (2.1) be compact from $l_{p,v}$ into $l_{q,u}$.

For $r \geq 1$, we introduce the following sequence:

$$\varphi_r = \{\varphi_{r,j}\}_{j=1}^\infty : \quad \varphi_{r,j} = \frac{f_{r,j}}{\|f_r\|_{p,v}},$$

where $f_r = \{f_{r,j}\}_{j=1}^\infty$: $\quad f_{r,j} = \begin{cases} \left(a_{r,j}^{(\gamma)}\right)^{p'-1} v_j^{-p'}, & 1 \leq j \leq r, \\ 0, & j > r. \end{cases}$

It is obvious that $\|\varphi_r\|_{p,v} = 1$. Since the operator (2.1) is compact from $l_{p,v}$ into $l_{q,u}$, the set $\{uA^+\varphi, \|\varphi\|_{p,v} = 1\}$ is precompact in l_q. Hence, from the criterion of precompactness of the sets in l_p (see Theorem C) we conclude that

$$\lim_{r \to \infty} \sup_{\|\varphi\|_{p,v}=1} \left(\sum_{i=r}^\infty u_i^q \left(A^+\varphi\right)_i^q \right)^{\frac{1}{q}} = 0. \tag{2.58}$$

Moreover, by using (2.11) we have that

$$\sup_{\|\varphi\|_{p,v}=1} \left(\sum_{i=r}^\infty u_i^q \left(A^+\varphi\right)_i^q \right)^{\frac{1}{q}} \geq \left(\sum_{i=r}^\infty u_i^q \left(A^+\varphi_r\right)_i^q \right)^{\frac{1}{q}} \tag{2.59}$$

$$= \left(\sum_{i=r}^\infty u_i^q \left(\sum_{j=1}^i a_{i,j}^{(n)} \frac{f_{r,j}}{\|f_r\|_{p,v}} \right)^q \right)^{\frac{1}{q}}$$

$$\geq \left(\sum_{i=r}^\infty u_i^q \left(\sum_{j=1}^r a_{i,j}^{(n)} \frac{f_{r,j}}{\|f_r\|_{p,v}} \right)^q \right)^{\frac{1}{q}}$$

$$= \left(\sum_{i=r}^\infty u_i^q \left(\sum_{j=1}^r a_{i,j}^{(n)} \left(a_{r,j}^{(\gamma)}\right)^{p'-1} v_j^{-p'} \right)^q \right)^{\frac{1}{q}} \left(\sum_{j=1}^r \left(a_{r,j}^{(\gamma)}\right)^{p'} v_j^{-p'} \right)^{-\frac{1}{p}}$$

$$\geq \left(\sum_{i=r}^\infty u_i^q (b_{i,r}^{n,\gamma})^q \right)^{\frac{1}{q}} \left(\sum_{j=1}^r \left(a_{r,j}^{(\gamma)}\right)^{p'} v_j^{-p'} \right)^{\frac{1}{p'}} = (\mathcal{B}_{\gamma,n})_r.$$

Since inequality (2.59) holds for all $\gamma = 0, 1, \ldots, n$ and from the validity of (2.58) we obtain

$$\lim_{r \to \infty} (\mathcal{B}_n)_r = 0$$

The proof of the necessity is complete.

Sufficiency. Let the matrix of operator (2.1) belong to the class $\mathcal{O}_n^+$, $n \geq 0$. Assume that at least one of the conditions (2.53) and (2.54) is valid. Then, by

Theorem 2.10, the operator (2.1) is bounded from $l_{p,v}$ into $l_{q,u}$. Consequently, the set $\{uA^+f, \ \|f\|_{p,v} \le 1\}$ is bounded in l_q. Let us show that this set is precompact in l_q. By the criterion on precompactness of the sets in l_q (see Theorem C), the bounded set $\{uA^+f, \ \|f\|_{p,v} \le 1\}$ is compact in l_q, if

$$\lim_{r\to\infty} \sup_{\|f\|_{p,v}\le 1} \left(\sum_{i=r}^{\infty} u_i^q |\left(A^+f\right)_i|^q \right)^{\frac{1}{q}} = 0. \tag{2.60}$$

For $r > 1$ we assume that $\widetilde{u} = \{\widetilde{u}_i\}_{i=1}^{\infty}$: $\quad \widetilde{u}_i = \begin{cases} 0, & 1 \le i \le r-1 \\ u_i, & r \le i. \end{cases}$

Then, by Theorem 2.10, we have that

$$\sup_{\|f\|_{p,v}\le 1} \left(\sum_{i=r}^{\infty} u_i^q |\left(A^+f\right)_i|^q \right)^{\frac{1}{q}} = \sup_{\|f\|_{p,v}\le 1} \left(\sum_{i=1}^{\infty} \widetilde{u}_i^q |(A^+f)_i|^q \right)^{\frac{1}{q}} \ll \widetilde{B}_n(r), \tag{2.61}$$

where

$$\widetilde{B}_n(r) = \sup_{k\ge 1} \max_{0\le\gamma\le n} \left(\sum_{i=k}^{\infty} \left(b_{i,k}^{n,\gamma}\right)^q \widetilde{u}_i^q \right)^{\frac{1}{q}} \left(\sum_{j=1}^{k} \left(a_{k,j}^{(\gamma)}\right)^{p'} v_j^{-p'} \right)^{\frac{1}{p'}}.$$

Since $\widetilde{u}_i = 0$ when $1 \le i \le r-1$ we have

$$\begin{aligned} \widetilde{B}_n(r) &= \sup_{k\ge r} \max_{0\le\gamma\le n} \left(\sum_{i=k}^{\infty} \left(b_{i,k}^{n,\gamma}\right)^q u_i^q \right)^{\frac{1}{q}} \left(\sum_{j=1}^{k} \left(a_{k,j}^{(\gamma)}\right)^{p'} v_j^{-p'} \right)^{\frac{1}{p'}} \\ &= \sup_{k\ge r} (B_n)_k, \end{aligned} \tag{2.62}$$

By (2.53), (2.54), (2.57), and (2.62) we deduce

$$\lim_{r\to\infty} \widetilde{B}_n(r) = \lim_{r\to\infty} \sup_{k\ge r} (B_n)_k = \lim_{r\to\infty} (B_n)_r = 0.$$

Hence, by using (2.61) we obtain (2.60) and the proof is complete.

Theorem 2.14 is proven in a similar way.

2.6   Boundedness and compactness of a class of matrix operators, the case $1 < p \leq q < \infty$. Proof of the main result.

In this section we give criteria on boundedness and compactness of the matrix operators A^+ and A^- from the weighted $l_{p,v}$ space into the $l_{q,u}$ space when the corresponding matrices belong to class $\mathcal{O}_m^+ \cup \mathcal{O}_m^-$, $m \geq 0$.

Theorem 2.15. *Suppose that $1 < p \leq q < \infty$. Let the matrix $(a_{i,j})$ in (2.1) belong to the class $\mathcal{O}_m^+ \cup \mathcal{O}_m^-$, $m \geq 0$. Let A^+ be the operator defined in (2.1). Then the following statements hold:*

(i) A^+ is bounded from $l_{p,v}$ into $l_{q,u}$ if and only if at least one of the conditions $\mathcal{B}^+ < \infty$ and $\mathcal{B}^- < \infty$ holds. Moreover $\mathcal{B}^+ \approx \mathcal{B}^- \approx C$, where C is the best constant in (2.3).

(ii) A^+ is compact from $l_{p,v}$ into $l_{q,u}$ if and only if at least one of the conditions $\lim\limits_{k\to\infty} \left(\mathcal{B}_{p,q}^+\right)_k = 0$ and $\lim\limits_{k\to\infty} \left(\mathcal{B}_{p,q}^-\right)_k = 0$ holds.

Theorem 2.16. *Suppose that $1 < p \leq q < \infty$. Let the matrix $(a_{i,j})$ in (2.2) belong to the class $\mathcal{O}_m^+ \cup \mathcal{O}_m^-$, $m \geq 0$. Let A^- be the operator defined in (2.2). Then the following statements hold:*

(j) A^- is bounded from $l_{p,v}$ into $l_{q,u}$ if and only if at least one of the conditions $\mathcal{A}^+ < \infty$ and $\mathcal{A}^- < \infty$ holds. Moreover $\mathcal{A}^+ \approx \mathcal{A}^- \approx C$, where C is the best constant in (2.3).

(jj) A^- is compact from $l_{p,v}$ into $l_{q,u}$ if and only if at least one of the conditions $\lim\limits_{k\to\infty} \left(\mathcal{A}_{p,q}^+\right)_k = 0$ and $\lim\limits_{k\to\infty} \left(\mathcal{A}_{p,q}^-\right)_k = 0$ holds.

Now based on the results of Sections 2.4 and 2.5 we prove Theorem 2.15. The proof of Theorem 2.16 can be carried out by the same method as in the proof of Theorem 2.15. Hence we give the proof of Theorem 2.15.

Proof of Theorem 2.15.

(i) If the matrix of (2.1) belong to $\mathcal{O}_m^+$, $m \geq 0$, then the statement *(i)* of Theorem 2.15 directly follows from Theorem 2.10. Suppose that the matrix $(a_{i,j}) = (a_{i,j}^{(m)})$ of (2.1) belongs to $\mathcal{O}_m^-$, $m \geq 0$. It is known that the boundedness of the operator (2.1) from $l_{p,v}$ into $l_{q,u}$ is equivalent to the boundedness of the conjugate operator from $l_{q',u^{-1}}$ into $l_{p',v^{-1}}$, which coincides with operator (2.2). From the condition that $1 < p \leq q < \infty$ it follows that $1 < q' \leq p' < \infty$. Then by Theorem 2.11 and the identities $\left(\mathcal{A}_{q',p'}^+\right)_k = \left(\mathcal{B}_{p,q}^+\right)_k$ and $\left(\mathcal{A}_{q',p'}^-\right)_k = \left(\mathcal{B}_{p,q}^-\right)_k$, the boundedness of the operator defined by (2.2) from $l_{q',u^{-1}}$ into $l_{p',v^{-1}}$ is equivalent to the conditions of the statement *(i)* of Theorem 2.15. Hence the statement *(i)* of Theorem 2.15 is also valid in the case when the matrix of (2.1) belongs to $\mathcal{O}_m^-$, $m \geq 0$. Thus the proof of the statement *(i)* of Theorem 2.15 is complete.

(ii) Let the matrix of (2.1) belong to $\mathcal{O}_m^+$, $m \geq 0$. Then the statement *(ii)* of Theorem 2.15 follows from Theorem 2.13. If the matrix $(a_{i,j}) = (a_{i,j}^{(m)})$ of (2.1) belongs to $\mathcal{O}_m^-$, $m \geq 0$, then by arguing with Theorem 2.14 as above statement *(ii)* of Theorem 2.15 follows. Thus the proof of Theorem 2.15 is complete.

2.7 Boundedness criteria of a class of matrix operators, the case $q < p$.

In this Section we obtain necessary and sufficient conditions for the boundedness of the matrix operators A^+ and A^- from the weighted $l_{p,v}$ space into the weighted $l_{q,u}$ space in case $1 < q < p < \infty$.

In this Section we consider inequality (2.3) under the Assumption A (see condition (2.4)).

We note that from (2.4) it easily follows that

$$da_{i,j} \geq a_{i,k}, \tag{2.63}$$

$$da_{i,j} \geq b_{k,j}\omega_i, \tag{2.64}$$

for $i \geq k \geq j \geq 1$.

Theorem 2.17. *Let $1 < q < p < \infty$. Let the entries of the matrix $(a_{i,j})$ satisfy Assumption A. Then estimate (2.3) for the operator defined by (2.2) holds if and only if $F = \max\{F_1, \ F_2\} < \infty$, where*

$$F_1 = \left(\sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} b_{i,j}^q u_j^q \right)^{\frac{p}{p-q}} \left(\sum_{k=i}^{\infty} \omega_k^{p'} v_k^{-p'} \right)^{\frac{p(q-1)}{p-q}} \omega_i^{p'} v_i^{-p'} \right)^{\frac{p-q}{pq}}$$

and

$$F_2 = \left(\sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} u_j^q \right)^{\frac{q}{p-q}} \left(\sum_{k=i}^{\infty} a_{k,i}^{p'} v_k^{-p'} \right)^{\frac{q(p-1)}{p-q}} u_i^q \right)^{\frac{p-q}{pq}}.$$

Moreover, $F \approx C$, where C is the best constant in (2.3).

Proof of Theorem 2.17. *Necessity.* Let us assume that (2.3) holds for a finite constant C. Let $m \geq 1$. Then we take a test sequence $\widetilde{f_m} = \{\widetilde{f}_{m,k}\}_{k=1}^{\infty}$ such that

$$\widetilde{f}_{m,k} = \begin{cases} \left(\sum_{j=1}^{k} b_{k,j}^q u_j^q \right)^{-\frac{1}{p-q}} \left(\sum_{i=k}^{m} \omega_i^{p'} v_i^{-p'} \right)^{\frac{q-1}{p-q}} \omega_k^{p'-1} v_k^{-p'}, & \text{if } 1 \leq k \leq m, \\ 0, & \text{if } k > m. \end{cases}$$

Then

$$
\|\widetilde{f}_m\|_{p,v} = \left(\sum_{k=1}^{\infty} \widetilde{f}_{m,k}^p v_k^p \right)^{\frac{1}{p}} \tag{2.65}
$$

$$
= \left(\sum_{k=1}^{m} \left(\sum_{j=1}^{k} b_{k,j}^q u_j^q \right)^{\frac{p}{p-q}} \left(\sum_{i=k}^{m} \omega_i^{p'} v_i^{-p'} \right)^{\frac{p(q-1)}{p-q}} \omega_k^{p'} v_k^{-p'} \right)^{\frac{1}{p}}.
$$

Substituting $\widetilde{f}_m$ in the left hand side of inequality (2.3) and using (2.8) and (2.64), we deduce that

$$
\|A^- \widetilde{f}_m\|_{q,u}^q \gg \sum_{k=1}^{m} \sum_{j=k}^{m} a_{j,k} \widetilde{f}_{m,j} \left(\sum_{i=j}^{m} a_{i,k} \widetilde{f}_{m,i} \right)^{q-1} u_k^q
$$

$$
= \sum_{j=1}^{m} \widetilde{f}_{m,j} \sum_{k=1}^{j} u_k^q a_{j,k} \left(\sum_{i=j}^{m} a_{i,k} \widetilde{f}_{m,i} \right)^{q-1}
$$

$$
\gg \sum_{j=1}^{m} \widetilde{f}_{m,j} \omega_j \sum_{k=1}^{j} u_k^q b_{j,k}^q \left(\sum_{i=j}^{m} \omega_i \widetilde{f}_{m,i} \right)^{q-1}
$$

$$
= \sum_{j=1}^{m} \widetilde{f}_{m,j} \omega_j \sum_{k=1}^{j} u_k^q b_{j,k}^q \left(\sum_{i=j}^{m} \omega_i \left(\sum_{s=1}^{i} b_{i,s}^q u_s^q \right)^{\frac{1}{p-q}} \right.
$$

$$
\times \left. \left(\sum_{k=i}^{m} \omega_k^{p'} v_k^{-p'} \right)^{\frac{q-1}{p-q}} \omega_i^{p'-1} v_i^{-p'} \right)^{q-1}
$$

$$
\gg \sum_{j=1}^{m} \widetilde{f}_{m,j} \omega_j \sum_{k=1}^{j} u_k^q b_{j,k}^q \left(\sum_{s=1}^{j} b_{j,s}^q u_s^q \right)^{\frac{q-1}{p-q}}
$$

$$
\times \left(\sum_{i=j}^{m} \omega_i^{p'} v_i^{-p'} \left(\sum_{k=i}^{m} \omega_k^{p'} v_k^{-p'} \right)^{\frac{q-1}{p-q}} \right)^{q-1}
$$

$$
\gg \sum_{j=1}^{m} \widetilde{f}_{m,j} \omega_j \left(\sum_{s=1}^{j} b_{j,s}^q u_s^q \right)^{\frac{p-1}{p-q}} \left(\sum_{i=j}^{m} \omega_i^{p'} v_i^{-p'} \right)^{\frac{(p-1)(q-1)}{p-q}}
$$

$$
= \sum_{j=1}^{m} \left(\sum_{s=1}^{j} b_{j,s}^q u_s^q \right)^{\frac{p}{p-q}} \left(\sum_{i=j}^{m} \omega_i^{p'} v_i^{-p'} \right)^{\frac{p(q-1)}{p-q}} \omega_j^{p'} v_j^{-p'},
$$

which implies that

$$
\|A^- \widetilde{f}_m\|_{q,u} \gg \left(\sum_{j=1}^{m} \left(\sum_{s=1}^{j} b_{j,s}^q u_s^q \right)^{\frac{p}{p-q}} \left(\sum_{i=j}^{m} \omega_i^{p'} v_i^{-p'} \right)^{\frac{p(q-1)}{p-q}} \omega_j^{p'} v_j^{-p'} \right)^{\frac{1}{q}}. \tag{2.66}
$$

From (2.3), (2.65) and (2.66) it follows that

$$\left(\sum_{j=1}^{m} \left(\sum_{s=1}^{j} b_{j,s}^{q} u_{s}^{q} \right)^{\frac{p}{p-q}} \left(\sum_{i=j}^{m} \omega_{i}^{p'} v_{i}^{-p'} \right)^{\frac{p(q-1)}{p-q}} \omega_{j}^{p'} v_{j}^{-p'} \right)^{\frac{p-q}{pq}} \ll C$$

for all $m \geq 1$. Since $m \geq 1$ is arbitrary we have that

$$F_1 \ll C. \tag{2.67}$$

We know that inequality (2.3) holds if and only if the following dual inequality

$$\|A^* g\|_{p', v^{-1}} \leq C \|g\|_{q', u^{-1}}, \qquad g \in l_{q', u^{-1}} \tag{2.68}$$

holds for the conjugate operator A^*, which is defined by (2.1). Moreover, the best constants in (2.3) and (2.68) coincide.

Now let $m \geq 1$. By taking a test sequence $\widetilde{g}_m = \{\widetilde{g}_{m,k}\}_{k=1}^{\infty}$ such that

$$\widetilde{g}_{m,k} = \begin{cases} \left(\sum_{j=1}^{k} u_{j}^{q} \right)^{\frac{q-1}{p-q}} \left(\sum_{i=k}^{m} a_{i,k}^{p'} v_{i}^{-p'} \right)^{\frac{(q-1)(p-1)}{p-q}} u_{k}^{q} & \text{for} \quad 1 \leq k \leq m, \\ 0 & \text{for} \quad k > m. \end{cases}$$

we have that

$$\|\widetilde{g}_m\|_{q', u^{-1}} = \left(\sum_{k=1}^{m} \left(\sum_{j=1}^{k} u_{j}^{q} \right)^{\frac{q}{p-q}} \left(\sum_{i=k}^{m} a_{i,k}^{p'} v_{i}^{-p'} \right)^{\frac{q(p-1)}{p-q}} u_{k}^{q} \right)^{\frac{1}{q'}}. \tag{2.69}$$

By using (2.7) and (2.63) we deduce that

$$\begin{aligned} \|A^* \widetilde{g}_m\|_{p', v^{-1}}^{p'} &\geq \sum_{i=1}^{m} \left(\sum_{j=1}^{i} a_{i,j} \widetilde{g}_{m,j} \right)^{p'} v_{i}^{-p'} \\ &\gg \sum_{i=1}^{m} \sum_{j=1}^{i} a_{i,j} \widetilde{g}_{m,j} \left(\sum_{k=1}^{j} a_{i,k} \widetilde{g}_{m,k} \right)^{p'-1} v_{i}^{-p'} \\ &\geq \sum_{j=1}^{m} \widetilde{g}_{m,j} \sum_{i=j}^{m} a_{i,j} \left(\sum_{k=1}^{j} a_{i,k} \widetilde{g}_{m,k} \right)^{p'-1} v_{i}^{-p'} \\ &\gg \sum_{j=1}^{m} \widetilde{g}_{m,j} \sum_{i=j}^{m} a_{i,j}^{p'} v_{i}^{-p'} \left(\sum_{k=1}^{j} \widetilde{g}_{m,k} \right)^{p'-1} \end{aligned}$$

$$= \sum_{j=1}^{m} \widetilde{g}_{m,j} \sum_{i=j}^{m} a_{i,j}^{p'} v_i^{-p'} \left(\sum_{k=1}^{j} \left(\sum_{s=1}^{k} u_s^q \right)^{\frac{q-1}{p-q}} \right.$$

$$\left. \times \left(\sum_{i=k}^{m} a_{i,k}^{p'} v_i^{-p'} \right)^{\frac{(q-1)(p-1)}{p-q}} u_k^q \right)^{p'-1}$$

$$\gg \sum_{j=1}^{m} \widetilde{g}_{m,j} \sum_{i=j}^{m} a_{i,j}^{p'} v_i^{-p'} \left(\sum_{i=j}^{m} a_{i,j}^{p'} v_i^{-p'} \right)^{\frac{q-1}{p-q}} \left(\sum_{k=1}^{j} u_k^q \left(\sum_{s=1}^{k} u_s^q \right)^{\frac{q-1}{p-q}} \right)^{p'-1}$$

$$\gg \sum_{j=1}^{m} \widetilde{g}_{m,j} \left(\sum_{i=j}^{m} a_{i,j}^{p'} v_i^{-p'} \right)^{\frac{p-1}{p-q}} \left(\sum_{k=1}^{j} u_k^q \right)^{\frac{1}{p-q}}$$

$$= \sum_{j=1}^{m} \left(\sum_{k=1}^{j} u_k^q \right)^{\frac{q}{p-q}} \left(\sum_{i=j}^{m} a_{i,j}^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} u_j^q,$$

which implies that

$$\|A^* \widetilde{g}_m\|_{p',v^{-1}} \gg \left(\sum_{j=1}^{m} \left(\sum_{k=1}^{j} u_k^q \right)^{\frac{q}{p-q}} \left(\sum_{i=j}^{m} a_{i,j}^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} u_j^q \right)^{\frac{1}{p'}}. \qquad (2.70)$$

Since $m \geq 1$ is arbitrary, then (2.68), (2.69), (2.70) imply that $F_2 \ll C$. Hence, (2.67) implies that

$$F \ll C. \qquad (2.71)$$

The proof of the necessity is thus complete.

Sufficiency. Let $F < \infty$ and $0 \leq f \in l_{p,v}$.

For all $j \geq 1$ we define the following set:

$$T_j = \{k \in \mathbb{Z} : (d+1)^{-k} \leq (A^- f)_j\},$$

where d is the constant from (2.4) and $\mathbb{Z}$ is the set of integers. We assume that $\inf T_j = \infty$, if $T_j = \emptyset$ and $k_j = \inf T_j$, if $T_j \neq \emptyset$. We can clearly assume that $(A^- f)_1 \neq 0$. Without loss of generality, we may assume that $a_{i,j}$ is non-increasing in j, otherwise we take $a_{i,j} \approx \widetilde{a}_{i,j} = \sup_{j \leq k \leq i} a_{i,k}$. Therefore $k_j < k_{j+1}$. If $k_j < \infty$, then

$$(d+1)^{-k_j} \leq (A^- f)_j < (d+1)^{-(k_j-1)}, \qquad j \geq 1. \qquad (2.72)$$

Let $m_1 = 0$, $k_1 = k_{m_1+1}$ and $M_1 = \{j \in \mathbb{N} : k_j = k_1 = k_{m_1+1}\}$, where $\mathbb{N}$ is the set of natural numbers. Suppose that m_2 is such that $\sup M_1 = m_2$. Obviously $m_2 > m_1$ and if the set M_1 is upper bounded, then $m_2 < \infty$ and $m_2 = \max M_1$. We now define inductively the numbers $0 = m_1 < m_2 < \ldots < m_s < \infty$, $s \geq 1$. We set $m_{s+1} = \sup M_s$, where $M_s = \{j \in N : k_j = k_{m_s+1}\}$.

Let $N_0 = \{s \in \mathbb{N} : m_s < \infty\}$. Further, we assume that $k_{m_{s+1}} = n_{s+1}$, $s \in N_0$. From the definition of m_s and from (2.72) it follows that

$$(d+1)^{-n_{s+1}} \leq \left(A^- f\right)_j < (d+1)^{-n_{s+1}+1}, \quad m_s + 1 \leq j \leq m_{s+1} \qquad (2.73)$$

for all $s \in N_0$. Then

$$\mathbb{N} = \bigcup_{s \in N_0} [m_s + 1, m_{s+1}], \quad \text{where} \quad [m_s + 1, m_{s+1}] \cap [m_l + 1, m_{l+1}] = \emptyset, \ s \neq l.$$

Therefore

$$\|A^- f\|_{q,u}^q = \sum_{s \in N_0} \sum_{j=m_s+1}^{m_{s+1}} \left(A^- f\right)_j^q u_j^q. \qquad (2.74)$$

We assume that $\displaystyle\sum_{j=m_s+1}^{m_{s+1}} = 0$, if $m_s = \infty$.

There are two possible cases: $N_0 = \mathbb{N}$ and $N_0 \neq \mathbb{N}$.

1. If $N_0 = \mathbb{N}$, then we estimate the left hand side of (2.3) in the following way.

Clearly inequalities $n_{s+1} < n_{s+2} < n_{s+3}$ imply that $-n_{s+3} + 1 \leq -n_{s+1} - 1$ for all $s \in \mathbb{N}$. Hence, (2.73), (2.4) imply that

$$
\begin{aligned}
(d+1)^{-n_{s+1}-1} &= (d+1)^{-n_{s+1}} - d(d+1)^{-n_{s+1}-1} & (2.75)\\
&\leq (d+1)^{-n_{s+1}} - d(d+1)^{-n_{s+3}+1} \\
&< \left(A^- f\right)_{m_{s+1}} - d\left(A^- f\right)_{m_{s+3}}
\end{aligned}
$$

$$= \sum_{i=m_{s+1}}^{\infty} a_{i,m_{s+1}} f_i - d \sum_{i=m_{s+3}}^{\infty} a_{i,m_{s+3}} f_i$$

$$\leq \sum_{i=m_{s+1}}^{m_{s+3}} a_{i,m_{s+1}} f_i + \sum_{i=m_{s+3}}^{\infty} [a_{i,m_{s+1}} - d a_{i,m_{s+3}}] f_i$$

$$\leq \sum_{i=m_{s+1}}^{m_{s+3}} a_{i,m_{s+1}} f_i$$

$$\quad + \sum_{i=m_{s+3}}^{\infty} [d(a_{i,m_{s+3}} + b_{m_{s+3},m_{s+1}} \omega_i) - d a_{i,m_{s+3}}] f_i$$

$$= \sum_{i=m_{s+1}}^{m_{s+3}} a_{i,m_{s+1}} f_i + d b_{m_{s+3},m_{s+1}} \sum_{i=m_{s+3}}^{\infty} \omega_i f_i.$$

Now, by using (2.73) and (2.75), we can estimate the summand on the left hand side in (2.3) in the following way:

$$\sum_{s \in \mathbb{N}} \sum_{j=m_s+1}^{m_{s+1}} \left(A^- f\right)_j^q u_j^q < \sum_{s \in \mathbb{N}} \sum_{j=m_s+1}^{m_{s+1}} (d+1)^{(-n_{s+1}+1)q} u_j^q \qquad (2.76)$$

$$= (d+1)^{2q} \sum_{s \in \mathbb{N}} (d+1)^{(-n_{s+1}-1)q} \sum_{j=m_s+1}^{m_{s+1}} u_j^q$$

$$\ll \sum_{s \in \mathbb{N}} \left(\sum_{i=m_{s+1}}^{m_{s+3}} a_{i,m_{s+1}} f_i + d b_{m_{s+3},m_{s+1}} \sum_{i=m_{s+3}}^{\infty} \omega_i f_i \right)^q \sum_{j=m_s+1}^{m_{s+1}} u_j^q$$

$$\ll \sum_{s \in \mathbb{N}} \left(\sum_{i=m_{s+1}}^{m_{s+3}} a_{i,m_{s+1}} f_i \right)^q \sum_{j=m_s+1}^{m_{s+1}} u_j^q$$

$$\quad + \sum_{s \in \mathbb{N}} b_{m_{s+3},m_{s+1}}^q \left(\sum_{i=m_{s+3}}^{\infty} \omega_i f_i \right)^q \sum_{j=m_s+1}^{m_{s+1}} u_j^q := S_1 + S_2,$$

where

$$S_1 = \sum_{s \in \mathbb{N}} \left(\sum_{i=m_{s+1}}^{m_{s+3}} a_{i,m_{s+1}} f_i \right)^q \sum_{j=m_s+1}^{m_{s+1}} u_j^q,$$

and

$$S_2 = \sum_{s \in \mathbb{N}} b_{m_{s+3},m_{s+1}}^q \left(\sum_{i=m_{s+3}}^{\infty} \omega_i f_i \right)^q \sum_{j=m_s+1}^{m_{s+1}} u_j^q.$$

To estimate S_1, we apply the Hölder's inequality in the inner summand with the powers p, p' and in the outer summand with the powers $\frac{p}{p-q}$, $\frac{p}{q}$, and we

obtain that

$$
\begin{aligned}
S_1 &\leq \sum_{s \in \mathbb{N}} \left(\sum_{i=m_{s+1}}^{m_{s+3}} a_{i,m_{s+1}}^{p'} v_i^{-p'} \right)^{\frac{q}{p'}} \left(\sum_{i=m_{s+1}}^{m_{s+3}} |v_i f_i|^p \right)^{\frac{q}{p}} \sum_{j=m_s+1}^{m_{s+1}} u_j^q \quad (2.77) \\
&\leq \left(\sum_{s \in \mathbb{N}} \left(\sum_{i=m_{s+1}}^{m_{s+3}} a_{i,m_{s+1}}^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} \left(\sum_{j=m_s+1}^{m_{s+1}} u_j^q \right)^{\frac{p}{p-q}} \right)^{\frac{p-q}{p}} \\
&\qquad\qquad \times \left(\sum_{s \in \mathbb{N}} \sum_{i=m_{s+1}}^{m_{s+3}} |v_i f_i|^p \right)^{\frac{q}{p}} \ll (\widetilde{F_2})^{\frac{p-q}{p}} \|f\|_{p,v}^q.
\end{aligned}
$$

By (2.8) and (2.63) we can estimate $\widetilde{F_2}$ as follows:

$$
\begin{aligned}
\widetilde{F_2} &= \sum_{s \in \mathbb{N}} \left(\sum_{i=m_{s+1}}^{m_{s+3}} a_{i,m_{s+1}}^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} \left(\sum_{j=m_s+1}^{m_{s+1}} u_j^q \right)^{\frac{p}{p-q}} \quad (2.78) \\
&\ll \sum_{s \in \mathbb{N}} \left(\sum_{i=m_{s+1}}^{m_{s+3}} a_{i,m_{s+1}}^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} \sum_{j=m_s+1}^{m_{s+1}} \left(\sum_{k=j}^{m_{s+1}} u_k^q \right)^{\frac{q}{p-q}} u_j^q \\
&\ll \sum_{s \in \mathbb{N}} \sum_{j=m_s+1}^{m_{s+1}} \left(\sum_{k=1}^{m_{s+1}} u_k^q \right)^{\frac{q}{p-q}} \left(\sum_{i=m_{s+1}}^{\infty} a_{i,j}^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} u_j^q \\
&\leq \sum_{j=1}^{\infty} \left(\sum_{k=1}^{j} u_k^q \right)^{\frac{q}{p-q}} \left(\sum_{i=j}^{\infty} a_{i,j}^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} u_j^q = F_2^{\frac{pq}{p-q}}.
\end{aligned}
$$

By (2.77) and (2.78) we deduce that

$$
S_1 \ll F_2^q \|f\|_{p,v}^q. \quad (2.79)
$$

Next we introduce the sequence $\{\Delta_j\}_{j=1}^{\infty}$ such that $\Delta_j = b_{m_{s+3},m_{s+1}}^q \sum_{i=m_s+1}^{m_{s+1}} u_i^q$, $j = m_{s+3}$ and $\Delta_j = 0$, $j \neq m_{s+3}$, $s \in \mathbb{N}$. Hence, we can rewrite S_2 in the following form:

$$
S_2 = \sum_{s \in \mathbb{N}} \left(\sum_{i=m_{s+3}}^{\infty} \omega_i f_i \right)^q b_{m_{s+3},m_{s+1}}^q \sum_{i=m_s+1}^{m_{s+1}} u_i^q = \sum_{j=1}^{\infty} \left(\sum_{i=j}^{\infty} \omega_i f_i \right)^q \Delta_j. \quad (2.80)
$$

Thus, by Theorem B, we have that

$$
S_2 \ll \widetilde{H^q} \|f\|_{p,v}^q, \quad (2.81)
$$

where

$$\widetilde{H} = \left(\sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} \Delta_i \right)^{\frac{p}{p-q}} \left(\sum_{j=k}^{\infty} \omega_j^{p'} v_j^{-p'} \right)^{\frac{p(q-1)}{p-q}} \omega_k^{p'} v_k^{-p'} \right)^{\frac{p-q}{pq}}. \qquad (2.82)$$

By Assumption A, $b_{i,j}$ is almost non-decreasing in i and almost non-increasing in j, and accordingly,

$$\sum_{i=1}^{k} \Delta_i = \sum_{m_{s+3} \leq k} b_{m_{s+3}, m_{s+1}}^q \sum_{j=m_s+1}^{m_{s+1}} u_j^q \qquad (2.83)$$

$$\ll \sum_{m_{s+3} \leq k} \sum_{j=m_s+1}^{m_{s+1}} b_{k,j}^q u_j^q \leq \sum_{j=1}^{k} b_{k,j}^q u_j^q.$$

By combining (2.81), (2.82) and (2.83), we obtain

$$S_2 \ll F_1^q \|f\|_{p,v}^q. \qquad (2.84)$$

Thus, from (2.74), (2.76), (2.79) and (2.84) it follows that

$$\|A^- f\|_{q,u} \ll F \|f\|_{p,v} \qquad f \geq 0. \qquad (2.85)$$

This means that inequality (2.3) holds and that $C \ll F$, where C is the best constant for which (2.3) holds.

2. If $N_0 \neq \mathbb{N}$, i.e. $\max N_0 < \infty$ and $N_0 = \{1, 2, ..., s_0\}$, $s_0 \geq 1$. Therefore $m_{s_0} < \infty$ and $m_{s_0+1} = \infty$. We assume that $\sum_{s=k}^{n} = 0$, if $k > n$ and $\sum_{s=k}^{n} = \sum_{s=1}^{n}$, if $k \leq 0$. We have two possible cases: $n_{s_0+1} < \infty$ and $n_{s_0+1} = \infty$. We consider such cases separately:

1) If $n_{s_0+1} < \infty$, then from (2.74) it follows that

$$\|A^- f\|_{q,u}^q = \sum_{s \in N_0} \sum_{j=m_s+1}^{m_{s+1}} \left(A^- f\right)_j^q u_j^q = \sum_{s=1}^{s_0} \sum_{j=m_s+1}^{m_{s+1}} \left(A^- f\right)_j^q u_j^q \qquad (2.86)$$

$$= \sum_{s=1}^{s_0-3} \sum_{j=m_s+1}^{m_{s+1}} \left(A^- f\right)_j^q u_j^q + \sum_{s=s_0-2}^{s_0} \sum_{j=m_s+1}^{m_{s+1}} \left(A^- f\right)_j^q u_j^q = I_1 + I_2.$$

If $I_1 \neq 0$ then we estimate I_1 using (2.75) and the previous proof for the case $N_0 = N$. Hence, we obtain

$$I_1 \ll F^q \|f\|_{p,v}^q. \qquad (2.87)$$

By using (2.73) and applying the Hölder's inequality with the powers p, p' and with the powers $\frac{p}{p-q}$, $\frac{p}{q}$, we obtain the following inequality

$$
\begin{aligned}
I_2 &= \sum_{s=s_0-2}^{s_0} \sum_{j=m_s+1}^{m_{s+1}} \left(A^- f\right)_j^q u_j^q \qquad\qquad (2.88)\\
&< \sum_{s=s_0-2}^{s_0} \sum_{j=m_s+1}^{m_{s+1}} (d+1)^{(-n_{s+1}+1)q} u_j^q \\
&= (d+1)^q \sum_{s=s_0-2}^{s_0} (d+1)^{-n_{s+1}q} \sum_{j=m_s+1}^{m_{s+1}} u_j^q \\
&\ll \sum_{s=s_0-2}^{s_0} \left(A^- f\right)_{m_{s+1}}^q \sum_{j=m_s+1}^{m_{s+1}} u_j^q \\
&= \sum_{s=s_0-2}^{s_0} \left(\sum_{i=m_{s+1}}^{\infty} a_{i,m_{s+1}} f_i\right)^q \sum_{j=m_s+1}^{m_{s+1}} u_j^q \\
&\leq \sum_{s=s_0-2}^{s_0} \left(\sum_{i=m_{s+1}}^{\infty} a_{i,m_{s+1}}^{p'} v_i^{-p'}\right)^{\frac{q}{p'}} \left(\sum_{i=m_{s+1}}^{\infty} |v_i f_i|^p\right)^{\frac{q}{p}} \sum_{j=m_s+1}^{m_{s+1}} u_j^q \\
&\leq \left(\sum_{s=s_0-2}^{s_0} \left(\sum_{i=m_{s+1}}^{\infty} a_{i,m_{s+1}}^{p'} v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}} \left(\sum_{j=m_s+1}^{m_{s+1}} u_j^q\right)^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}} \\
&\quad \times \left(\sum_{s=s_0-2}^{s_0} \sum_{i=m_{s+1}}^{\infty} |v_i f_i|^p\right)^{\frac{q}{p}} \ll \left(\widehat{F_2}\right)^{\frac{p-q}{p}} \|f\|_{p,v}^q.
\end{aligned}
$$

Using (2.8) and (2.63) we can estimate $\widehat{F_2}$ as follows:

$$
\begin{aligned}
\widehat{F_2} &= \sum_{s=s_0-2}^{s_0} \left(\sum_{i=m_{s+1}}^{\infty} a_{i,m_{s+1}}^{p'} v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}} \left(\sum_{j=m_s+1}^{m_{s+1}} u_j^q\right)^{\frac{p}{p-q}} \qquad\qquad (2.89)\\
&\ll \sum_{s=s_0-2}^{s_0} \left(\sum_{i=m_{s+1}}^{\infty} a_{i,m_{s+1}}^{p'} v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}} \sum_{j=m_s+1}^{m_{s+1}} \left(\sum_{k=j}^{m_{s+1}} u_k^q\right)^{\frac{q}{p-q}} u_j^q \\
&\ll \sum_{s=s_0-2}^{s_0} \sum_{j=m_s+1}^{m_{s+1}} \left(\sum_{k=1}^{m_{s+1}} u_k^q\right)^{\frac{q}{p-q}} \left(\sum_{i=m_{s+1}}^{\infty} a_{i,j}^{p'} v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}} u_j^q \\
&\leq \sum_{j=1}^{\infty} \left(\sum_{k=1}^{j} u_k^q\right)^{\frac{q}{p-q}} \left(\sum_{i=j}^{\infty} a_{i,j}^{p'} v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}} u_j^q = F_2^{\frac{pq}{p-q}}.
\end{aligned}
$$

From (2.88) and (2.89) we obtain

$$I_2 \ll F_2^q \|f\|_{p,v}^q. \tag{2.90}$$

From (2.86), (2.87) and (2.90) we deduce (2.85).

2) If $n_{s_0+1} = \infty$, which means that $k_{m_{s_0}+1} = \infty$, then we have $k_j = \infty$ and $T_j = \emptyset$, if $j \geq m_{s_0} + 1$, i.e. $(A^- f)_j = 0$, if $j \geq m_{s_0+1}$ and $(A^- f)_j = \sum_{i=j}^{m_{s_0}} a_{i,j} f_i$, $1 \leq j \leq m_{s_0}$. Therefore $m_2 < \infty$ and $s_0 \geq 2$. Then from (2.74) we have

$$\|A^- f\|_{q,u}^q = \sum_{s \in N_0} \sum_{j=m_s+1}^{m_{s+1}} \left(A^- f \right)_j^q u_j^q = \sum_{s=1}^{s_0-1} \sum_{j=m_s+1}^{m_{s+1}} \left(A^- f \right)_j^q u_j^q \tag{2.91}$$

Similarly, we can exploit (2.91) to prove (2.85). Then (2.85) together with (2.71) implies that $C \approx F$ and thus the proof is complete.

It is known that inequality (2.3) for the operator (2.2) holds if and only if the dual inequality defined by (2.68) holds for the conjugate operator A^*, which coincides with operator defined by (2.1). Moreover, the best constants in (2.3) and (2.68) coincide.

Therefore by using Theorem 2.17 with p', q', v^{-1} and u^{-1} replaced by q, p, u and v, respectively, we obtain the following dual version of Theorem 2.17:

Theorem 2.18. *Let $1 < q < p < \infty$. Let the entries of the matrix $(a_{i,j})$ satisfy Assumption A. Then estimate (2.3) for the operator defined by (2.1) holds if and only if $F^* = \max\{F_1^*,\ F_2^*\} < \infty$, where*

$$F_1^* = \left(\sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} b_{i,j}^{p'} v_j^{-p'} \right)^{\frac{q(p-1)}{p-q}} \left(\sum_{k=i}^{\infty} \omega_k^q u_k^q \right)^{\frac{q}{p-q}} \omega_i^q u_i^q \right)^{\frac{p-q}{pq}},$$

$$F_2^* = \left(\sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} v_j^{-p'} \right)^{\frac{p(q-1)}{p-q}} \left(\sum_{k=i}^{\infty} a_{k,i}^q u_k^q \right)^{\frac{p}{p-q}} v_i^{-p'} \right)^{\frac{p-q}{pq}}.$$

Moreover, $F^* \approx C$, where C is the best constant in (2.3).

Chapter 3

Weighted Hardy type inequalities on the cone of monotone sequences

3.1 Weighted estimates for a class of matrices on the cone of monotone sequences, the case $1 < p \le q < \infty$.

In this Section we consider weighted Hardy type inequalities restricted to the cone of monotone sequences under conditions which are weaker than those known in the literature.

We consider an inequality of the following form

$$\left(\sum_{i=1}^{\infty} u_i^q \left(\sum_{j=1}^{i} a_{i,j} f_j \right)^q \right)^{\frac{1}{q}} \le C \left(\sum_{i=1}^{\infty} v_i^p f_i^p \right)^{\frac{1}{p}} \tag{3.1}$$

on the cone of non-negative and non-increasing sequences $f = \{f_i\}_{i=1}^{\infty}$ of $l_{p,v}$, where C is a positive constant independent of f and $(a_{i,j})$ is a non-negative triangular matrix with entries $a_{i,j} \ge 0$ for $i \ge j \ge 1$ and $a_{i,j} = 0$ for $i < j$.

In this Section, we consider inequality (3.1) restricted to the cone of monotone sequences when the corresponding matrices belong to the classes $\mathcal{O}_m^+ \cup \mathcal{O}_m^-$, $m \geq 0$ for $1 < p \leq q < \infty$.

In [45] R. Oinarov and S.Kh. Shalgynbaeva have proved a statement which allows to reduce inequality (3.1) on the cone of monotone sequences to a corresponding inequality on the cone of non-negative sequences from $l_{p,v}$. Now we give this statement in a form convenient for us (see also Theorem 1.11 for an equivalent statement).

Theorem E. [45] *Let* $1 < p$, $q < \infty$. *Let* $V_k = \sum_{i=1}^{k} v_i^p$, $\forall k \geq 1$. *Then inequality (3.1) on the cone of non-negative and non-increasing sequences* $f \in l_{p,v}$ *is equivalent to the inequality*

$$\left(\sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} \sum_{i=j}^{\infty} a_{i,j} g_i \right)^{p'} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{1}{p'}} \leq \widetilde{C} \left(\sum_{i=1}^{\infty} g_i^{q'} u_i^{-q'} \right)^{\frac{1}{q'}} \quad (3.2)$$

for all non-negative sequences $g = \{g_i\}_{i=1}^{\infty}$, *if* $V_\infty = \lim_{k \to \infty} V_k = \infty$, *and to the inequality*

$$\left(\sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} \sum_{i=j}^{\infty} a_{i,j} g_i \right)^{p'} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{1}{p'}}$$
$$+ \left(\sum_{j=1}^{\infty} \sum_{i=j}^{\infty} a_{i,j} g_i \right) \left(\sum_{k=1}^{\infty} v_k^p \right)^{-\frac{1}{p}} \leq \overline{C} \left(\sum_{i=1}^{\infty} g_i^{q'} u_i^{-q'} \right)^{\frac{1}{q'}} \quad (3.3)$$

for all non-negative sequences $g = \{g_i\}_{i=1}^{\infty}$, *if* $V_\infty < \infty$.

Moreover, $\widetilde{C} \approx C$ *if* $V_\infty = \infty$, *and* $\overline{C} \approx C$ *if* $V_\infty < \infty$, *where* C, $\widetilde{C}$ *and* $\overline{C}$ *are the best constants in (3.1), (3.2), (3.3), respectively.*

We define

$$V_k = \sum_{i=1}^{k} v_i^p, \quad A_{ik} = \sum_{j=1}^{k} a_{i,j}, \quad E_1 = \sup_{s \geq 1} V_s^{-\frac{1}{p}} \left(\sum_{i=1}^{s} A_{ii}^q u_i^q \right)^{\frac{1}{q}},$$

$$E_2 = \sup_{s \geq 1} \left(\sum_{k=1}^{s} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \left(\sum_{i=s}^{\infty} A_{ik}^q u_i^q \right)^{\frac{p'}{q}} \right)^{\frac{1}{p'}},$$

$$E_3 = \sup_{s \geq 1} \left(\sum_{k=s}^{\infty} u_k^q \left(\sum_{i=1}^{s} A_{ki}^{p'} \left(V_i^{-\frac{p'}{p}} - V_{i+1}^{-\frac{p'}{p}} \right) \right)^{\frac{q}{p'}} \right)^{\frac{1}{q}}.$$

Theorem 3.1. *Let $1 < p \leq q < \infty$. Let the matrix $(a_{i,j})$ in (3.1) belong to the class $\mathcal{O}_m^+ \cup \mathcal{O}_m^-$, $m \geq 0$. Then the inequality (3.1) on the cone of non-negative and non-increasing sequences $f \in l_{p,v}$ holds if and only if at least one of the conditions $E_{12} = \max\{E_1, E_2\} < \infty$ and $E_{13} = \max\{E_1, E_3\} < \infty$ holds. Moreover, $E_{12} \approx E_{13} \approx C$, where C is the best constant in (3.1).*

Proof of Theorem 3.1. Let the matrix $(a_{i,j})$ in (3.1) belong to the class $\mathcal{O}_m^+ \cup \mathcal{O}_m^-$, $m \geq 0$. We consider two cases separately: $V_\infty = \infty$ and $V_\infty < \infty$.

1. We first consider case $V_\infty = \infty$. Then by Theorem E inequality (3.1) on the cone of non-negative and non-increasing sequences $f \in l_{p,v}$ holds if and only if the following inequality holds

$$\left(\sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} \sum_{i=j}^{\infty} a_{i,j}^{(m)} g_i \right)^{p'} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{1}{p'}} \leq \widetilde{C} \left(\sum_{i=1}^{\infty} g_i^{q'} u_i^{1-q'} \right)^{\frac{1}{q'}} \quad (3.4)$$

holds for all non-negative sequences $g = \{g_i\}_{i=1}^{\infty}$. Moreover, $\widetilde{C} \approx C$, where C is the best constant in (3.1).

a) Let the matrix $(a_{i,j}) = (a_{i,j}^{(m)})$ of (3.1) belong to the class $\mathcal{O}_m^+$, $m \geq 0$. Since $a_{i,j}^{(m)}$, g_i are non-negative we have

$$\sum_{j=1}^{k} \sum_{i=j}^{\infty} a_{i,j}^{(m)} g_i = \sum_{j=1}^{k} \sum_{i=j}^{k} a_{i,j}^{(m)} g_i + \sum_{j=1}^{k} \sum_{i=k+1}^{\infty} a_{i,j}^{(m)} g_i \quad (3.5)$$

$$\approx \sum_{i=1}^{k} A_{ii}^{(m)} g_i + \sum_{i=k}^{\infty} A_{ik}^{(m)} g_i.$$

Therefore,

$$\left(\sum_{j=1}^{k} \sum_{i=j}^{\infty} a_{i,j}^{(m)} g_i \right)^{p'} \approx \left(\sum_{i=1}^{k} A_{ii}^{(m)} g_i \right)^{p'} + \left(\sum_{i=k}^{\infty} A_{ik}^{(m)} g_i \right)^{p'}.$$

Substituting the last inequality in the left hand side of inequality (3.4) we

obtain the following inequality

$$\left(\sum_{k=1}^{\infty} \left[\left(\sum_{i=1}^{k} A_{ii}^{(m)} g_i \right)^{p'} + \left(\sum_{i=k}^{\infty} A_{ik}^{(m)} g_i \right)^{p'} \right] \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{1}{p'}} \tag{3.6}$$
$$\leq C_0 \left(\sum_{i=1}^{\infty} g_i^{q'} u_i^{-q'} \right)^{\frac{1}{q'}}$$

for all non-negative sequences $g = \{g_i\}_{i=1}^{\infty}$, which is equivalent to the inequality (3.4). Moreover, $\widetilde{C} \approx C_0$.

Inequality (3.6) holds if and only if the following inequalities hold simultaneously

$$\left(\sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} A_{ii}^{(m)} g_i \right)^{p'} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{1}{p'}} \leq C_1 \left(\sum_{i=1}^{\infty} g_i^{q'} u_i^{-q'} \right)^{\frac{1}{q'}}, \tag{3.7}$$

$$\left(\sum_{k=1}^{\infty} \left(\sum_{i=k}^{\infty} A_{ik}^{(m)} g_i \right)^{p'} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{1}{p'}} \leq C_2 \left(\sum_{i=1}^{\infty} g_i^{q'} u_i^{-q'} \right)^{\frac{1}{q'}} \tag{3.8}$$

for all non-negative sequences $g = \{g_i\}_{i=1}^{\infty}$. Moreover,

$$\widetilde{C} \approx \max\{C_1, \ C_2\}. \tag{3.9}$$

Inequality (3.7) is a Hardy type inequality. Hence, by Theorem A inequality (3.7) holds if and only if the following condition holds

$$\sup_{s \geq 1} \left(\sum_{k=s}^{\infty} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{1}{p'}} \left(\sum_{i=1}^{s} \left(A_{ii}^{(m)} \right)^q u_i^q \right)^{\frac{1}{q}}$$
$$= \sup_{s \geq 1} V_s^{-\frac{1}{p}} \left(\sum_{i=1}^{s} \left(A_{ii}^{(m)} \right)^q u_i^q \right)^{\frac{1}{q}} = E_1 < \infty. \tag{3.10}$$

Moreover,

$$E_1 \approx C_1. \tag{3.11}$$

In (3.8) by passing to the dual inequality we obtain

$$\left(\sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} A_{ki}^{(m)} \varphi_i \right)^q u_k^q \right)^{\frac{1}{q}} \leq C_2 \left(\sum_{k=1}^{\infty} \varphi_k^p \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right)^{-\frac{p}{p'}} \right)^{\frac{1}{p}} \tag{3.12}$$

for all non-negative sequences $\varphi = \{\varphi_i\}_{i=1}^{\infty}$.

Suppose that $A_{ki}^{(m)} = 0$ for $i > k$. Since $(a_{k,i}^{(m)}) \in \mathcal{O}_m^+$, then we have $(A_{ki}^{(m)}) \in \mathcal{O}_m^+$, $m \geq 0$. Indeed, for $k \geq s \geq i$

$$A_{ki}^{(m)} = \sum_{j=1}^{i} a_{k,j}^{(m)} \approx \sum_{j=1}^{i} \sum_{\gamma=0}^{m} b_{k,s}^{m,\gamma} a_{s,j}^{(\gamma)} = \sum_{\gamma=0}^{m} b_{k,s}^{m,\gamma} \sum_{j=1}^{i} a_{s,j}^{(\gamma)} = \sum_{\gamma=0}^{m} b_{k,s}^{m,\gamma} A_{si}^{(\gamma)}. \quad (3.13)$$

If $m = 0$ we see that $(A_{ki}^{(0)})$ belongs to the class $\mathcal{O}_0^+$. Assume that $(A_{ki}^{(m)}) \in \mathcal{O}_m^+$ for $m = 1, ..., r - 1$. Then by induction on $m = r$, (3.13) implies that $(A_{ki}^{(r)})$ belongs to the class $\mathcal{O}_r^+$.

Then by Theorem 2.15 inequality (3.12) holds if and only if one of the following conditions holds

$$\sup_{k \geq 1} \left(\sum_{j=1}^{k} \left(V_j^{-\frac{p'}{p}} - V_{j+1}^{-\frac{p'}{p}} \right) \left(\sum_{i=k}^{\infty} \left(A_{ij}^{(m)} \right)^q u_i^q \right)^{\frac{p'}{q}} \right)^{\frac{1}{p'}} = E_2 < \infty, \quad (3.14)$$

$$\sup_{k \geq 1} \left(\sum_{i=k}^{\infty} u_i^q \left(\sum_{j=1}^{k} \left(A_{ij}^{(m)} \right)^{p'} \left(V_j^{-\frac{p'}{p}} - V_{j+1}^{-\frac{p'}{p}} \right) \right)^{\frac{q}{p'}} \right)^{\frac{1}{q}} = E_3 < \infty. \quad (3.15)$$

Moreover,

$$C_2 \approx E_2 \approx E_3. \quad (3.16)$$

By (3.11) and (3.16) we deduce that inequalities (3.7), (3.8) hold if and only if at least one of the conditions $E_{12} = \max\{E_1, \ E_2\} < \infty$ and $E_{13} = \max\{E_1, \ E_3\} < \infty$ holds. Moreover, $E_{12} \approx E_{13} \approx \max\{C_1, \ C_2\}$, which implies that $E_{12} \approx E_{13} \approx C_0$. Since $C_0 \approx \widetilde{C}$, $\widetilde{C} \approx C$ we get $E_{12} \approx E_{13} \approx C$. The last equivalence proves the statement of Theorem 3.1 when $(a_{i,j}) \in \mathcal{O}_m^+$ in the case $V_{\infty} = \infty$.

b) Let the matrix $(a_{i,j}) = (a_{i,j}^{(m)})$ of (3.1) belong to the class $\mathcal{O}_m^-$, $m \geq 0$.

Since $a_{i,j}^{(m)}$, g_i are non-negative we have

$$
\begin{aligned}
\sum_{j=1}^{k}\sum_{i=j}^{\infty} a_{i,j}^{(m)} g_i &= \sum_{j=1}^{k}\sum_{i=j}^{k} a_{i,j}^{(m)} g_i + \sum_{j=1}^{k}\sum_{i=k+1}^{\infty} a_{i,j}^{(m)} g_i \qquad (3.17)\\
&\approx \sum_{i=1}^{k} A_{ii}^{(m)} g_i + \sum_{i=k}^{\infty} g_i \sum_{j=1}^{k} a_{i,j}^{(m)}\\
&\approx \sum_{i=1}^{k} A_{ii}^{(m)} g_i + \sum_{i=k}^{\infty} g_i \sum_{j=1}^{k}\sum_{\gamma=0}^{m} a_{i,k}^{(\gamma)} d_{k,j}^{\gamma,m}\\
&= \sum_{i=1}^{k} A_{ii}^{(m)} g_i + \sum_{\gamma=0}^{m} D_{kk}^{\gamma,m} \sum_{i=k}^{\infty} a_{i,k}^{(\gamma)} g_i,
\end{aligned}
$$

where $D_{kk}^{\gamma,m} = \sum\limits_{j=1}^{k} d_{k,j}^{\gamma,m}$. Therefore,

$$
\left(\sum_{j=1}^{k}\sum_{i=j}^{\infty} a_{i,j}^{(m)} g_i \right)^{p'} \approx \left(\sum_{i=1}^{k} A_{ii}^{(m)} g_i \right)^{p'} + \sum_{\gamma=0}^{m} \left(D_{kk}^{\gamma,m} \sum_{i=k}^{\infty} a_{i,k}^{(\gamma)} g_i \right)^{p'}.
$$

Substituting the last inequality in the left hand side of inequality (3.4) we obtain the following inequality

$$
\left(\sum_{k=1}^{\infty} \left[\left(\sum_{i=1}^{k} A_{ii}^{(m)} g_i \right)^{p'} + \sum_{\gamma=0}^{m} \left(D_{kk}^{\gamma,m} \sum_{i=k}^{\infty} a_{i,k}^{(\gamma)} g_i \right)^{p'} \right] \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{1}{p'}}
$$

$$
\leq C_0 \left(\sum_{i=1}^{\infty} g_i^{q'} u_i^{-q'} \right)^{\frac{1}{q'}} \qquad (3.18)
$$

for all non-negative sequences $g = \{g_i\}_{i=1}^{\infty}$, which is equivalent to the inequality (3.4). Moreover, $\widetilde{C} \approx C_0$.

Inequality (3.18) holds if and only if the following inequalities hold simultaneously

$$
\left(\sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} A_{ii}^{(m)} g_i \right)^{p'} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{1}{p'}}
$$

$$
\leq C_1 \left(\sum_{i=1}^{\infty} g_i^{q'} u_i^{-q'} \right)^{\frac{1}{q'}} \qquad \forall g \geq 0, \quad (3.19)
$$

$$\left(\sum_{k=1}^{\infty} \left(\sum_{i=k}^{\infty} a_{i,k}^{(\gamma)} g_i \right)^{p'} (D_{kk}^{\gamma,m})^{p'} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{1}{p'}}$$

$$\leq \widetilde{C}_\gamma \left(\sum_{i=1}^{\infty} g_i^{q'} u_i^{-q'} \right)^{\frac{1}{q'}} \qquad \forall g \geq 0 \quad (3.20)$$

for all $\gamma = 0, 1, ..., m$. Moreover,

$$\widetilde{C} \approx C_0 \approx \max\{C_1, \ \widetilde{C}_0, \ \widetilde{C}_1, ..., \ \widetilde{C}_m\}. \tag{3.21}$$

(3.19) is a Hardy type inequality. Hence, using Theorem A as in (3.10), we obtain that inequality (3.19) holds if and only if the following condition holds

$$E_1 = \sup_{s \geq 1} V_s^{-\frac{1}{p}} \left(\sum_{i=1}^{s} \left(A_{ii}^{(m)} \right)^q u_i^q \right)^{\frac{1}{q}} < \infty. \tag{3.22}$$

Moreover,

$$E_1 \approx C_1. \tag{3.23}$$

In (3.20) by passing to the dual inequality we obtain

$$\left(\sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} a_{k,i}^{(\gamma)} \varphi_i \right)^q u_k^q \right)^{\frac{1}{q}}$$

$$\leq \widetilde{C}_\gamma \left(\sum_{i=1}^{\infty} \varphi_i^p \ (D_{ii}^{\gamma,m})^{-p} \left(V_i^{-\frac{p'}{p}} - V_{i+1}^{-\frac{p'}{p}} \right)^{-\frac{p}{p'}} \right)^{\frac{1}{p}} \tag{3.24}$$

for all non-negative sequences $\varphi = \{\varphi_i\}_{i=1}^{\infty}$ and for all $\gamma = 0, 1, ..., m$.

Since $(a_{k,i}^{(\gamma)}) \in \mathcal{O}_\gamma^-$, $\gamma = 0, 1, ..., m$, by Theorem 2.15 inequality (3.24) holds if and only if one of the following conditions holds

$$\widetilde{B}_\gamma^+ = \sup_{k \geq 1} \left(\widetilde{B}_\gamma^+ \right)_k$$

$$= \sup_{k \geq 1} \left(\sum_{j=1}^{k} (D_{jj}^{\gamma,m})^{p'} \left(V_j^{-\frac{p'}{p}} - V_{j+1}^{-\frac{p'}{p}} \right) \left(\sum_{i=k}^{\infty} \left(a_{i,j}^{(\gamma)} \right)^q u_i^q \right)^{\frac{p'}{q}} \right)^{\frac{1}{p'}} < \infty, \quad (3.25)$$

$$\widetilde{B}_\gamma^- = \sup_{k \geq 1} \left(\widetilde{B}_\gamma^- \right)_k$$

$$= \sup_{k \geq 1} \left(\sum_{i=k}^\infty u_i^q \left(\sum_{j=1}^k \left(a_{i,j}^{(\gamma)} \right)^{p'} \left(D_{jj}^{\gamma,m} \right)^{p'} \left(V_j^{-\frac{p'}{p}} - V_{j+1}^{-\frac{p'}{p}} \right) \right)^{\frac{q}{p'}} \right)^{\frac{1}{q}} < \infty. \quad (3.26)$$

and

$$\widetilde{C}_\gamma \approx \widetilde{B}_\gamma^+ \approx \widetilde{B}_\gamma^-. \quad (3.27)$$

The expression $\left(\widetilde{B}_\gamma^+ \right)_k$ can be written in the following form.

$$\left(\widetilde{B}_\gamma^+ \right)_k = \left(\sum_{j=1}^k \left(V_j^{-\frac{p'}{p}} - V_{j+1}^{-\frac{p'}{p}} \right) \left(\sum_{i=k}^\infty \left(\sum_{s=1}^j a_{i,j}^{(\gamma)} d_{j,s}^{\gamma,m} \right)^q u_i^q \right)^{\frac{p'}{q}} \right)^{\frac{1}{p'}}.$$

Then we have

$$\sum_{\gamma=0}^m \left(\widetilde{B}_\gamma^+ \right)_k = \sum_{\gamma=0}^m \left(\sum_{j=1}^k \left(V_j^{-\frac{p'}{p}} - V_{j+1}^{-\frac{p'}{p}} \right) \left(\sum_{i=k}^\infty \left(\sum_{s=1}^j a_{i,j}^{(\gamma)} d_{j,s}^{\gamma,m} \right)^q u_i^q \right)^{\frac{p'}{q}} \right)^{\frac{1}{p'}}$$

$$\approx \left(\sum_{j=1}^k \left(V_j^{-\frac{p'}{p}} - V_{j+1}^{-\frac{p'}{p}} \right) \left(\sum_{i=k}^\infty \left(\sum_{s=1}^j \sum_{\gamma=0}^m a_{i,j}^{(\gamma)} d_{j,s}^{\gamma,m} \right)^q u_i^q \right)^{\frac{p'}{q}} \right)^{\frac{1}{p'}}$$

$$\approx \left(\sum_{j=1}^k \left(V_j^{-\frac{p'}{p}} - V_{j+1}^{-\frac{p'}{p}} \right) \left(\sum_{i=k}^\infty \left(\sum_{s=1}^j a_{i,s}^{(m)} \right)^q u_i^q \right)^{\frac{p'}{q}} \right)^{\frac{1}{p'}}$$

$$= \left(\sum_{j=1}^k \left(V_j^{-\frac{p'}{p}} - V_{j+1}^{-\frac{p'}{p}} \right) \left(\sum_{i=k}^\infty \left(A_{i,j}^{(m)} \right)^q u_i^q \right)^{\frac{p'}{q}} \right)^{\frac{1}{p'}}.$$

Therefore, we have $\sup_{k \geq 1} \sum_{\gamma=0}^m \left(\widetilde{B}_\gamma^+ \right)_k \approx E_2$. Since $\sup_{k \geq 1} \sum_{\gamma=0}^m \left(\widetilde{B}_\gamma^+ \right)_k \approx \sum_{\gamma=0}^m \widetilde{B}_\gamma^+$,

we obtain $\max_{0 \leq \gamma \leq m} \widetilde{B}_\gamma^+ \approx \sum_{\gamma=0}^m \widetilde{B}_\gamma^+ \approx E_2$. In the same way, we deduce that $\max_{0 \leq \gamma \leq m} \widetilde{B}_\gamma^- \approx$

$\sum_{\gamma=0}^m \widetilde{B}_\gamma^- \approx E_3$.

By (3.22), (3.25) and (3.26) we obtain that inequalities (3.19), (3.20) hold

if and only if one of the conditions $E_{12} = \max\{E_1, \ E_2\} < \infty$ and $E_{13} =$

$\max\{E_1,\ E_3\} < \infty$ holds. Moreover, we have $E_{12} \approx E_{13} \approx \max\{C_1, \widetilde{C}_0, \widetilde{C}_1, ...,$ $\widetilde{C}_m\}$, which implies that $E_{12} \approx E_{13} \approx \widetilde{C}$. Since $\widetilde{C} \approx C$, we get $E_{12} \approx E_{13} \approx C$. The last equivalence gives the statement of Theorem 3.1 when $(a_{i,j}) \in \mathcal{O}_m^-$, $m \geq 0$ in the case $V_\infty = \infty$.

2. Next we consider case $V_\infty < \infty$. By Theorem E inequality (3.1) on the cone of non-negative and non-increasing sequences $f \in l_{p,v}$ holds if and only if both inequality (3.4) and the inequality

$$\left(\sum_{k=1}^{\infty}\sum_{i=k}^{\infty} a_{i,k}^{(m)} g_i\right)\left(\sum_{i=1}^{\infty} v_i^p\right)^{-\frac{1}{p}} \leq \widehat{C}\left(\sum_{i=1}^{\infty} g_i^{q'} u_i^{-q'}\right)^{\frac{1}{q'}} \tag{3.28}$$

for all non-negative sequences $g = \{g_i\}_{i=1}^{\infty}$ hold. Here $\widehat{C}$ is the best constant for which (3.28) holds. Moreover, $C \approx \max\{\widetilde{C}, \widehat{C}\}$.

So in case $V_\infty < \infty$, inequality (3.4) holds if and only if one of the conditions $\max\{E_1',\ E_2\} < \infty$ and $\max\{E_1',\ E_3\} < \infty$ holds, where

$$E_1' = \sup_{s \geq 1}\left(V_s^{-\frac{p'}{p}} - V_\infty^{-\frac{p'}{p}}\right)^{\frac{1}{p'}}\left(\sum_{i=1}^{s}\left(A_{ii}^{(m)}\right)^q u_i^q\right)^{\frac{1}{q}}.$$

Indeed, if $V_\infty < \infty$, (3.10) implies that inequality (3.7) (and accordingly inequality (3.19)) holds if and only if $E_1' < \infty$. Moreover, $E_1' \approx C_1$.

Since $a_{i,j}^{(m)}$, g_i are non-negative, changing the order of summation in the left hand side of (3.28) we obtain

$$\left(\sum_{i=1}^{\infty} A_{ii}^{(m)} g_i\right) \leq \widehat{C} V_\infty^{\frac{1}{p}}\left(\sum_{i=1}^{\infty} g_i^{q'} u_i^{-q'}\right)^{\frac{1}{q'}} \quad \forall g \geq 0.$$

By the reverse Hölder's inequality we have

$$\left(\sum_{i=1}^{\infty}\left(A_{ii}^{(m)}\right)^q u_i^q\right)^{\frac{1}{q}} = \widehat{C} V_\infty^{\frac{1}{p}}.$$

Hence, inequality (3.28) holds if and only if

$$\widehat{C} = V_\infty^{-\frac{1}{p}}\left(\sum_{i=1}^{\infty}\left(A_{ii}^{(m)}\right)^q u_i^q\right)^{\frac{1}{q}} < \infty.$$

It is obvious that $\frac{1}{2}\left(E_1' + \widehat{C}\right) < E_1$. At the same time, for $s \geq 1$ we have

$$V_s^{-\frac{1}{p}} = \left(V_s^{-\frac{p'}{p}}\right)^{\frac{1}{p'}} = \left(V_s^{-\frac{p'}{p}} - V_\infty^{-\frac{p'}{p}} + V_\infty^{-\frac{p'}{p}}\right)^{\frac{1}{p'}} \leq \left(V_s^{-\frac{p'}{p}} - V_\infty^{-\frac{p'}{p}}\right)^{\frac{1}{p'}} + V_\infty^{-\frac{1}{p}},$$

which implies that $E_1 \leq \left(E_1' + \widehat{C} \right)$. Hence, we obtain that $\frac{1}{2} \left(E_1' + \widehat{C} \right) < E_1 \leq \left(E_1' + \widehat{C} \right)$. Therefore, inequalities (3.7) (and accordingly (3.19)) and (3.28) hold if and only if $E_1 < \infty$. Moreover, $E_1 \approx \max\{C_1, \ \widehat{C}\}$.

Now we see that $\max\{E_1, \ E_2\} \approx \max\{E_1, \ E_3\} \approx \max\{C_1, \ \widehat{C}, \ \widetilde{C}\} \approx C$ regardless of whether V_∞ is finite or infinite. Thus the proof is complete.

3.2 Two-sided estimates for matrix operators on the cone of monotone sequences, the case $q < p$.

In this Section we consider inequality (3.1) on the cone of monotone sequences for $1 < q < p < \infty$.

We define

$$V_k = \sum_{i=1}^{k} v_i^p, \quad W_k = \sum_{i=1}^{k} \omega_i, \quad A_{ik} = \sum_{j=1}^{k} a_{i,j}, \quad B_{ik} = \sum_{j=1}^{k} b_{i,j},$$

$$\mathbb{F}_1 = \left(\sum_{k=1}^{\infty} V_k^{\frac{q}{q-p}} \left(\sum_{i=1}^{k} A_{ii}^q u_i^q \right)^{\frac{q}{p-q}} A_{kk}^q u_k^q \right)^{\frac{p-q}{pq}},$$

$$\mathbb{F}_2 = \left(\sum_{k=1}^{\infty} \left(\sum_{j=k}^{\infty} b_{j,k}^q u_j^q \right)^{\frac{p}{p-q}} \left(\sum_{i=1}^{k} W_i^{p'} \left(V_i^{-\frac{p'}{p}} - V_{i+1}^{-\frac{p'}{p}} \right) \right)^{\frac{p(q-1)}{p-q}}$$
$$\times W_k^{p'} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{p-q}{pq}},$$

$$\mathbb{F}_3 = \left(\sum_{k=1}^{\infty} \left(\sum_{j=k}^{\infty} u_j^q \right)^{\frac{q}{p-q}} \left(\sum_{i=1}^{k} A_{ki}^{p'} \left(V_i^{-\frac{p'}{p}} - V_{i+1}^{-\frac{p'}{p}} \right) \right)^{\frac{q(p-1)}{p-q}} u_k^q \right)^{\frac{p-q}{pq}},$$

$$\mathfrak{F}_2 = \left(\sum_{k=1}^{\infty} \left(\sum_{i=k}^{\infty} w_i^q u_i^q \right)^{\frac{p}{p-q}} \left(\sum_{j=1}^{k} B_{jj}^{p'} \left(V_j^{-\frac{p'}{p}} - V_{j+1}^{-\frac{p'}{p}} \right) \right)^{\frac{p(q-1)}{p-q}}$$
$$\times B_{kk}^{p'} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{p-q}{pq}},$$

$$\mathfrak{F}_3 = \left(\sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} j^{p'} b_{k,j}^{p'} \left(V_j^{-\frac{p'}{p}} - V_{j+1}^{-\frac{p'}{p}} \right) \right)^{\frac{q(p-1)}{p-q}} \left(\sum_{i=k}^{\infty} w_i^q u_i^q \right)^{\frac{q}{p-q}} w_k^q u_k^q \right)^{\frac{p-q}{pq}},$$

$$\mathfrak{F}_4 = \left(\sum_{k=1}^{\infty} \left(\sum_{i=k}^{\infty} a_{i,k}^q u_i^q \right)^{\frac{p}{p-q}} \left(\sum_{j=1}^{k} j^{p'} \left(V_j^{-\frac{p'}{p}} - V_{j+1}^{-\frac{p'}{p}} \right) \right)^{\frac{p(q-1)}{p-q}} \right.$$

$$\left. \times k^{p'} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{p-q}{pq}}.$$

Theorem 3.2. *Let $1 < q < p < \infty$. Let the entries of the matrix $(a_{i,j})$ satisfy assumption (1.17). Then inequality (3.1) on the cone of non-negative and non-increasing sequences $f \in l_{p,v}$ holds if and only if $\mathbb{F}_0 = \max\{\mathbb{F}_1,\ \mathbb{F}_2,\ \mathbb{F}_3\} < \infty$. Moreover, $\mathbb{F}_0 \approx C$, where C is the best constant in (3.1).*

Theorem 3.3. *Let $1 < q < p < \infty$. Let the entries of the matrix $(a_{i,j})$ satisfy Assumption A. Then inequality (3.1) on the cone of non-negative and non-increasing sequences $f \in l_{p,v}$ holds if and only if $\mathfrak{F}_0 = \max\{\mathbb{F}_1,\ \mathfrak{F}_2,\ \mathfrak{F}_3,\ \mathfrak{F}_4\} < \infty$. Moreover, $\mathfrak{F}_0 \approx C$, where C is the best constant in (3.1).*

Proof of Theorem 3.2. Let the entries of the matrix $(a_{i,j})$ satisfy assumption (1.17). We consider two cases separately: $V_\infty = \infty$ and $V_\infty < \infty$.

1. Let $V_\infty = \infty$. Then by Theorem E inequality (3.1) on the cone of non-negative and non-increasing sequences $f \in l_{p,v}$ holds if and only if the following inequality

$$\left(\sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} \sum_{i=j}^{\infty} a_{i,j} g_i \right)^{p'} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{1}{p'}} \leq \widetilde{C} \left(\sum_{i=1}^{\infty} g_i^{q'} u_i^{-q'} \right)^{\frac{1}{q'}} \quad (3.29)$$

holds for all non-negative sequences $g = \{g_i\}_{i=1}^{\infty}$. Moreover, $\widetilde{C} \approx C$, where C is the best constant in (3.1).

Since $a_{i,j},\ g_i$ are non-negative we have

$$\sum_{j=1}^{k} \sum_{i=j}^{\infty} a_{i,j} g_i = \sum_{j=1}^{k} \sum_{i=j}^{k} a_{i,j} g_i + \sum_{j=1}^{k} \sum_{i=k+1}^{\infty} a_{i,j} g_i \approx \sum_{i=1}^{k} A_{ii} g_i + \sum_{i=k}^{\infty} A_{ik} g_i. \quad (3.30)$$

Therefore,

$$\left(\sum_{j=1}^{k} \sum_{i=j}^{\infty} a_{i,j} g_i \right)^{p'} \approx \left(\sum_{i=1}^{k} A_{ii} g_i \right)^{p'} + \left(\sum_{i=k}^{\infty} A_{ik} g_i \right)^{p'}.$$

Substituting the last inequality in the left hand side of inequality (3.29) we have the following inequality

$$\left(\sum_{k=1}^{\infty} \left[\left(\sum_{i=1}^{k} A_{ii}g_i \right)^{p'} + \left(\sum_{i=k}^{\infty} A_{ik}g_i \right)^{p'} \right] \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{1}{p'}}$$
$$\leq \widetilde{C}_0 \left(\sum_{i=1}^{\infty} g_i^{q'} u_i^{-q'} \right)^{\frac{1}{q'}} \quad (3.31)$$

for all non-negative sequences $g = \{g_i\}_{i=1}^{\infty}$, which is equivalent to the inequality (3.29). Moreover, $\widetilde{C} \approx \widetilde{C}_0$.

Inequality (3.31) holds if and only if the following inequalities hold simultaneously

$$\left(\sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} A_{ii}g_i \right)^{p'} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{1}{p'}} \leq \widetilde{C}_1 \left(\sum_{i=1}^{\infty} g_i^{q'} u_i^{-q'} \right)^{\frac{1}{q'}}, \quad (3.32)$$

$$\left(\sum_{k=1}^{\infty} \left(\sum_{i=k}^{\infty} A_{ik}g_i \right)^{p'} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{1}{p'}} \leq \widetilde{C}_2 \left(\sum_{i=1}^{\infty} g_i^{q'} u_i^{-q'} \right)^{\frac{1}{q'}} \quad (3.33)$$

for all non-negative sequences $g = \{g_i\}_{i=1}^{\infty}$. Moreover,

$$\widetilde{C} \approx \max\{\widetilde{C}_1, \ \widetilde{C}_2\}. \quad (3.34)$$

Inequality (3.32) is a Hardy type inequality. Hence by Theorem A inequality (3.32) holds if and only if the following condition holds

$$\left(\sum_{k=1}^{\infty} V_k^{\frac{q}{q-p}} \left(\sum_{i=1}^{k} A_{ii}^q u_i^q \right)^{\frac{q}{p-q}} A_{kk}^q u_k^q \right)^{\frac{p-q}{pq}} = \mathbb{F}_1 < \infty. \quad (3.35)$$

Moreover,

$$\mathbb{F}_1 \approx \widetilde{C}_1. \quad (3.36)$$

By passing to the dual inequality in (3.33) we obtain

$$\left(\sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} A_{ki}\varphi_i \right)^q u_k^q \right)^{\frac{1}{q}} \leq \widetilde{C}_2 \left(\sum_{k=1}^{\infty} \varphi_k^p \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right)^{-\frac{p}{p'}} \right)^{\frac{1}{p}} \quad (3.37)$$

for all non-negative sequences $\varphi = \{\varphi_i\}_{i=1}^{\infty}$.

The entries of the matrix (A_{ki}) for $k \geq s \geq i$ satisfy the following condition.

$$A_{ki} = \sum_{j=1}^{i} a_{k,j} \approx \sum_{j=1}^{i} (b_{k,s}\omega_j + a_{s,j}) = b_{k,s}W_i + \sum_{j=1}^{i} a_{s,j} = b_{k,s}W_i + A_{si}, \quad (3.38)$$

which asserts that the entries of the matrix (A_{ki}) satisfy assumption (1.17). Therefore, by Theorem 1.9 inequality (3.37) holds if and only if the following conditions hold

$$\left(\sum_{k=1}^{\infty} \left(\sum_{j=k}^{\infty} b_{j,k}^q u_j^q \right)^{\frac{p}{p-q}} \left(\sum_{i=1}^{k} W_i^{p'} \left(V_i^{-\frac{p'}{p}} - V_{i+1}^{-\frac{p'}{p}} \right) \right)^{\frac{p(q-1)}{p-q}} \right.$$
$$\left. \times W_k^{p'} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{p-q}{pq}} = \mathbb{F}_2 < \infty, \quad (3.39)$$

$$\left(\sum_{k=1}^{\infty} \left(\sum_{j=k}^{\infty} u_j^q \right)^{\frac{q}{p-q}} \left(\sum_{i=1}^{k} A_{ki}^{p'} \left(V_i^{-\frac{p'}{p}} - V_{i+1}^{-\frac{p'}{p}} \right) \right)^{\frac{q(p-1)}{p-q}} u_k^q \right)^{\frac{p-q}{pq}}$$
$$= \mathbb{F}_3 < \infty, \quad (3.40)$$

and

$$\widetilde{C}_2 \approx \max\{\mathbb{F}_2, \ \mathbb{F}_3\}. \quad (3.41)$$

By (3.35) and (3.39), (3.40) we obtain that inequalities (3.32) and (3.37) hold if and only if $\mathbb{F}_0 = \max\{\mathbb{F}_1, \ \mathbb{F}_2, \ \mathbb{F}_3\} < \infty$. Moreover, $\mathbb{F}_0 \approx \max\{\widetilde{C}_1, \ \widetilde{C}_2\}$, which implies that $\mathbb{F}_0 \approx \widetilde{C}$. Since $\widetilde{C} \approx C$, we get $\mathbb{F}_0 \approx C$. The last equivalence gives the statement of Theorem 3.2 in the case $V_{\infty} = \infty$.

2. Let $V_{\infty} < \infty$. By Theorem E inequality (3.1) on the cone of non-negative and non-increasing sequences $f \in l_{p,v}$ holds if and only if both inequality (3.29) and the inequality

$$\left(\sum_{k=1}^{\infty} \sum_{i=k}^{\infty} a_{i,k}g_i \right) \left(\sum_{i=1}^{\infty} v_i^p \right)^{-\frac{1}{p}} \leq \widehat{C} \left(\sum_{i=1}^{\infty} g_i^{q'} u_i^{-q'} \right)^{\frac{1}{q'}} \quad (3.42)$$

for all non-negative sequences $g = \{g_i\}_{i=1}^{\infty}$ hold. Here $\widehat{C}$ is the best constant for which (3.42) holds. Moreover, $C \approx \max\{\widetilde{C}, \widehat{C}\}$.

Since $a_{i,j}$, g_i are non-negative, changing the order of summation in the left hand side of (3.42) we obtain

$$\left(\sum_{i=1}^{\infty} g_i A_{ii} \right) \le \widehat{C} V_{\infty}^{\frac{1}{p}} \left(\sum_{i=1}^{\infty} g_i^{q'} u_i^{-q'} \right)^{\frac{1}{q'}} \quad \forall g \ge 0.$$

By the reverse Hölder's inequality we have

$$\left(\sum_{i=1}^{\infty} A_{ii}^q u_i^q \right)^{\frac{1}{q}} = \widehat{C} V_{\infty}^{\frac{1}{p}},$$

and accordingly

$$V_{\infty}^{-\frac{1}{p}} \left(\sum_{i=1}^{\infty} A_{ii}^q u_i^q \right)^{\frac{1}{q}} = \widehat{C}. \tag{3.43}$$

Therefore, inequality (3.42) holds if and only if $\widehat{C} < \infty$.

As proved above, inequality (3.29) holds if and only if the inequalities (3.32) and (3.33) hold simultaneously.

In the same way as in the case $V_{\infty} = \infty$ we prove that inequality (3.33) holds if and only if $\mathbb{F}_2 < \infty$, $\mathbb{F}_3 < \infty$.

By Theorem A inequality (3.32) holds if and only if the following condition holds

$$\mathbb{F}_1' = \left(\sum_{k=1}^{\infty} \left(\sum_{j=k}^{\infty} (V_j^{-\frac{p'}{p}} - V_{j+1}^{-\frac{p'}{p}}) \right)^{\frac{q(p-1)}{p-q}} \left(\sum_{i=1}^{k} A_{ii}^q u_i^q \right)^{\frac{q}{p-q}} A_{kk}^q u_k^q \right)^{\frac{p-q}{pq}}$$

$$= \left(\sum_{k=1}^{\infty} \left(V_k^{-\frac{p'}{p}} - V_{\infty}^{-\frac{p'}{p}} \right)^{\frac{q(p-1)}{p-q}} \left(\sum_{i=1}^{k} A_{ii}^q u_i^q \right)^{\frac{q}{p-q}} A_{kk}^q u_k^q \right)^{\frac{p-q}{pq}} < \infty.$$

Thus, in case $V_{\infty} < \infty$, inequality (3.1) on the cone of non-negative and non-increasing sequences $f \in l_{p,v}$ holds if and only if $\max \left\{ \mathbb{F}_1', \ \mathbb{F}_2, \ \mathbb{F}_3, \ \widehat{C} \right\} < \infty$. If we prove that $\mathbb{F}_1 \approx \max \left\{ \mathbb{F}_1', \ \widehat{C} \right\}$, then we obtain that inequality (3.1) on the cone of non-negative and non-increasing sequences $f \in l_{p,v}$ holds if and only if $\mathbb{F}_0 < \infty$ and $\mathbb{F}_0 \approx C$.

It is obvious that $\mathbb{F}_1' \le \mathbb{F}_1$. Now we show that $\widehat{C} \ll \mathbb{F}_1$.

Note that

$$
\begin{aligned}
\left(\sum_{i=1}^{\infty} A_{ii}^q u_i^q\right)^{\frac{1}{q}} &= \left[\left(\sum_{i=1}^{\infty} A_{ii}^q u_i^q\right)^{\frac{p}{p-q}}\right]^{\frac{p-q}{pq}} \\
&= \left[\left(\sum_{k=1}^{\infty}\left[\left(\sum_{i=1}^{k} A_{ii}^q u_i^q\right)^{\frac{p}{p-q}} - \left(\sum_{i=1}^{k-1} A_{ii}^q u_i^q\right)^{\frac{p}{p-q}}\right]\right)\right]^{\frac{p-q}{pq}} \\
&\ll \left(\sum_{k=1}^{\infty}\left(\sum_{i=1}^{k} A_{ii}^q u_i^q\right)^{\frac{q}{p-q}}\left(\sum_{i=1}^{k} A_{ii}^q u_i^q - \sum_{i=1}^{k-1} A_{ii}^q u_i^q\right)\right)^{\frac{p-q}{pq}} \\
&\ll \left(\sum_{k=1}^{\infty}\left(\sum_{i=1}^{k} A_{ii}^q u_i^q\right)^{\frac{q}{p-q}} A_{kk}^q u_k^q\right)^{\frac{p-q}{pq}},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\widehat{C} = V_\infty^{-\frac{1}{p}}\left(\sum_{i=1}^{\infty} A_{ii}^q u_i^q\right)^{\frac{1}{q}} &\ll V_\infty^{-\frac{1}{p}}\left(\sum_{k=1}^{\infty}\left(\sum_{i=1}^{k} A_{ii}^q u_i^q\right)^{\frac{q}{p-q}} A_{kk}^q u_k^q\right)^{\frac{p-q}{pq}} \\
&\leq \left(\sum_{k=1}^{\infty} V_k^{\frac{q}{q-p}}\left(\sum_{i=1}^{k} A_{ii}^q u_i^q\right)^{\frac{q}{p-q}} A_{kk}^q u_k^q\right)^{\frac{p-q}{pq}} = \mathbb{F}_1.
\end{aligned}
$$

Therefore, we deduce that $\max\{\mathbb{F}_1', \widehat{C}\} \ll \mathbb{F}_1$. The reverse relation follows by using Minkowski inequality in the following way.

$$
\begin{aligned}
\mathbb{F}_1 &= \left(\sum_{k=1}^{\infty}\left(V_k^{-\frac{p'}{p}} - V_\infty^{-\frac{p'}{p}} + V_\infty^{-\frac{p'}{p}}\right)^{\frac{q(p-1)}{p-q}}\left(\sum_{i=1}^{k} A_{ii}^q u_i^q\right)^{\frac{q}{p-q}} A_{kk}^q u_k^q\right)^{\frac{p-q}{pq}} \\
&\leq \left(\sum_{k=1}^{\infty}\left(V_k^{-\frac{p'}{p}} - V_\infty^{-\frac{p'}{p}}\right)^{\frac{q(p-1)}{p-q}}\left(\sum_{i=1}^{k} A_{ii}^q u_i^q\right)^{\frac{q}{p-q}} A_{kk}^q u_k^q\right)^{\frac{p-q}{pq}} \\
&\quad + V_\infty^{-\frac{1}{p}}\left(\sum_{k=1}^{\infty}\left(\sum_{i=1}^{k} A_{ii}^q u_i^q\right)^{\frac{q}{p-q}} A_{kk}^q u_k^q\right)^{\frac{p-q}{pq}} \\
&\ll \mathbb{F}_1' + \widehat{C} \leq 2\max\{\mathbb{F}_1', \widehat{C}\}.
\end{aligned}
$$

Hence, we obtain that $\mathbb{F}_1 \approx \max\left\{\mathbb{F}_1', \widehat{C}\right\}$.

Therefore, we deduce that inequality (3.1) on the cone of non-negative and non-increasing sequences $f \in l_{p,v}$ holds if and only if $\mathbb{F}_0 < \infty$ and

$C \approx \mathbb{F}_0 = \max\{\mathbb{F}_1, \ \mathbb{F}_2, \ \mathbb{F}_3\}$ regardless of whether V_∞ is finite or infinite. Thus, the proof is complete.

Now we prove Theorem 3.3.

Proof of Theorem 3.3. Let the entries of the matrix $(a_{i,j})$ satisfy Assumption A. We consider two cases separately: $V_\infty = \infty$ and $V_\infty < \infty$.

1. Let $V_\infty = \infty$. Then based on Theorem E inequality (3.1) on the cone of non-negative and non-increasing sequences $f \in l_{p,v}$ holds if and only if inequality (3.29) holds for all non-negative sequences $g = \{g_i\}_{i=1}^\infty$. Moreover, $\widetilde{C} \approx C$, where C and $\widetilde{C}$ are the best constants in (3.1) and (3.29), respectively.

Since $a_{i,j}$, g_i are non-negative and according to Assumption A we have

$$
\begin{aligned}
\sum_{j=1}^k \sum_{i=j}^\infty a_{i,j} g_i & = \sum_{j=1}^k \sum_{i=j}^k a_{i,j} g_i + \sum_{j=1}^k \sum_{i=k+1}^\infty a_{i,j} g_i \qquad (3.44) \\
& \approx \sum_{i=1}^k A_{ii} g_i + \sum_{i=k}^\infty g_i \sum_{j=1}^k a_{i,j} \\
& \approx \sum_{i=1}^k A_{ii} g_i + k \sum_{i=k}^\infty a_{i,k} g_i + B_{kk} \sum_{i=k}^\infty \omega_i g_i.
\end{aligned}
$$

Therefore,

$$
\left(\sum_{j=1}^k \sum_{i=j}^\infty a_{i,j} g_i \right)^{p'} \approx \left(\sum_{i=1}^k A_{ii} g_i \right)^{p'} + \left(k \sum_{i=k}^\infty a_{i,k} g_i \right)^{p'} + \left(B_{kk} \sum_{i=k}^\infty \omega_i g_i \right)^{p'}.
$$

Substituting the last inequality in the left hand side of inequality (3.29) we obtain the following inequality

$$
\left(\sum_{k=1}^\infty \left[\left(\sum_{i=1}^k A_{ii} g_i \right)^{p'} + \left(k \sum_{i=k}^\infty a_{i,k} g_i \right)^{p'} + \left(B_{kk} \sum_{i=k}^\infty \omega_i g_i \right)^{p'} \right] \right.
$$
$$
\left. \times \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{1}{p'}} \leq \widetilde{C}_0 \left(\sum_{i=1}^\infty g_i^{q'} u_i^{-q'} \right)^{\frac{1}{q'}} \qquad (3.45)
$$

for all non-negative sequences $g = \{g_i\}_{i=1}^\infty$, which is equivalent to inequality (3.29). Moreover, $\widetilde{C} \approx \widetilde{C}_0$.

Inequality (3.45) holds if and only if the following inequalities hold simultaneously

$$\left(\sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} A_{ii} g_i \right)^{p'} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{1}{p'}} \le \widetilde{C}_1 \left(\sum_{i=1}^{\infty} g_i^{q'} u_i^{-q'} \right)^{\frac{1}{q'}}, \quad (3.46)$$

$$\left(\sum_{k=1}^{\infty} \left(k \sum_{i=k}^{\infty} a_{i,k} g_i \right)^{p'} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{1}{p'}} \le \widetilde{C}_2 \left(\sum_{i=1}^{\infty} g_i^{q'} u_i^{-q'} \right)^{\frac{1}{q'}}, \quad (3.47)$$

$$\left(\sum_{k=1}^{\infty} \left(B_{kk} \sum_{i=k}^{\infty} \omega_i g_i \right)^{p'} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{1}{p'}} \le \widetilde{C}_3 \left(\sum_{i=1}^{\infty} g_i^{q'} u_i^{-q'} \right)^{\frac{1}{q'}} \quad (3.48)$$

for all non-negative sequences $g = \{g_i\}_{i=1}^{\infty}$. Moreover,

$$\widetilde{C} \approx \max\{\widetilde{C}_1, \ \widetilde{C}_2, \ \widetilde{C}_3\}. \tag{3.49}$$

By passing to the dual inequalities in (3.47) and (3.48), we obtain

$$\left(\sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} a_{k,i} \varphi_i \right)^{q} u_k^{q} \right)^{\frac{1}{q}} \le \widetilde{C}_2 \left(\sum_{i=1}^{\infty} \varphi_i^{p} \ i^{-p} \left(V_i^{-\frac{p'}{p}} - V_{i+1}^{-\frac{p'}{p}} \right)^{-\frac{p}{p'}} \right)^{\frac{1}{p}}, \quad (3.50)$$

$$\left(\sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} \varphi_i \right)^{q} \omega_k^{q} u_k^{q} \right)^{\frac{1}{q}} \le \widetilde{C}_3 \left(\sum_{k=1}^{\infty} \varphi_k^{p} B_{kk}^{-p} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right)^{-\frac{p}{p'}} \right)^{\frac{1}{p}} \quad (3.51)$$

for all non-negative sequences $\varphi = \{\varphi_i\}_{i=1}^{\infty}$.

Inequalities (3.46) and (3.51) are Hardy type inequalities. Hence, by Theorem A inequalities (3.46), (3.51) hold if and only if the following conditions hold, respectively.

$$\left(\sum_{k=1}^{\infty} V_k^{\frac{q}{q-p}} \left(\sum_{i=1}^{k} A_{ii}^q u_i^q \right)^{\frac{q}{p-q}} A_{kk}^q u_k^q \right)^{\frac{p-q}{pq}} = \mathbb{F}_1 < \infty, \tag{3.52}$$

$$\left(\sum_{k=1}^{\infty} \left(\sum_{i=k}^{\infty} w_i^q u_i^q \right)^{\frac{p}{p-q}} \left(\sum_{j=1}^{k} B_{jj}^{p'} \left(V_j^{-\frac{p'}{p}} - V_{j+1}^{-\frac{p'}{p}} \right) \right)^{\frac{p(q-1)}{p-q}} \right.$$

$$\left. \times B_{kk}^{p'} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{p-q}{pq}} = \mathfrak{F}_2 < \infty. \quad (3.53)$$

Moreover,

$$\mathbb{F}_1 \approx \widetilde{C}_1, \quad \mathfrak{F}_2 \approx \widetilde{C}_3. \tag{3.54}$$

The entries of the matrix $(a_{k,i})$ satisfy Assumption A. Therefore, by Theorem 2.18 inequality (3.50) holds if and only if the following conditions hold

$$\left(\sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} j^{p'} b_{k,j}^{p'} \left(V_j^{-\frac{p'}{p}} - V_{j+1}^{-\frac{p'}{p}} \right) \right)^{\frac{q(p-1)}{p-q}} \left(\sum_{i=k}^{\infty} w_i^q u_i^q \right)^{\frac{q}{p-q}} w_k^q u_k^q \right)^{\frac{p-q}{pq}}$$
$$= \mathfrak{F}_3 < \infty, \quad (3.55)$$

$$\left(\sum_{k=1}^{\infty} \left(\sum_{i=k}^{\infty} a_{i,k}^q u_i^q \right)^{\frac{p}{p-q}} \left(\sum_{j=1}^{k} j^{p'} \left(V_j^{-\frac{p'}{p}} - V_{j+1}^{-\frac{p'}{p}} \right) \right)^{\frac{p(q-1)}{p-q}}$$
$$\times k^{p'} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{p-q}{pq}} = \mathfrak{F}_4 < \infty \quad (3.56)$$

and

$$\widetilde{C}_2 \approx \max\{\mathfrak{F}_3, \ \mathfrak{F}_4\}. \tag{3.57}$$

By (3.52), (3.53) and (3.55), (3.56) we obtain that inequalities (3.46), (3.50) and (3.51) hold if and only if $\mathfrak{F}_0 = \max\{\mathbb{F}_1, \ \mathfrak{F}_2, \ \mathfrak{F}_3, \ \mathfrak{F}_4\} < \infty$. Moreover, $\mathfrak{F}_0 \approx \max\{\widetilde{C}_1, \ \widetilde{C}_2, \ \widetilde{C}_3\}$, which implies that $\mathfrak{F}_0 \approx \widetilde{C}$. Since $\widetilde{C} \approx C$ we get $\mathfrak{F}_0 \approx C$. The last equivalence gives the statement of Theorem 3.3 in the case $V_\infty = \infty$.

2. Let $V_\infty < \infty$. By Theorem E inequality (3.1) on the cone of non-negative and non-increasing sequences $f \in l_{p,v}$ holds if and only if both inequality (3.29) and inequality (3.42) for all non-negative sequences $g = \{g_i\}_{i=1}^{\infty}$ hold. Moreover, $C \approx \max\{\widetilde{C}, \widehat{C}\}$.

As in the proof of Theorem 3.2 in case $V_\infty < \infty$ we obtain that inequality (3.42) holds if and only if $\widehat{C} < \infty$.

As proved above, inequality (3.29) holds if and only if the inequalities (3.46), (3.47) and (3.48) hold simultaneously.

In the same way as in the case $V_\infty = \infty$ we prove that inequality (3.47) holds if and only if $\mathfrak{F}_3 < \infty$, $\mathfrak{F}_4 < \infty$ and that inequality (3.48) holds if and only if $\mathfrak{F}_2 < \infty$.

By Theorem A it follows that inequality (3.46) holds if and only if

$$\mathbb{F}'_1 = \left(\sum_{k=1}^{\infty} \left(V_k^{-\frac{p'}{p}} - V_\infty^{-\frac{p'}{p}} \right)^{\frac{q(p-1)}{p-q}} \left(\sum_{i=1}^{k} A_{ii}^q u_i^q \right)^{\frac{q}{p-q}} A_{kk}^q u_k^q \right)^{\frac{p-q}{pq}} < \infty$$

Thus, in case $V_\infty < \infty$, inequality (3.1) on the cone of non-negative and non-increasing sequences $f \in l_{p,v}$ holds if and only if $\max\left\{ \mathbb{F}'_1,\ \mathfrak{F}_2,\ \mathfrak{F}_3,\ \mathfrak{F}_4,\ \widehat{C} \right\} < \infty$.

However, in the proof of Theorem 3.2 we have already proved that $\mathbb{F}_1 \approx \max\left\{ \mathbb{F}'_1,\ \widehat{C} \right\}$. Therefore, we obtain that inequality (3.1) on the cone of non-negative and non-increasing sequences $f \in l_{p,v}$ holds if and only if $\mathfrak{F}_0 = \max\{\mathbb{F}_1,\ \mathfrak{F}_2,\ \mathfrak{F}_3,\ \mathfrak{F}_4\} < \infty$ and $C \approx \max\{\widetilde{C}, \widehat{C}\} \approx \mathfrak{F}_0$ regardless of whether V_∞ is finite or infinite. Thus, the proof is complete.

Chapter 4

Applications of the main results

4.1 Boundedness and compactness criteria of compositions of matrix operators

In this Section we consider boundedness and compactness problems of the composition of matrix operators in weighted spaces of sequences when the corresponding matrices belong to the classes $\mathcal{O}_n^+ \cup \mathcal{O}_n^-$, $n \geq 0$ for $1 < p \leq q < \infty$.

We define

$$\left(\Sigma^+ f\right)_i := \sum_{j=1}^{i} \sigma_{i,j} f_j, \qquad i \geq 1, \tag{4.1}$$

$$\left(\Sigma^- g\right)_j := \sum_{i=j}^{\infty} \sigma_{i,j} g_i, \qquad j \geq 1. \tag{4.2}$$

Remark 4.1. *If we consider operator defined by (2.1) and (4.1), then for non-negative $a_{i,j}$, $\sigma_{j,k}$ and g_k we have*

$$\left(A^+ \circ \Sigma^+\right)(g)_i \equiv \left(A^+\left(\Sigma^+ g\right)\right)_i = \sum_{j=1}^{i} a_{i,j} \sum_{k=1}^{j} \sigma_{j,k} g_k$$

$$= \sum_{k=1}^{i}\left(\sum_{j=k}^{i} a_{i,j}\sigma_{j,k}\right) g_k = \sum_{k=1}^{i} w_{i,k} g_k.$$

Therefore, if $(a_{i,j}) \in \mathcal{O}_n^+$, $(\sigma_{j,k}) \in \mathcal{O}_m^+$, then according to Lemma 2.6 the matrix $(w_{i,k})$ of the operator $A^+ \circ \Sigma^+$ belongs to the class $\mathcal{O}_{m+n+1}^+$.

In the same way using Lemma 2.7 one can prove that if $(a_{i,j}) \in \mathcal{O}_n^-$, $(\sigma_{j,k}) \in \mathcal{O}_m^-$, then the matrix of the operator $A^+ \circ \Sigma^+$ belongs to the class $\mathcal{O}_{m+n+1}^-$.

In general, if the matrices $(a_{i,j}^k)$ of the operators $\left(A_k^+ f\right)_i = \sum\limits_{j=1}^{i} a_{i,j}^k f_j$ belong to the classes $\mathcal{O}_{m_k}^\pm$, $k = 1, \ldots, n$, then the matrix of the operator $\mathbb{A}_n^+ \equiv A_1^+ \circ A_2^+ \circ \cdots \circ A_n^+$ belongs to the class $\mathcal{O}_m^\pm$, where $m = \sum\limits_{k=1}^{n} m_k + n - 1$. Thus, according to Theorem 2.15 we can obtain criteria of boundedness and compactness of the matrix operator $\mathbb{A}_n^+$ from the weighted $l_{p,v}$ space into the weighted $l_{q,u}$ space for $1 < p \le q < \infty$.

Similarly, if the matrices $(a_{i,j}^k)$ of the operators $\left(A_k^- g\right)_j = \sum\limits_{i=j}^{\infty} a_{i,j}^k g_i$ belong to the classes $\mathcal{O}_{m_k}^\pm$, $k = 1, \ldots, n$, then by Lemmas 2.6 and 2.7, and Theorem 2.16 we obtain necessary and sufficient conditions for the boundedness and compactness of operator $\mathbb{A}_n^- = A_1^- \circ A_2^- \circ \cdots \circ A_n^-$ from $l_{p,v}$ into $l_{q,u}$ for $1 < p \le q < \infty$.

We define

$$(D_1)_s = \left(\sum_{k=1}^{s} v_k^{-p'} \left(\sum_{i=s}^{\infty} \left(\sum_{j=1}^{k} a_{i,j} \sigma_{k,j} \right)^q u_i^q \right)^{\frac{p'}{q}} \right)^{\frac{1}{p'}},$$

$$(D_2)_s = \left(\sum_{i=s}^{\infty} u_i^q \left(\sum_{k=1}^{s} \left(\sum_{j=1}^{k} a_{i,j} \sigma_{k,j} \right)^{p'} v_k^{-p'} \right)^{\frac{q}{p'}} \right)^{\frac{1}{q}},$$

$$(D_3)_s = \left(\sum_{i=1}^{s} u_i^q \left(\sum_{k=s}^{\infty} \left(\sum_{j=1}^{i} a_{i,j} \sigma_{k,j} \right)^{p'} v_k^{-p'} \right)^{\frac{q}{p'}} \right)^{\frac{1}{q}},$$

$$(D_4)_s = \left(\sum_{k=s}^{\infty} v_k^{-p'} \left(\sum_{i=1}^{s} \left(\sum_{j=1}^{i} a_{i,j} \sigma_{k,j} \right)^q u_i^q \right)^{\frac{p'}{q}} \right)^{\frac{1}{p'}},$$

$$(G_1)_s = \left(\sum_{j=1}^{s} v_j^{-p'} \left(\sum_{k=s}^{\infty} \left(\sum_{i=k}^{\infty} a_{i,j} \sigma_{i,k} \right)^q u_k^q \right)^{\frac{p'}{q}} \right)^{\frac{1}{p'}},$$

$$(G_2)_s = \left(\sum_{k=s}^{\infty} u_k^q \left(\sum_{j=1}^{s} \left(\sum_{i=k}^{\infty} a_{i,j} \sigma_{i,k} \right)^{p'} v_j^{-p'} \right)^{\frac{q}{p'}} \right)^{\frac{1}{q}},$$

$$(G_3)_s = \left(\sum_{k=1}^{s} u_k^q \left(\sum_{j=s}^{\infty} \left(\sum_{i=j}^{\infty} a_{i,j} \sigma_{i,k} \right)^{p'} v_j^{-p'} \right)^{\frac{q}{p'}} \right)^{\frac{1}{q}},$$

$$(G_4)_s = \left(\sum_{j=s}^{\infty} v_j^{-p'} \left(\sum_{k=1}^{s} \left(\sum_{i=j}^{\infty} a_{i,j} \sigma_{i,k} \right)^{q} u_k^q \right)^{\frac{p'}{q}} \right)^{\frac{1}{p'}}.$$

Then we set

$$D_1 = \sup_{s \geq 1}(D_1)_s, \quad D_2 = \sup_{s \geq 1}(D_2)_s, \quad D_3 = \sup_{s \geq 1}(D_3)_s, \quad D_4 = \sup_{s \geq 1}(D_4)_s,$$

$$G_1 = \sup_{s \geq 1}(G_1)_s, \quad G_2 = \sup_{s \geq 1}(G_2)_s, \quad G_3 = \sup_{s \geq 1}(G_3)_s, \quad G_4 = \sup_{s \geq 1}(G_4)_s$$

and

$$D_{13} = \max\{D_1, \ D_3\}, \quad D_{14} = \max\{D_1, \ D_4\},$$

$$D_{23} = \max\{D_2, \ D_3\}, \quad D_{24} = \max\{D_2, \ D_4\},$$

$$G_{13} = \max\{G_1, \ G_3\}, \quad G_{14} = \max\{G_1, \ G_4\},$$

$$G_{23} = \max\{G_2, \ G_3\}, \quad G_{24} = \max\{G_2, \ G_4\}.$$

Theorem 4.2. *Suppose that $1 < p \leq q < \infty$. Let the matrix $(a_{i,j})$ in (2.1) belong to the class $\mathcal{O}_n^+ \cup \mathcal{O}_n^-$, $n \geq 0$. Let the matrix $(\sigma_{i,j})$ in (4.2) belong to the class $\mathcal{O}_m^+ \cup \mathcal{O}_m^-$, $m \geq 0$. Then the following statements hold.*

(i) *The operator $A^+ \circ \Sigma^-$ is bounded from $l_{p,v}$ into $l_{q,u}$ if and only if at least one of the suprema D_{13}, D_{14}, D_{23} and D_{24} is finite. Moreover, $\|A^+ \circ \Sigma^-\|_{l_{p,v} \to l_{q,u}} \approx D_{13} \approx D_{14} \approx D_{23} \approx D_{24}$.*

(ii) *The operator $A^+ \circ \Sigma^-$ is compact from $l_{p,v}$ into $l_{q,u}$ if and only if at least one of the following conditions holds*

1) $\lim\limits_{s \to \infty} (D_1)_s = 0, \ \lim\limits_{s \to \infty} (D_3)_s = 0;$ *2)* $\lim\limits_{s \to \infty} (D_2)_s = 0, \ \lim\limits_{s \to \infty} (D_4)_s = 0;$

3) $\lim\limits_{s \to \infty} (D_1)_s = 0, \ \lim\limits_{s \to \infty} (D_4)_s = 0;$ *4)* $\lim\limits_{s \to \infty} (D_2)_s = 0, \ \lim\limits_{s \to \infty} (D_3)_s = 0.$

Theorem 4.3. *Suppose that $1 < p \leq q < \infty$. Let the matrix $(a_{i,j})$ in (2.1) belong to the class $\mathcal{O}_n^+ \cup \mathcal{O}_n^-$, $n \geq 0$. Let the matrix $(\sigma_{i,j})$ in (4.2) belong to the class $\mathcal{O}_m^+ \cup \mathcal{O}_m^-$, $m \geq 0$. Then the following statements hold.*

(j) *The operator $\Sigma^- \circ A^+$ is bounded from $l_{p,v}$ into $l_{q,u}$ if and only if at least one of the suprema G_{13}, G_{14}, G_{23} and G_{24} is finite. Moreover, $\|\Sigma^- \circ A^+\|_{l_{p,v} \to l_{q,u}} \approx G_{13} \approx G_{14} \approx G_{23} \approx G_{24}$.*

(jj) *The operator $\Sigma^- \circ A^+$ is compact from $l_{p,v}$ into $l_{q,u}$ if and only if at least one of the following conditions holds*

1) $\lim\limits_{s \to \infty} (G_1)_s = 0$, $\lim\limits_{s \to \infty} (G_3)_s = 0$;   *2)* $\lim\limits_{s \to \infty} (G_2)_s = 0$, $\lim\limits_{s \to \infty} (G_4)_s = 0$;
3) $\lim\limits_{s \to \infty} (G_1)_s = 0$, $\lim\limits_{s \to \infty} (G_4)_s = 0$;   *4)* $\lim\limits_{s \to \infty} (G_2)_s = 0$, $\lim\limits_{s \to \infty} (G_3)_s = 0$.

Next we define

$$(W^+ f)_i = \sum_{k=1}^{i} f_k \sum_{j=1}^{k} a_{i,j} \sigma_{k,j}, \quad i \geq 1, \tag{4.3}$$

$$(\Phi^- f)_i = \sum_{k=i}^{\infty} f_k \sum_{j=1}^{i} \sigma_{k,j} a_{i,j}, \quad i \geq 1. \tag{4.4}$$

By exploiting the results of Sections 2.2 and 2.6 we obtain.

Theorem 4.4. *Suppose that $1 < p \leq q < \infty$. Let the matrix $(a_{i,j})$ of the operator W^+ belong to the class $\mathcal{O}_n^+ \cup \mathcal{O}_n^-$, $n \geq 0$. Let the matrix $(\sigma_{i,j})$ of the operator W^+ be non-negative. Then the following statements hold.*

(i) *The operator W^+ is bounded from $l_{p,v}$ into $l_{q,u}$ if and only if at least one of the conditions $D_1 < \infty$ and $D_2 < \infty$ holds. Moreover, $\|W^+\|_{l_{p,v} \to l_{q,u}} \approx D_1 \approx D_2$.*

(ii) *The operator W^+ is compact from $l_{p,v}$ into $l_{q,u}$ if and only if at least one of the limits $\lim\limits_{s \to \infty} (D_1)_s$ and $\lim\limits_{s \to \infty} (D_2)_s$ equals zero.*

Theorem 4.5. *Suppose that $1 < p \leq q < \infty$. Let the matrix $(\sigma_{i,j})$ of the operator Φ^- belong to the class $\mathcal{O}_m^+ \cup \mathcal{O}_m^-$, $m \geq 0$. Let the matrix $(a_{i,j})$ of the operator Φ^- be non-negative. Then the following statements hold.*

(i) *The operator Φ^- is bounded from $l_{p,v}$ into $l_{q,u}$ if and only if at least one of the conditions $D_3 < \infty$ and $D_4 < \infty$ holds. Moreover, $\|\Phi^-\|_{l_{p,v} \to l_{q,u}} \approx D_3 \approx D_4$.*

(ii) *The operator Φ^- is compact from $l_{p,v}$ into $l_{q,u}$ if and only if at least one of the limits $\lim\limits_{s \to \infty} (D_3)_s$ and $\lim\limits_{s \to \infty} (D_4)_s$ equals zero.*

Proof of Theorem 4.4. The boundedness of the operator W^+ is equivalent to the validity of the following inequality

$$\|W^+ f\|_{q,u} \le C\|f\|_{p,v} \tag{4.5}$$

for all $f \in l_{p,v}$ with a positive finite constant C independent of f.

We set $\varpi_{i,k} = \sum\limits_{j=1}^{k} a_{i,j} \sigma_{k,j}$, $i \ge k$. Then we have

$$(W^+ f)_i = \sum_{k=1}^{i} f_k \sum_{j=1}^{k} a_{i,j} \sigma_{k,j} = \sum_{k=1}^{i} \varpi_{i,k} f_k, \quad i \ge 1.$$

1. Let $(a_{i,j}) \in \mathcal{O}_n^+$, $n \ge 0$. Let $(\sigma_{i,j})$ be an arbitrary non-negative matrix. Then according to Lemma 2.8 it follows that $(\varpi_{i,k}) \in \mathcal{O}_n^+$, $n \ge 0$. Therefore, by Theorem 2.15, inequality (4.5) holds if and only if at least one of the following two conditions holds

$$\sup_{s \ge 1} \left(\sum_{k=1}^{s} v_k^{-p'} \left(\sum_{i=s}^{\infty} \varpi_{i,k}^q u_i^q \right)^{\frac{p'}{q}} \right)^{\frac{1}{p'}} = \sup_{s \ge 1}(D_1)_s = D_1 < \infty, \tag{4.6}$$

$$\sup_{s \ge 1} \left(\sum_{i=s}^{\infty} u_i^q \left(\sum_{k=1}^{s} \varpi_{i,k}^{p'} v_k^{-p'} \right)^{\frac{q}{p'}} \right)^{\frac{1}{q}} = \sup_{s \ge 1}(D_2)_s = D_2 < \infty. \tag{4.7}$$

Moreover,

$$D_1 \approx D_2 \approx C. \tag{4.8}$$

In addition, by Theorem 2.15 operator W^+ is compact from $l_{p,v}$ into $l_{q,u}$ if and only if at least one of the limits $\lim\limits_{s \to \infty} (D_1)_s$ and $\lim\limits_{s \to \infty} (D_2)_s$ equals zero.

2. Let $(a_{i,j}) \in \mathcal{O}_n^-$, $n \geq 0$. Let $(\sigma_{i,j})$ be an arbitrary non-negative matrix. Then according to Lemma 2.8 it follows that $(\varpi_{i,k}) \in \mathcal{O}_n^- \beta$, $n \geq 0$, where $\beta = \{\beta_k\}_{k=1}^\infty$ and $\beta_k = \sum\limits_{j=1}^{k} \sigma_{k,j}$.

The membership $(\varpi_{i,k}) \in \mathcal{O}_n^- \beta$, $n \geq 0$ means that $\varpi_{i,k} \approx \beta_k \widetilde{\varpi}_{i,k}$, where $(\widetilde{\varpi}_{i,k}) \in \mathcal{O}_n^-$, $n \geq 0$. Substituting this equivalence in the left hand side of inequality (4.5), we obtain that

$$\left(\sum_{i=1}^{\infty} u_i^q \left| \sum_{k=1}^{i} \beta_k \widetilde{\varpi}_{i,k} f_k \right|^q \right)^{\frac{1}{q}} \leq \widetilde{C} \left(\sum_{i=1}^{\infty} v_i^p |f_i|^p \right)^{\frac{1}{p}} \tag{4.9}$$

holds for all $f \in l_{p,v}$, which is equivalent to inequality (4.5). Moreover, $\widetilde{C} \approx C$.

By taking $g_k = \beta_k f_k$ we see that the validity of inequality (4.9) is equivalent to the validity of the following inequality

$$\left(\sum_{i=1}^{\infty} u_i^q \left| \sum_{k=1}^{i} \widetilde{\varpi}_{i,k} g_k \right|^q \right)^{\frac{1}{q}} \leq \widetilde{C} \left(\sum_{i=1}^{\infty} \beta_i^{-p} v_i^p |g_i|^p \right)^{\frac{1}{p}}. \tag{4.10}$$

By inequality (4.10), we see that the boundedness of the operator W^+ from $l_{p,v}$ into $l_{q,u}$ is equivalent to the boundedness of operator $\widetilde{W}^+$ from $l_{p,v\beta^{-1}}$ into $l_{q,u}$, where $\widetilde{W}^+ = \sum\limits_{k=1}^{i} \widetilde{\varpi}_{i,k} g_k$.

Since $(\widetilde{\varpi}_{i,k}) \in \mathcal{O}_n^-$, $n \geq 0$, Theorem 2.15 implies that operator $\widetilde{W}^+$ is bounded from $l_{p,v\beta^{-1}}$ into $l_{q,u}$ if and only if one of the following conditions holds

$$\sup_{s \geq 1} \left(\sum_{k=1}^{s} v_k^{-p'} \beta_k^{p'} \left(\sum_{i=s}^{\infty} \widetilde{\varpi}_{i,k}^q u_i^q \right)^{\frac{p'}{q}} \right)^{\frac{1}{p'}}$$

$$\approx \sup_{s \geq 1} \left(\sum_{k=1}^{s} v_k^{-p'} \left(\sum_{i=s}^{\infty} \varpi_{i,k}^q u_i^q \right)^{\frac{p'}{q}} \right)^{\frac{1}{p'}} = D_1 < \infty,$$

$$\sup_{s \geq 1} \left(\sum_{i=s}^{\infty} u_i^q \left(\sum_{k=1}^{s} \widetilde{\varpi}_{i,k}^{p'} \beta_k^{p'} v_k^{-p'} \right)^{\frac{q}{p'}} \right)^{\frac{1}{q}}$$

$$\approx \sup_{s \geq 1} \left(\sum_{i=s}^{\infty} u_i^q \left(\sum_{k=1}^{s} \varpi_{i,k}^{p'} v_k^{-p'} \right)^{\frac{q}{p'}} \right)^{\frac{1}{q}} = D_2 < \infty.$$

As a consequence, the operator W^+ is bounded from $l_{p,v}$ into $l_{q,u}$ if and only if at least one of the conditions $D_1 < \infty$ and $D_2 < \infty$ holds. Moreover,

$$D_1 \approx D_2 \approx \widetilde{C} \approx C. \tag{4.11}$$

In addition, by Theorem 2.15 the operator $\widetilde{W}^+$ is compact from $l_{p,v\beta^{-1}}$ into $l_{q,u}$ if and only if at least one of the limits $\lim\limits_{s\to\infty} (D_1)_s$ and $\lim\limits_{s\to\infty} (D_2)_s$ equals zero. Thus, the validity of at least one of these conditions is necessary and sufficient for the compactness of the operator W^+ from $l_{p,v}$ into $l_{q,u}$.

Hence, the proof of Theorem 4.4 is complete.

The proof of Theorem 4.5 follows directly from Theorem 2.16 by using Lemma 2.8.

Proof of Theorem 4.2.

(*i*) *Proof of boundedness.Necessity.* Let $(a_{i,j}) \in \mathcal{O}_n^+ \cup \mathcal{O}_n^-$, $n \geq 0$ and $(\sigma_{i,j}) \in \mathcal{O}_m^+ \cup \mathcal{O}_m^-$, $m \geq 0$. Let the operator $A^+ \circ \Sigma^-$ be bounded from $l_{p,v}$ into $l_{q,u}$. Then the following inequality

$$\|(A^+ \circ \Sigma^-)f\|_{q,u} \leq C\|f\|_{p,v} \quad \forall f \in l_{p,v} \tag{4.12}$$

holds with a positive finite constant C independent of f.

Since $a_{i,j}$ and $\sigma_{k,j}$ are non-negative, we note that for all non-negative $f \in l_{p,v}$ we have the following.

$$
\begin{aligned}
(A^+ \circ \Sigma^-)(f)_i &= (A^+(\Sigma^- f))_i = \sum_{j=1}^{i} a_{i,j} \sum_{k=j}^{\infty} \sigma_{k,j} f_k \tag{4.13} \\
&\approx \sum_{j=1}^{i} a_{i,j} \sum_{k=j}^{i} \sigma_{k,j} f_k + \sum_{j=1}^{i} a_{i,j} \sum_{k=i}^{\infty} \sigma_{k,j} f_k \\
&= \sum_{k=1}^{i} f_k \sum_{j=1}^{k} a_{i,j} \sigma_{k,j} + \sum_{k=i}^{\infty} f_k \sum_{j=1}^{i} \sigma_{k,j} a_{i,j} \\
&= \left(W^+ f\right)_i + \left(\Phi^- f\right)_i,
\end{aligned}
$$

where the operators W^+ and Φ^- are defined by (4.3) and (4.4), respectively.

Therefore, for all $f \geq 0$ we obtain the following.

$$\left(\sum_{i=1}^{\infty} u_i^q \left[\left(A^+ \circ \Sigma^- \right) f \right]_i^q \right)^{\frac{1}{q}}$$

$$\gg \max \left\{ \left(\sum_{i=1}^{\infty} u_i^q \left(W^+ f \right)_i^q \right)^{\frac{1}{q}}, \ \left(\sum_{i=1}^{\infty} u_i^q \left(\Phi^- f \right)_i^q \right)^{\frac{1}{q}} \right\}. \quad (4.14)$$

We consider the following four cases separately.

1. Suppose that the matrix $(a_{i,j})$ belongs to the class $\mathcal{O}_n^+$, $n \geq 0$ and that the matrix $(\sigma_{i,j})$ belongs to the class $\mathcal{O}_m^+ \cup \mathcal{O}_m^-$, $m \geq 0$. Then by Lemma 2.8, we have that $(\varpi_{i,k}) \in \mathcal{O}_n^+$, $n \geq 0$, where $\varpi_{i,k} = \sum_{j=1}^{k} a_{i,j} \sigma_{k,j}$, $i \geq k$. The membership $(\varpi_{i,k}) \in \mathcal{O}_n^+$, $n \geq 0$ means that there exist matrices $(\varpi_{i,k}) \in \mathcal{O}_\gamma^+$, $\gamma = 0, ..., n-1$ and matrices $(d_{i,k}^{n,\gamma})$ such that

$$\varpi_{i,k} \equiv \varpi_{i,k}^{(n)} \approx \sum_{\gamma=0}^{n} d_{i,j}^{n,\gamma} \varpi_{j,k}^{(\gamma)} \quad (4.15)$$

for all $i \geq j \geq k \geq 1$, where $d_{i,j}^{n,n} \equiv 1$.

For $r \geq 1$ we introduce the following sequence.

$$f_r = \{f_{r,k}\}_{k=1}^{\infty}: \quad f_{r,k} = \begin{cases} \left(\varpi_{r,k}^{(\gamma)} \right)^{p'-1} v_k^{-p'}, & 1 \leq k \leq r, \\ 0, & k > r. \end{cases} \quad (4.16)$$

Applying f_r to the right hand side of inequality (4.12) we obtain

$$\|f_r\|_{p,v} = \left(\sum_{k=1}^{\infty} |f_{r,k}|^p v_k^p \right)^{\frac{1}{p}} = \left(\sum_{k=1}^{r} \left(\varpi_{r,k}^{(\gamma)} \right)^{p'} v_k^{-p'} \right)^{\frac{1}{p}}. \quad (4.17)$$

Since f_r is the non-negative sequence, by (4.14) we have the following.

$$\|(A^+ \circ \Sigma^-) f_r\|_{q,u} \gg \|W^+ f_r\|_{q,u} = \left(\sum_{i=1}^{\infty} \left(\sum_{k=1}^{i} \varpi_{i,k}^{(n)} f_{r,k} \right)^q u_i^q \right)^{\frac{1}{q}}$$

$$\gg \left(\sum_{i=r}^{\infty} \left(d_{i,r}^{n,\gamma} \right)^q u_i^q \right)^{\frac{1}{q}} \left(\sum_{k=1}^{r} \left(\varpi_{r,k}^{(\gamma)} \right)^{p'} v_k^{-p'} \right). \quad (4.18)$$

By (4.12), (4.17) and (4.18) we deduce that

$$C \gg \left(\sum_{i=r}^{\infty} \left(d_{i,r}^{n,\gamma} \right)^q u_i^q \right)^{\frac{1}{q}} \left(\sum_{k=1}^{r} \left(\varpi_{r,k}^{(\gamma)} \right)^{p'} v_k^{-p'} \right)^{\frac{1}{p'}} \equiv \left(\mathcal{D}_{\gamma,n} \right)_r. \quad (4.19)$$

Since inequality (4.19) holds for all $\gamma = 0, 1, \ldots, n$, we obtain

$$C \gg \max_{0 \leq \gamma \leq n} (\mathcal{D}_{\gamma,n})_r \equiv (\mathcal{D}_n)_r \,.$$

Moreover,

$$(\mathcal{D}_n)_r = \max_{0 \leq \gamma \leq n} (\mathcal{D}_{\gamma,n})_r \approx \sum_{\gamma=0}^{n} (\mathcal{D}_{\gamma,n})_r \approx (D_1)_r \approx (D_2)_r. \tag{4.20}$$

Since $r \geq 1$ is arbitrary, we deduce

$$C \gg \sup_{r \geq 1} (\mathcal{D}_n)_r \approx \sup_{r \geq 1}(D_1)_r \approx \sup_{r \geq 1}(D_2)_r,$$

and accordingly, we have $C \gg D_1 \approx D_2$.

2. Suppose that the matrix $(a_{i,j})$ belongs to the class $\mathcal{O}_n^-$, $n \geq 0$ and that the matrix $(\sigma_{i,j})$ belongs to the class $\mathcal{O}_m^+ \cup \mathcal{O}_m^-$, $m \geq 0$. Then by Lemma 2.8, we have that $(\varpi_{i,k}) \in \mathcal{O}_n^- \beta$, $n \geq 0$. The membership $(\varpi_{i,k}) \in \mathcal{O}_n^+ \beta$, $n \geq 0$ means that there exist a non-negative sequence $\{\beta_k\}_{k=1}^{\infty}$ and matrices $(\widetilde{\varpi}_{i,k}) \in \mathcal{O}_n^-$ such that

$$\varpi_{i,k} \equiv \varpi_{i,k}^{(n)} \approx \widetilde{\varpi}_{i,k}^{(n)} \beta_k \quad \text{and} \quad \widetilde{\varpi}_{i,k}^{(n)} \approx \sum_{\gamma=0}^{n} \widetilde{\varpi}_{i,j}^{(\gamma)} \widetilde{d}_{j,k}^{\gamma,n} \tag{4.21}$$

for all $i \geq j \geq k \geq 1$, where $\widetilde{d}_{j,k}^{n,n} \equiv 1$.

For $r \geq 1$ we introduce the following sequence.

$$\widetilde{f}_r = \{\widetilde{f}_{r,k}\}_{k=1}^{\infty} : \quad \widetilde{f}_{r,k} = \begin{cases} \left(\widetilde{d}_{r,k}^{\gamma,n} \beta_k\right)^{p'-1} v_k^{-p'}, & 1 \leq k \leq r, \\ 0, & k > r. \end{cases} \tag{4.22}$$

Taking into account inequality (4.14), we apply $\widetilde{f}_r$ to inequality (4.12).

$$\begin{aligned} C\|\widetilde{f}_r\|_{p,v} &= C\left(\sum_{k=1}^{r} \left(\widetilde{d}_{r,k}^{\gamma,n} \beta_k\right)^{p'} v_k^{-p'}\right)^{\frac{1}{p}} \geq \|(A^+ \circ \Sigma^-)\widetilde{f}_r\|_{q,u} \\ &\gg \|W^+ \widetilde{f}_r\|_{q,u} = \left(\sum_{i=1}^{\infty} \left(\sum_{k=1}^{i} \varpi_{i,k}^{(n)} \widetilde{f}_{r,k}\right)^q u_i^q\right)^{\frac{1}{q}} \\ &\gg \left(\sum_{i=r}^{\infty} u_i^q \left(\widetilde{\varpi}_{i,r}^{(\gamma)}\right)^q\right)^{\frac{1}{q}} \left(\sum_{k=1}^{r} \left(\widetilde{d}_{r,k}^{\gamma,n} \beta_k\right)^{p'} v_k^{-p'}\right), \end{aligned}$$

which implies that

$$C \gg \left(\sum_{i=r}^{\infty} u_i^q \left(\widetilde{\varpi}_{i,r}^{(\gamma)} \right)^q \right)^{\frac{1}{q}} \left(\sum_{k=1}^{r} \left(\widetilde{d}_{r,k}^{\gamma,n} \beta_k \right)^{p'} v_k^{-p'} \right)^{\frac{1}{p'}} \equiv \left(\widetilde{\mathcal{D}}_{\gamma,n} \right)_r. \qquad (4.23)$$

Since inequality (4.23) holds for all $\gamma = 0, 1, \ldots, n$, we obtain

$$C \gg \max_{0 \le \gamma \le n} \left(\widetilde{\mathcal{D}}_{\gamma,n} \right)_r \equiv \left(\widetilde{\mathcal{D}}_n \right)_r.$$

Since

$$(\widetilde{\mathcal{D}}_n)_r = \max_{0 \le \gamma \le n} \left(\widetilde{\mathcal{D}}_{\gamma,n} \right)_r \approx \sum_{\gamma=0}^{n} \left(\widetilde{\mathcal{D}}_{\gamma,n} \right)_r \approx (D_1)_r \approx (D_2)_r, \qquad (4.24)$$

and since $r \ge 1$ is arbitrary, we deduce

$$C \gg \sup_{r \ge 1} \left(\widetilde{\mathcal{D}}_n \right)_r \approx \sup_{r \ge 1}(D_1)_r \approx \sup_{r \ge 1}(D_2)_r.$$

Hence, we have that $C \gg D_1 \approx D_2$.

It is known that the inequality (4.12) holds if and only if the following dual inequality

$$\|(A^+ \circ \Sigma^-)^* f\|_{p',v^{-1}} \le C\|f\|_{q',u^{-1}}, \qquad f \in l_{q',u^{-1}}, \qquad (4.25)$$

holds for the conjugate operator $(A^+ \circ \Sigma^-)^*$, which has the following from

$$\left[\left(A^+ \circ \Sigma^- \right)^* f \right]_k = \sum_{j=1}^{k} \sigma_{k,j} \sum_{i=j}^{\infty} a_{i,j} f_i. \qquad (4.26)$$

Moreover, the best constants in (4.12) and (4.25) coincide.

Since $a_{i,j}$, $\sigma_{k,j}$ are non-negative, for all non-negative $f \in l_{q',u^{-1}}$ we have the following.

$$\begin{aligned}
(A^+ \circ \Sigma^-)^*(f)_k &= \sum_{j=1}^{k} \sigma_{k,j} \sum_{i=j}^{\infty} a_{i,j} f_i \\
&\approx \sum_{j=1}^{k} \sigma_{k,j} \sum_{i=j}^{k} a_{i,j} f_i + \sum_{j=1}^{k} \sigma_{k,j} \sum_{i=k}^{\infty} a_{i,j} f_i \\
&= \sum_{i=1}^{k} f_i \sum_{j=1}^{i} a_{i,j} \sigma_{k,j} + \sum_{i=k}^{\infty} f_i \sum_{j=1}^{k} a_{i,j} \sigma_{k,j} \\
&\equiv \left(\Phi^+ f \right)_k + \left(W^- f \right)_k,
\end{aligned} \qquad (4.27)$$

where the operators Φ^+ and W^- are the conjugate operators to the operators Φ^- and W^+, respectively.

$$\left(\Phi^+ f\right)_k = \sum_{i=1}^k f_i \sum_{j=1}^i a_{i,j}\sigma_{k,j}, \quad k \geq 1, \tag{4.28}$$

$$\left(W^- f\right)_k = \sum_{i=k}^\infty f_i \sum_{j=1}^k a_{i,j}\sigma_{k,j}, \quad k \geq 1. \tag{4.29}$$

By (4.27) for all $f \geq 0$ we obtain that

$$\left(\sum_{k=1}^\infty v_k^{-p'} \left[\left(A^+ \circ \Sigma^-\right)^* f\right]_k^{p'}\right)^{\frac{1}{p'}}$$

$$\gg \max\left\{\left(\sum_{k=1}^\infty v_k^{-p'} \left(\Phi^+ f\right)_k^{p'}\right)^{\frac{1}{p'}}, \left(\sum_{k=1}^\infty v_k^{-p'} \left(W^- f\right)_k^{p'}\right)^{\frac{1}{p'}}\right\}. \tag{4.30}$$

3. Suppose that the matrix $(a_{i,j})$ belongs to the class $\mathcal{O}_n^+ \cup \mathcal{O}_n^-$, $n \geq 0$ and that the matrix $(\sigma_{i,j})$ belongs to the class $\mathcal{O}_m^+$. Then by Lemma 2.8, we have that $(\varphi_{k,i}) \in \mathcal{O}_m^+$, $m \geq 0$, where $\varphi_{k,i} = \sum_{j=1}^i \sigma_{k,j}a_{i,j}$, $k \geq i$. The membership $(\varphi_{k,i}) \in \mathcal{O}_m^+$, $m \geq 0$ means that there exist matrices $(\varphi_{k,i}) \in \mathcal{O}_\gamma^+$, $\gamma = 0, ..., m-1$ and matrices $e_{k,j}^{m,\gamma}$ such that

$$\varphi_{k,i} \equiv \varphi_{k,i}^{(m)} \approx \sum_{\gamma=0}^m e_{k,j}^{m,\gamma}\varphi_{j,i}^{(\gamma)} \tag{4.31}$$

for all $k \geq j \geq i \geq 1$, where $e_{k,j}^{m,m} \equiv 1$.

For $r \geq 1$ we introduce the following sequence.

$$\widehat{f}_r = \{\widehat{f}_{r,i}\}_{i=1}^\infty : \quad \widehat{f}_{r,i} = \begin{cases} \left(\varphi_{r,i}^{(\gamma)}\right)^{q-1} u_i^q, & 1 \leq i \leq r, \\ 0, & i > r. \end{cases} \tag{4.32}$$

Applying $\widehat{f}_r$ to the right hand side of inequality (4.25), we obtain

$$\|\widehat{f}_r\|_{q',u^{-1}} = \left(\sum_{i=1}^\infty |\widehat{f}_{r,i}|^{q'} u_i^{-q'}\right)^{\frac{1}{q'}} = \left(\sum_{i=1}^r \left(\varphi_{r,i}^{(\gamma)}\right)^q u_i^q\right)^{\frac{1}{q'}}. \tag{4.33}$$

Since $\widehat{f}_r$ is non-negative, by using (4.30) we have the following.

$$\|(A^+ \circ \Sigma^-)^* \widehat{f}_r\|_{p',v^{-1}} \gg \|\Phi^+ \widehat{f}_r\|_{p',v^{-1}} \tag{4.34}$$

$$= \left(\sum_{k=1}^{\infty} v_k^{-p'} \left(\sum_{i=1}^{k} \varphi_{k,i}^{(m)} \widehat{f}_{r,i} \right)^{p'} \right)^{\frac{1}{p'}}$$

$$\gg \left(\sum_{k=r}^{\infty} v_k^{-p'} \left(e_{k,r}^{m,\gamma} \right)^{p'} \right)^{\frac{1}{p'}} \left(\sum_{i=1}^{r} \left(\varphi_{r,i}^{(\gamma)} \right)^q u_i^q \right).$$

Hence, by (4.25), (4.33) and (4.34) we deduce that

$$C \gg \left(\sum_{k=r}^{\infty} v_k^{-p'} \left(e_{k,r}^{m,\gamma} \right)^{p'} \right)^{\frac{1}{p'}} \left(\sum_{i=1}^{r} \left(\varphi_{r,i}^{(\gamma)} \right)^q u_i^q \right)^{\frac{1}{q}} \equiv (\mathbb{D}_{\gamma,m})_r. \tag{4.35}$$

Since inequality (4.35) holds for all $\gamma = 0, 1, \ldots, m$, we obtain

$$C \gg \max_{0 \le \gamma \le m} (\mathbb{D}_{\gamma,m})_r \equiv (\mathbb{D}_m)_r.$$

Moreover,

$$(\mathbb{D}_m)_r = \max_{0 \le \gamma \le n} (\mathbb{D}_{\gamma,m})_r \approx \sum_{\gamma=0}^{m} (\mathbb{D}_{\gamma,m})_r \approx (D_3)_r \approx (D_4)_r. \tag{4.36}$$

Since $r \ge 1$ is arbitrary, we deduce that

$$C \gg \sup_{r \ge 1} (\mathbb{D}_m)_r \approx \sup_{r \ge 1} (D_3)_r \approx \sup_{r \ge 1} (D_4)_r,$$

and accordingly, we have that $C \gg D_3 \approx D_4$.

4. Suppose that the matrix $(a_{i,j})$ belongs to the class $\mathcal{O}_n^+ \cup \mathcal{O}_n^-$, $n \ge 0$ and that the matrix $(\sigma_{i,j})$ belongs to the class $\mathcal{O}_m^-$. Then by Lemma 2.8, we have that $(\varphi_{k,i}) \in \mathcal{O}_m^- \alpha$, $m \ge 0$. The membership $(\varphi_{k,i}) \in \mathcal{O}_m^- \alpha$, $m \ge 0$ means that there exist a non-negative sequence $\{\alpha_i\}_{i=1}^{\infty}$ and matrices $(\widetilde{\varphi}_{k,i}) \in \mathcal{O}_m^-$ such that

$$\varphi_{k,i} \equiv \varphi_{k,i}^{(m)} \approx \widetilde{\varphi}_{k,i}^{(m)} \alpha_i \quad \text{and} \quad \widetilde{\varphi}_{k,i}^{(m)} \approx \sum_{\gamma=0}^{m} \widetilde{\varphi}_{k,j}^{(\gamma)} \widetilde{e}_{j,i}^{\gamma,m} \tag{4.37}$$

for all $k \ge j \ge i \ge 1$, where $\widetilde{e}_{j,i}^{m,m} \equiv 1$.

For $r \ge 1$ we introduce the following sequence.

$$\overline{f}_r = \{\overline{f}_{r,i}\}_{i=1}^{\infty} : \quad \overline{f}_{r,i} = \begin{cases} \left(\widetilde{e}_{r,i}^{\gamma,m} \alpha_i \right)^{q-1} u_i^q, & 1 \le i \le r, \\ 0, & i > r. \end{cases} \tag{4.38}$$

Taking into account inequality (4.30), we apply $\overline{f}_r$ to inequality (4.25)

$$
\begin{aligned}
C\|\overline{f}_r\|_{q',u^{-1}} &= C\left(\sum_{i=1}^{\infty} |\overline{f}_{r,i}|^{q'} u_i^{-q'}\right)^{\frac{1}{q'}} = C\left(\sum_{i=1}^{r} \left(\widetilde{e}_{r,i}^{\gamma,m}\alpha_i\right)^q u_i^q\right)^{\frac{1}{q'}} \\
&\geq \|(A^+ \circ \Sigma^-)^* \overline{f}_r\|_{p',v^{-1}} \gg \|\Phi^+ \overline{f}_r\|_{p',v^{-1}} \\
&= \left(\sum_{k=1}^{\infty} v_k^{-p'} \left(\sum_{i=1}^{k} \varphi_{k,i}^{(m)}\overline{f}_{r,i}\right)^{p'}\right)^{\frac{1}{p'}} \\
&\gg \left(\sum_{k=r}^{\infty} \left(\varphi_{k,r}^{(\gamma)}\right)^{p'} v_k^{-p'}\right)^{\frac{1}{p'}} \left(\sum_{i=1}^{r} \left(\widetilde{e}_{r,i}^{\gamma,m}\alpha_i\right)^q u_i^q\right),
\end{aligned}
$$

which implies that

$$
C \gg \left(\sum_{k=r}^{\infty} \left(\varphi_{k,r}^{(\gamma)}\right)^{p'} v_k^{-p'}\right)^{\frac{1}{p'}} \left(\sum_{i=1}^{r} \left(\widetilde{e}_{r,i}^{\gamma,m}\alpha_i\right)^q u_i^q\right)^{\frac{1}{q}} \equiv \left(\widetilde{\mathbb{D}}_{\gamma,m}\right)_r. \qquad (4.39)
$$

Since inequality (4.39) holds for all $\gamma = 0, 1, \ldots, m$ we obtain

$$
C \gg \max_{0 \leq \gamma \leq n} \left(\widetilde{\mathbb{D}}_{\gamma,m}\right)_r \equiv \left(\widetilde{\mathbb{D}}_m\right)_r.
$$

Since

$$
\left(\widetilde{\mathbb{D}}_m\right)_r = \max_{0 \leq \gamma \leq m} \left(\widetilde{\mathbb{D}}_{\gamma,m}\right)_r \approx \sum_{\gamma=0}^{m} \left(\widetilde{\mathbb{D}}_{\gamma,m}\right)_r \approx (D_3)_r \approx (D_4)_r, \qquad (4.40)
$$

and since $r \geq 1$ is arbitrary, we deduce

$$
C \gg \sup_{r \geq 1} \left(\widetilde{\mathbb{D}}_m\right)_r \approx \sup_{r \geq 1}(D_3)_r \approx \sup_{r \geq 1}(D_4)_r.
$$

Hence, we have that $C \gg D_3 \approx D_4$.

Finally, we have that $C \gg D_1 \approx D_2$ and $C \gg D_3 \approx D_4$, which imply that

$$
\begin{aligned}
C \gg \max\{D_1, \ D_3\}, \ C \gg \max\{D_2, \ D_3\}, \\
C \gg \max\{D_1, \ D_4\}, \ C \gg \max\{D_2, \ D_4\}.
\end{aligned} \qquad (4.41)
$$

Thus, the proof of necessity is complete.

Sufficiency. Let the matrix $(a_{i,j})$ in (2.1) belong to the class $\mathcal{O}_n^+ \cup \mathcal{O}_n^-$, $n \geq 0$. Let the matrix $(\sigma_{i,j})$ in (4.2) belong to the class $\mathcal{O}_m^+ \cup \mathcal{O}_m^-$, $m \geq 0$. Let at least one of the suprema D_{13}, D_{14}, D_{23} and D_{24} be finite. Now we show

that the operator $A^+ \circ \Sigma^-$ is bounded from $l_{p,v}$ into $l_{q,u}$. It means that we have to prove that inequality (4.12) holds for all $f \in l_{p,v}$.

However, since $a_{i,j}$, $\sigma_{i,j}$ are non-negative, we have the following.

$$
\begin{aligned}
\|(A^+ \circ \Sigma^-)f\|_{q,u} &= \left(\sum_{i=1}^{\infty} u_i^q \left| \sum_{j=1}^{i} a_{i,j} \sum_{k=j}^{\infty} \sigma_{k,j} f_k \right|^q \right)^{\frac{1}{q}} \\
&\leq \left(\sum_{i=1}^{\infty} u_i^q \left(\sum_{j=1}^{i} a_{i,j} \sum_{k=j}^{\infty} \sigma_{k,j} |f_k| \right)^q \right)^{\frac{1}{q}}.
\end{aligned}
\tag{4.42}
$$

Therefore, if we prove that inequality (4.12) holds for all non-negative $f \in l_{p,v}$, by (4.42) we obtain that inequality (4.12) holds for all $f \in l_{p,v}$. Actually, we have to prove the validity of the following inequality

$$
\left(\sum_{i=1}^{\infty} u_i^q \left(\sum_{j=1}^{i} a_{i,j} \sum_{k=j}^{\infty} \sigma_{k,j} f_k \right)^q \right)^{\frac{1}{q}} \leq \widetilde{C} \left(\sum_{i=1}^{\infty} v_i^p f_i^p \right)^{\frac{1}{p}} \quad 0 \leq f \in l_{p,v}. \tag{4.43}
$$

Moreover, we have

$$
C \leq \widetilde{C}, \tag{4.44}
$$

where C and $\widetilde{C}$ are the best constants in (4.12) and (4.43), respectively.

By using (4.13) for all non-negative f we have the following.

$$
\begin{aligned}
\sum_{i=1}^{\infty} u_i^q &\left(\sum_{j=1}^{i} a_{i,j} \sum_{k=j}^{\infty} \sigma_{k,j} f_k \right)^q \\
&\approx \sum_{i=1}^{\infty} u_i^q \left(\sum_{k=1}^{i} f_k \sum_{j=1}^{k} a_{i,j} \sigma_{k,j} + \sum_{k=i}^{\infty} f_k \sum_{j=1}^{i} \sigma_{k,j} a_{i,j} \right)^q \\
&\approx \sum_{i=1}^{\infty} u_i^q \left(\sum_{k=1}^{i} f_k \sum_{j=1}^{k} a_{i,j} \sigma_{k,j} \right)^q + \sum_{i=1}^{\infty} u_i^q \left(\sum_{k=i}^{\infty} f_k \sum_{j=1}^{i} \sigma_{k,j} a_{i,j} \right)^q \\
&= \sum_{i=1}^{\infty} u_i^q \left(W^+ f \right)_i^q + \sum_{i=1}^{\infty} u_i^q \left(\Phi^- f \right)_i^q.
\end{aligned}
$$

Substituting the last inequality to the left hand side of inequality (4.43) we obtain

$$
\left(\sum_{i=1}^{\infty} u_i^q \left(W^+ f \right)_i^q + \sum_{i=1}^{\infty} u_i^q \left(\Phi^- f \right)_i^q \right)^{\frac{1}{q}} \leq C_0 \left(\sum_{i=1}^{\infty} v_i^p f_i^p \right)^{\frac{1}{p}} \quad 0 \leq f \in l_{p,v}, \tag{4.45}
$$

which is equivalent to the inequality (4.43). Moreover, $\widetilde{C} \approx C_0$.

Inequality (4.45) holds if and only if the following inequalities hold simultaneously

$$\left(\sum_{i=1}^{\infty} u_i^q \left(W^+ f \right)_i^q \right)^{\frac{1}{q}} \leq C_1 \left(\sum_{i=1}^{\infty} v_i^p f_i^p \right)^{\frac{1}{p}}, \quad 0 \leq f \in l_{p,v}, \qquad (4.46)$$

$$\left(\sum_{i=1}^{\infty} u_i^q \left(\Phi^- f \right)_i^q \right)^{\frac{1}{q}} \leq C_2 \left(\sum_{i=1}^{\infty} v_i^p f_i^p \right)^{\frac{1}{p}}, \quad 0 \leq f \in l_{p,v}. \qquad (4.47)$$

Moreover,

$$C_0 \approx \max\{C_1, \ C_2\}. \qquad (4.48)$$

According to statement of Theorem 4.2 $(a_{i,j})$ belong to the class $\mathcal{O}_n^+ \cup \mathcal{O}_n^-$, $n \geq 0$. Then by Theorem 4.4 inequality (4.46) holds if and only if one of the conditions $D_1 < \infty$ and $D_2 < \infty$ holds. Moreover,

$$D_1 \approx D_2 \approx C_1. \qquad (4.49)$$

In addition, since $(\sigma_{i,j})$ belong to the class $\mathcal{O}_m^+ \cup \mathcal{O}_m^-$, $m \geq 0$, by Theorem 4.5 we have that inequality (4.47) holds if and only if one of the conditions $D_3 < \infty$ and $D_4 < \infty$ holds. Moreover,

$$D_3 \approx D_4 \approx C_2. \qquad (4.50)$$

By (4.49), (4.50) and (4.48) we obtain that inequalities (4.46) and (4.47), accordingly inequality (4.43) hold for all non-negative $f \in l_{p,v}$ if and only if one of the following conditions holds

$$D_{13} = \max\{D_1, \ D_3\} < \infty, \quad D_{14} = \max\{D_1, \ D_4\} < \infty, \qquad (4.51)$$

$$D_{23} = \max\{D_2, \ D_3\} < \infty, \quad D_{24} = \max\{D_2, \ D_4\} < \infty.$$

Moreover,

$$D_{13} \approx D_{14} \approx D_{23} \approx D_{24} \approx \max\{C_1, \ C_2\} \approx C_0 \approx \widetilde{C}. \qquad (4.52)$$

Since $C \leq \widetilde{C}$, we obtain that $C \ll D_{13} \approx D_{14} \approx D_{23} \approx D_{24}$, which together with (4.41) implies that $C \approx D_{13} \approx D_{14} \approx D_{23} \approx D_{24}$.

Thus, the proof of boundedness is complete.

(*ii*) *Proof of compactness. Necessity.* Let the operator $A^+ \circ \Sigma^-$ be compact from $l_{p,v}$ to $l_{q,u}$. We introduce the following set:

$$M = \left\{ g = \{g_k\}_{k=1}^{\infty} : \ g_k = \frac{f_k}{\|f\|_{p,v}}, \quad 0 \leq f = \{f_k\}_{k=1}^{\infty} \in l_{p,v} \right\}.$$

Clearly, $\|g\|_{p,v} = 1$ for all $g \in M$. Since the operator $A^+ \circ \Sigma^-$ is compact from $l_{p,v}$ into $l_{q,u}$, the set $\{u(A^+ \circ \Sigma^-)g, \ \ g \in M\}$ is precompact in l_q. Hence, by Theorem C we conclude that

$$\lim_{r \to \infty} \sup_{g \in M} \left(\sum_{i=r}^{\infty} u_i^q \left[\left(A^+ \circ \Sigma^- \right) g \right]_i^q \right)^{\frac{1}{q}} = 0. \tag{4.53}$$

However, since $g = \{g_k\}_{k=1}^{\infty}$ is non-negative, by (4.13) we have that

$$\left(\sum_{i=r}^{\infty} u_i^q \left[\left(A^+ \circ \Sigma^- \right) g \right]_i^q \right)^{\frac{1}{q}} \gg \max \left\{ \left(\sum_{i=r}^{\infty} u_i^q \left(W^+ g \right)_i^q \right)^{\frac{1}{q}}, \ \left(\sum_{i=r}^{\infty} u_i^q \left(\Phi^- g \right)_i^q \right)^{\frac{1}{q}} \right\}$$

for all $g \in M$. Therefore,

$$\lim_{r \to \infty} \sup_{g \in M} \left(\sum_{i=r}^{\infty} u_i^q \left(W^+ g \right)_i^q \right)^{\frac{1}{q}} = 0 \tag{4.54}$$

and

$$\lim_{r \to \infty} \sup_{g \in M} \left(\sum_{i=r}^{\infty} u_i^q \left(\Phi^- g \right)_i^q \right)^{\frac{1}{q}} = 0. \tag{4.55}$$

1. Suppose that the matrix $(a_{i,j})$ belongs to the class $\mathcal{O}_n^+$, $n \geq 0$ and that the matrix $(\sigma_{i,j})$ belongs to the class $\mathcal{O}_m^+ \cup \mathcal{O}_m^-$, $m \geq 0$. Then by Lemma 2.8, we have that $(\varpi_{i,k}) \in \mathcal{O}_n^+$, $n \geq 0$, where $\varpi_{i,k} = \sum_{j=1}^{k} a_{i,j} \sigma_{k,j}$, $i \geq k$. Since $(\varpi_{i,k}) \in \mathcal{O}_n^+$, $n \geq 0$, there exist matrices $(\varpi_{i,k}) \in \mathcal{O}_\gamma^+$, $\gamma = 0, ..., n - 1$ and matrices $(d_{i,k}^{n;\gamma})$ such that (4.15) holds for all $i \geq j \geq k \geq 1$, where $d_{i,j}^{n,n} \equiv 1$.

Let $f_r = \{f_{r,k}\}_{k=1}^{\infty}$, $r \geq 1$ be the sequences defined by (4.16).

Let $M_1 = \left\{ g_r = \{g_{r,k}\}_{k=1}^{\infty} : \ g_{r,k} = \dfrac{f_{r,k}}{\|f_r\|_{p,v}}, \ r \geq 1 \right\}$. Since $M_1 \subset M$, we have

$$\lim_{r \to \infty} \sup_{g_r \in M_1} \left(\sum_{i=r}^{\infty} u_i^q \left(W^+ g_r \right)_i^q \right)^{\frac{1}{q}} = 0. \tag{4.56}$$

Moreover, as in (2.59), we obtain

$$\sup_{g_r \in M_1} \left(\sum_{i=r}^{\infty} u_i^q \left(W^+ g_r \right)_i^q \right)^{\frac{1}{q}} \geq \left(\sum_{i=r}^{\infty} u_i^q \left(W^+ g_r \right)_i^q \right)^{\frac{1}{q}} \tag{4.57}$$

$$= \left(\sum_{i=r}^{\infty} u_i^q \left(\sum_{k=1}^{i} \varpi_{i,k}^{(n)} \frac{f_{r,k}}{\|f_r\|_{p,v}} \right)^q \right)^{\frac{1}{q}}$$

$$\geq \left(\sum_{i=r}^{\infty} u_i^q \left(\sum_{k=1}^{r} \varpi_{i,k}^{(n)} \frac{f_{r,k}}{\|f_r\|_{p,v}} \right)^q \right)^{\frac{1}{q}}$$

$$= \left(\sum_{i=r}^{\infty} u_i^q \left(\sum_{k=1}^{r} \varpi_{i,k}^{(n)} \left(\varpi_{r,k}^{(\gamma)} \right)^{p'-1} v_k^{-p'} \right)^q \right)^{\frac{1}{q}} \left(\sum_{k=1}^{r} \left(\varpi_{r,k}^{(\gamma)} \right)^{p'} v_k^{-p'} \right)^{-\frac{1}{p}}$$

$$\gg \left(\sum_{i=r}^{\infty} u_i^q \left(d_{i,r}^{n,\gamma} \right)^q \right)^{\frac{1}{q}} \left(\sum_{k=1}^{r} \left(\varpi_{r,k}^{(\gamma)} \right)^{p'} v_k^{-p'} \right)^{\frac{1}{p'}} \equiv \left(\mathcal{D}_{\gamma,n} \right)_r .$$

Since inequality (4.57) holds for all $\gamma = 0, 1, \ldots, n$, the limiting relation (4.56) implies that

$$\lim_{r \to \infty} \left(\mathcal{D}_n \right)_r \equiv \lim_{r \to \infty} \max_{0 \leq \gamma \leq n} \left(\mathcal{D}_{\gamma,n} \right)_r = 0.$$

Therefore, (4.20) implies that

$$\lim_{r \to \infty} \left(D_1 \right)_r = 0 \ \ \text{and} \ \ \lim_{r \to \infty} \left(D_2 \right)_r = 0. \tag{4.58}$$

2. Suppose that the matrix $(a_{i,j})$ belongs to the class $\mathcal{O}_n^-$, $n \geq 0$ and that the matrix $(\sigma_{i,j})$ belongs to the class $\mathcal{O}_m^+ \cup \mathcal{O}_m^-$, $m \geq 0$. Then by Lemma 2.8, we have that $(\varpi_{i,k}) \in \mathcal{O}_n^- \beta$, $n \geq 0$. Therefore, for the matrix $(\varpi_{i,k}) \in \mathcal{O}_n^+ \beta$, $n \geq 0$ there exist a non-negative sequence $\{\beta_k\}_{k=1}^{\infty}$ and matrices $(\widetilde{\varpi}_{i,k}) \in \mathcal{O}_n^-$ such that (4.21) holds for all $i \geq j \geq k \geq 1$, where $\widetilde{d}_{j,k}^{n,n} \equiv 1$.

Let $\widetilde{f}_r = \{\widetilde{f}_{r,k}\}_{k=1}^{\infty}$, $r \geq 1$ be the sequence, which is defined by (4.22).

Let $M_2 = \left\{ \widetilde{g}_r = \{\widetilde{g}_{r,k}\}_{k=1}^{\infty} : \ \widetilde{g}_{r,k} = \dfrac{\widetilde{f}_{r,k}}{\|\widetilde{f}_r\|_{p,v}}, \ r \geq 1 \right\}$. Clearly, $M_2 \subset M$.

Hence, we have

$$\lim_{r \to \infty} \sup_{\widetilde{g}_r \in M_2} \left(\sum_{i=r}^{\infty} u_i^q \left(W^+ \widetilde{g}_r \right)_i^q \right)^{\frac{1}{q}} = 0. \tag{4.59}$$

Moreover, taking into account that $(\varpi_{i,k}) \in \mathcal{O}_n^- \beta$, $n \geq 0$, we obtain

$$\sup_{\widetilde{g}_r \in M_2} \left(\sum_{i=r}^{\infty} u_i^q \left(W^+ \widetilde{g}_r \right)_i^q \right)^{\frac{1}{q}} \geq \left(\sum_{i=r}^{\infty} u_i^q \left(W^+ \widetilde{g}_r \right)_i^q \right)^{\frac{1}{q}} \tag{4.60}$$

$$\geq \left(\sum_{i=r}^{\infty} u_i^q \left(\sum_{k=1}^{r} \varpi_{i,k}^{(n)} \left(\widetilde{d}_{r,k}^{\gamma,n} \beta_k \right)^{p'-1} v_k^{-p'} \right)^q \right)^{\frac{1}{q}} \left(\sum_{k=1}^{r} \left(\widetilde{d}_{r,k}^{\gamma,n} \beta_k \right)^{p'} v_k^{-p'} \right)^{-\frac{1}{p}}$$

$$\gg \left(\sum_{i=r}^{\infty} u_i^q \left(\widetilde{\varpi}_{i,r}^{(\gamma)} \right)^q \right)^{\frac{1}{q}} \left(\sum_{k=1}^{r} \left(\widetilde{d}_{r,k}^{\gamma,n} \beta_k \right)^{p'} v_k^{-p'} \right)^{\frac{1}{p'}} \equiv \left(\widetilde{\mathcal{D}}_{\gamma,n} \right)_r.$$

Since inequality (4.60) holds for all $\gamma = 0, 1, \ldots, n$ and by (4.59), we obtain

$$\lim_{r \to \infty} \left(\widetilde{\mathcal{D}}_n \right)_r \equiv \lim_{r \to \infty} \max_{0 \leq \gamma \leq n} \left(\widetilde{\mathcal{D}}_{\gamma,n} \right)_r = 0.$$

Moreover, by using (4.24), we obtain (4.58).

Next we consider the conjugate operator to the operator $A^+ \circ \Sigma^-$, which is defined by (4.26). Since the operator $A^+ \circ \Sigma^-$ is compact from $l_{p,v}$ to $l_{q,u}$, the conjugate operator $(A^+ \circ \Sigma^-)^*$ is compact from $l_{q',u^{-1}}$ to $l_{p',v^{-1}}$.

We introduce the following set:

$$\mathcal{M} = \left\{ g = \{g_k\}_{k=1}^{\infty} : \ g_k = \frac{f_k}{\|f\|_{q',u^{-1}}}, \quad 0 \leq f = \{f_k\}_{k=1}^{\infty} \in l_{q',u^{-1}} \right\}.$$

Clearly, $\|g\|_{q',u^{-1}} = 1$ for all $g \in \mathcal{M}$. Since the operator $(A^+ \circ \Sigma^-)^*$ is compact from $l_{q',u^{-1}}$ to $l_{p',v^{-1}}$, the set $\{v^{-1}(A^+ \circ \Sigma^-)^* g, \ g \in \mathcal{M}\}$ is precompact in $l_{p'}$. Therefore, by Theorem C we obtain that

$$\lim_{r \to \infty} \sup_{g \in \mathcal{M}} \left(\sum_{k=r}^{\infty} v_k^{-p'} \left[\left(A^+ \circ \Sigma^- \right)^* g \right]_k^{p'} \right)^{\frac{1}{p'}} = 0. \tag{4.61}$$

Since $\{g_i\}_{i=1}^{\infty}$ is non-negative, by using (4.27) we obtain that

$$\left(\sum_{k=r}^{\infty} v_k^{-p'} \left[\left(A^+ \circ \Sigma^- \right)^* g \right]_k^{p'} \right)^{\frac{1}{p'}}$$

$$\gg \max \left\{ \left(\sum_{k=r}^{\infty} v_k^{-p'} \left(\Phi^+ g \right)_k^{p'} \right)^{\frac{1}{p'}}, \ \left(\sum_{i=r}^{\infty} v_k^{-p'} \left(W^- g \right)_k^{p'} \right)^{\frac{1}{p'}} \right\}$$

for all $g \in \mathcal{M}$, where Φ^+ and W^- are operators defined by (4.28) and (4.29), respectively. Therefore, (4.61) implies that

$$\lim_{r \to \infty} \sup_{g \in \mathcal{M}} \left(\sum_{k=r}^{\infty} v_k^{-p'} \left(\Phi^+ g \right)_k^{p'} \right)^{\frac{1}{p'}} = 0 \qquad (4.62)$$

and

$$\lim_{r \to \infty} \sup_{g \in \mathcal{M}} \left(\sum_{i=r}^{\infty} v_k^{-p'} \left(W^- g \right)_k^{p'} \right)^{\frac{1}{p'}} = 0. \qquad (4.63)$$

3. Suppose that the matrix $(a_{i,j})$ belongs to the class $\mathcal{O}_n^+ \cup \mathcal{O}_n^-$, $n \geq 0$ and that the matrix $(\sigma_{i,j})$ belongs to the class $\mathcal{O}_m^+$. Then by Lemma 2.8, we have that $(\varphi_{k,i}) \in \mathcal{O}_m^+$, $m \geq 0$, where $\varphi_{k,i} = \sum_{j=1}^{i} \sigma_{k,j} a_{i,j}$, $k \geq i$. Since $(\varphi_{k,i}) \in \mathcal{O}_m^+$, $m \geq 0$, there exist matrices $(\varphi_{k,i}) \in \mathcal{O}_\gamma^+$, $\gamma = 0, ..., m-1$ and matrices $e_{k,j}^{m,\gamma}$ such that (4.31) holds for all $k \geq j \geq i \geq 1$, where $e_{k,j}^{m,m} \equiv 1$.

Let $\widehat{f}_r = \{\widehat{f}_{r,i}\}_{i=1}^{\infty}$, $r \geq 1$ be the sequence defined by (4.32).

Let $\mathcal{M}_1 = \left\{ \widehat{g}_r = \{\widehat{g}_{r,i}\}_{i=1}^{\infty} : \widehat{g}_{r,i} = \dfrac{\widehat{f}_{r,i}}{\|\widehat{f}_r\|_{q',u^{-1}}}, \; r \geq 1 \right\}$. Since $\mathcal{M}_1 \subset \mathcal{M}$, we have

$$\lim_{r \to \infty} \sup_{\widehat{g}_r \in \mathcal{M}_1} \left(\sum_{k=r}^{\infty} v_k^{-p'} \left(\Phi^+ \widehat{g}_r \right)_k^{p'} \right)^{\frac{1}{p'}} = 0. \qquad (4.64)$$

Moreover,

$$\sup_{g_r \in \mathcal{M}_1} \left(\sum_{k=r}^{\infty} v_k^{-p'} \left(\Phi^+ \widehat{g}_r \right)_k^{p'} \right)^{\frac{1}{p'}} \geq \left(\sum_{k=r}^{\infty} v_k^{-p'} \left(\Phi^+ \widehat{g}_r \right)_k^{p'} \right)^{\frac{1}{p'}} \qquad (4.65)$$

$$\geq \left(\sum_{k=r}^{\infty} v_k^{-p'} \left(\sum_{i=1}^{r} \varphi_{k,i}^{(m)} \frac{\widehat{f}_{r,i}}{\|\widehat{f}_r\|_{q',u^{-1}}} \right)^{p'} \right)^{\frac{1}{p'}}$$

$$= \left(\sum_{k=r}^{\infty} v_k^{-p'} \left(\sum_{i=1}^{r} \varphi_{k,i}^{(m)} \left(\varphi_{r,i}^{(\gamma)} \right)^{q-1} u_i^q \right)^{p'} \right)^{\frac{1}{p'}} \left(\sum_{i=1}^{r} \left(\varphi_{r,i}^{(\gamma)} \right)^q u_i^q \right)^{-\frac{1}{q'}}$$

$$\gg \left(\sum_{k=r}^{\infty} v_k^{-p'} \left(e_{k,r}^{m,\gamma} \right)^{p'} \right)^{\frac{1}{p'}} \left(\sum_{i=1}^{r} \left(\varphi_{r,i}^{(\gamma)} \right)^q u_i^q \right)^{\frac{1}{q}} \equiv \left(\mathbb{D}_{\gamma,m} \right)_r.$$

Since inequality (4.65) holds for all $\gamma = 0, 1, \dots, m$, the limiting relation (4.62) implies that

$$\lim_{r \to \infty} \left(\mathbb{D}_m \right)_r \equiv \lim_{r \to \infty} \max_{0 \leq \gamma \leq m} \left(\mathbb{D}_{\gamma,m} \right)_r = 0.$$

Therefore, by (4.36) we deduce that

$$\lim_{r\to\infty}(D_3)_r = 0 \quad \text{and} \quad \lim_{r\to\infty}(D_4)_r = 0. \tag{4.66}$$

4. Suppose that the matrix $(a_{i,j})$ belongs to the class $\mathcal{O}_n^+ \cup \mathcal{O}_n^-$, $n \geq 0$ and that the matrix $(\sigma_{i,j})$ belongs to the class $\mathcal{O}_m^-$. Then by Lemma 2.8, we have that $(\varphi_{k,i}) \in \mathcal{O}_m^- \alpha$, $m \geq 0$. Therefore, for the matrix $(\varphi_{k,i}) \in \mathcal{O}_m^- \alpha$, $m \geq 0$ there exist a non-negative sequence $\{\alpha_i\}_{i=1}^\infty$ and matrices $(\widetilde{\varphi}_{k,i}) \in \mathcal{O}_m^-$ such that (4.37) holds for all $k \geq j \geq i \geq 1$, where $\widetilde{e}_{j,i}^{m,m} \equiv 1$.

Let $\overline{f}_r = \{\overline{f}_{r,i}\}_{i=1}^\infty$, $r \geq 1$ be the sequence defined by (4.38).

Let $\mathcal{M}_2 = \left\{ \overline{g}_r = \{\overline{g}_{r,i}\}_{i=1}^\infty : \overline{g}_{r,i} = \dfrac{\overline{f}_{r,i}}{\|\overline{f}_r\|_{q',u^{-1}}}, \ r \geq 1 \right\}$. Since $\mathcal{M}_2 \subset \mathcal{M}$, we have

$$\lim_{r\to\infty}\sup_{\overline{g}_r\in\mathcal{M}_2}\left(\sum_{k=r}^\infty v_k^{-p'}\left(\Phi^+\overline{g}_r\right)_k^{p'}\right)^{\frac{1}{p'}} = 0. \tag{4.67}$$

Moreover,

$$\sup_{\overline{g}_r\in\mathcal{M}_2}\left(\sum_{k=r}^\infty v_k^{-p'}\left(\Phi^+\overline{g}_r\right)_k^{p'}\right)^{\frac{1}{p'}} \geq \left(\sum_{k=r}^\infty v_k^{-p'}\left(\Phi^+\overline{g}_r\right)_k^{p'}\right)^{\frac{1}{p'}} \tag{4.68}$$

$$\geq \left(\sum_{k=r}^\infty v_k^{-p'}\left(\sum_{i=1}^r \varphi_{k,i}^{(m)}\frac{\overline{f}_{r,i}}{\|\overline{f}_r\|_{q',u^{-1}}}\right)^{p'}\right)^{\frac{1}{p'}}$$

$$= \left(\sum_{k=r}^\infty v_k^{-p'}\left(\sum_{i=1}^r \varphi_{k,i}^{(m)}\left(\widetilde{e}_{r,i}^{\gamma,m}\alpha_i\right)^{q-1}u_i^q\right)^{p'}\right)^{\frac{1}{p'}}\left(\sum_{i=1}^r\left(\widetilde{e}_{r,i}^{\gamma,m}\alpha_i\right)^q u_i^q\right)^{-\frac{1}{q'}}$$

$$\gg \left(\sum_{k=r}^\infty\left(\varphi_{k,r}^{(\gamma)}\right)^{p'}v_k^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{i=1}^r\left(\widetilde{e}_{r,i}^{\gamma,m}\alpha_i\right)^q u_i^q\right)^{\frac{1}{q}} \equiv \left(\widetilde{\mathbb{D}}_{\gamma,m}\right)_r.$$

Since inequality (4.68) holds for all $\gamma = 0, 1, \ldots, m$ and from the validity of (4.67) we obtain

$$\lim_{r\to\infty}\left(\widetilde{\mathbb{D}}_m\right)_r \equiv \lim_{r\to\infty}\max_{0\leq\gamma\leq m}\left(\widetilde{\mathbb{D}}_{\gamma,m}\right)_r = 0.$$

Hence, by using (4.40) we obtain (4.66).

Thus, the proof of the necessity is complete.

Sufficiency. Let $(a_{i,j}) \in \mathcal{O}_n^+ \cup \mathcal{O}_n^-$, $n \geq 0$ and $(\sigma_{i,j}) \in \mathcal{O}_m^+ \cup \mathcal{O}_m^-$, $m \geq 0$. Assume that the first condition of statement (ii) of Theorem 4.2 holds. Then, according to statement (i) of Theorem 4.2, the operator $A^+ \circ \Sigma^-$ is bounded from $l_{p,v}$ into $l_{q,u}$. Therefore, the set $\{u(A^+ \circ \Sigma^-)f, \ \|f\|_{p,v} \leq 1\}$ is bounded in l_q. Now we show that this set is precompact in l_q. By Theorem C the bounded set $\{u(A^+ \circ \Sigma^-)f, \ \|f\|_{p,v} \leq 1\}$ is compact in l_q provided that

$$\lim_{r\to\infty} \sup_{\|f\|_{p,v}\leq 1} \left(\sum_{i=r}^{\infty} u_i^q | \left[(A^+ \circ \Sigma^-)f \right]_i |^q \right)^{\frac{1}{q}} = 0. \tag{4.69}$$

For $r > 1$ we assume that $\widetilde{u} = \{\widetilde{u}_i\}_{i=1}^{\infty}$: $\widetilde{u}_i = \begin{cases} 0, & 1 \leq i \leq r-1 \\ u_i, & r \leq i. \end{cases}$

Then, by statement (i) of Theorem 4.2, we have that

$$\sup_{\|f\|_{p,v}\leq 1} \left(\sum_{i=r}^{\infty} u_i^q | \left[(A^+ \circ \Sigma^-)f \right]_i |^q \right)^{\frac{1}{q}}$$

$$= \sup_{\|f\|_{p,v}\leq 1} \left(\sum_{i=1}^{\infty} \widetilde{u}_i^q | \left[(A^+ \circ \Sigma^-)f \right]_i |^q \right)^{\frac{1}{q}} \ll \widetilde{D}_{13}(r), \tag{4.70}$$

where

$$\widetilde{D}_{13}(r) = \max\{\widetilde{D}_1(r), \ \widetilde{D}_3(r)\}$$

and

$$\begin{aligned} \widetilde{D}_1(r) &= \sup_{s\geq 1} \left(\sum_{k=1}^{s} v_k^{-p'} \left(\sum_{i=s}^{\infty} \left(\sum_{j=1}^{k} a_{i,j}\sigma_{k,j} \right)^q \widetilde{u}_i^q \right)^{\frac{p'}{q}} \right)^{\frac{1}{p'}} \\ &= \sup_{s\geq r} \left(\sum_{k=1}^{s} v_k^{-p'} \left(\sum_{i=s}^{\infty} \left(\sum_{j=1}^{k} a_{i,j}\sigma_{k,j} \right)^q u_i^q \right)^{\frac{p'}{q}} \right)^{\frac{1}{p'}} \\ &= \sup_{s\geq r}(D_1)_s, \end{aligned} \tag{4.71}$$

$$\widetilde{D}_3(r) = \sup_{s \geq 1} \left(\sum_{i=1}^{s} \widetilde{u}_i^q \left(\sum_{k=s}^{\infty} \left(\sum_{j=1}^{i} a_{i,j} \sigma_{k,j} \right)^{p'} v_k^{-p'} \right)^{\frac{q}{p'}} \right)^{\frac{1}{q}} \qquad (4.72)$$

$$= \sup_{s \geq r} \left(\sum_{i=r}^{s} u_i^q \left(\sum_{k=s}^{\infty} \left(\sum_{j=1}^{i} a_{i,j} \sigma_{k,j} \right)^{p'} v_k^{-p'} \right)^{\frac{q}{p'}} \right)^{\frac{1}{q}}$$

$$\leq \sup_{s \geq r} (D_3)_s,$$

By the first condition of statement (ii) of Theorem 4.2 and by conditions (4.71), (4.72), we obtain

$$\lim_{r \to \infty} \widetilde{D}_1(r) = \lim_{r \to \infty} \sup_{s \geq r} (D_1)_s = \lim_{r \to \infty} (D_1)_r = 0,$$

$$\lim_{r \to \infty} \widetilde{D}_3(r) \leq \lim_{r \to \infty} \sup_{s \geq r} (D_3)_s = \lim_{r \to \infty} (D_3)_r = 0.$$

Hence, by inequality (4.70), we obtain (4.69).

The other cases of statement (ii) follow from the equivalences $(D_1)_s \approx (D_2)_s$ and $(D_3)_s \approx (D_4)_s$ for $s \geq 1$.

Thus the proof of Theorem 4.2 is complete.

The proof of Theorem 4.3 can be carried out by using Lemma 2.9 and by the same method of the proof of Theorem 4.2.

4.2 Three-weighted inequality of Hardy type.

Our main results can be used to derive other inequalities. We consider an additive estimate of the form

$$\|A^+ f\|_{q,u} \leq C \left(\|f\|_{p,v} + \|A_0^+ f\|_{p,\rho} \right) \tag{4.73}$$

for all non-negative sequences $f = \{f_i\}_{i=1}^{\infty}$, where the matrix operator A^+ is defined by (2.1) and the Hardy operator A_0^+ is defined by $\left(A_0^+ f \right)_i := \sum\limits_{j=1}^{i} f_j, \quad i \geq 1$.

We assume that the weighted sequences v and ρ satisfy the following conditions

$$v_k > 0, k \geq 1, \sum_{k=1}^{\infty} \rho_k < \infty.$$

We denote by $\Delta \varphi_i = \varphi_i - \varphi_{i-1}$ and for $n \geq 1$ we define

$$\varphi_n = \left\{ \min_{1 \leq k \leq n} \left[\left(\sum_{i=k}^{n} v_i^{-p'} \right)^{-\frac{1}{p'}} + \left(\sum_{i=k}^{\infty} \rho_i^p \right)^{\frac{1}{p}} \right] \right\}^{-1}, \quad \varphi_0 = 0.$$

Next we introduce the following result of R. Oinarov [120] on the equivalence of inequalities (4.73) and (2.3) which we exploit below.

Theorem F. *Let $1 < p, q < \infty$ and the entries of the matrix $(a_{k,i})$ of the operator A^+ are non-negative and non-increasing in i, which means that $a_{k,i+1} \leq a_{k,i}$ for $k \geq 1$, $i \geq 1$. Then inequality (4.73) holds for all non-negative sequences $f = \{f_i\}_{i=1}^{\infty}$ if and only if the inequality*

$$\left(\sum_{k=1}^{\infty} u_k^q \left(\sum_{i=1}^{k} a_{k,i} f_i \right)^q \right)^{\frac{1}{q}} \leq \widetilde{C} \left(\sum_{k=1}^{\infty} f_k^p \left(\varphi_k^{p'} - \varphi_{k-1}^{p'} \right)^{1-p} \right)^{\frac{1}{p}} \tag{4.74}$$

holds for all non-negative sequences $f = \{f_i\}_{i=1}^{\infty}$. Moreover, $C \approx \widetilde{C}$, where C and $\widetilde{C}$ are the best constants in (4.73) and (4.74), respectively.

By exploiting Theorem F, we obtain the following statement.

Theorem 4.6. *Suppose that $1 < p \leq q < \infty$. Let the matrix $(a_{i,j})$ in (4.73) belong to the class $\mathcal{O}_m^-$, $m \geq 0$. Then inequality (4.73) holds for all non-negative sequences $f = \{f_i\}_{i=1}^\infty$ if and only if at least one of the conditions $\mathcal{D}^+ < \infty$ and $\mathcal{D}^- < \infty$ holds, where*

$$\mathcal{D}^+ = \sup_{k \geq 1} \left(\sum_{j=1}^{k} \Delta \varphi_j^{p'} \left(\sum_{i=k}^{\infty} a_{i,k}^q u_i^q \right)^{\frac{p'}{q}} \right)^{\frac{1}{p'}}$$

and

$$\mathcal{D}^- = \sup_{k \geq 1} \left(\sum_{i=k}^{\infty} u_i^q \left(\sum_{j=1}^{k} a_{i,j}^{p'} \Delta \varphi_j^{p'} \right)^{\frac{q}{p'}} \right)^{\frac{1}{q}}.$$

Moreover, $\mathcal{D}^+ \approx \mathcal{D}^- \approx C$, where C is the best constant in (4.73).

Theorem 4.7. *Let $1 < q < p < \infty$. Let the entries of the matrix $(a_{i,j})$ satisfy Assumption A. Then inequality (4.73) holds for all non-negative sequences $f = \{f_i\}_{i=1}^\infty$ if and only if $E = \max\{E^+,\ E^-\} < \infty$, where*

$$E^+ = \left(\sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} b_{i,j}^{p'} \Delta \varphi_j^{p'} \right)^{\frac{q(p-1)}{p-q}} \left(\sum_{k=i}^{\infty} \omega_k^q u_k^q \right)^{\frac{q}{p-q}} \omega_i^q u_i^q \right)^{\frac{p-q}{pq}},$$

$$E^- = \left(\sum_{i=1}^{\infty} \Delta \varphi_i^{\frac{pq}{p-q}} \left(\sum_{k=i}^{\infty} a_{k,i}^q u_k^q \right)^{\frac{p}{p-q}} \right)^{\frac{p-q}{pq}}.$$

Moreover, $E \approx C$, where C is the best constant in (4.73).

Proof of Theorem 4.6. Let the matrix $(a_{i,j})$ in (4.73) belong to the class $\mathcal{O}_m^-$, $m \geq 0$. Therefore $(a_{i,j})$ is a matrix which is non-negative and non-increasing in the second index for all $i \geq j \geq 1$. Then according to Theorem F inequality (4.73) holds if and only if the inequality

$$\left(\sum_{k=1}^{\infty} u_k^q \left(\sum_{i=1}^{k} a_{k,i}^{(m)} f_i \right)^q \right)^{\frac{1}{q}} \leq C_1 \left(\sum_{k=1}^{\infty} f_k^p \left(\Delta \varphi_k^{p'} \right)^{1-p} \right)^{\frac{1}{p}} \qquad \forall f \geq 0 \quad (4.75)$$

holds. Moreover, $C_1 \approx C$, where C and C_1 are the best constants in (4.73) and (4.75), respectively.

Inequality (4.75) is equivalent to inequality (4.73). Then by Theorem 2.15 inequality (4.75), and accordingly (4.73) holds if and only if at least one of the conditions $\mathcal{D}^+ < \infty$ and $\mathcal{D}^- < \infty$ holds. Thus the proof is complete.

Proof of Theorem 4.7. We set $\sup\limits_{j \leq k \leq i} a_{i,k} = \widetilde{a}_{i,j}$. Obviously,

$$a_{i,j} \leq \widetilde{a}_{i,j}. \tag{4.76}$$

According to Assumption A we have

$$da_{i,j} \geq \sup\limits_{j \leq k \leq i} a_{i,k} = \widetilde{a}_{i,j}. \tag{4.77}$$

From (4.76) and (4.77) it follows that $a_{i,j} \approx \widetilde{a}_{i,j}$. The matrix operator $\left(\widetilde{A}^+ f\right)_i = \sum\limits_{j=1}^{i} \widetilde{a}_{i,j} f_j$, $i \geq 1$ is equivalent to the operator A^+, namely, $(A^+ f)_i \leq \left(\widetilde{A}^+ f\right)_i \leq d\,(A^+ f)_i$ or $(A^+ f)_i \approx \left(\widetilde{A}^+ f\right)_i$ for all $f \geq 0$, $i \geq 1$. Then inequality (4.73) is equivalent to

$$\|\widetilde{A}^+ f\|_{q,u} \leq C_1 \left(\|f\|_{p,v} + \|A_0^+ f\|_{p,\rho} \right) \qquad \forall f \geq 0. \tag{4.78}$$

Moreover, $C \approx C_1$, where C and C_1 are the best constants in (4.73) and (4.78), respectively. It is easy to see that the entries of the matrix $(\widetilde{a}_{i,j})$ satisfy the following condition $\widetilde{a}_{i,j} \geq \widetilde{a}_{i,k}$, $i \geq k \geq j \geq 1$. Then according to Theorem F inequality (4.78) holds if and only if the inequality

$$\left(\sum\limits_{k=1}^{\infty} u_k^q \left(\sum\limits_{i=1}^{k} \widetilde{a}_{k,i} f_i \right)^q \right)^{\frac{1}{q}} \leq C_2 \left(\sum\limits_{k=1}^{\infty} f_k^p \left(\Delta \varphi_k^{p'} \right)^{1-p} \right)^{\frac{1}{p}} \qquad \forall f \geq 0, \tag{4.79}$$

holds. Moreover, $C_1 \approx C_2$, where C_2 is the best constant in (4.79).

Since (4.78) is equivalent to inequality (4.73), inequality (4.79) is equivalent to inequality (4.73). By Theorem 2.18 inequality (4.79) (and, thus, (4.78) and (4.73)) holds if and only if $E = \max\{E^+,\ E^-\} < \infty$.

Hence, the proof is complete.

4.3 Applications of the main results for summable matrices.

In the theory of series it is very important to obtain estimates for the norms of summable matrices. Note that a lower triangular matrix $\widetilde{A} = (\widetilde{a}_{i,j})$ is called a summable matrix if $\widetilde{a}_{i,j} \geq 0$ and $\sum\limits_{j=1}^{i} \widetilde{a}_{i,j} = 1$. If $(a_{i,j}) \in \mathcal{O}_n^+$ or $(a_{i,j}) \in \mathcal{O}_n^-$, $n \geq 0$ then the matrix $(\widetilde{a}_{i,j}) = \left(\frac{a_{i,j}}{A_{ii}}\right)$ satisfies all conditions of summable matrix, where $A_{ii} = \sum\limits_{j=1}^{i} a_{i,j}$. If we consider inequality (2.3) for matrix $(\overline{a}_{i,j}) = (\beta_i a_{i,j})$, then we obtain inequality of the type (2.3)

$$\left(\sum_{i=1}^{\infty} u_i^q \beta_i^q \left|\sum_{j=1}^{i} a_{i,j} f_j\right|^q\right)^{\frac{1}{q}} \leq C \left(\sum_{i=1}^{\infty} v_i^p |f_i|^p\right)^{\frac{1}{p}}.$$

Therefore using the results of Chapter 2 and 3 we obtain two-sided estimates for the matrix $(\overline{a}_{i,j}) = (\beta_i a_{i,j})$ in $l_{p,v}$ and on the cone of monotone sequences. Consequently, we can estimate summable matrices, and in particular Hölder and Cesàro matrices.

Now we define

$$J_0^+ = \sup_{s \geq 1} \left(\sum_{j=1}^{s} v_j^{-p'} \left(\sum_{i=s}^{\infty} \left(\frac{i!(i-j+k-1)!}{(i-j)!(k+i)!}\right)^q u_i^q\right)^{\frac{p'}{q}}\right)^{\frac{1}{p'}},$$

$$J_0^- = \sup_{s \geq 1} \left(\sum_{i=s}^{\infty} u_i^q \left(\frac{i!}{(k+i)!}\right)^q \left(\sum_{j=1}^{s} \left(\frac{(i-j+k-1)!}{(i-j)!}\right)^{p'} v_j^{-p'}\right)^{\frac{q}{p'}}\right)^{\frac{1}{q}},$$

$$J_1^+ = \sup_{s \geq 1} \left(\sum_{j=1}^{s} u_j^q \left(\sum_{i=s}^{\infty} \left(\frac{i!(i-j+k-1)!}{(i-j)!(k+i)!}\right)^{p'} v_i^{-p'}\right)^{\frac{q}{p'}}\right)^{\frac{1}{q}},$$

$$J_1^- = \sup_{s \geq 1} \left(\sum_{i=s}^{\infty} v_i^{-p'} \left(\frac{i!}{(k+i)!}\right)^{p'} \left(\sum_{j=1}^{s} \left(\frac{(i-j+k-1)!}{(i-j)!}\right)^q u_j^q\right)^{\frac{p'}{q}}\right)^{\frac{1}{p'}}.$$

Now by using the results of Chapter 2 we obtain the following statements for the Cesàro matrix.

Theorem 4.8. *Let $1 < p \leq q < \infty$. Let the matrix $(a_{i,j})$ in (2.1) be the Cesàro matrix of order k, $k \geq 1$. Then the inequality (2.3) for the operator defined by (2.1) holds if and only if at least one of the conditions $J_0^+ < \infty$ and $J_0^- < \infty$ holds. Moreover, $J_0^+ \approx J_0^- \approx C$, where C is the best constant in (2.3).*

Theorem 4.9. *Let $1 < p \leq q < \infty$. Let the matrix $(a_{i,j})$ in (2.2) be the Cesàro matrix of order k, $k \geq 1$. Then the inequality (2.3) for the operator defined by (2.2) holds if and only if at least one of the conditions $J_1^+ < \infty$ and $J_1^- < \infty$ holds. Moreover, $J_1^+ \approx J_1^- \approx C$, where C is the best constant in (2.3).*

Next we set

$$
V_k = \sum_{i=1}^k v_i^p, \quad \mathbb{E}_1 = \sup_{s \geq 1} V_s^{-\frac{1}{p}} \left(\sum_{i=1}^s \left(\frac{i! \cdot u_i}{(k+i)!} \right)^q \left(\sum_{j=1}^i \frac{(i-j+k-1)!}{(i-j)!} \right)^q \right)^{\frac{1}{q}},
$$

$$
\mathbb{E}_2 = \sup_{s \geq 1} \left(\sum_{l=1}^s \left(V_l^{-\frac{p'}{p}} - V_{l+1}^{-\frac{p'}{p}} \right) \left(\sum_{i=s}^\infty \left(\frac{i! \cdot u_i}{(k+i)!} \right)^q \left(\sum_{j=1}^l \frac{(i-j+k-1)!}{(i-j)!} \right)^q \right)^{\frac{p'}{q}} \right)^{\frac{1}{p'}},
$$

$$
\mathbb{E}_3 = \sup_{s \geq 1} \left(\sum_{i=s}^\infty \left(\frac{i! \cdot u_i}{(k+i)!} \right)^q \left(\sum_{l=1}^s \left(\sum_{j=1}^l \frac{(i-j+k-1)!}{(i-j)!} \right)^{p'} \left(V_l^{-\frac{p'}{p}} - V_{l+1}^{-\frac{p'}{p}} \right) \right)^{\frac{q}{p'}} \right)^{\frac{1}{q}}.
$$

By using the results of Chapter 3 we obtain the following statements for the Cesàro matrix.

Theorem 4.10. *Let $1 < p \leq q < \infty$. Then the inequality (3.1) on the cone of non-negative and non-increasing sequences $f \in l_{p,v}$ for the Cesàro matrix of order k, $k \geq 1$ holds if and only if at least one of the conditions $\mathbb{E}_{12} = \max\{\mathbb{E}_1, \mathbb{E}_2\} < \infty$ and $\mathbb{E}_{13} = \max\{\mathbb{E}_1, \mathbb{E}_3\} < \infty$ holds. Moreover, $\mathbb{E}_{12} \approx \mathbb{E}_{13} \approx C$, where C is the best constant in (3.1).*

Now we define

$$
\nu_{i,j}^{(n)} \equiv \sum_{k_n=j}^i \sum_{k_{n-1}=k_n}^i \frac{1}{k_{n-1}} \sum_{k_{n-2}=k_{n-1}}^i \frac{1}{k_{n-2}} \cdots \sum_{k_1=k_2}^i \frac{1}{k_1}, \quad n \geq 1,
$$

$$J_2^+ = \sup_{s \geq 1} \left(\sum_{j=1}^{s} v_j^{-p'} \left(\sum_{i=s}^{\infty} \left(\nu_{i,j}^{(n)} \right)^q \left(\frac{u_i}{i} \right)^q \right)^{\frac{p'}{q}} \right)^{\frac{1}{p'}},$$

$$J_2^- = \sup_{s \geq 1} \left(\sum_{i=s}^{\infty} \left(\frac{u_i}{i} \right)^q \left(\sum_{j=1}^{s} \left(\nu_{i,j}^{(n)} \right)^{p'} v_j^{-p'} \right)^{\frac{q}{p'}} \right)^{\frac{1}{q}},$$

By exploiting the main results of Chapter 2 we have the following statements for the Hölder's matrix.

Theorem 4.11. *Let $1 < p \leq q < \infty$. Then the inequality (2.3) for the Hölder's operator of order n, $n \geq 1$ holds if and only if at least one of the conditions $J_2^+ < \infty$ and $J_2^- < \infty$ holds. Moreover, $J_2^+ \approx J_2^- \approx C$, where C is the best constant in (2.3).*

We define

$$V_k = \sum_{i=1}^{k} v_i^p, \quad \mathcal{V}_{ik}^{(n)} = \sum_{j=1}^{k} \nu_{i,j}^{(n)}, \quad \mathcal{E}_1 = \sup_{s \geq 1} V_s^{-\frac{1}{p}} \left(\sum_{i=1}^{s} \left(\mathcal{V}_{ii}^{(n)} \right)^q \left(\frac{u_i}{i} \right)^q \right)^{\frac{1}{q}},$$

$$\mathcal{E}_2 = \sup_{s \geq 1} \left(\sum_{k=1}^{s} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \left(\sum_{i=s}^{\infty} \left(\mathcal{V}_{ik}^{(n)} \right)^q \left(\frac{u_i}{i} \right)^q \right)^{\frac{p'}{q}} \right)^{\frac{1}{p'}},$$

$$\mathcal{E}_3 = \sup_{s \geq 1} \left(\sum_{k=s}^{\infty} \left(\frac{u_k}{k} \right)^q \left(\sum_{i=1}^{s} \left(\mathcal{V}_{ki}^{(n)} \right)^{p'} \left(V_i^{-\frac{p'}{p}} - V_{i+1}^{-\frac{p'}{p}} \right) \right)^{\frac{q}{p'}} \right)^{\frac{1}{q}}.$$

By using the main results of Chapter 3 we have the following two-sided estimates for the Hölder's matrix on the cone of non-negative and non-increasing sequences.

Theorem 4.12. *Let $1 < p \leq q < \infty$. Then the inequality (3.1) for the Hölder's matrix of order n, $n \geq 1$ on the cone of non-negative and non-increasing sequences $f \in l_{p,v}$ holds if and only if at least one of the conditions $\mathcal{E}_{12} = \max\{\mathcal{E}_1, \mathcal{E}_2\} < \infty$ and $\mathcal{E}_{13} = \max\{\mathcal{E}_1, \mathcal{E}_3\} < \infty$ holds. Moreover, $\mathcal{E}_{12} \approx \mathcal{E}_{13} \approx C$, where C is the best constant in (3.1).*

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