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Ungauged and gauged Supergravity Black Holes: results on U-duality

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Abstract

Extremal black holes are states in the non-perturbative spectrum of Supergravity theories. They may be charged under abelian fields and in this case give rise to an effective potential for the scalars. This, in addition, is responsible for the occurrence of an attractor mechanism: extremal solutions are determined by critical points of the effective potential. This implies that their supersymmetric features rely only on the algebraic properties of the electric-magnetic duality group. In this thesis black holes solutions and their properties under duality transformations are analyzed in extended Supergravity theories in four dimensions.

An introductory part reviews the construction of electric-magnetic duality invariant theories and describes the symplectic covariant formalism both in the case of $\mathcal{N} = 2$ and $\mathcal{N} > 2$ Supergravity theories. The attractor mechanism and black hole first order formalism are also reviewed.

The thesis than proceeds with the discussion of original results. First I discuss the ungauged N=8 theory and show how the supersymmetric properties of black hole duality orbits are manifest, once the proper representations of vectors and scalar fields in different symplectic frames is chosen, according to the algebraic branching of the orbit with respect to the duality group. In particular, one of these cases corresponds to the Kaluza−Klein reduction of the theory from five dimensions, as it can be seen from the relation between the central charge in four and five dimensions. Explicit computations are possible if one restricts the solutions to the *stu*-truncation.

Then I present static dyonic black holes in the context of N=2 U(1) gauged supergravity in four dimensions, with AdS_4 asymptotic geometry. It is shown that the flow of scalar fields and metric warp factors is governed by first order equations that can be derived for a general U(1) gauging potential. Explicit examples are finally presented, which only preserve up to half of the supersymmetry and thus evade previous no-go theorems.

Riassunto

I buchi neri estremali sono stati dello spettro non perturbativo di teorie di Supergravità. Sono sistemi carichi rispetto a campi abeliani, la cui presenza introduce un potenziale effettivo per i campi scalari. Questo stesso potenziale è responsabile di un meccanismo attrattore per i campi scalari: le soluzioni estremali corrispondono infatti ai punti critici di un potenziale efficace. In questo modo le proprietà di supersimmetria della soluzione dipendono solamente dalla struttura algebrica del gruppo di dualità elettromagnetica. In questa tesi vengono analizzate soluzioni di buco nero in teorie di Supergravità estese in quattro dimensioni.

Una parte introduttiva presenta la costruzione delle teorie invarianti per dualità elettromagnetica, e descrive il formalismo covariante simplettico sia nel caso della Supergravità estesa $\mathcal{N} = 2$ che di quelle con $\mathcal{N} > 2$. Vengono anche descritti il meccanismo degli attrattori e il formalismo del prim'ordine per le soluzioni di buco nero.

La tesi procede poi con la discussione dei risultati originali. In questa parte si considera la teoria $\mathcal{N} = 8$ in assenza di gauging delle isometrie del gruppo di dualità. Viene mostrato come le proprietà di supersimmetria delle orbite del buco nero siano manifeste se si sceglie una rappresentazione opportuna per i campi vettoriali e scalari, a seconda del branching algebrico corrispondente all'orbita nel gruppo di dualità. In particolare, uno di questi casi corrisponde alla riduzione dimensionale di Kaluza−Klein da cinque dimensioni, come si può leggere dalla relazione tra la carica centrale in quattro e in cinque dimensioni. Per la troncazione al modello *stu* verranno mostrate soluzioni esplicite.

Nella parte finale vengono presentati configurazioni di buchi neri dionici nella teoria di Supergravità $\mathcal{N} = 2$ con gauging $U(1)$ in quattro dimensioni. Queste soluzioni ammettono una geometria asintotica di tipo AdS_4. Viene mostrato come il flusso radiale dei campi scalari e il warp factor della metrica sono governati da equazioni del prim'ordine, che si possono ricavare per un generico potenziale di gauging. Sono presentati, infine, alcuni esempi espliciti di soluzioni di buco nero che preservano non più della metà di supersimmetrie, e quindi possono evadere teoremi di inesistenza presenti in letteratura.

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Chapter 1

Introduction

Black holes are solutions of General Relativity, the theory that describes classical gravitational interactions, or of its classical extensions, like Supergravity. They are characterized by the presence of an event horizon hiding a space-time singularity. For their intrinsic connection to the limits of validity of General Relativity, they have always been an interesting playground for the investigation of possible extensions to quantum theories of gravity.

Non linearity of General Relativity equations of motion implies, in fact, that smooth initial data can evolve into a singular field configuration [1], thus, the horizon protects the outer region of spacetime from causally interacting with the inner part, in contact with the singularity, because no particles can classically come out from the horizon. A naked singularity would instead cause the breakdown of the theory. As soon as one starts taking into account the semiclassical behavior, however, under very general assumptions, black holes behave as a thermal state, emitting particles with a black body spectrum [2]. This means that the emitted radiation carries no information about the matter that caused the formation of, or simply fell into the black hole, and was then emitted thermally [3]. This causes the breakdown of unitary evolution of states, leading to the "information loss" problem.

As thermal states, black holes have an entropy associated to them. From a semiclassical computation, Bekenstein and Hawking [4],[5],[6] found that it is proportional to the area of the event horizon, and is a topological invariant quantity. It is a fundamental question for any quantum extension of gravity, if this entropy has a statistical interpretation in terms of fundamental, quantum degrees of freedom. Up to now, the only theory that has been able to microscopically describe a black hole is String Theory.

Examples have been built, where the degrees of freedom are strings and branes, and give rise to a black hole in the low energy, effective theory. The microstates counting has been carried out at weak coupling $g_s \to 0$, and takes into account the BPS states of the corresponding D-brane worldvolume theory [7]. Because of supersymmetry, the number of states has to remain the same also at strong coupling, and the result of microscopic counting actually agrees with the semi-classical computation of black hole entropy, given by the Bekenstein−Hawking formula. What is interesting to our purposes, is that these low energy, effective configurations, in the large charges approximation, are precisely Supergravity black holes.

We will be dealing with the subset of extremal black holes that are charged massive states with vanishing thermal temperature. The charge and mass of a state in general relativity and all its classical extensions satisfy the BPS bound $M \geq Q$ [8]: extremal black holes are those saturating this bound. They are stable states and have non vanishing entropy, and this ensures them a regular near horizon geometry.

String theory and its low energy Supergravity limits base their formulation on a fundamental ingredient, which is Supersymmetry.

Supersymmetry is a candidate for a fundamental new symmetry of particles and fields. It relates fermions to bosons and organizes fields in supermultiplets. It is being intensively tested in these days at the LHC collider at CERN, in Geneva. Together with the search of the Higgs, the missing particle from the Standard Model of gauge interactions, testing the existence of supersymmetric partners of known particles, at accessible energies, is one of the most intriguing challenges of high energy physics of the XXI century. Supergravity theories, moreover, are theories of gravity which are also Supersymmetry invariant, their field content contains Standard Model-like fields but also other matter fields with spin less then two, and treat the rank-2 symmetric tensor of the metric as the only spin-2 field in the theory. Invariance under Supersymmetry determines completely the action and the coupling of the fields. It reveals, however, that the action possesses other symmetries and invariances, which follow from the specific structure of the terms in the Lagrangian allowed by supersymmetry.

It is the aim of this thesis to exploit the richness of symmetries and duality invariances that extended Supergravity theories manifest, to study their extremal black holes solutions. The relevant invariance in their description is electric-magnetic duality. Not only this will help in the determination of the black hole metric, but it will also allow for an algebraic classification of the solutions in terms of electric and magnetic orbits.

As charged states, black holes satisfy the BPS bound. In supersymmetric invariant theories, this is related to Supersymmetry, in the sense that (a part of the) Supersymmetry is preserved by a state, when the bound is saturated. Selecting a particular configuration of black hole electric and magnetic charges, indeed, fixes the duality orbit of the solution. For each orbit, black holes have an entropy which is given by a duality invariant expression of the charges, a geometric quantity of the duality group that encodes the supersymmetric properties of the orbit in the duality group.

Each orbit is obtained by solving the corresponding black hole "attractor equations". As it has been found in [9]-[13], the horizon of extremal black holes in Supergravity manifests an attractor behavior: an extremal black hole attractor is associated to a critical point of a suitably defined black hole effective potential, and it describes a scalar field configuration stabilized at the event horizon, purely in terms of conserved electric and magnetic charges, regardless the value of scalar fields at spatial infinity. This ensures that the entropy does not depend on continuous parameters of the theory, like the scalars v.e.v. at asymptotic infinity, consistently with a microscopic interpretation of entropy as the log of the number of fundamental degrees of freedom, giving rise to the statistical black hole-state.

More precisely, in this thesis, we explicitly parametrize orbits of ungauged $\mathcal{N} = 2$ and $\mathcal{N} = 8$ Supergravity, studying how the Supersymmetry features are encoded in the form of the central charge matrix at the attractor point. We derive the entropy and the explicit symplectic sections, which are suitable parametrizations of different branching of the fields representations, with respect to the maximal subgroups contained in the duality group.

In gauged Supergravity some of the global isometries of the scalar manifolds are made local, and the scalars are charged under the action of gauge fields. There is however a case, precisely the gauging of the diagonal $U(1)$ group in the $\mathcal{N} = 2$ theory, in which the only modification is the appearance of a scalar potential. This behaves as a position-dependent cosmological constant, thus allowing for asymptotically curved space-time solutions. The presence of this additional potential strictly constrains the scalar dynamics. Only recently, in fact, the standard lore of the non-existence of Supergravity black holes, in asymptotically Anti de Sitter (AdS) space-time in four dimensions, has been demonstrated wrong, by the construction of a magnetically charged black hole solution in the mentioned $\mathcal{N} = 2$ supergravity. We extended such formulation to render it duality invariant; in particular, in this framework, it is possible to introduce also magnetic gauging. We show how to recover an attractor flow and we identify the corresponding super-potential.

Finally, it is important to mention that, in more recent years, black holes have become a powerful tool in applications of the AdS/CFT correspondence to condensed matter and nuclear physics. The construction of black hole solutions for gauged Supergravity theories may give the opportunity to extend such results for *AdS* black holes in presence of multiple charges and nontrivial scalars profiles, thus investigate more complex dual configurations.

This thesis is organized as follows. Chapter 2 reviews electric-magnetic duality transformations in theories of vectors, scalars and fermions coupled to gravity. In Chapter 3 we introduce the geometric formulation of extended Supergravity theories with coset scalar manifold, exploiting the symplectic structure of their scalar manifold. We also show how a symplectic covariant formalism allows to clarify the connection between electric-magnetic duality and U-duality invariance of the theory. Attractor equations are derived for a static black hole solutions of Supergravity in Chapter 4. We then analyze some $\mathcal{N} = 8$ Supergravity specific configurations which capture representatives of both BPS and non-BPS orbits, in Chapter 5, and we focus on the properties of black holes in 4-dimensional theory that arise from dimensional reduction of 5-dim $\mathcal{N} = 8$ Supergravity in Chapter 6.

We dedicate Chapter 7 to the construction of dyonic black holes in $U(1)$-gauged Supergravity, and present examples of different charge configurations that can be related by duality or symplectic transformations.

Chapter 2

Electric-magnetic duality in Supergravity

Supersymmetry invariance constrains the form of the action, and thus determine the possible couplings among the fields of a given theory. However, the resulting Lagrangian shows additional interesting symmetries and invariances, which are collectively described as dualities. This, in part, reflects the geometric nature of the supersymmetric field content. For example, when the maximally extended theory of Supergravity was constructed, by Cremmer and Julia, a non-compact duality invariance under the action of $E_{7(7)}$ emerged. Gaillard and Zumino then considered this invariance for general theories, extending the duality of Maxwell electrodynamics to several abelian gauge fields; supersymmetric theories are just a subset of those.

Duality transformations rotate among themselves the abelian field strengths, and correspond to an invariance of the theory if they do not affect the equations of motion and Bianchi identities. The Lagrangian in general transforms under duality, and the action is not invariant. By exploiting the covariant transformation of equations of motion and Bianchi identities, one can study their properties in a unified framework, and clarify how solutions of different models can be mapped onto each others.

It is interesting how the supersymmetric feature of a theory automatically selects the couplings among the fields, in such a way that a nontrivial duality invariance remains, acting on the vector sector. We will see how this happens in Chapter 3, where we will discuss the details of the scalar manifold of Supergravity theories and in particular of its symplectic embedding. Before that, we will derive the constraints imposed by duality invariance for generic theories of vectors, coupled to scalars and fermionic fields, and we will follow in that the study of M.K. Gaillard and B. Zumino [14].

2.1 Gaillard–Zumino construction

One of the most interesting properties of charged solutions of Supergravity is their invariance under electric-magnetic duality rotations. Consider the bosonic sector of Supergravity, described by the following action

$$\mathcal{S} = \int \sqrt{-g}\, d^4x \left(-\frac{1}{2} R + \mathrm{Im}\mathcal{N}_{\Lambda\Gamma} F^{\Lambda}_{\mu\nu} F^{\Gamma,\ \mu\nu} + \frac{1}{2\sqrt{-g}} \mathrm{Re}\mathcal{N}_{\Lambda\Gamma} \epsilon^{\mu\nu\rho\sigma} F^{\Lambda}_{\mu\nu} F^{\Gamma}_{\rho\sigma} + \right.$$
$$\left. + \frac{1}{2} G_{rs}(\phi) \partial_\mu \phi^r \partial^\mu \phi^s \right) . \quad (2.1.1)$$

This is a theory of gravity coupled to n_v abelian fields ($\Lambda, \Gamma = 1, .., n_v$), and scalar fields which are described by a nonlinear σ-model with target space $\mathcal{M}_{scalar}$. They couple to the vector fields by the metric

$$\mathcal{N}_{\Lambda\Gamma} = \mathcal{N}_{\Lambda\Gamma}(\phi) . \quad (2.1.2)$$

Except for some cases in $\mathcal{N} = 2$ theories, the scalar manifold of extended Supergravity is a symmetric homogenous space of the form $\mathcal{M}_{scalar} = \mathcal{G}/\mathcal{H}$, where $\mathcal{G}$ is the duality group acting on the electric and magnetic field strengths, and $\mathcal{H}$ is its maximal compact subgroup. Of course, the action in (2.2.1) needs to be completed with terms containing fermionic fields for the theory to be supersymmetric. However, when we specify to black holes solutions, fermions decouple from the bosonic equations of motion, thus (2.2.1) is sufficient to find the solution for the metric.

In a seminal work of 1981 [14], M. K. Gaillard and B. Zumino considered the most general 2-derivatives action of bosonic and fermionic fields, with invariance under electric magnetic duality rotations, and showed that the group $\mathcal{G}$ must be embedded in the symplectic group $Sp(2n, \mathbb{R})$, where n is the number of vector fields in the theory. More exactly, the duality invariance of electromagnetism can be extended to the interaction with the gravitational field, but it is violated by electromagnetic couplings of the minimal type, and there is no non-abelian generalization of duality rotations that leave the pure Yang-Mills equations invariant [15]. However, generalizations to non minimal (e.g. magnetic moment type) couplings is possible, even to non abelian group.

In the following, the main procedure to derive such an action will be outlined. In particular, it will be shown that the most general group which can be realized, given n field strength, is the real symplectic group $Sp(2n, \mathbb{R})$, which has $U(n)$ as its maximal compact subgroup. $U(n)$ is indeed the largest group of duality transformations, in absence of scalar fields, and this is related to non-linear transformations of the scalars. In the examples we will touch in this thesis, the group of duality transformations is smaller than the whole symplectic group. In the case of $\mathcal{N} = 8$, for example, there are 28 field strengths, the duality group is $E_{7(7)}$, a subgroup of $Sp(56, \mathbb{R})$, whose maximal compact subgroup is $SU(8) \subset U(28)$.

Duality invariance in Maxwell-Einstein theory

The simplest theory in which electric-magnetic duality is realized is Maxwell theory of electromagnetism coupled to gravity, in which an abelian gauge field $A_\mu(x)$ is the connection of a $U(1)$ gauge bundle over the 4-dimensional space-time manifold, with metric $g_{\mu\nu}$. It is described by the action

$$S_{EM} = \frac{1}{16\pi G} \int \sqrt{-g} \{R - F_{\mu\nu} F^{\mu\nu}\} \ , \qquad (2.1.3)$$

leading to the equations of motion

$$\partial_\mu F^{\mu\nu} = 0 \ , \qquad (2.1.4)$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -8\pi G T_{\mu\nu} \ , \qquad (2.1.5)$$

and Bianchi identities

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \qquad \Longleftrightarrow \qquad \partial_{[\mu} F_{\nu\rho]} = 0 \ , \qquad (2.1.6)$$

where

$$^\star F = \frac{1}{2} \tilde{F}^{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{2\sqrt{-g}} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} dx^\mu \wedge dx^\nu \ , \qquad (2.1.7)$$

is the hodge dual field strength of the vector field. The stress-energy tensor is

$$T_{\mu\nu} = \frac{1}{4\pi} \left[F_{\mu\alpha} F^\alpha_\nu - \frac{1}{4} g_{\mu\nu} F^2 \right] \ . \qquad (2.1.8)$$

This theory is manifestly duality invariant, in the sense that the set of equations (2.1.4) is unaffected by the following transformations on the vector field strength

$$F'^{\mu\nu} = (\cos\alpha + j \sin\alpha) F^{\mu\nu} \ , \qquad \alpha \in \mathbb{R} \ , \qquad (2.1.9)$$

where the j "duality" operator is such that $jF = {}^*F$, corresponding to the following $U(1) \simeq SO(2)$ rotation

$$\begin{pmatrix} E' \\ H' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} \ . \qquad (2.1.10)$$

The Lagrangian of the vector field is written in terms of the field strengths and their duals, or explicitly

$$T = \frac{1}{4\pi}\left(E^2 - H^2\right) \ , \tag{2.1.11}$$

which is obviously not invariant for $SO(2)$ rotations acting on the vector (E, H). It is important to stress that duality rotations are not defined as transformations on the vector fields but on their field strengths, they are an invariance of the equations of motions and not symmetries of the action. Notice that duality transforms electric and magnetic charges, thus relating among them different configurations. On the other hand, the metric $g_{\mu\nu}$ remains a solution of the Einstein equations 2.1.5, in the new frame.

The Lagrangian will not be invariant but transforms in a specific way, that we will analyze in generalizations of Maxwell duality.

Duality invariance in a theory of vector fields

It is possible to extend duality invariance to the case of a theory of n interacting vector fields, coupled to other fields χ^i, both fermionic and bosonic, described by a Lagrangian of the form

$$\mathcal{L} = \mathcal{L}(F^a, \chi^i, \chi^i_\mu) \, , \tag{2.1.12}$$

where F^a, $(a = 1, ..., n)$ are abelian vector field strengths

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \tag{2.1.13}$$

and $\chi^i_\mu \equiv \partial_\mu \chi^i$. We define a *dual* electromagnetic curvature

$$\tilde{G}^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} G^{a\rho\sigma} \equiv 2\frac{\partial L}{\partial F^{a\,\mu\nu}} \ , \tag{2.1.14}$$

so that the equations of motion derived from (2.1.12) can be simply written as

$$\partial^\mu \tilde{G}^a_{\mu\nu} = 0 \ , \tag{2.1.15}$$

while Bianchi identities still hold in the form

$$\partial_\mu F^{a\,\mu\nu} = 0 \tag{2.1.16}$$

Since these equations are linear in the fields strength F, G, the infinitesimal transformation that leave these equations and (2.1.15) invariants, must act on the fields as

$$\delta \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} , \qquad (2.1.17)$$
$$\delta\chi^i = \xi^i(\chi) ,$$
$$\delta(\partial_\mu \chi^i) = \partial_\mu \xi^i = \partial_\mu \chi^j \frac{\partial \xi^i}{\partial \chi^j} ,$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is an arbitrary real $2n \times 2n$ matrix, and the functions $\xi^i(\chi)$ do not contain derivatives of the fields. We define the *duality group* as the one that acts linearly on the vectors of the field strengths and their duals, not affecting the dynamical equations of the theory; their covariance, indeed, put constraints on the possible duality transformations among the general linear ones.

Constraining the duality group

Given the above transformation, the generic variation of a Lagrangian of the form (2.1.12) is

$$\delta\mathcal{L} = \left[\xi^i \frac{\partial \mathcal{L}}{\partial \chi^j} + \chi^j_\mu \frac{\partial}{\partial \chi^i_\mu} + (F^c A^{bc} + G^c B^{bc}) \frac{\partial}{\partial F^b} \right] \mathcal{L} \, ; \qquad (2.1.18)$$

differentiating again with respect to F^a and using again the transformations in (2.1.17) gives

$$2 \frac{\partial}{\partial F^a} \delta\mathcal{L} = \frac{1}{2} \frac{\partial}{\partial F^a} (F C \tilde{F} + G B^T \tilde{G}) + 2(D^{ab} + A^{ba}) \frac{\partial \mathcal{L}}{\partial F^b}$$
$$+ \frac{1}{2} \left[(C^{ab} - C^{ba}) \tilde{F}^b + \frac{\partial G^c}{\partial F^a} (B^{bc} - B^{cb}) \tilde{G}^b \right] . \qquad (2.1.19)$$

The requirement that the r.h.s. is a derivative with respect to F^a gives

$$C = C^T \, , \qquad B = B^T \, , \qquad D^{ab} + A^{ba} = \eta \delta^{ab} \, , \qquad (2.1.20)$$

thus the Lagrangian must satisfy

$$\frac{\partial}{\partial F^a} \delta\mathcal{L} = \frac{\partial}{\partial F^a} \left(\frac{1}{4} F C \tilde{F} + \frac{1}{4} G B^T \tilde{G} + \eta \mathcal{L} \right) . \qquad (2.1.21)$$

Moreover, the covariance of the equations of motion for lower spin fields χ^i yields the condition

$$\left(\frac{\partial}{\partial \chi^i} - \partial_\mu \frac{\partial}{\partial \chi^i_\mu} \right) (\delta\mathcal{L} - \frac{1}{4} G B \tilde{G}) = 0 \, , \qquad (2.1.22)$$

which is consistent with (2.1.21) for $\eta = 0$, $D = -A^T$. This finally restricts the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ to be an element of $Sp(2n, \mathbb{R})$. We find the variation of the Lagrangian

$$\delta L = \frac{1}{4}(FC\tilde{F} + GB\tilde{G}) \; . \qquad (2.1.23)$$

Notice that this is not a total derivative, G being a curl only because of the equations of motion (2.1.15).

2.1.1 Construction of the Lagrangian

From the variation (2.1.23), given by the transformation (2.1.17), satisfying the constraints (2.1.20), we can write more simply $\delta\mathcal{L} = \frac{1}{4}\delta(F\tilde{G})$. We begin to write the functional as

$$L = \frac{1}{4}F\tilde{G} + \mathcal{L}_{inv}(F^a, \chi^i, \chi^i_\mu) \; ,$$

where $\mathcal{L}_{inv}$ is written as a function of invariants of the duality group. But in the general case where this group is $Sp(2n, R)$ and the field strengths $\begin{pmatrix} F \\ G \end{pmatrix}$ transform as a vector in the fundamental representation, the only possible invariant coupling of F and G to the fields ξ^s is built out of two Lorentz invariant tensors

$$(H_{\mu\nu}(\chi), I_{\mu\nu}(\chi)) \; , \qquad (2.1.24)$$

transforming as the vector (F, G) under duality. Then the Lagrangian whose equations of motions are invariant under duality has the form

$$\mathcal{L}_{inv.}(F, G, \chi^i, \chi^i_\mu) = \frac{1}{4}(FI - GH) + \mathcal{L}_{inv.}(\chi^i, \chi^i_\mu) \; ,$$

where $L_{inv.}$ is now an invariant functional of the χ^i fields only, so that it does not affect the equations of motion, and I, H form a vector in the fundamental representation of the Symplectic group.

By definition one has $\frac{\delta\mathcal{L}}{\delta F} = \frac{1}{2}\tilde{G}$, and this is actually a constraint on I and H

$$\tilde{G} - I = (F + \tilde{H})\frac{\partial\tilde{G}}{\partial F} \; . \qquad (2.1.25)$$

the operator j introduced in the previous section, giving a field strength $T_{\mu\nu}$, satisfies

$$j\, T_{\mu\nu} = \tilde{T}_{\mu\nu} \; ,$$
$$(j)^2 = -1 \; .$$

We can write (2.1.25) as

$$j\,G - I = (F + jH)\frac{\partial \tilde{G}}{\partial F} \ .$$

whose general solution is

$$\begin{aligned} jG - I &= -K(\chi)(F + jH) \ ; \\ &\Downarrow \\ jG &= I - K(\chi)(F + jH) \ . \end{aligned} \tag{2.1.26}$$

Thus the effect of an infinitesimal duality transformation of $Sp(2n, R)$, (2.1.17), is determined by the tranformations on (F, G) and (H, I) the vectors of the fundamental representation. We find

$$\delta K(\chi) = -jC - jKBK + DK - KA \ , \tag{2.1.27}$$

which restricts the form of the Lagrangian to

$$\mathcal{L} = -\frac{1}{4}FKF + \frac{1}{2}F(I - jKH) + \frac{1}{4}jH(I - jKH) + \mathcal{L}_{inv.}(\chi) \ . \tag{2.1.28}$$

Compact Duality Rotations

The case $K(\chi) = 1$ implies $\delta K = 0$. From (2.1.27) the constraints on the coefficients of the duality rotation are

$$B = -C = B^T \ , \qquad A = D = -A^T \ ,$$

which restrict the duality group to the maximal compact subgroup $U(n) \subset Sp(2n, R)$. This appears even more manifest in a complex basis of the fundamental representation, namely using the self and anti-self dual vectors

$$\begin{aligned} F^+ &\equiv F + iG \ , \\ F^- &\equiv F - iG \ , \end{aligned}$$

which allow to write (2.1.17) in the form

$$\delta \begin{pmatrix} F^+ \\ F^- \end{pmatrix} = \begin{pmatrix} T & 0 \\ 0 & T^* \end{pmatrix} \begin{pmatrix} F^+ \\ F^- \end{pmatrix} \ , \tag{2.1.29}$$

with $T = A - iB = -T^\dagger$. The complex basis which uses F^+ and F^- allows a symple physical interpretation: spin-1 fields of opposite helicity transform according to conjugate representations of the duality group, just as massless fermions do under chiral transformations.

It is easy to give a formulation that easily generalizes to the non compact case. The role of the imaginary complex element i in the definitions of the complex basis (2.1.29) is played in general by the "duality" operator

$$F \pm iG \to F \pm jG \ .$$

With the replacement $i \to j$, and the block-diagonal T-matrix becomes

$$Re\ T \pm i\, Im\, T \to Re\ T \pm j\ Im\ T \ .$$

Also, defining a complexified coupling $jH_\pm \equiv (H \pm jI)$, functions of the fields χ, the Lagrangian (2.1.28) can be written as

$$\mathcal{L} = -\frac{1}{4}F^2 + \frac{1}{2}FH_+ - \frac{1}{8}H_+^2 - \frac{1}{8}H_+H_- + \mathcal{L}_{inv.}(\chi) \ .$$

The field H_- has no dynamical meaning, since it does not appear in any of the couplings of F, and can be set to zero, meaning it is reabsorbed in $\mathcal{L}_{inv.}(\chi)$. Then, for $H = jI$, from (2.1.26) it follows that

$$I = (1 + K(\chi))^{-1} \left[K(\chi)F^2 + G^2 + FjG - K(\chi)GjF\right] \ . \tag{2.1.30}$$

Notice that, in this way, in the compact $K(\chi) = 1$ case, the invariant bilinear $FI - GH$ is

$$(FI - GH) = = \frac{1}{2}(F^2 + G^2) = \frac{1}{2}(F - iG)(F + iG) \ ,$$

which is manifestly invariant under linear unitary transformations among F and G.

Non-compact transformations and non-linear realizations on scalars

By now we have all the ingredients to describe the theory of interacting fields with invariance under a compact subgroup of $Sp(2n, \mathbb{R})$, but we need to generalize the description to non compact duality groups. The solution is to introduce in the theory scalar fields described by a nonlinear sigma model, taking values in the quotient space of group $\mathcal{G}$ with respect to its maximal compact subgroup $\mathcal{K}$, being the semisimple group $\mathcal{G}$ the duality group.

The scalars are described by a group element $g(x) \in \mathcal{G}$, in some representation of the duality group, but two elements are equivalent if they differ by right-action of the maximal compact subgroup $\mathcal{H}$ of $\mathcal{G}$. This equivalence, and thus the coset space structure, is implemented by the requirement that the Lagrangian is invariant under gauge transformations that can be written as

$$g(x) \to g(x)[k(x)]^{-1} \ , \tag{2.1.31}$$

together with rigid transformations

$$g(x) \to g_0 g(x) \ , \tag{2.1.32}$$

with $g_0 \in \mathcal{G}$. The covariant derivative is built from the $\mathcal{H}$ group connection Q_μ as $D_\mu g = \partial_\mu g - g Q_\mu$. Notice that $g^{-1} D_\mu g$ is invariant under the global transformation (2.1.32), thus the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\, Tr \left(g^{-1} D_\mu g\right)^2 \ ,$$

is invariant under both gauge and rigid transformations on $\mathcal{G}$. Defining $P_\mu = g^{-1} D_\mu g$, the equations of motions for the scalar of the non linear σ-model can be written as

$$D_\mu P_\mu \equiv \partial_\mu P_\mu - [P_\mu, Q_\mu] = 0 \ . \tag{2.1.33}$$

Given the structure of non the linear σ-model, it is possible to solve (2.1.26) for $\mathcal{G}$ non compact. The scalars of the coset can be represented by an $Sp(2n, \mathbb{R})$ matrix (symplectic embedding), which is easily expressed in a complex basis as[1]

$$g = \begin{pmatrix} \phi_0 & \phi_1^* \\ \phi_1 & \phi_0^* \end{pmatrix} \ , \tag{2.1.34}$$

where ϕ_0 and ϕ_1 are $n \times n$ matrices satisfying

$$\phi_0^\dagger \phi_0 - \phi_1^\dagger \phi^1 = 1 \ ; \tag{2.1.35}$$

scalar fields transform under the action of $Sp(2n, \mathbb{R})$ as

$$\delta g = \begin{pmatrix} T & V^* \\ V & T^* \end{pmatrix} g \ , \tag{2.1.36}$$

T and V are related to the block elements of the transformation matrix in (2.1.17) by

$$\begin{aligned} T &= \frac{A - iB}{2} + \frac{D + iC}{2} \\ V &= \frac{A - iB}{2} - \frac{D + iC}{2} \ . \end{aligned} \tag{2.1.37}$$

[1]For the derivation of complex coset representatives see the Appendix A of [14].

The solution to the transformation law of K in (2.1.27) is

$$\delta K = (\phi_0^\dagger + \phi_1^\dagger)^{-1}(\phi_0^\dagger - \phi_1^\dagger) \ , \tag{2.1.38}$$

where again complex numbers have to be reinterpreted as having i replaced by j, so for instance

$$\phi_i = \mathrm{Re}\phi_i + j\mathrm{Im}\phi_i \ . \tag{2.1.39}$$

Notice that in case one chooses the particular choice of gauge

$$g = \begin{pmatrix} \phi_0 & \phi_1^* \\ \phi_1 & \phi_0^* \end{pmatrix} = \exp\begin{pmatrix} 0 & P^* \\ P & 0 \end{pmatrix} \ , \tag{2.1.40}$$

because of the symmetry of the noncompact generators P, the matrices ϕ_0, ϕ_1 satisfy

$$\phi_0 = \phi_0^\dagger \ , \qquad \phi_1 = \phi_1^T \ . \tag{2.1.41}$$

A remark is in order. The conserved currents associated with the noncompact generators can be constructed only when the interactions with the scalar fields is present. It is possible to follow what happens in the decoupling limit, simply by dimensional analysis of the Lagrangian terms. Interestingly, in this limit, the noncompact part of the duality group becomes abelian, corresponding to a contraction of the original group to the $U(n)$-scalar free case, thus the compact case can be recovered smoothly from the noncompact construction.

Moreover, the structure underlying duality invariance presented above holds for generic theories. There is indeed an unspecified $\mathcal{L}_{inv}$ and an antisymmetric tensor which couples to fermions in the form $H_{\mu\nu}(\psi)$ which are completely free, up to now. In supergravity theories, these quantities, in fact the field content itself, are fixed by supersymmetry.

2.2 Duality rotations and covariance for the supergravity action in $d = 4$

As stated at the beginning of the chapter, any N-extended Supergravity theory in $d = 4$ has a bosonic sector described by the action

$$\mathcal{S} = \int \sqrt{-g}\, d^4x \left(-\frac{1}{2} R + \mathrm{Im}\mathcal{N}_{\Lambda\Gamma} F^\Lambda_{\mu\nu} F^{\Gamma,\ \mu\nu} + \frac{1}{2\sqrt{-g}} \mathrm{Re}\mathcal{N}_{\Lambda\Gamma} \epsilon^{\mu\nu\rho\sigma} F^\Lambda_{\mu\nu} F^\Gamma_{\rho\sigma} + \right.$$
$$\left. + \frac{1}{2} g^s c_{IJ}(\phi) \partial_\mu \phi^I \partial^\mu \phi^J \right) \ . \tag{2.2.1}$$

The matrix $\mathcal{N}_{\Lambda\Sigma}(\Phi)$ is a symmetric matrix $n \times n$, with n number of vector fields, depending on their representation of Gaillard−Zumino Symplectic group. Different Supergravity theories thus have different scalar manifolds and number of vector multiplets, and, since for $\mathcal{N} \geq 2$ vector multiplets contain scalar fields, the action of the vector isometry group $\mathcal{M}_{scalar}$ is deeply connected to the transformation of scalars. This results in the embedding of the isometry group in the duality group, whose explicit form relies on the specific Supergravity theory we are considering. Once we have this correspondence, we find the matrix $\mathcal{N}$ in its explicit form.

It is crucial, then, to study duality transformations in details, in the form of a linear action on the (abelian) vector field strengths and their dual forms. As stressed before, these transformations leave Bianchi Identities and equations of motions invariant, and generalize electromagnetic duality. For the purposes of studying black hole configurations, it is important to underline that $g_{\mu\nu}$, the four dimensional space-time metric, does not transform under duality. This means, in particular, that when $g_{\mu\nu}$ is a black hole metric, duality transformations will map black holes into other black holes. More generally, any solution for $g_{\mu\nu}$ depends on scalar fields and charges in a symplectic invariant way.

In what follows we will see the Gaillard−Zumino construction at work in the Supergravity framework.

Duality rotations and symplectic covariance.

We deal with a theory of vectors and scalar fields which is invariant under the action of a duality group, in $d = 4$. The gauge fields are n_V abelian fields A^Λ_μ, whose dynamic is described by the field strengths in the action (2.2.1). We can separately write the dual and anti-dual field strength

$$\begin{aligned} F^\pm &= \frac{1}{2}(F \pm i\,{}^\star F) \ , \\ {}^\star F^\pm &= \mp i F^\pm \ , \end{aligned} \tag{2.2.2}$$

and rewrite the vector part of the action as

$$\begin{aligned} \mathcal{L}_{vec} &= i \ \left[F^{-,T}\bar{\mathcal{N}}F^- - F^{+,T}\mathcal{N}F^+\right] = \\ &= -i \begin{pmatrix} F^{+\ T} \ , \ F^{-\ T} \end{pmatrix} \begin{pmatrix} \mathcal{N} & 0 \\ 0 & -\bar{\mathcal{N}} \end{pmatrix} \begin{pmatrix} F^+ \\ F^- \end{pmatrix} \ . \end{aligned} \tag{2.2.3}$$

Following the Gaillard$-$Zumino construction we introduce the tensor $G^{\Lambda}_{\mu\nu}$ defined as

$$ {}^{\star}G^{\Lambda}_{\mu\nu} \equiv \frac{1}{2}\frac{\partial \mathcal{L}}{\partial F^{\Lambda}_{\mu\nu}} \ , \tag{2.2.4} $$

that, for the theory under examination, is

$$ {}^{\star}G_{\Lambda\,\mu\nu} = \mathrm{Im}\ \mathcal{N}_{\Lambda\Sigma}\ F^{\Sigma}_{\mu\nu} + \mathrm{Re}\mathcal{N}_{\Lambda\Sigma}\,{}^{\star}F^{\Sigma}_{\mu\nu} \ . \tag{2.2.5} $$

The equations of motion and Bianchi identities are

$$ \begin{cases} \nabla^{\mu\star}F^{\Lambda}_{\mu\nu} = 0 \ , \\ \nabla^{\mu\star}G_{\Lambda\,\mu\nu} = 0 \ , \end{cases} \\ \Downarrow \\ \begin{cases} \nabla^{\mu}\mathrm{Im}F^{\pm\ \Lambda} = 0 \ , \\ \nabla^{\mu}\mathrm{Im}G^{\pm}_{\Lambda\,\mu\nu} = 0 \ , \end{cases} \tag{2.2.6} $$

where we also write $G_{\Lambda\,\mu\nu}$ separating its self-dual and anti self-dual part

$$ \begin{aligned} G^{\pm} &= \frac{1}{2}(G \pm i\,{}^{\star}G) \ , \\ {}^{\star}G^{\pm} &= \mp i G^{\pm} \ , \end{aligned} \tag{2.2.7} $$

whose relation on the field strength F is given by

$$ \begin{aligned} G^{+} &= \mathcal{N}F^{+} \ , \\ G^{-} &= \bar{\mathcal{N}}F^{-} \ . \end{aligned} \tag{2.2.8} $$

The vector part of the Lagrangian, if written in terms of F and G as in (2.2.4), takes the compact form

$$ \begin{aligned} \mathcal{L}_{vec} &= i\left[F^{-\,T}G^{-} - F^{+\,T}G^{+}\right] = \\ &= -i\begin{pmatrix} F^{+\ T} \ , \ F^{-\ T} \end{pmatrix}\begin{pmatrix} G^{+} \\ G^{-} \end{pmatrix} \ . \end{aligned} \tag{2.2.9} $$

Moreover, we introduce the $n+n$ components vector of 2-forms

$$ \mathbf{V} \equiv \begin{pmatrix} {}^{\star}F \\ {}^{\star}G \end{pmatrix} \ , $$

and we get equations of motion, from the variation of the vector fields, in the form

$$ \mathrm{d}\,\mathbf{V} = 0 \ , \tag{2.2.10} $$

considering differentiation with respect to space-time coordinates. Duality transformations are then simply described by

$$\mathbf{V}' = \mathcal{S}\,\mathbf{V}\ , \qquad \mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\ . \tag{2.2.11}$$

$\mathcal{S}$ is a priori a matrix in $GL(2n_v, \mathbb{R})$, and, since we always required duality invariance during the construction, the equations of motion for the vector $\mathbf{V}'$ are still given by $\mathrm{d}\mathbf{V}' = 0$.

Restriction to symplectic group from the scalar sector

While the duality rotation (2.2.11) acts on field strengths and corresponding duals, also the scalar fields are subject to the action of some diffeomorphism $\xi \in \mathrm{Diff}(\mathcal{M}_{scalar})$ of the scalar manifold, transforming the matrix of couplings $\mathcal{N}_{\Lambda\Sigma}$. We thus assume that, given $\mathcal{M}_{scalar}$ the manifold of the nonlinear sigma model, there exists a homomorphism of the form

$$\iota_\delta :\ \mathrm{Diff}\left(\mathcal{M}_{scalar}\right) \longrightarrow GL(2n, \mathbb{R}) \tag{2.2.12}$$

so that

$$\begin{aligned} &\forall \quad \xi \quad \in \mathrm{Diff}\left(\mathcal{M}_{scalar}\right)\ :\ \phi^I \xrightarrow{\ \xi\ } \phi^{I\prime} \\ &\exists\ \iota_\delta(\xi) = \begin{pmatrix} A_\xi & B_\xi \\ C_\xi & D_\xi \end{pmatrix} \in GL(2n, \mathbb{R})\ . \end{aligned} \tag{2.2.13}$$

The action of this homomorphism describes the transformation of all the fields, that can be expressed as

$$\xi\ :\ \begin{cases} \phi \longrightarrow \xi(\phi) \\ \mathbf{V} \longrightarrow \iota_\delta(\xi)\,\mathbf{V} \\ \mathcal{N}(\phi) \longrightarrow \mathcal{N}'(\xi(\phi)) \end{cases} \tag{2.2.14}$$

In particular, the transformation of the Lagrangian is

$$\mathcal{L}'_{vec} = \mathrm{i}\left[\mathcal{F}^{-T}\left(A+B\bar{\mathcal{N}}\right)^T\bar{\mathcal{N}}'(A+B\bar{\mathcal{N}})\mathcal{F}^- - \mathcal{F}^{+T}\left(A+B\mathcal{N}\right)^T\mathcal{N}'(A+B\mathcal{N})\mathcal{F}^+\right] \tag{2.2.15}$$

Consistency with the definition of G^+ requires that the matrix $\mathcal{N}_{\Lambda\Sigma}$ transforms as

$$\mathcal{N}' \equiv \mathcal{N}'(\xi(\phi)) = \left(C + D\mathcal{N}(\phi)\right)\left(A + B\mathcal{N}(\phi)\right)^{-1} \tag{2.2.16}$$

while consistency with the definition of G^- imposes the analogue transformation on the complex conjugate symplectic matrix

$$\bar{\mathcal{N}}' \equiv \bar{\mathcal{N}}'(\xi(\phi)) = \left(C + D\bar{\mathcal{N}}(\phi)\right)\left(A + B\bar{\mathcal{N}}(\phi)\right)^{-1} \tag{2.2.17}$$

Finally, by requiring that the transformed matrix $\mathcal{N}'$ be again symmetric it simply follows that the matrix $\Lambda \equiv \iota_\delta(\xi)$ must obey

$$\Lambda^T \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \Lambda = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} , \tag{2.2.18}$$

that is $\Lambda \in Sp(2n, \mathbb{R})$. Consequently, the homomorphism of eq. 2.2.12 becomes

$$\iota_\delta : \text{Diff}\,(\mathcal{M}_{scalar}) \longrightarrow Sp(2n, \mathbb{R}) \tag{2.2.19}$$

Clearly, since $Sp(2n, \mathbb{R})$ is a finite dimensional Lie group, while Diff $(\mathcal{M}_{scalar})$ is infinite–dimensional, the homomorphism ι_δ can never be an isomorphism.

Also notice that the transformation Λ is not a symmetry of the action, unless for a very restricted subset among the matrices with $C = B = 0$. However, the diffeomorphism ξ, defined by the correspondence $\Lambda = \iota_\delta(\xi)$, is a diffeomorphisms on the scalar manifold for which the Lagrangian is invariant, since it is just an isometry of the scalar manifold metric g^{sc}_{IJ}. In fact, if $\xi^* : T\mathcal{M}_{sc} \to T\mathcal{M}_{sc}$ is the push-forward of the diffeomorphism ξ, then $\forall X, Y \in T\mathcal{M}$, $g^{sc}(X, Y) = g^{sc}(\xi^* X, \xi * Y)$, meaning that ξ is an exact global symmetry of the scalar part of the Lagrangian (2.2.1). In connection to the previous construction of duality rotations, it is important to stress that these symmetries of the scalar sector are not guaranteed to admit an extension to symmetries of the whole action, but they can instead be extended to symmetries of the field equations of motion and Bianchi identities, that is to duality symmetries, as defined in the first part of this Chapter. To achieve this, the group of isometries of the scalar metric $\mathcal{I}(\mathcal{M}_{scalar})$ needs to be suitably embedded in the duality group $Sp(2n, \mathbb{R})$ while the matrix $\mathcal{N}_{\Lambda\Sigma}$ needs to be a scalar under transformations of $\mathcal{M}_{scalar}$ coordinates.

The description of this embedding, and the properties of $\mathcal{N} \geq 2$ supergravity theories, will be then the subject of the next Chapter.

Chapter 3

Symplectic structure of extended Supergravity

The aim of this Chapter is to provide the geometric formulation which describes extended Supergravities in four dimensions in presence of electric and magnetic sources, keeping manifest the underlying duality symmetries of the theory. We will emphasize the symplectic structure of $\mathcal{N} \geq 2$ extended Supergravity, focusing on the $\mathcal{N} = 8$ case. This can be seen as a consequence of the existence of a flat symplectic bundle on the scalar manifold. We then discuss, in the context of $\mathcal{N} = 2$, the modifications induced by the gauging of a subgroup of scalar manifold isometries, exploiting the notion of *momentum map.*

3.1 $\mathcal{N} = 2$ Supergravity and special geometry

Four dimensional $\mathcal{N} = 2$ extended Supergravity is particularly interesting due to its interpretation in connection to string theory, because it can be realized as the low energy effective theory of string compactifications on Calabi−Yau manifolds. The theory contains the gravity multiplet, and can be coupled to vector and hyper- matter multiplets

$$\begin{pmatrix} g_{\mu\nu} \\ \psi^A_\mu \\ A^0_\mu \end{pmatrix}, \qquad \begin{pmatrix} A^i_\mu \\ \lambda^i_A \\ z^i \end{pmatrix}, \qquad \begin{pmatrix} \zeta^\alpha \\ q^u \end{pmatrix}, \tag{3.1.1}$$

$$\text{gravity}, \qquad n_V \ \text{vector's} \qquad n_H \ \text{hyper's}$$

Since in $\mathcal{N} = 2$ the gravity multiplet contains the graviphoton, the total number of abelian vector fields for a given (ungauged) supergravity theory will be $n = n_V + 1$. Scalars are present both in the vector and in the hypervector sector, and in this case the non-linear σ model, defined by the Lagrangian kinetic terms of the scalars, is a tensor product of a Special Kähler with a Quaternionic Kähler manifold

$$\mathcal{M} = \mathcal{M}_{SK} \otimes \mathcal{M}_{QK} \; . \tag{3.1.2}$$

The scalars present in the vector multiplets are coordinates of $\mathcal{M}_{SK}$, while the hyperscalars parametrize $\mathcal{M}_{QK}$. This is a consequence of the factorized action of the R-symmetry group on the scalar manifold, namely $U(2) = U(1) \times SU(2)$. As it is shown in the action (2.2.1), supersymmetry of the action requires that the abelian field strength couple to the scalars through the matrix $\mathcal{N}_{\Lambda\Sigma}$, which is built from geometrical quantities determined by the particular SK manifold of the theory under consideration. In particular, they do not depend on hyperscalars. This will become important when dealing with black holes solutions, because we will look for configurations where hyperscalars are consistently set to zero.

Given any $\mathcal{N} = 2$ Supergravity, scalars of the vector multiplets span the complex $\mathcal{M}_{SK}$. Special Kähler means that the manifold $\mathcal{M}^{SK}$ is a Kähler−Hodge manifold endowed with an extra symplectic structure, where a Kähler manifold $\mathcal{M}$ is a Hodge manifold if and only if there exists a $U(1)$ bundle $\mathcal{L} \longrightarrow \mathcal{M}$ such that its first Chern class equals the cohomology class of the Kähler 2-form K:

$$c_1(\mathcal{L}) \; = \; [\, K \,] \; . \tag{3.1.3}$$

For local coordinates $z^i, \bar{z}^{\bar{\imath}}$, we can write

$$K \; = \; \mathrm{i}\, g_{i\bar{\jmath}}\, dz^i \wedge d\bar{z}^{\bar{\jmath}} \, , \tag{3.1.4}$$

where z^i are the $n = n_V + 1$ holomorphic coordinates on $\mathcal{M}^{SK}$, the scalar fields in the vector multiplets, and $g_{i\bar{\jmath}}$ its metric.
In this case the $U(1)$ Kähler connection is given by:

$$\mathcal{Q} = -\frac{\mathrm{i}}{2} \left(\partial_i \mathcal{K} dz^i - \partial_{\bar{\imath}} \mathcal{K} d\bar{z}^{\bar{\imath}} \right) \, , \tag{3.1.5}$$

where $\mathcal{K}$ is the Kähler potential, so that $K = d\mathcal{Q}$.

A Special Kähler manifold is not necessarily a coset manifold. It is possible, however, to exploit the symplectic covariant construction of Gaillard and Zumino also in the case of $\mathcal{N} = 2$ Supergravity.

3.1.1 Special geometry in Supergravity

Duality rotations and symplectic structure act already on charged BPS states of $\mathcal{N}=2$ rigid supersymmetric Yang-Mills theory [16]. When we couple it to gravity, the moduli space of $\mathcal{N}=2$ Supergravity acquires a Kähler−Hodge structure over the symplectic manifold, introducing a $U(1)$ connection that modifies the constraints of special geometry of the rigid formulation [17]. One striking difference is that, in presence of coupling to gravity, the prepotential of the theory may not exist. It is possible, however, to give a prepotential-independent formulation of special geometry in the Supergravity case.

Suppose the theory is coupled to n_V vector multiplets, then the total number of vectors, including the graviphton, is $n = n_V + 1$ and the scalar fields of the vector multiplets parametrize $G/H \in Sp(2n, \mathbb{R})$. Consider the $2n$ sections

$$V = (L^\Lambda, M_\Lambda) \ , \quad \Lambda = 0, 1, .., n_V \ . \tag{3.1.6}$$

The local Special Kähler geometry is defined by the following relations, which define a flat connection on the symplectic bundle,

$$\begin{aligned} U_i &= (\mathcal{D}_i L^\Lambda, \mathcal{D}_i M_\Lambda) \equiv (f_i^\Lambda, h_{\Lambda i}), \\ \mathcal{D}_i U_j &= i C_{ijk} g^{k\bar{l}} \overline{U}_{\bar{l}} \ , \\ \mathcal{D}_i \overline{U}_{\bar{\jmath}} &= g_{i\bar{\jmath}} \overline{V} \ , \\ \mathcal{D}_i \overline{V} &= 0 \ . \end{aligned} \tag{3.1.7}$$

Latin indices run over $1, 2, ..., n$, and $\mathcal{D}_i$ is the covariant derivative with respect to the Levi-Civita connection and the Kähler connection $\mathcal{Q}$, meaning that under a Kähler transformation given by $K \to K + f + \overline{f}$, a section over the $U(1)$ line bundle transforms accordingly as

$$\psi^i \to \exp^{-\frac{1}{2}(2pf + \bar{p}\bar{f})} \psi^i \ . \tag{3.1.8}$$

Its covariant derivative is then

$$\mathcal{D}_i \psi^j = \partial_i \psi^j + \Gamma^i_{jk} \psi^k + \frac{p}{2} \partial_i K \psi^j \ , \tag{3.1.9}$$

and analogously for $\overline{\mathcal{D}}_{\bar{\imath}}$, with the substitution $p \to \bar{p}$. $(p, \bar{p})$ are the Kähler weights of the line bundle section ψ.

In the rest of this section we introduce relations and formulae of Special Geometry that will be needed to construct solutions in $\mathcal{N}=2$ theories, and to analyze their supersymmetric properties.

We start with the coset parametrization. Symplectic sections in (3.1.6) have Kähler weights $(1,-1)$, conventionally, and they are are covariantly holomorphic, in the sense that $\overline{\mathcal{D}}_{\bar{i}}V = 0$. They are normalized as

$$i(\overline{L}^\Lambda M_\Lambda - L^\Lambda \overline{M}_\Lambda) = i\langle V, \overline{V}\rangle = 1 \; . \tag{3.1.10}$$

It is then convenient to introduce holomorphic sections $X^\Lambda = e^{-K/2}L^\Lambda$, $F_\Lambda = e^{-K/2}M_\Lambda$, satisfying by construction

$$\mathcal{K} = -\log\left[i(\overline{X}^\Lambda F_\Lambda - X^\Lambda \overline{F}_\Lambda)\right] \; . \tag{3.1.11}$$

In all cases when the integrability constraints[1] (3.1.7) can be solved in terms of a pre-potential, that is a holomorphic function homogeneous of degree two $F(X^\Lambda)$, then the holomorphic sections are determined by

$$F_\Lambda = \frac{\partial}{\partial X^\Lambda} F(X^\Sigma) \; , \tag{3.1.13}$$

while in general it holds that

$$M_\Lambda = \mathcal{N}_{\Lambda\Sigma} L^\Sigma \qquad h_{\Lambda i} = \overline{\mathcal{N}}_{\Lambda\Sigma} f_i^\Sigma \; ; \tag{3.1.14}$$

introducing the two matrices

$$h_{\Lambda I} = (h_{\Lambda 0}, h_{\Lambda i}) \; , \quad f_I^\Lambda = (f_0^\Lambda, f_i^\Lambda) \; , \quad h_{\Lambda 0} \equiv M_\Lambda \quad f_0^\Lambda \equiv L^\Lambda \; , \tag{3.1.15}$$

the symplectic matrix $\mathcal{N}_{\Lambda\Sigma}$ giving the couplings between scalars and gauge fields is determined explicitly in terms of (L^Λ, M_Λ) as

$$\mathcal{N}_{\Lambda\Sigma} = h_{\Lambda I}(f^{-1})^I_\Sigma \; . \tag{3.1.16}$$

The action of the duality group is the same in the rigid and in the local case and acts as

$$\begin{pmatrix} L^\Lambda \\ M_\Lambda \end{pmatrix} \to \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} L^\Lambda \\ M_\Lambda \end{pmatrix} \; , \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathcal{S} \in Sp(2n, \mathbb{R}) \; , \tag{3.1.17}$$

[1]In general, from the same integrability conditions for a special Kähler manifold, the curvature of the manifold is given by

$$R_{i\bar{j}k\bar{l}} = g_{i\bar{j}}g_{l\bar{k}} + g_{i\bar{k}}g_{l\bar{j}} - C_{ilq}C_{\bar{j}\bar{k}\bar{q}}g^{q\bar{q}} \; , \tag{3.1.12}$$

which is determined by the covariantly holomorphic $(2,-2)$ tensor C_{ijk} of (3.1.7).

so the vector of symplectic section transforms as the symplectic vector of (anti)self dual field strengths $(\mathcal{F}^{-\Lambda}_{\mu\nu},\ G^{\mu\nu}_{-\Lambda})$, with $\mathcal{F}^{-\Lambda}_{\mu\nu}=\mathcal{F}^{\Lambda}_{\mu\nu}+\frac{i}{2}\epsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\Lambda}_{\rho\sigma}$, and the dual field strength $G^{\mu\nu}_{-\Lambda}=i\delta\mathcal{L}/\delta\mathcal{F}^{-\Lambda}_{\mu\nu}$ for the bosonic Supergravity action is given by

$$G_{\Lambda\,\mu\nu}=\mathrm{Re}\mathcal{N}_{\Lambda\Sigma}\,F^{\Sigma}_{\mu\nu}-\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\,{}^{\star}F^{\Sigma}_{\mu\nu}\ . \tag{3.1.18}$$

The symplectic embedding describing the isomorphism

$$\iota : Diff(\mathcal{M}_{scalar})\to Sp(2n,\mathbb{R}) \tag{3.1.19}$$

for $\mathcal{N}\geq 2$ theories is simply stated in terms of the sections (f,h) defined above, as

$$\begin{pmatrix} f \\ h \end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix} A-iB \\ C-iD \end{pmatrix}\ ,\qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix}\in Sp(2n,\mathbb{R}) \tag{3.1.20}$$

implying on the sections the following normalization relations

$$\begin{aligned} i(f^{\dagger}h-h^{\dagger}f) &= 1\ ,\\ f^{t}h-h^{t}f &= 0\ . \end{aligned} \tag{3.1.21}$$

In this thesis we will study particular solutions to Supergravity equations, namely those in which the metric field describes a static charged black hole. The black hole is a dyonic state whose charges are defined as the fluxes of electric-and magnetic field strengths on a 2-sphere at spatial infinity

$$p^{\Lambda}=\frac{1}{4\pi}\int_{S^2}F^{\Lambda}\ ,\qquad q_{\Lambda}=\frac{1}{4\pi}\int_{S^2}G_{\Lambda}\ . \tag{3.1.22}$$

In the presence of scalar fields, the physical field strengths (T^{+},T^{-}) are dressed with the scalars, as it results from the supersymmetry variations of the fermions

$$T^{\pm}=h_{\Lambda}F^{\pm\Lambda}-f^{\Lambda}G^{\pm}_{\Lambda}\ . \tag{3.1.23}$$

They satisfy

$$T^{+}=h_{\Lambda}F^{+\Lambda}-f^{\Lambda}G^{+}_{\Lambda}=0 \tag{3.1.24}$$

so that $T=T^{-}$ (and $\bar{T}=\bar{T}^{+}$). The physical charges, given by the central charges and the matter charges, are now defined as the integrals over a S^2 of the physical graviphoton and matter vectors

$$Z=\int_{S^2}T=\int_{S^2}(h_{\Lambda}F^{\Lambda}-f^{\Lambda}G_{\Lambda})=h_{\Lambda}(z,\bar{z})p^{\Lambda}-f^{\Lambda}(z,\bar{z})q_{\Lambda} \tag{3.1.25}$$

with $z^i, \bar{z}^{\bar{\imath}}$ being the v.e.v. of the moduli fields for a given background. Notice that, since $V = (f^\Lambda, h_\Lambda)$, we have that

$$Z_i = h_{\Lambda\,i} p^\Lambda - f^{\Lambda\,i} q_\Lambda = \nabla_i Z \ . \tag{3.1.26}$$

In particular, a BPS solution with $Z_i = \nabla_i Z = 0$ would then have a minimum mass, since it would satisfy

$$\nabla_i Z = 0 \to \partial_i |Z|^2 = 0. \tag{3.1.27}$$

The symplectic structure of the manifold implies, for special geometry, the following sum rules

$$|Z|^2 \pm |Z_i|^2 \equiv |Z|^2 \pm Z_i g^{i\bar{\jmath}} \bar{Z}_{\bar{\jmath}} = -\frac{1}{2} Q^t \mathcal{M}_\pm Q \tag{3.1.28}$$

where

$$\mathcal{M}_+ = \begin{pmatrix} \mathbb{1} & -\mathrm{Re}\mathcal{N} \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathrm{Im}\mathcal{N} & 0 \\ 0 & \mathrm{Im}\mathcal{N}^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ -\mathrm{Re}\mathcal{N} & \mathbb{1} \end{pmatrix} \tag{3.1.29}$$

$$\mathcal{M}_- = \begin{pmatrix} \mathbb{1} & -\mathrm{Re}\mathcal{F} \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathrm{Im}\mathcal{F} & 0 \\ 0 & \mathrm{Im}\mathcal{F}^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ -\mathrm{Re}\mathcal{F} & \mathbb{1} \end{pmatrix} \tag{3.1.30}$$

$$Q = \left(p^\Lambda, q_\Lambda\right) \ . \tag{3.1.31}$$

Notice that the new symplectic metrics $\mathcal{M}_\pm$ are related by the exchange of $\mathcal{N} \leftrightarrow \mathcal{F}$, where $\mathcal{F}$ relies upon the existence of a prepotential F and is $\mathcal{F} = \frac{\partial^2 F}{\partial X^\Lambda \partial X^\Sigma} = F_{\Lambda\Sigma}$. Notice also that, while $\mathrm{Im}\mathcal{N}$ is always negative definite, this does not hold for $\mathrm{Im}\mathcal{F}$ in general.

Discussion on the existence of a prepotential

When a prepotential F exists, the degree-2 homogeneity of F requires that the holomorphic sections X^Λ, F_Λ transform under duality action of the matrix $\mathcal{S}$ like

$$\begin{aligned} \tilde{X}^\Lambda(X) &= (A^\Lambda{}_\Sigma + B^{\Lambda\Delta} F_{\Delta\Sigma}) X^\Sigma \ , \\ \tilde{F}_\Lambda(X) &= (C_{\Lambda\Sigma} + D_\Lambda{}^\Delta F_{\Delta\Sigma}) X^\Sigma \ , \end{aligned} \tag{3.1.32}$$

where $\mathcal{F} = F_{\Lambda\Sigma} = \partial_\Lambda \partial_\Sigma F$. It can be shown that the in new duality frame, a prepotential exists such that

$$\tilde{F}_\Lambda = \frac{\partial F}{\partial X^\Lambda} \ , \tag{3.1.33}$$

whenever the map $X^\Lambda \to \tilde{X}^\Lambda$ is invertible.

When working in Supergravity, special coordinates are defined such that $t^i = X^i/X^0$; assuming that $\mathcal{D}_i(X^\Lambda/X^0)$ is an invertible matrix, then it is possible to choose a frame such that $\partial_i(X^\Lambda/X^0) = \delta_i^\Lambda$. This is only possible if X^Λ are unconstrained (linearly independent) variables, and so $F_\Lambda(X)$. From the relations above follows, however, that whenever the $(n+1)\times(n+1)$matrix $A + B\mathcal{F}$ is not invertible, in the rotated frame the variables $\tilde{X}^\Lambda$ and $\tilde{F}_\Lambda$ still forms a good symplectic section, since the symplectic transformation matrix $\mathcal{S}$ is always invertible, but there will be no function $\tilde{F} = \tilde{F}(\tilde{X})$ such that $\tilde{F}^\Lambda = \partial\tilde{F}(\tilde{X})/\partial\tilde{X}^\Lambda$.

None of the formulae needed in this section to build special geometry use the existence of a prepotential (i.e. the functional dependence of F_Λ on the X^Λ), but all quantities are symplectic invariant or covariant, thus well defined in any duality frame. For example, in order to compute the electric-magnetic coupling matrix $\mathcal{N}$ in the rotated basis, the formula of the transformed matrix

$$\tilde{\mathcal{N}}_{\Lambda\Sigma}(\tilde{X},\tilde{F}) = \mathcal{N}(\tilde{X},\tilde{F}) = (C + D\mathcal{N}(X))(A + B\mathcal{N}(X))^{-1} \tag{3.1.34}$$

it is only needed that $A+B\mathcal{N}$ is invertible, but this is ensured, as in the rigid case, by the conventional negative definiteness of the matrix $\mathrm{Im}\mathcal{N}$. However, since in the local case $A+B\mathcal{N}$ is no more related to the change of coordinate, now being $\partial\tilde{X}/\partial X = (A+B\mathcal{F})$, the existence of a prepotential, as stressed, is no more guaranteed.

This formulation of special geometry, independent from a prepotential, is relevant for Supergravity theories obtained as low energy limits of heterotic String theory, for which a prepotential may not exists (see, for example, the discussion in [17]).

3.2 Symplectic embedding for coset scalar manifolds

We now deal with the cases in which the scalar σ-model is a coset space $\mathcal{G}/\mathcal{H}$, and $\mathcal{N} \geq 2$ arbitrary. $\mathcal{G}$ is a non-compact group acting as an isometry group on the scalar manifold, $\mathcal{H}$ is the isotropy subgroup

$$\mathcal{H} = \mathcal{H}_{Aut} \otimes \mathcal{H}_{matter} \ , \tag{3.2.1}$$

$\mathcal{H}_{Aut}$ is the automorphism group of the supersymmetry algebra, and $\mathcal{H}_{matter}$ is a local gauge invariance on the scalar manifold related to the presence of matter multiplets. Theories with $\mathcal{N} \geq 4$ Supersymmetry and field content of spin at most 2 have, of course, $\mathcal{H}_{matter} = \mathbb{1}$. As discussed from the beginning, the duality group $\mathcal{G}$ acts linearly on the field strengths $F^\Lambda_{\mu\nu}$, where generically Λ is the index spanning the representation of $\mathcal{G}$ in which the vector fields transform. We notice here that, since solutions of Supergravity are a large-charge approximation of the fundamental charged dyons, suitable at low energy with a classical coupling to gravity, the true duality symmetry (U-duality) acts instead on integral quantized electric and magnetic charges, and is nothing but the restriction of $\mathcal{G}$ to the integers, thus the moduli spaces of these theories is $\mathcal{G}(\mathbb{Z})\backslash\mathcal{G}/\mathcal{H}$.

The geometry of the coset manifold $\mathcal{M}_{scalar} = \mathcal{G}/\mathcal{H}$ determines the Supergravity theory. All its properties are indeed fixed in terms of the coset representatives $L(\phi)$, which transform according to

$$L(\phi') = gL(\phi)h(g,\phi) \ , \tag{3.2.2}$$

for change of coordinates $\phi'(\phi)$ on the scalar manifold ($g \in \mathcal{G}$, $h \in \mathcal{H}$). The kinetic and axionic metric $\mathcal{N}_{\Lambda\Sigma}$ for the 2-forms F^Λ are fixed in terms of L and the physical field strengths of the interacting theory are "dressed" with scalar fields in terms of the coset representatives. In this way, central charges associated to the 1-forms in the gravitational multiplet are determined by the geometrical structure of the moduli space (analogously, the 1-forms of matter multiplets give rise to related central charges).

If we use the self-dual and anti-self dual decomposition of vector field strengths

$$F^\pm = \frac{1}{2}(F \mp i\,{}^\star F) \ , \tag{3.2.3}$$

the kinetic part of the vector Lagrangian becomes

$$\mathcal{L}_{kin} = i\overline{\mathcal{N}}_{\Lambda\Sigma}F^{-\Lambda}F^{-\Sigma} + h.c. \tag{3.2.4}$$

The duality group action is given by

$$\mathcal{S}\begin{pmatrix} F^{-\Lambda} \\ G^-_\Lambda \end{pmatrix} = \begin{pmatrix} F^{-\Lambda} \\ G^-_\Lambda \end{pmatrix}' \ , \tag{3.2.5}$$

where:

$$\begin{aligned} G^-_\Lambda &= \overline{\mathcal{N}}_{\Lambda\Sigma}F^{-\Sigma} \ , \\ G^+_\Lambda &= \mathcal{N}_{\Lambda\Sigma}F^{+\Sigma} \ , \end{aligned} \tag{3.2.6}$$

$$\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{G} \subset Sp(2n,\mathbb{R}) \ . \tag{3.2.7}$$

If $L(\phi)$ is the coset representative of $\mathcal{G}$ for a given representation, then $\mathcal{S}$ corresponds to the embedded coset representative belonging to $Sp(2n,\mathbb{R})$, and A, B, C, D are built, in any theory, in terms of $L(\phi)$. It is useful, dealing also with complex coset spaces, to use a complex basis in $Sp(2n)$, and then the matrix in $USp(n)$

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} f + \mathrm{i}h & \bar{f} + \mathrm{i}\bar{h} \\ f - \mathrm{i}h & \bar{f} - \mathrm{i}\bar{h} \end{pmatrix} = \mathcal{A}^{-1}\mathcal{S}\mathcal{A} \tag{3.2.8}$$

the relation between f, h and $\mathcal{S}$ is the same as in (3.1.20)

$$\begin{aligned} i(f^\dagger h - h^\dagger f) &= 1\!\!1 \ , \\ f^t h - h^t f &= 0 \ , \end{aligned} \tag{3.2.9}$$

and

$$\mathcal{A} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \ , \tag{3.2.10}$$

however, the submatrices f, h can now be decomposed with respect to the isotropy group $\mathcal{H}_{Aut} \otimes \mathcal{H}_{matter}$ as

$$\begin{aligned} f &= (f^\Lambda_{AB}, f^\Lambda_I) \ , \\ h &= (h_{\Lambda\, AB}, h_{\Lambda\, I}) \end{aligned} \tag{3.2.11}$$

where AB are indices in the antisymmetric representation of $\mathcal{H}_{Aut} = SU(N) \times U(1)$ and I is an index of the fundamental representation of $\mathcal{H}_{matter}$ (upper $SU(N)$ indices label objects in the complex conjugate representation of $SU(N)$ $(f^\Lambda_{AB})^* = f^{\Lambda AB}$ etc.) Notice that $(f^\Lambda_{AB}, h_{\Lambda AB})$ and $(f^\Lambda_I, h_{\Lambda I})$ are symplectic sections of a $Sp(2n, \mathbb{R})$ bundle over $\mathcal{G}/\mathcal{H}$, which is actually a flat bundle. The real embedding given by $\mathcal{S}$ is appropriate for duality transformations of $F^\pm$ and their duals $G^\pm$, according to equations (3.2.7), (3.2.6), while the complex embedding in the matrix U is appropriate in writing down the fermion transformation laws and supercovariant field strengths. The kinetic matrix $\mathcal{N}$, according to Gaillard−Zumino construction, turns out to be:

$$\mathcal{N} = hf^{-1}, \qquad \mathcal{N} = \mathcal{N}^t \tag{3.2.12}$$

and, as stated already, transforms projectively under $Sp(2n, \mathbb{R})$ duality rotations:

$$\mathcal{N}' = (C + D\mathcal{N})(A + B\mathcal{N})^{-1} \ . \tag{3.2.13}$$

Due to this symplectic embedding, the physical field strengths appearing in the gravitino, dilatino and gaugino supersymmetry transformations are dressed by the scalars and result in

$$\begin{aligned}
T^-_{AB} &= \mathrm{i}(\bar{f}^{-1})_{AB\Lambda}F^{-\Lambda} = f^\Lambda_{AB}(\mathcal{N}-\overline{\mathcal{N}})_{\Lambda\Sigma}F^{-\Sigma} = h_{\Lambda AB}F^{-\Lambda} - f^\Lambda_{AB}G^-_\Lambda \ , && (3.2.14)\\
T^-_I &= \mathrm{i}(\bar{f}^{-1})_{I\Lambda}F^{-\Lambda} = f^\Lambda_I(\mathcal{N}-\overline{\mathcal{N}})_{\Lambda\Sigma}F^{-\Sigma} = h_{\Lambda I}F^{-\Lambda} - f^\Lambda_I G^-_\Lambda \ , && (3.2.15)\\
T^+_{AB} &= h_{\Lambda AB}F^{+\Lambda} - f^\Lambda_{AB}G^+_\Lambda = 0 \ , && (3.2.16)\\
T^+_I &= h_{\Lambda I}F^{+\Lambda} - f^\Lambda_I G^+_\Lambda = 0 \ . && (3.2.17)
\end{aligned}$$

The dressed central charges are then given by

$$\begin{aligned}
Z_{AB} &= \int_{S^2} T_{AB} = \int_{S^2}(T^+_{AB}+T^-_{AB}) = \int_{S^2} T^-_{AB} = h_{\Lambda AB}p^\Lambda - f^\Lambda_{AB}q_\Lambda \ ,\\
Z_I &= \int_{S^2} T_I = \int_{S^2}(T^+_I+T^-_I) = \int_{S^2} T^-_I = h_{\Lambda I}m^\Lambda - f^\Lambda_I q_\Lambda \quad (\mathcal{N}\le 4) \ .
\end{aligned} \qquad (3.2.18)$$

Using the embedded coset representative U, it is possible to derive the differential relations between central and matter charges, using Maurer−Cartan equations [18]. The connection on the symplectic bundle is the $USp(n,n)$ Lie algebra left invariant one form $\Gamma = U^{-1}dU$ satisfying:

$$d\Gamma + \Gamma\wedge\Gamma = 0 \ . \qquad (3.2.19)$$

This integrability condition means that Γ is a flat connection on the symplectic fiber bundle constructed on $\mathcal{G}/\mathcal{H}$. The dependence of Γ on (f,h) is given by

$$\Gamma \equiv U^{-1}dU = \begin{pmatrix} \mathrm{i}(f^\dagger dh - h^\dagger df) & \mathrm{i}(f^\dagger d\bar{h} - h^\dagger d\bar{f}) \\ -\mathrm{i}(f^t dh - h^t df) & -\mathrm{i}(f^t d\bar{h} - h^t d\bar{f}) \end{pmatrix} \equiv \begin{pmatrix} \Omega^{(H)} & \bar{\mathcal{P}} \\ \mathcal{P} & \bar{\Omega}^{(H)} \end{pmatrix} \qquad (3.2.20)$$

where the $n\times n$ subblocks $\Omega^{(H)}$ and $\mathcal{P}$ embed the $\mathcal{H}$ connection and the vielbein of $\mathcal{G}/\mathcal{H}$ respectively . This identification follows from the Cartan decomposition of the $Usp(n,n)$ Lie algebra. A further decomposition of the embedded vielbein

$$\mathcal{P} = \begin{pmatrix} P_{ABCD} & P_{ABJ} \\ P_{ICD} & P_{IJ} \end{pmatrix} \qquad (3.2.21)$$

reflects the decompositions of (3.2.11). Here the sub-blocks are related to the vielbein of $\mathcal{G}/\mathcal{H}$, $P = L^{-1}\nabla^{(H)}L$, since they are written in terms of the indices of $H_{Aut}\times H_{matter}$, they are used to write the differential relations among the central and matter charges

$$\begin{aligned}
\nabla(\omega)Z_{AB} &= \bar{Z}_I P^I_{AB} + \frac{1}{2}\bar{Z}^{CD}P_{ABCD}\\
\nabla(\omega)Z_I &= \frac{1}{2}\bar{Z}^{AB}P_{ABI} + \bar{Z}_J P^J_I \ .
\end{aligned} \qquad (3.2.22)$$

Notice that, since f belongs to the unitary matrix U, then $(f^\Lambda_{AB}, f^\Lambda_I)^\star = (\bar{f}^{\Lambda AB}, \bar{f}^{\Lambda I})$.

For $\mathcal{N} > 4$ no matter coupling is allowed, for a theory with spin content not higher than 2, then $\mathcal{P}$ coincides with the vielbein P_{ABCD} of the relevant $\mathcal{G}/\mathcal{H}$.

Besides the differential relations (3.2.22), the charges also satisfy sum rules quite analogous to those of the $\mathcal{N} = 2$ Special Geometry case.

The sum rule has the following form:

$$\frac{1}{2} Z_{AB} \bar{Z}^{AB} + Z_I \bar{Z}^I = -\frac{1}{2} P^t \mathcal{M}(\mathcal{N}) P \tag{3.2.23}$$

where $\mathcal{M}(\mathcal{N})$ and Q are:

$$\mathcal{M} = \begin{pmatrix} \mathbb{1} & -\mathrm{Re}\mathcal{N} \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} Im\mathcal{N} & 0 \\ 0 & \mathrm{Im}\mathcal{N}^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ -\mathrm{Re}\mathcal{N} & \mathbb{1} \end{pmatrix} \tag{3.2.24}$$

$$Q = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} \tag{3.2.25}$$

and follow from the identities

$$f f^\dagger = -\mathrm{i} \left(\mathcal{N} - \overline{\mathcal{N}} \right)^{-1} \tag{3.2.26}$$

$$h h^\dagger = -\mathrm{i} \left(\overline{\mathcal{N}}^{-1} - \mathcal{N}^{-1} \right)^{-1} \equiv -\mathrm{i}\mathcal{N} \left(\mathcal{N} - \overline{\mathcal{N}} \right)^{-1} \overline{\mathcal{N}} \tag{3.2.27}$$

$$h f^\dagger = \mathcal{N} f f^\dagger \tag{3.2.28}$$

$$f h^\dagger = f f^\dagger \overline{\mathcal{N}} \tag{3.2.29}$$

The matrix $\mathcal{M}$ is a symplectic tensor and can be written as

$$\mathcal{M}(\mathcal{N}) = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} f \\ h \end{pmatrix} \begin{pmatrix} f & h \end{pmatrix}^\dagger \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \tag{3.2.30}$$

where $\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} f \\ h \end{pmatrix}$ is the embedded object corresponding to the coset representative L satisfying

$$\frac{1}{2} L_{AB\Lambda} L^{AB}{}_{\Sigma} - L_{I\Sigma} L^I{}_{\Sigma} = \mathcal{N}_{\Lambda\Sigma} \ . \tag{3.2.31}$$

This formalism , valid for $D = 4$, $\mathcal{N} > 2$ theories is completely determined by the embedding of the coset representative of $\mathcal{G}/\mathcal{H}$ in $Sp(2n, \mathbb{R})$ and by the $Usp(n, n)$ embedded Maurer−Cartan equations .

This formalism, and in particular the identities (3.2.9), the differential relations among charges (3.2.22) and the sum rules (3.2.23), are completely analogous to the

Special Geometry relations of $\mathcal{N} = 2$ matter coupled supergravity, thanks essentially from the fact that, though the scalar manifold $\mathcal{M}_{N=2}$ of the $\mathcal{N} = 2$ theory is not in general a coset manifold, nevertheless it has a symplectic structure identical to the $\mathcal{N} > 2$ theories.

3.3 $\mathcal{N} = 8$ theory

Cremmer and Julia [19]-[20] in 1978-1979 built four dimensional $\mathcal{N} = 8$ Supergravity, invariant under U-duality action of the noncompact group $E_{7(7)}$ and under local $SU(8)$ action. They constructed it from dimensional reduction of $\mathcal{N} = 1$ Supergravity in 11 dimensions, and used the duality covariance to derive the complete supersymmetry transformations. It is a theory based on a massless supermultiplet of physical states

$$\begin{array}{c}\Big(g_{\mu\nu} \; , \;\; \psi^i_\mu \; , \;\; A^{IJ}_\mu \; , \;\; \chi^{klm} \; , \;\; \phi^{pqrs} \;\Big) \\ [1] \quad\; [8] \quad\; [28] \quad\; [56] \quad\; [70] \end{array} \tag{3.3.1}$$

corresponding to a vierbein e^a_μ, 8 Rarita-Schwinger spin 3/2-fields ψ^i_μ, 28 abelian gauge fields A^{IJ}_μ, 56 Majorana spinors χ^{klm} and 70 real scalars ϕ^{pqrs} in the irreducible anti-symmetric representation of $SU(8)$, parametrizing the coset

$$\mathcal{M}_{scalar} = \frac{E_{7(7)}}{SU(8)} \; , \tag{3.3.2}$$

which has real dimension $133 - 63 = 70$, indeed. In this thesis, we will be investigating solutions of the attractor equations for static black holes, which correspond to critical points of an effective potential V_{BH}. We want to anticipate that, in order to solve the equations for $\mathcal{N} = 8$ Supergravity, it is convenient to exploit the language of $\mathcal{N} = 2$ special geometry, generalized to any $\mathcal{N} > 2$ by constructing a flat symplectic bundle [18], as detailed in the previous section. This indeed allows to find a set of simple algebraic equations for the $\mathcal{N} = 8$ BPS and non-BPS black holes [21]. Moreover, properties of $\mathcal{N} = 2$ vector multiplets can be embedded into $\mathcal{N} = 8$ Supergravity.

With reference to the construction of the previous section, then, and specializing to the $\mathcal{N} = 8$ case, where in particular the matter sector is not present, the symplectic embedding is automatically realized in terms of the **56** representation of E_7, embedded in $USp(28, 28)$, and it is given by the usual coset element (3.2.8) where

$$f + \mathrm{i}h \equiv f^{\Lambda\Sigma}{}_{AB} + \mathrm{i}h_{\Lambda\Sigma AB} \tag{3.3.3}$$

$$\bar{f} - \mathrm{i}\bar{h} \equiv \bar{f}^{\Lambda\Sigma AB} - \mathrm{i}\bar{h}_{\Lambda\Sigma}{}^{AB} \tag{3.3.4}$$

$\Lambda\Sigma, AB$ are couples of antisymmetric indices, with Λ,Σ,A,B running from 1 to 8 . The supercovariant field-strengths and coset manifold vielbein, which also depend on fermionic fields, are

$$\hat{F}^{\Lambda\Sigma} = dA^{\Lambda\Sigma} + [f^{\Lambda\Sigma}_{AB}(a_1\bar{\psi}^A\psi^B + a_2\bar{\chi}^{ABC}\gamma_a\psi_C V^a) + h.c.] \tag{3.3.5}$$

$$\hat{P}_{ABCD} = P_{ABCD} - \bar{\chi}_{[ABC}\psi_{D]} + h.c. \tag{3.3.6}$$

where the vielbein satisfy $P_{ABCD} = \frac{1}{4!}\epsilon_{ABCDEFGH}\bar{P}^{EFGH} \equiv (L^{-1}\nabla^{SU(8)}L)_{AB|CD} = P_{ABCD,i}d\phi^i$ (ϕ^i coordinates of G/H). The fermion transformation laws are given by:

$$\delta\psi_A = D\epsilon_A + a_3 T^-_{AB|ab}\Delta^{abc}\epsilon^B V_c + \cdots \tag{3.3.7}$$

$$\delta\chi_{ABC} = a_4 P_{ABCD,i}\partial_a\phi^i\gamma^a\epsilon^D + a_5 T^-_{[AB|ab}\gamma^{ab}\epsilon_{C]} + \cdots \tag{3.3.8}$$

with dressed field strengths

$$\begin{aligned} T_{AB} &= -\frac{\mathrm{i}}{2}(\bar{f}^{-1})_{\Lambda\Sigma AB}F^{\Lambda\Sigma} = \frac{1}{4}(\mathcal{N}-\bar{\mathcal{N}})_{\Lambda\Sigma,\Gamma\Delta}f^{\Lambda\Sigma}_{AB}F^{\Gamma\Delta} \\ &= \frac{1}{2}(h_{\Lambda\Sigma AB}F^{\Lambda\Sigma} - f^{\Lambda\Sigma}_{AB}G_{\Lambda\Sigma}) \end{aligned} \tag{3.3.9}$$

the duality relations among the symplectic sections determine

$$\mathcal{N}_{\Lambda\Sigma,\Gamma\Delta} = \frac{1}{2}h_{\Lambda\Sigma AB}(f^{-1})^{AB}_{\Gamma\Delta} \tag{3.3.10}$$

$$G_{\Lambda\Sigma} = -\mathrm{i}/2\frac{\partial\mathcal{L}}{\partial F^{\Lambda\Sigma}} \ . \tag{3.3.11}$$

The central charges are

$$Z_{AB} = \frac{1}{2}(h_{\Lambda\Sigma AB}g^{\Lambda\Sigma} - f^{\Lambda\Sigma}_{AB}e_{\Lambda\Sigma}), \tag{3.3.12}$$

and satisfy the differential relations

$$\nabla^{SU(8)}Z_{\ AB} = \frac{1}{2}\bar{Z}^{\ CD}P_{ABCD} \tag{3.3.13}$$

and sum rule

$$\frac{1}{2}Z_{AB}\bar{Z}^{AB} = -\frac{1}{8}(p^{\Lambda\Sigma}, q_{\Lambda\Sigma})\mathcal{M}(\mathcal{N})_{\Lambda\Sigma,\Gamma\Delta}\begin{pmatrix} p^{\Gamma\Delta} \\ q_{\Gamma\Delta}\end{pmatrix} \ . \tag{3.3.14}$$

These relations will determine the particular form of the attractor equations in the folowing chapters.

3.4 Gauged Supergravities

It is possible to introduce a further gauge invariance on the scalar manifold of the theory, exploiting the already existent U-duality, if the gauge group is a subgroup of $\mathcal{M}_{scalar}$. We discuss here the gauging for the $\mathcal{N} = 2$ case, involving in general both scalars of the vector multiplets and hypermultiplets, for this is what will be used in Chapter 7. Gauging of $\mathcal{N} > 2$-extended theories is also possible and it is particularly interesting in the study of flux compactifications of String theory, for which we refer the reader to the review in [22]. For completeness we mention that the gauging procedure for $\mathcal{N} = 2$ rigid supersymmetric theories has a related construction to that of the supergravity case, which can be found in [23].

Gauging and the momentum map construction

The structure detailed up to now is that one underlying an abelian, ungauged supergravity. We restrict now to the bosonic Lagrangian of $\mathcal{N} = 2$ Supergravity coupled to n_v abelian vector multiplets with complex scalars and m hypermultiplets

$$\begin{aligned}\mathcal{L}_{ungauged} = \sqrt{-g}\Big[\, R[g] \,+\, g^{sc}_{i\bar{\jmath}}(z,\bar{z})\,\partial^\mu z^i\,\partial_\mu \bar{z}^{\bar{\jmath}} \,+\, h_{uv}(q)\,\partial^\mu q^u\,\partial_\mu q^v \\ +\,\mathrm{i}\,\Big(\bar{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}_{\mu\nu}\mathcal{F}^{-\Sigma|\mu\nu} \,-\, \mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{+\Lambda}_{\mu\nu}\mathcal{F}^{+\Sigma|\mu\nu}\Big)\,\Big]\end{aligned} \tag{3.4.1}$$

where the n_v complex fields z^i span a *special Kähler manifold* $\mathcal{SM}$ and the $4m$ real fields q^u span a quaternionic manifold $\mathcal{HM}$, whose metrics are respectively $g^{sc}_{i\bar{\jmath}}$ and h_{uv} . The period matrix $\mathcal{N}_{IJ}$ depends only on the special manifold coordinates $z^i\,,\bar{z}^{\bar{\imath}}$ as already stated, and is expressed through the symplectic sections of the flat symplectic bundle as

$$\mathcal{N}_{\Lambda\Sigma} = h_{\Lambda\,i} f^{-1\ i}_{\Sigma}\ . \tag{3.4.2}$$

In the theory there are no electric or magnetic currents, and there is on shell symplectic covariance. Using the homomorphism (2.2.19), any diffeomorphism of the scalar manifold can be lifted to a symplectic transformation on the electric-magnetic field strengths. Under this lifting any isometry of the scalar manifold becomes a symmetry of the differential system comprehending equations of motion and Bianchi identities. The problem of gauging the $\mathcal{N} = 2$ theory consits in identifying the gauge group $\tilde{\mathcal{G}}$ as a subgroup of the isometries of the product space

$$\mathcal{SM} \times \mathcal{HM}\ . \tag{3.4.3}$$

The gauging effect is to charge the scalars under the gauge fields, thus introducing covariant derivatives in the action, together with adding new terms which can be fermion-fermion bilinears with scalars dependent coupling coefficients, and, interestingly for the purposes of the black holes solutions, a scalar potential $V_{gauging}$.

All the modifications induced by the gauging to the Lagrangian and supersymmetry equations can be obtained via a geometric construction proper of any Lie group action on manifolds endowed with a symplectic structure: the momentum map. In supersymmetry, indeed, this geometric notion corresponds exactly to gauge multiplets auxiliary fields (D-fields).

Momentum map for a scalar Special Kähler manifold

The Lagrangian giving the action in (2.2.1) contains kinetic terms for the scalars that are of course invariant under continuous isometries of the scalar manifold metric $G_{\alpha\beta}$, which decomposes, because of the scalar manifold structure of $\mathcal{N} = 2$, into Special Kähler metric $g_{i\bar{\jmath}}$ and quaternionic metric h_{uv}. Scalars can however appear, as we have seen, through sections of vector bundles over $\mathcal{M}$, in the period matrix $\mathcal{N}$, which won't be left invariant.

Let us focus at first on the isometries of $g_{i\bar{\jmath}}$. Suppose then, that the holomorphic coordinates of $\mathcal{SM}$ change under an isometry generated by a Killing vector field

$$z^i \to z^i + \epsilon^\Lambda k^i_\Lambda(z) \ , \tag{3.4.4}$$

with $\Lambda = 0, 1, .., \dim \tilde{\mathcal{G}}$, and k^i_Λ holomorphic Killing vector

$$\partial_{\bar{\jmath}} k^i_\Lambda(z) = 0 \leftrightarrow \partial_j k^{\bar{\imath}}_\Lambda(\bar{z}) = 0 \tag{3.4.5}$$

satisfying the equation, in holomorphic indices ($k_{\Lambda\, i} = g_{i\bar{\jmath}} k^{\bar{\jmath}}_\Lambda$)

$$\nabla_i k_{\Lambda\, j} + \nabla_j k_{\Lambda\, i} = 0 \ ; \quad \nabla_{\bar{\imath}} k_{\Lambda\, j} + \nabla_j k_{\Lambda\, \bar{\imath}} = 0 \ . \tag{3.4.6}$$

Because of the Kähler structure of the manifold, whose metric is defined by differentiating a more fundamental Kähler potential, also the Killing vectors are built from a real prepotential $\mathcal{P}_\Lambda$ as

$$k^i_\Lambda = i g^{i\bar{\jmath}} \partial_{\bar{\jmath}} \mathcal{P}_\Lambda, \quad \mathcal{P}^*_\Lambda = \mathcal{P}_\Lambda \tag{3.4.7}$$

This means that, in order to find the isometries of the manifold, it is sufficient to find a real function $\mathcal{P}_\Lambda$ such that

$$\partial_{\bar{k}}(g^{i\bar{\jmath}} \partial_{\bar{\jmath}} \mathcal{P}_\Lambda) = 0 \ , \tag{3.4.8}$$

in order to require holomorphicity of the Killing vector, while eq. (3.4.6) is automatically satisfied by the definition (3.4.7).

This procedure has a geometrical origin which reveals a deep connection with supersymmetry, and involves the momentum map. This is a map from the symplectic manifold to the dual of the Lie algebra of the group acting on it, in our case the group $\tilde{\mathcal{G}} \in \mathcal{M}$ whose isometries we want to gauge. The group $\tilde{\mathcal{G}}$ actually acts on $\mathcal{M}$ and its action is called "symplectic reduction", whose definition actually involves the momentum map. The details of the construction are reviewed in Appendix 3.A. What is needed to our gauging procedure is the existence of this map, guaranteed by the symplectic structure underlying the Kähler manifold, whose Poissonian structure is based on the Kähler closed 2-form (3.1.4). In fact, the momentum map associates a function $\mathcal{P}_\Lambda \in C^\infty(\mathcal{M})$ to any generator of the Lie algebra $\tilde{\mathfrak{g}}$ of $\tilde{\mathcal{G}}$, as from (3.4.7) and the Poisson bracket of two $\mathcal{P}_\Lambda, \mathcal{P}_\Sigma$ is defined as

$$\{\mathcal{P}_\Lambda, \mathcal{P}_\Sigma\} \equiv 4\pi K(\vec{k}_\Lambda, \vec{k}_\Sigma) \ . \tag{3.4.9}$$

It has been demonstrated in [24] that, for any Lie algebra such that $H^2(\mathfrak{g}) = 0$, which is satisfied in particular by any semi-simple Lie algebra, the following identity holds

$$\{\mathcal{P}_\Lambda, \mathcal{P}_\Sigma\} = f_{\Lambda\Sigma}{}^\Gamma \mathcal{P}_\Gamma \ . \tag{3.4.10}$$

By the above definition of Poisson brackets, we have, in components,

$$\frac{i}{2} g_{i\bar{\jmath}}(k^i_\Lambda k^{\bar{\jmath}}_\Sigma - k^i_\Sigma k^{\bar{\jmath}}_\Lambda) = \frac{1}{2} f_{\Lambda\Sigma}{}^\Gamma \mathcal{P}_\Gamma \ ; \tag{3.4.11}$$

finally, by definition of momentum map (3.4.7) and Kähler form (3.1.4), we can derive the form of $\mathcal{P}_\Lambda$ in terms of derivatives of Kähler potential $\mathcal{K}$,

$$\mathrm{i}\,\mathcal{P}_\Lambda = \frac{1}{2}\left(k^i_\Lambda\,\partial_i\mathcal{K} - k^{i^\star}_\Lambda\,\partial_{i^\star}\mathcal{K}\right) = k^i_\Lambda\,\partial_i\mathcal{K} = -k^{i^\star}_\Lambda\,\partial_{i^\star}\mathcal{K} \ . \tag{3.4.12}$$

In $\mathcal{N}=2$ Supergravity, with respect to the rigid supersymmetry case, the manifold is not only Hodge-Kähler but also Special Kähler. This allows to have an expression of $\mathcal{P}_\Lambda$ in terms of symplectic invariants. In this case, also the isometry subgroup admits a symplectic embedding, and the formula for $\mathcal{P}_\Lambda$ is

$$\mathcal{P}_\Lambda = e^{\mathcal{K}}\left(F_\Delta\, f^\Delta{}_{\Lambda\Sigma}\,\bar{X}^\Sigma + \bar{F}_\Delta\, f^\Delta{}_{\Lambda\Sigma}\,X^\Sigma\right) \tag{3.4.13}$$

Triholomorphic momentum map for quaternionic manifolds

Let us now discuss the gauging procedure for the hypermultiplet scalars sector. For applications to $\mathcal{N}=2$ theories one has to assume that the same Lie group $\mathcal{G}$ of isometries acts both on $\mathcal{S}M$ and $\mathcal{H}M$. This means that Killing vectors on $\mathcal{HM}$

$$\vec{k}_\Lambda = k^u_\Lambda \frac{\vec{\partial}}{\partial q^u} \tag{3.4.14}$$

satisfy the same Lie algebra as the corresponding ones on $\mathcal{SM}$. We can express the Killing vectors on the full $\mathcal{M}_{scalar}$ as

$$\hat{\vec{k}}_\Lambda = k^i_\Lambda \vec{\partial}_i + k^{i^*}_\Lambda \vec{\partial}_{i^*} + k^u_\Lambda \vec{\partial}_u \tag{3.4.15}$$

according to the tensorial decomposition of the metric

$$\hat{g} = \begin{pmatrix} g_{ij^*} & 0 \\ 0 & h_{uv} \end{pmatrix} \tag{3.4.16}$$

defined on the product manifold $\mathcal{SM} \otimes \mathcal{HM}$.

Recall that supersymmetry requires an $SU(2)$-bundle over the HyperKähker manifold, thus leaving an $SU(2)$ rotations invariance of the HyperKähler structure, which implies triholomorphicity of the Killing vectors. This means that we can associate to each Killing vector a triplet of prepotentials $\mathcal{P}^x_\Lambda$ according to

$$\mathbf{i}_\Lambda K^x = -\nabla \mathcal{P}^x_\Lambda \equiv -(d\mathcal{P}^x_\Lambda + \epsilon^{xyz}\omega^y \mathcal{P}^z_\Lambda) \tag{3.4.17}$$

where ∇ denotes the $SU(2)$ covariant exterior derivative. One imposes an equivariance condition also in the quaternionic case

$$\mathbf{X} \circ \mathcal{P}_{\mathbf{Y}} \;=\; \mathcal{P}_{[\mathbf{X},\mathbf{Y}]} \tag{3.4.18}$$

and a tri-holomorphic Poisson bracket

$$\{\mathcal{P}_\Lambda, \mathcal{P}_\Sigma\}^x \equiv 2K^x(\Lambda,\Sigma) - \lambda\, \varepsilon^{xyz}\, \mathcal{P}^y_\Lambda\, \mathcal{P}^z_\Sigma \tag{3.4.19}$$

yielding a tri-holomorphic Poissonian realization of the Lie algebra

$$\{\mathcal{P}_\Lambda, \mathcal{P}_\Sigma\}^x \;=\; f^\Delta{}_{\Lambda\Sigma}\, \mathcal{P}^x_\Delta \tag{3.4.20}$$

which in components reads

$$K^x_{uv}\, k^u_\Lambda\, k^v_\Sigma \;-\; \frac{\lambda}{2}\, \varepsilon^{xyz}\, \mathcal{P}^y_\Lambda\, \mathcal{P}^z_\Sigma \;=\; \frac{1}{2}\, f^\Delta{}_{\Lambda\Sigma}\, \mathcal{P}^x_\Delta \tag{3.4.21}$$

Scalar potential

Once the supersymmetry variations have been adjusted to the new gauge couplings and derivatives, a set of Ward identities fixes the form of the scalar potential [25],[26]. For only electric gauging, the result is

$$V = (g_{ij^\star} k^i_\Lambda k^{j^\star}_\Sigma + 4h_{uv} k^u_\Lambda k^v_\Sigma)\bar{L}^\Lambda L^\Sigma + (U^{\Lambda\Sigma} - 3\bar{L}^\Lambda L^\Sigma)\mathcal{P}^x_\Lambda \mathcal{P}^x_\Sigma \ , \tag{3.4.22}$$

where $U^{\Lambda\Sigma}$ is defined as

$$U^{\Lambda\Sigma} \equiv f^\Lambda_i \, f^\Sigma_{j^\star} \, g^{ij^\star} = -\frac{1}{2}\,(\mathrm{Im}\mathcal{N})^{-1|\Lambda\Sigma} - \bar{L}^\Lambda L^\Sigma \ . \tag{3.4.23}$$

The first two terms in the scalar potential are actually related to the gauging of isometries of $\mathcal{M}_{scalar} = \mathcal{SK} \otimes \mathcal{Q}$, the last term is the gravitino mass contribution, and the term containing $U^{\Lambda\Sigma}$ is the contribution coming from the gaugino shift due to the quaternionic prepotential. We can recast the expression for the scalar potential as

$$V = (k_\Lambda, k_\Sigma)\bar{L}^\Lambda L^\Sigma + (U^{\Lambda\Sigma} - 3\bar{L}^\Lambda L^\Sigma)(\mathcal{P}^x_\Lambda \mathcal{P}^x_\Sigma - \mathcal{P}_\Lambda \mathcal{P}_\Sigma) \ , \tag{3.4.24}$$

where we used the scalar product of Killing vectors

$$(k_\Lambda, k_\Sigma) = \begin{pmatrix} k^i_\Lambda, \, k^{i^\star}_\Lambda, \, k^u_\Lambda \end{pmatrix} \begin{pmatrix} 0 & g_{ij^\star} & 0 \\ g_{i^\star j} & 0 & 0 \\ 0 & 0 & 2h_{uv} \end{pmatrix} \begin{pmatrix} k^j_\Sigma \\ k^{j^\star}_\Sigma \\ k^v_\Sigma \end{pmatrix} \ , \tag{3.4.25}$$

their definition in terms of the prepotential $\mathcal{P}_\Lambda$, and the relations from special geometry

$$k^i_\Lambda L^\Lambda = k^{i^\star}_\Lambda \bar{L}^\Lambda = \mathcal{P}_\Lambda L^\Lambda = \mathcal{P}_\Lambda \bar{L}^\Lambda = 0 \ . \tag{3.4.26}$$

For the purpose of building black holes solutions and attractor flows in gauged supergravity, we will restrict in this thesis to the case where the gauged isometries group $\tilde{\mathcal{G}}$ is an abelian group. It is possible that the scalar potential still remains nonzero, due to the presence of so called Fayet-Iliopoulos terms

$$\mathcal{P}^x_\Lambda = \xi^x_\Lambda \ ; \quad \epsilon^{xyz}\xi^y_\Lambda \xi^z_\Sigma = 0 \ . \tag{3.4.27}$$

We are left, in this case, with

$$V(z,\bar{z}) = (U^{\Lambda\Sigma} - 3\bar{L}^\Lambda L^\Sigma)\xi^x_\Lambda \xi^x_\Sigma \ . \tag{3.4.28}$$

This potential is the new term that appears in the Lagrangian of (2.2.1), for $U(1)-$gauging of $\mathcal{N} = 2$ Supergravity, the case that we are going to consider in Chapter 7. We will construct, in fact, solutions of static black holes, starting from a duality invariant generalisation of (3.4.28) . In particular, we will consider the minimal gauging in which the group $\tilde{\mathcal{G}}$ is the diagonal $U(1)$ in the tensor product of $\mathcal{SK} \otimes \mathcal{Q}$, which is also the largest subgroup of R-symmetry group $U(2)$ one can gauge in $\mathcal{N} = 2$ Supergravity, thus we will often call the corresponding theory, the $U(1)$-gauged $\mathcal{N} = 2$ theory, or $\mathcal{N} = 2$ Supergravity with R-symmetry gauging.

Appendix 3.A Symplectic reduction and momentum map

We will now give a more technical overview of the momentum map, involved in the symplectic reduction, determined by the action of a Lie group on a symplectic manifold.

The momentum map is a map from the symplectic manifold to the dual of the Lie algebra of the group acting on it[2]. As an historical note, momentum maps and symplectic reduction appeared in many examples from classical mechanics, and were defined in general only later by Konstant and Souriau in 1965. Special cases of momentum maps are, for example, the conserved linear and angular momentum. Symplectic structures appear naturally in classical mechanics, the phase space of a system is indeed a symplectic manifold. The Hamiltonian is the function defined on this space which generates the dynamics. A symmetry of the system is merely an action of a group on the phase space which leaves the symplectic form and the Hamiltonian invariant.

In the case under consideration, namely the gauging of a Supergravity theory, we are interested in the symplectic reduction of the scalar manifold $\mathcal{M}_{scalar}$ (which is itself a Lie group) under the action of the subgroup of gauged isometries $\tilde{\mathcal{G}}$ whose action on $\mathcal{M}_{scalar}$ is trivial.

Momentum map for the action of a compact Lie group on Kähler manifolds

Lemma (Invariance of the symplectic form). Let $\mathcal{M}$ be a Kähler manifold of dimension $2n$, and let G be a compact Lie group acting on $\mathcal{M}$ with an action that preserves the complex structure J of $\mathcal{M}$ (i.e. Killing vectors are holomorphic with respect to J). Then, these vectors also preserve the Kähler 2 form K.

Proof. Denote $\mathcal{L}_{\mathbf{X}}$ and $i_{\mathbf{X}}$ the Lie derivative along the Killing vector field $\mathbf{X}$ and the

[2]It is sometimes equivalently called moment map.

contraction (of forms) with it, respectively, then we have

$$\left.\begin{array}{l}\mathcal{L}_{\mathbf{X}}g=0\leftrightarrow\nabla_{(\mu}X_{\nu)}=0\\ \mathcal{L}_{\mathbf{X}}J=0\end{array}\right\}\Rightarrow 0=\mathcal{L}_{\mathbf{X}}K=i_{\mathbf{X}}dK+d(i_{\mathbf{X}}K)=d(i_{\mathbf{X}}K) \tag{3.A.1}$$

since the Kähler form is closed.

Definition (Momentum map). If a Lie group G acts on a symplectic manifold $(\mathcal{M},\omega)$ leaving the symplectic form ω invariant, then the action is called *Hamiltonian* if there exists a smooth, equivariant map

$$\mu:\mathcal{M}\rightarrow\mathfrak{g}^* \ , \tag{3.A.2}$$

such that for all $\mathbf{X}\in\mathfrak{g}$,

$$d\mu_{\mathbf{X}}\ =\ -i_{\mathbf{X}}\omega \ ; \tag{3.A.3}$$

the function $\mu_{\mathbf{X}}$ is defined by

$$\mu_{\mathbf{X}}(m)\ =\ \langle\mu(m),\mathbf{X}\rangle \ , \tag{3.A.4}$$

for $\mathbf{X}\in\mathfrak{g}$ and $m\in\mathcal{M}$. The map μ is called a *momentum map* for the action. Recall that on the Kähler manifold $\mathcal{M}$ the action of the isometries subgroup $\tilde{\mathcal{G}}$ is Hamiltonian, since the existence of the momentum map is ensured by (3.A.1).

Definition (Equivariance). Equivariance of the momentum map with respect to the coadjoint action of G on $\mathfrak{g}^*$ is defined as

$$\langle Ad^*(g)\mu(m),Y\rangle\ =\ \langle\mu(m),Ad^{-1}(g)Y\rangle \ , \tag{3.A.5}$$

for all $g\in G, m\in\mathcal{M}$ and $Y\in\mathfrak{g}$, and, infinitesimally, is given by the action of $\mathfrak{g}$ on $\mathfrak{g}^*$

$$\langle ad^*(X)\mu(m),Y\rangle\ =\ \langle\mu(m),-[X,Y]\rangle \tag{3.A.6}$$

for all $X,Y\in\mathfrak{g}$ and $m\in\mathcal{M}$, or

$$ad^*(X)\mu\ =\ -\mu_{[X,Y]} \ . \tag{3.A.7}$$

Remark (Uniqueness of momentum map). If μ and ν are momentum maps for the same action, then $\forall\ X\in\mathfrak{g}$, by definition,

$$d(\mu_X-\nu_X)\ =\ 0. \tag{3.A.8}$$

If $\mathcal{M}$ is connected (as in the cases we will be dealing with in $\mathcal{N} = 2$ Supergravity), this imply that

$$\mu_X - \nu_X = c_X \tag{3.A.9}$$

with c_X constant function on $\mathcal{M}$. From the definition of momentum map, c_X depends linearly on X, thus there is actually an element $\xi \in \mathfrak{g}^*$ such that

$$\mu - \nu = \xi \ . \tag{3.A.10}$$

Equivariance of the momentum map then fixes the element ξ by the coadjoint action of G on $\mathfrak{g}^*$. In fact, given the action of G on $(\mathcal{M}, \omega)$, the space of elements of g^*, that are fixed by the coadjoint action, parametrizes the set of all momentum maps associated to the G-action.

There is another, constructive, definition of momentum map, which is suitable in the case of gauging a duality subgroup of isometries, given in terms of Hamiltonian vector fields and Poisson brackets. We will define it here in the case the Hamiltonian vectors are the Killing vector fields corresponding to the isometries of $\tilde{\mathcal{G}}$, that we denote as k_Λ. If we expand the vector field in a basis of k_Λ's as $\mathbf{X} = a^\Lambda k_\Lambda$ such that

$$[k_\Lambda, k_\Sigma] = f_{\Lambda\Sigma}{}^{\Delta} k_\Delta \ , \tag{3.A.11}$$

then we have also $\mu_{\mathbf{X}} = a^\Lambda \mu_\Lambda$, and each μ_Λ is an element of $C^\infty(\mathcal{M})$. The Poisson bracket of μ_Λ and μ_Σ is defined as

$$\{\mu_\Lambda, \mu_\Sigma\} \equiv \omega(k_\Lambda, k_\Sigma) \ , \tag{3.A.12}$$

which, in the case of a Kähler manifold becomes $\{\mu_\Lambda, \mu_\Sigma\} \equiv K(k_\Lambda, k_\Sigma)$. It can be shown that a map satisfying (3.A.3) is equivariant if and only if it is an *anti-Poisson map*, which in our particular case means that

$$\{\mu_X, \mu_Y\} \equiv -\mu_{[X,Y]} \ . \tag{3.A.13}$$

This form of the equivariance condition corresponds exactly to eq. (3.4.10), or, in components to (3.4.11), that allowed us to solve for an expression of $\mathcal{P}_\Lambda$ in terms of the killing vectors and the Kähler potential.

Chapter 4

Black Holes in Supergravity: the attractor mechanism

In the classical description of Einstein−Maxwell theory, black holes can be considered solitons of general relativity. Zero-temperature black holes are stable against thermal emission of particles. In principle, loss of angular momentum or charge would cause instability, thus any non-rotating system in a theory whose elementary fields are not charged can be considered, at least classically, stable. These are the black holes that we will deal with in this thesis, as solutions of extended Supergravity theories. They generalize charged black holes of General Relativity, which are represented, in the static case, by the Reissner−Nördstrom metric. In particular, in the same way GR solutions obey the cosmic censorship conjecture and a no hair theorem holds, also the Supergravity extremal solutions obey these conditions. In fact, in Supergravity, the BPS bound $M \geq |Q|$ has to be replaced with the condition $M \geq |Z|$, with Z central charge, obtained by dressing the abelian charges of the theory with the scalar fields. The BPS bound, in this form, comes from the closure of the supersymmetry algebra and implies that the cosmic censorship is verified, thus there cannot exist naked singularities but they always have to be hidden, by the event horizon, from an observer at infinity. Moreover, in the extremal case, the near horizon geometry is a conformally flat, Bertotti−Robinson type metric, as in the Reissner−Nördstrom case, and the mass parameter only depends on the charge configuration and not on the scalar fields.

We can say that the extremal black hole looses memory of the scalar "hair" at the horizon. Notice, however, that this is not implied by the no-hair theorem, which, indeed, does not hold for such solutions. The "no-hair theorem", in general relativity, states that a charged black hole solution only depends on the observable parameters of angular momentum, charge and mass. In particular, the entropy is determined by the same quantities. However, to completely define the extremal black hole solution in Supergravity, one has to specify the v.e.v.'s that scalars acquire at infinity, and on which the configuration depends.

The scalar dependence actually cancels in the expression of the entropy, because of the requirement that the differential equations governing the scalar dynamics reach a critical point of the flow at the black hole horizon (*attractor mechanism*). As it will be shown in this Chapter, this is related to the condition on the horizon to be a regular point for the radial dynamics of scalar fields. The attractor mechanism is then stronger than the no-hair theorem, since for Supergravity black holes no physical principle would *a priori* imply that the dependence on scalar fields drops from the computation of physical quantities associated to the solution, such as the area of the horizon.

Since the black hole entropy is given by the horizon area, according to the Bekenstein − Hawking formula, for extremal black holes it is a topological quantity depending on electric and magnetic charges. This would be consistent with a microscopic interpretation of black hole entropy in terms of fundamental degrees of freedom, since charges are quantized as integer numbers. In Supergravity, however, charged systems are classical configurations, which correspond to a large charges approximation, where they take continuous real values.

We already pointed out that the fermions (and hypermultiplet, for $\mathcal{N} = 2$) decouple from the black hole dynamics. It is consistent, then, to look for solutions where all fermions and hyperscalars are set to zero. One can further exploit staticity and spherical symmetry, to write a general ansatz for the metric of the extremal black hole, as we are going to discuss, in a theory described by the action (2.2.1).

We restrict the attention to dynamics and field equations for the bosonic sector of Supergravity theories, that is to massless scalars and n vector fields coupled to gravity. The scalars describe a non linear σ-model over a manifold $\mathcal{G}/\mathcal{H}$, the vector fields transform according to a certain representation of the global symmetry group $\mathcal{G}$.

If the solution is stationary, then the space-time admits a time-like Killing vector field. We can use this field to perform a dimensional reduction of the action down to 3 dimensions. This would introduce a Kaluza−Klein field, the warp factor, in the form of a new scalar field that enlarges the scalar non linear σ-model. If the black hole is non-rotating, i.e. it has zero angular momentum, it is possible to further reduce the action down to a one dimensional system, subject to a Hamiltonian constraint, as it will be shown in Section 4.1.

4.1 Dimensional reduction and equations of motion

We derive the three dimensional effective metric in the case of static spherically symmetric non-extremal black holes, for a non linear sigma model coupled to gravity, which yields a regular metric in the extremal limit.

Let us consider at first a 4-dimensional space-time manifold Σ with metric $g_{\alpha\beta}$ coupled to scalars. The action is given by the Einstein-Hilbert term and the non linear sigma model of the scalar fields

$$S_\phi = \int_\Sigma \sqrt{g} dx \left[-\frac{1}{2} R(x) + \frac{1}{2} g^{\alpha\beta} \partial_\alpha \bar{\phi}^i \partial_\beta \bar{\phi}^j \gamma_{ij} \right] . \tag{4.1.1}$$

We take into account the contribution of vector fields only later, adding the stress-energy tensor to the Einstein equations and the black hole effective potential to the equations of motion for the scalars. The dynamics corresponding to the above action is described by

$$R_{\alpha\beta} - \gamma_{ij}\, \partial_\alpha \bar{\phi}^i \partial_\beta \bar{\phi}^j = 0 \, , \tag{4.1.2}$$

$$D^\alpha \partial_\alpha \bar{\phi}(x) = 0 \, . \tag{4.1.3}$$

Notice that solutions to (4.1.3) are harmonic maps from the (pseudo) Riemannian manifold $(\Sigma\,, g_{ij})$ to $(\mathcal{G}/\mathcal{H}\,, G_{ij})$ Stationary solutions are those admitting everywhere a time-like Killing vector field, which is orthogonal to the reduced 3 dimensional space Σ_3 and allows $SO(3)$ (spherical) symmetry. The metric decomposes as

$$g_{\alpha\beta} = \begin{pmatrix} e^{2U} & 0 \\ 0 & -e^{-2U} h_{ab} \end{pmatrix} . \tag{4.1.4}$$

h_{ab}, the metric on Σ_3, can be parametrized in terms of a function $f(r)$ so that

$$\begin{aligned} ds^2 &= -e^{2U} dt^2 + e^{-2U} \left(dr^2 + f(r)^2 (d\theta^2 + \sin^2\theta d\phi^2) \right) \, , \\ &\equiv -e^{2U} dt^2 + e^{-2U} h_{ab} dx^a dx^b \, . \end{aligned} \tag{4.1.5}$$

The effective Lagrangian for the reduced three dimensional system is

$$\frac{1}{2} \hat{R} - \frac{1}{2} \gamma^{mn} \partial_m \phi^a \partial_n \phi^b G_{ab} - c^2 \, , \tag{4.1.6}$$

where $c = \frac{\kappa A}{4\pi} = 2ST$, and G_{ab} is now the metric of the enlarged scalar manifold, so $\phi^a = (U, \bar{\phi}^a, \psi^\Lambda, \chi_\Lambda)$.

The equations of motion in this case are

$$\begin{aligned} f^{-2}\frac{d}{dr}\left(f^2\frac{d\phi^i}{dr}\right)+\Gamma^i_{jk}(\phi)\frac{d\phi^j}{dr}\frac{d\phi^k}{dr}&=0\ , \\ R_{rr}=-2f^{-1}\frac{d^2f}{dr^2}&=G_{ij}(\phi)\frac{d\phi^j}{dr}\frac{d\phi^k}{dr}\ , \\ \sin^2\theta R_{\phi\phi}=R_{\theta\theta}=f^{-2}\left(\frac{d}{dr}f\frac{df}{dr}-1\right)&=0\ . \end{aligned} \tag{4.1.7}$$

From the last one we find

$$f(r)^2=(r-r_0)^2+\tilde{c}\ , \tag{4.1.8}$$

thus, if we define the harmonic function on (Σ_3, h)

$$\tau(r)\equiv-\int_r^\infty f^{-2}(s)ds\ , \tag{4.1.9}$$

then being

$$f^{-2}(r)=-\frac{d\tau}{dr}\ , \tag{4.1.10}$$

we find that the first in (4.1.7) is

$$-\left(\frac{d\tau}{dr}\right)^2\frac{d}{dr}\left(f^2\frac{dr}{d\tau}\frac{d}{d\tau}\phi^i\right)+\Gamma^i_{jk}(\phi)\frac{d\phi^j}{d\tau}\frac{d\phi^k}{d\tau}\left(\frac{d\tau}{dr}\right)^2=0\ , \tag{4.1.11}$$

that is, the geodesic equation

$$\frac{d^2\phi(\tau)}{d\tau^2}+\Gamma^i_{jk}(\phi)\frac{d\phi^j}{d\tau}\frac{d\phi^k}{d\tau}=0\ . \tag{4.1.12}$$

The geodesic map ϕ satisfies the condition

$$G_{ij}(\phi)\frac{d\phi^i}{d\tau}\frac{d\phi^j}{d\tau}=2c^2\ ; \tag{4.1.13}$$

comparing with the general solution for $f(r)$ in (4.1.8), we can set $\tilde{c}=-c^2$.

To write the metric in (4.1.5) using τ coordinate we compute, from the definition (4.1.9)

$$\begin{aligned} (r-r_0)^2-c^2&=\frac{c^2}{\sinh^2(c\tau)} \\ &\Downarrow \\ (r-r_0)^2&=c^2\coth(c\tau)\ , \\ dr^2&=\frac{c^4}{\sinh^4(c\tau)}d\tau^2\ , \\ f^2(r(\tau))&=\frac{c^2}{\sinh^2(c\tau)}\ , \end{aligned} \tag{4.1.14}$$

so that we finally arrive to

$$ds^2 = -e^{2U}dt^2 + e^{-2U}\left[\frac{c^4 d\tau^2}{\sinh^4(c\tau)} + \frac{c^2}{\sinh^2(c\tau)}(d\theta^2 + \sin^2\theta d\phi^2)\right] . \quad (4.1.15)$$

This is the most general metric for a static, non-extremal spherically symmetric black hole. The extremal limit is obtained sending $c \to 0$.

The vector sector

Let us now complete the derivation by taking into account the vector sector. As discussed in the previous Chapters, the bosonic action is given by

$$\begin{aligned}\mathcal{S} &= \mathcal{S}_{EH} + \mathcal{S}_{scalar} + \mathcal{S}_V = \\ &= \int \sqrt{-g}\, d^4x \left(-\frac{1}{2}R + \frac{1}{2}\gamma_{rs}(\phi)\partial_\mu\phi^r\partial^\mu\phi^s - \frac{1}{4}F_{\alpha\beta}\left(\mu F^{\alpha\beta} - \nu\, {}^*F^{\alpha\beta}\right)\right) ,\end{aligned} \quad (4.1.16)$$

where $\mu_{\Lambda\Sigma} = -\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}$, $\nu_{\Lambda\Sigma} = -\mathrm{Re}\mathcal{N}_{\Lambda\Sigma}$ are the real symmetric matrices defining the coupling of scalars to the vector fields. To write the contribution of S_V to Einstein equations we need to compute the energy-momentum tensor

$$T_V^{\mu\nu} = \frac{2}{\sqrt{-g}}\left[\frac{\partial(\sqrt{-g}L_V)}{\partial g_{\mu\nu}} - \partial_\lambda \frac{\partial(\sqrt{-g}L_V)}{\partial(\partial_\lambda g_{\mu\nu})}\right] . \quad (4.1.17)$$

By definition of Hodge-star duality we have

$${}^*F^{\Lambda\alpha\beta} = \frac{1}{2\sqrt{-g}}\epsilon^{\alpha\beta\gamma\delta}F^\Lambda_{\gamma\delta} , \quad (4.1.18)$$

which gives

$$\frac{\partial}{\partial g_{\mu\nu}}(F^\Lambda_{\alpha\beta}\,{}^*F^{\Sigma\alpha\beta}) = F^\Lambda_{\alpha\beta}\frac{\partial\,{}^*F^{\Sigma\alpha\beta}}{\partial g_{\mu\nu}} , \quad (4.1.19)$$

and

$$\frac{\partial\,{}^*F^{\Lambda\alpha\beta}}{\partial g_{\mu\nu}} = \frac{1}{2}g^{\mu\nu}\,{}^*F^{\Lambda\alpha\beta} . \quad (4.1.20)$$

We then have

$$\frac{1}{2}\sqrt{-g}T^{\mu\nu} = -\frac{1}{2}\sqrt{-g}g^{\mu\nu}L_V + \sqrt{-g}\left[-\frac{1}{2}F^{\Lambda\sigma}_\mu \mu_{\Lambda\Sigma}F^\Sigma_{\nu\sigma} + \frac{1}{2}\cdot\frac{1}{4}g^{\mu\nu}F^\Lambda_{\alpha\beta}\nu_{\Lambda\Sigma}\,{}^*F^{\Sigma\alpha\beta}\right] , \quad (4.1.21)$$

and finally

$$T^{\mu\nu} = \frac{1}{4} g^{\mu\nu} F^{\Lambda}_{\alpha\beta} \mu_{\Lambda\Sigma} F^{\Sigma\,\alpha\beta} - F^{\Lambda}_{\mu\sigma} \mu_{\Lambda\Sigma} F^{\Sigma\,\sigma}_{\nu} \ . \tag{4.1.22}$$

The modification of scalar geodesic equations due to the vector fields coupling is found by considering

$$\frac{\delta L_V}{\delta \phi^i} = -\frac{1}{4}\sqrt{-g} F_{\alpha\beta} \left(\frac{\delta \mu}{\delta \phi^i} F^{\alpha\beta} - \frac{\delta \nu}{\delta \phi}\, {}^*F^{\alpha\beta} \right) \ . \tag{4.1.23}$$

Due to the specific form of $\mathcal{S}_V$, required by electric-magnetic duality invariance, the expressions of (4.1.22) and (4.1.23) can be recast in a compact form. Let us consider, indeed, the dual field strength defined as in (3.1.18)

$$-{}^*G_{\Lambda\,\mu\nu} = \mu_{\Lambda\Sigma} F^{\Sigma}_{\mu\nu} + \nu_{\Lambda\Sigma}\, {}^*F^{\Sigma}_{\mu\nu} \ , \tag{4.1.24}$$

and the symplectic vector $\mathcal{F} = \left(F^{\Lambda}\,,\, G_{\Lambda}\right)$. We can write the stress energy tensor and the contribution to scalar equations of motion from the vector Lagrangian in a manifestly symplectic way, simply introducing the matrix

$$\mathcal{M} = \begin{pmatrix} \mu + \nu\mu^{-1}\nu & \nu\mu^{-1} \\ \mu^{-1}\nu & \mu^{-1} \end{pmatrix} \tag{4.1.25}$$

The expressions in (4.1.22) and (4.1.23) then become

$$T^{\mu\nu} = -\frac{1}{2} \mathcal{F}^{\Lambda}_{\mu\gamma} \mathcal{M}_{\Lambda\Sigma} \mathcal{F}^{\Sigma\,\gamma}_{\nu} \ , \tag{4.1.26}$$

and

$$\frac{\delta L_V}{\delta \phi^i} = -\frac{1}{8}\sqrt{-g} \mathcal{F}^{\Lambda}_{\mu\nu} \frac{\delta \mathcal{M}_{\Lambda\Sigma}}{\delta \phi^i} F^{\Sigma\mu\nu} \ . \tag{4.1.27}$$

Spherical symmetry has played no role, so far, in the derivation of the equations of motion. We are going to see, in the next section, that it allows to further reduce the Lagrangian to describe a one dimensional system, together with an Hamiltonian constraint.

4.2 One dimensional Lagrangian for static configurations

In the case of time-independent solutions that preserve spherical symmetry, one can perform the integration of the 4-dimensional action over $\mathbb{R}_t \times S^2$. In order to do that, it is convenient to specify a consistent ansatz for the vector field strengths. Given the metric as in (4.1.15), we can take a potential $A^\Lambda = \chi^\Lambda(r)dt - p^\Lambda \cos\theta d\phi$, so that $F^\Lambda = dA^\Lambda$ leads to

$$F^\Lambda = \frac{1}{2} p^\Lambda \sin\theta d\theta \wedge d\phi - \chi^{\Lambda\prime}(r)\, dt \wedge dr \ . \tag{4.2.1}$$

Since χ^Λ only appears in the original action (4.1.16) under derivatives, one can integrate it out performing a Légendre transform on the action. From the equations of motion for χ^Λ one gets

$$\chi^{\Lambda\prime}(r) = e^{2U} \mathcal{I}^{-1\,\Lambda\Sigma}(q_\Sigma - \mathcal{R}_{\Sigma\Gamma} p^\Gamma) \ , \tag{4.2.2}$$

here and in the following we will use $\mathcal{I}_{\Lambda\Sigma} = \mathrm{Im}\mathcal{N}_{\Lambda\Sigma}$, $\mathcal{R}_{\Lambda\Sigma} = \mathrm{Re}\mathcal{N}_{\Lambda\Sigma}$.

Due to spherical symmetry, all the fields depend only on the radial variable so $\phi^i = \phi^i(r)$, $U = U(r)$, etc. . In particular, given the ansatz for the field strength as above, integration over angular variables of the vector sector $\mathcal{S}_v$ yields the effective black hole potential

$$V_{BH} = \frac{1}{2} Q^{T\,\Lambda} \mathcal{M}_{\Lambda\Sigma} Q^\Sigma \ , \tag{4.2.3}$$

where

$$Q^\Lambda = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} \ , \tag{4.2.4}$$

is the vector of charges

$$\begin{aligned} p^\Lambda &= \frac{1}{4\pi} \int_{S^2} F^\Lambda \ , \\ q_\Lambda &= \frac{1}{4\pi} \int_{S^2} G_\Lambda \ , \end{aligned} \tag{4.2.5}$$

The resulting effective action is given by [13]

$$\mathcal{L} = \left(\frac{dU}{d\tau}\right)^2 + G_{ab} \frac{d\phi^a}{d\tau} \frac{d\phi^b}{d\tau} + \mathrm{e}^{2U} V_{BH} - c^2 \ . \tag{4.2.6}$$

This holds quite general for any 4-dimensional theory whose bosonic sector is given as in (2.2.1). The explicit form of the effective potential actually selects the theory under consideration.

The dimensionally reduced system of equations of motion, however, is not completely equivalent to the four dimensional one. In order for the two sets of equations of motions to be equivalent, in fact, one has supplement (4.2.6) with the Hamiltonian constraint:

$$\left(\frac{dU}{d\tau}\right)^2 + G_{ab} \frac{d\phi^a}{d\tau} \frac{d\phi^b}{d\tau} - \mathrm{e}^{2U} V_{BH} = c^2 \ . \tag{4.2.7}$$

This, together with the equations of motion derived from the Lagrangian (4.2.6)

$$\frac{d^2U}{d\tau^2} = 2\mathrm{e}^{2U} V_{BH}(\phi, p, q), \tag{4.2.8}$$

$$\frac{D^2\phi^a}{D\tau^2} = \mathrm{e}^{2U}\frac{\partial V_{BH}}{\partial \phi^a}, \tag{4.2.9}$$

completely determines the solution. The constant is $c = 2ST$, where S is the entropy and T the temperature of the black hole is the non-extremality parameter: extremal black holes have zero temperature and can now equivalently be characterized by $c = 0$.

4.3 Near horizon dynamics and the attractor condition

The metric of the static spherically symmetric system can be described by

$$ds^2 = -e^{2U}dt^2 + e^{-2U}\left[\frac{d\tau^2}{\tau^4} + \frac{1}{\tau^2}d\Omega_2\right] , \tag{4.3.1}$$

where the horizon is located at negative infinity in terms of the coordinate τ. If it has a finite area then the term e^{-2U} has to behave as

$$e^{-2U} \to \left(\frac{A}{4\pi}\right)\tau^2 , \qquad as \;\; \tau \to -\infty . \tag{4.3.2}$$

The scalar term in the Lagrangian remains finite near the horizon if

$$G_{ij}\partial_m\phi^i\partial_n\phi^j\gamma^{mn} < \infty , \tag{4.3.3}$$

that is, in our coordinates,

$$G_{ij}\frac{d\phi^i}{d\tau}\frac{d\phi^j}{d\tau}e^{2U}\tau^4 < \infty . \tag{4.3.4}$$

The near horizon behavior is then given by

$$G_{ij}\frac{d\phi^i}{d\tau}\frac{d\phi^j}{d\tau}\left(\frac{4\pi}{A}\right)\tau^2 \to X^2 , \qquad \text{as} \;\; \tau \to -\infty , \tag{4.3.5}$$

that gives the condition, substituting in the constraint (4.2.7) in the extremal case $c = 0$, near the horizon,

$$A \leq 4\pi V_{BH}(p, q, \phi_H) , \tag{4.3.6}$$

and the metric is

$$ds^2 \approx -\frac{4\pi}{A\tau^2}dt^2 + \left(\frac{A}{4\pi}\right)\left[\frac{d\tau^2}{\tau^4} + \frac{1}{\tau^2}d\Omega_2\right] . \tag{4.3.7}$$

The $AdS_2 \times S^2$ horizon geometry of the extremal black hole appears explicitly once the metric is written in terms of the coordinate

$$\omega = \log \rho \ , \qquad \rho = -\frac{1}{\tau} \ , \tag{4.3.8}$$

since the metric becomes

$$ds^2 \approx -\frac{4\pi}{A} e^{2\omega} dt^2 + \left(\frac{A}{4\pi}\right) d\omega^2 + \left(\frac{A}{4\pi}\right)(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{4.3.9}$$

The condition (4.3.5) is now

$$G_{ij}\frac{d\phi^i}{d\omega}\frac{d\phi^j}{d\omega}\left(\frac{4\pi}{A}\right) \to X^2 \ , \qquad \text{as} \ \ \omega \to \infty \ ; \tag{4.3.10}$$

the only allowed value of X^2 is then $X^2 = 0$, in order for the moduli dynamic to be regular at the horizon, since a non-zero constant value of $\frac{d\phi^a}{d\omega}$

$$\frac{d\phi^a}{d\omega} = const. \qquad \text{as} \ \ \omega \to \infty \ , \tag{4.3.11}$$

provides a linear dependence on ω that would prevent regular moduli dynamics at the horizon. The only possibility is then

$$\frac{d\phi^a}{d\omega} = 0 \ , \tag{4.3.12}$$

so that the constraint (4.2.7) in the extremal case now strictly requires

$$\frac{A}{4\pi} = V_{BH}(p, q, \phi_H) \ . \tag{4.3.13}$$

In the case of constant scalar fields the black hole is double-extremal, its area is still given by V_{BH}, following immediately from (4.2.7), and it is equal to the area of an extremal black hole with the same electric and magnetic charges

$$A_{extr}(p, q) = A_{double-extr}(p, q) = 4\pi V_{BH}(p, q, \phi_\infty) \ . \tag{4.3.14}$$

The behavior of the scalars near the horizon, taking into account that $\frac{d\phi^a}{d\omega} = 0$, follows from the equation of motion (4.2.9) for which

$$\frac{d^2\phi^a}{d\tau^2} \to \frac{1}{2}\frac{\partial V_{BH}}{\partial \phi^a}\left(\frac{4\pi}{A\tau^2}\right) \ , \tag{4.3.15}$$

whose solution, recalling that a linear dependence on τ coordinates would give a singular dilaton field at the horizon, is

$$\phi^a \approx \phi^a_H + \left(\frac{2\pi}{A}\right)\frac{\partial V_{BH}}{\partial \phi^a} \log \tau \ . \tag{4.3.16}$$

The regularity requirement now gives the following extremum condition on the potential

$$\left(\frac{\partial V_{BH}}{\partial \phi^a}\right)_{hor} = 0 \ . \tag{4.3.17}$$

In this picture the black hole is a solution corresponding to dynamical trajectories in the moduli space $\mathcal{M}_\phi$ from the asymptotic point ϕ_∞ to the critical point ϕ_h.

Black hole entropy and duality orbits

Condition (4.3.13) means that attractor values of the black hole potential corresponds to the black hole entropy. The entropy, for theories based on moduli spaces given by symmetric manifolds $\mathcal{G}/\mathcal{H}$, is a duality invariant. This invariant has the form of a generalized metric on the moduli space, that determines the orbit of the solution. In the same way, in fact, that a four-vector in space-time can be time-like, space-like or light-like, depending on the values of his invariant norm, defined by the metric on the space-time 4-manifold, also the symplectic vector of charges has a different nature depending on the value of the scalar manifold invariant $\mathcal{I}(p^\Lambda, q_\Lambda) > 0$, $\mathcal{I}(p^\Lambda, q_\Lambda) < 0$, or $\mathcal{I}(p^\Lambda, q_\Lambda) = 0$. $\mathcal{I}$ is a U-duality invariant expression depending on the representation of the group $\mathcal{G}$ of $\mathcal{G}/\mathcal{H}$ under which electric and magnetic charges transform. All $\mathcal{N} = 2$ theories with symmetric space based on cubic prepotential, as well as $\mathcal{N} = 4, 6, 8$ theories, have a quartic invariant $\mathcal{I}_4$. The entropy is proportional to the square root of the invariant[1]

$$S_{\text{B-H}} \propto \sqrt{|I_4|}. \tag{4.3.18}$$

BPS solutions have $I_4 > 0$ while the non-BPS ones (with non vanishing central charge) have instead $I_4 < 0$. $\mathcal{N} = 2$ theories with quadratic prepotentials, $\mathcal{N} = 3$ and $\mathcal{N} = 5$ theories have only a quadratic invariant, and the entropy is

$$S_{\text{B-H}} \propto |I_2|. \tag{4.3.19}$$

The BPS solution has $I_2 > 0$, while the non-BPS one has vanishing central charge and $I_2 < 0$.

These solutions fall in the class of large black holes, which have $\mathcal{S}_{BH} \neq 0$, and thus for these configurations $\mathcal{I} \neq 0$. Solutions with $\mathcal{I} = 0$ do exist but they do not correspond to classical attractors since in this case the classical entropy/area formula vanishes. These are the so called small black holes, and to discuss their entropy one has to take into account quantum corrections, and include higher curvature terms in the action.

[1] See [27] and references therein.

4.4 First order BPS and non-BPS flow

The properties of black holes in Supergravity theories depend on the values ϕ_∞ of the massless scalar fields parametrizing the different vacua of the theory. The entropy of the black hole, $S = \frac{A}{4}$, however, in order to be consistent with the microstate counting interpretation in string theory, has to be independent, in the extreme case, of the particular ground state being determined only by the conserved electric and magnetic charges (dyonic black hole). The attractor equations correspond to algebraic constraints on the scalars, which fixes them in terms of the electric-magnetic charges, in such a way that, in the entropy formula, their dependence drops out. They are a horizon boundary condition of a radial flow, yielding a first-order description of black hole dynamics for BPS configurations. This is expected, since the supersymmetric state is actually a solution of the supersymmetry equations, which only contain first order derivatives. It was interestingly unexpected that the first orders formalism can be conveniently used to describe also non-BPS attractor flows of $d = 4$ extremal black holes [28], including solutions corresponding to non-BPS branes configurations in string theory [29].

The attractor flow we describe in this Chapter is valid for static black holes. Due to the nature of Einstein equations, it is possible to construct black hole solutions with multiple centers, which however require a stationary metric. BPS configurations are possible, and we refer to [30] and [31] for they description, where the first full solutions have been found.

4.4.1 BPS flow equations

Using symmetries and suitable ansatze we have been able to reduce a four dimensional system of scalar and vectors coupled to the space-time metric, to a one dimensional problem of solving equations of motions for scalars fields on a manifold, subject to a potential V_{BH}. These equations, although simpler than the original problem, still are second order differential equations. Exploiting supersymmetry, but more in general the form of the Lagrangian and the Hamiltonian constraint [28], we can express the actual fields dynamics through first order equations and then find explicit solutions. Let us notice here that, if we look for solutions in which fermions are identically set to zero, then the supersymmetry variations of bosonic fields, containing fermions in each term, are automatically satisfied. For the analogous reason, the requirement that the solution is supersymmetric, implies that the supersymmetry equations of fermionic fields are indeed first order differential equations for the scalars, under the assumptions of the states being a BPS configuration.

For example, the equations we get from gravitino and gaugino variations are

$$0 = \nabla_\mu \xi_A + \epsilon_{AB} T^-_{\mu\nu} \gamma^\nu \xi^B \,, \tag{4.4.1}$$

$$0 = \mathrm{i}\, \nabla_\mu z^i \gamma^\mu \xi^A + \frac{\mathrm{i}}{2} g^{i\bar{\jmath}} \bar{T}^-_{\bar{\jmath}|\mu\nu} \gamma^{\mu\nu} \epsilon^{AB} \xi_B \,, \tag{4.4.2}$$

where the Killing spinor $\xi_A(r)$ is of the form of a single radial function times a constant spinor satisfying

$$\begin{aligned} \xi_A(r) &= e^{f(r)} \chi_A \qquad \chi_A = \text{constant} \\ \gamma_0 \chi_A &= \mathrm{i}\, \frac{Z}{|Z|}\, \epsilon_{AB} \chi^B \end{aligned} \tag{4.4.3}$$

Notice that the condition (4.4.3) halves the number of supercharges preserved by the solution. Substituting in the supersymmetry equations the ansatz for the metric and the field strength, one finds that the supersymmetry equations (4.4.1,4.4.2) are solved for

$$\begin{cases} U' = -e^U |Z|, \\ z^{i\prime} = -2\, e^U g^{i\bar{\jmath}} \partial_{\bar{\jmath}} |Z| \,, \end{cases} \tag{4.4.4}$$

so the central charge is the superpotential that drives the flow of scalar fields along the radial direction.

4.4.2 Scalar charges and Black Hole asymptotic moduli dependence

The expansion of the scalar fields at spatial infinity

$$\phi^a = \phi^a_\infty + \frac{\Sigma^a}{r} + O(\frac{1}{r^2}) \,, \tag{4.4.5}$$

defines the scalar charges $\Sigma^a = \Sigma^a(A, q_\Lambda, p^\Lambda, \phi^a_\infty)$. In the presence of scalar fields, the first law of thermodynamics for a static dyonic black hole has to be replaced by

$$dM = T dA + \psi^\Lambda dq_\Lambda + \chi_\Lambda dp^\Lambda + \left(\frac{\partial M}{\partial \phi^a}\right) d\phi^a \,, \tag{4.4.6}$$

where the black hole temperature is $T = \frac{\kappa}{2\pi}$, and ψ^Λ, χ_Λ are electric and magnetic scalar potentials, respectively.

The potential $V(\phi, p, q)$ defines a symmetric tensor that satisfies the convexity condition

$$V_{ab} \equiv \nabla_a \nabla_b V \geq 0 \,, \tag{4.4.7}$$

on the scalar manifold M_ϕ. Moreover, if V_{ab} is strictly positive and the scalar charges vanish, the scalar fields have to be frozen to $\phi^a(\tau) = \phi^a_\infty$.

The mass of the black hole, by comparison with the asymptotic Gravitational potential, is given by

$$M = \left(\frac{dU}{d\tau}\right)_{\tau=0} \tag{4.4.8}$$

and this substitution in the constraint (4.2.7) evaluated at spatial infinity ($\tau = 0$) leads to

$$M^2 + G_{ab}(\phi_\infty)\Sigma^a\Sigma^b - V(\phi_\infty, p, q) = 4S^2T^2 \ . \tag{4.4.9}$$

The scond term on the left is the contribution

$$\left(\frac{\partial M}{\partial \phi^a}\right) = -G_{ab}(\phi_\infty)\Sigma^b \tag{4.4.10}$$

in expression (4.4.6). The right hand side is related to the black hole configuration described by the metric (4.1.15) by

$$c = 2ST \ . \tag{4.4.11}$$

For extremal black holes, the attractor mechanism fixes the moduli at the horizon in terms of electric and magnetic charges

$$\phi_{H,extr} = \phi_{fix}(p,q) \ , \tag{4.4.12}$$

and the extreme point can be found, for a given charge configuration, as

$$\left.\frac{\partial M_{extr}}{\partial \phi}\right|_{\phi=\phi_{extr}} = 0 \ . \tag{4.4.13}$$

In particular, the entropy of the extremal black hole is independent on ϕ_∞, being

$$S = \frac{A}{4} = \pi V_{BH}(\phi_{fix}, p, q) \ . \tag{4.4.14}$$

The scalar charge is not conserved, the flux of the gradient vanishes at the horizon, and it reveals that it resides entirely outside the horizon. Equivalently, moduli at infinity or the scalar charges have to be added to the mass M, the charges (q, p) and, in the non static case, to the angular momentum J to completely characterize the black hole solution.

4.4.3 First order formalism for $d = 4$ Extremal Black Holes

For $d = 4$ supergravities a general formula for a black hole effective potential holds,

$$V_{BH} = \frac{1}{2} Z_{AB} \overline{Z}^{AB} + Z_I \overline{Z}^I, \tag{4.4.15}$$

where $Z_{AB} = Z_{[AB]}$ $(A, B = 1, ..., \mathcal{N})$ is the central charge matrix, and Z_I $(I = 1, ..., n)$ are the matter charges, $n \in \mathbb{N}$ is the number of matter multiplets. One can in general rewrite this potential, in the *first order formalism* , as

$$V_{BH} = \mathcal{W}^2 + 4G^{i\bar{j}} (\partial_i \mathcal{W}) \overline{\partial}_{\bar{j}} \mathcal{W} = \mathcal{W}^2 + 4G^{i\bar{j}} (\nabla_i \mathcal{W}) \overline{\nabla}_{\bar{j}} \mathcal{W}, \tag{4.4.16}$$

where $\mathcal{W}$ is the moduli-dependent *first order superpotential*, and ∇ is the covariant differential operator.

In fact, the second order equations of motion (4.2.8) and (4.2.9) can be derived by a first order system, by performing the ansatz

$$\dot{U} = e^U \mathcal{W}(\phi, \tau) \ , \tag{4.4.17}$$

where $\dot{U} = \frac{dU}{d\tau}$. At the horizon we get the condition

$$\partial_\tau \mathcal{W} = 0 \ . \tag{4.4.18}$$

Differentiating equation (4.4.17) with respect to τ gives the equation of motion for the field $U(\tau)$ and the identification of

$$V_{BH} = \mathcal{W}^2 + e^{-U} \dot{\phi}^a \partial_a \mathcal{W} \ . \tag{4.4.19}$$

It follows from the constraint (4.2.7) that

$$\ddot{U} - \dot{U}^2 = \frac{1}{2} \dot{\phi}^a \dot{\phi}^b G_{ab} = \dot{\phi}^a \partial_a \mathcal{W} e^U \ , \tag{4.4.20}$$

which, disregarding contributions that do not affect the entropy, is solved by

$$\dot{\phi}^a = 2e^U g^{rs} \partial_s \mathcal{W} \ , \tag{4.4.21}$$

where the last equation is a first order type BPS-like condition. The effective potential becomes, as stated above,

$$V_{BH} = \mathcal{W}^2 + 2G^{ab} \partial_a \mathcal{W} \partial_b \mathcal{W} \ . \tag{4.4.22}$$

Extremization of V_{BH} corresponds to

$$\partial_a V_{BH} = 2\partial_b \mathcal{W} (\mathcal{W} \delta_a^b + 2G^{bc} \nabla_a \partial_c \mathcal{W}) = 0 \ , \tag{4.4.23}$$

which means that in the first order formalism the attractor point for scalar fields at the horizon of extremal black holes is directly related to the extrema of $\mathcal{W}$. This formalism leads to first order equations which imply second order equations of motion, but does not rely on supersymmetry. As was first shown in [28], remarkably, whenever the black hole effective potential admits a rewriting in the form (4.4.22), then the black hole equations reduce to the first order differential system

$$\begin{cases} U' = -e^U \mathcal{W}, \\ z^{i\prime} = -2\, e^U g^{i\bar{\jmath}}\, \overline{\partial}_{\bar{\jmath}} \mathcal{W}. \end{cases} \tag{4.4.24}$$

Obviously, the BPS case is recovered for $\mathcal{W} = |Z|$. However, notice that the expression (4.4.22) can be satisfied for different $\mathcal{W}$'s. This is why the first formalism is useful: it allows us to find the critical points of the black hole potential which are not critical points of the central charge. Attractor mechanism, then, tells us that those point correspond to extremal black holes, that, in the case $\mathcal{W} \neq |Z|$, are simply the non-BPS extremal solutions. For them, because of lack of supersymmetry, $\mathcal{W}$ is called *fake* superpotential.

4.4.4 General properties of attractors for $\mathcal{N} = 2$ Supergravity

For $\mathcal{N} = 2$ Supergravity, the black hole potential at the attractor point is given by one of the quadratic invariants of the scalar manifold

$$V_{BH} = I_1 = |Z|^2 + |D_i Z|^2 \ , \tag{4.4.25}$$

where D is the Kähler covariant derivative with respect to the complex holomorphic coordinates on the special Kähler manifold. The horizon is an attractor point and this condition requires, as we have seen, that it is also a critical point of a suitable black hole effective potential V_{BH}, namely

$$\partial_i V_{BH}|_h = 0 \ . \tag{4.4.26}$$

This is an algebraic equation on the central charge and its covariant derivatives; in fact, one has that

$$\begin{aligned} \partial_i V_{BH} &= \partial_i(|Z|^2 + |D_i Z|^2) = \\ &= \bar{Z} D_i Z + G^{k\bar{l}}(D_i D_k Z D_{\bar{l}} \bar{Z}) + G^{k\bar{l}} D_k Z D_i D_{\bar{l}} \bar{Z} \ . \end{aligned} \tag{4.4.27}$$

The extremum condition is satisfied whenever at the horizon

- $D_i Z = 0$, $Z \neq 0$, BPS ;
- $D_i Z \neq 0$, $Z = 0$, non-BPS ;

which refer to a supersymmetric and non-supersymmetric black hole solution, respectively.

Black hole entropy at the attractor points is given by the absolute value of the second quadratic invariant of the symmetric space,

$$S = |I_2| = \left||Z|^2 - |D_i Z|^2\right| \ , \tag{4.4.28}$$

where $|D_i Z|^2 = G^{i\bar{i}} D_i Z \, D_{\bar{i}} \bar{Z}$. In the BPS case, $D_i Z = 0$ yields

$$S_{N=2-Symm-BPS} = |Z|^2 \ . \tag{4.4.29}$$

The attractor point corresponds, in the BPS case, to a minimum of the potential. In fact, the Hessian metric for the black hole potential is

$$\begin{aligned} V_{BH\ \bar{j}i} &= \left. 2D_{\bar{j}}\bar{Z} D_i Z + 2\bar{Z} D_{\bar{j}} D_i Z \right|_{Z_i=0} = \\ &= \left. 2\bar{Z} D_{\bar{j}} D_i Z \right|_{Z_i=0} \ , \end{aligned} \tag{4.4.30}$$

and, from the relations of special geometry,

$$V_{BH\ \bar{j}i} = 2G_{\bar{j}i}|Z|^2 \ . \tag{4.4.31}$$

Since the metric is positive defined, this matrix has no null-eigenvalues, which means that there are no "flat directions" for the scalar fields.

Black hole entropy at the horizon, as well as the black hole potential, are invariant expressions of the charges, and can be written as

$$V_{BH} = -\frac{1}{2} Q^t \mathcal{M}(\mathcal{N}) Q \ , \tag{4.4.32}$$

$$S_{BH} = \left.\frac{1}{2} Q^t \mathcal{M}(\mathcal{F}) Q\right|_{\phi^i = \phi^i_{hor}} \ , \tag{4.4.33}$$

where $\mathcal{N}$ is the matrix in the vector fields kinetic term, and $\mathcal{F} \equiv F_{\Lambda\Sigma} = \partial_\Lambda \partial_\Sigma F(X)$. Equations (4.4.33) depends on the scalars through $\mathcal{F}$, and thus only holds at the attractor point, the horizon, while the expression for V_{BH} is valid along the flow.

4.4.5 Attractors for $\mathcal{N} = 8$ Supergravity

The black hole potential for $\mathcal{N} = 8$, d=4 supergravity is given by [20],[18]

$$\mathcal{V}_{BH}(\phi, Q) = Z_{AB} Z^{*AB} = \langle Q, V_{AB} \rangle \langle Q, \bar{V}^{AB} \rangle \qquad A, B = 1, \ldots, 8 \tag{4.4.34}$$

where Z_{AB} (and its conjugate Z^{*AB}) is the central charge matrix

$$Z_{AB}(\phi, Q) = \langle Q, V_{AB} \rangle = f^{\Lambda\Sigma}_{AB} e_{\Lambda\Sigma} - h_{\Lambda\Sigma,AB} m^{\Lambda\Sigma} \ . \tag{4.4.35}$$

Q is the charge vector, in the fundamental representation 56 of $E_{7(7)}$ and the symplectic section $(f^{\Lambda\Sigma}_{AB}(\phi), h_{\Lambda\Sigma,AB}(\phi))$ is an element of the coset space $\frac{E_{7(7)}}{SU(8)}$ which connects the real 56 representation to the complex 28 indices of $[AB]$. There are 70 real scalars ϕ^i (the local $SU(8)$ symmetry removes 63 scalars from the 133 parameters of $E_{7(7)}$), we sum over the indices $AB, \Lambda\Sigma$ for $A < B$ and $\Lambda < \Sigma$ in (4.4.34), (4.4.35).

Maurer−Cartan equations for the coset space define the covariant derivative of the central charge matrix

$$\mathcal{D}_i Z_{AB} = \frac{1}{2} P_{i,[ABCD]}(\phi) Z^{*CD}(\phi, Q) \tag{4.4.36}$$

where we introduced the 70×70 vielbein of the $E_{7(7)}/SU(8)$ coset space $P_{i,[ABCD]} d\phi^i$, $i = 1, \dots, 70$, whose self-dual real is

$$P_{i,[ABCD]} = \frac{1}{4!} \epsilon_{ABCDEFGH} (P_{i,[ABCD]})^* \ , \tag{4.4.37}$$

while D_i is the SU(8) covariant derivative [18].

The attractor condition, expressed as minimization of the black hole potential with respect to the scalars, is given by

$$\begin{aligned} 0 = \partial_i \mathcal{V} &= \frac{1}{2} \Big(\mathcal{D}_i Z_{AB} Z^{*AB} + Z_{AB} \mathcal{D}_i Z^{*AB} \Big) \\ &= \frac{1}{4} \Big(P_{i,[ABCD]} Z^{*AB} Z^{*CD} + P_i^{[ABCD]} Z_{AB} Z_{CD} \Big) \\ &= \frac{1}{4} P_{i,[ABCD]} \Big[Z^{*[CD} Z^{*AB]} + \frac{1}{4!} \epsilon^{CDABEFGH} Z_{EF} Z_{GH} \Big] \end{aligned} \tag{4.4.38}$$

thanks to the self duality condition.

It is important to notice that the vielbein $P_{i,[ABCD]}$ is invertible, thus one can multiply the previous equation by $(P_{i,[A'B'C'D']})^{-1}$ and still get a necessary and sufficient condition on the critical points of the black hole potential, that now is simply written as

$$Z^{*[AB} Z^{*CD]} + \frac{1}{4!} \epsilon^{ABCDEFGH} Z_{EF} Z_{GH} = 0 \ . \tag{4.4.39}$$

Moreover, one can rotate the central charge matrix to its normal form [32] where it has only the non-vanishing complex skew-eigenvalues $z_1 = Z_{12}, z_2 = Z_{34}, z_3 = Z_{56}, z_4 = Z_{78}$, and in such basis the attractor equations are

$$\begin{aligned} z_1 z_2 + z^{*3} z^{*4} &= 0 \\ z_1 z_3 + z^{*2} z^{*4} &= 0 \\ z_2 z_3 + z^{*1} z^{*4} &= 0 \end{aligned} \tag{4.4.40}$$

$SU(8)$ symmetry of Z_{AB} allows to further reduce all 4 complex eigenvalues to the following normal form

$$z_i = \rho_i e^{i\varphi/4} \qquad i = 1, 2, 3, 4. \tag{4.4.41}$$

so that only 5 real parameters are left independent: the 4 absolute values ρ_i and an overall phase, φ (the relative phase of z_i's can be changed but not the overall phase). The form of the quartic invariant J_4 is then [33]

$$J_4 = \left[(\rho_1 + \rho_2)^2 - (\rho_3 + \rho_4)^2\right]\left[(\rho_1 - \rho_2)^2 - (\rho_3 - \rho_4)^2\right] + 8\rho_1\rho_2\rho_3\rho_4(\cos\varphi - 1) \tag{4.4.42}$$

The central charge matrix in normal form is written as

$$Z_{AB} = \begin{pmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_3 & 0 \\ 0 & 0 & 0 & \rho_4 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{i\varphi/4} \tag{4.4.43}$$

we can order ρ's as $\rho_1 \geq \rho_2 \geq \rho_3 \geq \rho_4$ so that the first term in J_4 is positive, null or negative depending whether $\rho_1 - \rho_2 \geq \rho_3 - \rho_4$ or $\rho_1 - \rho_2 \leq \rho_3 - \rho_4$ while the last term is negative or null. It is easily seen, then, that the 1/8 BPS attractor orbit, characterized by $J_4 > 0$, corresponds to $\rho_2 = \rho_3 = \rho_4 = 0$, and the non-BPS one to $\rho_i = \rho$ and $\varphi = \pi$, which indeed implies $J_4 < 0$.

Notice that the non-BPS critical points of the potential the matrix of the second derivative is not guaranteed to be positive definite [34], and a critical point of the potential may not be its minimum.

The attractor equations (4.4.40) have thus 2 solutions that correspond to black holes with finite horizons

- 1/8 BPS orbit

$$z_1 = \rho_{BPS} e^{i\varphi_1} \neq 0 \qquad z_2 = z_3 = z_4 = 0 \qquad J_4^{BPS} = \rho_{BPS}^4 > 0 \tag{4.4.44}$$

  The black hole entropy-area of the corresponding 1/8-BPS black holes is

$$\frac{S_{BPS}(Q)}{\pi} = \frac{A_{BPS}(Q)}{4\pi} = \sqrt{J_4^{BPS}(Q)} = \rho_{BPS}^2 \tag{4.4.45}$$

- non-BPS orbit

$$z_i = \rho\, e^{i\frac{\pi}{4}} \qquad J_4^{nonBPS} = -16\rho_{nonBPS}^4 \tag{4.4.46}$$

  In the non-BPS case, the black hole entropy-area is given by

$$\frac{S_{nonBPS}(Q)}{\pi} = \frac{A_{nonBPS}(Q)}{4\pi} = \sqrt{-J_4^{BPS}(Q)} = 4\rho_{nonBPS}^2 \tag{4.4.47}$$

4.4.6 The gauged flow

The flow equations and derivation of the one dimensional Lagrangian, we discussed so far in the chapter, have been the subject of an extensive research over the past 15 years. However, as we have seen, the gauging of a supergravity theory, even in the minimal case, introduces a scalar potential which changes the action (2.2.1) we started from. The momentum map construction ensures the duality covariance of the gauging, however, new BPS flow equations and a new ansatz for the metric have to be considered.

Let us consider the Fayet-Iliopoulos gauging in $\mathcal{N} = 2$ supergravity. We can choose The main points that will allow us to find new solutions are

- The introduction of a second warp factor in the metric ansatz (4.1.15), for the angular part, which in general give a nontrivial holonomy on the 2-space orthogonal to the radial direction, and yielding a metric for static and spherically symmetric solutions of the form

$$ds^2 = -e^{2U} dt^2 + e^{-2U} dr^2 + e^{2U-2\psi}(d\theta^2 + \sin\theta^2 d\phi^2) \; ; \tag{4.4.48}$$

  where $U = U(r)$ and $\psi = \psi(r)$ have only radial dependence.

- If a black hole solution whose spherical horizon is regular, thus has nonzero area, also preserves some supersymmetry, then its electric-magnetic charges satisfy a constraint dictated by the gauging parameters $\mathcal{G} = (\xi^\Lambda, \xi_\Lambda)$ that can be expressed as

$$\langle \mathcal{G}, \mathcal{Q} \rangle = -1 \; . \tag{4.4.49}$$

These solutions are built for a gauging which also involves tensor fields, which allow the duality to be restored introducing a set of dual gauging parameters ξ^Λ in addition to the ξ_Λ. We notice however that in $U(1)$ gauged supergravity the fields are not charged under the gauge field, and the only modification to the action is the introduction of the scalar potential as in (3.4.28). It is possible, then, using the duality-complete vector of gauging $\mathcal{G}$ defined above, to give a duality invariant definition of the scalar potential as

$$V_{gauging} = |\mathcal{D}_i \mathcal{L}|^2 - 3|\mathcal{L}|^2 \; , \tag{4.4.50}$$

by means of a new symplectic invariant quantity which is analogous to the central charge, defined as

$$\mathcal{L} = \langle \mathcal{G}, \mathcal{V} \rangle \; , \tag{4.4.51}$$

with $\mathcal{V} = (X^\Lambda, F_\Lambda)$ the normalized symplectic sections. It is then straightforward to generalize the superpotential, driving the BPS flow in the ungauged theory, to the gauged case as

$$W = e^U |\mathcal{Z} - ie^{2A}\mathcal{L}| \tag{4.4.52}$$

where it is convenient to define the combination $A = \psi - U$. The BPS flow is

$$\begin{aligned} U' &= -e^{-2(A+U)} \left(W - \partial_A W\right), \\ A' &= e^{-2(A+U)}\, W, \\ z^{i\prime} &= -2e^{-2(A+U)}\, g^{i\bar{\jmath}}\, \partial_{\bar{\jmath}} W. \end{aligned} \tag{4.4.53}$$

With respect to the ungauged case the non linear sigma model is now enlarged to 2 new fields, instead of just one, the warp factor U and the new function ψ. Another feature that differs both from the ungauged case and the previously known, zero area, solutions of gauged supergravity [35],[36], is that the static black hole with regular horizon, that we will describe, only preserves 2 out of the 8 supercharges of the theory along the flow. It is then a 1/4-BPS solution, unlike the usual 1/2-BPS nature of the solution in the ungauged supergravity. The conditions that impose the supersymmetry preservation are related to those found in [37] for a massless supersymmetric state there called "cosmic monopole".

Chapter 5

Duality orbits in N=8 Supergravity

The $\mathcal{N} = 8$ supergravity theory in $d = 4$ [20] and $d = 5$ [38] dimensions is a remarkable theory which unifies the gravitational fields with other lower spin particles in a rather unique way, due to the high constraints of local $\mathcal{N} = 8$ supersymmetry, the maximal one realized in a 4d Lagrangian field theory. These theories, particularly in four dimensions, are supposed to enjoy exceptional ultraviolet properties. For this reason, 4d supergravity has been advocated not only as the simplest quantum field theory [39] but also as a potential candidate for a finite theory of quantum gravity, even without its completion into a larger theory [40].

Maximal supergravity in highest dimensions has a large number of classical solutions [41] which may survive at the quantum level. Among them, there are black p-branes of several types [42] and interestingly, 4d black holes of different nature. On the other hand, theories with lower supersymmetries (such as $\mathcal{N} = 2$) emerging from Calabi-Yau compactifications of M-theory or superstring theory, admit extremal black hole solutions that have been the subject of intense study, because of their wide range of classical and quantum aspects.

For asymptotically flat, stationary and spherically symmetric extremal black holes, the attractor behavior [9]-[13], has played an important role not only in determining universal features of fields flows toward the horizon, but also to explore dynamical properties such as wall crossing [43] and split attractor flows [44], the connections with string topological partition functions [45] and relations with microstates counting [46] . Therefore, it has become natural to study the properties of extremal black holes not only in the context of $\mathcal{N}=2$, but also in theories with higher supersymmetries, up to $\mathcal{N}=8$ [47]-[48].

5.1 Introduction

It has been known for some time [49] that extremal BPS black hole states coming from string and M theory compactifications to four and five dimensions, preserving various fractions of the original $\mathcal{N}=8$ supersymmetry, can be invariantly classified in terms of orbits of the fundamental representations of the exceptional groups $E_{7(7)}$ and $E_{6(6)}$. These are the duality groups of the low energy actions, whose discrete subgroups appear as symmetries of the non-perturbative spectrum of BPS states [50]. These orbits, which have been further studied in [33, 51, 52], correspond to well defined categories of allowed entropies of extremal black holes in $d=5$ and in $d=4$, given in terms of the cubic $E_{6(6)}$ invariant $\mathcal{I}_3$ [49, 51, 53] and the quartic $E_{7(7)}$ invariant $\mathcal{I}_4$ [54, 55, 21]. There are three types of orbits depending on whether the black hole background preserves 1/2, 1/4 or 1/8 of the original supersymmetry. Only 1/8 BPS states have non vanishing entropy and regular horizons, while 1/4 and 1/2 BPS configurations lead to vanishing classical entropy.

It has been shown in the previous chapters, how to solve the criticality condition for the $\mathcal{N}=8$ attractors black hole effective potential, extending the lore of $\mathcal{N}=2$ special Kähler geometry.

In this Chapter we focus on some specific simple configurations in $\mathcal{N}=8$, $d=4$ supergravity, which capture some representatives of the *"large"* BPS and non-BPS charge orbits of the theory, corresponding to regular extremal black holes, with non-vanishing classical entropy. One is the Reissner−Nördstrom (RN) dyonic black hole, with electric and magnetic charge e and m respectively, and Bekenstein−Hawking entropy (in unit of Planck mass) [8]

$$S_{RN}=\pi\left(e^2+m^2\right). \tag{5.1.1}$$

Another one is the Kaluza−Klein dyonic black hole, with a KK monopole charge p and a KK momentum q, which is dual to a $D0-D6$ brane configuration in Type II A supergravity. Its Bekenstein−Hawking entropy reads

$$S_{KK}=\pi\left|pq\right|. \tag{5.1.2}$$

One more interesting example is the extremal axion-dilaton black hole, a subsector of pure $\mathcal{N}=4$ supergravity in $d=4$ which was considered in the past [56]-[64].

We will show how the entropies of these black holes can be obtained in the context of $\mathcal{N} = 8$, $d = 4$ supergravity by exploiting the attractor mechanism [9]-[11],[13], for extremal BPS and non BPS black holes . Earlier studies for some specific cases where examined in [65], [66].

The black hole charge configuration with entropy given by (5.1.1) is 1/8 BPS [8], while the entropy (5.1.2) is related to a non BPS one. Indeed, the $E_{7(7)}$ quartic invariant $\mathcal{I}_4$ on these configurations reduces to

$$\sqrt{\mathcal{I}_4^{RN}} = e^2 + m^2; \tag{5.1.3}$$

$$\sqrt{-\mathcal{I}_4^{KK}} = |pq| \,. \tag{5.1.4}$$

In particular we note that, if the magnetic (or electric) charge is switched off, the Reissner−Nördstrom black hole remains regular, whereas the Kaluza−Klein black hole reaches zero entropy ($\mathcal{I}_4 = 0$) and becomes 1/2 BPS [33].

The simplest way to obtain these configurations is to observe that the BPS and non-BPS charge orbits with $\mathcal{I}_4 \neq 0$ in $\mathcal{N} = 8$, $d = 4$ supergravity are given by [49]

$$\mathcal{O}_{1/8-BPS} : \frac{E_{7(7)}}{E_{6(2)}}, \qquad \mathcal{I}_4 > 0; \tag{5.1.5}$$

$$\mathcal{O}_{non-BPS} : \frac{E_{7(7)}}{E_{6(6)}}, \qquad \mathcal{I}_4 < 0. \tag{5.1.6}$$

The moduli spaces corresponding to the above disjoint orbits are [67]

$$\begin{aligned} \mathcal{M}_{1/8-BPS} &= \frac{E_{6(2)}}{SU(6) \times SU(2)} \\ \mathcal{M}_{non-BPS} &= \frac{E_{6(6)}}{USp(8)} \,. \end{aligned} \tag{5.1.7}$$

Hence, a convenient representative of these orbits is given by the (unique) E_6-singlets in the decomposition of the fundamental representation **56** of $E_{7(7)}$ into the two relevant non-compact real forms of E_6:

$$RN \qquad \mathcal{O}_{1/8-BPS} : \begin{cases} E_{7(7)} \to E_{6(2)} \times U(1)\,; \\ \\ \mathbf{56} \to (\mathbf{27}, 1) + (\mathbf{1}, 3) + \left(\overline{\mathbf{27}}, -1\right) + (\mathbf{1}, -3)\,; \end{cases} \tag{5.1.8}$$

$$KK \qquad \mathcal{O}_{non-BPS} : \begin{cases} E_{7(7)} \to E_{6(6)} \times SO(1,1)\,; \\ \\ \mathbf{56} \to (\mathbf{27}, 1) + (\mathbf{1}, 3) + (\mathbf{27}', -1) + (\mathbf{1}', -3)\,, \end{cases} \tag{5.1.9}$$

where the $U(1)$ charges and $SO(1,1)$ weights are indicated, and the prime denotes the contravariant representations. Notice that, consistently with the group factors $U(1)$ and $SO(1,1)$, **27** is complex for $E_{6(2)}$, whereas it is real for $E_{6(6)}$. Both $E_{6(2)} \times U(1)$ and $E_{6(6)} \times SO(1,1)$ are maximal non-compact subgroups of $E_{7(7)}$, with symmetric embedding.

The results from the algebraic analysis can be stated as follows. The two extremal black hole charge configurations determining the embedding of RN and KK extremal black holes into $\mathcal{N}=8$, $d=4$ supergravity with entropies (5.1.1) and (5.1.2), are given by the two E_6-singlets in the decompositions (5.1.8) and (5.1.9).

The two situations can be efficiently associated to two different parametrizations of the real symmetric scalar manifold $\frac{E_{7(7)}}{SU(8)}$ ($dim_{\mathbb{R}}=70$, $rank$= 7) of $\mathcal{N}=8$, $d=4$ supergravity.

For the branching (5.1.8), pertaining to the RN extremal black hole, the relevant parametrization is the $SU(8)$-covariant one. This corresponds to the Cartan's decomposition basis, where the coset coordinates ϕ_{ijkl} $(i=1,...8)$ sit in the four-fold antisymmetric self-real irrep **70** of $SU(8)$. The attractor mechanism implies that at the horizon

$$\phi_{ijkl,H}=0, \tag{5.1.10}$$

i.e. the scalar configuration at the event horizon of the 1/8-BPS extremal black hole is given by the origin of $\frac{E_{7(7)}}{SU(8)}$. Some care should be taken with regards to flat directions [55], [67]. The $\frac{1}{8}$-BPS attractor solutions has a moduli space $\frac{E_{6(2)}}{SU(6)\times SU(2)}$, with dimension $dim_{\mathbb{R}}=40$, and rank= 4, which leaves 40 scalar degrees of freedom out of 70 undetermined, at the event horizon of the given $\frac{1}{8}-$BPS RN extremal black hole. In other words, 40 real scalar degrees of freedom, spanning the quaternionic symmetric coset $\frac{E_{6(2)}}{SU(6)\times SU(2)}$ (which is the *c-map* [68] of the vector multiplets' scalar manifold of $\mathcal{N}=2$, $d=4$ *"magic"* supergravity based on $J_3^{\mathbb{C}}$), can be set to any real value, without affecting the RN black hole entropy (5.1.1).

It should be noticed that, consistently with the Gaillard−Zumino formulation of electric-magnetic duality in presence of scalar fields, the solution (5.1.10) to the attractor equations is the only one allowed in presence of a compact underlying symmetry (in this case $U(1)$).

The branching (5.1.9), pertaining to the KK extremal black hole, on the other hand, is parametrized by the KK radius

$$r_{KK}\equiv\mathcal{V}^{1/3}\equiv e^{2\varphi}, \tag{5.1.11}$$

by the 42 real scalars ψ_{ijkl} $(i = 1, ...8)$ sitting in the **42** of $USp\,(8)$, and by the 27 real *axions* a^I $(I = 1, \ldots, 27)$ sitting in the **27** of $USp\,(8)$ (or equivalently, in the **27** of $E_{6(6)}$).

In virtue of the attractor mechanism, the KK radius is stabilized as [69]

$$r^3_{KK,H} \equiv \mathcal{V}_H \equiv e^{6\varphi_H} = 4\left|\frac{q}{p}\right|, \tag{5.1.12}$$

and has vanishing axions

$$a^I_H = 0. \tag{5.1.13}$$

The 42 real scalars ψ_{ijkl} are actually undetermined at the event horizon of the non-BPS KK black hole, without affecting its entropy (5.1.2). Indeed, they span the moduli space $\frac{E_{6(6)}}{USp(8)}$ $(dim_{\mathbb{R}} = 42,\ rank = 6)$ of the non-BPS attractor solutions, which is the real symmetric scalar manifold of $\mathcal{N} = 8$, $d = 5$ supergravity [67].

Thus, it follows from this discussion that the possibility of having a non-vanishing scalar stabilized at the horizon of the KK extremal black hole is related to the presence of a singlet in the corresponding decomposition of the 70 scalars. This is related to the existence of an underlying non-compact symmetry ($SO\,(1,1)$ in the present case), admitting no compact sub-symmetry.

An alternative way to obtain eqs. (5.1.1) and (5.1.2) is to use appropriate truncations for the bare charges in the expression of the quartic invariant $\mathcal{I}_4$, which is related to the Bekenstein−Hawking entropy by the formula

$$S = \sqrt{|\mathcal{I}_4|}. \tag{5.1.14}$$

The manifestly $SU(8)$-invariant expression of $\mathcal{I}_4$ reads as follows:

$$\mathcal{I}_4 = Tr\left(ZZ^\dagger\right)^2 - \frac{1}{4}Tr^2\left(ZZ^\dagger\right) + 8RePf\,(Z)\,, \tag{5.1.15}$$

where $Z \equiv Z_{AB}\,(\phi)$ is the central charge 8×8 skew-symmetric matrix. Since (5.1.15) is moduli-independent, it can be evaluated at $\phi = 0$ without loss of generality, and in such a case Z_{AB} is replaced by Q_{AB}, the *bare* charge matrix in the $SU\,(8)$ basis.

Considering the RN black hole, we will see that a suitable truncation of the $\mathcal{N} = 8$ *bare* charge matrix Q_{AB} $(A, B = 1, \ldots 8)$, reduces it to the form

$$Q^{RN}_{AB} \to (z\epsilon_{ab}, 0)\,,\ \ z \equiv e + im, \tag{5.1.16}$$

where $a, b = 1, 2$ and $\epsilon^T = -\epsilon$. Thus one obtains

$$\mathcal{I}_4 = |z|^4 = \left(e^2 + m^2\right)^2, \tag{5.1.17}$$

which is nothing but Eq. (5.1.3) and it is also the same result as in *pure* $\mathcal{N}=2$, $d=4$ supergravity, which has a $U(1)$ global $\mathcal{R}$-symmetry [8].

In the case of the Kaluza−Klein orbit solution, one has to consider the manifestly $E_{6(6)}$-invariant expression of $\mathcal{I}_4$ in terms of the cubic invariant $\mathcal{I}_3$, as function of the bare electric and magnetic charges [49, 70, 52]:

$$\mathcal{I}_4 = -\left(p^0 q_0 + p^i q_i\right)^2 + 4\left[q_0 \mathcal{I}_3\left(p\right) - p^0 \mathcal{I}_3\left(q\right) + \left\{\mathcal{I}_3\left(p\right), \mathcal{I}_3\left(q\right)\right\}\right]. \tag{5.1.18}$$

The corresponding truncation is given by the choice of the fluxes

$$p^i = 0 = q_i, \tag{5.1.19}$$

so thatt one obtains ($p^0 \equiv p$, $q_0 \equiv q$)

$$\mathcal{I}_4 = -\left(pq\right)^2, \tag{5.1.20}$$

which now coincides with Eq. (5.1.4).

We will show that there is yet another way to obtain the two entropies for RN and KK black holes (5.1.1) and (5.1.2) . This consists in using the attractor equations for the effective black hole potential $\frac{\partial V_{BH}}{\partial \phi} = 0$ and the expression of the entropy as the value of such potential at the critical point,

$$S = \pi \left. V_{BH}\right|_{crit}. \tag{5.1.21}$$

In the following, we will first consider various bases of $\mathcal{N}=8$, $d=4$ supergravity, namely the $SL\left(8,\mathbb{R}\right)$, $SU\left(8\right)$- and $USp(8)$-covariant ones, exploiting the relevant branchings of the U-duality group $E_{7(7)}$. We will analyze the fundamental quantities for the geometry of the scalar manifold $\frac{E_{7(7)}}{SU(8)}$ in the $SL\left(8,\mathbb{R}\right)$-covariant basis, first, and then move to the $E_{6(6)}$-covariant basis; with the goal of exhibiting the connection with $\mathcal{N}=8$, $d=5$ supergravity, we will recast the $d=4$ effective black hole potential in a manifestly $d=5$ covariant form. The charge configurations of this potential leading to vanishing axion fields are studied along with the corresponding attractor solutions. The embedding of the axion-dilaton extremal black hole in $\mathcal{N}=8$, $d=4$ supergravity, through an intermediate embedding into $\mathcal{N}=4$, $d=4$ theory with 6 vector multiplets will be presented.

5.2 Symplectic Frames

The de Wit-Nicolai [71] formulation of $\mathcal{N}=8$, $d=4$ supergravity is based on a symplectic frame where the maximal non-compact symmetry of the Lagrangian is $SL(8,\mathbb{R})$ [72], according to the decomposition

$$\begin{gathered} E_{7(7)} \to SL(8,\mathbb{R}), \\ \mathbf{56} \to \mathbf{28} + \mathbf{28}', \end{gathered} \tag{5.2.1}$$

where $SL(8,\mathbb{R})$ is a maximal non-compact subgroup of $E_{7(7)}$, and **28** is its two-fold antisymmetric irreducible representation. Since there is no matter coupling, the $SU(8)$ $\mathcal{R}$-symmetry, is the stabilizer of the scalar manifold. It is not a symmetry of the Lagrangian, but only of the equations of motion, the maximal compact symmetry of the Lagrangian is indeed the intersection of $SL(8,\mathbb{R})$ with $SU(8)$, which is $SO(8)$, the maximal compact subgroup of $SL(8,\mathbb{R})$ itself.

Another symplectic frame corresponds to the decomposition (5.1.9). In this case, the maximal non-compact symmetry of the Lagrangian is $E_{6(6)} \times SO(1,1) \ltimes T_{27}$, with T_{27} standing for the 27-dimensional Abelian subgroup of $E_{7(7)}$. The maximal compact symmetry is now $USp(8)$, which is also the maximal compact symmetry of the Lagrangian. Note that all terms in the Lagrangian are $SU(8)$ invariant, with the exception of the vector kinetic terms, which are $SU(8)$-invariant only on-shell.

Let us decompose $E_{7(7)}$ along two different maximal non-compact subgroups according to the following diagram:

$$\begin{array}{ccc} E_{7(7)} & \longrightarrow & SL(8,\mathbb{R}) \\ \downarrow & & \downarrow \\ E_{6(6)} \times SO(1,1) & \longrightarrow & SL(6,\mathbb{R}) \times SL(2,\mathbb{R}) \times SO(1,1)\,. \end{array} \tag{5.2.2}$$

If one goes first horizontally, the **56** of $E_{7(7)}$ decomposes as

$$\mathbf{56} \to \mathbf{28} + \mathbf{28}' \to \begin{cases} (\mathbf{15},\mathbf{1},1) + (\mathbf{6},\mathbf{2},-1) + (\mathbf{1},\mathbf{1},-3) + \\ + (\mathbf{15}',\mathbf{1},-1) + (\mathbf{6}',\mathbf{2},1) + (\mathbf{1},\mathbf{1},3)\,. \end{cases} \tag{5.2.3}$$

Alternatively, one can first go downward, and use that

$$\begin{aligned} &E_{6(6)} \to SL(6,\mathbb{R}) \times SL(2,\mathbb{R})\,; \\ &\mathbf{27} \to (\mathbf{15},\mathbf{1}) + (\mathbf{6}',\mathbf{2})\,, \\ &\mathbf{1} \to (\mathbf{1},\mathbf{1})\,, \end{aligned} \tag{5.2.4}$$

thus obtaining:

$$\mathbf{56} \to (\mathbf{27},1) + (\mathbf{1},3) + \big(\mathbf{27}',-1\big) + (\mathbf{1},-3) \to \begin{cases} (\mathbf{15},\mathbf{1},1) + (\mathbf{6}',\mathbf{2},1) + (\mathbf{1},\mathbf{1},3) + \\ \\ + (\mathbf{15}',\mathbf{1},-1) + (\mathbf{6},\mathbf{2},-1) + (\mathbf{1},\mathbf{1},-3)\,. \end{cases} \tag{5.2.5}$$

Therefore, either way on the diagram and irrespectively of the intermediate decomposition, one obtains the same irreducible representations of $SL\,(6,\mathbb{R})\times SL\,(2,\mathbb{R})\times SO\,(1,1)$, which enjoyes a unique embedding in the U-duality group $E_{7(7)}$. In particular, one sees that the singlets are indeed the same in the two cases, and the alternative decompositions are related by the interchange of $(\mathbf{15},\mathbf{1},1)$ with $(\mathbf{15}',\mathbf{1},-1)$. We can thus conclude that these two formulations, corresponding to two different symplectic frames, can be interchanged by dualizing 15 out of the 28 vector fields.

An analogous argument holds if one decomposes $E_{7(7)}$ according two two different maximal compact subgroups along the diagram

$$\begin{array}{ccc} E_{7(7)} & \longrightarrow & SU(8) \\ \downarrow & & \downarrow \\ E_{6(2)} \times U(1) & \longrightarrow & SU(6)\times SU(2)\times U(1)\,. \end{array} \tag{5.2.6}$$

This time, going first horizontally along the diagram, the result reads:

$$\mathbf{56} \to \mathbf{28} + \overline{\mathbf{28}} \to \begin{cases} (\mathbf{15},\mathbf{1},1) + (\mathbf{6},\mathbf{2},-1) + (\mathbf{1},\mathbf{1},-3) + \\ \\ + \big(\overline{\mathbf{15}},\mathbf{1},-1\big) + \big(\overline{\mathbf{6}},\mathbf{2},1\big) + (\mathbf{1},\mathbf{1},3)\,. \end{cases} \tag{5.2.7}$$

Equivalently, one can first go vertically on the diagram and use

$$E_{6(2)} \to SU(6) \times SU(2);$$
$$\mathbf{27} \to (\mathbf{15},\mathbf{1}) + \left(\overline{\mathbf{6}},\mathbf{2}\right), \tag{5.2.8}$$
$$\mathbf{1} \to (\mathbf{1},\mathbf{1}),$$

thus obtaining:

$$\mathbf{56} \to (\mathbf{27},1) + \left(\overline{\mathbf{27}},-1\right) + (\mathbf{1},3) + (\mathbf{1},-3) \to \begin{cases} (\mathbf{15},\mathbf{1},1) + \left(\overline{\mathbf{6}},\mathbf{2},1\right) + (\mathbf{1},\mathbf{1},3) + \\ \\ + \left(\overline{\mathbf{15}},\mathbf{1},-1\right) + (\mathbf{6},\mathbf{2},-1) + (\mathbf{1},\mathbf{1},-3). \end{cases} \tag{5.2.9}$$

Again, either of the two alternative branchings in (5.2.6) , which are related by the interchange of $(\mathbf{15},\mathbf{1},1)$ with $\left(\overline{\mathbf{15}},\mathbf{1},-1\right)$, yield the same decomposition into irreducible representations of $SU(6) \times SU(2) \times U(1)$. Moreover, the $U(1)$ singlet which commutes with $SU(6) \times SU(2)$ is the same as the one which commute with $E_{6(2)}$.

Let us now turn to the scalar sector. As mentioned above, the coordinate system for the scalar manifold $\frac{E_{7(7)}}{SU(8)}$ based on the Cartan decomposition, the real scalars ϕ_{ijkl} sit in the $\mathbf{70}$ (four-fold antisymmetric and self-real irreducible representation) of $SU(8)$ with $i = 1, \ldots, 8$. The embedding of the RN extremal black hole is related to the further decomposition

$$SU(8) \to SU(6) \times SU(2) \times U(1),$$
$$\mathbf{70} \to (\mathbf{20},\mathbf{2},0) + (\mathbf{15},\mathbf{1},-2) + \left(\overline{\mathbf{15}},\mathbf{1},2\right). \tag{5.2.10}$$

On the other hand, for describing the KK extremal black hole one decomposes $SU(8)$ under its maximal subgroup $USp(8)$:

$$SU(8) \to USp(8),$$
$$\mathbf{70} \to \mathbf{42} + \mathbf{27} + \mathbf{1}, \tag{5.2.11}$$

where $\mathbf{42}$ and $\mathbf{27}$ are respectively the four-fold and two-fold antisymmetric irreducible representations (both skew-traceless and self-real) of $USp(8)$.

The crucial difference between (5.2.10) and (5.2.11) is that the latter decomposition contains a real singlet, whereas the first one does not. This is related to an underlying maximal compact ($U(1)$ symmetry which is present for (5.2.10) and not for (5.2.11).

This feature explains the different behavior of the two solutions at the attractor point: the RN solution has the behavior (5.1.10) while the KK solution is given by (5.1.12)-(5.1.13).

5.3 $SL(8,\mathbb{R})$-Basis

We now turn to discuss the details of the symplectic formalism for extended supergravities reviewed in the introductory section, and the original formulation of $\mathcal{N}=8$ supergravity of [71] for some of the key geometrical objects that are relevant for the present investigation.

We start by considering the coset representative for $E_{7(7)/SU(8)}$, which is parametrized as [71]

$$\mathcal{V}=\begin{pmatrix} u_{ij}^{IJ} & v_{ijKL} \\ v^{klIJ} & u^{kl}_{KL}\,. \end{pmatrix} \tag{5.3.1}$$

The sub-matrices u and v carry indices of both $E_{7(7)}$ and $SU(8)$ ($I=1,\dots,8$, $I=1,\dots,8$) but one can choose a suitable $SU(8)$ gauge for the fields, and then retain only manifest invariance with respect to the rigid diagonal subgroup of $E_{7(7)}\times SU(8)$, without distinction among the two types of indices. Comparing the notation of [71] (in particular the appendix B) with the symplectic formalism of [14],[27], we can identify

$$\begin{cases} \phi_0 \equiv u \\ \phi_1 \equiv v \end{cases} \quad \to \quad \begin{aligned} u_{ij}{}^{kl} &= (P^{-1/2})_{ij}{}^{kl}\,, \\ v^{ijkl} &= -(\bar{P}^{-1/2})^{ij}{}_{mn}\bar{y}^{mnkl} \end{aligned}$$

so that

$$\begin{cases} \mathbf{f} = \frac{1}{\sqrt{2}}(\phi_0+\phi_1) = \frac{1}{\sqrt{2}}(u+v) \\ i\mathbf{h} = \frac{1}{\sqrt{2}}(\phi_0-\phi_1) = \frac{1}{\sqrt{2}}(u-v) \end{cases} . \tag{5.3.2}$$

Since sections are sub-matrices of the symplectic representation, relatively to electric and magnetic subgroups, their explicit indices components are given by

$$\begin{aligned} f_{ij}{}^{kl} &= \frac{1}{\sqrt{2}}\left((P^{-1/2})_{ij}{}^{kl} - (\bar{P}^{-1/2})^{ij}{}_{mn}\bar{y}^{mnkl}\right)\,, \\ h_{ij,kl} &= \frac{-i}{\sqrt{2}}\left((P^{-1/2})_{ij}{}^{kl} + (\bar{P}^{-1/2})^{ij}{}_{mn}\bar{y}^{mnkl}\right)\,, \end{aligned} \tag{5.3.3}$$

where, in matrix notation,

$$P = 1 - YY^\dagger\,, \qquad Y = B\frac{\tanh\sqrt{B^\dagger B}}{\sqrt{B^\dagger B}}\,, \qquad B_{ij,kl} = -\frac{1}{2\sqrt{2}}\phi_{ijkl}\,, \tag{5.3.4}$$

the last definition coming from the choice of the symmetric gauge for the coset representative in Eq. (B.1) of [71]. If one defines

$$\tilde{P} = 1 - Y^\dagger Y \ , \tag{5.3.5}$$

and uses the identity

$$(\tilde{P}^{-1/2})Y^\dagger = Y^\dagger(P^{-1/2})\,, \tag{5.3.6}$$

the following simple expressions for $\mathbf{f}$ and $\mathbf{h}$ are finally achieved:

$$\mathbf{f} = \frac{1}{\sqrt{2}}\left[P^{-1/2} - (\tilde{P}^{-1/2})Y^\dagger\right] = \frac{1}{\sqrt{2}}[1 - Y^\dagger]\frac{1}{\sqrt{1 - YY^\dagger}} \ , \tag{5.3.7}$$

$$\mathbf{h} = -\frac{i}{\sqrt{2}}\left[P^{-1/2} + (\tilde{P}^{-1/2})Y^\dagger\right] = -\frac{i}{\sqrt{2}}[1 + Y^\dagger]\frac{1}{\sqrt{1 - YY^\dagger}} \ . \tag{5.3.8}$$

The above notations are such that

$$\begin{aligned} P^{1/2} &= \sqrt{1 - YY^\dagger} \qquad \rightarrow \qquad P_{ij}{}^{kl} = \delta_{ij}^{kl} - y_{ijmn}\bar{y}^{mnkl} \\ \tilde{P}^{1/2} &= \sqrt{1 - Y^\dagger Y} \qquad \rightarrow \qquad \bar{P}^{kl}{}_{ij} = \delta_{ij}^{kl} - \bar{y}^{klmn}y_{mnij} \end{aligned} \tag{5.3.9}$$

It is easily checked that the symplectic sections satisfy the usual relations

$$\begin{aligned} i(\mathbf{f}^\dagger\mathbf{h} - \mathbf{h}^\dagger\mathbf{f}) &= 1 \ , \\ \mathbf{h}^T\mathbf{f} - \mathbf{f}^T\mathbf{h} &= 0 \ . \end{aligned} \tag{5.3.10}$$

These are obtained writing the symplectic sections as in (5.3.7) and (5.3.8), and using the identity

$$Y\tilde{P}^{-1} = P^{-1}Y \ . \tag{5.3.11}$$

The kinetic matrix is given in terms of the symplectic sections by [27]

$$\mathcal{N} = \mathbf{h}\mathbf{f}^{-1} \ . \tag{5.3.12}$$

Therefore, Eqs. (5.3.7) and (5.3.8) yield

$$\begin{aligned} \mathcal{N} &= -i\,[1 + Y^\dagger]\frac{1}{\sqrt{1 - YY^\dagger}}\sqrt{1 - YY^\dagger}\frac{1}{1 - Y^\dagger} = \\ &= -i\,\frac{1 + Y^\dagger}{1 - Y^\dagger} \\ &\Downarrow \\ \mathcal{N}_{ij|kl} &= -i(\delta_{mn}^{kl} + \bar{y}^{mnkl})(\delta_{ij}^{mn} - \bar{y}^{ijmn})^{-1} \ . \end{aligned} \tag{5.3.13}$$

We now turn to the central charge function, which is defined by

$$Z_{ij} = f_{ij}{}^{kl} q_{kl} - h_{ij|kl} p^{kl} \ , \tag{5.3.14}$$

where electric and magnetic charges are in the same $SO(8)$ adjoint representation as vector fields. Using the definitions in (5.3.3), one obtains[1]

$$\begin{aligned} Z_{ij} &= \frac{1}{\sqrt{2}}\left((P^{-1/2})_{ij}{}^{kl} - (\bar{P}^{-1/2})^{ij}{}_{mn}\bar{y}^{mnkl}\right) q_{kl} + \frac{i}{\sqrt{2}}\left((P^{-1/2})_{ij}{}^{kl} + (\bar{P}^{-1/2})^{ij}{}_{mn}\bar{y}^{mnkl}\right) p^{kl} = \\ &= (P^{-1/2})_{ij}{}^{kl} Q_{kl} - (\bar{P}^{-1/2})^{ij}{}_{mn}\bar{y}^{mnkl}\bar{Q}_{kl} = \\ &= \frac{1}{\sqrt{2}}\left[\left(\frac{1}{\sqrt{1-Y\overline{Y}}}\right)_{ij}^{\ \ kl} Q_{kl} - \left(\frac{1}{\sqrt{1-\overline{Y}Y}}\right)^{ij}_{\ \ mn} \bar{Y}^{mnkl}\bar{Q}_{kl}\right] \ , \end{aligned} \tag{5.3.15}$$

where the complex charges

$$Q_{ij} \equiv \frac{1}{\sqrt{2}}(q_{ij} + ip^{ij}) \tag{5.3.16}$$

have been introduced.

Then one can also give an expression for the black hole potential, which is given by

$$\begin{aligned} V_{BH} &= \frac{1}{2} Z_{ij}\overline{Z}^{ij} = \\ &= \frac{1}{4}\Big[(1-Y\overline{Y})^{-1\,ijkl} Q_{kl}\bar{Q}_{ij} + \\ &\quad - \left(\sqrt{1-Y\overline{Y}}\right)^{-1\,ab}_{ij} Q_{ab}\left(\sqrt{1-\overline{Y}Y}\right)^{-1\,ij}_{cd} Y_{cdkl}Q_{kl} + \\ &\quad - \left(\sqrt{1-\overline{Y}Y}\right)^{-1\,ij}_{ab} \overline{Y}^{abkl}\bar{Q}_{kl}\left(\sqrt{1-Y\overline{Y}}\right)^{-1\ \ cd}_{ij} \bar{Q}_{cd} + \\ &\quad + (1-\overline{Y}Y)^{-1}{}_{ijkl}\overline{Y}^{ijab} Y_{klmn}\bar{Q}_{ab}Q_{mn}\Big] \ . \end{aligned} \tag{5.3.17}$$

Thus, in the expansion around the zero field configuration, the black hole receives contribution from the term

$$V_{BH}(\phi = 0) = \frac{1}{4} Q_{ij}\bar{Q}^{ij} \ . \tag{5.3.18}$$

The linear term in the expansion of the black hole potential near the point $\phi = 0$ receives contributions from the second and third row of Eq. (5.3.17), yielding the condition

$$Q_{ij}\phi_{ijkl}Q_{kl} - \bar{Q}_{ij}\bar{\phi}^{ijkl}\bar{Q}_{kl} = 0 \ , \tag{5.3.19}$$

$$\Downarrow$$

$$Q_{ij}Q_{kl}\delta^{mnpq}_{ijkl} - \frac{1}{4!}\bar{Q}_{ij}\bar{Q}_{kl}\epsilon^{ijklmnpq} = 0 \ . \tag{5.3.20}$$

[1] The expression with explicit indices is given by

$$\bar{P}^{ij}{}_{kl} = (\tilde{P})_{kl}{}^{ij}$$

The configuration corresponding to charges Q_{AB} in the singlet of $SU(2) \times SU(6)$ trivially satisfies condition (5.3.20). Furthermore, it sets to zero the linear term for all values of ϕ, implying the $\phi = 0$ point to be an attractor point for this configuration.

5.4 $E_{6(6)}$-Basis and Relation to $d = 5$

This section is aimed to establish the relation between the $\mathcal{N} = 8$, $d = 4$ theory and $\mathcal{N} = 8$, $d = 5$ supergravity ([73, 74]), especially for what concerns the effective black hole potential.

In our normalizations the kinetic Lagrangian for vector fields in the $\mathcal{N} = 2$ theory reads (with $\mathcal{F}_{\mu\nu} \equiv \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) = \partial_{[\mu} A_{\nu]}$) [75],[18]

$$\mathcal{L} = \ldots - \mathrm{Im}\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^\Lambda \mathcal{F}^\Sigma - \mathrm{Re}\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^\Lambda \, {}^*\mathcal{F}^\Sigma \,, \tag{5.4.1}$$

where $\mathcal{N}_{\Lambda\Sigma}$ is the $d = 4$ vector kinetic matrix, with $\Lambda, \Sigma = 0, 1, ..., 27$. The effective black hole potential is given by [13]

$$V_{BH} = -\frac{1}{2} Q^T \mathcal{M}(\mathcal{N}) Q \,, \tag{5.4.2}$$

where Q is the symplectic charge vector $Q = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix}$, and the matrix $\mathcal{M}$ reads [13]

$$\mathcal{M}(\mathcal{N}) = \begin{pmatrix} \mathrm{Im}\mathcal{N} + \mathrm{Re}\mathcal{N}(\mathrm{Im}N)^{-1}\mathrm{Re}\mathcal{N} & -\mathrm{Re}\mathcal{N}(\mathrm{Im}\mathcal{N})^{-1} \\ -(\mathrm{Im}\mathcal{N})^{-1}\mathrm{Re}\mathcal{N} & (\mathrm{Im}\mathcal{N})^{-1} \end{pmatrix} . \tag{5.4.3}$$

The $d = 5$ U-duality group $E_{6(6)}$ acts linearly on the 27 vectors $\hat{A}^I_{\hat{\mu}}$, with $\hat{\mu} = 1, \ldots, 5$ and $I = 1, \ldots, 27$. The $d = 5$ vector kinetic matrix $\hat{\mathcal{N}}_{IJ}$ is a function of the scalar fields spanning the $d = 5$ scalar manifold $\frac{E_{6(6)}}{USp(8)}$ ($dim_{\mathbb{R}} = 42$, $rank$= 6).

According to the splitting $\Lambda = \{0, I\}$, the $d = 4$ kinetic vector matrix assumes the block form

$$\mathcal{N}_{\Lambda\Sigma} = \left(\begin{array}{c|c} \mathcal{N}_{00} & \mathcal{N}_{0\,J} \\ \hline \mathcal{N}_{I\,0} & \mathcal{N}_{I\,J} \end{array}\right) . \tag{5.4.4}$$

By using to the formulæ obtained in [76] which determine $\mathcal{N}_{\Lambda\,\Sigma}$ in terms of five-dimensional quantities, in a normalization[2] that is suitable for comparison to $\mathcal{N} = 2$, one obtains

$$\mathcal{N}_{\Lambda\,\Sigma} = \left(\begin{array}{c|c} \frac{1}{3}d_{IJK}a^I a^J a^K - i\left(e^{2\phi}a_{IJ}a^I a^J + e^{6\phi}\right) & -\frac{1}{2}d_{IJK}a^I a^K + ie^{2\phi}a_{KJ}a^K \\ \hline -\frac{1}{2}d_{IKL}a^K a^L + ie^{2\phi}a_{IK}a^K & d_{IJK}a^K - ie^{2\phi}a_{IJ} \end{array}\right) \tag{5.4.5}$$

Since the d_{IJK} tensor, the a^I fields, the $d = 5$ vector kinetic matrix a_{IJ} and the field ϕ are real, the expressions for Im$\mathcal{N}$ and Re$\mathcal{N}$ are given by

$$\mathrm{Im}\,\mathcal{N}_{\Lambda\,\Sigma} = -e^{6\phi}\left(\begin{array}{c|c} 1 + e^{-4\phi}a_{IJ}\,a^I a^J & -e^{-4\phi}a_{KJ}\,a^K \\ \hline -e^{-4\phi}a_{IK}\,a^K & e^{-4\phi}a_{IJ} \end{array}\right) ; \tag{5.4.6}$$

$$\mathrm{Re}\,\mathcal{N}_{\Lambda\,\Sigma} = \left(\begin{array}{c|c} \frac{1}{3}d_{KLM}a^K a^L a^M & -\frac{1}{2}d_{JLM}a^L a^M \\ \hline -\frac{1}{2}d_{ILM}a^L a^M & d_{IJK}a^K \end{array}\right) = \begin{pmatrix} \frac{1}{3}d & -\frac{1}{2}d_J \\ -\frac{1}{2}d_I & d_{IJ} \end{pmatrix} , \tag{5.4.7}$$

[2] Compared to the notation of [76], here we use $\mathcal{N}_{\Lambda\Sigma} \to 4\mathcal{N}_{\Lambda\Sigma}$, $2\hat{\mathcal{N}}_{IJ} \to a_{IJ}$, $d_{IJK} \to -d_{IJK}/4$ and $a^I \to -a^I$.

where the following shorthand notation has been introduced:

$$d \equiv d_{IJK}a^I a^J a^K \quad , \quad d_I \equiv d_{IJK}a^J a^K \quad , \quad d_{IJ} \equiv d_{IJK}a^K \; . \qquad (5.4.8)$$

The inverse matrix $(\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma})^{-1} \equiv \mathrm{Im}\,\mathcal{N}^{\Lambda\Sigma}$ can be determined by noticing the block structure of (5.4.6). Then, by performing computations analogous to those of [69], one finds

$$(\mathrm{Im}\mathcal{N}^{-1})^{\Lambda\,\Sigma} = -e^{-6\phi}\left(\begin{array}{c|c} 1 & a^J \\ \hline a^I & a^I a^J + e^{4\phi}a^{IJ} \end{array}\right), \qquad (5.4.9)$$

where $a^{IJ} \equiv (a_{IJ})^{-1}$. Inserting the above expressions into Eq. (6.2.1), the $\mathcal{N} = 8$, $d = 4$ effective black hole potential can finally be rewritten in a $d = 5$ language:

$$\begin{aligned} V_{BH} = &\, (p^0)^2\left[\frac{1}{2}e^{2\phi}a_{IJ}a^I a^J + \frac{1}{2}e^{6\phi} + \frac{1}{8}e^{-6\phi}\left(\frac{d^2}{9} + e^{4\phi}a^{IJ}d_I d_J\right)\right] + \\ &+ p^0 p^I\left[-e^{2\phi}a_{IJ}a^J - \frac{1}{4}e^{-6\phi}\left(\frac{1}{3}dd_I + 2e^{4\phi}a^{KJ}d_K d_{JI}\right)\right] + \\ &+ p^I p^J\left[\frac{1}{2}e^{2\phi}a_{IJ} + \frac{1}{8}e^{-6\phi}\left(d_I d_J + 4e^{4\phi}a^{KL}d_{IK}d_{LJ}\right)\right] + \\ &+ \frac{1}{6}q_0 p^0\, e^{-6\phi}d + \frac{1}{6}q_I p^0\, e^{-6\phi}\left[d\, a^I + 3e^{4\phi}a^{KI}d_K\right] + \\ &- \frac{1}{2}q_0 p^I\, e^{-6\phi}d_I - \frac{1}{2}q_I p^J\, e^{-6\phi}\left[d_J a^I + 2e^{4\phi}a^{KI}d_{JK}\right] + \\ &+ \frac{1}{2}(q^0)^2 e^{-6\phi} + q_0 q_I e^{-6\phi}a^I + \frac{1}{2}q_I q_J\, e^{-6\phi}\left[a^I a^J + e^{4\phi}a^{IJ}\right] \; . \end{aligned} \qquad (5.4.10)$$

Notice that this formula becomes identical to the corresponding one of [69] concerning (purely cubic) $\mathcal{N} = 2$ geometries [77],[78], where $a_{IJ} = 4e^{4\phi}g_{ij}$ and $\mathcal{V} \equiv e^{6\phi}$.

The potential (5.4.10), because of the definitions (6.2.5), can be seen to be a polynomial of degree up to sixth in the axion fields, whose general solutions are hard to determine. However, one can consider in particular attractor solutions with vanishing axion fields. These are given by specific charge configurations that solve the following attractor equations:

$$\left.\frac{\partial V_{BH}}{\partial a^I}\right|_{a^J=0} = -e^{2\phi}p^0 p^K a_{KI} - e^{-2\phi}q_J p^K d_{ILK}a^{JL} + q_0 q_I e^{-6\phi} = 0 \; . \qquad (5.4.11)$$

Therefore, the black hole charge configurations $Q = (p^0, p^I, q_0, q_I)$ supporting axion–free solutions fall into three classes:

$$\begin{array}{llll} a) & Q_e = (p^0, 0, 0, q_I) & \text{Electric black hole}\,; \\ b) & Q_m = (0, p^I, q_0, 0) & \text{Magnetic black hole}\,; \\ c) & Q_0 = (p^0, 0, q_0, 0) & \text{KK charged black hole}\ . \end{array} \tag{5.4.12}$$

In each of these classes, we now specify the black hole potential by setting to zero the appropriate charge configuration in (5.4.10):

a) Electric black hole:

$$V_{BH}(\phi, p^0, q_I)\big|_{a^I=0} = \frac{1}{2}e^{6\phi}(p^0)^2 + \frac{1}{2}e^{-2\phi}a^{IJ}q_I q_J\,. \tag{5.4.13}$$

b) Magnetic black hole:

$$V_{BH}(\phi, q_0, p^J)\big|_{a^I=0} = \frac{1}{2}e^{-6\phi}(q_0)^2 + \frac{1}{2}e^{2\phi}a_{IJ}p^I p^J\,. \tag{5.4.14}$$

c) Black hole charged with respect to the KK vector:

$$V_{BH}(\phi, q_0, p^0)\big|_{a^I=0} = \frac{1}{2}e^{-6\phi}(q_0)^2 + \frac{1}{2}e^{6\phi}(p^0)^2\ . \tag{5.4.15}$$

In order to recover the complete attractor solution, one also has to stabilize e^{ϕ}. For the KK charged black hole one gets,

$$\frac{\partial V^{KK}_{BH}(\phi, q_0, p^0)}{\partial\phi}\big|_{a^I=0} = 0 \quad \Longleftrightarrow \quad e^{6\phi} = \left|\frac{q_0}{p^0}\right|\ , \tag{5.4.16}$$

thus yielding

$$V^{KK}_{BH}(q_0, p^0)\big|_{a^I=0} = |q_0 p^0|\ . \tag{5.4.17}$$

In the electric case it holds that

$$\frac{\partial V^e_{BH}}{\partial\phi}\big|_{a^I=0} = 0 \quad \Longleftrightarrow \quad e^{2\phi} = \left(\frac{a^{IJ}q_I q_J}{3(p^0)^2}\right)^{\frac{1}{4}}\ , \tag{5.4.18}$$

implying the critical value

$$V^e_{BH}(q_I, p^0)|_{a^I=0} = 2|p^0|^{1/2}\left(\frac{a^{IJ}q_I q_J}{3}\right)^{3/4}\ . \tag{5.4.19}$$

Analogously, for the magnetic black hole one finds

$$\frac{\partial V^m_{BH}}{\partial\phi}\big|_{a^I=0} = 0 \quad \Longleftrightarrow \quad e^{2\phi} = \left(\frac{a_{IJ}p^I p^J}{3q_0^2}\right)^{-\frac{1}{4}}\ , \tag{5.4.20}$$

yielding

$$V^m_{BH}(q_0, p^I)|_{a^I=0} = 2|q_0|^{1/2}\left(\frac{a_{IJ}p^Ip^J}{3}\right)^{3/4} \quad . \tag{5.4.21}$$

In virtue of the Bekenstein−Hawking entropy-area formula, the above expressions for the critical electric and magnetic black hole potentials must be compared with appropriate powers of the $E_{6(6)}$ cubic invariants $\mathcal{I}_3(p) \equiv \frac{1}{3!}d_{IJK}p^Ip^Jp^K$ and $\mathcal{I}_3(q) \equiv \frac{1}{3!}d^{IJK}q_Iq_Jq_K$. Indeed, in $d = 5$ it must hold that [10]

$$S \sim V^{3/4}|_{crit} \sim |\mathcal{I}_3|^{1/2}\,, \tag{5.4.22}$$

Defining the electric and magnetic $d = 5$ effective potentials respectively as

$$V^e_5 = a^{IJ}q_Iq_J \quad , \quad V^m_5 = a_{IJ}p^Ip^J \tag{5.4.23}$$

one obtains

$$V^e_{crit} = 2|p^0|^{1/2}\left(\frac{V^e_5}{3}\right)^{3/4}|_{crit} \tag{5.4.24}$$

and

$$V^m_{crit} = 2|q^0|^{1/2}\left(\frac{V^m_5}{3}\right)^{3/4}|_{crit}\,. \tag{5.4.25}$$

By comparison with $\mathcal{N} = 2$ symmetric $d-$geometries having

$$V^e_5|_{crit} = |\mathcal{I}_3(q)|^{2/3} = |q_1q_2q_3|\,, \tag{5.4.26}$$

one obtains the expressions for the critical potential of the four dimensional electric and magnetic black holes:

$$V^e_{BH\,crit}(q_I, p^0) = 2\sqrt{\frac{|p^0d^{IJK}q_Iq_Jq_K|}{3!}}\,, \tag{5.4.27}$$

and

$$V^m_{BH\,crit}(q_0, p^I) = 2\sqrt{\frac{|q_0d_{IJK}p^Ip^Jp^K|}{3!}}\,. \tag{5.4.28}$$

More generally, these solutions can be compared with the embedding of the $\mathcal{N} = 2$ purely cubic supergravities into $\mathcal{N} = 8$ supergravity, and using the above critical values of the black hole potential in (5.1.21), one finds for the three family of configurations under exam the correct result:

$$\frac{S_{BH}}{\pi} = \sqrt{|\mathcal{I}_4|}\,. \tag{5.4.29}$$

It is interesting to remark that the KK black hole can be connected to the RN solution by performing an analytic continuation of the charges, as one can see from the redefinition

$$p^0 \to p + iq \ ,$$
$$q_0 \to p - iq \ ,$$

which allows one to recover the RN entropy

$$S_{RN} = \pi(\ p^2 + q^2) \ . \qquad (5.4.30)$$

We conclude this Section by pointing out that the 70 scalars of $\mathcal{N} = 8$, $d = 4$ supergravity have been decomposed according to representations of $USp\,(8)$ (maximal compact subgroup of $E_{6(6)} \times SO\,(1,1)$) as follows:

$$\mathbf{70} \to \mathbf{42} + \mathbf{27} + \mathbf{1} \ . \qquad (5.4.31)$$

The 42 unstabilized fields are the coordinates of the corresponding moduli space [67]. The non-compact form of the exceptional group, $E_{6(6)}$, in fact, enters in the expression of the coset

$$\frac{E_{6(6)}}{USp(8)} \ , \qquad (5.4.32)$$

which is the moduli space of the $d = 4$ non-BPS, $Z_{AB} \neq 0$ extremal black holes, whose orbit is precisely

$$\mathcal{O} = \frac{E_{7(7)}}{E_{6(6)}} \ . \qquad (5.4.33)$$

Indeed, the KK black hole is a non supersymmetric solution.

5.5 Embedding of the Axion-Dilaton Extremal black hole

The embedding of the axion-dilaton black hole in $\mathcal{N} = 8$, $d = 4$ supergravity can be performed by a three step supersymmetry reduction, which can be schematically indicated as

$$\mathcal{N} = 8 \to \mathcal{N} = 4,\ n_V = 6 \to pure\,\mathcal{N} = 4 \to \mathcal{N} = 2 \quad quadratic,\ n_V = 1, \qquad (5.5.1)$$

where n_V denotes the number of vector multiplets coupled to the supergravity multiplet. More precisely, the first step consists in truncating $\mathcal{N} = 8$ supergravity to an $\mathcal{N} = 4$

theory interacting with six matter (vector) multiplets. In the second step, $\mathcal{N}=4$ reduces to the pure theory, while in the last reduction one obtains $\mathcal{N}=2$ supergravity quadratic [79] theory with a single vector multiplet.

We are now going to examine more precisely each intermediate step.

1) In the first step, the $\mathcal{N}=8$ central charge matrix Z_{AB} assumes the block form ($a,b=1,..,4$, $i,j=1,...,4$):

$$Z_{AB} \rightarrow \begin{pmatrix} Z_{ab} & 0 \\ \\ 0 & i\overline{Z}_{ij} \end{pmatrix}. \tag{5.5.2}$$

where Z_{ab} is the $\mathcal{N}=4$ central charge matrix and Z_{ij} are the matter charges of the 6 vector multiplets (sitting in the two-fold antisymmetric of $SU(4)$, or equivalently in the vector representation of $SO(6) \sim SU(4)$).

Consequently, the $\mathcal{N}=8$ scalar manifold $\frac{E_{7(7)}}{SU(8)}$, reduces to

$$\frac{SL(2,\mathbb{R})}{U(1)} \times \frac{SO(6,6)}{SO(6)\times SO(6)} = \frac{SL(2,\mathbb{R})}{U(1)} \times \frac{SO(6,6)}{SU(4)\times SU(4)}, \tag{5.5.3}$$

which admits three orbits. This is the scalar manifold for $\mathcal{N}=4$ supergravity coupled to 6 vector multiplets, .

2) In the second step, the 6 vector multiplets are eliminated and $Z_{ij}=0$; this corresponds to retaining only states which are singlets with respect to the second $SU(4)$ in the stabilizer of the coset (5.5.3)), and the theory becomes pure $=4$, with U-duality $SL(2,\mathbb{R}) \times SU(4)$:

$$\begin{pmatrix} Z_{ab}\epsilon & 0 \\ \\ 0 & i\overline{Z}_{ij}\epsilon \end{pmatrix} \rightarrow \begin{pmatrix} Z_{ab} & 0 \\ \\ 0 & 0 \end{pmatrix}, \tag{5.5.4}$$

with $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Accordingly, the scalar manifold reduces to $\frac{SL(2,\mathbb{R})}{U(1)}$. Notice that, the presence of the axion-dilaton s spanning $\frac{SL(2,\mathbb{R})}{U(1)}$, in the $\mathcal{N}=4$ supergravity multiplet, only an $SU(4)$ out of the whole (local) $\mathcal{N}=4$ $\mathcal{R}$-symmetry $U(4)$ gets promoted to (global) U-duality symmetry .

3) In the last step, 4 out of 6 graviphotons drop out, reducing the overall gauge symmetry from $U(1)^6$ to $U(1)^2$, with resulting U-duality $SL(2,\mathbb{R}) \times U(1)$. Thus, the framework becomes $\mathcal{N}=2$-supersymmetric, with the two skew-eigenvalues (Z_1, Z_2) of

Z_{ab} related to the $\mathcal{N}=2$ central and matter charges $(Z, D_s Z)$:

$$Z_{ab} \rightarrow \begin{pmatrix} Z & 0 \\ & \\ 0 & i\,\overline{D}_{\bar{s}}\bar{Z} \end{pmatrix}. \tag{5.5.5}$$

Therefore, at the $\mathcal{N}=2$ level one can have both BPS attractors ($D_s Z = 0$) and the non-BPS ($Z=0$) ones [52].

On a group theoretical side, this step correspond to performing the decomposition

$$\begin{aligned} & SU(4) \rightarrow SU(2)\times SU(2)\times U(1)\,, \\ & \mathbf{4} \rightarrow \left(\mathbf{2},\mathbf{1},\tfrac{1}{2}\right)+\left(\mathbf{1},\mathbf{2},-\tfrac{1}{2}\right), \\ & \mathbf{6} \rightarrow (\mathbf{2},\mathbf{2},0)+(\mathbf{1},\mathbf{1},1)+(\mathbf{1},\mathbf{1},-1)\,, \end{aligned} \tag{5.5.6}$$

and to retaining only the singlets of $SU(2)\times SU(2)$.

The above three step reduction can be viewed from the point of view of the classification of large charge orbits [27],[80]. One starts with the $\mathcal{N}=8$ scalar manifold $E_{7(7)}/SU(8)$ admitting the two regular orbits (5.1.5) and (5.1.6). The large charge orbits of $\mathcal{N}=4$, $d=4$ supergravity coupled to 6 vector multiplets are:

$$\begin{cases} \mathcal{O}_{1/4\,BPS}: & SL(2,\mathbb{R})\times \frac{SO(6,6)}{SO(2)\times SO(6,4)}; \\ \mathcal{O}_{non\,BPS,\ Z_{ab}=0}: & SL(2,\mathbb{R})\times \frac{SO(6,6)}{SO(2)\times SO(6,4)}; \\ \mathcal{O}_{non\,BPS,\ Z_{ab}\neq 0}: & SL(2,\mathbb{R})\times \frac{SO(6,6)}{SO(1,1)\times SO(5,5)}, \end{cases} \tag{5.5.7}$$

where the coincidence of the first two orbits is due to the symmetry between the gravity and the matter sector.

The corresponding moduli spaces for the $\mathcal{N}=4$, $n=6$ attractor solutions, exploiting the hidden symmetries of the above charge orbits, are given by:

$$\begin{cases} \mathcal{M}_{BPS} = \frac{SO(6,4)}{SU(4)\times SU(2)\times SU(2)}; \\ \mathcal{M}_{non\,BPS, Z_{ab}=0} = \frac{SO(6,4)}{SO(6)\times SO(4)}; \\ \mathcal{M}_{non\,BPS, Z_{ab}\neq 0} = SO(1,1)\times \frac{SO(5,5)}{SO(5)\times SO(5)} = SO(1,1)\times \frac{SO(5,5)}{USp(4)\times USp(4)}. \end{cases} \tag{5.5.8}$$

Notice that $\mathcal{M}_{1/4\,BPS}$ (and $\mathcal{M}_{non-BPS,Z_{ab}=0}$) are homogeneous symmetric quaternionic manifolds, as in the $\mathcal{N}=4\rightarrow\mathcal{N}=2$ reduction they become the hypermultiplets' scalar manifold [27].

The truncation of the $\mathcal{N}=8$ theory into $\mathcal{N}=4$ is based on the decomposition

$$E_{7(7)}\rightarrow SL(2,R)\times SO(6,6) \tag{5.5.9}$$

and on the following group embeddings

$$SO(6,4)\times SO\left(2\right)\subsetneq E_{6(2)}; \tag{5.5.10}$$

$$SO(5,5)\times SO\left(1,1\right)\subsetneq E_{6(6)}. \tag{5.5.11}$$

Therefore, one can readily establish that the orbits 1/4 BPS and non BPS, $Z_{ab}=0$ descend from the $\mathcal{N}=8$, BPS orbit $\frac{E_{7(7)}}{E_{6(2)}}$, whereas the orbit $\mathcal{O}_{nonBPS,\;Z_{ab}\neq0}$ comes from the $\mathcal{N}=8$, non-BPS orbit $\frac{E_{7(7)}}{E_{6(6)}}$.

There is also another way to interpret the three step reduction (5.5.1), that is in terms of U-duality invariant representations. At group level, the embedding of the axion-dilaton extremal black hole into $\mathcal{N}=8$, $d=4$ supergravity is based on the decomposition of $E_{7(7)}\rightarrow SU(8)$ and

$$\begin{aligned}
&SU\left(8\right)\rightarrow SU\left(4\right)\times SU\left(4\right)\times U\left(1\right),\\
&\mathbf{8}\rightarrow\left(\mathbf{4},\mathbf{1},\tfrac{1}{2}\right)+\left(\mathbf{1},\mathbf{4},-\tfrac{1}{2}\right),\\
&\mathbf{28}\rightarrow\left(\mathbf{4},\mathbf{4},0\right)+\left(\mathbf{6},\mathbf{1},1\right)+\left(\mathbf{1},\mathbf{6},-1\right),\\
&\overline{\mathbf{28}}\rightarrow\left(\overline{\mathbf{4}},\overline{\mathbf{4}},0\right)+\left(\mathbf{6},\mathbf{1},-1\right)+\left(\mathbf{1},\mathbf{6},1\right),
\end{aligned} \tag{5.5.12}$$

where $SU\left(4\right)\times SU\left(4\right)\times U\left(1\right)$ is a maximal subgroup of $SU\left(8\right)$.

Then, the first truncation ($\mathcal{N}=8\rightarrow\mathcal{N}=4, n=6$) consists in setting

$$\left(\mathbf{4},\mathbf{4},0\right)=0=\left(\overline{\mathbf{4}},\overline{\mathbf{4}},0\right), \tag{5.5.13}$$

which gives rise to the decomposition (5.5.2).

We recall that the quartic invariant of the U-duality group $SL\left(2,\mathbb{R}\right)\times SO\left(6,n\right)$ of $\mathcal{N}=4$, $d=4$ supergravity coupled to n vector multiplets is [55]

$$\mathcal{I}_4=\mathcal{S}_1^2-\left|\mathcal{S}_2\right|^2, \tag{5.5.14}$$

where the three $SO(6,n)$ invariants $\mathcal{S}_1$, $\mathcal{S}_2$ and $\overline{\mathcal{S}_2}$ are defined by ($a, b = 1, ..., 4$, $I = 1, ..., n$):

$$\mathcal{S}_1 \equiv \frac{1}{2} Z_{ab}\overline{Z}^{ab} - Z_I\overline{Z}^I; \tag{5.5.15}$$

$$\mathcal{S}_2 \equiv \frac{1}{4}\epsilon^{abcd} Z_{ab}Z_{cd} - \overline{Z}_I\overline{Z}^I. \tag{5.5.16}$$

The case $n = 6$ is remarkably symmetric, as the symmetry of the gravity and matter sector is the same and furthermore, due to the isomorphism $SU(4) \sim SO(6)$, the $SO(6)$-vector Z_I of matter charges can be equivalently represented as the $SU(4)$-antisymmetric tensor $i\overline{Z}_{ij}$ ($i, j = 1, ..., 4$). Consequently, for $n = 6$ we have

$$\mathcal{S}_{1,n=6} \equiv \frac{1}{2} Z_{ab}\overline{Z}^{ab} - \frac{1}{2}\overline{Z}_{ij}Z^{ij}; \tag{5.5.17}$$

$$\mathcal{S}_{2,n=6} \equiv \frac{1}{4}\epsilon^{abcd} Z_{ab}Z_{cd} - \frac{1}{4}\epsilon_{ijkl}Z^{ij}Z^{kl}. \tag{5.5.18}$$

Notice that $\mathcal{O}_{1/4BPS}$ and $\mathcal{O}_{nonBPS,\ Z_{ab}=0}$ in Eq. (5.5.7) correspond to the two disconnected branches of the same manifold, classified by the sign of the real $SO(6,6)$-invariant [27] Indeed, $\mathcal{S}_{1,n=6} > 0$ for $\mathcal{O}_{1/4BPS}$ and $\mathcal{S}_{1,n=6} < 0$ for $\mathcal{O}_{nonBPS,\ Z_{ab}=0}$.

By a suitable $U(1)\times SU(4)\times SU(4)$ transformation, one can reach the normal frame for both gravity sector and matter sector, such that the two matrices Z_{ab} and Z_{ij} are simultaneously skew-diagonalized, obtaining

$$Z_{ab} \longrightarrow \begin{pmatrix} Z_1 & \\ & Z_2 \end{pmatrix} \otimes \epsilon; \tag{5.5.19}$$

$$Z_{ij} \longrightarrow e^{i\theta}\begin{pmatrix} Z_3 & \\ & Z_4 \end{pmatrix} \otimes \epsilon, \tag{5.5.20}$$

where $Z_1, Z_2 \in \mathbb{R}^+$, and $Z_3, Z_4 \in \mathbb{R}^+$, $\theta \in [0, 2\pi)$. In the normal frame, one obtains

$$\mathcal{S}_{1,n=6} \equiv |Z_1|^2 + |Z_2|^2 - |Z_3|^2 - |Z_4|^2; \tag{5.5.21}$$

$$\mathcal{S}_{2,n=6} \equiv 2\left(Z_1Z_2 - \bar{Z}_3\bar{Z}_4\right); \tag{5.5.22}$$

$$\begin{aligned}\mathcal{I}_{4,n=6} &= \mathcal{S}^2_{1,n=6} - |\mathcal{S}_{2,n=6}|^2 = \\ &= \sum_{i=1}^{4}|Z_i|^4 - 2\sum_{i<j=1}^{4}|Z_i|^2|Z_j|^2 + 4\left(\prod_{i=1}^{4} Z_i + \prod_{i=1}^{4}\overline{Z_i}\right).\end{aligned} \tag{5.5.23}$$

Eq. (5.5.23) coincides with the expression of the quartic invariant of $\mathcal{N} = 8$, $d = 4$ supergravity, as given by [54] (see also [33]) Considering now the second step of the reduction, where one reaches the pure $\mathcal{N} = 4$ theory, one sets $Z_{ij} = 0$, or equivalently

$Z_3 = 0 = Z_4$ in the normal frame (that is, retaining only states which are singlets with respect to the second $SU(4)$ in the stabilizer of the coset (5.5.3)). Notice that, by doing so, $\mathcal{I}_{4,n=0}$ becomes a perfect square:

$$\mathcal{I}_{4,n=0} = \mathcal{S}_{1,n=0}^2 - |\mathcal{S}_{2,n=0}|^2 = \left(|Z_1|^2 - |Z_2|^2\right)^2 = \left(Z_1^2 - Z_2^2\right)^2 . \tag{5.5.24}$$

Eq. (5.5.24) implies that $\mathcal{I}_{4,n=0}$ is (weakly) positive, and as a consequence an unique class of large attractor exists, namely the 1/4-BPS one. The (weak) positivity of $\mathcal{I}_{4,n=0}$ is consistent with the known expression of $\mathcal{I}_{4,n=0}$ in terms of the magnetic and electric charges $\left(p^\Lambda, q_\Lambda\right)$ $(\Lambda = 1, ..., 6)$:

$$\mathcal{I}_{4,n=0} = 4\left[p^2 q^2 - (p\cdot q)^2\right], \tag{5.5.25}$$

where here $p^2 \equiv p^\Lambda p^\Sigma \delta_{\Lambda\Sigma}$, $q^2 \equiv q_\Lambda q_\Sigma \delta^{\Lambda\Sigma}$ and $p\cdot q \equiv p^\Lambda q_\Sigma \delta_\Lambda^\Sigma$. Notice that in the basis of *bare* charges $\mathcal{I}_{4,n=0}$, as given by Eq. (5.5.25), is (weakly) positive due to the Schwarz inequality, and not because it is a non-trivial perfect square of an expression of the bare magnetic and electric charges [81].

Notice that $\sqrt{\mathcal{I}_{4,n=0}}$ (with $\mathcal{I}_{4,n=0}$ given by Eq. (5.5.25)) must coincide with the value of the effective black hole potential of the pure $\mathcal{N} = 4$ theory at its critical points. This can be understood (see the recent discussion given in [27] and [82]) because this potential reads as follows $(\Lambda = 1, ..., 6)$:

$$\begin{aligned} V_{BH,pure\mathcal{N}=4}\left(\phi,\, a,\, p^\Lambda,\, q_\Lambda\right) &= e^{2\phi}(sp_\Lambda - q_\Lambda)(\bar{s}p^\Lambda - q^\Lambda) = \\ &= (e^{2\phi}a^2 + e^{-2\phi})p^2 + e^{2\phi}q^2 - 2ae^{2\phi}p\cdot q, \end{aligned} \tag{5.5.26}$$

where the complex (axion-dilaton) field

$$s \equiv a + ie^{-2\phi} \tag{5.5.27}$$

parametrizes the coset $\frac{SU(1,1)}{U(1)}$ of $\mathcal{N} = 4$, $d = 4$ pure supergravity [83]. By computing the criticality conditions of $V_{BH,pure\mathcal{N}=4}$, one obtains the following stabilization equations for the axion a and the dilaton ϕ at criticality, $(\phi, a) = (\phi_H(p,q),\, a_H(p,q))$ [27]

$$\frac{\partial V_{BH}(\phi,a,p,q)}{\partial a}\Big|_{crit} = 0 \Longleftrightarrow a_H(p,q) = \frac{p\cdot q}{p^2}; \tag{5.5.28}$$

$$\begin{aligned} \frac{\partial V_{BH}(\phi,\,a,\,p,\,q)}{\partial \phi}\Big|_{crit} &= -e^{-4\phi}p^2 + q^2 - a_H(p,q)p\cdot q = -e^{-4\phi}p^2 + q^2 - \frac{(p\cdot q)^2}{p^2} = 0; \\ &\Updownarrow \\ e^{-2\phi_H(p,q)} &= \frac{\sqrt{p^2q^2 - (p\cdot q)^2}}{p^2}. \end{aligned} \tag{5.5.29}$$

The Bekenstein–Hawking black hole entropy is computed to be

$$\begin{aligned} S_{BH}(p,q) &= \frac{A_H(p,q)}{4} = \pi V_{BH}(\phi_H(p,q), a_H(p,q), p, q) \\ &= 2\pi\sqrt{p^2q^2-(p\cdot q)^2} = \pi\sqrt{\mathcal{I}_{4,n=0}}. \end{aligned} \tag{5.5.30}$$

The third and last step, when the pure $\mathcal{N}=4$ theory reduces to the $\mathcal{N}=2$ quadratic theory with $n_V=1$, is performed through the truncation $(U(1))^6 \to (U(1))^2$ of the overall Abelian gauge invariance ($\Lambda = 1,...,6 \to \Lambda = 1,2$). In this case, $\mathcal{I}_{4,n=0,(U(1))^6\to(U(1))^2}$ is a perfect square in both the basis of Z_{ab} and in the basis of charges $\left(p^\Lambda, q_\Lambda\right)$, and it actually is the square of the quadratic invariant $\mathcal{I}_{2(n=1)}$ of the axion-dilaton system:

$$\mathcal{I}_{4,n=0,(U(1))^6\to(U(1))^2} = \left(|Z_1|^2-|Z_2|^2\right)^2 = 4\left(p^1q_2-p^2q_1\right)^2 = \mathcal{I}^2_{2(n=1)}; \tag{5.5.31}$$

$$\Updownarrow$$

$$\mathcal{I}_{2(n=1)} = \pm 2\left|p^1q_2-p^2q_1\right|, \tag{5.5.32}$$

implying that the axion-dilaton system exhibits two types of attractors: the $\frac{1}{2}$-BPS one ($\mathcal{I}_{2(n=1)}>0$) and the non-BPS $Z=0$ one ($\mathcal{I}_{2(n=1)}<0$).

By further putting

$$p^1 = 0 = q_2,\ p^2 \equiv p,\ q_1 \equiv q \tag{5.5.33}$$

($\Rightarrow p\cdot q = 0$), one obtains:

$$\mathcal{I}^*_{4(n=0,U(1)^6\to U(1)^2)} = \mathcal{I}^{2*}_{2(n=1)} = 4(pq)^2; \tag{5.5.34}$$

$$\Updownarrow$$

$$\mathcal{I}^*_{2(n=1)} = \pm 2|pq|, \tag{5.5.35}$$

where $\mathcal{I}^*$ means the evaluation along Eq. (5.5.33).

The similarity between the r.h.s.'s of Eqs. (5.1.4) and (5.5.35) is only apparent. In fact, the KK extremal black hole has $\sqrt{-\mathcal{I}_{4,KK}}$, which necessarily implies that it is non-BPS ($Z_{AB}\neq 0$ in $\mathcal{N}=8$ and $Z\neq 0$ in $\mathcal{N}=2$). On the other hand, the axion-dilaton extremal black hole has $\mathcal{I}^*_{2(n=1)}$ and a "$\pm$" in the r.h.s., so that it can be both $\frac{1}{2}$-BPS and non-BPS $Z=0$ in $\mathcal{N}=2$. Moreover, the choice (5.5.33) leads to vanishing axion a (see Eq. (5.5.28)), and this explains that Eqs. (5.5.35) has $SO(1,1)$ symmetry, as Eq. (5.1.4).

5.5.1 Truncations of the scalar sector

As reported *e.g.* in Sects. 6 and 7 of [82], one can see that the attractor mechanism stabilizes the complex axion-dilaton s at the event-horizon of the axion-dilaton extremal black hole itself, while, as given by Eqs. (5.1.12) and (5.1.13) within the branching (5.2.11), only one real scalar degree of freedom, namely the KK radius r_{KK} defined by Eq. (5.1.11), is stabilized at the event horizon of the extremal KK black hole.

The relevant branching of the scalar sector for the embedding of the axion-dilaton extremal black hole into $\mathcal{N}=8$, $d=4$ supergravity is given by:

$$\begin{aligned} & SU(8) \to SU(4) \times SU(4) \times U(1), \\ & \mathbf{70} \to (\mathbf{1},\mathbf{1},2) + (\mathbf{1},\mathbf{1},-2) + (\mathbf{6},\mathbf{6},0) + \left(\overline{\mathbf{4}},\mathbf{4},1\right) + \left(\mathbf{4},\overline{\mathbf{4}},-1\right). \end{aligned} \tag{5.5.36}$$

Eq. (5.5.36) is the analogue of Eqs. (5.2.10) and (5.2.11), holding respectively for the ($\mathcal{N}=8$, $d=4$ embedding of the) RN and KK $d=4$ extremal (and asymptotically flat) black holes.

A remarkable feature characterizing the branchings (5.2.10), (5.2.11) and (5.5.36) is the possible presence of a singlet in their r.h.s.'s. The decomposition (5.5.36) contains two $SU(4)$ ($\times SU(4)$) singlets, whereas the decomposition (5.2.11) contains a real singlet, and the decomposition (5.2.10) does not contain any singlet. The presence of the singlet may lead to an underlying maximal compact symmetry ($U(1)$ for (5.2.10), absent for (5.2.11), and $SU(4)$ for (5.5.36)).

1. The first truncation ($\mathcal{N}=8 \to \mathcal{N}=4, n_V=6$) corresponds to setting[3]

$$\left(\overline{\mathbf{4}},\mathbf{4},1\right) = 0 = \left(\mathbf{4},\overline{\mathbf{4}},-1\right). \tag{5.5.37}$$

Indeed, by applying the condition (5.5.37), one obtains the correct quantum numbers of the scalar manifold $\frac{SL(2,\mathbb{R})}{U(1)} \times \frac{SO(6,6)}{SO(6)\times SO(6)}$ of the $\mathcal{N}=4$, $d=4$ supergravity coupled to 6 vector multiplets.

2. The second truncation ($\mathcal{N}=4, n_V=6 \to$ *pure* $\mathcal{N}=4$) simply consists in implementing the condition

$$(\mathbf{6},\mathbf{6},0) = 0, \tag{5.5.38}$$

which is consistently symmetric under the exchange of the gravity sector and the matter sector. Through condition (5.5.38), one achieves the correct quantum numbers of the scalar manifold $\frac{SL(2,\mathbb{R})}{U(1)}$ of the *pure* $\mathcal{N}=4$, $d=4$ supergravity.

[3]Notice the difference with respect to the analogue truncation condition (5.5.13) for the decomposition of the $\mathbf{28}$ and $\overline{\mathbf{28}}$ of $SU(8)$.

3. The third and last step (*pure* $\mathcal{N}=4 \to \mathcal{N}=2$ quadratic, $n_V=1$) does not change anything with respect to the previous one. Indeed, the scalar sector is unaffected by this third truncation, and the scalar manifold remains $\frac{SL(2,\mathbb{R})}{U(1)}$.

Appendix 5.A Truncation of $\mathcal{N}=8, d=5$ supergravity to the $d=5$ uplift of the *stu* model

The bosonic sector of the $\mathcal{N}=8$, $d=5$ supergravity theory consists in the metric $g_{\mu\nu}$ ($\mu,\nu=1,...,5$), 27 vectors A^{Λ}_{μ} and 42 scalars ϕ_{abcd} parametrizing the coset $\frac{E_{6(6)}}{USp(8)}$. The index $\Lambda=1,\dots 27$ is in the **27** of $E_{6(6)}$, and it can be traded for a couple of flat antisymmetric indices (ab) of $USp(8)$. Thus, the vectors A^{ab}_{μ} transform in the **27** of $USp(8)$, that is

$$\mathbf{27} \text{ of } E_{6(6)} \longrightarrow \mathbf{27} \text{ of } USp(8)\,. \tag{5.A.1}$$

The 42 scalars ϕ_{abcd} are in the traceless self-real 4-fold antisymmetric representation **42** of $USp(8)$.

Upon performing the $d=5 \to d=4$ reduction, one gets 70 scalars, which split into the following irreps. of $USp(8)$:

$$\mathbf{70} = \mathbf{42} + \mathbf{27} + \mathbf{1}\,. \tag{5.A.2}$$

Here **27** accounts for the axions coming from the A^{ab}_5 vectors of $E_{6(6)}$, **1** is the KK radius r_{KK} (see the definition (5.1.11)), and **42** corresponds to the scalars in $\frac{E_{6(6)}}{USp(8)}$.

In order to extract the *stu* model, we notice that its $d=5$ uplift is the $(SO(1,1))^2$ model with cubic hypersurface [77],[78] (see e.g. the treatment given in [69])

$$\widehat{\lambda}^1\widehat{\lambda}^2\widehat{\lambda}^3 = 1. \tag{5.A.3}$$

The $\mathcal{N}=8 \longrightarrow \mathcal{N}=2$, $d=5$ supersymmetry reduction corresponds, at the level of $E_{6(6)}$, to taking the decomposition

$$E_{6(6)} \longrightarrow SO(1,1)\times SO(5,5) \longrightarrow (SO(1,1))^2 \times SO(4,4)\,, \tag{5.A.4}$$

so that (weights with respect to $SO(1,1)$'s are disregarded, irrelevant for our purposes)

$$\mathbf{27} \to \mathbf{1} + \mathbf{16} + \mathbf{10} \to \mathbf{1} + \mathbf{8}_s + \mathbf{8}_c + \mathbf{1} + \mathbf{1} + \mathbf{8}_v. \tag{5.A.5}$$

Thus, three $SO(4,4)$-singlets are generated; they correspond to the three Abelian vector fields of the $d=5$ uplift of the *stu* model. By further reducing to $d=4$, one gets a further vector from the KK vector (*alias* the $d=4$ graviphoton). This can be easily seen by completing the decomposition (5.A.4) starting from the U-duality group $E_{7(7)}$ of $d=4$ maximal supergravity:

$$E_{7(7)} \longrightarrow SO(1,1) \times E_{6(6)} \longrightarrow (SO(1,1))^2 \times SO(5,5) \longrightarrow (SO(1,1))^3 \times SO(4,4)\,, \tag{5.A.6}$$

so that Eq. (5.A.5) gets completed as (again, neglecting weights with respect to $SO(1,1)$)

$$\mathbf{28} \to \mathbf{27}+\mathbf{1} \to \mathbf{1}+\mathbf{16}+\mathbf{10}+\mathbf{1} \to \mathbf{1}+\mathbf{8}_s+\mathbf{8}_c+\mathbf{1}+\mathbf{1}+\mathbf{8}_v+\mathbf{1}, \tag{5.A.7}$$

containing four $SO(4,4)$ singlets in the last term.

It is worth pointing out that at $d=4$ the $(SO(1,1))^3$ commuting with $SO(4,4)$ gets enhanced to $(SL(2,\mathbb{R}))^3$. By further decomposing

$$SO(4,4) \to (SL(2,\mathbb{R}))^4\,, \tag{5.A.8}$$

this yields the $(SL(2,\mathbb{R}))^7$, used for the seven qubit entanglement in quantum information theory [84],[85].

Notice that the presence of three different $\mathbf{8}$'s of $SO(4,4)$ in the r.h.s. of the decomposition (5.A.5) (as well as of (5.A.7)) is the origin of the *triality* symmetry [86],[87] of the *stu* model [88].

The $(SO(1,1))^2$ factor in the r.h.s. of the branching (5.A.4) is nothing but the scalar manifold of the $d=5$ counterpart of the *stu* model (spanned by $\widehat{\lambda}^1$, $\widehat{\lambda}^2$ and $\widehat{\lambda}^3$ satisfying the cubic constraint (5.A.3)). On the other hand, the $(SO(1,1))^3$ factor in the r.h.s. of the branching (5.A.7) is spanned by the unconstrained, strictly positive, $d=4$ dilatons $\lambda^1 \equiv -Im(s)$, $\lambda^2 \equiv -Im(t)$ and $\lambda^3 \equiv -Im(u)$. They are related to their hatted counterparts by $\lambda^i \equiv r_{KK}\widehat{\lambda}^i$, $i=1,2,3$, implying (see Eqs. (5.A.3) and Eq. (5.1.11); see also *e.g.* [69]))

$$\lambda^1\lambda^2\lambda^3 = r_{KK}^3 \equiv \mathcal{V}. \tag{5.A.9}$$

The decomposition of the $d=5$ stabilizer (analogue to the decomposition (5.A.4) of the U-duality group of the $d=5$ maximal supergravity) reads as follows:

$$\begin{aligned} USp(8) &\to USp(4)\times USp(4) = Spin(5)\times Spin(5) \to \\ &\to Spin(4)\times Spin(4) = (SU(2))^2 \times (SU(2))^2\,, \end{aligned} \tag{5.A.10}$$

yielding the following decomposition of the fundamental $\mathbf{8}$ of $USp(8)$:

$$\mathbf{8} \to (\mathbf{4},\mathbf{1})+(\mathbf{1},\mathbf{4}) \to (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})+(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})\,. \tag{5.A.11}$$

This allows one to compute the corresponding branchings of the $\mathbf{27} = (\mathbf{8}\times\mathbf{8})_{A,0}$ and $\mathbf{42} = (\mathbf{8}\times\mathbf{8}\times\mathbf{8}\times\mathbf{8})_{A,0}$ (the subscript "$A,0$" standing for "antisymmetric traceless") of $USp\,(8)$ (the intermediate decompositions with respect to $USp(4)\times USp(4)$ are omitted, because irrelevant for our purposes):

$$\begin{aligned}\mathbf{27}\rightarrow\ &(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})+(\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})+(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2})+(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})+\\&+(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})+(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})\ +3\,(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1});\end{aligned} \tag{5.A.12}$$

$$\begin{aligned}\mathbf{42}\rightarrow\ &(\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2})+(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})+(\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})+(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2})+(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})+\\&+(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})+(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})\ +2\,(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\end{aligned} \tag{5.A.13}$$

Consistently with previous statements, the three $(SU(2))^4$-singlets in the r.h.s. of the decomposition (5.A.12) and the two $(SU(2))^4$-singlets in in the r.h.s. of the decomposition (5.A.13) respectively are the three Abelian vector fields (including the $d=5$ graviphoton) and the two independent real scalars (say, $\widehat{\lambda}^1$ and $\widehat{\lambda}^2$) in the bosonic spectrum of the $(SO\,(1,1))^2$ model, which is the $d=5$ uplift of the *stu* model.

Reducing to $d=4$, the six real scalar degrees of freedom of the *stu* model are the radius r_{KK} (see Eqs. (5.1.11) and (5.A.9)), the two scalars $\widehat{\lambda}^1$ and $\widehat{\lambda}^2$, and the three axions (coming from the fifth component A_5^I $(I=1,2,3)$ of the three $d=5$ vectors). As previously mentioned, the four $d=4$ vectors come from the three $d=5$ vectors and from the KK vector $g_{5\mu}$ $(\mu=1,...,4)$.

Finally, it should be notice that $\lambda^1\lambda^2\lambda^3$ (defining the volume of the $d=5$ cubic hypersurface through Eqs. (5.1.11) and (5.A.9)) can be obtained through a consistent truncation of the $E_{6(6)}$-invariant expression $(\Lambda,\Sigma,\Delta=1,...,27)$

$$\frac{1}{3!}d_{\Lambda\Sigma\Delta}\lambda^\Lambda\lambda^\Sigma\lambda^\Delta \tag{5.A.14}$$

to $(SO\,(1,1))^2$, by retaining only the three singlets of $SO\,(4,4)$ (see the decompositions (5.A.4) and (5.A.5) above).

Appendix 5.B Discussion

We have considered, in this chapter, some examples of extremal black hole configurations in the framework of black hole attractors of $\mathcal{N}=8$ supergravity.

The effective black hole potential has been computed in different bases, namely in the manifestly $SU\,(8)$-covariant basis, as well as in the $USp\,(8)$-covariant one. The former is suitable to describe the (BPS) Reissner−Nördstrom extremal black hole with its $U(1)$ symmetry, as a consequence of the attractor point to be the origin of the $d = 4$ scalar manifold $\frac{E_{7(7)}}{SU(8)}$. The latter has an origin in $d = 5$, and it is appropriate in order to describe the non-BPS Kaluza−Klein extremal black hole, with its $SO(1,1)$ symmetry arising from the non-trivial attractor value of the KK radial mode.

We have also considered the axion-dilaton system, whose BPS or non-BPS nature depends on whether it is embedded in $\mathcal{N} = 2$ quadratic or in $\mathcal{N} = 4$, $d = 4$ supergravity. The axion-dilaton extremal black hole is obtained as a particular case of the attractor equations of the maximal $d = 4$ theory. In that case, all 70 scalars other than the $SU(4) \times SU(4)$-singlets in the decomposition 5.5.36 are set to vanish, and correspondingly only 12 graviphoton electric and magnetic charges are taken to be nonzero (see Eq. (5.5.12)). At the level $\mathcal{N} = 2$, this attractor solution is obtained by retaining only 4 (2 electric and 2 magnetic) non-vanishing charges, according to the decomposition (5.5.6) of $SU(4)$.

Chapter 6

5d/4d U-dualities for $\mathcal{N}=8$ black holes

The connection between the U-duality groups in $d=5$ and $d=4$ is used here to derive properties of the $\mathcal{N}=8$ black hole potential and its critical points (attractors). This approach allows to study and compare the supersymmetry features of different solutions.

6.1 Introduction

In $\mathcal{N}=8$ supergravity, in the Einsteinian approximation, there is a nice relation between the classification of large black holes which undergo the attractor flow and charge orbits which classify, in a duality invariant manner, the properties of the dyonic vector of electric and magnetic charges $Q=(p^\Lambda,\,q_\Lambda)$ ($\Lambda=0,...,27$ in $d=4$) [33],[49]. The attractor points are given by extrema of the $4d$ black hole potential, as discussed in Section 4.4,

$$V_{BH}=\frac{1}{2}Z_{AB}Z^{*\,AB}=\langle Q,V_{AB}\rangle\,\langle Q,\overline{V}^{AB}\rangle\ , \qquad (6.1.1)$$

where the central charge is the antisymmetric matrix $(A,B=1,...,8)$

$$Z_{AB}=\langle Q,V_{AB}\rangle=Q^T\,\Omega\,V_{AB}=f^\Lambda{}_{AB}\,q_\Lambda-h_{\Lambda\,AB}\,p^\Lambda\ , \qquad (6.1.2)$$

the symplectic sections are

$$V_{AB}=(f^\Lambda{}_{AB},h_{\Lambda\,AB})\ , \qquad (6.1.3)$$

and Ω is the symplectic invariant metric.

An important role is played by the Cartan quartic invariant I_4 [89, 20] in that it only depends on Q and not on the asymptotic values of the 70 scalar fields φ. This means that if we construct I_4 as a combination of quartic powers of the central charge matrix $Z_{AB}(q,p,\varphi)$ [54], the φ dependence drops out from the final expression

$$\frac{\partial}{\partial\varphi} I_4(Z_{AB}) = 0 \, . \tag{6.1.4}$$

Analogue (cubic) invariants I_3 exist for black holes and/or (black) strings in $d=5$ [12], [33]. These are given by

$$I_3(p^I) = \frac{1}{3!} d_{IJK} p^I p^J p^K \, , \tag{6.1.5}$$

$$I_3(q_I) = \frac{1}{3!} d^{IJK} q_I q_J q_K \, , \tag{6.1.6}$$

where d_{IJK}, d^{IJK} are the $(27)^3$ $E_{6(6)}$ invariants. Consequently, the $d=4$ $E_{7(7)}$ quartic invariant takes the form

$$I_4(Q) = -(p^0 q_0 + p^I q_I)^2 + 4\left[-p^0 I_3(q) + q_0 I_3(p) + \frac{\partial I_3(q)}{\partial q_I}\frac{\partial I_3(p)}{\partial p^I}\right] \, . \tag{6.1.7}$$

On the other hand, in terms of the central charge matrices $Z_{ab}(\phi,q)$ (in $d=5$ this is the **27** representation of $USp(8)$) and $Z_{AB}(\phi,p,q)$ (in $d=4$ this is the **28** of $SU(8)$), their expression is

$$I_3(q) = Z_{ab}\Omega^{bc} Z_{cd}\Omega^{dq} Z_{qp}\Omega^{pa} \, , \qquad Z_{ab}\Omega^{ab} = 0 \, , \tag{6.1.8}$$

$$I_4(p,q) = \frac{1}{4}\left[4\,Tr(ZZ^\dagger ZZ^\dagger) - (Tr\,ZZ^\dagger)^2 + 32\,\mathrm{Re}\,(Pf\,Z_{AB})\right] \, , \tag{6.1.9}$$

where $ZZ^\dagger = Z_{AB}\bar{Z}^{CB}$, Ω^{ab} is the $5d$ symplectic invariant metric, and the Pfaffian of the central charge is [20]

$$Pf\,(Z_{AB}) = \frac{1}{2^4 4!}\epsilon^{ABCDEFGH} Z_{AB} Z_{CD} Z_{EF} Z_{GH} \, . \tag{6.1.10}$$

In fact, these are simply the (totally symmetric) invariants which characterize the 27 dimensional representation of $E_{6(6)}$ and the 56 dimensional representation of $E_{7(7)}$, which are the U-duality [50] symmetries of $\mathcal{N}=8$ supergravity in $d=5$ and $d=4$, respectively.

When charges are chosen such that I_4 and I_3 are not vanishing, one has large black holes and in the extremal case the attractor behavior may occur. However, while at $d=5$ there is a unique ($\frac{1}{8}$-BPS) attractor orbit with $I_3 \neq 0$, associated to the space [49], [90]

$$\mathcal{O}_{d=5} = \frac{E_{6(6)}}{F_{4(4)}} \, , \tag{6.1.11}$$

at $d=4$ two orbits emerge, the BPS one

$$\mathcal{O}_{d=4,\,BPS} = \frac{E_{7(7)}}{E_{6(2)}}\,, \tag{6.1.12}$$

and the non BPS one with different stabilizer

$$\mathcal{O}_{d=4,\,non-BPS} = \frac{E_{7(7)}}{E_{6(6)}}\,. \tag{6.1.13}$$

Such orbits have further ramifications in theories with lower supersymmetry , but but we will confine our attention to the $\mathcal{N}=8$ theory.

In this Chapter, extending a previous result for $\mathcal{N}=2$ theories [69], we elucidate the connection between these configurations and we relate the critical points of the $\mathcal{N}=8$ black hole potential of the $5d$ and $4d$ theories. To achieve this goal we use a formulation of $4d$ supergravity in a $E_{6(6)}$ duality covariant basis [76], which is appropriate to discuss a $4d/5d$ correspondence. This is not the same as the Cremmer-Julia [20] or de Wit-Nicolai [71] manifest $SO(8)$ (and $SL(8,\mathbb{R})$) covariant formulation, but it is rather related to the Sezgin-Van Nieuwenhuizen $5d/4d$ dimensional reduction [74]. These two formulations are related to one another by dualizing several of the vector fields and therefore they interchange electric and magnetic charges of some of the 28 vector fields of the final theory, as we have seen in the previous chapter.

6.2 4d/5d relations for the $\mathcal{N}=8$ extremal black hole potential

Using known identities [21], [75], the black hole potential can be written as a quadratic form in terms of the charge vector Q and the symplectic 56×56 matrix $\mathcal{M}(\mathcal{N})$, related to the $4d$ vector kinetic matrix $\mathcal{N}_{\Lambda\Sigma}$

$$V_{BH} = -\frac{1}{2}Q^T\mathcal{M}(\mathcal{N})Q\,, \tag{6.2.1}$$

where $\mathcal{M}$ is

$$\mathcal{M}(\mathcal{N}) = \begin{pmatrix} \mathrm{Im}\mathcal{N}+\mathrm{Re}\mathcal{N}(\mathrm{Im}N)^{-1}\mathrm{Re}\mathcal{N} & -\mathrm{Re}\mathcal{N}(\mathrm{Im}\mathcal{N})^{-1} \\ \\ -(\mathrm{Im}\mathcal{N})^{-1}\mathrm{Re}\mathcal{N} & (\mathrm{Im}\mathcal{N})^{-1} \end{pmatrix}\,. \tag{6.2.2}$$

The indices $\Lambda\,,\Sigma$ of $\mathcal{N}_{\Lambda\Sigma}$ are now split as $(0,I)$, according to the decomposition of $4d$ charges with respect to $5d$ ones, thus $\mathcal{N}_{\Lambda\Sigma}$ assumes the block form

$$\mathcal{N}_{\Lambda\,\Sigma} = \left(\begin{array}{c|c} \mathcal{N}_{00} & \mathcal{N}_{0\,J} \\ \hline \mathcal{N}_{I\,0} & \mathcal{N}_{I\,J} \end{array}\right)\,, \tag{6.2.3}$$

The kinetic matrix depends on the 70 scalars of the $\mathcal{N}=8$ theory, which are given, in the $5d/4d$ KK reduction, by the 42 scalars of the $5d$ theory (encoded in the $5d$ vector kinetic matrix $a_{IJ}=a_{JI}$), by the 27 axions a^I and the dilaton field e^ϕ. In a normalization that is suitable for comparison to $\mathcal{N}=2$, it has the form

$$\mathcal{N}_{\Lambda\Sigma}=\left(\begin{array}{c|c} \frac{1}{3}d-i\left(e^{2\phi}a_{IJ}a^Ia^J+e^{6\phi}\right) & -\frac{1}{2}d_J+ie^{2\phi}a_{KJ}a^K \\ \hline -\frac{1}{2}d_I+ie^{2\phi}a_{IK}a^K & d_{IJ}-ie^{2\phi}a_{IJ}\end{array}\right)\ , \tag{6.2.4}$$

where

$$d\equiv d_{IJK}a^Ia^Ja^K \quad , \quad d_I\equiv d_{IJK}a^Ja^K \quad , \quad d_{IJ}\equiv d_{IJK}a^K\ . \tag{6.2.5}$$

The black hole potential, computed from (6.2.1) using the above formulas, can be rearranged as

$$\begin{aligned} V_{BH}=&\frac{1}{2}\left(p^0e^\phi a^I\right)a_{IJ}\left(p^0e^\phi a^J\right)+\frac{1}{2}\left(p^0e^{3\phi}\right)^2+\frac{1}{2}\left(\frac{d}{6}p^0e^{-3\phi}\right)^2+\\ &+\frac{1}{2}\left(\frac{1}{2}e^{-\phi}p^0d_I\right)a^{IJ}\left(\frac{1}{2}e^{-\phi}p^0d_J\right)+\frac{1}{2}\times 2\left(-p^0e^\phi a_I\right)a_{IJ}\left(p^Je^\phi\right)+\\ &+\frac{1}{2}\times 2\left(\frac{d}{6}p^0e^{-3\phi}\right)\left(-\frac{1}{2}p^Id_Ie^{-3\phi}\right)-\frac{1}{2}\times 2\left(\frac{1}{2}p^0e^{-\phi}d_I\right)a^{IJ}\left(p^Kd_{KJ}e^{-\phi}\right)+\\ &+\frac{1}{2}\left(e^\phi p^I\right)a_{IJ}\left(e^\phi p^J\right)+\frac{1}{2}\left(\frac{1}{2}e^{-3\phi}p^Kd_K\right)^2+\\ &+\frac{1}{2}\left(e^{-\phi}p^Kd_{KI}\right)a^{IJ}\left(e^{-\phi}p^Ld_{JL}\right)+\frac{1}{2}\times 2\left(q_0e^{-3\phi}\right)\left(\frac{d}{6}p^0e^{-3\phi}\right)+\\ &+\frac{1}{2}\times 2\left(q_Ia^Ie^{-3\phi}\right)\left(\frac{d}{6}p^0e^{-3\phi}\right)+\frac{1}{2}\times 2\left(q_Ie^{-\phi}\right)a^{IJ}\left(\frac{1}{2}p^0d_Je^{-\phi}\right)+\\ &-\frac{1}{2}\times 2\left(q_0e^{-3\phi}\right)\left(\frac{1}{2}p^Id_Ie^{-3\phi}\right)-\frac{1}{2}\times 2\left(q_Ia^Ie^{-3\phi}\right)\left(\frac{1}{2}p^Jd_Je^{-3\phi}\right)+\\ &-\frac{1}{2}\times 2\left(q_Ie^{-\phi}\right)a^{IJ}\left(p^Kd_{KJ}e^{-\phi}\right)+\frac{1}{2}\left(q_0e^{-3\phi}\right)^2+\frac{1}{2}\times 2\left(q_0e^{-3\phi}\right)\left(q_Ia^Ie^{-3\phi}\right)+\\ &+\frac{1}{2}\left(q_Ia^Ie^{-3\phi}\right)^2+\frac{1}{2}\left(q_Ie^{-\phi}\right)a^{IJ}\left(q_Je^{-\phi}\right)\ , \end{aligned} \tag{6.2.6}$$

with $a^{IJ}=a^{-1}_{IJ}$. This form shows that it can be written in terms of squares of electric and magnetic components as

$$V_{BH}=\frac{1}{2}(Z^e_0)^2+\frac{1}{2}\left(Z^0_m\right)^2+\frac{1}{2}Z^e_Ia^{IJ}Z^e_J+\frac{1}{2}Z^I_ma_{IJ}Z^J_m\ , \tag{6.2.7}$$

provided one defines,

$$\begin{aligned} Z_0^e &= e^{-3\phi}q_0 + e^{-3\phi}q_I a^I + e^{-3\phi}\frac{d}{6}p^0 - \frac{1}{2}e^{-3\phi}p^I d_I \ , \\ Z_m^0 &= e^{3\phi}p^0 \ , \\ Z_I^e &= \frac{1}{2}e^{-\phi}p^0 d_I - p^J d_{IJ}e^{-\phi} + q_I e^{-\phi} \ , \\ Z_m^I &= e^{\phi}p^I - e^{\phi}p^0 a^I \ . \end{aligned} \tag{6.2.8}$$

In order to get the symplectic embedding of the four dimensional theory, we still need to complexify the central charges. To this end, we define the two complex vectors

$$\begin{aligned} Z_0 &\equiv \frac{1}{\sqrt{2}}(Z_0^e + iZ_m^0) \ , \\ Z_a &\equiv \frac{1}{\sqrt{2}}(Z_a^e + iZ_m^a) \ , \end{aligned} \tag{6.2.9}$$

where

$$Z_a^e = Z_I^e(a^{-1/2})_a^I \ , \qquad Z_m^a = Z_m^I(a^{1/2})_I^a \tag{6.2.10}$$

such that

$$V_{BH} = |Z_0|^2 + Z_a\bar{Z}_a \ , \tag{6.2.11}$$

where now $a = 1, ..., 27$ is a flat index, which can be regarded as a $USp(8)$ antisymmetric traceless matrix.

The potential at the critical point gives the black hole entropy corresponding to the given solution, which in $d = 4$ reads

$$\frac{S_{BH}}{\pi} = \sqrt{|I_4|} = V_{BH}^{crit.} \ , \tag{6.2.12}$$

while in $d = 5$ it is [53]

$$\frac{S_{BH}}{\pi} = 3^{3/2}|I_3|^{1/2} = \left(3\,V_5^{crit}\right)^{3/4} \ , \tag{6.2.13}$$

where I_4 and I_3 are the invariants of the $\mathcal{N} = 8$ theory in $d = 4$ and $d = 5$ respectively.

6.2.1 Symplectic sections

In virtue of the previous discussion, we can trade the central charge (6.1.2)for the 28-component vector

$$Z_A = f^{\Lambda}{}_A q_\Lambda - h_{\Lambda A}p^\Lambda \ , \tag{6.2.14}$$

where f and h are symplectic sections satisfying

a) $\mathcal{N}_{\Lambda\Sigma} = h_{\Lambda A}(f^{-1})^A{}_\Sigma$,

b) $i(\mathbf{f}^\dagger \mathbf{h} - \mathbf{h}^\dagger \mathbf{f}) = \mathbf{Id}$,

c) $\mathbf{f}^T \mathbf{h} - \mathbf{h}^T \mathbf{f} = 0$.

Notice that one still has the freedom of a further transformation

$$\begin{aligned} h &\to hM \ , \\ f &\to fM \ , \end{aligned} \tag{6.2.15}$$

as it leaves invariant the vector kinetic matrix $\mathcal{N}$, as well as relations $a) - c)$, when M is a unitary matrix

$$MM^\dagger = 1 \ . \tag{6.2.16}$$

Indeed, when the central charge transforms as

$$\begin{aligned} Z &\to ZM \ , \\ ZZ^\dagger &\to ZMM^\dagger Z^\dagger = ZZ^\dagger \ , \end{aligned} \tag{6.2.17}$$

the black hole potential

$$V_{BH} \equiv ZZ^\dagger \tag{6.2.18}$$

is left invariant. In our case, we rearrange the 28 indices into a single complex vector index, to be identified, for a suitable choice of M, with the two-fold antisymmetric representation of $SU(8)$, according to the decomposition $\mathbf{28} \to \mathbf{27} + \mathbf{1}$ of $SU(8) \to USp(8)$; we thus have

$$\begin{aligned} Z_0 &= f^\Lambda{}_0 q_\Lambda - h_{\Lambda\,0} p^\Lambda = \\ &= f^0{}_0 q_0 + f^J{}_0 q_J - h_{0\,0} p^0 - h_{J\,0} p^J \ , \\ Z_a &= f^\Lambda{}_a q_\Lambda - h_{\Lambda\,a} p^\Lambda = \\ &= f^0{}_a q_0 + f^J{}_a q_J - h_{0\,a} p^0 - h_{J\,a} p^J \ ; \end{aligned} \tag{6.2.19}$$

which, from the definition in (6.2.9) yields

$$\begin{aligned} Z_0 &= \frac{1}{\sqrt{2}} \left[e^{-3\phi} q_0 + e^{-3\phi} a^I q_I + \left(e^{-3\phi} \frac{d}{6} + i e^{3\phi} \right) p^0 - \frac{1}{2} \left(e^{-3\phi} d_I \right) p^I \right] \ , \\ Z_a &= \frac{1}{\sqrt{2}} \left[e^{-\phi} q_I (a^{-1/2})^I{}_a + \left(\frac{1}{2} e^{-\phi} d_I (a^{-1/2})^I{}_a - i e^{\phi} a^J (a^{1/2})_J{}^a \right) p^0 + \right. \\ &\quad \left. - \left(e^{-\phi} d_{IJ} (a^{-1/2})^I{}_a - i e^{\phi} (a^{1/2})_J{}^a \right) p^J \right] \ . \end{aligned} \tag{6.2.20}$$

Thus we consider

$$f^{\Lambda}{}_{A}=\frac{1}{\sqrt{2}}\left(\begin{array}{c|c} e^{-3\phi} & 0 \\ \hline e^{-3\phi}a^{I} & e^{-\phi}(a^{-1/2})^{I}{}_{a} \end{array}\right), \tag{6.2.21}$$

$$h_{\Lambda\,A}=\frac{1}{\sqrt{2}}\left(\begin{array}{c|c} -e^{-3\phi}\frac{d}{6}-ie^{3\phi} & -\frac{1}{2}e^{-\phi}d_{K}(a^{-1/2})^{K}{}_{a}+ie^{\phi}a^{K}(a^{1/2})_{K}{}^{a} \\ \hline \frac{1}{2}e^{-3\phi}d_{I} & e^{-\phi}d_{IJ}(a^{-1/2})^{J}{}_{a}-ie^{\phi}(a^{1/2})_{I}{}^{a} \end{array}\right). \tag{6.2.22}$$

From $\mathbf{f}^{-1}$

$$(f^{-1})_{\Lambda}{}^{A}=\sqrt{2}\left(\begin{array}{c|c} e^{3\phi} & 0 \\ \hline -e^{\phi}a^{I}(a^{1/2})_{I}{}^{a} & e^{\phi}(a^{1/2})_{I}{}^{a} \end{array}\right), \tag{6.2.23}$$

by matrix multiplication, we find that relations $a)$ $b)$ and $c)$ are fulfilled by $\mathbf{f}$ and $\mathbf{h}$, that we now recognize to be the symplectic sections.

We finally perform the transformation $f'=fM$ (where $M=f^{-1}f'=h^{-1}h'$), with M unitary matrix, in virtue of identities $a)$, $b)$ and $c)$, valid for both (f,h) and (f',h'). A model independent formula for M valid for any $\mathcal{N}=2$ d-geometry (in particular, for any truncation of $\mathcal{N}=8$ to an $\mathcal{N}=2$ geometry, such as the models treated in this paper) is given by the matrix [91]

$$M=A^{1/2}\hat{M}G^{-1/2}\,, \tag{6.2.24}$$

with

$$A = \left(\begin{array}{c|c} 1 & 0...0 \\ \hline 0 & \\ . & a_{IJ} \\ . & \\ 0 & \end{array} \right) , \qquad G = \left(\begin{array}{c|c} 1 & 0...0 \\ \hline 0 & \\ . & g_{IJ} \\ . & \\ 0 & \end{array} \right) , \qquad g_{IJ} = \frac{1}{4} e^{-4\phi} a_{IJ} \ , \quad (6.2.25)$$

with $\hat{M}$ given by

$$\hat{M} = \frac{1}{2} \begin{pmatrix} 1 & \partial_{\bar{J}} K \\ -i\lambda^I e^{-2\phi} & e^{-2\phi} \delta^I_{\bar{J}} + i e^{-2\phi} \lambda^I \partial_{\bar{J}} K \end{pmatrix} , \qquad (6.2.26)$$

where "$-\lambda^I$" are the imaginary parts of the complex moduli $z^I = a^I - i\lambda^I$, and K is the Kähler potential $K = -\ln(8\mathcal{V})$, with $\mathcal{V} = \frac{1}{3!} d_{IJK} \lambda^I \lambda^J \lambda^K$; the matrix $\hat{M}$ satisfies the properties

$$\begin{aligned} A \hat{M} G^{-1} \hat{M}^\dagger &= Id \ , \\ G^{-1} \hat{M}^\dagger A \hat{M} &= Id \ . \end{aligned} \qquad (6.2.27)$$

For the models considered below, this matrix M does indeed reproduce, for the given special configurations, the formula in eq. (6.4.7).

Note that $\hat{M}$ performs the change of basis between the central charges defined as

$$\begin{aligned} Z_0 &= \frac{1}{\sqrt{2}} (Z_0^e + i Z_m^0) \ , \\ Z_I &= \frac{1}{\sqrt{2}} (Z_I^e + i a_{IJ} Z_m^J) \ , \end{aligned} \qquad (6.2.28)$$

and the special geometry charges $(Z, \mathcal{D}_{\bar{I}} \overline{Z})$, that is the charges in "curved" rather than the "flat" indices.

6.3 Attractors in the 5 dimensional theory

It was shown in [33] that the cubic invariant of the five dimensions can be written as

$$I_3 = Z_1^5 Z_2^5 Z_3^5 \ , \qquad (6.3.1)$$

where Z^5_a's are related to the skew eigenvalues of the $USp(8)$ central charge matrix in the normal frame

$$e_{ab} = \begin{pmatrix} Z^5_1 + Z^5_2 - Z^5_3 & 0 & 0 & 0 \\ 0 & Z^5_1 + Z^5_3 - Z^5_2 & 0 & 0 \\ 0 & 0 & Z^5_2 + Z^5_3 - Z^5_1 & 0 \\ 0 & 0 & 0 & -(Z^5_1 + Z^5_2 + Z^5_3) \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{6.3.2}$$

We consider a configuration of only three non-vanishing electric charges (q_1, q_2, q_3), that we can take all non-negative. We further confine to two moduli λ_1, λ_2, describing a geodesic submanifold $SO(1,1)^2 \in E_{6(6)}/USp(8)$ whose special geometry is determined by the constraint

$$\frac{1}{3!} d_{IJK} \hat{\lambda}^I \hat{\lambda}^J \hat{\lambda}^K = \hat{\lambda}^1 \hat{\lambda}^2 \hat{\lambda}^3 = 1 \ , \tag{6.3.3}$$

where $\hat{\lambda}^I = \mathcal{V}^{-1/3} \lambda^I$, defining the $stu-$model [69].

The metric a_{IJ}, restricted to this surface, takes the diagonal form

$$a_{IJ} = - \frac{\partial^2}{\partial \hat{\lambda}^I \partial \hat{\lambda}^J} \log \mathcal{V} \big|_{\mathcal{V}=1} = \begin{pmatrix} \frac{1}{\hat{\lambda}_1^2} & 0 & 0 \\ 0 & \frac{1}{\hat{\lambda}_2^2} & 0 \\ 0 & 0 & \frac{1}{\hat{\lambda}_3^2} = \hat{\lambda}_1^2 \hat{\lambda}_2^2 \end{pmatrix} , \tag{6.3.4}$$

and the five dimensional black hole potential for electric charges is[1]

$$V^e_5 = q_I a^{IJ} q_J = \sum_{a=1}^{3} Z^5_a(q) Z^5_a(q) \ , \tag{6.3.5}$$

with $Z^5_a(q) = (a^{-1/2})^I{}_a \, q_I$; the moduli at the attractor point of the 5-dimensional solution are (see eq. 4.4 and 4.7 of [69])

$$\hat{\lambda}^I_{crit} = \frac{I_3^{1/3}}{q^I} \ , \tag{6.3.6}$$

and

$$\begin{aligned} V^{crit}_5 &= 3|q_1 q_2 q_3|^{2/3} = 3 I_3^{2/3} \ , \\ a^{IJ}_{crit} &= \frac{I_3^{2/3}}{q_I^2} \delta^{IJ} \end{aligned} \tag{6.3.7}$$

[1] In an analogous way, the black hole potential for magnetic charges, $V^m_5 = \sum_{a=1}^{3} Z^5_a(p) Z^5_a(p)$, is obtained by replacing $q_I \to p^I$ and $a^{IJ} \to a_{IJ}$ [69, 53], with $Z^5_a(p) = p^I (a^{1/2})_I{}^a$.

with no sum over repeated indices. We find

$$Z_a^{5,\,crit} = I_3^{1/3} \;, \qquad I_3 = Z_1^5 Z_2^5 Z_3^5 \;. \tag{6.3.8}$$

These relations also allow to connect the potential in (6.3.5)

$$V_5 = (Z_1^5)^2 + (Z_2^5)^2 + (Z_3^5)^2 \;, \tag{6.3.9}$$

with the form given in terms of the central charges [53], where it is the trace of the square matrix

$$V_5 = \frac{1}{2} Z_{ab}^5 Z^{5\,ab} \;. \tag{6.3.10}$$

The eigenvalues of Z_{ab}^5 are written in (6.3.2) in terms of Z_1^5, Z_2^5, Z_3^5. The 5d central charge matrix in the normal frame at the attractor point thus becomes

$$e_{ab} = \begin{pmatrix} I_3^{1/3}\epsilon & 0 & 0 & 0 \\ 0 & I_3^{1/3}\epsilon & 0 & 0 \\ 0 & 0 & I_3^{1/3}\epsilon & 0 \\ 0 & 0 & 0 & -3I_3^{1/3}\epsilon \end{pmatrix} , \tag{6.3.11}$$

which shows the breaking $USp(8) \to USp(6) \times USp(2)$.

6.4 Attractors in the 4 dimensional theory

In this section we reconsider the attractor solutions in terms of the present formalism based on central charges. We separately examine the three “axion free” configurations.

6.4.1 Electric solution $Q = (p^0\,, q_i)$

Let us first compute the 4dim central charge for the electric charge configuration with vanishing axions; using (6.2.20) we find

$$Z_0 = \frac{i}{\sqrt{2}} e^{3\phi} p^0 \;, \qquad Z_a = \frac{1}{\sqrt{2}} e^{-\phi} q_I (a^{-1/2})^I{}_a \;. \tag{6.4.1}$$

The 4-dim potential is

$$V_{BH} = \frac{1}{2} e^{-2\phi} V_5^e + \frac{1}{2} e^{6\phi} (p^0)^2 \;, \tag{6.4.2}$$

(where ϕ is connected to the volume used in ref. [69] by the formula $\mathcal{V} = e^{6\phi}$) and has the same critical points of the 5 dimensional potential, since

$$\frac{\partial V_{BH}}{\partial \lambda^I} = 0 \iff \frac{\partial V_5^e}{\partial \hat{\lambda}^I} = 0 \;, \qquad \forall\; I = 1, 2 \,. \tag{6.4.3}$$

The attractor values of $\hat{\lambda}^I$ are still given by (6.3.6), while the ϕ field at the critical point is [69]

$$e^{8\phi}|_{crit.} = I_3^{2/3}(p^0)^{-2} \ . \tag{6.4.4}$$

This fixes the central charges at the attractor point to be

$$\begin{aligned} Z_0{}^{attr} &= \frac{i}{\sqrt{2}}|p^0 q_1 q_2 q_3|^{1/4} sign(p^0) = \frac{i}{2}|I_4|^{1/4} sign(p^0) \ , \\ Z_a{}^{attr} &= \frac{1}{\sqrt{2}} I_3^{-1/12}(p^0)^{1/4} q_I \frac{I_3^{1/3}}{q_I} = \frac{1}{2}|I_4|^{1/4} \ , \end{aligned} \tag{6.4.5}$$

where the quartic invariant is $I_4 = -4\, p^0 q_1 q_2 q_3$. So we find

$$Z_1^{crit} = Z_2^{crit} = Z_3^{crit} = \frac{1}{2}|I_4|^{1/4} \equiv Z \ , \qquad Z_0^{crit} = \frac{i}{2}|I_4|^{1/4} sign(p^0) \equiv iZ_0 \ . \tag{6.4.6}$$

Let us define the 4d central charge matrix as

$$2Z_{AB} = e_{AB} - iZ^0\Omega \ , \tag{6.4.7}$$

where e_{AB} is the matrix in (6.3.2) in which, instead of Z_1^5, Z_2^5, Z_3^5 of the 5d theory, we now write the 4d Z_a's defined in (6.2.20). it can be readily seen that for axion free solutions eq. (6.4.7) correctly gives

$$V_{BH} = \sum_i |z_i|^2 = |Z_0|^2 + \sum_a |Z_a|^2 \tag{6.4.8}$$

where z_i's, for $i = 1, .., 4$, are the (complex skew-diagonal) elements of Z_{AB}. We then have

$$\begin{aligned} 2Z_{AB} &= \begin{pmatrix} Z\epsilon & 0 & 0 & 0 \\ 0 & Z\epsilon & 0 & 0 \\ 0 & 0 & Z\epsilon & 0 \\ 0 & 0 & 0 & -3Z\epsilon \end{pmatrix} + \begin{pmatrix} Z_0\epsilon & 0 & 0 & 0 \\ 0 & Z_0\epsilon & 0 & 0 \\ 0 & 0 & Z_0\epsilon & 0 \\ 0 & 0 & 0 & Z_0\epsilon \end{pmatrix} = \\ &= \begin{pmatrix} (Z+Z_0)\epsilon & 0 & 0 & 0 \\ 0 & (Z+Z_0)\epsilon & 0 & 0 \\ 0 & 0 & (Z+Z_0)\epsilon & 0 \\ 0 & 0 & 0 & (-3Z+Z_0)\epsilon \end{pmatrix} . \end{aligned} \tag{6.4.9}$$

Since (6.4.5) and (6.4.6) yield that $Z = |Z_0|$, depending on the choice $p^0 > 0$ or $p^0 < 0$, two different solutions arise. In fact,

$$Z + Z_0 = 0 \quad \rightarrow \quad Z_{AB} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2Z_0 \end{pmatrix} \otimes \epsilon \ , \tag{6.4.10}$$

gives the $\frac{1}{8}$-BPS solution when $p^0 < 0$ and shows $SU(6) \times SU(2)$ symmetry. Conversely,

$$Z = Z_0 \quad \rightarrow \quad Z_{AB} = \begin{pmatrix} Z_0 & 0 & 0 & 0 \\ 0 & Z_0 & 0 & 0 \\ 0 & 0 & Z_0 & 0 \\ 0 & 0 & 0 & -Z_0 \end{pmatrix} \otimes \epsilon \ , \tag{6.4.11}$$

is the non-BPS solution that corresponds to the choice $p^0 > 0$, with residual $USp(8)$ symmetry.

6.4.2 Magnetic solution $Q = (p_i \, , q^0)$

This case is symmetric to the electric solution of Section 6.4.1. If we take all positive magnetic charges, then the cubic invariant is $I_3 = p^1 p^2 p^3$, the quartic invariant is $I_4 = 4\, q_0\, p^1 p^2 p^3$ and the values of the critical 5d moduli are now (see eq. (5.3) of [69])

$$\hat{\lambda}^I = \frac{p^I}{I_3^{1/3}} \ . \tag{6.4.12}$$

The central charges for this configuration are, from (6.2.20),

$$Z_0 = \frac{1}{\sqrt{2}} e^{-3\phi} q_0 \ , \qquad\qquad Z_a = \frac{i}{\sqrt{2}} e^{\phi} p^I (a^{1/2})_I{}^a \ , \tag{6.4.13}$$

and the black hole potential is

$$V_{BH} = \frac{1}{2} e^{2\phi} V_5^m + \frac{1}{2} e^{-6\phi} (q_0)^2 \ . \tag{6.4.14}$$

This gives the attractor value of the ϕ field as

$$e^{8\phi}|_{crit.} = I_3^{-2/3} (q_0)^2 \ . \tag{6.4.15}$$

At the attractor point $(a^{1/2}_{crit.})_{IJ} = (\hat{\lambda}^I)^{-1} \delta_{IJ}$, and the magnetic central charges are

$$Z_a^{crit} = \frac{i}{\sqrt{2}} (I_3)^{1/4} |q_0|^{1/4} = \frac{i}{2} |I_4|^{1/4} \equiv iZ \ , \qquad a = 1, 2, 3 \ . \tag{6.4.16}$$

We can then write the central charge matrix corresponding to the **27** representation in the normal frame as

$$e_{AB} = \begin{pmatrix} Z\epsilon & 0 & 0 & 0 \\ 0 & Z\epsilon & 0 & 0 \\ 0 & 0 & Z\epsilon & 0 \\ 0 & 0 & 0 & -3Z\epsilon \end{pmatrix} . \tag{6.4.17}$$

To describe the four dimensional solution we need the electric central charge, that at the attractor point is

$$Z_0^{crit} = \frac{1}{\sqrt{2}}(I_3)^{1/4}|q_0|^{1/4}\, sign(q_0) = \frac{1}{2}|I_4|^{1/4}\, sign(q_0) \equiv Z_0 \ .$$

Then, using the definition(6.4.7) the complete 4d central charge matrix is

$$2Z_{AB} = i \begin{pmatrix} Z\epsilon & 0 & 0 & 0 \\ 0 & Z\epsilon & 0 & 0 \\ 0 & 0 & Z\epsilon & 0 \\ 0 & 0 & 0 & -3Z\epsilon \end{pmatrix} - i \begin{pmatrix} Z_0\epsilon & 0 & 0 & 0 \\ 0 & Z_0\epsilon & 0 & 0 \\ 0 & 0 & Z_0\epsilon & 0 \\ 0 & 0 & 0 & Z_0\epsilon \end{pmatrix} =$$

$$= e^{i\pi/2} \begin{pmatrix} (Z-Z_0)\epsilon & 0 & 0 & 0 \\ 0 & (Z-Z_0)\epsilon & 0 & 0 \\ 0 & 0 & (Z-Z_0)\epsilon & 0 \\ 0 & 0 & 0 & (-3Z-Z_0)\epsilon \end{pmatrix} . \tag{6.4.18}$$

The $sign(q_0)$ determines whether the solution is supersymmetric or not. We may have

$$q_0 > 0 \quad \rightarrow \quad Z = Z_0 \ ,$$

$$Z_{AB} = e^{i\pi/2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2Z_0 \end{pmatrix} \otimes \epsilon \tag{6.4.19}$$

which is a magnetic $\frac{1}{8}$-BPS solutions with $SU(6)\times SU(2)$ symmetry, or

$$q_0 < 0 \quad \to \quad Z = -Z_0 \ ,$$

$$Z_{AB} = e^{i\pi/2}\begin{pmatrix} -Z_0 & 0 & 0 & 0 \\ 0 & -Z_0 & 0 & 0 \\ 0 & 0 & -Z_0 & 0 \\ 0 & 0 & 0 & Z_0 \end{pmatrix}\otimes \epsilon \tag{6.4.20}$$

which is the non-BPS solution with $USp(8)$ symmetry. These solutions have the same Z_0 as the electric ones, but now the choice of positive q_0 charge leads to the supersymmetric solution while the negative q_0 charge gives the non-supersymmetric one, in contrast with what happened for the choice of p^0 in the electric case in eq. (6.4.10) and (6.4.11).

6.4.3 KK dyonic solution $Q = (p^0\,, q_0)$

This charge configuration also has vanishing axions, and the only non-zero charges give

$$\begin{gathered} Z_0^e = e^{-3\phi} q_0\ , \quad Z_m^0 = e^{3\phi} p^0\ , \\ \Downarrow \\ Z_0 = \tfrac{1}{\sqrt{2}}(e^{-3\phi}q_0 + i e^{3\phi}p^0)\ . \end{gathered} \tag{6.4.21}$$

Since none of the 5 dimensional charges are turned on, the four dimensional black hole potential is

$$V_{BH} = \frac{1}{2}\left[e^{-6\phi}q_0^2 + e^{6\phi}(p^0)^2\right]\ , \tag{6.4.22}$$

which is extremized at the horizon by the value of the ϕ field

$$e^{6\phi}|_{crit.} = \left|\frac{q_0}{p^0}\right|\ . \tag{6.4.23}$$

We only focus on the case $p^0 > 0$ and $q_0 > 0$, since all the other choices are related to this by a duality rotation. Evaluating the central charge at the attractor point we find

$$Z_0^{crit} = \sqrt{|p^0 q_0|}\frac{1+i}{\sqrt{2}} = \sqrt{|p^0 q_0|}e^{i\pi/4}\ . \tag{6.4.24}$$

Following the prescription in (6.4.7) we find that at the attractor point

$$\begin{aligned} 2Z_{AB} &= -iZ_0\Omega = \\ &= -ie^{i\pi/4}\begin{pmatrix} \sqrt{|p^0q_0|}\epsilon & 0 & 0 & 0 \\ 0 & \sqrt{|p^0q_0|}\epsilon & 0 & 0 \\ 0 & 0 & \sqrt{|p^0q_0|}\epsilon & 0 \\ 0 & 0 & 0 & \sqrt{|p^0q_0|}\epsilon \end{pmatrix} \end{aligned} \tag{6.4.25}$$

that gives a non-BPS 4 dimensional black hole with $I_4 = -(p^0 q_0)^2$.

Note that eqs. (6.4.11), (6.4.20) and (6.4.25) imply that the sum of the phases of the four complex skew entries is π, as appropriate to a non-BPS $\mathcal{N} = 8$ solution [21]. Also, in all cases, $V_{BH}|_{crit.} = \sqrt{|I_4|}$.

6.4.4 $\mathcal{N} = 8$ and $\mathcal{N} = 2$ attractive orbits at $d = 5$ and $d = 4$

We now compare the different interpretations in the $\mathcal{N} = 8$ and $\mathcal{N} = 2$ theories of the critical points of the very same black hole $4d$ potential, in terms of the axion-free electric solution (sec. 6.4.1) as discussed in this paper and in ref. [69].

Since the "normal frame" solution is common to all symmetric spaces (with rank three), it can be regarded as the generating solution of any model. So we confine our attention to the exceptional $\mathcal{N} = 2$ (octonionic) $E_{7(-25)}$ model [78] which has a charge vector in $5d$ and $4d$ of the same dimension as in $\mathcal{N} = 8$ supergravity. At $d = 5$ the duality group is $E_{6(-26)}$, with moduli space of vector multiplets $E_{6(-26)}/F_4$.

It is known [49], [51] that in $d = 5$ there are two different charge orbits,

$$\mathcal{O}^{\mathcal{N}=2}_{d=5,\, BPS} = \frac{E_{6(-26)}}{F_4} , \tag{6.4.26}$$

the BPS one, and the non BPS one

$$\mathcal{O}^{\mathcal{N}=2}_{d=5,\, non-BPS} = \frac{E_{6(-26)}}{F_{4(-20)}} , \tag{6.4.27}$$

The latter one precisely corresponds to the non supersymmetric solution and to $(++\,-)$, $(--\,+)$ signs of the q_1, q_2, q_3, charges (implying $\partial Z \neq 0$). For charges of the same sign $(+++)$, $(---)$ one has the $\frac{1}{8}$BPS solution ($\partial Z = 0$), as discussed in [69].

It is easy to see that in the $\mathcal{N} = 8$ theory all these solutions just interchange Z_1, Z_2, Z_3 and $Z_4 = -3Z_3$ but always give a normal frame matrix of the form

$$Z_{ab} = \begin{pmatrix} Z\epsilon & 0 & 0 & 0 \\ 0 & Z\epsilon & 0 & 0 \\ 0 & 0 & Z\epsilon & 0 \\ 0 & 0 & 0 & -3Z\epsilon \end{pmatrix} , \tag{6.4.28}$$

which has $USp(6) \times USp(2) \in F_{4(4)}$ as maximal symmetry. Another related observation is that while $E_{6(-26)}$ contains both F_4 and $F_{4(-20)}$, so that one expects two orbits and two classes of solution, in the $\mathcal{N} = 8$ case $E_{6(6)}$ contains only the non compact $F_{4(4)}$, thus only one class of solutions is possible.

These orbits and critical points at $d=5$ have a further story when used to study the $d=4$ critical points with axion free solutions as it is the case for the electric (p^0, q_1, q_2, q_3) configuration. Since in this case $I_4 = -4p^0 q_1 q_2 q_3$, in the $\mathcal{N}=8$ case, once one choose q_1, q_2, $q_3 > 0$, the $I_4 > 0$, $p^0 < 0$ solution is BPS, while the $I_4 < 0$, $p^0 > 0$ is non BPS.

Things again change in $\mathcal{N}=2$ [52], when now we consider the solution embedded in the Octonionic model with $4d$ moduli space $E_{7(-25)}/E_6 \times U(1)$. A new non BPS orbit in $d=4$ is generated, corresponding to $Z=0$ ($\partial Z \neq 0$) solution, so three $4d$ orbits exist in this case depending whether the $(+++)$ and $(++-)$ solutions are combined with $-p^0 \lessgtr 0$. So

$$(+,+++) \quad \text{is BPS with } I_4 > 0 \ , \quad \mathcal{O} = \frac{E_{7(-25)}}{E_6} \ , \tag{6.4.29}$$

$$(-,-++) \quad \text{is non BPS with } I_4 > 0 \ , \quad \mathcal{O} = \frac{E_{7(-25)}}{E_{6(-14)}} \ , \tag{6.4.30}$$

$$(+,-++) \ \text{ or } (-,+++) \quad \text{is non BPS with } I_4 < 0 \ , \quad \mathcal{O} = \frac{E_{7(-25)}}{E_{6(-26)}} \ . \tag{6.4.31}$$

6.5 Maurer−Cartan equations of the four dimensional theory

Let us call Maurer−Cartan equations [18] those which give the derivative of the central charges (coset representatives) with respect to the moduli ϕ, a^I, λ^i. Using (6.2.8) we have

$$\begin{aligned} \partial_\phi Z_0^e = -3 Z_0^e \ , \qquad \partial_\phi Z_m^0 = 3 Z_m^0 \ , \\ \partial_\phi Z_I^e = -Z_I^e \ , \qquad \partial_\phi Z_m^I = Z_m^I \ , \end{aligned} \tag{6.5.1}$$

and

$$\begin{aligned} \frac{\partial Z_0^e}{\partial a^I} = e^{-2\phi} Z_I^e \ , \qquad \frac{\partial Z_m^0}{\partial a^I} = 0 \ , \\ \frac{\partial Z_m^I}{\partial a^J} = -\delta_J^I e^{-2\phi} Z_e^0 \ , \qquad \frac{\partial Z_I^e}{\partial a^J} = -e^{-2\phi} d_{IJK} Z_m^K \ . \end{aligned} \tag{6.5.2}$$

In our notation the 5d metric a_{IJ}, $(I, J = 1, .., 27)$ can also be rewritten with a pair of antisymmetric (traceless) indices

$$a_{\Lambda\Sigma\,,\Delta\Gamma} = L^{ab}{}_{\Lambda\Sigma} L_{\Delta\Gamma\, ab} \ , \tag{6.5.3}$$

where $L^{ab}{}_{\Lambda\Sigma}$ is the coset representative; in a fixed gauge (where a, b and Λ, Σ indices are identified)

$$L_I{}^a = (a^{1/2})_I{}^a \ , \qquad (\bar{L}_{Ia} = L^T_{Ia}) \tag{6.5.4}$$

The object $\mathbb{P}_i \equiv a^{1/2}\partial_i a^{-1/2}$ can be regarded as the Maurer−Cartan connection (see reference [74]). In fact, by reminding that $Z^e_a = Z^e_I (a^{-1/2})^I{}_a$, we have $\partial_i Z^e_a = (\partial_i a^{-1/2})^I{}_a Z^e_I$ (since $\partial_i Z^e_I = 0$). Since we can also write

$$\partial_i Z^e_a = (\partial_i a^{-1/2})^I{}_a (a^{1/2})_I{}^b Z^e_b \tag{6.5.5}$$

we find that $\mathbb{P}_{i,a}{}^b$ is such that

$$\partial_i Z^e_a = \mathbb{P}_{i,a}{}^b Z^e_b \; . \tag{6.5.6}$$

Notice that using $\mathbb{P}_{i,a}{}^b = Q_{i,a}{}^b + V_{i,a}{}^b$, we identify a connection which satisfies

$$\nabla_i Z^e_a = V_a{}^b Z^e_b \; , \tag{6.5.7}$$

with $\nabla_i = \partial_i - Q_i$.

6.5.1 Attractor equations from Maurer−Cartan equations

We can now use this formalism to write the attractor equations for the potential

$$V_{BH} = \frac{1}{2}(Z^e_0)^2 + \frac{1}{2}(Z^0_m)^2 + \frac{1}{2} Z^e_I a^{IJ} Z^e_J + \frac{1}{2} Z^I_m a_{IJ} Z^J_m \; . \tag{6.5.8}$$

By differentiating with respect to ϕ, a^I, λ^i, we get

$$\partial_\phi V_{BH} = -3(Z^e_0)^2 + 3(Z^0_m)^2 - Z^e_I a^{IJ} Z^e_J + Z^I_m a_{IJ} Z^J_m = 0 \; , \tag{6.5.9}$$

$$\partial_{a^I} V_{BH} = e^{-2\phi}\left[Z^e_0 Z^e_I - Z^e_J a^{JK} d_{IKL} Z^L_m - Z^0_m a_{IJ} Z^J_m\right] = 0 \; , \tag{6.5.10}$$

$$\partial_{\lambda^i} V_{BH} \equiv \partial_i V_{BH} = \frac{1}{2} Z^e_I \, \partial_i a^{IJ} \, Z^e_J + \frac{1}{2} Z^I_m \, \partial_i a_{IJ} \, Z^J_m = 0 \; . \tag{6.5.11}$$

From (6.5.10) we see that a solution with $a^I = 0$ implies

$$\partial_{a^I} V_{BH}\big|_{a^I=0} = 0 = e^{-2\phi}\left[e^{-4\phi} q_0 q_I - q_J a^{JK} d_{IKL} p^L - e^{4\phi} p^0 a_{IJ} p^J\right] = 0 \; , \tag{6.5.12}$$

which is trivially satisfied if we set $\neq 0$ (q_0, p^0) or (q_0, p^I) or (p^0, q_I).

From (6.5.9) we see that for an axion-free solution, if $Z^e_0, Z^I_m = 0$, we get

$$3(Z^0_m)^2 = Z^e_I a^{IJ} Z^e_J \; , \tag{6.5.13}$$

and if a_{IJ} is diagonal, $I = J = 1, 2, 3$, we obtain

$$3(Z^0_m)^2 = (Z^e_1)^2 a^{11} + (Z^e_2)^2 a^{22} + (Z^e_3)^2 a^{33} \; , \tag{6.5.14}$$

which is compatible with $Z^e_1 = Z^e_2 = Z^e_3 = \pm Z^0_m$.

The derivative with respect to the 5d moduli λ^i, $i=1,..,42$ for $\mathcal{N}=8$ theory, only receives contributions from the matrix a_{IJ}. Indeed since Z^e_I, Z^I_m do not depend on the λ^i(see eq.6.2.8), one finds

$$\partial_i V_4 = 0 = Z^e_I\,\partial_i a^{IJ}\,Z^e_J + Z^I_m\,\partial_i a_{IJ}\,Z^J_m\ . \tag{6.5.15}$$

By rewriting the charges multiplied by $(a^{-1/2})^I{}_a$ and $(a^{1/2})_I{}^a$ so that

$$Z^e_a \equiv Z^e_I\,(a^{-1/2})^I{}_a\ , \qquad Z^a_m = Z^I_m(a^{1/2})^a{}_I\ , \tag{6.5.16}$$

we have

$$\begin{aligned}\partial_i Z^e_a &= \mathbb{P}_{i,a}{}^b Z^e_b\ , \qquad \mathbb{P}_{i,a}{}^b = \partial_i(a^{-1/2})^I{}_a(a^{1/2})_I{}^b\ ,\\ \partial_i Z^a_m &= \mathbb{P}_i{}^a{}_b Z^b_m\ , \qquad \mathbb{P}_i{}^a{}_b = \partial_i(a^{1/2})_I{}^a(a^{-1/2})^I{}_b\ ,\end{aligned} \tag{6.5.17}$$

where $\mathbb{P}^a_{i\ b} = -\mathbb{P}_{i\,b}{}^a$ since $\partial_i(Z^e_a Z^a_m)=0$. Then we also have

$$\begin{aligned}\partial_i(Z^e_a Z^e_a) &= Z^e_a(\mathbb{P}_{ia}{}^b)Z^e_b =\\ &= Z^e_a\mathbb{P}_{i,ab}Z^e_b =\\ &= Z^e_a\mathbb{P}_{i\,(ab)}Z^e_b = 0\ ,\end{aligned} \tag{6.5.18}$$

and if we split $\mathbb{P}_{i,ab} = Q_{i\,[ab]} + V_{i\,(ab)}$, with

$$\begin{aligned}\mathbb{P}_i{}^a{}_b &= Q_i{}^a{}_b + V_i{}^a{}_b\ ,\\ \mathbb{P}_{i,a}{}^b &= Q_{i,a}{}^b - V_{i,a}{}^b\ ,\end{aligned} \tag{6.5.19}$$

the critical condition implies

$$\partial_i(Z^e Z^e) = Z^e_a V_{i\,(ab)} Z^e_b = 0\ , \tag{6.5.20}$$

and the analogue equation for magnetic charges

$$\partial_i(Z^m Z^m) = Z^a_m V_{i\,(ab)} Z^b_m = 0\ , \tag{6.5.21}$$

so that only the vielbein $V_{i\,,ab}$ enters in the equations of motion.

The criticality condition on the potential of eq. (6.5.15) now gives

$$\partial_i V_{BH} = 0 \quad \to \quad Z^e_a V_i{}^{ab} Z^e_b + Z^a_m V_{i,\,ab} Z^b_m = 0\ , \tag{6.5.22}$$

thus, for electric configurations ($Z^b_m = 0$) with $a^I = 0$,

$$Z^e_a V_i{}^{ab} Z^e_b = 0\ . \tag{6.5.23}$$

Comparing results of [53] with our formulæ we see that $V_1,\ V_2,\ V_3$, with $V_1+V_2+V_3=0$, in the case where the metric a_{IJ} is diagonal, correspond to

$$(a^{-1/2})^I{}_a\partial_i(a^{1/2})_J{}^a=(a^{-1/2})^I\partial_i(a^{1/2})_I=\mathbb{P}_i{}^I{}_I=V_i{}^I{}_I\equiv V_i{}^I\ , \tag{6.5.24}$$

where $(a^{-1/2})^I{}_I\equiv(a^{-1/2})^I$, $(a^{1/2})_I{}^I\equiv(a^{1/2})_I$, $I=1,2,3$, and using (6.3.4) we find

$$\begin{aligned} V_1^I &= \left(\frac{1}{\hat\lambda_1}\,,0\,,-\frac{1}{\hat\lambda_1}\right)\ ,\\ V_2^I &= \left(0\,,\frac{1}{\hat\lambda_2}\,,-\frac{1}{\hat\lambda_2}\right)\ . \end{aligned} \tag{6.5.25}$$

Indeed,

$$\sum_{i=1,2,3}V_i^I=0\ , \tag{6.5.26}$$

so, by using eq. (2.31)-(2.33) of ref. [53] one gets the desired result. In fact, using the definitions of $\mathbb{P}_1^I$ and $\mathbb{P}_2^I$ we get from the $\hat\lambda^i$ equations of motion

$$\sum_I Z_I^eV_i^IZ_I^e=0\ , \tag{6.5.27}$$

which explicitly gives

$$\begin{aligned} Z_1^eZ_1^e-Z_3^eZ_3^e &= 0\ ,\\ Z_2^eZ_2^e-Z_3^eZ_3^e &= 0\ , \end{aligned} \tag{6.5.28}$$

whose solution, combined with eq. (6.5.14), gives

$$\begin{gathered} (Z_1^e)^2=(Z_2^e)^2=(Z_3^e)^2\ =(Z_m^0)^2\ ,\\ \Downarrow\\ Z_1^e=Z_2^e=Z_3^e=\pm Z_m^0\ , \end{gathered} \tag{6.5.29}$$

all the other sign choices being equivalent in the 5d theory.

Chapter 7

Black holes in gauged Supergravity

7.1 Introduction

We now discuss the investigation of black hole solutions of 4-dimensional $N = 2$ gauged supergravity theories, where the matter content is given by vector multiplets and the U(1) gauging is obtained by Fayet–Iliopoulos terms. The main motivation for considering these toy models is the analysis of the attractor mechanism and of the entropy formula in the case of extremal solutions in theories where there may be a non-trivial cosmological constant and the moduli cannot be freely changed in the solution. Generically, an Anti–de Sitter (AdS) vacuum stabilizes all the scalar fields and therefore a black hole in AdS may only appear for values of the dilaton such that one cannot extrapolate between strong and weak coupling.

Supersymmetric static black hole solutions in theories with a negative cosmological constant have already been considered in [92, 93, 94], where it was shown that they usually lead to naked singularities, unless higher order derivative corrections are added to the Lagrangian. For this reason, most subsequent approaches to this problem considered extremal non-BPS configurations [35, 95, 96, 97]. One strong limitation of the work in [92, 93, 94], however, was the requirement that the scalar fields remained constant along the solution. If there is some sort of attractor mechanism at work, the AdS_4 vacuum may in fact require a definite value for the scalars that differs from the one required by the construction of a supersymmetric $AdS_2 \times S^2$ horizon geometry. Hence the appearance of singular geometries. However, if the scalars are allowed to flow, supersymmetry can be restored and regular geometries can be obtained. An important step forward in this direction was obtained by the authors of [98], who considered a setup like the one of this paper and where supersymmetric black hole configurations were explicitly constructed, though mostly with a hyperbolic horizon.

Although we use [98] as an important basis, we will extend their results in two main directions. Since the electric gauging procedure breaks the electric–magnetic duality that a generic 4-dimensional Einstein–Maxwell theory possess, the approach presented in [98] has the limitation that for the same supergravity model only part of the black hole solutions are accessible, whenever the prepotential defining the scalar σ-model is fixed. We will present a completely covariant approach by considering a general U(1) gauged supergravity, where also magnetic gaugings are allowed. We are also going to describe the black hole solutions by means of first order flow equations driven by a superpotential W, which is a function of the scalar fields and the warp factors. This clearly mimics the flow equations of black holes in ungauged supergravity, where the superpotential is the absolute value of the central charge for supersymmetric configurations [10] or a duality invariant function for non-supersymmetric extremal configurations [28] and gives both the ADM mass at infinity and the horizon area. However, the different metric ansatz and the presence of a non-trivial cosmological constant usually forbid a direct relation between W and S and/or the mass of the black hole. As we will show, the general construction of this superpotential proves a very effective procedure in order to obtain explicit solutions.

Before presenting our results, we would like to introduce one last important motivation to the analysis of black hole solutions to gauged supergravity theories: flux compactifications. It is well known that flux compactifications provide an efficient tool to address the moduli problem in string compactifications. Fluxes provide a non-trivial source for a potential in the effective theory, as well as deformations leading to gauged supergravity models (see for instance [99, 100, 101]). It is therefore of vital importance for this scenario to understand if there is still an attractor mechanism at work in the case of black hole configurations in gauged supergravities, because their generation may destabilize the vacuum [102, 103]. In fact, the presence of a charged black hole may drive the value of the moduli fields to a new value at the horizon, different from the one obtained by the potential generated by flux compactification and eventually catalyze the production of new vacuum bubbles within the original setup [104].

We should point out that we expect realistic scenarios of flux compactification to require the presence of hypermultiplets. This means that our analysis should be extended to the case where also this type of scalars is allowed to acquire a non-trivial profile. In fact, in contrast with the case of ungauged theories, where hyperscalars are moduli of black hole solutions, in gauged supergravity black holes, the hypermultiplet scalars may be charged and hence actively participate to the solution. A very interesting development in this direction is given by the work of [105], where the authors constructed new solutions in gauged supergravities with non-trivial hypermultiplets, embedding known solutions to the ungauged theories. A general treatment in terms of a superpotential would be desirable for these cases, too, generalizing the construction we will explain below.

We should also mention that supersymmetric black holes in gauged supergravities were also analyzed in [106], [107], though there the authors focussed on non-abelian configurations.

7.2 BPS flow equations for dyonic configurations

7.2.1 Notations and setup

We are interested in dyonic black hole solutions of $\mathcal{N} = 2$ U(1) gauged supergravity. For this reason we are going to consider supergravity models coupled to n_V vector multiplets, a linear combination of which is going to gauge a U(1) factor via suitable Fayet–Iliopoulos (FI) terms. The bosonic Lagrangian of this class of models is

$$\mathscr{L} = \frac{R}{2} - g_{i\bar{\jmath}}\,\partial_\mu z^i \partial^\mu \bar{z}^{\bar{\jmath}} + \frac{1}{4}\,\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}\,F^\Lambda_{\mu\nu}\,F^{\Sigma\,\mu\nu} + \frac{1}{4}\,\mathrm{Re}\mathcal{N}_{\Lambda\Sigma}\,F^\Lambda_{\mu\nu}\,\frac{\epsilon^{\mu\nu\rho\sigma}}{2\sqrt{-g}}F^\Sigma_{\rho\sigma} - V_g. \quad (7.2.1)$$

The index $\Lambda = 0, 1, \ldots, n_V$ runs over the n_V vectors of the vector multiplets and the graviphoton, z^i denote the complex scalar fields sitting in the vector multiplets and V_g is the scalar potential of the theory generated by the FI terms. The scalar fields parameterize a special-Kähler σ-model and all the relevant quantities in the Lagrangian and in the supersymmetry transformations can be written in terms of special geometry. The σ-model metric $g_{i\bar{\jmath}}(z, \bar{z})$ can be derived from the second mixed derivatives of the Kähler potential, which in turn is a function of the covariantly holomorphic symplectic sections $\mathcal{V} \equiv e^{K/2}\left(X^\Lambda(z), F_\Lambda(z)\right)$, as follows from

$$1 = i\,\langle\mathcal{V}, \overline{\mathcal{V}}\rangle, \quad (7.2.2)$$

where the brackets denote the symplectic scalar product $\langle A, B\rangle = A^T \Omega B = A_\Lambda B^\Lambda - A^\Lambda B_\Lambda$, where Ω is the $\mathrm{Sp}(2n_v + 2)$ metric. The vector kinetic matrix $\mathcal{N}_{\Lambda\Sigma}(z)$ is then a complex and symmetric function of the scalar fields and the scalar potential

$$V_g = g^{i\bar{j}}\, D_i\mathcal{L}\overline{D}_{\bar{j}}\overline{\mathcal{L}} - 3|\mathcal{L}|^2 \qquad (\text{where} \quad D_i\mathcal{L} \equiv \partial_i\mathcal{L} + 1/2\,\partial_i K\,\mathcal{L}) \tag{7.2.3}$$

can be obtained in terms of the superpotential

$$\mathcal{L} = \langle \mathcal{G}, \mathcal{V}\rangle = e^{K/2}\left(X^\Lambda g_\Lambda - F_\Lambda g^\Lambda\right), \tag{7.2.4}$$

where $\mathcal{G} = (g^\Lambda, g_\Lambda)$ denote the FI terms. One should not be confused by the fact that we have introduced both electric and magnetic gaugings because in consistent models the electric-magnetic duality group will always allow one to reduce to the case where only electric gaugings are turned on (i.e. $g^\Lambda = 0$). However, this also implies a rotation of the symplectic sections and the choice of a somewhat preferred basis. We therefore prefer to maintain duality covariance and allow for generic FI terms $\mathcal{G}$.

A duality covariant action for generic gauging has been recently built in [108] for $\mathcal{N} = 2$ conformal Supergravity, using the embedding tensor formalism, and these results can be extended beyond the conformal approach. As shown in [109],[110], whenever one introduces magnetic gaugings, tensor fields have to be introduced. In the case of supergravity coupled to vector multiplets, one has therefore to improve couplings to vector-tensor multiplets. In [111] the authors worked out the supersymmetry transformations and scalar potential for supergravity coupled to vector-tensor multiplets and for a generic gauging, although in the case of vanishing FI terms. However, the extension to non-trivial FI terms is straightforward [108] and, taking a pragmatic approach, we will use the action (7.2.1) as our starting point, as this is going to be the relevant sector for our solutions because we will always consider vanishing tensor fields anyway.

We seek static dyonic black hole configurations. Hence we will consider the metric ansatz

$$ds^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}\left(dr^2 + e^{2\psi(r)}d\Omega^2\right), \tag{7.2.5}$$

where $d\Omega^2$ is going to be the line element of a 2-sphere for most of the applications considered in this paper and appropriate profiles for the vector fields so that

$$\int_{S^2} F^\Lambda = 4\pi p^\Lambda, \qquad \int_{S^2} G_\Lambda = 4\pi q_\Lambda, \qquad \left(\text{with} \quad G_\Lambda = \frac{\delta\mathscr{L}}{\delta F^\Lambda}\right) \tag{7.2.6}$$

where $Q \equiv (p^\Lambda, q_\Lambda)$ are the black hole magnetic and electric charges, respectively. We also assume that the scalar fields have only a radial dependence $z^i = z^i(r)$. Although we look for static configurations and preserve an SO(3) isometry group along the solutions, the metric ansatz (7.2.5) differs from the one of asymptotically flat static configurations because of the additional factor depending on $\psi(r)$. We inserted this additional factor, because, as we will see, it will be necessary to compensate for the additional curvature contributions to the Einstein equations coming from the (varying) non-trivial cosmological constant.

Once we plug these ansatze in the action (7.2.1) we obtain an effective 1-dimensional theory for the scalar fields and the warp factors $U(r)$ and $\psi(r)$

$$S_{1d} = \int dr \left\{ e^{2\psi} \left[(U' - \psi')^2 + 2\psi'^2 + g_{i\bar{\jmath}} z^{i\prime} \bar{z}^{\bar{\jmath}\prime} + e^{2U-4\psi} V_{BH} + e^{-2U} V_g + 2\psi'' - U'' \right] - 1 \right\}, \tag{7.2.7}$$

which, after an integration by parts, can be written as

$$\begin{aligned} S_{1d} = & \int dr \left\{ e^{2\psi} \left[U'^2 - \psi'^2 + g_{i\bar{\jmath}} z^{i\prime} \bar{z}^{\bar{\jmath}\prime} + e^{2U-4\psi} V_{BH} + e^{-2U} V_g \right] - 1 \right\} \\ & + \int dr \frac{d}{dr} \left[e^{2\psi} (2\psi' - U') \right]. \end{aligned} \tag{7.2.8}$$

Primes denote derivatives with respect to the radial coordinate and the black hole potential

$$V_{BH} = |D\mathcal{Z}|^2 + |\mathcal{Z}|^2 \tag{7.2.9}$$

is a function of the central charge

$$\mathcal{Z} \equiv \langle Q, \mathcal{V} \rangle. \tag{7.2.10}$$

It is also useful to rewrite the black hole potential as

$$V_{BH} = -\frac{1}{2} Q^T \mathcal{M} Q, \tag{7.2.11}$$

where $\mathcal{M}$ is the symplectic matrix defined as (3.1.29).

7.2.2 BPS rewriting of the action

Since we are interested in analyzing supersymmetric configurations, we have to impose the vanishing of the supersymmetry transformation rules on our background, in addition to solving the equations of motion. This analysis was performed in this way for generic half-supersymmetric configurations in [112] and applied to a black hole similar to ours in [98], though only for electric gaugings. The resulting first order differential equations provide solutions to both the supersymmetry conditions and the equations of motion. We will now extend this work for configurations obtained in the duality-symmetric setup given by (7.2.7).

As a first step in this process, we will show that one can rewrite the action (7.2.7) as a sum of squares of first-order differential equations as long as a specific constraint between the black hole charges and the FI parameters is satisfied. This rewriting then guarantees the solution of the equations of motion of the effective action. An important outcome of this rewriting is the existence of an additional constraint on the field configurations that may lead to consistent BPS solutions, which will be identified with the defining equation for a phase factor $\alpha(r)$. Then we will show how the first-order equations derived here follow from a real superpotential, which is the norm of a complex quantity whose phase is α, and we finally give a direct analysis of the supersymmetry transformations, which give the same result. Following a strategy similar to the one used in the ungauged BPS case in [31], we can rewrite the action (7.2.7) as a sum of BPS squares by using a series of special geometry identities. In particular, we can use the negative-definite matrix $\mathcal{M}$ as a "metric" for a set of symplectic covariant first-order equations. In order to do so, we will use several special geometry identities. A basic identity, which will be repeatedly used, is

$$\frac{1}{2}\left(\mathcal{M} - i\Omega\right) = \Omega\overline{\mathcal{V}}\,\mathcal{V}\Omega + \Omega U_i\, g^{i\bar{\jmath}}\,\overline{U}_{\bar{\jmath}}\Omega\,, \tag{7.2.12}$$

which leads to

$$\mathcal{M}\mathcal{V} = i\,\Omega\mathcal{V}, \qquad \mathcal{M}U_i = -i\Omega U_i, \tag{7.2.13}$$

from which follows that

$$\overline{\mathcal{V}}^T\mathcal{M}\mathcal{V} = i\langle\overline{\mathcal{V}},\mathcal{V}\rangle = -1 \tag{7.2.14}$$

and

$$U_i^T\mathcal{M}\overline{U}_{\bar{\jmath}} = i\langle U_i,\overline{U}_{\bar{\jmath}}\rangle = -g_{i\bar{\jmath}}. \tag{7.2.15}$$

The first step is to rewrite the kinetic term for the scalar fields and the scalar potentials V_g and V_{BH} in terms of symplectic sections using

$$-\,\mathcal{V}'^T\mathcal{M}\overline{\mathcal{V}}' = g_{i\bar{\jmath}}z^{i\,\prime}\bar{z}^{\bar{\jmath}\,\prime} + \mathcal{A}_r^2\,, \tag{7.2.16}$$

where

$$\mathcal{A}_r \equiv \frac{i}{2}\left(\bar{z}^{\bar{\jmath}\prime}\,\overline{\partial}_{\bar{\jmath}}K - z^{i\prime}\,\partial_i K\right) \tag{7.2.17}$$

is a composite connection. Given the properties of the symplectic sections, we can also introduce a phase factor, which we will see related to the spinor projector one imposes in order to solve the supersymmetry equations (see the Appendix), so that

$$-\,\mathrm{Im}(e^{i\alpha}\mathcal{V}'^T)\mathcal{M}\mathrm{Im}(e^{i\alpha}\mathcal{V}') = \frac{1}{2}g_{i\bar{\jmath}}z^{i\,\prime}\bar{z}^{\bar{\jmath}\,\prime} + \frac{1}{2}\mathcal{A}_r^2\,, \tag{7.2.18}$$

and once more obtain new identities:

$$\mathrm{Re}(e^{i\alpha}\mathcal{V})^T\mathcal{M}\mathrm{Re}(e^{i\alpha}\mathcal{V}) = \mathrm{Im}(e^{i\alpha}\mathcal{V}^T)\mathcal{M}\mathrm{Im}(e^{i\alpha}\mathcal{V}) = -\frac{1}{2} \ , \tag{7.2.19}$$

$$\mathrm{Im}(e^{i\alpha}\mathcal{V}^T)\mathcal{M}\mathrm{Re}(e^{i\alpha}\mathcal{V}) = 0 \ , \tag{7.2.20}$$

$$\mathrm{Im}(e^{i\alpha}\mathcal{V}') = \mathrm{Im}(e^{i\alpha}z^{i\prime}U_i) - \mathcal{A}_r\,\mathrm{Re}(e^{i\alpha}\mathcal{V}) \ , \tag{7.2.21}$$

$$\mathrm{Im}(e^{i\alpha}\mathcal{V}^T)\mathcal{M}\,Q = \mathrm{Re}(e^{i\alpha}\mathcal{Z}) \ , \quad \mathrm{Re}(e^{i\alpha}\mathcal{V}^T)\mathcal{M}\,Q = -\mathrm{Im}(e^{i\alpha}\mathcal{Z}) \ , \tag{7.2.22}$$

$$\mathrm{Im}(e^{i\alpha}\mathcal{V}')\mathcal{M}\,Q = -\mathrm{Re}(e^{i\alpha}\mathcal{Z}') + 2\,\mathcal{A}_r\,\mathrm{Im}(e^{i\alpha}Z) \ . \tag{7.2.23}$$

After some long, but straightforward manipulations, the action (7.2.7) can then be rewritten as

$$\begin{aligned} S_{1d} = \int dr \Big\{ & -\frac{1}{2}e^{2(U-\psi)}\mathcal{E}^T\mathcal{M}\mathcal{E} - e^{2\psi}\left[(\alpha' + \mathcal{A}_r) + 2e^{-U}\,\mathrm{Re}(e^{-i\alpha}\mathcal{L})\right]^2 \\ & - e^{2\psi}\left[\psi' - 2e^{-U}\,\mathrm{Im}(e^{-i\alpha}\mathcal{L})\right]^2 - (1 + \langle\mathcal{G}, Q\rangle) \\ & - 2\frac{d}{dr}\left[e^{2\psi-U}\,\mathrm{Im}(e^{-i\alpha}\mathcal{L}) + e^{U}\,\mathrm{Re}(e^{-i\alpha}\mathcal{Z})\right]\Big\} \ , \end{aligned} \tag{7.2.24}$$

where we introduced

$$\mathcal{E}^T \equiv 2e^{2\psi}\left(e^{-U}\mathrm{Im}(e^{-i\alpha}\mathcal{V})\right)'^{\,T} - e^{2(\psi-U)}\mathcal{G}^T\Omega\mathcal{M}^{-1} + 4e^{-U}(\alpha' + \mathcal{A}_r)\mathrm{Re}(e^{-i\alpha}\mathcal{V})^T + Q^T . \tag{7.2.25}$$

A simple inspection of (7.2.24) shows that we succeeded in rewriting the action (7.2.7) as a sum of squares of first order differential conditions and a boundary term provided the charges fulfill the constraint

$$\langle\mathcal{G}, Q\rangle = -1. \tag{7.2.26}$$

Once this is satisfied we obtain that BPS configurations have to satisfy three sets of equations

$$\mathcal{E} = 0, \tag{7.2.27}$$

$$\psi' = 2\,e^{-U}\,\mathrm{Im}(e^{-i\alpha}\mathcal{L}), \tag{7.2.28}$$

$$\alpha' + \mathcal{A}_r = -2e^{-U}\ \mathrm{Re}(e^{-i\alpha}\mathcal{L}). \tag{7.2.29}$$

The first set of conditions contains both the flow equations for the scalar field as well as the equation for the warp factor U. Equation (7.2.28) describes the evolution of the other warp factor ψ. Finally, (7.2.29) gives the condition on the phase α.

Some comments are in order here. First of all, we can see that the first set of equations reduces to the known BPS equations of the ungauged case as presented in [31] whenever $\mathcal{G} = 0$ (and then $\mathcal{L} = 0$). In such a case, however, we would get an inconsistency from the constraint (7.2.26). This implies that the BPS configurations we find by solving such a system are solitonic [98]. Actually, the BPS rewriting in the $\mathcal{G} = 0$ case can be achieved by rewriting the second line of (7.2.24) as a new squared first order equation and a boundary term

$$-\left(e^{\psi}\psi - 1\right)^2 - \left(2e^{\psi}\right)', \tag{7.2.30}$$

which leads to the identification of $e^{\psi(r)} = r$ and hence to reducing the metric ansatz to the known one of the asymptotically flat configurations. Then we see that the equations we derived are all symplectic covariant or invariant. This means that once we obtain some solution in a given frame, for a specific choice of charges Q and FI terms $\mathcal{G}$, we can map it to a different solution for a different set of charges and FI terms related to the original ones by a duality transformation. We can also compare our BPS equations with those found in [98] by identifying $b = e^{-i\alpha - U}$ and setting the magnetic FI terms to zero $g^{\Lambda} = 0$. The two sets of conditions match and therefore we can also conclude that our BPS conditions imply also the full 4-dimensional equations of motion. Finally, we would like to point out that the BPS rewriting of the effective action and the derivation of the first order equations (7.2.27)–(7.2.29) can be trivially extended to the case of flat or hyperbolic horizons and yields the same results, but for the charge constraint (7.2.26), which becomes $\langle\mathcal{G}, Q\rangle = 0$ or $\langle\mathcal{G}, Q\rangle = 1$ in the flat and hyperbolic case, respectively.

7.2.3 Superpotentials and flow equations

Although the BPS square rewriting of the effective 1-dimensional action already led to a set of first-order differential equations for the scalar field dependent symplectic sections $\mathcal{V}$ and the warp factors, we now provide an explicit expression for the resulting flow equations for the actual scalar fields z^i. This rewriting will lead to the identification of a proper superpotential function driving the BPS flow.

The equation (7.2.27) is actually a complex symplectic vector of equations whose information can be extracted by appropriate projections with all possible independent sections. We first discuss the projections of the BPS equations $\mathcal{E} = 0$ on the symplectic sections $\mathcal{V}$ and their derivatives U_i and then pass to the possible contractions with the charges Q and FI terms $\mathcal{G}$. From the contraction

$$\langle \mathcal{E}, \mathrm{Re}(e^{-i\alpha}\mathcal{V})\rangle = 0 \qquad (7.2.31)$$

we obtain the flow equation for the warp factor $U(r)$:

$$U' = -e^{U-2\psi}\,\mathrm{Re}(e^{-i\alpha}\mathcal{Z}) + e^{-U}\,\mathrm{Im}(e^{-i\alpha}\mathcal{L}). \qquad (7.2.32)$$

The contraction

$$\langle \mathcal{E}, \mathrm{Im}(e^{-i\alpha}\mathcal{V})\rangle = 0 \qquad (7.2.33)$$

produces once more an equation for the phase

$$\alpha' + \mathcal{A}_r = -e^{U-2\psi}\,\mathrm{Im}(e^{-i\alpha}\mathcal{Z}) - e^{-U}\,\mathrm{Re}(e^{-i\alpha}\mathcal{L}). \qquad (7.2.34)$$

Finally, the contraction along the covariant derivatives of the sections

$$\langle \mathcal{E}, U_i\rangle = 0 \qquad (7.2.35)$$

leads to the scalar fields flow equations

$$z^{i\prime} = -e^{i\alpha}g^{i\bar{\jmath}}\left(e^{U-2\psi}\overline{D}_{\bar{\jmath}}\overline{\mathcal{Z}} + i\,e^{-U}\overline{D}_{\bar{\jmath}}\overline{\mathcal{L}}\right). \qquad (7.2.36)$$

Contractions with Q and/or $\mathcal{G}$ give identities once (7.2.32), (7.2.34), (7.2.36) and (7.2.29) are used. The first thing we notice is that the flow equation for the phase (7.2.34) differs from the one derived directly from the action, namely (7.2.29). Consistency of the two equations then implies the following constraint:

$$e^{U-2\psi}\,\mathrm{Im}(e^{-i\alpha}\mathcal{Z}) = e^{-U}\,\mathrm{Re}(e^{-i\alpha}\mathcal{L}). \qquad (7.2.37)$$

The constraint arises as a consequence of the fact that in the BPS rewriting we introduced an additional degree of freedom $\alpha(r)$ that was not present in the reduced action. We can actually rewrite this constraint as an expression that identifies the phase as

$$e^{2i\alpha} = \frac{\mathcal{Z} - i\,e^{2(\psi-U)}\mathcal{L}}{\overline{\mathcal{Z}} + i\,e^{2(\psi-U)}\overline{\mathcal{L}}}. \qquad (7.2.38)$$

We can see that this phase gets identified with the phase of $\mathcal{Z}$ in the limit where the gauging goes to zero (or, better, $e^{2i\alpha} = e^{2i\phi_{\mathcal{Z}}}$; we will come back on this issue later on). Another interesting remark is that, by using (7.2.38), it is straightforward to check that the phase equation (7.2.34) is identically satisfied if the BPS equations associated to the scalar fields and to the warp factor are used.

The other important outcome of this analysis is that we can now realize the BPS condition as flow equations for the effective scalar degrees of freedom U, ψ, z^i. Once we define a superpotential

$$W \equiv e^U \,\mathrm{Re}(e^{-i\alpha}\mathcal{Z}) + e^{-U+2\psi}\,\mathrm{Im}(e^{-i\alpha}\mathcal{L}), \tag{7.2.39}$$

or, by using the phase constraint (7.2.38),

$$W = e^U \,|\mathcal{Z} - i\, e^{2(\psi-U)}\mathcal{L}|, \tag{7.2.40}$$

we can rewrite the flow equations as

$$U' = -g^{UU}\,\partial_U W, \tag{7.2.41}$$

$$\psi' = -g^{\psi\psi}\,\partial_\psi W, \tag{7.2.42}$$

$$z^{i\prime} = -2\,\widetilde{g}^{i\bar{\jmath}}\,\partial_{\bar{\jmath}} W, \tag{7.2.43}$$

where $g_{UU} = -g_{\psi\psi} = e^{2\psi}$, $\widetilde{g}_{i\bar{\jmath}} = e^{2\psi} g_{i\bar{\jmath}}$ and we used the constraint (7.2.37) in the derivation of the last equation. It is remarkable that W looks precisely like the norm of a complex quantity whose phase is given by α and that it reduces to the supersymmetric superpotential for $\mathcal{G} = 0$.

Although the structure of the flow equations looks rather neat in these variables, for the subsequent discussion it is useful to rewrite them by introducing a different parameterization for the warp factors. In detail, we can introduce

$$A = \psi - U, \tag{7.2.44}$$

so that the metric ansatz becomes

$$ds^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}dr^2 + e^{2A(r)}d\Omega^2. \tag{7.2.45}$$

By using these variables

$$W = e^U \,|\mathcal{Z} - i\, e^{2A}\mathcal{L}| \tag{7.2.46}$$

and the flow equations become

$$\begin{aligned} U' &= -e^{-2(A+U)}\,(W - \partial_A W)\,, \\ A' &= e^{-2(A+U)}\,W, \\ z^{i\prime} &= -2e^{-2(A+U)}\,g^{i\bar{\jmath}}\,\partial_{\bar{\jmath}} W. \end{aligned} \tag{7.2.47}$$

7.3 Attractors

One of the key properties of extremal black hole solutions is the so-called attractor mechanism. We will now show that such an attractor mechanism is at work also for supersymmetric black holes in U(1) gauged supergravity: we will show that one can write the equations defining the value of the scalar fields at the black hole horizon in terms of a set of algebraic conditions on the charges and the symplectic sections. We stress, that despite formal similarities, the situation is fundamentally different from the one of asymptotically flat solutions. In fact, AdS_4 solutions already fix the asymptotic value of the moduli, which are then driven to the horizon value by the attractor mechanism. This means that, although the existence of a black hole horizon specifies the values of the moduli fields in terms of the charges, this attractor cannot be reached from a generic point in moduli space because of the asymptotic constraint in terms of the gauging parameters.

7.3.1 Near horizon limit

When approaching the horizon of a supersymmetric extremal black hole we expect the metric (7.2.5) to approach that of an $AdS_2 \times S^2$ spacetime:

$$ds^2 = -\frac{r^2}{R_A^2}dt^2 + \frac{R_A^2}{r^2}dr^2 + R_S^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{7.3.1}$$

where R_S and R_A are the radii of the 2-dimensional sphere and of the 2-dimensional Anti-de Sitter spacetime, respectively. In the framework of the metric ansatz proposed in (7.2.5), this is obtained by imposing

$$U = \log\frac{r}{R_A}, \qquad \text{and} \qquad \psi = \log\frac{rR_S}{R_A}, \tag{7.3.2}$$

or, in terms of the alternative variables for the warp factors,

$$A = \log R_S. \tag{7.3.3}$$

This means that

$$A' = 0 \qquad \Leftrightarrow \qquad W = 0 \tag{7.3.4}$$

at the horizon. We also expect the scalar fields to be constant $z^{i\prime} = 0$ at the horizon and therefore we expect

$$\partial_i|\mathcal{Z} - i\,e^{2A}\mathcal{L}| = 0 \qquad \Leftrightarrow \qquad D_i\mathcal{Z} - i\,e^{-2A}D_i\mathcal{L} = 0. \tag{7.3.5}$$

The attractor equations can then be obtained by using special geometry identities to expand the moduli independent quantity $Q + i\, e^{2A}\, \mathcal{G}$ and then use the horizon conditions (7.3.5). When we multiply from the left the charge combination just mentioned by $\Omega\mathcal{M} + i$ we get

$$\Omega\mathcal{M}Q + iQ + i\, e^{2A}\, \Omega\mathcal{M}\mathcal{G} - e^{2A}\mathcal{G} = 2\left(\mathcal{Z} + i\, e^{2A}\, \mathcal{L}\right)\, \overline{\mathcal{V}} + 2\left(\overline{D}_{\bar{\imath}}\overline{\mathcal{Z}} + i\, e^{2A}\, \overline{D}_{\bar{\imath}}\mathcal{L}\right) U^{\bar{\imath}}. \quad (7.3.6)$$

This is a general expansion valid at any point of the moduli space. However, at the attractor point the last term vanishes and we therefore obtain that

$$Q + e^{2A}\, \Omega\mathcal{M}\mathcal{G} = -2\mathrm{Im}(\overline{\mathcal{Z}}\mathcal{V}) + 2\, e^{2A}\, \mathrm{Re}(\overline{\mathcal{L}}\mathcal{V}), \quad (7.3.7)$$

which is the attractor equation. Once again, for $\mathcal{G} = 0$, we can see that it reduces to the known attractor equation $Q = -2\mathrm{Im}(\overline{\mathcal{Z}}\mathcal{V})$. Since this equation only gives the value of the scalar fields at the attractor point, but we also need to fix the value of A in order to obtain the right geometry, one has to supplement the conditions just derived with the $W = 0$ condition, namely

$$|\mathcal{Z} - i\, e^{2A}\mathcal{L}| = 0. \quad (7.3.8)$$

Although this is a real condition, it is easy to see that the request that e^A be a real number gives as an outcome that

$$e^{2A} = -i\, \frac{\mathcal{Z}}{\mathcal{L}} = R_S^2. \quad (7.3.9)$$

This equation was also derived in [98], as a horizon condition. Summarizing, the BPS attractors in a U(1) gauged supergravity are

$$Q + e^{2A}\, \Omega\mathcal{M}\mathcal{G} = -2\mathrm{Im}(\overline{\mathcal{Z}}\mathcal{V}) + 2\, e^{2A}\, \mathrm{Re}(\overline{\mathcal{L}}\mathcal{V}), \quad (7.3.10)$$

$$e^{2A} = -i\, \frac{\mathcal{Z}}{\mathcal{L}} = R_S^2. \quad (7.3.11)$$

From the last condition we also learn that the phases of the central charge and of the superpotential of the gauging are related at the horizon, so that

$$\phi_{\mathcal{Z}} = \phi_{\mathcal{L}} + \frac{\pi}{2}. \quad (7.3.12)$$

If we plug this information in the definition of the phase factor α we obtain that $e^{2i\alpha} = e^{2i\phi_{\mathcal{Z}}}$

$$\alpha = \phi_{\mathcal{Z}} + k\,\pi, \quad k \in \mathbb{Z}, \quad (7.3.13)$$

at the horizon. This is an important consistency requirement, in order to obtain spherical horizons, because we can see from inserting the near horizon limits for the warp factors in the flow equations that at the fixed point

$$e^{-i\alpha}\mathcal{Z} = -\frac{R_S^2}{2R_A} < 0 \tag{7.3.14}$$

and this is possible only if the phase α at the horizon is identified with $\phi_{\mathcal{Z}} + \pi$. A different attractor equation was proposed in [98], which depends only on the moduli fields. This equation can be obtained from ours by plugging (7.3.11) into (7.3.10), but it looses the information on the horizon area, which instead is governed by (7.3.11).

Although the attractor equations (7.3.10)–(7.3.11) are $2n_V+4$ conditions for $2n_V+1$ variables (the $2n_V$ scalar fields and the warp factor A), we can see that not all of them are independent. In fact, if we contract (7.3.10) with $\mathcal{V}$ we obtain an identity and we can therefore argue that it is equivalent to (7.3.5), which one recovers by contracting (7.3.10) with U_i. In order to have a spherical horizon these conditions have to be supplemented by the constraint (7.2.26), which can at times overconstrain the system, as we will show in a while.

More information on the attractor point can also be obtained by further contracting the attractor equation (7.3.10) by the charges of the gauging or of the black hole and by using (7.3.11). In the first case we obtain that

$$e^{-2A} = 2\left(|D_i\mathcal{L}|^2 - |\mathcal{L}|^2\right), \tag{7.3.15}$$

while in the second case we get that

$$e^{2A} = 2\left(|D_i\mathcal{Z}|^2 - |\mathcal{Z}|^2\right). \tag{7.3.16}$$

These equations are very interesting because they can be related to the second symplectic invariant

$$I_2(Q) = |\mathcal{Z}|^2 - |D_i\mathcal{Z}|^2 = -\frac{1}{2}Q\mathcal{M}(F)Q, \tag{7.3.17}$$

where $\mathcal{M}(F)$ is a matrix constructed using $\mathrm{Re}\, F_{\Lambda\Sigma}$ and $\mathrm{Im}\, F_{\Lambda\Sigma}$ rather than $\mathrm{Re}\, \mathcal{N}_{\Lambda\Sigma}$ and $\mathrm{Im}\, \mathcal{N}_{\Lambda\Sigma}$. We can also see that if we start from an AdS_4 vacuum $D_i\mathcal{L} = 0$ and we try to obtain a black hole solution by keeping the scalars constant, we get to an immediate contradictory result, because (7.3.15) implies that $e^{-2A} = -2|\mathcal{L}|^2 < 0$. This excludes the possibility of spherical horizons in an asymptotically AdS geometry while keeping scalars fixed and therefore explains the results of [92, 93, 94]. More in general, the second attractor equation (7.3.11) can also be written as

$$e^{2A} = -\frac{\mathrm{Im}(\overline{\mathcal{Z}}\mathcal{L})}{|\mathcal{L}|^2}, \tag{7.3.18}$$

which, for $D_i\mathcal{L} = 0$, is equivalent to

$$e^{2A} = \frac{1}{2}\frac{\langle \mathcal{G}, Q\rangle}{|\mathcal{L}|^2}\,. \tag{7.3.19}$$

We then see that this is positive only for hyperbolic horizons, while for spherical horizons $\langle \mathcal{G}, Q\rangle = -1 < 0$.

7.4 Supersymmetry equations

In order to explicitly prove that the configurations discussed so far are supersymmetric, we now analyze in detail the supersymmetry variations of $\mathcal{N} = 2$ U(1) gauged supergravity. For simplicity we will discuss the case without magnetic gauging parameters, but the extension to the full case is straightforward. The relevant variations are then

$$\delta\psi_{\mu\, A} = D_\mu\epsilon_A - \varepsilon_{AB}\, T^-_{\mu\nu}\,\gamma^\nu\,\epsilon^B - \frac{i}{2}\,\mathcal{L}\,\delta_{AB}\,\gamma^\nu\,\eta_{\mu\nu}\,\epsilon^B\;, \tag{7.4.1}$$

$$\delta\lambda^{iA} = -i\,\partial_\mu z^i\,\gamma^\mu\,\epsilon^A - G^{-i}_{\mu\nu}\,\gamma^{\mu\nu}\,\varepsilon^{AB}\,\epsilon_B + \overline{D}^i\overline{\mathcal{L}}\,\delta^{AB}\,\epsilon_B\;, \tag{7.4.2}$$

where the covariant derivative is defined as

$$D_\mu\epsilon_A \equiv \partial_\mu\epsilon_A - \frac{1}{4}\,\omega^{ab}_\mu\gamma_{ab}\epsilon_A + \frac{i}{2}\,\mathcal{A}_\mu\epsilon_A + g_\Lambda\,A^\Lambda_\mu\,\delta_{AC}\varepsilon^{CB}\epsilon_B, \tag{7.4.3}$$

and $\mathcal{A}_\mu$ is the composite connection for the Kähler transformations:

$$\mathcal{A}_\mu \equiv \frac{i}{2}\,\left(\partial_\mu\bar{z}^{\bar{\jmath}}\,\overline{\partial}_{\bar{\jmath}}K - \partial_\mu z^i\,\partial_i K\right). \tag{7.4.4}$$

We also have that the vector field strengths $F^\Lambda_{\mu\nu} = 2\partial_{[\mu}A^\Lambda_{\nu]}$ appear via their (anti)self–dual combinations

$$F^-_{\mu\nu} \equiv \frac{1}{2}\left(F_{\mu\nu} - \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\right), \tag{7.4.5}$$

dressed by the scalar fields

$$T^-_{\mu\nu} = 2i\,\mathcal{I}_{\Lambda\Sigma}\,L^\Sigma\,F^{\Lambda -}_{\mu\nu} \qquad\qquad G^{-i}_{\mu\nu} = \overline{D}^i\,\bar{L}^\Gamma\,\mathcal{I}_{\Gamma\Lambda}\,F^{\Lambda -}_{\mu\nu}\,. \tag{7.4.6}$$

The ansatz for the field strengths is

$$F^\Lambda_{tr} = \frac{e^{2U-2\psi}}{2}\,(\mathcal{I}^{-1})^{\Lambda\Sigma}\,\left(\mathcal{R}_{\Sigma\Gamma}\,p^\Gamma - q_\Sigma\right)\;, \tag{7.4.7}$$

$$F^\Lambda_{\theta\phi} = -\frac{1}{2}p^\Lambda\sin\theta\;, \tag{7.4.8}$$

which, in the combinations (7.4.6), reconstruct the central charge $\mathcal{Z}$ and its derivatives.

Once the metric ansatz (7.2.5), the vector field strengths ansatz (7.4.7) and the requirement that the scalar fields depend only on the radial coordinate is used in the supersymmetry transformations above, we should be able to reproduce the flow equations (7.2.27)–(7.2.29) by requiring the existence of some Killing spinors.

The first variation we analyze is the time component of the gravitino $\delta\psi_{tA} = 0$. This gives the condition

$$\frac{1}{2}e^{2U}U'\gamma^{01}\epsilon_A + \frac{1}{2}\,A_t^\Lambda g_\Lambda \delta_{AC}\varepsilon^{CB}\epsilon_B + \frac{i}{2}\,e^{3U-2\psi}\,\mathcal{Z}\,\gamma^1\varepsilon_{AB}\epsilon^B - \frac{i}{2}\,e^U\,\mathcal{L}\,\delta_{AB}\gamma^0\epsilon^B = 0, \quad (7.4.9)$$

where we assumed that $\partial_t\epsilon_A = 0$. Since this equation contains both chiralities of the 4-dimensional supersymmetry parameters, we need to impose a projector condition that relates them. We can actually identify the required projectors by rewriting the above equation as

$$U'\epsilon_A = e^{-2U}\,A_t^\Lambda g_\Lambda\,\delta_{AC}\,\gamma^1\gamma^0\varepsilon^{CB}\epsilon_B + i\,e^{U-2\psi}\,\mathcal{Z}\,\gamma^0\varepsilon_{AB}\epsilon^B - i\,e^{-U}\mathcal{L}\,\delta_{AB}\gamma^1\epsilon^B. \quad (7.4.10)$$

If we introduce two distinct projectors relating the spinor components as

$$\gamma^0\epsilon_A = i\,e^{i\alpha}\,\varepsilon_{AB}\epsilon^B \quad (7.4.11)$$

and

$$\gamma^1\epsilon_A = e^{i\alpha}\,\delta_{AB}\epsilon^B, \quad (7.4.12)$$

we can rewrite the $\delta\psi_{t\,A} = 0$ condition as a single differential equation multiplying the same spinor ϵ_A. This is proved also using

$$\gamma^0\epsilon^A = -ie^{-i\alpha}\varepsilon^{AB}\epsilon_B \quad \text{and} \quad \gamma^1\epsilon^A = e^{-i\alpha}\delta^{AB}\epsilon_B, \quad (7.4.13)$$

which follow from (7.4.11)–(7.4.12) by consistency. The resulting time component of the gravitino variation gives

$$\left(-U' + ie^{-2U}\,A_t^\Lambda g_\Lambda - \,e^{U-2\psi}\,e^{-i\alpha}\mathcal{Z}\, - i\,e^{-U}\,e^{-i\alpha}\mathcal{L}\right)\epsilon_A = 0, \quad (7.4.14)$$

which is satisfied only if the quantity within brackets vanishes. Identifying the real and imaginary parts of the resulting differential equation, one gets that

$$U' = -\,e^{U-2\psi}\,\mathrm{Re}(e^{-i\alpha}\mathcal{Z}) + e^{-U}\mathrm{Im}(e^{-i\alpha}\mathcal{L}) \quad (7.4.15)$$

and

$$e^U\,A_t^\Lambda g_\Lambda = e^{-U}\mathrm{Re}(e^{-i\alpha}\mathcal{L}) + e^{U-2\psi}\mathrm{Im}(e^{-i\alpha}\mathcal{Z}). \quad (7.4.16)$$

We can now analyze the radial component of the gravitino variation $\delta\psi_{rA} = 0$, which gives

$$\partial_r \epsilon_A + \frac{i}{2}\mathcal{A}_r \epsilon_A - \frac{i}{2R^2} e^{U-2\psi} \mathcal{Z}\gamma^0 \varepsilon_{AB}\epsilon^B - \frac{i}{2}\mathcal{L}\,\delta_{AB}\gamma^1 e^{-U}\epsilon^B = 0. \tag{7.4.17}$$

By using the projectors (7.4.11)–(7.4.12) and the supersymmetry conditions (7.4.15)–(7.4.16), this reduces to

$$\partial_r \epsilon_A - \frac{1}{2}\left(U' - i\widetilde{\mathcal{A}}\right)\epsilon_A = 0, \tag{7.4.18}$$

where we introduced

$$\widetilde{\mathcal{A}} = \mathcal{A}_r + \left(e^{U-2\psi}\,\mathrm{Im}(e^{-i\alpha}\mathcal{Z}) + e^{-U}\,\mathrm{Re}(e^{-i\alpha}\mathcal{L})\right). \tag{7.4.19}$$

This equation is readily solved by

$$\epsilon_A = e^{\frac{U}{2} - \frac{i}{2}\int \widetilde{\mathcal{A}}\,dr}\chi_A, \tag{7.4.20}$$

for a spinor χ_A that is r independent. Consistency with the projector conditions defined above also imply that

$$\alpha + \int \widetilde{\mathcal{A}}\,dr = 0 \tag{7.4.21}$$

and hence

$$\alpha' + \mathcal{A}_r = -e^{U-2\psi}\,\mathrm{Im}(e^{-i\alpha}\mathcal{Z}) - e^{-U}\,\mathrm{Re}(e^{-i\alpha}\mathcal{L}), \tag{7.4.22}$$

reproducing the phase equation (7.2.34).

We are then left with the angular components of the gravitino variations and the dilatino. From the θ direction we get that

$$\partial_\theta \epsilon_A - \frac{1}{2}\,e^{\psi}(U' - \psi')\gamma^{12}\epsilon_A - \frac{1}{2}\,e^{U-\psi}\,\mathcal{Z}\,\gamma^3\varepsilon_{AB}\epsilon^B - \frac{i}{2}\,e^{-U+\psi}\,\mathcal{L}\,\delta_{AB}\gamma^2\epsilon^B = 0. \tag{7.4.23}$$

Once more, using the projectors above as well as the supersymmetry conditions derived so far, we can simplify this equation to

$$\partial_\theta \epsilon_A = \frac{1}{2}\,e^{\psi}\left[\psi' - 2e^U\,\mathrm{Im}(e^{-i\alpha}\mathcal{L}) + i\left(e^{U-2\psi}\,\mathrm{Im}(e^{-i\alpha}\mathcal{Z}) - e^{-U}\,\mathrm{Re}(e^{-i\alpha}\mathcal{L})\right)\right]\gamma^{21}\epsilon_A. \tag{7.4.24}$$

Since the radial dependence is fixed on both sides of the equation by (7.4.20), we need to require that both the real and imaginary parts of the quantities between square brackets vanish. This leads to the flow equation for ψ

$$\psi' = 2e^U\,\mathrm{Im}(e^{-i\alpha}\mathcal{L}) \tag{7.4.25}$$

and to the constraint

$$e^{U-2\psi}\,\mathrm{Im}(e^{-i\alpha}\mathcal{Z}) = e^{-U}\,\mathrm{Re}(e^{-i\alpha}\mathcal{L}) \tag{7.4.26}$$

This condition now fixes the ansatz for the time component of the vector fields

$$A_t^{\Lambda} g_{\Lambda} = 2\,e^{U}\,\mathrm{Re}(e^{-i\alpha}\mathcal{L}). \tag{7.4.27}$$

We also get that the Killing spinors ϵ_A should not depend on θ:

$$\partial_\theta \epsilon_A = 0. \tag{7.4.28}$$

A similar analysis can be performed for the other angular direction, which gives the same set of flow equations and leaves the following condition on the Killing spinors:

$$\partial_\phi \epsilon_A = \frac{1}{2}\cos\theta\,\gamma^{32}\epsilon_A - \frac{i}{2}\langle\mathcal{G}, Q\rangle\,\cos\theta\,\gamma^{01}\epsilon_A. \tag{7.4.29}$$

This is solved by requiring that

$$\partial_\phi \epsilon_A = 0 \tag{7.4.30}$$

and that

$$\langle\mathcal{G}, Q\rangle + 1 = 0. \tag{7.4.31}$$

The only supersymmetry equation remaining is the dilatino variation $\delta\lambda^{iA} = 0$. By using once more the projector conditions (7.4.11)–(7.4.12) and the other supersymmetry constraints obtained above we eventually find the flow equations for the scalar fields:

$$z^{i\prime} = -e^{i\alpha} g^{i\bar{\jmath}}\left[e^{U-2\psi}\,\overline{D}_{\bar{\jmath}}\overline{\mathcal{Z}} + i\,e^{-U}\,\overline{D}_{\bar{\jmath}}\overline{\mathcal{L}}\right]. \tag{7.4.32}$$

Summarizing, the analysis of the supersymmetry transformations reproduces the flow equations (7.2.27)–(7.2.29) for a Killing spinor of the form

$$\epsilon_A = e^{\frac{U}{2} + \frac{i}{2}\int \widetilde{\mathcal{A}}\,dr}\chi_A, \tag{7.4.33}$$

where χ_A is a constant spinor fulfilling

$$\gamma^0\chi_A = i\,\varepsilon_{AB}\chi^B, \qquad \gamma^1\chi_A = \delta_{AB}\chi^B. \tag{7.4.34}$$

Since we imposed two independent projector conditions, the resulting configurations will be 1/4 BPS (each projector halving the number of preserved supersymmetries).

7.5 Examples of dyonic solutions

We now turn to the analysis of the full flow equations and to the construction of explicit solutions, as an example of how the flow equations work and especially of the fact that now we can obtain in a single duality frame all possible black hole solutions for a given gauged supergravity model. As explained above, in order to have a regular black hole solution in an asymptotically AdS spacetime, the scalar fields have to flow according to the attractor mechanism discussed in the previous section. We will now analyze some examples where this is required. Actually, we will first show that there may be models that do not admit at all such flows, because the AdS_4 vacua and the $\mathrm{AdS}_2 \times S^2$ can never appear simultaneously for any given set of charges. We will then investigate the STU model, which is known to admit spherical horizons for special values of the charges [98].

7.5.1 Constant scalar flows

As already explained, we cannot have regular flows with constant scalars, unless the horizon is not spherical, but for instance hyperbolic [92, 93, 94]. In this case one can have regular solutions by using our flow equations together with the constraint $\langle \mathcal{G}, Q\rangle = 1$. If we assume that the scalar fields are fixed at the horizon value, we can impose that

$$e^{-i\alpha}\mathcal{Z} = -\frac{R_H^2}{2R_A}, \quad \text{and} \quad e^{-i\alpha}\mathcal{L} = \frac{i}{2R_A}. \tag{7.5.1}$$

Once inserted in the superpotential we get that

$$W = \frac{e^U}{2R_A}\left(e^{2A} - R_H^2\right). \tag{7.5.2}$$

This implies that the equations for the warp factor reduce to

$$U' = \frac{e^{-U}}{2R_A}\left(1 + R_H^2 e^{-2A}\right), \tag{7.5.3}$$

$$A' = \frac{e^{-U}}{2R_A}\left(1 - R_H^2 e^{-2A}\right). \tag{7.5.4}$$

A trivial solution is for constant A

$$e^A = R_H, \qquad e^U = \frac{r}{R_A}, \tag{7.5.5}$$

which reproduces the $\mathrm{AdS}_2 \times H^2$ horizon solution. More generally, we can solve these equations first in terms of the variables A and ψ, with the equation for ψ being

$$\psi' = A' + U' = \frac{e^{A-\psi}}{R_A}. \tag{7.5.6}$$

In fact, introducing now

$$C = e^{2A} - R_H^2, \tag{7.5.7}$$

the differential equations for A and ψ can be used to write

$$C' = C\psi', \tag{7.5.8}$$

which is readily solved by

$$C = k\, e^{\psi} \quad \Leftrightarrow \quad e^{2A} = R_H^2 + k\, e^{\psi}, \tag{7.5.9}$$

where $k = 0$ should give back the $\mathrm{AdS}_2 \times H^2$ metric. Plugging the solution into the equation for ψ (7.5.6), we get that

$$(e^{\psi})' = \frac{\sqrt{R_H^2 + k\, e^{\psi}}}{R_A}, \tag{7.5.10}$$

which is solved by

$$e^{\psi} = k\, \frac{r^2}{4R_A^2} + \frac{\sqrt{R_S^2 + k\, \alpha}}{R_A}\, r + \alpha, \tag{7.5.11}$$

where we chose the integration constant so that the limit $k \to 0$ is well-defined.

If we set $\alpha = 0$, we get that the asymptotic behavior of the warp factor is

$$r \to 0: \qquad e^{2A} \to R_H^2, \quad e^{2U} \to \frac{r^2}{R_A^2}, \tag{7.5.12}$$

which leads to the $\mathrm{AdS}_2 \times H^2$ metric

$$ds^2 = -\frac{r^2}{R_A^2} dt^2 + \frac{R_A^2}{r^2} dr^2 + R_H^2 ds_{H^2}^2, \tag{7.5.13}$$

and

$$r \to \infty: \qquad e^{2A} \to \frac{k^2}{4R_A^2} r^2, \quad e^{2U} \to \frac{r^2}{k}, \tag{7.5.14}$$

which leads to a metric that differs from AdS_4 by $1/r$ terms in the limit.

7.5.2 One modulus case

One of the simplest special Kähler moduli spaces is given by the geometry defined by the prepotential

$$F = -i\, X^0 X^1\,. \tag{7.5.15}$$

This space has only one modulus and the σ-model metric can be obtained from the Kähler potential

$$K = -\log 2(z+\bar{z}), \tag{7.5.16}$$

which requires that Re$z > 0$. The gauging potential is determined by

$$\mathcal{L} = e^{K/2}\left(g_0 + i\, g^1 + (g_1 + i g^0) z\right), \tag{7.5.17}$$

which gives a supersymmetric AdS_4 extremum at

$$z = \frac{g_0 g_1 + g^0 g^1 + i\,(g_0 g^0 - g_1 g^1)}{(g_1)^2 + (g^0)^2}. \tag{7.5.18}$$

This is in the allowed region of the moduli space if and only if

$$g_0 g_1 + g^0 g^1 > 0. \tag{7.5.19}$$

For such a simple model the second derivatives of the prepotential (7.5.15) are constant and therefore the second symplectic invariant I_2 is a constant function of the charges at every point of the moduli space:

$$I_2(\mathcal{G}) = |\mathcal{G}|^2 - |D_i\mathcal{G}|^2 = -\frac{1}{2}\mathcal{G}\mathcal{M}(F)\mathcal{G} = g_0 g_1 + g^0 g^1. \tag{7.5.20}$$

Since at the horizon $e^{-2A} = -I_2(\mathcal{G})$, we immediately see that the requirement to have a regular solution would require

$$g_0 g_1 + g^0 g^1 < 0, \tag{7.5.21}$$

in direct contradiction with the requirement to have a supersymmetric AdS vacuum. Hence we conclude that for such a model there are no regular spherical black holes with an AdS asymptotic geometry. This also implies that the AdS_4 vacua of this model will not be destabilized by the presence of supersymmetric black holes.

7.5.3 The STU model

The STU model is defined by various prepotentials, according to the choice of symplectic frame. Since our formalism is duality covariant, we can fix a symplectic basis where the prepotential has the classic form

$$F = \frac{X^1 X^2 X^3}{X^0}. \tag{7.5.22}$$

In this basis the Kähler potential is

$$K = -\log[-i(s-\bar{s})(t-\bar{t})(u-\bar{u})], \tag{7.5.23}$$

where we introduced normal coordinates $s = X^1/X^0$, $t = X^2/X^0$ and $u = X^3/X^0$. The symplectic vector $\mathcal{V}$ for such a prepotential is given by

$$\mathcal{V} = e^{K/2}\,(1, s, t, u, -stu, tu, su, st)^T. \tag{7.5.24}$$

From [98] we know that the STU model admits spherical horizon solutions for electric gaugings $\mathcal{G} = (0, g_\Lambda)$ and magnetic charges $Q = (p^\Lambda, 0)$, but in the symplectic frame defined by the prepotential

$$F_{CK} = \sqrt{X^0 X^1 X^2 X^3}. \tag{7.5.25}$$

The Kähler potentials of the two models are obviously the same, but the symplectic sections $\mathcal{V}$ for the square root prepotential F_{CK} are now

$$\mathcal{V}_{CK} = e^{K/2}\,(1, -tu, -su, -st, -stu, s, t, u)^T. \tag{7.5.26}$$

The two frames are therefore related by a symplectic transformation

$$S = \begin{pmatrix} 1 & & & & & & & \\ & & & & & -1 & & \\ & & & & & & -1 & \\ & & & & & & & -1 \\ & & & & 1 & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \end{pmatrix}, \tag{7.5.27}$$

so that $\mathcal{V}_{CK} = S\mathcal{V}$. We should stress that such a transformation is an allowed change of frame, but it is not a duality transformation. In fact, the duality transformations for the STU model are only a subset of the full symplectic group: $\mathrm{SU}(1,1)^3 \subset \mathrm{Sp}(8,\mathbb{R})$. Their form can be computed explicitly (see for instance [87]) and the matrix S does not belong to any of their combinations. However, the effective 1-dimensional model we started from (7.2.7) is fully constructed out of symplectic invariant quantities. This means that a solution to the model where, for instance, the gauging potential is obtained from $\mathcal{L}_{CK} = \langle \mathcal{G}_{CK}, \mathcal{V}_{CK}\rangle$, can be mapped to a solution of a different system where $\mathcal{L} = \langle \mathcal{G}, \mathcal{V}\rangle$, with $\mathcal{V}_{CK} = S\mathcal{V}$ and $\mathcal{G} = S^{-1}\mathcal{G}_{CK}$. Hence we should be able to reproduce solutions with a spherical horizon for our model, with non-trivial gauging charges $\mathcal{G} = (0, \tilde{g}^1, \tilde{g}^2, \tilde{g}^3, g_0, 0, 0, 0)^T$ and black hole charges $Q = (p^0, 0, 0, 0, 0, q_1, q_2, q_3)$.

Notice that, given our framework, however, we can do more than this. Since our formalism allows for the introduction of arbitrary electric and magnetic charges both for the gauging as well as for the black hole, once we have fixed a solution, like the one above, we can generate new ones by means of duality transformations. We actually know that the gauging breaks the duality group SU(1,1)^3 to a U(1) related to the isometry of the scalar manifold that is gauged by the graviphoton and the 3 vector fields, which couple to the 4 independent charges of the gauging among the 8 parameters $\mathcal{G}$. This means, however, that we can still act with this symmetry on the scalar fields and the gauging and black hole charges. In particular, we could now generate solutions with non-trivial axions, by using the representation of the three U(1) $\subset$ SU(1,1) duality transformations, which act as follows:

$$z^i \to \frac{\cos\theta_i\, z^i + \sin\theta_i}{-\sin\theta_i\, z^i + \cos\theta_i}. \tag{7.5.28}$$

The action on the charges can be then deduced by the corresponding symplectic transformations derived, for instance, in [87].

The electric dyonic configuration

The model and symplectic frame are defined by (7.5.22)-(7.5.24), and the prepotential then becomes $F = stu$. The symplectic sections are

$$\mathcal{V} = \left(L^\Lambda\, , M_\Lambda\right)\ , \tag{7.5.29}$$

where

$$L^\Lambda = e^{\mathcal{K}/2}\begin{pmatrix}1\\ s\\ t\\ u\end{pmatrix}\ , \qquad M_\Lambda = e^{\mathcal{K}/2}\begin{pmatrix}-stu\\ tu\\ su\\ st\end{pmatrix}\ , \tag{7.5.30}$$

and the Kähler potential is $\mathcal{K} = -\log(8\lambda_1\lambda_2\lambda_3)$ so that $e^{\mathcal{K}/2} = 1/(2\sqrt{2}\sqrt{\lambda_1\lambda_2\lambda_3})$, which requires to be on the branch of positive λ_i's.

In our framework, the superpotential for such a model is given by

$$W = e^{K/2}|q_1 s + q_2 t + q_3 u + p^0 stu - ie^{2A}(g_0 - g^1 tu - g^2 su - g^3 st)|. \tag{7.5.31}$$

By using the flow equations we can immediately check that we can consistently fix the axions $\mathrm{Re}\, s = \mathrm{Re}\, t = \mathrm{Re}\, u = 0$ along the whole solution, for the charge configuration we consider. For the remaining flow equations we can then use an ansatz similar to the one proposed in [98], namely (where now $z^i = (s,t,u)$)

$$\mathrm{Im}\, z^i = \sqrt{\frac{\frac{1}{2}\,|\epsilon_{ijk}| H_j H_k}{H^0 H_i}}\,, \quad \psi = \log(ar^2 + c)\,, \quad U = -\frac{1}{4}\log 4\, H^0 H_1 H_2 H_3\ , \tag{7.5.32}$$

and

$$H^0 = \frac{\alpha^0 r + \beta^0}{ar^2 + c}\ , \qquad H_i = \frac{\alpha_i r + \beta_i}{ar^2 + c}\ , \quad i = 1,2,3\ . \tag{7.5.33}$$

The symplectic sections for our configuration are

$$L^\Lambda = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1/\sqrt{\lambda_1\lambda_2\lambda_3} \\ -i\sqrt{\lambda_1/\lambda_2\lambda_3} \\ -i\sqrt{\lambda_2/\lambda_1\lambda_3} \\ -i\sqrt{\lambda_3/\lambda_1\lambda_2} \end{pmatrix}\ , \qquad M_\Lambda = \frac{1}{2\sqrt{2}}\begin{pmatrix} -i\sqrt{\lambda_1\lambda_2\lambda_3} \\ -\sqrt{\lambda_2\lambda_3/\lambda_1} \\ -\sqrt{\lambda_1\lambda_3/\lambda_2} \\ -\sqrt{\lambda_1\lambda_2/\lambda_3} \end{pmatrix}\ . \tag{7.5.34}$$

These determine the symplectic matrix

$$\mathcal{N}_{\Lambda\Sigma} = i\mathcal{I}_{\Lambda\Sigma}\ , \qquad \mathcal{I}_{\Lambda\Sigma} = \begin{pmatrix} -\lambda_1\lambda_2\lambda_3 & & & \\ & -\lambda_2\lambda_3/\lambda_1 & & \\ & & -\lambda_1\lambda_3/\lambda_2 & \\ & & & -\lambda_1\lambda_2/\lambda_3 \end{pmatrix}, \tag{7.5.35}$$

then the matrix $\mathcal{M}$ is simply given by

$$\mathcal{M} = \begin{pmatrix} \mathcal{I} & \\ & \mathcal{I}^{-1} \end{pmatrix}\ . \tag{7.5.36}$$

Finally, the central black hole and gauge charges are

$$\mathcal{Z} = \langle Q, \mathcal{V}\rangle = \frac{i}{2\sqrt{2}}\left(p^0\sqrt{\lambda_1\lambda_2\lambda_3} - q_1\sqrt{\frac{\lambda_1}{\lambda_2\lambda_3}} - q_2\sqrt{\frac{\lambda_2}{\lambda_1\lambda_3}} - q_3\sqrt{\frac{\lambda_3}{\lambda_1\lambda_2}}\right), \tag{7.5.37}$$

$$\mathcal{L} = \langle \mathcal{G}, \mathcal{V}\rangle = \frac{1}{2\sqrt{2}}\left(\frac{g_0}{\sqrt{\lambda_1\lambda_2\lambda_3}} + \tilde{g}^1\sqrt{\frac{\lambda_2\lambda_3}{\lambda_1}} + \tilde{g}^2\sqrt{\frac{\lambda_1\lambda_3}{\lambda_2}} + \tilde{g}^3\sqrt{\frac{\lambda_1\lambda_2}{\lambda_3}}\right)\ . \tag{7.5.38}$$

Knowing these central charges we can easily compute the phase α from

$$e^{2i\alpha} = \frac{\mathcal{Z} - ie^{2A}\mathcal{L}}{\bar{\mathcal{Z}} + ie^{2A}\bar{\mathcal{L}}}\ , \tag{7.5.39}$$

and its value at the asymptotic AdS_4 where

$$\lambda_1 = \sqrt{\frac{g_0\tilde{g}^1}{\tilde{g}^2\tilde{g}^3}}\ , \quad \lambda_2 = \sqrt{\frac{g_0\tilde{g}^2}{\tilde{g}^1\tilde{g}^3}}\ , \quad \lambda_3 = \sqrt{\frac{g_0\tilde{g}^3}{\tilde{g}^1\tilde{g}^2}}\ , \tag{7.5.40}$$

since the zero axions configuration has $\dot{\alpha} = 0$, the phase is fixed at $\alpha = -\pi/2$.

Equations of motion

The BPS equations relative to this configurations are

$$2e^{2\psi}\left(e^{-U}\mathrm{Re}\mathcal{V}\right)' + e^{2(\psi-U)}\Omega\mathcal{M}\mathcal{G} + \mathcal{Q} = 0 \,,$$
$$(e^{\psi})' = 2e^{\psi-U}\mathrm{Re}\mathcal{L}\,, \tag{7.5.41}$$

now consider that

$$\Omega\mathcal{M}\mathcal{G} = \begin{pmatrix} g_0/\lambda_1\lambda_2\lambda_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\tilde{g}^1\lambda_2\lambda_3/\lambda_1 \\ -\tilde{g}^2\lambda_1\lambda_3/\lambda_2 \\ -\tilde{g}^3\lambda_1\lambda_2/\lambda_3 \end{pmatrix} = 8\begin{pmatrix} g_0(L^0)^2 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\tilde{g}^1(M_1)^2 \\ -\tilde{g}^2(M_2)^2 \\ -\tilde{g}^3(M_3)^2 \end{pmatrix}\,, \tag{7.5.42}$$

so if we define four positive functions[1]

$$H^0 = 2L^0 e^{-U}\,,\quad H_i = -2M_i e^{-U}\,, \tag{7.5.43}$$

we can rewrite (7.5.41) as

$$e^{2\psi}\begin{pmatrix} \partial_r H^0 + 2g_0(H^0)^2 \\ -\partial_r H_1 - 2\tilde{g}^1(H_1)^2 \\ -\partial_r H_2 - 2\tilde{g}^2(H_2)^2 \\ -\partial_r H_3 - 2\tilde{g}^3(H_3)^2 \end{pmatrix} = \begin{pmatrix} -p^0 \\ -q_1 \\ -q_2 \\ -q_3 \end{pmatrix}\,,\qquad \psi' = g_0 H^0 + \tilde{g}^1 H_1\,, \tag{7.5.44}$$

As already stated, we follow the assumptions of [98] and make the ansatz

$$H^0 = e^{-\psi}(\alpha^0 r + \beta^0)\,,\quad H_i = e^{-\psi}(\alpha_i r + \beta_i)\,,\quad \psi = \log(ar^2 + c)\,. \tag{7.5.45}$$

We look for $-c = r_h^2$, so that $\psi = \log(a\,r^2 - r_h^2)$. The equations (7.5.44) now become algebraic equations

$$p^0 = \alpha^0 r_h^2 - 2g_0(\beta^0)^2 \qquad \alpha^0 = \frac{a}{2g_0}\,,$$
$$q_i = -\alpha_i r_h^2 + 2\tilde{g}^i(\beta_i)^2 \qquad \alpha_i = \frac{a}{2\tilde{g}^i} \quad \forall\, i = 1,2,3\,,$$
$$g_0\beta^0 + \sum_{i=1}^{3}\tilde{g}^i\beta_i = 0\,, \tag{7.5.46}$$

[1] These functions are analogous to those defined in [98] up to a factor of 2 and, for the H_i's, an overall minus sign.

and we don't have to forget the constraint

$$\langle \mathcal{G}, Q \rangle = g_0 p^0 - \sum_{i=1}^{3} \tilde{g}^i q_i = -1 \ . \tag{7.5.47}$$

t^3-black hole

We take $a = 1$, and we look for the simple solution which has all quantities with i-indices equal; from (7.5.46) e (7.5.47) we are left then with the system of 4 equations

$$\begin{aligned} p^0 &= \frac{r_h^2}{2g_0} - 2g_0(\beta^0)^2 \qquad & q = -\frac{r_h^2}{2\tilde{g}} + 2\tilde{g}(\beta)^2 \\ 0 &= g_0\beta^0 + 3\tilde{g}\beta \qquad & g_0 p^0 - 3\tilde{g}q = -1 \end{aligned} \tag{7.5.48}$$

and 7 unknowns $\{q, p^0, \tilde{g}, g^0, \beta, \beta^0, r_h\}$; we choose to parametrize the solution with $q, \tilde{g}$ and g^0. Moreover, we see that if we define the hatted quantities

$$\hat{q} \equiv q \cdot \tilde{g} \qquad \hat{p}^0 \equiv p^0 \cdot g_0 \qquad \hat{\beta} \equiv \beta \cdot \tilde{g} \qquad \hat{\beta}^0 \equiv \beta^0 \cdot g_0 \ , \tag{7.5.49}$$

choosing $g^0 > 0$, $\tilde{g} > 0$, the equations become simply

$$\begin{aligned} \hat{p}^0 &= \frac{r_h^2}{2} - 2(\hat{\beta}^0)^2 \qquad & \hat{q} = -\frac{r_h^2}{2} + 2(\hat{\beta})^2 \\ 0 &= \hat{\beta}^0 + 3\hat{\beta} \qquad & \hat{p}^0 - 3\hat{q} = -1 \end{aligned} \tag{7.5.50}$$

and we choose to parametrize the solution of these system with $\hat{q}$. We then have

$$\hat{p}^0 = 3\hat{q} - 1 \qquad \hat{\beta} = -\frac{\sqrt{1-4\hat{q}}}{4} \qquad \hat{\beta}^0 = \frac{3}{4}\sqrt{1-4\hat{q}} \qquad r_h = \frac{\sqrt{1-12\hat{q}}}{2} \ , \tag{7.5.51}$$

in fact one can show that a regular solution with all positive gauge charges cannot have $\beta > 0$. We also have to check that the functions in (7.5.43) are well defined, in particular that they are positive throughout the flow; this imply, given $\hat{\beta} < 0$, that $r_h > -2\hat{\beta}$ which results in

$$\hat{q} < 0 \quad \Rightarrow \quad q < 0 \ \cup \ p^0 < 0 \ . \tag{7.5.52}$$

To summarize, we have a black hole solution whose scalars and metric warp factors are parametrized by the functions in (7.5.43, 7.5.45), with α^0, α_i given in (7.5.46) and the other parameters are[2].

$$p^0 = \frac{3\,\tilde{g}\,q - 1}{g_0} \qquad \beta = -\frac{\sqrt{1 - 4\,\tilde{g}\,q}}{4\tilde{g}} \qquad \beta^0 = \frac{3}{4g^0}\sqrt{1 - 4\,\tilde{g}\,q} \qquad r_h = \frac{\sqrt{1 - 12\,\tilde{g}\,q}}{2} \ , \tag{7.5.53}$$

[2]Confronting the value for the β parameter found here with the one in eq. (4.24) of [113], we see that the different factor of 2 is consistent with the same rescaling factor in the definition of H^0 and H_i's, since we previously follows the notation of [98]

we are left with the freedom to choose $q<0$, $g^0>0$ and $g>0$. The scalar is

$$\lambda = \sqrt{\frac{H_i}{H^0}} = \lambda_\infty \sqrt{\frac{2r-\sqrt{1-4\,\tilde{g}\,q}}{2r+3\sqrt{1-4\,\tilde{g}\,q}}} \ , \tag{7.5.54}$$

where we defined $\lambda_\infty = \sqrt{g_0/\tilde{g}}$.

The value of the scalar field at the horizon is

$$\lambda_h = \frac{\lambda_\infty}{\sqrt{2}} \sqrt{\frac{-1+6\,\tilde{g}q+\sqrt{1-16\,\tilde{g}\,q+48\,\tilde{g}^2q^2}}{1-3\,\tilde{g}\,q}} \ , \tag{7.5.55}$$

the entropy is given by the warp factor $e^{2A}|_h = e^{2\psi(r_h)-2U(r_h)}$, with

$$e^{2U(r)} = 1/(2\sqrt{H_0H_1H_2H_3}) \ , \tag{7.5.56}$$

thus we get

$$e^{2A(r_h)} = \frac{1}{4g^2\lambda_\infty} \sqrt{1-3(1-4\,\tilde{g}\,q)^2+2(1-4\,\tilde{g}\,q)\sqrt{1-16\,\tilde{g}\,q+48\,\tilde{g}^2q^2}} \ . \tag{7.5.57}$$

We recall that the asymptotically AdS_4 metric, solution of the STU-model in $U(1)$-gauged $\mathcal{N}=2$ supergravity with $AdS_2\times S^2$ horizon is

$$ds^2 = -e^{2U}dt^2 + e^{-2U}dr^2 + e^{-2U+2\psi}(d\theta^2+\sin\theta^{\,2}\,d\phi^2) \ , \tag{7.5.58}$$

where the warp factors are

$$\begin{aligned} e^{2\psi(r)} &= (r^2-r_h^2)^2 \ , \\ e^{2U} &= \frac{2\sqrt{g^0(\tilde{g})^3}(r^2-r_h^2)^2}{\left(r-\frac{1}{2}\sqrt{1-4\,\tilde{g}\,q}\right)^{3/2}\left(r+\frac{3}{2}\sqrt{1-4\,\tilde{g}\,q}\right)^{1/2}} \\ &= \frac{2\sqrt{g^0(\tilde{g})^3}(r^2-r_h^2)^2}{\left(r-\sqrt{r_h^2-2\,\tilde{g}\,q}\right)^{3/2}\left(r+3\sqrt{r_h^2-2\,\tilde{g}\,q}\right)^{1/2}} \ . \end{aligned} \tag{7.5.59}$$

7.5.4 Purely electric black hole in four dimensional AdS_4

The work of [114] and [115] presents an holographic renormalization approach to the computation of the black hole mass in asymptotically AdS space, in various dimensions. It is possible to use their results for black holes in AdS_4, in the framework we are presenting in this Chapter. In order to make contact with the setup of those works, one has to consider a black hole solution in a frame where all abelian charges are electric and the gauging is purely magnetic.

This is another configuration which is easily constructed from the previous one. It can be obtained, indeed, by performing S-duality on he scalars from the solution of [98] $s \to -\frac{1}{s}$, $t \to -\frac{1}{t}$, $u \to -\frac{1}{u}$.

Consider the stu-model with prepotential

$$F = -2\sqrt{-X^0 X^1 X^2 X^3}\ , \tag{7.5.60}$$

and the vector of non-normalized symplectic sections

$$v = \left(1, \frac{1}{tu}, \frac{1}{su}, \frac{1}{st}, \frac{i}{stu}, \frac{i}{s}, \frac{i}{t}, \frac{i}{u}\right)\ , \tag{7.5.61}$$

giving the Kähler potential

$$\mathcal{K} = \log\left[-\frac{|stu|^2}{(s+\bar{s})(t+\bar{t})(u+\bar{u})}\right]\ . \tag{7.5.62}$$

The zero axion configuration in this case will be given by the choice of real negative scalars, thus a suitable parametrization of the scalars is $s = -x^1 + i\lambda^1$, $t = -x^2 + i\lambda^2$, $u = -x^3 + i\lambda^3$.

The normalized sections are $\mathcal{V} = (L^\Lambda, M_\Lambda)$

$$L^\Lambda = \frac{-i|stu|}{\sqrt{(s+\bar{s})(t+\bar{t})(u+\bar{u})}}\begin{pmatrix} 1 \\ \frac{1}{tu} \\ \frac{1}{su} \\ \frac{1}{st}\end{pmatrix}\ , \qquad M_\Lambda = \frac{|stu|}{\sqrt{(s+\bar{s})(t+\bar{t})(u+\bar{u})}}\begin{pmatrix} \frac{1}{stu} \\ 1/s \\ 1/t \\ 1/u\end{pmatrix}\ . \tag{7.5.63}$$

Zero axions solution

The electric black hole solution is supported by zero axion configuration. From now on we will restrict to the branch

$$s = -x^1\ , \quad t = -x^2\ , \quad u = -x^3\ , \qquad x^1\ ,\ x^2\ ,\ x^3\ > 0\ , \tag{7.5.64}$$

that in particular implies

$$\mathcal{K} = -\log\left[\frac{8}{x^1x^2x^3}\right]\ , \qquad L^\Lambda = \frac{1}{2\sqrt{2}}\begin{pmatrix}\sqrt{x^1x^2x^3} \\ \sqrt{\frac{x^1}{x^2x^3}} \\ \sqrt{\frac{x^2}{x^1x^3}} \\ \sqrt{\frac{x^3}{x^1x^2}}\end{pmatrix}\ , \qquad M_\Lambda = \frac{i}{2\sqrt{2}}\begin{pmatrix} 1/\sqrt{x^1x^2x^3} \\ \sqrt{\frac{x^2x^3}{x^1}} \\ \sqrt{\frac{x^1x^3}{x^2}} \\ \sqrt{\frac{x^1x^2}{x^3}}\end{pmatrix}\ . \tag{7.5.65}$$

The black hole charges are

$$\mathcal{Q} = (0,0,0,0,q_0,q_1,q_2,q_3) \ , \tag{7.5.66}$$

the gauging is magnetic

$$\mathcal{G} = \left(g^0,g^1,g^2,g^3,0,0,0,0\right) , \tag{7.5.67}$$

the black hole and the "gauging" central charges are

$$\begin{aligned} \mathcal{Z} &= \frac{1}{2\sqrt{2}\sqrt{x^1x^2x^3}}\left(q_0x^1x^2x^3 + q_1x^1 + q_2x^2 + q_3x^3\right) \ , \\ \mathcal{L} &= \frac{-i}{2\sqrt{2}\sqrt{x^1x^2x^3}}\left(g^0 + g^1x^2x^3 + g^2x^1x^3 + g^3x^1x^2\right) \ , \end{aligned} \tag{7.5.68}$$

so that $\alpha = 0$. The symplectic matrix $\mathcal{N}_{\Lambda\Sigma}$ and the black hole potential quadratic form $\mathcal{M}$ are

$$\begin{aligned} \mathcal{N}_{\Lambda\Sigma} = i\mathcal{I}_{\Lambda\Sigma} \ , \qquad \mathcal{I}_{\Lambda\Sigma} &= \begin{pmatrix} -\frac{1}{x^1x^2x^3} & 0 & 0 & 0 \\ 0 & -\frac{x^2x^3}{x^1} & 0 & 0 \\ 0 & 0 & -\frac{x^1x^3}{x^2} & 0 \\ 0 & 0 & 0 & -\frac{x^1x^2}{x^3} \end{pmatrix} , \\ \mathcal{M} &= \begin{pmatrix} \mathcal{I} & \\ & \mathcal{I}^{-1} \end{pmatrix} . \end{aligned} \tag{7.5.69}$$

There is a supersymmetric AdS_4 minimum at asymptotic infinity

$$D_i\mathcal{L}_\infty = 0 \quad \to \quad x^1_\infty = \sqrt{\frac{g^0g^1}{g^2g^3}} \ , \quad x^2_\infty = \sqrt{\frac{g^0g^2}{g^1g^3}} \ , \quad x^3_\infty = \sqrt{\frac{g^0g^3}{g^1g^2}} \ . \tag{7.5.70}$$

BPS equations of motion

The equation of motion are in this case

$$\begin{aligned} e^{2\psi}\partial_r \mathrm{Im}\left(e^{-U}\mathcal{V}\right) + e^{2(\psi-U)}\Omega\mathcal{M}\mathcal{G} + Q &= 0 \ , \\ \psi' &= 2e^{-U}\mathrm{Im}\mathcal{L} \ , \end{aligned} \tag{7.5.71}$$

and we have

$$\Omega\mathcal{M}\mathcal{G} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{x^1x^2x^3}g^0 \\ -g^1x_2x_3/x^1 \\ -g^2x^1x^3/x^2 \\ -g^3x^1x^2/x^3 \end{pmatrix} = 8\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ g^0(M_0)^2 \\ g^1(M_1)^2 \\ g^2(M_2)^2 \\ g^3(M_3)^2 \end{pmatrix} , \tag{7.5.72}$$

thus we can define

$$H_\Lambda = -2iM_\Lambda e^{-U} \;, \qquad \Lambda = 0,1,2,3 \tag{7.5.73}$$

so that the eom.s become

$$e^{2\psi}\left(\partial_r H_\Lambda + 2g^\Lambda (H_\Lambda)^2\right) = -q_\Lambda \;, \qquad \psi' = -\sum_{\Lambda=0}^{3} g^\Lambda H_\Lambda \;, \tag{7.5.74}$$

where no sum is intended in the first of the above equations. Once again we proceed with the ansatz

$$H_\Lambda = e^{-\psi}(\alpha_\Lambda r + \beta_\Lambda) \;, \quad \psi = \log(ar^2 - r_H^2) \;, \tag{7.5.75}$$

turning the BPS equations to the algebraic ones

$$\alpha_\Lambda r_H^2 + 2g^\Lambda(\beta_\Lambda)^2 = -q_\Lambda \;, \qquad a + 2g^\Lambda \alpha_\Lambda = 0 \;, \qquad \sum_{\Lambda=0}^{3} g_\Lambda \beta^\Lambda = 0 \;. \tag{7.5.76}$$

Electric t^3 model

Taking into account the constraint (7.5.47), we choose $a = 1$ and find a solutions for $\beta_i \equiv \beta$ and $q_i \equiv q$, thus $g^i \equiv g$, and we take all $g^\Lambda < 0$, in order to recover the previous electric solution. Notice that the existence of an asymptotic AdS point is not effected by an overall rotation of the gauge charges in this configuration by a same phase, in particular a minus sign. Again, it's easier to rewrite the equations in terms of $\hat\beta = |g^i|\beta^i$, $\hat\beta_0 = |g^0|\beta_0$, and so on. We get the system

$$\begin{aligned} \hat q^0 &= -\frac{r_H^2}{2} + 2(\hat\beta^0)^2 & \qquad \hat q &= -\frac{r_H^2}{2} + 2\hat\beta^2 \\ 0 &= \hat\beta^0 + 3\hat\beta & \qquad \hat q^0 + 3\hat q &= -1 \end{aligned} \tag{7.5.77}$$

in the unknowns $\{\hat\beta, \hat\beta^0, \hat q, \hat q^0, r_H\}$. We choose to parametrize the solution with $\hat q$, which gives the same solution of the previous case in the hatted coordinates

$$\hat q_0 = -3\hat q - 1 \qquad \hat\beta = -\frac{\sqrt{1-4\hat q}}{4} \qquad \hat\beta_0 = \frac{3}{4}\sqrt{1-4\hat q} \qquad r_h = \frac{\sqrt{1-12\hat q}}{2} \;, \tag{7.5.78}$$

with $\hat q < 0$. The explicit dependence of the parameters on the gauge charges is then

$$q_0 = \frac{1-3g\,q}{g^0} \qquad \beta = \frac{\sqrt{1+4g\,q}}{4g} \qquad \beta_0 = -\frac{3\sqrt{1+4g\,q}}{4g^0} \qquad r_h = \frac{\sqrt{1+12g\,q}}{2} \;, \tag{7.5.79}$$

where we still have the freedom to choose the charges $q < 0$, $g^0 < 0$, $g < 0$. The scalar fields are

$$x = \sqrt{\frac{H_1}{H_0}} = x_\infty \sqrt{\frac{r - 2g\beta}{r - 2g^0\beta_0}} = x_\infty \sqrt{\frac{2r - \sqrt{1+4gq}}{2r + 3\sqrt{1+4gq}}} \;,$$
$$x_\infty = \sqrt{\frac{g^0}{g^1}} \qquad e^\psi = \frac{4r^2 - 1 - 12g\,q}{4} \;, \tag{7.5.80}$$

$$e^{-2U} = 2\sqrt{H_0H_1H_2H_3} = \frac{e^{-2\psi}}{2\sqrt{g^0g^3}} \prod_{\Lambda=0}^{3} \sqrt{r - 2\beta_\Lambda g^\Lambda} =$$
$$= \frac{\sqrt{2r + 3\sqrt{1+4gq}}\,\sqrt{(2r - \sqrt{1+4g\,q})^3}}{8\sqrt{g^0g^3}\,(r^2 - r_H^2)^2} \;, \tag{7.5.81}$$

the metric solution is

$$ds^2 = -\frac{8\sqrt{g^0(g)^3}\,(r^2 - r_H^2)^2}{\sqrt{2r + 3\sqrt{1+4gq}}\,\sqrt{(2r - \sqrt{1+4g\,q})^3}} dt^2 + \frac{\sqrt{2r + 3\sqrt{1+4gq}}\,\sqrt{(2r - \sqrt{1+4g\,q})^3}}{8\sqrt{g^0(g)^3}\,(r^2 - r_H^2)^2} dr^2$$
$$+ \frac{\sqrt{2r + 3\sqrt{1+4gq}}\,\sqrt{(2r - \sqrt{1+4g\,q})^3}}{8\sqrt{g^0(g)^3}} (d\theta^2 + \sin\theta^2 d\phi^2) \;. \tag{7.5.82}$$

7.5.5 Confronting previous solutions in gauged supergravities

The examples we have presented in this section rely on the metric ansatz (7.2.5). It is convenient, however, to compare our conventions to the previous literature, see for example [92, 36, 35, 93], where Supergravity black holes in asymptotic AdS space have been discussed, before a regular horizon solution was shown to exist in [98] .

We define

$$\tilde{H} = 1 - \frac{\tilde{Q}}{r} \;, \quad \tilde{Q} = \sqrt{1+4gq} \;, \quad \tilde{H}_0 = 1 + \frac{\tilde{Q}_0}{r} \;, \quad \tilde{Q}_0 = 3\tilde{Q} \;, \quad \mathcal{I}(G) = 2\sqrt{g^0g^1g^2g^3} \;, \tag{7.5.83}$$

and rewrite the metric as

$$ds^2 = -\mathcal{I}(G)\,r^2 \left(1 - \frac{r_H^2}{r^2}\right)^2 \left(\sqrt{\tilde{H}_0(\tilde{H})^3}\right)^{-1} dt^2 + \mathcal{I}(G)^{-1} \left(1 - \frac{r_H^2}{r^2}\right)^{-2} \sqrt{\tilde{H}_0(\tilde{H})^3}\,\frac{dr^2}{r^2} +$$
$$+ \frac{r^2}{\mathcal{I}(G)} \sqrt{\tilde{H}_0(\tilde{H})^3}\,(d\theta^2 + \sin\theta^2 d\phi^2) \;, \tag{7.5.84}$$

whose asymptotical behavior is

$$ds^2_\infty \sim -\mathcal{I}(G)\,r^2 dt^2 + \frac{1}{\mathcal{I}(G)}\frac{dr^2}{r^2} + \frac{r^2}{\mathcal{I}(G)}(d\theta^2 + \sin\theta^2 d\phi^2) \;. \tag{7.5.85}$$

One can define a function $f(r)$

$$f(r) = r^2\left(1-\frac{r_H^2}{r^2}\right)^2 \ , \tag{7.5.86}$$

so that the metric takes the form

$$\begin{aligned} ds^2 &= -e^{2A}f(r)dt^2 + e^{-2A}\left(\frac{dr^2}{f(r)} + r^2 d\Omega_2^2\right) \ , \\ e^{2A} &= \frac{\mathcal{I}(G)}{\sqrt{\tilde{H}_0\tilde{H}_1\tilde{H}_2\tilde{H}_3}} \ , \end{aligned} \tag{7.5.87}$$

which is analogous to the metric used in [35]. Notice that, with respect to the ansatz of the same paper where $f(r)^{Duff-Liu} = 1-(r/r_0)+g^2r^2$, we now have

$$f(r) = -2r_H^2 + (r_H^2/r)^2 + r^2 \ , \tag{7.5.88}$$

in which the minus sign of the constant is crucial to find a regular solution. Also, a factor $\mathcal{I}(G)$ is added, with respect to previous ansatz, and changes the asymptotic behavior of the metric.

7.5.6 The magnetic solution of Cacciatori and Klemm

For completeness, we give here the details of a purely magnetic black hole solution that Cacciatori and Klemm in [98] demonstrated to exist, and to exihbit a regular spherical horizon. The solution can be found in the t^3 model, for which $p^1 = p^2 = p^3 \equiv p$, $H^1 = H^2 = H^3$, $g_1 = g_2 = g_3 \equiv g$, and it is

$$\begin{aligned} ds^2 &= -e^{2U(r)}dt^2 + e^{-2U(r)}\left[dr^2 + e^{2\psi(r)}d\Omega^2\right] \ , \\ e^{2\psi(r)} &= (r^2-r_H^2)^2 \ , \qquad e^{-2U(r)} = 2\sqrt{H_0H_1H_2H_3} \ , \\ H_{(i)} &= e^{-\phi}(\alpha_i r + \beta_i) \ . \end{aligned} \tag{7.5.89}$$

It can be parametrized by $g_0\ ,\ g>0\ ,\ p>0$ as

$$\alpha = \frac{1}{2g}\ , \quad \alpha^0 = \frac{1}{2g_0}\ , \quad \beta = -\frac{\sqrt{1+4gp}}{4g}\ , \quad \beta^0 = \frac{3\sqrt{1+4gp}}{4g_0}\ , \tag{7.5.90}$$

with a black hole horizon given by

$$r_H = \frac{1}{2}\sqrt{1+12gp} \ . \tag{7.5.91}$$

Moreover, the constraint of spherical horizon gives

$$p^0 = -\frac{1+3gp}{g_0} \ . \tag{7.5.92}$$

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