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BERNSTEIN MARKOV PROPERTIES AND APPLICATIONS

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Abstract

The *Bernstein Markov Property* for a compact set E and a positive finite measure μ supported on E is a strong comparability assumption between L^2_μ and uniform norms on E of polynomials (or other nested families of functions) as their degree tends to infinity.

Admissible meshes are sequences of sampling sets $A_k \subset E$ whose cardinality is growing sub-exponentially with respect to k and for which there exists a positive finite constant C such that $\max_E |p| \leq C \max_{A_k} |p|$ for any polynomial of degree at most k.

These two mathematical objects have several applications and motivations from Approximation Theory and Pluripotential Theory, the study of plurisubharmonic functions in several complex variables.

The properties of Bernstein Markov measures and admissible meshes for a given compact set E are very similar, indeed they may be seen as the continuous and the discrete approach to the same problem.

This work is concerned on providing sufficient conditions for some different instances of the Bernstein Markov property and explicitly constructing admissible meshes.

As first problem, we study sufficient conditions for a version of the Bernstein Markov property for rational functions on the complex plane and its relation with the polynomial Bernstein Markov property.

In Chapter 5, we consider the case of a compact subset E of an algebraic pure m-dimensional subset A of $\mathbb{C}^n$ and we prove a sufficient condition for the Bernstein Markov property for the traces of polynomials on E.

To this aim, we provide two new results in Pluripotential Theory regarding the convergence and the comparability of the relative capacities, the relative and global extremal functions and the Chebyshev constants on a (possibly non-smooth) pure m-dimensional algebraic variety in $\mathbb{C}^n$, which are of independent interest.

In the last part of the dissertation, we provide some construction procedures for admissible meshes on some classes of real compact sets.

Finally, we present some algorithms, based on admissible meshes, for the numerical approximation of the most relevant objects in Pluripotential Theory, namely the transfinite diameter, the Siciak Zaharjuta extremal function and the pluripotential equilibrium measure.

Sunto

La proprietà di *Bernstein Markov* per un compatto E ed una misura positiva finita μ avente supporto in E è un' assunzione di comparabilità asintotica tra le norme uniformi ed L^2_μ dei polinomi di grado al più k (o altre famiglie innestate di funzioni) al tendere all' infinito di k.

Le *Admissible Meshes* sono sequenze di sottoinsiemi finiti A_k del compatto E la cui cardinalità cresce in modo subesponenziale rispetto a k e per i quali esiste una costante positiva C tale che $\max_E |p| \leq C \max_{A_k} |p|$ per ogni polinomi di grado al più k.

Questi due oggetti matematici hanno molte appliicazioni e motivazioni provenienti dalla Teoria dell' Approssimazione e dalla Teoria del Pluripotenziale, lo studio delle funzioni plurisubarmoniche in più variabili complesse.

Le proprietà delle misure di Bernstein Markov e delle admissible meshes per un dato compatto E sono molto simili, infatti le due definizioni possono essere viste come gli approcci rispettivamente continuo e discreto dello stesso problema.

Questo lavoro si concentra nel fornire condizioni sufficienti per la proprietà di Bernstein Markov in diverse situazioni e nella costruzione esplicita di admissible meshes.

Come primo problema vengono studiate condizioni sufficienti per una versione della proprietà di Bernstein Markov per successioni di funzioni razionali nel piano complesso in relazione alla stessa proprietà per i polinomi.

Nel Capitolo 5 viene considerato il caso di un compatto E sottoinsieme di una varietà algebrica $A \subset \mathbb{C}^n$ di dimensione pura $m < n$ ed irriducibile e quindi provata una condizione sufficiente per la proprietà di Bernstein Markov per le tracce dei polinomi su E.

A questo scopo vengono provati due risultati nuovi in Teoria del Pluripotenziale riguardanti la convergenza e la comparabilità della capacità relativa (di Monge Ampère), delle funzioni plurisubarmoniche estremali globali e relative e delle costanti di Chebyshev per sottoinsiemi E_j di un dato compatto E della varietà algebrica A, anche nel caso A sia singolare. Tali risultati sono di interesse indipendente.

Nell'ultima parte della tesi vengono provate ed illustrate alcune procedure per la costruzione di admissible meshes per alcune classi di compatti reali.

In ultimo vengono presentati alcuni nuovi algoritmi, basati sulle admissible meshes, per l' approssimazione numerica delle più rilevanti grandezze in Teoria del Pluripotenziale: il diametro transfinito, la funzione estremale di Siciak-Zaharjuta e la misura di equilibrio pluripotenziale.

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List of Symbols

$\mathscr{P}^k(\mathbb{C}^n)$	The space of polynomials of n complex variables of total degree not larger than k.
N_k	Denotes $\dim \mathscr{P}^k(\mathbb{C}^n)$, the dimension of $\mathscr{P}^k(\mathbb{C}^n)$, where n would be clarified by the context.
$\mathrm{VDM}_k(z_1, \dots, z_{N_k})$	The k-degree Vandermonde matrix with respect to the monomial basis of $\mathscr{P}^k(\mathbb{C}^n)$.
$\mathrm{VDM}_{k,Q}(z_1, \dots, z_{N_k})$	The k-degree weighted Vandermonde matrix with respect to the monomial basis of $\mathscr{P}^k(\mathbb{C}^n)$ and the weight function Q.
$\delta(E), \delta_Q(E)$	The transfinite diameter and the weighted transfinite diameter of the compact set E.
$\mathcal{M}^+(E)$	The space of non-negative Borel finite measures with support in E endowed with the weak* topology.
$\mathcal{M}_1^+(E)$	The space of Borel probability measures with support in E endowed with the weak* topology.
$G_k^Q(\mu)$	The Gram determinant of $\mathscr{P}^k(\mathbb{C}^n)$ with respect to the scalar product of L^2_μ and the graded lexicographical ordered monomial basis.
$B_k^\mu(z),\ B_k^{Q,\mu}(z)$	The Bergman function of the space $\mathscr{P}^k(\mathbb{C}^n)$ with respect to the scalar product of L^2_μ, and its weighted version.
$\mathrm{cap}(E)$	The logarithmic capacity of the compact set $E \subset \mathbb{C}$.
$I[\mu]$	The logarithmic energy of the compactly supported Borel measure μ.
$U^\mu(z)$	The logarithmic potential of the compactly supported Borel measure μ.
$V_E^*(z),\ V_{E,Q}^*(z)$	The Siciak Zaharjuta extremal plurisubharmonic function related to the compact set $E \subset \mathbb{C}^n$ and its weighted version.
$U_{E,D}^*(z)$	The relative extremal plurisubharmonic function related to the compact subset E of the open set $D \subset \mathbb{C}^n$.
$(\mathrm{dd^c}\,\cdot)^n$	The (complex) Monge Ampere operator acting on locally bounded plurisubharmonic functions in $\mathbb{C}^n$.

μ_E, $\mu_{E,Q}$	The (pluri-)potential equilibrium measure and the weighted pluripotential equilibrium measure related to the compact set E and the weight Q. One has $\mu_E = (\mathrm{dd^c}\, V_E^*)^n$, $\mu_{E,Q} = (\mathrm{dd^c}\, V_{E,Q}^*)^n$.
A	In general will denote a pure m dimensional algebraic set in $\mathbb{C}^n$.
$z \mapsto (z', z'')$	The canonical projections of $\mathbb{C}^n$ onto $\mathbb{C}^n$ and $\mathbb{C}^{n-m}$ respectively, where the choice of $\mathbb{C}^n$ coordinates is done accordingly to the so called Rudin coordinates of A.
π	Usually denotes the coordinate projection $z \mapsto z'$.
$\Omega(z_0, r)$	The pseudoball on A centered at z_0 of radius r, $\Omega := \{z \in A : \lvert z' - z_0' \rvert < r\}$; we use also $\Omega := \Omega(0, 1)$.
$(\mathrm{dd^c}\, \cdot)^m$	The (complex) Monge Ampere operator acting on locally bounded plurisubharmonic functions defined on a pure m-dimensional algebraic set in $\mathbb{C}^n$.
$V_E^*(z, A)$	The extremal plurisubharmonic function related to the compact subsett E of the pure m-dimensional algebraic subset A of $\mathbb{C}^n$.
$U_{E,\Omega}^*(z, A)$	The relative extremal plurisubharmonic function related to the compact subset E of the pseudoball $\Omega \subset A$.
$\mathrm{Cap}(E, D)$	The Monge Ampere capacity of the compact set E relative to the open set D of $\mathbb{C}^n$ or of a pure dimensional algebraic subset A of $\mathbb{C}^n$.
$T(E, A)$	The Chebyshev constant with normalization in $\Omega \subset A$.
$\{A_k\}$, A_k	An admissible mesh or a weakly admissible mesh.
M_k	The cardinality of the considered mesh.
μ_k	The uniform probability measure on an admissible mesh.

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Introduction

É dunque sognando a occhi aperti,
io credo, che vivi intensamente; ed
é ancora con l'immaginazione che
puoi trovarti a competere persino
con l'inattuabile. E qualche volta
ne esci anche vincitore.

Walter Bonatti

The framework of our study

The Bernstein Markov Property. Orthogonal polynomials and their computation is a classical topic having its roots in the 19-th Century and going back to the work of Hermite, Jacobi, Laguerre, Bessel and Chebyshev. They were motivated by possible applications in several branches of Mathematics as Approximation Theory, Operator Theory, differential equations and Mathematical Physics; see for instance [**95**], [**96**] and references therein.

The first study on the *asymptotic of orthogonal polynomials* of one complex variable is due to Szegö, [**98**]. His work gave even more impulse to this developing area since he pointed out the deep connections among orthogonal polynomials, Potential Theory and classical Complex Analysis. These mathematical relations and their extensions are still being investigated, we refer to [**97**] and [**99**] for an extensive treatment.

During the last decades, a new non linear potential theory related to plurisubharmonic functions in $\mathbb{C}^n$ and the complex Monge Ampere operator $(dd^c u)^n = c_n \det\left[\partial_i \bar\partial_j u\right]$, namely *Pluripotential Theory*, has been developed, see [**13**], [**12**] and [**59**].

As a consequence, the well established topic of asymptotic of orthogonal polynomials started attracting new attention since *all the relations among orthogonal polynomials of one complex variable and Potential Theory have their counterpart relating the asymptotic of orthogonal polynomials of several complex variables and Pluripotential Theory;* see for instance [**20**] and [**44**].

One of the most important open problem in Pluripotential Theory was the asymptotic of Fekete points, i.e., points that maximize the modulus of the Vandermonde determinant

$$\det \mathrm{VDM}_k(z_1, \dots, z_{N_k}) := \det[z_i^{\alpha_j}]_{1\leq i\leq N_k, 0\leq|\alpha_j|\leq k}, \quad N_k := \dim \mathscr{P}^k(\mathbb{C}^n),$$

on a given compact set $E \subset \mathbb{C}^n$; here $\mathscr{P}^k$ is the space of polynomials of n complex variables of total degree not larger than k.

It was first conjectured by Leja that the behaviour of Fekete points for $E \subset \mathbb{C}^n$ should be similar to the one of Fekete points for $E \subset \mathbb{C}$, that is, they should converge weak* to a finite positive measure supported on E being the unique minimizer of an energy problem.

After many years this was finally proved by Berman Boucksom and Nyström [**15**], [**16**]. They proved that the sequence $\{\mu_k\}$ of uniform probability measures associated to any sequence of Fekete points for E is converging weak* to the *pluripotential equilibrium measure* μ_E. One has $\mu_E := (\mathrm{dd^c}\, V_E^*)^n$ where V_E^* is the *pluricomplex Green function* introduced by Siciak [**94**] and Zaharjuta and enjoys in this theory the role that the Green function does in one complex variable.

The work by Berman Boucksom and Nysrtöm pointed out (among many other deep facts) that,

- *provided a strong asymptotic comparability of L^2_μ and $L^\infty(E)$ norms of polynomials holds, one can equivalently work with uniform or L^2_μ maximization of Vandermonde determinants, the arising asymptotic quantities being the same.*

They termed such an assumption the *Bernstein Markov Property* of the measure μ for the set E. Precisely, they assume that there exists a sequence $\{M_k\}$ of positive real numbers such that $\limsup_k M_k^{1/k} \leq 1$ and for each polynomial p of degree not larger than k one has

$$\max_E |p| \leq M_k \|p\|_{L^2_\mu}.$$

There are several variants of this property: one can consider weighted polynomials $pe^{-\deg pQ}$ for an admissible weight function Q (see [**91**]) or rational functions or even exponentials of Riesz potentials [**29**].

It is worth to say that, under a further assumption on E, the class of Bernstein Markov measures for E corresponds to the class of *measures with regular asymptotic behaviour* first introduced in [**97**] for $E \subset \mathbb{C}$ and generalized to several variables in [**20**].

Note that one can take $M_k = \max_E \left(\sum_{j=1}^{N_k} |q_j(z,\mu)|^2\right)^{1/2} =: \max_E (B_k^\mu(z))^{1/2}$, where $q_j(z,\mu)$ are the orthonormal polynomials with respect to μ obtained by Gram-Schmidt orthonormalization applied to monomials sorted by graded lexicographical ordering, and B_k^μ is termed the *Bergman function*. Therefore, the Bernstein Markov property is equivalent to $\limsup_k \|B_k^\mu\|_E^{1/2k} \leq 1$.

Several lines of research generated by the results by Berman Boucksom and Nysrtöm: asymptotic of orthogonal polynomials can be used to derive probabilistic results concerning random polynomials and random matrices [**27**],[**28**], large deviations for certain sequences of random arrays, and vector energy problems [**30**].

Also, the Bernstein Markov property and the asymptotic of orthogonal polynomials can be used in an Approximation Theory context, it turns out, see [**68**], that one has a several variables L^2 version of the classical Bernstein Walsh Lemma [**100**]: the k-th root asymptotic of L^2_μ approximation numbers of a given continuous function is related to its maximum radius of holomorphic extension, provided the Bernstein Markov property holds.

Admissible Meshes. Very recently, Calvi and Levenberg [**37**] introduced the definition of *admissible meshes*. These are sequences $\{A_k\}$ of finite subsets of a given compact set $E \subset \mathbb{C}^n$ such that the sampling inequality

$$\max_E |p| \leq C \max_{A_k} |p| \ , \quad \forall p \in \mathscr{P}^k$$

holds for some positive finite C and whose cardinality is increasing sub-exponentially as $k \to \infty$.

Their original aim was to give sufficient conditions for the polynomial discrete least squares projection $\Lambda_k[f]$ of a given continuous function $f \in \mathscr{C}(E)$ being uniformly convergent to f on E. Indeed their approach is very profitable: they can prove a theoretical bound [**37**, Th. 5] on $\|f - \Lambda_k[f]\|_E$, moreover numerical experiments show that the "typical" behaviour (on some trial cases) of the error is even better than the a priori estimate; see [**32**].

Admissible meshes have been shown [**31**] to be very interesting also because one can always extract from them a sequence of "good" points for multivariate polynomial interpolation by standard numerical linear algebra. More relevant, it turns out that any of these sequences of points is a nice approximation of true Fekete points, the two sequences having the same asymptotic of the Lebesgue constant and having the same weak* limit μ_E.

Since admissible meshes has been shown to be well promising both from the theoretical and computational point of view, much study has been done in order to be able to construct explicitly these sets starting by a rather "general" compact set E; see [**69**] and references therein.

It is possible to show that admissible meshes are actually a *nice discrete model of Bernstein Markov measures* in the sense that

- *the orthogonal polynomials with respect to the uniform probability measure μ_k on the admissible mesh A_k enjoy the same asymptotic properties* *of the orthogonal polynomials with respect to a fixed Bernstein Markov measure* μ*.*

- *All the fundamental quantities in Pluripotential Theory can be recovered by* L^2 *methods based either on a Bernstein Markov measure* μ *for* E *or on an admissible mesh* $\{A_k\}$ *for* E*.*

We discuss this in Section 6.3.

Our findings

In this thesis we consider some problems in the framework that we described above. Part I is concerned on several instances of the Bernstein Markov property, in particular on providing sufficient conditions for it in different contexts. Part II take into account the problem of constructing admissible meshes on some classes of compact sets and presents algorithms, based on admissible meshes, to approximate the most relevant object in Pluripotential Theory.

In view of the above discussion, these two topics can be actually seen as the *continuous* and the *discrete* approach to the same problem.

In **Chapter 2** we study the Bernstein Markov property for rational functions of one complex variable with restricted poles in $P \subset \mathbb{C}$, $P \cap E = \emptyset$, *rational Bernstein Markov property* for short; the results of this chapter are essentially the ones of [**76**].

First, we relate (**Proposition 2.3.2** and **Proposition 2.3.3**) the rational Bernstein Markov property to the Bernstein Markov property for weighted polynomials with respect to a specific class of weights (defined by P), then we use this connection to prove that, under certain additional hypothesis, the polynomial Bernstein Markov property implies the rational Bernstein Markov property; see **Theorem 2.3.5**.

In particular, it follows that the classical *mass density sufficient condition* for the Bernstein Markov property stated by Stahl and Totik, i.e., there exists $t > 0$

such that

$$\operatorname{cap}(\operatorname{supp}\mu) = \lim_{r\to 0^+} \operatorname{cap}\left(\{z \in \operatorname{supp}\mu : \mu(B(z,r)) > r^t\}\right)$$

implies the rational Bernstein Markov property as well, see **Theorem 2.4.6**. Here $\operatorname{cap}(\cdot)$ is the logarithmic capacity and E is assumed to coincide with $\operatorname{supp}\mu$.

We consider sequences of Green functions $g_{E_j}(z, a_j)$ for $\mathbb{C} \setminus E_j$, $E_j \subset E$ with arbitrary logarithmic poles $P \ni a_j \to a \in P$ and we prove (**Theorem 2.2.4**) that the convergence of the logarithmic capacities of E_j to the one of E is equivalent to the local uniform convergence of $g_{E_j}(\cdot, a_j)$ to $g_E(\cdot, a)$. This result is used to give an alternative direct proof of Theorem 2.4.6 and of its extension Theorem 2.4.7.

Moreover, in **Theorem 2.5.8** we show that it is possible to formulate a L^2 version of the Bernstein Walsh Lemma in the complex plane for rational approximations to a meromorphic function.

The central part of this work is concerned on Pluripotential Theory on an algebraic set $A \subset \mathbb{C}^n$. This is an extension of the "flat" case $A = \mathbb{C}^m$ mainly due to Bedford [**11**] Zeriahi [**107**] and Sadullaev [**88**]; in **Chapter 3** and the appendices to Part I we provide the definitions and all the tools we need later on.

In **Chapter 4** we present two original results. We consider the Chebyshev constant $T(E, A)$ with respect to certain *pseudoball* $\Omega \subset A$ for all compact sets $E \subset \Omega$,

$$m_j(E) := \inf\{\|p\|_E : p \in \mathscr{P}^j(\mathbb{C}^n), \|p\|_{\overline{\Omega}} \geq 1\},$$
$$T(E, A) := \inf_{j\geq 0} m_j(E)^{1/j} = \lim_j m_j(E)^{1/j}.$$

Also we consider the *relative capacity* $\operatorname{Cap}(E, \Omega)$ of E with respect to Ω,

$$\operatorname{Cap}(E, \Omega) := \sup\left\{\int_E (dd^c u)^m, u \in \mathrm{PSH}(\Omega, [0, 1])\right\},$$

where $(dd^c \cdot)^m$ is the complex Monge Ampere operator defined in Section 3.1 and PSH is the set of plurisubharmonic functions.

In **Theorem 4.2.1** we prove the following comparability. There exist positive finite constants c_1, c_2 such that

$$\exp\left[-\left(\frac{c_1}{\mathrm{Cap}(E,\Omega)}\right)^{1/m}\right] \geq T(E,A) \geq \exp\left(-\frac{c_2}{\mathrm{Cap}^2(E,\Omega)}\right).$$

This result extends [**2**, Th. 2.1] to our setting.

Then we consider the *relative extremal function*

$$U^*_{E,\Omega}(z) := \limsup_{\zeta\to z}\ \sup\{u(\zeta), u \in \mathrm{PSH}(\Omega, [-1,0]), u|_E \leq -1\}$$

and a sequence $\{E_j\}$ of subsets of $E \subset \Omega$.

In **Theorem 4.3.2** we show that the following properties of the sequence $\{E_j\}$ are equivalent.

(1) $\mathrm{Cap}(E_j, B) \to \mathrm{Cap}(E, B)$,

(2) $U^*_{E_j,B} \to U^*_{E,B}$,

(3) $\mu_{E_j} \to \mu_E$,

(4) $V^*_{E_j}(\cdot, A) \to V^*_E(\cdot, A)$,

where the mode of convergence depends on possible further assumptions on E, E_j. Here $V^*_E(\cdot, A)$ is the extremal plurisubharmonic function, see Subsection 3.3.2, whose definition is the natural extension to our setting of the Siciak Zaharjuta function $V^*_E(z)$ for $E \subset A := \mathbb{C}^m$.

As an application, we consider the problem of finding a sufficient condition for the Bernstein Markov property for the traces of polynomials on a compact subset of an algebraic set.

We show in **Theorem 5.1.1** that the classical *mass density sufficient condition* for the Bernstein Markov property, introduced by Stahl and Totik [**97**] in $\mathbb{C}$ and extended by Bloom and Levenberg to $\mathbb{C}^n$ [**24**], can be slightly modified to work in our setting of E being a compact subset of an algebraic set. Precisely, our mass density condition reads as follows.

$$\operatorname{Cap}(E,\Omega)=\lim_{r\to 0^+}\operatorname{Cap}\left(\left\{z\in E: d(z',Y)>2r \text{ and } \mu(\Omega_{j(z)}(z,r))>r^t\right\},\Omega\right).$$

Here we assume $E = \operatorname{supp}\mu$; Y is a certain analytic subset of $\mathbb{C}^m$, $\Omega_{j(z)}(z,r)$ is one of the components the pseudoball $\Omega(z,r)$, and $z' := (z_1,\dots z_m)$ for any $z \in A \subset \mathbb{C}^n$. Here the choice of coordinates in $\mathbb{C}^n$ is relevant, precisely it needs to be performed according to the Rudin's characterization of algebraic subsets of $\mathbb{C}^n$, [**86**]; see Appendix A.

In **Chapter 6** we give a brief introduction to the theory of admissible meshes that has been developed during the last seven years, presenting their theoretical properties and showing why they should be understood as Bernstein Markov sequences of discrete measures. In particular we present algorithms for the approximation of

(1) the plurisubharmonic extremal function V_E^*,
(2) the pluripotential equilibrium measure μ_E,
(3) the transfinite diameter $\delta(E)$, i.e., the asymptotic of the modulus of the Vandermonde determinant computed on its maximizers.

We test our approximation of the transfinite diameter and extremal function on some trial cases, some of the few instances where they are known analytically. To the author's knowledge these are the first produced algorithms for the numerical computation of these quantities in the several variables setting.

Finally, in **Chapter 7** we present some results from our articles [**80**], [**79**] and [**75**] on the existence of optimal admissible meshes (i.e., having cardinality increasing at the optimal rate $\mathcal{O}(k^n)$) on some classes of real sets. Since the proofs are fully constructive they define computing algorithms as well. Precisely we construct

(1) Optimal admissible meshes on the closure of a star shaped bounded domain in $\mathbb{R}^n$.
(2) Optimal admissible meshes on the closure of a $\mathscr{C}^{1,1}$ domain in $\mathbb{R}^n$.

Also, we provide some numerical examples of explicit computations of admissible meshes.

Part I

Continuous Approach: Bernstein Markov Property

CHAPTER 1

Motivations for the Study of the Bernstein Markov Property

I think it all comes down to motivation. If you really want to do something, you will work hard for it.

Edmund Hillary

In this chapter we present the main reasons of interest for studying the Bernstein Markov property. The reader is invited to compare this to Section 6.3, where we reprise these motivations in a discrete fashion, i.e., for certain sequences of measures with finite support. For a survey on the Bernstein Markov property we refer to [**29**].

1.1. From Approximation Theory

Let E be a compact polynomial determining subset of $\mathbb{C}^n$ (i.e., if $p(z) = 0$ for a polynomial p and all $z \in E$ then $p \equiv 0$) and f be any bounded function defined on E. Assume that a Borel finite positive measure μ with support in E (we use the notation $\mu \in \mathcal{M}^+(E)$) is given. We can define the *least squares* projection Λ_k on the space $\mathscr{P}^k(\mathbb{C}^n)$ of polynomials of degree not greater of k in n complex variables as $\Lambda_k[f](z) := \sum_{j=1}^{N_k} \langle f, q_j \rangle_{L^2_\mu} q_j(z)$, where $\{q_j\}_{j=1,\dots,N_k}$ is an orthonormal basis of $\mathscr{P}^k$ with respect to the scalar product $\langle f, g \rangle_{L^2_\mu} := \int f(z)\bar{g}(z) d\mu(z)$. If we denote by $B_k^\mu := \sum_{j=1}^{N_k} |q_j(z)|^2$ the *Bergman function* of the Hilbert space $\mathscr{P}^k_\mu := \left(\mathscr{P}^k, \langle \cdot ; \cdot \rangle_{L^2_\mu}\right)$ of polynomials of degree not greater than k, these estimates follow easily.

$$
\begin{aligned}
\|\Lambda_k[f]\|_E &\leq \|f\|_{L^2_\mu} \sqrt{\|B_k^\mu\|_E}, \\
\|\Lambda_k[f] - f\|_E &\leq \|\Lambda_k[f] - p_k + p_k - f\|_E \leq \|p_k - f\|_E + \|\Lambda_k(p_k - f)\|_E \\
&\leq \|p_k - f\|_E + \|f - p_k\|_{L^2_\mu} \sqrt{\|B_k^\mu\|_E} \\
&\leq \|p_k - f\|_E \left(1 + \sqrt{\mu(E)\|B_k^\mu\|_E}\right)
\end{aligned}
$$

$$= d_k(f, E)\left(1 + \sqrt{\mu(E)\|B_k^\mu\|_E}\right).$$

Here we denoted by p_k the best uniform polynomial approximation to f on E of degree not greater than k and by $d_k(f, E) = \|f - p_k\|_E$ its error.

In general, the factor $\|B_k^\mu\|_E$ may grow very fast as $k \to \infty$, in this case it may be not possible to approximate by $\Lambda_k[\cdot]$ even of very *"nice functions"* (e.g., functions whose approximation numbers $d_k(f, E)$ decay rather fast to 0); from the Approximation Theory point of view it is then important to find sufficient conditions to ensure that $\|B_k^\mu\|_E$ has a moderate growth.

Given a Borel finite measure μ of compact support $\operatorname{supp}\mu \subseteq E$ where E is any compact subset of $\mathbb{C}^n$, we say that (E, μ) has the *Bernstein Markov property* if there exists a sequence of positive numbers $\{M_k\}$ such that

$$\|p\|_E \leq M_k \|p\|_{L^2_\mu} \ \forall p \in \mathscr{P}^k(\mathbb{C}^n), \tag{1.1.1}$$

$$\limsup_k M_k^{1/k} \leq 1. \tag{1.1.2}$$

It is not difficult to see (by Parseval Identity) that

$$\sqrt{B_k^\mu(z)} = \sup_{0 \neq p \in \mathscr{P}^k} \frac{|p(z)|}{\|p\|_{L^2_\mu}},$$

thus the best possible factors M_k in the Bernstein Markov inequality (1.1.1) are precisely the numbers $\|B_k^\mu\|_E^{1/2}$ and vice versa if (E, μ) has the Bernstein Markov property, then we have $\limsup_k \|B_k^\mu\|_E^{1/2k} \leq 1$. This is a first motivation of interest for the Bernstein Markov property.

If $E \subset \mathbb{C}$ is compact, regular (i.e., $\mathbb{C} \setminus E$ has a classical Green function g_E with logarithmic pole at infinity, see Section 2.2) and polynomially convex (e.g, $\hat{E} = E$, where $\hat{E} := \{z \in \mathbb{C} : |p(z)| \leq \|p\|_E \text{ for all polynomials}\}$) Walsh [**100**] showed that a striking phenomenon takes place, he termed this *overconvergence*. Given a function $f \in \mathscr{C}(E)$ the k-th roots of the approximation numbers $d_k(f, E)^{1/k}$ tends to $1/R$ for some $R > 1$ if and only if f is actually the restriction to E of $\tilde{f}$ which is a holomorphic function on certain sub-level set D_R of the Green function with logarithmic pole at infinity, if this is the case, the best uniform polynomial approximation on E indeed converges *locally uniformly* on D_R.

This theorem is even more interesting because it goes precisely to the several complex variables setting, replacing the Green function with logarithmic pole at infinity by the *pluricomplex Green function* V_E^*. Such a function, commonly termed the Siciak Zaharjuta extremal function, is defined as follows

$$\begin{aligned} V_E(z) &:= \sup\{u(z), u \in \mathcal{L}(\mathbb{C}^n),\ \|u\|_E \leq 1\} \\ V_E^*(z) &:= \limsup_{\zeta\to z} V_E^*(\zeta). \end{aligned} \tag{1.1.3}$$

Here $\mathcal{L}(\mathbb{C}^n)$ is the Lelong class of plurisubharmonic functions[1] with logarithmic pole at infinity.

Two situation may occur: either V_E^* is a locally bounded plurisubharmonic function, or $V_E^* \equiv +\infty$. The latter situation occurs when E is a *pluripolar* set, i.e., there exists a plurisubharmonic function $u \not\equiv -\infty$ such that $E \subseteq \{u = -\infty\}$. If E is non pluripolar and V_E^* is a continuous function, then E is said to be a *regular* (or $\mathcal{L}$-regular) set.

Precisely, the $\mathbb{C}^n$ statement of the Bernstein Walsh Siciak Lemma, see [**68**], reads as follows. Let E be a compact regular polynomially convex subset of $\mathbb{C}^n$, $f \in \mathscr{C}(E)$ and $R > 1$, then one has

$$\limsup_k d_k(f, E)^{1/k} = 1/R \quad \Leftrightarrow \quad f = \tilde{f}|_E, \tilde{f} \in hol(D_R), \tag{1.1.4}$$

where $D_R := \{V_E^* < \log R\}$ and $hol(D_R)$ is the class of holomorphic functions on D_R.

Let us assume that $E = \operatorname{supp}\mu$ for a finite Borel positive measure μ. If (E, μ) has the Bernstein Markov property (1.1.1) the Bernstein-Walsh-Siciak Lemma can be rephrased in a L^2 fashion. Namely, the following properties can be shown to be equivalent; see [**66**].

i) $f = \tilde{f}|_E$, for certain $\tilde{f} \in hol(D_R)$.
ii) $\limsup_k d_k(f, E)^{1/k} = 1/R$.
iii) $\limsup_k \|f - \Lambda_k[f]\|_E^{1/k} = 1/R$.
iv) $\limsup_k \|f - \Lambda_k[f]\|_{L^2_\mu}^{1/k} = 1/R$.

In Section 2.5 we prove an extension of this result in the complex plane for the rational approximations to a meromorphic function whose poles are away from E, provided the measure μ satisfies a Bernstein Markov property for rational functions.

[1]Plurisubharmonic functions are function that are globally upper semi-continuous and subharmonic along each complex line.

1.2. From Pluripotential Theory

Let us briefly recall the definitions of some important quantities in weighted Pluripotential Theory, we refer to [**25**] for details.

We start by a closed, possibly unbounded, set $E \subset \mathbb{C}^n$ and an *admissible weight* Q, this means that

- $Q : E \to \mathbb{R} \cup \{\infty\}$ is lower semi-continuous,
- the set $\{z \in E : Q(z) < \infty\}$ is not pluripolar (see Section 3.1 below (3.1.2)) and
- in the case E is unbounded we assume $\liminf_{E \ni z \to \infty}(Q(z) - |z|) = +\infty$.

We introduce the *weighted extremal function* $V^*_{E,Q}$, defined by

$$V_{E,Q}(z) := \sup\{u(z) \in \mathcal{L}(\mathbb{C}^n), u|_E \leq Q\} \tag{1.2.1}$$

$$V^*_{E,Q}(z) := \limsup_{\zeta \to z} V_{E,Q}(\zeta).$$

It turns out that the *weighted equilibrium measure*

$$\mu_{E,Q} := \left(\mathrm{dd^c}\, V^*_{E,Q}\right)^n \tag{1.2.2}$$

has support $S_{E,Q} \subseteq E$ and more precisely we have

$$S_{E,Q} = S^*_{E,Q} \setminus F, \text{ where } S^*_{E,Q} := \{z \in E : V_{E,Q} \geq Q\}, F \text{ is pluripolar.}$$

Also, one has a easier representation of $V^*_{E,Q}$: the defining upper envelope (1.2.1) can be taken among weighted polynomials instead of functions in the $\mathcal{L}(\mathbb{C}^n)$ class.

$$V_{E,Q}(z) = \sup\left\{\frac{1}{\deg p}\log|p(z)|, p \in \mathscr{P}(\mathbb{C}^n), \|pe^{-\deg p Q}\|_E \leq 1\right\}.$$

In the weighted theory in $\mathbb{C}^n$ one has (at least) two different notions of weighted transfinite diameter (see [**26**, Prop 2.1 and lines above]), here we will consider only one of them, the other notion being equivalent up to a normalization constant $\exp\left(\frac{1}{n}\int_E Q d\mu_{E,Q}\right)$.

One defines first the *Vandermonde determinant*

$$\mathrm{VDM}_k(z_1, \ldots, z_{N_k}) := \det\left[z_i^{\alpha_j}\right]_{i,|\alpha_j|=1,\ldots,N_k}.$$

and the *weighted Vandermonde determinant*

$$\mathrm{VDM}_{k,Q}(z_1, \ldots, z_{N_k}) := \det\left[z_i^{\alpha_j} e^{-|\alpha_j|Q(z_i)}\right]_{i,|\alpha_j|=1,\ldots,N_k}.$$

Here the multi-indexes α_j are ordered by the graded lexicographical ordering. Then we define the *Fekete points* for E as any N_k-tuple $(\hat{z}_1, \ldots, \hat{z}_{N_k})$ which maximizes $|\operatorname{VDM}_k(z_1, \ldots, z_{N_k})|$ among $(z_1, \ldots, z_{N_k}) \in E^{N_k}$. Similarly, *weighted Fekete points* for E with respect to Q are any N_k-tuple $(\hat{z}_1, \ldots, \hat{z}_{N_k})$ which maximizes $|\operatorname{VDM}_{k,Q}(z_1, \ldots, z_{N_k})|$ among $(z_1, \ldots, z_{N_k}) \in E^{N_k}$.

The *transfinite diameter* $\delta(E)$ is defined as

$$\delta(E) := \lim_{k\to\infty} \left|\operatorname{VDM}_k(z^{(k)})\right|^{\frac{n+1}{nkN_k}}, \tag{1.2.3}$$

where $\{z^{(k)}\} = \{(z_1^{(k)}, \ldots, z_{N_k}^{(k)})\}$ is any sequence of Fekete points. The existence of the limit in this several variables setting has been proved by Zaharjuta, see [**105**].

Finally the *weighted transfinite diameter* $\delta_Q(E)$ is defined as follows

$$\delta_Q(E) := \lim_{k\to\infty} \left|\operatorname{VDM}_{k,Q}(z^{(k)})\right|^{\frac{n+1}{nkN_k}}, \tag{1.2.4}$$

where $\{z^{(k)}\} = \{(z_1^{(k)}, \ldots, z_{N_k}^{(k)})\}$ is any sequence of weighted Fekete points. where the limit has been shown to exist in [**26**].

The results in the next subsections can be found, for instance, in [**68**] and [**67**], with detailed proofs and a explanation of the relevance of the involved quantities.

1.2.1. Recovering the weighted transfinite diameter.

For a given admissible weight function $Q : E \to \mathbb{R} \cup \{\infty\}$ for any finite positive Borel measure $\mu \in \mathcal{M}^+(E)$ and for any $k \in \mathbb{N}$ we introduce the scalar product $\langle f; g\rangle_{\mu,kQ} := \int f(z)\bar{g}(z)e^{-2kQ}d\mu$.

We consider as basis for $\mathscr{P}^k$ the graded lexicographical ordered[2] set of monomials $\{z^{\alpha_j}\}_{j=1,\ldots,N_k}$, where $N_k := \dim \mathscr{P}^k$ and $\alpha_j \in \mathbb{N}^n$ has length $|\alpha_j|$ at most k. Then we form the Gram matrix $G_k^Q(\mu)$ of the space of weighted polynomials in this basis, we have

$$G_k^Q(\mu) := \left[\langle z^{\alpha_i}; z^{\alpha_j}\rangle_{\mu,kQ}\right]_{i,j=1,\ldots,N_k}.$$

Let us denote by $\{q_j(z,\mu)\}_{j=1,2,\ldots,N_k}$ the orthonormal basis with respect to the scalar product $\langle f; g\rangle_{\mu,kQ}$ obtained by the Gram Schmidt orthonormalization procedure starting by the monomial basis. The function

$$K^{\mu,kQ}(z,\zeta) := \sum_{j=1}^{N_k} q_j(z,\mu)\overline{q_j(\zeta,\mu)}$$

[2]This means to use the following order relation among multi-indexes $\alpha_i > \alpha_j$ if $|\alpha_i| > |\alpha_j|$ or $|\alpha_i| = |\alpha_j|$ and there exists $m \in \{1, 2, \ldots, n\}$ such that $\alpha_i^m > \alpha_j^m$ and $\alpha_i^l = \alpha_j^l$ for all $l < m$.

is the reproducing kernel of the Hilbert space $\mathcal{H}_\mu^{kQ} := \left(\mathscr{P}^k(\mathbb{C}^n), \langle\cdot;\cdot\rangle_{\mu,kQ}\right)$ and we define as customary its *Bergman function* $B_k^{Q,\mu}(z)$ setting

$$B_k^{Q,\mu}(z) := K^{\mu,kQ}(z,z)e^{-2kQ(z)} := \sum_{j=1}^{N_k} |q_j(z,\mu)|^2 e^{-2kQ(z)}.$$

The most interesting property of the Bergman function is its extremality, indeed by Parseval Identity one has

$$B_k^{Q,\mu}(z) = \sup_{0\neq p\in\mathscr{P}^k} \frac{|p(z)|e^{-kQ(z)}}{\|p\|_{\mathcal{H}_\mu^{kQ}}}.$$

Let us assume that $[E,\mu,Q]$ has the Weighted Bernstein Markov property, that is, there exists a positive sequence of numbers $\{M_{k,Q}\}$ such that

$$\|pe^{-kQ}\|_E \leq M_{k,Q}\|p\|_{\mathcal{H}_k^Q}\ \forall p\in\mathscr{P}^k(\mathbb{C}^n), \tag{1.2.5}$$

$$\limsup_k M_{k,Q}^{1/k} \leq 1. \tag{1.2.6}$$

By the lines above, *the weighted Bernstein Markov property for the triple* $[E,\mu,Q]$ *is equivalent to*

$$\limsup_k \left(\|B_k^{Q,\mu}\|_E\right)^{1/2k} \leq 1. \tag{1.2.7}$$

Also we denote by $V_k^Q(\mu)$ the *generalized Vandermonde matrix* of the measure μ with respect to the weight Q and the degree k, that is

$$V_k^Q(\mu) := \left[\langle z^{\alpha_i}; q_j(z,\mu)\rangle_{\mu,kQ}\right]_{i,j=1,\dots,N_k}.$$

Note that for μ being the probability measure canonically associated to an array of unisolvent interpolation points of degree k, $V_k^Q(\mu)$ is precisely the standard weighted Vandermonde matrix divided by $\sqrt{N_k}$.

It is not difficult to see that (denoting by A^{H} the conjugate transpose of A) we have

$$G_k^Q(\mu) = V_k^Q(\mu)\left(V_k^Q(\mu)\right)^{\mathrm{H}}.$$

An important property that a measure $\mu\in\mathcal{M}^+(E)$ may have is *to lead to the weighted transfinite diameter,* that is

$$\lim_k \det G_k^Q(\mu)^{\frac{n+1}{2nkN_k}} = \lim_k \left|\det V_k^Q(\mu)\right|^{\frac{n+1}{nkN_k}} = \delta_Q(E). \tag{1.2.8}$$

Note that, for any finite measure $\nu \in \mathcal{M}^+(E)$ one has (see [**26**])

$$\det G_k^Q(\nu) = \frac{\int_{E^{N_k}} |\operatorname{VDM}_{k,Q}(\zeta_1, \ldots, \zeta_{N_k})|^2 d\nu(\zeta_1) \ldots d\nu(z_{N_k})}{N_k!}.$$

Here $\operatorname{VDM}_k(\zeta_1, \ldots, \zeta_{N_k})$ stands for the Vandermonde determinant of degree k with respect to the basis $\{z^{\alpha_j}\}$ computed at the points $(\zeta_1, \ldots, \zeta_{N_k})$.

In the case $[E, \mu, Q]$ has the weighted Bernstein Markov property (1.2.5), we have

$$\begin{aligned}
Z_k^Q(\mu) :=& \int_{E^{N_k}} \left|\operatorname{VDM}_{k,Q}(\zeta_1, \ldots, \zeta_{N_k})\right|^2 d\mu(\zeta_1) \ldots d\mu(\zeta_{N_k}) \\
\geq& \frac{1}{\|B_k^{Q,\mu}\|_E} \int_{E^{N_k-1}} \max_{z_1 \in E} \left|\operatorname{VDM}_{k,Q}(z_1, \zeta_2, \ldots, \zeta_{N_k})\right|^2 d\mu(\zeta_2) \ldots d\mu(\zeta_{N_k}) \\
\geq& \frac{1}{\|B_k^{Q,\mu}\|_E^2} \int_{E^{N_k-1}} \max_{z_1 \in E} \left|\operatorname{VDM}_{k,Q}(z_1, z_2, \zeta_3 \ldots, \zeta_{N_k})\right|^2 d\mu(\zeta_2) \ldots d\mu(\zeta_{N_k}) \\
\geq& \frac{1}{\|B_k^{Q,\mu}\|_E^{N_k}} \max_{z \in E^{N_k}} \left|\operatorname{VDM}_{k,Q}(z_1, \ldots, z_{N_k})\right|^2
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\liminf_k \det G_k^Q(\mu)^{\frac{n+1}{2nkN_k}} &= \liminf_k \left(\frac{1}{N_k!}\right)^{\frac{n+1}{2nkN_k}} Z_k^Q(\mu)^{\frac{n+1}{2nkN_k}} \\
\geq \left(\frac{1}{N_k!}\right)^{\frac{n+1}{2nkN_k}} & \frac{1}{\|B_k^{Q,\mu}\|_E^{\frac{n+1}{2nk}}} \left(\max_{z \in E^{N_k}} \left|\operatorname{VDM}_{k,Q}(z_1, \ldots, z_{N_k})\right|\right)^{\frac{n+1}{nkN_k}}.
\end{aligned}$$

Notice that $\left(\frac{1}{N_k!}\right)^{\frac{n+1}{2nkN_k}} \to 1$ as $k \to \infty$ (use the Stirling Formula) and $\|B_k^{Q,\mu}\|_E^{\frac{n+1}{2nk}} \to 1$ since $[E, \mu, Q]$ has the Weighted Bernstein Markov property, thus

$$\liminf_k \det G_k^Q(\mu)^{\frac{n+1}{2nkN_k}} \geq \delta_Q(E).$$

On the other hand, since μ is a finite measure, one can use the trivial inequality $\max_{z \in E} |f(z)| \geq (\mu(E)^{-1/2} \|f\|_{L^2_\nu}$ for all upper semi-continuous bounded functions f to get

$$\limsup_k \det G_k^Q(\mu)^{\frac{n+1}{2nkN_k}} \leq \delta_Q(E).$$

Therefore we proved the following.

Proposition 1.2.1 (Recovering the transfinite diameter; [**26**]). *Let $E \subset \mathbb{C}^n$ be any compact set and $Q : E \to \mathbb{R} \cup \{\infty\}$ any admissible weight. For any weighted Bernstein Markov Measure $\mu \in \mathcal{M}^+(E)$ one has*

$$\lim_k \det G_k^Q(\mu)^{\frac{n+1}{2nkN_k}} = \delta_Q(E).$$

If E is compact and non pluripolar and $Q \equiv 0$ the same conclusion holds true for any measure $\mu \in \mathcal{M}^+(E)$ such that (E, μ) has the Bernstein Markov property.

1.2.2. Recovering the extremal function.

THEOREM 1.2.1 (Bernstein Markov k-th root asymptotic). *Let $E \subset \mathbb{C}^n$ be a compact regular set and $\mu \in \mathcal{M}^+(E)$. Suppose that (E, μ) has the Bernstein Markov property, then*

$$\lim_k \frac{1}{2k} \log B_k^\mu(z) = V_E^*(z) \quad \textit{uniformly in } \mathbb{C}^n.$$

PROOF. Without loss of generality we can assume $\mu \in \mathcal{M}_1^+(E)$. Let us introduce the following sequence of functions and families

$$\begin{aligned} f_\mu^{(k)}(z) \quad &:= \sup\left\{\frac{1}{k}\log|p(z)|, \deg p \leq k, \|p\|_{L^2_\mu} \leq 1\right\} \\ &=: \sup\left\{\frac{1}{k}\log|p(z)|, p \in \mathcal{F}_\mu^{(k)}\right\} \\ \log \Phi_E^{(k)}(z) \quad &:= \sup\left\{\frac{1}{k}\log|p(z)|, \deg p \leq k, \|p\|_E \leq 1\right\} \\ &=: \sup\left\{\frac{1}{k}\log|p(z)|, p \in \mathcal{F}_E^{(k)}\right\}. \end{aligned}$$

The sequence of function $\Phi_E^{(k)}$ has been introduced by Siciak and has been shown to converge to $\exp V_E$, locally uniformly if $V_E = V_E^*$ is continuous; [**94**], see also Subsection 3.3.2.

Now notice that, due to the Parseval Identity, we have

$$B_k^\mu(z) = \sup_{p \in \mathcal{F}_\mu^{(k)}} |p(z)|^2, \quad \text{thus we have } \; f_\mu^{(k)}(z) = \frac{1}{2k}\log B_k^\mu(z).$$

Let us pick $p \in \mathcal{F}_\mu^{(k)}$, we have $\|p\|_E \leq \sqrt{\|B_k^\mu\|_E}\|p\|_{L^2_\mu}$ for the reason above, thus $q := p\|B_k^\mu\|_E^{-1/2} \in \mathcal{F}_E^{(k)}$.

Hence

$$\log \Phi_E^{(k)}(z) \geq \frac{1}{k}\log|q(z)| = \frac{1}{k}\log|p(z)| - \frac{1}{2k}\log\|B_k^\mu\|_E, \forall p \in \mathcal{F}_\mu^{(k)}.$$

It follows that

$$\log \Phi_E^{(k)}(z) + \frac{1}{2k}\log\|B_k^\mu\|_E \geq f_\mu^{(k)}(z).$$

On the other hand, since μ is a probability measure, we have $\|p\|_E \geq \|p\|_{L^2_\mu}$ for any polynomial. Hence if $p \in \mathcal{F}_E^{(k)}$ it follows that $p \in \mathcal{F}_\mu^{(k)}$. Thus $f_\mu^{(k)}(z) \geq \log \Phi_E^{(k)}(z)$. Therefore we have

$$\log \Phi_E^{(k)}(z) + \frac{1}{2k} \log \|B_k^\mu\|_E \geq f_\mu^{(k)}(z) \geq \log \Phi_E^{(k)}(z).$$

Note that the Bernstein Markov property in particular implies $\limsup_k \|B_k^\mu\|_E^{1/2k} \leq 1$, hence we can conclude that locally uniformly we have

$$\begin{aligned} V_E^*(z) &\leq \liminf_k \left(\log \Phi_E^{(k)}(z) - \frac{1}{2k} \log \|B_k^\mu\|_E \right) \\ &\leq \liminf f_\mu^{(k)}(z) \leq \limsup f_\mu^{(k)}(z) \\ &\leq \limsup_k \log \Phi_E^{(k)}(z) = V_E^*(z). \end{aligned}$$

□

1.2.3. Recovering the weighted equilibrium measure. The following is a deep result by Berman Boucksom and Nymstrom regarding the asymptotic of the Bergman function for Bernstein Markov measures; see [**16**], [**15**]. We refer to [**22**] as well. Recall that the weighted extremal measure has been defined above in (1.2.2).

Theorem 1.2.2 (Strong Bergman asymptotic; [**16**]). *Let $E \subset \mathbb{C}^n$ be a closed (possibly unbounded) set, Q an admissible weight on E and $\mu \in \mathcal{M}^+(E)$. Suppose that $[E, \mu, Q]$ has the weighted Bernstein Markov property, then*

$$\frac{B_k^{\mu,Q}}{N_k} \mu \rightharpoonup^* \mu_{E,Q}.$$

The same conclusion holds true for $Q \equiv 0$, provided that E is compact non pluripolar.

It is worth to stress that Theorem 1.2.2 is achieved by an argument that does need the weighted setting, even if one aims to prove it for $Q \equiv 0$.

1.2.4. Further motivations. Finally, it is worth to recall that the (weighted) Bernstein Markov property it is a key tool in a series of probabilistic results regarding zeros of random polynomials and eigenvalues of random matrices, vector energy problems in the complex plane and large deviations of random arrays generated by a determinantal point process.

The history of this line of research goes back to Kac [**57**], [**58**] and Szegö and is still developing recently; see for instance [**106**], [**54**], [**28**], [**30**] and references therein.

CHAPTER 2

Bernstein Markov Properties in $\mathbb{C}$

When people say, "it can't be done" or "you don't have what it takes", it makes the task all more interesting.

Lynn Hill

In this chapter we introduce the Bernstein Markov Property for polynomials in $\mathbb{C}$ and some variants concerning weighted polynomials and sequences of rational functions with restricted poles; we essentially base our exposition on [**76**]. In Section 1 we present these properties also by some examples. In Section 2, after recalling some standard facts in Logarithmic Potential Theory, we establish some convergence results for sequences of Green functions, this will be a tool later. In Section 3 we compare the different Bernstein Markov properties finding out some conditions for the polynomial Bernstein Markov Property to imply the rational one. In Section 4 we give a sufficient condition for a finite Borel measure of compact support to satisfy the rational Bernstein Markov Property on its support. Finally, in Section 5 we give an application of the rational Bernstein Markov Property: we relate the L^2_μ approximation numbers of a given continuous function f to the property of being the restriction to $K := \operatorname{supp}\mu$ of a meromorphic function on a certain specific domain related to K, this extends the classical result of Bernstein and Walsh.

2.1. Polynomial, Weighted and Rational Bernstein Markov Properties in $\mathbb{C}$

2.1.1. Definitions. Let $K \subset \mathbb{C}$ be compact and have infinitely many points. In such a case $\|p\|_K := \max_{z\in K} |p(z)|$ is a norm on the space $\mathscr{P}^k$ of polynomials of degree not greater than k for any $k \in \mathbb{N}$.

Let us pick a positive finite Borel measure μ supported on K. When $\|\cdot\|_{L^2_\mu(K)}$ is a norm on $\mathscr{P}^k$ we can compare it with the uniform norm on K. In fact, since $\mathscr{P}^k$ is a finite dimensional normed vector space, there exist positive constants c_1, c_2

depending only on (K,μ,k) such that

$$c_1\|p\|_{L^2_\mu} \leq \|p\|_K \leq c_2\|p\|_{L^2_\mu} \quad \forall p \in \mathscr{P}^k.$$

Notice that there exists such a c_1 because the measure μ is finite (one can take $c_1 = \mu(K)^{-1/2}$) while c_2 is finite precisely when μ induces a norm.

The Bernstein Markov property is a quantitative asymptotic growth assumption on c_2 as $k \to \infty$. Namely, the couple (K,μ) is said to enjoy the *Bernstein Markov Property* if for any sequence $\{p_k\} : p_k \in \mathscr{P}^k$ we have

$$\limsup_k \left(\frac{\|p_k\|_K}{\|p_k\|_{L^2_\mu}}\right)^{1/k} \leq 1. \tag{2.1.1}$$

We remark that the class of measures having the Bernstein Markov property is very close to the **Reg** class studied in the monograph [**97**] (later generalized to the multidimensional case in [**20**]). Precisely, if we restrict our attention to measures μ whose support $\operatorname{supp}\mu$ is a *regular* set for the Dirichlet problem for the Laplace operator (i.e., $\mathbb{C}\setminus\operatorname{supp}\mu$ admits a classical Green function g with logarithmic pole at infinity such that $g|_{\partial\Omega_K} \equiv 0$, where Ω_K is the unbounded component of $\mathbb{C}\setminus K$) the two notions coincide.

We define the following classes of sequences of rational functions, in order to study a slightly modified Bernstein Markov Property.

$$\begin{aligned}
\mathcal{R}(P) &:= \left\{\{p_k/q_k\} \;:\; p_k, q_k \in \mathscr{P}^k, Z(q_k) \subseteq P \;\forall k \in \mathbb{N}\right\} \quad \text{and}\\
Q(P) &:= \left\{\{p_k/q_k\} \;:\; p_k, q_k \in \mathscr{P}^k, \deg q_k = k, Z(q_k) \subseteq P \;\forall k \in \mathbb{N}\right\},
\end{aligned}$$

where we set $Z(p) := \{z \in \mathbb{C} : p(z) = 0\}$ and where $P \subset \mathbb{C}$ is any compact set that from now on we suppose to have empty intersection with K.

Let us introduce the following definition.

Definition 2.1.1 (Rational Bernstein Markov Property). *Let $K, P \subset \mathbb{C}$ be compact disjoint sets and $\mu \in \mathcal{M}^+(K)$.*

(i) (Rational Bernstein Markov Property.) *If*

$$\limsup_k \left(\frac{\|r_k\|_K}{\|r_k\|_{L^2_\mu}}\right)^{1/k} \leq 1 \quad \forall \{r_k\} \in \mathcal{R}(P), \tag{2.1.2}$$

then (K,μ,P) is said to enjoy the rational Bernstein Markov Property.

(ii) (sub-diagonal Rational Bernstein Markov Property.) *If*

$$\limsup_k \left(\frac{\|r_k\|_K}{\|r_k\|_{L^2_\mu}} \right)^{1/k} \leq 1 \quad \forall \{r_k\} \in Q(P), \tag{2.1.3}$$

then (K, μ, P) is said to enjoy the sub-diagonal rational Bernstein Markov property.

Another modification of the classical Bernstein Markov Property is the following.

Definition 2.1.2 (Weighted Bernstein Markov Property). *Let $K \subset \mathbb{C}$ be a closed set and $w : K \to [0, +\infty[$ be an upper semicontinuous function, let $\mu \in \mathcal{M}^+(K)$, then the triple $[K, \mu, w]$ is said to have the weighted Bernstein Markov property if for any sequence of polynomials $p_k \in \mathscr{P}^k$ we have*

$$\limsup_k \left(\frac{\|p_k w^k\|_K}{\|p_k w^k\|_{L^2_\mu}} \right)^{1/k} \leq 1. \tag{2.1.4}$$

One motivation to study such properties is given by the discretization of a quite general class of vector energy problems performed in [**30**]. Bloom, Levenberg and Wielonsky introduce a probability Prob$(\cdot)$ on the space of sequences of arrays of points $\{z^{(1)}, \dots, z^{(m)}\}$, where $z^{(l)} = \{z_0^{(l)}, \dots, z_k^{(l)}\} \in (K^{(l)})^{k+1}$, on a vector of compact sets $\{K^{(1)}, \dots, K^{(m)}\}$ in the complex plane based on a vector of probability measures $\mu^{(i)} \in \mathcal{M}_1^+(K^{(i)})$ such that $(K^{(i)}, \mu^{(i)}, \cup_{j \neq i} K^{(j)})$ has the rational Bernstein Markov property. In [**30**] the authors actually deal with *strong* rational Bernstein Markov measures, which is a variant of rational Bernstein Markov property where weighted rational function are considered instead of standard ones, however their paper can be read in the *un-weighted* setting picking (in their notation) $Q \equiv 0$. Then they prove a Large Deviation Principle (LDP) for measures canonically associated to arrays of points randomly generated according to Prob. Also, they show that the validity of the LDP is not affected by the particular choice of $\{\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(m)}\}$ that are only required to form a vector of rational Bernstein Markov measures.

Measures having the rational Bernstein Markov property are worth to be studied also from the approximation theory point of view. In fact, for such measures it turns out that the radius of maximum meromorphic extension with exactly m poles of a function $f \in \mathscr{C}(K)$ is related to the asymptotic of its L^2_μ approximation numbers

$$\left(\min_{\deg p \leq k, \deg q = m} \|f - p/q\|_{L^2_\mu} \right)^{1/k}.$$

The reader is referred to Section 2.5 for a precise statement.

2.1.2. Examples. Let us illustrate some significantly different situations which can occur by providing some easy examples where we are able to perform explicit computations.

We recall that, given an orthonormal basis $\{q_j\}_{j=1,2,\dots}$ of a separable Hilbert space H (endowed with its induced norm $\|\cdot\|_H$) of continuous functions on a given compact set, the *Bergman Function* $B_k(z)$ of the subspace $H_k := \operatorname{span}\{q_1, q_2, \dots, q_k\}$ is

$$B_k(z) := \sum_{j=1}^{k} |q_j(z)|^2.$$

It follows by its definition and by Parseval Identity that for any function $f \in H_k$ one has $|f(z)| \leq \sqrt{B_k(z)}\|f\|_H$, while the function $f(z) := \sum_{j=1}^k \bar{q}_j(z_0) q_j(z)$ achieves the equality at the point z_0, thus

$$B_k(z) = \max_{f \in H_k \setminus \{0\}} \left(\frac{|f(z)|}{\|f\|_H} \right)^2. \tag{2.1.5}$$

Example 2.1.1. *(a) Let μ be the arc length measure on the boundary $\partial\mathbb{D}$ of the unit disk. Let $K = \partial\mathbb{D}$ and $P = \{0\}$.*

Let us take a sequence $\{r_k\} = \left\{\frac{p_{l_k}}{z^k}\right\}$ in $\mathcal{R}(P)$ where $\deg p_{l_k} = l_k \leq k$, then we have

$$\begin{aligned} \|r_k\|_K = \left\| \frac{p_{l_k}}{z^k} \right\|_K = \|p_{l_k}\|_K \leq \\ \|B^\mu_{l_k}\|_K{}^{1/2} \|p_{l_k}\|_{L^2(\mu)} = \|B^\mu_{l_k}\|_K{}^{1/2} \|r_k\|_{L^2(\mu)}. \end{aligned} \tag{2.1.6}$$

Here we indicated by $B^\mu_k(z)$ the Bergman function of the space $\left(\mathscr{P}^k, \langle\cdot,\cdot\rangle_{L^2_\mu}\right)$.

For this choice of μ the orthonormal polynomials $q_k(z,\mu)$ are simply the normalized monomials $\left\{\frac{z^k}{\sqrt{2\pi}}\right\}$, thus we have

$$(\max_K B^\mu_k)^{1/2k} = \left(\max_K \frac{\sum_{j=0}^k |z|^{2j}}{2\pi} \right)^{1/2k} = \left(\frac{k+1}{2\pi} \right)^{1/2k}. \tag{2.1.7}$$

It follows by (2.1.6) *and* (2.1.7) *that (K, μ, P) has the rational Bernstein Markov Property. A similar computation shows that actually any ν such that (K, ν) has the Bernstein Markov Property is such that (K, ν, P) has the rational Bernstein Markov Property.*

(b) On the other hand, the same measure μ does not enjoy the sub-diagonal rational Bernstein Markov Property in the triple (K, μ, P) with $K = \{1/2 \leq |z| \leq 1\}$

and $P = \{0\}$ as the sequence of functions $\{1/z^k\}$ clearly shows: $\|z^{-k}\|_K = 2^k$, $\|z^{-k}\|_{L^2_\mu} = 1$. A fortiori the rational Bernstein Markov Property is not satisfied by (K, μ, P).

(c) *On the contrary, the arc length measure on the inner boundary of $K = \{1/2 \leq |z| \leq 1\}$ and $P = \{0\}$ has the sub-diagonal rational Bernstein Markov Property, equation* (2.1.3)*, but neither the rational Bernstein Markov Property equation* (2.1.2)*, nor the polynomial one, equation* (2.1.1)*, as is shown by the sequence $\left\{z^k\right\}$. Notice that*

$$\left(\int_{\frac{1}{2}\partial\mathbb{D}} |z|^{2k}\, ds\right)^{1/2} = \sqrt{\pi}2^{-k} \text{ and } \|z^k\|_K = 1, \text{ thus}$$
$$\left(\frac{\|z^k\|_K}{\|z^k\|_{L^2_\mu}}\right)^{1/k} = 2\pi^{-1/2k} \to 2 \not\leq 1.$$

In fact, in these last two examples the support of μ is not the whole set K, however we can provide a similar example also under the restriction $\operatorname{supp}\mu = K$.

(d) *Let us take a dense sequence $\{z_j\}$ in $K = \{1/2 \leq |z| \leq 1\}$ and a summable sequence of positive numbers $c := \{c_j\}$ such that $\sum_{j=1}^\infty c_j = 1$, we define*

$$\mu_c := \frac{1}{4\pi} ds|_{\partial\mathbb{D}} + \frac{1}{2}\sum_{j=1}^\infty c_j \delta_{z_j} \in \mathcal{M}_1^+(K).$$

Notice that $\operatorname{supp}\mu = K$. *It is well known that $ds|_{\partial\mathbb{D}}$ has the Bernstein Markov property for $\overline{\mathbb{D}}$, so does the measure μ_c have.*

On the other hand, we can show that $(K, \mu_c, \{0\})$ does not have the rational Bernstein Markov property, provided a suitable further assumption on c and z_j.

Precisely, let $\{c_j\} \in \ell^1$ and a sequence $\{n_k\}$ of natural numbers be such that

$$\begin{aligned} &\liminf_k \left(1 + \sum_{j=k+1}^\infty c_j |z_j|^{2n_k}\right)^{1/2n_k} = 1 \\ &0 \leq k \leq n_k \\ &\lim_k k/n_k < 1. \end{aligned} \tag{2.1.8}$$

We construct a sequence $\{\tilde{r}_k\} \in \mathcal{Q}(\{0\})$ of rational functions for which (2.1.3) *does not hold with $\mu = \mu_c$ and $P = \{0\}$; hence we show that (K, μ_c, P) does not have the sub-diagonal rational Bernstein Markov property.*

Let us define $r_{n_k}(z) := \frac{p_k(z)}{z^{n_k}} = \frac{\prod_{l=1}^k z-z_l}{z^{n_k}}$. *We notice that*

$$
\begin{aligned}
\|r_{n_k}\|_K &= \max\{2^{n_k}\|p_k\|_{1/2\partial\mathbb{D}}, \|p_k\|_{\partial\mathbb{D}}\} \geq 2^{n_k}\|p_k\|_{1/2\partial\mathbb{D}},\\
\|r_{n_k}\|_{L^2_{\mu_c}} &= \left(\frac{1}{4\pi}\int_{\partial\mathbb{D}}|p_k|^2 ds + \frac{1}{2}\sum_{j=k+1}^{\infty}\frac{c_j}{|z_j|^{2n_k}}|p_k(z_j)|^2\right)^{1/2}\\
&\leq \frac{\|p_k\|_{\partial\mathbb{D}}}{\sqrt{2}}\left(1+\sum_{j=k+1}^{\infty}\frac{c_j}{|z_j|^{2n_k}}\right)^{1/2}\\
&\leq 2^{-1/2+k}\|p_k\|_{1/2\partial\mathbb{D}}\left(1+\sum_{j=k+1}^{\infty}\frac{c_j}{|z_j|^{2n_k}}\right)^{1/2}.
\end{aligned}
$$

Here we used the second equation in (2.1.8) *and the classical Bernstein Walsh Inequality for* $1/2\partial\mathbb{D}$ *twice, e.g.* $|p(z)| \leq \|p\|_{1/2\partial\mathbb{D}}\exp(\deg p \log^+(2|z|))$. *It follows that*

$$
\left(\frac{\|r_{n_k}\|_K}{\|r_{n_k}\|_{L^2_{\mu_c}}}\right)^{1/n_k} \geq 2^{1-\frac{k}{n_k}+\frac{1}{2n_k}}\frac{1}{\left(1+\sum_{j=k+1}^{+\infty}c_j|z_j|^{-2n_k}\right)^{1/2n_k}}.
$$

We can construct the sequence $\{\tilde{r}_m\}$ *above setting* $\tilde{r}_m = r_{n_k}$ *for any* m *for which it exists* k *with* $m = n_k$ *and picking any other rational function with at most* m *zeros and a* m*-order pole at* 0 *for other values of* m. *Now we use the assumptions* (2.1.8) *and properties of* $\limsup$ *to get*

$$
\begin{aligned}
\limsup_m\left(\frac{\|r_m\|_K}{\|r_m\|_{L^2_{\mu_c}}}\right)^{1/m} &\geq \limsup_k\left(\frac{\|r_{n_k}\|_K}{\|r_{n_k}\|_{L^2_{\mu_c}}}\right)^{1/n_k}\\
&> \frac{1}{\liminf_k\left(1+\sum_{j=k+1}^{\infty}c_j|z_j|^{2n_k}\right)^{1/2n_k}} = 1.
\end{aligned}
$$

Thus $(K, \mu_c, \{0\})$ *does not have the rational sub-diagonal Bernstein Markov property, since the rational Bernstein Markov is a stronger property.*

(e) *Lastly, the measure* $d\mu := d\mu_1 + d\mu_2 := 1/2\, ds|_{\partial\mathbb{D}} + 1/2\, ds|_{1/2\partial\mathbb{D}}$ *(here* ds *denotes the standard arc length measure and* $1/2\partial\mathbb{D} := \{z : |z| = 1/2\}$*) has the rational Bernstein Markov property for* $K = \partial\mathbb{D} \cup 1/2\partial\mathbb{D}$, $P = \{0\}$.

In order to show that, we pick any sequence of polynomials $\{p_k\}$ *of degree not greater than* k *and* $\{m_k\}$ *where* $m_k \in \{0, 1, \dots, k\}$, *we consider the Bergman* *function for μ_1 and μ_2 and using* (2.1.5) *we get*

$$\left\|\frac{p_k}{z^{m_k}}\right\|_{L^2_\mu} = \|p_k\|_{L^2_{\mu_1}} + 2^{m_k}\|p_k\|_{L^2_{\mu_2}} \geq$$

$$(B_k^{\mu_1}(z_1))^{-1/2}|p_k(z_1)|\Big|_{z_1\in\partial\mathbb{D}} + 2^{m_k}(B_k^{\mu_2}(z_2))^{-1/2}|p_k(z_2)|\Big|_{z_2\in 1/2\partial\mathbb{D}} =$$

$$\left(\left(\frac{2\pi}{\sum_{j=0}^k |z_1^j|^2}\right)^{1/2}|p_k(z_1)|\right)_{z_1\in\partial\mathbb{D}} + 2^{m_k}\left(\frac{2\pi}{\sum_{j=0}^k 2|z_2{}^j|^2}\right)^{1/2} p_k(z_2)|_{z_2\in\partial\mathbb{D}}.$$

Now we pick $z_1 \in \partial\mathbb{D}$ and $z_2 \in 1/2\partial\mathbb{D}$ maximizing $|p_k|$ and we get

$$\left\|\frac{p_k}{z^{m_k}}\right\|_{L^2_\mu} \geq \sqrt{\frac{2\pi}{k+1}}\|p_k\|_{\partial\mathbb{D}} + 2^{m_k}\sqrt{\frac{3\pi}{4^{k+1}-1}}\|p_k\|_{1/2\partial\mathbb{D}} \geq$$

$$\sqrt{\frac{3\pi}{4^{k+1}-1}}\cdot\left(\|p_k\|_{\partial\mathbb{D}} + 2^{m_k}\|p_k\|_{1/2\partial\mathbb{D}}\right) =$$

$$\sqrt{\frac{3\pi}{4^{k+1}-1}}\left(\left\|\frac{p_k}{z^{m_k}}\right\|_{\partial\mathbb{D}} + \left\|\frac{p_k}{z^{m_k}}\right\|_{1/2\partial\mathbb{D}}\right) \geq$$

$$\sqrt{\frac{3\pi}{4^{k+1}-1}}\left\|\frac{p_k}{z^{m_k}}\right\|_K.$$

It follows that, denoting p_k/z^k by r_k, we have

$$\limsup_k \left(\frac{\|r_k\|_K}{\|r_k\|_{L^2_\mu}}\right)^{1/k} \leq \lim_k \left(\frac{4^{k+1}-1}{3\pi}\right)^{1/(2k)} = 1,$$

hence $(K,\mu,\{0\})$ has the rational Bernstein Markov property.

The relation between these three properties is a little subtle: the examples above show that different aspects come in play from the geometry of K and P and the classes $\mathcal{R}(P), \mathcal{Q}(P)$. It will be clear later that the measure theoretic and potential theoretic features are important as well.

2.2. Logarithmic Potential Theory in $\mathbb{C}$: Convergence of Capacities and Green Functions

2.2.1. Preliminaries. In this section we briefly recall for the reader's convenience some classical results about Logarithmic Potential Theory on the complex plane; we refer to [**85**] and [**91**] for proofs and details.

Let $\Omega \subset \mathbb{C}$ be a domain and $h \in \mathscr{C}^2(\Omega)$, we say that u is *harmonic* in Ω if $\Delta h = \frac{\partial^2 h}{\partial z \partial \bar{z}}(z) \equiv 0$.

Let $u : \Omega \to [-\infty, +\infty[$ be a upper semicontinuous function such that

$$u(z_0) \leq \frac{1}{2\pi r} \int_{\partial B(z_0,r)} u(\zeta) d\sigma(\zeta), \ \forall z_0 \in \Omega : B(z_0, r) \subseteq \Omega.$$

Then u is said to be *subharmonic*. One can equivalently require u to be upper semicontinuous and one of the following property to hold true.

- $u(z_0) \leq \frac{1}{\pi r^2} \int_{B(z_0,r)} u(\zeta) dm(\zeta), \ \forall z_0 \in \Omega : B(z_0, r) \subseteq \Omega.$
- For any domain $\Omega' \subset \Omega$ and any h harmonic on Ω such that $h|_{\partial\Omega'} \geq u|_{\partial\Omega'}$ one has $h \geq u$ in Ω'.

We denote the set of subharmonic functions on Ω by $shm(\Omega)$.

Given a subharmonic function u we can consider the Laplacian Δu in the distributional sense. It follows by the properties of subharmonic functions above that this distribution is positive and thus it is a positive measure.

Using the Green Identities one can prove that $\Delta \log |z| = \delta_0$ in the sense of distributions, it follows that, given a positive measure μ one has

$$\Delta(\log |\cdot| * \mu) = \mu.$$

Indeed, the Riesz Decomposition Theorem states that any subharmonic function can be expressed as the sum of an harmonic one and a term of the type $\log |\cdot| * \mu$ for some positive Borel measure.

The *logarithmic potential* is defined (up to the sign) as the convolution above, that is, for any positive Borel measure μ of compact support S_μ one sets

$$U^\mu(z) := \int \log \frac{1}{|z - \zeta|} d\mu(\zeta).$$

Two situation may occur, either U^μ is identically $+\infty$, or $-U^\mu$ is a subharmonic function on $\mathbb{C}$ that is harmonic on $\mathbb{C} \setminus S_\mu$.

To the log kernel it is attached a variational problem (representing the electrostatic in the plane):

$$\begin{aligned} &\text{Minimize } I[\mu] := \int \int \log \frac{1}{|z - \zeta|} d\mu(\zeta) d\mu(z) \\ &\text{among } \mu \in \mathcal{M}_1^+(K). \end{aligned}$$

Here $\mathcal{M}_1^+(K)$ is the set of Borel probability measures on K endowed with the weak* topology.

This classical problem can be solved by the *Direct Method*. One shows first that the functional $I[\cdot]$ is lower semicontinuous, then observes that the domain is a convex locally compact space, this proves the existence of minimizers, possibly having infinite energy.

Then the strict convexity is showed and this leads to the unicity of the minimizer, provided the class of the measures supported in K having finite energy is non empty. Therefore one can have

A) either $I[\mu] = +\infty$ for all $\mu \in \mathcal{M}_1^+(K)$,
B) or there exists a unique $\mu_K \in \mathcal{M}_1^+(K)$ such that $I[\mu_K] = \inf_{\mu\in\mathcal{M}_1^+(K)} I[\mu]$.

When situation (A) above occurs, we say that the compact set K is *polar*. It turns out that K is polar if and only if $K \subseteq \{u = -\infty\}$ for some subharmonic function (not identically $-\infty$) defined in a neighbourhood of K.

If a property holds outside of a polar set we will say that such property holds *quasi everywhere*, *q.e.* for short.

When situation (B) occurs we term the unique minimizer μ_K the *equilibrium measure* of K and U^{μ_K} its equilibrium potential.

The quantity $I[\mu_K] = \inf_{\mu\in\mathcal{M}_1^+(K)} I[\mu]$ is termed the Wiener constant of K and usually denoted by W_K. The number

$$\operatorname{cap}(K) := \exp(-W_K) = \exp\left(-\inf_{\mu\in\mathcal{M}_1^+(K)} I[\mu]\right)$$

is called the *logarithmic capacity* of the set K, note that the condition $\operatorname{cap}(K) = 0$ characterize polar sets by definition.

In the following we will make repeated use of these properties of logarithmic potentials.

Theorem 2.2.1 (Principle of Descent). *Let $K \subset \mathbb{C}$ be compact and $\{\mu_j\}$ be a sequence in $\mathcal{M}_1^+(K)$ weak* converging to $\mu \in \mathcal{M}_1^+(K)$. For any sequence $\mathbb{C} \ni z_j \to \hat{z}$ we have*

$$U^{\mu}(\hat{z}) \leq \liminf_j U^{\mu_j}(z_j).$$

If U^{μ} is a continuous function the inequality $U^{\mu} \leq \liminf_j U^{\mu_j}$ holds locally uniformly in $\mathbb{C}$.

Theorem 2.2.2 (Principle of Domination). *Let μ, ν be finite Borel measures with compact support. Suppose that $I[\mu] < \infty$ and $\nu(\mathbb{C}) \leq \mu(\mathbb{C})$. Then if*

$$U^{\mu} \leq c + U^{\nu} \quad \mu\text{-a.e.},$$

it follows that $U^{\mu} \leq c + U^{\nu}$ in $\mathbb{C}$.

Also the characterization of the equilibrium measure in terms of its potential is very useful.

THEOREM 2.2.3 (Frostman). *Let $K \subset \mathbb{C}$ be a compact set and $\mu \in \mathcal{M}_1^+(K)$, then $\mu = \mu_K$ if and only if we have for some constant $c > 0$*

$$U^\mu(z) \le c \quad \forall z \in \mathbb{C}$$
$$U^\mu(z) \ge c \quad q.e.\ in\ K.$$

Then necessarily $c = I[\mu_K] = -\log \operatorname{cap}(K)$.

For any compact set $K \subset \mathbb{C}$ there exists a standard splitting of $\mathbb{C}_\infty \setminus K$. Namely, one considers the (countable) collection of its connected components, due to the compactness of K, only one of these contains $\{\infty\}$, this is the only unbounded connected component of $\mathbb{C}_\infty \setminus K$ and it is usually denoted by Ω_K while the others by $\Omega^j, \in \mathbb{N}$.

Given a compact set $K \subset \mathbb{C}$ the *polynomial hull* of K is denoted by $\hat{K}$ and defined as

$$\hat{K} := \{z \in \mathbb{C} : |p(z)| \le \|p\|_K \ \forall p \in \mathscr{P}\}. \tag{2.2.1}$$

Here $\mathscr{P}$ denotes the set of all polynomials.

It follows by the Maximum Modulus Theorem and the Runge Theorem that actually one has $\hat{K} = \mathbb{C} \setminus \Omega_K = K \bigcup \left(\cup_j \Omega^j\right)$.

Given a proper sub-domain $D \subset \mathbb{C}_\infty$ the *Green function* with logarithmic pole at w is defined (when it does exist) as the unique function $G_D : D \times D \to \mathbb{R} \cup \{+\infty\}$ such that

i) $G_D(z, w)$ is harmonic with respect to the variable z in $D \setminus \{w\}$ and bounded out of each neighbourhood of w.
ii) $G_D(w, w) = +\infty$ and
$$\begin{cases} \lim_{z\to\infty} G_D(z, w) - \log|z| = 0 & \text{if } w = \infty, \\ \lim_{z\to w} G_D(z, w) + \log|z - w| = 0 & \text{if } w \ne \infty. \end{cases}$$
iii) For q.e. $z_0 \in \partial D$ we have $\lim_{z\to z_0} G_D(z, w) = 0$ for all $w \in D$.

In the rest of the chapter we will deal with Greens functions for the domain Ω_K for a given compact non polar set K, we use a specific notation for the extension of such a function:

$$g_K(z,w) := \begin{cases} G_{\Omega_K}(z,w) & z \in \Omega_K \\ \limsup_{K \not\ni \zeta \to z} G_{\Omega_K}(\zeta,w) & z \in \partial K \\ 0 & z \in K \setminus \partial K \end{cases} \quad , \; w \in \mathbb{C} \setminus K$$

If $g_K(\cdot, w)$ is a continuous function (obviously one needs to check this only at ∂K) we say that the compact set K is *regular*. Note that $g_K(\cdot, w)$ is *globally subharmonic* and *locally bounded*.

These type of Green function are usually expressed in terms of the upper semi-continuous regularization of a Perron Bremermann upper envelope

$$V_K(z)^* := \limsup_{\zeta \to z} \sup\{u(\zeta) : u \in \mathcal{L}(\mathbb{C}), u|_K \le 0\}. \tag{2.2.2}$$

Here $\mathcal{L}(\mathbb{C})$ is the *Lelong class* of all subharmonic function u having a logarithmic pole at infinity, e.g., for any neighbourhood of ∞ the function $u - \log|z|$ is bounded above. Moreover, it is very useful to our aims to recall that one can replace the upper envelope of (2.2.2) with the following.

$$V_K = \sup\left\{\frac{1}{\deg p}\log|p|, p \in \mathscr{P}, \|p\|_K \le 1\right\} =: \exp \Phi_K(z).$$

Hence in particular one has the *Bernstein Walsh Inequality*

$$|p(z)| \le \|p\|_K \exp(\deg p \, V_K(z)). \tag{2.2.3}$$

It turns out (see for instance [**85**, Ch. 4] of [**68**, Sec. 3]) that the *extremal subharmonic function* $V_K^*(z) := \limsup_{\zeta \to z} V_K(\zeta)$ coincides with $g_K(z, \infty)$. Moreover, it follows by the Frostman Theorem that

$$g_K(z,\infty) = -U^{\mu_K}(z) - \log \operatorname{cap}(K).$$

Thus in particular $\Delta g_K(z,\infty) = \mu_K$.

Finally we recall some nice properties of g_K under mappings. Let $w \in \Omega_K \setminus \{\infty\}$ and set $\eta_w(z) := \frac{1}{z-w}$, we have $G_{\Omega_K}(z,w) = G_{\eta_w(\Omega_K)}(z,\infty)$. More in general one has

$$G_{D'}(f(z), f(w)) \ge G_D(z,w)$$

for any meromorphic function f of the domain D to the domain D' and any $z, w \in D$. *Equality* holds for conformal mappings.

2.2.2. Convergence of capacities and Green functions. The aim of this subsection is to relate the convergence of logarithmic capacities of a sequence of compact subsets of a given compact set K to the convergence of Green functions, where we allow the poles to move in a compact set $P \subset \Omega_K$; precisely, we have the following.

THEOREM 2.2.4. *Let $K \subset \mathbb{C}$ be a regular compact set and P a compact subset of Ω_K. Then there exists an open bounded set D such that $K \subset D$ and $P \cap \overline{D} = \emptyset$, such that for any sequence $\{K_j\}$ of compact subsets of K the following are equivalent.*

$$\begin{array}{lll} (i) & \lim_j \operatorname{cap}(K_j) = & \operatorname{cap}(K). \\ (ii) & \lim_j g_{K_j}(z,a) = & g_K(z,a) \quad \textit{loc. unif. for } z \in D\ ,\ \textit{unif. for } a \in P \quad . \end{array}$$

In order to prove Theorem 2.2.4 we need the following proposition.

PROPOSITION 2.2.1. *Let $K \subset \mathbb{C}$ be a regular compact set and $\{K_j\}$ a sequence of compact subsets of K, let D be a smooth bounded domain such that $K \subset D$ and $f : D \to \mathbb{C}$ a bi-holomorphism on its image. Suppose that* $\lim_j \operatorname{cap}(K_j) = \operatorname{cap}(K)$. *Then*

i) $g_{K_j}(z,\infty) \to g_K(z,\infty)$ *locally uniformly,*
ii) $g_{f(K_j)}(z,\infty) \to g_{f(K)}(z,\infty)$ *locally uniformly and*
iii) $\lim_j \operatorname{cap}(f(K_j)) = \operatorname{cap}(f(K))$.

PROOF. It follows by the hypothesis on convergence of capacities that $\mu_{K_j} \rightharpoonup^* \mu_K$, see for instance [**97**, Proof of Th. 4.2.3].

Let us pick any sequence $\{z_j\}$ of complex numbers converging to $\hat{z} \in \mathbb{C}$, it follows by the Principle of Descent [**91**, Th. 6.8], see Theorem 2.2.1, that

$$\limsup_j -U^{\mu_{K_j}}(z_j) \leq -U^{\mu_K}(\hat{z}).$$

On the other hand, due to regularity of K, the fact that $K_j \subset K$ for all j and since by assumption the sequence $-\log \operatorname{cap}(K_j)$ does have limit, we have

$$\begin{array}{rl} & g_K(\hat{z},\infty) \\ = & \liminf_j g_K(z_j,\infty) \leq \liminf_j g_{K_j}(z_j,\infty) \leq \limsup_j g_{K_j}(z_j,\infty) \\ = & \limsup_j -U^{\mu_{K_j}}(z_j) - \log \operatorname{cap}(K_j) \\ = & \limsup_j -U^{\mu_{K_j}}(z_j) - \log \operatorname{cap}(K) \\ \leq & -U^{\mu_K}(\hat{z}) - \log \operatorname{cap}(K) = g_K(\hat{z},\infty). \end{array} \tag{2.2.4}$$

Thus equality holds, moreover, since the sequence and the limit point are arbitrary we get $g_{K_j}(\cdot,\infty) \to g_K(\cdot,\infty)$ locally uniformly in $\mathbb{C}$. Indeed, we can pick any compact set $L \subset \mathbb{C}$ and any maximizing[1] sequence $\{z_j\}$ of points in L for $|g_{K_j}(z,\infty) - g_K(z,\infty)|$, i.e., $g_{K_j}(z_j,\infty) - g_K(z_j,\infty) = \max_{z\in L} g_{K_j}(z,\infty) - g_K(z,\infty)$, and notice that extracting a converging subsequence of $z_{j_k} \to \hat{z} \in L$ and relabelling indexes we have

$$\begin{aligned}
&\limsup_j \|g_{K_j}(z,\infty) - g_K(z,\infty)\|_L = \limsup_j |g_{K_j}(z_j,\infty) - g_K(z_j,\infty)| \\
&\leq \limsup_j |g_{K_j}(z_j,\infty) - g_K(\hat{z},\infty)| + \limsup_j |g_K(z_j,\infty) - g_K(\hat{z},\infty)| \\
&= \limsup_j |g_{K_j}(z_j,\infty) - g_K(\hat{z},\infty)| = 0.
\end{aligned}$$

Here we used both the continuity of $g_K(\cdot,\infty)$ and (2.2.4).

Now we introduce some tools that are classical in (pluri-)potential theory in several complex variables. The one variable counterparts of these notions are just normalizations by a negative scaling factor: this leads to consider sup in place of inf and superharmonic functions in place of subharmonic. We choose this setting because it is easier to provide a proof of the above statement in this notation; we refer the reader to [**91**, Ch. II.5] for the one variable definitions and properties.

We pick a domain D containing K and we define the relative extremal subharmonic function

$$U^*_{K,D}(z) := \limsup_{\zeta\to z} \sup\{u(\zeta) \in shm(D), u \leq 0,\ u|_K \leq -1\}. \tag{2.2.5}$$

Here $shm(D)$ stands for the set of subharmonic functions on D. This is a subharmonic function on D whose distributional Laplacian is a positive measure supported on K, moreover $U^*_{K,D}(z) = -1$ q.e. on K for an arbitrary compact set K and $U^*_{K,D}|_K \equiv -1$ for any regular compact set K; see [**13**]. The reader is invited to compare this to the Green potential of the condenser $(K, \partial D)$ in [**91**, Ch. II.5].

The function $U^*_{K,D} - 1$ is a maximizer for the following variational problem that defines the *relative capacity* of K in D.

$$\operatorname{cap}(K,D) := \sup\left\{\int_K \Delta u : u \in shm(D,[0,1])\right\}, \tag{2.2.6}$$

namely one has $\operatorname{cap}(K,D) = \int_K \Delta U^*_{K,D} = \int_K -U^*_{K,D}\Delta U^*_{K,D}$.

[1]Notice that $g_{K_j}(z,\infty) \leq g_K(z,\infty)$ at any $z \in \mathbb{C}$ and, by the continuity of $g_K(\cdot,\infty)$, the function $|g_{K_j}(z_j,\infty) - g_K(z_j,\infty)| = g_{K_j}(z_j,\infty) - g_K(z_j,\infty)$ is upper semi continuous, thus it achieves its maximum on L.

Now we show that $U^*_{K_j,D} \to U^*_{K,D}$ uniformly on D.

On one hand, by the definition (2.2.5) above, we have

$$U^*_{K,D}(z) \le U^*_{K_j,D}(z),$$
$$U^*_{K_j,D}(z) - \max_K U^*_{K_j,D} - 1 \le U^*_{K,D}(z)\ \forall z \in D$$

and thus

$$0 \le U^*_{K_j,D}(z) - U^*_{K,D}(z) \le \max_K U^*_{K_j,D} + 1\ \forall z \in D. \tag{2.2.7}$$

On the other hand, by the estimate $g_{K_j}(z,\infty) \ge \inf_{\zeta\in\partial D} g_{K_j}(\zeta,\infty)(U^*_{K_j,D}(z)+1)$ for all $z \in D$ (see [**59**, Prop. 5.3.3]), it follows that

$$-1 \le U^*_{K_j,D}(z) \le \frac{g_{K_j}(z,\infty)}{\inf_{\zeta\in\partial D} g_{K_j}(\zeta,\infty)} - 1\ \forall z \in D.$$

Note that the right hand side of the above inequality converges uniformly on K to $-1 \equiv U^*_{K,D}$ since we proved that $g_{K_j}(z,\infty) \to g_K(z,\infty)$ locally uniformly and hence $\inf_{w\in\partial D} g_{K_j}(w,\infty) \to \inf_{w\in\partial D} g_K(w,\infty)$. We get $\max_K U^*_{K_j,D} + 1 \to 0$ locally uniformly on K and finally, due to (2.2.7), $U^*_{K_j,D} \to U^*_{K,D}$ locally uniformly on D.

It follows by the above convergence that $\mathrm{cap}(K_j, D) \to \mathrm{cap}(K, D)$ as well. To show that we simply pick $\varphi \in \mathscr{C}^\infty_c(D, [0,1])$ such that $\varphi \equiv 1$ in a neighbourhood of K and we write

$$\begin{aligned}\mathrm{cap}(K,D) &= \int_K \Delta U^*_{K,D} = \int_D \varphi\Delta U^*_{K,D} = \int_D \varphi\Delta U^*_{K,D}\\ &= \lim_j \int_D \varphi\Delta U^*_{K_j,D} = \lim_j \mathrm{cap}(K_j, D).\end{aligned}$$

Now we note that, given a biholomorphism f of D on the smooth domain $f(D) = \Omega \subset \mathbb{C}$ there is a one to one correspondence between functions in $\{u \in shm(D) : u \le 0, u|_G \le -1\}$ and $\{v \in shm(\Omega) : v \le 0, v|_{f(G)} \le -1\}$ for any compact set $G \subset D$. For this reason, setting $F = f(K)$ and $F_j = f(K_j)$, one has $U^*_{F_j,\Omega} \equiv U_{K_j,D} \circ f$ and $U^*_{F,\Omega} \equiv U_{K,D} \circ f$. Therefore we have

$$U^*_{F_j,\Omega} \to U^*_{F,\Omega} \text{ locally uniformly in } \Omega. \text{ and}$$

$$\mathrm{cap}(F_j,\Omega) \to \mathrm{cap}(F,\Omega).$$

Let us recall that we can find a constant $A > 0$ such that $\sup_\Omega g_{F_j}(z,\infty) \le \frac{A}{\mathrm{cap}(F_j,\Omega)}$ for each subset F_j of the compact set F; see [**2**]. Thus we can pick j_0 such that, for $j \ge j_0$, we have $\sup_\Omega g_{F_j}(z,\infty) \le \frac{2A}{\mathrm{cap}(F,\Omega)} = M$.

It follows by the definition of relative extremal function that we have

$$0 \le \frac{g_{F_j}(z,\infty)}{M} - 1 \le U^*_{F_j,\Omega}(z), \forall j > j_0, \forall z \in \Omega.$$

But since the right hand side converges uniformly to -1 on F we get that $g_{F_j}(z,\infty) \to 0$ uniformly on F. Note that the same reasoning shows that in particular $g_F(z,\infty) \equiv 0$ on F, that is F is regular.

In particular for any $\epsilon > 0$ we can pick j_ϵ such that for any $j > j_\epsilon$ we have $g_{F_j}(z,\infty) - \epsilon \le 0 \equiv g_F(z,\infty)$ for any $z \in F$.

Hence, we get $g_{F_j}(z,\infty) - \epsilon \le g_F(z,\infty), \forall j > j_\epsilon, z \in \mathbb{C}$.

On the other hand, $g_F(z,\infty) \le g_{F_j}(z,\infty), \forall j \in \mathbb{N}, z \in \mathbb{C}$, since $F_j \subset F$. Therefore we have $g_{F_j}(z,\infty) \to g_F(z,\infty)$ locally uniformly in $\mathbb{C}$.

It follows by this uniform convergence that $\mu_{F_j} \rightharpoonup^* \mu_F$ (note that $\mu_F = \Delta g_F(z,\infty)$ and the distributional Laplacian, by linearity, is continuous under the local uniform convergence) and thus $U^{\mu_F} = \lim_j U^{\mu_{F_j}}$ uniformly on compact sets of $\mathbb{C} \setminus f(K)$ (by the uniform continuity of the log kernel away from 0), thus in particular $U^{\mu_F}(\hat{z}) = \lim_j U^{\mu_{F_j}}(\hat{z})$ for any given $\hat{z} \in \mathbb{C} \setminus F$.

Now we have, for any $\hat{z} \in \mathbb{C} \setminus F$

$$\begin{aligned} & -\log \operatorname{cap}(F_j) \\ =& g_{F_j}(\hat{z},\infty) + U^{\mu_{F_j}}(\hat{z}) \to g_F(\hat{z},\infty) + U^{\mu_F}(\hat{z}) \\ =& -\log \operatorname{cap}(F). \end{aligned}$$

□

Proof of Theorem 2.2.4. By Hilbert Lemniscate Theorem for any $\epsilon < d(K,P) := \inf_{z\in K} d(z,P)$ we can pick a polynomial q such that

$$\hat{K} \subset D := \{|q| < \|q\|_K\} \subset \hat{K}^\epsilon \ , \ \hat{K}^\epsilon \cap P = \emptyset.$$

Let D be fixed in such a way.

We introduce a more concise notation for the Green functions involved in the proof: we denote by $g(z,a)$ the Green function with pole at a for the set Ω_K, we omit the pole when $a = \infty$, we add a subscript j if K is replaced by K_j and a superscript b if K or K_j are replaced by $\eta_b(K)$ or $\eta_b(K_j)$, where $\eta_b(z) := 1/(z-b)$.

In symbols

$$\begin{aligned} g(z) &:= g_K(z,\infty) \quad , \quad g_j(z,a) := g_{K_j}(z,a), \\ g_j(z) &:= g_{K_j}(z,\infty) \quad , \quad g^b(z,a) := g_{\eta_b K}(z,a), \\ g(z,a) &:= g_K(z,a) \quad , \quad g_j^b(z,a) := g_{\eta_b K_j}(z,a). \end{aligned}$$

Moreover we set $E_j := \eta_{a_j}(K_j)$ and $E := \eta_{\hat a}(K)$.

Proof of $(i) \Rightarrow (ii)$. In order to prove the local uniform convergence of $g_j(\cdot,a)$ to $g(\cdot,a)$, uniformly with respect to $a \in P$, we pick any converging sequence $P \ni a_j \to \hat a$, we set $\tilde D := \eta_{\hat a}(D)$ and we prove

$$g_j^{a_j} \to g^a \ \text{ loc. unif. in } \tilde D. \tag{2.2.8}$$

Finally we notice that $g_j(\cdot,a_j) = g_j^{a_j} \circ \eta_{a_j}^{-1} \to g^a \circ \eta_{\hat a}^{-1} = g(\cdot,\hat a)$ loc. unif. in D hence the result follows.

We proceed along the following steps:

$$\lim_j \operatorname{cap}(E_j) = \operatorname{cap}(E). \tag{S1}$$

$$\mu_{E_j} \rightharpoonup^* \mu_E. \tag{S2}$$

$$\lim_j g_{E_j}(z,\infty) = g_E(z,\infty), \ \text{ loc. unif. in } \mathbb{C}. \tag{S3}$$

By the above argument, (S3) implies in particular (ii).

To prove (S1) we use [**85**, Th. 5.3.1] applied to the set of maps $\varphi_j := \eta_{a_j} \circ \eta_{\hat a}^{-1}$ and $\psi_j := \varphi_j^{-1}$ together with the assumption (i). Each map is bi-holomorphic on a neighbourhood of $\tilde D$, moreover we have

$$\begin{aligned} \|\varphi_j'\|_{\eta_{\hat a}(K)} &= \max_{\zeta \in \eta_{\hat a}(K)} \frac{1}{\left|1 + (\hat a - a_j)\zeta\right|^2} \\ &\le \max_K \frac{|z-\hat a|^2}{|z-a_j|^2} \le 1 + \frac{|\hat a - a_j|^2}{|\operatorname{dist}(K,P)|^2} =: L_j. \end{aligned} \tag{2.2.9}$$

$$\begin{aligned} \|\psi_j'\|_{\eta_{\hat a_j}(K_j)} &= \left(\min_{\zeta \in \eta_{a_j}(K_j)} \left|1 + (a_j - \hat a)\zeta\right|\right)^{-2} \\ &\le \max_{K_j} \frac{|z-a_j|^2}{|z-\hat a|^2} \le 1 + \frac{|\hat a - a_j|^2}{|\operatorname{dist}(K,P)|^2} = L_j. \end{aligned} \tag{2.2.10}$$

We denoted by $\operatorname{dist}(K,H) := \inf\{\epsilon > 0 : K^\epsilon \supseteq H\,,\, H^\epsilon \supseteq K\}$ the Hausdorff distance of K and H. Notice that $L_j \to 1$ as $j \to \infty$.

We recall that $\operatorname{cap}(f(E)) \le \operatorname{Lip}_E(f)\operatorname{cap}(E)$, where $\operatorname{Lip}_E(f) := \inf\{L : |f(x) - f(y)| < L|x-y|\, \forall x,y \in E\}$ for any Lipschitz mapping $f : E \to \mathbb{C}$; [**85**][Th. 5.3.1].

Therefore, due to (2.2.9) and (2.2.10), we have the following upper bounds.

$$\operatorname{cap}(E_j) = \operatorname{cap}(\varphi_j(\eta_{\hat{a}}(K_j))) \leq L_j \operatorname{cap}(\eta_{\hat{a}}(K_j)),$$
$$\operatorname{cap}(\eta_{\hat{a}}(K_j)) = \operatorname{cap}(\eta_{\hat{a}} \circ \eta_{a_j}^{-1}(E_j)) = \operatorname{cap}(\psi_j(E_j)) \leq L_j \operatorname{cap}(E_j).$$

Thus, using $\lim_j L_j = 1$, we have

$$\liminf_j \operatorname{cap}(E_j) \geq \liminf_j \frac{1}{L_j} \operatorname{cap}(\eta_{\hat{a}}(K_j)) = \liminf_j \operatorname{cap}(\eta_{\hat{a}}(K_j)), \tag{2.2.11}$$
$$\limsup_j \operatorname{cap}(E_j) \leq \limsup_j L_j \operatorname{cap}(\eta_{\hat{a}}(K_j)) \leq \limsup_j \operatorname{cap}(\eta_{\hat{a}}(K_j)). \tag{2.2.12}$$

Now we use Proposition 2.2.1 to get $\lim_j \operatorname{cap}(\eta_{\hat{a}}(K_j)) = \operatorname{cap}(\eta_{\hat{a}}(K))$ and thus

$$\frac{1}{L_j} \liminf_j \operatorname{cap}(E_j) \geq \lim_j \operatorname{cap}(\eta_{\hat{a}}(K_j)) \geq \frac{1}{L_j} \limsup_j \operatorname{cap}(E_j).$$

But since $L_j \to 1$ all inequalities are equality and $\lim_j \operatorname{cap}(E_j) = \operatorname{cap}(E)$; this concludes the proof of (S1).

The proof of (S2) is by the Direct Method of Calculus of Variation. More explicitly, let $\mu_j := \mu_{E_j}$ be the sequence of equilibrium measures, i.e., the minimizers of $I[\cdot]$ among the classes $\mu \in \mathcal{M}_1(E_j)$. From (S1) it follows that $\liminf_j I[\mu_j] = I[\mu_E]$. Therefore, if μ is any weak* closure point of the sequence, by lower semicontinuity of I, we get $I[\mu] \leq I[\mu_E]$.

Notice that without loss of generality we can assume K_j, and thus E_j, to be not polar, since $\operatorname{cap}(K_j) > 0$ for j large enough.

If $\operatorname{supp} \mu \subseteq E$, by the strict convexity of the energy functional, we have that $\mu = \mu_E$ and the whole sequence is converging to μ_E; see [**91**, Part I, Th. 1.3]. Then we are left to prove $\operatorname{supp} \mu \subseteq E$, this follows by the uniform convergence of η_{a_j} to $\eta_{\hat{a}}$ and by properties of weak* convergence of measures.

To this aim, we suppose by contradiction $\operatorname{supp} \mu \cap (\mathbb{C} \setminus E) \neq \emptyset$. It follows that there exists a Borel set $B \subset \mathbb{C} \setminus E$ with $\mu(B) > 0$. Since μ is Borel we can find a closed set $C \subset B$ still having positive measure. Being $\mathbb{C}$ a metric space and we can find an open neighbourhood A of C disjoint by E with $\mu(A) > 0$.

Due to the Portemanteau Theorem (see for instance [**17**, Th. 2.1]) we have

$$0 < \mu(A) \leq \liminf_j \mu_j(A).$$

Therefore $C \subseteq A \subset E_{j_m}$ for an increasing subsequence j_m.

By the uniform convergence $\eta_{a_{j_m}} \to \eta_{\hat{a}}$ it follows that $C \subseteq A \subseteq E$, a contradiction since we assumed $C \cap E = \emptyset$.

Let us prove (S3).

First, we recall (see for instance [**91**, pg. 53]) that for any compact set $M \subset \mathbb{C}$ we have $g_M(z,\infty) = -\log \operatorname{cap}(M) - U^{\mu_M}(z)$. Hence it follows that

$$g_{E_j}(\zeta,\infty) = -\log \operatorname{cap}(E_j) - U^{\mu_j}(\zeta). \tag{2.2.13}$$

Due to (S2) and by the Principle of Descent 2.2.1 for any $\zeta \in \mathbb{C}$ we have

$$\limsup_j -U^{\mu_j}(\zeta) \leq -U^{\mu_E}(\zeta). \tag{2.2.14}$$

It follows by (S1),(2.2.13) and (2.2.14) that

$$\limsup_j g_{E_j}(\zeta,\infty) \leq g_E(\zeta,\infty),\ \forall \zeta \in \mathbb{C}.$$

The sequence of subharmonic functions $\{g_{E_j}(\zeta,\infty)\}$ is locally uniformly bounded above and non negative, therefore we can apply the Hartog's Lemma. For each $\epsilon > 0$ there exists $j(\epsilon) \in \mathbb{N}$ such that

$$\|g_{E_j}(\zeta,\infty)\|_E \leq \|g_E(\zeta,\infty)\|_E + \epsilon = \epsilon.$$

Here the last equality is due to the regularity of K and thus of E (e.g. $g_E(\zeta,\infty) \equiv 0$ $\forall \zeta \in E$). Therefore we have

$$g_{E_j}(\zeta,\infty) - \epsilon \leq g_E(\zeta,\infty)\ ,\ \forall \zeta \in E. \tag{2.2.15}$$

By the extremal property of the Green function (see (2.2.2) and lines below) and the upper bound (2.2.15) it follows that

$$g_{E_j}(\zeta,\infty) - \epsilon \leq g_E(\zeta,\infty)\ ,\ \forall \zeta \in \mathbb{C}, j \geq j(\epsilon). \tag{2.2.16}$$

Since $g_E(\cdot,\infty)$ is continuous (hence uniformly continuous on a compact neighbourhood M of E containing all E_j) for any $\epsilon > 0$ we can pick $\delta > 0$ such that $g_E(\zeta,\infty) \leq \epsilon$ for any $\zeta \in E^\delta$.

Let us set $j'(\epsilon) := \min\{\bar{j} : E_j \subseteq E^\delta \forall j \geq \bar{j}\}$, notice that $j'(\epsilon) \in \mathbb{N}$ for any (sufficiently small) $\epsilon > 0$ since

$$E_j \subset \eta_{a_j}(K) \subseteq L_j \eta_{\hat{a}}(K) = L_j E \subseteq E^{(L_j-1)\|z\|_E},$$

where L_j is defined in equations (2.2.9) (2.2.10) and $L_j \to 1$.

It follows by this choice that

$$\|g_E(\zeta,\infty)\|_{E_j} \leq \epsilon,\ \forall j \geq j'(\epsilon).$$

Therefore, again by the extremal property of $g_{E_j}(\zeta,\infty)$, we have

$$g_E(\zeta,\infty)-\epsilon\leq g_{E_j}(\zeta,\infty),\ \ \forall\zeta\in\mathbb{C}, j\geq j'(\epsilon). \tag{2.2.17}$$

Now simply observe that (2.2.17) and (2.2.16) imply

$$g_E(\zeta,\infty)-\epsilon\leq g_{E_j}(\zeta,\infty)\leq g_E(\zeta,\infty)+\epsilon\ ,\ \ \forall j\geq\max\{j(\epsilon),j'(\epsilon)\}.$$

Therefore $g_{E_j}(\cdot,\infty)$ converges locally uniformly to $g_E(\cdot,\infty)$.

To conclude the proof of $(i)\Rightarrow(ii)$ let us pick any compact subset L of D.

$$\begin{aligned}&\|g_j^{a_j}-g^{\hat a}\|_L=\|g_{E_j}(\eta_{a_j}(z),\infty)-g_E(\eta_{\hat a}(z),\infty)\|_L\leq\\&\|g_{E_j}(\eta_{a_j}(z),\infty)-g_E(\eta_{a_j}(z),\infty)\|_L+\|g_E(\eta_{a_j}(z),\infty)-g_E(\eta_{\hat a}(z),\infty)\|_L\to 0\end{aligned}$$

Here we used the continuity of $g_E(z,\infty)$ and the local uniform convergence of η_{a_j} to $\eta_{\hat a}$. By the arbitrariness of the sequence of poles $\{a_j\}$ (ii) follows.

Proof of $(ii)\Rightarrow(i)$. Fix any pole $a\in P$ and set $\eta_a(z):=\frac{1}{z-a}$, $E:=\eta_a(K)$, $E_j:=\eta_a(K_j)$, by our assumption we have $g_j^a\to g^a$ locally uniformly in $\mathbb{C}$ thus

$$g_{E_j}(\cdot,\infty)\to g_E(\cdot,\infty),$$

uniformly on some neighbourhood D of E (where η_a^{-1} is a biholomorphism on its image).

It follows that $\mu_{E_j}\rightharpoonup^*\mu_E$. Let us pick a point $\hat z\in D\setminus E$, by uniform continuity of the log kernel away from 0 we have $U^{\mu_{E_j}}(\hat z)\to U^{\mu_E}(\hat z)$. On the other hand $g_{E_j}(\hat z,\infty)\to g_E(\hat z,\infty)$, therefore we have

$$\begin{aligned}&\lim_j(-\log\operatorname{cap}(E_j))=\lim_j(g_{E_j}(\hat z,\infty)+U^{\mu_{E_j}}(\hat z))\\=&g_E(\hat z,\infty)+U^{\mu_E}(\hat z)=-\log\operatorname{cap}(E),\end{aligned}$$

where existence of the limit is part of the statement and follows by the existence of the limits of the two terms of the sum.

We apply Proposition 2.2.1 with $f:=\eta_a^{-1}$ to get $-\log\operatorname{cap}(K_j)\to-\log\operatorname{cap}(K)$. $\square$

2.3. Relations among Bernstein Markov Properties

It is rather natural to ask which are the relations among the different Bernstein Markov properties we defined. In this section we relate the sub-diagonal rational Bernstein Markov property and the rational Bernstein Markov property to the weighted Bernstein Markov property with respect to a specific class of weights in Proposition 2.3.2 and 2.3.3. Namely, for any compact set P we introduce the following notation

$$\begin{aligned}\mathcal{W}(P) &:= \{e^{U^\sigma} : \sigma \in \mathcal{M}^+(P), 0 \le \sigma(P) < \infty\}\,,\\ \mathcal{W}_1(P) &:= \{e^{U^\sigma} : \sigma \in \mathcal{M}_1^+(P)\},\end{aligned}$$

where $U^\sigma(z) := -\int \log|z-\zeta| d\sigma(\zeta)$ is the logarithmic potential of the measure σ and we set by definition $U^0 \equiv 0$. This approach will allow to prove (see Theorem 2.3.5) that, under certain further assumption on K and P, the Bernstein Markov Property implies the rational one.

Proposition 2.3.2. *Let $K \subset \mathbb{C}$ be a non polar compact set, $\mu \in \mathcal{M}^+(K)$ and P any compact set disjoint by K. Then the following are equivalent*

(i) $\forall w \in \mathcal{W}_1(P)$ the triple $[K, \mu, w]$ has the weighted Bernstein Markov Property.

(ii) (K, μ, P) has the sub-diagonal rational Bernstein Markov Property.

Proof of (i) implies (ii). Let us pick a sequence $\{r_k\} = \{p_k/q_k\}$ in $\mathcal{Q}(P)$, where $q_k := \prod_{j=1}^k (z - z_j)$, and let us set $\sigma_k := \frac{1}{k}\sum_{j=1}^k \delta_{z_j}$. Then we can notice that

$$U^{\sigma_k} = \int \log \frac{1}{|z-\zeta|} d\sigma_k(\zeta) = \frac{1}{k}\sum_{j=1}^k \log\frac{1}{|z-z_j|} = -\frac{1}{k}\log|q_k|.$$

Thus, setting $U_k := U^{\sigma_k}$, we have

$$a_k := \left(\frac{\|r_k\|_K}{\|r_k\|_{L^2_\mu}}\right)^{1/k} = \left(\frac{\|p_k e^{(kU_k)}\|_K}{\|p_k e^{(kU_k)}\|_{L^2_\mu}}\right)^{1/k}. \tag{2.3.1}$$

Now we pick any maximizing subsequence $j \mapsto k_j$ for a_k, that is $\limsup_k a_k = \lim_j a_{k_j}$. Let us pick any weak* limit $\sigma \in \mathcal{M}_1^+(P)$ and a subsequence $l \mapsto j_l$ such that $\tilde\sigma_l := \sigma_{k_{j_l}} \rightharpoonup^* \sigma$. Moreover $\lim_l b_l := \lim_l a_{k_{j_l}} = \limsup_k a_k$.

Let us notice that $U := U^\sigma$ and all $U_l := U^{\tilde\sigma_l}$ are harmonic functions on $\mathbb{C} \setminus P$, moreover, due to [**91**, Th. 6.9 I.6], $\{U_l\}$ converges quasi everywhere to U. Notice that $U^{\tilde\sigma_l} := -E * \tilde\sigma_l$, where $E(z) := \log|z|$ is a locally absolutely continuous function on $\mathbb{C} \setminus \{0\}$, hence weak convergence of measures supported on P implies local uniform convergence of potentials on $\mathbb{C} \setminus P$.

We can exploit this uniform convergence as follows. For any $\epsilon > 0$ there exists l_ϵ such that for any $l > l_\epsilon$ we have

$$U - \epsilon \le U_l \le U + \epsilon \ \text{ uniformly on } K. \tag{2.3.2}$$

Now we denote k_{j_l} by $\tilde{k}_l$ and $p_{\tilde{k}_l}$ by $\tilde{p}_l$. It follows by (2.3.2) that for l large enough

$$\begin{aligned}
&\|\tilde{p}_l e^{\tilde{k}_l U_l}\|_K \leq \|\tilde{p}_l e^{\tilde{k}_l (U+\epsilon)}\|_K \leq e^{\tilde{k}_l \epsilon}\|\tilde{p}_l e^{\tilde{k}_l U}\|_K,\\
&\|\tilde{p}_l e^{\tilde{k}_l U_l}\|_{L^2_\mu} \geq \|\tilde{p}_l e^{\tilde{k}_l (U-\epsilon)}\|_{L^2_\mu} \geq e^{-\epsilon \tilde{k}_l}\|\tilde{p}_l e^{\tilde{k}_l U}\|_{L^2_\mu} \text{ and thus}\\
&\frac{\|\tilde{p}_l e^{\tilde{k}_l U_l}\|_K}{\|\tilde{p}_l e^{\tilde{k}_l U_l}\|_{L^2_\mu}} \leq e^{2\tilde{k}_l \epsilon}\frac{\|\tilde{p}_l e^{\tilde{k}_l U}\|_K}{\|\tilde{p}_l e^{\tilde{k}_l U}\|_{L^2_\mu}}.
\end{aligned}$$

Hence, exploiting $w := e^U \in \mathcal{W}_1(P)$ and μ having the weighted Bernstein Markov property for such a weight, we have

$$\begin{aligned}
\limsup_k a_k &= \lim_l \left(\frac{\|\tilde{p}_l e^{\tilde{k}_l U_l}\|_K}{\|\tilde{p}_l e^{\tilde{k}_l U_l}\|_{L^2_\mu}}\right)^{1/\tilde{k}_l} \leq e^{2\epsilon}\lim_l \left(\frac{\|\tilde{p}_l e^{\tilde{k}_l U}\|_K}{\|\tilde{p}_l e^{\tilde{k}_l U}\|_{L^2_\mu}}\right)^{1/\tilde{k}_l}\\
&\leq e^{2\epsilon}\lim_l \left(\frac{\|\tilde{p}_l w^{\tilde{k}_l}\|_K}{\|\tilde{p}_l w^{\tilde{k}_l}\|_{L^2_\mu}}\right)^{1/\tilde{k}_l} = e^{2\epsilon} \longrightarrow 1 \quad \text{as } \epsilon \to 0.
\end{aligned}$$

□

To prove the reverse implication we need the following fact. *Let P be a compact set in $\mathbb{C}$ and σ a Borel measure supported on it having total mass equal to* 1. *There exists a sequence of arrays $\{(z_1^{(k)}, \dots, z_k^{(k)})\}$ of points of P such that we get*

$$\sigma_k := \frac{1}{k}\sum_{j=1}^k \delta_{z_j^{(k)}} \rightharpoonup^* \sigma. \tag{2.3.3}$$

To show that one picks a countable dense basis of $\mathscr{C}(K)$ made of functions uniformly bounded by one, then first produces a sequence of measures $\tilde{\sigma}_k := \sum_{j=1}^k \tilde{b}_j^k \delta_{\eta_j}$ with $\tilde{b}_j^k \in \mathbb{R}^+$ such that $\sum_{j=1}^k \tilde{b}_j^k = 1$ and $\int f_j d\mu = \int f_j d\tilde{\sigma}_k$ for all $j \leq k$. Then a weakly* converging subsequence can be extracted and it is possible to show that the limit coincides with σ. Finally the arrays $\{(z_1^{(k)}, \dots, z_k^{(k)})\}$ are constructed repeating each η_j m_j^k times such that $m_j^k/(\sum_{j=1}^k m_j^k)$ approximate $\tilde{b}_j^k$.

Proof of (ii) implies (i). Suppose by contradiction that there exists $\sigma \in \mathcal{W}_1(P)$ such that $[K, \mu, \exp U^\sigma]$ does not have the weighted Bernstein Markov Property.

We pick $\{z_1^{(k)}, \dots, z_k^{(k)}\}_{k=1,\dots}$ and $\sigma_k = \frac{1}{k}\sum_{j=1}^k \delta_{z_j^{(k)}}$ as in (2.3.3).

Let us set $w = \exp U^\sigma$, $w_k = \exp U^{\sigma_k}$. We can perform the same reasoning as above, using the absolute continuity of the log kernel away from 0, to get $U^{\sigma_k} \to U^\sigma$ uniformly on K. Thus for any $\epsilon > 0$ we have $U^{\sigma_k} - \epsilon \leq U^\sigma \leq U^{\sigma_k} + \epsilon$ uniformly on K for k large enough. That is

$$w_k e^{-\epsilon} \leq w \leq w_k e^{\epsilon} \quad \text{uniformly on } K \text{ for } k \text{ large enough.} \tag{2.3.4}$$

Notice that given any sequence $\{p_k\}$ such that $p_k \in \mathscr{P}^k$ we have

$$\{r_k\} := \{p_k w_k^k\} = \left\{\frac{p_k}{\prod_{j=1}^k (z - z_j)}\right\} \in Q(P).$$

Since we assumed that $[K, \mu, w]$ does not have the weighted Bernstein Markov property we can pick p_k such that, using (2.3.4),

$$1 < \limsup_k \left(\frac{\|p_k w^k\|_K}{\|p_k w^k\|_{L^2_\mu}}\right)^{1/k} \leq \limsup_k e^{2\epsilon} \left(\frac{\|p_k w_k^k\|_K}{\|p_k w_k^k\|_{L^2_\mu}}\right)^{1/k}$$
$$\leq e^{2\epsilon} \to 1 \text{ as } \epsilon \to 0.$$

This is a contradiction. □

We can prove the following variant of the previous proposition by some minor modifications of the proof.

Proposition 2.3.3. *Let $K \subset \mathbb{C}$ be a non polar compact set, $\mu \in \mathcal{M}^+(K)$ and P any compact set disjoint by K. Then the following are equivalent*

(i) $\forall w \in \mathcal{W}(P)$ the triple $[K, \mu, w]$ has the weighted Bernstein Markov Property.
(ii) (K, μ, P) has the rational Bernstein Markov Property.

Proof of (i) implies (ii). We pick an extremal sequence in $\mathcal{R}(P)$ (i.e., for a_k as in (2.3.1)) $r_k := \frac{p_{l_k}}{q_{m_k}}$, where $\deg p_{l_k} = l_k \leq k$ and $\deg q_{m_k} = m_k \leq k$.

We notice that

$$r_k = p_{l_k} e^{\left(m_k U^{\sigma_{m_k}}\right)} = p_{l_k} e^{\left(k U^{\frac{m_k}{k}\sigma_{m_k}}\right)} =: p_{l_k} e^{\left(k U^{\hat{\sigma}_k}\right)}, \text{ where}$$

σ_k are as in the previous proof. Notice that the sequence of measures $\{\hat{\sigma}_k\} := \{\frac{m_k}{k}\sigma_{m_k}\}$ has the property $\int_P d\hat{\sigma}_k \leq \int_P d\sigma_{m_k} = 1$ since $m_k/k \leq 1$.

By the local sequential compactness we can extract a subsequence (relabeling indeces) converging to any weak* closure point σ that necessarily is a Borel measure such that $\int_P d\sigma \leq 1$. Notice that σ can be also the zero measure: here is the main difference between this case and Proposition 2.3.2 where each weak* limit has the same positive mass.

Notice that $U^{\hat{\sigma}_k}$ converges to U^σ uniformly on K as in the previous proof, hence for any $\epsilon > 0$ we can pick k_ϵ such that for any $k > k_\epsilon$ we have

$$U^{\hat{\sigma}_k} - \epsilon \leq U^\sigma \leq U^{\hat{\sigma}_k} + \epsilon.$$

Therefore, seetting $w := U^\sigma$ we have

$$\begin{aligned} r_k e^{-k\epsilon} = p_{l_k} e^{\left(kU^{\hat\sigma_k}\right)} e^{(-k\epsilon)} \leq p_{l_k} e^{(kU^\sigma)} = p_{l_k} w^k \\ \leq p_{l_k} e^{\left(kU^{\hat\sigma_k}\right)} e^{(k\epsilon)} = r_k e^{k\epsilon}. \end{aligned} \tag{2.3.5}$$

The result follows by the same lines as in proof of Proposition 2.3.2, using the weighted Bernstein Markov property of $[K,\mu,w]$ $\forall w \in \mathcal{W}(P)$. □

Proof of (ii) implies (i). Pick σ such that $U^\sigma \in \mathcal{W}(P)$. If $\sigma = 0$ we notice that the rational Bernstein Markov property is stronger than the usual Bernstein Markov property.

If σ is not the zero measure we set $c := \int_P d\sigma$, $\hat\sigma = \sigma/c \in \mathcal{M}_1^+(P)$, and we pick a sequence of natural numbers $0 \leq m_k \leq k$ such that $\lim_k m_k/k = c$. We find $\sigma_k \in \mathcal{M}_1^+(P)$, $\sigma_k := (1/m_k)\sum_{j=1}^{m_k} \delta_{z_j^{(m_k)}}$ such that $\sigma_k \to^* \hat\sigma$ as in the previous proof, thus $\frac{m_k}{k}\sigma_k \to^* \sigma$.

It follows that

$$m_k U^{\sigma_k} + k\epsilon = k(\frac{m_k}{k}U^{\sigma_k} - \epsilon) \leq kU^\sigma \leq k(\frac{m_k}{k}U^{\sigma_k} - \epsilon) = m_k U^{\sigma_k} - k\epsilon, \tag{2.3.6}$$

for k large enough.

We can work by contradiction supposing that $[K,\mu,U^\sigma]$ does not satisfy the weighted Bernstein Markov property and following the same lines of the proof of (*ii*) implies (*i*) of the previous proposition using (2.3.6) instead of (2.3.4). □

Remark 2.3.1. *The combination of the two previous propositions proves in particular that if (K,μ,P) has the sub-diagonal rational Bernstein Markov property and (K,μ) has the Bernstein Markov property, it follows that (K,μ,P) has the rational Bernstein Markov property.*

On the other hand if (K,μ,P) has the sub-diagonal rational Bernstein Markov property but not the rational Bernstein Markov property, it follows that (K,μ) does not satisfy the Bernstein Markov property.

According to Proposition 2.3.3, our original question boils down to *whether the Bernstein Markov property implies the weighted Bernstein Markov property for any weight in the class $\mathcal{W}(P)$*. In the next theorem we give two possible sufficient conditions for that, corresponding to two different situations that are rather extremal in a sense. The reader is invited to compare them with situation of Example 1(a) and 1(b).

We denote by S_K the *Shilov boundary* of K with respect to the uniform algebra $\mathcal{P}(K)$ of functions that are uniform limits on K of entire functions (or equivalently polynomials). We recall that S_K is defined as the smallest closed subset B of K such that $\max_{z\in K}|f(z)| = \max_{z\in B}|f(z)|$ for all $f \in \mathcal{P}(K)$.

THEOREM 2.3.5. *Let $K \subset \mathbb{C}$ be a compact non polar set and $\mu \in \mathcal{M}^+(K)$ be such that* $\operatorname{supp}\mu = K$ *and (K,μ) has the Bernstein Markov Property. For a compact set $P \subset \mathbb{C}$ such that $K \cap P = \emptyset$, suppose that one of the following occurs.*

Case a: $$S_K = K.$$

Case b: $$\hat{K} \cap P = \emptyset.$$

Then the triple $[K,\mu,w]$ has the weighted Bernstein Markov Property with respect to any weight $w \in \mathcal{W}(P)$ and thus (K,μ,P) has the rational Bernstein Markov Property.

PROOF. Let us pick $\sigma \in \mathcal{M}^+(P)$ and set $w = \exp U^\sigma$, also we pick a sequence $\{p_k\}$, where $p_k \in \mathscr{P}^k$. We show that in both cases $[K,\mu,w]$ has the weighted Bernstein Markov Property with respect to any weight $w \in \mathcal{W}_1(P)$, the rest following by Proposition 2.3.3.

Case a. We first recall (see [**97**, Lemma 3.2.4 pg. 70]) that the set $\{|g| : g \in \mathscr{P}\}$ is dense in the cone of positive continuous functions on S_K, which w belongs to.

For any $\epsilon > 0$ we can pick $g_\epsilon \in \mathscr{P}^{m_\epsilon}$ such that

$$(1-\epsilon)|g_\epsilon| \leq w \leq (1+\epsilon)|g_\epsilon|. \tag{2.3.7}$$

Notice that $|g_\epsilon|^k = |g_\epsilon^k| = |\tau_{\epsilon,k}|$, where $\tau_{\epsilon,k} \in \mathscr{P}^{m_\epsilon k}$.

If for any $p_k \in \mathscr{P}^k$ we set $\tilde{p}_k := \tau_{\epsilon,k} p_k \in \mathscr{P}^{(m_\epsilon+1)k}$, then we have

$$\begin{aligned}
\|p_k w^k\|_K \leq &(1+\epsilon)^k \|\tau_{\epsilon,k} p_k\|_K = \|\tilde{p}_k\|_K,\\
\|p_k w^k\|_{L^2_\mu} \geq &(1-\epsilon)^k \|\tau_{\epsilon,k} p_k\|_{L^2_\mu} = \|\tilde{p}_k\|_{L^2_\mu}, \text{ and thus}\\
\left(\frac{\|p_k w^k\|_K}{\|p_k w^k\|_{L^2_\mu}}\right)^{1/k} \leq &\frac{1+\epsilon}{1-\epsilon}\left[\left(\frac{\|\tilde{p}_k\|_K}{\|\tilde{p}_k\|_{L^2_\mu}}\right)^{\frac{1}{(m_\epsilon+1)k}}\right]^{m_\epsilon+1}.
\end{aligned} \tag{2.3.8}$$

Using the polynomial Bernstein Markov property of (K,μ) and the arbitrariness of $\epsilon > 0$ we can conclude that $\limsup_k \left(\frac{\|p_k w^k\|_K}{\|p_k w^k\|_{L^2_\mu}}\right)^{1/k} \leq 1$.

Case b. Suppose first that $\hat{K}$ is connected, then it follows that there exists an open neighbourhood D of $\hat{K}$ which is a simply connected domain and $P \cap D = \emptyset$. We recall that any harmonic function on a simply connected domain is the real part of a holomorphic one. Hence, being U^σ harmonic on D, we can pick f holomorphic on D such that

$$w = \exp U^\sigma = \exp \Re f = |\exp f|. \tag{2.3.9}$$

Since $g := \exp f$ is an holomorphic function on D, by Runge Theorem, we can uniformly approximate it by polynomials g_ϵ on $\hat{K} := \{z \in \mathbb{C}, |p(z)| \leq \|p\|_K \, \forall p \in \mathscr{P}(\mathbb{C})\}$. Now we can conclude the proof by the same argument (2.3.8) and (2.3.9) of the *Case a* above.

If otherwise $\hat{K}$ is not known to be connected, we apply the following version of the Hilbert Lemniscate Theorem [**56**, Th. 16.5.6], given any open neighbourhood U of $\hat{K}$ not intersecting P we can pick a polynomial $s \in \mathscr{P}$ such that $|s(z)| > \|s\|_{\hat{K}} = \|s\|_K$ for any $z \in \mathbb{C} \setminus U$.

It follows that, picking a suitable positive δ, the set $E := \{|s| \leq \|s\|_K + \delta\}$ is a closed neighbourhood of $\hat{K}$ not intersecting P.

Notice that the set E has at most $\deg s$ connected components E_j and by definition it is polynomially convex. Moreover the Maximum Modulus Theorem implies that each $D_j := \operatorname{int} E_j$ is simply connected or the disjoint union of a finite number of simply connected domains that we do not relabel.

For any $j = 1, 2, \ldots, \deg s$ we set $w_j := w|_{D_j}$. We can find holomorphic functions f_j and g_j on D_j, continuous up to its boundary, such that $w_j = |\exp f_j| = |g_j|$.

Now notice that the function $g(z) = g_j(z) \, \forall z \in D_j$ is holomorphic on D and continuous on E, since D is the disjoint union of the sets D_j's. Hence we can apply the Mergelyan Theorem to find for any $\epsilon > 0$ a polynomial g_ϵ such that

$$(1 - \epsilon)|g_\epsilon(z)| \leq w(z) \leq (1 + \epsilon)|g_\epsilon(z)| \quad \forall z \in E \supseteq K.$$

We are back to the *Case a* and the proof can be concluded by the same lines. □

2.4. Mass Density Sufficient Condition for the Rational Bernstein Markov Property in $\mathbb{C}$

In the case of $K = \operatorname{supp} \mu$ being a regular set for the Dirichlet problem, the Bernstein Markov Property for (K, μ) is equivalent (cfr. [**20**, Th. 3.4]) to $\mu \in$ **Reg**. A positive Borel measure is in the class **Reg** or *has regular n-th root asymptotic* *behaviour* if for any sequence of polynomials $\{p_k\}$ one has

$$\limsup_k \left(\frac{|p_k(z)|}{\|p_k\|_{L^2_\mu}}\right)^{1/\deg p_k} \leq 1 \ \text{ for } z \in K \setminus N,\ N \subset K, N \text{ is polar.} \tag{2.4.1}$$

However, the definition can be given in terms of other equivalent conditions, see [**97**, Th. 3.1.1, Def. 3.1.2].

Moreover in [**97**, Th. 4.2.3] it has been proven that any Borel compactly supported finite measure having regular support $K \subset \mathbb{C}$ and enjoying a *mass density condition* (Λ^*-criterion [**97**, pag. 132]) is in the class **Reg**, consequently (K, μ) has the Bernstein Markov property. In order to fulfil such Λ^* condition a measure needs (roughly speaking) to be thick in a measure-theoretic sense on a subset of its support which has full logarithmic capacity. Precisely, the positive finite Borel measure μ having compact support K is said to satisfy the *mass density condition* Λ^* if there exists $t > 0$ such that

$$\lim_{r \to 0^+} \operatorname{cap}\left(\{z \in K : \mu(B(z, r)) > r^t\}\right) = \operatorname{cap}(K).$$

It is worth to say that, even if this Λ^* criterion is not known to be necessary for the Bernstein Markov property, in [**97**] authors show that the criterion has a kind of sharpness property and no counterexamples to the conjecture of Λ^* being necessary for the Bernstein Markov property are known. Moreover, this mass density sufficient condition has been extended (here the logarithmic capacity has been substituted by the relative Monge-Ampere capacity with respect to a ball containing the set K) to the case of several complex variables by Bloom and Levenberg [**24**].

Here we observe that under the hypothesis of Theorem 2.3.5 this condition turns out to be sufficient for the rational Bernstein Markov property as well; we state this in Theorem 2.4.6 then we generalize this result in Theorem 2.4.7.

Theorem 2.4.6 (Mass-density sufficient condition I). *Let $K \subset \mathbb{C}$ be a compact regular set and $P \subset \Omega_K$ be compact. Let $\mu \in \mathcal{M}^+(K)$, $\operatorname{supp} \mu = K$ and suppose that there exists $t > 0$ such that*

$$\lim_{r \to 0^+} \operatorname{cap}\left(\{z \in K : \mu(B(z, r)) \geq r^t\}\right) = \operatorname{cap}(K). \tag{2.4.2}$$

Then (K, μ, P) has the rational Bernstein Markov Property.

Short proof of Theorem 2.4.6. By [**97**, Th. 4.2.3] it follows that (K, μ) has the Bernstein Markov property, by Theorem 2.3.5 *Case b* we can conclude that the rational Bernstein Markov property holds for (K, μ, P) for any $P \subset \Omega_K$ as well. $\square$

We provide a direct proof of Theorem 2.4.6 by the convergence of Green functions result of Theorem 2.2.4.

Direct proof of Theorem 2.4.6. The proof follows the idea of [**97**, Th. 4.2.3], except for the lack of the Bernstein Walsh Inequality (2.2.3) which is not available for rational functions.

In place of it we use the following variant due to Blatt [**18**, eqn. 2.2] which holds for any rational function r_k of the form

$$r_k(\zeta) = \frac{p_k(\zeta)}{q_k(\zeta)} = \frac{c_k \prod_{j=0}^{m_k}(\zeta - z_j^{(k)})}{\prod_{j=0}^{n_k}(\zeta - a_j^{(k)})}.$$

For $\zeta \notin \{a_1, \ldots, a_{n_k}\}$ we have

$$|r_k(\zeta)| \leq \|r_k\|_K \exp\left(\sum_{j=1}^{n_k} g_K(\zeta, a_j) + (m_k - n_k) g_K(\zeta, \infty)\right). \tag{2.4.3}$$

Thus in particular we have

$$|r_k(\zeta)| \leq \|r_k\|_{K_j} \exp\left(n_k \max_{a\in P} g_{K_j}(\zeta, a) + (m_k - n_k) g_{K_j}(\zeta, \infty)\right) \ \forall \zeta \in \mathbb{C} \setminus P.$$

Notice that, for any sequence $K_j \subset K$ such that $\operatorname{cap} K_j \to \operatorname{cap} K$, from Theorem 2.2.4 it follows that

$$\max_{a\in P} g_{K_j}(\zeta, a) \to \max_{a\in P} g_K(\zeta, a) \ \text{ locally uniformly in } \mathbb{C} \setminus P.$$

Moreover, from Proposition 2.2.1 we have

$$g_{K_j}(\zeta, \infty) \to g_K(\zeta, \infty) \ \text{ locally uniformly in } \mathbb{C}.$$

Pick any $\{r_k\} \in \mathcal{R}(P)$. By the regularity of K and the compactness of P for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$g_K(\zeta, a) \leq \epsilon \ \ \forall \zeta : \operatorname{dist}(\zeta, K) \leq \delta, \ \forall a \in P$$
$$g_K(\zeta, \infty) \leq \epsilon \quad \forall \zeta : \operatorname{dist}(\zeta, K) \leq \delta.$$

Let us pick $\epsilon > 0$, it follows by (2.4.3) that there exists $\delta > 0$ such that $\forall \zeta$: $\operatorname{dist}(\zeta, K) \leq \delta$ we have

$$|r_k(\zeta)| \leq \|r_k\|_K e^{(n_k \max_{a\in P} g_K(\zeta, a) + (m_k - n_k) g_K(\zeta, \infty))} \leq e^{(k\epsilon)} \|r_k\|_K. \tag{2.4.4}$$

By Theorem 2.2.4 (possibly shrinking δ) we have, for any $A \subset K$, with $\operatorname{cap}(A) > \operatorname{cap}(K) - \delta$ and locally uniformly in $\mathbb{C} \setminus P$,

$$\max_{w \in P} g_A(\zeta, w) \leq \max_{w \in P} g_K(\zeta, w) + \epsilon \;, \tag{2.4.5}$$

$$g_A(\zeta, \infty) \leq g_K(\zeta, \infty) + \epsilon \;. \tag{2.4.6}$$

Using (2.4.4) and (2.4.6) we have

$$|r_k(\zeta)| \leq e^{(2\epsilon k)} \|r_k\|_A \quad \forall \zeta \in K^\delta,\ \forall A \subset K \text{ with } \operatorname{cap}(A) > \operatorname{cap}(K) - \delta. \tag{2.4.7}$$

Let $\zeta_0 \in A$ be such that $\|r_k\|_A = |r_k(\zeta_0)|$, we show that a lower bound for $|r_k|$ holds in a ball centred at ζ_0. By the Cauchy Inequality we have $|r_k'(\zeta)| < \frac{\|r_k\|_{\overline{B(\zeta_0, s)}}}{s} \leq \frac{e^{(2\epsilon k)}\|r_k\|_A}{s}$, for any $|\zeta - \zeta_0| < s$, $s < \delta$. Taking $s = \delta/2$ we can integrate such an estimates as follows $\forall z \in \overline{B(\zeta_0, \delta/2)}$

$$\|r_k\|_A = |r_k(\zeta_0)| = \left| r_k(z) + \int_{[z, \zeta_0]} r_k'(\zeta) d\zeta \right| \leq |r_k(z)| + |z - \zeta_0| \frac{e^{(2\epsilon k)} \|r_k\|_A}{\delta/2}.$$

It follows by the above estimate that

$$\min_{\overline{z \in B(\zeta_0, \frac{\delta e^{(-2\epsilon k)}}{4})}} |r_k(z)| \geq \frac{\|r_k\|_A}{2} \quad \forall A \subset K \text{ with } \operatorname{cap}(A) > \operatorname{cap}(K) - \delta. \tag{2.4.8}$$

Now we provide a lower bound for L^2_μ norms of r_k by integrating the last inequality on a (possibly smaller ball) and picking $A \subset K$ according to the mass density condition (2.4.13).

Precisely, set $\rho_k := e^{(-3k\epsilon)}$, by the hypothesis we can pick $t > 0$ and $A_k \subset K$ with $\operatorname{cap}(A_k) > \operatorname{cap}(K) - \delta$ such that $\mu(B_k) := \mu(\overline{B(\eta, \rho_k)}) \geq \rho_k^t \ \forall \eta \in A_k$. We pick $k \geq \bar{k}$ such that $\rho_k < \frac{\delta e^{(-2\epsilon k)}}{4}$, thus using (2.4.8) we get

$$\begin{aligned} \|r_k\|^2_{L^2_\mu} &\geq \int_{B_k} |r_k|^2 d\mu \geq \min_{z \in B_k} |r_k(z)|^2 \mu(B_k) \geq \frac{\|r_k\|^2_{A_k}}{4} \rho_k^t \\ &\geq \frac{e^{(-3tk\epsilon)}}{4} \|r_k\|^2_{A_k} \geq \frac{e^{(-(4+3t)k\epsilon)}}{4} \|r_k\|^2_K. \end{aligned}$$

It follows that $\left(\frac{\|r_k\|_K}{\|r_k\|_{L^2_\mu}} \right)^{1/k} \leq 4^{1/k} e^{((4+3t)\epsilon)}$, by arbitrariness of $\epsilon > 0$ we can conclude that

$$\limsup_k \left(\frac{\|r_k\|_K}{\|r_k\|_{L^2_\mu}} \right)^{1/k} \leq 1$$

□

If we remove the hypothesis $P \subset \Omega_K$, then Theorem 2.3.5 is no more applicable. We go around such a difficulty in the case $K \subset \Omega_P$ by a suitable conformal mapping f of a neighbourhood of $K \cup P$ given by the Proposition 2.4.4 below.

We recall, for the reader's convenience, the definitions of *Fekete points* and *transfinite diameter*. Given any compact set K in the complex plane, for any positive integer k, a set of Fekete points of order k is an array $\boldsymbol{z_k} = \{z_0, \dots, z_k\} \in K^k$ that maximizes the product of distances of its points among all such arrays, that is

$$V_k(\boldsymbol{z_k}) := \prod_{1 \le i < j \le k} |z_i - z_j| = \max_{\zeta \in K^k} \prod_{1 \le i < j \le k} |\zeta_i - \zeta_j|.$$

Notice that such maximizing array does not need to be unique.

It turns out that, denoting by $\delta_k(K) := \left(\max_{\zeta \in K^k} V_k(\zeta)\right)^{\frac{2}{k(k+1)}}$ the k-th diameter of K, we have

$$\lim_k \delta_k(K) =: \delta(K) = \operatorname{cap}(K), \tag{2.4.9}$$

where $\delta(K)$ is the *transfinite diameter* of K (existence of the limit being part of the statement). We refer the reader to [**85, 91, 90**] for further details.

Recall that we indicate by $\hat{E}$ the polynomial hull of the set E, see (2.2.1).

Proposition 2.4.4. *Let $K, P \subset \mathbb{C}$ be compact sets, where $K \cap \hat{P} = \emptyset$. Then there exist $w_1, w_2, \dots, w_m \in \mathbb{C} \setminus (K \cup \hat{P})$ and $R_2 > R_1 > 0$ such that denoting by f the function $z \mapsto \frac{1}{\prod_{j=1}^m (z - w_j)}$ we have*

$$K \subset\subset \{|f| < R_1\},$$
$$P \subset\subset \{R_1 < |f| < R_2\}.$$

Proof. We first suppose P to be not polar.

Moreover we show that we can suppose without loss of generality that

$$\log \delta(P) < \min_K g_P(\cdot, \infty). \tag{2.4.10}$$

To do that, consider $0 < \lambda < \frac{1}{\delta(P)}$ and notice that

$$\log \delta(\lambda P) = \log \lambda \delta(P) < 0.$$

On the other hand one has $g_{\lambda P}(z, \infty) = g_P(\frac{z}{\lambda}, \infty)$, thus it follows that

$$\min_{z \in K} g_P(z, \infty) = \min_{z \in \lambda K} g_{\lambda P}(z, \infty) > 0 > \log \delta(\lambda P),$$

where the first inequality is due to the assumption $K \cap \hat{P} = \emptyset$.

If we build $\tilde{f}$ as in the proposition for the sets $P' := \lambda P$ and $K' := \lambda K$, then $f := \tilde{f} \circ \frac{1}{\lambda}$ enjoys the right properties for the original sets P, K. Hence in the following we can suppose (2.4.10) to hold.

Let us pick $0 < \rho < \bar{\rho} := d(\hat{P}, K)/2$, where $d(A, B) := \inf_{x \in A, y \in B} |x - y|$, and consider the set $\hat{P}^\rho$.

For the sake of an easier notation we denote by $g(z)$ and $g_\rho(z)$ the functions $g_P(z, \infty)$ and $g_{\hat{P}^\rho}(z, \infty)$.

For any $k \in \mathbb{N}$ let us pick any set $Z_k(\rho) := \{z_1^{(k)}, \ldots, z_k^{(k)}\}$ of Fekete points for $\hat{P}^\rho$, moreover we denote the polynomial $\prod_{j=1}^k (z - z_j^{(k)})$ by q_k. Notice that $Z_k(\rho) \subset (\partial \hat{P}^\rho)^k \subset (\mathbb{C} \setminus (K \cup P))^k$, hence $\{z_1^{(k)}, \ldots, z_k^{(k)}\}$ is an admissible tentative choice for $w_1, w_2, \ldots, w_k$.

Let us set

$$\begin{aligned} a(\rho) &:= \min_K g_\rho, \\ a &:= \min_{\rho \in [0, \bar{\rho}]} a(\rho) = a(\bar{\rho}), \\ b &:= \max_{\rho \in [0, \bar{\rho}]} \max_K g_\rho = \max_K g. \end{aligned}$$

We recall that (see [**91**, III Th. 1.8])

$$\lim_k \frac{1}{k} \log^+ |q_k| = g_\rho, \quad \text{locally uniformly on } \mathbb{C} \setminus \hat{P}^\rho.$$

Thus for any $\epsilon > 0$ we can choose $m(\epsilon) \in \mathbb{N}$ such that

$$\left\| \frac{1}{m} \log^+ |q_m| - g \right\|_{B(\rho)} < \epsilon \quad \forall m \geq m(\epsilon),$$

where $B(\rho) := \{z \in \mathbb{C} : a \leq g_\rho(z) \leq b\}$, notice that $\hat{P}^\rho \cap B(\rho) = \emptyset$.

Then, taking $\epsilon < a$ we have $\forall m \geq m(\epsilon)$

$$\begin{aligned} K \subset \left\{ a(\rho) - \epsilon \leq \frac{1}{m} \log^+ |q_m| \leq b + \epsilon \right\} = \\ \left\{ e^{m(a(\rho) - \epsilon)} \leq |q_m| \leq e^{m(b+\epsilon)} \right\} =: A(\epsilon, \rho, m). \end{aligned} \tag{2.4.11}$$

On the other hand, exploiting the extremal property of Fekete polynomials [**85**, Th. 5.5.4 (b)], we have $\|q_m\|_{\hat{P}^\rho} \leq \delta_m(\hat{P}^\rho)^m$, where $\delta_m(E)$ is the m-th order diameter of E. In other words

$$P \subset \left\{ |q_m| \leq \delta_m(\hat{P}^\rho)^m \right\} =: D(\rho, m).$$

In order to prove that $A(\epsilon,\rho,m)\cap D(\rho,m)=\emptyset$, for suitable $\epsilon>0$, $\operatorname{dist}(K,\hat{P})>\rho>0$ and $m>m(\epsilon)$, we need to show that for such values of parameters

$$\log\delta_m(\hat{P}^\rho)<a(\rho)-\epsilon. \tag{2.4.12}$$

In such a case the function $f(z):=\frac{1}{q_m(z)}$ satisfies the properties of the proposition since

$$\|f\|_K\leq e^{(-m(a(\rho)-\epsilon))}<\delta_m(\hat{P}^\rho)^{-m}\leq\min_P|f|.$$

To conclude, we are left to prove that we can choose admissible $m,\rho>0$ and $\epsilon>0$ such that (2.4.12) holds. To do that we recall that, since $P=\cap_{l\in\mathbb{N}}P_{\frac{1}{l}}$, by [**85**, Th. 5.1.3] we have

$$\delta(P)=\lim_l\delta(P_{\frac{1}{l}})=\lim_l\lim_m\delta_m(P_{\frac{1}{l}}).$$

By the same reason $g_{1/m}$ is uniformly converging by the Dini's Lemma to g on a neighbourhood of K not intersecting $P_{\bar{\rho}}$.

Therefore, it follows by (2.4.11) and (2.4.10) that possibly shrinking ϵ to get

$$0<\epsilon<\min\{a,\min_K g-\log\delta(P)\}\quad\text{we have}$$

$$\lim_l\lim_m\log\delta_m(P_{\frac{1}{l}})=\log\delta(P)\;<\;\min_K g-\epsilon=\lim_m\min_K g_{1/m}-\epsilon.$$

Hence (possibly taking $\epsilon'<\epsilon$) there exists a increasing subsequence $k\mapsto l_k$ with

$$\lim_m\log\delta_m(P_{1/l_k})<\lim_m\min_K g_{1/m}-\epsilon'\text{ for any }\;k\in\mathbb{N}.$$

In the same way we can pick a subsequence $k\to m_k$ such that $\log\delta_{m_k}(P_{1/l_k})<\min_K g_{1/m_k}-\epsilon''$ for all $k\in\mathbb{N}$. Taking k large enough to get $m_k>m(\epsilon'')$ and setting $m:=m_k,\rho:=1/l_k$ suffices.

In the case of P being a polar subset of $\mathbb{C}$ we observe that for any positive ρ the set $\hat{P}^\rho$ is not polar since it contains at least one disk. Moreover notice that $\lim_m\delta_m(P_{1/m})=\log\delta(P)=-\infty$ whereas the sequence of harmonic (on a fixed suitable neighbourhood of K) functions $g_{1/m}$ is positive and increasing. Equation (2.4.11) is then satisfied for m large enough. The rest of the proof is identical. □

We use the standard notation $f_*\mu(A):=\int_{f^{-1}(A)}d\mu$ for any Borel set $A\subset\mathbb{C}$.

If we use Proposition 2.4.4 and set $E:=f(K)$, $Q:=f(P)$ we can see that $\widehat{E}\cap Q=\emptyset$ thus E,Q are precisely in the same relative position as in the Theorem 2.4.6. Therefore we are now ready to state a sufficient condition for the rational Bernstein

Markov property under more general hypothesis, where we do not assume $\hat{K} \cap P = \emptyset$.

THEOREM 2.4.7 (Mass-density sufficient condition II). *Let $K, P \subset \mathbb{C}$ be compact disjoint sets where K is regular with respect to the Dirichlet problem and $\hat{P} \cap K = \emptyset$. Let $\mu \in \mathcal{M}^+(K)$ be such that $\operatorname{supp} \mu = K$ and suppose that there exist $t > 0$ and f as in Proposition 2.4.4 such that the following holds*

$$\lim_{r \to 0^+} \operatorname{cap}\left(\{z \in f(K) : f_*\mu(B(z, r)) \geq r^t\}\right) = \operatorname{cap}(f(K)). \tag{2.4.13}$$

Then (K, μ, P) has the rational Bernstein Markov Property.

PROOF. By Theorem 2.4.6 it follows that the triple $(E, f_*\mu, Q)$ has the rational Bernstein Markov Property.

To conclude the proof it is sufficient to notice that for any sequence $\{r_k\}$ in $\mathcal{R}(P)$, the sequence $\{\tilde{r}_j\}$ defined by

$$\tilde{r}_j := r_{\lfloor j/m \rfloor} \circ f \quad j = 1, 2, \ldots$$

is an element of $\mathcal{R}(Q)$. Moreover by the rational Bernstein Markov property of $(E, f_*\mu, Q)$ we can pick $c_j > 0$ such that $\limsup_j c_j^{1/j} \leq 1$ and

$$\|r_k\|_K = \|\tilde{r}_{mk}\|_E \leq c_{mk} \|\tilde{r}_{mk}\|_{L^2(f_*\mu)} \leq c_{mk} \|r_k\|_{L^2(\mu)}.$$

Thus we have

$$\left(\frac{\|r_k\|_K}{\|r_k\|_{L^2(\mu)}}\right)^{1/k} \leq \left(c_{mk}^{1/(mk)}\right)^m \to 1^m = 1.$$

□

We can also state the above result in a simpler way, thought not completely equivalent.

COROLLARY 2.4.1. *Let $K, P \subset \mathbb{C}$ be compact sets where K is regular with respect to the Dirichlet problem and $\hat{P} \cap K = \emptyset$. Let $\mu \in \mathcal{M}^+(K)$ be such that $\operatorname{supp} \mu = K$ and suppose that there exist $t > 0$ and f as in Proposition 2.4.4 such that the following holds*

$$\lim_{r \to 0^+} \operatorname{cap}\left(f\left(\{\zeta \in K : \mu(B(\zeta, r)) \geq r^t\}\right)\right) = \operatorname{cap}(f(K)). \tag{2.4.14}$$

Then (K, μ, P) has the rational Bernstein Markov Property.

PROOF. Let $L := \mathrm{Lip}_K f = \inf\{L : |f(x) - f(y)| < L|x - y|,\ \text{for all } x, y \in K\}$, we set

$$A_r := \{\zeta \in K :\ \mu(B(\zeta, r/L)) \geq r^t\}$$
$$D_r := \{z \in f(K) :\ f_*\mu(B(z, r)) \geq r^t\}.$$

We observe that if $\zeta_0 \in A_r$ then $z_0 := f(\zeta_0)$ lies in D_r. For, notice that

$$f_*\mu(B(z_0, r)) = \int_{f^{-1}(B(z_0,r))} d\mu \geq \int_{B(\zeta_0,r/L)} d\mu$$

since $f(B(\zeta_0, r/L)) \subseteq B(z_0, r)$. Therefore $f(A_r) \subseteq D_r$.

If we suppose that $\mathrm{cap}(f(A_r)) \to \mathrm{cap}(f(K))$, then it follows that $\mathrm{cap}(D_r) \to \mathrm{cap}(f(K))$ as well by the inequality $\mathrm{cap}(f(K)) \geq \mathrm{cap}(D_r) \geq \mathrm{cap}(f(A_r)) \to \mathrm{cap}(f(K))$.

Now consider the set $B_r := \{\zeta \in K : \mu(B(\zeta, r)) \geq r^{t'}\}$, for some $t' > t$, condition (2.4.14) says $\lim_{s\to 0^+} \mathrm{cap}(f(B_s)) = \mathrm{cap}(f(K))$. Now take $s = r/L$ and notice that for small r we have $\left(\frac{r}{L}\right)^{t'} \geq r^t$, thus by condition (2.4.14) it follows that $\lim_{r\to 0^+} \mathrm{cap}(f(A_r)) = \mathrm{cap}(f(K))$. By the previous argument condition (2.4.13) follows and Theorem 2.4.7 applies. □

2.4.1. Further examples.

EXAMPLE 2.4.2. *We go back to the case of the Example 1 (e) to show that the same conclusion follows by applying Corollary 1. Let us recall the notation. We consider the annulus* $A := \{z : 1/2 \leq |z| \leq 1\}$, *set* $K := \partial A$, $P := \{0\}$ *and* $\mu := 1/2 ds|_{\partial\mathbb{D}} + 1/2 ds|_{\frac{1}{2}\partial\mathbb{D}}$, *where* ds *is the standard arc length measure.*

We proceed as in Proposition 2.4.4 to build the map f*: we take* $\rho = 0.1$ *and for each* $m \in \mathbb{N}$ *we pick a set of Fekete points for* $P^\rho = \{|z| \leq 0.1\}$.

In this easy example $m = 2$ *suffices to our aim, so we can choose* $w_1 = 0.1$, $w_2 = -0.1$, $f(z) = \frac{1}{(z-w_1)(z-w_2)} = \frac{1}{z^2-0.01}$.

We notice that f *is a holomorphic map of a neighbourhood* K^δ *of* K *and we can compute its Lipschitz constant* $\mathrm{Lip}_K(f) := \inf\{L > 0 : |f(x) - f(y)| \leq L|x - y|, \forall x \neq y \in K\}$ *as follows.*

$$L_\delta := \mathrm{Lip}_{K^\delta}(f) = \|f'\|_{K^\delta} = \max_{z\in K^\delta} \left|\frac{-2z}{(z^2 - 0.01)^2}\right|.$$

For instance, taking $\delta = 0.1$ *we get* $L_\delta = \frac{4(1-2\delta)}{1-4\delta} = 5.\overline{3}$.

For any $\zeta \in \partial\mathbb{D}$ and $r < 1/2$ we have

$$\mu\left(B(\zeta, r)\right) = \frac{1}{2}\int_{B(\zeta,r)\cap\partial\mathbb{D}} ds = \frac{1}{2}\int_{\arg(\zeta)-\arcsin(r)}^{\arg(\zeta)+\arcsin(r)} 1\, d\theta$$
$$= \arcsin\left(r\right),$$

similarly for any $\zeta \in 1/2\partial\mathbb{D}$ we have

$$\mu\left(B(\zeta, r)\right) = \frac{1}{2}\int_{B(\zeta,r)\cap 1/2\partial\mathbb{D}} ds = \frac{1}{2}\int_{\arg\zeta-2\arcsin(r)}^{\arg\zeta+2\arcsin(r)} 1\,\frac{d\theta}{2}$$
$$= \arcsin(r).$$

Notice that taking $t = 1$ and $r < 1/2$ (2.4.14) *is satisfied since $\{\zeta \in K : \mu(B(\zeta, r)) \geq r\} = K$ for all $0 < r < 1/2$.*

Finally we notice that also (A, μ, P) has the rational Bernstein Markov property (as we observed in Example 1 (e)) since any rational function having poles on P achieves the maximum of its modulus on K.

It is worth to notice that a measure μ can satisfy (2.4.14) even if the mass of balls of radius r decays very fast (e.g. faster than any power of r) as $r \to 0$ at some points of the support of μ. This is the case of the following example.

Example 2.4.3. *Let us consider the measure μ, where*

$$\frac{d\mu}{d\theta} := \exp\left(\frac{-1}{1-\left(\frac{\theta}{\pi}\right)^2}\right),\ -\pi \leq \theta \leq \pi$$

defined on the unit circle $\partial\mathbb{D}$ and pick as pole set $P := \{0\}$.

$$\mu(B(e^{i\theta}, r)) = \int_{\theta-2\arcsin r/2}^{\theta+2\arcsin r/2} \exp\left(\frac{-\pi^2}{\pi^2-u^2}\right) du \tag{2.4.15}$$

$$\geq \begin{cases} 4\arcsin r/2 \exp\left(\frac{-\pi^2}{\pi^2-(\theta+2\arcsin r/2)^2}\right) & ,0 \leq \theta < \pi - 2\arcsin r/2 \\ 4\arcsin r/2 \exp\left(\frac{-\pi^2}{\pi^2-(\theta-2\arcsin r/2)^2}\right) & ,-\pi + 2\arcsin r/2 \leq \theta \leq 0. \end{cases} \tag{2.4.16}$$

We try to test condition (2.4.14) *using $t = 1$ and the map $f(z) := \frac{1}{z-0.01}$ which is a biholomorphism of a neighbourhood of $\partial\mathbb{D}$. Therefore the condition* $\lim_{r\to 0^+} \operatorname{cap}(f(K_r)) = \operatorname{cap}(f(K))$ *of Corollary 1 for sets $K_r \subseteq K$ is equivalent to* $\lim_{r\to 0^+} \operatorname{cap} K_r = \operatorname{cap} K$ *and we are reduce to test the simpler condition*

$$\lim_{r\to 0^+} \operatorname{cap}\left(\{z \in \partial\mathbb{D} : \mu(B(z, r)) \geq r\}\right) =: \lim_{r\to 0^+} \operatorname{cap} K_r = \operatorname{cap}(\partial\mathbb{D}). \tag{2.4.17}$$

It is not difficult to see by (2.4.16) *that*

$$
\begin{aligned}
&K_r \supset \\
&\left\{e^{i\theta} : \theta \in [0, \pi - 2\arcsin r/2[\,, \exp\left(\frac{-\pi^2}{\pi^2 - (\theta + 2\arcsin r/2)^2}\right) \geq \frac{r}{4\arcsin r/2}\right\} \bigcup \\
&\ \left\{e^{i\theta} : \theta \in]-\pi + 2\arcsin r/2, 0]\,, \exp\left(\frac{-\pi^2}{\pi^2 - (\theta - 2\arcsin r/2)^2}\right) \geq \frac{r}{4\arcsin r/2}\right\} = \\
&K_r^1 \cup K_r^2 =: \tilde{K}_r,
\end{aligned}
$$

where

$$
\begin{aligned}
K_r^i &= \{e^{i\theta}, \theta \in [a_i, b_i]\} \\
a_1 &= \max\left\{0, 2\arcsin r/2 - \pi\sqrt{1 + \frac{1}{\log\frac{4\arcsin r/2}{2}}}\right\} \\
b_1 &= \min\left\{\pi - 2\arcsin r/2, \pi\sqrt{1 - \frac{1}{\log\frac{2\arcsin r}{r}}} - \arcsin r\right\} \\
a_2 &= \min\left\{-\pi + 2\arcsin r/2, \pi\sqrt{1 - \frac{1}{\log\frac{2\arcsin r}{r}}} - \arcsin r\right\} \\
b_2 &= \max\left\{0, -2\arcsin r/2 + \pi\sqrt{1 + \frac{1}{\log\frac{4\arcsin r/2}{2}}}\right\}.
\end{aligned}
$$

It is not difficult to see that for $r \to 0^+$ *we have* $[a_1, b_1] = [0, \pi - 2\arcsin r/2]$, $[a_2, b_2] = [-\pi + 2\arcsin r/2, 0]$, *hence* $K_r \supseteq \{e^{i\theta}, \theta \in [-\pi + 2\arcsin r/2, \pi - 2\arcsin r/2]$

We recall that the logarithmic capacity of an arc of circle of radius 1 *and length* α *is* $\sin(\alpha/4)$*; see* [**85**, pg. 135]. *Therefore we have*

$$
\begin{aligned}
&\operatorname{cap}(\partial\mathbb{D}) \geq \lim_{r\to 0^+} \operatorname{cap}(K_r) \geq \lim_{r\to 0^+} \operatorname{cap}(\tilde{K}_r) = \\
&\lim_{r\to 0^+} \sin\left(\frac{2\pi - 4\arcsin r/2}{4}\right) = 1 = \operatorname{cap}(\partial\mathbb{D}),
\end{aligned}
\tag{2.4.18}
$$

this proves (2.4.17) *and since we considered a bi-holomorphic map* f (2.4.14) *follows. By Corollary 2.4.1 we can conclude that* $\{\partial\mathbb{D}, \mu, \{0\}\}$ *has the rational Bernstein Markov property.*

2.5. A L^2 Meromorphic Version of the Bernstein Walsh Lemma

For a given compact set $K \subset \mathbb{C}$ we denote by D_r the set $\{z \in \mathbb{C} : g_K(z, \infty) < \log r\}$ and by $\mathscr{M}_n(D_r)$ the class of meromorphic functions having precisely n poles (counted with their multiplicities) in D_r. Let us denote by $\mathcal{R}_{k,n}$ the class of rational functions having at most k zeroes and at most n poles (each of them counted with its multiplicity).

It follows by the work of Walsh [**100**], Saff [**89**] and Gonchar [**52**] that, given a function $f \in \mathscr{C}(K)$, where K is a compact regular set, f admits a meromorphic extension $\tilde{f} \in \mathscr{M}_n(D_r)$ if and only if one has the *overconvergence* of the best uniform norm approximation by rational functions with n poles, that is

$$\limsup_k d_{n,k}(f,K)^{1/k} := \limsup_k \inf_{r \in \mathcal{R}_{k,n}} \|f - r\|_K^{1/k} \leq 1/r. \tag{2.5.1}$$

In the case of a finite measure μ having compact support K and such that (K, μ, P) has the rational Bernstein Markov property for any compact set P, $P \cap K = \emptyset$, one can rewrite such a theorem checking the overconvergence of best L^2_μ rational approximations instead of best uniform ones. Notice that if $K = \hat{K}$ any Bernstein Markov measure supported on K has such a property. More precisely, we can prove the following in the spirit of [**68**, Prop. 9.4], where we use the notation $\text{Poles}(f)$ to denote the set of poles of the function f.

Theorem 2.5.8 (L^2 Meromorphic Bernstein Walsh Lemma). *Let K be a compact regular subset of $\mathbb{C}$, let $f \in \mathscr{C}(K)$ and let $r > 1$. The following are equivalent.*

i) There exists $\tilde{f} \in \mathscr{M}_n(D_r)$ such that $\tilde{f}|_K \equiv f$.

ii) $\limsup_k d_{k,n}^{1/k}(f, K) \leq 1/r$.

iii) For any finite Borel measure μ such that $\operatorname{supp} \mu = K$ *and (K, μ, P) has the rational Bernstein Markov property for any compact set P such that $P \cap K = \emptyset$, denoting by $r^\mu_{k,n}$ a best L^2_μ approximation to f in $\mathcal{R}_{k,n}$, one has*

$$\limsup_k \left(\|f - r^\mu_{k,n}\|_K \right)^{1/k} \leq 1/r,$$

provided that $\overline{\{\text{Poles}(r_{k,n})\}_k} \cap K = \emptyset$.

iv) With the same hypothesis and notations as in iii) *we have*

$$\limsup_k \left(\|f - r^\mu_{k,n}\|_{L^2_\mu} \right)^{1/k} \leq 1/r,$$

provided that $\overline{\{\text{Poles}(r_{k,n})\}_k} \cap K = \emptyset$.

Proof. (Proof of $i \Leftrightarrow ii$.) The theorem has been proven in [**52**], see also [**55**]. ($ii \Rightarrow iii$.) Let us pick $\rho > r$, we find $C > 0$ such that

$$d_{k,n}^{1/k}(f, K) \leq C/\rho^k, \quad \forall k.$$

Let us pick $s_{k,n} \in \mathcal{R}_{k,n}$ such that $\|f - s_{k,n}\|_K = d_{k,n}(f, K)$ and set $P_\infty = \overline{\{\text{Poles}(s_{k,n})\}_k}$. Notice that

$$\|f - r^\mu_{k,n}\|_{L^2_\mu} \leq \|f - s_{k,n}\|_{L^2_\mu} \leq \mu(K)^{-1/2}\|f - s_{k,n}\|_K$$
$$= \mu(K)^{-1/2} d_{k,n}(f, K) \leq \mu(K)^{-1/2} C/\rho^k. \tag{2.5.2}$$

In particular it follows that

$$\|r^\mu_{k,n} - r^\mu_{k-1,n}\|_{L^2_\mu} \leq \|f - r^\mu_{k,n}\|_{L^2_\mu} + \|f - r^\mu_{k-1,n}\|_{L^2_\mu} \leq \frac{\mu(K)^{-1/2}C(1+\rho)}{\rho^k}.$$

We apply the rational Bernstein Markov property to (K, μ, P), with $P := P_\infty \cup P_2$, $P_2 = \overline{\{\text{Poles}(r_{k,n})\}_k}$, in the following equivalent formulation, for any $\epsilon > 0$ there exists $M = M(\epsilon, K, \mu, P)$ such that $\|s\|_K \leq M(1+\epsilon)^k \|s\|_{L^2_\mu}$ for any $s \in \mathcal{R}_{k,n}$, Poles $s \subset P$, $n \leq k$, $\forall k$. Notice that $P_\infty \cap K = \emptyset$ follows by the assumption $\limsup_k d^{1/k}_{k,n}(f, K) \leq 1/r$; [**100**]. We get

$$\|r^\mu_{k,n} - r^\mu_{k-1,n}\|_K \leq M\mu(K)^{-1/2}C(1+\rho)\left(\frac{1+\epsilon}{\rho}\right)^k. \tag{2.5.3}$$

By equation (2.5.2) $r^\mu_{k,n} \to f$ in L^2_μ, therefore some subsequence converges almost everywhere with respect to μ. By equation (2.5.3) we can show that the sequence of functions $\{r_{k,n}\}$ is a Cauchy sequence in $\mathscr{C}(K)$ thus it has a uniform continuous limit g. Therefore $f \equiv g$ and the whole sequence is uniformly converging to f on K. Notice that $f \equiv g$ on a carrier of μ, thus on a dense subset of the support K of μ.

Now notice that

$$\|f - r_{k,n}\|_K \leq \left\| \sum_{j=k+1}^\infty r^\mu_{j,n} - r^\mu_{j-1,n} \right\|_K \leq \sum_{j=k+1}^\infty \|r^\mu_{j,n} - r^\mu_{j-1,n}\|_K$$
$$\leq M\mu(K)^{-1/2}C(1+\rho)\sum_{j=k+1}^\infty \left(\frac{1+\epsilon}{\rho}\right)^j = M\mu(K)^{-1/2}C(1+\epsilon)\frac{1+\rho}{\rho-1}\left(\frac{1+\epsilon}{\rho}\right)^k.$$

Therefore we have

$$\limsup_k \|f - r_{k,n}\|_K^{1/k} \leq \limsup_k \left(\frac{M\mu(K)^{-1/2}C(1+\epsilon)(1+\rho)}{\rho-1}\right)^{1/k} \frac{1+\epsilon}{\rho} = \frac{1+\epsilon}{\rho}.$$

The thesis follows letting $\epsilon \to 0^+$ and $\rho \to r^+$.

($iii \Rightarrow ii$.) By definition one has

$$1/r \geq \limsup_k \left(\|f - r^\mu_{k,n}\|_K\right)^{1/k} \geq \limsup_k \left(\inf_{r \in \mathcal{R}_{k,n}} \|f - r\|_K\right)^{1/k}$$

$$= \limsup_k d_{k,n}^{1/k}(f,K).$$

($iii \Rightarrow iv$.) Simply notice that

$$\begin{aligned}1/r \geq \limsup_k \left(\|f - r^{\mu}_{k,n}\|_K\right)^{1/k} &\geq \limsup_k \left(\mu(K)^{1/2}\|f - r^{\mu}_{k,n}\|_{L^2}\right)^{1/k}\\ &= \limsup_k \left(\|f - r^{\mu}_{k,n}\|_{L^2}\right)^{1/k}.\end{aligned}$$

($iv \Rightarrow iii$.) This implication can be proven using a similar reasoning to the one of ($ii \Rightarrow iii$).

The sequence $r_{k,n}$ is converging to f in L^2_μ by assumption, then there exists a subsequence converging to f almost everywhere.

Due to the rational Bernstein Markov property of μ with respect to K and P_2 we have

$$\|r_{k,n} - r_{k-1,n}\|_K \leq M(1+\epsilon)^k \|r_{k,n} - r_{k-1,n}\|_{L^2_\mu}$$

and we can estimate the right hand side as follows

$$\|r_{k,n} - r_{k-1,n}\|_{L^2_\mu} \leq \|r_{k,n} - f\|_{L^2_\mu} + \|f - r_{k-1,n}\|_{L^2_\mu} \leq C/\rho^k(1+\rho)$$

for a suitable $C > 0$ and $\rho > r$. Thus the sequence $r_{k,n}$ has a uniform limit coinciding μ-a.e. with the continuous function f and hence the whole sequence is uniformly converging to f, being the two continuous function equal on a carrier of μ which needs to be dense in $K = \operatorname{supp}\mu$.

Now notice, as above, that

$$\begin{aligned}&\|f - r_{k,n}\|_K \leq \left\| \sum_{j=k+1}^{\infty} r^{\mu}_{j,n} - r^{\mu}_{j-1,n} \right\|_K \leq \sum_{j=k+1}^{\infty} \|r^{\mu}_{j,n} - r^{\mu}_{j-1,n}\|_K\\ &\leq M\mu(K)^{1/2}C(1+\rho)\sum_{j=k+1}^{\infty}\left(\frac{1+\epsilon}{\rho}\right)^j = M\mu(K)^{1/2}C(1+\epsilon)\frac{1+\rho}{\rho-1}\left(\frac{1+\epsilon}{\rho}\right)^k.\end{aligned}$$

Therefore we have

$$\limsup_k \|f - r_{k,n}\|_K^{1/k} \leq \limsup_k \left(\frac{M\mu(K)^{1/2}C(1+\epsilon)(1+\rho)}{\rho-1}\right)^{1/k}\frac{1+\epsilon}{\rho} = \frac{1+\epsilon}{\rho}.$$

The thesis follows letting $\epsilon \to 0^+$ and $\rho \to r^+$.

□

CHAPTER 3

Pluripotential Theory on Algebraic Sets: a Toolkit

Non esistono montagne impossibili, esistono solo uomini che non sono capaci di salirle.

Cesare Maestri

The aim of this chapter is to provide the definitions and the main tools that we are going to use in Chapter 4 for proving some original results.

We recall the definition of the Monge Ampere operator acting on plurisubharmonic locally bounded functions on an irreducible pure m-dimensional algebraic variety and the extension of all standard notions of the classical Pluripotential Theory in $\mathbb{C}^n$. Much of what follows can be extended to the case of weakly plurisubharmonic (see Definition C.1.1) functions or to plurisubharmonic functions on more general spaces (e.g., Stein spaces) [**42**], or even in both directions, that is on weakly plurisubharmonic functions on Stein spaces with a parabolic potential; see [**11, 107**]. Here we chose to deal with this easier case both to simplify the proofs and because this is the setting we need to work with in the rest of the thesis. In particular we provide in the context of pure dimensional irreducible algebraic sets:

- the definition of the Monge Ampere operator acting on locally bounded plurisubharmonic functions,
- definitions of global and local extremal plurisubharmonic functions, extremal measures, relative capacity and pluripolar sets,
- continuity of the Monge Ampere operator under point-wise decreasing limits,
- some integration by parts formulas for wedge powers of terms of the type $dd^c u$ for u plurisubharmonic locally bounded function,
- Lelong Jensen Poisson formula.

3.1. Definition of the Monge Ampere Operator

In [**13**], see also [**12, 14**], authors introduce the generalized complex Monge Ampere operator $(\mathrm{dd^c}\, u)^k$, $1 \le k \le n$, for $u \in \mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}$ as a positive (k,k) current, where Ω is a domain of $\mathbb{C}^n$ (or any complex manifold) by an inductive procedure. We refer to Appendix B for the definition and the main properties of currents.

We briefly recall their procedure. The term $\mathrm{dd^c}\, u$ is a well defined closed positive $(1,1)$ current for any plurisubharmonic function u.

Let us suppose that $(\mathrm{dd^c}\, u)^k$ has been defined as a closed positive (k,k) current of order zero i.e., acting on compactly supported *continuous* forms. Note that for any locally bounded measurable function v the current $v(\mathrm{dd^c}\, u)^k$ is well defined. They set

$$\langle(\mathrm{dd^c}\, u)^{k+1}, \varphi\rangle := \langle u(\mathrm{dd^c}\, u)^k, \mathrm{dd^c}\, \varphi\rangle \quad \forall \varphi \in \mathcal{D}^{n-k-1,n-k-1}(\Omega). \tag{3.1.1}$$

Then they prove, using the properties of $(\mathrm{dd^c}\, u)^k$, that the above formula is a closed positive $(k+1, k+1)$ current, therefore it is of order zero and has measure coefficients.

Note that for $u \in \mathscr{C}^2$ one has

$$(\mathrm{dd^c}\, u)^n = 4^n n! \det\left[\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right]_{i,j} \beta_n^n,$$

where $\beta_n := \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\bar{z}_j$ is the standard Kähler form on $\mathbb{C}^n$.

It turns out, see for instance [**59**], that the following inequality, known as the *Chern Levine Nirenberg Estimate*, holds for bounded plurisubharmonic functions on open bounded sets of $\mathbb{C}^n$. For any compact set $K \subset \Omega$ there exists a positive finite constant $c_k(K, \Omega)$ such that for all bounded $u_1, \dots, u_k \in \mathrm{PSH}(\Omega)$, $k = 1, 2, \dots, n$ we have

$$\int_\Omega \mathrm{dd^c}\, u_1 \wedge \mathrm{dd^c}\, u_2 \wedge \cdots \wedge \beta_n^{n-k} \le c_k(K, \Omega) \prod_{i=1}^k \sup_\Omega |u_i|. \tag{3.1.2}$$

In the Pluripotential Theory of $\mathbb{C}^n$ the notion of *pluripolar set* plays an important role. A set $E \subset \mathbb{C}^n$ is locally pluripolar if for each $z_0 \in E$ there exists a neighbourhood B_{z_0} of z_0 such that $E \cap B_{z_0}$ is contained in the set $\{u_{z_0} = -\infty\}$ for some $u_{z_0} \in \mathrm{PSH}(B_{z_0})$, $u_{z_0} \not\equiv -\infty$. If this property can be satisfied globally, with the same u for each point, the set is said to be *pluripolar*. In the $\mathbb{C}^n$ case the two notions coincide.

For Ω open bounded subset of $\mathbb{C}^n$ Bedford and Taylor [**13**] introduced the relative capacity of any Borel subset E of Ω as

$$\mathrm{Cap}(E,\Omega) := \sup\left\{\int_{E}(\mathrm{dd^c}\, u)^n, u \in \mathrm{PSH}(\Omega,[0,1])\right\}$$

and showed that, defining the *relative extremal function*

$$U^*_{E,\Omega}(z) := \limsup_{\zeta\to z}\sup\{u(\zeta) : u \in \mathrm{PSH}(\Omega), u \le 0, u|_E \le -1\},$$

one has

$$\mathrm{Cap}(E,\Omega) = \int_{E}(\mathrm{dd^c}\, U^*_{E,\Omega})^n = \int_{\Omega}(\mathrm{dd^c}\, U^*_{E,\Omega})^n = \int_{\Omega} -U^*_{E,\Omega}(\mathrm{dd^c}\, U^*_{E,\Omega})^n.$$

Moreover the property

$$\mathrm{Cap}^*(E,\Omega) := \inf\{\mathrm{Cap}(O,\Omega), E \subseteq O \subset \Omega, O \text{ open}\} = 0$$

characterize (see [**13**]) the pluripolar Borel subsets of Ω, thus in particular $(\mathrm{dd^c}\, u)^n$ *puts no mass on pluripolar sets for any locally bounded plurisubharmonic function* u.

It follows by the definition of analytic subsets that if A is a analytic subset of $\mathbb{C}^n$ it is locally pluripolar and thus pluripolar in $\mathbb{C}^n$, for, one considers the logarithm of the modulus of the product of a set of local defining functions for A; we refer to Appendix A for definitions and main properties of analytic and algebraic sets.

Despite any analytic subset A of $\mathbb{C}^n$ is pluripolar, the set A_{reg} of its regular point is a complex manifold and therefore the complex Monge Ampere operator is well defined on it.

Indeed, the aim of this section is to extend the inductive definition of the Monge Ampere operator to plurisubharmonic locally bounded functions on algebraic subsets of $\mathbb{C}^n$. The procedure to do that is the same used by Lelong [**64**] to show that the current of integration on A_{reg} extends to A. That is, one first shows that the considered current is locally finite at any neighbourhood of a bounded subset of A_{sing}, then uses this property to show that *its extension by zero* is well defined and preserves the properties of the original current, e.g. is closed and positive[1].

The first step is contained in Lemma 3.1.1 below. The proof is essentially the same as [**107**, Lemma 1.7], where only the m-th wedge power instead of the k-th one, with $k \le m$, and the more general case of weakly plurisubharmonic functions

[1]A current T of bi degree (k,k) is closed if $\langle dT;\varphi\rangle := \langle T; d\varphi\rangle = 0$ for all test form $\varphi \in \mathcal{D}^{m-k,m-k}$; we refer the reader to Appendix B for the definitions of test forms, currents and positivity.

instead of plurisubharmonic ones is considered. For the second step, we prefer to rely on a general theorem by El Mir, see Theorem 3.1.1 below.

We recall here, for the reader's convenience, the definition of plurisubharmonic and weakly plurisubharmonic functions on a open subset Ω of an algebraic set A in $\mathbb{C}^n$.

A function $u : \Omega \to \mathbb{R} \cup [-\infty, +\infty[$ is said to be plurisubharmonic on Ω if for any $z_0 \in \Omega$ there exists an open neighbourhood D_{z_0} of z_0 in $\mathbb{C}^n$ and a plurisubharmonic function $\tilde{u}_{z_0}$ on D_{z_0} such that $\tilde{u}_{z_0} \equiv u$ on $D_{z_0} \cap \Omega$. In such a case we write $u \in \mathrm{PSH}(\Omega)$.

Instead $u : \Omega \to \mathbb{R} \cup [-\infty, +\infty[$ is said to be weakly plurisubharmonic on Ω if $u|_{\Omega \cap A_{\mathrm{reg}}}$ is plurisubharmonic as function defined on a complex manifold and u is locally bounded on Ω. We denote such a property by $u \in \widetilde{\mathrm{PSH}}(\Omega)$.

Lemma 3.1.1. *Let A be a pure m-dimensional irreducible algebraic set in $\mathbb{C}^n$ and $u^j \in \mathrm{PSH}(A) \cap L^\infty_{loc}(A)$, then the currents $dd^c(u^1|_{A_{reg}}) \wedge \cdots \wedge dd^c(u^k|_{A_{reg}})$, $k = 1, 2, \ldots, m$ have locally finite mass near A_{sing}, that is, for any open relatively compact set $O \subset A$*

$$\sup\Big\{ \int_{O \setminus A_{sing}} \psi \wedge dd^c(u^1|_{A_{reg}}) \wedge \cdots \wedge dd^c(u^k|_{A_{reg}}),\\ \psi \in \mathcal{D}_c^{m-k,m-k}(O \cap A_{reg}),\ \|\psi\|_O \le 1 \Big\} < \infty. \tag{3.1.3}$$

Here and later on we denote by $\|\psi\|_O$ the quantity $\sup_{z \in O} \max_{I,J} |\psi_{I,J}|$ for any form $\psi := \sum'_{I,J} \psi_{I,J} dz^I \wedge d\bar{z}^J$. The key idea of the proof is to define a family of locally bounded plurisubharmonic functions u_I on $\mathbb{C}^m$ obtained by composing the given $u \in \mathrm{PSH}(A) \cap L^\infty_{\mathrm{loc}}$ with the projections on each possible coordinate plane of complex dimension m and then use the standard $\mathbb{C}^m$ theory to show that equation 3.1.3 is satisfied.

The previous lemma used in conjunction with the following theorem allow us to define wedge powers of currents of type $dd^c u$ for $u \in \mathrm{PSH}(A) \cap L^\infty_{\mathrm{loc}}$. We recall that a set $P \subset \Omega$ is said *complete pluripolar* in Ω if there exists a (not identically $-\infty$) function $u \in \mathrm{PSH}(\Omega)$ such that $P = \{u = -\infty\}$. Note that in particular an algebraic subset of $\mathbb{C}^n$ is complete pluripolar in $\mathbb{C}^n$ and, given an algebraic set A in $\mathbb{C}^n$ we can always find an open neighbourhood $\tilde{\Omega}$ of A in $\mathbb{C}^n$ such that A is a complete pluripolar subset of $\tilde{\Omega}$.

Theorem 3.1.1 (Extension of closed positive currents; [**46**]). *Let S be a closed complete pluripolar subset of an open set Ω in $\mathbb{C}^n$ and T a closed positive (k, k)*

current on $\Omega_1 := \Omega \setminus S$ of locally finite mass on Ω, that is for any open set $O \subset \Omega$ we have

$$\sup\left\{\langle T, \psi\rangle,\ \psi \in \mathcal{D}_c^{m-k,m-k}(O \cap \Omega_1),\ \|\psi\|_O \leq 1\right\} < \infty. \tag{3.1.4}$$

Then the extension $\tilde{T}$ of T to 0 on S is a closed positive current, where

$$\langle \tilde{T}, \psi\rangle := \lim_{r\to 0^+} \langle T, \eta_r \psi\rangle \ \ \forall \psi \in \mathcal{D}_0^{m-k,m-k}(\Omega). \tag{3.1.5}$$

Here $0 \leq \eta_r \leq 1$ is any sequence of $\mathscr{C}_c(\Omega_1)$ functions such that for any compact set $K \subset \Omega_1$ $\eta_r|_K \equiv 1$ for all $r < r_0(K)$.

Corollary 3.1.1 (Extension of the Monge Ampere operator). *Let A be an irreducible pure m-dimensional algebraic set in $\mathbb{C}^n$ and $u^1, \ldots, u^k$ be plurisubharmonic functions on A, the current $\mathrm{dd^c}\, u^1|_{A_{reg}} \wedge \ldots \mathrm{dd^c}\, u^k|_{A_{reg}}$ extends to a closed positive current on $\mathbb{C}^n$ supported on A that we denote by $\mathrm{dd^c}\, u^1 \wedge \cdots \wedge \mathrm{dd^c}\, u^k$ by setting for any continuous $(m-k, m-k)$ form ψ compactly supported on $\mathbb{C}^n$*

$$\langle \mathrm{dd^c}\, u^1 \wedge \cdots \wedge \mathrm{dd^c}\, u^k, \psi\rangle := \lim_{r\to 0^+} \int \mathrm{dd^c}\, u^1|_{A_{reg}} \wedge \ldots \mathrm{dd^c}\, u^k|_{A_{reg}} \mathcal{I}^* \psi \eta_r, \tag{3.1.6}$$

where η_r are as in (3.1.5) and $\mathcal{I}$ is the inclusion of A_{reg} in $\mathbb{C}^n$.

Proof. The statement is local, so we can pick an open relatively compact subset Ω of A and an open subset $\tilde{\Omega}$ of $\mathbb{C}^n$ such that $\Omega = \tilde{\Omega} \cap A$ and prove the statement on Ω.

Let us denote by T_k the current $\mathrm{dd^c}\, u^1|_{A_{\mathrm{reg}}} \wedge \ldots \mathrm{dd^c}\, u^k|_{A_{\mathrm{reg}}}$ acting on $\Omega \setminus A_{\mathrm{sing}}$. We notice that T_k extends canonically to a closed positive current on $\tilde{\Omega} \setminus A_{\mathrm{sing}}$ that we denote by T_k^1. For, we use the fact that $\Omega \setminus A_{\mathrm{sing}}$ is a complex submanifold of $\tilde{\Omega} \setminus A_{\mathrm{sing}}$, let $\mathcal{I} : \Omega \setminus A_{\mathrm{sing}} \to \tilde{\Omega} \setminus A_{\mathrm{sing}}$ be the inclusion map which is smooth and proper. Then we set

$$T_k^1 := \mathcal{I}_* T_k.$$

Here $\mathcal{I}_* T_k$ is the push-forward of the current T_k, i.e., $\langle \mathcal{I}_* T_k, \varphi\rangle = \langle T_k, \mathcal{I}^* \varphi\rangle$ for any $\varphi \in \mathcal{D}_c^{n-k,n-k}(\tilde{\Omega} \setminus A_{\mathrm{sing}})$ and $\mathcal{I}^*$ is the usual pull-back of differential forms.

Positivity and closedness of T_k^1 follows easily by the properties of exterior derivatives under pull-back by smooth proper maps.

Now we notice that $S := A_{\mathrm{sing}} \cap \tilde{\Omega}$ is a complete pluripolar subset of $\tilde{\Omega}$, being an algebraic subset, moreover it follows by Lemma 3.1.1 that the hypothesis (3.1.4) is satisfied by T_k^1 on $\tilde{\Omega} \setminus S$.

We can apply Theorem 3.1.1 to extend T_k^1 to a closed positive current on $\tilde{\Omega}$, its support being necessarily on Ω. $\square$

REMARK 3.1.1. *We will refer to the integration as in the right hand side of* (3.1.6) *as* improper integration *over A_{reg}. If ψ is a continuous compactly supported (m,m) form on an open set $\Omega \subset \mathbb{C}^n$ we set*

$$\int_{\Omega\cap A} \psi := \int_{\Omega\cap A_{reg}} \psi = \lim_{r\to 0^+} \int_{\Omega\cap A_{reg}} \eta_r \psi.$$

3.1.1. The operator $d \wedge d^c \wedge (\mathrm{dd^c})^{m-1}$. In the following we will sometimes use the operator $d \wedge d^c \wedge (\mathrm{dd^c})^{m-1}$ acting on locally bounded plurisubharmonic functions.

We notice that for non negative u, by convexity of $x \to x^2$, the function u^2 is a locally bounded plurisubharmonic function for any $u \in \mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}$. If u is smooth, we have $\mathrm{dd^c}\, u^2 = 2u\, \mathrm{dd^c}\, u + 2du \wedge d^c u$, we use this to define the term $du \wedge d^c u$

$$du \wedge d^c u := \frac{1}{2}\, \mathrm{dd^c}\, u^2 - u\, \mathrm{dd^c}\, u \quad \forall u \in \mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}.$$

In a similar way we introduce $d(u+v) \wedge d^c(u+v)$.

Now notice that for any smooth $(m-1, m-1)$ form ψ and $u, v \in \mathscr{C}^2$ one has

$$du \wedge d^c v \wedge \psi = dv \wedge d^c u \wedge \psi.$$

We use this to introduce $\forall u, v \in \mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}$

$$du \wedge d^c v \wedge \psi := \frac{1}{2}(d(u+v) \wedge d^c(u+v) - du \wedge d^c u - dv \wedge d^c v) \wedge \psi.$$

Again we can notice that the right hand side makes sense not only for ψ smooth form but even for any positive $(m-1, m-1)$ current of locally finite mass, hence we can introduce the following operator for any $w \in \mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}$.

$$\begin{aligned}
&du \wedge d^c v \wedge (\mathrm{dd^c}\, w)^{m-1} \\
:=&\frac{1}{2}(d(u+v) \wedge d^c(u+v) - du \wedge d^c u - dv \wedge d^c v) \wedge (\mathrm{dd^c}\, w)^{m-1} \\
=&\frac{1}{2}\Big(\frac{1}{2}\, \mathrm{dd^c}(u+v)^2 - (u+v)\, \mathrm{dd^c}(u+v) - \frac{1}{2}\, \mathrm{dd^c}\, u^2 + \\
&\qquad u\, \mathrm{dd^c}\, u - \frac{1}{2}\, \mathrm{dd^c}\, v^2 + v\, \mathrm{dd^c}\, v\Big) \wedge (\mathrm{dd^c}\, w)^{m-1} \\
=&\frac{1}{2}\left[\frac{1}{2}\left(\mathrm{dd^c}(u+v)^2 - \mathrm{dd^c}\, u^2 - \mathrm{dd^c}\, v^2\right) - (u\, \mathrm{dd^c}\, v + v\, \mathrm{dd^c}\, u)\right] \wedge (\mathrm{dd^c}\, w)^{m-1}.
\end{aligned}$$

3.2. The Domination and Comparison Principles

PROPOSITION 3.2.2 (Domination Principle for open sets [**107**]). *Let Ω be an open set of the pure m dimensional algebraic set A and $u, v \in \widetilde{\mathrm{PSH}}(\Omega) \cap L^\infty_{loc}$ such that*

(1) $\limsup_{z \to \partial\Omega} |u^*(z) - v^*(z)| = 0$ *and*

(2) $\int_{\{u^* < v\}} (\mathrm{dd^c}\, u)^m = 0.$

Then one has

$$v^* \leq u^* \quad on\ \Omega.$$

It is worth to notice that in the case of A being irreducible and u continuous the above statement improves a bit (regarding the points in A_{sing}). In particular it follows that for $u, v \in \mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}$ and u continuous if conditions 1 and 2 above hold then $v \leq v^* \leq u$ on Ω.

THEOREM 3.2.2 (Comparison Principle; [**11**]). *Let $u, v \in \mathrm{PSH}(A) \cap L^\infty_{loc}$, where A is a pure m dimensional irreducible algebraic subset of $\mathbb{C}^n$, be such that $\{u \leq v\} \subset\subset A$. Then we have*

$$\int_{\{u<v\}} (\mathrm{dd^c}\, v)^m \leq \int_{\{u<v\}} (\mathrm{dd^c}\, u)^m \,. \tag{3.2.1}$$

We recall that, following [**107**], we denote by $\mathcal{L}(A)$ the *Lelong class* of plurisubharmonic functions on A with respect to the *parabolic potential* $(z, w) \to \log |z|$, where (z, w) is a system of Rudin coordinates for A; see Proposition A.0.2. That is $u \in \mathcal{L}(A)$ if $u \in \mathrm{PSH}(A)$ and there exists a constant $C = C_u$ such that

$$u(z, w) \leq C_u + \log^+ |z|. \tag{3.2.2}$$

We introduce also the class $\mathcal{L}^+(A)$ of functions $u \in \mathcal{L}(A)$ such that there exists a constant C'_u such that $C'_u + \log^+ |z| \leq u(z, w)$. We need also this modified version of [**14**, Lemma 6.5].

THEOREM 3.2.3 (Global Domination Principle). *Let $u \in \mathcal{L}(A)$ and $v \in \mathcal{L}^+(A)$. Suppose that $u \leq v$ $(\mathrm{dd^c}\, v)^m$-a.e., then $u(\zeta) \leq v(\zeta)$ for any $\zeta \in A_{reg}$.*

PROOF. We refer to the original proof of [**14**, Lemma 6.5], the only modification being the improper integration over A_{reg}. □

REMARK 3.2.2. *We stress that, under the additional hypothesis of*

$$u(z_0) = \limsup_{A_{reg} \ni z \to z_0} u(z)\ ,\ v(z_0) = \limsup_{A_{reg} \ni z \to z_0} v(z)\ \forall z_0 \in A_{sing}$$

the conclusion of Theorem 3.2.3 holds at each point of A.

3.3. Capacities and Extremal Functions

3.3.1. Relative capacity and extremal function. Let A be an irreducible pure m-dimensional set in $\mathbb{C}^n$, we introduce the Monge-Ampere *relative capacity* following [**107**] and [**11**]. For any open set $\Omega \subset A$ and any Borel set $E \subset \Omega$ we set

$$\mathrm{Cap}(E,\Omega) := \sup\left\{\int_{E\cap A_{\mathrm{reg}}} (dd^c u)^m, u \in \mathrm{PSH}(\Omega),\ 0 \le u \le 1\right\}. \tag{3.3.1}$$

Also we define the *outer relative capacity* by setting

$$\mathrm{Cap}^*(E,\Omega) := \inf\{\mathrm{Cap}(O,\Omega), E \subseteq O \text{ open}\}.$$

It will be useful to use also the so called $(m-k)$*-relative capacities* for any $k = 1, 2, \ldots m-1$, we set

$$\underset{m-k}{\mathrm{Cap}}(E,\Omega) := \sup\left\{\int_{E\cap A_{\mathrm{reg}}} (dd^c u)^{m-k} \wedge \beta_m^k, u \in \mathrm{PSH}(\Omega),\ 0 \le u \le 1\right\}, \tag{3.3.2}$$

where β_m is the standard Kahler form induced by $\mathbb{C}^n$ on A_{reg}. We also introduce the outer $(m-k)$-relative capacities as above, that is

$$\mathrm{Cap}^*_{m-k}(E,\Omega) := \inf\{\underset{m-k}{\mathrm{Cap}}(O,\Omega), E \subseteq O \text{ open}\}.$$

It is worth to notice that, as in the case of Ω being a domain in $\mathbb{C}^n$ one has for any Borel subset E of the open set Ω

$$\mathrm{Cap}^*_{m-k}(E,\Omega) \le A_\Omega \mathrm{Cap}^*(E,\Omega). \tag{3.3.3}$$

Here the positive finite constant A_Ω depends only on Ω; see [**102**, pg. 458].

We will use the following definition.

Definition 3.3.1 (Pluripolar sets). *Let A be an algebraic set in $\mathbb{C}^n$ and $E \subset A$ be a Borel set. The set E is said to be* pluripolar *in A if $E \cap A_{reg}$ is pluripolar in the usual sense in the complex manifold A_{reg}.*

It is worth to say that usually one defines also *locally pluripolar* subsets P of a complex space X as sets such that for each $z_0 \in P$ there exists a neighbourhood O of z_0 in X such that $P \cap O \subseteq \{u = -\infty\}$ for a plurisubharmonic function on O not identically $-\infty$; the notion not a priori coinciding with being globally pluripolar.

However Bedford [**11**] showed that in our setting (and even in more general ones) the local and global definitions coincide.

REMARK 3.3.3. *Let us stress that A_{sing} is pluripolar in A by definition, since it does not contain any regular point.*

Let $u \in \mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}$ for some open set $\Omega \subset A$, we say that u is maximal if for any open relatively compact set $G \subset \Omega$ and any $v \in \mathrm{PSH}(G) \cap L^\infty_{\mathrm{loc}}$ such that $\liminf_{\zeta \to \partial G}(u(\zeta) - v(\zeta)) \geq 0$ we have $u \geq v$ in $G \cap A_{\mathrm{reg}}$. Note that in this sense *maximal plurisubharmonic functions* enjoy the role of harmonic functions in one complex variable.

REMARK 3.3.4. *We stress that, due to the Global Domination Principle Theorem 3.2.3, a locally bounded plurisubharmonic function satisfying the generalized Monge Ampere equation in Ω, that is $\int \varphi\, (\mathrm{dd^c}\, u)^m = 0$ for all $\varphi \in \mathscr{C}_c(\Omega)$, is necessarily maximal.*

As in the flat case we can introduce[2] the *relative extremal function* $U^*_{E,\Omega}$ as follows

$$U_{E,\Omega}(z, A) := \sup\{u(z) : u \in \mathrm{PSH}(\Omega),\ u \leq 0,\ u|_E \leq -1\} \tag{3.3.4}$$

$$U^*_{E,\Omega}(z, A) := \limsup_{A_{\mathrm{reg}} \ni \zeta \to z} U_{E,\Omega}(\zeta, A). \tag{3.3.5}$$

If A is clarified by the context we may drop it from the notation.

We stress that, due to Theorem C.1.1, and the properties of plurisubharmonic functions under upper envelopes, $U^*_{E,\Omega}(z, A)$ is either a plurisubharmonic function and identically -1 on $E \setminus N$, where N is a pluripolar set, or identically 0 in Ω. The latter situation occurs if and only if E is pluripolar in A. This follows by the original methods of Bedford and Taylor [**13**], see also [**107**], applied to $E \cap A_{\mathrm{reg}}$ in $\Omega \cap A_{\mathrm{reg}}$. Indeed, we could even define $U^*_{E,\Omega}(z, A)$ by the upper semi-continuous regularization $\limsup_{A_{\mathrm{reg}} \ni \zeta \to z} U_{E \cap A_{\mathrm{reg}}, \Omega \cap A_{\mathrm{reg}}}(\zeta)$ and this would lead to the same function.

THEOREM 3.3.4 (Extremal property of $U^*_{E,\Omega}$; [**107**]). *Let A, E, Ω be as above, then $U^*_{E,\Omega}(\cdot, A)$ is a maximal plurisubharmonic function on $\Omega \setminus E$ and we have*

[2]Notice that we are taking an upper envelope only among $\mathrm{PSH}(\Omega)$ functions. In [**107**] such an envelope is taken among $\widetilde{\mathrm{PSH}}(\Omega)$ functions but in our setting the two classes coincide, up to performing a upper semi-continuous regularization among A_{reg}.

$\left(dd^c\, U^*_{E,\Omega}\right)^m = 0$ *in* $\Omega \setminus \overline{E}$ *and*

(3.3.6)
$$\mathrm{Cap}(E,\Omega) = \int_{\overline{E}} \left(dd^c\, U^*_{E,\Omega}\right)^m = \int_{\Omega} \left(dd^c\, U^*_{E,\Omega}\right)^m = \int_{\Omega} -U^*_{E,\Omega} \left(dd^c\, U^*_{E,\Omega}\right)^m .$$

We will refer to the measure $\left(dd^c\, U^*_{E,\Omega}\right)^m$ as the *relative equilibrium measure* of E with respect to Ω and denote it by $\mu_{E,\Omega}$.

As in the case of $A = \mathbb{C}^n$ it is convenient to introduce the class of *hyperconvex* open sets. An open bounded subset Ω of the algebraic set A is said to be hyperconvex if there exists a function $\rho \in \mathrm{PSH}(\Omega, [-\infty, 0[)$ such that $\{z \in \Omega : \rho(z) < c\} \subset\subset \Omega\ \forall c < 0$, i.e., if there exists a negative plurisubharmonic *exhaustion function* ρ. Notice that one necessarily has $\lim_{\Omega \ni \zeta \to z} \rho(\zeta) = 0$ for all $z \in \partial\Omega$, existence of the limit being part of the statement.

The use of this class of open sets is easy to see: if Ω is a hyperconvex open subset of the pure m dimensional algebraic set A, then one has

$$\lim_{\Omega \ni \zeta \to z} U^*_{E,\Omega}(\zeta) = 0, \quad \forall z \in \partial\Omega, \forall E \subset \Omega \text{ compact.}$$

Proposition 3.3.3 ([**11**]). *Let A be a m dimensional algebraic subset of $\mathbb{C}^n$, then for any open bounded set Ω we have*

$$\mathrm{Cap}^*(A_{sing} \cap \Omega, \Omega) = 0, \tag{3.3.7}$$

$$\mathrm{Cap}^*_{m-k}(A_{sing} \cap \Omega, \Omega) = 0, \forall k = 1, 2, \ldots, m-k. \tag{3.3.8}$$

We remark that in [**11**] one can find only the proof of equation (3.3.7), while equation (3.3.8) follows by equation (3.3.3).

3.3.2. Global extremal functions. Let us introduce two other extremal functions mimicking the case of $\mathbb{C}^n$. Let A be a **analytic set** in $\mathbb{C}^n$ and E a compact subset of it, then we set

$$\log \Phi_E(z, A) := \sup \left\{ \frac{1}{\deg p} \log |p(z)|,\ p \text{ polynomial }, \|p\|_E \leq 1 \right\} \tag{3.3.9}$$

$$\log \Phi^*_E(z, A) := \limsup_{A_{\mathrm{reg}} \ni \zeta \to z} \log \Phi_E(\zeta, A). \tag{3.3.10}$$

We refer to $\log \Phi^*_E(z, A)$ as the Siciak extremal function; [**93**],[**94**],[**92**]. We also introduce the Zaharjuta-Sadullaev type extremal function $S^*_E(z, A)$, see [**105**], [**103**],[**104**] and [**88**], for A being an **analytic subset** of $\mathbb{C}^n$. For, let us denote by $\mathcal{L}(\mathbb{C}^n)$ the *Lelong class* of functions $u \in \mathrm{PSH}(\mathbb{C}^n)$ of logarithmic growth as $A \ni z \to \infty$.

For $z \in A$ we define

$$S_E(z, A) := \sup\{u(z),\ u \in \mathcal{L}(\mathbb{C}^n), \|u\|_E \le 1\} \tag{3.3.11}$$

$$S^*_E(z, A) := \limsup_{A_{\mathrm{reg}} \ni \zeta \to z} S_E(\zeta, A). \tag{3.3.12}$$

Also, for an *algebraic* pure dimensional irreducible set A we consider the *Lelong class* $\mathcal{L}(A)$ with respect to the parabolic potential $(z', z'') \to \log|z|$, where (z', z'') are Rudin coordinates for A, see Proposition A.0.2, and define the Zaharjuta-Zeriahi extremal function

$$V_E(z, A) := \sup\{u(z),\ u \in \mathcal{L}(A), \|u\|_E \le 1\} \tag{3.3.13}$$

$$V^*_E(z, A) := \limsup_{A_{\mathrm{reg}} \ni \zeta \to z} V_E(\zeta, A). \tag{3.3.14}$$

Note that a priori one has $V^*_E \ge S^*_E \ge \log \Phi^*_E$ by the obvious inclusion of the classes where we took upper envelopes.

The following characterization of algebraic sets due to Sadullaev is of main importance for our aims.

Theorem 3.3.5 (Characterization of algebraic sets by $\log \Phi^*_E(\cdot, A)$; [**88**]). *Let A be an irreducible pure m-dimensional analytic set in $\mathbb{C}^n$. The set A is an algebraic set (i.e., it is a subset of a pure m-dimensional algebraic subset $\tilde{A}$ of $\mathbb{C}^n$) if and only if the following condition holds.*

(3.3.15)

There exists a compact $E \subset A$ such that $S_E(\cdot, A)$ is locally bounded on A.

In such a case this holds for any compact non pluripolar set $E \subset A$. Moreover $S^*_E(\cdot, A)$ is a maximal plurisubharmonic function on $\tilde{A} \setminus E$, locally bounded on A for any non pluripolar compact set $E \subset A$.

Remark 3.3.5. *Notice that the intersection A of two irreducible pure m dimensional subsets $\tilde{A}_1$ and $\tilde{A}_2$ of $\mathbb{C}^m$ is either pluripolar in both $\tilde{A}_1$ and $\tilde{A}_2$ or coincides with A_1 and A_2. It follows that, in the case when the condition* (3.3.15) *is satisfied, the conclusions of the theorem hold true for any pure m dimensional algebraic set A_1 containing A and in particular on* ***the*** *pure m dimensional algebraic subset $\tilde{A}$ of $\mathbb{C}^n$ containing A. From here on we use the notation $\tilde{A}$ for such an algebraic subset of $\mathbb{C}^n$.*

In the flat case of $\mathbb{C}^n$ and E being compact one has $S^*_E \equiv V^*_E$ by definition, while Siciak and Zaharjuta showed that $S^*_E \equiv \log \Phi^*_E$.

The following result was proved for $m = 1$ by Sadullaev [**88**, Prop. 3.4], while just stated as a consequence of a conjecture (proved later by Bedford and Taylor) for $1 \leq m \leq n-1$, see Remark 3.5 of the same paper. Finally Zeriahi gave a explicit proof in [**107**], see also [**108**].

Proposition 3.3.4 (Siciak-Zaharjuta extremal function). *Let A be a pure irreducible m-dimensional algebraic subset of $\mathbb{C}^n$, then for each compact non pluripolar set $E \subset A$ we have*

$$\log \Phi^*_E(z, A) \equiv V^*_E(z, A) \equiv S^*_E(z, A). \tag{3.3.16}$$

3.3.2.1. *Bernstein Walsh Type Inequality.* We recall here for future use the following estimate of Bernstein Walsh type; see [**107**].

$$|p(z)| \leq \|p\|_E \exp(kV_E(z, A))\ \forall z \in A, p \in \mathscr{P}^k(\mathbb{C}^n). \tag{3.3.17}$$

Also, it is worth to mention that one can replace $V^*_E(z, A)$ in equation (3.3.17) by $V_E(z, A)$. This follows by the fact that $V_E(z, A)$ can be expressed by the supremum of a family of continuous function, thus it is lower semicontinuous on A. Therefore the replacement of the value of $V_E(z_0, A)$ at a singular point $z_0 \in A_{\text{sing}}$ by $\limsup_{A_{\text{reg}} \ni \zeta \to z_0} V_E(\zeta, A)$ will preserve the above inequality.

3.3.3. Regularity of a compact set.

Definition 3.3.2 (Regular set). *Let E be a compact (non pluripolar) subset of a pure m-dimensional algebraic set $A \subset \mathbb{C}^n$. The set E is said to be* regular *if V^*_E is continuous on E.*

We refer to [**101**, Sec. 3] for a discussion on a possible different definition that in our setting coincide with the one above.

It follows by adapting the argument as in [**59**, Prop. 5.3.3 and below] that one can equivalently define regular sets as the sets for which $U^*_{E,\Omega}$ is continuous on E for all open sets $\Omega \supseteq \hat{E}$, where $\hat{E}$ is the polynomial convex hull of E in A.

It is possible to get a stronger result. One can use the Domination Principle [**107**, 1.10] to show that if for some open neighbourhood Ω of E in A one has $U^*_{E,\Omega}|_E = -1$, then V^*_E is continuous on E, i.e., $V^*_E|_E = 0$ and vice-versa. Note that, in order to do that, one needs to know that μ_E (respectively $\mu_{E,\Omega}$) puts no mass on pluripolar sets, but this can be proven by the Chern Levine Nirenberg type estimate [**42**, Th. 2.2] using the local boundedness of V_E^* (respectively $U_{E,\Omega}^*$).

3.3.4. Chebyshev constant. Let A be an irreducible pure m-dimensional algebraic subset of $\mathbb{C}^n$ and let us considered embedded in $\mathbb{C}^n$ with a set of Rudin coordinates $\mathbb{C}^n \ni z = (z', z'') \in \mathbb{C}^m \times \mathbb{C}^{n-m}$, see Proposition A.0.2. We use the following notation

$$\Omega := \{z \in A : |z'| \leq 1\}$$

and we refer to Ω as the *pseudoball* of radius 1, we denote by $\overline{\Omega}$ the closure of Ω in A.

We introduce the Chebyshev constant $T(E, A)$ of $E \subset \Omega$ in A (relative to these coordinates) as follows:

$$m_j(E) := \inf\{\|p\|_E : p \in \mathscr{P}^j(\mathbb{C}^n), \|p\|_{\overline{\Omega}} \geq 1\}$$

$$T(E, A) := \inf_{j \geq 0} m_j(E)^{1/j} = \lim_j m_j(E)^{1/j}.$$

In the case of $A \equiv \mathbb{C}^n$ it has been proven by Siciak [**94**] that one has

$$T(E) := T(E, \mathbb{C}^n) = \exp\left(-\|V_E^*\|_{\overline{\Omega}}\right).$$

It turns out that the same holds true in a pure dimensional irreducible algebraic set.

Proposition 3.3.5 ([**94**], [**108**]). *Let A be a pure m-dimensional irreducible algebraic set, then for any compact subset E of Ω one has*

$$T(E, A) = \exp\left(-\|V_E^*\|_{\overline{\Omega}}\right).$$

3.4. Continuity Property of $(\mathrm{dd^c})^k$ Operator under Monotone Limits

In [**13**] authors introduce the operator $\mathscr{L}^k$ (generalizing $(\mathrm{dd^c})^k$) mapping $\mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}$ in $\left(\mathcal{D}_0^{m-k,m-k}(\Omega)\right)'$, the space of (k,k) currents of order zero, see Appendix B, where

$$\mathscr{L}^k(u^0, \dots u^k)[\psi] := \int u^0 \, \mathrm{dd^c}\, u^1 \wedge \cdots \wedge \mathrm{dd^c}\, u^k \wedge \psi, \ \forall \psi \in \mathcal{D}^{m-k,m-k}(\Omega)$$

and Ω is any domain in $\mathbb{C}^n$.

They show the continuity under decreasing monotone limits of plurisubharmonic locally bounded functions both of $(\mathrm{dd^c})^k$ and $\mathscr{L}^k$. This result has been extended to a much more general context by Bedford in [**11**]. We recall such a result here for using it later on, while we offer the proof of a slightly weaker version in Appendix D.

Following Bedford [**11**] we introduce the following notation.

DEFINITION 3.4.3 ($\mathscr{A}^k(A)$). *Let $\mathscr{A}^k(A)$, $k \leq m$ be the linear space of wedge products of factors of the type*

a) smooth forms θ on A

b) currents of the form du, $d^c u$ or $\mathrm{dd^c}\, u$ for $u \in \mathrm{PSH}(A) \cap L^\infty_{loc}$,

such that the total sum of the bi-degrees does not exceed k.

We also recall that, given an open subset Ω of the pure m-dimensional algebraic set A, a sequence of Borel functions f_j is said to converge *quasi uniformly* to the Borel function f on Ω if

- they are locally uniformly bounded on Ω, uniformly in j,
- $f_j \to f$ almost everywhere with respect to β_m^m,
- for each $\epsilon > 0$ there exists an open set $O_\epsilon \subset \Omega$ such that $\mathrm{Cap}(O_\epsilon, \Omega) < \epsilon$ and $f_j \to f$ uniformly on $\Omega \setminus O_\epsilon$.

THEOREM 3.4.6 (Continuity under monotone limits; [**11**]). *Let Ω be an open set of the pure m dimensional algebraic set A, and let*

(1) $\{f_j\}$ be a sequence of functions converging quasi uniformly to f,
(2) ψ_j^l be sequences of smooth (p_l, q_l) forms on Ω converging locally uniformly to the forms ψ^l,
(3) $\{u_j^l\}$ be sequences of function in $\mathrm{PSH}(\Omega) \cap L^\infty_{loc}$ converging monotonically almost everywhere to the functions $u^l \in \mathrm{PSH}(\Omega) \cap L^\infty_{loc}$.

Then the sequence of Radon measures

$$\mu_j := f_j \psi_j^1 \wedge \cdots \wedge \psi_j^{l_1} \wedge du_j^{l_1+1} \wedge d^c u_j^{l_1+2} \wedge \cdots \wedge \mathrm{dd^c}\, u_j^{l_2} \wedge \ldots \mathrm{dd^c}\, u_j^{l_3},$$

where the bi-degrees are such that $\mu_j \in \mathscr{A}^{2m}(\Omega)$, i.e.

$$\sum_{l=1}^{l_1} p_l + (l_2 - l_1 + 1) + 2(l_3 - l_2 + 1) = m,$$

$$\sum_{l=1}^{l_1} q_l + (l_2 - l_1 + 1) + 2(l_3 - l_2 + 1) = m$$

converges weak on* Ω *to*

$$f\psi^1 \wedge \cdots \wedge \psi^{l_1} \wedge du^{l_1+1} \wedge d^c u^{l_1+2} \wedge \cdots \wedge \mathrm{dd^c}\, u^{l_2} \wedge \ldots \mathrm{dd^c}\, u^{l_3}.$$

3.5. Integration by Parts on Algebraic Varieties

3.5.1. The Stokes Theorem and its consequences. An important feature of any pure dimensional analytic (and in particular algebraic) set A is that a version of the Stokes Theorem holds on it when one considers *compactly supported* forms on A with differentiable coefficients; see [**38**, Ch. 14 sec. 3]. We recall that a $\mathscr{C}^s$ form η on a open set $\Omega \subset \mathbb{C}^n$ is said to have compact support on the algebraic set A if $\operatorname{supp} \eta \cap A$ is a compact set, in such a case we say that η is of class $\mathscr{C}^s_c(A)$.

The combination of the Stokes Theorem, the existence of smooth decreasing approximations to plurisubharmonic functions and Theorem 3.4.6 allows to prove the following.

Proposition 3.5.6 (Integration by parts formula for smooth forms). *Let A be a pure m dimensional algebraic set in $\mathbb{C}^n$, $u_1, u_2, \ldots, u_m$ locally bounded plurisubharmonic functions on A and η a $\mathscr{C}^2_c(A)$ function, then we have*

$$\int \eta \,\mathrm{dd^c}\, u^1 \wedge \cdots \wedge \mathrm{dd^c}\, u^m = \int u^1 \,\mathrm{dd^c}\, \eta \wedge \mathrm{dd^c}\, u^2 \wedge \cdots \wedge \mathrm{dd^c}\, u^m. \tag{3.5.1}$$

Proof. We simply pick a sequence of smooth (possibly non plurisubharmonic) monotonically decreasing approximations u^l_j converging to u^l for each $l = 1, 2, \ldots, m$ as in Lemma C.2.1. Note that $u^1_j \to u^1$ quasi uniformly due to Proposition C.2.2, thus

$$\mathrm{dd^c}\, u^1_j \wedge \mathrm{dd^c}\, u^2_j \cdots \wedge \mathrm{dd^c}\, u^m_j \rightharpoonup^* \mathrm{dd^c}\, u^1 \wedge \mathrm{dd^c}\, u^2 \cdots \wedge \mathrm{dd^c}\, u^m$$

and

$$u^1_j \,\mathrm{dd^c}\, u^2_j \wedge \cdots \wedge \mathrm{dd^c}\, u^m_j \rightharpoonup^* u^1 \,\mathrm{dd^c}\, u^2 \wedge \cdots \wedge \mathrm{dd^c}\, u^m.$$

Therefore we have

$$\int \eta \,\mathrm{dd^c}\, u^1 \wedge \cdots \wedge \mathrm{dd^c}\, u^m = \lim_j \int \eta \,\mathrm{dd^c}\, u^1_j \wedge \cdots \wedge \mathrm{dd^c}\, u^m_j$$

$$\lim_j \int u^1_j \wedge \mathrm{dd^c}\, \eta \wedge \mathrm{dd^c}\, u^2_j \cdots \wedge \mathrm{dd^c}\, u^m_j = \int u^1 \,\mathrm{dd^c}\, \eta \wedge \mathrm{dd^c}\, u^2 \wedge \cdots \wedge \mathrm{dd^c}\, u^m.$$

□

3.5.2. Stokes Theorem for currents and integration by parts formulas for plurisubharmonic functions. A stronger statement of the Stokes Theorem is proved in [**11**]; we recall that given $\eta \in \mathscr{A}^{2m-1}(A)$ (see Definition 3.4.3) is said to have compact support if there exists a compact set $S \subset A$ such that for each smooth form ψ compactly supported in $A \setminus S$ one has $\langle d\psi, \eta\rangle = 0$ and $\langle d^c\psi, \eta\rangle = 0$.

THEOREM 3.5.7 (Stokes in $\mathscr{A}^{2m-1}(A)$; [**11**]). *Let $\eta \in \mathscr{A}^{2m-1}(A)$ have compact support, then*

$$\int d\eta = 0.$$

The above theorem is very important to our aims since it allows to prove the following integration by parts formulas, the first and the second being partial extensions of Theorems [**39**, Th. 3.1 and 3.3].

THEOREM 3.5.8 (Integration by parts for plurisubharmonic functions I). *Let Ω be a open bounded hyperconvex subset of the pure m-dimensional algebraic set A in $\mathbb{C}^n$, $w \in \mathrm{PSH}(\Omega) \cap L^\infty_{loc}$ and $u, v \in \mathrm{PSH}(\Omega) \cap L^\infty_{loc}$ be negative functions. Assume that $u \equiv v$ on $\Omega \setminus K$ for a compact set $K \subset \Omega$. Then*

$$\int_\Omega u\, \mathrm{dd^c}\, v \wedge (\mathrm{dd^c}\, w)^{m-1} = \int_\Omega v\, \mathrm{dd^c}\, u \wedge (\mathrm{dd^c}\, w)^{m-1}. \tag{3.5.2}$$

PROOF. We consider the current $\eta \in \mathscr{A}^{2m-1}$, $\eta := [(u-v)d^c v - v d^c(u-v)] \wedge (\mathrm{dd^c}\, w)^{m-1}$ and we claim that it is compactly supported. For, we pick any $\varphi \in \mathscr{C}^\infty_c(\Omega \setminus K)$ and we compute $\langle \eta, d\varphi\rangle$ using smooth approximations u_j, v_j to u, v as in Lemma C.2.1 (produced relaying on the same covering and the same partition of unity for u and v) and the continuity property Theorem 3.4.6.

$$\langle \eta, d\varphi\rangle = \lim_j \int d\varphi \wedge [(u_j - v_j)d^c v_j - v_j d^c(u_j - v_j)] \wedge (\mathrm{dd^c}\, w_j)^{m-1}.$$

Note that since the support S of φ is compactly contained in the set $\{z \in \Omega : u \equiv v\}$, we have $S \subset \{z \in \Omega : u_j \equiv v_j\}$ for j large enough. Therefore the right hand side of the above equation vanishes identically for $j > j_\varphi$, thus the limit is zero.

A similar approximation argument shows, in particular, that $(du \wedge d^c v - dv \wedge d^c u) \wedge (\mathrm{dd^c}\, w)^{m-1}$ is the zero current.

Thus Theorem 3.5.7 implies

$$\begin{aligned} 0 = \int d\eta &= \int d[(u-v)d^c v - v d^c(u-v)] \wedge (\mathrm{dd^c}\, w)^{m-1} \\ &= \int [d(u-v) \wedge d^c v - dv \wedge d^c(u-v)] \wedge (\mathrm{dd^c}\, w)^{m-1} + \\ &\quad + \int [(u-v)\, \mathrm{dd^c}\, v - v\, \mathrm{dd^c}\,(u-v)] \wedge (\mathrm{dd^c}\, w)^{m-1} \\ &= \int (du \wedge d^c v - dv \wedge d^c u) \wedge (\mathrm{dd^c}\, w)^{m-1} + \end{aligned}$$

$$+\int u\,\mathrm{dd^c}\,v\wedge(\mathrm{dd^c}\,w)^{m-1}-\int v\,\mathrm{dd^c}\,u\wedge(\mathrm{dd^c}\,w)^{m-1}$$
$$=\int u\,\mathrm{dd^c}\,v\wedge(\mathrm{dd^c}\,w)^{m-1}-\int v\,\mathrm{dd^c}\,u\wedge(\mathrm{dd^c}\,w)^{m-1}.$$

□

THEOREM 3.5.9 (Integration by parts for plurisubharmonic functions II). *Let Ω be a open bounded hyperconvex subset of the pure m-dimensional algebraic set A in $\mathbb{C}^n$, $w\in\mathrm{PSH}(\Omega)\cap L^\infty_{loc}$ and $u,v\in\mathrm{PSH}(\Omega)$ be bounded functions.*

a) Assume that v is a negative exhaustion function for Ω and $\int_\Omega \mathrm{dd^c}\,v\wedge(\mathrm{dd^c}\,w)^{m-1}<\infty$, then

$$\int_\Omega u\,\mathrm{dd^c}\,v\wedge(\mathrm{dd^c}\,w)^{m-1}\geq\int_\Omega v\,\mathrm{dd^c}\,u\wedge(\mathrm{dd^c}\,w)^{m-1}. \tag{3.5.3}$$

b) Equality holds if both u and v are negative exhaustion functions for Ω and $\int_\Omega \mathrm{dd^c}\,u\wedge(\mathrm{dd^c}\,w)^{m-1}<\infty$, $\int_\Omega \mathrm{dd^c}\,v\wedge(\mathrm{dd^c}\,w)^{m-1}<\infty$.

PROOF. Let us pick $\epsilon>0$ and set $u_j:=\max\{u-\epsilon, jv\}$. By the assumptions on v we have $u_j\equiv jv$ on some neighbourhood of the boundary, moreover the sequence u_j decreases point-wise to $u-\epsilon$. It follows by the Monotone Convergence Theorem and the assumption $\int_\Omega \mathrm{dd^c}\,v\wedge(\mathrm{dd^c}\,w)^{m-1}<\infty$ that

$$\int_\Omega(u-\epsilon)\,\mathrm{dd^c}\,v\wedge(\mathrm{dd^c}\,w)^{m-1}=\lim_j\int_\Omega u_j\,\mathrm{dd^c}\,v\wedge(\mathrm{dd^c}\,w)^{m-1}.$$

On the other hand by Theorem 3.5.8 we have

$$\int_\Omega u_j\,\mathrm{dd^c}\,v\wedge(\mathrm{dd^c}\,w)^{m-1}=\int_\Omega v\,\mathrm{dd^c}\,u_j\wedge(\mathrm{dd^c}\,w)^{m-1}.$$

Now notice that the measure $v\,\mathrm{dd^c}\,u_j\wedge(\mathrm{dd^c}\,w)^{m-1}$ is negative for each j and hence

$$\int_\Omega v\,\mathrm{dd^c}\,u_j\wedge(\mathrm{dd^c}\,w)^{m-1}\leq\int_\Omega\varphi v\,\mathrm{dd^c}\,u_j\wedge(\mathrm{dd^c}\,w)^{m-1},\forall\varphi\in\mathscr{C}_c(\Omega,[0,1]).$$

Thus

$$\int_\Omega u\,\mathrm{dd^c}\,v\wedge(\mathrm{dd^c}\,w)^{m-1}-\epsilon\int_\Omega \mathrm{dd^c}\,v\wedge(\mathrm{dd^c}\,w)^{m-1}$$
$$=\int_\Omega(u-\epsilon)\,\mathrm{dd^c}\,v\wedge(\mathrm{dd^c}\,w)^{m-1}=\lim_j\int_\Omega u_j\,\mathrm{dd^c}\,v\wedge(\mathrm{dd^c}\,w)^{m-1}$$
$$=\lim_j\int_\Omega v\,\mathrm{dd^c}\,u_j\wedge(\mathrm{dd^c}\,w)^{m-1}\leq\int_\Omega\varphi v\,\mathrm{dd^c}\,u_j\wedge(\mathrm{dd^c}\,w)^{m-1}$$
$$=\int_\Omega\varphi v\,\mathrm{dd^c}\,u\wedge(\mathrm{dd^c}\,w)^{m-1}.$$

Here we used that $v\,dd^c u_j \wedge (dd^c w)^{m-1} \rightharpoonup^* v\,dd^c u \wedge (dd^c w)^{m-1}$ by Theorem 3.4.6.

Since we assumed $\int_\Omega dd^c v \wedge (dd^c w)^{m-1} < \infty$, letting $\epsilon \to 0^+$ we get

$$\int_\Omega u\,dd^c v \wedge (dd^c w)^{m-1} \le \int_\Omega \varphi v\,dd^c u \wedge (dd^c w)^{m-1}, \ \forall \varphi \in \mathscr{C}_c(\Omega, [0,1]).$$

Finally, by the inner regularity of the Borel measure $-v\,dd^c u \wedge (dd^c w)^{m-1}$ we get

$$\int_\Omega u\,dd^c v \wedge (dd^c w)^{m-1} \le \int_\Omega v\,dd^c u \wedge (dd^c w)^{m-1}.$$

The equality case is obtained by the same procedure interchanging u and v. □

Using the integration by parts Theorem 3.5.9, Theorem 3.4.6 and Corollary D.0.1 one can extend to the case of irreducible pure dimensional algebraic sets a useful estimate holding in $\mathbb{C}^n$, we refer the reader to [**19**, Th. 2.1.8] for a detailed proof.

Proposition 3.5.7 ([**19**]). *Let Ω be a bounded hyperconvex subset of the pure m dimensional irreducible algebraic set $A \subset \mathbb{C}^n$ and K any compact subset of Ω. Let u, v, w be bounded plurisubharmonic functions on Ω such that*

i) $\lim_{\zeta \to \partial\Omega}(v(\zeta) - u(\zeta)) = 0$ *and*
ii) $v \ge u$ *on* Ω.

Then the following holds for any $p \in \mathbb{N}$, $p > m$.

$$\int_\Omega (v-u)^p\,(dd^c w)^m \le \frac{p!}{(p-m)!}\|w\|_\Omega^m \int_\Omega (v-u)^{p-m}\,(dd^c u)^m . \tag{3.5.4}$$

3.6. The Poisson Jensen Lelong Formula

Let $\varphi \in \mathrm{PSH}(A)$ be a non-positive continuous exhaustive function for A, $-\infty < R < 0$ such that $\Omega_R := \{z \in A : \varphi(z) < R\} \subset\subset A$, for any $-\infty < r < R$ we denote by φ_r the plurisubharmonic function $\max\{\varphi, r\}$. Following Demailly we introduce the family of measures

$$\mu_r := (dd^c \varphi_r)^m - \chi_{A \setminus \Omega_r}(dd^c \varphi)^m .$$

Here χ_S is the characteristic function of the set S.

Theorem 3.6.10 (Poisson Jensen Lelong Formula; [**42**],[**43**]). *Let A be a m dimensional algebraic set of $\mathbb{C}^n$, $u \in \mathrm{PSH}(A) \cap L^\infty_{loc}(A)$ then $u \in L^1(\mu_r)$ for any*

$-\infty < r < R$ *and for such* r *we have*

$$\begin{aligned}\int u d\mu_r &= \int_{\Omega_r} u\,(\mathrm{dd^c}\,\varphi_r)^m + \int_{-\infty}^{r}\int_{\Omega_t} \mathrm{dd^c}\,u \wedge (\mathrm{dd^c}\,\varphi)^{m-1} dt \\ &= \int_{\Omega_r} u\,(\mathrm{dd^c}\,\varphi_r)^m + \int_{\Omega_r} (r-\varphi)\,\mathrm{dd^c}\,u \wedge (\mathrm{dd^c}\,\varphi)^{m-1}.\end{aligned} \tag{3.6.1}$$

CHAPTER 4

Two New Results in Pluripotential Theory on Algebraic Sets

"Why did you want to climb Mount Everest?"
Because it's there.

George Mallory[a]

[a]This question was asked of George Leigh Mallory, who was with both expeditions toward the summit of the world' s highest mountain, in 1921 and 1922. He plans to go again in 1924, and he gave as the reason for persisting in these repeated attempts to reach the top, "Because it's there."

4.1. Introduction

The aim of this Chapter is to prove two new results in the context of Pluripotential Theory on algebraic subsets of $\mathbb{C}^n$. We refer to Chapter 3 for all the definitions and the results about Pluripotential Theory both in $\mathbb{C}^n$ and on its algebraic subsets. In Pluripotential Theory in $\mathbb{C}^n$ many capacities have been introduced as *relative capacity*, *projective capacity*, *transfinite diameter*, *Chebychev constant* or *Siciak capacity* see for instance [**65**], [**60**], [**1**], [**87**] and [**26**]. In contrast with the case of $\mathbb{C}$, if $n > 1$ these capacities are not in general equal, but they have been proved

- to be comparable and therefore
- to characterize pluripolar sets, i.e. $C_\alpha(E) = 0$ is equivalent to E being pluripolar, for any of these capacities C_α.

Note that the generalization of the notions of Chebyshev constant and transfinite diameter to algebraic varieties is a current subject of research, some interesting progress has been done in [**71**] and [**5**]. It is worth to underline that the definition of Chebyshev constant in the aforementioned papers differs from the one of this Chapter.

Here we deal with a given irreducible pure m-dimensional algebraic subset A of $\mathbb{C}^n$, for any $m < n$.

Section 4.2 is dedicated to prove the comparability of the Chebyshev constant $T(E, A)$ with normalization on the pseudo-ball

$$\Omega := \{z \in A : |z'| < 1\},$$

where $\mathbb{C}^n \ni z \to (z', z'') \in \mathbb{C}^m \times \mathbb{C}^{n-m}$ is a system of Rudin coordinates for A, see Proposition A.0.2, and the relative capacity $\mathrm{Cap}(E, \Omega)$ (with respect to the same pseudo-ball) for any compact subset E of Ω. This result extends [**2**, Theorem 2.1] proved by Alexander and Taylor in the case $A \equiv \mathbb{C}^n$.

We state such result in Theorem 4.2.1 and give the proof in Subsection 4.2.2.

Our main motivation for the study of the comparability of Chebyshev constant and relative capacity is given by the fact that this allows to compare the maximum of the Siciack-Zaharjuta extremal plurisubharmonic function $V_E^*(z, A)$ (see Subsection 3.3.2) on the closure of the pseudo-ball Ω (see equation 4.2.1 below) with the relative capacity of E with respect to the same pseudo-ball. Since the plurisubharmonic function $v := V_E^*(z, A)\|V_E^*(z, A)\|_{\overline{\Omega}}^{-1} - 1$ is a competitor for the upper envelope defining the relative extremal function $U_{E,\Omega}^*(z)$ (see 3.3.1), the comparability of $T(E, A)$ with $\mathrm{Cap}(E, \Omega)$ boils down to a comparability of $V_E^*(z, A)$ with $U_{E,\Omega}^*(z)$.

In **Section** 4.3 we use this comparabilities to study the relationship among the following properties that a sequence $\{E_j\}$ of subsets of E may have (the mode of convergence will be specified later and depends on the assumptions on E)

(1) $\mathrm{Cap}(E_j, \Omega) \to \mathrm{Cap}(E, \Omega)$,
(2) $U_{E_j,\Omega}^* \to U_{E,\Omega}^*$,
(3) $\mu_{E_j} \to \mu_E$,
(4) $V_{E_j}^*(\cdot, A) \to V_E^*(\cdot, A)$.

Here $\mu_E := \left(dd^c\, U_{E,\Omega}^*\right)^m$ is the relative Pluripotential equilibrium measure of E with respect to Ω and $(dd^c)^m$ is the Monge Ampere operator on A; see Theorem 3.3.4 and Section 3.1.

This study has been done in the "flat" case of $\mathbb{C}^n$ by Bloom and Levenberg [**24**], including also the Robin Function. In Theorem 4.3.2 we state that the above properties (1)-(4) are equivalent with a mode of convergence depending on the further assumptions we may do on E. This is the analogue of [**24**, Th. 1.1,Th. 1.2] in our setting.

4.2. Comparison Theorem for Relative Capacity and Chebyshev Constant

4.2.1. Statement of the Result. Here we consider a pure m dimensional irreducible algebraic subset A of $\mathbb{C}^n$, $n > m$, that we suppose to be endowed in $\mathbb{C}^n = \mathbb{C}^m \times \mathbb{C}^{n-m}$ by a set of Rudin coordinates $(z', z'') \in \mathbb{C}^m \times \mathbb{C}^{n-m}$, see Proposition A.0.2.

It is convenient to introduce some further notations. We denote by $\pi : A \to \mathbb{C}^m$ the coordinate projection $z \mapsto z'$ and use the following symbols for the pseudo-balls

$$
\begin{aligned}
\Omega(z_0, r) &:= \{(z', z'') \in A : |z' - z_0'| < r\}, \\
\Omega(r) &:= \{(z', z'') \in A : |z'| < r\}, \\
\Omega &:= \Omega(1).
\end{aligned}
\tag{4.2.1}
$$

Let us notice that each of the above pseudo-balls is a hyperconvex set (see Subsection 3.3.1 below Theorem 3.3.4), being $\rho_{r,z_0}(z) := |z' - z_0|^2 - r^2$ a negative plurisubharmonic exhaustion function for it.

We recall here for the reader's convenience the definitions of Chebyshev constant, relative capacity and relative extremal function; see Chapter 3.

$$
m_j(E) := \inf\{\|p\|_E : p \in \mathscr{P}^j(\mathbb{C}^n), \|p\|_{\overline{\Omega}} \geq 1\},
$$

$$
T(E, A) := \inf_{j \geq 0} m_j(E)^{1/j} = \lim_j m_j(E)^{1/j}.
$$

$$
\operatorname{Cap}(E, \Omega) := \sup \left\{ \int_{E \cap A_{\mathrm{reg}}} (dd^c u)^m, u \in \mathrm{PSH}(\Omega),\ 0 \leq u \leq 1 \right\}.
$$

$$
U_{E,\Omega}(z, A) := \sup\{u(z) : u \in \mathrm{PSH}(\Omega),\ u \leq 0,\ u|_E \leq -1\},
$$

$$
U^*_{E,\Omega}(z, A) := \limsup_{A_{\mathrm{reg}} \ni \zeta \to z} U_{E,\Omega}(\zeta, A).
$$

One can a priori consider any open bounded hyperconvex set $B \subset A$ in place of Ω.

We also stress that we may drop A from the definition of $U^*_{E,\Omega}(z, A)$ when it is clear by the context or even replace it by $\mathbb{C}^m$ when we want to consider the (standard) relative extremal function of some compact subset of the unit ball in $\mathbb{C}^m$.

Here is our main result of this section.

Theorem 4.2.1 (Comparison of Chebyshev Constant and Relative Capacity). *Let A be a irreducible pure m-dimensional algebraic subset of $\mathbb{C}^n$. For any $0 < r < 1$ there exist two positive constants c_1, c_2 (depending only on A and r) such that for* *any compact non pluripolar $E \subset \Omega(r)$ we have*

$$\exp\left[-\left(\frac{c_1}{\mathrm{Cap}(E,\Omega)}\right)^{1/m}\right] \geq T(E,A), \tag{4.2.2}$$

$$T(E,A) \geq \exp\left(-\frac{c_2}{\mathrm{Cap}^2(E,\Omega)}\right). \tag{4.2.3}$$

In particular

$$\max_{\overline{\Omega}} V_E(\cdot, A) \leq \frac{c_2}{\mathrm{Cap}^2(E,\Omega)}. \tag{4.2.4}$$

It is worth to compare Theorem 4.2.1 with its $\mathbb{C}^n$ analogue [**2**, Th. 2.1]. The two statements are equivalent except for of the exponent in the right hand side of (4.2.3). In our inequality the capacity is squared while in the Alexander and Taylor version such an exponent is one. This allow them to prove certain optimality property of the bound itself that we can not prove for the reason above.

This difference is intrinsic in the strategy of the proof, where one compares (extremal functions and capacities of) the compact set E with its projection $\pi(E)$ on the first m coordinates z' and with the lifting (back to A) $\pi^{-1} \circ \pi(E)$ of the projection. In this sense our proof plays the same procedure of [**2**, Th. 2.1] *twice*, however working on a (non smooth) algebraic set in place of an euclidean space causes several technical obstacles that we need to overcome.

4.2.2. Proof of Theorem 4.2.1.

Proof of (4.2.4). We notice that, given equation (4.2.3), the estimate (4.2.4) follows by

$$T(E,A) = -\log(\sup_{\Omega} V_E(\cdot, A)), \tag{4.2.5}$$

see Proposition 3.3.5, [**108**]. □

Proof of (4.2.2). The proof is equivalent to the one of [**2**], but uses the Comparison Principle for complex spaces, see Theorem 3.2.2, in lieu of the one for $\mathbb{C}^n$, see [**13**].

Precisely, one takes a big pseudoball $\Omega(R)$ containing Ω and for each $\epsilon > 0$ picks $A(\epsilon)$ such that, setting $v_\epsilon(z) := (1-\epsilon)\log^+|z'| + A(\epsilon)$, we have $\liminf_{A \ni z \to \partial\Omega(R)}(V_E^*(z,A) - v_\epsilon(z)) \geq 0$ and $V_E^*(z,A) \leq v_\epsilon(z)$ q.e. on E.

Then, by Comparison Principle, we obtain

$$\int_E (\mathrm{dd^c}\, v_\epsilon)^m = \int_{V_E^*(z,A)<v_\epsilon} (\mathrm{dd^c}\, v_\epsilon)^m \leq \int_{V_E^*(z,A)<v_\epsilon} \left(\mathrm{dd^c}\, V_E^*(z,A)\right)^m = \int_E \left(\mathrm{dd^c}\, V_E^*(z,A)\right)^m .$$

Hence in particular

$$\int_E \left(\mathrm{dd^c}\, V_E^*(z,A)\right)^m \geq (1-\epsilon)^m \int_A (\mathrm{dd^c} \log^+ |z'|)^m .$$

Then we repeat the same argument, but we consider the functions

$$v := \frac{V_E^*(z,A)}{\|V_E^*(z,A)\|_\Omega} - 1 \ , \quad u := (1+\epsilon) U_{E,\Omega}^*(z,A)$$

to obtain

$$\begin{aligned} &\mathrm{Cap}(E,\Omega) = \\ =&\frac{1}{(1+\epsilon)^m}\int_\Omega (\mathrm{dd^c}\, u)^m \geq \left(\frac{1}{(1+\epsilon)\|V_E^*(z,A)\|_\Omega}\right)^m \int_A \left(\mathrm{dd^c}\, V_E^*(z,A)\right)^m \\ \geq&\left(\frac{1}{\|V_E^*(z,A)\|_\Omega}\right)^m \int_A (\mathrm{dd^c} \log^+ |z'|)^m . \end{aligned}$$

The proof is concluded using (4.2.5). □

The proof of (4.2.2) is quite long and technical, hence we prefer to split it in some lemmata.

We set

$$K = \pi(E)\ , \quad H = \pi^{-1} \circ \pi(E).$$

Also we consider the positive $(1,1)$ form

$$\beta_m := \mathrm{dd^c}\, |z'|^2 = \frac{1}{2}\sum_{j=1}^m dz'^j \wedge d\bar{z}'^j.$$

Notice that $\beta_m = \mathrm{dd^c}\, \rho_r$ for any positive r. We will denote by $\mathcal{I}$ the canonical inclusion of A_{reg} in $\mathbb{C}^n$ that needs to be used to define integrations of global forms on A_{reg}. To avoid an heavier notation we will sometimes identify (for instance) $\mathcal{I}^*\beta_m$ with β_m, since the domain of integration will clarify it.

Lemma 4.2.1. *Let $\tilde{\Omega}$ be a bounded hyperconvex open set of $\mathbb{C}^m$ and let $\Omega = \pi^{-1}\tilde{\Omega}$ (in particular it is a bounded open hyperconvex subset of A). Let $\theta \in \mathcal{D}^{1,1}(\Omega)$ depend only on z', then there exists a positive constant C depending only on n, m, A*

such that

$$(C\,\|\theta\|_{\Omega}\,\beta_m \;\pm\; \theta\,) \quad \textit{is a strongly positive form on } A_{reg}. \tag{4.2.6}$$

Proof. This can be proved modifying the proof of [**59**, Prop. 3.2.7] and taking into account that θ is depending only on z'. Notice that only positivity needs to be proved since for $(1,1)$ forms the notion of positivity coincides with the strong one. □

We need the following specific version of the Chern Levine Nirenberg Estimate [**42**, Th. 2.2], [**59**].

Proposition 4.2.2 (Chern Levine Nirenberg type Estimate). *Let D be an open bounded subset of A, for any compact subset E of the open relatively compact domain $D' \subset D$ there exists a constant C depending on D, D' such that $\forall u \in$ $\mathrm{PSH}(D, [-\infty, 0[) \cap L^{\infty}_{loc}$,*

$$\int_E (\mathrm{dd^c}\, u)^m \le C^{m-1} \|u\|_{D'}^{m-1} \int_{D'} \mathrm{dd^c}\, u \wedge \beta_m^{m-1}. \tag{4.2.7}$$

Proof. Let us pick a increasing sequence of compact subsets H_j, $j = 0, 1, \dots, m$ of $\pi D'$ with $H_0 = K = \pi E$. Also we pick a sequence of smooth cut-off functions η_j such that $\forall j = 0, \dots, m-1$

$$\begin{aligned} &\eta_j \in \mathscr{C}^{\infty}_c(H_{j+1}, [0,1]),\\ &\eta_j|_{H_j} \equiv 1. \end{aligned}$$

We denote the lifting $\eta_j \circ \pi$ of η_j still by η_j, while we set $G_j := \pi^{-1} H_j$.

Here we use Proposition 3.5.6 and the constant C is chosen accordingly to Lemma 4.2.1.

$$\begin{aligned} \int_E (\mathrm{dd^c}\, u)^k \wedge \beta_m^{m-k} &\le \int_{G_1} \eta_0 (\mathrm{dd^c}\, u)^k \wedge \beta_m^{m-k} \\ = \int_{G_1} & u (\mathrm{dd^c}\, u)^{k-1} \wedge \mathrm{dd^c}\, \eta_0 \wedge \beta_m^{m-k} \\ = \int_{G_1} & u (\mathrm{dd^c}\, u)^{k-1} \wedge (\mathrm{dd^c}\, \eta_0 + C \|\mathrm{dd^c}\, \eta_0\|_{G_1} \beta_m) \wedge \beta_m^{m-k} + \\ & + C \|\mathrm{dd^c}\, \eta_0\|_{G_1} \int_{G_1} -u (\mathrm{dd^c}\, u)^{k-1} \wedge \beta_m^{m-k+1}. \end{aligned}$$

Since u is negative and $(\mathrm{dd^c}\, u)^{k-1} \wedge (\mathrm{dd^c}\, \eta_0 + C\|\mathrm{dd^c}\, \eta_0\|_{G_1} \beta_m) \wedge \beta_m^{m-k}$ is positive by Lemma 4.2.1, the term $\int_{G_1} u (\mathrm{dd^c}\, u)^{k-1} \wedge (\mathrm{dd^c}\, \eta_0 + C\|\mathrm{dd^c}\, \eta_0\|_{G_1} \beta_m) \wedge \beta_m^{m-k}$ is negative, thus we have

$$\int_E (\mathrm{dd^c}\, u)^k \wedge \beta_m^{m-k} \leq C \|\,\mathrm{dd^c}\,\eta_0\|_{G_1} \int_{G_1} -u(\mathrm{dd^c}\, u)^{k-1} \wedge \beta_m^{m-k+1}$$

$$\leq C \|\,\mathrm{dd^c}\,\eta_0\|_{G_1} \|u\|_{G_1} \int_{G_1} (\mathrm{dd^c}\, u)^{k-1} \wedge \beta_m^{m-k+1}.$$

Now we perform the second step replacing E by G_1, η_0 by η_1 and G_1 by G_2, so we get

$$\int_{G_1} (\mathrm{dd^c}\, u)^{k-1} \wedge \beta_m^{m-k+1} \leq C \|\,\mathrm{dd^c}\,\eta_1\|_{G_2} \|u\|_{G_2} \int_{G_2} (\mathrm{dd^c}\, u)^{k-2} \wedge \beta_m^{m-k+2}.$$

After $k-1$ steps we get

$$\int_E (\mathrm{dd^c}\, u)^k \wedge \beta_m^{m-k} \leq C^{k-1} \left(\prod_{l=1}^{k-1} \|u\|_{G_l} \|\,\mathrm{dd^c}\,\eta_{l-1}\|_{G_l} \right) \int_{G_{k-1}} \mathrm{dd^c}\, u \wedge \beta_m^{m-1}.$$

Since $G_{k-1} \subset D'$ we have

$$\int_E (\mathrm{dd^c}\, u)^k \wedge \beta_m^{m-k} \leq C^{k-1} \|u\|_{D'}^{k-1} \left(\prod_{l=1}^{k-1} \|\,\mathrm{dd^c}\,\eta_{l-1}\|_{D'} \right) \int_{D'} \mathrm{dd^c}\, u \wedge \beta_m^{m-1}.$$

Finally we take $k = m$ and we get (4.2.7). □

Corollary 4.2.1. *Let $E \subset \Omega(r)$, $r < 1$ and $z_0 \in \Omega(r)$, then there exists $0 < C < +\infty$ not depending on z_0 such that we have*

$$\mathrm{Cap}(E, \Omega(z_0, 3)) \leq C_1 \int_{\Omega(z_0,2)} \mathrm{dd^c}\, U^*_{E,\Omega(z_0,3)} \wedge \beta_m^{m-1}. \tag{4.2.8}$$

Proof. Simply apply Proposition 4.2.2 with $D = \Omega(z_0, 3)$, $D' = \Omega(z_0, 2)$, $u = U^*_{E,\Omega(z_0,3)}$. □

Proposition 4.2.3. *Let $0 < r' < r$ and $u \in \mathrm{PSH}(\Omega(z_0, r), [-\infty, 0]) \cap L^\infty_{loc}$, then we have*

$$\int_{\Omega(z_0,r')} \mathrm{dd^c}\, u \wedge \beta_m^{m-1} \leq \frac{1}{(r^2 - r'^2)} \int_{\Omega(z_0,r)} -u\beta_m^m. \tag{4.2.9}$$

Proof. Fix $R > r$. We use the Poisson Jensen Lelong formula [**42**], see Theorem 3.6.10, applied to the defining function $\rho_R(z) := |z' - z_0'|^2 - R^2$ for some $R \geq 3$. Notice that $\mathrm{dd^c}\,\rho_R = \mathrm{dd^c}\,|z'|^2 = \beta_m$ and $\Omega(z_0, r) = \{\rho_R < r^2 - R^2\}$. We have

$$\int u d\mu_{r^2-R^2} + \int_{\Omega(z_0,r)} -u(\mathrm{dd^c}\,\rho_R)^m = \int_{-\infty}^{r^2-R^2} \int_{\Omega(z_0,t)} \mathrm{dd^c}\, u \wedge (\mathrm{dd^c}\,\rho_R)^{m-1} dt.$$

By the remark above and being u negative and $\mu_{r^2-R^2}$ a positive measure we get

$$\int_{\Omega(z_0,r)} -u\beta_m^m \geq \int_{-R^2}^{r^2-R^2} \int_{\Omega(z_0,t)} \mathrm{dd^c}\, u \wedge \beta_m^{m-1} dt. \tag{4.2.10}$$

Let us focus on the right hand side. Set $n(t) := \int_{\Omega(z_0,t)} \mathrm{dd^c}\, u \wedge \beta_m^{m-1}$, notice that $n(t)$ is positive and increasing in $[-R^2, r^2 - R^2]$ hence

$$\int_{-R^2}^{r^2-R^2} n(t)dt \geq \int_{r'^2-R^2}^{r^2-R^2} n(t)dt \geq n(r'^2 - R^2)(r^2 - r'^2). \tag{4.2.11}$$

Due to equations (4.2.10) and (4.2.11) one has

$$\int_{\Omega(z_0,r)} -u\beta_m^m \geq (r^2 - r'^2) \int_{\Omega(z_0,r')} \mathrm{dd^c}\, u \wedge \beta_m^{m-1}$$

and the thesis follows. □

Applying Corollary 4.2.1 and Proposition 4.2.3 we get the following.

Corollary 4.2.2. *In the above hypothesis we have*

$$\mathrm{Cap}(E, \Omega(z_0, 3)) \leq C_2 \int_{\Omega(z_0,2)} -U^*_{E,\Omega(z_0,3)}\beta_m^m. \tag{4.2.12}$$

Now we start comparing the relative extremal functions for E with respect to a pseudo-ball in A with the one for K in the honest $\mathbb{C}^m$ ball of the same radius.

Lemma 4.2.4. *Let $0 < r < 1$, $E \subset \Omega(r)$ and $z_0 \in \Omega(r)$, there exists a positive finite constant C_3 not depending on E or z_0 such that, setting $B := \pi\Omega(z_0, 3) = B(z_0', 3)$, we have*

$$\mathrm{Cap}(E, \Omega(z_0, 3)) \leq -C_3 U^*_{K,B}(z_0'). \tag{4.2.13}$$

Proof. We set $u(z) := U^*_{K,B}(z')$, it follows that

$$u(z) \leq U^*_{E,\Omega(z_0,3)}(z) \ \ \forall z \in \Omega(z_0, 3),$$

since u is an element of the upper envelope defining $U_{E,\Omega(z_0,3)}$.

In particular

$$u(z_0) \leq U^*_{E,\Omega(z_0,3)}(z_0). \tag{4.2.14}$$

Let us recall (see Theorem A.0.3) that there exists an algebraic subset Y of $B(z_0', 3)$ such that $Y \supseteq \pi(A_{\mathrm{sing}} \cap \Omega(z_0, 3))$ and for some positive integer l

$$\tilde{\pi} := \pi : \Omega(z_0, 3) \setminus \pi^{-1}(Y) \to B(z_0', 3) \setminus Y$$

has l holomorphic inverses π_l^{-1} that are local coordinates on each component of $\Omega(z_0, 3) \setminus \pi^{-1}(Y)$. Also, we notice that $V := \pi^{-1}(Y)$ is a pluripolar set.

Now we consider $\beta_m^m = \left(\mathrm{dd^c}\,|z' - z_0'|^2\right)^m$ and we notice that, being $|z' - z_0'|^2$ a locally bounded plurisubharmonic function, $\left(\mathrm{dd^c}\,|z' - z_0'|^2\right)^m$ does not charge any pluripolar subset of $\Omega(z_0, 2)$; this follows from the Chern Levine Nirenberg estimate [**42**]; see equation 3.1.2 and lines below. Therefore

$$\int_{\Omega(z_0,2)} -u\beta_m^m = \int_{\Omega(z_0,2)\setminus V} -u\beta_m^m. \tag{4.2.15}$$

Now we use Corollary 4.2.2, equation (4.2.14) and equation (4.2.15) to get

$$\begin{aligned}\mathrm{Cap}(E, \Omega(z_0, 3)) \leq C_2 \int_{\Omega(z_0,2)} -U^*_{E,\Omega(z_0,3)}\beta_m^m \leq C_2 \int_{\Omega(z_0,2)} -u\beta_m^m\\ =C_2 \int_{\{\zeta\in\mathbb{C}^m:|\zeta - z_0'|<2\}\setminus Y} -U^*_{K,B}(\zeta)\,\mathrm{Card}(\pi^{-1}(\zeta))\beta_m^m \leq -lC_2 \int_{\{\zeta\in\mathbb{C}^m:|\zeta - z_0'|<2\}} U^*_{K,B}(\zeta)d\lambda_m(\zeta).\end{aligned}$$

Here λ_m denotes the Lebesgue measure in $\mathbb{C}^m$.

Now notice that since $U^*_{K,B}$ is plurisubharmonic we have

$$\int_{|\zeta - z_0'|<2} U^*_{K,B}(\zeta)d\lambda_m(\zeta) \geq \lambda_m(\{|\zeta - z_0'| < 2\})U^*_{K,B}(z_0')$$

and this conclude the proof since one can take $C_3 := \frac{lC_2}{\lambda_m(\{|\zeta - z_0'|<2\})}$ □

Lemma 4.2.5. *Let $B := \{\zeta \in \mathbb{C}^m : |\zeta - z_0'| < 3\}$, $K = \pi(E)$, $H := \pi^{-1}(K)$, Y, V as in Lemma 4.2.4. Moreover we set*

$$\begin{aligned}
\tilde v(\zeta) &:= \max\nolimits_{z'=\zeta} V_E^*(z, A) && \forall \zeta \in B \setminus Y,\\
\tilde v^*(\zeta) &:= \limsup\nolimits_{(B\setminus Y)\ni\xi\to\zeta} \tilde v(\xi) && \forall \zeta \in B,\\
v(\zeta) &:= \frac{\tilde v^*(\zeta)}{\|V_E(\cdot, A)\|_H + \|V_K \circ \pi^{-1}\|_{\Omega(z_0,3)}} - 1 && \forall \zeta \in B,\\
u(\zeta) &:= U^*_{K,B}(\zeta) && \forall \zeta \in B,\\
U(z) &:= U^*_{E,\Omega(z_0,3)}(z) && \forall z \in \Omega(z_0, 3).
\end{aligned}$$

We have

$$v \circ \pi \leq u \circ \pi \leq U \quad \text{on } \Omega(z_0, 3). \tag{4.2.16}$$

Proof. Let us notice that the second inequality has already been proved in the proof of Lemma 4.2.4, see equation (4.2.14). For the first we notice that $\tilde v \in \mathrm{PSH}(B \setminus Y)$ being the projection a proper map with finite fibers and holomorphic inverses on $B \setminus Y$; see Theorem A.0.3. Therefore $\tilde{v}^*$ is a plurisubharmonic function on B.

Moreover it follows by the definition that $V_E^*(z, A) - \|V_E^*(\cdot, A)\|_H \leq V_H^*(z, A)$, hence

$$\max_{z'=\zeta} V_E^*(z, A) - \|V_E^*(\cdot, A)\|_H \leq \max_{z'=\zeta} V_H^*(z, A) \leq V_K^*(\zeta).$$

It follows in particular that

$$\tilde{v}^*(\zeta) \leq \|V_E^*(\cdot, A)\|_H + V_K^*(\zeta)$$

Therefore $M := \|\tilde{v}^*\|_B \leq \|V_E^*(\cdot, A)\|_H + \|V_K^*\|_B$.

It follows by the definition as upper envelope of u that any function $f \in \mathrm{PSH}(B)$ with $f \leq -1$ on K and $f \leq 0$ has the property $f \leq u$ on B. The function $v = M^{-1}\tilde{v}^* - 1$ has been constructed to satisfy such assumptions, indeed $\tilde{v}^* \leq 0$ on K and $\tilde{v}^* \leq M$ on B. It follows that $v \leq u$ on B and thus $v \circ \pi \leq u \circ \pi$ on $\Omega(z_0, 3)$. □

Lemma 4.2.6. *For any $0 < r < 1$ there exists a positive finite constant $C_4 := C_4(r)$ such that for any compact set $E \subset \Omega(z_0, 3)$ and any $z_0 \in \overline{\Omega(r)}$ we have*

$$\mathrm{Cap}(E, \Omega) \leq C_4 \,\mathrm{Cap}(E, \Omega(z_0, 3)). \tag{4.2.17}$$

Proof. The proof is similar to the one of [**2**, Lemma 3.5], one needs just to replace the use of the Comparison Principle of [**13**] with its version for complex spaces, see Theorem 3.2.2. □

Recall that $B := \{\zeta \in \mathbb{C}^m : |\zeta - z_0'| < 3\}$.

Corollary 4.2.3. *For any $0 < r < 1$, there exist a constant $C_5 = C_3 \cdot C_4$ (above) such that, for any $E \subset \Omega(r)$ and any $z_0 \in H = \pi^{-1}\pi(E)$ such that $\max_{z'=z_0'} V_E(z, A) = \|V_E(\cdot, A)\|_H$, we have*

$$\mathrm{Cap}(E, \Omega) \leq C_5 \frac{\|V_K^*\|_B}{\|V_E(\cdot, A)\|_{\Omega(z_0,3)}}. \tag{4.2.18}$$

Proof. Using lemmata 4.2.4, 4.2.5 and 4.2.6 and the extremality property of z_0 we get

$$\begin{aligned}
\mathrm{Cap}(E, \Omega) &\leq C_4 \,\mathrm{Cap}(E, \Omega(z_0, 3)) \leq C_4 \cdot C_3(-U_{K,B}^*(z_0)) \\
&\leq C_4 \cdot C_3(-v(z_0')) = C_4 \cdot C_3 \frac{\|V_E^*(\cdot, A)\|_H + \|V_K^*\|_B - \tilde{v}^*(z_0')}{\|V_E^*(\cdot, A)\|_H + \|V_K^*\|_B} \\
&= C_5 \frac{\|V_K^*\|_B}{\|V_E^*(\cdot, A)\|_H + \|V_K^*\|_B} \leq C_5 \frac{\|V_K^*\|_B}{\|V_E^*(\cdot, A)\|_{\Omega(z_0,3)}}.
\end{aligned}$$

□

LEMMA 4.2.7. *For any $0 < r < 1$ there exists a positive finite constant C_6 such that for any compact subset E of $\Omega(r)$ and any $z_0 \in \overline{\Omega(r)}$ we have*

$$\mathrm{Cap}(E, \Omega(z_0, 3)) \le \frac{C_6}{\|V_K\|_B} \tag{4.2.19}$$

PROOF. The proof can be provided repeating the whole argument of this section but considering other quantities. More precisely, we pick $z_1 \in \partial\Omega(z_0, 3)$ such that $\sup_{z\in\Omega(z_0,3)} V_K^*(z') = V_K^*(z_1')$.

Using (an analogous version of) Lemma 4.2.6 we can find C_4' such that

$$\mathrm{Cap}(E, \Omega(z_0, 3)) \le C_4' \,\mathrm{Cap}(E, \Omega(z_1, 9)).$$

By the same argument as in 4.2.2 (and Proposition before) we can find a positive finite constant C_2' such that

$$\mathrm{Cap}(E, \Omega(z_1, 9)) \le C_2' \int_{\Omega(z_1,6)} -U^*_{E,\Omega(z_1,9)} \beta_m^m.$$

Following the lines of Lemma 4.2.4 we can find a positive finite constant C_3' such that

$$\mathrm{Cap}(E, \Omega(z_1, 9)) \le -C_3' U^*_{E,\Omega(z_1,9)}(z_1).$$

Now we set $B_1 := \pi\Omega(z_1, 9) = B(z_1', 9)$ and introduce the function

$$W(\zeta) := \frac{V_K^*(\zeta)}{\|V_K^*\|_{\overline{B}} + \|V_{\overline{B}}^*\|_{\overline{B_1}}} - 1.$$

Following the proof of Lemma 4.2.5 it is not difficult to see that $W \in \mathrm{PSH}(B_1)$, $W \le 0$ and $W|_K \le -1$, hence we have $W(z) \le U^*_{K,B_1}(z)$. In particular, due to the extremal property of z_1, we have

$$-U^*_{K,B_1}(z_1') \le -W(z_1') = \frac{\|V_{\overline{B}}^*\|_{\overline{B_1}}}{\|V_K^*\|_{\overline{B}} + \|V_{\overline{B}}^*\|_{\overline{B_1}}} \le \frac{\|V_{\overline{B}}^*\|_{\overline{B_1}}}{\|V_K^*\|_{\overline{B}}} = \frac{\left\|\log^+ \frac{|z_0'-\zeta|}{3}\right\|_{\overline{B}_1}}{\|V_K^*\|_{\overline{B}}} \le \frac{\log 2}{\|V_K^*\|_{\overline{B}}}.$$

On the other hand $U^*_{K,B_1} \circ \pi \le U^*_{E,\Omega(z_1,9)}$ since the former function is in the upper envelope defining the latter.

Finally we combine the inequalities above to get

$$\mathrm{Cap}(E, \Omega(z_0, 3)) \le C_3' \cdot C_4' \cdot \log 2 \frac{1}{\|V_K\|_{\overline{B}}} =: \frac{C_6}{\|V_K\|_{\overline{B}}}.$$

□

PROOF OF (4.2.3). To conclude the proof we use Corollary 4.2.3 and Lemma 4.2.7. We have

$$\operatorname{Cap}(E,\Omega) \le C_5 \frac{\|V_K\|_B}{\|V_E(\cdot,A)\|_{\Omega(z_0,3)}} \le \frac{C_5 \cdot C_6}{\operatorname{Cap}(E,\Omega(z_0,3))\|V_E(\cdot,A)\|_{\Omega(z_0,3)}}.$$

Thus, using Lemma 4.2.6,

$$\operatorname{Cap}(E,\Omega)^2 \le C_4 \operatorname{Cap}(E,\Omega) \cdot \operatorname{Cap}(E,\Omega(z_0,3)) \le \frac{C_4 \cdot C_5 \cdot C_6}{\|V_E(\cdot,A)\|_{\Omega(z_0,3)}} \le \frac{C_4 \cdot C_5 \cdot C_6}{\|V_E(\cdot,A)\|_{\Omega}}.$$

Taking $c_2 := C_4 \cdot C_5 \cdot C_6$ this proves (4.2.4). $\square$

4.3. Convergence Theorem for Relative Capacity and Extremal Functions

4.3.1. Statement of the Result. In the case of a domain Ω in $\mathbb{C}^n$ it has been proved in [**13**] that the Monge Ampere operator is continuous under any monotone sequence of locally bounded plurisubharmonic functions. This results extends to the setting of $\Omega \subset A$, where A is an algebraic set, see Theorem D.0.1, and even to more general settings [**11**].

Using this continuity it is not difficult to see that $\mu_{E_j,\Omega} \to^* \mu_{E,\Omega}$ for any increasing sequence of compact subsets E_j of E such that $\mathrm{Cap}(E_j, \Omega) \to \mathrm{Cap}(E, \Omega)$, where E is a compact set in the open hyperconvex set $\Omega \subset A$ and A is a irreducible algebraic set.

The aim of this section is to investigate, following the idea of Bloom and Levenberg, [**24**], the relation of the convergence of the relative capacities $\mathrm{Cap}(E_j, B) \to \mathrm{Cap}(E, B)$ and the convergences $U_{E_j,B} \to U_{E,B}$ and $\mu_{E_j,B} \to^* \mu_{E,B}$, without any monotonicity assumption on the sequence $\{E_j\}$, where $E_j \subset E \subset \Omega$ and Ω is a pseudo-ball (see equations 4.2.1) in the pure m-dimensional irreducible algebraic subset A of $\mathbb{C}^n$.

A main tool in this Chapter is the notion of *convergence in capacity*.

Definition 4.3.1 (Convergence in capacity). *Let D a open set in A and $v_j, v \in$ PSH(D), $j = 1, 2, \ldots$. The sequence $\{v_j\}$ is said to converge* in capacity *to v if for any compact subset $K \subset\subset D$ and any $\delta > 0$ we have*

$$\lim_j \mathrm{Cap}(\{z \in K : |v_j - v| > \delta\}, D) = 0. \tag{4.3.1}$$

We use the following notations, $\mathbb{C}^n \supset A \ni z = (z', z'') \in \mathbb{C}^m \times \mathbb{C}^{n-m}$ where the choice of global coordinates is done accordingly to Proposition A.0.2, $\Omega := \{z \in A : |z'| < 1\}$. Also, we **warn the reader** that throughout this section we use the following short notations, which slightly differ from the ones previously introduced.

$$u_j(z) := U^*_{E_j,\Omega}(z, A)\ , \quad u(z) := U^*_{E,\Omega}(z, A)$$
$$v_j(z) := V^*_{E_j}(z, A)\ , \quad v(z) := V^*_{E}(z, A).$$

We prove a result which is analogous to the ones achieved in [**24**, Th 1.1 and Th. 1.2] for the flat case $A \equiv \mathbb{C}^n$.

THEOREM 4.3.2. *Let $A \subset \mathbb{C}^n$ be a pure m-dimensional irreducible algebraic subset of $\mathbb{C}^n$, and $E \subset \Omega$ be a compact set. Let $\{E_j\}$ be a sequence of Borel subsets of E. Then the following are equivalent.*

i) $\lim_j \mathrm{Cap}(E_j, \Omega) = \mathrm{Cap}(E, \Omega)$
ii) $\lim_j u_j = u$ *in capacity and* $\left(\mathrm{dd^c}\, u_j\right)^m \rightharpoonup^* (\mathrm{dd^c}\, u)^m$.
iii) $\lim_j u_j = u$ *point-wise on* Ω.
iv) $\lim_j v_j = v$ *point-wise on* A.

If we furthermore suppose E to be regular and E_j to be compact for any j, then equations (iii) *and* (iv) *can be replaced by*

v) $\lim_j u_j = u$ *uniformly on* Ω.
vi) $\lim_j v_j = v$ *uniformly on* A.

REMARK 4.3.1. *There exists a one to one correspondence between functions in $\mathcal{L}(\mathbb{C}^n)$ and of so-called ω-plurisubharmonic functions on projective manifolds. Thus, in the case of A being smooth, it may be possible to prove results related to Theorem 4.2.1 and Theorem 4.3.2 by the techniques developed in this setting, see for instance* [**53**] *and* [**36**].

4.3.2. Proof of Theorem 4.3.2. The proof is provided by following the lines of Bloom and Levenberg [**24**], adapting the steps to the context of algebraic sets by using our findings of Section 4.2. Along the proof we need also an additional property that the sequence u_j may have:

(weak iii) $$\lim_j u_j = u \text{ point-wise on } \Omega \cap A_{\mathrm{reg}}.$$

The proof of Theorem 4.3.2 is provided showing that

A (i) implies (ii),
B (ii) implies (i),
C (i) implies (weak iii), assuming A,
D (weak iii) implies (ii),
E (i) implies (iv), assuming A,
F (iv) implies (iii).

Notice that, as suggested by the name, trivially (iii) implies (weak iii), thus (see Figure 4.3.1) proving the above implications will conclude the proof of the first statement of Theorem 4.3.2.

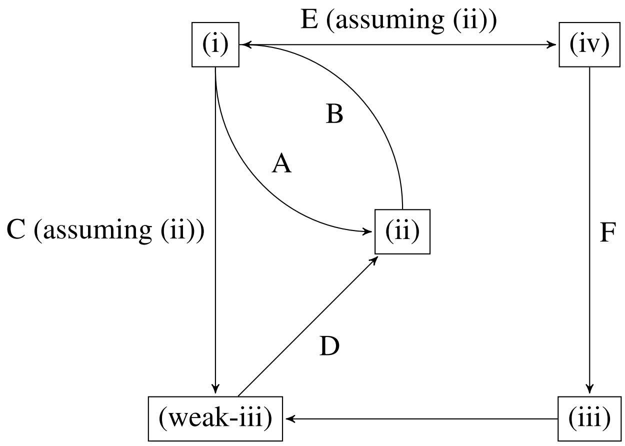

FIGURE 4.3.1. The diagram of the proof of Theorem 4.3.2.

Finally we show that, under the additional hypothesis of E being regular and E_js compact, one can has

- (i) implies (vi).

The replacement of (iii) by (v) is similar.

We start with some preliminary results. First we extend [**24**, Prop. 1.1] to our setting.

PROPOSITION 4.3.8. *In the notations and under hypothesis of Theorem 4.3.2 suppose that* (i) *holds, then*

$$\left.\begin{array}{ll} \lim_j \int_\Omega u_j(dd^c u_j)^k \wedge (dd^c u)^{m-k} & k=0,1,\dots,m \\ \lim_j \int_\Omega u(dd^c u_j)^k \wedge (dd^c u)^{m-k} & k=0,1,\dots,m \end{array}\right\} = \int_\Omega u\,(dd^c u)^m . \tag{4.3.2}$$

PROOF. We use the integration by parts formula of Theorem 3.5.9 and $u_j \geq u$ (by definition as upper envelopes) to get

$$\begin{aligned} -\operatorname{Cap}(E_j,\Omega) &= \int_\Omega u_j \left(dd^c u_j\right)^m \geq \int_\Omega u \left(dd^c u_j\right)^m = \int_\Omega u_j\, dd^c u \wedge (dd^c u_j)^{m-1} \\ &\geq \int_\Omega u\, dd^c u \wedge (dd^c u_j)^{m-1} = \int_\Omega u_j (dd^c u)^2 \wedge (dd^c u_j)^{m-2} \dots\dots \\ &\geq -\operatorname{Cap}(E,\Omega). \end{aligned}$$

Note that the hypothesis of Theorem 3.5.9 are satisfied since each term $\int_\Omega (dd^c u_j)^k \wedge (dd^c u)^{m-k}$ is finite by the definition of capacity.

Since by the hypothesis (i) the first term converges to the last, the same holds true for each term in between. ◻

To relate the convergence $u_j \to u$ in capacity to the convergence of the corresponding Monge Ampere measures we need the following proposition that can be derived by minor modifications of [**102**, Th. 1]. Notice that the proof by Xing relies on two facts: the quasi-continuity of locally bounded plurisubharmonic functions [**13**, Th. 3.5] and the fact (see for instance [**102**, pg. 458]) that for any bounded hyperconvex domain Ω there exists a constant A_Ω such that for any compact set $K \subset \Omega$ $\mathrm{Cap}_{m-1}(K,\Omega) \leq A_\Omega \,\mathrm{Cap}(K,\Omega)$; see Subsection 3.3.1 for the definitions. We stress that both these facts go directly to our setting. For the quasi-continuity this follows directly by the definition of relative capacity on algebraic sets, notice that a plurisubharmonic function on Ω is in particular a plurisubharmonic function on the complex manifold $\Omega \cap A_{\mathrm{reg}}$. For the inequality between capacities this can be proved exactly as in the flat case, using the expansion of $\left(\mathrm{dd^c}\, u + |z|^2\right)^m$. For another proof, see [**61**, Th. 1.1.1].

Proposition 4.3.9 ([**102**]). *Let Ω be an open hyperconvex domain in the pure m dimensional irreducible algebraic subset A of $\mathbb{C}^n$ and $v_j, v \in \mathrm{PSH}(\Omega) \cap L^\infty_{loc}$. Suppose that $v_j \to v$ in capacity. Then $\left(\mathrm{dd^c}\, v_j\right)^m \to (\mathrm{dd^c}\, v)^m$.*

The first step of the proof of Theorem 4.3.2 is equivalent to the original version in [**24**]. However we use the estimate Equation (3.5.4) instead of [**102**, Th. 2].

(A) Proof of (i) implies (ii). Let us pick any compact set $F \subset \Omega$ and $\delta > 0$: we aim to estimate

$$\mathrm{Cap}(F \cap \{u_j > u + \delta\}, \Omega) := \sup_{w \in \mathrm{PSH}(\Omega,[0,1])} \int_{F \cap \{u_j > u+\delta\}} (\mathrm{dd^c}\, w)^m .$$

It is more convenient to pick a large R such that $\Omega \subset \Omega(R) =: \Omega'$ and notice that

$$\begin{aligned}\mathrm{Cap}(F \cap \{u_j > u + \delta\}, \Omega) &\leq C_R \,\mathrm{Cap}(F \cap \{u_j > u + \delta\}, \Omega') \\ =&C_R \sup_{w \in \mathrm{PSH}(\Omega',[0,1])} \int_{F \cap \{u_j > u+\delta\}} (\mathrm{dd^c}\, w)^m .\end{aligned}$$

See the proof of Theorem 4.2.1.

First we modify u_j away from F to make it agree with u on a neighbourhood of $\partial\Omega$ in Ω, for, we set

$$u_j^\epsilon := \max\{u_j - \epsilon, u\}.$$

Now we use an argument which is taken from [**13**, Proof of Th. 3.4] and used in [**102**, Proof of Th. 2], our variant relies on some integration by parts formulas in the generalized sense that follow by Theorem 3.5.7, note that all considered currents are compactly supported in Ω since $u_j^\epsilon - u$ is, and lie in $\mathscr{A}^{2m}$ (see Definition 3.4.3). We do not offer proofs of such formulas since they are analogous to the one of Theorem 3.5.8.

$$
\begin{aligned}
&\mathrm{Cap}\Big(F\cap\{u_j>u+\delta\},\Omega'\Big) = \sup_{w\in \mathrm{PSH}(\Omega',[0,1])}\int_{F\cap\{u_j>u+\delta\}}(\mathrm{dd^c}\, w)^m\\
=&\sup_{w\in \mathrm{PSH}(\Omega',[0,1])}\int_{F\cap\{u_j^\epsilon>u+\delta-\epsilon\}}(\mathrm{dd^c}\, w)^m\\
\leq&\sup_{w\in \mathrm{PSH}(\Omega',[0,1])}\int_{F\cap\{F\cap\{u_j^\epsilon>u+\delta-\epsilon\}}\frac{u_j^\epsilon-u}{\delta-\epsilon}(\mathrm{dd^c}\, w)^m\\
\leq&\frac{1}{\delta-\epsilon}\sup_{w\in \mathrm{PSH}(\Omega',[0,1])}\int_\Omega (u_j^\epsilon-u)\,(\mathrm{dd^c}\, w)^m\\
=&\frac{1}{\delta-\epsilon}\sup_{w\in \mathrm{PSH}(\Omega',[0,1])}-\int_\Omega d(u_j^\epsilon-u)\wedge d^c w\wedge(\mathrm{dd^c}\, w)^{m-1}\\
\leq&\frac{1}{\delta-\epsilon}\sup_{w\in \mathrm{PSH}(\Omega',[0,1])}\left(-\int_\Omega d(u_j^\epsilon-u)\wedge d^c(u_j^\epsilon-u)\wedge(\mathrm{dd^c}\, w)^{m-1}\right)^{1/2}\times\\
&\qquad\left(-\int_\Omega dw\wedge d^c w\wedge(\mathrm{dd^c}\, w)^{m-1}\right)^{1/2}\\
\leq&C(\overline{\Omega}',\Omega)^{1/2}\frac{1}{\delta-\epsilon}\sup_{w\in \mathrm{PSH}(\Omega',[0,1])}\left(-\int_\Omega d(u_j^\epsilon-u)\wedge d^c(u_j^\epsilon-u)\wedge(\mathrm{dd^c}\, w)^{m-1}\right)^{1/2}
\end{aligned}
$$

Here We used the Cauchy Schwarz inequality for currents, [**13**], and the Chern Levine Nirenberg Estimate [**13**, Th 2.10 (iii)] stating that

$$C(\overline{\Omega}',\Omega) := \sup_{w\in \mathrm{PSH}(\Omega,[0,1])}\left|-\int_{\Omega'} dw\wedge d^c w\wedge(\mathrm{dd^c}\, w)^{m-1}\right|$$

is bounded and depends only on Ω' and Ω. These results are originally stated for domains in $\mathbb{C}^n$, not in our setting. However, we already shown in Proposition 4.2.2 that an analogous of [**13**, Th 2.10 (i)] holds and the extension to our setting of its variant [**13**, Th 2.10 (iii)] can be done precisely in the same way.

Now we perform a further integration by parts and we get

$$
\begin{aligned}
&\mathrm{Cap}\Big(F\cap\{u_j>u+\delta\},\Omega\Big)\\
\leq&\frac{C(\overline{\Omega}',\Omega)}{\delta-\epsilon}\sup_{w\in \mathrm{PSH}(\Omega',[0,1])}\left(\int_\Omega (u_j^\epsilon-u)\,\mathrm{dd^c}\,(u_j^\epsilon-u)\wedge(\mathrm{dd^c}\, w)^{m-1}\right)^{1/2}\\
\leq&\frac{C(\overline{\Omega}',\Omega)}{\delta-\epsilon}\sup_{w\in \mathrm{PSH}(\Omega',[0,1])}\left(\int_\Omega (u_j^\epsilon-u)(\mathrm{dd^c}\, u_j^\epsilon+\mathrm{dd^c}\, u)\wedge(\mathrm{dd^c}\, w)^{m-1}\right)^{1/2}.
\end{aligned}
$$

Then we repeat the procedure $m-1$ times and we end up with an inequality of the form

$$\begin{aligned}&\mathrm{Cap}\left(F \cap \{u_j > u + \delta\}, \Omega\right) \leq \\ &\quad \leq \frac{C}{\delta - \epsilon}\left(\int_\Omega (u_j^\epsilon - u)\,(\mathrm{dd^c}\, u)^m - (u_j^\epsilon - u)\left(\mathrm{dd^c}\, u_j^\epsilon\right)^m\right)^{1/2^m}.\end{aligned}$$

Note that in the first $m-1$ step we replace a factor $\mathrm{dd^c}\, u_j^\epsilon - \mathrm{dd^c}\, u$ by $\mathrm{dd^c}\, u_j^\epsilon + \mathrm{dd^c}\, u$, while in the last step we do not.

We consider only the term $\frac{1}{\delta-\epsilon}\int_\Omega (u_j^\epsilon - u)\,(\mathrm{dd^c}\, u)^m - (u_j^\epsilon - u)\left(\mathrm{dd^c}\, u_j^\epsilon\right)^m$.

Note that the measure $(u_j^\epsilon - u)\left(\mathrm{dd^c}\, u_j^\epsilon\right)^m$ is positive because $u_j^\epsilon \geq u$ by definition, hence

$$\int_\Omega (u_j^\epsilon - u)\,(\mathrm{dd^c}\, u)^m - (u_j^\epsilon - u)\left(\mathrm{dd^c}\, u_j^\epsilon\right)^m \leq \int_\Omega (u_j^\epsilon - u)\,(\mathrm{dd^c}\, u)^m.$$

Now we let $\epsilon \to 0^+$ and by Monotone Convergence Theorem we get

$$\mathrm{Cap}\left(F \cap \{u_j > u + \delta\}, \Omega\right) \leq \frac{C}{\delta}\left(\int_\Omega (u_j - u)\,(\mathrm{dd^c}\, u)^m\right)^{1/2^m}.$$

By Proposition 4.3.8 this last term converges to 0 as $j \to \infty$. Thus $\mathrm{Cap}(\{u_j - u > \delta\} \cap F, \Omega) \to 0$.

The convergence of relative equilibrium measures follows by Proposition 4.3.9.

□

(B) Proof of (ii) implies (i). It suffices to pick $\varphi \in \mathscr{C}_c^\infty(\tilde{D})$, where $\tilde{D}$ is an open neighbourhood of E in $\mathbb{C}^n$ and $\tilde{D} \cap A \subset\subset \Omega$ such that $\varphi \equiv 1$ on E and notice that, since both $(\mathrm{dd^c}\, u)^m$ and $\left(\mathrm{dd^c}\, u_j\right)^m$ are supported on E, we have

$$\mathrm{Cap}(E, \Omega) = \int_\Omega \varphi\,(\mathrm{dd^c}\, u)^m = \lim_j \int_\Omega \varphi\left(\mathrm{dd^c}\, u_j\right)^m = \lim_j \mathrm{Cap}(E_j, \Omega).$$

□

We recall this property of subharmonic functions, for which we use the standard notation $\mathrm{shm}(D)$ for any domain $D \subset \mathbb{C}^m$.

Lemma 4.3.10 (Lemma 1.1 in [**24**]). *Let $0 < s < r$ and $a \leq b \leq r$. There exists $\delta := \delta(a, b, r, s) > 0$ such that $\forall v \in \mathrm{shm}(B(\zeta_0, r)), v(z_0) \geq b$ we have*

$$\lambda_m\left(\{\zeta \in B(\zeta_0, s) : v(\zeta) > a\}\right) > \delta. \tag{4.3.3}$$

Here we used the notation $B(\zeta_0, r) := \{\zeta \in \mathbb{C}^m : |\zeta - \zeta_0| < r\}$.

(C) PROOF OF (i) IMPLIES (weak iii). The proof is by contradiction. We assume that there exists $z_0 \in \Omega \cap A_{\mathrm{reg}}$ such that for some subsequence (that we relabel) $\lim_j u_j(z_0) \geq b - 1 > u(z_0)$.

Let us denote by $\mathcal{I}$ the set of all choices of m distinct increasing indexes in $\{1, 2, \dots, n\}$ and, for each $I \in \mathcal{I}$ denote by π_I the canonical projection on the coordinate plane $\{z_j = 0, \forall j \notin I\}$.

Being A algebraic (see Proposition A.0.3) for each $I \in \mathcal{I}$ we can find an analytic subset Y_I of $B(\pi_I(z_0), r)$ such that π_I has a finite number of holomorphic inverses $\pi_{I,l}^{-1}$ on $B(\pi_I(z_0), r) \setminus Y_I$, we also set $S_I := \pi_I^{-1}(Y_I)$. Moreover, for each $z_0 \in A_{\mathrm{reg}}$, there exists $\hat{I} \in \mathcal{I}$ such that $z_0 \notin S_{\hat{I}}$.

Now pick a, a' such that $u(z_0) < a' < a < b$ and find a neighbourhood U of z_0 in $A_{\mathrm{reg}} \setminus S_{\hat{I}}$ such that $v(z) < a'$ for all $z \in U$. Also, possibly further shrinking U, we can assume U to be of the form $\pi_{\hat{I},l}^{-1}(B(\pi_{\hat{I}}(z_0), r))$.

Let us introduce the functions $v := u \circ \pi_{\hat{I},l}^{-1} + 1$ and $v_j := u_j \circ \pi_{\hat{I},l}^{-1} + 1$, notice that $v_j \geq v$ by definition and all of them is a (pluri-) subharmonic function on $B(\pi_{\hat{I}}(z_0), r)$ bounded above by 1. We can apply Lemma 4.3.10 to these v_js and v and we get for a given $0 < s < r$

$$\lambda_m\Big(\{\zeta \in B(\pi_{\hat{I}}(z_0), s) : v_j(z) > a\}\Big) > \delta \ \ \forall j = 1, \dots. \tag{4.3.4}$$

Now we claim that

$$\mathrm{Cap}(G, \Omega) \geq \lambda_m(\pi_{\hat{I}} G), \ \forall G \subset\subset \pi_{\hat{I},l}^{-1} B(\pi_{\hat{I}}(z_0), s). \tag{4.3.5}$$

From this claim it follows that $\forall j$

$$\begin{aligned} &\mathrm{Cap}(\{z \in \Omega : u_j(z) - u(z) > a - a'\}, \Omega) \geq C\,\mathrm{Cap}(\{z \in \Omega : u_j(z) > a\}, \Omega) \\ \geq &\lambda_m(\pi_{\hat{I}}\{z \in B : u_j(z) - u(z) > a - a'\}) \geq \delta > 0. \end{aligned}$$

This contradicts the assumption (ii).

In order to conclude the proof we are left to prove the claim (4.3.5). To do that, simply notice that

$$\begin{aligned} \lambda_m(\pi_{\hat{I}} G) = &\int_{\pi_{\hat{I}} G} \left(dd^c \frac{|\pi_{\hat{I}} z|^2}{2}\right)^m \leq \int_{G \cap A_{\mathrm{reg}}} \left(dd^c \frac{|z|^2}{2}\right)^m \\ \leq &2^{-m} \sup\left\{ \int_{G \cap A_{\mathrm{reg}}} (dd^c v)^m, \ v \in \mathrm{PSH}(\Omega), 0 \leq v \leq 1 \right\} = 2^{-m}\,\mathrm{Cap}(G, \Omega). \end{aligned}$$

$\square$

(D) PROOF OF (weak iii) IMPLIES (ii). Let us notice that

$$u \le u_j \le w_j := \sup_{k \ge j} u_k \le w_j^*$$

and $w_j^* \in \mathrm{PSH}(\Omega, [-1,0])$. Moreover since $u_j \to u$ on $A_{\mathrm{reg}} \cap \Omega$ we have $w_j^* \to u$ on $(A_{\mathrm{reg}} \cap \Omega) \setminus P$ for a negligible, hence pluripolar in $\Omega \cap A_{\mathrm{reg}}$ set P. Therefore, setting $w := \lim_j w_j^* = \inf_j w_j^*$, we have $w = u$ quasi everywhere on Ω.

Now (for any compact set $F \subset\subset \Omega$) we can repeat the argument we used for proving (A) to get the following estimate

$$\begin{aligned} &\mathrm{Cap}(\{u_j > u + \delta\} \cap F, \Omega) \le \mathrm{Cap}(\{w_j^* > u + \delta\} \cap F, \Omega) \\ \le &\frac{C}{\delta}\left(\int_\Omega (w_j^* - u)\,(\mathrm{dd^c}\,u)^m\right)^{1/2^m} \to \frac{C}{\delta}\left(\int_\Omega (w - u)\,(\mathrm{dd^c}\,u)^m\right)^{1/2^m}. \end{aligned}$$

Here we use the Monotone Convergence Theorem, note that w_j^* is a decreasing sequence.

Finally, since u is locally bounded, $(\mathrm{dd^c}\,u)^m$ does not charge pluripolar sets, thus we have

$$\int_\Omega (w - u)\,(\mathrm{dd^c}\,u)^m = \int_{\Omega \setminus P} (w - u)\,(\mathrm{dd^c}\,u)^m = 0$$

since $u \equiv w$ on $\Omega \setminus P$. □

Let us recall for the reader's convenience that, given an open subset D of A, $f : D \to [-\infty, +\infty[$ is said to be weakly plurisubharmonic if $f|_{D \cap A_{\mathrm{reg}}}$ is plurisubharmonic as function on a complex manifold and f is locally bounded on D. We denote such a property by $f \in \widetilde{\mathrm{PSH}}(D)$. We refer the reader to Appendix C for further details.

In order to distinguish between regularization at points of A_{reg} from the regularization on A from A_{reg} we introduce a new notation. Precisely, for any function f on an algebraic set A we define

$$f^\star(z) := \limsup_{A_{\mathrm{reg}} \ni \zeta \to z} f(\zeta) \;\; (\text{i.e., } (f|_{A_{\mathrm{reg}}})^*(z)) , \quad \forall z \in A_{\mathrm{reg}},$$

recall that

$$f^*(z) = \limsup_{A_{\mathrm{reg}} \ni \zeta \to z} f(\zeta) , \; \forall z \in A.$$

In particular, we notice that, if $f \in \widetilde{\mathrm{PSH}}(A)$, then $f^\star \equiv f|_{A_{\mathrm{reg}}} = (f^*)|_{A_{\mathrm{reg}}}$. Moreover if A is irreducible and $f \in \widetilde{\mathrm{PSH}}(A)$ then f^* is a plurisubharmonic locally bounded function coinciding with f on A_{reg}. Let us recall a useful lemma.

LEMMA 4.3.11 (Pag. 494 [**88**]). *Let $\{f_j\}$ be a decreasing sequence of weakly plurisubharmonic functions on the irreducible pure m-dimensional algebraic set $A \subset \mathbb{C}^n$. Let us set $f := \inf f_j = \lim_j f_j$, then we have*

$$f^*(z) = \lim f_j^*(z), \ \forall z \in A_{sing} \tag{4.3.6}$$

COROLLARY 4.3.4. *$\{f_j\}$ be a locally uniformly bounded decreasing sequence of plurisubharmonic functions on the irreducible pure m-dimensional algebraic set $A \subset \mathbb{C}^n$. Let us assume that each f_j has the following property.*

$$\limsup_{\zeta \to z} f_j(\zeta) = f_j(z)\ , \quad \forall z \in A_{sing}. \tag{4.3.7}$$

Then, setting $f := \inf f_j = \lim_j f_j$, we have

$$f \in \mathrm{PSH}(A) \cap L^\infty_{loc} \text{ and } \quad f^* \equiv f \ \text{ on } A.$$

PROOF. By a standard argument, on any complex manifold the decreasing limit of locally uniformly bounded plurisubharmonic functions is a locally bounded plurisubharmonic function, thus we have $f|_{A_{\mathrm{reg}}} \in \mathrm{PSH}(A_{\mathrm{reg}})$. In particular, due to this plurisubharmonicity of f the upper semi continuous regularization does not change its values on A_{reg}. Hence $f^\star = f^*|_{A_{\mathrm{reg}}} \equiv f|_{A_{\mathrm{reg}}} \in \mathrm{PSH}(A_{\mathrm{reg}})$.

Now notice that, being plurisubharmonic on A_{reg} and locally bounded, $f \in \widetilde{\mathrm{PSH}}(A)$. By Lemma 4.3.11 we get $f^*(z_0) = \lim_j f_j^*(z_0)$ at any $z_0 \in A_{\mathrm{sing}}$. We use our assumption on f_js to get:

$$f^*(z_0) = \lim_j f_j^*(z_0) = \lim_j \limsup_{A_{\mathrm{reg}} \ni \zeta \to z_0} f_j(\zeta) = \lim_j f_j(z_0), \ \forall z_0 \in A_{\mathrm{sing}}.$$

Thus $f^* \equiv f$ on A.

By [**42**] and being $f \in \widetilde{\mathrm{PSH}}(A)$ and A irreducible, $f^* \in \mathrm{PSH}(A)$, but since $f \equiv f^*$ we actually have $f \in \mathrm{PSH}(A) \cap L^\infty_{\mathrm{loc}}$. □

(E) PROOF OF (i) IMPLIES (iv). Let us pick j_0 such that for any $j > j_0$ we have $\mathrm{Cap}(E_j, B) \geq 1/2\, \mathrm{Cap}(E, B)$.

Now we use Equation 4.2.4 in Theorem 4.2.1: there exists a positive constant C such that

$$\sup_{z \in \Omega} V^*_{E_j}(z, A) \leq \frac{C}{\mathrm{Cap}^2(E_j, \Omega)} \leq \frac{4C}{\mathrm{Cap}^2(E, \Omega)}, \ \forall j \geq j_0. \tag{4.3.8}$$

It follows by the definitions of relative and global extremal functions and by $E_j \subset E$ that

$$v(z) \le v_j(z) = V^*_{E_j}(z, A) \le \frac{4C}{\mathrm{Cap}(E,\Omega)^2}(u_j(z)+1) \ \ \forall z \in \Omega. \tag{4.3.9}$$

We already proved that (i) implies (weak iii), hence $u_j \to u$ in $\Omega \cap A_{\mathrm{reg}}$. Since $u = -1$ and $v = 0$ q.e. in E it follows that

$$v_j \to 0, \text{ q.e. in } E.$$

We introduce the sequence of functions $w_j := \sup_{k\ge j} v_k$, by a standard argument $w_j^\star(z) \in \mathrm{PSH}(A_{\mathrm{reg}})$. Moreover $w_j^*|_{A_{\mathrm{reg}}} \equiv w_j^\star$ and is uniformly locally bounded, since

$$v(z) \le v_j(z) \le \frac{4C}{\mathrm{Cap}^2(E,B)} + \log^+|z'|, \ \forall j > j_0$$

due to the uniform bound (4.3.8). Thanks to [**42**] $(w_j^\star)^*$ is a locally bounded plurisubharmonic function on A, Notice that $\tilde{w}_j := (w_j^\star)^*$ satisfy (4.3.7) by definition and $\tilde{w}_j \ge \sup_{k\ge j} v_k$ on A thanks to the lower semicontinuity of w_j.

Also we define $w(z) := \lim_j \tilde{w}_j(z) =_{\text{q.e.}} \limsup_j v_j(z)$. Notice that w is a decreasing limit of plurisubharmonic functions satisfying (4.3.8). Due to Corollary 4.3.4, $w \in \mathrm{PSH}(A) \cap L^\infty_{\mathrm{loc}}$.

It follows by (4.3.9) and the convergence $u_j \to -1$ q.e. on E that $w \equiv 0$ q.e. on E. In particular $w \le v$ $(dd^c v)^m$-almost everywhere.

On the other hand, again by (4.3.9) it follows that $w \in \mathcal{L}(A)$; see Section 3.2. We use the Domination Principle, see Theorem 3.2.3, to get $w \equiv w^* \le v^* \equiv v$ on A; here the first $\equiv$ sign is due to Corollary 4.3.4, while the second is by definition of $v = V_E^*$.

Now we have

$$w(z) \le v(z) \le \liminf_j v_j(z) \le \limsup_j v(z) \le \lim_j \tilde{w}_j(z) = w(z),$$

thus equality holds and $v_j \to v$ point-wise on A. □

We need the following lemma, the proof is identical to the flat case, thus we omit it and refer the reader to [**59**, Prop. 5.3.3].

Lemma 4.3.12. *Let $E \subset \Omega$ be a non pluripolar set, then we have*

$$V_E^*(z, A) \ge \inf_{\zeta \in \partial\Omega} V_E^*(\zeta, A)(U^*_{E,\Omega}(z) + 1), \ \forall z \in \Omega.$$

(F) PROOF OF (iv) IMPLIES (iii). By Lemma 4.3.12 above we have

$$v_j(z) \geq \inf_{\zeta\in\partial\Omega} v_j(\zeta)\,(u_j(z)+1) \geq \inf_{\zeta\in\partial\Omega} v(\zeta)\,(u_j(z)+1).$$

By assuming (iv) we get $u_j \to -1$ on E. Now we set $w_j := \sup_{s\geq j} u_s$ and we get

$$u(z) \leq u_j(z) \leq w_j(z) \;\; \forall z \in \Omega.$$

Also, set $w(z) := \lim_j w_j(z)$ and notice that $w = -1$ on E, thus $w = w^* \leq u^* = u$ (see *Proof of* (i) *implies* (iv) for a detailed explanation) on Ω due to the Domination Principle [**107**, 1.10].

It follows that for any $z \in \Omega$

$$w(z) \leq w^*(z) \leq u(z) \leq \liminf u_j(z) \leq \limsup_j u_j(z) \leq w(z),$$

thus equality holds and $u_j \to u$ on Ω. □

We now consider the case when E is a regular subset of Ω and E_j is compact for each $j > 0$. Our main tool for replacing (iii) and (iv) by the stronger properties (v) and (vi) is the Hartogs Lemma on plurisubharmonic functions, see [**88**, 1.4 pg 495] for analytic varieties and [**107**] for the statement for weakly plurisubharmonic functions on complex spaces. We give the proof of (i) implies (vi), as the proof of (i) implies (v) is analogous.

PROOF OF (i) IMPLIES (vi) UNDER THE ADDITIONAL HYPOTHESIS. We already shown that $v_j \to v$ point-wise, in particular $\limsup_j v_j(z) \leq v(z) \equiv 0$ for any $z \in E$, because E is regular. Since the sequence $\{v_j\}$ is locally uniformly bounded, it follows by the Hartogs Lemma that for any $\epsilon > 0$ there exists $j_\epsilon \in \mathbb{N}$ such that $v_j(z) \leq \epsilon$ for any $j \geq j_\epsilon$ and $z \in E$. Now we notice that $v_j - \epsilon \in \mathcal{L}(A)$ and $v_j - \epsilon \leq 0$ on E. Hence we have

$$v(z) - \epsilon \leq v_j(z) - \epsilon \leq v(z), \;\; \forall j \geq j_\epsilon, \;\; \forall z \in A.$$

Therefore $\sup_A |v_j - v_j| \leq \epsilon$ for all $j \geq j_0$, that is $v_j \to v$ uniformly on A. □

CHAPTER 5

Mass-Density Sufficient Condition to the Berstein Markov Property on Algebraic Sets

> *A mathematician who can only generalise is like a monkey who can only climb up a tree, and a mathematician who can only specialise is like a monkey who can only climb down a tree. In fact neither the up monkey nor the down monkey is a viable creature. A real monkey must find food and escape his enemies and so must be able to incessantly climb up and down. A real mathematician must be able to generalise and specialise.*
>
> George Pólya

Let E be any compact subset of A and μ be a positive Borel finite measure on A such that $\operatorname{supp}\mu \subseteq E$. Suppose that for any sequence of polynomials $\{p_k\}$ in n complex variables we have

$$\limsup_k \left(\frac{\|p_k\|_E}{\|p_k\|_{L^2_\mu}} \right)^{1/\deg p_k} \leq 1,$$

then we say that (E, μ) has the Bernstein Markov Property. We stress that in the above formula we considered $\deg p_k$, the total degree of the polynomial p_k, and not the degree of it over A.

The aim of this chapter is to prove a sufficient condition he Bernstein Markov property for a measure with compact support in an algebraic m dimensional set $A \subset \mathbb{C}^n$ extending [**24**, Th. 2.2].

We assume from now on that $A \subset \mathbb{C}^n$ is irreducible and has pure dimension m; see Section A for the definition.

We recall here (see Proposition A.0.2) that, possibly after a linear unitary change of coordinates of $\mathbb{C}^n$, the canonical projection π from A to $\mathbb{C}^m$ is a proper map, moreover it is an *analytic covering*, see Theorem A.0.3. Precisely, there exists an analytic subset Y of $\mathbb{C}^m$ such that, setting $S := \pi^{-1}(Y)$, the restriction $\tilde{\pi}$ of π

to $A \setminus S$ is a holomorphic s-sheeted covering of $A \setminus S$ on $\mathbb{C}^m \setminus Y$, i.e., $\tilde{\pi}$ has holomorphic inverses π_j^{-1} $j = 1, 2, \dots, s$. We will refer to these coordinates as Rudin coordinates and use the notation

$$z = (z', z'') \in \mathbb{C}^m \times \mathbb{C}^{n-m}\ ,\ \ z' = \pi(z).$$

Given $z \in A \setminus S$ we denote by $j(z)$ the unique index $j \in \{1, 2, \dots, s\}$ such that $z = \pi_j^{-1}(\pi(z))$, that is the sheet number of z.

We recall the notation for *pseudoballs* in A as in the previous chapter

$$\Omega(z, r) := \pi^{-1}(B(z', r)),\ \forall z \in A,$$

and we introduce the following notation for the piece of $\Omega(z, r)$ containing z

$$\Omega_{j(z)}(z, r) := \pi_{j(z)}^{-1}(B(z', r)),\ \forall z \in A : d(z', Y) > r. \tag{5.0.10}$$

Here $d(z', Y) := \inf_{w \in Y} |z' - w|$ is the standard $\mathbb{C}^m$ distance.

In order to simplify the notation we make few additional assumptions that can be removed by adapting the statement of the main result of this chapter in the obvious way. Namely we will denote by Ω the *unit pseudoball* $\pi^{-1}(B(0, 1)) = \Omega(\pi^{-1}(0), 1)$ and we will always assume that E is a compact subset of Ω; notice that $\pi^{-1}(0)$ is always non empty.

5.1. Mass Density Sufficient Condition for the Polynomial Bernstein Markov Property on an Algebraic Irreducible Set in $\mathbb{C}^n$.

Theorem 5.1.1 (Mass-density sufficient condition on algebraic sets). *Let A be a pure m dimensional irreducible algebraic set in $\mathbb{C}^n$, $n > m$. Let E be a compact regular subset of Ω and $\mu \in \mathcal{M}^+(E)$ such that* $\operatorname{supp}\mu = E$. *Suppose that there exists* $t > 0$ *such that the following mass density condition holds*

$$\operatorname{Cap}(E, \Omega) = \lim_{r \to 0^+} \operatorname{Cap}\left(\left\{z \in E : d(z', Y) > 2r \text{ and } \mu(\Omega_{j(z)}(z, r)) > r^t\right\}, \Omega\right). \tag{5.1.1}$$

Then (E, μ) has the Bernstein Markov property for the restriction of polynomials to A.

Proof. Let us denote by E_r the subset of E on the right hand side of the mass density condition (5.1.1), i.e.,

$$E_r := \left\{z \in E : d(z', Y) > 2r \text{ and } \mu(\Omega_{j(z)}(z, r)) > r^t\right\}.$$

We pick $\epsilon > 0$, by the regularity of E we can pick an open neighbourhood O_ϵ of E in Ω such that $V^*_E(z, A) \le \epsilon/2$ for each $z \in O_\epsilon$.

By the condition (5.1.1) and using "(i) implies (vi)" in Theorem 4.3.2 we can find $r_0 > 0$ such that

$$V^*_{E_r}(z, A) \le V^*_E(z, A) + \epsilon/2 \quad \forall z \in A, 0 < r < r_0,$$

Hence in particular we have $V^*_{E_r}(z, A) \le \epsilon \ \forall z \in O_\epsilon$. By the Bernstein Walsh inequality (see 3.3.17) we get, for any polynomial p of degree at most k,

$$\|p\|_{O_\epsilon} \le \|p\|_E e^{k\epsilon/2} \le \|p\|_{E_r} e^{k\epsilon}. \tag{5.1.2}$$

Now let us pick, for any such polynomial p, a point $\hat{z} \in E_r$ such that $|p(\hat{z})| = \|p\|_{E_r}$.

We note that, since $\hat{z} \in E_r$, we have $d(\hat{z}', Y) > 2r$ and thus $B(\hat{z}', s)$ is an open subset of $\mathbb{C}^m \setminus Y$ where $\pi^{-1}_{j(\hat{z})}$ is well defined and holomorphic for any $s < 2r$. Possibly shrinking r_0, we get

$$O_\epsilon \supseteq \pi^{-1}_{j(\hat{z})}(B(\hat{z}', r)) = \Omega_{j(\hat{z})}(\hat{z}, r). \tag{5.1.3}$$

Let us pick $w \in \Omega_{j(\hat{z})}(\hat{z}, s)$ with $s := \frac{r}{4} e^{-2k\epsilon}$ and define the function

$$f(t) := p \circ \pi^{-1}_{j(\hat{z})}\left(\hat{z}' + t\frac{w' - \hat{z}'}{|w' - \hat{z}'|}\right).$$

Note that f is function of one complex variable, holomorphic in $B(0, 2r). = \{t \in \mathbb{C} : |t| < 2r\}$ thanks to the above discussion on $\pi^{-1}_{j(\hat{z})}$. We furthermore have

$$f(0) = p(\hat{z}) = \|p\|_{E_r}, \quad f(|w' - \hat{z}'|) = p(w)$$

The next estimates follows by the above equations, equations (5.1.2) and (5.1.3) and by the Cauchy Inequality for the derivative of a holomorphic function.

$$\begin{aligned}
|p(w)| = |f(|w' - \hat{z}'|)| &\ge |f(0)| - \left|\int_{[0,|w'-\hat{z}'|]} f'(s)ds\right| \ge \|p\|_{E_r} - \int_{[0,|w'-\hat{z}'|]} |f'|(s)ds \\
&\ge \|p\|_{E_r} - |\hat{z}' - w'| \sup_{[0,|w'-\hat{z}'|]} |f'| \ge \|p\|_{E_r} - |\hat{z}' - w'| \sup_{s\in[0,|w'-\hat{z}'|]} \frac{2}{r}\|f\|_{B(0,r/2)} \\
&\ge \|p\|_{E_r} - |\hat{z}' - w'| \sup_{\Omega_{j(\hat{z})}(\hat{z},r)} \frac{2}{r}|p| \ge \|p\|_{E_r} - \frac{r}{4}e^{-2k\epsilon} \sup_{O_\epsilon} \frac{2}{r}|p| \\
&\ge \|p\|_E e^{-\frac{k}{2}\epsilon}\left(1 - \frac{1}{2}e^{-k\epsilon}\right) \\
&\ge \frac{1}{2}\|p\|_E e^{-\frac{k}{2}\epsilon}.
\end{aligned}$$

Hence we have

$$\min_{\Omega_{j(\hat{z})}(\hat{z},r/4e^{-2k\epsilon})} |p| \geq \frac{1}{2}\|p\|_E e^{-\frac{k}{2}\epsilon}. \tag{5.1.4}$$

Now we use the assumption on the measure of $\Omega_{j(\hat{z})}(\hat{z}, r/4e^{-2k\epsilon})$ with respect to the t power of its radius.

$$\begin{aligned}\|p\|^2_{L^2_\mu} &\geq \|p\|^2_{L^2_\mu(\Omega_{j(\hat{z})}(\hat{z},r/4e^{-2k\epsilon}))} \geq \mu\left(\Omega_{j(\hat{z})}(\hat{z}, r/4e^{-2k\epsilon})\right) \min_{\Omega_{j(\hat{z})}(\hat{z},r/4e^{-2k\epsilon})} |p|^2 \\ &\geq \frac{1}{4}\|p\|^2_E e^{-k\epsilon}\left(r/4\, e^{-2k\epsilon}\right)^t\end{aligned}$$

Now we pick a sequence of r_k each of them for the degree k, namely $r_k := e^{-3k\epsilon}$ and we use the above estimates obtained for each k and each $p_k \in \mathscr{P}^k$ to get

$$\begin{aligned}\limsup_k \left(\frac{\|p_k\|_E}{\|p\|_{L^2_\mu}}\right)^{1/k} &\leq \limsup_k \left(4^{t+1} e^{k\epsilon(5t+1)}\right)^{1/k} = e^{\epsilon(5t+1)} \limsup_k 4^{\frac{t+1}{k}} \\ &\leq e^{\epsilon(5t+1)}.\end{aligned}$$

By arbitrariness of $\epsilon > 0$ we conclude that $\limsup_k \left(\frac{\|p_k\|_E}{\|p\|_{L^2_\mu}}\right)^{1/k} \leq 1$ for any sequence of polynomials $\{p_k\}$ of degree at most k. □

5.2. A Motivating Example: from Real Points of the Complex Sphere to a Weighted Bernstein Markov Measure on the Complex Plane.

In this section we consider the problem of finding a weighted Bernstein Markov measure for a closed possibly unbounded subset C of the complex plane $\mathbb{C}$ with respect to the weight

$$w(\zeta) := (1 + |\zeta|^2)^{-1} = e^{-\log Q(\zeta)}\ , \quad Q(\zeta) := \log(1 + |\zeta|^2).$$

We will see that our result of Theorem 5.1.1 can be used in order to construct such a measure.

The weight Q is a classical *admissible weight* in the sense of [**91**] on any closed non polar subset of $\mathbb{C}$. Namely, Q does satisfy the growth assumption

$$\liminf_{\zeta\to\infty}(Q(\zeta) - \log|\zeta|) = +\infty \tag{5.2.1}$$

that characterize admissible weights. Also we note that Q is continuous function on $\mathbb{C}$.

Our <u>first step</u> is the *compactification* of the problem.

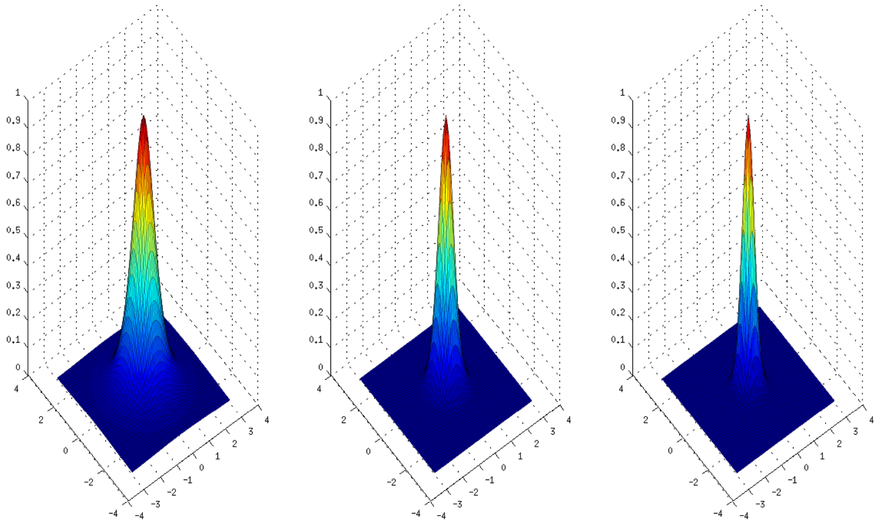

FIGURE 5.2.1. Plots of (from left to right) w, w^2 and w^3.

We use the *stereographic projection* $\Psi : \overline{\mathbb{C}} \to S := \{x \in \mathbb{R}^3, x_1^2 + x_2^2 + (x_3 - 1/2)^2 - 1/4 = 0\} = \frac{1}{2}e_1 + \frac{1}{2}\mathbb{S}^2$, where

$$\Psi(\zeta) := \left(\frac{\Re\zeta}{1+|\zeta|^2}, \frac{\Im\zeta}{1+|\zeta|^2}, \frac{|\zeta|^2}{1+|\zeta|^2}\right) =: (x_1, x_2, x_3).$$

Here $\mathbb{S}^2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$.

Note that $\zeta = \frac{x_1 + ix_2}{1-x_3}$ and $(1 + |\zeta|^2) = \frac{1}{w(\zeta)} = \frac{1}{1-z_3}$.

We embed S in $\mathbb{C}^3$ in the natural way, hence we write z_1, z_2, z_3 in place of x_1, x_2, x_3.

To any weighted polynomial in one complex variable $p(\zeta)w^k$ of degree k we can associate a polynomial in three complex variables.

$$p(\zeta)w(\zeta)^k := w(\zeta)^k \sum_{j=0}^{k} c_j \zeta^j = (1 - z_3)^k \sum_{j=0}^{k} c_j \left(\frac{z_1 + iz_2}{1 - z_3}\right)^j \tag{5.2.2}$$

$$= \sum_{j=0}^{k} c_j (z_1 + iz_2)^j (1 - z_3)^{k-j} =: \tilde{p}(z_1, z_2, z_3). \tag{5.2.3}$$

On S we have $pw^k \circ \Psi^{-1} \equiv \tilde{p}$.

It is clear that, if $(\Psi(C), \nu)$ has the Bernstein Markov Property, then, setting $\mu := \Psi_*^{-1}\nu$ (the pull-back measure), then $[C, \mu, w]$ has the weighted Bernstein Markov property.

The second step in our construction is to *embed S in an algebraic variety* and use the *mass density condition* of Theorem 5.1.1

We consider the complex sphere $\mathcal{H}_{1/2} := \{z \in \mathbb{C}^3 : z_1^2+z_2^2+(z_3-1/2)^2-1/4 = 0\}$ of center $(0,0,1/2)$ and radius $1/2$ and we look at S as a compact set in it, indeed S is the set of all real points of such a complex sphere. In order to further simplify the computations we will consider a slightly modified version of this setting and prove the following.

Proposition 5.2.1 (Surface area has the mass density condition). *Let $\mathcal{H} := \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^2 + z_2^2 + z_3^2 - 1 = 0\}$, $\mathbb{S}^2 := \mathcal{H} \cap \mathbb{R}^3$ and σ the standard surface area on $\mathbb{S}^2$. Then σ enjoys the mass density condition* (5.1.1) *on $\mathbb{S}^2$ with respect to $\mathcal{H}$ and the pseudoball of radius* 2*, i.e., $\{z \in \mathcal{H} : |z_1|^2 + |z_2|^2 < 4\}$.*

Therefore $(\sigma, \mathbb{S}^2)$ has the Bernstein Markov property, thanks to Theorem 5.1.1.

It clearly follows by Proposition 5.2.1 that the same holds true for the surface area on S as a subset of $\mathcal{H}_{1/2}$.

Proof. Let us note that $\mathcal{H}$ is a pure 2 dimensional irreducible algebraic subset of $\mathbb{C}^3$. We consider the canonical projection π on the first two coordinates, this is an analytic covering of $\mathcal{H}$ onto $\mathbb{C}^2$ with branching locus $Y := \{(z_1, z_2) \in \mathbb{C}^2 : z_1^2 + z_2^2 = 1\}$. Given a point $x_0 := (x_{1,0}, x_{2,0}, x_{3,0}) \in \mathbb{S}^2$ we need to compute

$$d(\pi(x_0), Y) := \left(\min_{(z_1,z_2)\in Y} |z_1 - x_{1,0}|^2 + |z_2 - x_{2,0}|^2\right)^{1/2}.$$

Notice that, a priori, the minimizer does not need to be a real point, i.e. a point of the real circle $x_1^2 + x_2^2 = 1$.

To do that we can use the Lagrange Multipliers to solve the problem

$$\begin{cases} \text{Minimize } |z_1 - x_{1,0}|^2 + |z_2 - x_{2,0}|^2 \\ \text{under } z_1^2 + z_2^2 = 1 \\ z_1, z_2 \in \mathbb{C} \end{cases}$$

that can be re-written in real coordinates

$$\begin{cases} \text{Minimize } (x_1 - x_{1,0})^2 + (x_2 - x_{2,0})^2 + y_1^2 + y_2^2 \\ \text{under } x_1^2 + x_2^2 - y_1^2 - y_2^2 = 1 \\ \text{and } x_1y_1 + x_2y_2 = 0 \\ x_1, x_2, y_1, y_2 \in \mathbb{R} \end{cases}. \tag{5.2.4}$$

The Lagrange Multiplier system is

$$(5.2.5)\qquad \begin{cases} -2x_{1,0} + \lambda_1 y1 + 2x_1\lambda_2 = 0 \\ -2x_{2,0} + \lambda_1 y_2 + 2x_2\lambda_2 = 0 \\ 4y_1 + \lambda_1 x_1 - 2\lambda_2 y_1 = 0 \\ 4y_2 + \lambda_1 x_2 - 2\lambda_2 y_2 = 0 \\ x_1^2 + x_2^2 - y_1^2 - y_2^2 - 1 = 0 \\ x_1y_1 + x_2y_2 = 0 \end{cases} .$$

There are only two *real* solutions, namely

$$(5.2.6)\qquad \mathrm{I} := \begin{cases} x_1 & = -\frac{x_{1,0}}{\sqrt{x_{1,0}^2+x_{2,0}^2}} \\ x_2 & = -\frac{x_{2,0}}{\sqrt{x_{1,0}^2+x_{2,0}^2}} \\ y_1 & = 0 \\ y_2 & = 0 \\ \lambda_1 & = 0 \\ \lambda_2 & = -\sqrt{x_{1,0}^2 + x_{2,0}^2} \end{cases} ,\qquad \mathrm{II} := \begin{cases} x_1 & = \frac{x_{1,0}}{\sqrt{x_{1,0}^2+x_{2,0}^2}} \\ x_2 & = \frac{x_{2,0}}{\sqrt{x_{1,0}^2+x_{2,0}^2}} \\ y_1 & = 0 \\ y_2 & = 0 \\ \lambda_1 & = 0 \\ \lambda_2 & = \sqrt{x_{1,0}^2 + x_{2,0}^2} \end{cases}$$

Substituting the solution in the object of the minimization we get

$$\begin{aligned} & d(\pi(x_0), Y) \\ =& \left(\min\left\{1 + x_{1,0}^2 + x_{2,0}^2 + 2\sqrt{x_{1,0}^2 + x_{2,0}^2}, 1 + x_{1,0}^2 + x_{2,0}^2 - 2\sqrt{x_{1,0}^2 + x_{2,0}^2}\right\}\right)^{1/2} \\ =& \sqrt{1 + x_{1,0}^2 + x_{2,0}^2 - 2\sqrt{x_{1,0}^2 + x_{2,0}^2}} = \sqrt{\left(1 - \sqrt{x_{1,0}^2 + x_{2,0}^2}\right)^2} = \left|1 - \sqrt{x_{1,0}^2 + x_{2,0}^2}\right| \\ =& 1 - \sqrt{x_{1,0}^2 + x_{2,0}^2}, \end{aligned}$$

i.e, the minimizer correspond to the solution (II). In particular the distance of $\pi(x_0)$ from Y is precisely the distance from its real points, i.e., the distance from the real unit circle.

Therefore we can easy characterize $F_r := \{x \in \mathbb{S}^2 : d(\pi(x), Y) > 2r\}$ as simply the lifting of the disk $\{(x_1, x_2) : x_1^2 + x_2^2 < (1-2r)^2\}$ to the real points of both pieces of $\mathcal{H}$. Namely,

$$\begin{aligned} F_r = & \left\{(x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}) : x_1^2 + x_2^2 < (1 - 2r)^2\right\} \cup \\ & \cup \left\{(x_1, x_2, -\sqrt{1 - x_1^2 - x_2^2}) : x_1^2 + x_2^2 < (1 - 2r)^2\right\}. \end{aligned}$$

Note that $F_r \uparrow \mathbb{S}^2 \setminus V$ as $r \to 0^+$. Now we pick any point $x_0 \in F_r$, we can compute the set $\Omega_{j(x_0)}(x_0, r)$ defined in equation (5.0.10) and taken into account in the mass density condition equation (5.1.1). Precisely we get

$$\Omega_{j(x_0)}(x_0, r) = \left\{(x_1, x_2, \operatorname{sgn} x_{0,3} \sqrt{1 - x_1^2 - x_2^2}) : (x_1 - x_{1,0})^2 + (x_2 - x_{2,0})^2 < r^2\right\}.$$

Therefore we can estimate the measure $\sigma(\Omega_{j(x_0)}(x_0, r))$ as follows. Recall that the surface area of the real unit sphere is $d\sigma = \frac{1}{\sqrt{1-x_1^2-x_2^2}} dx_1 dx_2$.

$$\sigma(\Omega_{j(x_0)}(x_0, r)) = \int\int_{B((x_{1,0}, x_{2,0}), r)} \frac{1}{\sqrt{1 - x_1^2 - x_2^2}} dx_1 dx_2 \geq \pi r^2 \min_{B((x_{1,0}, x_{2,0}), r)} \frac{1}{\sqrt{1 - x_1^2 - x_2^2}}$$
$$\geq \pi r^2 \frac{1}{\sqrt{1 - ((x_{1,0}^2 + x_{2,0}^2)^{1/2} + r)^2}} \geq \pi r^2 \frac{1}{\sqrt{1 - (1 - r)^2}} \geq \frac{\pi}{2} r^{3/2}.$$

Therefore, setting $E_r := \{x \in F_r : \sigma(\Omega_{j(x_0)}(x, r)) > r^2\}$, we have $F_r \equiv E_r$. In order to conclude the proof we are left to prove that

$$\mathbb{S}^2 \text{ is a regular subset of } \mathcal{H}, \tag{5.2.7}$$

$$\operatorname{Cap}(E_r, \{z \in \mathcal{H} : |\pi z| < 2\}) \to \operatorname{Cap}(\mathbb{S}^2, \{z \in \mathcal{H} : |\pi z| < 2\}). \tag{5.2.8}$$

The property (5.2.7) can be shown by direct computation, indeed we shown in [**33**, Prop. 4.1] that

$$V_{\mathbb{S}^2}(z, \mathcal{H}) = V_{\overline{B(0,1)}}(z) = \frac{1}{2} \log\left(|z|^2 + \sqrt{1 - |z|^4}\right),$$

which is a continuous function on $\mathcal{H}$.

The property (5.2.8) can be achieved easily by using the sub additivity of the relative capacity and the fact that $K = E \setminus (\cup_{r>0} E_r)$ is a subset of Y and thus has zero outer capacity and the monotonicity of the sequence E_r. □

Thus we have proved the following.

Proposition 5.2.2 (Weighted Bernstein Markov measure on $\mathbb{C}$). *Let*

$$\mu := \Psi_*^{-1} \sigma_{1/2,1/2},$$

(i.e., $\mu(B) := \sigma_{1/2,1/2}(\Psi(B))$ for any Borel set B) where $\sigma_{1/2,1/2}$ is the surface area measure on $1/2 e_1 + 1/2 \mathbb{S}^2$ and $w(\zeta) := (1 + |\zeta|^2)^{-1}$. Then the triple $[\mathbb{C}, \mu, w]$ has the weighted Bernstein Markov property.

Differentiating the map Ψ and the parametrization of $S = 1/2e_1 + 1/2\mathbb{S}^2$ we can compute μ explicitly. We have

$$d\sigma_{1/2,1/2} = \left(1 + \frac{4(x_1^2 + x_2^2)}{|1 - 4(x_1^2 + x_2^2)|}\right)^{1/2} dx_1 dx_2 = \frac{1 + |\zeta|^2}{1 - |\zeta|^2} \circ \Psi^{-1} dx_1 dx_2,$$

while

$$dx_1 dx_2 = \left(\frac{1 - |\zeta|^4}{(1 + |\zeta|^2)^2}\right) \frac{i}{2} d\zeta \wedge d\bar{\zeta}.$$

Therefore we can compute

$$d\mu = \left(\frac{1}{(1 + |\zeta|^2)^2}\right) \frac{i}{2} d\zeta \wedge d\bar{\zeta} = w^2(\zeta) dm(\zeta).$$

As we could expect the density is radial, moreover it coincides with w^2. This density is rather fast decreasing to 0 as shown by the second graph of Figure 5.2.1.

Remark 5.2.1. *Our computations show actually more. Let us pick a continuous weight $Q : \mathbb{C} \to \mathbb{R}$ with a slightly modified growth assumption, we suppose that*

$$-\infty < Q(\zeta) - \log(1 + |\zeta|^2) < +\infty \quad \forall \zeta \in \mathbb{C}. \tag{5.2.9}$$

Let us assume for simplicity that Q is radial, i.e. depends only on $|\zeta|$. Then any weighted polynomial $pe^{-k \log Q}$ in one complex variable of degree k can be re-written as a weighted polynomial on the sphere $\tilde{p}e^{-k \log \tilde{Q}}$ of the same degree k with respect to the weight $\tilde{Q}(x_1, x_2, x_3) := Q(\sqrt{\frac{x_3}{1-x_3}}) - \log|1 - x_3|$. This follows by the same computation as in (5.2.2) *and below.*

Due to the growth assumption (5.2.9)*, we get that $\tilde{w} := e^{-\tilde{Q}}$ is a positive (bounded) continuous function on the real sphere. It is then possible, for any $\epsilon > 0$, to find a homogeneous polynomial q (say of degree l) such that*

$$(1 - \epsilon)|q| \le \tilde{w} \le (1 + \epsilon)|q|.$$

Therefore we get, for any sequence of polynomials p_k of degree k,

$$\limsup_k \left(\frac{\|p_k \tilde{w}^k\|_{\mathbb{S}^2}}{\|p_k \tilde{w}^k\|_{L^2_\sigma}}\right)^{1/k} \le \limsup_k \frac{1 + \epsilon}{1 - \epsilon} \left(\frac{\|p_k q^k\|_{\mathbb{S}^2}}{\|p_k q^k\|_{L^2_\sigma}}\right)^{1/k}$$

$$= \frac{1 + \epsilon}{1 - \epsilon} \limsup_k \left[\left(\frac{\|p_k q^k\|_{\mathbb{S}^2}}{\|p_k q^k\|_{L^2_\sigma}}\right)^{1/((l+1)k)}\right]^{l+1} = \frac{1 + \epsilon}{1 - \epsilon} \longrightarrow_{\epsilon \to 0} 1.$$

Therefore the measure σ has the weighted Bernstein Markov property with respect to the weight $\tilde{Q}$ on $\mathbb{S}^2$ and thus $\frac{1}{(1+|\zeta|^2)^2} \frac{i}{2} d\zeta \wedge d\bar{\zeta}$ has the weighted Bernstein Markov property for the weight Q on $\mathbb{C}$.

Indeed, by minor modification of the technique we propose here, it is possible to manage even weights that do not satisfy the condition (5.2.9) but still are admissible in the classical sense. In contrast, we cannot deal with weights that are just weakly admissible, i.e., one has $\liminf_{z\to\infty} Q(z) - \log|z| > -\infty$, as considered for instance in [**54**].

Part II

Discrete Approach: Weakly Admissible Meshes

CHAPTER 6

Introducing (Weakly) Admissible Meshes

Se mi fosse dato di vivere senza la possibilitá di sognare e di lottare per un sogno, bello quanto inutile, sarei un uomo finito.

Giusto Gervasutti

In this chapter we introduce a tool of growing interest during the last years, namely (weakly) admissible meshes; see for instance [**37**], [**31**], [**77**], [**75**], [**78**], [**80**], [**62**], [**63**], [**84**] and references therein. The study of admissible meshes is motivated both by polynomial approximation (by discrete least squares) and by the quest for "good interpolation points" for a given compact subset of $\mathbb{C}^n$. Moreover, as we will point out in Section 6.3, admissible meshes constitute a *good discrete model* for Bernstein Markov measures, since they share some of their properties and thus can be used to reconstruct certain important quantities in Pluripotential Theory by L^2 methods. However, in the case of admissible meshes, all involved computations can be performed by sampling polynomials on a finite number of points (for each given degree), therefore it is possible to implement these procedures providing approximation algorithms with a strong theoretical motivation.

Along this chapter and Chapter 7 we will also present some examples, figures and numerical computations. We stress that all used MATLAB software is free downloadable at CAA software webpage; a presentation the MATLAB package for working with weakly admissible meshes, WAM package, can be found in [**69**].

6.1. Definitions and Main Properties

6.1.1. Definitions. Let us denote by $\mathscr{P}^k(\mathbb{C}^n)$ the space of polynomials of n complex variables having degree at most k. We recall that a compact set $E \subset \mathbb{R}^n$ (or $\mathbb{C}^n$) is said to be polynomial determining if any polynomial vanishing on E is necessarily the null polynomial.

Let us consider a polynomial determining compact set $E \subset \mathbb{R}^n$ (or $\mathbb{C}^n$) and let A_k be a subset of E. If there exists a positive constant C_k such that for any polynomial $p \in \mathscr{P}^k(\mathbb{C}^n)$ the following inequality holds

$$\|p\|_E \leq C_k \|p\|_{A_k}, \tag{6.1.1}$$

then A_k is said to be a *norming set* for $\mathscr{P}^k(\mathbb{C}^n)$.

Let $\{A_k\}$ be a sequence of norming sets for $\mathscr{P}^k(\mathbb{C}^n)$ with constants $\{C_k\}$, suppose that both C_k and $\mathrm{Card}(A_k)$ grow at most polynomially with k (i.e., $\max\{C_k, \mathrm{Card}(A_k)\} = O(k^s)$ for a suitable $s \in \mathbb{N}$), then $\{A_k\}$ is said to be a *weakly admissible mesh* (WAM) for E; see[1] [**37**]. Observe that necessarily

$$\mathrm{Card}\, A_k \geq N_k := \dim \mathscr{P}^k(\mathbb{C}^n) = \binom{k+n}{k} = O(k^n) \tag{6.1.2}$$

since a (W)AM A_k is $\mathscr{P}^k(\mathbb{C}^n)$-determining by definition.

If $C_k \leq C\ \forall k$, then $\{A_k\}_{\mathbb{N}}$ is said to be an *admissible mesh* (AM) for E; in the sequel, with a little abuse of notation, we term (weakly) admissible mesh not only the whole sequence but also its k-th element A_k. When $\mathrm{Card}(A_k) = O(k^n)$, following Kroó [**62**], we refer to $\{A_k\}$ as an *optimal admissible mesh*, since this grow rate for the cardinality is the minimal one in view of equation (6.1.2).

6.1.2. Basic properties. Let E be a compact polynomial determining subset of $\mathbb{R}^n$ (or $\mathbb{C}^n$) and $\{A_k\}$ a (weakly) admissible mesh for E with constants $\{C_k\}$, the following properties can be derived directly from the above definition.

(1) AFFINE MAPPING. If T is any affine mapping and $K := T(E)$ then $B_k := T(A_k)$ is a (weakly) admissible mesh for K with constant $\tilde{C}_k := C_k$.
(2) If $B_k \supseteq A_k$ and $\mathrm{Card}(B_k)$ grows polynomially with respect to k and A_k is a (weakly) admissible mesh for E of constant C_k, then B_k is a (weakly) admissible mesh for E having constant $\tilde{C}_k \leq C_k$.
(3) UNION. If $A_{k,j}$ is a (weakly) admissible mesh of constant $C_{k,j}$ for the polynomial determining set E_j then $B_k := \cup_{j\in\mathcal{J}} A_{k,j}$ is a (weakly) admissible mesh for $E := \cup_{j\in\mathcal{J}} E_j$ for any finite set $\mathcal{J}$, being $\max_{j\in\mathcal{J}} C_k^j$ the constant of B_k.
(4) CARTESIAN PRODUCT. If $A_{k,j}$ is a (weakly) admissible mesh of constant $C_{k,j}$ for the polynomial determining set E_j then $B_k := \prod_{j\in\mathcal{J}} A_{k,j}$ is a (weakly) admissible mesh for $E := \prod_{j\in\mathcal{J}} E_j$ for any finite set $\mathcal{J}$, being $\prod_{j\in\mathcal{J}} C_{k,j}$ the constant of B_k.

[1]The original definition in [**37**] is actually a little weaker (sub-exponential growth instead of polynomial growth is allowed), here we prefer to use the present one which is now the most common in the literature.

(5) POLYNOMIAL MAPPING If P_m is any polynomial mapping of degree at most m and $K := P_m(E)$ then $B_k := P(A_{m\cdot k})$ is a (weakly) admissible mesh for K with constant $\tilde{C}_k := C_{m\cdot k}$.

(6) GOOD INTERPOLATION POINTS. Any set of unisolvent[2] interpolation points whose Lebesgue constant Λ_k grows polynomially with respect to the considered degree is a weakly admissible mesh of constant $C_k = \Lambda_k$.

Despite their simplicity these properties are rather useful to construct an admissible mesh in several instances. For example, the Chebyshev Lobatto nodes $X_k := \left\{\cos\left(\frac{j\pi}{k}\right)\right\}_{j=0,1,\dots,k}$ are good interpolation points on the standard interval $[-1,1]$ in the sense that their Lebesgue constants L_k grows as $O(\log k)$. Therefore, due to property (6), X_k is a weakly admissible mesh of constant L_k. Now we can apply property (4) to get a weakly admissible mesh $A_k := X_k \times X_k$ for the square $E := [-1,1]^2$ having constant $C_k := L_k^2$. We introduce the Duffy transformation

$$\mathcal{D}_{a,b,c,d}(x,y) := \frac{1}{4}[(1-x)(1+y)a + (1+x)(1-y)b + \dots$$
$$+(1+x)(1+y)c + (1-x)(1+y)d],$$

mapping the square onto the convex quadrangle $Q_{a,b,c,d}$ with vertices $a,b,c,d \in \mathbb{R}^2$; note that $\mathcal{D}_{a,b,c,d}$ is a bilinear map, and, if we take two of the parameters a,b,c,d to be equal, then $Q_{a,b,c,d}$ is a triangle. For any choice of the parameters a,b,c,d using property (5) we can construct a weakly admissible mesh $\tilde{A}_k^{a,b,c,d} := \mathcal{D}_{a,b,c,d}(A_{2\cdot k})$ for $Q_{a,b,c,d}$ of constant $C'_k := C_{2k} = L_{2k}^2$.

Finally, if we consider a polygon P, we can split it in a finite union of convex quadrangles and triangles Q_{a_j,b_j,c_j,d_j}, $j = 1,2,\dots,M$ and, due to property (3), we have that $B_k := \cup_{j=1}^M \tilde{A}_k^{a_j,b_j,c_j,d_j}$ is a weakly admissible mesh for P of constant C'_k.

Similarly, we could start with an admissible mesh $\tilde{X}_k$ for the interval ($\tilde{X}_k = X_{m\cdot k}$ with $m > 1$ would suffice, [**45**]) and end up with an admissible mesh for P. Figure 6.1.1 shows how the final admissible mesh (built in this way by a particular triangulation algorithm) for a star shape looks like.

The problem of constructing WAMs on more general class of compact sets will be the subject of the next chapter.

[2]An array of points $A_k := \{x_k^1, \dots, x_k^{N_k}\} \subset E$ is said to be unisolvent of degree k if for any set of values $\{y_k^1, \dots, y_k^{N_k}\} \in \mathbb{C}^{N_k}$ there exists a unique interpolating polynomial $p_k \in \mathscr{P}^k$ such that $p_k(x_k^j) = y_k^j$ for all $j = 1,2,\dots,N_k$.

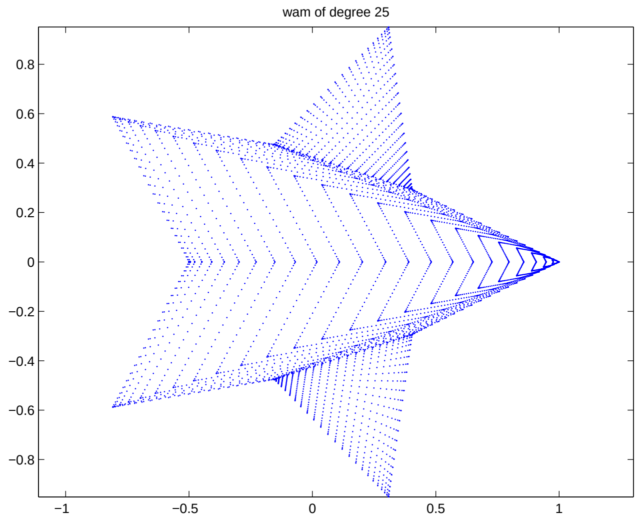

FIGURE 6.1.1. An admissible mesh of degree 25 for a star shape.

6.1.3. Further properties. It is worth to recall other nice properties of (weakly) admissible meshes that are more complicated to show. Namely, they enjoy a stability property under smooth mapping and small perturbations. In order to state such results we need first to recall the *Markov polynomial Inequality.* Let E be a compact subset of $\mathbb{C}^n$, we will say that E enjoy the Markov inequality of constant $M \geq 0$ and exponent $r \geq 1$ if for any $k \in \mathbb{N}$ and any polynomial $p \in \mathscr{P}^k$ one has

$$\|\nabla p\|_E \leq M k^r \|p\|_E. \tag{6.1.3}$$

In such a case we equivalently say that E is a Markov compact set of parameters (M, r). Several variants of this inequality have been studied as *tangential Markov Inequality* and *Markov brothers Inequality*; it is probably worth to say that, not surprisingly, the parameters in such inequalities are intimately related with the pluripotential theoretic aspects of the considered compact set E, in particular the smoothness properties of the plurisubharmonic extremal function (for definition see Subsection 3.3.2); see [**9**], [**10**], [**82**] and [**34**].

THEOREM 6.1.1 (Smooth mapping of WAMs; [**77**]). *Let $E \subset \mathbb{C}^n$ be a compact set, φ a analytic mapping of a neighbourhood of $\hat{E}$ (i.e., its polynomial hull) onto the Markov compact set $K := \varphi(E)$ and let $\{A_k\}$ be a (weakly) admissible mesh for E. Then there exists a sequence of natural numbers $j(k) = \mathcal{O}(\log(k))$ such that $\{\tilde{A}_k\} := \{\varphi(A_{k \cdot j(k)})\}$ is a (weakly) admissible mesh for K.*

There are some improvements and consequences of this theorem, see for instance [**78**, Cor. 2 case (B)] for mappings of finite smothness and [**84**].

THEOREM 6.1.2 (Small perturbations of WAMs,[**78**]). *Let $E \subset \mathbb{C}^n$ be a Markov compact set of parameters (M, r) and $\{A_k\}$ a weakly admissible mesh of constants $\{C_k\}$, let $t \in (0, \hat{t})$, where $\hat{t}$ solves $t\exp(t/2) = 1$, and consider any finite set $\tilde{A}_k \subset E$ such that*

$$d_{\mathscr{H}}(\tilde{A}_k, A_k) \leq \frac{t}{Mn^r(1 + C_k)},$$

where $d_{\mathscr{H}}(A, B)$ denotes the Hausdorff distance between A and B.

Then $\{\tilde{A}_k\}$ is a weakly admissible mesh for E of constants $\{\tilde{C}_k\}$, $\tilde{C}_k \leq \frac{C_k}{1-tn\exp(tn/2)}$, provided that $\operatorname{Card}(\tilde{A}_k) = O(k^s)$ *for some finite s.*

We remark that Theorem 6.1.2 is useful if one aims to numerically compute a weakly admissible mesh for a given Markov compact set: roughly speaking, if the numerical computations are performed with sufficient accuracy the computed mesh is, indeed, weakly admissible.

6.2. Main Motivations

Here we sketch the main motivations for the study of weakly admissible meshes.

The first motivation for the study of (weakly) admissible meshes is given by the good behaviour of discrete polynomial least squares approximation produced by sampling on an admissible mesh. Calvi and Levenberg noted that given a weakly admissible mesh $\{A_k\}$ for the compact polynomial determining set E the following estimates hold true. Here we denote by $\Lambda_k : \mathscr{C}(E) \to \mathscr{P}^k$ the discrete least square projection onto $\mathscr{P}^k(\mathbb{C}^n)$ with respect to the inner product $\langle f, g\rangle_{A_k} := \sum_{x\in A_k} f(x)\overline{g}(x)$ canonically associated with A_k.

$$\|\Lambda_k f\|_E \leq C_k \left(\|f\|_E + \sqrt{\operatorname{Card} A_k}\, d_k(f, E)\right),$$
$$\|f - \Lambda_k f\|_E \leq \left(1 + C_k\left(1 + \sqrt{\operatorname{Card} A_k}\right)\right) d_k(f, E).$$

Here $d_k(f, E) := \min_{p\in\mathscr{P}^k(\mathbb{C}^n)} \|f - p\|_E$. Roughly speaking, if it is possible to well uniformly approximate on E the continuous function f (e.g., one has additional smoothness properties and/or good properties of E as for instance the Jackson property [**83**]), then *the discrete least squares projection on any admissible mesh provides an effective way to compute a uniform approximation to f whose behaviour in term of error is not too far from the best possible one.*

The other most interesting feature of weakly admissible meshes is that it is possible to extract good interpolation arrays from them by standard numerical linear algebra. In [**31**] authors present two algorithms, *approximate Fekete points* (AFP for short) and *discrete Leja sequences* (DLS for short), based on the QR and the LU factorizations of Vandermonde matrices respectively, that extract unisolvent arrays from a weakly admissible mesh in a nearly optimal way in the sense of Theorem 6.2.3 below.

The core idea in both algorithms is the following, instead of optimizing the modulus of the Vandermonde determinant on the continuous set E^{N_k} for any k, they perform a optimization on the finite set $A_k^{N_k}$, then the problem (still numerically very hard) is boiled down to a numerical linear algebra one by an heuristic. For instance, the AFP algorithm uses the QR factorization with column pivoting to solve a undetermined system of equations, this leads to the extraction of a maximum rank square sub-matrix of the transpose of the rectangular Vandermonde matrix whose determinant is nearly maximum among all the possible choices. Instead, the DLS algorithm uses the LU factorization by Gaussian elimination with row pivoting.

Theorem 6.2.3 (Discrete extremal sets; [**31**]). *Let E be any compact polynomially convex non pluripolar and regular set and $\{A_k\}$ a weakly admissible mesh for E. Let $\{z_1^{(k)}, \dots, z_{N_k}^{(k)}\}_k$ be extremal sets of degree k computed starting by A_k either the by the AFP or DLS algorithm, then the following hold.*

i) $\lim_k \left|\mathrm{VDM}_k(z_1^{(k)}, \dots, z_N^{(k)})\right|^{\frac{n+1}{nkN_k}} = \delta(E)$.
ii) $\frac{1}{N_k}\sum_{j=1}^{N_k} \delta_{z_j^{(k)}} \rightharpoonup^* \mu_E$.

See Chapter 1 for the definitions of $\delta(E)$ and μ_E.

We stress that, for $n > 1$, finding unisolvent arrays for total degree polynomial interpolation on a given compact set is a non trivial issue by itself. This *discrete extremal arrays* have been shown to enjoy the much stronger property of *leading to the transfinite diameter* (i.e., (i) above), moreover numerical experiments show that the Lebesgue constant of these interpolation arrays is slowly growing with the degree k. Note that the Lebesgue constant of true Fekete points of order k extracted from an admissible mesh A_k having constant C is bounded above by CN_k; [**23**].

6.3. Relations with the Bernstein Markov Property and Pluripotential Theory

We illustrate the analogies with Bernstein Markov measures, thus the reader is invited to compare this section and the next one to Section 1.1 and 1.2 respectively.

As we claimed before admissible meshes are *nice discrete models* for Bernstein Markov measures. Let us suppose $E \subset \mathbb{C}^n$ to be any polynomial determining compact set and $\{A_k\}$ be an admissible mesh of constant C for it. We can canonically associate to A_k the discrete probability measure $\mu_k \in \mathcal{M}^+(E)$ setting

$$\mu_k := \frac{1}{\operatorname{Card} A_k} \sum_{x \in A_k} \delta_x.$$

Note that by definition we have $\|p_k\|_E \leq C\|p_k\|_{L^\infty_{\mu_k}}$ for any $p_k \in \mathscr{P}^k$. On the other hand, for any such a polynomial we have

$$\|p_k\|^2_{L^2_{\mu_k}} = \frac{1}{\operatorname{Card} A_k} \sum_{j=1}^{\operatorname{Card} A_k} |p(x_j)|^2 \geq \frac{\|p_k\|^2 L^\infty_{\mu_k}}{\operatorname{Card} A_k}.$$

Therefore, recalling that $\operatorname{Card} A_k$ is growing at most polynomially in k, for any sequence of polynomial $\{p_k\}$ with $\deg p_k \leq k$ we have

$$\limsup_k \left(\frac{\|p_k\|_E}{\|p_k\|_{L^2_{\mu_k}}} \right)^{1/k} \leq \lim_k (C\sqrt{\operatorname{Card} A_k})^{1/k} = 1. \tag{6.3.1}$$

The same holds true if we start by a weakly admissible mesh A_k.

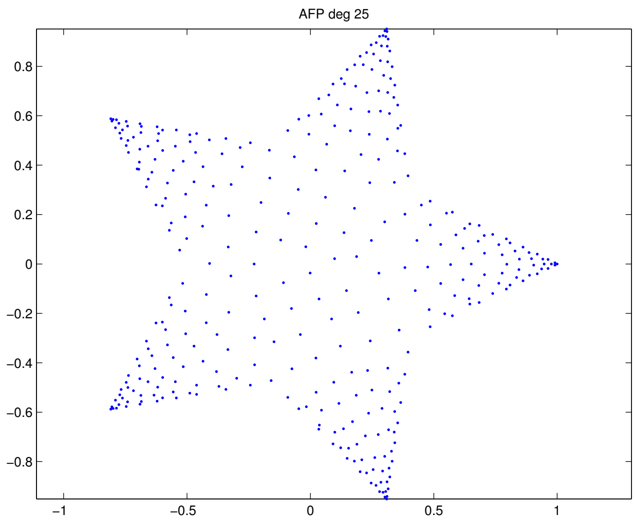

FIGURE 6.2.2. Approximate Fekete points of degree 25 for the star shape, extracted from the admissible mesh of Figure 6.1.1.

This inequality not only *closely resembles the Bernstein Markov property*, also it can be explicitly used to build approximation algorithms as we will see in Theorem 6.3.4 and Proposition 6.3.1.

Moreover, we can show that given a weakly admissible mesh $\{A_k\}$ for the compact set $E \subset \mathbb{C}^n$ we can always construct a Bernstein Markov measure with countable carrier in E. Indeed, in the above notation, it suffices to define

$$\mu := \sum_{k=0}^{\infty} \frac{1}{2^k} \mu_k,$$

where the convergence of the series is to be intended in the weak star sense. Suppose that $\max\{C_k, \operatorname{Card} A_k\} \leq C k^s$ for k large enough, then for any p_k as above we have

$$\begin{aligned}
\|p_k\|^2_{L^2_\mu} &\geq \sum_{j=k}^{\infty} \frac{1}{2^j} \int |p_k|^2 d\mu_j \geq \sum_{j=k}^{\infty} \frac{1}{2^j \operatorname{Card} A_j} \|p_k\|^2_{L^\infty_{\mu_j}} \\
&\geq \|p_k\|_E \sum_{j=k}^{\infty} \frac{1}{2^j \operatorname{Card} A_j C_j^2} \geq \|p_k\|_E \frac{1}{C}^3 \sum_{j=k}^{\infty} \frac{1}{2^j j^{3s}}.
\end{aligned}$$

Therefore

$$\limsup_k \left(\frac{\|p_k\|_E}{\|p_k\|_{L^2_\mu}} \right)^{1/k} \leq \limsup_k C^{3/2k} \left(\sum_{j=k}^{\infty} \frac{1}{2^j j^{3s}} \right)^{-1/2k}.$$

It is not difficult to see that for each $\epsilon > 0$ there exists k_ϵ such that

$$\sum_{j=k}^{\infty} \frac{1}{2^j j^{3s}} = \sum_{j=k}^{\infty} \frac{1}{2^{j+3s\log_2 j}} \geq \sum_{j=k}^{\infty} \frac{1}{2^{(1+\epsilon)j}} \;\; \forall k \geq k_\epsilon,$$

hence we have

$$\begin{aligned}
\limsup_k \left(\frac{\|p_k\|_E}{\|p_k\|_{L^2_\mu}} \right)^{1/k} &\leq \left(\sum_{j=k}^{\infty} \frac{1}{2^{(1+\epsilon)j}} \right)^{-1/2k} \\
= \lim_k \left(\frac{2^{-k(1+\epsilon)}}{1 - 2^{-(1+\epsilon)}} \right)^{-1/2k} &= 2^{\epsilon/2}, \;\; \forall \epsilon > 0.
\end{aligned}$$

Taking $\epsilon \to 0^+$ we show that μ is a Bernstein Markov measure for E.

Let us recall that, a Berstein Markov measure μ on E can be used to recover the extremal function V_E^* the transfinite diameter $\delta(E)$ and the pluripotential equilibrium measure μ_E, see Chapter 1. Precisely, one has the following asymptotic properties.

$$\lim_k \frac{1}{2k} \log B_k^\mu = V_E \quad \text{uniformly in } \mathbb{C}^n$$

$$\lim_k \det G_k(\mu)^{\frac{n+1}{2nkN_k}} = \delta(E)$$

$$\frac{B_k^\mu}{N_k}\mu \rightharpoonup^* \mu_E$$

Here

$$B_k^\mu(z) := \sum_{j=1}^{N_k} |q_j(z,\mu)|^2,$$

where $\{q_j(z,\mu)\}$ is an orthonormal basis of $\mathscr{P}^k$ with respect to the scalar product induced by L^2_μ.

An examination of the proofs of these results, see Section 1.2, shows that the same holds true if we replace μ by a *sequence of asymptotically Bernstein Markov measures*, i.e., a sequence $\{\mu_k\}$ such that

(1) $\mu_k \in \mathcal{M}_1^+(E)$ for each k
(2) for any sequence of polynomials $\{p_k\}$ with $\deg p_k \leq k$ we have

$$\limsup_k \left(\frac{\|p_k\|_E}{\|p_k\|_{L^2_{\mu_k}}} \right)^{1/k} \leq 1.$$

We notice (see equation 6.3.1 above) that *the sequence of uniform probability measures canonically associated to a weakly admissible mesh enjoys the properties (1) and (2) above*. Therefore, just repeating the arguments of Proposition 1.2.1, Theorem 1.2.1 and Theorem 1.2.2 we obtain the following.

Theorem 6.3.4 (Bergman asymptotic for weakly admissible meshes). *Let $E \subset \mathbb{C}^n$ be a compact regular non pluripolar set and $\{A_k\}$ be a weakly admissible mesh for it. Let us denote by μ_k the uniform probability measure on A_k, that is*

$$\mu_k := \frac{1}{\operatorname{Card} A_k} \sum_{x \in_k} \delta_x.$$

Then the following asymptotic properties hold true.

$$\lim_k V_{E,k}^{(1)} := \lim_k \frac{1}{2k} \log B_k^{\mu_k} = V_E \quad \textit{uniformly in } \mathbb{C}^n \tag{6.3.2}$$

$$\lim_k \det G_k(\mu_k)^{\frac{n+1}{2nkN_k}} = \delta(E) \tag{6.3.3}$$

$$\frac{B_k^\mu}{N_k}\mu \rightharpoonup^* \mu_E \tag{6.3.4}$$

We stress that Theorem 6.3.4 indeed provides three approximation algorithms for three of the most important objects in Pluripotential Theory.

Also we can prove a variant of equation 6.3.2 in the above theorem. Instead of considering the Bergman function $B_k^{\mu_k}$, i.e., the diagonal of the reproducing kernel of $\left(\mathscr{P}^k(\mathbb{C}^n), \langle\cdot;\cdot\rangle_{\mu_k}\right)$, we look at the asymptotic behaviour of the k-th root of the $L^1_{\mu_k}$ norm of the kernel itself

$$K_k^{\mu_k}(z,w) := \sum_{j=1}^{N_k} q_j(z,\mu_k)\overline{q_j(w,\mu_k)}.$$

Proposition 6.3.1 (k-th root asymptotic for the reproducing kernel of weakly admissible meshes). *Let E be a regular compact subset of $\mathbb{C}^n$ and $\{A_k\}$ a weakly admissible mesh for E of cardinality* $\operatorname{Card} A_k =: M_k$. *Then we have*

$$\lim_k V_{E,k}^{(2)} := \lim_k \frac{1}{k}\log\int |K_k^{\mu_k}(z,\zeta)|d\mu_k(\zeta) = V_E(z), \tag{6.3.5}$$

locally uniformly in $\mathbb{C}^n$.

Note that $\int |K_k^{\mu_k}(z,\zeta)|d\mu_k(\zeta) = \frac{1}{M_k}\sum_{i=1}^{M_k}\left|\sum_{j=1}^{N_k} q_j(z,\mu_k)\overline{q_j(\zeta_i,\mu_k)}\right|$ *for* $A_k := \{\zeta_1,\dots,\zeta_{M_k}\}$.

Proof. On one hand we have

$$\begin{aligned}
&\frac{1}{M_k}\sum_{i=1}^{M_k}\left|\sum_{j=1}^{N_k} q_j(z,\mu_k)\overline{q_j(\zeta_i,\mu_k)}\right|\\
\le&\frac{1}{M_k}\sum_{i=1}^{M_k}\left[\left(\sum_{j=1}^{N_k}|q_j(z,\mu_k)|^2\right)^{1/2}\times\left(\sum_{j=1}^{N_k}|q_j(\zeta_i,\mu_k)|^2\right)^{1/2}\right]\\
\le&\left(\sum_{j=1}^{N_k}|q_j(z,\mu_k)|^2\right)^{1/2}\sum_{i=1}^{M_k}\frac{\left(\sum_{j=1}^{N_k}|q_j(\zeta_i,\mu_k)|^2\right)^{1/2}}{M_k}\\
=&(B_k^{\mu_k}(z))^{1/2}\int (B_k^{\mu_k})^{1/2}(\zeta)d\mu_k(\zeta) = \|(B_k^{\mu_k})^{1/2}\|_{L^1_{\mu_k}} B_k^{\mu_k}(z)^{1/2}\\
\le&\|(B_k^{\mu_k})^{1/2}\|_{L^2_{\mu_k}} B_k^{\mu_k}(z)^{1/2} \le \sqrt{N_k}B_k^{\mu_k}(z)^{1/2}
\end{aligned}$$

Here we used the Cauchy Schwarz Inequality, the Holder Inequality and the fact that

$$\int B_k^{\mu_k}(\zeta)d\mu_k(\zeta) = \sum_{j=1}^{N_k}\int |q_j(\zeta,\mu_k)|^2 d\mu_k(\zeta) = N_k.$$

On the other hand, for any $p \in \mathscr{P}^k$ we have

$$|p(z)| = \left|\langle K_k^{\mu_k}(z,\zeta); p(\zeta)\rangle_{L^2_{\mu_k}}\right| = \left|\int K_k^{\mu_k}(z,\zeta)p(\zeta)d\mu_k(\zeta)\right|$$

$$\leq \|p\|_{L^\infty_{\mu_k}} \left| \int K_k^{\mu_k}(z,\zeta) d\mu_k(\zeta) \right|$$
$$\leq \|p\|_E \int |K_k^{\mu_k}(z,\zeta)| d\mu_k(\zeta),$$

hence, using the definition of Siciak function,

$$\int |K_k^{\mu_k}(z,\zeta)| d\mu_k(\zeta) \geq \sup_{p\in\mathscr{P}^k\setminus\{0\}} \frac{|p(z)|}{\|p\|_E} = (\Phi_E^{(k)})^k.$$

Here $\Phi_E^{(k)} := \sup\{|p(z)|^{\frac{1}{k}}, \deg p \leq k, \|p\|_E \leq 1\}$ is the Siciak extremal function (see [**94**]) and one has $\log \Phi_E^{(k)} \to V_E^*$ locally uniformly, see the proof of Theorem 1.2.1. Finally we have

$$\log \Phi_E^{(k)}(z) \leq \frac{1}{k} \log \int |K_k^{\mu_k}(z,\zeta)| d\mu_k(\zeta) \leq \frac{1}{2k} \log B_k^{\mu_k}(z) + \log N_k^{1/k}.$$

The proof is concluded since, due to Theorem 6.3.4, $\frac{1}{2k} \log B_k^{\mu_k}(z) \to V_E^*$ locally uniformly and $N_k^{1/2k} \to 1$ since $N_k = O(k^n)$. □

6.4. Numerical Approximation of the Transfinite Diameter and the Extremal Function

Despite the strong theoretical motivation, the algorithms provided by Theorem 6.3.4 may lead to the typical drawbacks appearing when one tries to approximate a highly non linear problem, as slow convergence and ill-conditioning. Below we present some examples to show howto cope with ill-conditioning; [**81**].

6.4.1. Computing the transfinite diameter. We consider a *real* compact set $E \subset \mathbb{R}^2$ for which we are able to compute an admissible mesh on it; we aim to calculate $\delta(E)$. We can always assume without loss of generality that $E \subseteq [-1,1]^2$, this is because translations do not affect $\delta(E)$, while $\delta(\lambda E) = \lambda\delta(E)$; note the homogeneity of the definition of the transfinite diameter equation (1.2.4).

We introduce the *Chebyshev basis*

$$\mathcal{T}_k := \{T_i(x)T_j(y), 0 \leq i + j \leq k\} =: \{t_\alpha(x,y), |\alpha| \leq k\}, \tag{6.4.1}$$

where $T_h(x) := \cos(h \arccos(x))$ is the Chebyshev polynomial of the first kind and we choose the *graded lexicographical ordering* on the muilti-index α (i.e., $(i,j) \succ (l,k)$ if $i + j > j + k$ or $i + j = l + k$ and $i > l$).

This choice is motivated by the fact that this basis has good stability properties, that is, experimentally the Vandermonde matrices computed in this basis are better conditioned than (for instance) the one computed with respect to the monomial basis $\mathcal{M}_k$, where

$$\mathcal{M}_k := \{x^i y^j, 0 \le i + j \le k\} =: \{m_\alpha(x, y), |\alpha| \le k\}.$$

We denote by $V_k = V_k(A_k, \mathcal{T}_k)$ the Vandermonde matrix of degree k with respect the mesh $A_k := \{(x_1, y_1), \ldots, (x_{M_k}, y_{M_k})\}$ and the basis $\mathcal{T}_k$, that is

$$V_k := [t_\alpha(x_h, y_h)]_{h=1,\ldots,M_k,|\alpha|\le k}\,,$$

similarly we define $W_k := V_k(A_k, \mathcal{M}_k)$ where the chosen reference basis is the monomial one.

Now we notice that, setting $M_k := \operatorname{Card} A_k$,

$$\langle m_\alpha, m_\beta\rangle_{L^2_{\mu_k}} = M_k^{-1} \sum_{h=1}^{M_k} (W_k)_{\alpha,h}(W_k)_{h,\beta},$$

thus we have

$$\det G_k(\mu_k) = \det \frac{W_k^{\mathrm{T}} W_k}{M_k}.$$

The direct application of this procedure leads to a unstable computation that actually does not converge.

On the other hand, the computation of the Gram determinant in the Chebyshev basis,

$$\det \tilde{G}_k(\mu_k) := \det \frac{V_k^{\mathrm{T}} V_k}{M_k},$$

is more stable and we have

$$\begin{aligned}(\det G_k(\mu_k))^{\frac{n+1}{2nkN_k}} &= \left(\det \frac{W_k^{\mathrm{T}} W_k}{M_k}\right)^{\frac{n+1}{2nkN_k}} \\ = \left(\det \frac{P_k^{\mathrm{T}} V_k^{\mathrm{T}} V_k P_k}{M_k}\right)^{\frac{n+1}{2nkN_k}} &= (\det(P_k))^{\frac{n+1}{nkN_k}} \det \tilde{G}_k(\mu)^{\frac{n+1}{2nkN_k}}.\end{aligned}$$

Here the matrix P_k is the matrix of the change of basis. Again the numerical computation of $\det P_k$ becomes severely ill-conditioned as k grows large.

Instead, our approach is based on noticing that P_k does not depend on the particular choice of E, thus we can compute the term $(\det(P_k))^{\frac{n+1}{nkN_k}}$ once we know $(\det G_k(\hat{\mu}_k))^{\frac{n+1}{2nkN_k}}$ and $(\det \tilde{G}_k(\hat{\mu}_k))^{\frac{n+1}{2nkN_k}}$ for a particular $\hat{\mu}_k$ which is a Bernstein Markov measure for $\hat{E} \subseteq [-1, 1]^2$ as

$$(\det(P_k))^{\frac{n+1}{nkN_k}} = \left(\frac{\det G_k(\hat{\mu}_k)}{\det \tilde{G}_k(\hat{\mu}_k)}\right)^{\frac{n+1}{2nkN_k}}.$$

Also we can introduce a further approximation, since $\det G_k(\hat{\mu}_k)^{\frac{n+1}{2nkN_k}} \to \delta(\hat{E})$, we replace in the above formula $\det G_k(\hat{\mu}_k)^{\frac{n+1}{2nkN_k}}$ by $\delta(\hat{E})$. Finally, we pick $\hat{E} := [-1, 1]^2$ and $\hat{\mu}_k$ uniform probability measure on an admissible mesh for the square, for instance the Chebyshev Lobatto grid with $(2k + 1)^2$ points, thus our approximation formula becomes

$$\delta(E) \approx \frac{1}{2}\left(\det \tilde{G}_k(\mu_k)\frac{1}{\det \tilde{G}_k(\hat{\mu}_k)}\right)^{\frac{n+1}{2nkN_k}}, \tag{6.4.2}$$

where we used $\delta([-1, 1]^2) = 1/2$; [**21**].

Finally to compute the determinants of $\tilde{G}_k(\mu_k)$ and $G_k(\hat{\mu}_k)$ we use the QR algorithm to orthogonalize the matrices W_k^T relative to μ_k and $\hat{\mu}_k$ then consider the product of the squares of the diagonal elements in the R matrices, see the code below.

```
function tdiam = transfinitediam(k,wam);
%---------------------------------------------------------------
% INPUT
%      k      considered polynomial degree
%      wam    Mx2 matrix of points of the admissible mesh for E
% OUTPUT
%      tdiam  approximation of the transfinite diameter
%---------------------------------------------------------------
% WAM of the square
j=(0:2*k); t=cos(j*pi/(2*k));[x,y]=meshgrid(t);
wamS=[x(:) y(:)];
% Chebyshev-Vandermonde matrices at the WAM points
VS=chebvand(k,wamS,[-1 1 -1 1])/sqrt(length(wamS(:,1)));
V=chebvand(k,wam,[-1 1 -1 1])/sqrt(length(wam(:,1)));
% computing the determinant of V and VS
% orthogonalization
[Q,R]=qr(V,0);
[QS,RS]=qr(VS,0);
% dimension of the polynomial space
s=(k+1)*(k+2)/2;
% approximating the transfinite diameter
d=prod(abs(diag(R)).^(3/(2*k*s)));
```

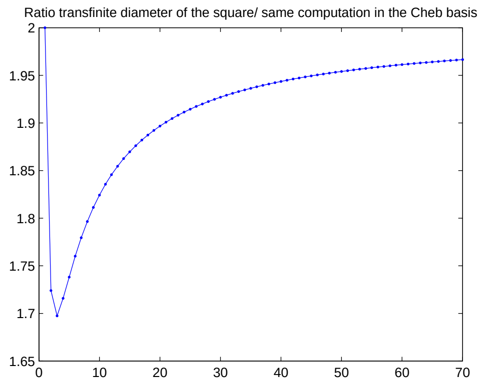

FIGURE 6.4.3. Numerical computation of the factor $\frac{1}{2}\left(\frac{1}{\det \tilde{G}_k(\hat{\mu}_k)}\right)^{\frac{n+1}{2nkN_k}}$ for (6.4.2).

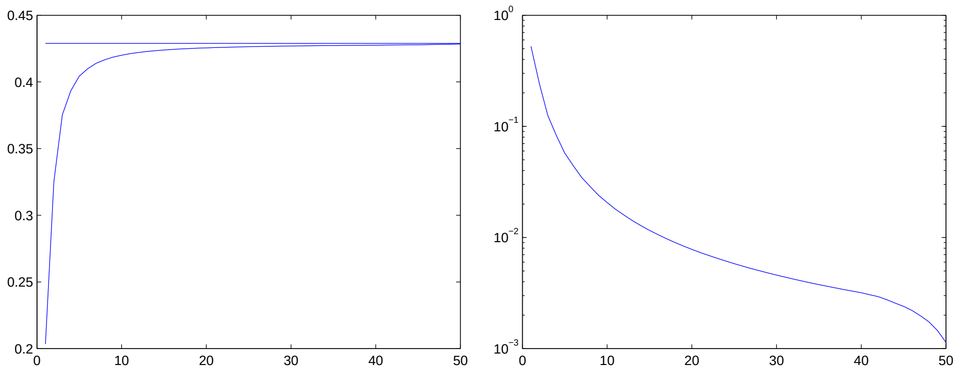

FIGURE 6.4.4. Numerical approximation of $\delta(\overline{B(0,1)})$ by formula (6.4.2). Left: result compared with the true value (straight line), right: relative error.

```
dS=prod(abs(diag(RS)).^(3/(2*k*s)));
coeff=1/(2*dS);
tdiam=d*coeff;
```

As an example, we compute by (6.4.2) the transfinite diameter of the real unit disk centred at 0 which has been shown to be equal to $(2e)^{-1/2} \approx 0.428881\ldots$; [**21**]. Figures 6.4.3 and 6.4.4 illustrate the results of the experiment which shows a good profile of convergence.

REMARK 6.4.1. *It is worth to say that we may use another algorithm in order to compute the transfinite diameter since we can relay on Theorem 6.2.3 instead of Theorem 6.3.4.*

Precisely we can compute the approximate Fekete or Leja points A_k of degree k and associate to them the uniform probability measure μ_k supported to A_k. Then we use the same procedure illustrated above.

6.4.2. Computing the extremal function. The extremal plurisubharmonic function

$$V_E^*(z) = \limsup_{\zeta \to z} \sup \{u(z), u \in \mathcal{L}(\mathbb{C}^n), u|_E \leq 1\}$$

associated to a given compact (non pluripolar) set $E \subset \mathbb{C}^n$ is explicitly known in very few instances, as for example when E is a polydisk, a ball, a real cube or the image under a polynomial mapping of one of these sets. Indeed, finding explicit formulas for more general instances seems to be a very difficult problem in analysis.

Also, the main differential properties of V_E^*, that are

- being plurisubharmonic and
- $(dd^c\, V_E^*)^n = 0$ on $\mathbb{C}^n \setminus E$,

as well as its geometric property of

- being *maximal*[3] in $\mathbb{C}^n \setminus E$,

are not that much of help when one aims to compute V_E^*.

This is because the properties of having Monge Ampere measure $(dd^c\, v)^n$ vanishing on E, logarithmic pole at infinity, and satisfying $v = 0$ quasi everywhere on E (i.e., on $E \setminus F$ for some pluripolar set F), do not fully characterize the function V_E^*. Consider for instance $\mathbb{B} := \{z \in \mathbb{C}^n : |z| \leq 1\}$ we have $V_{\mathbb{B}}^*(z) = \log^+ |z|$, but the function $\log |z|$ does satisfy the above properties as well.

Nevertheless, we are able to provide an approximation algorithm, whose implementation is based on Theorem 6.3.4 and Proposition 6.3.1, which has strong theoretical motivations and whose performances are rather good, if one takes in account the high difficulty of the problem.

We believe that, tough the convergence on the test cases is slow, having qualitative results may be interesting, since it can lead to formulate conjectures and to have more insight on the behaviour of V_E^* for rather general compact sets E.

Lastly, we stress that, to the author's knowledge, *there are no other available algorithms for the numerical approximate solution of this problem.*

[3]The plurisubharmonic function v on $\mathbb{C}^n \setminus E$ is said maximal in $\mathbb{C}^n \setminus E$ if for any open bounded $\Omega \subset \mathbb{C}^n \setminus E$ and any plurisubharmonic function u on Ω such that $\limsup_{\Omega \ni z \to \partial\Omega}(v(z) - u(z)) \geq 0$ we have $u \leq v$ in Ω.

We assume that E is a compact set where a procedure to explicitly construct an admissible, or weakly admissible mesh is available; this need, and other computational issues, suggested us to consider, so far, only real sets $E \subset \mathbb{R}^2$ as test cases.

The implementation of our algorithm, that computes the discrete extremal functions

$$
\begin{aligned}
V^{(1)}_{E,k}(x) &:= \frac{1}{2k}\log\sum_{j=1}^{N_k}|q_j(x,\mu_k)|^2\\
V^{(2)}_{E,k}(x) &:= \frac{1}{k}\log\frac{1}{M_k}\sum_{h=1}^{M_k}\left|\sum_{j=1}^{N_k}q_j(x,\mu_k)q_j(x_h,\mu_k)\right|
\end{aligned}
$$

is based on the following operations and choices.

- We choose as "stable" basis of $\mathscr{P}^k$ for our computations a suitable scaling of the basis $\mathcal{T}^k$ above, see equation (6.4.1). This is the result of the composition of Chebyshev polynomials with an affine map, mapping $[-1,1]^2$ onto the smallest closed coordinate rectangle containing E.
- We pick a bounded rectangular equispaced evaluation grid $X \subset \mathbb{R}^2$.
- We produce a starting admissible mesh $\{\tilde{A}_k\}$ for E.
- We extract a *discrete extremal set* (of Fekete or Leja type) F_k from $\tilde{A}_k$ and we set $A_k := F_{2k}$. This heuristic is motivated by the aim of controlling the oscillations of the Bergman function $B_k^{\mu_k}$. It turns out that this choice is very effective in this sense.
- We set μ_k uniform probability measure on A_k.
- We compute the orthonormal basis $\{q_j(x_i,\mu_k)\}$ for $j = 1,\dots,N_k$ and for each $x_i \in X$. This computation can't be performed by straightforward orthonormalization, we need to use first two change of basis, starting by $\mathcal{T}^k$, these change of basis are computed by the QR with pivoting algorithm. Finally, for the evaluation of the $q_j(z,\mu_k)$'s on the points in X, we compute the values of $\mathcal{T}_k$ on X using three terms recursion that provides a well defined and stable algorithm for any X. See the code below.

Also, we introduce two more possible approximations to V_E^* based on the following heuristic. We notice that for each given k we have

$$\log\Phi_E^{(k)}(z) \le V^{(1)}_{E,k}(z) \le \|V^{(1)}_{E,k}\|_E + V_E^*(z),\ \forall z\in\mathbb{C}^n.$$

Here the first inequality is part of Theorem 6.3.4, while the second follows by the fact that $V_{E,k}^{(1)}$ is plurisubharmonic and has logarithmic pole at infinity. Note that $g_k(z) := V_{E,k}^{(1)}(z) - \|V_{E,k}^{(1)}\|_E \in \mathcal{L}(\mathbb{C}^n)$ and $g_k(z) \leq 0$ on E, thus $g_k \leq V_E^*$ on $\mathbb{C}^n$ by definition.

Therefore, we can consider the "centred difference"

$$V_{E,k}^{(3)}(z) := V_{E,k}^{(1)}(z) - \frac{\|V_{E,k}^{(1)}\|_E}{2}$$

instead of $f_k^{(1)}(z)$. Also we consider the "average centred difference"

$$V_{E,k}^{(4)}(z) := V_{E,k}^{(1)}(z) - (\mathrm{Card}(X \cap E))^{-1} \frac{\sum_{x \in X \cap E} V_{E,k}^{(1)}(x)}{2}.$$

We report the matlab code for this algorithm.

```
function [siciakL,siciakB] = SEF(deg,wam,pts)
%-------------------------------------------------------------
% computes discrete versions of Siciak Extremal Function
% by a Weakly Admissible Mesh in R^2
%-------------------------------------------------------------
% INPUT
% deg: polynomial degree
% wam: 2-column array of mesh points for degree deg
% pts: 2-column array of target points
% CALL (see below)
%      chebvandr
%      cheb2poly
% OUTPUT
% siciakL,siciakB: 1-column array of values of the extremal
% functions at pts
%-------------------------------------------------------------
% FUNCTION BODY
% rectangle containing the mesh wam
rect=[min(wam(:,1)) max(wam(:,1)) min(wam(:,2)) max(wam(:,2))];
% Chebyshev-Vandermonde matrix at the mesh points
V=chebvandr(deg,wam,rect);
% 2-step orthogonalization
[Q1,R1]=qr(V,0);
[Q,R2]=qr(V/R1,0);
% discrete orthonormal polynomials computed at the target
% points
DOP=chebvandr(deg,pts,rect)/R1/R2;
% discrete Siciak estremal functions
% via the Lebesgue function of discrete LS
phiL=((sum(abs(Q*DOP'))).^(1/deg))';
siciakL=log(phiL);
% via the Bergman function
phiB=sqrt(sum(DOP.^2,2)*length(DOP(:,1))).^(1/deg);
siciakB=log(phiB);
%--------------------------------------------------------------
```

Here are the called functions

```
function V = chebvandr(deg,gmesh,rect);
% computes the bivariate Chebyshev-Vandermonde matrix (with the
% Chebyshev basis defined on the rectangle rect) at the target
% points gmesh by recurrence
%---------------------------------------------------------------
% INPUT:
% deg = polynomial degree
% gmesh = 2-column array of mesh point coordinates
% rect = 4-component vector such that the rectangle
% [rect(1),rect(2)] x [rect(3),rect(4)] contains the mesh
% OUTPUT:
% V = Chebyshev-Vandermonde matrix at gmesh
%---------------------------------------------------------------
% FUNCTION BODY
% rectangle containing the mesh
if isempty(rect)
    rect=[min(gmesh(:,1)) max(gmesh(:,1))...
               ... min(gmesh(:,2)) max(gmesh(:,2))];
end;
% couples with length less or equal to deg
% graded lexicographical order
j=(0:1:deg);
[j1,j2]=meshgrid(j);
dim=(deg+1)*(deg+2)/2;
couples=zeros(dim,2);
for s=0:deg
    good=find(j1(:)+j2(:)==s);
    couples(1+s*(s+1)/2:(s+1)*(s+2)/2,:)=[j1(good) j2(good)];
end
% mapping the mesh in the square [-1,1]^2
a=rect(1);b=rect(2);c=rect(3);d=rect(4);
map=[(2*gmesh(:,1)-b-a)/(b-a) (2*gmesh(:,2)-d-c)/(d-c)];
% Chebyshev-Vandermonde matrix on the mesh
T1=chebpoly(deg,map(:,1));
```

```
T2=chebpoly(deg,map(:,2));V=T1(:,couples(:,1)+1)...
                ... .*T2(:,couples(:,2)+1);
%------------------------------------------------------------

function T=chebpoly(deg,x)
% computes the Chebyshev-Vandermonde matrix on the real line
% by recurrence
%------------------------------------------------------------
% INPUT:
% deg = maximum polynomial degree
% x = 1-column array of abscissas
% OUTPUT
% T: Chebyshev-Vandermonde matrix at x,
%    T(i,j+1)=T_j(x_i), j=0,...,deg
%-----------------------------------------------------------
T=zeros(length(x),deg+1);
t0=ones(length(x),1);
T(:,1)=t0;
t1=x;
T(:,2)=t1;
for j=2:deg
t2=2*x.*t1-t0;
T(:,j+1)=t2;
t0=t1;
t1=t2;
%------------------------------------------------------------
```

We stress that all involved computations can be done with MatLab using the free downloadable WAM package.

6.4.2.1. *Test case 1: real regular polygons.* Convex symmetric real sets are probably the neatest example of test cases for our algorithm, since in such a case the extremal plurisubharmonic function is explicitly known.

The following result is due to Baran, see [**7**] and [**8**].

For a convex real symmetric set $E \subset \mathbb{R}^n$, $0 \in E$, one defines the *polar set* $E^* := \{y \in \mathbb{R}^n : \langle x, y\rangle \leq 1\}$. Also, we recall that the set $\text{Extr}E^*$ of *extremal points* of E^* is the set of all points in E^* that are not the mid point of any non-trivial segment lining in E^*. Then we have the following formula

$$V_E^*(z) = \sup_{w \in \mathrm{Extr} E^*} \log |h(\langle z, w\rangle)|.$$

Here $h(z) := z + \sqrt{z^2 - 1}$ is the *inverse Joukowsky map*, where the square root is to be intended as its principal branch.

If we consider a regular polygon E_l having l vertex inscribed in the real unit circle, then E^* is precisely the *dual polygon* that can be obtained by E by a π/l rotation centred at 0. Moreover $\mathrm{Extr}E^*$ is simply the set of vertex of E^*.

We look at the traces of our solution and of $V_{E_l}^*$ on X, thus the Baran's formula can be further simplified. Indeed, we have

$$\begin{aligned}
v_i &:= \left(\cos \frac{2\pi(j-1)}{l}, \sin \frac{2\pi(j-1)}{l}\right) \\
E_l &:= conv(v_1, \ldots, v_l) \\
v_i^* &:= \left(\cos \frac{2\pi(j-1/2)}{l}, \sin \frac{2\pi(j-1/2)}{l}\right) \\
E_l^* &:= conv(v_1^*, \ldots, v_l^*) \\
\mathrm{Extr}E_l^* &:= (v_1^*, \ldots, v_l^*) \\
V_{E_l}^*(x) &:= \max_{i=1,\ldots,l} \log |h(\langle x, v_i^*\rangle)|.
\end{aligned}$$

We perform some numerical tests to compare our approximate solutions to the exact one, we consider the regular pentagon, hexagon and octagon. In Figure 6.4.5 we illustrate the performance of the approximation by $V_{E_l,k}^{(2)}$ and $V_{E_l,k}^{(4)}$, $l = 5, 6, 8$ in terms of absolute error.

Finally we consider the following approximation of the L^1 norm relative error of the approximations with respect to the Lebesgue measure restricted to $D := [-5, 5]^2 \setminus E$.

$$e_k^{(h)} = \frac{\sum_{x \in \tilde{X}} |V_{E_5}(x) - V_{E,k}^{(h)}(x)|}{\sum_{x \in \tilde{X}} V_{E_5}(x)} =: \frac{\|V_{E_5} - V_{E,k}^{(h)}\|_{l^1(\tilde{X})}}{\|V_{E_5}\|_{l^1(\tilde{X})}} \approx \frac{\|V_{E_5} - V_{E,k}^{(h)}\|_{L^1(D)}}{\|V_{E_5}\|_{L^1(D)}}.$$

Here $h \in \{1, 2, 3\}$ and $\tilde{X} := X \cap D$.

We report the behaviour of $e_k^{(1)}, e_k^{(2)}$ and $e_k^{(3)}$ in Figure 6.4.6.

6.4.2.2. *Test case 2: real unit disk.* We consider also the case of E being the unit real disk $\overline{\mathbb{S}}^2 := \{x \in \mathbb{R}^2 : |x| \leq 1\} \subset \mathbb{C}^2$, in this case the formula for the

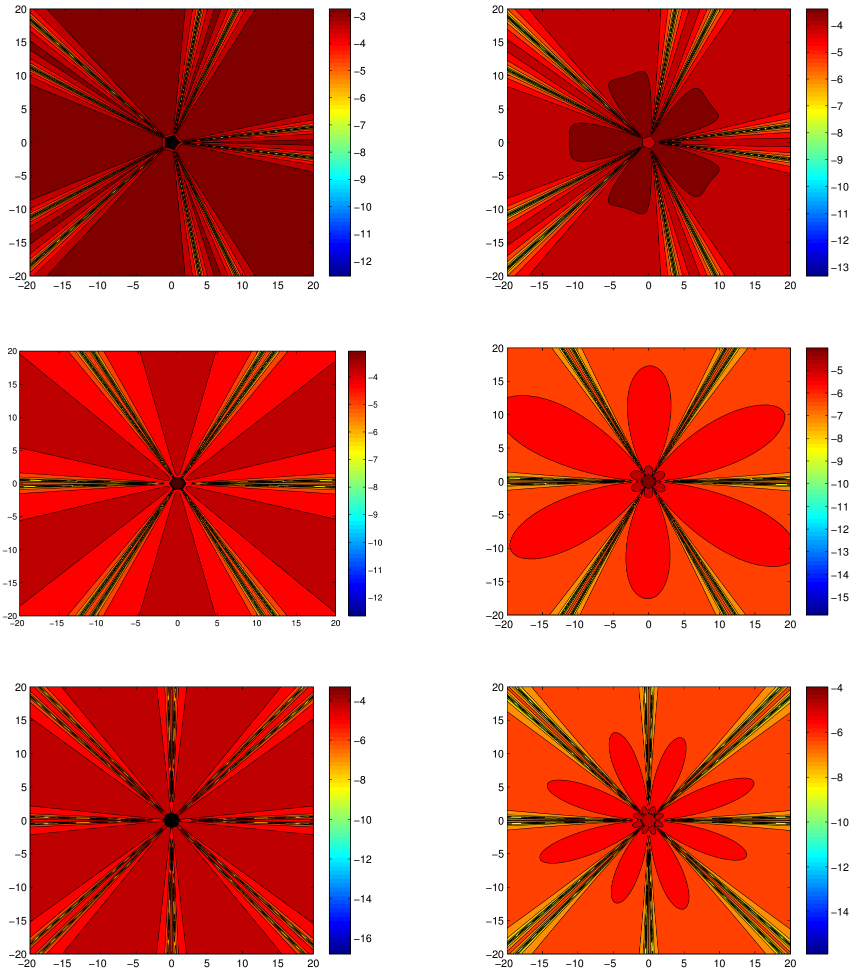

FIGURE 6.4.5. Contour plots of the absolute error of the approximation of $V^*_{E_5}$, $V^*_{E_6}$ and $V^*_{E_8}$ on a square domain $[-20, 20]^2$ with $k = 30$, by $V^{(4)}_{E_l,30}$ on the left and by $V^{(2)}_{E_l,30}$ on the right for $l = 5, 6, 8$.

extremal function is due to Lundin. Precisely we have

$$V^*_{\bar{\mathbb{S}}^2}(z) = \log |h(|z|^2 + |\langle z, \bar{z}\rangle - 1|)|, \ \ \forall z \in \mathbb{C}^2.$$

Again we test the performance of our four approximations as k increases: we report the results on the relative (approximated) L^1 error in Figure 6.4.7 and the absolute error for $k = 25$ in Figure 6.4.8.

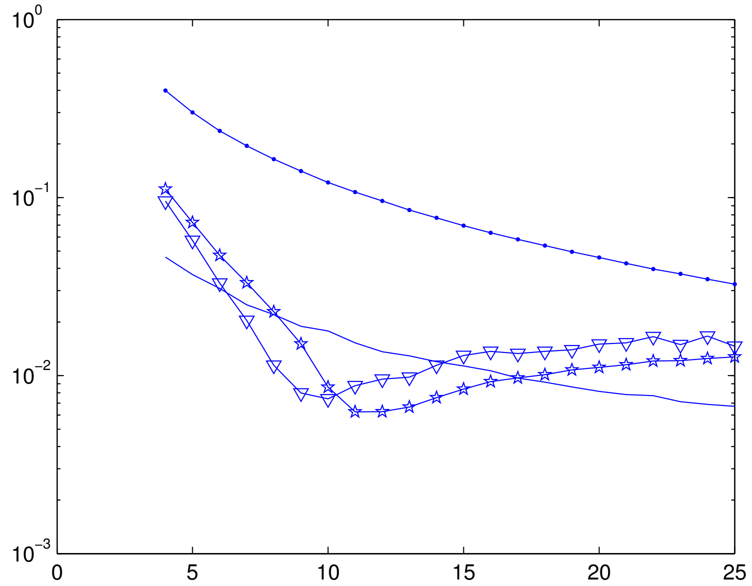

FIGURE 6.4.6. The approximation of the $L^1(D)$ relative errors of the approximation of $V_{E_5}^*$ by $V_{E,k}^{(1)}$ (dots and line), $V_{E,k}^{(2)}$ (line), $V_{E,k}^{(3)}$ (stars and line), and $V_{E,k}^{(4)}$ (triangles and line).

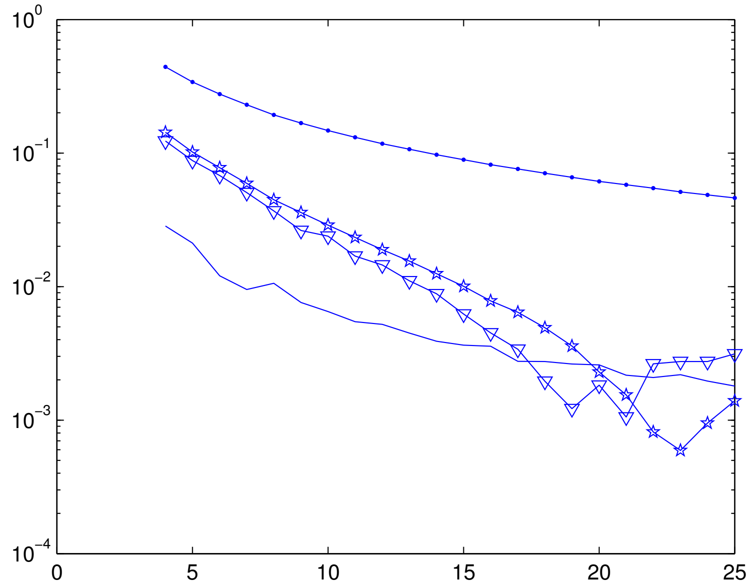

FIGURE 6.4.7. The approximation of the $L^1([-5,5]^2 \setminus \overline{\mathbb{S}}^2)$ relative errors of the approximation of $V^*_{\overline{\mathbb{S}}^2}$ by $V^{(1)}_{E,k}$ (dots and line), $V^{(2)}_{E,k}$ (line), $V^{(3)}_{E,k}$ (stars and line), and $V^{(4)}_{E,k}$ (triangles and line).

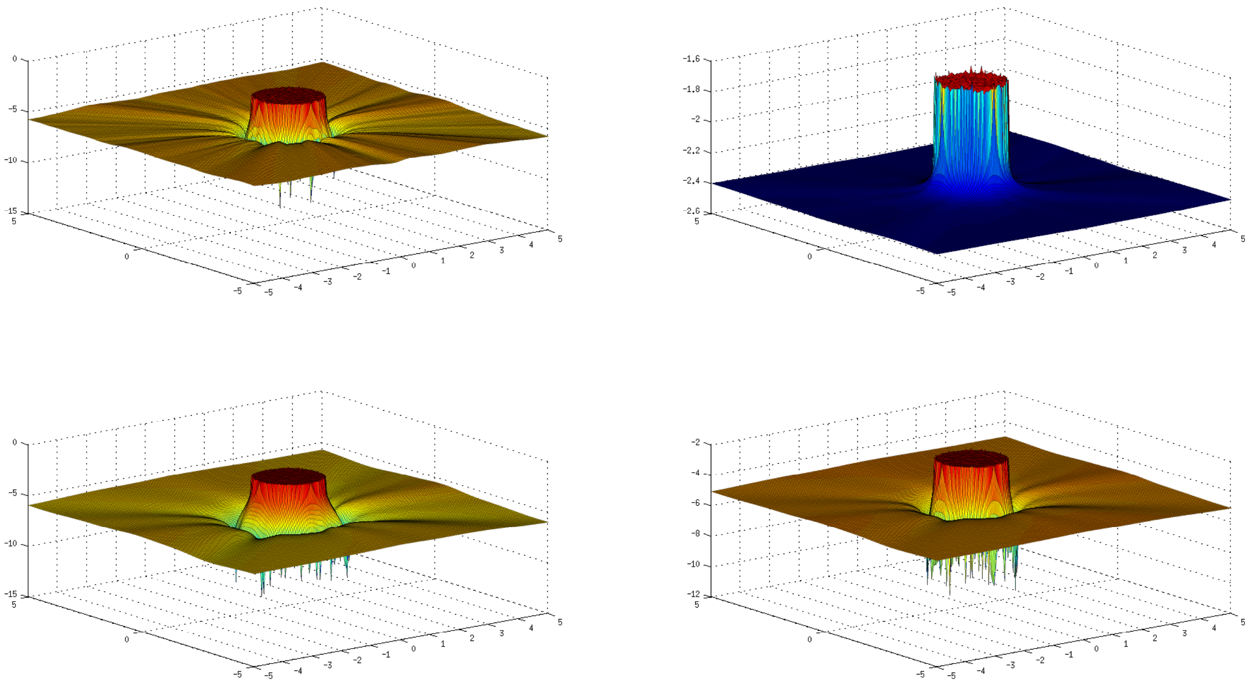

FIGURE 6.4.8. Surface plot of the absolute error in approximating $V^*_{\mathbb{S}^2}$ on $[-5,5]^2$ by by $V^{(1)}_{E,k}$ (above right), $V^{(2)}_{E,k}$ (above left), $V^{(3)}_{E,k}$ (below left), and $V^{(4)}_{E,k}$ (below right).

From these examples it seems that $V^{(2)}_{E,k}$ is the best choice in terms of lowest error in a wide range of considered ks. However, it is worth to say, that the computation of $V^{(2)}_{E,k}$ is more expensive than all the others $V^{(h)}_{E,k}$ in terms of computing time, especially if one wants to compute the approximation on a very large grid X.

CHAPTER 7

Constructing Good Admissible Meshes

If I feel unhappy, I do Mathematics to feel happy. If I feel happy I do Mathematics to keep happy.

Alfred Renyi

The aim of this chapter is to investigate some possible constructions for (weakly) admissible meshes, our exposition is based on the preprint [**74**] and the article [**79**].

We recall that it is possible to construct an admissible mesh with $O(k^{rn})$ points on any real compact set satisfying a Markov Inequality (see equation 6.1.3) with exponent r. The mesh can be obtained by intersecting the compact set with a uniform grid having $O(k^{-r})$ step size by [**37**, Thm. 5].

Indeed, the hypothesis of [**37**, Thm. 5] are not too restrictive. For instance one has a Markov Inequality with exponent 2 for any compact set $E \subset \mathbb{R}^n$ satisfying a uniform cone condition [**6**]. Thus also for the closure of any bounded Lipschitz domain. However the Markov Inequality holds with an exponent possibly greater than 2 even for more general classes of sets; see [**72**] and [**73**] for details.

The cardinality growth order of admissible meshes built by this procedure, however, causes severe computational drawbacks already for $n = 2$. This gives a strong practical motivation to construct low-cardinality admissible meshes, in particular optimal ones.

It has been proved in [**23**], see [**66**] as well, that for any compact polynomial determining set $E \subset \mathbb{C}^n$ there exists an admissible mesh with $O((k \log k)^n)$ cardinality, unfortunately the method relies on the determinations of Fekete points, which are not known in general and whose construction is an extremely hard task.

In order to build meshes with nearly optimal cardinality growth order one can restrict his attention to sets with simple geometry as simplices, squares, balls and their images under any polynomial map (see for instance [**31**]) or can look at some specific geometric-analytic classes of sets; here we follow the latter idea.

7.1. Optimal Admissible Meshes on the Closure of a Star Shaped Bounded Domain in $\mathbb{R}^n$

7.1.1. Statement of the result. In this section we build an optimal mesh for the closure E of a star shaped Lipschitz bounded domain Ω (see the lines before Proposition 7.1.1) having complement of positive reach in the sense of Federer, see Appendix E, by the following technique.

First, we consider the hypersurfaces given by the images of the boundary of the domain under a one parameter family of homotheties, being the parameters chosen as Chebyshev points scaled to a suitable interval. We prove that this family of hypersurfaces is a norming set for the given compact.

The second key element is that on each such hypersurface we can use a Markov Tangential Inequality

$$\left|\frac{\partial p(x)}{\partial v}\right| \leq M_k \|v\| \|p\|_S\,,\ \forall p \in \mathscr{P}^k, x \in S, v \in \mathcal{T}_x S$$

for certain spheres S that laying in E, note that spheres of radius r enjoy such inequality with parameter $M_k = \frac{k}{r}$, see [**34**].

THEOREM 7.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded star-shaped Lipschitz domain such that $\complement\Omega$ has positive reach (see Definition E.1.1), then $E := \overline{\Omega}$ has an optimal polynomial admissible mesh.*

This result should be compared to the recent article [**63**, Theorem 3]. Here the author works in a little more general context, still his results do not cover the case of a Lipschitz domain with complement having Positive Reach but not being $\mathscr{C}^{1,1-2/n}, n \geq 2$ globally smooth. This discrepancy is due to the fact that inward pointing corners and cusps are allowed in our setting, while they are not in [**63**].

Theorem 7.1.1 is formulated in a rather general way, here we provide two corollaries that specialize such result.

It has been shown (see [**3**]) that $\mathscr{C}^{1,1}$ domains (see Definition E.2.5) of $\mathbb{R}^n$ are characterized by the so called *uniform double sided ball condition*, that is, Ω is a $\mathscr{C}^{1,1}$ domain iff there exists $r > 0$ such that for any $x \in \partial\Omega$ there exist $v \in \mathbb{S}^{n-1}$ such that we have $B(x + rv, r) \subseteq \Omega$ and $B(x - rv, r) \subseteq \complement\overline{\Omega}$, this property in particular says that $\complement\Omega$ (and Ω itself) has positive reach, see definition E.1.1. Therefore the following is a straightforward corollary of our main result.

Corollary 7.1.1. *Let Ω be a bounded star-shaped $\mathscr{C}^{1,1}$ domain, then its closure has an optimal AM.*

It is worth recalling that such domains can also be characterized by the behaviour of the oriented distance function of the boundary (i.e. $b_\Omega(x) := d(x,\Omega) - d(x,\complement\Omega)$, where $d(x,F) := \inf_{y\in F}|x-y|$ for any set $F \subset \mathbb{R}^n$). For any such $\mathscr{C}^{1,1}$ domain there exists a (double sided) tubular neighbourhood of the boundary where the oriented distance function has the same regularity of the boundary, this condition characterizes $\mathscr{C}^{1,1}$ domains too. This framework is widely studied in [**41**] and [**40**].

In the planar case a similar result holds under slightly weaker assumptions.

Theorem 7.1.2 ([**79**]). *Let Ω be a bounded star-shaped domain in $\mathbb{R}^2$ satisfying a* Uniform Interior Ball Condition *UIBC (see Definition E.1.4), then $E := \overline{\Omega}$ has an optimal polynomial admissible mesh.*

A comparison of the statements of Theorem 7.1.1 and Theorem 7.1.2 reveals that actually in the second one we are dropping two assumptions, first the domain is no longer required to be Lipschitz, second we ask the weaker condition UIBC instead of complement of positive reach.

The first property is assumed to hold in the proof of the general case to make possible the construction of the geodesic mesh with a control on the asymptotics of the cardinality. In $\mathbb{R}^2$ the boundary of a bounded domain satisfying the UIBC is rectifiable; see [**51**]. Therefore, the geodesic mesh can be created by equally spaced (with respect to arc-length) points.

On the other hand the role of the second missing property is recovered by a deep fact in measure theory. If a set has the UIBC then then the set of points where the normal space (see Definition E.1.2) has dimension greater or equal to d has locally finite $n-d$ Hausdorff measure; [**48**], [**70**]. In our bi-dimensional (i.e., $n = 2$) case this result reads as follow: the normal space has dimension greater or equal to $d = 2$ on a subset having $0-$Hausdorff measure equal to 0, that is a finite set; [**48**]. Moreover it can be proved that, apart from this small set, the *single valued* normal space is Lipschitz.

7.1.2. Proof of Theorem 7.1.1. In order to prove Theorem 7.1.1 we need to introduce some notations and preliminary results.

In approximation theory it is customary to consider as mesh parameter the *fill distance* $h(Y)$ of a given finite set of points Y with respect to a compact subset X of $\mathbb{R}^n$.

$$h(Y) := \sup_{x\in X}\inf_{y\in Y}|x-y|. \tag{7.1.1}$$

In this definition it is not important whether the segment $[x,y]$ lies in X or not. If one wants to control the minimum length of paths joining x to y and supported in X then one may consider the following straightforward extension of the concept of fill distance given above.

Definition 7.1.1 (Geodesic Fill-Distance). *Let Y be a finite subset of the set $X \subset \mathbb{R}^n$, then we set*

$$\mathscr{A}_{x,y}(X) := \{\gamma \in \mathscr{C}([0,1],X) : \gamma(0)=x, \gamma(1)=y, \mathrm{Var}[\gamma]<\infty\}$$

and define

$$h_X(Y) := \sup_{x\in X}\inf_{y\in Y}\inf_{\gamma\in\mathscr{A}_{x,y}}\mathrm{Var}[\gamma], \tag{7.1.2}$$

the geodesic fill distance of Y over X.

Here and later on we denote by $\mathrm{Var}[\gamma]$ the total variation of the curve γ,

$$\mathrm{Var}[\gamma] := \sup_{N\in\mathbb{N}}\ \sup_{0=t_0<t_1\cdots<t_N=1}\sum_{i=1}^{N}|\gamma(t_i)-\gamma(t_{i-1})|.$$

Notice that, if we make the further assumption of the local completeness of X, then there exists a minimizer in $\mathscr{A}_{x,y}(X)$ for (7.1.2), provided $\mathscr{A}_{x,y}(X)$ is not empty. That is, if there exists a rectifiable curve ψ connecting any x and y in X such that $\mathrm{Var}[\psi] \leq L < \infty$. Thus if X has finite geodesic diameter, which will be the case of all instances considered later on, then we can replace $\inf_{\gamma\in\mathscr{A}_{x,y}}\mathrm{Var}[\gamma]$ by $\min_{\gamma\in\mathscr{A}_{x,y}}\mathrm{Var}[\gamma]$ in (7.1.2).

Now we want to build a mesh on the boundary of a bounded Lipschitz domain having a given geodesic fill distance but keeping as small as possible the cardinality of the mesh. Then we use such a “geodesic" mesh to build an optimal admissible mesh for the closure of the domain.

For the reader’s convenience we recall here that a domain $\Omega \subset \mathbb{R}^n$ is termed a (uniformly) *Lipschitz domain* if there exist $0 < L < \infty$, $r > 0$ and an open neighbourhood B of 0 in $\mathbb{R}^{n-1}$ such that for any $x \in \partial\Omega$ there exists $\varphi_x : B \to]-r, r[$

and a rotation $R_x \in SO_n$ such that $\varphi_x(0) = 0$, $\mathrm{Lip}(\varphi_x) \leq L$ and

$$R_x^{-1}\left(\Omega \cap (x + R_x(B\times]-r, r[)) - x\right) = \operatorname{epi}\varphi_x := \{(\xi, t) : \xi \in B, t \in]-R, \varphi_x(t)[\}.$$

The following result, despite its rather easy proof, is a key element in our construction. For a bounded Lipschitz domain the euclidean and geodesic (on the boundary) distances restricted to the boundary are equivalent.

Proposition 7.1.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, then there exists $\bar{h} > 0$ such that there exists $Y_h \subset X := \partial\Omega$, $0 < h < \bar{h}$ and the following hold:*

(i) $\mathrm{Card}\, Y_h = O\left(h^{1-n}\right)$ *as* $h \to 0$.

(ii) $h_X(Y_h) \leq h$.

Proof. Here we denote by $B^s_\infty(x_0, r)$ the s dimensional ball of radius r centred at x_0 with respect to the norm $|x|_\infty := \max_{i\in\{1,2,\dots,s\}} |x_i|$, i.e. the coordinate cube centred at x_0 and having sides of length $2r$.

Since Ω is a Lipschitz domain using the above notation we can write

$$(\, x + R_x B^n_\infty(0, r)\,) \cap \partial\Omega = R_x \,\mathrm{Graph}(\varphi_x).$$

Let us denote the *graph* function of φ_x by $g_x : B^{n-1}_\infty(0, r) \longrightarrow \mathbb{R}^n$, that is $B^{n-1}_\infty(0, r) \ni \boldsymbol{\xi} \mapsto \{\xi_1, \xi_2, \dots, \xi_{n-1}\varphi_x(\boldsymbol{\xi})\} = g_x(\boldsymbol{\xi})$.

By compactness we can pick $x_1, x_2, \dots, x_{M(r)} \in \partial\Omega$ such that

$$\partial\Omega \subseteq \cup_{i=1}^{M(r)} X_i =: \cup_{i=1}^{M(r)} \left(\, (x_i + R_{x_i} B^n_\infty(0, r)) \cap \partial\Omega\,\right).$$

Let $\bar{h} := r\sqrt{1 + L^2}$, take any $0 < h \leq \bar{h}$ and let us consider the grid of step-size $\frac{h}{\sqrt{(1+L^2)}}$ in the $d-1$ dimensional cube

$$Z_h := \left(\left\{-r + \frac{jh}{\sqrt{(1 + L^2)}}\right\}_{j=0,1,\dots,\lceil \frac{2r\sqrt{1+L^2}}{h}\rceil}\right)^{n-1} \subset B^{n-1}_\infty(0, r),$$

where $\lceil\cdot\rceil$ is the ceil operator. Set

$$\begin{aligned} Y_h^i &:= x_i + R_{x_i}\left(g_{x_i}(Z_h)\right), \\ Y_h &:= \cup_{i=1}^{M(r)} Y_h^i. \end{aligned}$$

Now notice that

$$\mathrm{Card}\, Y_h \leq \sum_{i=1}^{M(r)} \mathrm{Card}\, Y_h^i = M(r)\,\mathrm{Card}\, Z_h$$

$$= \quad M(r)\left(1+\left\lceil\frac{2r\sqrt{1+L^2}}{h}\right\rceil\right)^{n-1}$$

$$= \quad O(h^{1-n}).$$

In order to verify the *(ii)* for any $x \in \partial\Omega$ we explicitly find $y \in Y_h$ and build a curve γ_x connecting x to y whose variation gives an upper bound for the geodesic distance of x from Y_h.

Take any $x \in \partial\Omega$, then there exist (at least one) $i \in \{1, 2, \dots, M(r)\}$ such that $x \in X_i$. Let us pick such an i.

Let us denote by proj_i the canonical projection on the first $n-1$ coordinates acting from $R_{x_i}^{-1}\ ((x_i + R_{x_i}B_\infty^n(0,r)) \cap \partial\Omega\ -\ x_i\ )$ onto $B_\infty^{n-1}(0,r)$.

Let $x' := \mathrm{proj}_i(x)$, by the very construction we can find $y' \in Z_h$ such that $|x'-y'| \leq \frac{h}{\sqrt{1+L^2}} =: h'$, moreover the whole segment $[x', y']$ lies in $B_\infty^{n-1}(0,r)$.

We consider the curve $\alpha_x : \xi \mapsto x' + \xi\frac{y'-x'}{|y'-x'|}, \xi \in [0, h']$ and we set $\gamma_x(\xi) := x_i + g_{x_i}(\alpha(\xi))$ the curve that joins x to $y := x_i + g_{x_i}(y') \in Y_h$ obtained by mapping the segment $[x', y']$ under g_{x_i}.

Now we use *Area Formula* [**49**] [**47**][Th. 1 pg. 96] to compute the length of the Lipschitz curve γ_x.

$$\mathrm{Var}[\gamma_x] = \int_0^{h'} \mathrm{Jac}[\gamma](t)dt = \tag{7.1.3}$$

$$= \int_0^{h'}\left[\sum_{i=1}^{n-1}\left(\frac{y_i'-x_i'}{|y'-x'|}\right)^2 + \dots + \left(\nabla\varphi_x\left(x' + t\frac{y_i'-x_i'}{|y'-x'|}\right)\cdot\left(t\frac{y_i'-x_i'}{|y'-x'|}\right)\right)^2\right]^{\frac{1}{2}} nt$$

$$= \int_0^{h'}\left[\left|\frac{y'-x'}{|y'-x'|}\right|^2 + L\left|\frac{y'-x'}{|y'-x'|}\right|^2\right]^{\frac{1}{2}} nt \leq \sqrt{1+L^2}h' = h.$$

Here Jac is the Jacobian of a Lipschitz mapping, see [**47**][pg. 101].

We take the maximum over $x \in \partial\Omega$ using (7.1.2), notice that our γ_x by the construction is an element of $\mathscr{A}_{x,y}$,

$$h_{\partial\Omega}(Y_h) = \sup_{x\in X}\inf_{y\in Y_h}\inf_{\eta\in\mathscr{A}_{x,y}}\mathrm{Var}[\eta] \leq \sup_{x\in X}\mathrm{Var}[\gamma_x] \leq h.$$

□

PROOF OF THEOREM 7.1.1. We can suppose without loss of generality the center of the star to be 0 by stability of admissible meshes under euclidean isometries [**35**].

Let us set $b_k^i(r) := \frac{r}{2}(1+\cos\frac{\pi(2k-i)}{2k})$ for any $r > 0$ $i = 1, 2, \dots 2k+1$. By a well known result ([**45**]) the set $G_k(r)$ of all $b_k^i(r)$'s (varying the index i) is an admissible mesh of degree k and constant $\sqrt{2}$ for the interval $[0, r]$:

$$\|p\|_{[0,r]} \leq \sqrt{2}\|p\|_{G_k(r)} \;\; \forall p \in \mathscr{P}^k. \tag{7.1.4}$$

Let us take any $x \in X := \partial E$ and consider the set $\tilde{G}_k(x) := xG_k(1)$, notice that $\tilde{G}_k(x) \subset E$ because E is star-shaped.

One can set $Z_k := \cup_{x\in X}\tilde{G}_k(x)$, i.e., Z_k is the union of the images of X under the homotheties having parameters $\cos\frac{\pi(2k-i)}{2k}$.

Notice that the restriction of any polynomial of degree at most k in n variables to any segment is a univariate polynomial of degree at most k, then due to (7.1.4) Z_k are norming sets for E, that is

$$\|p\|_E \leq \sqrt{2}\|p\|_{Z_k} \;\; \forall p \in \mathscr{P}^k. \tag{7.1.5}$$

Therefore we are reduced to finding an admissible polynomial mesh of degree k for Z_k.

Let us consider any[1] Lipschitz curve $\gamma : [0, 1] \to X$, by Proposition E.1.1 for a.e. $s \in]0, 1[$ there exists $v \in \mathbb{S}^n$ such that

(1) $B(\gamma(s) + rv, r) \subseteq E$ and
(2) $\gamma'(s) \in \mathcal{T}_{\gamma(s)}\partial B(\gamma(s) + rv, r)$.

Hereafter $\mathcal{T}_pM$ is, as customary, the tangent space to M at $p \in M$.

Since the boundary of the ball is a compact algebraic manifold, it admits *Markov Tangential Inequality* of degree 1 (see [**34**] and the references therein), moreover the constant of such an inequality is the inverse of the radius of the ball:

$$\left|\frac{\partial p}{\partial v}(x)\right| \leq \frac{|v|}{r}k\|p\|_{B(x_0,r)} \;\; \forall p \in \mathscr{P}^k \;,\; \forall v \in \mathcal{T}_x\partial B(x_0, r). \tag{7.1.6}$$

Let us recall (see for instance [**4**][Lemma 1.1.4]) that any Lipschitz curve γ can be re-parametrized by arclength by the inversion of $t \mapsto \mathrm{Var}[\gamma|_{[0,t]}]$, obtaining a Lipschitz curve

$$\begin{aligned} \tilde{\gamma} : [0, \mathrm{Var}[\gamma]] &\to X \\ \mathrm{Var}[\tilde{\gamma}] &= \mathrm{Var}[\gamma] \\ \mathrm{Lip}[\tilde{\gamma}] &= 1 =_{\text{a.e.}} |\tilde{\gamma}'| \end{aligned}$$

[1]Notice that X is compact connected, non empty and consists of an infinite number of points, obviously it contains an infinite number of Lipschitz curves.

Therefore (using Rademacher Theorem, see for instance [**47**][Th.2 pg 81]) for a.e. $s \in]0,1[$ we have

$$\left|\frac{\partial(p\circ\tilde{\gamma})}{\partial t}(t)\right| = |\nabla p(\tilde{\gamma}(t))\cdot\tilde{\gamma}'(t)| \tag{7.1.7}$$

$$\leq \frac{|\tilde{\gamma}'(t)|k}{r}\|p\|_{B(\tilde{\gamma}(t)+rv,r)} \leq \frac{n}{r}\|p\|_E. \tag{7.1.8}$$

By Proposition 7.1.1 we can pick subsets $Y_{\frac{r}{2k}}$ on X such that $h_X\left(Y_{\frac{r}{2k}}\right) \leq \frac{r}{2k}$ and Card $Y_{\frac{r}{2k}} = O(k^{n-1})$. For notational convenience we write Y_k in place of $Y_{\frac{r}{2k}}$.

Let us now pick any $x \in X$ and consider γ, an arc connecting a closest point y_k^i of Y_k to x and x itself such that $\mathrm{Var}[\gamma] \leq \frac{r}{2k}$, parametrized in the arclength.

By the Lebesgue Fundamental Theorem of Calculus for any $p \in \mathscr{P}^k$ one has

$$\begin{aligned}
|p(x)| &\leq |p(y_k^i)| + \left|\int_0^{\mathrm{Var}[\gamma]} \frac{\partial(p\circ\gamma)}{\partial\xi}(\xi)d\xi\right| \\
&\leq |p(y_k^i)| + \int_0^{\mathrm{Var}[\gamma]} \left|\nabla p(\gamma(\xi))\cdot\gamma'(\xi)\right| d\xi \\
&\leq |p(y_k^i)| + \int_0^{r/2k} \frac{k}{r}\|p\|_E d\xi \leq |p(y_k^i)| + \frac{1}{2}\|p\|_E
\end{aligned}$$

where in the last line we used (7.1.8). Thus we have

$$\|p\|_X \leq \|p\|_{Y_k} + \frac{1}{2}\|p\|_E. \tag{7.1.9}$$

By the properties of rescaling, setting $b_k^i := b_k^i(1) = \frac{1+\cos(i\pi/k)}{2}$, we have also

$$\|p\|_{b_k^i X} \leq \|p\|_{b_k^i Y_k} + 1/2\|p\|_{b_k^i E} \leq \|p\|_{b_k^i Y_k} + \frac{1}{2}\|p\|_E,$$

for, consider the homothety $\Theta_k^i : \mathbb{R}^n \to \mathbb{R}^n$, where $\Theta_k^i(x) := \frac{x}{b_k^i}$ and write the inequality (7.1.9) for each $q_{i,k} := p\circ\Theta_k^i$.

Therefore, taking the union over $i = 0, 1, 2k$ and using $x\tilde{G}_k = \cup_{i=0}^{m_k} b_k^i x$ and $Z_k = \cup_{x\in X} x\tilde{G}_k$, we have

$$\|p\|_{Z_k} = \|p\|_{\cup_{x\in X}(\cup_i b_k^i x)} \leq \|p\|_{\cup_i b_k^i Y_k} + \frac{1}{2}\|p\|_E.$$

Hence, setting $X_k := \cup_{i=0}^{2k} b_k^i Y_k$, we can write

$$\|p\|_{Z_k} \leq \|p\|_{X_k} + \frac{1}{2}\|p\|_E.$$

Now we can use (7.1.5) to get $\|p\|_E \leq \sqrt{2}\left(\|p\|_{X_k} + \frac{1}{2}\|p\|_E\right)$ and hence

$$\|p\|_E \leq \frac{2\sqrt{2}}{2-\sqrt{2}}\|p\|_{X_k} = 2(\sqrt{2}+1)\|p\|_{X_k}.$$

Thus X_k is an admissible polynomial mesh for E. The set X_k is the disjoint union of $2k+1$ sets $b_k^i Y_k$,thus

$$\operatorname{Card} X_k = (2k+1)O(k^{n-1}) = O(k^n),$$

therefore X_k is an optimal admissible mesh of constant $2(\sqrt{2}+1)$. □

7.2. Optimal Admissible Meshes on the Closure of a $\mathscr{C}^{1,1}$ Bounded Domain in $\mathbb{R}^n$

7.2.1. Statement of the result. In [**62**] the author conjectures that any real compact set admits an optimal admissible mesh, in this section we prove that this holds at least for any real compact set E which is the the closure of a bounded $\mathscr{C}^{1,1}$ domain Ω, see Definition E.2.5. Precisely we have the following.

Theorem 7.2.3. *Let Ω be a bounded $\mathscr{C}^{1,1}$ domain in $\mathbb{R}^n$, then there exists an optimal admissible mesh for $K := \overline{\Omega}$.*

We sketch of the overall geometric construction and introduce some notations here, the proof is postponed to Subsection 7.2.3, after achieving some technical preliminary results in Subsection 7.2.2.

We denote by $d_{\complement\Omega}(\cdot)$ the distance function w.r.t. the complement $\complement\Omega$ of Ω, i.e.

$$d_{\complement\Omega}(x) := \inf_{y\in\complement\Omega} |y-x|, \tag{7.2.1}$$

and by $\operatorname{proj}_{\complement\Omega}(\cdot)$ the *metric projection* onto $\complement\Omega$ i.e., $\operatorname{proj}_{\complement\Omega}(x)$ is the set of all minimizer of (7.2.1). We continue to use the same notation as in the previous section for the closure and the boundary of Ω, namely $X := \partial\Omega$ and $E := \overline{\Omega}$.

First for a given $\mathscr{C}^{1,1}$ domain Ω we take $0 < \delta < 2r_\Omega$, where r_Ω is the maximum radius of the ball of the uniform interior ball condition satisfied by Ω.

We can split $E := \overline{\Omega}$ as follows

$$\begin{aligned} \overline{\Omega} &= E_\delta \cup \overline{\Omega^\delta} \quad \text{where} \\ E_\delta &:= \{x \in \Omega : d_{\complement\Omega}(x) \le \delta\} \text{ and} \\ \Omega^\delta &= \Omega \setminus E_\delta. \end{aligned}$$

To construct an admissible mesh of degree k on $\overline{\Omega}$ we work separately on E_δ and $\overline{\Omega^\delta}$ to obtain inequalities of the type

$$\|p\|_{E_\delta} \le \|p\|_{Z_{k,\delta}} + \frac{1}{\lambda}\|p\|_E \,, \lambda > 1 \text{ and}$$

$$\|p\|_{\Omega^\delta} \quad \leq \quad 2\|p\|_{Y_{k,\delta}} + \frac{2}{\theta}\|p\|_E \ , \mu > 1,$$

for $p \in \mathscr{P}^k$, where $Z_{k,\delta} \subset E_\delta$ and $Y_{k,\delta} \subset \Omega^\delta$ are suitably chosen finite sets.

In the case of E_δ this is achieved by the trivial observation $x \in E_\delta$ implies $\overline{B(x,\delta)} \subseteq \overline{\Omega}$ and therefore one can bound any directional derivative of a given polynomial using the univariate Bernstein Inequality (see Theorem 7.2.4 below). The resulting inequality is a variant of a Markov Inequality with exponent 1 which is convenient and allow us to build a low cardinality mesh by a modification of the reasoning in [**37**].

The construction of an admissible mesh on $\overline{\Omega^\delta}$ is more complicated. The resulting mesh is given by points lining on some properly chosen level surfaces of $d_{\complement\Omega}$. The result is proved using the regularity property of the function $d_{\complement\Omega}$ in a small tubular neighbourhood of X and the Markov Tangential Inequality for the sphere.

7.2.2. Bernstein-like Inequalities and polynomial estimates via the distance function. For the reader's convenience we recall here the Bernstein Inequality.

THEOREM 7.2.4 (Bernstein Inequality). *Let $p \in \mathscr{P}^k$, then for any $a < b \in \mathbb{R}$ we have*

$$|p'(x)| \leq \frac{k}{\sqrt{(x-a)(b-x)}}\|p\|_{[a,b]}, \ x \in]a,b[. \tag{7.2.2}$$

Let us introduce the following notation

$$l(x) \quad := \quad \min_{y \in \mathrm{proj}_{\complement\Omega}(x)} \inf\left\{\lambda > 0 : y + \lambda\frac{x-y}{|x-y|} \notin \Omega\right\} \quad x \in \Omega \tag{7.2.3}$$

$$l_\Omega \quad := \quad \inf_{x \in \Omega} l(x). \tag{7.2.4}$$

REMARK 7.2.1. *In the case when Ω is a $\mathscr{C}^{1,1}$ domain one has the estimate $l_\Omega \geq 2r$ where $r <$ Reach$(\partial\Omega)$ see Definition E.1.1 and thereafter.*

The following consequence of *Bernstein Inequality* will play a central role in our construction.

PROPOSITION 7.2.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ and let us introduce the sequence of functions*

$$\varphi_k(x) := \begin{cases} \frac{k}{\sqrt{d_{\complement\Omega}(x)(l_\Omega - d_{\complement\Omega}(x))}}, & \text{if } d_{\complement\Omega}(x) < l_\Omega \\ \frac{k}{d_{\complement\Omega}(x)} \ , & \text{otherwise} \end{cases}. \tag{7.2.5}$$

For any $x \in \Omega$ let $v \in \{\frac{x-y}{|x-y|} : y \in \operatorname{proj}_{\complement\Omega}(x)\}$, then for any $p \in \mathscr{P}^k$ we have

$$|\partial_v p(x)| \leq \varphi_k(x)\|p\|_E. \tag{7.2.6}$$

If moreover we have $l_\Omega > 0$, let us pick any $0 < \delta < l_\Omega$ and define the sequence of functions

$$\varphi_{k,\delta}(x) := \begin{cases} \frac{k}{\sqrt{d_{\complement\Omega}(x)(\delta - d_{\complement\Omega}(x))}}, & \text{if } d_{\complement\Omega}(x) < \delta \\ \frac{k}{d_{\complement\Omega}(x)}, & \text{otherwise} \end{cases}. \tag{7.2.7}$$

Then the above polynomial estimate (7.2.6) *still holds when φ_k is replaced by $\varphi_{k,\delta}$.*

Proof. Pick $p \in \mathscr{P}^k$. Let us take $x \in \Omega$ such that $d_{\complement\Omega}(x) < l_\Omega$. We denoted by $S_v(x)$ the segment $x + [-d_{\complement\Omega}(x), l_\Omega - d_{\complement\Omega}(x)]v$, where v is as above and $x \in S_v(x)$ due to $d_{\complement\Omega}(x) < l_\Omega$. The restriction of p to this segment is an univariate polynomial $q(\xi) := p(x + v\xi)$ of degree not exceeding k, then we can use the Bernstein Inequality 7.2.4 to get

$$\left|\frac{\partial q}{\partial \xi}(\xi)\right| \leq \frac{k}{\sqrt{(\xi + d_{\complement\Omega}(x))(l_\Omega - d_{\complement\Omega}(x) - \xi)}}\|p\|_{S_v(x)},$$

evaluating at $\xi = 0$ we get

$$|\partial_v p(x)| \leq \frac{k\|p\|_{S_v(x)}}{\sqrt{d_{\complement\Omega}(x)(l_\Omega - d_{\complement\Omega}(x))}} \leq \frac{k\|p\|_K}{\sqrt{d_{\complement\Omega}(x)(l_\Omega - d_{\complement\Omega}(x))}}, \tag{7.2.8}$$

thus establishing the first case of (7.2.7).

Let x be such that $d_{\complement\Omega}(x) \geq l_\Omega$. Notice that $B(x, d_{\complement\Omega}(x)) \subseteq \Omega$ and hence $\forall \eta \in \mathbb{S}^{n-1}$ (the standard unit $n-1$ dimensional sphere) we can pick a segment in the direction of η having length $d_{\complement\Omega}(x)$ lying in E and having x as midpoint. The Bernstein Inequality gives

$$|\partial_v p(x)| \leq \max_{\eta \in \mathbb{S}^{n-1}} |\partial_\eta p(x)| \leq \frac{k}{d_{\complement\Omega}(x)}\|p\|_{B(x, d_{\complement\Omega}(x))} \leq \frac{k}{d_{\complement\Omega}(x)}\|p\|_E. \tag{7.2.9}$$

The last statement follows directly by the special choice of $\delta < l_\Omega$. The right hand side in (7.2.7) dominates (case by case) the r.h.s. in (7.2.5) when cases are chosen accordingly to (7.2.7). □

Actually the above proof proves also the following corollary, it suffices to take (7.2.7) and substitute $\frac{k}{d_{\complement\Omega}(x)}$ by $\frac{k}{\delta}$ in the second case.

Corollary 7.2.2. *Let Ω be an open bounded domain and δ a positive number such that $E_\delta := \{x \in \Omega : d_{\complement\Omega}(x) \geq \delta\} \neq \emptyset$. Then for any $v \in \mathbb{S}^{n-1}$ we have $\forall p \in \mathscr{P}^k$*

$$\|\partial_v p\|_{E_\delta} \leq \frac{k}{\delta}\|p\|_E. \tag{7.2.10}$$

We introduce the following in the spirit of [?]. Let us denote by $ds(\cdot)$ the standard length measure in $\mathbb{R}^n$.

Proposition 7.2.3. *Let Ω be a bounded domain in $\mathbb{R}^n$ such that $l_\Omega > 0$ and let $0 < \delta \leq l_\Omega$. Then*

(i) for any $x \in \Omega$ the map

$$\operatorname*{proj}_{\complement\Omega}(x) \ni y \mapsto \int_{[y,x]} \varphi_{k,\delta}(\xi)ds(\xi)$$

is constant, let $F_{k,\delta}(x)$ be its value.

(ii) We have

$$F_{k,\delta}(x) = \begin{cases} k \arccos(1 - \frac{2d_{\complement\Omega}(x)}{\delta}), & \text{if } d_{\complement\Omega}(x) < \delta \\ k\left(\pi + \ln \frac{d_{\complement\Omega}(x)}{\delta}\right) \quad , & \text{otherwise}. \end{cases} \tag{7.2.11}$$

In particular $F_{k,\delta}$ extends continuously to $\overline{\Omega}$.

(iii) $F_{k,\delta}$ is constant on any level set of $d_{\complement\Omega}(\cdot)$ and $\sup_{\Omega\setminus E_\delta} F_{k,\delta} = k\pi$.

Let us set $a^i_{k\delta} := \frac{ik\pi}{m_k}$ where $i = 0, 1, \dots m_k$ and m_k is any positive integer greater than $2k\pi$, we denote by $\Gamma^i_{k,\delta}$ the $a^i_{k,\delta}$-level set of $F_{k,\delta}$.

(iv) We have

$$\begin{aligned} \Gamma^i_{k,\delta} &= \{x \in E : d_{\complement\Omega}(x) = d^i_{k,\delta}\} \ , \text{ where} \\ d^i_{k,\delta} &:= \frac{\delta}{2}\left(1 - \cos\left(\frac{i\pi}{m_k}\right)\right). \end{aligned}$$

(v) Let $\Gamma_{k,\delta} := \cup_{i=0}^{m_k}\Gamma^i_{k,\delta}$, then for any $p \in \mathscr{P}^k$ we have

$$\|p\|_E \leq \max\{2\|p\|_{\Gamma_{k,\delta}}, \|p\|_{E_\delta}\}. \tag{7.2.12}$$

Proof. (i) The function $\varphi_{k,\delta}(\cdot)$ depends on its argument only by the distance function, $\varphi_{k,\delta}(x) =: g_{k,\delta}(d_{\complement\Omega}(x))$. The length of the segment $[y, x]$ is clearly constant when y varies in the set $\operatorname{proj}_{\complement\Omega}(x)$.

Moreover for any $y, z \in \operatorname{proj}_{\complement\Omega}(x)$ let us denote by $R_{y,z}$ an euclidean isometry that maps $[y, x]$ onto $[z, x]$, one trivially has $d_{\complement\Omega}(\xi) = d_{\complement\Omega}(R_{y,z}\xi)$ for any $\xi \in [y, x]$.

This is because $\mathrm{proj}_{\complement\Omega}(\xi) \ni y$ for any $\xi \in [x, y]$ by the Triangle Inequality and thus $d_{\complement\Omega}(\xi) = |\xi - y|$.

Thus we have

$$\begin{aligned}
&\int_{[y,x]} \varphi_{k,\delta}(\xi) ds(\xi) = \int_{[y,x]} g_{k,\delta}(d_{\complement\Omega}(\xi)) ds(\xi) \\
= &\int_{[y,x]} g_{k,\delta}(d_{\complement\Omega}(R_{y,z}\xi)) ds(\xi) = \int_0^1 g_{k,\delta}\left(d_{\complement\Omega}\left(R_{y,z}\left(y + t\frac{x-y}{|x-y|}\right)\right)\right) dt \\
= &\int_0^1 g_{k,\delta}\left(d_{\complement\Omega}\left(z + t\frac{z-x}{|z-x|}\right)\right) dt = \int_{[z,x]} \varphi_{k,\delta}(\eta) ds(\eta).
\end{aligned}$$

(ii) Let us parametrize the segment as $y + s\frac{x-y}{|x-y|}$, then we have

$$F_{k,\delta}(x) = \begin{cases} \int_0^{d_{\complement\Omega}(x)} \frac{k}{\sqrt{s(\delta-s)}} ds, & \text{if } d_{\complement\Omega}(x) < \delta \\ \int_0^{\delta} \frac{k}{\sqrt{s(\delta-s)}} ds + \int_\delta^{d_{\complement\Omega}(x)} \frac{k}{s} ds \quad , & \text{otherwise.} \end{cases} \tag{7.2.13}$$

The first integral can be solved by substitution: $s = \frac{\delta}{2}(1 - \cos\theta)$. The integration domain becomes $[0, \theta_x]$ where $\frac{\delta}{2}(1 - \cos(\theta_x)) = d_{\complement\Omega}(x)$, while the integral itself becomes $\int_0^{\theta_x} d\theta = \theta_x$, thus the first case in (7.2.11) is proven.

The second integral has an immediate primitive. $F_{k,\delta}$ depends on x only by the distance function, moreover we notice that

$$\lim_{s\to\delta^-} \arccos\left(1 - \frac{2s}{\delta}\right) = \pi = \lim_{s\to\delta^+}\left(\pi + \ln\frac{s}{\delta}\right),$$

hence $F_{k,\delta}$ is a continuous function of the distance function. Since $d_{\complement\Omega}$ is well known to be 1−Lipschitz $F_{k,\delta}$ is continuous on Ω.

Since $d_{\complement\Omega}$ extends continuously to $\overline{\Omega}$, then $F_{k,\delta}$ does. Actually we must take $F_{k,\delta}|_{\partial\Omega} \equiv 0$.

(iii) We already used that $F_{k,\delta}$ depends on x only by the distance function and hence $F_{k,\delta}|_{d^{\leftarrow}_{\complement\Omega}(a)} = \text{constant}$[2], moreover the functions $\arccos\left(1 - \frac{2s}{\delta}\right)$ and $\left(\pi + \ln\frac{s}{\delta}\right)$ are both increasing in $[0, \max_{x\in\overline{\Omega}} d_{\complement\Omega}(x)]$, see Figure **??**, hence any level set of $F_{k,\delta}$ must coincide with a suitable level set of the distance function.

(iv) The conclusion follows immediately by inverting the equation

$$k \arccos\left(1 - \frac{2d^i_{k,\delta}}{\delta}\right) = a^i_{k,\delta}.$$

[2]We denote by $f^{\leftarrow}(a)$ the inverse image under $f : D \to \mathbb{R}$ of the number $a \in \mathrm{Range}[f]$, i.e., $\{x \in D : f(x) = a\}$ that, in general, is a set.

(v) Let $p \in \mathscr{P}^k$ be fixed, let us pick $x \in E$, then two possibilities can occur. In the first case $x \in E_\delta$. In this case we have $|p(x)| \leq \|p\|_{E_\delta}$. In the second we suppose $x \notin E_\delta$, let us consider $y \in \mathrm{proj}_{\complement\Omega}(x)$. The segment $[y, x]$ cuts $\Gamma^i_{k,\delta}$ for every i such that $d^i_{k,\delta} \leq d_{\complement\Omega}(x)$, moreover $[y, x] \cap \Gamma^i_{k,\delta} = \{y^i\}$, due to the monotonicity of $F_{k,\delta}$ along any segment where $d_{\complement\Omega}$ is monotone.

Let $i(x) := \max\{i : d^i_{k,\delta} \leq d_{\complement\Omega}(x)\}$ and let $y^{i(x)+1}$ be the unique intersection of $\Gamma^{i(x)+1}_{k,\delta}$ and the ray starting from x and having direction $\frac{x-y}{|x-y|}$.

Let $s(\cdot)$ be the arc length parametrization of the segment $[y^{i(x)}, y^{i(x)+1}]$ now we have

$$\begin{aligned}
|p(x)| &\leq |p(y^{i(x)})| + \int_0^{s^{-1}(x)} \left|\frac{\partial(p \circ s)}{\partial t}(t)\right| dt \\
&\leq |p(y^{i(x)})| + \int_0^{1} \left|\frac{\partial(p \circ s)}{\partial t}(t)\right| dt \\
&= |p(y^{i(x)})| + \int_0^{1} \|p\|_E \varphi_{k,\delta}(s(t)) dt \\
&= |p(y^{i(x)})| + \int_{[y^{i(x)}, y^{i(x)+1}]} \|p\|_E \varphi_{k,\delta}(\xi) ds(\xi) \\
&\leq |p(y^{i(x)})| + \frac{\|p\|_E}{m_k} \int_{[y^0, y^{m_k}]} \varphi_{k,\delta}(\xi) ds(\xi) \\
&\leq \|p\|_{\Gamma^{i(x)}_{k,\delta}} + \frac{F_{k,\delta}(y^{m_k})}{m_k} \|p\|_E \leq \|p\|_{\Gamma^{i(x)}_{k,\delta}} + \frac{1}{2}\|p\|_E,
\end{aligned}$$

where we used (7.2.6) in the third line while the special choice of $a^i_{k,\delta}$ (and thus y^i) as equally spaced points in the image of $F_{k,\delta}$ and the choice of $m_k > 2k\pi$ has been used in the last two lines.

To conclude we take the maximum of the above estimates among $x \in E$ thus letting i varying among $0, 1, \ldots, m_k - 1$ and considering both cases $x \in E_\delta$ and $x \notin E_\delta$. □

Proposition 7.2.4. *Let Ω be a bounded $\mathscr{C}^{1,1}$ domain, $0 < r < \mathrm{Reach}(\partial\Omega)$ $0 < \delta \leq r$ and let $m_k > 2k\pi$, then*

(i) *For any $i = 1, \ldots m_k$ $\Gamma^i_{k,\delta}$ is a $\mathscr{C}^{1,1}$ hypersurface.*

(ii) *For any $p \in \mathscr{P}^k(\mathbb{R}^n)$ any $x \in \Gamma^i_{k,\delta}$ and any $v \in \mathbb{S}^{n-1} \cap \mathcal{T}_x\Gamma^i_{k,\delta}$ where $i = 0, 1, \ldots, m_k$ we have*

$$|\partial_v p(x)| \leq \begin{cases} \frac{k}{\delta}\|p\|_E & i = 0 \\ \frac{2k}{\delta}\|p\|_E & i = 1, 2, \ldots, m_k \end{cases}. \tag{7.2.14}$$

Proof. (i) Notice that we have, due to E.2.1,

$$0 < \min\{\mathrm{Reach}(\Omega), \mathrm{Reach}(\complement\Omega)\} = \mathrm{Reach}(\partial\Omega).$$

If $i > 0$ due to (E.2.1) and Theorem E.2.1. We have $\forall x \in \Gamma^i_{k,\delta}$

$$\nabla d_{\complement\Omega}(x) = -\nabla b_\Omega(x) = \frac{x - \mathrm{proj}_{\partial\Omega}(x)}{|x - \mathrm{proj}_{\partial\Omega}(x)|},$$

moreover this is a Lipschitz function when restricted to $\{|b_\Omega(x)| < \delta\}$ for any $0 < \delta < \min\{\mathrm{Reach}(\Omega), \mathrm{Reach}(\complement\Omega)\}$.

Also we have $b_\Omega|_{\overline{\Omega}} \equiv -d_{\complement\Omega}$.

We notice that $\nabla d_{\complement\Omega}(x) \neq 0$, therefore any level-set of $d_{\complement\Omega}$ contained in $\Omega \setminus K_\delta$ is a $\mathscr{C}^{1,1}$ $d-1$ dimensional manifold by the Implicit Function Theorem.

(ii) If $i = 0$ Theorem E.2.1 tells that for any x in $\Gamma^i_{k,\delta}$ we have $B_x := B(x + \delta\nabla b_\Omega(x), \delta) \subseteq \Omega$, moreover $\mathcal{T}_x\Gamma^i_{k,\delta} = \mathcal{T}_x\partial B_x$. Therefore we can apply the Markov Tangential Inequality to the ball B_x : for any polynomial $p \in \mathscr{P}^k$ and any $u \in \mathcal{T}_x\Gamma^i_{k,\delta} = \mathcal{T}_x\partial B_x$ we have

$$|\partial_u p(x)| \leq \frac{k}{\delta}\|p\|_{B_x} \leq \frac{k}{\delta}\|p\|_E. \tag{7.2.15}$$

Where the last inequality follows from $\overline{B}_x \subseteq E$.

Now we focus on $i > 0$. Let us take $x \in \Gamma^i_{k,\delta}$, then $y = \mathrm{proj}_{\complement\Omega}(x) \Rightarrow \nabla b_\Omega(y) = \nabla b_\Omega(x)$ and hence we have $\mathcal{T}_x\Gamma^i_{k,\delta} = \mathcal{T}_y X, i = 0, 1, \dots, m_k$

Moreover we notice that

$$B^i_x := \begin{cases} B\left(y + \frac{d^i_{k,\delta}}{2}\nabla b_\Omega(x), \frac{d^i_{k,\delta}}{2}\right) \subset \Omega & d^i_{k,\delta} \geq \delta/2 \\ B\left(y + (d^i_{k,\delta} + \frac{2\delta - d^i_{k,\delta}}{2})\nabla b_\Omega(x), \frac{2\delta - d^i_{k,\delta}}{2}\right) \subset \Omega & d^i_{k,\delta} < \delta/2. \end{cases} \tag{7.2.16}$$

and

$$\mathcal{T}_x\Gamma^i_{k,\delta} = \mathcal{T}_x B^i_x.$$

Now we notice that the radius of B^i_x can be bounded below uniformly in i by $\delta/2$. Therefore The Markov Tangential Inequality for the ball gives us the following: $\forall p \in \mathscr{P}^k$ and $\forall v \in \mathcal{T}_x\Gamma^i_{k,\delta}, |v| = 1$ we have

$$|\partial_v p(x)| \leq \frac{k}{\delta/2}\|p\|_{B^i_x}.$$

Now due to $\mathcal{T}_x\Gamma^i_{k,\delta} = \mathcal{T}_x B^i_x$ and $B^i_x \subset \Omega$ we have $\forall p \in \mathscr{P}^k, v \in \mathcal{T}_x\Gamma^i_{k,\delta}, |v| = 1, \forall i = 0, 1, m_k$

$$|\partial_v p(x)| \leq \frac{k}{\delta/2}\|p\|_E.$$

□

7.2.3. Proof of Theorem 7.2.3. We developed all required tools to prove Theorem 7.2.3. The idea of its constructive proof is mixing the technique of Theorem 7.1.1 with an improvement of the one being used in [**37**][Th. 5]. More precisely, the hypersurfaces Z_k of Theorem 7.1.1 here are replaced by the level sets $\Gamma^i_{k,\delta}$ which, together with the set $E_\delta = \{x \in E : d_{\complement\Omega}(x) \geq \delta\}$, are shown to form a norming set for E.

Proof. Notice that we have $0 < \min\{\mathrm{Reach}(\Omega), \mathrm{Reach}(\complement\Omega)\} = \mathrm{Reach}\,\partial\Omega$ due to E.2.1 we fix $\delta \leq r < \mathrm{Reach}\,\partial\Omega$

Let us recall the above notation

$$\begin{aligned} E_\delta &:= \{x \in E : d_{\complement\Omega}(x) \geq \delta\}, \\ \Gamma_{k,\delta} &:= \cup_i \Gamma^i_{k,\delta} \text{ where} \\ \Gamma^i_{k,\delta} &:= \{x \in E : d_{\complement\Omega}(x) = d^i_{k,\delta}\}, \\ d^i_{k,\delta} &:= \frac{\delta}{2}\left(1 - \cos\left(\frac{i\pi}{m_k}\right)\right) \text{, where we can take} \\ m_k &:= \lceil 2k\pi \rceil + 1. \end{aligned}$$

Let $p \in \mathscr{P}^k(\mathbb{R}^n)$.

• **Claim 1.** *For any $\lambda > 1$ there exists $Z_{k,\delta,\lambda} \subset E_\delta$ such that*

$$\|p\|_{E_\delta} \leq \|p\|_{Z_{k,\delta,\lambda}} + \frac{1}{\lambda}\|p\|_E \text{ and} \tag{7.2.17}$$

$$\mathrm{Card}\, Z_{k,\delta,\lambda} = \mathcal{O}(k^n). \tag{7.2.18}$$

• **Proof of Claim 1.** Let us consider for any $\lambda > 1$ a mesh $Z_{k,\delta,\lambda}$ such that its fill distance

$$h(Z_{k,\delta,\lambda}) \leq \frac{\delta}{\lambda k + 1/2} =: h \text{ , see (7.1.1).}$$

Let us define $Z_{k,\delta,\lambda} \subset E_\delta$ as the intersection of E with a grid G with a step-size $\frac{h}{\sqrt{n}}$ on a suitable n-dimensional cube containing E. It follows that $\mathrm{Card}(Z_{k,\delta,\lambda}) = \left(\frac{\sqrt{n}}{h}\right)^n = \mathcal{O}(k^n)$.

Now pick any $x \in E_\delta$ and find $y \in Z_{k,\delta,\lambda}$ such that $|x-y| \leq h$ and define $v := \frac{x-y}{|x-y|}$ and notice that

$$\begin{aligned} & |p(x)| \\ \leq\ & |p(y)| + \left|\int_0^{|x-y|} \partial_v p(x + sv)ds\right| \leq \|p\|_{Z_{k,\delta,\lambda}} + |x-y|\|p\|_{[x,y]} \\ \leq\ & \|p\|_{Z_{k,\delta,\lambda}} + \|\partial_v p\|_{B(E_\delta,h/2)}. \end{aligned}$$

Where we used $\min_{\xi\in[x,y]} \operatorname{dist}(\xi, E_\delta) \geq h/2$ due to the Triangle Inequality for the euclidean distance $\operatorname{dist}(\cdot, E_\delta)$ from E_δ.

By the observation $B(E_\delta, h/2) \subseteq E_{\delta-h/2}$ we can apply inequality (7.2.10) where δ is replaced by $\delta - h/2$.

$$|p(x)| \leq |p(y)| + h\frac{k}{\delta - h/2}\|p\|_E.$$

Taking maximum over $x \in E_\delta$ and using the particular choice $h := \frac{\delta}{\lambda k+1/2}$ we are done.

• **Claim 2.** *For any $2 < \mu$ there exist finite sets $Y^i_{k,\delta} \subset \Gamma^i_{k,\delta}$, $i = 0, 1, ..m_n$, such that if we set $Y_{k,\delta} := \cup_i Y^i_{k,\delta}$ we get*

$$\|p\|_{\cup_i\Gamma^i_{k,\delta}} \leq \|p\|_{Y_{k,\delta}} + \frac{1}{\mu}\|p\|_E \text{ and} \tag{7.2.19}$$
$$\operatorname{Card} Y_{k,\delta} = O(k^n). \tag{7.2.20}$$

•**Proof of Claim 2.** Let us pick $Y^i_{k,\delta} \subset \Gamma^i_{k,\delta}$ such that

$$h_{\Gamma^i_{k,\delta}}(Y^i_{k,\delta}) \leq \begin{cases} \frac{\delta}{\mu k} & i = 0 \\ \frac{\delta}{2\mu k} & i = 1, 2, \ldots, m_k \end{cases} \quad \text{(see Definition 7.1.2).} \tag{7.2.21}$$

Now fix any $i \in \{0, 1, \ldots, m_k\}$, by (7.2.21) for any $x \in \Gamma^i_{k,\delta}$ there exist a point $y \in Y^i_{k,\delta}$ and a Lipschitz curve[3] γ lying in $\Gamma^i_{k,\delta}$, connecting x to y and such that $\operatorname{Var}[\gamma] \leq h_{\Gamma^i_{k,\delta}}(Y_{k,\delta})$. Let us denote the arclength re-parametrization of γ by $\tilde{\gamma}$, then we have

$$\begin{aligned}
|p(x)| &\leq |p(y)| + \int_0^{\operatorname{Var}[\gamma]} \frac{d(p\circ\tilde{\gamma})}{dt}(t)dt \\
&\leq \|p\|_{Y^i_{k,\delta}} + h_{\Gamma^i_{k,\delta}}(Y_{k,\delta}) \max_{\xi\in\Gamma_{k,\delta}, v\in\mathbb{S}^{n-1}\cap\mathcal{T}_\xi\Gamma^i_{k,\delta}} |\partial_v p(\xi)| \\
&\leq \|p\|_{Y^i_{k,\delta}} + \frac{1}{\mu}\|p\|_E.
\end{aligned}$$

Here, in the 3-rd line, we used the inequality (7.2.14). Let us take the maximum w.r.t. x varying in $\Gamma^i_{k,\delta}$ and i varying over $\{0, 1, \ldots, m_k\}$, we obtain $\|p\|_{\Gamma_{k,\delta}} \leq \|p\|_{Y_{k,\delta}} + \frac{1}{\mu}\|p\|_E$.

We are left to prove that we can pick $Y^i_{k,\delta}$ such that $\operatorname{Card}(Y_{k,\delta}) = O(k^n)$.

[3]Notice that $\Gamma^i_{k,\delta}$ are compact $\mathscr{C}^{1,1}$ hypersurfaces, thus in particular they are locally complete with respect the geodesic distance. Therefore there exists a curve γ realizing the infimum in the definition of geodesic fill distance.

When $i = 0$ Proposition 7.1.1 ensures (X is a $\mathscr{C}^{1,1}$ hypersurface and a fortiori is Lipshitz) the existence of such an $Y^0_{k,\delta}$ with $h_{\Gamma^0_{k,\delta}}(Y^0_{k,\delta}) \leq \frac{\delta}{\mu k}$ and $\operatorname{Card}(Y^0_{k,\delta}) = O(k^{n-1})$. Let us study the case $i > 0$.

Now let us notice that by *(v)* in Theorem E.2.1 one has
$\operatorname{proj}_{\partial\Omega}|_{b_\Omega=\rho}$ is an injective function for any $0 < \rho < \operatorname{Reach}(\partial\Omega)$. Since ∇b_Ω constant along metric projections we can also notice that $\nabla b_\Omega(x) = \nabla b_\Omega(\operatorname{proj}_{\partial\Omega}(x))$. Moreover by *(iii)* in Theorem E.2.1 if $x \in \Gamma^i_{n,\delta}$, $y = \operatorname{proj}_{\complement\Omega(x)}$ then

$$\begin{aligned} y = \operatorname*{proj}_{\complement\Omega}(x) &= x - |x - \operatorname*{proj}_{\complement\Omega}(x)|\nabla b_\Omega(x) \\ &= x - d^i_{k,\delta}\nabla b_\Omega(x) = x - d^i_{k,\delta}\nabla b_\Omega(\operatorname*{proj}_{\partial\Omega}(y)) \\ &= x - d^i_{k,\delta}\nabla b_\Omega(y). \end{aligned}$$

Thus we can introduce the family of inverse maps $f_i := \left(\operatorname{proj}_{\complement\Omega}|_{\Gamma^i_{k,\delta}}\right)^{-1}$

$$\begin{aligned} f_i : \Gamma^0_{k,\delta} &\longrightarrow \Gamma^i_{k,\delta} \\ x &\longmapsto x + d^i_{k,\delta}\nabla b_\Omega(x). \end{aligned}$$

Notice that $\nabla b_\Omega|_{\partial\Omega}$ is a Lipschitz function, see Theorem E.2.1 *(iii)*. Let us denote L its Lipschitz constant.

Therefore $\{f_i\}_{i=1,2,\dots,m_k}$ is a family of equi-continuous functions of Lipschitz constant

$$\max_{i=1,2,\dots,m_k}(1 + Ld^i_{k,\delta}) \leq (1 + L\delta).$$

Now the *Area Formula* says that f_i (being $1 + L\delta$ Lipschitz) maps a mesh of $\Gamma^0_{k,\delta}$ with geodesic fill distance $\frac{h}{1+\delta L}$ onto a mesh in $\Gamma^i_{k,\delta}$ having geodesic fill distance bounded by h. We already used this property and explained its application in more detail in the proof of Theorem 7.1.1, see (7.1.3) and thereafter.

Thanks to Proposition 7.1.1 we can pick the mesh $\tilde{Y}^i_{k,\delta} \subset \Gamma^0_{k,\delta}$ such that $h_{\Gamma^0_{k,\delta}}(\tilde{Y}^i_{k,\delta}) \leq \frac{\delta}{2\mu k(1+\delta L)}$ with the cardinality bound $\operatorname{Card}(\tilde{Y}^i_{k,\delta}) = O(\left(\frac{k}{h}\right)^{n-1})$ where we denote $\frac{\delta}{2\mu(1+\delta L)}$ by h. Let us set $Y^i_{k,\delta} := \{f_i(y), y \in \tilde{Y}^i_{k,\delta}\}$. Now we can notice that

$$\operatorname{Card}(Y_{k,\delta}) = \sum_{i=0}^{m_k} \operatorname{Card} Y^i_{k,\delta} = k^{n-1} + \sum_{i=1}^{m_k} O\left(\left(\frac{k}{h}\right)^{n-1}\right) = O(k^n).$$

• **Claim 3:** $A_{k,\delta} := Y_{k,\delta} \cup Z_{k,\delta,\lambda}$ *is an optimal admissible mesh for E.*

• **Proof of Claim 3.** By the special choice of $\delta < r \leq l_\Omega/2$ we can use jointly (7.2.12), (7.2.17) and (7.2.19) and we obtain

$$\|p\|_E \leq \max\{2\|p\|_{Y_{k,\delta}} + 2\frac{1}{\mu}\|p\|_E\ ,\ \|p\|_{Z_{k,\delta,\lambda}} + \frac{1}{\lambda}\|p\|_E\}.$$

By the elementary properties of max we have

$$\|p\|_E \leq \max\{\frac{2\mu}{\mu-2}\ ,\ \frac{1}{\lambda-1}\}\|p\|_{Y_{k,\delta}\cup Z_{k,\delta,\lambda}}. \tag{7.2.22}$$

Thus $Y_{k,\delta} \cup Z_{k,\delta,\lambda} =: A_{k,\delta}$ satisfies

$$\|p\|_E \leq C(\delta, \lambda, \mu)\|p\|_{A_{k,\delta}}\ \forall p \in \mathscr{P}^k(\mathbb{R}^n)\ \forall k \in \mathbb{N} \tag{7.2.23}$$

and has the correct cardinality growth order. □

Appendices

APPENDIX A

Analytic and Algebraic Subsets of $\mathbb{C}^n$

In this section we recall all the definitions and the properties concerning analytic and algebraic affine subsets of $\mathbb{C}^n$ that we use. For the proofs of the statements and an extensive treatment of the subject we refer the reader to [**38**].

Definition A.0.1 (Analytic subset). *Let $D \subseteq \mathbb{C}^n$ be a domain and $A \subset D$. If for any $a \in D$ there exists a open neighbourhood U of a in D and a set of holomorphic maps $\{f_1, f_2, \dots, f_k\}$ in U such that*

$$A \cap U = \{z \in U : f_1(z) = f_2(z) = \cdots = f_k(z) = 0\}, \tag{A.0.24}$$

then A is said to be a analytic subset *of D.*

Notice that it follows by definition that any analytic subset of D is closed in D. It is sometimes more convenient to do not require A to be close, so we introduce another terminology. We say that the set A is (locally) a *analytic set* if there exists a neighbourhood of each point $a \in A$ (not $a \in D$) such that (A.0.24) holds. In particular it follows that there exists a neighbourhood of A such that A is an analytic subset of it.

It is worth to notice that from the definition the following topological properties can be derived.

Proposition A.0.1 (Some topological properties). *Let D be a domain in $\mathbb{C}^n$.*

(1) Let $A \subset D$ be a analytic set. Suppose that A contains a open non empty subset of D, then $A \equiv D$.
(2) Let A be a proper analytic subset of D, then $D \setminus A$ is arc-wise connected.

One can, roughly speaking, think to analytic sets as complex manifolds with singularities, this would be clear after the next definitions.

Definition A.0.2 (A_{reg} and A_{sing}). *Let $A \subset D$ be an analytic set and $a \in A$. If there exists an open neighbourhood U of a in D such that $A \cap U$ is a complex* *manifold then we say that a is a* regular *point of A, we further set*

$$A_{reg} := \{z \in A : \ z \text{ is a regular point}\},$$
$$A_{sing} := A \setminus A_{reg}.$$

Two fundamental facts about A_{reg} and A_{sing} are that A_{reg} is **dense** in A while A_{sing} is **closed and nowhere dense**, moreover it turns out that A_{sing} is an **analytic set** itself.

Definition A.0.3 (Dimension). *Let A be an analytic set, for any $a \in A_{reg}$ the set A is coinciding near a with a m_a dimensional complex manifold. We set*

$$\dim_a(A) := \begin{cases} m_a & , a \text{ is a regular point} \\ \limsup_{A_{reg} \ni b \to a} m_b & , \text{otherwise} \end{cases}.$$

The dimension of A is defined as $\dim(A) := \max_{a \in A} \dim_a(A)$.

Definition A.0.4 (Pure dimensional analytic sets). *An analytic set A is of* pure dimension *m if* $\dim_a(A) \equiv m$.

In the sequel we will deal only with *irreducible* analytic sets.

Definition A.0.5 (Irreducibility).

i) Let A be an analytic subset *of the domain $D \subset \mathbb{C}^n$, A is said to be* irreducible *if it can not be represented as $A = A_1 \cup A_2$ where A_1, A_2 are non empty analytic subsets of D. An* analytic set *$A \subset D$ (here D is any domain) is said to be* irreducible *if it is irreducible as analytic subset of a domain D' (in which A is necessarily closed).*

ii) We say that a irreducible analytic subset $A_1 \subset A$ of D is a irreducible component *of the analytic set A if for any analytic set A_2 such that $A_1 \subset A_2 \subset A$ we have that A_2 is reducible (i.e., it is not irreducible).*

We collect the most important facts about the irreducible components of an analytic set in the following theorem.

Theorem A.0.1 (Splitting in irreducible components).

i) If S is a connected component of A_{reg} then $\mathrm{Clos}_A(S)$ *is a irreducible component of A.*

ii) Any irreducible component of A has the form $\mathrm{Clos}_A(S)$ *for a connected component S of A_{reg}. In particular, any irreducible analytic set has pure dimension.*

iii) If $A_{reg} = \cup_{j\in J} S_j$ is the splitting in connected components, then $A = \cup_{j\in J} \mathrm{Clos}_A(S_j)$ is the splitting of A in irreducible components.

iv) The above splitting is at most countable and it is locally finite.

The use of the existence of proper projections will be one of the main tools in the next sections.

Theorem A.0.2 (Proper projections). *Let A be an analytic set in $\mathbb{C}^n$ and $a \in A$. Then there exist a neighbourhood U of a in $\mathbb{C}^n$, a choice of of orthonormal coordinates in $\mathbb{C}^n$, an analytic subset Y of $V := \pi(U)$, where π is the projection on the first m coordinates, and a natural number $k \leq m$ such that*

(1) π is a proper map and the restriction $\pi : A \cap U \setminus \pi^{-1}(Y) \to V \setminus Y$ is a locally bi-holomorphic k-sheeted covering (i.e., a holomorphic map having k holomorphic inverses).

(2) $\pi^{-1}(Y)$ is nowhere dense in $A \cap U$ and contains $A_{sing} \cap U$.

We will concern only about the smaller class of algebraic sets.

Definition A.0.6 (Algebraic set). *Let $A \subset D$ be a analytic subset* (resp. analytic set) *in the domain D, it is an algebraic subset of D* (resp. algebraic set in D) *if all the defining functions in equation* (A.0.24) *are polynomials.*

This fundamental characterization of algebraic sets is due to Rudin.

Proposition A.0.2 (Rudin coordinates). *Let A be a pure m dimensional analytic set in $\mathbb{C}^n$, then A is algebraic if and only if there exist a unitary change of coordinates of $\mathbb{C}^n$ and a constants $0 < C, s < \infty$ such that in this coordinates $A \subset \{z = (z', z'') \in \mathbb{C}^m \times \mathbb{C}^{n-m} : |z''| \leq C(1 + |z'|)^s$. We will refer to such choice of coordinates as a* system of Rudin coordinates *for A.*

The main tool in defining the Monge-Ampere operator on algebraic sets will be the following: for algebraic sets one has a much more sharp version of Theorem A.0.2, obtained by embedding $\overline{A}$ in $\mathbb{P}^n$ and using the Chow Theorem and the Wirtinger Theorem, see for instance [**38**].

Theorem A.0.3 (Proper projections for algebraic sets). *Let A be a pure m-dimensional analytic set in $\mathbb{C}^n$, then it is an algebraic set if and only if there exists a linear automorphism L of $\mathbb{C}^n$ such that all the projections of $L(A)$ on each m-dimensional coordinates plane are proper maps.*

In particular, for each set of distinct indexes $I := (I_1, \dots, I_m)$ *in* $\{1, \dots, n\}$ *there exists a algebraic set* $Y_I \subset \mathbb{C}^m$ *such that the map* $\pi_I : A \setminus \pi_I^{-1}(Y_I) \to C_I \setminus Y_I$ *is a locally bi-holomorphic m-sheeted covering,* $\pi_I^{-1}(Y_I)$ *is nowhere dense in A and contains* A_{sing}.

APPENDIX B

Differential Forms and Currents

We recall the most relevant definitions and facts about differential forms, differentiation and currents on complex domains and manifold, and their generalization to algebraic sets; for an extensive treatment we refer the reader to [**59**], [**43**] and [**42**].

B.1. Differential forms

Let us introduce some notations. We use the symbol $\Lambda^r(V, W)$ to indicate the space of $\mathbb{R}$-multilinear alternating mappings of the finite dimensional complex vector space V on W, where $W = \mathbb{R}$ or $\mathbb{C}$.

The standard splitting of $\Lambda^r(\mathbb{C}^n, \mathbb{C})$ is the following

$$\Lambda^r(\mathbb{C}^n, \mathbb{C}) = \bigoplus_{p+q=r,\, p,q\in\mathbb{N}} \Lambda^{p,q}(\mathbb{C}^n, \mathbb{C})$$

$$\Lambda^{p,q}(\mathbb{C}^n, \mathbb{C}) := \operatorname{span}\left\{dz^\alpha \wedge d\bar{z}^\beta,\ \alpha, \beta,\ \text{increasing}, \operatorname{Card}\alpha = p, \operatorname{Card}\beta = q\right\}$$

We endow $\mathbb{C}^n$ with the standard Käler metric $\beta_n := \sum_{j=0}^n dz^j \wedge d\bar{z}^j$ and the associated volume form $d\operatorname{Vol}_n = (\beta_n)^n$,

$$d\operatorname*{Vol}_n = \left(\frac{i}{2} dz \wedge d\bar{z}\right)^n = \left(\frac{i}{2}\right)^n dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n.$$

Positivity later will play a central role. We say that a $\omega \in \Lambda^{n,n}(\mathbb{C}^n, \mathbb{C})$ form is *positive* if $\omega = \lambda d\operatorname{Vol}_n$ for a non-negative (real) constant λ. We say that $\omega \in \Lambda^{p,p}(\mathbb{C}^n, \mathbb{C})$ is *elementary strongly positive* if there exist linearly independent $\mathbb{C}$ linear mappings $\eta_1, \ldots, \eta_p$ such that $\omega = \left(\frac{i}{2}\eta_1 \wedge \bar{\eta}_1\right) \wedge \cdots \wedge \left(\frac{i}{2}\eta_p \wedge \bar{\eta}_p\right)$. We say that ω is *strongly positive* if it is in the convex positive cone $SP^{p,p}$ generated by the elementary strongly positive forms. Finally we say that $\omega \in \Lambda^{p,p}(\mathbb{C}^n, \mathbb{C})$ is *positive* if for any $\theta \in SP^{n-p,n-p}$ the form $\omega \wedge \theta$ is positive; we can even check this property for θ just elementary strongly positive since $SP^{n-p,n-p}$ has a basis of elementary strongly positive forms. We recall also that the *standard Kähler form* $\beta_n := \sum_{j=1}^n \frac{i}{2} dz^j \wedge d\bar{z}^j$ belongs to the interior of $SP^{1,1}$.

Let $\Omega \subset \mathbb{C}^n$ be a domain, a function of class $\mathscr{C}^k(\Omega, \Lambda^{p,q}(\mathbb{C}^n, \mathbb{C}))$ is termed a *differential form* of type (p,q) and class $\mathscr{C}^k$ We introduce the following complex differential operators on $\mathscr{C}^1(\Omega, \mathbb{C}) = \mathscr{C}^1(\Omega, \Lambda^0(\mathbb{C}^n, \mathbb{C}))$

$$\partial = \sum_{j=1}^{n} \frac{\partial}{\partial z^j} dz^j \ , \qquad \bar{\partial} = \sum_{j=1}^{n} \frac{\partial}{\partial \bar{z}^j} d\bar{z}^j$$

$$d := \partial + \bar{\partial} \ , \qquad d^c := i(-\partial + \bar{\partial}).$$

Then we extend them to differential forms $\omega = \sum_{\alpha,\beta} \omega_{\alpha,\beta} \wedge dz^\alpha \wedge d\bar{z}^\beta \in \mathscr{C}^1(\Omega, \Lambda^{p,q}(\mathbb{C}^n, \mathbb{C}))$ by setting for example

$$\partial\omega = \sum_{\alpha,\beta} \partial\omega_{\alpha,\beta} \wedge dz^\alpha \wedge d\bar{z}^\beta.$$

Let us notice that by definition we have $\mathrm{dd^c} = 2i\partial\bar{\partial}$.

For $u \in \mathscr{C}^2(\Omega)$ the *complex Monge Ampere operator* is defined as

$$(\mathrm{dd^c}\, u)^n = \mathrm{dd^c}\, u \wedge \cdots \wedge \mathrm{dd^c}\, u = 4^n n! \det \left[\frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} \right]_{1 \le i,j \le n} d \operatorname*{Vol}_n .$$

B.2. Currents

We use the notation $\mathcal{D}^{p,q}(\Omega)$ for the space $\mathscr{C}_c^\infty(\Omega, \Lambda^{p,q}(\mathbb{C}^n, \mathbb{C}))$ of (p, q) differential forms with smooth coefficients and compact support in Ω and $\mathcal{D}_0^{p,q}(\Omega)$ for he space of (p, q) differential forms with continuous coefficients and compact support in Ω.

This spaces are endowed with the strict inductive limit topology generated by the topology of local uniform convergence on an increasing sequence of subdomains $\Omega_j \subset \Omega$ such that $\cup_j \Omega_j = \Omega$. Roughly speaking, this means that $\mathcal{D}^{p,q}(\Omega) \ni \psi_j \to \psi$ if and only if there exists a compact set $K \subset \Omega$ such that $\operatorname{supp} \psi_j \subset\subset K$ for all j and $\psi_j \to \psi$ together with all partial derivatives of the coefficients, uniformly on K. The statement for the convergence in $\mathcal{D}_0^{p,q}$ can be formulated in an analogous way.

The topological dual spaces $(\mathcal{D}^{p,q}(\Omega))'$ and $(\mathcal{D}_0^{p,q}(\Omega))'$ are termed the space of $(n - p, n - q)$ *currents* on Ω and the space of $(n - p, n - q)$ *currents of order 0* on Ω, we endow these spaces with the weak* topologies induced by $\mathcal{D}^{p,q}(\Omega)$ and $\mathcal{D}_0^{p,q}(\Omega)$ respectively.

It is worth to notice that for any locally integrable (p, q) form ψ one can associate a current of order 0 defined by $T_\psi(\varphi) := \int \psi \wedge \varphi \ \forall \varphi \in \mathcal{D}_0^{n-p,n-q}(\Omega)$. On the other hand any current can be represented by *a differential form with distributional* *coefficients*. For any multi-indexes $\alpha = (\alpha_1, \ldots, \alpha_p)$, $\beta = (\beta_1, \ldots, \beta_q)$ we denote by α^c and β^c the increasing complements of α and β, also we choose $a_{\alpha,\beta}$ such that $a_{\alpha,\beta} dz^\alpha \wedge d\bar{z}^\beta \wedge dz^{\alpha^c} \wedge d\bar{z}^{\beta^c} = d\,\mathrm{Vol}_n$. Then we define the distributions

$$T_{\alpha,\beta}(\varphi) := a_{\alpha,\beta} T(\varphi dz^{\alpha^c} \wedge d\bar{z}^{\beta^c})$$

where φ is any function in $\mathscr{C}_c^\infty(\Omega)$ if $T \in (\mathcal{D}^{p,q}(\Omega))'$ or $\mathscr{C}_c(\Omega)$ if $T \in (\mathcal{D}_0^{p,q}(\Omega))'$. Finally we can represent

$$T = \sum\nolimits'_{\alpha,\beta} T_{\alpha,\beta} dz^\alpha \wedge d\bar{z}^\beta.$$

We say that a (p, q) current T is *positive* if for any $\omega \in \mathscr{C}_c^\infty(\Omega, SP^{n-p,n-q})$ we have $T(\omega) \geq 0$.

The fundamental fact, due to distributions theory, for the definition of the *generalized* Monge Ampere operator is the following.

- *Any positive current has complex measure coefficients hence is a current of order zero.*

We stress that it follows by this statement (in the above notation) that for any locally bounded function u and any positive current T the current

$$\langle uT, \varphi \rangle := \sum_{\alpha,\beta} T_{\alpha,\beta}(u\varphi)$$

is a well defined current of order zero.

B.3. Submanifolds

All the definitions and facts of this section can be extended easily to the case of a m-dimensional submanifold M of $\mathbb{C}^n$. One simply considers the canonical inclusion map $\mathcal{I} : M \to \mathbb{C}^n$ and observes that the pull-back of ∂ and $\bar{\partial}$ on M by $\mathcal{I}$ coincides (by definition of submanifold) with the operators ∂_M and $\bar{\partial}_M$ defined using the local coordinates for M. It is customary to state such a property simply saying that *exterior differentiation commutes with pull-back*.

The same holds true for β_n: we refer to $\mathcal{I}^*\beta_n = \sum_{j=1}^m d\zeta^j \wedge d\bar{\zeta}^j =: \beta_m$, where ζ^js are holomorphic coordinates for M as the standard Kähler for on M and to $\beta_m^m = \beta_m \wedge \cdots \wedge \beta_m = I^*\beta_n^n$ as the standard volume form on M.

Once we fixed the volume form on M we can define the set of positive (m, m) forms as in the case of (n, n) forms on $\mathbb{C}^n$. Similarly we can do with elementary strongly positive forms and strongly positive forms. Notice that these definitions can be equivalently given by considering pull-backs of forms in the ambient space, being the pull-back operator surjective.

For the space of currents (being a dual space) one needs to use the push-forward. If $T \in (D^{p,q}(M))'$ we define $\mathcal{I}_*T$ simply setting $\mathcal{I}_*T(\varphi) := T(\mathcal{I}^*\varphi)$ for any $D^{p,q}(\mathbb{C}^n)$.

The positivity notion for currents defined on a submanifold is the same as above, that is a current $T \in (\mathcal{D}^{p,p}(M))'$ is positive if for any set of smooth compactly supported functions $(\eta_1, \dots, \eta_p)$ one has $T(d\eta_1 \wedge d\bar{\eta}_1 \wedge \cdots \wedge d\eta_p \wedge d\bar{\eta}_p) \geq 0$. It follows that for any positive $T \in (\mathcal{D}^{p,p}(M))'$ the current $\mathcal{I}_*T$ is positive.

The property of positive currents in $\mathbb{C}^n$ of being representable by differential forms with measure coefficients (and thus being currents of order zero) is preserved in this geometric setting.

APPENDIX C

Plurisubharmonic Functions

We use the usual notation $\mathrm{PSH}(D)$ to denote the class of plurisubharmonic function on the domain (or complex manifold) $D \subset \mathbb{C}^n$, that is, functions which are subharmonic along each complex line (thus along each analytic disk) and upper semicontinuous in D.

Given an analytic set A in $\mathbb{C}^n$ the class of *plurisubharmonic* functions on A consists of functions $u : A \to [-\infty, +\infty[$ such that for each point a of A there exists a open neighbourhood B of a in $\mathbb{C}^n$ and a plurisubharmonic function $\tilde{u} : B \to [-\infty, +\infty[$ such that $\tilde{u}(z) = u(z)\ \forall z \in A \cap B$. Such class is usually denoted by $\mathrm{PSH}(A)$.

There exists another definition which is involving a priori weaker assumptions on the functions, that is, we require $u : A \to [-\infty, +\infty[$ to be globally upper semicontinuous and subharmonic along each analytic disc in A (i.e., $u \circ \Psi$ is subharmonic for any analytic function $\Psi : \mathbb{D} \to A$, where $\mathbb{D}$ is the unit disc in $\mathbb{C}$).

By a deep theorem [**50**] of Fornaess and Narasimhan the latter class coincides with $\mathrm{PSH}(A)$, even in more general contexts than the one we are considering, e.g. A being an algebraic set in $\mathbb{C}^n$.

C.1. Plurisubharmonic vs weakly plurisubharmonic functions

It is not difficult to notice that, since A_{reg} is a submanifold of $\mathbb{C}^n$, one can consider the class of functions that are plurisubharmonic on A_{reg} as complex manifold, notice that in particular any $u \in \mathrm{PSH}(A)$ lies in this class. It turns out that, requiring the local boundedness on A, this lead to a profitable definition.

Definition C.1.1 (Weakly plurisubharmonic functions). *Let A be an analytic set in $\mathbb{C}^n$ and $u : A \to [-\infty, +\infty[$ be a locally bounded function. If $u|_{A_{reg}} \in \mathrm{PSH}(A_{reg})$ we say that u is a* weakly plurisubharmonic *function on A. We denote such a class by $\widetilde{\mathrm{PSH}}(A)$.*

DEFINITION C.1.2 (usc regularizartion). *Let A be a analytic set in $\mathbb{C}^n$ and $u \in \widetilde{\mathrm{PSH}}(A)$ we define the* upper semicontinuous regularization *u^* of u as*

$$u^*(z) := \limsup_{A_{reg} \ni \zeta \to z} u(\zeta) \quad \forall z \in A. \tag{C.1.1}$$

In general if $u \in \mathrm{PSH}(A)$ it follows that $(u|_{A_{\mathrm{reg}}})^* \in \widetilde{\mathrm{PSH}}(A)$ while the converse does not hold in general. Due to the following result by Demailly if A is a locally irreducible analytic set the situation becomes simpler.

THEOREM C.1.1 ([**42**]). *Let A be an analytic set in $\mathbb{C}^n$ and $u \in \widetilde{\mathrm{PSH}}(A)$. If A is locally irreducible then $u^* \in \mathrm{PSH}(A) \cap L^\infty_{loc}$.*

Due to this theorem, in the next sections we will concern only on locally bounded plurisubharmonic functions, since we consider only the case of a pure m-dimensional irreducible analytic set A in $\mathbb{C}^n$.

C.2. Approximation of plurisubharmonic fiunctions

We recall a useful smoothing procedure for plurisubharmonic functions introduced by Bedford [**11**].

LEMMA C.2.1 (Smooth decreasing approximation of psh functions). *Let $u \in \mathrm{PSH}(A) \cap L^\infty_{loc}$ where A is a irreducible pure m dimensional algebraic set in $\mathbb{C}^n$, $n > m$. Then there exists $u_j \in \mathscr{C}^\infty(A)$ such that $u_j \downarrow u$ point-wise*

The functions u_j are constructed as follows. One first take a locally finite open countable covering of A by means of open balls B^α in $\mathbb{C}^n$ chosen in a way that u has a plurisubharmonic bounded extension u^α to B^α, also we pick a partition of unity $\{\chi^\alpha\}$ for A adapted to the covering such that $\chi^\alpha \in \mathscr{C}^\infty_c(B^\alpha)$.

Then we pick a family $\{\rho_\epsilon\}_{\epsilon>0}$ of radial convolution kernels in $\mathbb{C}^n$ and a sequence $\epsilon_j \downarrow 0$ and set $u_j^\alpha := u^\alpha * \rho_{\epsilon_j}$, finally we set $u_j := \sum_\alpha \chi_\alpha u_j^\alpha$.

REMARK C.2.1. *It is worth to notice that the smooth approximations u_j in general are not plurisubharmonic.*

Indeed, combining Lemma C.2.1 with the quasi continuity of plurisubharmonic functions [**13**, Sec. 3], one has a stronger result.

Note that if $u \in \mathrm{PSH}(D)$ for some open bounded subset D of an algebraic set A, in particular $u \in \mathrm{PSH}(D \cap A^{(l)}_{\mathrm{reg}})$ ($A^{(l)}_{\mathrm{reg}}$ denoting any connected component of A_{reg}) thus, by the argument of [**13**] for any $\epsilon > 0$ one can find an open set $O^{(l)}_\epsilon$ such that u

is continuous on $(D \cap A^{(l)}_{\mathrm{reg}}) \setminus O^{(l)}_{\epsilon}$ and $\mathrm{Cap}^*(O^{(l)}_{\epsilon}, A^{(l)}_{\mathrm{reg}}) < \epsilon$. It follows that $u_j - u$ is a uniformly bounded decreasing sequence of continuous functions on $(D \cap A^{(l)}_{\mathrm{reg}}) \setminus O^{(l)}_{\epsilon}$ converging point-wise to 0, by the Dini Lemma such a convergence is indeed local uniform. We state this in a proposition to be able to refer to it.

Proposition C.2.2. *Let $A \subset \mathbb{C}^n$ be a pure dimensional algebraic set $u \in \mathrm{PSH}(A)$ and $u_j \in \mathscr{C}^\infty(A)$ be as in Lemma C.2.1, then $u_j \to u$ locally quasi uniformly on A.*

APPENDIX D

Proof of a Continuity Property of the Monge Ampere Operator

THEOREM D.0.1 (Continuity under decreasing limits; [**13**],[**11**]). *Let A be a pure m-dimensional irreducible algebraic subset of the open set $\tilde{\Omega} \subset \mathbb{C}^n$, set $\Omega := \tilde{\Omega} \cap A$ and pick $k \in \{1, 2, \dots, m\}$. Let $u^0_j, u^1_j, \dots, u^k_j \in \mathrm{PSH}(\Omega) \cap L^\infty_{loc}(\Omega)$ and $u^0, u^1, \dots, u^k \in \mathrm{PSH}(\Omega) \cap L^\infty_{loc}(\Omega)$ such that $u^s_j \downarrow u^s$ for all $s = 0, 1, \dots, k$ on $\Omega \cap A_{reg}$, then*

i) $\mathrm{dd^c}\, u^1_j \wedge \cdots \wedge \mathrm{dd^c}\, u^k_j \to \mathrm{dd^c}\, u^1 \wedge \cdots \wedge \mathrm{dd^c}\, u^k$ *as currents of order* 0*, that is,* $\forall \psi \in \mathcal{D}^{m-k,m-k}_c(\tilde{\Omega})$*, denoting the restriction (pull-back by the canonical inclusion map) of* ψ *to* A_{reg} *by* ψ *itself, we have*

$$\lim_j \int_\Omega \mathrm{dd^c}\, u^1_j \wedge \cdots \wedge \mathrm{dd^c}\, u^k_j \wedge \psi = \int_\Omega \mathrm{dd^c}\, u^1 \wedge \cdots \wedge \mathrm{dd^c}\, u^k \wedge \psi. \tag{D.0.1}$$

ii) $\mathscr{L}^k(u^0_j, \dots, u^k_j)[\psi] \to \mathscr{L}^k(u^0, \dots, u^k)[\psi]$*, for any* ψ *as above.*

Recall that $\mathscr{L}^k(u^0, \dots, u^k) := u^0\, \mathrm{dd^c}\, u^1 \wedge \mathrm{dd^c}\, u^2 \wedge \cdots \wedge \mathrm{dd^c}\, u^k$.

PROOF. The proof is given generalizing the original result. We provide only the proof of *i)* since the proof of *ii)* is completely equivalent.

For any given sequence of cut-off functions $\eta_r \in \mathscr{C}^\infty_c(\Omega \cap A_{\mathrm{reg}})$ as in equation (3.1.5) we can define the following sequences of real numbers.

$$a_{j,r} := \int_\Omega \mathrm{dd^c}\, u^1_j \wedge \cdots \wedge \mathrm{dd^c}\, u^k_j \wedge \psi\eta_r,$$

$$a_r := \int_\Omega \mathrm{dd^c}\, u^1 \wedge \cdots \wedge \mathrm{dd^c}\, u^k \wedge \psi\eta_r,$$

$$a_j := \int_\Omega \mathrm{dd^c}\, u^1_j \wedge \cdots \wedge \mathrm{dd^c}\, u^k_j \wedge \psi,$$

$$a := \int_\Omega \mathrm{dd^c}\, u^1 \wedge \cdots \wedge \mathrm{dd^c}\, u^k \wedge \psi$$

It follows by [**13**] and by the definition of improper integration (defined by equation (3.1.5)) that

$$\lim_r \lim_j a_{j,r} = \lim_{r\to 0^+} a_r = a. \tag{D.0.2}$$

On the other hand $\lim_{r\to 0^+} a_{j,r} = a_j$.

If we show that $\lim_r a_{j,r} = a_j$ holds uniformly with respect to j, then it is not difficult to prove that a_j is a Cauchy sequence and that the limit is necessarily a; this will conclude the proof.

Now we pick a open relatively compact subset D of Ω such that $\operatorname{supp}\psi := S \subset D$ and, for any $\epsilon > 0$, a open neighbourhood O_ϵ of $S \cap A_{\text{sing}}$ in D such that $\mathrm{Cap}^*(O_\epsilon, D) < \epsilon$. Notice that such a O_ϵ exists due to Proposition 3.3.1.

We recall that, by the definition of η_r we can find r_ϵ such that $\eta_r|_{S\setminus O_\epsilon} \equiv 1$ for all $r < r_\epsilon$. Thus for any such r we have

$$\sup_j |a_{j,r} - a_j| = \sup_j \left|\langle \mathrm{dd^c}\, u_j^1 \wedge \cdots \wedge \mathrm{dd^c}\, u_j^k, \psi(1-\eta_r)\rangle\right|$$
$$= \sup_j \left| \int_{S\cap O_\epsilon} \mathrm{dd^c}\, u_j^1 \wedge \cdots \wedge \mathrm{dd^c}\, u_j^k \wedge (\psi(1-\eta_r))\right|$$

Let us set $S_\epsilon := (O_\epsilon \cap S) \setminus A_{\text{sing}}$.

Now we use the fact [**59**] that there exists a constant $C > 0$ such that for any continuous compactly supported on $S_\epsilon \cap A_{\text{reg}}$ $(m-k, m-k)$-form θ the forms $C\|\theta\|_{S_\epsilon}\beta_m^{m-k} \pm \theta$ are positive. It follows that for a positive (k,k) current of order zero T we have

$$\left|\int_{S_\epsilon} T \wedge \theta\right| \le C\|\theta\|_{S_\epsilon} \int T \wedge \beta_m^{m-k}.$$

Hence we get

$$\sup_j \left| \int_{S\cap O_\epsilon} \mathrm{dd^c}\, u_j^1 \wedge \cdots \wedge \mathrm{dd^c}\, u_j^k \wedge (\psi(1-\eta_r))\right|$$
$$\le C\|\psi(1-\eta_r)\|_{S_\epsilon} \int_{S_\epsilon} \mathrm{dd^c}\, u_j^1 \wedge \cdots \wedge \mathrm{dd^c}\, u_j^k \wedge \beta_m^{m-k}$$
$$\le C\|\psi(1-\eta_r)\|_{S_\epsilon} \prod_{l=1}^k \|u_j^l\|_{\overline{D}} \int_{S_\epsilon} \mathrm{dd^c}\, \frac{u_j^1}{\|u_j^1\|_{\overline{D}}} \wedge \cdots \wedge \mathrm{dd^c}\, \frac{u_j^k}{\|u_j^k\|_{\overline{D}}} \wedge \beta_m^{m-k}$$
$$\le C\|\psi(1-\eta_r)\|_{S_\epsilon} \prod_{l=1}^k \|u_j^l\|_{\overline{D}} \overset{*}{\underset{m-k}{\mathrm{Cap}}}(S_\epsilon, D)$$

$$\leq C\|\psi\|_D \max_l \|u_1^l\|_{\overline{D}}\epsilon \to 0.$$

Here we used that u_j^k are upper semicontinuous functions point-wise decreasing to the locally bounded function u_j and that $\overline{D}$ is a compact subset of Ω.

Therefore the order of the limits can be exchanged, we have

$$\begin{aligned}
&\lim_j \int_\Omega \mathrm{dd^c}\, u_j^1 \wedge \cdots \wedge \mathrm{dd^c}\, u_j^k \wedge \psi \\
=&\lim_j \lim_{r\to 0^+} a_{j,r} = \lim_{r\to 0^+} \lim_j a_{j,r} = \lim_{r\to 0^+} a_r \\
=&\int_\Omega \mathrm{dd^c}\, u^1 \wedge \cdots \wedge \mathrm{dd^c}\, u^k \wedge \psi.
\end{aligned}$$

□

Since operator $du\wedge d^c v\wedge(\mathrm{dd^c}\, w)^{m-1}$, with $u, v, w \in \mathrm{PSH}(\Omega)\cap L^\infty_{\mathrm{loc}}$, is defined by means of terms of the type $\mathrm{dd^c}\, u$ and $u\, \mathrm{dd^c}\, v$ Theorem D.0.1 implies the following.

Corollary D.0.1 (Continuity property of $d \wedge d^c \wedge (dd^c)^{m-1}$). *Let A be a pure m-dimensional irreducible algebraic subset of the open set $\tilde{\Omega} \subset \mathbb{C}^n$, set $\Omega := \tilde{\Omega}\cap A$. Let $u_j, v_j, w_j^0, w_j^1, \ldots, w_j^{m-1} \in \mathrm{PSH}(\Omega)\cap L^\infty_{loc}(\Omega)$ and $u, v, w^1, \ldots, w^{m-1} \in \mathrm{PSH}(\Omega)\cap L^\infty_{loc}(\Omega)$ such that $u_j, v_j, w_j^s \downarrow u, v, w^s$ for all $s = 0, 1, \ldots, m-1$ on $\Omega \cap A_{reg}$. Then*

$$du_j \wedge d^c v_j \wedge \mathrm{dd^c}\, w_j^1 \wedge \cdots \wedge \mathrm{dd^c}\, w_j^{m-1} \to du \wedge d^c v \wedge \mathrm{dd^c}\, w^1 \wedge \cdots \wedge \mathrm{dd^c}\, w^{m-1}$$

as currents of order 0. That is, $\forall \psi \in \mathscr{C}_c(\tilde{\Omega})$, denoting the restriction (pull-back by the inclusion map) of ψ to A_{reg} by ψ itself, we have

$$\lim_j \int_\Omega \psi du_j\wedge d^c v_j\wedge \mathrm{dd^c}\, w_j^1\wedge\cdots\wedge \mathrm{dd^c}\, w_j^{m-1} = \int_\Omega \psi du\wedge d^c v\wedge \mathrm{dd^c}\, w^1\wedge\cdots\wedge \mathrm{dd^c}\, w^{m-1}. \tag{D.0.3}$$

APPENDIX E

Some Tools from Geometric Analysis

E.1. Sets of positive reach

Here we provide very concisely some essential tools that we use in the proofs of Chapter 7. Of course we do not even try to be exhaustive, since this is far from our aim.

We deal with Federer sets of positive reach, they were introduced in the well known article [**48**].

Definition E.1.1 (Reach of a Set). [**48**] *Let $A \subset \mathbb{R}^d$ be any set, we denote by* $\mathrm{proj}_A(x) = \{y \in A : |y-x| = d_A(x)\}$ *the metric projection onto A, where we denoted by* $d_A(x) := \inf_{y\in A}|x-y|$. *Moreover let* $Unp(A) := \{x \in \mathbb{R}^d : \exists! y \in A,\ \mathrm{proj}_A(x) = \{y\}\}$. *Then we define*

$$\text{(E.1.1)} \qquad \mathrm{Reach}(A,a) := \sup_{r>0}\{r : B(a,r) \subseteq Unp(A)\} \text{ for any } a \in A,$$

$$\text{(E.1.2)} \qquad \mathrm{Reach}(A) := \inf_{a\in A} \mathrm{Reach}(A,a).$$

The set A is said to be a set of positive reach if $\mathrm{Reach}(A) > 0$.

By this definition sets of reach $r > 0$ are precisely the subsets of $\mathbb{R}^d$ for which there exists a tubular neighborhood of radius r where the metric projection is unique and moreover this tubular neighborhood is maximal.

This class of sets was introduced by Federer in the study of *Steiner Polynomial* relative to a (very smooth) set, the polynomial that computed at $r > 0$ gives the d-dimensional measure of the r tubular neighborhood of the given set. The main interest on such a class of sets is that under this assumption (in place of high degree of smoothness) one can recover the coefficients of Steiner Polynomial as Radon measures, the Curvature Measures.

Sets with positive reach may be seen as a generalization of $\mathscr{C}^{1,1}$ bounded domains, in fact the latter can be characterized as domains such that the boundary has positive reach, a more restrictive condition. Moreover if Ω is a domain having positive reach it can be shown that the subset of $\partial\Omega$ where the distance function defines uniquely a normal vector field (as for $\mathscr{C}^{1,1}$ domains) is "big" in the right measure theoretic sense.

However, from our point of view the most relevant feature of sets of positive reach is the one concerning the regularity properties of the distance function $d_A(\cdot)$. They can be found in [**48**][Section 4]. If A has positive reach then $d_A(\cdot)$ is differentiable at any point of $\mathbb{R}^d \setminus A$ having unique projection and we have $\nabla d_A(x) = \frac{x-\mathrm{proj}_A(x)}{d_A(x)}$ and this is a Lipschitz function in any set of the type $\{x : 0 < s \le d_A(x) \le r < \mathrm{Reach}(A)\}$.

In the sequel we need to use a little of tangential calculus on non-smooth structures, so we introduce the following.

Definition E.1.2 (Tangent and Normal). *Let $A \subset \mathbb{R}^d$ be any set. Let $a \in A$ then we define respectively the tangent and the normal set to A at the point a as*

$$\begin{aligned}
\mathrm{Tan}(A,a) &:= \left\{u \in \mathbb{R}^d : \forall \epsilon > 0\, \exists x \in A : \ |x-a| < \epsilon, \left|\frac{u}{|u|} - \frac{x-a}{|x-a|}\right| < \epsilon\right\}\\
\mathrm{Nor}(A,a) &:= \left\{v \in \mathbb{R}^d : \langle v, u\rangle \le 0 \ \forall u \in \mathrm{Tan}(A,a)\right\}.
\end{aligned}$$

Here the idea is to take all possible sequences $x_n \in A$ approaching a and take the limit of $\frac{x_n-a}{|x_n-a|}$. For the normal set in the above definition the $\le$ is preferred to the equality sign to allow to consider the non-smooth case and to work with more flexibility. The set $\mathrm{Nor}(A,a)$ actually is in general a cone given by the intersection of all half spaces *dual*[1] to a vector of $\mathrm{Tan}(A,a)$.

The notion of normal vector we introduced should be compared with other possible notions, the most relevant one is that of *proximal calculus.*

Definition E.1.3 (Proximal Normal). *Let $A \subset \mathbb{R}^d$ and $x \in \partial A$. The vector $v \in \mathbb{S}^{d-1}$ is said to be a proximal normal to A at x (and we write $v \in N^P_A(x)$) iff there exists $r > 0$ such that*

$$\left\langle v, \frac{y-x}{|y-x|}\right\rangle \le \frac{1}{2r}|y-x|, \ \forall y \in \partial A. \tag{E.1.3}$$

Notice that the inequality E.1.3 implies that the boundary of A lies outside of $B(x + r\frac{v}{|v|}, r)$. If we focus on the boundary of a closed set the property of having non

[1]Hereafter the word dual must be intended in the following sense [**48**], u is dual to $N \subset \mathbb{R}^d$ iff $\langle u, v\rangle \le 0$ for any $v \in N$.

empty proximal normal set to the complement at each point of the boundary, i.e.

$$N^P_{\complement\Omega}(x) \neq \emptyset \ \forall x \in \partial\Omega$$

is known as **Uniform Interior Ball Condition (UIBC)** and it is usually stated in the following (equivalent) way

Definition E.1.4. *Let $\Omega \subset \mathbb{R}^d$ be a domain, suppose that for any $x \in \partial\Omega$ there exists $y \in \Omega$ such that $B(y, r) \cap \complement\Omega = \emptyset$ and $x \in \partial B(y, r)$. Then Ω is said to admit the uniform Interior Ball Condition.*

Such a condition (and some variants) appears in the literature also as External Sphere Condition (w.r.t. the complement of the set) in the context of the study of some properties of Minimum Time function in Optimal Control [**70**], while the previous nomenclature is more frequently used in the framework of regularity theory of PDE.

It is worthwile recalling that positive reach is a strictly stronger condition when compared to UIBC. Actually if a set A has positive reach, then it satisfies the UIBC at each point a of its boundary and *in any direction* of $\mathrm{Nor}(A, a)$.

We will use several times the following easy fact.

Proposition E.1.1. *Let $A \subset \mathbb{R}^d$, $\gamma : [0, 1] \to \partial A$ a Lipschitz curve, $r > 0$ and let us suppose* $\mathrm{Reach}(A) > r$*. Then we have for a.e. $s \in]0, 1[$ there exists $v \in \mathbb{S}^{d-1}$ such that*

(i) $B_s := B(\gamma(s) + rv, r) \subseteq A^c$,

(ii) $\gamma'(s) \in \mathcal{T}_{\gamma(s)} B_s$.

Proof. Let us consider the arclength re-parametrization $\tilde{\gamma}$ of γ that is a $1-$Lipschitz curve from $[0, \mathrm{Var}[\gamma]]$ to $\mathrm{supp}\,\gamma$. Notice that $\tilde{\gamma}$, being Lipschitz, is a.e. differentiable in $]0, \mathrm{Var}[\gamma][$, Let $\Sigma_{\tilde{\gamma}}$ be the set of singular points of $\tilde{\gamma}$ and let moreover t_0 be a point in $]0, \mathrm{Var}[\gamma][\setminus\Sigma_{\tilde{\gamma}}$.

First we claim that $\tilde{\gamma}'(t_0) \in \mathrm{Tan}(A, \tilde{\gamma}(t_0))$.

By differentiability of $\tilde{\gamma}$ at t_0 we have

$$\lim_{\substack{t \to t_0 \\ t \in [0, \mathrm{Var}[\gamma]] \setminus \Sigma_{\tilde{\gamma}}}} \frac{\tilde{\gamma}(t) - \tilde{\gamma}(t_0)}{t - t_0} = \tilde{\gamma}'(t_0). \tag{E.1.4}$$

Thus, recalling that $|\tilde{\gamma}'(t)| = 1 \neq 0$ in a neighborhood of t_0, we have

$$\lim_{\substack{t \to t_0 \\ t \in [0, \mathrm{Var}[\gamma]] \setminus \Sigma_{\tilde{\gamma}}}} \frac{\tilde{\gamma}(t) - \tilde{\gamma}(t_0)}{t - t_0} \frac{|t - t_0|}{|\tilde{\gamma}(t) - \tilde{\gamma}(t_0)|} = \frac{\tilde{\gamma}'(t_0)}{|\tilde{\gamma}'(t_0)|}.$$

Therefore we have

$$\lim_{\substack{t \to t_0 \\ t \in [t_0, \mathrm{Var}[\gamma]] \setminus \Sigma_{\tilde{\gamma}}}} \left| \frac{\tilde{\gamma}'(t_0)}{|\tilde{\gamma}'(t_0)|} - \frac{\tilde{\gamma}(t) - \tilde{\gamma}(t_0)}{|\tilde{\gamma}(t) - \tilde{\gamma}(t_0)|} \right| = 0.$$

Thus for any $\epsilon > 0$ we can build the point $x \in \operatorname{supp}\gamma$ of definition E.1.2 that realizes the vector $\tilde{\gamma}'(t_0)$ as a vector of $\mathrm{Tan}(A, a)$.

Moreover for a.e. s_0 in $]0, 1[$ the arc length $t_0 = t(s_0) := \mathrm{Var}[\gamma_{[0,s_0]}]$ is an element of $]0, \mathrm{Var}[\gamma][\setminus \Sigma_{\tilde{\gamma}}$ and $\frac{\gamma'}{|\gamma'|}(s_0) = \tilde{\gamma}'(t_0)$.

Now we recall [**48**] that since A has positive reach and $\gamma(s_0) \in \partial A$ then $\mathrm{Nor}(A, \gamma(s_0))$ is not $\{0\}$. Therefore $\exists v_0 \neq 0$ in $\mathbb{R}^d$ such that $\langle \gamma'(s_0), v_0 \rangle \leq 0$.

Now we can consider $\bar{\gamma}(s) := \gamma(1 - s)$ and $\bar{s}_0 := 1 - s_0$ and apply the same reasoning above to get

$$0 \leq \langle -\gamma'(s_0), v_0 \rangle = \langle \bar{\gamma}'(\bar{s}_0), v_0 \rangle \leq 0. \Rightarrow \gamma'(s_0) \in \langle v_0 \rangle^{\perp}.$$

Taking $v = \frac{v_0}{|v_0|}$ we are done. □

E.2. (Oriented) distance function and $\mathscr{C}^{1,1}$ domains

Now we switch to the case of a bounded $\mathscr{C}^{1,1}$ domain in $\mathbb{R}^d$. For the reader's convenience we clarify that here we are using the following definition, however several (essentially equivalent) variants are available.

Definition E.2.5. *Let $\Omega \subset \mathbb{R}^d$ be a domain, then it is said to be a $\mathscr{C}^{1,1}$ domain iff the following holds.*

There exist $r > 0$, $L > 0$ such that for any $x \in \partial\Omega$ there exist a coordinate rotation $R_x \in SO^d$ and $f_x \in \mathscr{C}^{1,1}\left(B^{d-1}(0, r),]-r, r[\right)$ (that is, a differentiable function having Lipschitz gradient) such that

$$\begin{aligned} f_x(0) &= 0 \\ \nabla f_x(0) &= 0 \\ \|f_x\|_{\mathscr{C}^{1,1}} &\leq L \end{aligned}$$

$$x + R_x \operatorname{Graph} f_x \quad = \quad \partial\Omega \cap (x + R_x B(x, r)),$$

where$\|f_x\|_{\mathscr{C}^{1,1}} := \max\{\sup_D |f|, \sup_D |\nabla f|, \operatorname{Lip}(\nabla f)\}$.

In the spirit of [**41**] and [**40**] one may study regularity properties of a domain Ω comparing it to the smoothness of the *Distance Function* and the *Oriented Distance Function*

$$b_\Omega(\cdot) := d_\Omega(\cdot) - d_{\complement\Omega}.$$

We collect all the properties we need of a $\mathscr{C}^{1,1}$ domain in $\mathbb{R}^d$ in the following theorem. Detailed proofs can be easily provided combining classical results that can be found in [**?**][Th. 5.1.9],[**48**],[**?**] and [**40**].

Theorem E.2.1. *Let $\Omega \subset \mathbb{R}^d$ be a $\mathscr{C}^{1,1}$ bounded domain. Then the following hold.*

(i) Both Ω and $\complement\Omega$ have positive reach,

$$\operatorname{Reach}(\partial\Omega) = \min\{\operatorname{Reach}(\Omega), \operatorname{Reach}(\complement\Omega)\}.$$

(ii) For any $0 < h <$ Reach$(\partial\Omega)$ $b_\Omega \in \mathscr{C}^1(U_h(\Omega))$ where $U_h(\Omega) := \{x \in \mathbb{R}^d : -h < b_\Omega(x) < h\}$.

(iii) For any $x \in U_h(\Omega)$, $0 < h <$ Reach$(\partial\Omega)$

$$\nabla b_\Omega(x) = -\frac{x - \operatorname{proj}_{\partial\Omega}(x)}{|x - \operatorname{proj}_{\partial\Omega}(x)|}, \tag{E.2.1}$$

where the right side is well defined also on $\partial\Omega$. Moreover ∇b_Ω is a Lipschitz function.

(iv) For any $x \in \partial\Omega$ we have $\operatorname{Tan}(x, \partial\Omega) = \mathcal{T}_x\partial\Omega$ *and* $\operatorname{Nor}(x, \Omega) = \langle \nabla b_\Omega(x) \rangle$.

(v) For all $x \in \partial\Omega$ an d for any $r <$ Reach$(\partial\Omega)$ we have

$$B(x - r\nabla b_\Omega(x), r) \quad \subseteq \quad \Omega \tag{E.2.2}$$

$$B(x + r\nabla b_\Omega(x), r) \quad \subseteq \quad \complement\Omega \tag{E.2.3}$$

Bibliography

[1] H. Alexander. A note on projective capacity. *Canad. J. Math.*, 34(6):1319–1329, 1982.

[2] H. J. Alexander and B. A. Taylor. Comparison of two capacities in $\mathbb{C}^n$. *Math. Z.*, 186:407–414, 1984.

[3] R. Alvarado, D. Brigham, V. Maz'ya, M. Mitrea, and E. Ziadé. On the regularity of domains satisfying a uniform hour-glass condition and a sharp version of the Hopf-Oleinik boundary point principle. *J. Math. Sci. (N. Y.)*, 176(3):281–360, 2011. Problems in mathematical analysis. No. 57.

[4] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, second edition, 2008.

[5] W. Baleikorocau and S. Ma'u. Chebyshev constants, transfinite diameter, and computation on complex algebraic curves. *Comput. Methods Funct. Theory*, 15(2):291–322, 2015.

[6] M. S. Baouendi and C. Goulaouic. Approximation polynomiale de fonctions C^∞ et analytiques. *Ann. Inst. Fourier (Grenoble)*, 21(4):149–173, 1971.

[7] M. Baran. Siciak's extremal function of convex sets in $\mathbf{C}^N$. *Ann. Polon. Math.*, 48(3):275–280, 1988.

[8] M. Baran. *Siciak's extremal function and complex equilibrium measures for compact sets of* $\mathbb{R}^n$. PhD thesis, Jagellonian University (Krakow). PhD Dissertation, 1989.

[9] M. Baran and L. Bialas-Ciez. Hölder continuity of the Green function and Markov brothers' inequality. *Constr. Approx.*, 40(1):121–140, 2014.

[10] M. Baran, L. Białas-Cież, and B. Milówka. On the best exponent in Markov inequality. *Potential Anal.*, 38(2):635–651, 2013.

[11] E. Bedford. The operator $(dd^c)^n$ on complex spaces. *Lecture Notes in Math. Springer, Berlin-New York*, 919:294–323, 1982.

[12] E. Bedford and B. A. Taylor. The Dirichlet problem for a complex Monge-Ampère equation. *Invent. Math.*, 37(1):1–44, 1976.

[13] E. Bedford and B. A. Taylor. A new capacity for plurisubharmonic functions. *Acta Mathematica*, 149(1):1–40, 1982.

[14] E. Bedford and B. A. Taylor. Plurisubharmonic functions with logarithmic singularities. *Annales de l institut Fourier*, 38(4):133–171, 1988.

[15] R. Berman and S. Boucksom. Growth of balls of holomorphic sections and energy at equilibrium. *Invent. Math.*, 181(2):337–394, 2010.

[16] R. Berman, S. Boucksom, and D. Witt Nyström. Fekete points and convergence towards equilibrium measures on complex manifolds. *Acta Math.*, 207(1):1–27, 2011.

[17] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[18] H.-P. Blatt. The influence of poles on equioscillation in rational approximation. *Ukraïn. Mat. Zh.*, 58(1):3–11, 2006.

[19] Z. Błochki. The complex Monge-Ampére operator in pluripotential theory. lecture notes based on graduate course at Jagiellonian University, 1997. Last update November 2002.

[20] T. Bloom. Orthogonal polynomials in $\mathbb{C}^n$. *Indiana Univ. Math. J.*, 46(2):427–452, 1997.

[21] T. Bloom, L. Bos, and N. Levenberg. The transfinite diameter of the real ball and simplex. *Ann. Polon. Math.*, 106:83–96, 2012.

[22] T. Bloom, L. Bos, N. Levenberg, and S. Waldron. On the convergence of optimal measures. *Constr. Approx.*, 32(1):159–179, 2010.

[23] T. Bloom, L. P. Bos, J. Calvi, and N. Levenberg. Approximation in $\mathbb{C}^n$. *Ann. Polon. Math.*, (106):53–81, 2012.

[24] T. Bloom and N. Levenberg. Capacity convergence results and applications to a Bernstein-Markov inequality. *Trans. Amer. Math. Soc.*, 351(12):4753–4767, 1999.

[25] T. Bloom and N. Levenberg. Weighted pluripotential theory in $\mathbf{C}^N$. *Amer. J. Math.*, 125(1):57–103, 2003.

[26] T. Bloom and N. Levenberg. Transfinite diameter notions in $\mathbb{C}^N$ and integrals of Vandermonde determinants. *Ark. Mat.*, 48(1):17–40, 2010.

[27] T. Bloom and N. Levenberg. Pluripotential energy and large deviation. *Indiana Univ. Math. J.*, 62(2):523–550, 2013.

[28] T. Bloom and N. Levenberg. Random polynomials and pluripotential-theoretic extremal functions. *Potential Anal.*, 42(2):311–334, 2015.

[29] T. Bloom, N. Levenberg, F. Piazzon, and F. Wielonsky. Bernstein-Markov: a survey. *DRNA Dolomites Research Notes on Approximation*, 8:under review, 2015.

[30] T. Bloom, N. Levenberg, and F. Wielonsky. Vector energy and large deviation. *J. Anal. Math.*, 125:139–174, 2015.

[31] L. Bos, J.-P. Calvi, N. Levenberg, A. Sommariva, and M. Vianello. Geometric weakly admissible meshes, discrete least squares approximations and approximate Fekete points. *Math. Comp.*, 80(275):1623–1638, 2011.

[32] L. Bos, A. Sommariva, and M. Vianello. Least-squares polynomial approximation on weakly admissible meshes: disk and triangle. *J. Comput. Appl. Math.*, 235(3):660–668, 2010.

[33] L. P. Bos, N. Levenberg, S. Maʿu, and F. Piazzon. A weighted extremal function and equilibrium measure. *Math. Scand., to appear*, 2015.

[34] L. P. Bos, N. Levenberg, P. D. Milman, and B. A. Taylor. Tangential Markov inequalities on real algebraic varieties. *Indiana Univ. Math. J.*, 47(4):1257–1272, 1998.

[35] L. P. Bos, S. D. Marchi, A. Sommariva, and M. Vianello. Weakly admissible meshes and discrete extremal sets. *Numer. Math. Theory Methods Appl.*, 41(1):1–12, 2011.

[36] S. Boucksom, P. Eyssidieux, V. Guedj, and A. Zeriahi. Monge-Ampère equations in big cohomology classes. *Acta Math.*, 205(2):199–262, 2010.

[37] J.-P. Calvi and N. Levenberg. Uniform approximation by discrete least squares polynomials. *J. Approx. Theory*, 152(1):82–100, 2008.

[38] H. Chirka. *Complex analytic sets*. Kluwer Accademic pubblishers, 1985.

[39] D. Coman. Integration by parts for currents and applications to the relative capacity and Lelong numbers. *Mathematica*, 32(62):45–57, 1997.

[40] M. C. Delfour and J. P. Zolesio. Shape analysis via distance functions. *J. Functional Analysis*, 123(1):129–201, 1994.

[41] M. C. Delfour and J. P. Zolesio. *Shapes and Geometry.* Metrics, Analysis, Differential Calculus, and Optimization, SIAM, 2001.

[42] J. P. Demailly. Mesures de Monge-Ampere et caracterisation geometrique des variete algebriques affines. *Mem. soc. Math. France*, 19:1–124, 1985.

[43] J. P. Demailly. *Complex Analytic and Differential Geometry*. Université de Grenoble I Institut Fourier, UMR 5582 du CNRS, 2012.

[44] C. F. Dunkl and Y. Xu. *Orthogonal polynomials of several variables*, volume 81 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2001.

[45] H. Ehlich and K. Zeller. Schwankung von Polynomen zwischen Gitterpunkten. *Math. Z.*, 86:41–44, 1964.

[46] H. El Mir. Sur le prolongement des courants positifs fermés. (french). *Acta Math.*, 153(1-2):1–45, 1984.

[47] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[48] H. Federer. Curvature measures. *Trans. Amer. Math. Soc*, 93(3):418–491, 1959.

[49] H. Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[50] J. E. Fornaess and R. Narasimhan. The Levi problem on complex spaces with singularities. *Math. Ann.*, 248, 1980.

[51] N. Fusco, M. S. Gelli, and G.Pisante. On a Bonnesian type inequality involving the spherical deviation. *J. Math Pures Appl.*, 98:616–632, 2012.

[52] A. A. Gonchar. On a theorem of Saff. *(Russian) Mat. Sb.*, 94(136):152–157, 1975.

[53] V. Guedj and A. Zeriahi. Intrinsic capacities on compact Kähler manifolds. *J. Geom. Anal.*, 15(4):607–639, 2005.

[54] A. Hardy. A note on large deviations for 2d coulomb gas with weakly confining potential. *Electronic Communications in Probability*, 17(19):1–12, 2012.

[55] M. B. Hernándeza and B. de la Calle Ysernb. Meromorphic continuation of functions and arbitrary distribution of interpolation points. *Journal of Mathematical Analysis and Applications*, 403(1):107–119, 1971.

[56] E. Hille. *Analytic function theory. Vol. II*. Introductions to Higher Mathematics. Ginn and Co., Boston, Mass.-New York-Toronto, Ont., 1962.

[57] M. Kac. On the average number of real roots of a random algebraic equation. *Bull. Amer. Math. Soc.*, 49:314–320, 1943.

[58] M. Kac. On the average number of real roots of a random algebraic equation. II. *Proc. London Math. Soc. (2)*, 50:390–408, 1949.

[59] M. Klimek. *Pluripotential Theory*. Oxford Univ. Press, 1991.

[60] S. Kołodziej. Capacities associated to the Siciak extremal function. *Ann. Polon. Math.*, 49(3):279–290, 1989.

[61] S. Kołodziej. The complex Monge-Ampère equation and pluripotential theory. *Mem. Amer. Math. Soc.*, 178(840):x+64, 2005.

[62] A. Kroó. On optimal polynomial meshes. *J. Approx. Theory*, 163(9):1107–1124, 2011.

[63] A. Kroó. Bernstein type inequalities on star-like domains in $\mathbb{R}^d$ with application to norming sets. *Bull. Math. Sci.*, 3(3):349–361, 2013.

[64] P. Lelong. Intégration sur un ensemble analytique complexe. *Bulletin de la Société Mathématique de France*, 85:239–262, 1957.

[65] N. Levenberg. *Capacities in Several Complex Variables*. PhD thesis, University of Michigan. PhD Dissertation, 1984.

[66] N. Levenberg. Approximation in $\mathbb{C}^N$. *Surv. Approx. Theory*, 2:92–140, 2006.

[67] N. Levenberg. Weighted pluripotential theory results of Berman-Boucksom. http://arxiv.org/abs/1010.403, 2010.

[68] N. Levenberg. Ten lectures on weighted pluripotential theory. *Dolomites Notes on Approximation*, 5:1–59, 2012.

[69] S. D. Marchi, F. Piazzon, A. Sommariva, and M. Vianello. Polynomial meshes: Computation and approximation. *Proceedings of CMMSE*, pages 414–425, 2015.

[70] A. Marigonda, K. T. Nguyen, and D. Vittone. Some regularity results for a class of upper semi-continuous functions. *Indiana Univ. Math. J.*, 62:45–89, 2013.

[71] S. Maʿu. Chebyshev constants and transfinite diameter on algebraic curves in $\mathbb{C}^2$. *Indiana Univ. Math. J.*, 60(5):1767–1796, 2011.

[72] W. Pawłucki and W. Pleśniak. Markov's inequality and C^∞ functions on sets with polynomial cusps. *Math. Ann.*, 275(3):467–480, 1986.

[73] W. Pawłucki and W. Pleśniak. Markov's inequality and C^∞-functions on subanalytic sets. In *A. Haar memorial conference, Vol. I, II (Budapest, 1985)*, volume 49 of *Colloq. Math. Soc. János Bolyai*, pages 703–709. North-Holland, Amsterdam, 1987.

[74] F. Piazzon. Optimal polynomial admissible meshes on some classes of compact subsets of $\mathbb{R}^d$. submitted to Journal on Approximation Theory.

[75] F. Piazzon. Optimal polynomial admissible meshes on compact subsets of $\mathbb{R}^d$ with mild boundary regularity. *preprint submitted to JAT, arXiv:1302.4718*, 2013.

[76] F. Piazzon. Some results on the rational Bernstein Markov property in the complex plane. *Computational Methods and Function Theory (under review)*, 2015.

[77] F. Piazzon and M. Vianello. Analytic transformations of admissible meshes. *East J. Approx.*, 16(4):389–398, 2010.

[78] F. Piazzon and M. Vianello. Small perturbations of polynomial meshes. *Appl. Anal.*, 92(5):1063–1073, 2013.

[79] F. Piazzon and M. Vianello. Constructing optimal polynomial meshes on planar starlike domains. *Dolomites Res. Notes Approx. DRNA*, 7:22–25, 2014.

[80] F. Piazzon and M. Vianello. Suboptimal polynomial meshes on planar Lipschitz domains. *Numer. Funct. Anal. Optim.*, 35(11):1467–1475, 2014.

[81] F. Piazzon and M. Vianello. Numerical approximation in pluripotential theory. *in preparation*, 2015.

[82] W. Pleśniak. Multivariate polynomial inequalities via pluripotential theory and subanalytic geometry methods. In *Approximation and probability*, volume 72 of *Banach Center Publ.*, pages 251–261. Polish Acad. Sci., Warsaw, 2006.

[83] W. Pleśniak. Multivariate Jackson inequality. *J. Comput. Appl. Math.*, 233(3):815–820, 2009.

[84] W. Pleśniak. Nearly optimal meshes in subanalytic sets. *Numer. Algorithms*, 60(4):545–553, 2012.

[85] T. Ransford. *Potential theory in the complex plane*, volume 28 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1995.

[86] W. Rudin. A geometric criterion for algebraic varieties. *J. Math. Mech.*, 17:671–683, 1967/1968.

[87] A. Sadullaev. The operator $(dd^c u)^n$ and condenser capacities. *Dokl. Akad. Nauk SSSR*, 251(1):44–47, 1980.

[88] A. Sadullaev. An estimates for polynomials on analytic sets. *Math. URSS Izvestiya*, 20(3):493–502, 1982.

[89] E. Saff. Regions of meromorphy determined by the degree of best rational approximation. *Proc. Amer. Math. Soc.*, 29:30âĂŞ38, 1971.

[90] E. B. Saff. Logarithmic potential theory with applications to approximation theory. *Surveys in Approximation Theory*, 5:165–200, 2010.

[91] E. B. Saff and V. Totik. *Logarithmic potentials with external fields*. Springer-Verlag Berlin, 1997.

[92] J. Siciak. On some extremal functions and their application to the theory of analytic functions of several complex variables. *Trans. of AMS*, 105(2):322–357, 1962.

[93] J. Siciak. Extremal plurisubharmonic functions in $\mathbb{C}^n$. *Proceedings of the First Finnish-Polish Summer School in Complex Analysis (Podlesice, 1977)*, 319:115–152, 1978.

[94] J. Siciak. Extremal plurisubharmonic functions in $\mathbb{C}^n$. *Ann Polon. Math.*, 319:175–211, 1981.

[95] B. Simon. *Orthogonal polynomials on the unit circle. Part 1*, volume 54 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2005. Classical theory.

[96] B. Simon. *Orthogonal polynomials on the unit circle. Part 2*, volume 54 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2005. Spectral theory.

[97] H. Stahl and V. Totik. *General orthogonal polynomials*, volume 43 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992.

[98] G. Szegö. *Orthogonal Polynomials*. American Mathematical Society, New York, 1939. American Mathematical Society Colloquium Publications, v. 23.

[99] W. Van Assche. *Asymptotics for orthogonal polynomials*, volume 1265 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987.

[100] J. L. Walsh. *Interpolation and approximation by rational function on complex domains*. AMS, 1929.

[101] F. Wikstrom. Continuity of the relative extremal function on analytic varieties in $\mathbb{C}^n$. *Annales Polonici Mathematici*, 3(83), 2005.

[102] Y. Xing. Continuity of the complex Monge Ampere operator. *Proceedings of the American Mathematical Society*, 124(2), 1996.

[103] V. P. Zaharjuta. Extremal plurisubharmonic functions, Hilbert scales, and the isomorphism of spaces of analytic functions of several variables. i, (russian). *Teor. Funkciĭ Funkcional. Anal. i Priložen.*, 127(19):133–157, 1974.

[104] V. P. Zaharjuta. Extremal plurisubharmonic functions, hilbert scales, and the isomorphism of spaces of analytic functions of several variables. ii, (russian). *Teor. Funkciĭ Funkcional. Anal. i Priložen.*, 127(21):65–83, 1974.

[105] V. P. Zaharjuta. Transfinite diameter, Chebyshev constant and capacity for compacta in $\mathbb{C}^n$. *Math. USSR Sb.*, 25(350), 1975.

[106] O. Zeitouni and S. Zelditch. Large deviations of empirical measures of zeros of random polynomials. *Int. Math. Res. Not. IMRN*, (20):3935–3992, 2010.

[107] A. Zeriahi. Fonction de Green pluriclomplexe à pole à l'infini sur un espace de Stein parabolique et application. *Mathematica Scandinavica*, 69:89–126, 1991.

[108] A. Zeriahi. Pluripotential theory on analytic sets and applications to algebraicity. *Acta Mathematica Vietnamica*, 27(3):407–424, 2002.