Sede Amministrativa: Università degli Studi di Padova

Dipartimento di Matematica "Tullio Levi-Civita"
Scuola di Dottorato di ricerca in Scienze Matematiche
Ciclo xxxiv

Triangulated equivalences for commutative noetherian rings

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30 Settembre 2021

Summary

This thesis exhibits a class of t-structures, the intermediate *restrictable* ones, in the derived category of a commutative noetherian ring R, whose hearts are derived and coderived equivalent to the category of R-modules. To this end, tools to prove some equivalences on the derived level are developed and applied. For the coderived side, results from the literature are extended to apply to these hearts, and the derived equivalences are lifted.

Riassunto

La presente tesi esibisce una classe di t-strutture, ovvero le intermedie *restringibili*, nella categoria derivata di un anello commutativo noetheriano, i cui cuori sono derivato- e coderivato-equivalenti alla categoria degli R-moduli. A questo scopo, vengono sviluppati e applicati degli strumenti per dimostrare alcune equivalenze a livello derivato. Per il lato coderivato, vengono estesi dei risultati presenti in letteratura per includere anche questi cuori, e si dimostra che le equivalenze derivate trovate in precedenza inducono equivalenze coderivate.

Introduction

The central topic of the present thesis are equivalences between some triangulated categories. Given an abelian category $\mathcal{A}$, such as the category $\mathsf{Mod}(R)$ of (right) modules over a ring R, or the category $\mathsf{Qcoh}(X)$ of quasi-coherent sheaves over a scheme X, one can often construct a handful of interesting triangulated categories. There are the *bounded* and *unbounded derived categories* $\mathsf{D}^b(\mathcal{A})$ and $\mathsf{D}(\mathcal{A})$; there is also the bigger *coderived category* $\mathsf{D}^{\mathsf{co}}(\mathcal{A})$. Inside the derived categories of $\mathcal{A}$, the category $\mathcal{A}$ itself can be recovered as the *heart* of the *standard t-structure*. t-Structures are ways to decompose the objects of a triangulated category into orthogonal parts, and each of them has a heart, which is an abelian category. The standard t-structure has, by construction, the special property that the derived category of its heart is the ambient category $\mathsf{D}(\mathcal{A})$. The natural question arises, whether other t-structures in $\mathsf{D}(\mathcal{A})$ have this property. Namely, given a t-structure $\mathbb{T}$ with heart $\mathcal{H}$, we ask whether there is an equivalence of triangulated categories $\mathsf{D}(\mathcal{H}) \simeq \mathsf{D}(\mathcal{A})$, which maps the standard t-structure of $\mathsf{D}(\mathcal{H})$ onto $\mathbb{T}$. Such a functor, if it exists, will be called a *realisation functor* for $\mathbb{T}$ (see §2.6); and we will say that $\mathbb{T}$ *induces derived equivalence*. If this is the case, since, in the spirit of category theory, we may consider $\mathsf{D}(\mathcal{A})$ up to equivalence, clearly $\mathbb{T}$ is as good a t-structure as the standard t-structure.

When finding that the t-structure $\mathbb{T}$ in $\mathsf{D}(\mathcal{A})$ induces derived equivalence, new information can be obtained both on the heart $\mathcal{H}$ of $\mathbb{T}$ and on $\mathcal{A}$. The literature on derived invariants, i.e. properties shared by derived equivalent categories $\mathcal{H}$ and $\mathcal{A}$, is huge. In the thesis there are some instances of this as well. For example, if $\mathcal{A} \simeq \mathsf{Mod}(R)$, its special properties, such as having a projective finitely presented generator, reflect onto $\mathsf{D}(\mathcal{A})$ (which in this case is compactly generated), and therefore onto $\mathcal{H}$. This direction is exploited in Chapter 4. Conversely, the heart $\mathcal{H}$ may have some additional properties, which shed new light on $\mathsf{D}(\mathcal{A})$. For an example of this, see Example 3.2.15.

The problem of determining whether a t-structure $\mathbb{T}$ induces derived equivalence is far from trivial. The theory of classical *tilting* objects of $\mathsf{D}(\mathcal{A})$ deals with the cases in which $\mathcal{H}$ is itself a category of modules; the more general theory of (big) tilting objects, together with the dual *cotilting* objects, covers a more general situation, in which $\mathcal{H}$ has enough projectives (respectively, injectives).

The main approach we use to produce t-structures inducing derived equivalences is to take a t-structure with this property and "deform" it into a new one. The tool for this is the *HRS-tilting procedure*: given a t-structure $\mathbb{T}$ with heart $\mathcal{H}$, and a *torsion pair* $\mathbf{t}$ in $\mathcal{H}$, one constructs a new t-structure $\mathbb{T}'$, whose heart $\mathcal{H}'$ is a "deformation" of $\mathcal{H}$ along $\mathbf{t}$ (see §2.7). The main point is that to check whether $\mathbb{T}'$ still induces derived equivalence, a criterion on $\mathbf{t}$ exists: we further investigate and specialise it, obtaining a straightforward result, of (relatively) easy application:

Theorem (Theorem 2.9.7)**.** *Let $\mathcal{A}$ be an abelian category, and let $\mathbb{T}$ be a t-structure in $\mathsf{D}(\mathcal{A})$ inducing derived equivalence. Assume that the heart $\mathcal{H}$ of $\mathbb{T}$ is a Grothendieck category, with generator G. Let $\mathbf{t}$ be a hereditary torsion pair in the heart $\mathcal{H}$ of $\mathbb{T}$, and $\mathbb{T}_{\mathbf{t}}$ the t-structure obtained by HRS-tilting with respect to $\mathbf{t}$. Then $\mathbb{T}_{\mathbf{t}}$ induces a (bounded) derived equivalence if and only if the object $G/\mathsf{tr}_{fG}G$ is torsion, where $\mathsf{tr}_{fG}G$ denotes the trace of the torsion-free part of G in G.*

Chapter 3 has the application of this criterion as its final goal. We restrict our setting to $\mathcal{A} = \mathsf{Mod}(R)$, for a commutative noetherian ring R. This choice is motivated by the abundant literature describing many aspects of $\mathsf{Mod}(R)$ and $\mathsf{D}(R)$ in this situation. Many classification results exist, for the objects of our interest: namely, *hereditary* torsion pairs in $\mathsf{Mod}(R)$ and *compactly generated* t-structures in $\mathsf{D}(R)$ (see §3.1). Our first step is to extend the classification of the former from $\mathsf{Mod}(R)$ to the heart of a wide class of t-structures, in order to gain some insight helping in the application of the criterion (§3.2). The result is summarised by the following:

Proposition (Proposition 3.2.2)**.** *Let R be a commutative noetherian ring. Let $\mathbb{T}$ be a non-degenerate smashing t-structure in $\mathsf{D}(R)$, with heart $\mathcal{H}$. There is a subset $\mathcal{P}_{\mathcal{H}} \subseteq 2^{\mathsf{Spec}(R)}$ for which there is a bijection*

$$\begin{aligned} \{\textit{Hereditary torsion pairs in } \mathcal{H}\} &\longleftrightarrow \mathcal{P}_{\mathcal{H}} \\ (\mathcal{T}, \mathcal{F}) &\longmapsto \mathsf{supp}(\mathcal{T}) \\ (\mathsf{supp}^{-1}(V), \mathcal{F}) &\longleftarrow\!\shortmid V. \end{aligned}$$

This description gives a way to check whether a given object is torsion with respect to a hereditary torsion pair in $\mathcal{H}$: and therefore it is very useful for our purposes. After some work, involving an iterated application of the criterion above, we manage to prove the main result of Chapter 3:

Theorem (Theorem 3.3.15)**.** *Let R be a commutative noetherian ring, and $\mathbb{T}$ an intermediate t-structure in $\mathsf{D}(R)$. If $\mathbb{T}$ restricts to a t-structure in $\mathsf{D}^b(\mathsf{mod}(R))$, then it induces a derived equivalence.*

This result adds new interest to the restrictability property of a t-structure, and it is the starting point of Chapter 4.

The hearts of intermediate restrictable t-structures in $\mathsf{D}(R)$ are locally coherent Grothendieck categories, which is a first generalisation of the notion of locally noetherian Grothendieck categories (such as $\mathsf{Mod}(R)$). The fact that they are derived equivalent to $\mathsf{Mod}(R)$ makes them even nicer: for example, their derived category is in particular compactly generated. This allows us to extend to this situation some results for locally noetherian categories, namely the existence of a *recollement* with the coderived category $\mathsf{D}^{\mathsf{co}}(\mathcal{H}) := \mathsf{K}(\mathsf{Inj}(\mathcal{H}))$ in the middle (see §4.1). Another special property proved on the way is a characterisation of the objects of $\mathsf{D}^b(\mathsf{fp}(\mathcal{H}))$ as the compact objects of $\mathsf{D}^b(\mathcal{H})$, which needs not to be true in general. At this point we have two recollements

$$\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(R)) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathsf{K}(\mathsf{Inj}(R)) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathsf{D}(R)$$

$$\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(\mathcal{H})) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathsf{K}(\mathsf{Inj}(\mathcal{H})) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathsf{D}(\mathcal{H}).$$

Moreover, Chapter 3 gives us an equivalence between the rightmost categories. It is therefore natural to ask whether this equivalence lifts to an equivalence between the two recollements. The rest of Chapter 4 is devoted to the explicit construction of a functor $\mathsf{K}(\mathsf{Inj}(\mathcal{H})) \to \mathsf{K}(\mathsf{Inj}(R))$, and to the proof that it gives an equivalence of recollements. As a side-effect, we obtain an equivalence between the *singularity categories* $\mathsf{S}(\mathcal{H}) := \mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(\mathcal{H}))$ of $\mathcal{H}$ and $\mathsf{S}(R) := \mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(R))$ of $\mathsf{Mod}(R)$.

Theorem (Theorem 4.3.10)**.** *Let R be a commutative noetherian ring and $\mathbb{T}$ an intermediate restrictable t-structure in $\mathsf{D}(R)$, with heart $\mathcal{H}$. Then there exists an equivalence of recollements:*

$$\begin{array}{ccccc}
\mathsf{S}(R) & \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow} & \mathsf{D}^{\mathsf{co}}(R) & \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow} & \mathsf{D}(R) \\
\downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq} \\
\mathsf{S}(\mathcal{H}) & \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow} & \mathsf{D}^{\mathsf{co}}(\mathcal{H}) & \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow} & \mathsf{D}(\mathcal{H}).
\end{array}$$

All the background notions and results needed to formulate and prove these theorems, together with related material, are collected in Chapter 1 (on abelian categories) and Chapter 2 (on triangulated categories).

Acknoledgments

I would like to warmly thank my Supervisor, Professor Alberto Tonolo, for everything he taught me, and even more for all the support, advice and direction he gave me in these three years. My gratitude also goes to Jorge Vitória and Michal Hrbek, with whom I had the pleasure to work on the results contained in this Thesis, for their help and friendship.

I have had the opportunity to engage in a really stimulating and welcoming environment, and there are too many people to which I owe something, to list them all. I will mention Lidia Angeleri-Hügel, Silvana Bazzoni, Alberto Facchini, Luisa Fiorot, Lorenzo Martini, Giovanna Le Gros, Rosie Laking, Francesco Mattiello, Leonid Positselski, Simone Virili, and I apologise to the others.

I want to thank Prof. Martino Bardi and Ms. Loretta Dalla Costa, for their help and guidance in the bureaucratic matters of PhD studies.

I would like to thank also the Scuola Grande di S. Teodoro, in Venice, for their support during my undergraduate and graduate studies.

Lastly, but more importantly, I must thank my parents and my brother, for everything I am.

Contents

Chapter 1

Abelian categories

1.1 Preliminaries on categories

Notation 1.1.1. In this Thesis, subcategories are always assumed to be full and strict (i.e. closed under isomorphism).

Definition 1.1.2. A category $\mathcal{C}$ is **small** if its class of objects is a set, and **skeletally small** if it is equivalent to a small category.

Definition 1.1.3. A non-empty small category I is **filtered** if

(f1) for every $i, j \in I$ there are $k \in I$ and morphisms $i \to k, j \to k$;

(f2) for every $i, j \in I$ and parallel morphisms $i \rightrightarrows j$ there is $k \in I$ and a morphism $j \to k$ such that the two compositions $i \rightrightarrows j \longrightarrow k$ coincide.

Example 1.1.4. A partially ordered set, considered as a category, is filtered when any pair of elements admits an upper bound.

Definition 1.1.5. Let $\mathcal{C}$ be a category. If I is a small category, one can form the category $\mathcal{C}^I$ of **(small) diagrams of shape** I in $\mathcal{C}$, whose objects are functors $I \to \mathcal{C}$ and morphisms are natural transformations. A diagram of shape I is also denoted by $(C_i \mid i \in I)$, with $C_i = F(i)$ and the morphisms being understood.

There is the **constant diagram functor** $\Delta^I : \mathcal{C} \to \mathcal{C}^I$, which maps an object C of $\mathcal{C}$ to the constant functor $i \mapsto C$, and sends a morphism $f \in \mathsf{Hom}_{\mathcal{C}}(C, D)$ to the natural transformation $\eta \colon \Delta^I(C) \Rightarrow \Delta^I(D)$ defined by $\eta_i = f$.

Given a diagram $F \in \mathcal{C}^I$, its **colimit** is a pair (C, η) of an object C of $\mathcal{C}$ and a natural transformation $\eta \colon F \Rightarrow \Delta^I(C)$, such that for every other such pair (D, ζ) there is a unique morphism $\gamma \colon C \to D$ such that $\zeta = \Delta^I(\gamma) \circ \eta$. A colimit (C, η) of F is denoted by $\operatorname{colim} F$ (more often, we will write $\operatorname{colim} F = C$, suppressing η).

If I is filtered, a diagram $F \in \mathcal{C}^I$ is called a **direct system** of objects of $\mathcal{C}$, and its colimit a *filtered colimit*; we will instead use the term **direct limit** with the same meaning, and denote it by the symbol $\varinjlim F$.

Definition 1.1.6. Dually, the limit in $\mathcal{C}$ of a diagram $F \in \mathcal{C}^I$ is defined as the colimit in $\mathcal{C}^{op}$ of the induced diagram $F^{op} \colon I^{op} \to \mathcal{C}^{op}$ of shape I^{op}, and it is denoted by $\lim F$. If I is filtered, F is called an **inverse system** of objects of $\mathcal{C}$, and its limit is called a *cofiltered limit*; we will instead use the term **inverse limit** with the same meaning, and denote it by $\varprojlim F$.

Definition 1.1.7. Given an object $C \in \mathcal{C}$, a **direct system of subobjects** of C is a filtered diagram $F \in \mathcal{C}^I$ which has a natural transformation $\iota \colon F \to \Delta^I(C)$ such that $\eta_i \colon F(i) \to C$ is a monomorphism for every $i \in I$. The direct limit of F is also called the **sum** of the subobjects $F(i)$ of C, and we will denote it by $\sum F(i) := \varinjlim F$. Even when it exists, is not a subobject of C, in general; compare with Lemma 1.2.2.

Remark 1.1.8. From the definition, it is clear that the colimit and the limit of a diagram, when they exist, are unique up to unique isomorphism. Therefore, if all the diagrams of shape I in $\mathcal{C}$ admit colimit (respectively, limit), this gives a functor $\operatorname{colim} \colon \mathcal{C}^I \to \mathcal{C}$ (respectively, $\lim \colon \mathcal{C}^I \to \mathcal{C}$). It is easy to see that colim (respectively, lim) is then the left (respectively, right) adjoint of the constant diagram functor $\Delta^I \colon \mathcal{C} \to \mathcal{C}^I$.

Example 1.1.9. (1) When I is a set viewed as discrete category (i.e., without any non-identity morphism), a diagram of shape I in $\mathcal{C}$ is just a family of objects of $\mathcal{C}$ indexed by I. Its colimit is their coproduct, its limit is their product.

(2) Assume $\mathcal{C}$ is additive. Consider the graphs

$$\ulcorner := \left\{ \begin{matrix} \bullet & \to & \bullet \\ \downarrow & & \\ \bullet & & \end{matrix} \right\} \qquad \lrcorner := \left\{ \begin{matrix} & & \bullet \\ & & \downarrow \\ \bullet & \to & \bullet \end{matrix} \right\}$$

viewed as categories. The colimit of a diagram of shape $\ulcorner$ is its pushout, the limit of a diagram of shape $\lrcorner$ is its pullback. In particular, for a morphism $f \colon C \to D$ in $\mathcal{C}$, the kernel and cokernel of f are obtained as

$$\ker f = \lim \begin{pmatrix} & & 0 \\ & & \downarrow \\ C & \xrightarrow{f} & D \end{pmatrix} \qquad \operatorname{coker} f = \operatorname{colim} \begin{pmatrix} C & \xrightarrow{f} & D \\ \downarrow & & \\ 0 & & \end{pmatrix}$$

Definition 1.1.10. A category $\mathcal{C}$ such that all small diagrams, of any shape, admit colimit (respectively, limit) is called **cocomplete** (respectively, **complete**).

1.2 Abelian categories

1.2.1 Notation

For the definition of abelian category, see for example Freyd's book [19, § 2].

We denote by $\mathsf{Gen}\,\mathcal{S}$ the full subcategory of epimorphic images of coproducts of objects of $\mathcal{S}$ (when they exist). We say that the objects of $\mathcal{S}$ **generate**, or are **generators** of, $\mathsf{Gen}\,\mathcal{S}$. Dually, $\mathsf{Cogen}\,\mathcal{S}$ will be the full subcategory of subobjets of existing products of objects of $\mathcal{S}$.

Given subcategories $\mathcal{B}, \mathcal{C}$ of $\mathcal{A}$, we denote by $\mathcal{B} * \mathcal{C}$ the full subcategory of objects X appearing in a short exact sequences

$$0 \to B \to X \to C \to 0$$

with $B \in \mathcal{B}$ and $C \in \mathcal{C}$.

Yoneda extensions. [82, §3.4] Let $\mathcal{A}$ be an abelian category. For every objects X, Y of $\mathcal{A}$ and integer $n \geq 1$, consider the class $\mathcal{E}^n_{\mathcal{A}}(X,Y)$ of exact sequences in $\mathcal{A}$ of the form

$$\varepsilon : \quad 0 \to Y \to Z_1 \to \cdots \to Z_n \to X \to 0$$

On $\mathcal{E}^n_{\mathcal{A}}(X,Y)$, consider the equivalence relation $\sim$ generated by pairs $(\varepsilon, \varepsilon')$ connected by a morphism of sequences, i.e. for which there is a commutative diagram

$$\begin{tikzcd}
\varepsilon : \arrow[d] & 0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Z_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & Z_n \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
\varepsilon' : & 0 \arrow[r] & Y \arrow[r] & Z'_1 \arrow[r] & \cdots \arrow[r] & Z'_n \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

Namely, two sequences $\varepsilon, \varepsilon'$ are equivalent if and only if there is a zig-zag of morphisms of sequences as above connecting them, $\varepsilon \to \varepsilon_1 \leftarrow \cdots \to \varepsilon_r \leftarrow \varepsilon'$.

The class of Yoneda extensions from X to Y is defined as $\mathsf{Ext}^n_{\mathcal{A}}(X,Y) := \mathcal{E}^n_{\mathcal{A}}(X,Y)/\sim$. In general it is a proper class (see [19, Exercise A, p. 131]). It is a set when $\mathcal{A}$ has enough injectives or enough projectives.

Notation 1.2.1. Given a family of objects $\mathcal{S}$ in $\mathcal{A}$ and a set $I \subseteq \mathbb{N}$, we denote

$$\mathcal{S}^{\perp_I} := \{X \in \mathcal{A} \colon \mathsf{Ext}^i_{\mathcal{A}}(S,X) = 0 \text{ for every } i \in I\}$$
$${}^{\perp_I}\mathcal{S} := \{X \in \mathcal{A} \colon \mathsf{Ext}^i_{\mathcal{A}}(X,S) = 0 \text{ for every } i \in I\},$$

where we adopt the convention that $\mathsf{Ext}^0_{\mathcal{A}} := \mathsf{Hom}_{\mathcal{A}}$. If $\mathcal{S}$ consists of a single object S, we write $S^{\perp_I}$ and ${}^{\perp_I}S$. Instead of I we may write a list of natural numbers or symbols like > 0, with the obvious meaning. If $I = \{0\}$, we simply write $\mathcal{S}^{\perp}, {}^{\perp}\mathcal{S}$.

1.2.2 ABn axioms

Let $\mathcal{A}$ be an abelian category. Grothendieck [24] introduced the following additional axioms:

(AB3) $\mathcal{A}$ has all coproducts (hence it is cocomplete by [46, dual of Theorem V.2.1]).

(AB4) $\mathcal{A}$ satisfies AB3 and coproducts are (left-)exact, i.e. the coproduct of a family of monomorphisms is a monomorphism.

(AB5) $\mathcal{A}$ satisfies AB3 and given a filtered system of subobjects $(A_i \mid i \in I)$ of an object A, for every other subobject B of A we have

$$\left(\sum A_i\right) \cap B = \sum (A_i \cap B).$$

This is equivalent to asking that $\mathcal{A}$ satisfies AB3 and (filtered) direct limits are (left-)exact ([24, Proposition 1.8] and [75, Proposition V.1.1]).

There are also the dual axioms ABn^*. We will also encounter the following relaxation of AB4 and AB4*, for integers $k \geq 0$, introduced by Roos [69]:

(AB4-k) (when $\mathcal{A}$ has enough projectives.) Let $(A_i \mid i \in I)$ be a family of objects of $\mathcal{A}$, and for each consider a projective resolution

$$P_i^\bullet \to A_i \to 0: \qquad \cdots \to P_i^{-n} \to \cdots \to P_i^{-1} \to P_i^0 \to A_i \to 0.$$

Then the sequence

$$\coprod P_i^\bullet: \qquad \cdots \to \coprod P_i^{-n} \to \cdots \to \coprod P_i^{-1} \to \coprod P_i^0 \to 0$$

is exact in degree $-n$ for every $n > k$.

(AB4*-k) ($\mathcal{A}$ has enough injectives.) Let $(A_i \mid i \in I)$ be a family of objects of $\mathcal{A}$, and for each consider an injective resolution

$$0 \to A_i \to E_i^\bullet: \qquad 0 \to A_i \to E_i^0 \to E_0^1 \to \cdots \to E_i^n \to \cdots$$

Then the sequence

$$\prod E_i^\bullet: \qquad 0 \to \prod E_i^0 \prod E_i^1 \to \cdots \to \prod E_i^n \to \cdots$$

is exact in degree n for every $n > k$.

Lemma 1.2.2. *Let $\mathcal{A}$ be an abelian category satisfying* AB4.

(1) Let $(A_i \mid i \in I)$ be a filtered system of subobjects of an object A. Then the canonical morphism $\sum A_i \to A$ is a monomorphism.

(2) Let A_i be family of objects. Then the canonical morphism $\coprod A_i \to \prod A_i$ is a monomorphism.

Proof. (1) Denote $j_i \colon : A_i \to \sum A_i$ and $\varphi \colon \sum A_i \to A$ the canonical morphisms. Then $\ker \varphi = \ker \varphi \cap \sum A_i = \sum(\ker \varphi \cap A_i) = \sum \ker(\varphi j_i) = \sum 0 = 0$.

(2) For every finite subset $J \subseteq I$, consider the subcoproduct $\coprod_{i \in J} A_i \simeq \prod_{i \in J} A_i \subseteq \prod_{i \in I} A_i$. Then their sum is $\sum_{J \subseteq I} \coprod_{i \in J} A_i \simeq \coprod_{i \in I} A_i$, and we can apply (1). □

Definition 1.2.3. An abelian category is **Grothendieck** if it satisfies AB5 and it has a generator, i.e. an object G such that $\mathsf{Gen}\, G = \mathcal{A}$.

Theorem 1.2.4. *Let $\mathcal{A}$ be a Grothendieck category. Then in addition to the axioms, it has the following properties:*

1. *$\mathcal{A}$ is **well-powered**, i.e. the subobjects of a given object of $\mathcal{A}$, up to isomorphism, form a set. Equivalently, the quotients of a given object of $\mathcal{A}$, up to isomorphism, form a set.*
2. *$\mathcal{A}$ has enough injectives [24, Théorème 1.10.1]. In fact, it has injective envelopes [75, Proposition V.2.5].*
3. *$\mathcal{A}$ has an injective cogenerator.*
4. *$\mathcal{A}$ is complete [75, Corollary X.4.4].*

Proof. These are standard facts. (1) comes from the existence of a generator [19, Proposition 3.35][1]. (3) follows from (1) and (2) (taking the product of the injective envelopes of the quotients of a generator). □

Example 1.2.5. The standard example for us is the category $\mathsf{Mod}(R)$ of right R-modules, for a ring R. Module categories are precisely the Grothendieck categories with a *progenerator* P, i.e. a small (in the sense that $\mathsf{Hom}_{\mathcal{A}}(P, -)$ commutes with coproducts) projective generator (see e.g. [19, Exercise F, p. 103]).

For an example of a Grothendieck category which is not a module category (without going into algebraic geometry), consider the category $\mathcal{T}$ of torsion abelian groups. It is a localising subcategory of $\mathsf{Mod}(\mathbb{Z})$ (see §1.3), and therefore it is Grothendieck (Lemma 1.3.13). However, it does not have any projective object. Indeed, by the Structure Theorem for abelian groups, its objects are products of finite cyclic groups of the form $\mathbb{Z}/p^n\mathbb{Z}$, for some prime p. If any such product were projective in $\mathcal{T}$, so would be its direct summands $\mathbb{Z}/p^n\mathbb{Z}$; but this is not the case, since there is an epimorphism $\mathbb{Z}/p^{n+1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ from an indecomposable object.

For an example of an abelian category (with a small projective generator) which is not Grothendieck (and therefore not a module category) consider the category $\mathsf{mod}(R)$ of finitely presented R-modules, for a noetherian ring R (see §1.2.3).

[1] Freyd notices on page 84 that the fact that a generator makes an abelian category well-powered is "electrifying".

1.2.3 Local finiteness properties.

Definition 1.2.6. Let $\mathcal{A}$ be a Grothendieck category. An object X of $\mathcal{A}$ is called **finitely generated** if for every direct system $(A_i \mid i \in I)$ of subobjects of an object A of $\mathcal{A}$ the canonical morphism

$$\sum \mathsf{Hom}_{\mathcal{A}}(X, A_i) \to \mathsf{Hom}_{\mathcal{A}}\left(X, \sum A_i\right)$$

is an isomorphism. X is called **finitely presented** if for every direct system $(A_i \mid i \in I)$ of objects of $\mathcal{A}$ the canonical morphism

$$\varinjlim \mathsf{Hom}_{\mathcal{A}}(X, A_i) \to \mathsf{Hom}_{\mathcal{A}}(X, \varinjlim A_i)$$

is an isomorphism. Equivalently, if X if finitely generated and every epimorphism $Y \to X$ with Y finitely generated has finitely generated kernel [75, Proposition V.3.4]. The full subcategory of finitely presented objects of $\mathcal{A}$ is denoted by $\mathsf{fp}(\mathcal{A})$. An object X is called **noetherian** if every ascending chain of subobjects of X is stationary.

Lemma 1.2.7. *For an object X of $\mathcal{A}$,*

$$\textit{noetherian} \Rightarrow \textit{finitely presented} \Rightarrow \textit{finitely generated.}$$

Lemma 1.2.8. *Let $\mathcal{A}$ be a Grothendieck category, and take a short exact sequence*

$$0 \to A \to B \to C \to 0.$$

(1) If B is finitely generated, C is finitely generated.

(2) If A is finitely generated and B is finitely presented, C is finitely presented.

(3) B is noetherian if and only if A and C are noetherian.

Definition 1.2.9. A Grothendieck category $\mathcal{A}$ is called **locally finitely presented** if $\mathsf{fp}(\mathcal{A})$ is skeletally small and $\varinjlim \mathsf{fp}(\mathcal{A}) = \mathcal{A}$, i.e. every object of $\mathcal{A}$ can be written as the direct limit of a direct system of finitely presented objects. $\mathcal{A}$ is called **locally coherent** if it is locally finitely presented and $\mathsf{fp}(\mathcal{A})$ is an abelian subcategory of $\mathcal{A}$. $\mathcal{A}$ is called **locally noetherian** if it is locally finitely presented and it has a set of noetherian generators.

Lemma 1.2.10. *A locally Grothendieck category $\mathcal{A}$ is finitely presented if and only if it has a set of finitely presented generators.*

Proof. $(\Rightarrow)$ Let $\mathcal{S}$ be a set of representatives of the isomorphism classes of the finitely presented objects of $\mathcal{A}$. Then for every object $X \in \mathcal{A}$, we write $X = \varinjlim S_i$ for a direct system $(S_i \mid i \in I)$ with $S_i \in \mathcal{S}$; and we have the canonical epimorphism $\coprod S_i \to \varinjlim S_i = X$, which shows that $\mathcal{S}$ generates X.

($\Leftarrow$) This is classical for module categories. Let $\mathcal{S}$ be a set of finitely presented generators; we first show that every object is the sum of its finitely generated subobjects. For any X in $\mathcal{A}$, and consider an epimorphism $\pi\colon \coprod S_i \to X$, with $S_i \in \mathcal{S}, i \in I$. For every finite subset $J \subseteq I$, let $X_J \subseteq X$ be the image of the finitely presented subcoproduct $\coprod_{i\in J} S_i$. Then the X_J are finitely generated, and $\sum X_J = X$. Now we show that the finitely generated objects form a set, up to isomorphism; and therefore in particular $\mathsf{fp}(\mathcal{A})$ is skeletally small. With the notation above, if X is finitely generated, the identity 1_X factors through some X_J, i.e. $X = X_J$ is a quotient of a finite coproduct $\coprod_{i\in J} S_i$. Since $\mathcal{A}$ is well-powered, such quotients form a set, up to isomorphism. Lastly, to see that $\varinjlim \mathsf{fp}(\mathcal{A}) = \mathcal{A}$, it is enough to show that every finitely generated object is the direct limit of a direct system of finitely presented objects. Let X be finitely generated, and consider an epimorphism $\pi\colon S \to X$ with S finitely presented. Let K be the kernel of π, and write $K = \sum K_i$, with $K_i \subseteq K$ a direct system of finitely generated subobjects. Then the objects S/K_i form a direct system of finitely presented objects, whose direct limit is X. $\square$

Lemma 1.2.11. *In a locally finitely presented Grothendieck category $\mathcal{A}$, every object is the sum of its finitely generated subobjects.*

Proof. A direct limit $\varinjlim X_i$ is the sum of the images of the morphisms $X_i \to \varinjlim X_i$. If the X_i are finitely presented, this images are finitely generated by Lemma 1.2.8(1). $\square$

Example 1.2.12. When $\mathcal{A} = \mathsf{Mod}(R)$ for a ring R, finitely generated and finitely presented coincide with the usual notions. We denote $\mathsf{mod}(R) := \mathsf{fp}(\mathsf{Mod}(R))$. $\mathsf{Mod}(R)$ is always locally finitely presented (by R). It is locally coherent precisely when R is a coherent ring (i.e. finitely generated ideals are finitely presented). It is locally noetherian precisely when R is noetherian.

Lemma 1.2.13. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then*

(1) $\mathcal{A}$ is locally coherent if and only if finitely generated subobjects of finitely presented objects are finitely presented.

(2) if $\mathcal{A}$ is locally noetherian, every finitely generated object is noetherian.

(3) if $\mathcal{A}$ is locally noetherian then it is locally coherent.

Proof. (1) $\mathsf{fp}(\mathcal{A})$ is always closed under extensions and cokernels; this condition is equivalent to closure under kernels, by the equivalent definition of finitely presented objects. (2) Easy; see [75, §V.4]. (3) follows from (1) and (2). $\square$

Proposition 1.2.14 ([75, Proposition V.4.3])**.** *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then $\mathcal{A}$ is locally noetherian if and only if any coproduct of injective objects is again injective.*

1.3 Localisation of abelian categories

In this section we follow Popescu [63, §4]; see also Gabriel [20, §III]. Let $\mathcal{A}$ be a an abelian category.

Definition 1.3.1. A subcategory $\mathcal{B}$ of $\mathcal{A}$ is called **Serre** (*dense* in [63], *épaisse* in [20]) if it is closed under extensions, subobjects and quotients .

Serre subcategories are the kernels of exact functors.

Proposition 1.3.2. *Let $\mathcal{B}$ be a Serre subcategory of $\mathcal{A}$. Then there exists an abelian category $\mathcal{A}/\mathcal{B}$ (called the **Serre quotient**) and an essentially surjective exact functor $T\colon \mathcal{A} \to \mathcal{A}/\mathcal{B}$ (called the **Serre localisation**), such that:*

(1) $\mathcal{B} = \operatorname{Ker}(T)$*;*

(2) for every other abelian category $\mathcal{C}$ and functor $F\colon \mathcal{A} \to \mathcal{C}$, if $\mathcal{B} \subseteq \operatorname{Ker}(F)$ then F factors uniquely through T.

Proof. The construction of the Serre localisation functor T is explained in [63, §4.3]. (1) is a consequence of [63, Lemma §4.3.4], applied to the identity morphisms of objects. (2) is [63, Corollary §4.3.11], which builds on [63, §4.1]. □

Remark 1.3.3. The fact that the morphisms in $\mathcal{A}/\mathcal{B}$ between two given objects form a set is guaranteed by the proof only if $\mathcal{A}$ is well-powered. In this section we will always implicitly assume this to happen, as it does in the cases we are interested in (namely, when $\mathcal{A}$ is a Grothendieck category).

Remark 1.3.4. In particular, notice that, for a morphism $f \in \mathsf{Hom}_{\mathcal{A}}(X,Y)$, Tf vanishes if and only if $\operatorname{im} f \in \mathcal{T}$ ([63, Lemma §4.3.4]), and Tf is an monomorphism (respectively, epimorphism) if and only if $\ker f \in \mathcal{B}$ (respectively, $\operatorname{coker} f \in \mathcal{B}$) ([63, Lemma §4.3.5]).

Definition 1.3.5. A subcategory $\mathcal{B}$ of $\mathcal{A}$ is called **localising** (respectively, **colocalising**) if it is Serre and the Serre localisation functor $\mathcal{A} \to \mathcal{A}/\mathcal{B}$ has a right (respectively left) adjoint.

We will use the following Lemma about adjunction.

Lemma 1.3.6 ([46, Theorem §4.3.1])**.** *Let $(L,R)\colon \mathcal{A} \rightleftarrows \mathcal{C}$ be an adjunction pair. Then R is fully faithful if and only if the counit $LR \Rightarrow \mathsf{id}_{\mathcal{C}}$ is a natural isomorphism. Dually, L is fully faithful if and only if the unit $1_{\mathcal{A}} \Rightarrow RL$ is a natural isomorphism.*

Lemma 1.3.7 ([63, Proposition §4.4.3(1)])**.** *Let $\mathcal{B}$ be a (co)localising subcategory of $\mathcal{A}$, and denote by $T\colon \mathcal{A} \to \mathcal{A}/\mathcal{B}$ the Serre localisation functor and by $S\colon \mathcal{A}/\mathcal{B} \to \mathcal{A}$ its right (respectively, left) adjoint. Then $\mathsf{id}_{\mathcal{A}/\mathcal{B}} \underset{nat}{\simeq} TS$. In particular, T is essentially surjective and S is fully faithful.*

Lemma 1.3.8 ([63, Theorem §4.4.9])**.** *Let $F\colon \mathcal{A} \to \mathcal{A}'$ be an exact functor between abelian categories, with a fully faithful adjoint (either left or right). Then there is an equivalence $\mathcal{A}' \simeq \mathcal{A}/\operatorname{Ker} F$, which identifies F with the Serre quotient functor.*

Definition 1.3.9. A subcategory $\mathcal{B}$ of $\mathcal{A}$ is called:

(1) **reflective** (respectively **coreflective**) if the inclusion $\mathcal{B} \subseteq \mathcal{A}$ has a left (respectively right) adjoint.

(2) **bireflective** if it is both reflective and coreflective.

If $\mathcal{A}$ is Grothendieck, $\mathcal{B}$ is called:

(3) **Giraud** if it is reflective and the left adjoint of the inclusion $\mathcal{B} \subseteq \mathcal{A}$ preserves kernels. In this case $\mathcal{B}$ is also a Grothendieck category [75, Proposition X.1.3].

Remark 1.3.10. Some remarks are in order.

(1) In the notation of Lemma 1.3.7, if $\mathcal{B}$ is (co)localising, then the fully faithful functor S identifies $\mathcal{A}/\mathcal{B}$ with a (co)reflective subcategory $\mathcal{C}$ of $\mathcal{A}$ (whose inclusion has ST as the adjoint). The inverse of the equivalence $S\colon \mathcal{A}/\mathcal{B} \to \mathcal{C}$ is the restriction of T to $\mathcal{C}$. The objects of $\mathcal{C}$ are called **$\mathcal{B}$-local** (respectively, **$\mathcal{B}$-colocal**).

(2) If $\mathcal{A}$ is Grothendieck and $\mathcal{B}$ is localising, $\mathcal{C}$ is a Giraud subcategory, because T is exact, $S\colon \mathcal{A}/\mathcal{B} \to \mathcal{C}$ is an equivalence and therefore the left adjoint ST of the inclusion $\mathcal{C} \subseteq \mathcal{A}$ is (left) exact.

(3) Assume $\mathcal{B}$ is localising, and let $\eta\colon \mathsf{id}_{\mathcal{A}} \Rightarrow ST$ be the unit of the adjunction (T, S). Then $T\eta$ is the natural isomorphism $T \Rightarrow TST$ given by the natural isomorphism $\mathsf{id}_{\mathcal{A}/\mathcal{B}} \simeq ST$. In particular, $\ker \eta_A$ and $\operatorname{coker} \eta_A$ belong to $\mathcal{B}$ for every $A \in \mathcal{A}$.

Lemma 1.3.11. *In the notation of Remark 1.3.10, we have $\mathcal{C} = \mathcal{B}^{\perp_{0,1}}$.*

Proof. ($\subseteq$) Let $B \in \mathcal{B}$ and $C \in \mathcal{C}$, i.e. $C = STX$ for some $X \in \mathcal{A}$. We have $\mathsf{Hom}_{\mathcal{A}}(B, C) = \mathsf{Hom}_{\mathcal{A}}(B, STX) \simeq \mathsf{Hom}_{\mathcal{A}/\mathcal{B}}(TB, TX) = 0$, since $TB = 0$.

Now consider a short exact sequence

$$0 \to C \xrightarrow{f} A \to B \to 0$$

with $A \in \mathcal{A}$. Since T is exact and $TB = 0$, then Tf is an isomorphism. Let $g \in \mathsf{Hom}_{\mathcal{A}/\mathcal{B}}(TA, TC)$ be the inverse, and consider $Sg \in \mathsf{Hom}_{\mathcal{C}}(STA, STC) \simeq \mathsf{Hom}_{\mathcal{C}}(STA, C)$. For the composition $(Sg)f \in \mathsf{Hom}_{\mathcal{C}}(C, C)$, we have that $T((Sg)f) = (TSg)Tf = gTf = \mathsf{id}_{TC}$ is an isomorphism, and therefore $(Sg)f$ is an isomorphism as well. This shows that f splits.

($\supseteq$) Let $A \in \mathcal{B}^{\perp_{0,1}}$: we shall show that $\eta_A \colon A \to STA$ is an isomorphism. Since $\ker \eta_A \in \mathcal{B}$ and $A \in \mathcal{B}^{\perp_0}$, we have that $\ker \eta_A = 0$. Consider then the short exact sequence

$$0 \to A \overset{\eta_A}{\to} STA \to \operatorname{coker} \eta_A \to 0.$$

Since $\operatorname{coker} \eta_A \in \mathcal{B}$ and $A \in \mathcal{B}^{\perp_1}$, this sequence splits, and therefore also $\operatorname{coker} \eta_A = 0$. □

Lemma 1.3.12 ([63, Proposition §4.6.3])**.** *Let $\mathcal{A}$ be an* AB3 *abelian category with injective envelopes. A subcategory $\mathcal{B} \subseteq \mathcal{A}$ is localising if and only if it is Serre and closed under coproducts.*

Lemma 1.3.13 ([63, Corollary §4.6.2])**.** *Let $\mathcal{A}$ be a Grothendieck category, and $\mathcal{B}$ a localising subcategory of $\mathcal{A}$. Then $\mathcal{B}$ and $\mathcal{A}/\mathcal{B}$ are Grothendieck.*

Definition 1.3.14. A **recollement** of abelian categories is a diagram of functors

$$\begin{tikzcd} \mathcal{B} \arrow[r, "i_*"] & \mathcal{A} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{C} \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are abelian categories and the six functors satisfy the following conditions:

(aR1) there are adjoint triples $(i^*, i_*, i^!)$ and $(j_!, j^*, j_*)$;

(aR2) the functors $i_*, j^!, j^*$ are fully faithful;

(aR3) $\operatorname{Im}(i_*) = \operatorname{Ker}(j^*)$.

Remark 1.3.15. (1) By (aR2) we can (and will) identify $\mathcal{B}$ with the bireflective subcategory $\operatorname{Im}(i_*)$ of $\mathcal{A}$.

(2) By Lemma 1.3.6, (aR2) is equivalent to asking that there are isomorphisms $j^*j_! \underset{nat}{\simeq} \mathsf{id}_{\mathcal{C}} \underset{nat}{\simeq} j^*j_*$ and $i^!i_* \underset{nat}{\simeq} \mathsf{id}_{\mathcal{B}} \underset{nat}{\simeq} i^*i_*$.

(3) By (2) and (aR1), j^* is essentially surjective and exact; by (aR3) and Lemma 1.3.8 it is (up to equivalence) the Serre localisation with respect to $\mathcal{B}$. Therefore, $\mathcal{B}$ is also localising and colocalising.

(4) It is clear then that to give a recollement is the same, up to equivalence, as to give a bireflective, localising and colocalising subcategory $\mathcal{B} \subseteq \mathcal{A}$.

1.4 Torsion pairs

Let $\mathcal{A}$ be an abelian category. Dickson [16] introduced the following notion.

Definition 1.4.1. A **torsion pair** in $\mathcal{A}$ is a pair of full subcategories $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ such that

(t1) $\mathsf{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$.

(t2) $\mathcal{T} * \mathcal{F} = \mathcal{A}$.

$\mathcal{T}$ is called the **torsion class**, $\mathcal{F}$ the **torsion-free class**. Their objects are called **torsion** and **torsion-free**, respectively. Given an object X of $\mathcal{A}$, there is a unique short exact sequence

$$0 \to T \to X \to F \to 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$, called the **approximation sequence** of X with respect to $\mathbf{t}$. T and F are called the **torsion** and the **torsion-free parts** of X, respectively. The assignment $X \mapsto T$ extends to a functor $t\colon \mathcal{A} \to \mathcal{T}$, which is the right adjoint to the inclusion $\mathcal{T} \subseteq \mathcal{A}$. The composition $t\colon \mathcal{A} \to \mathcal{T} \subseteq \mathcal{A}$ is called the **torsion radical** associated to $\mathbf{t}$. Similarly, the assignment $X \mapsto F$ extends to a functor $f\colon \mathcal{A} \to \mathcal{F}$, which is the left adjoint to the inclusion $\mathcal{F} \subseteq \mathcal{A}$. The composition $f\colon \mathcal{A} \to \mathcal{T} \subseteq \mathcal{A}$ is called the **torsion-free coradical** associated to $\mathbf{t}$.

Lemma 1.4.2. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$. Then:*

(1) $\mathcal{T}^{\perp} = \mathcal{F}$ and $\mathcal{T} = {}^{\perp}\mathcal{F}$.

(2) $\mathcal{T}$ is closed under extensions, quotients, and existing coproducts. $\mathcal{F}$ is closed under extensions, subobjects and existing products.

(3) $\mathcal{T}$ is closed under existing colimits, $\mathcal{F}$ is closed under existing limits.

(4) If $\mathcal{A}$ is complete, cocomplete and well-powered, torsion and torsion-free classes are characterised by the closure properties of (2) [75, Propositions VI.2.1, VI.2.2].

(5) If $\mathcal{A}$ satisfies AB4*, $\mathcal{F}$ is closed under coproducts.*

Proof. (1) is clear from the definition. (2) and (3) follow from (1). For (5), use Lemma 1.2.2(2) and (2). □

Remark 1.4.3. Let $\mathcal{A}$ be an AB4, well-powered abelian category with injective envelopes (e.g. a Grothendieck category). Then hereditary torsion classes and localising subcategories coincide in $\mathcal{A}$, by Lemma 1.4.2(4) and Lemma 1.3.12. In particular, if $\mathcal{T}$ is a hereditary torsion class, $\mathcal{T}^{\perp_{0,1}}$ is a Giraud subcategory of $\mathcal{A}$, by Lemma 1.3.11.

Proposition 1.4.4 ([16, §3])**.** *Let $\mathcal{A}$ be a complete, cocomplete and well-powered abelian category, and let $\mathcal{S}$ be a class of objects in $\mathcal{A}$. Then there are torsion pairs*

(1) $({}^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp})$, *called* ***generated*** *by* $\mathcal{S}$*; its torsion class is the intersection of all those containing* $\mathcal{S}$.

(2) $({}^{\perp}\mathcal{S}, ({}^{\perp}\mathcal{S})^{\perp})$, *called* ***cogenerated*** *by* $\mathcal{S}$*; its torsion-free class is the intesection of all those containing* $\mathcal{S}$.

Proof. Use Lemma 1.4.2(4). □

Remark 1.4.5. In the notation of the proposition above, let $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be the torsion pair generated by $\mathcal{S}$. Then clearly $\mathsf{Gen}\,\mathcal{S} \subseteq \mathcal{T}$; but this inclusion is strict in general, as $\mathsf{Gen}\,\mathcal{S}$ needs not to be closed under extensions. For example, $\mathsf{Gen}(\mathbb{Z}/2\mathbb{Z})$ inside $\mathsf{Mod}(\mathbb{Z}/4\mathbb{Z})$ does not contain $\mathbb{Z}/4\mathbb{Z}$. Similarly for the cogenerated case.

A torsion pair may satisfy some additional properties.

Definition 1.4.6. Let $\mathcal{A}$ be a Grothendieck category, and $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ a torsion pair in $\mathcal{A}$, with radical t. We call $\mathbf{t}$:

(1) **hereditary** if $\mathcal{T}$ is closed under subobjects. Equivalently, if t is left exact, or if $\mathcal{F}$ is closed under injective envelopes ([75, Propositions VI.3.1 and VI.3.2]). Hereditary torsion classes coincide with the localising subcategories of $\mathcal{A}$ by Lemma 1.3.12.

(2) **of finite type** if $\mathcal{F}$ is closed under direct limits. Since $\mathcal{T}$ is also closed under direct limits, and these are exact, this is equivalent to t commuting with direct limits.

If in addition $\mathcal{A}$ is locally coherent, we call $\mathbf{t}$

(3) **restrictable** if $\mathbf{t} \cap \mathsf{fp}(\mathcal{A}) := (\mathcal{T} \cap \mathsf{fp}(\mathcal{A}), \mathcal{F} \cap \mathsf{fp}(\mathcal{A}))$ is a torsion pair in $\mathsf{fp}(\mathcal{A})$. Equivalently, if the torsion part of a finitely presented object is finitely presented.

Lemma 1.4.7. *Let* $\mathcal{A}$ *be a Grothendieck category. A torsion pair* $\mathbf{t}$ *in* $\mathcal{A}$ *is hereditary if and only if it is cogenerated by an injective object.*

Proof. The proof is essentially the same as in [75, Proposition VI.3.7], (see also [31, Theorem 1.1]), adapted from the case $\mathcal{A} = \mathsf{Mod}(R)$. The implication $(\Leftarrow)$ is untouched. For $(\Rightarrow)$, take, instead of the cyclic modules, the quotients of a generator G of $\mathcal{A}$ (which form a set, because $\mathcal{A}$ is well-powered). The cogenerator is then the product of the injective envelopes of the torsion-free among those quotients. □

We notice the following facts about generating torsion pairs:

Lemma 1.4.8. *Let* $\mathcal{A}$ *be a Grothendieck category,* $\mathcal{S} \subseteq \mathcal{A}$ *a class of objects, and* $\mathbf{t}$ *the torsion pair generated by* $\mathcal{S}$*. Then:*

(1) If $\mathcal{S} \subseteq \mathsf{fp}(\mathcal{A})$*,* $\mathbf{t}$ *is of finite type.*

(2) If $\mathcal{S}$ *is closed under subobjects, then* $\mathbf{t}$ *is hereditary.*

Proof. (1) Trivial. (2) Let $\mathcal{S}$ be closed under subobjects, and let $\bar{\mathcal{S}}$ be its closure under quotients: we claim that $\bar{\mathcal{S}}$ is still closed under subobjects. Indeed, let $S \in \mathcal{S}$ and $\pi\colon S \to X$ be an epimorphism, so that $X \in \bar{\mathcal{S}}$. Given a subobject $X' \subseteq X$, we construct the pullback diagram

$$\begin{tikzcd} S' \arrow[r, dotted, "\pi_{|S'}"] \arrow[d, dotted, hook] & X' \arrow[d, hook] \\ S \arrow[r, "\pi"] & X. \end{tikzcd}$$

S' is a subobject of S, and therefore $S' \in \mathcal{S}$. Since π is an epimorphism, this is also a pushout diagram, and so $\pi_{|S'}$ is also an epimorphism. Therefore $X' \in \bar{\mathcal{S}}$.

Now, it is easy to see that $\mathcal{S}$ and $\bar{\mathcal{S}}$ generate the same torsion pair. Indeed, let $\mathcal{T}$ and $\bar{\mathcal{T}}$ be the smallest torsion classes containing $\mathcal{S}$ and $\bar{\mathcal{S}}$ respectively: since $\mathcal{S} \subseteq \bar{\mathcal{S}}$ then $\mathcal{T} \subseteq \bar{\mathcal{T}}$; and since $\mathcal{T}$ is closed under quotients, $\bar{\mathcal{S}} \subseteq \mathcal{T}$, which shows that $\bar{\mathcal{T}} \subseteq \mathcal{T}$. Now by [75, Proposition VI.3.3] the torsion pair generated by $\bar{\mathcal{S}}$ is hereditary. $\square$

Lemma 1.4.9 ([36, Lemma 2.3])**.** *Let* $\mathcal{A}$ *be a locally coherent Grothendieck category, and let* $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ *be a hereditary torsion pair of finite type in* $\mathcal{A}$*. Then* $\mathbf{t}$ *is generated by* $\mathcal{T} \cap \mathsf{fp}(\mathcal{A})$*. Moreover,* $\mathcal{T} = \varinjlim(\mathcal{T} \cap \mathsf{fp}(\mathcal{A}))$*.*

Lemma 1.4.10 ([14, §4.4])**.** *Let* $\mathcal{A}$ *be a locally coherent Grothendieck category, and let* $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ *be a torsion pair in* $\mathsf{fp}(\mathcal{A})$*. Then the torsion pair in* $\mathcal{A}$ *generated by* $\mathcal{T}$ *is* $\varinjlim \mathbf{t} := (\varinjlim \mathcal{T}, \varinjlim \mathcal{F})$*, and it is of finite type.*

We collect the previous lemmas in the following result.

Proposition 1.4.11. *Let* $\mathcal{A}$ *be a locally coherent Grothendieck category, and* $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ *a torsion pair in* $\mathcal{A}$*. Then:*

(1) If $\mathbf{t}$ *is generated by finitely presented objects, it is of finite type.*

(2) If $\mathbf{t}$ *is either hereditary or restrictable, the converse implication of (1) holds.*

Proof. (1) is Lemma 1.4.8(1). (2) for hereditary is Lemma 1.4.9; if $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ is restrictable, apply Lemma 1.4.10 to $\mathbf{t}' = (\mathcal{T}', \mathcal{F}') := (\mathcal{T} \cap \mathsf{fp}(\mathcal{A}), \mathcal{F} \cap \mathsf{fp}(\mathcal{A}))$, and notice that since $\mathbf{t}$ is of finite type, $\varinjlim \mathcal{T}' \subseteq \mathcal{T}$ and $\varinjlim \mathcal{F}' \subseteq \mathcal{F}$. It is then easy to see that $\mathbf{t} = \varinjlim \mathbf{t}'$, which is generated by $\mathcal{T}'$. $\square$

Proposition 1.4.12. *Let* $\mathcal{A}$ *be a locally noetherian Grothendieck category.*

(1) Every torsion pair in $\mathcal{A}$ *is restrictable.*

(2) *A torsion pair in $\mathcal{A}$ is of finite type if and only if it is generated by finitely presented objects.*

(3) *Every hereditary torsion pair in $\mathcal{A}$ is of finite type.*

Proof. Let $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$.

(1) For every finitely presented (hence noetherian) object, its torsion and torsion-free parts are again finitely presented. Therefore, $\mathbf{t}$ is restrictable. (2) follows from (1) and Proposition 1.4.11. (3) Every object in $\mathcal{A}$ is the sum of its finitely presented subobjects (Lemmas 1.2.11 and 1.2.13(3)). Therefore, if $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ is hereditary, $\mathcal{T}$ is the smallest torsion class containing $\mathcal{T} \cap \mathsf{fp}(\mathcal{A})$, i.e. this class of finitely presented objects generates $\mathbf{t}$. $\square$

Corollary 1.4.13. *Let $\mathcal{A}$ be a locally noetherian Grothendieck category. Then there is a bijection between hereditary torsion pairs in $\mathcal{A}$ and in $\mathsf{fp}(\mathcal{A})$, given by the assignments*

$$\mathbf{t} \textit{ in } \mathcal{A} \mapsto \mathbf{t} \cap \mathsf{fp}(\mathcal{A}) \textit{ in } \mathsf{fp}(\mathcal{A}), \qquad \mathbf{t}' \textit{ in } \mathsf{fp}(\mathcal{A}) \mapsto \varinjlim \mathbf{t}' \textit{ in } \mathcal{A}$$

Proof. First, the assignments are well defined. Indeed, if $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion pair in $\mathcal{A}$, it is restrictable by Proposition 1.4.12(1), and the torsion class $\mathcal{T} \cap \mathsf{fp}(\mathcal{A})$ of the restriction is still closed under subobjects. For the converse, if $\mathbf{t}' = (\mathcal{T}', \mathcal{F}')$ is a hereditary torsion pair in $\mathsf{fp}(\mathcal{A})$, the torsion pair $\varinjlim \mathbf{t}'$ of $\mathcal{A}$ (see Lemma 1.4.10) is generated by $\mathcal{T}'$, which is closed under subobjects, and therefore it is hereditary by Lemma 1.4.8(2).

Now, if we start from $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$, by Proposition 1.4.12(2-3) it is generated by the class of its torsion finitely presented objects $\mathcal{T} \cap \mathsf{fp}(\mathcal{A})$, and therefore it coincides with $\varinjlim(\mathbf{t} \cap \mathsf{fp}(\mathcal{A}))$. Conversely, given $\mathbf{t}' = (\mathcal{T}', \mathcal{F}')$ in $\mathsf{fp}(\mathcal{A})$, we show that $\mathcal{T}' = (\varinjlim \mathcal{T}') \cap \mathsf{fp}(\mathcal{A})$. Indeed, $X \in \mathsf{fp}(\mathcal{A})$ belongs to $\varinjlim \mathcal{T}'$ if and only if $0 = \mathsf{Hom}_{\mathcal{A}}(X, \varinjlim \mathcal{F}') \simeq \varinjlim \mathsf{Hom}_{\mathcal{A}}(X, \mathcal{F}')$, if and only if $X \in \mathcal{T}'$. $\square$

Remark 1.4.14. We will show that Proposition 1.4.12(2) is also true when $\mathcal{A}$ is the (locally finitely presented) heart of a compactly generated t-structure in the derived category of a commutative noetherian ring; see Corollary 3.1.20. On the other hand, those hearts will provide counterexamples to Proposition 1.4.12(3) (see Remark 3.2.14).

Example 1.4.15. (1) In an abelian category $\mathcal{A}$, there are always the trivial torsion pairs $(\mathcal{A}, 0)$ and $(0, \mathcal{A})$.

(2) In the locally noetherian Grothendieck category $\mathsf{Mod}(\mathbb{Z})$, there is the canonical torsion pair $(\mathcal{T}_{\mathsf{can}}, \mathcal{F}_{\mathsf{can}})$ of torsion and torsion-free abelian groups (a particular case of Proposition 3.1.9). It is restrictable (obviously), hereditary, and therefore of finite type, generated by the finite cyclic groups.

(3) Again in $\mathsf{Mod}(\mathbb{Z})$, there is the divisible-reduced torsion pair $(\mathcal{D}, \mathcal{R})$, where $\mathcal{D}$ consists of the groups G such that $nG = G$ for every $n \neq 0$, and $\mathcal{R}$ consists of the groups G such that $\cap_{n\neq 0} nG = 0$. It is not hereditary (as $\mathcal{R} \ni \mathbb{Z} \subseteq \mathbb{Q} \in \mathcal{D}$), and not of finite type (as $\mathcal{D} \ni \mathbb{Q} = \sum_{n\neq 0} \frac{1}{n}\mathbb{Z} \in \varinjlim \mathcal{R}$).

(4) In §3.2.2, we construct an example of a locally coherent Grothendieck category $\mathcal{H}_{\mathbf{t}}$, in which there are

   (a) a hereditary torsion pair which is not of finite type (Remark 3.2.14);

   (b) a hereditary torsion pair of finite type which is not restrictable (in Example 3.3.18, the torsion pair $\mathbf{s}$ is hereditary of finite type by Corollary 3.2.7 but not restrictable by Corollary 3.3.3).

Lemma 1.4.16. *Let $\mathcal{A}$ be a Grothendieck category, G a generator of $\mathcal{A}$ and $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ a torsion pair, with torsion-free coradical f. Then $\mathcal{F} \subseteq \mathsf{Gen}(fG)$.*

Proof. Since coproducts in $\mathcal{A}$ are exact and both $\mathcal{T}$ and $\mathcal{F}$ are closed under coproducts, it is easy to see that the coproduct of approximation sequences is an approximation sequence. Therefore, f commutes with coproducts. Now, let $F \in \mathcal{F}$ and I be a set, such that there is an epimorphism $\pi\colon G^{(I)} \to F$. Then π factors through $f(G^{(I)}) = (fG)^{(I)}$, showing that $F \in \mathsf{Gen}(fG)$. □

We mention more in depth the relation with the last subsection.

Definition 1.4.17. Let $\mathcal{A}$ be Grothendieck category. A **TTF-triple** in $\mathcal{A}$ is a triple of subcategories $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ such that $(\mathcal{C}, \mathcal{T})$ and $(\mathcal{T}, \mathcal{F})$ are torsion pairs.

Remark 1.4.18. If we have a recollement

$$\begin{tikzcd} \mathcal{B} \arrow[r, "i_*" description] & \mathcal{A} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*" description] & \mathcal{C} \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

then $\mathcal{B}$ is the middle class of a TTF-triple in $\mathcal{A}$. For the converse implication, under additional assumptions on $\mathcal{A}$, see [66].

We conclude this section with the lattice of torsion pairs; see e.g [15, 79].

Let $\mathcal{A}$ be a Grothendieck category, and consider the class (which is not a set in general, see [22, Theorem 4.1]) of torsion pairs, which we will denote by $\mathsf{tors}\mathcal{A}$. It can be partially ordered by inclusion of the torsion classes, setting $(\mathcal{T}, \mathcal{F}) \leq (\mathcal{T}', \mathcal{F}')$ if and only if $\mathcal{T} \subseteq \mathcal{T}'$. This partially ordered set is in fact a complete lattice: every subset $\mathcal{S} \subseteq \mathsf{tors}\mathcal{A}$ has a greatest lower bound (its **meet**) $\wedge\mathcal{S}$ and a smallest upper bound $\vee\mathcal{S}$ (its **join**). $\wedge\mathcal{S}$ is defined as the torsion pair having as torsion class the intersection $\cap_{(\mathcal{T},\mathcal{F})\in\mathcal{S}}\mathcal{T}$, while $\vee\mathcal{S}$ is defined as the torsion pair having as torsion-free class the intersection $\cap_{(\mathcal{T},\mathcal{F})\in\mathcal{S}}\mathcal{F}$ (in both cases, using Lemma 1.4.2(4)).

Lemma 1.4.19. *There are two partially ordered subclasses:*

(1) The partially ordered subclass $\mathsf{tors}_h\mathcal{A} \subseteq \mathsf{tors}\mathcal{A}$ *of hereditary torsion pairs is a set, and it is a complete sublattice;*

(2) The partially ordered subclass $\mathsf{tors}_f\mathcal{A} \subseteq \mathsf{tors}\mathcal{A}$ *of torsion pairs of finite type is closed under arbitrary meet.*

Proof. (1) The fact that $\mathsf{tors}_h\mathcal{A}$ is a set was notice already in [31, §1], when $\mathcal{A} = \mathsf{Mod}(R)$ for a ring R. In our situation, a hereditary torsion pair is determined by the torsion-free objects among a set of injectives (see Lemma 1.4.7 and its proof). For a torsion pair to be hereditary is a closure condition, either of the torsion class (under subobjects) or of the torsion-free class (under injective envelopes). It follows that the meet and the join of a set of hereditary torsion pairs is again hereditary.

(2) Clear. □

1.4.1 Torsion pairs in module categories

We now collect some facts about torsion pairs when $\mathcal{A} = \mathsf{Mod}(R)$ is the category of right R-modules, for a ring R. Fix a torsion pair $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ in $\mathsf{Mod}(R)$.

Definition 1.4.20. Consider the torsion approximation sequence for R,

$$0 \to tR \to R \to fR \to 0.$$

The right ideal tR is called the **torsion ideal** of R. $\mathbf{t}$ is called **faithful** if $tR = 0$, i.e. if $R \in \mathcal{F}$.

We want to deduce some properties a right ideal of R must satisfy, in order to be a torsion ideal. We start with a general and easy fact.

Lemma 1.4.21. *Let* $\mathcal{A}$ *be an abelian category,* X *an object and*

$$0 \to tX \to X \to fX \to 0$$

its approximation sequence with respect to a torsion pair in $\mathcal{A}$*. Then any endomorphism of* X *induces an endomorphism of* tX *by restriction.*

Proof. Let $g \in \mathsf{Hom}_{\mathcal{A}}(X, X)$, and consider the restriction $g_{|tX} \colon tX \to X \to X$: the composition $tX \xrightarrow{g_{|tX}} X \to fX$ vanishes, being a morphism from a torsion to a torsion-free; therefore $g_{|tX}$ must factor as $tX \xrightarrow{\bar{g}} tX \to X$. □

This has a useful consequence, already noted by Jans [31], and a curious one.

Corollary 1.4.22. *Let* $\mathbf{t}$ *be any torsion pair in* $\mathsf{Mod}(R)$*. Then*

(1) The torsion ideal tR *is two-sided.*

(2) If k is an R-algebra which is a division ring, it is either torsion or torsion-free.

Proof. (1) tR is closed under endomorphisms of R, i.e. under left multiplication by any element of R; therefore it is a left ideal as well. (2) Similarly, the torsion part tk of k is closed under left multiplication by elements of k, and therefore it is a left k-vector subspace of k. It follows that it is either 0 or the whole k. □

By item (1), the reflection morphism $R \to R/tR$ is a surjective ring epimorphism. The argument above can be modified slightly to prove:

Lemma 1.4.23. *tR is the intersection of all the annihilators of the torsion-free modules. In particular, $\mathcal{F} \subseteq \mathsf{Mod}(R/tR)$ (compare Lemma 1.4.16).*

Proof. For every torsion-free module $F \in \mathcal{F}$ and element $f \in F$, the morphism $f \cdot -: tR \to F$ vanishes, so tR is contained in the annihilator of every torsion-free module. For the converse inclusion, notice that tR is the annihilator of R/tR. □

Torsion ideals are characterised by the following property:

Lemma 1.4.24. *An ideal I is the torsion ideal with respect to a torsion pair if and only if $\mathsf{Hom}_R(I, R/I) = 0$, and with respect to a hereditary torsion pair if and only if $\mathsf{Hom}_R(xR, R/I) = 0$ for every $x \in I$.*

Proof. If I is the torsion ideal with respect to some torsion pair, R/I is torsion-free. Conversely, if $\mathsf{Hom}_R(I, R/I) = 0$, I and R/I are respectively torsion and torsion-free for the torsion pair generated by I. For the hereditary case, one argues similarly, using the set $\{J \leq I\}$ and Lemma 1.4.8(2) to generate a hereditary torsion pair. □

We now focus on hereditary torsion pairs in $\mathsf{Mod}(R)$.

Notation 1.4.25. Given an ideal $\mathfrak{a} \leq R$ and an element $a \in R$, we denote

$$(\mathfrak{a} : a) := \{r \in R \colon a \cdot r \in \mathfrak{a}\},$$

which is a right ideal. We will also consider various kinds of annihilators. For a right R-module M and a subset $S \subseteq M$, we write

$$\mathsf{ann}_r(S) = \{r \in R \colon mr = 0 \text{ for every } m \in S\}.$$

If $S = \{m\}$ we write $\mathsf{ann}_r(m) := \mathsf{ann}_r(\{m\})$. For every set S, $\mathsf{ann}_r(S)$ is a right ideal of R. If moreover $SR = S$, for example if S is a submodule of M, then $\mathsf{ann}_r(S)$ is also a two-sided ideal of R: in this case we write $\mathsf{Ann}_r(S) := \mathsf{ann}_r(S)$. Similary are defined the left ideal $\mathsf{ann}_l(T)$ and the two-sided ideal $\mathsf{Ann}_l(T)$ when N is a left R-module and $T \subseteq N$.

Notice also that when restricted to subsets of R, ann_l and ann_r give a Galois connection: if $S \subseteq T$, we have $\mathsf{ann}_*(S) \supseteq \mathsf{ann}_*(T)$; and for every subset S we have $S \subseteq \mathsf{ann}_r(\mathsf{ann}_l(S))$ and $S \subseteq \mathsf{ann}_l(\mathsf{ann}_r(S))$. In particular, for every $S \subseteq R$, we have $\mathsf{ann}_r(S) = \mathsf{ann}_r(\mathsf{ann}_l(\mathsf{ann}_r(S)))$.

Definition 1.4.26. A **Gabriel filter** on R is a collection $\mathfrak{F}$ of right ideals such that:

(GF1) If $\mathfrak{a} \subseteq \mathfrak{b}$ and $\mathfrak{a} \in \mathfrak{F}$, then $\mathfrak{b} \in \mathfrak{F}$.

(GF2) If $\mathfrak{a}, \mathfrak{b} \in \mathfrak{F}$, then $\mathfrak{a} \cap \mathfrak{b} \in \mathfrak{F}$.

(GF3) If $\mathfrak{a} \in \mathfrak{F}$ and $a \in R$, then $(\mathfrak{a} : a) \in \mathfrak{F}$.

(GF4) If $\mathfrak{b} \in \mathfrak{F}$ and $(\mathfrak{a} : b) \in \mathfrak{F}$ for every $b \in \mathfrak{b}$, then $\mathfrak{a} \in \mathfrak{F}$.

Proposition 1.4.27 ([75, Theorem VI.5.1])**.** *Let R be a ring. There is a bijection between hereditary torsion classes $\mathcal{T}$ in $\mathsf{Mod}(R)$ and Gabriel filters $\mathfrak{F}$ on R, given by the assignments:*

$$\mathcal{T} \mapsto \mathfrak{F} := \{\mathfrak{a} \leq R \colon R/\mathfrak{a} \in \mathcal{T}\}$$
$$\mathfrak{F} \mapsto \mathcal{T} := \{M \in \mathsf{Mod}(R) \colon \mathsf{ann}_r(m) \in \mathfrak{F} \textit{ for every } x \in X\}$$

We can now give a further new characterisation of torsion ideals with respect to a hereditary torsion pair, over noetherian rings.

Proposition 1.4.28. *Let R be a (two-sided-)noetherian ring, and $I \leq R$ a right ideal. Let $\mathbf{t}$ be a hereditary torsion pair, with Gabriel filter $\mathfrak{F}$. Then:*

(1) If I is two-sided, then it is torsion if and only if $\mathsf{Ann}_r(I) \in \mathfrak{F}$.

(2) I is the torsion ideal with respect to $\mathbf{t}$ if and only if $I = \mathsf{Ann}_l[\mathsf{Ann}_r(I)]^2$.

Proof. (1) ($\Rightarrow$) If I is torsion, by Proposition 1.4.27 $\mathsf{ann}_r(x) \in \mathfrak{F}$ for every $x \in I$. Since R is left-noetherian, we can write $I = \sum_{i=1}^n Rx_i$ for finitely many generators $x_i \in I$: it is then easy to see that $\mathsf{Ann}_r(I) = \cap_{i=1}^n \mathsf{ann}_r(x_i) \in \mathfrak{F}$ by (GF2). ($\Leftarrow$) This is a more general fact: if $\mathsf{Ann}_r(I) \in \mathfrak{F}$ then for every $x \in I$ also $\mathsf{Ann}_r(I) \subseteq \mathsf{ann}_r(x) \in \mathfrak{F}$ by (GF1), so I is torsion by Proposition 1.4.27.

(2) Now, assume that I is the torsion ideal for $\mathbf{t}$ (hence, two-sided by Corollary 1.4.22(1)). Then $\mathsf{Ann}_r(I) \in \mathfrak{F}$ by (1), and therefore also $[\mathsf{Ann}_r(I)]^2 \in \mathfrak{F}$ because $\mathfrak{F}$ is closed under products of ideals [75, Lemma 5.3]. Therefore, we also have $[\mathsf{Ann}_r(I)]^2 \subseteq \mathsf{Ann}_r\mathsf{Ann}_l[\mathsf{Ann}_r(I)]^2 \in \mathfrak{F}$ by (GF1). We conclude that $\mathsf{Ann}_l[\mathsf{Ann}_r(I)]^2$ is torsion, and therefore contained in I. The other inclusion is trivial.

Conversely, assume that $I = \mathsf{Ann}_l[\mathsf{Ann}_r(I)]^2$, and consider and element $x \in I$ and a morphism $f \in \mathsf{Hom}_R(xR, R/I)$. Let $r \in R$ be such that $f(x) = r + I$: then $0 = f(x \cdot \mathsf{ann}_r(x)) = f(x) \cdot \mathsf{ann}_r(x) = r \cdot \mathsf{ann}_r(x) + I$, i.e. $r \cdot \mathsf{ann}_r(x) \subseteq I$.

In particular, since $\mathsf{Ann}_r(I) \subseteq \mathsf{ann}_r(x)$, we have $r \cdot \mathsf{Ann}_r(I) \subseteq I$, and therefore $r \cdot [\mathsf{Ann}_r(I)]^2 \subseteq I \cdot \mathsf{Ann}_r(I) = 0$. This shows that $r \in \mathsf{Ann}_l[\mathsf{Ann}_r(I)]^2 = I$, and therefore that $f \equiv 0$. Now use Lemma 1.4.24. $\square$

We recall the following definition.

Definition 1.4.29. Let R be a ring, and $\mathfrak{F}$ a Gabriel filter on R. Denote by $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ the corresponding hereditary torsion pair and by $\mathcal{C} := \mathcal{T}^{\perp_{0,1}}$ the corresponding Giraud subcategory (see §1.3). $\mathfrak{F}$ and $\mathbf{t}$ are called **perfect** if $\mathcal{C}$ is coreflective (in addition to being reflective).

Remark 1.4.30. [75, Proposition XI.3.4] lists several conditions which are equivalent to this definition (which appears as item (b)). To ease the parsing of the notation, which is different from the one used here, we provide a dictionary. The ring is A; denote by $T\colon \mathsf{Mod}(A) \to \mathsf{Mod}(A)/\mathcal{T}$ the Serre localisation functor, and by $S\colon \mathcal{T} \to \mathsf{Mod}(A)$ its right adjoint, so that $\mathcal{C} = \mathrm{Im}(S)$.

(1) $\mathcal{A}_{\mathfrak{F}}$ is the target of the unit morphism $A \to A_{\mathfrak{F}}$ of the adjunction (T, S); it is shown that it is a ring. Moreover, $\mathcal{C} \simeq \mathsf{Mod}(A)/\mathcal{T}$ is naturally a full subcategory of $\mathsf{Mod}(A_{\mathfrak{F}})$, via the functor j.

(2) $\mathsf{Mod}(A, \mathfrak{F}) := \mathcal{C}$.

(3) We have functors $a := ST\colon \mathsf{Mod}(A) \to \mathsf{Mod}(A, \mathfrak{F})$ and $q = ja\colon \mathsf{Mod}(A) \to \mathsf{Mod}(A_{\mathfrak{F}})$.

Lemma 1.4.31 ([61, Lemma 4.10])**.** *Let R be a ring and let $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be a perfect torsion pair in $\mathsf{Mod}(R)$. Then $\mathcal{T}^{\perp_{0,1}} = \mathcal{T}^{\perp_{\geq 0}}$.*

Proof. Since $\mathbf{t}$ is hereditary, $\mathcal{F} = \mathcal{T}^{\perp}$ is closed under injective envelopes, so $\mathcal{T}^{\perp_{0,1}}$ is as well. Recall that $\mathcal{T}^{\perp_{0,1}} \simeq \mathsf{Mod}(R)/\mathcal{T}$ is abelian. Since $\mathbf{t}$ is perfect, the inclusion functor $\mathcal{T}^{\perp_{0,1}} \hookrightarrow \mathsf{Mod}(R)$ has a right adjoint: therefore, cokernels taken in $\mathcal{T}^{\perp_{0,1}}$ coincide with those in $\mathsf{Mod}(R)$. This shows that $\mathcal{T}^{\perp_{0,1}}$ is closed under cokernels in $\mathsf{Mod}(R)$. It follows that $\mathcal{T}^{\perp_{0,1}}$ is closed under cosygyzies, and one concludes by dimension shifting. $\square$

Chapter 2

Triangulated categories

2.1 Triangulated categories

A modern introduction to triangulated categories can be found in Chapter 1 of Neeman's book [56].

Definition 2.1.1. Let $\mathcal{D}$ be an additive category, and $\Sigma\colon \mathcal{D} \to \mathcal{D}$ an autoequivalence. A **candidate triangle** in $\mathcal{D}$ is a diagram

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

such that the compositions vu, wv and $(\Sigma u)w$ vanish. A diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle \Sigma a} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

with candidate triangles as its rows is called a **morphism of triangles** between its rows, and it is an isomorphism if a, b, c are.

Definition 2.1.2. A **triangulated category** is an additive category $\mathcal{D}$ together with an autoequivalence $\Sigma\colon \mathcal{D} \to \mathcal{D}$ (called the **suspension functor**, with its inverse Σ^{-1} called the **cosuspension functor**) and a family of **(distinguished) triangles**, i.e. a family of candidate triangles satisfying the following axioms:

(TR0) Candidate triangles isomorphic to distinguished triangles are distinguished. The candidate triangles of the form

$$X \xrightarrow{1_X} X \longrightarrow 0 \longrightarrow \Sigma X$$

are distinguished.

(TR1) Every morphism $X \xrightarrow{u} Y$ can be completed to a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X.$$

(TR2) A candidate triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is distinguished if and only if the shifted candidate triangle

$$Y \xrightarrow{-v} Z \xrightarrow{-w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$$

is distinguished.

(TR3) Every commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "a"] & Y \arrow[r] \arrow[d, "b"] & Z \arrow[r] & \Sigma X \arrow[d, "\Sigma a"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'
\end{tikzcd}$$

with distinguished triangles as rows can be completed to a morphism of triangles

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "a"] & Y \arrow[r] \arrow[d, "b"] & Z \arrow[r] \arrow[d, "c"] & \Sigma X \arrow[d, "\Sigma a"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'.
\end{tikzcd}$$

(TR4) Given composable morphisms $X \xrightarrow{u} Y \xrightarrow{v} Z$, there exists a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "v"] & A \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "vu"] \arrow[d] & Z \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & \Sigma X \arrow[d] \\
0 \arrow[r] \arrow[d] & C \arrow[r, equal] \arrow[d] & C \arrow[r] \arrow[d] & 0 \arrow[d] \\
\Sigma \arrow[r, "\Sigma u"] & \Sigma Y \arrow[r] & \Sigma A \arrow[r] & \Sigma^2 X
\end{tikzcd}$$

whose rows and columns are distinguished triangles.

A triangulated category $\mathcal{D}$ may satisfy the following additional axioms:

(TR5) the coproduct in $\mathcal{D}$ of every family of objects of $\mathcal{D}$ exists.

(TR5*) the product in $\mathcal{D}$ of every family of objects of $\mathcal{D}$ exists.

Remark 2.1.3. In Axiom (TR1), the object Z will be called a **cone of** f. Notice that it is not required to be unique. Similarly, in Axiom (TR3) the morphism c is not required to be unique. Using the Four Lemma (see Lemma 2.1.5

below), it is an exercise to prove that cones of a morphism are indeed unique, but only up to non unique isomorphism; in other words, the cone construction is not functorial.

Axiom (TR4) is called the *Octahedron Axiom*, and the diagram it features, an *octahedron diagram*, because of another way of writing it, which resembles this solid. An equivalent axiom called (TR4') was introduced by Neeman [52] (see also [56, §1.3]).

Notation 2.1.4. Let $\mathcal{D}$ be a triangulated category. Given subcategories $\mathcal{X}, \mathcal{Z} \subseteq \mathcal{D}$, we will denote by $\mathcal{X} * \mathcal{Z}$ the subcategory of objects Y for which there is a triangle

$$X \to Y \to Z \to \Sigma X$$

with $X \in \mathcal{X}$ and $Z \in \mathcal{Z}$. Given a set of indices $I \subseteq \mathbb{Z}$, we will write

$$\begin{aligned} \mathcal{X}^{\perp_I} &:= \{Y \in \mathcal{D} \colon \mathsf{Hom}_{\mathcal{D}}(X, \Sigma^i Y) = 0 \text{ for every } i \in I\} \\ {}^{\perp_I}\mathcal{X} &:= \{Y \in \mathcal{D} \colon \mathsf{Hom}_{\mathcal{D}}(Y, \Sigma^i X) = 0 \text{ for every } i \in I\} \end{aligned}$$

with the same conventions as in Notation 1.4.25. This notation for the left orthogonal is standard, chosen to be consistent with the one for Ext-orthogonals (see §2.3).

Lemma 2.1.5 (Four Lemma [56, Proposition 1.1.20])**.** *Consider a morphism of distinguished triangles*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\alpha"] & Y \arrow[r] \arrow[d, "\beta"] & Z \arrow[r] \arrow[d, "\gamma"] & \Sigma X \arrow[d, "\Sigma\alpha"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'. \end{tikzcd}$$

If two of α, β and γ are isomorphisms, then so is the third.

Lemma 2.1.6 (Verdier's 3×3 Lemma [8, Proposition 1.1.11])**.** *For any commutative diagram whose solid rows and columns are distinguished triangles*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dotted] & \Sigma X \arrow[d] \\ X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d, dotted] & \Sigma X' \arrow[d] \\ X'' \arrow[r, dotted] \arrow[d] & Y'' \arrow[r, dotted] \arrow[d] & \square \arrow[r, dotted] \arrow[d, dotted] & \Sigma X'' \arrow[d] \\ \Sigma X \arrow[r] & \Sigma Y \arrow[r] & \Sigma Z \arrow[r] & \Sigma^2 X \end{tikzcd}$$

there is a choice of an object Z'' to fill in the blank entry and of dotted morphisms so that the dotted row and column are distinguished triangles.

Remark 2.1.7. If one starts with the commutative square in the top left corner, while the cones X'', Y'', Z, Z' can be freely chosen, not every dotted morphism given by (T3) can be used to fill the diagram (for a counterexample see [52, Example 2.6]).

Lemma 2.1.8 ([56, dual of Proposition 1.2.1])**.** *Let $\mathcal{D}$ be a triangulated category, and $(X_i \to Y_i \to Z_i \to \Sigma X_i \mid i \in I)$ be a family of distinguished triangles, such that there exist the coproducts $\coprod X_i, \coprod Y_i, \coprod Z_i$ in $\mathcal{D}$. Then there is a distinguished triangle*

$$\coprod X_i \to \coprod Y_i \to \coprod Z_i \to \Sigma \coprod X_i.$$

2.1.1 Compactly generated and well-generated triangulated categories

Definition 2.1.9. Let $\mathcal{D}$ be a triangulated category. An object C in $\mathcal{D}$ is called **compact** if for every family $(X_i \mid i \in I)$ of objects whose coproduct in $\mathcal{D}$ exists, the canonical homomorphism of abelian groups

$$\coprod \mathsf{Hom}_{\mathcal{D}}(C, X_i) \to \mathsf{Hom}_{\mathcal{D}}(C, \coprod X_i)$$

is an isomorphism. The subcategory of $\mathcal{D}$ consisting of compact objects is denoted by $\mathcal{D}^c$.

Lemma 2.1.10. *Let $\mathcal{D}$ be a triangulated category. Then $\mathcal{D}^c$ is a **thick** subcategory of $\mathcal{D}$, i.e. a triangulated subcategory closed under direct summands.*

Proof. Easy. $\square$

Definition 2.1.11. A TR5 triangulated category $\mathcal{D}$ is called **compactly generated** if $\mathcal{D}^c$ is skeletally small and the following equivalent conditions hold:

(1) $\mathcal{D}$ is its own smallest triangulated subcategory closed under coproducts and containing $\mathcal{D}^c$;

(2) $(\mathcal{D}^c)^\perp = 0$.

Remark 2.1.12. It is rather easy to see that (1) $\Rightarrow$ (2); for the converse implication, see §2.5.1 (in particular Remark 2.5.20).

This classical notion is generalised by the following, introduced by Neeman in [56] (see also [57]) and reworked by Krause in [38].

Definition 2.1.13. Let $\mathcal{D}$ be a triangulated category, and α a cardinal. An object S of $\mathcal{D}$ is **α-small** if, for every family objects $(X_i \mid i \in I)$ whose coproduct in $\mathcal{D}$ exists, every morphism $S \to \coprod_I X_i$ factors through a subcoproduct $S \to \coprod_J X_i \subseteq \coprod_I X_i$, with $J \subseteq I$ of cardinality $< \alpha$.

Remark 2.1.14. Compact objects are the $\aleph_0$-small objects.

Definition 2.1.15 ([56, Definition 1.15] and [38, Theorem A])**.** A TR5 triangulated category $\mathcal{D}$ is called **well-generated** if there exists a set $\mathcal{S}$ of objects such that:

(1) there is a cardinal α for which the objects of $\mathcal{S}$ are α-small;

(2) $\mathcal{S}^{\perp} = 0$;

(3) for every set of morphisms $(f_i \colon X_i \to Y_i \mid i \in I)$ and object $S \in \mathcal{S}$, if the induced morphisms $\mathsf{Hom}_{\mathcal{D}}(S, f_i) \colon \mathsf{Hom}_{\mathcal{D}}(S, X_i) \to \mathsf{Hom}_{\mathcal{D}}(S, Y_i)$ are all surjective, then the induced morphism $\mathsf{Hom}_{\mathcal{D}}(S, \coprod f_i) \colon \mathsf{Hom}_{\mathcal{D}}(S, \coprod X_i) \to \mathsf{Hom}_{\mathcal{D}}(S, \coprod Y_i)$ is surjective.

Remark 2.1.16. In [56], the cardinal α is required to be *regular*. In [57, Remark 5.4] it is noted that this can be assumed freely, up to substituting α with its successor cardinal, which is automatically regular. This is also apparent from the characterisation of [38], which we used as the definition above, since α-small implies β-small for every $\beta > \alpha$.

While many natural triangulated categories are at least well-generated, there is the following source of counterexamples.

Proposition 2.1.17 ([56, Corollaries 1.18 and E.1.3])**.** *Let $\mathcal{D}$ be a compactly generated triangulated category. Then its opposite category $\mathcal{D}^{op}$ is* TR5 *but it is not well-generated.*

Remark 2.1.18. The proof of the referenced result [56, Corollary E.1.3] simplifies when $\mathcal{D}$ is compactly generated (and not α-compactly generated, for $\alpha > \aleph_0$). Indeed, in this case there is a theory of *purity* (see e.g. Krause's [37]), and there are *pure-injective* objects in $\mathcal{D}$. Without going into definitions, which we will not need, we record the following characterisation:

Proposition 2.1.19 ([37, Theorem 1.8])**.** *An object X in $\mathcal{D}$ is pure-injective if and only if for every set I the canonical morphism $C^{(I)} \to C$ factors through the canonical morphism $C^{(I)} \to C^I$.*

A consequence of this is that any pure-injective object in $\mathcal{D}$ is not α-small in $\mathcal{D}^{op}$, for any α: indeed, the morphism $\sigma_\alpha \in \mathsf{Hom}_{\mathcal{D}}(C^\alpha, C) = \mathsf{Hom}_{\mathcal{D}^{op}}(C, \coprod_\alpha C)$ factoring $C^{(\alpha)} \to C$ does not factor through less than α components by construction. For an example of pure-injective objects, in the derived category of a ring (§2.3) every bounded cosilting complex (see Definition 2.5.37) is pure-injective ([47, Proposition 3.10], which uses [76]).

Another counterexample is the following:

Proposition 2.1.20 ([56, Appendix E.3])**.** *The homotopy category of abelian groups $\mathsf{K}(\mathbb{Z})$ (see §2.3) is not well-generated, and neither is $\mathsf{K}(\mathbb{Z})^{op}$.*

2.1.2 Triangulated functors

Definition 2.1.21. Let $(\mathcal{D}, \Sigma), (\mathcal{D}', \Sigma')$ be triangulated categories. A **triangulated functor** is a pair $(F, \eta)\colon \mathcal{D} \to \mathcal{D}'$ consisting of an additive functor F and a natural transformation $\eta\colon F\Sigma \Rightarrow \Sigma' F$, such that for every

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

distinguished triangle of $\mathcal{D}$ there is a distinguished triangle of $\mathcal{D}'$

$$FX \xrightarrow{Fu} FY \xrightarrow{Fv} FZ \xrightarrow{\eta_X \circ Fw} \Sigma' FX.$$

Lemma 2.1.22 ([56, Lemma 5.3.6])**.** *Let $(F, \omega)\colon \mathcal{D} \to \mathcal{D}'$ be a triangulated functor, and $G\colon \mathcal{D}' \to \mathcal{D}$ a left (respectively, right) adjoint. Then (G, ζ) is triangulated, with ζ given by the adjunction. In particular, the quasi-inverse of a triangulated equivalence is triangulated.*

We will often drop the natural transformation η from the notation, when it is understood.

Lemma 2.1.23 (Double dévissage)**.** *Let $F\colon \mathcal{D} \to \mathcal{D}'$ be a triangulated functor, and let $\mathcal{S}$ be a subcategory of $\mathcal{D}$ such that $\Sigma\mathcal{S} = \mathcal{S}$ and F is fully faithful on $\mathcal{S}$. Then F is fully faithful on the smallest triangulated subcategory of $\mathcal{D}$ containing $\mathcal{S}$. If F preserves coproducts and both $\mathcal{S}$ and $F\mathcal{S}$ consist of compact objects, then F is fully faithful on the smallest triangulated subcategory of $\mathcal{D}$ containing $\mathcal{S}$ and closed under existing coproducts.*

Proof. This is a standard argument. Let $\bar{\mathcal{S}}$ be the smallest triangulated subcategory of $\mathcal{D}$ containing $\mathcal{S}$. For every objects $X, Y \in \mathcal{D}$, denote by $F_{X,Y}$ the morphism $\mathsf{Hom}_{\mathcal{D}}(X, Y) \to \mathsf{Hom}_{\mathcal{D}'}(FX, FY)$ induced by F. Let $\mathcal{Y}$ be the subcategory of $\mathcal{D}$ of objects Y such that $F_{S,Y}$ is an isomorphism for every $S \in \mathcal{S}$. Since $\Sigma\mathcal{S} = \mathcal{S}$, then $\Sigma\mathcal{Y} = \mathcal{Y}$; and $\mathcal{Y}$ is closed under extensions by the Five Lemma. Lastly, $\mathcal{S} \subseteq \mathcal{Y}$ by hypothesis, and therefore $\bar{\mathcal{S}} \subseteq \mathcal{Y}$. Now, let $\mathcal{X}$ be the subcategory of $\mathcal{D}$ of objects for which $F_{X,Y}$ is an isomorphism for every $Y \in \mathcal{Y}$; again, it is triangulated and $\mathcal{S} \subseteq \mathcal{X}$ by construction. Therefore, $\bar{\mathcal{S}} \subseteq \mathcal{X} \cap \mathcal{Y}$, i.e. F is fully faithful on $\bar{\mathcal{S}}$. Now assume moreover that F preserves coproducts and $\mathcal{S}, F\mathcal{S}$ consist of compact objects. In the notation above, $\mathcal{Y}$ is closed under existing coproducts: if $(Y_i \mid i \in I)$ is a set of objects of $\mathcal{Y}$ whose coproduct exists in $\mathcal{D}$, for every $S \in \mathcal{S}$ we have

$$\begin{tikzcd}
\mathsf{Hom}_{\mathcal{D}}(S, \coprod Y_i) \arrow[r, "F_{S,\coprod Y_i}"] \arrow[d, "\simeq"] & \mathsf{Hom}_{\mathcal{D}'}(FS, \coprod FY_i) \arrow[d, "\simeq"] \\
\coprod \mathsf{Hom}_{\mathcal{D}}(S, Y_i) \arrow[r, "\coprod F_{S,Y_i}", "\simeq"'] & \coprod \mathsf{Hom}_{\mathcal{D}'}(FS, Y_i).
\end{tikzcd}$$

On the other hand, $\mathcal{X}$ is automatically closed under existing coproducts, and we conclude. $\square$

Lemma 2.1.24. *Let $F\colon \mathcal{D} \to \mathcal{D}'$ be a triangulated functor. If F is full, then its essential image is a triangulated subcategory of $\mathcal{D}'$.*

Proof. Let $\mathcal{S}$ denote the essential image of F, as a (full) subcategory of $\mathcal{D}'$. Since F commutes with the suspension functors, $\Sigma'\mathcal{S} = \mathcal{S}$, so we only need to show closure under extensions. Take a distinguished triangle in $\mathcal{D}'$

$$FX \longrightarrow Y' \longrightarrow FZ \xrightarrow{w'} \Sigma' FX$$

with $X, Z \in \mathcal{D}$. Since $w' \in \mathsf{Hom}_{\mathcal{D}'}(FZ, \Sigma' FX) \simeq \mathsf{Hom}_{\mathcal{D}'}(FZ, F\Sigma X)$, by fullness of F we get $w' = Fw$, for some $w \in \mathsf{Hom}_{\mathcal{D}}(Z, \Sigma X)$. This in turns gives a distinguished triangle in $\mathcal{D}$

$$X \longrightarrow Y \longrightarrow Z \xrightarrow{w} \Sigma X.$$

Applying F to it, we obtain a diagram with distinguished triangles of $\mathcal{D}'$ as rows

$$\begin{tikzcd} FX \arrow[r] \arrow[d, equal] & FY \arrow[r] & FZ \arrow[r, "Fw"] \arrow[d, equal] & \Sigma' FX \arrow[d, equal] \\ FX \arrow[r] & Y' \arrow[r] & FZ \arrow[r, "w'"] & \Sigma' FX \end{tikzcd}$$

and by (TR3) and the Four Lemma this shows that $Y' \simeq FY \in \mathcal{S}$. □

Example 2.1.25. Notice that in general the image of a triangulated functor is not closed under extensions: for a counterexample, take any abelian category $\mathcal{A}$ with a subcategory $\mathcal{B}$ which is an abelian category, such that the inclusion $\mathcal{B} \subseteq \mathcal{A}$ is exact but $\mathcal{B}$ is not an abelian subcategory of $\mathcal{A}$ (i.e. it is not closed under extensions). A concrete choice is $\mathsf{Mod}(\mathbb{Z}/2\mathbb{Z}) \subseteq \mathsf{Mod}(\mathbb{Z}/4\mathbb{Z})$. Then this inclusion extends to a triangulated inclusion $\mathsf{D}(\mathcal{B}) \subseteq \mathsf{D}(\mathcal{A})$, which is not full, and whose image is not closed under extensions (otherwise $\mathcal{B}$ would be in $\mathcal{A}$).

Also, not every triangulated functor whose image is a triangulated subcategory is full: a counterexample is the localisation functor $\mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$, which is clearly essentially surjective but not full. This is expected: compare though with Proposition 2.6.11(3).

2.2 Localisation of triangulated categories

We now present the triangulated version of §1.3. Our main reference is Neeman's [56, §2.1]; see also Krause's [42]. Let $\mathcal{D}$ be a triangulated category.

Definition 2.2.1. A subcategory $\mathcal{S}$ of $\mathcal{D}$ is called **thick** (also *saturé* [80], *épaisse* [54]) if it is triangulated and closed under direct summands.

Thick subcategories are the kernels of triangulated functors.

Proposition 2.2.2. *Let $\mathcal{S}$ be a thick subcategory of $\mathcal{D}$. Then there exists a triangulated category $\mathcal{D}/\mathcal{S}$ (called the* ***Verdier quotient****) and an essentially surjective triangulated functor $Q\colon \mathcal{D} \to \mathcal{D}/\mathcal{S}$ (called the* ***Verdier localisation****) such that:*

(1) $\mathcal{S} = \mathrm{Ker}(Q)$;

(2) for every other triangulated category $\mathcal{C}$ and triangulated functor $F\colon \mathcal{D} \to \mathcal{C}$, if $\mathcal{S} \subseteq \mathrm{Ker}(F)$ then F factors uniquely through Q.

Proof. This result goes back to Verdier's thesis [80, §II.2.2.10]; see [56, Theorem 2.1.8]. □

Remark 2.2.3. The triangulated "category" $\mathcal{D}/\mathcal{S}$ is not guaranteed to be locally small, in general; i.e. the morphisms between two given objects may form a proper class. For example, Freyd ([19, Exercise A, p. 131]) constructed an abelian category whose derived category (see §2.3) exhibits this pathology. One way to make sure this does not happen is by showing that the localisation functor Q admits an adjoint, on either side (see Remark 2.2.9).

Definition 2.2.4. Let $\mathcal{D}$ be a TR5 (respectively, TR5*) triangulated category. A subcategory $\mathcal{S}$ of $\mathcal{D}$ is **localising** (respectively, **colocalising**), if it is closed under coproducts (respectively, closed under products). A localising (respectively, colocalising) subcategory $\mathcal{S}$ is **smashing** (respectively, **cosmashing**) if also $\mathcal{S}^{\perp}$ (respectively, ${}^{\perp}\mathcal{S}$) is localising (respectively, colocalising).

Definition 2.2.5. A subcategory $\mathcal{S}$ of $\mathcal{D}$ is called:

(1) **reflective** (respectively, **coreflective**) if the inclusion $\mathcal{S} \subseteq \mathcal{D}$ has a left (respectively, right) adjoint;

(2) **bireflective** if it is both reflective and coreflective.

Lemma 2.2.6. *Let $\mathcal{D}$ be a* TR5 *(respectively* TR5**) triangulated category. A coreflective (respectively, reflective) subcategory $\mathcal{S}$ of $\mathcal{D}$ is localising (respectively, colocalising).*

Proof. Easy: if $\mathcal{S}$ is coreflective, given a family of objects $(X_i \mid i \in I)$ of $\mathcal{S}$, the coreflection of the coproduct $\coprod X_i$ in $\mathcal{D}$ is the coproduct in $\mathcal{S}$. But then the inclusion $\mathcal{S} \subseteq \mathcal{D}$, being a left adjoint, preserves coproducts, and therefore $\coprod X_i$ is its own coreflection, i.e. it belongs to $\mathcal{S}$. Similarly for the dual case. □

Proposition 2.2.7 ([42, Proposition 4.9.1])**.** *Let $\mathcal{D}$ be a triangulated category and $\mathcal{S}$ thick subcategory of $\mathcal{D}$. Then the following are equivalent:*

(1) $\mathcal{S}$ is coreflective (respectively, reflective);

(2) the Verdier localisation functor $Q\colon \mathcal{D} \to \mathcal{D}/\mathcal{S}$ admits a right (respectively, left) adjoint, which is then fully faithful;

(3) $\mathcal{S} * \mathcal{S}^{\perp}$ (respectively ${}^{\perp}\mathcal{S} * \mathcal{S}$) is the whole $\mathcal{D}$.

Proof. We mention that the full faithfulness of the adjoint in (2) is proved in [42, Proposition 3.2.3], which is used to construct the adjoint in the proof of the referenced result. ☐

Corollary 2.2.8. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{S}$ a thick subcategory of $\mathcal{D}$. If $\mathcal{S}$ is coreflective (respectively, reflective) then $\mathcal{S}^{\perp}$ (respectively, ${}^{\perp}\mathcal{S}$) is reflective (respectively coreflective).*

Proof. Use item (3) of the proposition. ☐

Remark 2.2.9. If $Q\colon \mathcal{D} \to \mathcal{D}/\mathcal{S}$ admit a right (respectively, left) adjoint, then $\mathcal{D}/\mathcal{S}$ automatically has Hom-sets. Indeed, assume for example that Q has a right adjoint F. Then for every $Y = QX, Y' = QX'$ in $\mathcal{D}/\mathcal{S}$, we have $\mathsf{Hom}_{\mathcal{D}/\mathcal{S}}(QX, QX') \simeq \mathsf{Hom}_{\mathcal{D}}(X, FQX')$, which is a set.

Remark 2.2.10. In the literature, coreflective thick (and therefore localising, by Lemma 2.2.6) subcategories are sometimes called **Bousfield localising** ([42, 54]).

Remark 2.2.11. We spell out where the parallel with the abelian situation breaks: the triangulated case is simpler. Let $\mathcal{A}$ be an abelian category with $\mathcal{B} \subseteq \mathcal{A}$ a Serre subcategory, and $\mathcal{D}$ a triangulated category with $\mathcal{S}$ a thick subcategory. Implication $(*)$ holds if $\mathcal{A}$ is Grothendieck.

$$(\mathcal{B} \subseteq \mathcal{A} \text{ has a right adjoint}) \Rightarrow (\mathcal{B} \text{ is } \coprod\text{-closed}) \overset{(*)}{\Rightarrow} (\mathcal{A} \to \mathcal{A}/\mathcal{B} \text{ has a right adjoint})$$

$$(\mathcal{D} \to \mathcal{D}/\mathcal{S} \text{ has a right adjoint}) \Leftrightarrow (\mathcal{S} \subseteq \mathcal{D} \text{ has a right adjoint}) \Rightarrow (\mathcal{S} \text{ is } \coprod\text{-closed}).$$

As an analgous to implication $(*)$ in the triangulated setting, we have the following result; compare with Proposition 2.5.22.

Proposition 2.2.12 ([42, Theorem 5.1.1(2)])**.** *Let $\mathcal{D}$ be a* TR5 *triangulated category, and $\mathcal{S}$ be a localising subcategory of $\mathcal{D}$, which is compactly generated: i.e. it is the smallest localising subcategory of $\mathcal{D}$ containing a given set of compact objects. Then $\mathcal{S}$ is coreflective.*

Proof. Apply the referenced result to the inclusion $\mathcal{S} \subseteq \mathcal{D}$, which is triangulated because $\mathcal{S}$ is a triangulated subcategory, and preserves coproducts because $\mathcal{S}$ is localising. If $\mathcal{S}$ is compactly generated, it is in particular *perfectly generated.* ☐

Definition 2.2.13 ([39])**.** A **(co)localising sequence** is a diagram of functors

$$\begin{tikzcd} \mathcal{S} \arrow[r, "F"] & \mathcal{D} \arrow[r, "G"] \arrow[l, bend left, dotted, "F_\rho"] & \mathcal{R} \arrow[l, bend left, dotted, "G_\rho"] \end{tikzcd} \qquad \text{(respectively} \quad \begin{tikzcd} \mathcal{S} \arrow[r, "F"] & \mathcal{D} \arrow[r, "G"] \arrow[l, bend right, dotted, "F_\lambda"'] & \mathcal{R} \arrow[l, bend right, dotted, "G_\lambda"'] \end{tikzcd} \text{)}$$

such that

(1) there are adjunction pairs $(F, F_\rho), (G, G_\rho)$ (respectively, $(F_\lambda, F), (G_\lambda, G)$);

(2) F and G_ρ (respectively G_λ) are fully faithful;

(3) $\mathrm{Im}(F) = \mathrm{Ker}(G)$.

Remark 2.2.14. (1) Axiom (2) is equivalent to having natural isomorphisms $\mathsf{id}_{\mathcal{S}} \underset{nat}{\simeq} F_\rho F$ and $GG_\rho \underset{nat}{\simeq} \mathsf{id}_{\mathcal{R}}$ (respectively, $F_\lambda F \underset{nat}{\simeq} \mathsf{id}_{\mathcal{S}}$ and $\mathsf{id}_{\mathcal{R}} \underset{nat}{\simeq} GG_\lambda$). Then G is triangulated, full and essentially surjective, and therefore it is equivalent to the Verdier localisation of $\mathcal{D}$ with respect to $\mathrm{Ker}(G) = \mathrm{Im}(F)$.

(2) In view of Proposition 2.2.7, to give a localising (respectively colocalising) sequence is then the same as giving a coreflective (respectively, reflective) thick subcategory $\mathcal{S}$ of $\mathcal{D}$.

(3) Notice moreover that there is a duality between localising and colocalising sequences:

$$\begin{tikzcd} \mathcal{S} \arrow[r, "F"] & \mathcal{D} \arrow[l, dotted, bend left, "F_\rho"] \arrow[r, "G"] & \mathcal{R} \arrow[l, dotted, bend left, "G_\rho"] \end{tikzcd} \text{ is localising} \Leftrightarrow \begin{tikzcd} \mathcal{R} \arrow[r, "G_\rho"] & \mathcal{D} \arrow[l, dotted, bend right, "G"'] \arrow[r, "F_\rho"] & \mathcal{S} \arrow[l, dotted, bend right, "F"'] \end{tikzcd} \text{ is colocalising.}$$

The notion of recollement for triangulated categories was introduced by Beĭlinson, Bernstein and Deligne [8] (before the corresponding abelian notion).

Definition 2.2.15. A **recollement** of triangulated categories is a diagram of functors

$$\begin{tikzcd} \mathcal{S} \arrow[r, "i_*"] & \mathcal{D} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{R} \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

where $\mathcal{S}, \mathcal{D}$ and $\mathcal{R}$ are triangulated categories and the six functors are triangulated and satisfy the following conditions:

(tR1) there are adjoint triples $(i^*, i_*, i^!)$ and $(j_!, j^*, j_*)$;

(tR2) the functors $i_*, j^!, j^*$ are fully faithful;

(tR3) $\mathrm{Im}(i_*) = \mathrm{Ker}(j^*)$.

Remark 2.2.16. Notice that a recollement is just a localising and colocalising sequence $\mathcal{R} \to \mathcal{D} \to \mathcal{S}$.

Definition 2.2.17. Two recollements $\mathcal{R} \mathrel{\substack{\leftarrow \\ \rightarrow \\ \leftarrow}} \mathcal{D} \mathrel{\substack{\leftarrow \\ \rightarrow \\ \leftarrow}} \mathcal{S}$ and $\mathcal{R}' \mathrel{\substack{\leftarrow \\ \rightarrow \\ \leftarrow}} \mathcal{D}' \mathrel{\substack{\leftarrow \\ \rightarrow \\ \leftarrow}} \mathcal{S}'$ are **equivalent** if there are triangulated equivalences $F : \mathcal{D} \to \mathcal{D}'$ and $G : \mathcal{S} \to \mathcal{S}'$ such that the diagram

$$\begin{tikzcd} \mathcal{D} \arrow[r, "j^*"] \arrow[d, "F"', "\cong"] & \mathcal{S} \arrow[d, "G"', "\cong"] \\ \mathcal{D}' \arrow[r, "{j^*}'"] & \mathcal{S}' \end{tikzcd}$$

is commutative (up to a natural equivalence). Clearly, if this is the case the composition $\mathcal{R} \subseteq \mathcal{D} \xrightarrow{F} \mathcal{D}'$ factors through a functor $H : \mathcal{R} \to \mathcal{R}'$, which is a triangulated equivalence; and all of the six possible squares corresponding to the six different functors from the definition of recollement commute.

We will say more on recollements when we will have introduced t-structures: see Definition 2.5.11.

2.3 Categories of complexes

We now recall some standard material on various categories of complexes; for reference see e.g. Kashiwara and Schapira [33, §1.3-1.7] and Weibel [82, §1-2].

2.3.1 Category of cochain complexes

Let $\mathcal{A}$ be an abelian category. We will denote by $\mathsf{C}(\mathcal{A})$ the category of cochain complexes over $\mathcal{A}$ and cochain maps

$$X^\bullet = (X^i, d_X^i) = (\cdots \to X^i \xrightarrow{d_X^i} X^{i+1} \to \cdots) \qquad f^\bullet = (f^i \colon X^i \to Y^i) \colon X^\bullet \to Y^\bullet$$

$X_i \in \mathcal{A}$, $f_i \in \mathsf{Hom}_{\mathcal{A}}(X_i, Y_i)$, which is abelian. As usual, $\mathsf{C}^+(\mathcal{A})$ (respectively, $\mathsf{C}^-(\mathcal{A})$) will denote the subcategory of complexes such that $X^i = 0$ for every $i \ll 0$ (respectively, $i \gg 0$), which we will call (strictly) bounded below (respectively, above); $\mathsf{C}^b(\mathcal{A}) = \mathsf{C}^+(\mathcal{A}) \cap \mathsf{C}^-(\mathcal{A})$ is the category of (strictly) bounded complexes. They are all abelian categories; if $\mathcal{A}$ is (co)complete, so is $\mathsf{C}(\mathcal{A})$, with (co)limits computed componentwise. If $\mathcal{E} \subseteq \mathcal{A}$ is an exact subcategory (i.e. closed under extensions), we will denote by $\mathsf{C}^*(\mathcal{E})$, with $* \in \{\emptyset, +, -, b\}$, the subcategory of $\mathsf{C}^*(\mathcal{A})$ of complexes with terms in $\mathcal{E}$.

Example 2.3.1. (1) Let $(\mathcal{A}_i \mid i \in I)$ be a family of abelian categories, and let $\mathcal{A}$ be their cartesian product, which is again an abelian category. Then $\mathsf{C}(\mathcal{A})$ is the cartesian product of the $\mathsf{C}(\mathcal{A}_i)$.

(2) Let $\mathcal{A}$ be an abelian category, and I a small category. Then $\mathcal{A}^I$, the category of diagrams of shape I in $\mathcal{A}$, is abelian, and $\mathsf{C}(\mathcal{A}^I) \simeq \mathsf{C}(\mathcal{A})^I$.

Definition 2.3.2 ([82, 1.2.6])**.** Let $\mathcal{A}$ be a cocomplete abelian category. Consider the category $\mathsf{C}(\mathsf{C}(\mathcal{A}))$ of bicomplexes and morphisms

$$X_\bullet^\bullet = \begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \\ \cdots \arrow[r] & X_j^i \arrow[r, "d_j^i"] \arrow[d, "\delta_j^i"] & X_j^{i+1} \arrow[r] \arrow[d, "\delta_j^{i+1}"] & \cdots \\ \cdots \arrow[r] & X_{j-1}^i \arrow[r, "d_{j-1}^i"] \arrow[d] & X_{j-1}^{i+1} \arrow[r] \arrow[d] & \cdots \\ & \vdots & \vdots & \end{tikzcd} \qquad f_\bullet^\bullet = (f_j^i \colon X_j^i \to Y_j^i) \colon X_\bullet^\bullet \to Y_\bullet^\bullet$$

Given a bicomplex $X_\bullet^\bullet$, its **($\prod$-)totalisation** is the complex $\mathsf{Tot}(X_\bullet^\bullet)$ whose terms are defined by $\mathsf{Tot}(X_\bullet^\bullet)^n = \prod_{i-j=n} X_j^i$ and differentials are induced by the morphisms $d_j^i + (-1)^j \delta_j^i$. Similarly, its $\coprod$-totalisation is the complex $\mathsf{Tot}^{\coprod}(X_\bullet^\bullet)$ defined using coproducts.

The **shift functor** $-[1]$ is the autoequivalence of $\mathsf{C}(\mathcal{A})$ defined by $(X^i, d^i)[1] = (X^{i+1}, -d^{i+1})$ on complexes and by $(f^i)[1] = (f^{i+1})$ on morphisms. We write $-[n] := -[1]^n$, for $n \in \mathbb{Z}$. The **cone** of a morphism $f \colon X \to Y$ is the complex $\mathsf{cone}(f) := \mathsf{Tot}(f[1])$. There are degree-wise split morphisms $Y \to \mathsf{cone}(f)$ and $\mathsf{cone}(f) \to X[1]$. For a complex $X = (X^i, d^i)$, we denote its (complex) cohomologies as $H^i X = \ker d^i / \operatorname{im} d^{i-1}$. We will call X **acyclic** if $H^i X = 0$ for every $i \in \mathbb{Z}$. A morphism $f \colon X \to Y$ induces morphisms $H^i f \colon H^i X \to H^i Y$. We will call f a **quasi-isomorphism** if $H^i f$ is an isomorphism for every $i \in \mathbb{Z}$, or equivalently if $\mathsf{cone}(f)$ is acyclic.

Lemma 2.3.3 ([33, Proposition 14.1.3])**.** *If $\mathcal{A}$ is a Grothendieck category, then $\mathsf{C}(\mathcal{A})$ is a Grothendieck category.*

2.3.2 Homotopy categories

Given complexes X, Y and morphism $f, g \colon X \to Y$, a **homotopy** $s \colon f \Rightarrow g$ is a (not necessarily commutative) diagram

$$\begin{tikzcd} \cdots \arrow[r] & X^{i-1} \arrow[r] \arrow[dl, "s^{i-1}"'] & X^i \arrow[r] \arrow[dl, "s^i"'] & X^{i+1} \arrow[r] \arrow[dl, "s^{i+1}"'] & \cdots \arrow[dl, "s^{i+2}"'] \\ \cdots \arrow[r] & Y^{i-1} \arrow[r] & Y^i \arrow[r] & Y^{i+1} \arrow[r] & \cdots \end{tikzcd}$$

such that $f - g = sd_X + d_Y s$. We write $f \sim g$ if there exists a homotopy $f \Rightarrow g$; this defines an equivalence relation. A morphism $f \colon X \to Y$ is **nullhomotopic** if $f \sim 0$. A morphism $f \colon X \to Y$ is a **homotopy equivalence** if there exists a morphism $g \colon Y \to X$ such that $gf \sim 1_X$ and $fg \sim 1_Y$. A complex is contractible if and only if its identity is nullhomotopic; more generally, a morphism is nullhomotopic if and only if it factors through a contractible complex. A morphism is a homotopy equivalence if and only if its cone is contractible.

The **homotopy category** $\mathsf{K}(\mathcal{A})$ of $\mathcal{A}$ has the same objects as $\mathsf{C}(\mathcal{A})$, and Hom-sets defined by $\mathsf{Hom}_{\mathsf{K}(\mathcal{A})}(X,Y) := \mathsf{Hom}_{\mathsf{C}(\mathcal{A})}(X,Y)/\sim$. Notice that given an additive functor $F\colon \mathsf{C}(\mathcal{A}) \to \mathcal{C}$, for an additive category $\mathcal{C}$, F induces a functor $F\colon \mathsf{K}(\mathcal{A}) \to \mathcal{C}$ as soon as $F(f) = 0$ for every nullhomotopic morphism f, or equivalently, as soon as $F(C) = 0$ for every contractible complex C. In particular, given an additive functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories, it induces additive functors $F\colon \mathsf{C}(\mathcal{A}) \to \mathsf{C}(\mathcal{B})$ and $F\colon \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{B})$. Similarly, the shift functor $\mathsf{C}(\mathcal{A}) \to \mathsf{C}(\mathcal{A})$ induces a shift functor $\mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{A})$. $\mathsf{K}(\mathcal{A})$, with this functor and the family of distinguished triangles isomorphic to triangles of the form

$$X \xrightarrow{f} Y \to \mathsf{cone}(f) \to X[1]$$

is a triangulated category. Functors $F\colon \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{B})$ as above are triangulated. If $\mathcal{A}$ is cocomplete and it has coproducts, since (co)products of contractible complexes are again contractible, $\mathsf{K}(\mathcal{A})$ is TR5 (respectively, TR5*). As usual, $\mathsf{K}^*(\mathcal{A})$, with $* \in \{b,+,-\}$, denotes the triangulated subcategory with the same objects as $\mathsf{C}^*(\mathcal{A})$.

Definition 2.3.4. For $X, Y \in \mathsf{C}(\mathcal{A})$, consider the double complex $\mathcal{H}om^{\bullet;\bullet}_{\mathcal{A}}(X,Y)$

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathsf{Hom}_{\mathcal{H}}(X^j, Y^i) & \longrightarrow & \mathsf{Hom}_{\mathcal{H}}(X^j, Y^{i+1}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathsf{Hom}_{\mathcal{H}}(X^{j-1}, Y^i) & \longrightarrow & \mathsf{Hom}_{\mathcal{H}}(X^{j-1}, Y^{i+1}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & &
\end{array}$$

Assume that $\mathcal{A}$ is complete; we write $\mathcal{H}om_{\mathcal{A}}(X,Y) := \mathsf{Tot}(\mathcal{H}om^{\bullet;\bullet}_{\mathcal{A}}(X,Y))$ for the totalisation of this bicomplex. It gives an additive bifunctor $\mathsf{C}(\mathcal{A})^{op} \times \mathsf{C}(\mathcal{A}) \to \mathsf{C}(\mathbb{Z})$. Since it sends contractible complexes in contractible complexes, it also induces a (triangulated) bifunctor $\mathsf{K}(\mathcal{A})^{op} \times \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathbb{Z})$.

Lemma 2.3.5 ([82, §2.7.5])**.** *Let $\mathcal{A}$ be a complete abelian category. Then for every complexes X, Y over $\mathcal{A}$ we have $H^n(\mathcal{H}om_{\mathcal{A}}(X,Y)) \simeq \mathsf{Hom}_{\mathsf{K}(\mathcal{A})}(X, Y[n])$, for every $n \in \mathbb{Z}$.*

2.3.3 Derived categories

The subcategory of $\mathsf{K}(\mathcal{A})$ consisting of acyclic complexes is denoted by $\mathsf{K}_{\mathsf{ac}}(\mathcal{A})$; it is a thick subcategory. The **derived category** of $\mathcal{A}$ is the Verdier quotient $\mathsf{D}(\mathcal{A}) := \mathsf{K}(\mathcal{A})/\mathsf{K}_{\mathsf{ac}}(\mathcal{A})$. The Verdier localisation functor will be denoted by

$Q\colon \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$. Recall that $\mathsf{D}(\mathcal{A})$ may not be a locally small category (see Remark 2.2.3); recall also that to solve this problem it is enough to provide an adjoint to the localisation functor (Remark 2.2.9). We give the following definition, due to Spaltenstein [74].

Definition 2.3.6. Let $\mathcal{A}$ be an abelian category. A complex X over $\mathcal{A}$ is **homotopically injective** (originally *K-injective* [74]) if for every acyclic complex C the complex $\mathcal{H}om_{\mathcal{A}}(C, X)$ is acyclic; equivalently, if $\mathsf{Hom}_{\mathsf{K}(\mathcal{A})}(C, X) = 0$ for every acyclic complex C. Dually, X is **homotopically projective** (or *K-projective*) if for every acyclic complex C the complex $\mathcal{H}om_{\mathcal{A}}(X, C)$ is acyclic; equivalently, if $\mathsf{Hom}_{\mathsf{K}(\mathcal{A})}(X, C) = 0$ for every acyclic complex C.

Lemma 2.3.7. *We list some properties of the subcategories $\mathsf{K}_{\mathsf{ac}}(\mathcal{A})^{\perp}$ of homotopically injectives and ${}^{\perp}\mathsf{K}_{\mathsf{ac}}(\mathcal{A})$ of homotopically projectives:*

(1) $\mathsf{K}^{+}(\mathsf{Inj}(\mathcal{A})) \subseteq \mathsf{K}_{\mathsf{ac}}(\mathcal{A})^{\perp} \subseteq \mathsf{K}(\mathsf{Inj}(\mathcal{A}))$; $\mathsf{K}^{-}(\mathsf{Proj}(\mathcal{A})) \subseteq {}^{\perp}\mathsf{K}_{\mathsf{ac}}(\mathcal{A}) \subseteq \mathsf{K}(\mathsf{Proj}(\mathcal{A}))$;

(2) they are triangulated subcategory;

(3) $\mathsf{K}_{\mathsf{ac}}(\mathcal{A})^{\perp}$ *it is closed under existing products,* ${}^{\perp}\mathsf{K}_{\mathsf{ac}}(\mathcal{A})$ *is closed under existing coproducts.*

Given a complex X, a **homotopically injective resolution** of X is a quasi-isomorphism $X \to I$, with I homotopically injective. If every complex admits a homotopically injective resolution, we say that $\mathsf{K}(\mathcal{A})$ has **enough homotopically injective objects**. Dually, a **homotopically projective resolution** of X is a quasi-isomorphism $P \to X$, with P homotopically projective. If every complex admits a homotopically projective resolution, we say that $\mathsf{K}(\mathcal{A})$ has **enough homotopically projective objects**.

Proposition 2.3.8. *Let $\mathcal{A}$ be an abelian category. Then $\mathsf{K}(\mathcal{A})$ has enough homotopically injective objects as soon as one of the following holds:*

(1) $\mathcal{A}$ is Grothendieck [33, Corollary 14.1.8].

(2) $\mathcal{A}$ is AB4* *and it has enough injectives [9, Application 2.4'],[40, Lemma §5.1].*

We are interested in homotopically injective and projective resolutions for the following reason.

Lemma 2.3.9. *Let $\mathcal{A}$ be an abelian category. Then the following are equivalent:*

(1) $\mathsf{K}(\mathcal{A})$ has enough homotopically injective (respectively, projective) objects;

(2) the Verdier localisation functor $Q\colon \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ admits a right (respectively, left) adjoint.

If this is the case, given a complex X, the right (respectively, left) adjoint of Q applied to QX coincides with a (fixed) homotopically injective (respectively, projective) resolution of X.

Proof. Since a morphism in $\mathsf{C}(\mathcal{A})$ is a quasi-isomorphism if and only if its (co)cone is acyclic, the existence of homotopically injective (respectively, projective) resolutions of every object of $\mathsf{K}(\mathcal{A})$ can be restated by saying that $\mathsf{K}_{\mathsf{ac}}(\mathcal{A}) * \mathsf{K}_{\mathsf{ac}}(\mathcal{A})^{\perp} = \mathsf{K}(\mathcal{A})$ (respectively, ${}^{\perp}\mathsf{K}_{\mathsf{ac}}(\mathcal{A}) * \mathsf{K}_{\mathsf{ac}}(\mathcal{A}) = \mathsf{K}(\mathcal{A})$). Now apply Proposition 2.2.7 (respectively, its dual). $\square$

As we said, if $\mathcal{A}$ is (co)complete, then $\mathsf{K}(\mathcal{A})$ is TR5 (respectively TR5*). This is not necessarily the case for $\mathsf{D}(\mathcal{A})$. Nonetheless, as soon as $Q\colon \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ admits a (say, right) adjoint, then $\mathsf{D}(\mathcal{A}) \simeq \mathsf{K}_{\mathsf{ac}}(\mathcal{A})^{\perp}$; and this subcategory, being reflective, is closed under products in $\mathsf{K}(\mathcal{A})$, and it has coproducts, computed by reflecting those of $\mathsf{K}(\mathcal{A})$. This means that products and coproducts in $\mathsf{D}(\mathcal{A})$ exist, and they are computed by taking the componentwise products and coproducts of homotopically injective resolutions. This computation simplifies under additional hypotheses.

Lemma 2.3.10. *Let $\mathcal{A}$ be a* AB4 *(respectively,* AB4**) abelian category. Then* $\mathsf{D}(\mathcal{A})$ *is* TR5 *(respectively,* TR5**), and (co)products in* $\mathsf{D}(\mathcal{A})$ *are computed componentwise.*

Proof. This is essentially [56, Corollary 3.2.11], once we notice that if (co)products in $\mathcal{A}$ are exact, then $\mathsf{K}_{\mathsf{ac}}(\mathcal{A})$ is closed under (co)products in $\mathsf{K}(\mathcal{A})$. $\square$

We denote by $\mathsf{D}^*(\mathcal{A})$, for $* \in \{b, +, -\}$, the essential image under Q of the subcategory $\mathsf{K}^*(\mathcal{A})$. They are triangulated subcategories of $\mathsf{D}(\mathcal{A})$.

There is the following relation between certain morphisms in $\mathsf{D}(\mathcal{A})$ and Yoneda extensions in $\mathcal{A}$.

Proposition 2.3.11. *For every $X, Y \in \mathcal{A}$ and $n \geq 0$, there is a binatural isomorphism*

$$\theta\colon \mathsf{Ext}^n_{\mathcal{A}}(X,Y) \underset{nat}{\simeq} \mathsf{Hom}_{\mathsf{D}(\mathcal{A})}(X, Y[n])$$

where X, Y are identified with complexes concentrated in degree 0.

2.3.4 Derived functors

Let $F\colon \mathcal{A} \to \mathcal{B}$ an additive functor between abelian categories. We have seen that it induces an additive functor $F\colon \mathsf{C}(\mathcal{A}) \to \mathsf{C}(\mathcal{A})$ and a triangulated functor $F\colon \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{A})$, computed componentwise. If we assume F to be exact, then it sends acyclic complexes to acyclic complexes, and therefore it induces a triangulated functor $F\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$, again computed componentwise. This functor fits in a commutative diagram

$$\begin{tikzcd} \mathsf{K}(\mathcal{A}) \arrow[r, "F"] \arrow[d, "Q"] & \mathsf{K}(\mathcal{B}) \arrow[d, "Q"] \\ \mathsf{D}(\mathcal{A}) \arrow[r, "F"] & \mathsf{D}(\mathcal{B}) \end{tikzcd}$$

If $F\colon \mathcal{A} \to \mathcal{B}$ is not exact, then such a functor $F\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ cannot be constructed. It is possible, though, to try to "approximate" it, from the left and from the right.

Definition 2.3.12. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories.

(1) A **left derived functor** of F is a triangulated functor $\mathbb{L}F\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ with a natural transformation $\eta\colon \mathbb{L}FQ \Rightarrow QF$, such that for every other triangulated functor $G\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ and natural transformation $\psi\colon GQ \Rightarrow QF$ there exists a unique natural transformation $\gamma\colon G \Rightarrow \mathbb{L}F$ such that $\psi = \eta\gamma_Q$.

(2) A **right derived functor** of F is a triangulated functor $\mathbb{R}F\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ with a natural transformation $\varepsilon\colon QF \Rightarrow \mathbb{R}FQ$, such that for every other triangulated functor $G\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ and natural transformation $\psi\colon QF \Rightarrow GQ$ there exists a unique natural transformation $\gamma\colon \mathbb{R}F \Rightarrow G$ such that $\psi = \gamma_Q\varepsilon$.

Remark 2.3.13. If F is exact, then the induced functor $F\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ is both a left and a right derived functor of F.

Derived functors can be constructed using adjoints of $Q\colon \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$, if they exist.

Proposition 2.3.14. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor.*

(1) Assume that $Q\colon \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ has a left adjoint $p\colon \mathsf{D}(\mathcal{A}) \to \mathsf{K}(\mathcal{A})$. Then the composition $QFp\colon \mathsf{D}(\mathcal{A}) \to \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{B}) \to \mathsf{D}(\mathcal{B})$ is a left derived functor of F. The natural transformation $(QFp)Q \Rightarrow QF$ is $QF\eta$, where $\eta\colon pQ \Rightarrow \mathsf{id}_{\mathsf{K}(\mathcal{A})}$ is the counit of the adjunction.

(2) Assume that $Q\colon \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ has a right adjoint $i\colon \mathsf{D}(\mathcal{A}) \to \mathsf{K}(\mathcal{A})$. Then the composition $QFi\colon \mathsf{D}(\mathcal{A}) \to \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{B}) \to \mathsf{D}(\mathcal{B})$ is a right derived functor of F. The natural transformation $QF \Rightarrow (QFi)Q$ is $QF\varepsilon$, where $\varepsilon\colon \mathsf{id}_{\mathsf{K}(\mathcal{A})} \Rightarrow iQ$ is the unit of the adjunction.

Proof. We prove (1); (2) is dual. Given $G\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ with a natural transformation $\psi\colon GQ \Rightarrow QF$, we need to construct uniquely the natural transformation $\gamma\colon G \Rightarrow QFp$. Notice that $Qp \underset{\text{nat}}{\simeq} \mathsf{id}_{\mathsf{D}(\mathcal{A})}$: γ is then defined as $\psi_p\colon G \underset{\text{nat}}{\simeq} GQp \Rightarrow QFp$. $\square$

Left derived functors are usually constructed for right (respectively, left) exact functors $F\colon \mathcal{A} \to \mathcal{B}$. This is because of the following relation.

Proposition 2.3.15. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left (respectively, right) exact functor between abelian categories. Let $\mathbb{R}F\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ (respectively, $\mathbb{R}F\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$) be a right (respectively, left) derived functor for F. Then the following diagram commutes:*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "-[0]"'] & \mathcal{B} \\ \mathsf{D}(\mathcal{A}) \arrow[r, "\mathbb{R}F"] & \mathsf{D}(\mathcal{B}) \arrow[u, "H^0"'] \end{tikzcd} \qquad \textit{(respectively,} \qquad \begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "-[0]"'] & \mathcal{B} \\ \mathsf{D}(\mathcal{A}) \arrow[r, "\mathbb{L}F"] & \mathsf{D}(\mathcal{B}) \arrow[u, "H^0"'] \end{tikzcd} \;).$$

2.3.5 Coderived categories

Let $\mathcal{A}$ be a Grothendieck category. Inside $\mathsf{K}(\mathcal{A})$ there is the subcategory $\mathsf{K}(\mathsf{Inj}(\mathcal{A}))$ of complexes with injective terms. We give the following definition (see [78, Definition 6.7], and [6, 64] for the original definitions).

Definition 2.3.16. A complex X over $\mathcal{A}$ is **coacylic** if the complex $\mathcal{H}\mathrm{om}_{\mathcal{A}}(X, I)$ is acyclic for every $I \in \mathsf{C}(\mathsf{Inj}(\mathcal{A}))$. Equivalently, if $\mathsf{Ext}^1_{\mathsf{C}(\mathcal{A})}(X, I) = 0$ for every $I \in \mathsf{C}(\mathsf{Inj}(\mathcal{A}))$; equivalently, if $\mathsf{Hom}_{\mathsf{K}(\mathcal{A})}(X, I) = 0$ for every $I \in \mathsf{K}(\mathsf{Inj}(\mathcal{A}))$. The subcategory of $\mathsf{K}(\mathcal{A})$ consisting of coacyclic complexes is denoted by $\mathsf{K}_{\mathsf{coac}}(\mathcal{A})$.

Proposition 2.3.17 ([43, Corollary 7 and Example 5])**.** *The colocalising subcategory $\mathsf{K}(\mathsf{Inj}(\mathcal{A})) \subseteq \mathsf{K}(\mathcal{A})$ is reflective. The localising subcategory $\mathsf{K}_{\mathsf{coac}}(\mathcal{A}) \subseteq \mathsf{K}(\mathcal{A})$ is coreflective.*

Proof. For the claim about $\mathsf{K}_{\mathsf{coac}}(\mathcal{A})$ apply Corollary 2.2.8. □

We will denote by $I_\lambda\colon \mathsf{K}(\mathsf{Inj}(\mathcal{A})) \to \mathsf{K}(\mathcal{A})$ the reflection.

Definition 2.3.18 (Becker [6])**.** Let $\mathcal{A}$ be a Grothendieck category. The **coderived category** $\mathsf{D}^{\mathsf{co}}(\mathcal{A})$ of $\mathcal{A}$ is defined as the Verdier quotient $\mathsf{K}(\mathcal{A})/\mathsf{K}_{\mathsf{coac}}(\mathcal{A})$. It is equivalent to $\mathsf{K}(\mathsf{Inj}(\mathcal{A}))$, via the functor induced by I_λ.

Remark 2.3.19. There is a different definition of a coderived category in the literature, which is due to Positselski [64]. The two definitions are known to coincide in many situations, for example if the underlying Grothendieck category is locally noetherian [64, §3.7], but it seems to be an open question even for module categories whether they coincide in general (see e.g. [65, Example 2.5(3)]). However, as we will see in Corollary 4.2.7, for the locally Grothendieck categories we are most interest in, that is the hearts of intermediate restrictable t-structure over commutative noetherian rings, the two definitions of a coderived category are indeed equivalent, and so there is no need to distinguish them.

Coacyclic complexes are in particular acyclic (otherwise they would have a non-zero morphism to the injective envelopes of their non-zero cohomologies), so there is a localisation $\mathsf{D}^{\mathsf{co}}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$. The interplay between the functors appearing in this subsection is illustrated by the following commutative diagram of functors.

$$
\begin{tikzcd}
\mathsf{K}(\mathcal{A}) \arrow[r, "I_\lambda"] \arrow[rr, bend left, "Q"] \arrow[dr] & \mathsf{K}(\mathsf{Inj}(\mathcal{A})) \arrow[r, "Q"] \arrow[d, "\simeq"] & \mathsf{D}(\mathcal{A}). \\
& \mathsf{D}^{\mathsf{co}}(\mathcal{A}) \arrow[ur] &
\end{tikzcd}
$$

2.4 Derivators and homotopy (co)limits

Let $\mathcal{A}$ be a complete and cocomplete abelian category, and $\mathsf{D}(\mathcal{A})$ its derived category. While it is possible for $\mathsf{D}(\mathcal{A})$ to have categorical limits and colimits (e.g., when $\mathcal{A} = \mathsf{Vect}_k$, $\mathsf{D}(\mathcal{A}) \simeq \mathsf{Vect}_k^{(\mathbb{N})}$), what is more interesting are the notions of homotopy (co)limits. If they exist, these are the right (respectively left) derived functors of the functors $(\mathsf{co})\mathsf{lim}_I \colon \mathcal{A}^I \to \mathcal{A}$, for every small category I (compare with Remark 1.1.8). In particular, notice that if $\mathcal{A}$ has exact direct limits, then homotopy colimits in $\mathcal{A}$ are computed componentwise (see §2.3.4).

A more general way to define homotopy (co)limits is using the language of derivators, which we now briefly present. For an in-depth treatment see Groth's paper [23].

Notation 2.4.1. As it is customary, we will denote by Cat the 2-category of small categories, functors and natural transformations; I, J will denote small categories. Similarly, CAT will be the 2-"category" of all categories, functors and natural transformations (which is not a 2-category, since functors between two arbitary categories to not usually form a set). The bold natural number $\mathbf{n}$ will denote the partially ordered set $\{0 < 1 < \cdots < n-1\}$, viewed as a small category. In particular, $\mathbf{1}$ is the category with one object and only the identity morphism. Other small categories we will use are

$$
\ulcorner := \left\{ \begin{matrix} \bullet & \to & \bullet \\ \downarrow & & \\ \bullet & & \end{matrix} \right\}, \qquad \lrcorner := \left\{ \begin{matrix} & & \bullet \\ & & \downarrow \\ \bullet & \to & \bullet \end{matrix} \right\}, \qquad \square := \left\{ \begin{matrix} \bullet & \to & \bullet \\ \downarrow & & \downarrow \\ \bullet & \to & \bullet \end{matrix} \right\}
$$

There are obvious functors $i_\ulcorner \colon \ulcorner \to \square$ and $i_\lrcorner \colon \lrcorner \to \square$.

Definition 2.4.2. A **prederivator** is a strict 2-functor $\mathbb{D} \colon \mathsf{Cat}^{op} \to \mathsf{CAT}$. The category $\mathbb{D}(\mathbf{1})$ is called the **base** of $\mathbb{D}$.

Example 2.4.3. We present two motivating examples of prederivators:

(1) $I \mapsto \mathcal{A}^I$, for a complete and cocomplete abelian category $\mathcal{A}$;

(2) $\mathbb{D}_{\mathcal{A}} := \mathbb{D}_{\mathsf{C}(\mathcal{A})} \colon I \mapsto \mathsf{D}(\mathcal{A}^I)$, for a Grothendieck category $\mathcal{A}$.

The intuition we draw from these examples is that objects of $\mathbb{D}(I)$ should be considered **coherent diagrams** of shape I with terms in the base $\mathbb{D}(\mathbf{1})$, as opposed (in item (2)) to the **incoherent diagrams** in $\mathbb{D}(\mathbf{1})^I$. The functors $i\colon \mathbf{1} \to I$ corresponding to every object $i \in I$ give functors $\mathbb{D}(I) \to \mathbb{D}(\mathbf{1})$, and together they induce a functor $\mathsf{dia}_I \colon \mathbb{D}(I) \to \mathbb{D}(\mathbf{1})^I$, which assigns to each coherent diagram its **underlying** incoherent diagram.

Given a functor $u\colon I \to J$ between small categories, a prederivator $\mathbb{D}$ gives a functor $u^* := \mathbb{D}(u)\colon \mathbb{D}(J) \to \mathbb{D}(I)$. A derivator is a prederivator satisfying the additional axioms (Der1–Der4) below, which guarantee that these u^* have both a left and a right adjoint (Der3), called **Kan extensions**, and give formulae to compute them (Der4). Of particular interest is the case when $u = \mathsf{pt}_I \colon I \to \mathbf{1}$, because then the adjoints of u^* are functors $\mathsf{hocolim}_I \colon \mathbb{D}(I) \to \mathbb{D}(\mathbf{1})$ and $\mathsf{holim}_I \colon \mathbb{D}(J) \to \mathbb{D}(\mathbf{1})$.

Definition 2.4.4. A **derivator** is a prederivator $\mathbb{D}$ which satisfies the following axioms:

(Der1) For any $I, J \in \mathsf{Cat}$, the functor $\mathbb{D}(I \sqcup J) \to \mathbb{D}(I) \times \mathbb{D}(J)$ induced by the inclusions is an equivalence; the base $\mathbb{D}(\mathbf{1})$ is not the empty category.

(Der2) A morphism $f\colon X \to Y$ in J is an isomorphism if and only if $j^*(f)\colon j^*X \to j^*Y$ is an isomorphism in $\mathbb{D}(\mathbf{1})$ for every $j \in J$.

(Der3) For any $u\colon I \to J$ there are adjunction pairs $(u_!, u^*)$ and (u^*, u_*).

(Der4) For every functor $u\colon I \to J$ and object $j \in J$, the morphisms

$$\mathsf{hocolim}_{I_{/j}} \mathsf{pr}^*(X) \overset{\alpha_!}{\to} j^* u_!(X) \qquad j^* u_*(X) \overset{\alpha_*}{\to} \mathsf{holim}_{I_{j/}} \mathsf{pr}^*(X)$$

are isomorphisms for every $X \in \mathbb{D}(I)$.

We will not fully explain the notation of axiom (Der4), as we will not need the formulae, and refer to [23].

Example 2.4.5. The prederivators of Example 2.4.3 are all derivators.

The derivators $\mathbb{D}_{\mathcal{A}}$ are a special case of a more general kind of derivators, which have a canonical triangulated structure.

Definition 2.4.6. Let $\mathbb{D}$ be a derivator. Then:

(1) $\mathbb{D}$ is **strong** if the partial diagram functors $\mathsf{dia}_{\mathbf{2},J} \colon \mathbb{D}(\mathbf{2} \times J) \to \mathbb{D}(J)^{\mathbf{2}}$ are equivalences.

(2) $\mathbb{D}$ is **pointed** if $\mathbb{D}(\mathbf{1})$ has a zero object.

(3) $\mathbb{D}$ is **stable** if it is pointed and the essential images of the functors $(i_\ulcorner)_! \colon \mathbb{D}(\ulcorner) \to \mathbb{D}(\square)$ and $(i_\lrcorner)_* \colon \mathbb{D}(\lrcorner) \to \mathbb{D}(\square)$ coincide. Notice that in [23, Definition 4.6] the definition of "stable" requires also "strong".

Theorem 2.4.7 ([23, Theorem 4.15 and Corollary 4.19])**.** *Let $\mathbb{D}$ be a strong and stable derivator. Then $\mathbb{D}(I)$ has a canonical triangulated structure, for every small category I. Moreover, for every $u \colon I \to J$, the functors $u_!, u^*$ and u_* are triangulated with respect to this structure.*

Proposition 2.4.8 ([23, Proposition 1.36 and Example 4.2(i)])**.** *Let $\mathcal{A}$ be a Grothendieck category. Then the derivator $\mathbb{D}_{\mathcal{A}}$ is strong and stable, and the triangulated structure of $\mathsf{D}(\mathcal{A}^I)$ is the one of Theorem 2.4.7.*

We conclude this brief overview with a simple construction of some homotopy colimits.

Definition 2.4.9. Let $\mathcal{D}$ be a TR5 triangulated category, and $(X_0 \overset{f_0}{\to} X_1 \overset{f_1}{\to} X_2 \overset{f_2}{\to} \cdots)$ be an $\mathbb{N}$-diagram. Let $\mathsf{shift} \colon \coprod X_i \to \coprod X_i$ be the coproduct of the morphisms $f_i \colon X_i \to X_{i+1}$. The **Milnor colimit** of (X_i) is defined by the distinguished triangle

$$\coprod X_i \xrightarrow{1-\mathsf{shift}} \coprod X_i \longrightarrow \mathsf{Mcolim} X_i \longrightarrow \Sigma \coprod X_i.$$

Notice that this is not a functorial construction, although it computes the correct homotopy colimit, as shown by the following Proposition.

Proposition 2.4.10 ([34, Proposition 11.3])**.** *Let $\mathbb{D}$ be a strong and stable derivator, and $(X_0 \to X_1 \to X_2 \to \cdots)$ be an $\mathbb{N}$-diagram in $\mathbb{D}(\mathbf{1})$. Then there exists a coherent diagram $\mathcal{X} \in \mathbb{D}(\mathbb{N})$ with (X_i) as its underlying incoherent diagram, and $\mathsf{hocolim}_{\mathbb{N}} \mathcal{X} \simeq \mathsf{Mcolim} X_i$.*

There is also a notion of a morphism and equivalence between derivators, we refer the reader to [77, §5] and [23]. For our purposes, it will be enough to say that if $\mathcal{A}, \mathcal{B}$ are two Grothendieck categories, then a **morphism** of derivators $\eta : \mathbb{D}_{\mathcal{A}} \to \mathbb{D}_{\mathcal{B}}$ induces functors $\eta^I : \mathbb{D}_{\mathcal{A}}(I) \to \mathbb{D}_{\mathcal{B}}(I)$ such that for each morphism $u : I \to J$ the following square commutes (up to natural equivalence):

$$\begin{array}{ccc} \mathbb{D}_{\mathcal{A}}(J) & \xrightarrow{\mathbb{D}_{\mathcal{A}}(u)} & \mathbb{D}_{\mathcal{A}}(I) \\ {\scriptstyle \eta^J}\big\downarrow & & {\scriptstyle \eta^I}\big\downarrow \\ \mathbb{D}_{\mathcal{B}}(J) & \xrightarrow{\mathbb{D}_{\mathcal{B}}(u)} & \mathbb{D}_{\mathcal{B}}(I) \end{array} \tag{2.1}$$

The morphism of derivators η is an **equivalence** if all the functors η^I are equivalences. If this is the case, then η is an honest equivalence in a suitable category of derivators [23, Proposition 2.11], and all the equivalences η^I are triangle equivalences [77, Proposition 5.12]. Furthermore, if η is an equivalence then one can check by passing to adjoint functors that η is also compatible with left and right Kan extensions along any morphism u in Cat. In particular, we get the commutative square for any $I \in \mathsf{Cat}$:

$$\begin{array}{ccc} \mathbb{D}_{\mathcal{A}}(I) & \xrightarrow{\mathsf{hocolim}_I} & \mathbb{D}_{\mathcal{A}}(\star) \\ {\scriptstyle \eta^I}\downarrow{\scriptstyle \cong} & & {\scriptstyle \eta^\star}\downarrow{\scriptstyle \cong} \\ \mathbb{D}_{\mathcal{B}}(I) & \xrightarrow{\mathsf{hocolim}_I} & \mathbb{D}_{\mathcal{B}}(\star) \end{array} \tag{2.2}$$

Note that since cohomology is computed coordinate-wise, an object $\mathcal{X}$ of the bounded derived category $\mathsf{D}^b(\mathcal{A}^I)$ is an I-shaped diagram in $\mathsf{C}(\mathcal{A})$ such that the cohomologies of the coordinates $\mathcal{X}_i$ are uniformly bounded, that is, there are integers $l < m$ such that $H^j(\mathcal{X}_i) = 0$ for all $j < l$ or $j > m$ and all $i \in I$. By the exactness of direct limits in $\mathsf{C}(\mathcal{A})$, we see that for any small directed category I the homotopy colimit functor restricts to a functor $\mathsf{hocolim}_I : \mathsf{D}^b(\mathcal{A}^I) \to \mathsf{D}^b(\mathcal{A})$. We say that an equivalence of standard derivators $\eta : \mathbb{D}_{\mathcal{A}} \to \mathbb{D}_{\mathcal{B}}$ is **bounded** if for any small category I the triangle equivalence η^I restricts to a triangle equivalence $\eta^I : \mathsf{D}^b(\mathcal{A}^I) \to \mathsf{D}^b(\mathcal{B}^I)$. If I is directed, the above commutative square restricts to another one:

$$\begin{array}{ccc} \mathsf{D}^b(\mathcal{A}^I) & \xrightarrow{\mathsf{hocolim}_I} & \mathsf{D}^b(\mathcal{A}) \\ {\scriptstyle \eta^I}\downarrow{\scriptstyle \cong} & & {\scriptstyle \eta^\star}\downarrow{\scriptstyle \cong} \\ \mathsf{D}^b(\mathcal{B}^I) & \xrightarrow{\mathsf{hocolim}_I} & \mathsf{D}^b(\mathcal{B}) \end{array}$$

We remark that the bounded property can be naturally reformulated in terms of restriction to bounded standard derivators $\mathbb{D}^b_{\mathcal{A}}$, as it is done in [81]. These derivators are defined similarly to the standard derivators $\mathbb{D}_{\mathcal{A}}$, but one needs to replace Cat by the full subcategory of all suitably finite categories to reflect the fact that $\mathsf{D}^b(\mathcal{A}))$ is not (co)complete.

2.5 t-structures

The notion of t-structures, modeled on that of torsion pairs, was introduced by Beĭlinson, Bernstein and Deligne in their seminal paper [8].

Definition 2.5.1. Let $\mathcal{D}$ be a triangulated category. A ***t*-structure** in $\mathcal{D}$ is a pair of strictly full subcategories $\mathbb{T} := (\mathcal{U}, \mathcal{V})$ such that:

(tS1) $\Sigma\mathcal{U} \subseteq \mathcal{U}$;

(tS2) $\mathsf{Hom}_{\mathcal{D}}(\mathcal{U}, \mathcal{V}) = 0$;

(tS3) $\mathcal{U} * \mathcal{V} = \mathcal{D}$.

$\mathcal{U}$ is called the **aisle**, $\mathcal{V}$ the **coaisle** of the t-structure. The triangles in by (tS3) are called **approximation triangles** with respect to the t-structure.

Another notation we will use for the aisle and coaisle of $\mathbb{T}$ is $(\mathcal{U}, \mathcal{V}) = (\mathbb{T}^{\leq 0}, \mathbb{T}^{\geq 1})$.

Remark 2.5.2. Let $\mathbb{T} := (\mathcal{U}, \mathcal{V})$ be a t-structure in $\mathcal{D}$. We point out some immediate consequences of the definition.

(i) $\mathcal{U} = {}^{\perp}\mathcal{V}$ and $\mathcal{U}^{\perp} = \mathcal{V}$. The inclusions in one direction are (tS2); for the others, consider for example $X \in \mathcal{U}^{\perp}$ and its approximation triangle. It must split, and therefore $X \in \mathcal{V}$, and similarly to prove $\mathcal{U} \supseteq {}^{\perp}\mathcal{V}$.

(ii) $\mathcal{V} \subseteq \Sigma\mathcal{V}$. This comes from (tS1) and (i).

(iii) $\mathcal{U} = {}^{\perp_{\leq 0}}\mathcal{V}$ and $\mathcal{U}^{\perp_{\leq 0}} = \mathcal{V}$, using (i), (tS1) and (ii).

(iv) $\mathcal{U}$ is closed under extensions, cones and existing coproducts; $\mathcal{V}$ is closed under extensions, co-cones and existing products. Closure under extensions and (co)products comes from (i); (co-)cones then follow from (tS1) and (ii).

(v) If $\mathcal{D}$ is the base of a strong and stable derivator, $\mathcal{U}$ is closed under homotopy colimits; $\mathcal{V}$ is closed under homotopy limits ([73, Proposition 4.2]).

(vi) $\Sigma^n\mathbb{T} := (\Sigma^n\mathcal{U}, \Sigma^n\mathcal{V})$ is also a t-structure, for every $n \in \mathbb{Z}$. We will write $\Sigma^n\mathbb{T} = (\mathbb{T}^{\leq -n}, \mathbb{T}^{\geq -n+1})$.

The triangles in (tS3) are also functorial:

Remark 2.5.3 ([8, Proposition 1.3.3])**.** Let $\mathcal{D}$ be a triangulated category and $\mathbb{T} := (\mathcal{U}, \mathcal{V})$ a t-structure in $\mathcal{D}$. There exist adjunction pairs

$$(\subseteq, \tau_{\mathbb{T}}^{\leq 0})\colon \mathcal{U} \rightleftarrows \mathcal{D} \qquad \text{and} \qquad (\tau_{\mathbb{T}}^{\geq 1}, \supseteq)\colon\ \mathcal{D} \rightleftarrows \mathcal{V}$$

The counit and unit of these adjunction pairs produce the approximation triangles: given $X \in \mathcal{D}$, its approximation triangle is unique and it is

$$\tau_{\mathbb{T}}^{\leq 0}X \longrightarrow X \longrightarrow \tau_{\mathbb{T}}^{\geq 1}X \longrightarrow \Sigma\tau_{\mathbb{T}}^{\leq 0}X.$$

Definition 2.5.4. The functors $\tau_{\mathbb{T}}^{\leq 0}, \tau_{\mathbb{T}}^{\geq 1}$ are called the **left** (resp. **right**) **truncation** with respect to the t-structure. Using (vi) above, we denote by $\tau_{\mathbb{T}}^{\leq -n}, \tau_{\mathbb{T}}^{\geq 1-n}$ the truncation functors with respect to $\Sigma^n\mathbb{T}$, for every $n \in \mathbb{Z}$. Spelling it out, we have $\tau_{\mathbb{T}}^{\leq -n} = \Sigma^n\tau_{\mathbb{T}}^{\leq 0}\Sigma^{-n}$ and $\tau_{\mathbb{T}}^{\geq 1-n} = \Sigma^n\tau_{\mathbb{T}}^{\geq 1}\Sigma^{-n}$.

Remark 2.5.5 ([8, Proposition 1.3.5])**.** $\tau_{\mathbb{T}}^{\leq m}$ and $\tau_{\mathbb{T}}^{\geq n}$ naturally commute with each other for every $m, n \in \mathbb{Z}$, i.e. there are natural isomorphisms $\tau_{\mathbb{T}}^{\leq m}\tau_{\mathbb{T}}^{\geq n} \simeq \tau_{\mathbb{T}}^{\geq n}\tau_{\mathbb{T}}^{\leq m}$.

The main feature of a t-structure $\mathbb{T} := (\mathcal{U}, \mathcal{V})$ is its **heart**, which is defined as the subcategory $\mathcal{H}_{\mathbb{T}} := \mathcal{U} \cap \Sigma\mathcal{V}$. Since the truncation functors naturally commute, there is also a functor $H^0_{\mathbb{T}} := \tau^{\leq 0}_{\mathbb{T}} \tau^{\geq 0}_{\mathbb{T}} \colon \mathcal{D} \to \mathcal{H}_{\mathbb{T}}$, called the **cohomology functor** with respect to the t-structure. We write $H^n_{\mathbb{T}} := H^0_{\mathbb{T}}\Sigma^n$ for every $n \in \mathbb{Z}$.

Proposition 2.5.6 ([8, Théorème 1.3.6])**.** *Let $\mathcal{D}$ be a triangulated category, and $\mathbb{T} := (\mathcal{U}, \mathcal{V})$ be a t-structure in $\mathcal{D}$. Then its heart $\mathcal{H}_{\mathbb{T}}$ is an abelian category, whose short exact sequences are the sequences $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ such that in $\mathcal{D}$ there is a triangle $A \xrightarrow{f} B \xrightarrow{g} C \to \Sigma A$. The functor $H^0_{\mathbb{T}} \colon \mathcal{D} \to \mathcal{H}_{\mathbb{T}}$ is cohomological, i.e. it maps triangles to long exact sequences.*

Example 2.5.7. Let $\mathcal{D} = \mathsf{D}(\mathcal{A})$ be the derived category of an abelian category $\mathcal{A}$. There is the **standard** t-structure $\mathbb{D} := (\mathbb{D}^{\leq 0}, \mathbb{D}^{\geq 1})$, defined by

$$\mathbb{D}^{\leq 0} := \{X \in \mathsf{D}(\mathcal{A}) \colon H^n(X) = 0 \ \forall n > 0\}$$
$$\mathbb{D}^{\geq 1} := \{X \in \mathsf{D}(\mathcal{A}) \colon H^n(X) = 0 \ \forall n \leq 0\}.$$

By identifying the objects of $\mathcal{A}$ with the corresponding stalk complexes in degree zero, $\mathcal{A}$ is equivalent to the heart of $\mathbb{D}$. The truncation functors with respect to $\mathbb{D}$ are induced by the *smart truncations* of complexes, namely

$$\tau^{\leq 0}(\cdots \to X_{-1} \to X_0 \xrightarrow{d} X_1 \to \cdots) := (\cdots \to X_{-1} \to X_0 \to \operatorname{im} d \to 0 \to \cdots)$$
$$\tau^{\geq 0}(\cdots \to X_{-1} \to X_0 \xrightarrow{d} X_1 \to \cdots) := (\cdots \to 0 \to \ker d \to X_0 \to X_1 \to \cdots)$$

and defined on morphisms in the natural way. The cohomological functor $H^0_{\mathbb{D}} := \tau^{\leq 0}\tau^{\geq 0}$ is then the usual cohomology of complexes in degree zero (up to the identification $\mathcal{A} \simeq \mathcal{A}[0]$).

Definition 2.5.8. A t-structure $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ is **left** (respectively **right**) **non-degenerate** if $\cap_{n \in \mathbb{N}} \Sigma^n \mathcal{U} = 0$ (respectively, $\cap_{n \in \mathbb{N}} \Sigma^{-n}\mathcal{V} = 0$). It is **non-degenerate** if it is both left and right non-degenerate.

Lemma 2.5.9. *Let $\mathcal{D}$ be a triangulated category. The following are equivalent for a t-structure $\mathbb{T}$ in $\mathcal{D}$:*

(1) $\mathbb{T}$ is non-degenerate.

(2) If $H^n X = 0$ for every $n \in \mathbb{Z}$, then $X = 0$.

(3) $\mathcal{U} = \{X \in \mathcal{D} \colon H^n_{\mathbb{T}} X = 0 \ \forall n > 0\}$ and $\mathcal{V} = \{X \in \mathcal{D} \colon H^n_{\mathbb{T}} X = 0 \ \forall n \leq 0\}$.

Proof. ($1 \Rightarrow 2$) Assume first that $X \in \mathcal{U}$. Then we have a triangle

$$\tau^{\leq -1} X \to X \to H^0 X \to \Sigma\tau^{\leq -1} X$$

and since $H^0 X = 0$ by hypothesis, we conclude that $X \simeq \tau^{\leq -1} X \in \Sigma\mathcal{U}$. By iterating a similar argument, we obtain that $X \in \Sigma^n \mathcal{U}$ for every $n \geq 0$, and since $\mathbb{T}$ is non-degenerate, that $X = 0$. We can argue in the dual way if we start with $X \in \mathcal{V}$. Now for a general X with vanishing cohomologies, we consider its truncations $\tau^{\leq 0}X$ and $\tau^{\geq 1}X$: they have again vanishing cohomologies, and therefore they are zero by the previous discussion. Then $X = 0$ as well.

($2 \Rightarrow 3$) We prove the equality for the aisle. ($\subseteq$) clearly holds in general. For ($\supseteq$), let X be such that $H^n X = 0$ for every $n \geq 1$, and consider its approximation triangle

$$\tau^{\leq 0}X \to X \to \tau^{\geq 1}X \to \Sigma\tau^{\leq 0}.$$

We have $H^n \tau^{\geq 1}X = 0$ for every $n \in \mathbb{Z}$, and so $\tau^{\geq 1}X = 0$ by (2). We conclude that $X \simeq \tau^{\leq 0} \in \mathcal{U}$.

($3 \Rightarrow 1$) Let $X \in \cap_{n\geq 0}\Sigma^n\mathcal{U}$. By the description of the aisle, it follows that $H^n X = 0$ for every $n \in \mathbb{Z}$. But then $X \in \mathcal{V}$, and so $X \in \mathcal{U} \cap \mathcal{V} = 0$. A similar argument works if $X \in \cap_{n\geq 0}\Sigma^{-n}\mathcal{V}$. $\square$

Example 2.5.10. The standard t-structure of $\mathsf{D}(\mathcal{A})$ is non-degenerate. Examples of (even compactly generated, see Definition 2.5.37) t-structures which are only left or only right degenerate can be constructed in the derived category of a commutative noetherian ring using Lemma 3.1.18. An example of left and right degenerate t-structures is given by the following Definition.

Definition 2.5.11. A t-structure $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ is called **stable** if $\Sigma^{-1}\mathcal{U} \subseteq \mathcal{U}$ (and therefore $\Sigma\mathcal{V} \subseteq \mathcal{V}$). Together with (tS1) this means that $\Sigma\mathcal{U} = \mathcal{U}$, i.e. $\mathcal{U}$ and $\mathcal{V}$ are triangulated subcategories.

Lemma 2.5.12. *Let $\mathcal{D}$ be a triangulated category. The aisles of stable t-structures of $\mathcal{D}$ are precisely the coreflective thick subcategories of $\mathcal{D}$.*

Proof. The aisle $\mathcal{U}$ of a stable t-structure is by construction a thick subcategory (closure under direct summands comes from the fact that $\mathcal{U} = {}^{\perp}(\mathcal{U}^{\perp})$). It is also coreflective, the right adjoint to $\mathcal{U} \subseteq \mathcal{D}$ being the left truncation functor.

Conversely, let $\mathcal{S}$ be a coreflective thick subcategory of $\mathcal{D}$. Clearly $\Sigma\mathcal{S} = \mathcal{S}$; moreover, by Proposition 2.2.7(3), there exist approximation triangles with respect to the pair $(\mathcal{S}, \mathcal{S}^{\perp})$. The coaisle $\mathcal{S}^{\perp}$ is the essential image of the right adjoint of $\mathcal{D} \to \mathcal{D}/\mathcal{S}$. $\square$

It follows that a localising sequence as in Definition 2.2.13 gives rise to a stable t-structure.

Lemma 2.5.13. *Consider a localising sequence of triangulated categories*

$$\begin{tikzcd} \mathcal{S} \arrow[r, "F"] & \mathcal{D} \arrow[r, "G"] \arrow[l, bend left, dotted, "F_\rho"] & \mathcal{R} \arrow[l, bend left, dotted, "G_\rho"] \end{tikzcd}$$

Then we have a stable t-structure $(\mathrm{Im}(F), \mathrm{Im}(G_\rho))$ *in* $\mathcal{D}$*. In particular, given a recollement*

$$\begin{tikzcd} \mathcal{S} \arrow[r, "i_*" description] & \mathcal{D} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*" description] & \mathcal{R}. \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

We have stable t-structures $(\mathrm{Im}(i_*), \mathrm{Im}(j_*))$ *and* $(\mathrm{Im}(j_!), \mathrm{Im}(i_*))$ *in* $\mathcal{D}$*.*

Proof. $\mathcal{S}$ is a coreflective thick subcategory, so by applying Lemma 2.5.12 we obtain the stable t-structure $(\mathrm{Im}(F), \mathrm{Im}(G_\rho))$. The recollement is a localising and colocalising sequence, so it gives two localising sequences by Remark 2.2.14. The claim follows from the first part. $\square$

2.5.1 t-Structures generated by objects

By analogy to torsion pairs generated by objects in an AB4 abelian category $\mathcal{A}$, one would like to use a set of objects of a triangulated category $\mathcal{D}$ to construct a t-structure. Notice that by Proposition 1.4.4, given a set $\mathcal{S}$ of objects of $\mathcal{A}$, the torsion pair $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ generated by $\mathcal{S}$ can be described in two ways:

- $\mathcal{T}$ is the smallest subcategory containing $\mathcal{S}$ and closed under extensions, quotients and coproducts;
- $\mathcal{F} = \mathcal{S}^{\perp}$.

The first way is translated to the triangulated setting by the following notion.

Definition 2.5.14. Let $\mathcal{D}$ be a TR5 triangulated category. A subcategory $\mathcal{U} \subseteq \mathcal{D}$ is called a **preaisle** if it is closed under suspension, extensions and direct summands. It is called a **cocomplete** pre-aisle if in addition it is closed under coproducts.

Indeed, aisles are cocomplete preaisles, characterised as follows.

Proposition 2.5.15 ([35, §1])**.** *Let* $\mathcal{D}$ *be a* TR5 *triangulated category and* $\mathcal{U} \subseteq \mathcal{D}$ *a cocomplete pre-aisle. Then* $\mathcal{U}$ *is the aisle of a t-structure if and only the inclusion* $\mathcal{U} \hookrightarrow \mathcal{D}$ *admits a right adjoint.*

We then give the following definition.

Definition 2.5.16. Let $\mathcal{D}$ be a TR5 triangulated category, $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ be a t-structure in $\mathcal{D}$ and $\mathcal{S} \subseteq \mathcal{U}$ be a set of objects. We say that $\mathbb{T}$ is:

(1) **(strongly) generated** by $\mathcal{S}$ if $\mathcal{U}$ is the smallest cocomplete preaisle containing $\mathcal{S}$;

(2) **weakly generated** by $\mathcal{S}$ if $\mathcal{V} = \mathcal{S}^{\perp_{\leq 0}}$.

Lemma 2.5.17. *In the notation of the Definition, if $\mathbb{T}$ is strongly generated by $\mathcal{S}$, then it is also weakly generated by $\mathcal{S}$.*

Proof. Since $\mathcal{S} \subseteq \mathcal{U}$, clearly $\mathcal{V} = \mathcal{U}^{\perp_{\leq 0}} \subseteq \mathcal{S}^{\perp_{\leq 0}}$. To prove the converse inclusion $\mathcal{S}^{\perp_{\leq 0}} \subseteq \mathcal{V}$, we may equivalently prove that $\mathcal{U} = {}^{\perp}\mathcal{V} \subseteq {}^{\perp}(\mathcal{S}^{\perp_{\leq 0}})$; and the latter is easily seen to be a cocomplete preaisle which contains $\mathcal{S}$, which concludes the proof. □

Remark 2.5.18. Notice that a set of objects needs not, in general, generate a t-structure, either strongly or weakly. Therefore, the converse of Lemma 2.5.17 may fail, when a set $\mathcal{S}$ generates a t-structure weakly but not strongly.

We now collect some results on sets of objects which (strongly and therefore also weakly) generate a t-structure, in triangulated categories of varying generality, together with explicit descriptions of the objects of the aisles.

Proposition 2.5.19 ([2, Theorem A.1],[34, Appendix 2])**.** *Let $\mathcal{D}$ be a* TR5 *triangulated category, and $\mathcal{S} \subseteq \mathcal{D}^c$ a set of compact objects. Then the smallest cocomplete pre-aisle $\mathcal{U}$ containing $\mathcal{S}$ is an aisle (i.e., $\mathcal{S}$ generates a t-structure). Let $\overline{\mathcal{S}}$ be the full subcategory of finite extensions of coproducts of shifts of objects of $\mathcal{S}$. The objects of $\mathcal{U}$ can be described as:*

(1) cones of morphisms between coproducts of objects of $\overline{\mathcal{S}}$;

(2) Milnor colimits of objects of $\overline{\mathcal{S}}$, if $\mathsf{Hom}_{\mathcal{D}}(S, \Sigma^n S') = 0$ for every $S, S' \in \mathcal{S}$ and $n > 0$;

(3) homotopy colimits of $\mathbb{N}$-sequences of objects of $\overline{\mathcal{S}}$, if $\mathcal{D}$ is the base of a strong and stable derivator.

Remark 2.5.20. In view of this Proposition and Lemma 2.5.17, a t-structure is **compactly generated** (i.e. generated by a set $\mathcal{S}$ of compact objects) if and only if it is weakly generated by the same set $\mathcal{S}$. This shows that the two conditions of Definition 2.1.11 coincide, as they both amount to say that the trivial t-structure $(\mathcal{D}, 0)$ is compactly generated.

Proposition 2.5.21. *Let $\mathcal{D}$ be a* TR5 *triangulated category, admitting a suitable* model structure*, and let $\mathcal{S} \subseteq \mathcal{D}$ be any set objects. Then the smallest cocomplete pre-aisle containing $\mathcal{S}$ is an aisle.*

Proposition 2.5.22 ([58, Theorem 2.3])**.** *Let $\mathcal{D}$ be a well-generated triangulated category, and $\mathcal{S} \subseteq \mathcal{D}$ be any set of objects. Then the smallest cocomplete pre-aisle containing $\mathcal{S}$ is an aisle. Its objects are obtained as homotopy colimits of countable sequences, whose terms are α-extensions of coproducts of objects of $\mathcal{S}$, where α is the regular infinite cardinal appearing in the definition of well-generation (see also Remark 2.1.16).*

We mention the fact that Proposition 2.5.22 replaces Proposition 2.5.21. To conclude, we spend a few words about the natural dualisation of these results.

Definition 2.5.23. Let $\mathcal{D}$ be a triangulated category with products, $\mathcal{T} = (\mathcal{U}, \mathcal{V})$ a t-structure in $\mathcal{D}$ and $\mathcal{S} \subseteq \mathcal{V}$ a set of objects. We say that $\mathbb{T}$ is:

(1) **strongly cogenerated** by $\mathcal{S}$ if $\mathcal{V}$ is the smallest complete precoaisle, i.e. the smallest subcategory closed under cosuspension, extensions and products, containing $\mathcal{S}$;

(2) **weakly cogenerated** by $\mathcal{S}$ if $\mathcal{U} = {}^{\perp_{\leq 0}}\mathcal{S}$.

Notice that to say that $\mathcal{S}$ strongly (respectively, weakly) cogenerates $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ in $\mathcal{D}$ is equivalent to saying that $\mathcal{S}$ strongly (respectively, weakly) generated $\mathbb{T}^{op} = (\mathcal{V}^{op}, \mathcal{U}^{op})$ in the opposite triangulated category $\mathcal{D}^{op}$.

This allows to dualise Lemma 2.5.17, to see that that strong cogeneration is stronger than weak cogeneration.

We conclude with an easy observation, which often allows to reduce the number of objects used to weakly (co)generated to one.

Lemma 2.5.24. *Let $\mathcal{D}$ be a triangulated category, and $\mathcal{S}$ a set of objects.*

(1) If the coproduct $\coprod S_i$ of the objects in $\mathcal{S}$ exists, then $(\coprod S_i)^{\perp_{\leq 0}} = \mathcal{S}^{\perp_{\leq 0}}$;

(2) If the product $\prod S_i$ of the objects in $\mathcal{S}$ exists, then ${}^{\perp_{\leq 0}}(\prod S_i) = {}^{\perp_{\leq 0}}\mathcal{S}$;

Proof. Trivial. □

2.5.2 Bounded subcategory with respect to a t-structure

Definition 2.5.25. Let $\mathcal{D}$ be a triangulated category, and $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ a t-structure in $\mathcal{D}$. The **bounded subcategory of $\mathcal{D}$ with respect to $\mathbb{T}$** is the full subcategory

$$\mathcal{D}^b_{\mathbb{T}} := \bigcup_{n \in \mathbb{N}} (\Sigma^{-n}\mathcal{U} \cap \Sigma^n \mathcal{V}).$$

We have the following easy lemma.

Lemma 2.5.26. *Let $\mathcal{D}$ be a triangulated category and $\mathbb{T}$ be a t-structure in $\mathcal{D}$, with heart $\mathcal{H}$. Then $\mathcal{D}^b_{\mathbb{T}}$ is the smallest triangulated subcategory of $\mathcal{D}$ containing $\mathcal{H}$.*

Proof. Let $\mathcal{S}$ be the smallest triangulated subcategory of $\mathcal{D}$ containing $\mathcal{H}$. Since $\Sigma^i(\Sigma^{-n}\mathcal{U} \cap \Sigma^n\mathcal{V}) \subseteq \Sigma^{-n-|i|}\mathcal{U} \cap \Sigma^{n+|i|}\mathcal{V}$, the union $\mathcal{D}^b_{\mathbb{T}}$ is closed under suspension and cosuspension. Moreover, since both $\mathcal{U}$ and $\mathcal{V}$ are closed under extensions, $\mathcal{D}^b_{\mathbb{T}}$ is a triangulated subcategory of $\mathcal{D}$. Clearly it contains $\mathcal{H}$, so $\mathcal{S} \subseteq \mathcal{D}^b_{\mathbb{T}}$.

For the converse inclusion, arguing by induction on $n \geq 0$ and using approximation triangles, one sees that objects of $\Sigma^{-n}\mathcal{U} \cap \Sigma^n\mathcal{V}$ are $2n$-fold extensions of (co)suspensions of their cohomologies with respect to $\mathbb{T}$; therefore, they lie in $\mathcal{S}$. $\square$

Corollary 2.5.27. *Let $\mathcal{D}$ be a triangulated category, and $\mathbb{T}, \mathbb{T}'$ t-structures in $\mathcal{D}$, with hearts $\mathcal{H}_{\mathbb{T}}$ and $\mathcal{H}_{\mathbb{T}'}$ respectively. Then $\mathcal{D}^b_{\mathbb{T}} = \mathcal{D}^b_{\mathbb{T}'}$ if and only if $\mathcal{H}_{\mathbb{T}} \subseteq \mathcal{D}^b_{\mathbb{T}'}$ and $\mathcal{H}_{\mathbb{T}'} \subseteq \mathcal{D}^b_{\mathbb{T}}$.*

Definition 2.5.28. Let $\mathcal{D}$ be a triangulated category. A t-structure $\mathbb{T} := (\mathcal{U}, \mathcal{V})$ in $\mathcal{D}$ is called **bounded** if $\mathcal{D}^b_{\mathbb{T}} = \mathcal{D}$.

Example 2.5.29. (1) For the standard t-structure $\mathbb{D}$ in $\mathsf{D}(\mathcal{A})$ we have $\mathsf{D}(\mathcal{A})^b_{\mathbb{D}} = \mathsf{D}^b(\mathcal{A})$.

(2) For a t-structure $\mathbb{T}$ in $\mathcal{D}$, $\mathcal{D}^b_{\mathbb{T}} = 0$ if and only if $\mathbb{T}$ is stable.

It is easy to see that $\mathbb{T}$ induces a bounded t-structure $\mathbb{T} \cap \mathcal{D}^b_{\mathbb{T}} := (\mathcal{U} \cap \mathcal{D}^b_{\mathbb{T}}, \mathcal{V} \cap \mathcal{D}^b_{\mathbb{T}})$ on $\mathcal{D}^b_{\mathbb{T}}$. In fact, we have the following lemma.

Lemma 2.5.30. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{S}$ a triangulated subcategory of $\mathcal{D}$ and $\mathbb{T}$ a t-structure in $\mathcal{D}$. If $\mathbb{T} \cap \mathcal{S}$ is a bounded t-structure in $\mathcal{S}$, then $\mathcal{S} \subseteq \mathcal{D}^b_{\mathbb{T}}$.*

Proof. It follows immediately from the fact that $\mathcal{S} = \mathcal{S}^b_{\mathbb{T}\cap\mathcal{S}} \subseteq \mathcal{D}^b_{\mathbb{T}}$. $\square$

We now recall the following definition by Fiorot, Mattiello and Tonolo [17].

Definition 2.5.31. Let $\mathcal{D}$ be a triangulated category, and $\mathbb{T} := (\mathcal{U}, \mathcal{V}), \mathbb{T}' := (\mathcal{U}', \mathcal{V}')$ t-structures in $\mathcal{D}$. The **gap** of the pair $(\mathbb{T}, \mathbb{T}')$ is defined as

$$\mathsf{gap}(\mathbb{T}, \mathbb{T}') := \min\{m - n \colon m, n \in \mathbb{Z}, \Sigma^m\mathcal{U} \subseteq \mathcal{U}' \subseteq \Sigma^n\mathcal{U}\}$$

if this set is not empty, and $\mathsf{gap}(\mathbb{T}, \mathbb{T}') := \infty$ otherwise.

If $\mathcal{D} = \mathsf{D}(\mathcal{A})$ is a derived category, a t-structure which has finite gap from the standard t-structure is called **intermediate**.

Remark 2.5.32. Some remarks on this definition:

(1) If $\Sigma^m\mathcal{U} \subseteq \mathcal{U}' \subseteq \Sigma^n\mathcal{U}$, then we also have $\Sigma^{-n}\mathcal{U}' \subseteq \mathcal{U} \subseteq \Sigma^{-m}\mathcal{U}'$, so $\mathsf{gap}(\mathbb{T}, \mathbb{T}') = \mathsf{gap}(\mathbb{T}', \mathbb{T})$.

(2) Since aisles and coaisles are the orthogonal of each other, the definition implies that $\Sigma^m\mathcal{V} \supseteq \mathcal{V}' \supseteq \Sigma^n\mathcal{V}$. Moreover, any two of these four inclusions between aisles and coaisles imply the other two.

(3) If $\mathbb{T}, \mathbb{S}$ have finite gap, then $\mathcal{D}^b_{\mathbb{T}} = \mathcal{D}^b_{\mathbb{S}}$.

(4) The gap satisfies a triangular inequality: i.e. for any t-structures $\mathbb{T}, \mathbb{T}', \mathbb{T}''$ we have
$$\mathsf{gap}(\mathbb{T}, \mathbb{T}'') \leq \mathsf{gap}(\mathbb{T}, \mathbb{T}') + \mathsf{gap}(\mathbb{T}', \mathbb{T}'').$$

Example 2.5.33. In this example, we show that the converse of (3) is false, i.e. two t-structures may have the same bounded subcategory without having finite gap. On the way, we will also find an example of a strict inclusion between the bounded subcategories induced by two (non-degenerate) t-structures. We start by constructing the ambient triangulated category $\mathcal{D}$, in two consecutive steps. For every $n \in \mathbb{Z}$, let $\mathcal{A}_n$ be an abelian category. One can form their product $\mathcal{A} := \prod \mathcal{A}_n$, which is again an abelian category. Its objects are $\mathbb{Z}$-tuples $(A_n \mid n \in \mathbb{Z})$ with $A_n \in \mathcal{A}_n$, and morphisms are defined componentwise. The short exact sequences are also componentwise. The derived category of $\mathcal{A}$ is $\mathsf{D}(\mathcal{A}) \simeq \prod \mathsf{D}(\mathcal{A}_n)$. We introduce the following notation for its objects: they are complexes $X^\bullet = (\cdots \to X^i \to X^{i+1} \to \cdots)$ with $X^i = (A_n^i \mid n \in \mathbb{Z}) \in \mathcal{A}$, $A_n^i \in \mathcal{A}_n$. The equivalence $\mathsf{D}(\mathcal{A}) \simeq \prod \mathsf{D}(\mathcal{A}_n)$ is given by fixing n, to obtain complexes $A_n^\bullet := (\cdots \to A_n^i \to A_n^{i+1} \to \cdots) \in \mathsf{D}(\mathcal{A}_n)$.

Let $\mathbb{D}$ be the standard t-structure of $\mathsf{D}(\mathcal{A})$, and for every n, denote by $\mathbb{D}_n$ the standard t-structure of $\mathsf{D}(\mathcal{A}_n)$. It is easy to see that $\mathbb{D} = \prod \mathbb{D}_n$, in the sense that

$$\mathbb{D}^{\leq 0} = \{X \in \mathsf{D}(\mathcal{A})\colon A_n^\bullet \in \mathbb{D}_n^{\leq 0}\} \quad \text{and} \quad \mathbb{D}^{\geq 1} = \{X \in \mathsf{D}(\mathcal{A})\colon A_n^\bullet \in \mathbb{D}_n^{\geq 1}\}$$

using the notation for complexes introduced above. It is clear that in the same way we can define a non-degenerate t-structure $\mathbb{T} := \prod \Sigma^{-n}\mathbb{D}_n$ in $\mathsf{D}(\mathcal{A})$. Its heart is $\mathcal{H}_\mathbb{T} = \prod \mathcal{A}_n[-n] \simeq \prod \mathcal{A}_n = \mathcal{A}$. For every object of $\mathcal{A}$ we have that $H^n_\mathbb{T}(A_n \mid n \in \mathbb{Z}) = A_n$.

Now, inside $\mathcal{A}$, consider the subcategory

$$\mathcal{B} := \{(A_n \mid n \in \mathbb{Z}) \in \mathcal{A}\colon A_n = 0 \text{ for almost all } n \in \mathbb{Z}\}.$$

It is easily seen to be closed under extensions, subobjects and quotient, so it is an abelian subcategory of $\mathcal{A}$. Consider $\mathsf{D}(\mathcal{B}) \subseteq \mathsf{D}(\mathcal{A})$. The standard t-structure of $\mathsf{D}(\mathcal{B})$ is the restriction of $\mathbb{D}$. Similarly, $\mathbb{T}$ also restricts to a t-structure on $\mathsf{D}(\mathcal{B})$, which we denote by $\mathbb{T}'$. Notice that its heart is $\mathcal{H}_{\mathbb{T}'} = \mathcal{H}_\mathbb{T} \cap \mathsf{D}(\mathcal{B}) = \mathcal{H}_\mathbb{T} \simeq \mathcal{A}$. Since every object $(A_n \mid n \in \mathbb{Z}) \in \mathcal{B} \subseteq \mathcal{A}$ has finitely many non-zero A_n, it also has finitely many non-zero cohomologies with respect to $\mathbb{T}'$, so it belongs to $\mathsf{D}(\mathcal{B})^b_{\mathbb{T}'}$. We then have a chain of strict inclusions

$$\mathsf{D}^b(\mathcal{B}) = \mathsf{D}(\mathcal{B})^b_\mathbb{D} \subsetneq \mathsf{D}(\mathcal{B})^b_{\mathbb{T}'} \subsetneq \mathsf{D}(\mathcal{B}).$$

The first inclusion is strict because clearly $\mathcal{H}_{\mathbb{T}'} \not\subseteq \mathsf{D}^b(\mathcal{B})$, while the second is strict because of complexes such that $A_n^\bullet$ is unbounded for some n.

We let $\mathcal{D} := \mathsf{D}^b(\mathcal{B})$. Notice that $\mathbb{T}'$ restricts to $\mathsf{D}^b(\mathcal{B})$. Indeed, given $(A_n \mid n \in \mathbb{Z}) \in \mathcal{B}$, its truncations with respect to $\mathbb{T}'$ are its direct summands, in which some of the components are zeroed out; in particular, they still lie in $\mathsf{D}^b(\mathcal{B})$. By induction, the same holds for objects of $\mathsf{D}^b(\mathcal{B})$. By the above inclusions, the t-structure $\mathbb{T}''$ induced by $\mathbb{T}'$ on $\mathsf{D}^b(\mathcal{B})$ is bounded, i.e. $\mathsf{D}^b(\mathcal{B})^b_\mathbb{D} = \mathsf{D}^b(\mathcal{B})^b_{\mathbb{T}''}$. It is easy, however, to see that $\mathbb{T}'$ is not intermediate, as for every $0 \neq A_n \in \mathcal{A}_n$, its shift $A_n[-n]$ belongs to $\mathcal{H}_{\mathbb{T}''}$ but not to $\mathbb{D}^{<n}$.

Despite this example, it is often the case that if $\mathcal{T}^b_{\mathbb{T}} = \mathcal{T}^b_{\mathbb{S}}$ for two t-structure $\mathbb{T}$ and $\mathbb{S}$, then they have finite gap:

Lemma 2.5.34. *Let $\mathcal{D}$ be a triangulated category, and $\mathbb{T}, \mathbb{S}$ t-structures in $\mathcal{D}$. Assume that $\mathcal{D}^b_{\mathbb{T}} = \mathcal{D}^b_{\mathbb{S}}$. Then $\mathbb{T}$ and $\mathbb{S}$ have finite gap, as soon as one of the following sufficient conditions is verified:*

(1) $\mathbb{T}$ is weakly generated and weakly cogenerated by two objects of $\mathcal{D}^b_{\mathbb{T}} = \mathcal{D}^b_{\mathbb{S}}$.

(2) $\mathbb{T}, \mathbb{S}$ are weakly generated by two objects of $\mathcal{D}^b_{\mathbb{T}} = \mathcal{D}^b_{\mathbb{S}}$.

(3) $\mathbb{T}, \mathbb{S}$ are weakly cogenerated by two objects of $\mathcal{D}^b_{\mathbb{T}} = \mathcal{D}^b_{\mathbb{S}}$.

Proof. Let $\mathbb{T} := (\mathcal{U}, \mathcal{V}), \mathbb{S} := (\mathcal{U}', \mathcal{V}')$.

(1) Let S, C be the objects of $\mathcal{D}^b_{\mathbb{T}} = \mathcal{D}^b_{\mathbb{S}}$ which weakly generated and cogenerated $\mathbb{T}$, respectively. In particular, there exist n, m such that $S \in \Sigma^n \mathcal{U}'$ and $C \in \Sigma^m \mathcal{V}'$. It follows that $\mathcal{V} = S^{\perp_{\leq 0}} \supseteq (\Sigma^n \mathcal{U}')^{\perp_{\leq 0}} = \Sigma^n \mathcal{V}'$ and similarly $\mathcal{U} = {}^{\perp_{\leq 0}}C \supseteq {}^{\perp_{\leq 0}}(\Sigma^m \mathcal{V}') = \Sigma^m \mathcal{U}'$. Then $\mathbb{T}, \mathbb{S}$ have finite gap by Remark 2.5.32(2). (2) and its dual (3) are analogous. $\square$

We conclude with the interplay between triangulated functors and t-structures.

Definition 2.5.35. Let $\mathcal{D}, \mathcal{D}'$ be triangulated categories, and $\mathbb{T} := (\mathcal{U}, \mathcal{V}), \mathbb{T}' := (\mathcal{U}', \mathcal{V}')$ be t-structures in $\mathcal{D}$ and $\mathcal{D}'$ respectively, with hearts $\mathcal{H}$ and $\mathcal{H}'$. A triangulated functor $F\colon \mathcal{D} \to \mathcal{D}'$ is called ***t*-exact** with respect to $\mathbb{T}$ and $\mathbb{T}'$ if $F\mathcal{U} \subseteq \mathcal{U}'$ and $F\mathcal{V} \subseteq \mathcal{V}'$.

Lemma 2.5.36. *In the same notation of the definition, assume F is t-exact with respect to $\mathbb{T}$ and $\mathbb{T}'$. Then:*

(1) F restricts to an exact functor $F_{|\mathcal{H}}\colon \mathcal{H} \to \mathcal{H}'$.

(2) F commutes with truncations and cohomology, in the sense that there are canonical natural isomorphisms $F\tau^{\leq 0}_{\mathbb{T}} \simeq \tau^{\leq 0}_{\mathbb{T}'}F$, $\tau^{\geq 1}_{\mathbb{T}'}F \simeq F\tau^{\geq 1}_{\mathbb{T}}$ and $H^0_{\mathbb{T}'}F \simeq F_{|\mathcal{H}}H^0_{\mathbb{T}}$.

(3) Assume that $\mathbb{T}$ is non-degenerate. If F is an equivalence, then so is $F_{|\mathcal{H}}$.

Proof. (1) Since $\mathcal{H} = \mathcal{U} \cap \Sigma\mathcal{V}$, then $F\mathcal{H} \subseteq F\mathcal{U} \cap \Sigma' F\mathcal{V} \subseteq \mathcal{U}' \cap \Sigma'\mathcal{V} = \mathcal{H}'$, so F restricts to a functor $F_{|\mathcal{H}}\colon \mathcal{H} \to \mathcal{H}'$. Short exact sequences in $\mathcal{H}$ are triangles in $\mathcal{D}$, and the triangulated functor F sends them to triangles in $\mathcal{D}'$; since their terms lie in $\mathcal{H}'$, they are short exact sequences in $\mathcal{H}'$, so $F_{|\mathcal{H}}$ is exact.

(2) We first construct the canonical natural transformations $\eta\colon F\tau^{\leq 0}_{\mathbb{T}} \Rightarrow \tau^{\leq 0}_{\mathbb{T}'}F$ and $\varepsilon\colon \tau^{\geq 1}_{\mathbb{T}'}F \to F\tau^{\geq 1}_{\mathbb{T}}$. For $X \in \mathcal{D}$, consider its $\mathbb{T}$-approximation triangle

$$\tau^{\leq 0}_{\mathbb{T}}X \xrightarrow{u} X \xrightarrow{v} \tau^{\geq 1}_{\mathbb{T}}X \longrightarrow \Sigma\tau^{\leq 0}_{\mathbb{T}}X$$

By t-exactness of F, $F\tau_{\mathbb{T}}^{\leq 0}X \in \mathcal{U}'$, and by the adjunction $(\mathcal{U}' \subseteq \mathcal{D}', \tau_{\mathbb{T}'}^{\leq 0})$ the morphism Fu factors through a morphism $\eta_X \colon F\tau_{\mathbb{T}}^{\leq 0}X \to \tau_{\mathbb{T}'}^{\leq 0}FX$, which is natural in X. ε is obtained similarly, using the adjunction $(\tau_{\mathbb{T}'}^{\geq 1}, \mathcal{V}' \subseteq \mathcal{D}')$ to factor Fv. The fact that the morphisms η_X, ε_X are isomorphisms is the uniqueness, up to unique isomorphism, of the $\mathbb{T}'$-approximation triangle of FX (see Remark 2.5.3).

The natural transformation $\gamma \colon FH_{\mathbb{T}}^0 \Rightarrow H_{\mathbb{T}'}^0 F$ is obtained as the composition

$$\begin{aligned} FH_{\mathbb{T}}^0 = F_{|\mathcal{H}}\tau_{\mathbb{T}}^{\leq 0}\tau_{\mathbb{T}}^{\geq 0} &\overset{\eta_{\tau_{\mathbb{T}}^{\geq 0}}}{\Longrightarrow} \tau_{\mathbb{T}'}^{\leq 0}F\tau_{\mathbb{T}}^{\geq 0} = \tau_{\mathbb{T}'}^{\leq 0}F\Sigma\tau_{\mathbb{T}}^{\geq 1}\Sigma^{-1} \simeq \\ \simeq \tau_{\mathbb{T}'}^{\leq 0}\Sigma' F\tau_{\mathbb{T}}^{\geq 1}\Sigma^{-1} &\overset{\tau_{\mathbb{T}'}^{\leq 0}\Sigma'\varepsilon_{\Sigma^{-1}}^{-1}}{\Longrightarrow} \tau_{\mathbb{T}'}^{\leq 0}\Sigma'\tau_{\mathbb{T}'}^{\geq 1}F\Sigma^{-1} \simeq \tau_{\mathbb{T}'}^{\leq 0}\Sigma'\tau_{\mathbb{T}'}^{\geq 1}\Sigma'^{-1}F = H_{\mathbb{T}'}^0 F \end{aligned}$$

and it is therefore a natural isomorphism.

(3) By (1), we have a fully faithful functor $F_{\mathcal{H}} \colon \mathcal{H} \to \mathcal{H}'$, and we only have to prove that it is essentially surjective. Let $X' \in \mathcal{H}'$, and by essential surjectivity of F let $X \in \mathcal{D}$ be such that $FX \simeq X'$. By (2), for every integer $n \neq 0$ we have $F_{|\mathcal{H}}H_{\mathbb{T}}^n X \simeq H_{\mathbb{T}'}^n FX \simeq H_{\mathbb{T}'}^n X' = 0$, and since F is faithful we deduce that $H_{\mathbb{T}}^n X = 0$ for every $n \neq 0$. If we assume that $\mathbb{T}$ is non-degenerate, this implies that $X \in \mathcal{H}$, hence X' lies in the essential image of $F_{|\mathcal{H}}$. □

2.5.3 Properties of t-structures

We now list some additional properties that a t-structure may enjoy.

Definition 2.5.37. Let $\mathcal{D}$ be a triangulated category, and $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ be a t-structure.

(1) If $\mathcal{D}$ has coproducts, $\mathbb{T}$ is **k-smashing**, for $k \geq 0$, if the coproduct of a family of objects of $\mathcal{V}$ lies in $\Sigma^k\mathcal{V}$. In particular, for $k = 0$, $\mathcal{V}$ must be closed under coproducts; in this case we say that $\mathbb{T}$ is **smashing** (compare with Definition 2.2.4, and with Remark 2.5.2(ii)). If $\mathcal{D}$ has products, the notion of **(k-)cosmashing** is define dually (asking $\Sigma^{-k}\mathcal{U}$ to contain the products of objects of $\mathcal{U}$).

(2) If $\mathcal{D}$ is the base of a strong and stable derivator, $\mathbb{T}$ is **homotopically smashing** if $\mathcal{V}$ is closed under homotopy colimits (compare with Remark 2.5.2(v)).

(3) If $\mathcal{D}$ has coproducts, $\mathbb{T}$ is **compactly generated** if it is generated by a set of compact objects (see Proposition 2.5.19).

(4) If $\mathcal{D}$ has coproducts, $\mathbb{T}$ is **silting** if it is of the form $\mathbb{T} = (T^{\perp_{>0}}, T^{\perp_{\leq 0}})$, for an object T of $\mathcal{D}$ (which is then called **silting**). $\mathbb{T}$ and T are called **tilting** if $\mathsf{Add}(T) \subseteq T^{\perp_{>0}}$. If $\mathbb{T}$ is silting it is weakly generated by T.

(5) If $\mathcal{D}$ has products, $\mathbb{T}$ is **cosilting** if it is of the form $\mathbb{T} = ({}^{\perp_{\leq 0}}C, {}^{\perp_{>0}}C)$, for an object C of $\mathcal{D}$ (which is then called **cosilting**). $\mathbb{T}$ and C are called **cotilting** if $\mathsf{Prod}(C) \subseteq {}^{\perp_{\leq 0}}C$.

(6) If $\mathcal{D} = \mathsf{D}(\mathcal{A})$, for a locally coherent Grothendieck category $\mathcal{A}$, $\mathbb{T}$ is **restrictable** if the pair $\mathbb{T} \cap \mathsf{D}^b(\mathsf{fp}(\mathcal{A})) := (\mathcal{U} \cap \mathsf{D}^b(\mathsf{fp}(\mathcal{A})), \mathcal{V} \cap \mathsf{D}^b(\mathsf{fp}(\mathcal{A})))$ is a t-structure in $\mathsf{D}^b(\mathsf{fp}(\mathcal{A}))$ (equivalently, if $\mathsf{D}^b(\mathsf{fp}(\mathcal{A}))$ is closed under (one of) the truncation functors of $\mathbb{T}$).

Lemma 2.5.38 ([73, Proposition 5.6])**.** *Let $\mathcal{D}$ be a triangulated category. As soon as these notions are well-defined, there are strict implications for a t-structure*

$$\textit{compactly generated} \;\Rightarrow\; \textit{homotopically smashing} \;\Rightarrow\; \textit{smashing.}$$

Proof. If $\mathcal{D}$ has coproducts, it is easy to see from the definition of compact objects that "compactly generated" implies "smashing". If $\mathcal{D}$ is the base of a strong and stable derivator, the claim is the referenced result (which also points to counterexamples for the converse implications). □

These properties of a t-structure reflect on properties of its heart.

Lemma 2.5.39. *Let $\mathcal{D}$ be a* TR5 *(respectively* TR5**) triangulated category, and $\mathbb{T}$ a t-structure in $\mathcal{D}$, with heart $\mathcal{H}$. Then:*

(1) $\mathcal{H}$ is always AB3 *(respectively* AB3**); a coproduct (respectively, product) in $\mathcal{H}$ is computed as the cohomology of the same coproduct (respectively, product) computed in $\mathcal{D}$ [59, Proposition 3.2].*

(2) If $\mathbb{T}$ is smashing (respectively cosmashing), then $\mathcal{H}$ is AB4 *(respectively* AB4**) [59, Proposition 3.3]. The truncation and cohomology functors with respect to $\mathbb{T}$ commute with coproducts (respectively, products).*

Proof. (2) We mention the argument for the last statement, in the smashing case; the other one is dual. Denote by $\tau^{\leq 0}, \tau^{\geq 1}, H^0_{\mathbb{T}}$ the truncation and cohomology functors with respect to $\mathbb{T}$, and let $(X_i \mid i \in I)$ be a family of objects of $\mathcal{D}$. Since both the aisle and the coaisle are closed under coproducts, the triangle

$$\coprod \tau^{\leq 0} X_i \to \coprod X_i \to \coprod \tau^{\geq 1} X_i \to \Sigma \coprod \tau^{\leq 0} X_i$$

is an approximation triangle for $\coprod X_i$, which shows that the truncation functors commute with coproducts. It follows that $H^0_{\mathbb{T}}(\coprod X_i)$ is isomorphic to the coproduct of the $H^0_{\mathbb{T}}(X_i)$ in $\mathcal{D}$, and therefore *a fortiori* in $\mathcal{H}$. □

Proposition 2.5.40 ([73, Theorem B, Corollary 5.8])**.** *Let $\mathcal{D}$ be the base of strong and stable derivator, and $\mathbb{T}$ a homotopically smashing t-structure, with heart $\mathcal{H}$. Then $\mathcal{H}$ is* AB5*, and direct limits in $\mathcal{H}$ are computed as homotopy colimits in $\mathcal{D}$.*

Proposition 2.5.41 ([44, Theorem 4.6])**.** *Let $\mathcal{D}$ be the base of a strong and stable derivator, and assume it is compactly generated. Let $\mathbb{T}$ be a non-degenerate t-structure in $\mathcal{D}$, with heart $\mathcal{H}$. Then the following are equivalent:*

(1) $\mathbb{T}$ is homotopically smashing.

(2) $\mathbb{T}$ is smashing and $\mathcal{H}$ is Grothendieck.

(3) $\mathbb{T}$ is cosilting with respect to a pure injective cosilting object.

Proposition 2.5.42 ([72, Theorem 1.6])**.** *Let $\mathcal{D}$ be a* TR5 *triangulated category, and let $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ be a compactly generated t-structure in $\mathcal{D}$, with heart $\mathcal{H}$ and cohomological functor $H^0 \colon \mathcal{D} \to \mathcal{H}$. Write $\mathcal{U}_0 := \mathcal{U} \cap \mathcal{D}^c$. Then $\mathcal{H}$ is a locally finitely presented Grothendieck category, and $\mathsf{fp}(\mathcal{H}) = \mathsf{add}(H^0(\mathcal{U}_0))$.*

Proposition 2.5.43 ([48, Corollary 4.2])**.** *Let R be a noetherian ring, and $\mathbb{T}$ be a homotopically smashing intermediate t-structure, with heart $\mathcal{H}_{\mathbb{T}}$ (which is a Grothendieck category). Then the following are equivalent:*

(1) $\mathbb{T}$ is restrictable.

(2) $\mathcal{H}_{\mathbb{T}}$ is locally coherent with $\mathsf{fp}(\mathcal{H}_{\mathbb{T}}) = \mathcal{H}_{\mathbb{T}} \cap \mathsf{D}^b(\mathsf{mod}(R))$.

Proof. We mention the fact that if in addition R is commutative, the implication (1) $\Rightarrow$ (2) is [71, Theorem 6.3]. $\square$

2.6 Realisation functors

In this section, we deal with the problem of recognising when a triangulated category is a derived category. To this goal, we adopt the following point of view: A derived category is just a triangulated category with a standard t-structure. This leads us to the following definition.

Definition 2.6.1. Let $\mathcal{D}$ be a triangulated category $\mathcal{D}$, and $\mathbb{T}$ a t-structure, with heart $\mathcal{H}$. We will say that $\mathbb{T}$ is

(1) an **almost standard bounded t-structure** if there is a triangle equivalence $\mathsf{D}^b(\mathcal{H}) \simeq \mathcal{D}^b_{\mathbb{T}}$ which is t-exact with respect to the standard t-structure of $\mathsf{D}^b(\mathcal{H})$ and $\mathbb{T}$;

(2) an **almost standard t-structure** if there is a triangle equivalence $\mathsf{D}(\mathcal{H}) \simeq \mathcal{D}$ which is t-exact with respect to the standard t-structure of $\mathsf{D}(\mathcal{H})$ and $\mathbb{T}$.

If $\mathcal{D} = \mathsf{D}(\mathcal{A})$, for an abelian category $\mathcal{A}$, the usual terminology for $\mathbb{T}$ being almost standard (bounded) is that it **induces (bounded) derived equivalence**, as we get an equivalence $\mathsf{D}(\mathcal{H}) \simeq \mathsf{D}(\mathcal{A})$ (respectively, $\mathsf{D}^b(\mathcal{H}) \simeq \mathsf{D}^b(\mathcal{A})$).

From this viewpoint, it makes sense to consider all almost standard t-structures in a triangulated category $\mathcal{D}$ more or less equivalent. In particular, when $\mathcal{D} = \mathsf{D}(\mathcal{A})$ already is a derived category, the standard t-structure is almost standard; but we should try to reference it directly as less as possible. Therefore, we make the following observations, to show that some notions can be reformulated without referring to it.

Lemma 2.6.2. *Let R be a ring, $\mathsf{D}(R)$ its derived category and $\mathsf{D}(R)^{\mathsf{c}}$ the thick subcategory of compact objects of $\mathsf{D}(R)$.*

(1) $\mathsf{D}^b(R)$ consists of all the objects $X \in \mathsf{D}(R)$ such that for every compact $C \in \mathsf{D}(R)^{\mathsf{c}}$ the set $\{n \in \mathbb{Z} \colon \mathsf{Hom}_{\mathsf{D}(R)}(C, X[n]) \neq 0\}$ is finite.

(2) [70, Corollary 6.17] if R is noetherian, $\mathsf{D}^b(\mathsf{fp}(R)) = \mathsf{D}^b(R)^{\mathsf{c}}$.

Proof. (1) follows from the fact that $R \in \mathsf{D}(R)^{\mathsf{c}}$. □

Lemma 2.6.3. *Let R be a ring, $\mathsf{D}(R)$ its derived category and $\mathbb{T}$ a t-structure in $\mathsf{D}(R)$. Then the following properties of $\mathbb{T}$ can be characterised without reference to the standard t-structure.*

(1) $\mathbb{T}$ is compactly generated.

(2) $\mathbb{T}$ is intermediate.

(3) (if R is commutative noetherian) $\mathbb{T}$ restricts to $\mathsf{D}^b(\mathsf{fp}(R))$.

Proof. (1) is clear. (2) We claim that $\mathbb{T}$ is intermediate if and only if $\mathsf{D}(R)^b_{\mathbb{T}} = \mathsf{D}^b(R)$ (notice that this does not reference the standard t-structure by 2.6.2(1)). Indeed, the only non trivial implication ($\Leftarrow$) follows from 2.5.34(1), since the standard t-structure is both weakly generated by R and weakly cogenerated by an injective cogenerated W of $\mathsf{Mod}(R)$ (in fact this characterisation of intermediate t-structures holds in the derived category of any Grothendieck category $\mathcal{A}$, using a generator of $\mathcal{A}$ in place of R; see [67, Lemma 4.14]). (3) follows from Lemma 2.6.2(2). □

2.6.1 Realisation functors

Remark 2.6.4. Let $\mathcal{D}$ be a triangulated category and $\mathbb{T}$ an almost standard (bounded) t-structure, with heart $\mathcal{H}$. Notice that the t-exact equivalence $\mathsf{D}(\mathcal{H}) \to \mathcal{D}$ (respectively, $\mathsf{D}^b(\mathcal{H}) \to \mathcal{D}^b_{\mathbb{T}}$) automatically restricts to an equivalence between the hearts $\mathcal{H}$ of the standard t-structure of $\mathsf{D}(\mathcal{H})$ and $\mathcal{H}$ of $\mathbb{T}$.

Definition 2.6.5. Let $\mathcal{D}$ be a triangulated category, and $\mathbb{T}$ a t-structure in $\mathcal{D}$, with heart $\mathcal{H}$. A (respectively, **bounded**) **realisation functor** for $\mathbb{T}$ is a functor $\mathsf{real}_{\mathbb{T}} \colon \mathsf{D}(\mathcal{H}) \to \mathcal{D}$ (respectively, $\mathsf{real}^b_{\mathbb{T}} \colon \mathsf{D}^b(\mathcal{H}) \to \mathcal{D}^b_{\mathbb{T}}$) such that it restricts to an equivalence $\mathcal{H} \to \mathcal{H}$.

Remark 2.6.6. We point out that this definition differs from the classical definition of realisation functor, in that it does not require $\mathsf{real}_{\mathbb{T}}$ to induce an isomorphism $\mathcal{H} = \mathcal{H}$. This is only a slight generalisation: indeed, let $\mathsf{real}_{\mathbb{T}}\colon \mathsf{D}(\mathcal{H}) \to \mathcal{D}$ induce an equivalence $\alpha\colon \mathcal{H} \to \mathcal{H}$. Since α is exact, we can extend it to an equivalence $\bar{\alpha}\colon \mathsf{D}(\mathcal{H}) \to \mathsf{D}(\mathcal{H})$. If β is a quasi-inverse of $\bar{\alpha}$, the functor $\mathsf{real}_{\mathbb{T}} \circ \beta$ induces an isomorphism $\alpha\beta\colon \mathcal{H} = \mathcal{H}$, and therefore it is a realisation functor in the classical sense.

Our reason for this modified definition is that we want the following facts to be true:

(1) any triangle equivalence $\varphi\colon \mathsf{D}(\mathcal{A}) \to \mathcal{D}$ restricts to an equivalence $\mathcal{A} \simeq \varphi(\mathcal{A})$, and therefore it is a realisation functor for the image in $\mathcal{D}$ of the standard t-structure of $\mathsf{D}(\mathcal{A})$.

(2) More generally, applying an equivalence after a realisation functor always yields another realisation functor (for the pushed-forward t-structure).

Remark 2.6.7. In the notation above, we also point out briefly that a bounded realisation functor $\mathsf{real}^b_{\mathbb{T}}\colon \mathsf{D}^b(\mathcal{H}) \to \mathcal{D}^b_{\mathbb{T}}$ is automatically t-exact with respect to the standard t-structure $\mathbb{D}$ of $\mathsf{D}^b(\mathcal{H})$ and $\mathbb{T}$. Indeed, the aisle of $\mathbb{D}$ is the smallest subcategory closed under suspension and extensions and containing $\mathcal{H}$; so it gets sent to the aisle of $\mathbb{T}$, and similarly for the coaisle.

2.6.2 Existence of realisation functors

Given a triangulated category $\mathcal{D}$ and a t-structure $\mathbb{T}$ in $\mathcal{D}$, we give an account of results about the existence of bounded and unbounded realisation functors for $\mathbb{T}$.

The common feature of these results is that they rely on some additional structure over $\mathcal{D}$. A bounded realisation functor was already constructed by Beĭlinson, Bernstein and Deligne [8], in the case when $\mathcal{D}$ is a full triangulated subcategory of a bounded below derived category. They made use of the *filtered derived category*; their construction was then generalised by Psaroudakis and Vitória [67], leading to the notion of **f-enhancement** of the triangulated category $\mathcal{D}$. When they exist, f-enhancements produce each a realisation functor for a given t-structure in $\mathcal{D}$.

Proposition 2.6.8 ([8, §3], [67, §3])**.** *Let $\mathcal{D}$ be a triangulated category admitting an f-enhancement, and let $\mathbb{T}$ be a t-structure in $\mathcal{D}$, with heart $\mathcal{H}$. Then there exists a bounded realisation functor* $\mathsf{real}^b_{\mathbb{T}}\colon \mathsf{D}^b(\mathcal{H}) \to \mathcal{D}^b_{\mathbb{T}}$.

We mention that f-enhancements exist for example when $\mathcal{D}$ is the derived category of an abelian category (see [67, Example 3.2]), so the work of [67] extends that of [8] in this regard. Before proceding further, we also point out that

[67] defines bounded realisation functors as those produced by f-enhancements, rather than by their properties, as we do.

When $\mathcal{D}$ is the base of a strong and stable derivator, the notion of f-enhancement of $\mathcal{D}$ admits a description in the language of derivators. In Virili's paper [81], the bounded realisation functor of Proposition 2.6.8 was lifted to a morphism of prederivators, and an unbounded version was achieved.

Proposition 2.6.9 ([81, Theorem 4.13])**.** *Let $\mathbb{D}$ be a strong and stable derivator, and $\mathbb{T}$ a t-structure in $\mathbb{D}(\mathbf{1})$, with heart $\mathcal{H}$. Then there exists an exact morphism of prederivators $\mathfrak{real}^b_{\mathbb{T}}\colon \mathbb{D}^b_{\mathcal{H}} \to \mathbb{D}$ extending the inclusion $\mathcal{H} \subseteq \mathbb{D}(\mathbf{1})$.*

Proposition 2.6.10 ([81, Theorem 6.7])**.** *Let $\mathbb{D}$ be a strong and stable derivator, and $\mathbb{T}$ a t-structure in $\mathbb{D}(\mathbf{1})$, with heart $\mathcal{H}$. Assume that $\mathcal{H}$ has enough injectives and it is* AB4**-k, or that it has enough projectives and it is* AB4*-k. Then there exists an exact morphism of prederivators $\mathfrak{real}_{\mathbb{T}}\colon \mathbb{D}_{\mathcal{H}} \to \mathbb{D}$.*

2.6.3 When realisation functors are equivalences

Let $\mathcal{D}$ be a triangulated category and $\mathbb{T}$ be a t-structure in $\mathcal{D}$, with heart $\mathcal{H}$. Assume we can construct a (bounded or not) realisation functor for $\mathbb{T}$. We recall some results studying when it is an equivalence.

Proposition 2.6.11 ([8, Proposition 3.1.16], [67, Theorem 3.11][12, Theorem 2.9])**.** *Let $\mathsf{real}^b_{\mathbb{T}}\colon \mathsf{D}^b(\mathcal{H}) \to \mathcal{D}^b_{\mathbb{T}}$ be a bounded realisation functor for $\mathbb{T}$. Then the following are equivalent:*

(1) $\mathsf{real}^b_{\mathbb{T}}$ *is an equivalence;*

(2) $\mathsf{real}^b_{\mathbb{T}}$ *is fully faithful;*

(3) $\mathsf{real}^b_{\mathbb{T}}$ *is essentially surjective;*

(4) $\mathsf{real}^b_{\mathbb{T}}$ *induces isomorphisms $\mathsf{Hom}_{\mathsf{D}^b(\mathcal{H})}(X, Y[n]) \simeq \mathsf{Hom}_{\mathcal{D}}(X, \Sigma^n Y)$ for every $X, Y \in \mathcal{H}$ and $n \in \mathbb{Z}$ (or equivalently, for every $n \geq 2$);*

*(5) (*effaçabilité*) for every $X, Y \in \mathcal{H}$, $n \geq 2$ and morphism $f \in \mathsf{Hom}_{\mathcal{D}}(X, \Sigma^n Y)$, there exists an epimorphism $g\colon Z \to X$ in $\mathcal{H}$ such that $f \circ g = 0$.*

*(6) (co-*effaçabilité*) for every $X, Y \in \mathcal{H}$, $n \geq 2$ and morphism $f \in \mathsf{Hom}_{\mathcal{D}}(X, \Sigma^n Y)$, there exists a monomorphism $h\colon Y \to Z$ in $\mathcal{H}$ such that $\Sigma^n h \circ f = 0$.*

Proof. (1) $\Rightarrow$ (2) is obvious. (2) $\Rightarrow$ (3) If $\mathsf{real}^b_{\mathbb{T}}$ is full, its essential image is a triangulated subcategory (Lemma 2.1.24), and by definition of realisation functor it contains $\mathcal{H}$, therefore it coincides with $\mathcal{D}^b_{\mathbb{T}}$. (3) $\Rightarrow$ (1) is [12, Theorem 2.9]. (2) $\Rightarrow$ (4) is obvious; (4) $\Rightarrow$ (2) follows by double dévissage (apply Lemma 2.1.23 with $\mathcal{S} = \bigcup_{n \in \mathbb{Z}} \mathcal{H}[n]$). For an explicit proof that (4) $\Leftrightarrow$ (5) and (4) $\Leftrightarrow$ (6) see [67, Theorem 3.11]. $\square$

Remark 2.6.12. While the original results are formulated for a realisation functor in the classical sense, they hold also for our definition, by Remark 2.6.6.

Notice also that conditions (3) and (4) do not depend on the chosen bounded realisation functor: in this sense, for a t-structure $\mathbb{T}$ to be an almost standard bounded t-structure of $\mathcal{D}^b_{\mathbb{T}}$ is an intrinsic property.

If $\mathbb{T}$ is a (co)silting t-structure, there is an additional equivalent condition.

Proposition 2.6.13 ([67, Proposition 5.1])**.** *Assume that $\mathbb{T}$ is (co)silting, with respect to a (co)silting object M, and let $\mathsf{real}^b_{\mathbb{T}} \colon \mathsf{D}^b(\mathcal{H}) \to \mathcal{D}^b_{\mathbb{T}}$ be a bounded realisation functor for $\mathbb{T}$. Then the following are equivalent:*

(1) $\mathsf{real}^b_{\mathbb{T}}$ is an equivalence;

(2) M is (co)tilting.

Proof. Notice that in [67] the term "realisation functor" is used for those constructed from an f-enhancement. Nonetheless, the proof of the referenced result shows that M is (co)tilting if and only if $\mathbb{T}$ is effaçable, and then uses Proposition 2.6.11. Therefore it also applies to our definition of realisation functors. □

On the unbounded side, many known instances in which a realisation functor is an equivalence are encompassed by the following result.

Proposition 2.6.14 ([81, Theorem 6.8])**.** *Assume that $\mathcal{D}$ is the base of a strong and stable derivator $\mathbb{D}$, and that the heart $\mathcal{H}$ of $\mathbb{T}$ satisfies one of the two conditions of Proposition 2.6.10, so that we can construct the realisation functor $\mathfrak{real}_{\mathbb{T}} \colon \mathbb{D}_{\mathcal{H}} \to \mathbb{D}$. Let $\mathsf{real}_{\mathbb{T}} := \mathfrak{real}^{\mathbf{1}}_{\mathbb{T}}$. Then $\mathsf{real}_{\mathbb{T}}$ commutes with products and coproducts and is fully faithful if and only if the following hold:*

(1) $\mathbb{T}$ satisfies one of the equivalent conditions of Proposition 2.6.11;

(2) $\mathbb{T}$ is non-degenerate;

(3) (a) $\mathbb{T}$ is k-cosmashing for some $k \in \mathbb{N}$;

(b) $\mathbb{T}$ is smashing;

(c) $\mathcal{H}$ has enough injectives.

or

(a') $\mathbb{T}$ is cosmashing;

(b') $\mathbb{T}$ is k-smashing for some $k \in \mathbb{N}$;

(c') $\mathcal{H}$ has enough projectives.

Proposition 2.6.15 ([81, Theorems 7.7, 7.9])**.** *Assume $\mathcal{D}$ is the base of a strong and stable derivator, and that there is a classical tilting t-structure in $\mathcal{D}$ with finite gap from $\mathbb{T}$. Then if $\mathbb{T}$ is either tilting or cotilting, the realisation functor $\mathfrak{real}_{\mathbb{T}} \colon \mathbb{D}_{\mathcal{H}} \to \mathbb{D}$ exists and it is a bounded (see §2.4) equivalence of (pre)derivators.*

2.7 HRS-tilting

In this section we recall the HRS-tilting construction, originally introduced by Happel, Reiten and Smalø [25].

Definition 2.7.1. Let $\mathcal{D}$ be a triangulated category, $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ a t-structure in $\mathcal{D}$ and $\mathcal{H}$ its heart. Denote by $H^0_{\mathbb{T}}$ the cohomology functor with respect to $\mathbb{T}$. Let $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{H}$. Then the subcategories

$$\begin{aligned} \mathcal{U}_{\mathbf{t}} &:= \mathcal{U}[1] * \mathcal{T} = \{X \in \mathcal{U} \colon H^0_{\mathbb{T}} X \in \mathcal{T}\} \\ \mathcal{V}_{\mathbf{t}} &:= \mathcal{F} * \mathcal{V} \quad = \{X \in \mathcal{V}[1] \colon H^0_{\mathbb{T}} X \in \mathcal{F}\} \end{aligned}$$

form a t-structure $\mathbb{T}_{\mathbf{t}}$. Its heart is $\mathcal{H}_{\mathbf{t}} := \mathcal{F}[1] * \mathcal{T}$. $\mathbb{T}_{\mathbf{t}}$ and $\mathcal{H}_{\mathbf{t}}$ are said to be obtained by **HRS-tilting** $\mathbb{T}$ with respect to $\mathbf{t}$, or more shortly, to be **HRS-tilted** from $\mathbb{T}$.

Proof. The following proof is a straightforward adaptation of [25] to our slightly more general formulation.

Clearly $\mathcal{U}_{\mathbf{t}}[1] \subseteq \mathcal{U}_{\mathbf{t}}$ and $\mathcal{V}_{\mathbf{t}} \subseteq \mathcal{V}_{\mathbf{t}}[1]$. Denote by $\tau^{\leq 0}_{\mathbb{T}}, \tau^{\geq 1}_{\mathbb{T}}$ and $H^0_{\mathbb{T}}$ the truncation and cohomology functors with respect to $\mathbb{T}$. Since $\mathcal{U}_{\mathbf{t}} \subseteq \mathcal{U}$ and $\mathcal{V}_{\mathbf{t}} \subseteq \mathcal{V}[1]$, we have

$$\mathsf{Hom}_{\mathcal{D}}(\mathcal{U}_{\mathbf{t}}, \mathcal{V}_{\mathbf{t}}) \simeq \mathsf{Hom}_{\mathcal{D}}(H^0_{\mathbb{T}}(\mathcal{U}_{\mathbf{t}}), H^0_{\mathbb{T}}(\mathcal{V}_{\mathbf{t}})) \simeq \mathsf{Hom}_{\mathcal{D}}(\mathcal{T}, \mathcal{F}) = \mathsf{Hom}_{\mathcal{H}}(\mathcal{T}, \mathcal{F}) = 0.$$

It is left to verify (t3), i.e. the existence of approximation triangles. Let $X \in \mathcal{D}$, and consider the octahedral diagram

$$\begin{tikzcd} tH^0_{\mathbb{T}}X \arrow[r] \arrow[d, equal] & H^0X \arrow[r] \arrow[d] & fH^0_{\mathbb{T}}X \arrow[r] \arrow[d] & (tH^0_{\mathbb{T}})[1] \arrow[d, equal] \\ tH^0_{\mathbb{T}}X \arrow[r] \arrow[d] & (\tau^{\leq -1}_{\mathbb{T}}X)[1] \arrow[r] \arrow[d] & U[1] \arrow[r] \arrow[d, "\alpha[1]"] & (tH^0_{\mathbb{T}})[1] \arrow[d] \\ 0 \arrow[r] \arrow[d] & (\tau^{\leq 0}_{\mathbb{T}}X)[1] \arrow[r, equal] \arrow[d] & (\tau^{\leq 0}_{\mathbb{T}}X)[1] \arrow[r] \arrow[d] & 0 \arrow[d] \\ (tH^0_{\mathbb{T}}X)[1] \arrow[r] & (H^0X)[1] \arrow[r] & (fH^0_{\mathbb{T}}X)[1] \arrow[r] & (tH^0_{\mathbb{T}})[2] \end{tikzcd}$$

The top row is the $\mathbf{t}$-approximation sequence of $H^0_{\mathbb{T}}X$ in $\mathcal{H}$; the second column is (a shift of) the $\mathbb{T}$-approximation triangle of $\tau^{\leq 0}_{\mathbb{T}}X$. The object U defined by the above diagram belongs to $\mathcal{U}_{\mathbf{t}}$ by the second row. We claim that it is the left truncation of X with respect to $\mathbb{T}_{\mathbf{t}}$. Indeed, we have another octahedral diagram

$$\begin{tikzcd}
U \arrow[r, "\alpha"] \arrow[d, equal] & \tau_{\mathbb{T}}^{\leq 0}X \arrow[r] \arrow[d] & fH_{\mathbb{T}}^0X \arrow[r] \arrow[d] & U[1] \arrow[d] \\
U \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & U[1] \arrow[d] \\
0 \arrow[r] \arrow[d] & \tau_{\mathbb{T}}^{\geq 1}X \arrow[r, equal] \arrow[d] & \tau_{\mathbb{T}}^{\geq 1}X \arrow[r] \arrow[d] & 0 \arrow[d] \\
U[1] \arrow[r] & (\tau_{\mathbb{T}}^{\leq 0}X)[1] \arrow[r] & (fH_{\mathbb{T}}^0X)[1] \arrow[r] & U[2]
\end{tikzcd}$$

where the first row is (a shift of) the third column of the previous diagram, while the second column is the $\mathbb{T}$-approximation triangle of X. The third columns shows that $V \in \mathcal{V}_{\mathbf{t}}$, so the second row is a $\mathbb{T}_{\mathbf{t}}$-approximation triangle for X.

The claim that $\mathcal{H}_{\mathbf{t}} = \mathcal{F}[1] * \mathcal{T}$ is easy to prove. □

Remark 2.7.2. In the literature, there are two slightly but crucially different versions of this definition. The one above (e.g. [11, 17, 62]), works inside an arbitrary triangulated category $\mathcal{D}$, from a t-structure $\mathbb{T}$ with heart $\mathcal{H}$. The original one ([25], but also [12, 59]) only deals with $\mathcal{D} = \mathsf{D}(\mathcal{H})$ a derived category and $\mathbb{T}$ the standard t-structure. This leads to a possible ambiguity: given a t-structure $\mathbb{T}$ in $\mathcal{D}$, with heart $\mathcal{H}$, and a torsion pair $\mathbf{t}$ in $\mathcal{H}$, "HRS-tilting $\mathcal{H}$ with respect to $\mathbf{t}$" may mean "inside $\mathcal{D}$" or "inside $\mathsf{D}(\mathcal{H})$". The outcome is always a t-structure, but the ambient category differs; this has also consequences on the reversibility of HRS-tilting (see Remark 2.7.7). We will always use Definition 2.7.1. Sometimes we will want to apply results which only hold for the more restrictive definition: in those case, we will always be in the situation of the following Lemma and Remark.

Lemma 2.7.3. *Let $\mathcal{D}, \mathcal{D}'$ be triangulated categories, and $\mathbb{T}, \mathbb{T}'$ t-structures in $\mathcal{D}$ and $\mathcal{D}'$, respectively. Denote by $\mathcal{H}, \mathcal{H}'$ their hearts. Let $F\colon \mathcal{D} \to \mathcal{D}'$ be a triangulated equivalence, which is t-exact with respect to $\mathbb{T}$ and $\mathbb{T}'$. For any torsion pair $\mathbf{t}$ in $\mathcal{H}$, the equivalence $F_{|\mathcal{H}}\colon \mathcal{H} \to \mathcal{H}'$ gives a torsion pair $\mathbf{t}' = F\mathbf{t}$ in $\mathcal{H}'$. Let $\mathbb{T}_{\mathbf{t}}, \mathbb{T}'_{\mathbf{t}'}$ be the HRS-tilts of $\mathbb{T}, \mathbb{T}'$ with respect to $\mathbf{t}, \mathbf{t}'$ respectively; then F is t-exact with respect to $\mathbb{T}_{\mathbf{t}}$ and $\mathbb{T}'_{\mathbf{t}'}$.*

Proof. Obvious from the descriptions of the aisle and coaisle of the HRS-tilts. □

Remark 2.7.4. In particular, we will use this Lemma with $\mathcal{D}$ a triangulated category and $\mathbb{T}$ a t-structure, with heart $\mathcal{H}$, which is an almost standard bounded t-structure of $\mathcal{D}^b_{\mathbb{T}}$. Recall that this means that there is a bounded realisation functor $\mathsf{real}^b_{\mathbb{T}}\colon \mathsf{D}^b(\mathcal{H}) \to \mathcal{D}^b_{\mathbb{T}} = \mathcal{D}$ which is an equivalence; therefore we may take

$\mathcal{D}' = \mathsf{D}^b(\mathcal{H})$, $\mathbb{T}' = \mathbb{D}$ the standard t-structure and $F = \mathsf{real}_{\mathbb{T}}^{b\,-1}$. The Lemma then guarantees that $\mathsf{real}_{\mathbb{T}}^{b}$ identifies the two possible meaning of HRS-tilting $\mathcal{H}$ with respect to $\mathbf{t}$.

Remark 2.7.5. We point out two easy first observations.

(1) One sees immediately that $\mathbf{t}' := (\mathcal{F}[1], \mathcal{T})$ is a torsion pair in $\mathcal{H}_{\mathbf{t}}$, with the $\mathbf{t}'$-approximation sequences being the $\mathbb{T}$-approximation triangles.

(2) From the definition, it is clear that $\mathcal{U}[1] \subseteq \mathcal{U}_{\mathbf{t}} \subseteq \mathcal{U}$ and $\mathcal{V} \subseteq \mathcal{V}_{\mathbf{t}} \subseteq \mathcal{V}[1]$, i.e. $\mathsf{gap}(\mathbb{T}, \mathbb{T}_{\mathbf{t}}) \leq 1$.

(3) The HRS-tilt of $\mathbb{T}$ with respect to $\mathbf{t}$ in $\mathcal{D}$ induces by restriction the HRS-tilt of $\mathbb{T} \cap \mathcal{D}^b_{\mathbb{T}}$ with respect to $\mathbf{t}$ in $\mathcal{D}^b_{\mathbb{T}}$.

In fact, the following result by Polishchuk gives a converse of (2).

Proposition 2.7.6 ([62, Lemma 1.1.2])**.** *Let $\mathcal{D}$ be a triangulated category, and $\mathbb{T}, \mathbb{T}'$ be t-structures with gap ≤ 1. Then they are obtained from each other by HRS-tilting.*

Proof. We recall the idea of the proof. If $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ and $\mathbb{T}' = (\mathcal{U}', \mathcal{V}')$ have hearts $\mathcal{H}, \mathcal{H}'$ respectively, then one constructs torsion pairs $\mathbf{t} = (\mathcal{H} \cap \mathcal{U}', \mathcal{H} \cap \mathcal{V}')$ and $\mathbf{t}' = (\mathcal{H}' \cap \mathcal{U}, \mathcal{H}' \cap \mathcal{V})$ in $\mathcal{H}$ and $\mathcal{H}'$ respectively. HRS-tilting $\mathbb{T}$ with respect to $\mathbf{t}$ gives $\mathbb{T}'$, and viceversa. □

Remark 2.7.7. Notice that this result in particular says that the HRS-tilting construction is *reversible*; in fact, from the proof we see that the torsion pair $\mathbf{t}'$ in $\mathcal{H}_{\mathbf{t}}$ needed to reconstruct $\mathcal{H}$ is precisely that of Remark 2.7.5(1). This is not the case with the original definition of [25].

Remark 2.7.8. In Remark 2.7.5(1), if $\mathcal{H}_{\mathbf{t}}$ happens to be a Grothendieck category and $\mathcal{T}$ is a hereditary torsion class in $\mathcal{H}$, it can moreover be shown that $\mathcal{T}$ is in fact a TTF class in $\mathcal{H}_{\mathbf{t}}$. Since $\mathcal{H}_{\mathbf{t}}$ is Grothendieck and $\mathcal{T}$ is a torsionfree class, $\mathcal{T}$ is closed under coproducts and it only remains to see that it is closed under quotients. If $X \in \mathcal{T}$ and $f \colon X \to Z$ is an epimorphism in $\mathcal{H}_{\mathbf{t}}$, $H^{-1}_{\mathbb{T}}(Z)$ lies in $\mathcal{F}$ (because Z lies $\mathcal{H}_{\mathbf{t}}$) and, simultaneously, $H^{-1}_{\mathbb{T}}(Z)$ is a subobject (in $\mathcal{H}_{\mathbb{T}}$) of $\ker(f)$, which is an object of $\mathcal{T}$. Thus, we have $H^{-1}_{\mathbb{T}}(Z) = 0$ and Z lies in $\mathcal{T}$.

As can be expected, properties of a torsion pair translate to properties of the corresponding HRS-tilted t-structure.

Lemma 2.7.9. *Let $\mathcal{D}$ be a* TR5 *triangulated category, $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ a smashing t-structure in $\mathcal{D}$, with heart $\mathcal{H}$, and $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ a torsion pair in $\mathcal{H}$. Then the HRS-tilted t-structure $\mathbb{T}_{\mathbf{t}}$ is smashing.*

Proof. By Lemma 2.5.39(2), $\mathcal{H}$ is AB4; in particular, $\mathcal{F}$ is closed under coproducts (Lemma 1.4.2(5)). Since by hypothesis also $\mathcal{V}$ is closed under coproducts, so is the coaisle $\mathcal{F} * \mathcal{V}$ of $\mathbb{T}_{\mathbf{t}}$. □

Proposition 2.7.10 ([73, Propositions 6.1 and 6.4])**.** *Let $\mathcal{D}$ be the base of a strong and stable derivator, and $\mathbb{T}$ a homotopically smashing t-structure in $\mathcal{D}$, with heart $\mathcal{H}$. Let $\mathbf{t}$ be a torsion pair in $\mathcal{H}$, and $\mathbb{T}_{\mathbf{t}}$ the HRS-tilted t-structure, with heart $\mathcal{H}_{\mathbf{t}}$. Then:*

(1) $\mathbb{T}_{\mathbf{t}}$ is homotopically smashing if and only if $\mathbf{t}$ is of finite type.

(2) if $\mathbb{T}_{\mathbf{t}}$ is compactly generated, then $\mathbf{t}$ is generated by finitely presented objects.

Proposition 2.7.11 ([59, Theorem 4.9, Corollary 4.10])**.** *Let $\mathcal{D}$ be a triangulated category, and $\mathbb{T}$ an almost standard bounded t-structure in $\mathcal{D}$, with heart $\mathcal{H}$. Let $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{H}$ such that either*

(1) $\mathbf{t}$ is hereditary, or

(2) $\mathcal{T}$ cogenerates $\mathcal{H}$, or

(3) $\mathcal{F}$ generates $\mathcal{H}$.

Then the HRS-tilt of $\mathbb{T}$ with respect to $\mathbf{t}$ has Grothendieck heart if and only if $\mathbf{t}$ is of finite type.

Proof. We comment on the fact that the referenced results are formulated with $\mathcal{D} = \mathsf{D}(\mathcal{H}_{\mathbb{T}})$ and $\mathbb{T}$ the standard t-structure. We argue as anticipated in Lemma 2.7.3 and Remark 2.7.4: by Remark 2.7.5(3) the heart of the HRS-tilt of $\mathbb{T}$ with respect to $\mathbf{t}$ in $\mathcal{D}$ coincides with that of the HRS-tilt of $\mathbb{T} \cap \mathcal{D}^b_{\mathbb{T}}$ in $\mathcal{D}^b_{\mathbb{T}}$; this is equivalent via $\mathsf{real}^b_{\mathbb{T}}$ to the heart of the HRS-tilt of the standard t-structure of $\mathsf{D}^b(\mathcal{H})$ with respect to $\mathsf{real}^{b\,-1}_{\mathbb{T}}\mathbf{t}$. This in turn is again the heart of the HRS-tilt of the standard t-structure of $\mathsf{D}(\mathcal{H})$. Now we can apply the results of [59], to say that this heart is Grothendieck if and only if $\mathsf{real}^{b\,-1}_{\mathbb{T}}\mathbf{t}$ is of finite type, which is equivalent to $\mathbf{t}$ being of finite type. □

Proposition 2.7.12 ([71, Theorem 5.2])**.** *Let $\mathcal{D}$ be a triangulated category and $\mathbb{T}$ an almost standard bounded t-structure of $\mathcal{D}$ with locally coherent Grothendieck heart $\mathcal{H}$. Let $\mathbf{t}$ be a torsion pair in $\mathcal{H}$, and $\mathbb{T}_{\mathbf{t}}$ be the HRS-tilt of $\mathbb{T}$ with respect to $\mathbf{t}$, with heart $\mathcal{H}_{\mathbf{t}}$. Denote by $\mathcal{D}'$ the subcategory of $\mathcal{D}^b_{\mathbb{T}}$ consisting of the objects with finitely presented $\mathbb{T}$-cohomologies. Then the following are equivalent:*

(1) $\mathbb{T}_{\mathbf{t}}$ restricts to $\mathcal{D}'$ and $\mathcal{H}_{\mathbf{t}}$ is a locally coherent Grothendieck category with $\mathsf{fp}(\mathcal{H}_{\mathbf{t}}) = \mathcal{H}_{\mathbf{t}} \cap \mathcal{D}'$.

(2) $\mathbf{t}$ is of finite type and restrictable.

Proof. As for Proposition 2.7.11, the referenced result is stated for $\mathcal{D} = \mathsf{D}(\mathcal{H})$, $\mathbb{T}$ the standard t-structure and $\mathcal{D}' = \mathsf{D}^b(\mathsf{fp}(\mathcal{H}))$. A similar argument as used before applies, in view of the fact that $\mathsf{real}^b_{\mathbb{T}}$ identifies $\mathsf{D}^b(\mathsf{fp}(\mathcal{H}))$ with $\mathcal{D}'$. □

2.8 Iterated HRS-tilting

In this section, $\mathcal{D}$ is a fixed triangulated category.

The HRS-tilting procedure can obviously be iterated: starting from a t-structure $\mathbb{T}_0$ of $\mathcal{D}$, with heart $\mathcal{H}_0$, and a torsion pair $\mathbf{t}_0$ in $\mathcal{H}_0$, one can construct the HRS-tilt $\mathbb{T}_1$, with heart $\mathcal{H}_1$. Given a torsion pair $\mathbf{t}_1$ in $\mathcal{H}_1$, another HRS-tilt yields a t-structure $\mathbb{T}_2$ with heart $\mathcal{H}_2$, and so on. From a t-structure $\mathbb{T}_n$ with heart $\mathcal{H}_n$ and a torsion pair $\mathbf{t}_n$ in $\mathcal{H}_n$, one construct the HRS-tilt $\mathbb{T}_{n+1}$.

Definition 2.8.1. In the notation above, we will say that $\mathbb{T}_n$ is obtained by **n-iterated HRS-tilting** from $\mathbb{T}_0$. The sequence $(\mathbb{T}_0, \mathbb{T}_1, \dots, \mathbb{T}_n)$ will be called a **chain of HRS-tilts** from $\mathbb{T}_0$ to $\mathbb{T}_n$, of **length** n.

Remark 2.8.2. (1) By Remark 2.7.7, if $(\mathbb{T}_0, \mathbb{T}_1, \dots, \mathbb{T}_n)$ is a chain of HRS-tilts from $\mathbb{T}_0$ to $\mathbb{T}_n$, then $(\mathbb{T}_n, \mathbb{T}_{n-1}, \dots, \mathbb{T}_0)$ is a chain of HRS-tilts from $\mathbb{T}_n$ to $\mathbb{T}_0$.

(2) By Remarks 2.5.32(4) and 2.7.5(2), if $\mathbb{S}$ is obtained by n-iterated HRS-tilting from $\mathbb{T}$, then $\mathsf{gap}(\mathbb{T}, \mathbb{S}) \leq n$.

Given two t-structures $\mathbb{T}, \mathbb{S}$ with $\mathsf{gap}(\mathbb{T}, \mathbb{S}) \leq n$, it is natural two ask whether there is a chain of HRS-tilts of length n from one to the other. There are two candidates, first considered by Fiorot, Mattiello and Tonolo [17].

Definition 2.8.3. Let $\mathbb{T} = (\mathcal{U}, \mathcal{V}), \mathbb{T}' = (\mathcal{U}', \mathcal{V}')$ be t-structures with $\mathsf{gap}(\mathbb{T}, \mathbb{T}') \leq n$. The pair $(\mathbb{T}, \mathbb{T}')$ is called:

(1) **right filterable** if the complete precoaisles $\mathcal{V}_i := \mathcal{V}' \cap \Sigma^i \mathcal{V}$ are the coaisles of some t-structures, for every $i \in \mathbb{Z}$.

(2) **left filterable** if the cocomplete preaisles $\mathcal{U}_i := \mathcal{U}' \cap \Sigma^i \mathcal{U}$ are the aisles of some t-structures, for every $i \in \mathbb{Z}$.

Remark 2.8.4. In the notation of the Definition, notice that we have

$$\mathcal{V}_i = \begin{cases} \Sigma^i \mathcal{V} & \text{for } i \ll 0 \\ \mathcal{V}' & \text{for } i \gg 0 \end{cases} \quad \text{and} \quad \mathcal{U}_i = \begin{cases} \mathcal{U}' & \text{for } i \ll 0 \\ \Sigma^i \mathcal{U} & \text{for } i \gg 0 \end{cases}$$

Proposition 2.8.5. *Let $\mathcal{D}$ be a* TR5 *triangulated category, and let $\mathbb{T}, \mathbb{T}'$ be t-structures weakly generated by sets $\mathcal{S}, \mathcal{S}'$ of objects, with $\mathsf{gap}(\mathbb{T}, \mathbb{T}') < \infty$.*

(1) If $\mathcal{S}, \mathcal{S}' \subseteq \mathcal{D}^c$, then the pair $(\mathbb{T}, \mathbb{T}')$ is right filterable.

(2) *If $\mathcal{D}$ is well-generated, then the pair $(\mathbb{T}, \mathbb{T}')$ is right filterable.*

Proof. Let $\mathbb{T} = (\mathcal{U}, \mathcal{V}), \mathbb{T}' = (\mathcal{U}', \mathcal{V}')$. In both the cases of the Proposition, the idea is the same. Since $\mathcal{V} = \mathcal{S}^{\perp_{\leq 0}}$ and $\mathcal{V}' = \mathcal{S}'^{\perp_{\leq 0}}$, we have

$$\mathcal{V}_i = \mathcal{V}' \cap \Sigma^i \mathcal{V} = (\mathcal{S}^{\perp_{\leq 0}}) \cap \Sigma^i (\mathcal{S}'^{\perp_{\leq 0}}) = (\mathcal{S} \cup \Sigma^i \mathcal{S}')^{\perp_{\leq 0}}.$$

If $\mathcal{S}, \mathcal{S}' \subseteq \mathcal{D}^c$ or $\mathcal{D}$ is well-generated, by Propositions 2.5.19 and 2.5.22, this set $\mathcal{S} \cup \Sigma^i \mathcal{S}'$ generates a t-structure, and so $\mathcal{V}_i$ is a coaisle. We mention that (1) appeared as [50, Lemma 4.9]. $\square$

Proposition 2.8.6 ([17, Lemma 2.10])**.** *Let $\mathcal{D}$ be a triangulated category, and let $\mathbb{T} = (\mathcal{U}, \mathcal{V}), \mathbb{T}' = (\mathcal{U}', \mathcal{V}')$ be t-structures with $\mathsf{gap}(\mathbb{T}, \mathbb{T}') < \infty$. If $\mathcal{D}$ is* TR5*, the pair $(\mathbb{T}, \mathbb{T}')$ is right filterable as soon as one of the following conditions holds:*

(1) *$\mathcal{V}, \mathcal{V}'$ are closed under Milnor colimits;*

(2) *$\Sigma^i \mathcal{V}$ is closed under the right truncation functor with respect to $\mathbb{T}'$, for every $i \in \mathbb{Z}$.*

Dually, if $\mathcal{D}$ is TR5**, the pair $(\mathbb{T}, \mathbb{T}')$ is left filterable as soon as one of the following conditions holds:*

(1) *$\mathcal{U}, \mathcal{U}'$ are closed under Milnor limits (defined dually to Milnor colimits);*

(2) *$\Sigma^i \mathcal{U}$ is closed under the left truncation functor with respect to $\mathbb{T}'$, for every $i \in \mathbb{Z}$.*

Proposition 2.8.7 ([17, Theorem 2.13])**.** *Let $\mathbb{T}, \mathbb{T}'$ be t-structures such that $\mathsf{gap}(\mathbb{T}, \mathbb{T}') \leq n$ and that the pair $(\mathbb{T}, \mathbb{T}')$ is either left or right filterable. Then there is a chain of HRS-tilts from $\mathbb{T}$ to (a shift of) $\mathbb{T}'$.*

Proof. We show the right filterable case; the other one is similar. Let $\mathbb{T} = (\mathcal{U}, \mathcal{V}), \mathbb{T}' = (\mathcal{U}', \mathcal{V}')$, and assume that $\mathcal{V} \subseteq \mathcal{V}' \subseteq \Sigma^n \mathcal{V}$ (up to shifting $\mathbb{T}'$). If the pair $(\mathbb{T}, \mathbb{T}')$ is right filterable, for every $i \in \mathbb{Z}$ we have a t-structure $\mathbb{T}_i = (\mathcal{U}_i, \mathcal{V}_i = \mathcal{V}' \cap \Sigma^i \mathcal{V})$. By assumption, $\mathbb{T}_0 = \mathbb{T}$ and $\mathbb{T}_n = \mathbb{T}'$, as they have the same coaisles. Now we show that $\mathbb{T}_{i+1}$ is obtained by HRS-tilting $\mathbb{T}_i$, for every $i \in \mathbb{Z}$. Indeed, we have

$$\begin{array}{ccccc} \mathcal{V}_i & \subseteq & \mathcal{V}_{i+1} & \subseteq & \Sigma \mathcal{V}_i \\ \| & & \| & & \| \\ \mathcal{V}' \cap \Sigma^i \mathcal{V} & \subseteq & \mathcal{V}' \cap \Sigma^{i+1} \mathcal{V} & \subseteq & \Sigma(\mathcal{V}' \cap \Sigma^i \mathcal{V}) \end{array}$$

i.e. $\mathsf{gap}(\mathbb{T}_i, \mathbb{T}_{i+1}) \leq 1$. Therefore we conclude by Proposition 2.7.6. $\square$

2.9 HRS-tilting and derived equivalences

The material for this section is taken from joint work with J. Vitória [61], when not specified otherwise. Already in [25], the authors dealt with the question of derived equivalences arising via their HRS-tilting procedure. Given an abelian category $\mathcal{A}$ with enough injectives and a torsion pair $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$, they constructed a bounded realisation functor $\mathsf{real}^b_{\mathbb{D}_\mathbf{t}} \colon \mathsf{D}^b(\mathcal{H}_\mathbf{t}) \to \mathsf{D}^b(\mathcal{A})$ for the t-structure $\mathbb{D}_\mathbf{t}$, with heart $\mathcal{H}_\mathbf{t}$, obtained by HRS-tilting the standard t-structure $\mathbb{D}$ with respect to $\mathbf{t}$. The existence of this bounded realisation functor is for us a particular case of Proposition 2.6.9. Then they proved the following result.

Proposition 2.9.1 ([25, Theorem I.3.3(b)]). *In the situation above, if $\mathcal{T}$ cogenerates $\mathcal{A}$, then $\mathsf{real}^b_{\mathbb{D}_\mathbf{t}}$ is an equivalence.*

This particular case is vastly generalised by the following criterion, by Chen, Han and Zhou.

Proposition 2.9.2 ([12, Theorem 3.4]). *Let $\mathcal{D}$ be a triangulated category, and $\mathbb{T}$ a almost standard bounded t-structure in $\mathcal{D}$ with heart $\mathcal{H}$. Let $\mathbf{t}$ be a torsion pair in $\mathcal{H}$, and denote by $\mathbb{T}_\mathbf{t}$ the HRS-tilted t-structure, with heart $\mathcal{H}_\mathbf{t}$. Assume that a bounded realisation functor $\mathsf{real}^b_{\mathbb{T}_\mathbf{t}} \colon \mathsf{D}^b(\mathcal{H}_\mathbf{t}) \to \mathcal{D}^b_{\mathbb{T}_\mathbf{t}} = \mathcal{D}$ exists. Then the following are equivalent:*

(1) $\mathsf{real}^b_{\mathbb{T}_\mathbf{t}}$ is an equivalence, i.e. $\mathbb{T}_\mathbf{t}$ is an almost standard bounded t-structure for $\mathcal{D}$;

(2) (2-effaçabilité) For every $X, Y \in \mathcal{H}$ and morphism $f \in \mathsf{Hom}_\mathcal{D}(X, \Sigma^2 Y)$, there exists an epimorphism $g \colon Z \to X$ in $\mathcal{H}$ such that $f \circ g = 0$.

(3) (2-co-effaçabilité) For every $X, Y \in \mathcal{H}$ and morphism $f \in \mathsf{Hom}_\mathcal{D}(X, \Sigma^2 Y)$, there exists a monomorphism $h \colon Y \to Z$ in $\mathcal{H}$ such that $h[2] \circ f = 0$.

(4) Every object $X \in \mathcal{H}$ fits in an exact sequence in $\mathcal{H}$

$$\varepsilon_X : \quad 0 \to F_0 \to F_1 \to X \to T_0 \to T_1 \to 0$$

such that $T_i \in \mathcal{T}$, $F_i \in \mathcal{F}$ and ε_X represents the zero element of $\mathsf{Ext}^3_\mathcal{H}(T_1, F_0)$.

Proof. Again, the referenced result is stated for $\mathcal{H} = \mathcal{A}$ an abelian category, $\mathcal{D} = \mathsf{D}^b(\mathcal{A})$ its bounded derived category and $\mathbb{T} = \mathbb{D}$ the standard t-structure, and our formulation follows Lemma 2.7.3 and Remark 2.7.4. Assume that $\mathsf{real}^b_\mathcal{H} \colon \mathsf{D}^b(\mathcal{H}) \to \mathcal{D}$ is an equivalence, and let $\mathbf{t}'$ be the torsion pair in $\mathcal{H}$ such that $\mathsf{real}^b_\mathcal{H}\mathbf{t}' = \mathbf{t}$. Let $\mathbb{T}_\mathbf{t}$, with heart $\mathcal{H}_\mathbf{t}$, be the HRS-tilt of $\mathbb{T}$ with respect to $\mathbf{t}$ in $\mathcal{D}$, and let $\mathbb{D}_{\mathbf{t}'}$, with heart $\mathcal{H}_{\mathbf{t}'}$, be the HRS-tilt of the standard t-structure $\mathbb{D}$ of $\mathsf{D}^b(\mathcal{H})$ with respect to $\mathbf{t}'$. By the Lemma and the Remark,

$\mathsf{real}^b_{\mathbb{T}}$ is t-exact with respect to $\mathbb{D}_{\mathbf{t}'}$ and $\mathbb{T}_{\mathbf{t}}$, and therefore it induces an equivalence $\mathsf{real}^b_{\mathbb{T}|\mathcal{H}_{\mathbf{t}'}} : \mathcal{H}_{\mathbf{t}'} \to \mathcal{H}_{\mathbf{t}}$. Now, assume that a bounded realisation functor $\mathsf{real}^b_{\mathbb{T}_{\mathbf{t}}} : \mathsf{D}^b(\mathcal{H}_{\mathbf{t}}) \to \mathcal{D}$ exists; then the diagram

$$\begin{tikzcd} \mathsf{D}^b(\mathcal{H}_{\mathbf{t}'}) \arrow[r, dotted, "\mathsf{real}^b_{\mathbb{D}_{\mathbf{t}'}}"] \arrow[d, "\mathsf{real}^b_{\mathbb{T}|\mathcal{H}}"', "\simeq"] & \mathsf{D}^b(\mathcal{H}) \arrow[d, "\simeq"', "\mathsf{real}^b_{\mathbb{T}}"] \\ \mathsf{D}^b(\mathcal{H}_{\mathbf{t}}) \arrow[r, "\mathsf{real}^b_{\mathbb{T}_{\mathbf{t}}}"] & \mathcal{D} \end{tikzcd}$$

defines a functor $\mathsf{real}^b_{\mathbb{D}_{\mathbf{t}'}} : \mathsf{D}^b(\mathcal{H}_{\mathbf{t}'}) \to \mathsf{D}^b(\mathcal{H})$, which is easily seen to be a realisation functor for $\mathbb{D}_{\mathbf{t}'}$. Now one can apply the original theorem [12, Theorem 3.4] to this functor $\mathsf{real}^b_{\mathbb{D}_{\mathbf{t}'}}$. Clearly, it is an equivalence if and only if $\mathsf{real}^b_{\mathbb{T}_{\mathbf{t}}}$ is. Via the equivalence $\mathsf{real}^b_{\mathbb{T}}$, conditions (2-3) are true for $\mathcal{H}$ in $\mathsf{D}^b(\mathcal{H})$ if and only if they are true for $\mathcal{H}$ in $\mathcal{D}$. Again via the same equivalence, condition (4) is true for $\mathbf{t}$ in $\mathcal{H}$ if and only if it is for $\mathbf{t}'$ in $\mathcal{H}$, and we conclude.

We comment on the fact that (2) and (3) above are an equivalent way to express item (3) of the referenced result, which states that the canonical morphisms $\mathsf{Ext}^2_{\mathcal{H}_{\mathbb{T}}}(X,Y) \to \mathsf{Hom}_{\mathsf{D}(\mathcal{A})}(X,Y[2])$ are isomorphisms (see [8, 67]). $\square$

Remark 2.9.3. Item (4) takes place inside $\mathcal{A}$, without references to $\mathsf{D}(\mathcal{A})$; therefore the fact that $\mathsf{real}^b_{\mathbb{T}_{\mathbf{t}}}$ is an equivalence is intrinsic to the torsion pair $\mathbf{t}$. Given an object X of $\mathcal{A}$, a sequence ε_X as above, if it exists, will be called a **CHZ-sequence** for X.

This criterion can be further simplified when $\mathcal{A}$ has some additional properties. We present here a different proof with respect to that of [61].

Lemma 2.9.4. *Consider morphisms $B \overset{u}{\leftarrow} A \overset{v}{\to} C$, and their pushout D*

$$\begin{tikzcd} 0 \arrow[r] & \ker u \arrow[r, "i"] \arrow[d, dotted, "v_|"] & A \arrow[r, "u"] \arrow[d, "v"] & B \arrow[r, "w"] \arrow[d, "\bar v"] & \operatorname{coker} u \arrow[r] \arrow[d, dotted, "\simeq"] & 0 \\ 0 \arrow[r] & \ker \bar u \arrow[r, "\bar i"] & C \arrow[r, "\bar u"] & D \arrow[r, "\bar w"] & \operatorname{coker} \bar u \arrow[r] & 0 \end{tikzcd}$$

Then $\operatorname{coker} u \simeq \operatorname{coker} \bar u$, and $v_|$ is an epimorphism.

Proof. The morphism $\operatorname{coker} u \to \operatorname{coker} \bar u$ factors $\bar w \bar v$ through $\operatorname{coker} u$. Its inverse factors through $\operatorname{coker} \bar u$ the morphism $D \to \operatorname{coker} u$, which in turns factors the morphisms $B \overset{w}{\to} \operatorname{coker} u$ and $C \overset{0}{\to} \operatorname{coker} u$.

The pushout square gives an exact sequence

$$A \xrightarrow{\left[\begin{smallmatrix} -u \\ v \end{smallmatrix}\right]} B \oplus C \xrightarrow{[\,\bar v\ \bar u\,]} D \longrightarrow 0$$

Any $y \in \ker \bar u$ gives an element $(0,y) \in \ker [\,\bar v\ \bar u\,] = \operatorname{im} \left[\begin{smallmatrix} -u \\ v \end{smallmatrix}\right]$, so there exists $x \in A$ such that $(0,y) = (-u(x), v(x))$, i.e. $x \in \ker u$ and $y = v(x) = v_|(x)$. $\square$

Proposition 2.9.5 ([61, Proposition 5.4])**.** *Let $\mathcal{A}$ be an* AB4 *abelian category with a set of generators $(G_i \mid i \in I)$. Then there exist CHZ-sequences for every object of $\mathcal{A}$ if and only if there is a CHZ-sequence for G_i, for every $i \in I$.*

Proof. We prove the only non-trivial implication ($\Leftarrow$). First of all, since $\mathcal{A}$ is AB4, both $\mathcal{T}$ and $\mathcal{F}$ are closed under coproducts (Lemma 1.4.2), and coproducts of exact sequences are exact. Lastly, for any family of exact sequences

$$\varepsilon_i \colon 0 \to A_i \to B_i \to C_i \to D_i \to E_i \to 0$$

representing the zero elements of $\mathsf{Ext}^3_{\mathcal{A}}(E_i, A_i)$, their coproduct $\coprod \varepsilon_i$ also represents the zero element of $\mathsf{Ext}^3_{\mathcal{A}}(\coprod E_i, \coprod A_i)$. This shows that there are CHZ-sequences for every coproduct of the generators. Given X in $\mathcal{A}$, let $f \colon G \to X$ be an epimorphism from such a coproduct G of the generators. Consider then the diagram

$$\begin{tikzcd}
\varepsilon_G\colon \arrow[d, Rightarrow, "\varphi"] & 0 \arrow[r] & F_0 \arrow[r, "a"] \arrow[d, dotted, "h"] & F_1 \arrow[r, "b"] \arrow[d, equal] & G \arrow[r, "c"] \arrow[d, "f"] & T_0 \arrow[r, "d"] \arrow[d, "g"] & T_1 \arrow[r] \arrow[d, dotted, "\simeq"] & 0 \\
\varepsilon\colon & 0 \arrow[r] & \ker fb \arrow[r] & F_1 \arrow[r, "fb"] & X \arrow[r, "\bar{c}"] & P \arrow[r, "\bar{d}"] & \operatorname{coker} \bar{c} \arrow[r] & 0
\end{tikzcd}$$

where the top row is a CHZ-sequence for G and P is the pushout of f and c. Since f is an epimorphism, g is as well, so $P \in \mathcal{T}$. The isomorphism $T_1 \simeq \operatorname{coker} \bar{c}$ follows from Lemma 2.9.4, and so does the exactness of the lower row ε in the degree where X is, using the epimorphism $f_{|} \colon \operatorname{im} b = \ker c \to \ker \bar{c}$.

We claim that ε is a CHZ-sequence for X; the only thing left to show is that it represents the zero element of $\mathsf{Ext}^3_{\mathcal{A}}(T_1, \ker fb)$. To this goal, consider the composition $f \circ [\varepsilon_G]$ of Yoneda extensions, which is represented by the second row of the following diagram:

$$\begin{tikzcd}
\varepsilon_G\colon & 0 \arrow[r] & F_0 \arrow[r, "a"] \arrow[d, "h"] & F_1 \arrow[r, "b"] \arrow[d, "h_1"] & G \arrow[r, "c"] \arrow[d, "h_2"] & T_0 \arrow[r, "d"] \arrow[d, "h_3"] & T_1 \arrow[r] \arrow[d, "h_4"] & 0 \\
f \circ \varepsilon_G\colon & 0 \arrow[r] & \ker fb \arrow[r] \arrow[d, equal] & Q_1 \arrow[r] \arrow[d, dotted] & Q_2 \arrow[r, "c"] \arrow[d, dotted] & Q_3 \arrow[r] \arrow[d, dotted] & Q_4 \arrow[r] \arrow[d, dotted] & 0 \\
\varepsilon\colon & 0 \arrow[r] & \ker fb \arrow[r] & F_1 \arrow[r, "fb"] & X \arrow[r, "\bar{c}"] & P \arrow[r, "\bar{d}"] & \operatorname{coker} \bar{c} \arrow[r] & 0
\end{tikzcd}$$

where each Q_i is the pushout of the square of which it is the south-east corner. Again by Lemma 2.9.4, h_4 is an isomorphism. The dotted morphisms to the terms of ε are such that they make the diagram commute, and at the same time they factor the morphisms $\varphi \colon \varepsilon_G \Rightarrow \varepsilon$. In particular, the morphism $Q_4 \to \operatorname{coker} \bar{c}$ is an isomorphism, and therefore $[\varepsilon] = [f \circ \varepsilon_G] = f \circ [\varepsilon_G] = f \circ 0 = 0$ as elements of $\mathsf{Ext}^3_{\mathcal{A}}(T_1, \ker fb)$. $\square$

Another simplification of the criterion happens when the torsion pair $\mathbf{t}$ is hereditary.

Lemma 2.9.6. *Let $\mathcal{A}$ be an abelian category and $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ a hereditary torsion pair in $\mathcal{A}$. Given an object X in $\mathcal{A}$, there exists a CHZ-sequence for X if and only if there exists a sequence of the form*

$$F_X \xrightarrow{b} X \to T_X \to 0 \tag{2.3}$$

with $F \in \mathcal{F}$ and $T \in \mathcal{T}$.

Proof. ($\Leftarrow$) By adding $\ker b \in \mathcal{F}$ to the sequence (2.3) we obtain a sequence

$$0 \to \ker b \to F \xrightarrow{b} X \to T \to 0 \to 0$$

which clearly represents the zero element of $\mathsf{Ext}^3_{\mathcal{A}}(0, \ker b)$, and therefore it is a CHZ-sequence for X. ($\Rightarrow$) Conversely, let

$$0 \to F_0 \to F_1 \to X \xrightarrow{c} T_0 \to T_1 \to 0$$

be a CHZ-sequence for X. Then $\operatorname{im} c \subseteq T_0$ belongs to $\mathcal{T}$, since $\mathbf{t}$ is hereditary, and we have a sequence as in (2.3)

$$F_1 \to X \to \operatorname{im} c \to 0. \qquad \square$$

Now we combine these two results. For objects X, Y in a cocomplete abelian category $\mathcal{A}$, define the **trace of** X **in** Y to be the image $\mathsf{tr}_X(Y)$ of the canonical morphism

$$X^{(\mathsf{Hom}_{\mathcal{A}}(X,Y))} \to Y.$$

If $\mathcal{A}$ is AB4, $\mathsf{tr}_X(Y)$ is equivalently defined as the sum of all the images of morphisms $X \to Y$.

Theorem 2.9.7 ([61, Theorem 5.6])**.** *Let $\mathcal{A}$ be an* AB4 *abelian category with a generator G, and let $\mathbf{t}$ be a hereditary torsion pair in $\mathcal{A}$. Denote by fG the torsion-free part of G with respect to $\mathbf{t}$. Then there exists a CHZ-sequence for every object of $\mathcal{A}$ if and only if $G/\mathsf{tr}_{fG}(G)$ belongs to $\mathcal{T}$.*

Proof. By Proposition 2.9.5 and Lemma 2.9.6, there are CHZ-sequences for every object of $\mathcal{A}$ if and only if there is an exact sequence

$$F \xrightarrow{b} G \to T \to 0 \tag{$*$}$$

with $F \in \mathcal{F}$ and $T \in \mathcal{T}$. ($\Leftarrow$) If $G/\mathsf{tr}_{fG}(G)$ belongs to $\mathcal{T}$, then since $\mathcal{F}$ is closed under coproducts (Lemma 1.4.2) the sequence

$$(fG)^{(\mathsf{Hom}_{\mathcal{A}}(fG,G))} \to G \to G/\mathsf{tr}_{fG}(G) \to 0$$

has the shape ($*$). ($\Rightarrow$) Conversely, assume that the sequence ($*$) exists. Then, by Lemma 1.4.16, there is an epimorphism $(fG)^{(I)} \to F$ for some set I, and we easily deduce that $\operatorname{im} b \subseteq \mathsf{tr}_{fG}(G)$. We then obtain an epimorphism $T \simeq \operatorname{coker} b \to G/\mathsf{tr}_{fG}(G)$, which shows that the latter belongs to $\mathcal{T}$. $\square$

We now specialise this result to the case $\mathcal{D} = \mathsf{D}(R)$, for a ring R, $\mathbb{T} = \mathbb{D}$ the standard t-structure and $\mathcal{A} = \mathsf{Mod}(R)$. Notice that for every torsion pair $\mathbf{t}$ in $\mathcal{A}$, if we denote by $\mathbb{D}_{\mathbf{t}}$ the corresponding HRS-tilt of $\mathbb{D}$ and by $\mathcal{H}_{\mathbf{t}}$ its heart, a bounded realisation $\mathsf{real}^b_{\mathbb{D}_{\mathbf{t}}} \colon \mathsf{D}^b(\mathcal{H}_{\mathbf{t}}) \to \mathsf{D}^b(R)$ exists by Proposition 2.6.9. Therefore we are in the situation to apply Proposition 2.9.2.

Lemma 2.9.8. *Let R be a ring and $\mathbf{t}$ be a hereditary torsion pair in $\mathsf{Mod}(R)$. Denote by tR the torsion ideal with respect to $\mathbf{t}$. Then $\mathsf{tr}_{R/tR}(R) = \mathsf{Ann}_l(tR)$.*

Proof. Since tR annihilates (from the right) every torsion-free module (see Lemma 1.4.23), it also annihilates the epimorphic image of a torsion-free module $\mathsf{tr}_{\mathcal{F}G}(G)$, and therefore $\mathsf{tr}_{\mathcal{F}G}(G) \subseteq \mathsf{Ann}_l(tR)$. Conversely, for every $r \in \mathsf{Ann}_l(tR)$, the morphism $r \cdot - \colon R \to R$ factors through R/tR, and therefore $r = r \cdot (1 + tR)$ belongs to $\mathsf{tr}_{R/tR}(R)$. □

Corollary 2.9.9. *Let R be a ring and $\mathbf{t}$ be a hereditary torsion pair in $\mathsf{Mod}(R)$. Denote by tR the torsion ideal with respect to $\mathbf{t}$. Let $\mathbb{D}_{\mathbf{t}}$ be the HRS-tilt of the standard t-structure of $\mathsf{D}(R)$ with respect to $\mathbf{t}$. Then $\mathbb{D}_{\mathbf{t}}$ is an almost standard bounded t-structure of $\mathsf{D}^b(R)$ if and only if $R/\mathsf{Ann}_l(tR) \in \mathcal{T}$.*

Proof. As said above, by Proposition 2.6.9 a bounded realisation for $\mathbb{D}_{\mathbf{t}}$ exists, so we can combine Proposition 2.9.2, Theorem 2.9.7 and Lemma 2.9.8. □

Definition 2.9.10 ([45, Definitions 10.8 and 10.15])**.** A ring R is **semiprime** if the only nilpotent two-sided ideal of R is 0: i.e. $I^k = 0 \implies I = 0$ for every two-sided ideal $I \leq R$.

Example 2.9.11 ([45, Examples 10.17])**.** Examples of semiprime rings are:

(1) the reduced rings, i.e. rings without nilpotent elements;

(2) the semisimple rings, i.e. products of rings of matrices over a division ring (Artin-Wedderburn Theorem);

(3) more generally, the von Neumann regular rings, i.e. rings such that for every element $r \in R$ there exists $s \in R$ such that $r = rsr$;

(4) any product of semiprime rings.

Lemma 2.9.12. *Let R be a semiprime ring, and I be a two-sided ideal. Then $\mathsf{Ann}_l(I) = \mathsf{Ann}_r(I)$.*

Proof. ($\subseteq$) I and $\mathsf{Ann}_l(I)$ are two-sided ideals, and therefore so is $I\mathsf{Ann}_l(I)$. We have $(I\mathsf{Ann}_l(I))^2 = I(\mathsf{Ann}_l(I)I)\mathsf{Ann}_l(I) = 0$, and therefore $I\mathsf{Ann}_l(I) = 0$. ($\supseteq$) The converse inclusion is similar. □

Corollary 2.9.13 ([61, Corollary 5.11]). *Let R be a left noetherian ring which is either commutative or semiprime. In the notation of Corollary 2.9.9, for every hereditary torsion pair $\mathbf{t}$ in $\mathsf{Mod}(R)$ the t-structure $\mathbb{D}_{\mathbf{t}}$ is an almost standard bounded t-structure.*

Proof. Let $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion pair in $\mathsf{Mod}(R)$, and denote by tR the torsion ideal with respect to $\mathbf{t}$. Be R commutative or semiprime, in either case $\mathsf{Ann}_l(tR) = \mathsf{Ann}_r(tR)$. If moreover R is left noetherian, tR is finitely generated as a left ideal, i.e. there are elements $x_1, \dots, x_n$ in R such that $tR = \sum Rx_i$. Now, consider the finite product $(tR)^n$, and the right R-linear morphism $R \to (tR)^n$ given by $1 \mapsto (x_1, \dots, x_n)$. Its kernel is $\mathsf{Ann}_r(tR)$. Indeed, if $r \in \mathsf{Ann}_r(tR)$, then $r \mapsto (x_1 r, \dots, x_n r) = 0$; conversely, if $(x_1 r, \dots, x_n r) = 0$, then also $tR \cdot r = (\sum Rx_i) \cdot r = \sum Rx_i r = 0$, so $r \in \mathsf{Ann}_r(tR)$. Therefore we obtain a monomorphism

$$R/\mathsf{Ann}_l(tR) = R/\mathsf{Ann}_r \hookrightarrow tR^n.$$

The latter being a finite product, i.e. also a coproduct, it belongs to $\mathcal{T}$, and since $\mathbf{t}$ is hereditary we deduce that $R/\mathsf{Ann}_l(tR) \in \mathcal{T}$ as well. $\square$

This corollary has a direct implication in silting theory for commutative noetherian rings.

Corollary 2.9.14. *Every two-term cosilting complex over a commutative noetherian ring is cotilting.*

Proof. If R is a commutative noetherian ring, then the HRS-tilting t-structure at any hereditary torsion pair in $\mathsf{Mod}(R)$ is a cosilting t-structure associated with a two-term cosilting complex ([3, Corollary 4.1, Lemma 4.2]). Since, by Corollary 2.9.13, this t-structure induces a derived equivalence, the two-term cosilting complex must be cotilting ([67, Corollary 5.2]). $\square$

Remark 2.9.15. When R is two-sided noetherian, and either commutative or semiprime, we can recover the previous Corollary as a consequence of Proposition 1.4.28. Indeed, let $\mathfrak{F}$ be the Gabriel filter associated to the hereditary torsion pair $\mathbf{t} = (\mathcal{T}, \mathcal{F})$. Since tR is torsion, then $\mathsf{Ann}_r(tR) \in \mathfrak{F}$ by item (1) of that result, and therefore by definition $R/\mathsf{Ann}_r(tR)$ belongs to $\mathcal{T}$.

The corollary above can fail if R is not assumed to be commutative or semiprime.

Example 2.9.16. Consider the quiver $\mathbb{A}_3 = (1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3)$, and let $R = k\mathbb{A}_3$. It is obviously not commutative, and also not semiprime, since the ideal I of paths of positive length is such that $I^3 = 0$.

The Auslander-Reiten quiver of R is well known to be

$$\begin{tikzcd}[column sep=tiny, row sep=tiny] & & {\begin{smallmatrix}1\\2\\3\end{smallmatrix}} \arrow[dr] & & \\ & {\begin{smallmatrix}2\\3\end{smallmatrix}} \arrow[ur] \arrow[dr] & & {\begin{smallmatrix}1\\2\end{smallmatrix}} \arrow[dr] & \\ 3 \arrow[ur] & & 2 \arrow[ur] & & 1 \end{tikzcd} = \begin{tikzcd}[column sep=tiny, row sep=tiny] & & \bullet \arrow[dr] & & \\ & \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[dr] & \\ \bullet \arrow[ur] & & \bullet \arrow[ur] & & \bullet \end{tikzcd}$$

$R = \langle e_1, e_2, e_3, \alpha, \beta, \alpha\beta \rangle$, where we use the $\langle\rangle$ notation to denote generation as a k-vector space. R is written as sum of indecomposable projectives as

$$R = {\scriptstyle 3} \oplus {\begin{smallmatrix}2\\3\end{smallmatrix}} \oplus {\begin{smallmatrix}1\\2\\3\end{smallmatrix}} = \langle e_3 \rangle \oplus \langle e_2, \beta \rangle \oplus \langle e_1, \alpha, \alpha\beta \rangle$$

By Corollary 1.4.13, hereditary torsion pairs in $\mathsf{Mod}(R)$ are in bijection with hereditary torsion pairs in $\mathsf{mod}(R)$: moreover, since every indecomposable has finite length, a hereditary torsion pair is determined by the simple modules it contains. We list them, with the convention that $\bullet$ represents torsion objects, $\circ$ torsion-free objects, and $\cdot$ the other objects.

(a) $\begin{tikzcd}[column sep=tiny, row sep=tiny] & & \circ \arrow[dr] & & \\ & \circ \arrow[ur] \arrow[dr] & & \circ \arrow[dr] & \\ \circ \arrow[ur] & & \circ \arrow[ur] & & \circ \end{tikzcd}$, (b) $\begin{tikzcd}[column sep=tiny, row sep=tiny] & & \cdot \arrow[dr] & & \\ & \cdot \arrow[ur] \arrow[dr] & & \circ \arrow[dr] & \\ \bullet \arrow[ur] & & \circ \arrow[ur] & & \circ \end{tikzcd}$, (c) $\begin{tikzcd}[column sep=tiny, row sep=tiny] & & \circ \arrow[dr] & & \\ & \circ \arrow[ur] \arrow[dr] & & \cdot \arrow[dr] & \\ \circ \arrow[ur] & & \bullet \arrow[ur] & & \circ \end{tikzcd}$,

(d) $\begin{tikzcd}[column sep=tiny, row sep=tiny] & & \circ \arrow[dr] & & \\ & \circ \arrow[ur] \arrow[dr] & & \circ \arrow[dr] & \\ \circ \arrow[ur] & & \circ \arrow[ur] & & \bullet \end{tikzcd}$, (e) $\begin{tikzcd}[column sep=tiny, row sep=tiny] & & \cdot \arrow[dr] & & \\ & \bullet \arrow[ur] \arrow[dr] & & \cdot \arrow[dr] & \\ \bullet \arrow[ur] & & \bullet \arrow[ur] & & \circ \end{tikzcd}$, (f) $\begin{tikzcd}[column sep=tiny, row sep=tiny] & & \cdot \arrow[dr] & & \\ & \cdot \arrow[ur] \arrow[dr] & & \circ \arrow[dr] & \\ \bullet \arrow[ur] & & \circ \arrow[ur] & & \bullet \end{tikzcd}$,

(g) $\begin{tikzcd}[column sep=tiny, row sep=tiny] & & \circ \arrow[dr] & & \\ & \circ \arrow[ur] \arrow[dr] & & \bullet \arrow[dr] & \\ \circ \arrow[ur] & & \bullet \arrow[ur] & & \bullet \end{tikzcd}$, (h) $\begin{tikzcd}[column sep=tiny, row sep=tiny] & & \bullet \arrow[dr] & & \\ & \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[dr] & \\ \bullet \arrow[ur] & & \bullet \arrow[ur] & & \bullet \end{tikzcd}$.

In particular, if we denote by t_x the corresponding torsion radicals, we have

$$\begin{aligned} &t_aR = t_cR = t_dR = t_gR = 0, \\ &t_bR = t_fR = {\scriptstyle 3} \oplus {\scriptstyle 3} \oplus {\scriptstyle 3} = \langle e_3, \beta, \alpha\beta \rangle, \\ &t_eR = {\scriptstyle 3} \oplus {\begin{smallmatrix}2\\3\end{smallmatrix}} \oplus {\begin{smallmatrix}2\\3\end{smallmatrix}} = \langle e_3, e_2, \beta, \alpha, \alpha\beta \rangle, \\ &t_hR = R \end{aligned}$$

$$\begin{aligned} &\mathsf{Ann}_l(t_aR) = \mathsf{Ann}_l(t_cR) = \mathsf{Ann}_l(t_dR) = \mathsf{Ann}_l(t_gR) = R, \\ &\mathsf{Ann}_l(t_bR) = \mathsf{Ann}_l(t_fR) = 0, \\ &\mathsf{Ann}_l(t_eR) = 0, \\ &\mathsf{Ann}_l(t_hR) = 0 \end{aligned}$$

It follows from Corollary 2.9.9 that the only hereditary torsion pairs whose HRS-tilt is an almost standard bounded t-structure of $\mathsf{D}^b(R)$ are the faithful ones (a,c,d,g) and the trivial one (h).

We give another application of the criterion of Lemma 2.9.6, which is extracted from the proof of [61, Theorem 6.3], and which will be used in proving Theorem 3.3.2.

Proposition 2.9.17. *Let $\mathcal{A}$ be a Grothendieck category, and $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ a hereditary torsion pair. Assume that there is a set of exact functors $(L_i \colon \mathcal{A} \to \mathcal{A}_i \mid i \in I)$, for abelian categories $\mathcal{A}_i$, such that $\mathcal{T} = \bigcap \ker L_i$. Let X be an object of $\mathcal{A}$, denote by fX its torsion-free part, and assume moreover that the group homomorphisms $L_{i\,fX,X} \colon \mathsf{Hom}_{\mathcal{A}}(fX, X) \to \mathsf{Hom}_{\mathcal{A}_i}(L_i fX, L_i X)$ are surjective, for every $i \in I$. Then there is a CHZ-sequence for X.*

Proof. Since $\mathbf{t}$ is hereditary, by Lemma 2.9.6 it is enough to show that there is an exact sequence $F \xrightarrow{b} X \to T \to 0$ with $F \in \mathcal{F}$ and $T \in \mathcal{T}$. We will set $F := fX^{(\mathsf{Hom}_{\mathcal{A}}(fX,X))}$, b to be the canonical morphism and therefore $T := X/\mathsf{tr}_{fX}(X)$, which we need to show to be torsion. Consider the $\mathbf{t}$-approximation sequence for X

$$0 \to tX \to X \xrightarrow{v} fX \to 0.$$

For every $i \in I$, since L_i is exact and $tX \in \ker L_i$, we have an isomorphism $L_i v \colon L_i X \xrightarrow{\cong} L_i fX$ in $\mathcal{A}_i$. By assumption, there must be a morphism u_i in $\mathsf{Hom}_{\mathcal{A}}(fX, X)$ such that $L_i u_i = (L_i v)^{-1} \in \mathsf{Hom}_{\mathcal{A}_i}(L_i fX, L_i X)$. Clearly, since L_i is exact, we have $L_i(\operatorname{coker} u_i) = \operatorname{coker}(L_i u_i) = 0$, because $L_i u_i$ is an isomorphism. Now consider the diagram

$$\begin{tikzcd}
fX^{(\mathsf{Hom}_{\mathcal{A}}(fX,X))} \arrow[r, "b"] & X \arrow[r, "c"] & X/\mathsf{tr}_{fX}(X) \arrow[r] & 0 \\
fX \arrow[u, "j"] \arrow[r, "u_i"] & X \arrow[u, equal] \arrow[r] & \operatorname{coker} u_i \arrow[u, dotted, "\gamma"] \arrow[r] & 0.
\end{tikzcd}$$

where j is the inclusion in the component indexed by $u_i \in \mathsf{Hom}_{\mathcal{A}}(fX, X)$. The dotted morphism γ is an epimorphism, because it factors the epimorphism c; and again, since L_i is exact, $L_i\gamma \colon 0 = L_i(\operatorname{coker} u_i) \to L_i(X/\mathsf{tr}_{fX}(X))$ is an epimorphism, which shows that the latter object is 0 as well. Since this happens for every $i \in I$, we deduce that $X/\mathsf{tr}_{fX}(X) \in \mathcal{T}$, which concludes the proof. $\square$

Chapter 3

Derived equivalences for commutative noetherian rings

The material in this Chapter is taken from joint work with J. Vitória [61].

3.1 Preliminaries: commutative noetherian rings

In this section, R denotes a commutative noetherian ring. The set of prime ideals of R, partially ordered by inclusion, will be denoted by $\mathsf{Spec}(R)$. For an ideal $I \leq R$, we write

$$\vee(I) := \{\mathfrak{p} \in \mathsf{Spec}(R) \colon I \subseteq \mathfrak{p}\} \quad \text{and} \quad \wedge(I) := \{\mathfrak{p} \in \mathsf{Spec}(R) \colon \mathfrak{p} \subseteq I\}$$

The set $\mathsf{Spec}(R)$ has a natural topology, whose closed subsets are the $\vee(I)$ for all ideals $I \leq R$. This is called the **Zariski topology** on $\mathsf{Spec}(R)$. This topological space turns out to encode significant information concerning the representation theory of R.

Definition 3.1.1. A subset $\mathcal{P}$ of $\mathsf{Spec}(R)$ is said to be **specialisation-closed** if for any $\mathfrak{p}$ in $\mathcal{P}$ we have that $\vee(\mathfrak{p})$ is contained in $\mathcal{P}$. Dually, the subset $\mathcal{P}$ is called **generalisation-closed** if for any $\mathfrak{p}$ in $\mathcal{P}$ we have that $\wedge(\mathfrak{p})$ is contained in $\mathcal{P}$.

Note that the complement of a specialisation-closed subset is generalisation-closed and vice-versa. We will denote the complement of a subset $\mathcal{P} \subseteq \mathsf{Spec}(R)$ by $\mathcal{P}^c$. From their definition, specialisation-closed subsets are (possibly infinite) unions of Zariski-closed subsets, and thus, generalisation-closed subsets are (possibly infinite) intersections of Zariski-open subsets. For a family $\mathcal{P} \subseteq \mathsf{Spec}(R)$, its **specialisation closure** is the smallest specialisation-closed set containing $\mathcal{P}$, namely $\vee(\mathcal{P}) := \bigcup_{\mathfrak{p}\in\mathcal{P}} \vee(\mathfrak{p})$.

The spectrum of R is deeply related to the structure of $\mathsf{Mod}(R)$, as we now review. We start with the following definition.

Definition 3.1.2 ([75, §VII.1])**.** Let R be a commutative noetherian ring, and M an R-module. The **set of associated primes of** M is defined as

$$\begin{aligned}\mathsf{Ass}(M) &:= \{\mathfrak{p} \in \mathsf{Spec}(R) \colon \text{there is a monomorphism } R/\mathfrak{p} \hookrightarrow M\} \\ &:= \{\mathfrak{p} \in \mathsf{Spec}(R) \colon \mathfrak{p} = \mathsf{Ann}(m) \text{ for some } m \in M\}.\end{aligned}$$

Lemma 3.1.3 ([75, Proposition VII.1.1])**.** *Let R be a commutative noetherian ring, and M an R-module. If $\mathsf{Ass}(M) = \emptyset$, then $M = 0$.*

Proof. Let $M \neq 0$, and consider the set $\{\mathsf{Ann}(m) \colon 0 \neq m \in M\} \neq \emptyset$, which consists of proper ideals. Since R is noetherian, by Zorn's Lemma this set has a maximal element $\mathfrak{p} = \mathsf{Ann}(m)$, and this must be prime. Indeed, let $I, J \leq R$ such that $IJ \subseteq \mathfrak{p}$. Assume that $I \nsubseteq \mathfrak{p}$, and let $r \in I \setminus \mathfrak{p}$. Then $J \subseteq \mathsf{Ann}(mr) = \mathfrak{p}$ by maximality of $\mathfrak{p}$. □

Example 3.1.4. For a prime $\mathfrak{p} \in \mathsf{Spec}(R)$ we have $\mathsf{Ass}(R/\mathfrak{p}) = \{\mathfrak{p}\}$. Moreover, if we denote by $E(M)$ the injective envelope of a module M, we have $\mathsf{Ass}(E(M)) = \mathsf{Ass}(M)$ ([75, Lemma 1.4]).

The spectrum of the R can be read from the category $\mathsf{Mod}(R)$, by the following theorem of Matlis.

Proposition 3.1.5 ([49, Proposition 3.1])**.** *Let R be a commutative noetherian ring. Then there is a bijection between $\mathsf{Spec}(R)$ and the set of isomorphism classes of indecomposable injective modules, given by the assignement*

$$\mathfrak{p} \mapsto E(R/\mathfrak{p}), \qquad \mathfrak{p}_E \mapsfrom E,$$

where $\mathfrak{p}_E$ is the only prime such that $\mathsf{Ass}(E) = \{\mathfrak{p}_E\}$ (see Example 3.1.4). Moreover, $\mathfrak{p} \leq \mathfrak{q}$ if and only if $\mathsf{Hom}_R(E(R/\mathfrak{p}), E(R/\mathfrak{q})) \neq 0$.

Proof. We add the proof of the last claim. If $\mathfrak{p} \leq \mathfrak{q}$, then we have a morphism $R/\mathfrak{p} \to R/\mathfrak{q}$ which gives a non-zero morphism $E(R/\mathfrak{p}) \to E(R/\mathfrak{q})$. Conversely, notice that for every non-zero element of $R/\mathfrak{p}$, its annihilator is precisely $\mathfrak{p}$. Now, given a non-zero morphism $f \colon E(R/\mathfrak{p}) \to E(R/\mathfrak{q})$, its image intersects the essential submodule $R/\mathfrak{q}$. The preimage of this intersection is a non-zero submodule of $E(R/\mathfrak{p})$, and therefore it also intersects the essential submodule $R/\mathfrak{p}$. Restricting the morphism to this intersection, we obtain a non-zero morphism $f_|$ from a non-zero submodule of $R/\mathfrak{p}$ to $R/\mathfrak{q}$. Let x be an element in the domain of this morphism which is not in the kernel: then $\mathfrak{p} = \mathsf{Ann}(x) \subseteq \mathsf{Ann}(f_|(x)) = \mathfrak{q}$. □

Proposition 3.1.6 ([49, Theorem 2.5])**.** *Let R be a right-noetherian ring. Then every injective right R-module is a coproduct of indecomposable injective right R-modules.*

3.1.1 Supports

Given a prime ideal $\mathfrak{p}$ of R, let $R_\mathfrak{p}$ denote the localisation of R at the complement of $\mathfrak{p}$ (see e.g. [30, §7.2-7.4]) and $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ the residue field of R at $\mathfrak{p}$. Consider the two left derived functors (see §2.3.4)

$$-_\mathfrak{p} := - \otimes_R R_\mathfrak{p} \colon \mathsf{D}(R) \longrightarrow \mathsf{D}(R) \qquad \text{and} \qquad - \otimes^{\mathbb{L}} k(\mathfrak{p}) \colon \mathsf{D}(R) \longrightarrow \mathsf{D}(R),$$

Since $R_\mathfrak{p}$ is a flat R-module (e.g. [30, Proposition 7.7]), for a complex X, $X_\mathfrak{p} := X \otimes_R R_\mathfrak{p}$ is the componentwise localisation of X as an object of $\mathsf{D}(R)$. In particular, we have $H^i(X_\mathfrak{p}) \simeq H^i(X)_\mathfrak{p}$ for all i in $\mathbb{Z}$.

Definition 3.1.7. Let R be a commutative noetherian ring. Given a complex X in $\mathsf{D}(R)$, we define the following subsets of $\mathsf{Spec}(R)$:

- $\mathsf{supp}(X) := \{\mathfrak{p} \in \mathsf{Spec}(R) \colon X \otimes^{\mathbb{L}}_R k(\mathfrak{p}) \neq 0\}$, the (small) **support** of X;
- $\mathsf{Supp}(X) := \{\mathfrak{p} \in \mathsf{Spec}(R) \colon X_\mathfrak{p} \neq 0\}$, the **big support** of X.

The (big) support of a subcategory $\mathcal{X}$ of $\mathsf{D}(R)$ is the union of the (big) supports of the objects in $\mathcal{X}$. Since localisation at $\mathfrak{p}$ commutes with standard cohomology, as noticed above, $\mathsf{Supp}(X) = \mathsf{Supp}(\coprod H^i(X))$. This set is therefore also called the **homological support** of X.

The following Lemma collects some facts about supports of objects of $\mathsf{D}(R)$.

Lemma 3.1.8. *Let R be a commutative noetherian ring.*

(1) For every prime $\mathfrak{p}$ of R we have:

(a) $\mathsf{supp}(k(\mathfrak{p})) = \{\mathfrak{p}\} = \mathsf{supp}(E(R/\mathfrak{p}))$;

(b) $\mathsf{supp}(R/\mathfrak{p}) = \bigvee(\mathfrak{p})$;

(c) $\mathsf{supp}(R_\mathfrak{p}) = \bigwedge(\mathfrak{p})$;

(2) For any X in $\mathsf{D}(R)$, $\mathsf{Supp}(X)$ is specialisation closed;

(3) For any X in $\mathsf{D}(R)$, $\mathsf{supp}(X) \subseteq \mathsf{Supp}(X)$;

(4) For any bounded below complex X in $\mathsf{D}^+(R)$, $\mathsf{supp}(X)$ coincides with the set of prime ideals $\mathfrak{p}$ for which the module $E(R/\mathfrak{p})$ is a summand of a module appearing in the minimal homotopically injective resolution of X;

(5) For any bounded below X in $\mathsf{D}^+(R)$, we have $\bigvee(\mathsf{supp}(X)) = \mathsf{Supp}(X)$;

(6) For every R-module M, $\mathsf{supp}(M) = \mathsf{Ass}(M) \cup \mathsf{supp}(E(M)/M)$; the minimal elements of the sets $\mathsf{Supp}(M), \mathsf{supp}(M)$ and $\mathsf{Ass}(M)$ coincide.

Proof. (1) is a computation.

(2) is proved for modules in [7, Lemma 2.1], and then follows in general, taking into account that the big support of a complex coincides with the union of the big supports of its cohomologies.

(3) is proved on page 158 of Foxby's [18].

(4) is proved when X is bounded in [18, Proposition 2.8 and Remark 2.9], and in our more general situation in [13, Proposition 2.1 and Remark 2.2].

For (5), notice that by (2) and (3) we already have that $\bigvee(\mathsf{supp}(X)) \subseteq \mathsf{Supp}(X)$. For the converse inclusion, let $X \to E(X)$ denote a minimal K-injective resolution of X in $\mathsf{D}(R)$ and let $\mathfrak{q}$ be a prime ideal of R. Then $\mathfrak{q}$ is not in $\bigvee(\mathsf{supp}(X))$ if and only if $\mathfrak{q}$ is not in $\bigvee(\mathfrak{p})$ for any $\mathfrak{p}$ in $\mathsf{supp}(X)$, i.e. if and only if, by [7, Lemma 2.2], $\mathfrak{q}$ is not in $\bigvee(\mathfrak{p}) = \mathsf{Supp}(E(R/\mathfrak{p}))$ for any $\mathfrak{p}$ in $\mathsf{supp}(X)$. Hence, by item (4), if $\mathfrak{q}$ does not lie in $\bigvee(\mathsf{supp}X)$ then $E(X)_{\mathfrak{q}} = 0$ (or, equivalently, $X_{\mathfrak{q}} = 0$) in $\mathsf{D}(R)$.

(6) Consider a minimal injective resolution of M, $E_0 \to E_1 \to \cdots$, and denote by $Z_i \subseteq E_i$ the i-th cycles, so that $Z_0 = M$ and $Z_1 = E(M)/M$. Then $E_i \to E_{i+1} \to \cdots$ is a minimal injective resolution of Z_i. By (4), we obtain that $\mathsf{supp}(Z_i) = \mathsf{supp}(E_i) \cup \mathsf{supp}(Z_{i+1}) = \mathsf{Ass}(E_i) \cup \mathsf{supp}(Z_{i+1})$, which gives the first claim. Now, showing that two sets of primes have the same minimal elements is the same as showing that they have the same specialisation closure. The fact that $\mathsf{Supp}(M)$ and $\mathsf{supp}(M)$ have the same minimal elements then follows from (5). For $\mathsf{supp}(M)$ and $\mathsf{Ass}(M)$, we already know that $\mathsf{Ass}(M) \subseteq \mathsf{supp}M \subseteq \bigvee(\mathsf{supp}(M))$; for the other direction, by Proposition 3.1.5 we have that $\mathsf{Hom}_R(E(R/\mathfrak{p}), E(R/\mathfrak{q})) = 0$ if $\mathfrak{q} \notin \bigvee(\mathfrak{p})$, so $\mathsf{supp}(Z_{i+1}) \subseteq \bigvee(\mathsf{supp}(E_i)) = \bigvee(\mathsf{Ass}(E_i))$, and therefore $\mathsf{supp}(M) = \mathsf{Ass}(E(M)) \cup \mathsf{supp}(Z_1) \subseteq \bigvee(\mathsf{Ass}(E(M)))$. □

For a subset $\mathcal{P} \subseteq \mathsf{Spec}(R)$, we write $\mathsf{supp}^{-1}(\mathcal{P})$ (respectively, $\mathsf{Supp}^{-1}(\mathcal{P})$) for the subcategory of $\mathsf{D}(R)$ whose objects have small (respectively, big) support contained in $\mathcal{P}$. By item (3) above, $\mathsf{Supp}^{-1}(\mathcal{P})$ is contained in $\mathsf{supp}^{-1}(\mathcal{P})$. If $\mathcal{P}$ is specialisation-closed, then item (5) of the lemma above guarantees that a bounded below complex belongs to $\mathsf{supp}^{-1}(\mathcal{P})$ if and only if it belongs to $\mathsf{Supp}^{-1}(\mathcal{P})$, i.e.

$$\mathcal{P} = \bigvee(\mathcal{P}) \quad \Longrightarrow \quad \mathsf{supp}^{-1}(\mathcal{P}) \cap \mathsf{D}^{+}(R) = \mathsf{Supp}^{-1}(\mathcal{P}) \cap \mathsf{D}^{+}(R) \tag{3.1}$$

3.1.2 Classification of localising subcategories

Via the assignement of support, $\mathsf{Spec}(R)$ controls the localising subcategories of both $\mathsf{Mod}(R)$ and $\mathsf{D}(R)$ (see §1.3 and §2.2), as we now explain.

Recall that since R is noetherian, by Proposition 1.4.27, localising subcategories of $\mathsf{Mod}(R)$ are in bijection with Gabriel filters. When R is also commutative, we have the following characterisation, originally from Gabriel [20].

Proposition 3.1.9 ([75, Theorem 3.4 and following Example])**.** *Let R be a commutative noetherian ring. Then there are bijections between:*

(1) Hereditary torsion classes $\mathcal{T}$ in $\mathsf{Mod}(R)$;

(2) Gabriel filters $\mathfrak{F}$ on R;

(3) Specialisation closed subsets V of $\mathsf{Spec}(R)$;

given by the assignements

$$
\begin{array}{lll}
(1 \leftrightarrow 3) & \mathcal{T} \mapsto \mathsf{Supp}(\mathcal{T}), & \mathsf{Supp}^{-1}(V) \leftarrow\!\shortmid V \\
(2 \leftrightarrow 3) & \mathfrak{F} \mapsto \{\mathfrak{p} \in \mathsf{Spec}(R) \colon \mathfrak{p} \in \mathfrak{F}\}, & \{I \leq R \colon \mathsf{V}(I) \subseteq V\} \leftarrow\!\shortmid V
\end{array}
$$

Notice that since this classification involves specialisation closed subsets V, we will equivalently use Supp or supp, as allowed by Equation (3.1). The following Lemma gives some more characterisations of hereditary torsion pairs and the support of their torsion class.

Lemma 3.1.10. *Let R be a commutative noetherian ring, and $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ a hereditary torsion pair in $\mathsf{Mod}(R)$. Then we have:*

(1) A prime $\mathfrak{p}$ lies in $\mathsf{supp}(\mathcal{T})$ if and only if $k(\mathfrak{p}) \in \mathcal{T}$;

(2) For any $\mathfrak{p} \in \mathsf{Spec}(R)$, either $k(\mathfrak{p}) \in \mathcal{T}$ or $k(\mathfrak{p}) \in \mathcal{T}^{\perp_{0,1}} \subseteq \mathcal{F}$.

(3)
$$
\begin{aligned}
\mathcal{F} &= \{M \in \mathsf{Mod}(R) \colon \mathsf{Ass}(M) \cap \mathsf{supp}(\mathcal{T}) = \emptyset\} \\
&= \mathsf{Cogen}(\mathsf{supp}^{-1}(\mathsf{supp}(\mathcal{T})^c) \cap \mathsf{Mod}(R))
\end{aligned}
$$

Proof. These are well-known statements. (1) follows from Proposition 3.1.9. For (2), one needs to check that if $\mathfrak{p}$ does not lie in $\mathsf{supp}(\mathcal{T})$, then we have $\mathsf{Hom}_R(T, k(\mathfrak{p})) = 0 = \mathsf{Ext}^1_R(T, k(\mathfrak{p}))$ for all T in $\mathcal{T}$. By Lemma 3.1.8(4), every injective module in the minimal injective resolution of $k(\mathfrak{p})$ is a coproduct of copies of $E(R/\mathfrak{p})$. The statement now follows from the fact that T has only maps to injective modules of the form $E(R/\mathfrak{q})$, with $\mathfrak{q}$ in $\mathsf{supp}(\mathcal{T})$ (see Proposition 3.1.5). (3) Since $\mathbf{t}$ is hereditary, $\mathcal{F}$ is closed under injective envelopes, i.e. a module M belongs to $\mathcal{F}$ if and only if $E(M) \in \mathcal{F}$. Now, the injectives of $\mathcal{F}$ are precisely those whose associated primes lie in $\mathsf{supp}(\mathcal{T})^c$. This shows both identities. □

We record another fact about hereditary torsion pairs over R.

Lemma 3.1.11 ([3, Lemma 4.2])**.** *Let R be a commutative noetherian ring. A torsion pair in $\mathsf{Mod}(R)$ is hereditary if and only if it is of finite type.*

Remark 3.1.12. This Lemma provides the converse of Proposition 1.4.12(3). Notice that without the commutativity assumption this is not true, as it is shown by the following example. In the notation of Example 2.9.16, consider the torsion pair of $\mathsf{Mod}(k\mathbb{A}_3)$ generated by the set $\{ \substack{2\\3}, \substack{1\\2\\3} \}$, which is automatically of finite type. Looking at the finitely presented objects it contains, we get a diagram

$$\begin{tikzcd}[column sep=small, row sep=small]
& & \bullet \arrow[dr] & & \\
& \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[dr] & \\
\circ \arrow[ur] & & \bullet \arrow[ur] & & \bullet,
\end{tikzcd}$$

which shows that the torsion pair is not hereditary.

In the derived category, a parametrisation of localising subcategories in terms of their supports is due to Neeman.

Theorem 3.1.13 ([53])**.** *Let R be a commutative noetherian ring. Then, the following statements hold.*

(1) Localising subcategories $\mathcal{L}$ of $\mathsf{D}(R)$ are coreflective; therefore $(\mathcal{L}, \mathcal{L}^{\perp})$ is a stable t-structure (see Definition 2.5.11).

(2) There is a bijection between localising subcategories $\mathcal{L}$ of $\mathsf{D}(R)$ and subsets $\mathcal{P}$ of $\mathsf{Spec}(R)$, given by

$$\mathcal{L} \mapsto \mathsf{supp}(\mathcal{L}), \qquad \mathsf{supp}^{-1}(\mathcal{P}) \leftarrow\!\shortmid \mathcal{P},$$

which restricts to a bijection between smashing subcategories of $\mathsf{D}(R)$ and specialisation closed subsets of $\mathsf{Spec}(R)$.

(3) For a localising subcategory $\mathcal{L}$ of $\mathsf{D}(R)$ we have that:

(a) a prime $\mathfrak{p}$ lies in $\mathsf{supp}(\mathcal{L})$ if and only if $k(\mathfrak{p})$ lies in $\mathcal{L}$;

(b) for any $\mathfrak{p}$ in $\mathsf{Spec}(R)$, then $k(\mathfrak{p})$ lies either in $\mathcal{L}$ or in $\mathcal{L}^{\perp}$;

(c) $\mathcal{L}$ is the smallest localising subcategory containing $\{k(\mathfrak{p}) \colon \mathfrak{p} \in \mathsf{supp}(\mathcal{L})\}$;

Proof. We spend a word on (1), which in [53, Theorem 2.6] is attributed to Bousfield “possibly after increasing the universe”. In fact, the whole section [53, §2] never uses this fact. Then item (3.c) shows that $\mathcal{L}$ is the smallest cocomplete preaisle containing the set $\mathcal{S}$ of all shift of residue fields of primes in $\mathsf{supp}(\mathcal{L})$; since $\mathsf{D}(R)$ is compactly generated, $\mathcal{L}$ is then the aisle of a (stable) t-structure, by Proposition 2.5.22. $\square$

Notice the parallel, albeit with some subtle differences, between the abelian and the derived classification results. The following result summarises the relation between the two theorems above.

Proposition 3.1.14. *Let R be a commutative noetherian ring, V a specialisation closed subset of $\mathsf{Spec}(R)$ and $\mathcal{L} = \mathsf{supp}^{-1}(V)$ the associated smashing subcategory of $\mathsf{D}(R)$. Then the localising subcategory $\mathcal{T}$ of $\mathsf{Mod}(R)$ associated to V is $\mathcal{L} \cap \mathsf{Mod}(R)$ and, moreover, $\mathcal{T}^{\perp_{\geq 0}} = \mathcal{L}^{\perp} \cap \mathsf{Mod}(R)$.*

Remark 3.1.15. Note that for a hereditary torsion class in $\mathsf{Mod}(R)$, it can be shown that $\mathcal{T}^{\perp_{\geq 0}}$ coincides with the Giraud subcategory $\mathcal{T}^{\perp_{0,1}}$ if and only if $\mathcal{T}$ is a perfect torsion class, i.e. if and only if the associated Gabriel topology is perfect (see Definition 1.4.29). In §3.2.2 we will prove this fact and make use of it to obtain a complete classification of hereditary torsion pairs in the HRS-tilt of $\mathsf{Mod}(R)$ with respect to a perfect torsion pair.

3.1.3 Classification of compactly generated t-structures

As it is the case with localising subcategories, the spectrum of R also classifies the compactly generated t-structures of $\mathsf{D}(R)$.

Definition 3.1.16. Let R be a commutative noetherian ring. A function φ from $\mathbb{Z}$ to the power set of $\mathsf{Spec}(R)$ is said to be an *sp-filtration of* $\mathsf{Spec}(R)$ if φ is a decreasing function between posets (i.e. if for all integers n, $\varphi(n) \supseteq \varphi(n+1)$) and $\varphi(n)$ is specialisation-closed, for all n.

Theorem 3.1.17 ([1, Theorem 4.10], [28, Theorem 1.1])**.** *Let R be a commutative noetherian ring. The following are equivalent for a t-structure $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ in $\mathsf{D}(R)$.*

(1) $\mathbb{T}$ is compactly generated;

(2) $\mathbb{T}$ is homotopically smashing;

(3) There is an sp-filtration of $\mathsf{Spec}(R)$ for which

$$\mathcal{U} = \{X \in \mathsf{D}(R) \colon \mathsf{Supp}(H^0(X[n])) \subseteq \varphi(n), \forall n \in \mathbb{Z}\}$$
$$\mathcal{V} = \{X \in \mathsf{D}(R) \colon \mathbb{R}\Gamma_{\varphi(n)}(X) \in \mathbb{D}^{\geq n+1}, \forall n \in \mathbb{Z}\},$$

where Γ_V denotes the (left exact) torsion radical of the hereditary torsion pair $(\mathsf{Supp}^{-1}(V), \mathcal{F}_V)$ of $\mathsf{Mod}(R)$, for a specialisation-closed set V (see Proposition 3.1.9).

We have the following easy lemma.

Lemma 3.1.18. *Let R be a commutative noetherian ring, and let $\mathbb{T}$ be compactly generated t-structure in $\mathsf{D}(R)$, with associated sp-filtration φ. Then*

(1) $\mathbb{T}$ is left non-degenerate if and only if $\cap_{n\in\mathbb{Z}}\varphi(n) = \emptyset$;

(2) $\mathbb{T}$ is right non-degenerate if and only if $\varphi(n) = \mathsf{Spec}(R)$ for every $n \ll 0$.

(3) $\mathbb{T}$ is intermediate if and only if there are integers $a < b$ such that $\varphi(n) = \mathsf{Spec}(R)$ for every $n \leq a$ and $\varphi(n) = \emptyset$ for every $n \geq b$,

Proof. Write $\mathbb{T} = (\mathcal{U}, \mathcal{V})$.

(1) Let $\mathfrak{p} \in \cap_{n\in\mathbb{Z}}\varphi(n)$; then $k(\mathfrak{p}) \in \mathcal{U}[n]$ for every $n \in \mathbb{Z}$, and therefore $\mathbb{T}$ is left degenerate. Conversely, assume $\cap_{n\in\mathbb{Z}}\varphi(n) = \emptyset$, and let $X \in \cap_{n\in\mathbb{Z}}\mathcal{U}[n]$. Then $\mathsf{Supp}(H^iX) \subseteq \cap_{n\in\mathbb{Z}}\varphi(n)$, which implies $H^iX = 0$. Therefore $X = 0$.

(2) First notice that $\varphi(n) = \mathsf{Spec}(R)$ for every $n \ll 0$ if and only if $\cup_{n\in\mathbb{Z}}\varphi(n) = \mathsf{Spec}(R)$: indeed, $\mathsf{Spec}(R)$ has only finitely many minimal elements (see e.g. [32, Theorem 88]). Now, if $\mathfrak{p} \in \mathsf{Spec}(R)\backslash\cup_{n\in\mathbb{Z}}\varphi(n)$, then $k(\mathfrak{p}) \notin \mathcal{U}[n]$ for every $n \in \mathbb{Z}$, and therefore $k(\mathfrak{p}) \in \mathcal{V}[n]$ for every $n \in \mathbb{Z}$ by [28, Lemma 2.7]. Hence $\mathbb{T}$ is right degenerate. Conversely, assume that $\varphi(n) = \mathsf{Spec}(R)$ for every $n \ll 0$: then $\mathcal{U}$ contains a shift of aisle of the standard t-structure. It follows that $\mathcal{V}$ is contained in the corresponding shift of the coaisle of the standard t-structure, and therefore $\cap_{n\in\mathbb{Z}}\mathcal{V}[n] \subseteq \cap_{n\in\mathbb{Z}}\mathbb{D}^{\geq n} = 0$. See also [27, Lemma 3.10].

(3) Trivial. □

Definition 3.1.19. A sp-filtration φ will be called **(left, right) non-degenerate, intermediate** if the corresponding t-structure is.

Combining the results above, we observe the following useful statement.

Corollary 3.1.20. *Let R be a commutative notherian ring, and $\mathcal{H}$ be the heart of a non-degenerate compactly generated t-structure $\mathbb{T}$ in $\mathsf{D}(R)$. Then, a torsion pair $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ in $\mathcal{H}$ is of finite type if and only if it is generated by a set of finitely presented objects of $\mathcal{H}$.*

Proof. By Lemma 1.4.8(1), we only need to prove the implication ($\Rightarrow$). If $\mathbf{t}$ is of finite type in $\mathcal{H}$, Proposition 2.7.10 shows that the HRS-tilt of $\mathbb{T}$ with respect to $\mathbf{t}$ is homotopically smashing; therefore by Theorem 3.1.17 it is compactly generated. Finally, another application of Proposition 2.7.10 shows that $\mathbf{t}$ is generated by finitely presented objects. □

Remark 3.1.21. Note that in the above corollary, $\mathcal{H}$ is not necessarily locally coherent — although, it is locally finitely presented by Proposition 2.5.42.

3.2 Hereditary torsion pairs in Grothendieck hearts

In this section we discuss hereditary torsion pairs in a given Grothendieck heart in the derived category of a commutative noetherian ring. Throughout, once again R will denote a commutative noetherian ring.

3.2.1 A characterisation by support

We begin by showing that hereditary torsion classes in the heart of a smashing non-degenerate t-structure of $\mathsf{D}(R)$ are completely determined by their support in $\mathsf{Spec}(R)$, similarly to what happens in $\mathsf{Mod}(R)$.

Lemma 3.2.1. *Let R be a commutative noetherian ring and let $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ be a non-degenerate t-structure in $\mathsf{D}(R)$ with heart $\mathcal{H}$ and cohomological functor $H^0_{\mathbb{T}}\colon \mathsf{D}(R) \to \mathcal{H}$. Then:*

(1) for each $\mathfrak{p}$ in $\mathsf{Spec}(R)$, there is an integer $n_{\mathfrak{p}}$ for which $k(\mathfrak{p})[n_{\mathfrak{p}}]$ lies in $\mathcal{H}$;

(2) $\mathsf{supp}(\mathcal{H}) = \mathsf{Spec}(R)$.

Proof. (1) It is shown in [28, Lemma 2.7] that the following two subsets form a partition of $\mathbb{Z}$:

$$A(\mathfrak{p}) := \{a \in \mathbb{Z}\colon k(\mathfrak{p}) \in \mathcal{U}[a]\} \qquad \text{and} \qquad B(\mathfrak{p}) := \{b \in \mathbb{Z}\colon k(\mathfrak{p}) \in \mathcal{V}[b]\}.$$

Since $\mathbb{T}$ is non-degenerate, this is a nontrivial partition. Moreover, since $\mathcal{U}$ (respectively, $\mathcal{V}$) is closed under positive (respectively, negative) shifts, if $m \leq n \in A(\mathfrak{p})$ then $m \in A(\mathfrak{p})$ (respectively, if $m \geq n \in B(\mathfrak{p})$ then $m \in B(\mathfrak{p})$). Hence, $A(\mathfrak{p})$ has a maximum, say α, and $B(\mathfrak{p})$ has a minimum, say β. Since these sets form a partition of $\mathbb{Z}$, we conclude that $\beta = \alpha + 1$ and, thus, $k(\mathfrak{p})$ lies in $\mathcal{U}[\alpha] \cap \mathcal{V}[\alpha + 1] = \mathcal{H}[\alpha]$. In other words, we have that $k(\mathfrak{p})[-\alpha]$ lies in $\mathcal{H}$, so it suffices to take $n_{\mathfrak{p}} = -\alpha$.

(2) Since $\mathsf{supp}(k(\mathfrak{p})[n]) = \{\mathfrak{p}\}$ for any integer n, it follows from (1) that $\mathsf{supp}(\mathcal{H}) = \mathsf{Spec}(R)$. □

Proposition 3.2.2. *Let R be a commutative noetherian ring and let $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ be a non-degenerate smashing t-structure in $\mathsf{D}(R)$ with heart $\mathcal{H}$ and cohomological functor $H^0_{\mathbb{T}}\colon \mathsf{D}(R) \to \mathcal{H}$. If $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion pair in $\mathcal{H}$, then:*

(1) $\mathcal{L}_{\mathbf{t}} := \{X \in \mathsf{D}(R)\colon H^0_{\mathbb{T}}(X[i]) \in \mathcal{T}$ for every $i \in \mathbb{Z}\}$ is a localising subcategory of $\mathsf{D}(R)$;

(2) $\mathsf{supp}(\mathcal{L}_{\mathbf{t}}) = \mathsf{supp}(\mathcal{T}) = \{\mathfrak{p} \in \mathsf{Spec}(R)\colon k(\mathfrak{p})[n_{\mathfrak{p}}] \in \mathcal{T}\}$

(3) $\mathcal{L}_{\mathbf{t}}$ is the smallest localising subcategory containing $\mathcal{T}$;

(4) $\mathcal{L}_{\mathbf{t}}^{\perp} \cap \mathcal{H} \subseteq \mathcal{T}^{\perp_{0,1}}$;

(5) for each $\mathfrak{p}$ in $\mathsf{Spec}(R)$, $k(\mathfrak{p})[n_{\mathfrak{p}}]$ lies in $\mathcal{T}$ or $k(\mathfrak{p})[n_{\mathfrak{p}}]$ lies in the Giraud subcategory $\mathcal{T}^{\perp_{0,1}}$;

(6) $\mathsf{supp}^{-1}(\mathsf{supp}(\mathcal{T})) \cap \mathcal{H} = \mathcal{T}$, i.e. $\mathcal{T}$ is completely determined by its support.

Proof. (1) Since $\mathcal{T}$ is closed under subobjects, extensions and quotient objects, it is easy to see that $\mathcal{L}_{\mathbf{t}}$ is a triangulated subcategory. Furthermore, since $\mathbb{T}$ is smashing, $H^0_{\mathbb{T}}$ commutes with coproducts (Lemma 2.5.39(2)); thus, since $\mathcal{T}$ is closed under coproducts, we conclude that $\mathcal{L}_{\mathbf{t}}$ is a localising subcategory.

(2) Let us denote by $\mathcal{P}$ the subset of $\mathsf{Spec}(R)$ consisting of the prime ideals $\mathfrak{p}$ for which $k(\mathfrak{p})[n_{\mathfrak{p}}]$ lies in $\mathcal{T}$. We prove our statement by showing that

$$\mathsf{supp}(\mathcal{L}_{\mathbf{t}}) \subseteq \mathcal{P} \subseteq \mathsf{supp}(\mathcal{T}) \subseteq \mathsf{supp}(\mathcal{L}_{\mathbf{t}}).$$

Let $\mathfrak{p}$ be a prime ideal of R. If $\mathfrak{p}$ lies in $\mathsf{supp}(\mathcal{L}_{\mathbf{t}})$ then $k(\mathfrak{p})$ lies in $\mathcal{L}_{\mathbf{t}}$ (see Theorem 3.1.13) and, thus, $k(\mathfrak{p})[n_{\mathfrak{p}}]$ lies in $\mathcal{T}$, i.e. $\mathfrak{p}$ lies in $\mathcal{P}$. If $\mathfrak{p}$ lies in $\mathcal{P}$, since $\mathsf{supp}(k(\mathfrak{p})[n_{\mathfrak{p}}]) = \{\mathfrak{p}\}$, then $\mathfrak{p}$ lies in $\mathsf{supp}(\mathcal{T})$. Finally, if $\mathfrak{p}$ lies in $\mathsf{supp}(\mathcal{T})$, since $\mathcal{T}$ is contained in $\mathcal{L}_{\mathbf{t}}$, it follows that $\mathfrak{p}$ lies in $\mathsf{supp}(\mathcal{L}_{\mathbf{t}})$.

(3) Since $\mathcal{T}$ is contained in $\mathcal{L}_{\mathbf{t}}$, the smallest localising subcategory containing $\mathcal{T}$ must be contained in $\mathcal{L}_{\mathbf{t}}$. Conversely, if $\mathcal{L}$ is an arbitrary localising subcategory containing $\mathcal{T}$, then $\mathsf{supp}(\mathcal{L})$ must contain $\mathsf{supp}(\mathcal{T}) = \mathsf{supp}(\mathcal{L}_{\mathbf{t}})$ and, thus, $\mathcal{L}$ must contain $\mathcal{L}_{\mathbf{t}}$.

(4) Given X in $\mathcal{L}_{\mathbf{t}}^{\perp} \cap \mathcal{H}$ and T in $\mathcal{T}$ (and, thus, in $\mathcal{L}_{\mathbf{t}}$), we have that $\mathsf{Hom}_{\mathsf{D}(R)}(T, X) = 0$ and $\mathsf{Ext}^1_{\mathcal{H}}(T, X) \simeq \mathsf{Hom}_{\mathsf{D}(R)}(T[-1], X) = 0$ since $T[-1]$ lies also in $\mathcal{L}_{\mathbf{t}}$. Thus, X lies in $\mathcal{T}^{\perp_{0,1}}$.

(5) Let $\mathfrak{p}$ be an arbitrary prime ideal of R and consider the object $k(\mathfrak{p})[n_{\mathfrak{p}}]$ of $\mathcal{H}$. By Theorem 3.1.13, this object either lies in $\mathcal{L}_{\mathbf{t}} \cap \mathcal{H} = \mathcal{T}$ or in $\mathcal{L}_{\mathbf{t}}^{\perp} \cap \mathcal{H} \subseteq \mathcal{T}^{\perp_{0,1}}$.

(6) From (3) and Theorem 3.1.13, it follows that that $\mathsf{supp}^{-1}(\mathsf{supp}(\mathcal{T}))) = \mathcal{L}_{\mathbf{t}}$. By definition of $\mathcal{L}_{\mathbf{t}}$, we have $\mathcal{L}_{\mathbf{t}} \cap \mathcal{H} = \mathcal{T}$, thus proving our claim. □

Note that, in particular, it follows that the hereditary torsion classes of the heart of a non-degenerate smashing t-structure form a set. Item (6) of the previous proposition motivates the following definition.

Definition 3.2.3. Let R be a commutative noetherian ring and $\mathbb{T}$ a non-degenerate smashing t-structure in $\mathsf{D}(R)$, with heart $\mathcal{H}$. A set $U \subseteq \mathsf{Spec}(R)$ is called a **$\mathcal{H}$-support** if it is the support of a hereditary torsion class in $\mathcal{H}$. This torsion class will then be $\mathsf{supp}^{-1}(U) \cap \mathcal{H}$.

As a side corollary of the proposition above, we deduce a relation between the support of a complex and that of its cohomologies with respect to a smashing t-structure.

Corollary 3.2.4. *Let R be a commutative noetherian ring, and let $\mathbb{T}$ be a non-degenerate smashing t-structure in $\mathsf{D}(R)$, with heart $\mathcal{H}$ and cohomology functor $H^0_{\mathbb{T}}$. Let $U \subseteq \mathsf{Spec}(R)$ be a $\mathcal{H}$-support: then, for every object X of $\mathsf{D}(R)$, we have*

$$\mathsf{supp}(X) \subseteq U \quad \textit{if and only if} \quad \mathsf{supp}(H^0_{\mathbb{T}}(X[n])) \subseteq U \ \forall n \in \mathbb{Z}$$

Proof. This is direct consequence of items (1) and (2) of Theorem 3.2.2. □

Remark 3.2.5. Note that this corollary recovers and extends the known relation (see [7, Corollary 5.3])

$$\bigvee(\mathsf{supp}(X)) = \bigvee(\mathsf{supp}(\bigoplus_{n\in\mathbb{Z}} H^0(X[n])))$$

by taking $\mathbb{T} = \mathbb{D}$ the standard t-structure, and using that both sides of the equation are $\mathsf{Mod}(R)$-supports (i.e. specialisation-closed subsets).

The following theorem provides some examples of $\mathcal{H}$-supports, for some particular kinds of hearts.

Theorem 3.2.6. *Let R be a commutative noetherian ring and let $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ be an intermediate compactly generated t-structure in $\mathsf{D}(R)$ with heart $\mathcal{H}$. The following statements hold.*

(1) If V is specialisation closed, then

(a) $\mathcal{T}_V := \mathsf{supp}^{-1}(V) \cap \mathcal{H}$ is a hereditary torsion class in $\mathcal{H}$;

(b) if, additionally, $\mathbb{T}$ induces a derived equivalence, then $\mathbf{t}_V = (\mathcal{T}_V, \mathcal{T}_V^{\perp})$ is a torsion pair of finite type and $\mathcal{T}_V^{\perp} = \mathsf{Cogen}(\mathsf{supp}^{-1}(V^c) \cap \mathcal{H})$.

(2) If $\mathbb{T}$ is restrictable, then for any hereditary torsion pair of finite type $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ in $\mathcal{H}$ we have that $\mathsf{supp}(\mathcal{T})$ is specialisation closed.

Proof. (1.a) We first show that $\mathcal{T}_V := \mathsf{supp}^{-1}(V) \cap \mathcal{H}$ is a hereditary torsion class in $\mathcal{H}$, whenever V is a specialisation closed subset of $\mathsf{Spec}(R)$. First note that, since $\mathbb{T}$ is intermediate, $\mathcal{H}$ is contained in $\mathsf{D}^b(R)$, and since V is specialisation closed, it follows from Lemma 3.1.8 that $\mathsf{supp}^{-1}(V) \cap \mathcal{H} = \mathsf{Supp}^{-1}(V) \cap \mathcal{H}$. Since $\mathbb{T}$ is compactly generated, it is homotopically smashing, i.e. both $\mathcal{U}$ and $\mathcal{V}$ are closed under directed homotopy colimits. From [28, Lemma 2.11] it follows that both $\mathcal{U}$ and $\mathcal{V}$ are closed under $- \otimes_R R_\mathfrak{p}$ and, therefore, $- \otimes_R R_\mathfrak{p}$ is exact in $\mathcal{H}$, for any $\mathfrak{p}$ in $\mathsf{Spec}(R)$. This shows that given a short exact sequence in $\mathcal{H}$ of the form

$$0 \to X \to Y \to Z \to 0$$

we have that $Y \otimes_R R_\mathfrak{p} = 0$ if and only if $X \otimes_R R_\mathfrak{p} = 0 = Z \otimes_R R_\mathfrak{p}$. In other words, we have that $\mathsf{Supp}(Y) = \mathsf{Supp}(X) \cup \mathsf{Supp}(Z)$ and, thus $\mathsf{Supp}(Y)$ is contained in V if and only if both $\mathsf{Supp}(X)$ and $\mathsf{Supp}(Z)$ are contained in V. This shows that $\mathcal{T}_V$ is closed under extensions, subobjects and quotient objects. Since it is also clearly closed under coproducts, $\mathcal{T}_V$ is a hereditary torsion class.

(1.b) Suppose now that $\mathbb{T}$ induces a derived equivalence. In this case we know that there is an isomorphism $\mathsf{Ext}^k_{\mathcal{H}}(X, Y) \simeq \mathsf{Hom}_{\mathsf{D}(R)}(X, Y[k])$ for any X and Y in $\mathcal{H}$ and $k \geq 0$. In particular, for a subcategory $\mathcal{S}$ of $\mathcal{H}$, there is no ambiguity when calculating the orthogonal $\mathcal{S}^{\perp_J}$: this Ext-orthogonal subcategory in $\mathcal{H}$ coincides with the intersection with $\mathcal{H}$ of the orthogonal computed in $\mathsf{D}(R)$.

We first show that $\mathcal{T}_V^{\perp_{\geq 0}} = \mathsf{supp}^{-1}(V^c) \cap \mathcal{H}$. It follows from Theorem 3.1.13 that $\mathcal{L}_V := \mathsf{supp}^{-1}(V)$ is a smashing subcategory of $\mathsf{D}(R)$ and, thus, $\mathcal{B}_V := \mathcal{L}_V^{\perp}$ is also localising with $\mathsf{supp}(\mathcal{B}_V) = V^c$. Since, from Proposition 3.2.2, $\mathcal{L}_V$ is the smallest localising subcategory containing $\mathcal{T}_V$ and since $\mathbb{T}$ induces a derived equivalence, it follows that $\mathcal{B}_V \cap \mathcal{H} = \mathcal{T}_V^{\perp_{\geq 0}}$. Finally, note that since $\mathcal{T}_V$ is hereditary, we have that $(\mathcal{T}_V, \mathcal{T}_V^{\perp}) = ({}^{\perp}E_V, \mathsf{Cogen}(E_V))$ for an injective object E_V in $\mathcal{H}$, and E_V lies in $\mathcal{T}^{\perp_{\geq 0}}$. It then follows that $\mathcal{T}_V^{\perp} = \mathsf{Cogen}(\mathcal{B}_V \cap \mathcal{H}) = \mathsf{Cogen}(\mathsf{supp}^{-1}(V^c) \cap \mathcal{H})$.

We now show that this torsion pair $\mathbf{t} = (\mathcal{T}_V, \mathcal{T}_V^{\perp})$ is of finite type. Notice that since V is specialisation-closed, $\mathcal{L}_V$ admits a description as $\{X \in \mathsf{D}(R) \colon \mathsf{Supp}(H^n X) \subseteq V\}$ (see Definition 3.1.7 and Lemma 3.1.8). Therefore, it is the aisle of a homotopically smashing t-structure, by Theorem 3.1.17; i.e. $\mathcal{B}$ is closed under homotopy colimits. By Proposition 2.5.40, since $\mathbb{T}$ is homotopically smashing, every direct limit in $\mathcal{H}$ is a directed homotopy colimit in $\mathsf{D}(R)$. Hence, $\mathcal{B}_V \cap \mathcal{H} = T^{\perp_{\geq 0}}$ is closed under direct limits in $\mathcal{H}$. Since direct limits are exact in $\mathcal{H}$ and $\mathcal{T}_V^{\perp} = \mathsf{Cogen}(\mathcal{T}^{\perp_{\geq 0}} \cap \mathcal{H})$, we get that $\mathcal{T}_V^{\perp}$ is closed under direct limits.

(2) By Proposition 2.5.43, since $\mathbb{T}$ is restrictable, $\mathcal{H}$ is locally coherent and $\mathsf{fp}(\mathcal{H}) = \mathcal{H} \cap \mathsf{D}^b(\mathsf{mod}(R))$. Since $\mathbf{t}$ is a hereditary torsion pair of finite type, it follows from Lemma 1.4.9 that $\mathcal{T} = \varinjlim(\mathcal{T} \cap \mathsf{fp}(\mathcal{H}))$. Let $\mathcal{L}$ be the smallest localising subcategory of $\mathsf{D}(R)$ containing $\mathcal{T} \cap \mathsf{fp}(\mathcal{H})$. Clearly, $\mathcal{L}$ is contained in the smallest localising subcategory containing $\mathcal{T}$, which we denote by $\mathcal{L}_{\mathbf{t}}$. Since $\mathcal{L}$ is the aisle of a t-structure (namely $(\mathcal{L}, \mathcal{L}^{\perp})$), $\mathcal{L}$ is closed under directed homotopy colimits. As above, since $\mathbb{T}$ is homotopically smashing, directed limits in $\mathcal{H}$ are directed homotopy colimits in $\mathsf{D}(R)$ and, thus, $\mathcal{T}$ is contained in $\mathcal{L}$, showing that $\mathcal{L} = \mathcal{L}_{\mathbf{t}}$. Therefore, we have that $\mathsf{supp}(\mathcal{T}) = \mathsf{supp}(\mathcal{L}_{\mathbf{t}}) = \mathsf{supp}(\mathcal{L})$. Now, by assumption, the t-structure $(\mathcal{L}, \mathcal{L}^{\perp})$ is generated by all shifts of $\mathcal{T} \cap \mathsf{fp}(\mathcal{H})$ which, by assumption is made of complexes in $\mathsf{D}^b(\mathsf{mod}(R))$. Now by [1, Theorem 3.10] this means that $\mathcal{L}$ is compactly generated and, therefore, smashing. This shows, by Theorem 3.1.13, that $\mathsf{supp}(\mathcal{T}) = \mathsf{supp}(\mathcal{L})$ is specialisation closed. $\square$

Notice that the theorem above provides an immediate generalisation of Proposition 3.1.9, which we will further simplify in Corollary 3.3.17.

Corollary 3.2.7. *Let R be a commutative noetherian ring and let $\mathbb{T}$ be a restrictable and intermediate compactly generated t-structure in $\mathsf{D}(R)$ inducing a derived equivalence. Then there is a bijection between hereditary torsion pairs of finite type in the heart of $\mathbb{T}$ and specialisation closed subsets of $\mathsf{Spec}(R)$.*

In fact, we can be more precise about the support of a hereditary torsion pair of finite type for non-degenerate compactly generated t-structures.

Proposition 3.2.8. *Let R be a commutative noetherian ring and let $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ be a non-degenerate compactly generated t-structure in $\mathsf{D}(R)$ with heart $\mathcal{H}$ and associated sp-filtration φ.*

(1) For each $\mathfrak{p}$ in $\mathsf{Spec}(R)$, let $\varphi_{max}(\mathfrak{p})$ denote the largest integer n for which $\mathfrak{p}$ belongs to $\varphi(n)$. Then, in the notation of Lemma 3.2.1, we have $n_\mathfrak{p} = -\varphi_{max}(\mathfrak{p})$ and, in particular, if $\mathfrak{p}$ is contained in a prime $\mathfrak{q}$, then $n_\mathfrak{p} \geq n_\mathfrak{q}$.

(2) If $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion pair of finite type in $\mathcal{H}$, then there is an sp-filtration ψ such that $\varphi(j+1) \subseteq \psi(j) \subseteq \varphi(j)$ for all j in $\mathbb{Z}$ and

$$\mathsf{supp}(\mathcal{T}) = \bigcup_{j \in \mathbb{Z}} [\psi(j) \setminus \varphi(j+1)] \,.$$

Proof. (1) Given the cohomological description of $\mathcal{U}$ (see Theorem 3.1.17), we know that the stalk complex $k(\mathfrak{p})[-\varphi_{max}(\mathfrak{p})]$ lies in $\mathcal{U}$ but $k(\mathfrak{p})[-(\varphi_{max}(\mathfrak{p})+1)]$ does not. By [28, Lemma 2.7], this means that $k(\mathfrak{p})[-\varphi_{max}(\mathfrak{p})-1]$ lies in $\mathcal{V}$ and, therefore, $k(\mathfrak{p})[-\varphi_{max}(\mathfrak{p})]$ lies in $\mathcal{V}[1] \cap \mathcal{U} = \mathcal{H}$, as wanted. Since an sp-filtration is a decreasing sequence of specialisation-closed subsets, we have that if $\mathfrak{p}$ is contained in a prime $\mathfrak{q}$, then $\varphi_{max}(\mathfrak{p}) \leq \varphi_{max}(\mathfrak{q})$ and, thus, $n_\mathfrak{p} \geq n_\mathfrak{q}$.

(2) Let $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion pair of finite type in $\mathcal{H}$. The t-structure obtained by HRS-tilting $\mathbb{T}$ with respect to $\mathbf{t}$ is compactly generated, since $\mathbf{t}$ is of finite type (see Proposition 2.7.10(1) and Theorem 3.1.17). Thus, it is determined by an sp-filtration ψ, satisfying $\varphi(j+1) \subseteq \psi(j) \subseteq \varphi(j)$. Let $\mathcal{U}_\psi$ be the aisle of the t-structure associated to ψ. Clearly, we have that $\mathcal{T} = \mathcal{U}_\psi \cap \mathcal{H}$. We need to check which shifted residue fields belong to $\mathcal{T}$ (following Proposition 3.2.2(2)). For any $\mathfrak{p}$ in $\mathsf{Spec}(R)$, by (1) $k(\mathfrak{p})[-j]$ lies in $\mathcal{H}$ if and only if $\mathfrak{p}$ lies in $\varphi(j) \setminus \varphi(j+1)$. Now, $k(\mathfrak{p})[-j]$ lies in $\mathcal{T}$ if and only if $k(\mathfrak{p})[-j]$ lies in $\mathcal{H} \cap \mathcal{U}_\psi$, i.e. $\mathfrak{p}$ lies in $\psi(j) \setminus \varphi(j+1)$. Thus $\mathsf{supp}(\mathcal{T})$ coincides with the union of all such sets $\psi(j) \setminus \varphi(j+1)$. □

3.2.2 A complete classification of hereditary torsion pairs in a special case

The previous section shows that when trying to classify the hereditary torsion pairs in the heart $\mathcal{H}$ of a non-degenerate compactly generated t-structure, one may equivalently describe the corresponding $\mathcal{H}$-supports. For example, by Theorem 3.2.6 we know that specialisation-closed sets are often $\mathcal{H}$-supports. While the problem of classifying all $\mathcal{H}$-supports remains, in general, open, we are able to provide a complete classification for some hearts. These occur as HRS-tilts of $\mathsf{Mod}(R)$ at a perfect torsion pair (see Definition 1.4.29).

Remark 3.2.9. Notice that the HRS-tilt of $\mathsf{Mod}(R)$ at a hereditary torsion pair, corresponding to a specialisation-closed subset $V \subseteq \mathsf{Spec}(R)$, is always compactly generated (for example by Propositions 2.7.10(1) and 1.4.12(3)). Indeed, its aisle corresponds to the sp-filtration $\cdots = \mathsf{Spec}(R) \supseteq V \supseteq \emptyset = \cdots$, with V in degree 0.

The following lemma gives examples of hearts $\mathcal{H}$ with $\mathcal{H}$-supports that are not specialisation-closed.

Lemma 3.2.10. *Let R be a ring and let $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be a perfect torsion pair in $\mathsf{Mod}(R)$. Denote the corresponding Giraud subcategory by $\mathcal{C} := \mathcal{T}^{\perp_{0,1}}$. Let $\mathcal{H}_{\mathbf{t}}$ be the heart of the HRS-tilt at $\mathbf{t}$. The following statements hold:*

(1) There is a TTF triple $(\mathcal{F}[1], \mathcal{T}, \mathcal{C}[1])$ in $\mathcal{H}_{\mathbf{t}}$;

(2) $\mathcal{C}[1]$ itself is a hereditary torsion class in $\mathcal{H}_{\mathbf{t}}$;

(3) If R is commutative noetherian, then $\mathsf{supp}(\mathcal{C}[1]) = V^c$. Hence, V^c is a (generalisation-closed) $\mathcal{H}_{\mathbf{t}}$-support.

Proof. (1) The fact that $\mathcal{T}$ is TTF class in $\mathcal{H}_{\mathbf{t}}$ follows from Remark 2.7.8. We only need to verify that the corresponding torsionfree class in $\mathcal{H}_{\mathbf{t}}$, denoted by $\mathcal{F}'$, coincides with $\mathcal{C}[1]$. Since $\mathsf{Hom}_{\mathcal{H}_{\mathbf{t}}}(T, C[1]) = \mathsf{Hom}_{\mathsf{D}(R)}(T, C[1]) \simeq \mathsf{Ext}^1_R(T, C) = 0$, for all T in $\mathcal{T}$ and C in $\mathcal{C}$, we have $\mathcal{C}[1] \subseteq \mathcal{F}'$. For the converse, let X be an object in $\mathcal{F}'$, and consider the triangle

$$F_X[1] \longrightarrow X \longrightarrow T_X \xrightarrow{w} F_X[2]$$

which corresponds to the short exact sequence in $\mathcal{H}_{\mathbf{t}}$ given by the torsion pair $(\mathcal{F}[1], \mathcal{T})$. Since $\mathcal{F}'$ is closed under subobjects in $\mathcal{H}_{\mathbf{t}}$, for every T in $\mathcal{T}$ we have $0 = \mathsf{Hom}_{\mathcal{H}_{\mathbf{t}}}(T, F_X[1]) = \mathsf{Hom}_{\mathsf{D}(R)}(T, F_X[1]) \simeq \mathsf{Ext}^1_R(T, F_X)$. This shows that in fact F_X lies in $\mathcal{C}$. Since the torsion pair $\mathbf{t}$ is perfect, $\mathcal{C} = \mathcal{T}^{\perp_{\geq 0}}$ (Lemma 1.4.31) and, therefore, $0 = \mathsf{Ext}^2_R(T_X, F_X) \simeq \mathsf{Hom}_{\mathsf{D}(R)}(T_X, F_X[2])$. Since w lies in the latter Hom-space, we conclude that $w = 0$ and that the triangle above splits; hence $T_X = 0$ and $X = F_X[1]$ lies in $\mathcal{C}[1]$.

(2) Since $\mathcal{C}[1]$ is a torsionfree class in $\mathcal{H}_{\mathbf{t}}$, it suffices to show that $\mathcal{C}[1]$ is closed under cokernels of monomorphisms in $\mathcal{H}_{\mathbf{t}}$. Consider then C and C' in $\mathcal{C}$ and a triangle

$$C[1] \longrightarrow C'[1] \longrightarrow X \longrightarrow C[2]$$

with X in $\mathcal{H}_{\mathbf{t}}$. Applying the functor $\mathsf{Hom}_{\mathsf{D}(R)}(T, -)$ for every T in $\mathcal{T}$, since $\mathsf{Hom}_{\mathsf{D}(R)}(T, C'[1]) \simeq \mathsf{Ext}^1_R(T, C') = 0$ and $\mathsf{Hom}_{\mathsf{D}(R)}(T, C[2]) \simeq \mathsf{Ext}^2_R(T, C) = 0$ (given that $\mathbf{t}$ is a perfect torsion pair), we have that $\mathsf{Hom}_{\mathsf{D}(R)}(T, X) = 0$. This shows that X indeed lies in $\mathcal{F}' = \mathcal{C}[1]$.

(3) If R is commutative noetherian, since $\mathsf{supp}(\mathcal{C}[1]) = \mathsf{supp}(\mathcal{C})$, it suffices to observe that $\mathsf{supp}(\mathcal{C}) = V^c$ (because $E(R/\mathfrak{p})$ lies in $\mathcal{C}$ for all $\mathfrak{p}$ not in V). The last assertion then follows from part (2). $\square$

Remark 3.2.11. The hereditary torsion pair $\mathbf{t}'$ with torsion class $\mathcal{C}[1]$ is not of finite type. Indeed, if this were the case, by Proposition 2.7.10(1) and Theorem 3.1.17, the HRS-tilt of $\mathbb{T}_{\mathbf{t}}$ at $\mathbf{t}'$ would correspond to a sp-filtration. The aisle of this HRS-tilt is easily seen to be

$$(\mathbb{D}^{\leq -1} * \mathcal{T})[1] * \mathcal{C}[1] = \mathbb{D}^{\leq -2} * (\mathcal{T} * \mathcal{C})[1] =: (*).$$

But $\mathcal{T} * \mathcal{C}$ is not a hereditary torsion class in $\mathsf{Mod}(R)$: otherwise, by support we would have $\mathcal{T} * \mathcal{C} = \mathsf{Mod}(R)$, and it would follow that $\mathcal{C} = \mathcal{F}$. This shows that $(*)$ is not the aisle of a compactly generated t-structure.

In fact, using this generalisation-closed $\mathcal{H}_{\mathbf{t}}$-support one can construct many more which are not specialisation-closed. Recall that hereditary torsion pairs in a Grothendieck category $\mathcal{H}$ form a lattice $\mathsf{tors}_h\mathcal{H}$ (§1.4).

Lemma 3.2.12. *Let R be a commutative noetherian ring and let $\mathcal{H}$ be the heart of a non-degenerate compactly generated t-structure in $\mathsf{D}(R)$. Let $\mathcal{S} \subseteq \mathsf{tors}_h\mathcal{H}$ be a set of hereditary torsion pairs in $\mathcal{H}$. Then:*

(1) $\mathsf{supp}(\bigwedge \mathcal{S}) = \cap_{\mathbf{t}\in\mathcal{S}} \mathsf{supp}(\mathbf{t})$*;*

(2) $\mathsf{supp}(\bigvee \mathcal{S}) = \cup_{\mathbf{t}\in\mathcal{S}} \mathsf{supp}(\mathbf{t})$*.*

In particular, arbitrary intersections and unions of $\mathcal{H}$-supports are again $\mathcal{H}$-supports.

Proof. Both the claims follow from Proposition 3.2.2, items (2) and (5). □

We are now able to classify supports in the heart $\mathcal{H}_{\mathbf{t}}$ of an HRS-tilt at a perfect torsion pair.

Proposition 3.2.13. *Let R be a commutative noetherian ring, $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ a perfect torsion pair in $\mathsf{Mod}(R)$ with $\mathsf{supp}(\mathcal{T}) = V$, and $\mathcal{H}_{\mathbf{t}}$ the associated heart by HRS-tilt. Then the $\mathcal{H}_{\mathbf{t}}$-supports are all the sets of the form $(W \cap V) \cup (W' \cap V^c)$, for $W, W' \subseteq \mathsf{Spec}(R)$ specialisation closed.*

Proof. We have already noted that $\mathcal{H}_{\mathbf{t}}$ is the heart of an intermediate compactly generated t-structure, so it follows from Theorem 3.2.6(1.a) that specialisation-closed subsets are $\mathcal{H}_{\mathbf{t}}$-supports. Since both V and V^c are $\mathcal{H}_{\mathbf{t}}$-supports (see Lemma 3.2.10), it follows from Lemma 3.2.12 that the subsets presented in the statement are indeed $\mathcal{H}_{\mathbf{t}}$-supports. To prove the converse, let U be a $\mathcal{H}_{\mathbf{t}}$-support and let $\mathbf{t}' = (\mathcal{T}', \mathcal{F}')$ be the hereditary torsion pair in $\mathcal{H}_{\mathbf{t}}$ with $\mathsf{supp}(\mathcal{T}') = U$. We first show that

(1) If $U \subseteq V$, then U is specialisation-closed;

(2) If $U \subseteq V^c$, then $U = \bigvee(U) \cap V^c$.

(1) If $U \subseteq V$, then we have that $\mathcal{T}' \subseteq \mathcal{T} \subseteq \mathsf{Mod}(R)$. We show that $\mathcal{T}'$ is a hereditary torsion class in $\mathsf{Mod}(R)$ as well, and so U is specialisation-closed. Clearly $\mathcal{T}'$ is closed under extensions and coproducts in $\mathsf{D}(R)$, and thus it is so in $\mathsf{Mod}(R)$ as well. Let now

$$(*)\colon \qquad 0 \to X \to Y \to Z \to 0$$

be a short exact sequence in $\mathsf{Mod}(R)$, with Y in $\mathcal{T}' \subseteq \mathcal{T}$. Since $\mathcal{T}$ is a hereditary torsion class in $\mathsf{Mod}(R)$, both X and Z belong to $\mathcal{T} \subseteq \mathcal{H}_{\mathbf{t}}$, so that $(*)$ is a short exact sequence in $\mathcal{H}_{\mathbf{t}}$ as well. Now since $\mathcal{T}'$ is a hereditary torsion class in $\mathcal{H}_{\mathbf{t}}$ we conclude that X and Z belong to $\mathcal{T}'$, as wanted.

(2) The non-trivial inclusion is $U \supseteq \bigvee(U) \cap V^c$. Let $\mathcal{C}$ denote the Giraud subcategory associated to $\mathbf{t}$, i.e. $\mathcal{C} = \mathcal{T}^{\perp_{0,1}}$. Given $\mathfrak{p}$ in $U \subseteq V^c$ and $\mathfrak{q}$ in V^c such that $\mathfrak{p} \subseteq \mathfrak{q}$, we will show that $\mathfrak{q}$ lies in U as well. Translating this in terms of objects of $\mathcal{H}_{\mathbf{t}}$, consider the stalk complexes $E(R/\mathfrak{p})[1]$ and $E(R/\mathfrak{q})[1]$ in $\mathcal{C}[1] \subseteq \mathcal{H}_{\mathbf{t}}$. By assumption we have $E(R/\mathfrak{p})[1]$ lies in $\mathcal{T}'$, and we want to prove that $E(R/\mathfrak{q})[1]$ lies in $\mathcal{T}'$ as well. Denote by $(**)$ the torsion sequence of $E(R/\mathfrak{q})[1]$ with respect to $\mathbf{t}'$. Since, by Lemma 3.2.10, $\mathcal{C}[1]$ is a hereditary torsion class in $\mathcal{H}_{\mathbf{t}}$, we deduce that $(**)$ has all its terms in $\mathcal{C}[1]$. Therefore, applying a shift to it, we obtain the exact sequence of modules (in solid arrows) with T in $\mathcal{T}'[-1]$ and F in $\mathcal{F}'[-1]$.

$$(**)[-1]\colon \qquad \begin{tikzcd} 0 \arrow[r] & T \arrow[r] \arrow[dr, dotted] & E(R/\mathfrak{q}) \arrow[r] \arrow[d, dotted, shift left] & F \arrow[r] & 0 \\ & & E(T) \arrow[u, dotted, shift left] & & \end{tikzcd}$$

In $\mathsf{Mod}(R)$, consider then the injective envelope $E(T)$ of T: by injectivity, we get the two dotted vertical arrows in the diagram above. Moreover, since the morphism $T \to E(T)$ is left minimal, we conclude that $E(T)$ is a direct summand of the indecomposable module $E(R/\mathfrak{q})$. Now, we use our hypothesis that $\mathfrak{p} \subseteq \mathfrak{q}$ to notice that there is a nonzero morphism $E(R/\mathfrak{p})[1] \to E(R/\mathfrak{q})[1]$. Since the source of this morphism is in $\mathcal{T}'$, the target cannot be in $\mathcal{F}'$, and therefore $T \neq 0$. Then, we must have an isomorphism $0 \neq E(T) \simeq E(R/\mathfrak{q})$, which means that

$$\{\mathfrak{q}\} = \mathsf{supp}(E(R/\mathfrak{q})) = \mathsf{supp}(E(T)) \subseteq \mathsf{supp}(T) = \mathsf{supp}(T[1]) \subseteq U.$$

Returning to the general case of an arbitrary $\mathcal{H}_{\mathbf{t}}$-support U, note that by Lemma 3.2.12 and Lemma 3.2.10, both $U \cap V$ and $U \cap V^c$ are $\mathcal{H}_{\mathbf{t}}$-supports. Set $W := U \cap V$ and $W' := \bigvee(U \cap V^c)$: the first is specialisation-closed by item (1) above, while the second is specialisation-closed by definition. Now by item (2) above it follows that $U = (U \cap V) \cup (U \cap V^c) = (W \cap V) \cup (W' \cap V^c)$. $\square$

Remark 3.2.14. By Lemma 3.1.11, in $\mathsf{Mod}(R)$ hereditary torsion pairs coincide with those of finite type. In the heart $\mathcal{H}_{\mathbf{t}}$ constructed in this subsection this is not the case. Indeed, by Corollary 2.9.13, $\mathcal{H}_{\mathbf{t}}$ is derived equivalent to $\mathsf{Mod}(R)$. On the other hand, since every torsion pair $\mathbf{t}$ in $\mathsf{Mod}(R)$ is restrictable, so is the t-structure $\mathbb{T}_{\mathbf{t}}$, by Theorem 2.7.12. Therefore, by Corollary 3.2.7, we conclude that in $\mathcal{H}_{\mathbf{t}}$ hereditary torsion pairs of finite type correspond bijectively to specialisation closed subsets of $\mathsf{Spec}(R)$. Proposition 3.2.13 then shows that if $\mathbf{t}$ is perfect then, in general, not every hereditary torsion pair is of finite type (since not all $\mathcal{H}_{\mathbf{t}}$-supports are specialisation-closed).

We conclude this subsection with an illustrating example.

Example 3.2.15. Let R be a commutative noetherian ring of Krull dimension 1. In this case, every hereditary torsion pair is perfect (see [41, Corollary 4.3] and [4, Corollary 4.10]). Let V denote the set of maximal ideals of R. It is, of course, a specialisation-closed subset of $\mathsf{Spec}(R)$; denote by $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ the associated hereditary torsion pair in $\mathsf{Mod}(R)$. Let $\mathcal{H}_{\mathbf{t}} := \mathcal{F}[1] * \mathcal{T}$ be the heart of the HRS-tilt of the standard t-structure with respect to $\mathbf{t}$. Following Proposition 3.2.13, the $\mathcal{H}_{\mathbf{t}}$-supports are the sets of primes of the form $(W \cap V) \cup (W' \cap V^c)$, for specialisation-closed subsets W and W' of $\mathsf{Spec}(R)$. However, it is quite easy to see that, since R has Krull dimension 1, any subset of $\mathsf{Spec}(R)$ is of this form. Interestingly, this means that in this case hereditary torsion pairs in $\mathcal{H}_{\mathbf{t}}$ are in bijection with localising subcategories of $\mathsf{D}(R)$ (not only the smashing ones, as it happens with hereditary torsion pairs in $\mathsf{Mod}(R)$). Concretely, following items (1) and (2) of Theorem 3.2.2, the bijection is

$$\begin{array}{rcl} \{\textit{hereditary torsion pairs in } \mathcal{H}_{\mathbf{t}}\} & \xleftrightarrow{1:1} & \{\textit{localising subcategories of } \mathsf{D}(R)\} \\ (\mathcal{T}, \mathcal{F}) & \longmapsto & \{X \in \mathsf{D}(R) \colon H^0_{\mathbf{t}}(X[i]) \in \mathcal{T} \ \forall i \in \mathbb{Z}\} \\ \mathcal{H}_{\mathbf{t}} \cap \mathcal{L} & \longleftarrow\!\shortmid & \mathcal{L} \end{array}$$

where $H^0_{\mathbf{t}} \colon \mathsf{D}(R) \to \mathcal{H}_{\mathbf{t}}$ is the cohomology functor. In particular, all localising subcategories of $\mathsf{D}(R)$ admit a cohomological description with respect to $\mathcal{H}_{\mathbf{t}}$. We will later prove that $\mathcal{H}_{\mathbf{t}}$ is derived equivalent to $\mathsf{Mod}(R)$ (as a consequence of Corollary 2.9.13). This means that we get different insights on the triangulated structure of this derived category, depending on the abelian category that we start with.

3.3 t-structures inducing derived equivalences

Now we will combine the insight given by the classification of hereditary torsion pairs of the previous section, with the results of §2.9. In particular, we aim to find sufficient conditions for a given intermediate compactly generated t-structure to induce a derived equivalence.

Remark 3.3.1. Recall that by Theorem 2.5.41, intermediate compactly generated t-structures in $\mathsf{D}(R)$ satisfy hypotheses (2) and (3) of Proposition 2.6.14. Therefore, their heart is derived equivalent to $\mathsf{Mod}(R)$ if and only if it is bounded derived equivalent. This includes the case of the HRS-tilt of an intermediate compactly generated t-structure at a torsion pair of finite type (which is homotopically smashing by item (1) of Proposition 2.7.10 and then compactly generated by Theorem 3.1.17). In the following we will use this fact without an explicit mention.

3.3.1 Sufficient conditions for derived equivalence

Let R be a commutative noetherian ring, and consider a hereditary torsion pair $\mathbf{t}$ in $\mathsf{Mod}(R)$. If $\mathbb{T}_\mathbf{t}$ denotes the t-structure obtained by HRS-tilting $\mathsf{Mod}(R)$ at $\mathbf{t}$, with heart $\mathcal{H}_\mathbf{t}$, we know that:

- $\mathbb{T}_\mathbf{t}$ is an intermediate compactly generated t-structure (Remark 3.2.9);
- $\mathcal{H}_\mathbf{t}$ is a locally coherent Grothendieck category (Proposition 2.7.12 and Lemma 3.1.11);
- $\mathcal{H}_\mathbf{t}$ is derived equivalent to $\mathsf{Mod}(R)$ (Corollary 2.9.13).

This subsection takles the question of whether we can proceed with a chain of HRS-tilts (§2.8) at suitable torsion pairs, so that all the obtained t-structures will retain these properties. For this purpose, we will use Proposition 2.9.2 together with Theorem 2.9.7.

Theorem 3.3.2. *Let $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ be an intermediate compactly generated t-structure in $\mathsf{D}(R)$ with heart $\mathcal{H}$, and suppose that $\mathbb{T}$ induces a derived equivalence. Let $V \subseteq \mathsf{Spec}(R)$ be a specialisation-closed set, let $\mathbf{t} = (\mathcal{T}_V, \mathcal{F}_V)$ be the corresponding hereditary torsion pair of finite type in $\mathcal{H}$ (see Theorem 3.2.6(1)).*

If there is a set of generators $\{G_\lambda : \lambda \in \Lambda\}$ of $\mathcal{H}$ such that the torsion-free parts fG_λ are finitely presented, then the HRS-tilted t-structure associated to $\mathbf{t}$ induces a derived equivalence.

Proof. In order to apply Proposition 2.9.2, we will construct CHZ-sequences for the generators G_λ and then use Proposition 2.9.5. These CHZ-sequences will be provided by Proposition 2.9.17.

First of all, recall that, as noted in the proof of Theorem 3.2.6, for every prime $\mathfrak{p}$ the functor $- \otimes R_\mathfrak{p}$ restricts to an exact functor $L_\mathfrak{p} \colon \mathcal{H} \to \mathcal{H}$. Now, since V is specialisation closed, by Lemma 3.1.8

$$\mathcal{T}_V = \mathcal{H} \cap \mathsf{supp}^{-1}(V) = \mathcal{H} \cap \mathsf{Supp}^{-1}(V) = \bigcap_{\mathfrak{p} \notin V} \mathrm{Ker}(- \otimes R_\mathfrak{p}).$$

Then we need to show that for every $\mathfrak{p} \notin V$ and $\lambda \in \Lambda$, the group homomorphism

$$L_{\mathfrak{p}\, fG_\lambda, G_\lambda} \colon \mathsf{Hom}_\mathcal{H}(fG_\lambda, G_\lambda) \to \mathsf{Hom}_\mathcal{H}(fG_\lambda \otimes R_\mathfrak{p}, G_\lambda \otimes R_\mathfrak{p})$$

is an epimorphism, and we will be done.

Since $R_\mathfrak{p}$ is flat, we can write it as a direct limit of free R-modules, i.e. $R_\mathfrak{p} = \varinjlim_{j\in J} R^{(n_j)}$. Since directed homotopy colimits in $\mathsf{D}(R)$ are computed as componentwise direct limits, one sees that

$$G_\lambda \otimes R_\mathfrak{p} = G_\lambda \otimes \varinjlim_J R^{(n_j)} = \mathsf{hocolim}_J G_\lambda^{(n_j)}$$

(see the proof of [27, Lemma 4.1] for the details). Now, by Proposition 2.5.40, this directed homotopy colimit of objects of $\mathcal{H}$ is a direct limit in $\mathcal{H}$. We can therefore use the hypothesis that fG_λ is finitely presented in $\mathcal{H}$ to write

$$\begin{aligned}\mathrm{Im}(L_{\mathfrak{p}\,fG_\lambda,G_\lambda}) = \mathsf{Hom}_\mathcal{H}(fG_\lambda, G_\lambda) \otimes_R R_\mathfrak{p} \simeq \\ \simeq \varinjlim_J \mathsf{Hom}_\mathcal{H}(fG_\lambda, G_\lambda)^{(n_j)} \simeq \mathsf{Hom}_\mathcal{H}(fG_\lambda, G_\lambda \otimes_R R_\mathfrak{p}) =: (*).\end{aligned}$$

Since $\mathsf{D}(R_\mathfrak{p})$ is a bireflective subcategory of $\mathsf{D}(R)$ with reflection functor $-\otimes_R R_\mathfrak{p}$, it follows that

$$(*) \simeq \mathsf{Hom}_\mathcal{H}(fG_\lambda \otimes_R R_\mathfrak{p}, G_\lambda \otimes_R R_\mathfrak{p}).$$

This shows that $L_{\mathfrak{p}\,fG_\lambda,G_\lambda}$ is surjective, and by Proposition 2.9.17 we conclude. □

Corollary 3.3.3. *Let $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ be an intermediate compactly generated t-structure in $\mathsf{D}(R)$ with a locally coherent heart $\mathcal{H}$, and suppose that $\mathbb{T}$ induces a derived equivalence. Let $\mathbf{t} = (\mathcal{T}_V, \mathcal{F}_V)$ be the hereditary torsion pair of finite type in $\mathcal{H}$ associated to a specialisation-closed V and suppose that $(\mathcal{T}_V, \mathcal{F}_V)$ is restrictable. Then the HRS-tilted t-structure associated to $\mathbf{t}$ induces a derived equivalence.*

Proof. Under the assumption that $\mathbf{t}$ is restrictable, for any set $\{G_\lambda : \lambda \in \Lambda\}$ of finitely presented generators of $\mathcal{H}$, the torsion-free parts fG_λ will also be finitely presented. Hence, the result follows from Theorem 3.3.2. □

Remark 3.3.4. Since we know that for a commutative noetherian ring R, every hereditary torsion pair in $\mathsf{Mod}(R)$ is restrictable, note that Corollary 2.9.13 follows immediately from the corollary above.

Remark 3.3.5. If $\mathbb{T}$ is as in Corollary 3.3.3, $\mathbf{t}$ is any torsion pair in $\mathcal{H}$ and we happen to know that $\mathcal{H}_\mathbf{t}$ is locally coherent, then the torsion pair $\mathbf{t}$ is restrictable if and only if there is a set $\{G_\lambda : \lambda \in \Lambda\}$ of finitely presented generators of $\mathcal{H}$ such that the torsion-free parts fG_λ are finitely presented (see [60, Remark 6.3(3)]). Therefore, knowing this information about $\mathcal{H}_\mathbf{t}$, the hypothesis of Corollary 3.3.3 is minimal to apply Theorem 3.3.2.

3.3.2 Intermediate compactly generated t-structures via iterated HRS-tilting

In this section we will use the notion of filterable pairs of t-structures, introduced in §2.8. We make the following observation; compare it with Proposition 2.8.5.

Lemma 3.3.6. *Let R be a commutative noetherian ring, and let $\mathbb{T}_1, \mathbb{T}_2$ be compactly generated t-structures in $\mathsf{D}(R)$, having finite gap. Then the pair $(\mathbb{T}, \mathbb{D})$ is left filterable.*

Proof. This is a consequence of Theorem 3.1.17. Write $\mathbb{T}_1 = (\mathcal{U}_1, \mathcal{V}_1), \mathbb{T}_2 = (\mathcal{U}_2, \mathcal{V}_2)$, and let φ_1, φ_2 be the sp-filtrations corresponding to $\mathbb{T}_1, \mathbb{T}_2$ respectively. We claim that for every $i \in \mathbb{Z}$ the subcategory $\mathcal{U}_1 \cap \mathcal{U}_2[i]$ is the aisle of a compactly generated t-structure. Indeed,

$$\mathcal{U}_1 \cap \mathcal{U}_2[i] =$$
$$= \{X \in \mathsf{D}(R)\colon \mathsf{Supp}(H^n X) \subseteq \varphi_1(n)\} \cap \{X \in \mathsf{D}(R)\colon \mathsf{Supp}(H^n X) \subseteq \varphi_2(n-i)\} =$$
$$\{X \in \mathsf{D}(R)\colon \mathsf{Supp}(H^n X) \subseteq \varphi_1(n) \cap \varphi_2(n-i)\}$$

which is the aisle of the compactly generated t-structure associated to the sp-filtration $\varphi_1 \wedge \varphi_2[i]\colon n \mapsto \varphi_1(n) \cap \varphi_2(n-i)$. $\square$

Now we focus on the case where $\mathbb{T}_1 = \mathbb{T}$ is an intermediate compactly generated t-structure, with heart $\mathcal{H}$, and $\mathbb{T}_2 = \mathbb{D}$ is the standard t-structure. In this situation, we can be more precise on the the torsion pairs involved in the chain of HRS-tilts, from $\mathsf{Mod}(R)$ to $\mathcal{H}$, given by Lemma 3.3.6.

Proposition 3.3.7. *Let φ be an intermediate sp-filtration, with $\varphi(0) \neq \varphi(1) = \emptyset$, and denote by $\mathcal{H}_\varphi$ the heart of the associated compactly generated t-structure $\mathbb{T}_\varphi = (\mathcal{U}_\varphi, \mathcal{V}_\varphi)$. Then $\mathcal{T}_0 := \mathcal{H}_\varphi \cap \mathsf{Mod}(R)$ is a TTF class in $\mathcal{H}_\varphi$. In particular, we have that $\mathcal{T}_0 = \mathsf{supp}^{-1}(\varphi(0)) \cap \mathcal{H}_\varphi = \mathsf{Supp}^{-1}(\varphi(0)) \cap \mathcal{H}_\varphi$.*

Proof. Denote by $\mathbb{D}$ the standard t-structure. We begin by noticing that

$$\mathcal{H}_\varphi \cap \mathbb{D}^{\geq 0} \overset{(1)}{=} \mathcal{T}_0 \overset{(2)}{=} \mathcal{U}_\varphi \cap \mathsf{Mod}(R) \overset{(3)}{=} \mathsf{Supp}^{-1}(\varphi(0)) \cap \mathsf{Mod}(R).$$

Indeed, equality (1) follows from $\mathcal{H}_\varphi \subseteq \mathcal{U}_\varphi \subseteq \mathbb{D}^{\leq 0}$, (2) follows from $\mathsf{Mod}(R) \subseteq \mathbb{D}^{\geq 0} \subseteq \mathcal{V}_\varphi[1]$, and (3) follows by definition of $\mathcal{U}_\varphi$. In particular, (3) shows that $\mathsf{supp}(\mathcal{T}_0) = \mathsf{Supp}(\mathcal{T}_0) = \varphi(0)$ by Proposition 3.1.9.

We now show that $(\mathcal{H}_\varphi \cap \mathbb{D}^{\leq -1}, \mathcal{T}_0) = (\mathcal{H}_\varphi \cap \mathbb{D}^{\leq -1}, \mathcal{H}_\varphi \cap \mathbb{D}^{\geq 0})$ is a torsion pair in $\mathcal{H}_\varphi$. First, $\mathsf{Hom}_{\mathcal{H}_\varphi}(\mathcal{H}_\varphi \cap \mathbb{D}^{\leq -1}, \mathcal{T}_0) = 0$ is clear. Now, let X be an object of $\mathcal{H}_\varphi$ and consider the truncation triangle with respect to the standard t-structure

$$\tau^{\leq -1} X \longrightarrow X \longrightarrow H^0(X) \longrightarrow (\tau^{\leq -1} X)[1].$$

By definition of $\mathcal{U}_\varphi$, it is closed under standard truncations, so all the vertices belong to $\mathcal{U}_\varphi$. Moreover, it is also clear that $H^0(X)$ lies in $\mathbb{D}^{\geq 0} \subseteq \mathcal{V}_\varphi[1]$, and therefore $H^0(X)$ lies in $\mathcal{H}_\varphi$. Lastly, since $\mathcal{V}_\varphi[1]$ is closed under taking co-cones, also $\tau^{\leq -1}X$ belongs to $\mathcal{V}_\varphi[1]$, and hence $\tau^{\leq -1}X$ lies also in $\mathcal{H}_\varphi$. The triangle above is then a short exact sequence in $\mathcal{H}_\varphi$ and it is the torsion decomposition of X with respect to the torsion pair $(\mathcal{H}_\varphi \cap \mathbb{D}^{\leq -1}, \mathcal{T}_0)$.

It remains to show that $\mathcal{T}_0$ is also a torsion class in $\mathcal{H}_\varphi$ or, equivalently, that $\mathcal{T}_0$ is closed under quotients in $\mathcal{H}_\varphi$. Consider a short exact sequence in $\mathcal{H}_\varphi$

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

where Y lies in $\mathcal{T}_0$. Since $\mathcal{T}_0$ is a torsionfree class in $\mathcal{H}_\varphi$, it follows that X is also in $\mathcal{T}_0$. Since $\mathsf{supp}(Z) \subseteq \mathsf{supp}(X[1]) \cup \mathsf{supp}(Y)$, it follows that $\mathsf{supp}(Z) \subseteq \varphi(0)$. It remains to show that Z lies in $\mathsf{Mod}(R)$. Applying the standard cohomology functor to the triangle induced by the short exact sequence above, we observe that, since $\mathcal{T}_0$ is a hereditary torsion class in $\mathsf{Mod}(R)$, $H^{-1}(Z)$ lies in $\mathcal{T}_0 \subseteq \mathcal{H}_\varphi$. Moreover, the above paragraph has also shown that $\tau^{\leq -1}Z \simeq H^{-1}(Z)[1]$ lies in $\mathcal{H}_\varphi$. But this means that $H^{-1}(Z) = 0$ and Z must then lie in $\mathcal{T}_0$.

Finally, the last statement follows from the fact that $\varphi(0)$ is specialisation closed, $\mathcal{H}_\varphi$ is contained in $\mathsf{D}^b(R)$ and hereditary torsion classes in $\mathcal{H}_\varphi$ are determined by their support (see Proposition 3.2.2). $\square$

Lemma 3.3.8. *Let φ and ψ be intermediate sp-filtrations such that $\varphi(1) = \psi(1) = \emptyset$ and such that $\psi(i) = \varphi(i+1)$ for every $i < 0$. Then the compactly generated t-structure $\mathbb{T}_\psi$ associated to ψ is obtained by HRS-tilting $\mathbb{T}_\varphi = (\mathcal{U}_\varphi, \mathcal{V}_\varphi)$ (with heart $\mathcal{H}_\varphi$) with respect to a hereditary torsion pair of finite type whose torsion class is $\mathsf{Supp}^{-1}(\psi(0)) \cap \mathcal{H}_\varphi$.*

Proof. Let $\mathcal{T}_i$ denote the hereditary torsion class in $\mathsf{Mod}(R)$ supported on $\varphi(i)$, for any integer i. Since $\psi(0) \subseteq \psi(-1) = \varphi(0)$, we have that $\mathcal{T}' := \mathcal{H}_\varphi \cap \mathsf{Supp}^{-1}(\psi(0)) \subseteq \mathcal{H}_\varphi \cap \mathsf{Supp}^{-1}(\varphi(0)) = \mathcal{T}_0$, the last equality following from Proposition 3.3.7. We know from Theorem 3.2.6 that $\mathcal{T}'$ is a hereditary torsion class in $\mathcal{H}_\varphi$. If we tilt $\mathcal{H}_\varphi$ with respect to $\mathcal{T}'$ we obtain a t-structure having aisle

$$\overline{\mathcal{U}} := \mathcal{U}_\varphi[1] * \mathcal{T}'.$$

Now, since we have that $\mathcal{U}_\varphi[1] \subseteq \mathbb{D}^{\leq -1}$ and that $\mathcal{T}' \subseteq \mathcal{T}_0 \subseteq \mathbb{D}^{\geq 0}$, this aisle $\overline{\mathcal{U}}$ consists of the objects X such that the standard truncation $\tau^{\leq -1}X$ lies in $\mathcal{U}_\varphi[1]$ and the standard truncation $\tau^{\geq 0}X$ lies in $\mathcal{T}'$, i.e.

$$\overline{\mathcal{U}} = \{X \in \mathsf{D}(R)\colon \mathsf{Supp}(H^iX) \subseteq \varphi(i+1) = \psi(i),\ \forall i < 0,\ \mathsf{Supp}(H^0X) \subseteq \psi(0)\}$$

In other words, we have that $\overline{\mathcal{U}}$ is the aisle of the t-structure determined by ψ. Moreover, since this is also a compactly generated t-structure, it follows that the hereditary torsion pair we have tilted at is of finite type (see Proposition 2.7.10(2)). $\square$

Notation 3.3.9. Note that in the above lemma, the sp-filtration φ can be recovered from ψ. We will denote this operation on sp-filtrations by writing $\varphi = \psi^{\langle 1\rangle}$. In the notation of [1, §5.3], we have that $\psi^{\langle 1\rangle}$ is a shift of ψ', i.e. $\psi^{\langle 1\rangle}(i) = \psi'(i-1)$. Moreover, starting with an sp-filtration ψ such that $\psi(1) = \emptyset$, we will denote the iterations of this process by $\psi^{\langle n\rangle}$, for $n \geq 1$:

$$\psi^{\langle n\rangle}(i) = \begin{cases} \emptyset & \text{if } i > 0 \\ \psi(i-n) & \text{if } i \leq 0. \end{cases}$$

Proposition 3.3.10. *Let φ be an intermediate sp-filtration such that $\varphi(1) = \emptyset$. Then the compactly generated t-structure $\mathbb{T}_\varphi$ associated to φ can be built from the standard t-structure by an iteration of HRS-tilts at hereditary torsion pairs of finite type having specialisation-closed support.*

Proof. Since φ is intermediate, we have $\mathsf{Spec}(R) = \varphi(-n) \supsetneq \varphi(-n+1)$ for some $n \geq 0$. The statement then follows by induction on n, using Lemma 3.3.8. $\square$

3.3.3 Restrictable t-structures and derived equivalences

We now turn to restrictable, intermediate and compactly generated t-structures, with the aim of establishing that they induce derived equivalences. We begin by reviewing what is known about how to characterise the sp-filtrations associated to the restrictable compactly generated t-structures (see [1]). The following condition turns out to play a significant role in that characterisation for some commutative rings.

Definition 3.3.11. An sp-filtration φ is said to satisfy the **weak Cousin condition** if whenever $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals such that $\mathfrak{p} \subsetneq \mathfrak{q}$ and $\mathfrak{p}$ is maximal under $\mathfrak{q}$ (i.e. there is no prime ideal $\mathfrak{t}$ such that $\mathfrak{p} \subsetneq \mathfrak{t} \subsetneq \mathfrak{q}$), then we have

$$\forall j \in \mathbb{Z},\ \mathfrak{q} \in \varphi(j) \Rightarrow \mathfrak{p} \in \varphi(j-1)$$

Theorem 3.3.12 ([1, Theorem 3.10 and 6.9, Corollary 4.5][71, Theorem 6.3])**.** *Let R be a commutative noetherian ring, $\mathscr{B}$ the set of t-structures in $\mathsf{D}^b(\mathsf{mod}(R))$ and $\mathscr{T}$ the set of compactly generated t-structures in $\mathsf{D}(R)$. There is an assignment $\Theta\colon \mathscr{B} \longrightarrow \mathscr{T}$, sending a t-structure $\mathbb{B} := (\mathcal{X}, \mathcal{Y})$ in $\mathsf{D}^b(\mathsf{mod}(R))$ to the t-structure generated by $\mathcal{X}$, namely $\Theta(\mathbb{B}) := ({}^{\perp}(\mathcal{X}^{\perp}), \mathcal{X}^{\perp})$. Moreover, for every $\mathbb{B}$ in $\mathscr{B}$, we have*

(1) $\Theta(\mathbb{B}) \cap \mathsf{D}^b(\mathsf{mod}(R)) = \mathbb{B}$ *(and, in particular, Θ is injective);*

(2) The sp-filtration associated to $\Theta(\mathbb{B})$ satisfies the weak Cousin condition;

(3) The heart of $\Theta(\mathbb{B})$ is locally coherent and its subcategory of finitely presented objects coincides with the heart of $\mathbb{B}$.

The image of Θ is, then, the set of restrictable compactly generated t-structures. Moreover, if R admits a dualising complex, then the t-structures in the image of Θ are those whose associated sp-filtrations satisfy the weak Cousin condition.

Definition 3.3.13. Let R be a commutative noetherian ring. We say that an sp-filtration φ in $\mathsf{Spec}(R)$ is *restrictable* if the associated compactly generated t-structure is restrictable (in other words, the associated t-structure is in the image of the assignment Θ).

Note that it follows easily from the definition that if φ is an sp-filtration with $\varphi(1) = \emptyset$ satisfying the weak Cousin condition, then $\varphi^{\langle n \rangle}$ also satisfies the weak Cousin condition, for every $n \geq 1$. In fact, the following related statement holds.

Proposition 3.3.14 ([1, Lemma 5.7])**.** *If an intermediate sp-filtration φ is restrictable, then $\varphi^{\langle n \rangle}$ is also (intermediate and) restrictable, for any $n \geq 1$.*

Theorem 3.3.15. *Let R be a commutative noetherian ring and let $\mathbb{T}$ be an intermediate restrictable compactly generated t-structure in $\mathsf{D}(R)$. Then $\mathbb{T}$ induces a derived equivalence.*

Proof. Up to shifting $\mathbb{T}$, we may assume that the the associated sp-filtration φ has $\varphi(1) = \emptyset$. Proposition 3.3.10 then shows that there is a chain of HRS-tilts from the standard t-structure to $\mathbb{T}$, each step with respect to a hereditary torsion pairs of finite type. Morevoer, by Proposition 3.3.14, each of the t-structures $\mathbb{D} = \mathbb{T}_0, \mathbb{T}_1, \mathbb{T}_2, \dots, \mathbb{T}_n = \mathbb{T}$ of this chain is restrictable. Now we show by induction on i that $\mathbb{T}_i$ induces derived equivalence. For $i = 0$ there is nothing to do. Assume that we have shown that $\mathbb{T}_i$ induced derived equivalence. Then we can apply Proposition 2.7.12: since $\mathbb{T}_{i+1}$ is restrictable, the torsion pair used in this HRS-tilt must also be restrictable. Then, by Corollary 3.3.3, $\mathbb{T}_{i+1}$ also induces derived equivalence. $\square$

Corollary 3.3.16. *Let R be a commutative noetherian ring. Every bounded cosilting object of $\mathsf{D}(R)$ whose t-structure is restrictable is cotilting.*

Proof. It follows from [47, Proposition 3.10] that every bounded cosilting object is pure-injective. The associated t-structure is then compactly generated by [28, Corollary 2.14]. Since the complex is bounded, the associated t-structure is an intermediate t-structure. The result then follows from Theorem 3.3.15 and from the fact that a cosilting t-structure induces a derived equivalence if and only if it is cotilting (Proposition 2.6.13). $\square$

Taking Theorem 3.3.15 into account, the assumption that $\mathbb{T}$ induces a derived equivalence in Corollary 3.2.7 is redundant, therefore leading to the following simplification.

Corollary 3.3.17. *Let R be a commutative noetherian ring and $\mathbb{T}$ an intermediate restrictable compactly generated t-structure in $\mathsf{D}(R)$ with heart $\mathcal{H}$. Then there is a bijection between hereditary torsion pairs of finite type in $\mathcal{H}$ and specialisation-closed subsets of $\mathsf{Spec}(R)$.*

At this point, one might speculate whether every intermediate compactly generated t-structure leads to a derived equivalence. We instead show an example of such a t-structure that does not induce a derived equivalence. In other words, this provides (implicitly) an example of a bounded (3-term) pure-injective cosilting complex which is not cotilting over a commutative noetherian ring. Note that by Corollary 2.9.14 such an example cannot be found among 2-term cosilting complexes.

Example 3.3.18. Recall the situation considered in Example 3.2.15 and assume, furthermore, that R is connected, i.e. that it has no non-trivial idempotent elements. With the same notation, $\mathcal{H}_{\mathbf{t}}$ is the heart of the t-structure corresponding to the sp-filtration $\mathsf{Spec}(R) \supseteq V \supseteq \emptyset$. By Lemma 3.2.10 we know that the set V also corresponds to a hereditary torsion pair (of finite type, by Theorem 3.2.6) in $\mathcal{H}_{\mathbf{t}}$, namely $\mathbf{s} = (\mathcal{T}, \mathcal{C}[1])$, where $\mathcal{C}$ is the Giraud subcategory associated to $\mathcal{T}$ in $\mathsf{Mod}(R)$. Consider the heart $\mathcal{H}_{\mathbf{s}}$ of the HRS-tilt of the t-structure with heart $\mathcal{H}_{\mathbf{t}}$ with respect to $\mathbf{s}$. The corresponding t-structure, by Lemma 3.3.8 is associated to the intermediate sp-filtration $\mathsf{Spec}(R) \supseteq V \supseteq V \supseteq \emptyset$. Notice that this filtration does not satisfy the weak Cousin condition and, hence, this t-structure is not restrictable.

By construction, we have $\mathcal{H}_{\mathbf{s}} = \mathcal{C}[2] * \mathcal{T}$. Notice that since $\mathbf{t}$ is perfect, for all objects T in $\mathcal{T}$ and C in $\mathcal{C}$ we have $\mathsf{Hom}_{\mathsf{D}(R)}(T, C[3]) \simeq \mathsf{Ext}^3_R(T, C) = 0$ and hence all triangles

$$C[2] \longrightarrow X \longrightarrow T \longrightarrow C[3]$$

split. In other words, the torsion pair $(\mathcal{C}[2], \mathcal{T})$ in $\mathcal{H}_{\mathbf{s}}$ is a split torsion pair. Moreover, the same argument shows that $\mathsf{Hom}_{\mathsf{D}(R)}(T, C[2]) \simeq \mathsf{Ext}^2_R(T, C) = 0$ and, thus, we have that in fact also $(\mathcal{T}, \mathcal{C}[2])$ is a torsion pair in $\mathcal{H}_{\mathbf{s}}$. In other words, $\mathcal{C}[2]$ and $\mathcal{T}$ are abelian subcategories of $\mathcal{H}_{\mathbf{s}}$ and $\mathcal{H}_{\mathbf{s}} \simeq \mathcal{C}[2] \times \mathcal{T}$.

Now, since R is connected, it follows that $\mathsf{D}(R)$ is an indecomposable triangulated category, i.e. it is not the product of two triangulated subcategories (see [10, Example 3.2]). However, it is clear that $\mathsf{D}(()\mathcal{H}_{\mathbf{s}})$ is not indecomposable, as it is equivalent to the product $\mathsf{D}(()\mathcal{C}[2]) \times \mathsf{D}(()\mathcal{T})$. As a consequence, $\mathcal{H}_{\mathbf{s}}$ cannot be derived equivalent to $\mathsf{Mod}(R)$. Note that, in particular, this provides an example of a cosilting (3-term) object of $\mathsf{D}(R)$ which is not cotilting.

We conclude the paper exploring some consequences for the hearts of t-structures of $\mathsf{D}^b(\mathsf{mod}(R))$.

Proposition 3.3.19. *Let R be a commutative noetherian ring, and $\mathbb{B}$ a bounded t-structure of $\mathsf{D}^b(\mathsf{mod}(R))$, with heart $\mathcal{B}$. Then $\mathcal{B}$ is the category of finitely* *presented objects of a locally coherent Grothendieck category which is derived equivalent to* $\mathsf{Mod}(R)$*. Moreover, Serre subcategories of* $\mathcal{B}$ *are in bijection with specialisation-closed subsets of* $\mathsf{Spec}(R)$*.*

Proof. Consider the compactly generated t-structure $\Theta(\mathbb{B})$. It is intermediate because so is $\mathbb{B}$, and it is restrictable by construction. Hence, its heart $\mathcal{H}$ is derived equivalent to $\mathsf{Mod}(R)$, by Theorem 3.3.15. Now, by Proposition 2.5.43, $\mathcal{H}$ is a locally coherent Grothendieck category with $\mathsf{fp}(\mathcal{H}) = \mathcal{H} \cap \mathsf{D}^b(\mathsf{mod}(R)) = \mathcal{B}$. Finally, since Serre subcategories of $\mathcal{B}$ are in bijection with hereditary torsion pairs of finite type $\mathcal{H}$ (see [27, 36]), and therefore with specialisation closed subsets of $\mathsf{Spec}(R)$ by Corollary 3.3.17. $\square$

Chapter 4

Coderived equivalences for commutative noetherian rings

The material of this Chapter is taken from joint work with M. Hrbek [29]. Let R be a commutative noetherian ring, and $\mathbb{T}$ an intermediate restrictable t-structure in $\mathsf{D}(R)$, with heart $\mathcal{H}$. We recall some examples:

Example 4.0.1. Let R be a commutative noetherian ring.

- The t-structure obtained by HRS-tilting the standard t-structure with respect to a hereditary torsion pair in $\mathsf{Mod}(R)$ is compactly generated and restrictable (see the previous Chapter). These t-structures correspond to sp-filtrations Φ such that $\Phi(n) = \mathsf{Spec}(R)$ for all $n < 0$ and $\Phi(n) = \emptyset$ for all $n > 0$ (Remark 3.2.9).

- Assume that d is a codimension function on $\mathsf{Spec}(R)$, that is, a function $\mathsf{d} : \mathsf{Spec}(R) \to \mathbb{Z}$ such that $\mathsf{d}(\mathfrak{q}) = \mathsf{d}(\mathfrak{p}) + 1$ whenever $\mathfrak{p} \subsetneq \mathfrak{q}$ are primes with $\mathfrak{q}$ minimal over $\mathfrak{p}$. Then the assignment $\Phi_{\mathsf{d}}(n) = \{\mathfrak{p} \in \mathsf{Spec}(R) \mid \mathsf{d}(\mathfrak{p}) > n\}$ defines an sp-filtration which satisfies the weak Cousin condition.

  Furthermore, any pointwise dualizing complex D induces a codimension function d_D [26, p. 287], and therefore a restrictable t-structure, see [1, §6].

- If R admits a dualizing complex D, the restrictable t-structure induced by the codimension function d_D has a particularly nice description, we follow [1, §6.4]. The functor $\mathbb{R}\mathsf{Hom}_R(-, D)$ induces a duality functor on the category $\mathsf{D}^b(\mathsf{mod}(R))$, and therefore it sends the standard t-structure to another t-structure on $\mathsf{D}^b(\mathsf{mod}(R))$, called the **Cohen-Macaulay t-structure**. This t-structure then naturally lifts to a restrictable t-structure in $\mathsf{D}(\mathsf{Mod}(R))$, see [48, §3], and coincides with the compactly generated t-structure corresponding to the sp-filtration Φ_{d_D}.

Our goal in this Chapter is to extend the derived equivalence $\mathsf{D}(\mathcal{H}) \simeq \mathsf{D}(R)$ of Theorem 3.3.15 to an equivalence between coderived categories.

4.1 Krause's recollement for locally coherent Grothendieck categories

In [39], Krause constructed the following recollement, for a locally noetherian Grothendieck category $\mathcal{A}$:

$$\begin{tikzcd} \mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(\mathcal{A})) \arrow[r, "\subseteq"] & \mathsf{K}(\mathsf{Inj}(\mathcal{A})) \arrow[l, bend right, "i_l"'] \arrow[l, bend left, "i_r"] \arrow[r, "Q"] & \mathsf{D}(\mathcal{A}). \arrow[l, bend right, "Q_l"'] \arrow[l, bend left, "Q_r"] \end{tikzcd}$$

The functor Q_r is the right adjoint of the Verdier localisation Q, given by homotopically injective resolutions (see §2.3.3); the existence of i_r follows (see Proposition 2.2.7). The main content of the recollement is the existence of the left adjoint Q_l; in order to prove it, it is shown that Q_r identifies the objects of $\mathsf{D}^b(\mathsf{fp}(\mathcal{A}))$ with the compact objects of $\mathsf{K}(\mathsf{Inj}(\mathcal{A}))$.

We are going to extend this construction to the case of $\mathcal{A}$ locally coherent Grothendieck category, with the additional property that $\mathsf{D}(\mathcal{A})$ is compactly generated. This was done by Šťovíček [78, Theorem 7.7] under the additional assumption that $\mathcal{A}$ admits a set of finitely presented generators of finite projective dimension (see [78, Hypothesis 7.1]). This assumption implies that $\mathsf{D}(\mathcal{A})$ is compactly generated, but it is strictly stronger: we demonstrate an example, which is a Happel-Reiten-Smalø tilt in the derived category of a commutative noetherian ring, in Example 4.2.8. Our approach here is closer to the original one of Krause, but relies on some of the results of Šťovíček [78, §6] (these do not depend on the aforementioned [78, Hypothesis 7.1]).

Our starting point is the following result of Šťovíček.

Proposition 4.1.1. *[78, Corollary 6.13] Let $\mathcal{A}$ be a locally coherent Grothendieck category. Then $\mathsf{K}(\mathsf{Inj}(\mathcal{A}))$ is compactly generated and the functor assigning to an object of $\mathsf{D}^b(\mathsf{fp}(\mathcal{A}))$ its injective resolution induces an equivalence $\mathsf{D}^b(\mathsf{fp}(\mathcal{A})) \cong \mathsf{K}(\mathsf{Inj}(\mathcal{A}))^c$.*

Corollary 4.1.2. *Let $\mathcal{A}$ be a locally coherent Grothendieck category. Then the functor $Q : \mathsf{K}(\mathsf{Inj}(\mathcal{A})) \to \mathsf{D}(\mathcal{A})$ admits a right adjoint Q_r.*

Furthermore, the equivalence $\mathsf{D}^b(\mathsf{fp}(\mathcal{A})) \cong \mathsf{K}(\mathsf{Inj}(\mathcal{A}))^c$ of Proposition 4.1.1 is induced by the restrictions of the adjoint functors Q_r and Q.

Proof. By Proposition 4.1.1, $\mathsf{K}(\mathsf{Inj}(\mathcal{A}))$ is compactly generated. Since $\mathcal{A}$ has exact coproducts, the functor Q preserves coproducts, and so [55, Theorem 4.1] applies and produces the desired right adjoint.

It follows directly from the adjunction that for any $X \in \mathsf{D}(\mathcal{A})$, $Q_r(X)$ is homotopy equivalent to a homotopically injective resolution of X (which exists by Proposition 2.3.8). By Proposition 4.1.1 we have that $Q_r(X)$ restricts to the equivalence $\mathsf{D}^b(\mathsf{fp}(\mathcal{A})) \cong \mathsf{K}(\mathsf{Inj}(\mathcal{A}))^c$ with the inverse equivalence being the restriction of Q to $\mathsf{K}(\mathsf{Inj}(\mathcal{A}))^c$. □

4.1.1 Compact objects of $\mathsf{D}(\mathcal{A})$ and the (small) singularity category

The main obstacle in extending Krause's proof to the locally coherent case is showing that any compact object of $\mathsf{D}(\mathcal{A})$ belongs to $\mathsf{D}^b(\mathsf{fp}(\mathcal{A}))$, and therefore represents a compact object also in $\mathsf{K}(\mathsf{Inj}(\mathcal{A}))$ via Q_r; the proof in the locally noetherian case [39, Lemma 4.1] does not generalize directly.

Following Gillespie [21], an object M of a Grothendieck category $\mathcal{A}$ is said to be *of type* FP_∞ if the functor $\mathsf{Ext}^i_{\mathcal{A}}(M, -)$ naturally preserves direct limits for all $i \geq 0$. It will be convenient for our purposes to extend this notion to any object of the bounded derived category.

Definition 4.1.3. Let $\mathcal{A}$ be a Grothendieck category. An object $X \in \mathsf{D}^b(\mathcal{A})$ is **of type** FP_∞ if for any direct system $(M_i \mid i \in I)$ in $\mathcal{A}$ and any $n \in \mathbb{Z}$ the natural map

$$\varinjlim_{i\in I} \mathsf{Hom}_{\mathsf{D}^b(\mathcal{A})}(X, M_i[n]) \to \mathsf{Hom}_{\mathsf{D}^b(\mathcal{A})}(X, \varinjlim_{i\in I} M_i[n])$$

is an isomorphism.

Not very surprisingly, Definition 4.1.3 admits a somewhat more internal characterization using homotopy colimits of bounded directed coherent diagrams (see §2.4), which in turn provides a "bounded" version of the following notion from the theory of stable derivators.

Definition 4.1.4 ([73, Definition 5.1])**.** Given a directed small category I, $X \in \mathsf{D}(\mathcal{A})$, and $\mathcal{Y} \in \mathsf{D}(\mathcal{A}^I)$, there is a natural map (see [77, Definition 6.5])

$$\varinjlim_{i\in I} \mathsf{Hom}_{\mathsf{D}(\mathcal{A})}(X, \mathcal{Y}_i) \to \mathsf{Hom}_{\mathsf{D}(\mathcal{A})}(X, \mathsf{hocolim}_I \mathcal{Y}).$$

An object $X \in \mathsf{D}(\mathcal{A})$ is called **homotopically finitely presented** if the map above is an isomorphism for any choice of I and $\mathcal{Y}$.

Lemma 4.1.5. *Let $\mathcal{A}$ be a Grothendieck category. An object $X \in \mathsf{D}^b(\mathcal{A})$ is of type FP_∞ if and only if for any directed small category I and any coherent diagram $\mathcal{Y} \in \mathsf{D}^b(\mathcal{A}^I)$ the natural map*

$$\varinjlim_{i\in I} \mathsf{Hom}_{\mathsf{D}^b(\mathcal{A})}(X, \mathcal{Y}_i) \to \mathsf{Hom}_{\mathsf{D}^b(\mathcal{A})}(X, \mathsf{hocolim}_I \mathcal{Y})$$

is an isomorphism.

Proof. Since $\mathcal{Y} \in \mathsf{D}^b(\mathcal{A}^I)$, the coherent diagram $\mathcal{Y}$ is represented by a direct system $(Y_i \mid i \in I)$ in $\mathsf{C}(\mathcal{A})$ such that the cohomology of the complexes Y_i is uniformly bounded. Therefore, there is $n \in \mathbb{Z}$ and $k \geq 0$ such that for all $i \in I$, the cohomology of Y_i vanishes outside of degrees $n, \dots, n+k$. If $k = 0$, by applying the soft truncation we may assume that $\mathcal{Y}$ is such that $(Y_i \mid i \in I)$ is a direct system of stalk complexes in degree n, and therefore the required isomorphism is provided by the definition of an object of type FP_∞. The general case follows by induction on $k > 0$. Indeed, applying $\mathsf{hocolim}_I$ to the soft truncation triangle of $\mathcal{Y}$ in $\mathsf{D}^b(\mathcal{A}^I)$ we obtain the triangle

$$\mathsf{hocolim}_I \tau^{\leq n}\mathcal{Y} \to \mathsf{hocolim}_I \mathcal{Y} \to \mathsf{hocolim}_I \tau^{>n}\mathcal{Y} \to (\mathsf{hocolim}_I \tau^{\leq n})[1]$$

in $\mathsf{D}^b(\mathcal{A})$. Notice that soft truncations commute naturally with the component functors $(-)_i$, and we have triangles in $\mathsf{D}^b(\mathcal{A})$

$$\tau^{\leq n}\mathcal{Y}_i \to \mathcal{Y}_i \to \tau^{>n}\mathcal{Y}_i \to (\tau^{\leq n}\mathcal{Y}_i)[1].$$

Then there is the following commutative diagram, in which the horizontal maps are induced by the two triangles above and the vertical ones are the natural maps:

$$\begin{array}{ccccccc}
\varinjlim_{i \in I} \mathsf{Hom}(X, \tau^{>n}\mathcal{Y}_i[-1]) & \longrightarrow & \varinjlim_{i \in I} \mathsf{Hom}(X, \tau^{\leq n}\mathcal{Y}_i) & \longrightarrow & \varinjlim_{i \in I} \mathsf{Hom}(X, \mathcal{Y}_i) & \to \\
\downarrow & & \downarrow & & \downarrow & \\
\mathsf{Hom}(X, \mathsf{hocolim}_I \tau^{>n}\mathcal{Y}[-1]) & \to & \mathsf{Hom}(X, \mathsf{hocolim}_I \tau^{\leq n}\mathcal{Y}) & \to & \mathsf{Hom}(X, \mathsf{hocolim}_I \mathcal{Y}) & \to
\end{array}$$

$$\begin{array}{ccccc}
\to & \varinjlim_{i \in I} \mathsf{Hom}(X, \tau^{>n}\mathcal{Y}_i) & \longrightarrow & \varinjlim_{i \in I} \mathsf{Hom}(X, \tau^{\leq n}\mathcal{Y}_i[1]) \\
& \downarrow & & \downarrow \\
\to & \mathsf{Hom}(X, \mathsf{hocolim}_I \tau^{>n}\mathcal{Y}) & \to & \mathsf{Hom}(X, \mathsf{hocolim}_I \tau^{\leq n}\mathcal{Y}[1])
\end{array}$$

Then the induction step follows directly by Five lemma, as both the coherent diagrams $\tau^{>n}\mathcal{Y}$ and $\tau^{\leq n}\mathcal{Y}$ are subject to the induction hypothesis for $k-1$. $\square$

Lemma 4.1.6. *Let $\mathcal{A}$ be a Grothendieck category. The objects of type FP_∞ of $X \in \mathsf{D}^b(\mathcal{A})$ form a thick subcategory of $\mathsf{D}^b(\mathcal{A})^c$.*

Proof. By exactness of coproducts in $\mathcal{A}$, the coproducts in $\mathsf{D}^b(\mathcal{A})$ are precisely the coproducts of collections of objects with uniformly bounded cohomology computed in $\mathsf{D}(\mathcal{A})$. Therefore, any coproduct in $\mathsf{D}^b(\mathcal{A})$ can be realized as a directed homotopy colimit of a suitable diagram of $\mathsf{D}^b(\mathcal{A}^I)$ whose components are finite subcoproducts. In this way Lemma 4.1.5 shows that any object of type FP_∞ in $\mathsf{D}^b(\mathcal{A})$ is compact in $\mathsf{D}^b(\mathcal{A})$. The fact that objects of type FP_∞ form a triangulated subcategory follows from the Five lemma similarly as in the proof of Lemma 4.1.5; the closure under retracts is clear. $\square$

Lemma 4.1.7. *Let $\mathcal{A}$ be a locally coherent Grothendieck category. An object $X \in \mathsf{D}^b(\mathcal{A})$ is of type FP_∞ if and only if $X \in \mathsf{D}^b(\mathsf{fp}(\mathcal{A}))$.*

Proof. An object $F \in \mathsf{fp}(\mathcal{A})$ is of type FP_∞ as an object in $\mathsf{D}^b(\mathcal{A})$, see [21, Theorem 3.21]. By Lemma 4.1.6, any object in the thick closure of $\mathsf{fp}(\mathcal{A})$ in $\mathsf{D}^b(\mathcal{A})$ is of type FP_∞, which shows that $X \in \mathsf{D}^b(\mathsf{fp}(\mathcal{A}))$ implies that X is of type FP_∞.

For the converse implication, let $X \in \mathsf{D}^b(\mathcal{A})$ be of type FP_∞ and let n be a maximal integer such that $H^n(X) \neq 0$. For any $M \in \mathcal{A}$ the soft truncation yields a natural isomorphism $\mathsf{Hom}_{\mathsf{D}^b(\mathcal{A})}(X, M[-n]) \cong \mathsf{Hom}_{\mathcal{A}}(H^n(X), M)$. Since X is of type FP_∞, it follows that the functor $\mathsf{Hom}_{\mathcal{A}}(H^n(X), -) : \mathcal{A} \to \mathsf{Ab}$ preserves direct limits, and so $H^n(X)$ belongs to $\mathsf{fp}(\mathcal{A})$. Using the previous paragraph and Lemma 4.1.6 we infer that the soft truncation $\tau^{<n}X$ is of type FP_∞. Continuing by finite induction we conclude that all cohomologies of X belong to $\mathsf{fp}(\mathcal{A})$, and so $X \in \mathsf{D}^b(\mathsf{fp}(\mathcal{A}))$, see e.g. [33, Theorem 15.3.1]. □

Remark 4.1.8. Combining Lemmas 4.1.7 and 4.1.6 we obtain the inclusion $\mathsf{D}^b(\mathsf{fp}(\mathcal{A})) \subseteq \mathsf{D}^b(\mathcal{A})^c$. We do not know whether the converse inclusion holds true in general for a locally coherent Grothendieck category such that $\mathsf{D}(\mathcal{A})$ is compactly generated. However, in §4.2, we will show that this these two subcategories coincide in case $\mathcal{A}$ is the heart of an intermediate cotilting t-structure over a commutative noetherian ring.

Proposition 4.1.9. *Let $\mathcal{A}$ be a locally coherent Grothendieck category. There is an inclusion $\mathsf{D}(\mathcal{A})^c \subseteq \mathsf{D}^b(\mathsf{fp}(\mathcal{A}))$.*

Proof. Let C be a compact object of $\mathsf{D}(\mathcal{A})$. For each $n \in \mathbb{Z}$ there is a natural map $C \to E(H^n(C))[-n]$ in $\mathsf{D}(\mathcal{A})$ to a shift of the injective envelope of $H^n(C)$. This induces a morphism $C \to \prod_{n\in\mathbb{Z}} E(H^n(C))[-n]$. Products in $\mathsf{D}(\mathcal{A})$ are computed as component-wise products of homotopically injective resolutions; so in this case, the compontent-wise product of the $E(H^n(C))[-n]$. In this particular case, it coincides with the component-wise coproduct. This is also the coproduct in $\mathsf{D}(\mathcal{A})$, since $\mathcal{A}$ has exact coproducts. Therefore we obtain a morphism $C \to \coprod_{n\in\mathbb{Z}} E(H^n(C))[-n]$ in $\mathsf{D}(\mathcal{A})$. By compactness of C, this map factors through a finite subcoproduct. It follows that C has finitely many non-zero cohomologies.

By [73, Proposition 5.4], C is homotopically finitely presented in $\mathsf{D}(\mathcal{A})$. In particular, C is of type FP_∞ in $\mathsf{D}^b(\mathcal{A})$. Therefore, $C \in \mathsf{D}^b(\mathsf{fp}(\mathcal{A}))$ by Lemma 4.1.7. □

Remark 4.1.10. Let $\mathcal{A}$ be a locally coherent Grothendieck category. Proposition 4.1.9 shows that $\mathsf{D}(\mathcal{A})^c$ is a thick subcategory of $\mathsf{D}^b(\mathsf{fp}(\mathcal{A}))$, and therefore we can form the Verdier quotient $\mathsf{D}^{\mathsf{sing}}(\mathcal{A}) = \mathsf{D}^b(\mathsf{fp}(\mathcal{A}))/\mathsf{D}(\mathcal{A})^c$. Following the locally noetherian case [39], we call $\mathsf{D}^{\mathsf{sing}}(\mathcal{A})$ the **(small) singularity category** of $\mathcal{A}$.

4.1.2 The left adjoint Q_l

Lemma 4.1.11. *Let $\mathcal{A}$ be a locally coherent Grothendieck category. For any $C \in \mathsf{D}(\mathcal{A})^c$ and any $Y \in \mathsf{K}(\mathsf{Inj}(\mathcal{A}))$, there is a natural isomorphism*

$$\mathsf{Hom}_{\mathsf{D}(\mathcal{A})}(C, QY) \cong \mathsf{Hom}_{\mathsf{K}(\mathsf{Inj}(\mathcal{A}))}(Q_r C, Y).$$

Proof. Consider the natural transformation

$$\eta_{C,Y} : \mathsf{Hom}_{\mathsf{K}(\mathsf{Inj}(\mathcal{A}))}(Q_r C, Y) \to \mathsf{Hom}_{\mathsf{D}(()\mathcal{A})}(QQ_r C, QY)$$

induced by Q. By Corollary 4.1.2, the functors Q_r and Q induce an equivalence $\mathsf{D}^b(\mathsf{fp}(\mathcal{A})) \cong \mathsf{K}(\mathsf{Inj}(\mathcal{A}))^c$. We see that $QQ_r C$ is naturally isomorphic to C and also, in view of Proposition 4.1.9, that $\eta_{C,Y}$ is an isomorphism whenever $Y \in \mathsf{K}(\mathsf{Inj}(\mathcal{A}))^c$. Consider the subcategory $\mathcal{K}$ of $\mathsf{K}(\mathsf{Inj}(\mathcal{A}))$ consisting of all objects Y such that $\eta_{C,Y}$ is an isomorphism for all $C \in \mathsf{D}(\mathcal{A})^c$. A standard argument shows that $\mathcal{K}$ is a triangulated subcategory. Since C is compact in $\mathsf{D}(\mathcal{A})$ and $Q_r C$ is compact in $\mathsf{K}(\mathsf{Inj}(\mathcal{A}))$, the subcategory $\mathcal{K}$ is closed under coproducts. Then $\mathcal{K}$ is a localizing subcategory of $\mathsf{K}(\mathsf{Inj}(\mathcal{A}))$ containing all compact objects, and therefore $\mathcal{K} = \mathsf{K}(\mathsf{Inj}(\mathcal{A}))$ by Proposition 4.1.1. □

Lemma 4.1.12. *Let $\mathcal{A}$ be a locally coherent Grothendieck category such that $\mathsf{D}(\mathcal{A})$ is compactly generated. Then the functor $Q : \mathsf{K}(\mathsf{Inj}(\mathcal{A})) \to \mathsf{D}(\mathcal{A})$ admits a left adjoint Q_l.*

Proof. Let $\mathcal{L}$ be the localizing subcategory of $\mathsf{K}(\mathsf{Inj}(\mathcal{A}))$ generated by $Q_r(\mathsf{D}(\mathcal{A})^c)$. Then $\mathcal{L}$ is a compactly generated triangulated category, and the restriction $Q_{\restriction \mathcal{L}} : \mathcal{L} \to \mathsf{D}(\mathcal{A})$ is a functor between compactly generated triangulated categories that preserves coproducts and by Corollary 4.1.2 restricts further to an equivalence $\mathcal{L}^c \cong \mathsf{D}(\mathcal{A})^c$. Then $Q_{\restriction \mathcal{L}}$ is an equivalence by Lemma 2.1.23, and so there is an inverse equivalence $P : \mathsf{D}(\mathcal{A}) \xrightarrow{\sim} \mathcal{L}$. We define Q_l as the composition of P and the inclusion $\iota : \mathcal{L} \hookrightarrow \mathsf{K}(\mathsf{Inj}(\mathcal{A}))$.

The inclusion ι of $\mathcal{L}$ into $\mathsf{K}(\mathsf{Inj}(\mathcal{A}))$ has a right adjoint $\tau : \mathsf{K}(\mathsf{Inj}(\mathcal{A})) \to \mathcal{L}$, see e.g. [55, Theorem 4.1]. It follows that $Q_l = \iota \circ P$ has a right adjoint $Q \circ \tau$. It remains to show that $Q \circ \tau$ is naturally equivalent to Q. Applying Q to the counit transformation $\iota \circ \tau \to \mathrm{id}_{\mathsf{K}(\mathsf{Inj}(\mathcal{A}))}$ we see that it is enough to show that any object of $\mathcal{L}^{\perp_0}$ is sent to zero by Q, i.e. $\mathcal{L}^{\perp_0} \subseteq \mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(\mathcal{A}))$. If $Y \in \mathcal{L}^{\perp_0}$ then $\mathsf{Hom}_{\mathsf{K}(\mathsf{Inj}(\mathcal{A}))}(Q_r C, Y) = 0$ for all $C \in \mathsf{D}(\mathcal{A})^c$. By Lemma 4.1.11, this implies $\mathsf{Hom}_{\mathsf{D}(\mathcal{A})}(C, QY) = 0$ for all $C \in \mathsf{D}(\mathcal{A})^c$, and since $\mathsf{D}(\mathcal{A})$ is compactly generated, we have $QY = 0$, as desired. □

We record the following auxiliary property of the adjoints of Q for later use.

Lemma 4.1.13. *In the setting of Lemma 4.1.12 we have an isomorphism $Q_r C \cong Q_l C$ for all $C \in \mathsf{D}(\mathcal{A})^c$.*

Proof. By Lemmas 4.1.12 and 4.1.11, there are natural isomorphisms for all $Y \in \mathsf{K}(\mathsf{Inj}(\mathcal{A}))$

$$\mathsf{Hom}_{\mathsf{K}(\mathsf{Inj}(\mathcal{A}))}(Q_l C, Y) \cong \mathsf{Hom}_{\mathsf{D}(\mathcal{A})}(C, QY) \cong \mathsf{Hom}_{\mathsf{K}(\mathsf{Inj}(\mathcal{A}))}(Q_r C, Y).$$

The isomorphism $Q_r C \cong Q_l C$ thus follows from the Yoneda lemma. $\square$

Theorem 4.1.14. *Let $\mathcal{A}$ be a locally coherent Grothendieck category such that $\mathsf{D}(\mathcal{A})$ is compactly generated. Then there is a recollement:*

$$\begin{tikzcd}
\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(\mathcal{A})) \arrow[r, "\subseteq"] & \mathsf{K}(\mathsf{Inj}(\mathcal{A})) \arrow[l, bend right, "i_l"'] \arrow[l, bend left, "i_r"] \arrow[r, "Q"] & \mathsf{D}(\mathcal{A}) \arrow[l, bend right, "Q_l"'] \arrow[l, bend left, "Q_r"]
\end{tikzcd}$$

Proof. Recall that the functor Q is a Verdier localization functor whose kernel is the full subcategory $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(\mathcal{A}))$. By Proposition 2.2.7, it is enough to establish that Q admits both left and right adjoint functors, which we showed in Corollary 4.1.2 and Lemma 4.1.12. $\square$

Corollary 4.1.15. *In the setting of Theorem 4.1.14, the category $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(\mathcal{A}))$ is compactly generated and the subcategory of compact objects $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(\mathcal{A}))^c$ is equivalent up to retracts to the singularity category $\mathsf{D}^{\mathsf{sing}}(\mathcal{A})$ of $\mathcal{A}$.*

Proof. This follows directly from [54, Theorem 2.1] applied to the situation of Theorem 4.1.14. $\square$

Corollary 4.1.16 (cf. [39, Corollary 4.4])**.** *Let $\mathcal{A}$ be a locally coherent Grothendieck category such that $\mathsf{D}(\mathcal{A})$ is compactly generated. Then any product of acyclic complexes of injective objects is acyclic.*

Remark 4.1.17. In the locally noetherian situation [39], the category $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(\mathcal{A}))$ is called the **stable derived category** of $\mathcal{A}$ and denoted by $\mathsf{S}(\mathcal{A})$, while other sources [6, 78] call it the **(large) singularity category** of $\mathcal{A}$. In the latter two citations, it is shown that $\mathsf{S}(\mathcal{A})$ is a homotopy category of $\mathsf{C}(\mathcal{A})$ endowed with a suitable abelian model structure. It is also explained in [78, §7] that $\mathsf{S}(\mathcal{A})$ naturally identifies with the subcategory of all acyclic complexes of the coderived category $\mathsf{D}^{\mathsf{co}}(\mathcal{A})$ via the equivalence $\mathsf{K}(\mathsf{Inj}(\mathcal{A})) \cong \mathsf{D}^{\mathsf{co}}(\mathcal{A})$), and the same equivalence identifies the recollement of Theorem 4.1.14 with the recollement of the form

$$\begin{tikzcd}
\mathsf{S}(\mathcal{A}) \arrow[r, "\subseteq"] & \mathsf{D}^{\mathsf{co}}(\mathcal{A}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \mathsf{D}(\mathcal{A}) \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

4.2 Restrictable t-structures

Recall from Theorem 3.1.17 that if R is a commutative noetherian ring and $\mathbb{T}$ is an intermediate cotilting t-structure with heart $\mathcal{H}$, then $\mathbb{T}$ is compactly generated and $\mathcal{H}$ is a locally finitely presentable Grothendieck category by Proposition 2.5.42. In view of the previous section, we are mostly interested in the case when $\mathcal{H}$ is in addition locally coherent. Therefore, in this section we consider the following setting.

Setting 4.2.1. *Let R be a commutative noetherian ring. Let $\mathbb{T}_C$ be a t-structure in $\mathsf{D}(R)$, with heart $\mathcal{H}_C$, such that:*

(C1) $\mathbb{T}_C$ is the cotilting t-structure associated to a cotilting object C.

(C2) $\mathbb{T}_C$ is intermediate.

(C3) $\mathcal{H}_C$ is a locally coherent Grothendieck category.

Condition (C2) is equivalent to the requirement that $C \in \mathsf{K}^b(\mathsf{Inj}(R))$, which is sometimes included in the definition of a cotilting object. The fact that C is cotilting provides us with a triangle equivalence

$$\mathsf{real}_C \colon \mathsf{D}(\mathcal{H}_C) \to \mathsf{D}(\mathsf{Mod}(R))$$

which restricts to the level of bounded derived categories and which lifts to an equivalence between the standard derivators, see Proposition 2.6.15.

The main goal of this section is to characterize Setting 4.2.1 using the restrictability of the t-structure $\mathbb{T}_C$. To do that, we first need to better understand the compact objects in the bounded derived category of $\mathcal{H}_C$. Recall from Remark 4.1.8 that we have an inclusion $\mathsf{D}^b(\mathsf{fp}(\mathcal{H}_C)) \subseteq \mathsf{D}^b(\mathcal{H}_C)^c$. We will use the derived equivalence to $\mathsf{Mod}(R)$ to show that this inclusion is an equality.

Lemma 4.2.2. *Let $\mathcal{A}$ and $\mathcal{E}$ be Grothendieck categories and $\eta : \mathbb{D}_{\mathcal{A}} \to \mathbb{D}_{\mathcal{E}}$ a bounded equivalence of derivators. Then for an object $X \in \mathsf{D}^b(\mathcal{A})$ is of type FP_∞ if and only if $\eta^{\mathbf{1}}(X)$ is of type FP_∞ in $\mathsf{D}^b(\mathcal{E})$.*

Proof. Let I be a directed small category and $\mathcal{Y} \in \mathsf{D}^b(\mathcal{A}_C^I)$. Then there is the following commutative square induced by application of the equivalence η between derivators, where all of the maps are the naturally induced ones:

$$\begin{array}{ccc} \varinjlim_{i\in I} \mathsf{Hom}_{\mathsf{D}(\mathcal{A})}(X, \mathcal{Y}_i) & \xrightarrow{\cong} & \varinjlim_{i\in I} \mathsf{Hom}_{\mathsf{D}(\mathcal{E})}(\eta^{\mathbf{1}}X, (\eta^I\mathcal{Y})_i) \\ \downarrow & & \downarrow \\ \mathsf{Hom}_{\mathsf{D}(\mathcal{A})}(X, \mathsf{hocolim}_I\mathcal{Y}) & \xrightarrow{\cong} & \mathsf{Hom}_{\mathsf{D}(\mathcal{E})}(\eta^{\mathbf{1}}X, \mathsf{hocolim}_I(\eta^I\mathcal{Y})) \end{array}$$

Note that both the horizontal isomorphisms are induced by the triangle equivalence $\eta^{\mathbf{1}}$. Indeed, this follows from the two canonical isomorphisms induced by the derivator equivalence η:

$$\mathsf{hocolim}_I(\eta^I \mathcal{Y}) \cong \eta^{\mathbf{1}}(\mathsf{hocolim}_I \mathcal{Y}) \text{ and } (\eta^I \mathcal{Y})_i \cong \eta^{\mathbf{1}}(\mathcal{Y}_i),$$

see Diagrams (2.1) and (2.2) at the end of §2.4. Since the equivalence η is bounded, $\eta^I \mathcal{Y} \in \mathsf{D}^b(\mathcal{E}^I)$. Therefore, if $\eta^{\mathbf{1}}$ is of type FP_∞ then the right vertical map is an isomorphism by Lemma 4.1.5. Then the square implies that the left vertical map is an isomorphism for any choice of $\mathcal{Y} \in \mathsf{D}^b(\mathcal{A}^I)$, and so X is of type FP_∞. The converse implication follows similarly using the fact that $\eta^{\mathbf{1}}$ and η^I are equivalences between the bounded derived categories. □

Lemma 4.2.3. *In Setting 4.2.1, we have* $\mathsf{D}^b(\mathcal{H}_C)^c = \mathsf{D}^b(\mathsf{fp}(\mathcal{H}_C))$. *In particular, the derived equivalence* real_C *restricts to an equivalence* $\mathsf{D}^b(\mathsf{fp}(\mathcal{H}_C)) \to \mathsf{D}^b(\mathsf{mod}(R))$.

Proof. Recall from Proposition 2.6.15 that the intermediate cotilting t-structure $\mathbb{T}$ induces a bounded equivalence $\mathfrak{real}_C : \mathbb{D}_{\mathcal{H}_C} \to \mathbb{D}_{\mathsf{Mod}(R)}$ of derivators. In particular, we have a triangle equivalence $\mathsf{D}^b(\mathcal{H}_C) \xrightarrow{\sim} \mathsf{D}^b(\mathsf{Mod}(R))$ obtained by restriction of $\mathfrak{real}_C^{\mathbf{1}} : \mathsf{D}(\mathcal{H}_C) \xrightarrow{\sim} \mathsf{D}(\mathsf{Mod}(R))$. Then $\mathfrak{real}_C^{\mathbf{1}}$ further restricts to an equivalence $\mathsf{D}^b(\mathcal{H}_C)^c \xrightarrow{\sim} \mathsf{D}^b(\mathsf{Mod}(R))^c$ between the categories of compact objects. Since R is noetherian, $\mathsf{D}^b(\mathsf{Mod}(R))^c = \mathsf{D}^b(\mathsf{mod}(R))$ (see Lemma 2.6.2(2)) and $\mathsf{D}^b(\mathsf{mod}(R))$ is also precisely the subcategory of $\mathsf{D}^b(\mathsf{Mod}(R))$ consisting of objects of type FP_∞, see Lemma 4.1.7. Then Lemma 4.2.2 applies and shows that $\mathsf{D}^b(\mathcal{H}_C)^c$ coincides with the subcategory of all objects of type FP_∞ of $\mathsf{D}^b(\mathcal{H}_C)$. But by Lemma 4.1.7 this is precisely the subcategory $\mathsf{D}^b(\mathsf{fp}(\mathcal{H}_C))$.

Finally, note that we proved the second statement along the way, since $\mathsf{real}_C = \mathfrak{real}_C^{\mathbf{1}}$. □

Corollary 4.2.4. *In Setting 4.2.1, the functor* real_C *induces a triangle equivalence* $\mathsf{D}^{\mathsf{sing}}(\mathcal{H}_C) \to \mathsf{D}^{\mathsf{sing}}(\mathsf{Mod}(R))$ *between singularity categories.*

Proof. By Lemma 4.2.3, the derived equivalnce $\mathsf{real}_C : \mathsf{D}(\mathcal{H}_C) \to \mathsf{D}(\mathsf{Mod}(R))$ restricts to an equivalence $\mathsf{D}^b(\mathsf{fp}(\mathcal{H}_C)) \to \mathsf{D}^b(\mathsf{mod}(R))$. Since real_C also restricts to an equivalence $\mathsf{D}(\mathcal{H}_C)^c \to \mathsf{D}(\mathsf{Mod}(R))^c$ between the subcategories of compact objects, the result follows formally by passing to Verdier quotients. □

Now we are ready to formulate the main result of this section, that is, to characterize the case in which the heart $\mathcal{H}_C$ is a locally coherent category. Our results can be seen as a refinement of the characterization of the locally coherent property of hearts due to Marks and Zvonareva [48, Corollary 4.2], but only in the special case of intermediate compactly generated t-structures in $\mathsf{D}(\mathsf{Mod}(R))$.

Theorem 4.2.5. *Let R be a commutative noetherian ring and $\mathbb{T}$ be an intermediate compactly generated t-structure in* $\mathsf{D}(\mathsf{Mod}(R))$ *with heart $\mathcal{H}$. Then the following are equivalent:*

(1) *we are in Setting 4.2.1, that is,* $\mathsf{real}^b_{\mathbb{T}}$ *is an equivalence and* $\mathcal{H}$ *is locally coherent;*

(2) *the t-structure* $\mathbb{T}$ *restricts to* $\mathsf{D}^b(\mathsf{mod}(R))$*.*

Proof. Recall that $\mathsf{real}^b_{\mathbb{T}}$ being an equivalence amounts to $\mathbb{T}$ being induced by a cotilting object C by Proposition 2.6.13, and therefore the description in (1) indeed corresponds to Setting 4.2.1.

The implication (2) $\Rightarrow$ (1) is proven in [48, Corollary 4.2] and Corollary 2.9.13.

It remains to show (1) $\Rightarrow$ (2). Assume now that $\mathcal{H}$ is locally coherent. To establish that $\mathbb{T}$ is restrictable, we just need to recall from Lemma 4.2.3 that the derived equivalence $\mathsf{real}_C : \mathsf{D}(\mathcal{H}) \xrightarrow{\sim} \mathsf{D}(\mathsf{Mod}(R))$ restricts to an equivalence $\mathsf{D}^b(\mathsf{fp}(\mathcal{H})) \xrightarrow{\sim} \mathsf{D}^b(\mathsf{mod}(R))$. The t-structure $\mathbb{T}$ corresponds under real to the standard t-structure on $\mathsf{D}(\mathcal{H})$, which clearly restricts to a t-structure in $\mathsf{D}^b(\mathsf{fp}(\mathcal{H}))$. $\square$

As another application of Lemma 4.2.3, we can show that the two versions of coderived categories of $\mathcal{H}_C$ due to Becker and Positselski coincide. Recall that an object $M \in \mathcal{H}_C$ is **fp-injective** if $\mathsf{Ext}^1_{\mathcal{H}_C}(F, M) = 0$ for all $F \in \mathsf{fp}(\mathcal{H}_C)$. Furthermore, $M \in \mathcal{H}_C$ is of **finite fp-injective dimension** if M is isomorphic in $\mathsf{D}(()\mathcal{H}_C)$ to a bounded complex of fp-injective objects concentrated in non-negative degrees.

Lemma 4.2.6. *In Setting 4.2.1, any object in* $\mathcal{H}_C$ *of finite fp-injective dimension is of finite injective dimension.*

Proof. Let $M \in \mathcal{H}_C$, put $X = \mathsf{real}_C(M) \in \mathsf{D}^b(R)$, and let us denote the converse equivalence to real_C as $\mathsf{real}_C^{-1} : \mathsf{D}(R) \xrightarrow{\sim} \mathsf{D}(\mathcal{H}_C)$. Since the t-structure $\mathbb{T}$ is intermediate, and using Lemma 4.2.3, there is an integer $n \in \mathbb{Z}$ such that $\mathsf{real}_C^{-1}(\mathsf{mod}(R)) \subseteq \mathsf{D}(\mathsf{fp}(\mathcal{H}_C)) \cap \mathbb{D}^{\geq n}$. If M is of finite fp-injective dimension then $\mathsf{Hom}_{\mathsf{D}(\mathcal{H}_C)}(\mathsf{D}(\mathsf{fp}(\mathcal{H}_C))^{\geq n}, M[i]) = 0$ for all $i \gg 0$. Applying real_C we therefore obtain $\mathsf{Hom}_{\mathsf{D}(R)}(\mathsf{mod}(R), X[i]) = 0$ for all $i \gg 0$, which amounts to $X \in \mathsf{D}^b(R)$ being of finite injective dimension in $\mathsf{D}(R)$, since R is noetherian. Equivalently, we have $\mathsf{Hom}_{\mathsf{D}(R)}(\mathbb{D}^{\geq 0}, X[i]) = 0$ for $i \gg 0$. But using the intermediacy of $\mathbb{T}$ again, we know that $\mathsf{real}_C \mathcal{H}_C[j] \subseteq \mathbb{D}^{\geq 0}$ for $j \ll 0$, and so it follows by applying real_C^{-1} that $\mathsf{Hom}_{\mathsf{D}(\mathcal{H}_C)}(\mathcal{H}_C, M[i+j]) = 0$ for $i + j \gg 0$, which in turn implies that M is of finite injective dimension in $\mathcal{H}_C$. $\square$

Corollary 4.2.7. *In Setting 4.2.1, the coderived category* $\mathsf{K}(\mathsf{Inj}(\mathcal{H}_C))$ *(in Becker's sense) is equivalent to the coderived category in Positselski's sense.*

Proof. This follows directly from [64, §3.7, Theorem] in view of Lemma 4.2.6
. $\square$

We finish this section with an example of a locally coherent Grothendieck category which does not satisfy [78, Hypothesis 7.1] even though its derived category is compactly generated. In fact, we obtain it as a heart $\mathcal{H}_{\mathbb{T}}$ in $\mathsf{D}(R)$ induced by a compactly generated, intermediate and restrictable t-structure.

Example 4.2.8. Let $(R, \mathfrak{m})$ be a commutative and noetherian local ring, of dimension 1, which is not Cohen-Macaulay; for example, take R to be the localisation of $k[x,y]/(x^2, xy)$, for an algebraically closed field k, at the maximal ideal $\mathfrak{m} = (x, y)$. In particular, we have $1 = \mathsf{dim}(R) > \mathsf{depth}(R) = 0$; and then, by the Auslander-Buchsbaum formula, every non-zero finitely generated module is projective or has infinite projective dimension (in other words, the small finitistic global dimension of R is 0). Moreover, since R is not Cohen-Macaulay, $\mathfrak{m}$ is an associated prime of R; and the other primes are minimal, so they are associated as well, i.e. $\mathsf{Ass}(R) = \mathsf{Spec}(R)$. Therefore, every cyclic module $R/\mathfrak{p}R$ for a prime $\mathfrak{p}$ is a subobject of a projective module (R itself). It follows from Matlis' Theorem and [5, Theorem 7.1] that the finitistic injective global dimension of R and, by duality, also the finitistic weak global dimension of R are 0. We recall that this means that any R-module of finite flat dimension is automatically flat.

Let $V = \{\mathfrak{m}\}$, consider the associated hereditary torsion pair $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ in $\mathsf{Mod}(R)$, and let $\mathcal{H}$ be the HRS-tilt of $\mathsf{Mod}(R)$ with respect to $\mathbf{t}$; namely, $\mathcal{H} = \mathcal{F}[1] * \mathcal{T}$. Notice that since $\mathsf{D}(\mathcal{H}) \cong \mathsf{D}(R)$ (by Corollary 2.9.13, together with Remark 3.3.1) the former is compactly generated. Also, $\mathcal{H}$ is the heart of the Happel-Reiten-Smalø t-structure corresponding to the torsion pair $(\mathcal{T}, \mathcal{F})$, and this is an intermediate t-structure which is compactly generated and restrictable (Remark 3.2.9 and Proposition 2.7.12).

Nonetheless, we shall show that there are no non-zero finitely presented objects of finite projective dimension in $\mathcal{H}$, and therefore [78, Hypothesis 7.1] is not satisfied.

Since R has dimension 1, every subset of $\mathsf{Spec}(R)$ is coherent, and therefore V corresponds to a flat ring epimorphism $R \to S$; given our choice of V, S will be a regular ring of dimension 0. In $\mathcal{H}$, there is a hereditary torsion pair $\mathbf{s} = (\mathcal{T}, \mathsf{Mod}(S)[1])$ (see Lemma 3.2.10(1)).

Let X be a finitely presented object of $\mathcal{H}$, i.e. $X \in \mathcal{H} \cap \mathsf{D}^b(\mathsf{mod}(R))$, and assume it has finite projective dimension. Consider its approximation sequence with respect to $\mathbf{s}$ in $\mathcal{H}$, i.e. the triangle

$$T \to X \to L[1] \to T[1]$$

with $T \in \mathcal{T}$ and L an S-module. In particular, since $\mathrm{gl.dim}(S) = 0$, L is a projective S module; since S is a flat R-module, it has finite projective dimension over R [68, Seconde partie, Corollaire 3.2.7], and then so does L. From the triangle above, we deduce that T has finite projective dimension as well. Then, its flat dimension in $\mathsf{Mod}(R)$ is also finite, and since the finitistic weak global dimension of R is 0, T is a flat R-module. Now, we claim that this implies $T = 0$. Indeed, consider a presentation

$$0 \to K \to F \to T \to 0$$

with $F = R^{(\alpha)}$ a free R-module. Since T is flat, this sequence is pure exact, and therefore the torsion radical t of $\mathbf{t}$ gives a short exact sequence

$$0 \to tK \to tF \to T \to 0.$$

By construction, tR is supported on $V = \{\mathfrak{m}\}$, and since it is finitely generated, this means that $V(\mathsf{ann}(tR)) = \{\mathfrak{m}\}$. Hence $\mathfrak{m} = \sqrt{\mathsf{ann}(tR)}$, and since R is noetherian it follows that there exists n such that $\mathfrak{m}^n tR = 0$. Therefore, $tR, tF = (tR)^{(\alpha)}$ and also T are $R/\mathfrak{m}^n$-modules. T is also flat over $R/\mathfrak{m}^n$, and since this is an artinian local ring, T is free, i.e. $T \cong (R/\mathfrak{m}^n)^{(\beta)}$. But then, if $T \neq 0$, its direct summand $R/\mathfrak{m}^n$ should be a finitely presented flat R-module, and therefore projective, which is a contradiction because it would force R to be artinian (and therefore 0-dimensional).

It follows that our finitely presented object X of $\mathcal{H}$ is isomorphic to $L[1]$. But then, L is a finitely presented R-module of finite projective dimension, hence it is projective, hence free. Now, since L is also an S-module, if $L \neq 0$ this would imply that $R \in \mathsf{Mod}(S)$. In particular, R would be torsion-free with respect to $\mathbf{t}$, which is not the case since $\mathfrak{m} \in \mathsf{Ass}(R)$. We conclude that $X \cong L[1] = 0$.

4.3 The equivalence of recollements

Let R be a commutative noetherian ring. By Theorem 4.2.5, Setting 4.2.1 characterises the case in which we have an intermediate compactly generated restrictable t-structure $\mathbb{T}$.

Consider now the following seemingly new situation.

Setting 4.3.1. *Let $\mathcal{H}$ be a locally coherent Grothendieck category, and assume that there exists an object T in $\mathsf{D}(\mathcal{H})$ such that:*

(T1) T is compact tilting.

(T2) T has finite projective dimension, i.e. $\mathsf{Hom}_{\mathsf{D}(\mathcal{H})}(T, \mathcal{H}[i]) = 0$ for $i \gg 0$.

(T3) $\mathsf{End}_{\mathsf{D}(\mathcal{H})}(T)$ is isomorphic to a commutative noetherian ring R.

Condition (T1) ensures that $\mathsf{D}(\mathcal{H})$ is compactly generated. Since T is compact, it belongs to $\mathsf{D}^b(\mathcal{A})$, see Proposition 4.1.9. Under this assumption, similarly to before, condition (T2) is equivalent to requiring the tilting t-structure

$\mathbb{T}_T$ of $\mathsf{D}(\mathcal{H})$ associated to T to be intermediate. Conditions (T1) and (T3) imply that its heart $\mathcal{H}_T$ is isomorphic to $\mathsf{Mod}(R)$, and we have a triangle equivalence

$$\mathsf{real}_T\colon \mathsf{D}(R) = \mathsf{D}(\mathcal{H}_T) \to \mathsf{D}(\mathcal{H}).$$

Using the equivalences real_C and real_T, we see that these two settings are the two sides of the same picture: starting from Setting 4.2.1, the choices $\mathcal{H} := \mathcal{H}_C$ and $T := \mathsf{real}_C^{-1}(R)$ fit Setting 4.3.1; conversely, taking $C := \mathsf{real}_T^{-1}(W)$ for an injective cogenerator W of $\mathcal{H}$, one obtains the t-structure $\mathbb{T}_C$ as the pullback along real_T of the standard t-structure of $\mathsf{D}(\mathcal{H})$. In the following we will work with Setting 4.3.1, with Setting 4.2.1 serving as motivation.

Since $\mathcal{H}$ is locally coherent and $\mathsf{D}(\mathcal{H})$ is compactly generated (by T), we have the recollement of Theorem 4.1.14; and there is also Krause's recollement for R:

$$\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(\mathcal{H})) \leftrightarrows \mathsf{K}(\mathsf{Inj}(\mathcal{H})) \leftrightarrows \mathsf{D}(\mathcal{H}) \qquad \mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(R)) \leftrightarrows \mathsf{K}(\mathsf{Inj}(R)) \leftrightarrows \mathsf{D}(R)$$

Our goal is to construct an equivalence between these two recollements (see Definition 2.2.17). In order to do that, we replace the derived equivalence real_C by another one which we are able to lift to the coderived level. We start by fixing a convenient resolution of T.

Lemma 4.3.2. *Up to shift, T admits a resolution $T := (F_{-n} \to F_{-n+1} \to \cdots \to F_0)$ with finitely presented objects $F_i \in \mathsf{fp}(\mathcal{H})$.*

Proof. Since T is compact, by Proposition 4.1.9 it belongs to $\mathsf{D}^b(\mathsf{fp}(\mathcal{H}))$, so it is quasi-isomorphic to a complex over the abelian category $\mathsf{fp}(\mathcal{H})$. By taking soft truncations this complex can be made strictly bounded. $\square$

Now we consider the functor $\mathcal{H}\mathrm{om}(T, -)\colon \mathsf{C}(\mathcal{H}) \to \mathsf{C}(\mathbb{Z})$, defined as the totalisation of the bicomplex $\mathcal{H}\mathrm{om}^{\bullet,\bullet}(T, -)$. Notice that this bicomplex is always bounded along the direction of T (because we chose a strictly bounded resolution of T).

Since R is commutative, $\mathsf{D}(\mathcal{H}) \cong \mathsf{D}(R)$ is an R-linear category, and then so is $\mathcal{H}$. The bicomplex $\mathcal{H}\mathrm{om}^{\bullet,\bullet}(T, -)$ and its totalisation $\mathcal{H}\mathrm{om}(T, -)$ have therefore terms in $\mathsf{Mod}(R)$ and R-linear differentials; this gives us a functor

$$\Psi := \mathcal{H}\mathrm{om}(T, -)\colon \mathsf{C}(\mathcal{H}) \to \mathsf{C}(R). \tag{4.1}$$

Moreover, if $X \in \mathsf{C}(\mathcal{H})$ is contractible, then the rows of $\mathcal{H}\mathrm{om}^{\bullet,\bullet}(T, X)$ are also contractible, since $\mathsf{Hom}_{\mathcal{H}}(F_i, -)$ is an additive functor for all $-n \leq i \leq 0$. It follows that $\mathcal{H}\mathrm{om}(T, X) \in \mathsf{C}(R)$ is also contractible, which gives us a functor

$$\Psi := \mathcal{H}\mathrm{om}(T, -)\colon \mathsf{K}(\mathcal{H}) \to \mathsf{K}(R). \tag{4.2}$$

In particular, by restriction of the domain, Ψ induces functors on the subcategories $\mathsf{K}(\mathsf{Inj}(\mathcal{H})) \subseteq \mathsf{K}(\mathsf{fpInj}{-}\mathcal{H}) \subseteq \mathsf{K}(\mathcal{H})$, which we will continue to denote by Ψ.

We record immediately that Ψ induces a derived equivalence $\mathsf{D}(\mathcal{H}) \cong \mathsf{D}(R)$.

Lemma 4.3.3. *The functor* $\mathbb{R}\mathsf{Hom}_{\mathcal{H}}(T, -) := Q\Psi Q_r \colon \mathsf{D}(\mathcal{H}) \to \mathsf{D}(R)$ *is an equivalence. Moreover, it restricts to an equivalence* $\mathsf{D}^b(\mathcal{H}) \to \mathsf{D}^b(R)$*, and also to an equivalence* $\mathsf{D}^b(\mathsf{fp}(\mathcal{H})) \to \mathsf{D}^b(\mathsf{mod}(R))$.

Proof. By (T1) and (T3) we have $\mathbb{R}\mathsf{Hom}_{\mathcal{H}}(T,T) \cong \mathsf{End}_{\mathsf{D}(\mathcal{H})}(T) \cong R$, so the functor $\mathbb{R}\mathsf{Hom}_{\mathcal{H}}(T,-)$ sends a compact generator of $\mathsf{D}(\mathcal{H})$ to a compact generator of $\mathsf{D}(R)$. Moreover, since $\mathbb{R}\mathsf{Hom}_{\mathcal{H}}(T,-)$ is R-linear on Hom-sets, it must induce the isomorphism $\mathsf{End}_{\mathsf{D}(\mathcal{H})}(T) \cong \mathsf{End}_R(R) = R$ of endomorphism rings. Since T is a compact generator of $\mathsf{D}(\mathcal{H})$ and R is a compact generator of $\mathsf{D}(R)$, a standard arguments shows that $\mathbb{R}\mathsf{Hom}_{\mathcal{H}}(T,-)$ induces an equivalence $\mathsf{D}(\mathcal{H})^c \xrightarrow{\sim} \mathsf{D}(R)^c$ between the categories of compact objects (see e.g. [51, Proposition 6]). Lastly, $\mathbb{R}\mathsf{Hom}_{\mathcal{H}}(T,-)$ preserves coproducts, since T is compact. Then, the derived equivalence is established by double *dévissage* (Lemma 2.1.23).

For the claim about the bounded equivalence, let $X \in \mathsf{D}(\mathcal{H})$. Then its image $\mathbb{R}\mathsf{Hom}_{\mathcal{H}}(T,X)$ belongs to $\mathsf{D}^b(R)$ if and only if $\mathsf{Hom}_{\mathsf{D}(\mathcal{H})}(T, X[i]) = 0$ for all but finitely many $i \in \mathbb{Z}$; and this means that X has finitely many cohomologies with respect to $\mathbb{T}_T$. Since $\mathbb{T}_T$ is intermediate, this is equivalent to X belonging to $\mathsf{D}^b(\mathcal{H})$. Therefore, $\mathbb{R}\mathsf{Hom}_{\mathcal{H}}(T,-)$ restricts to an equivalence $\mathsf{D}^b(\mathcal{H}) \xrightarrow{\sim} \mathsf{D}^b(R)$, and therefore also to an equivalence $\mathsf{D}^b(\mathcal{H})^c \xrightarrow{\sim} \mathsf{D}^b(R)^c$ between compact objects of the bounded derived categories. By Lemma 4.2.3, this last equivalence is the same as the desired equivalence $\mathsf{D}^b(\mathsf{fp}(\mathcal{H})) \xrightarrow{\sim} \mathsf{D}^b(\mathsf{mod}(R))$. $\square$

Lemma 4.3.4. $\Psi \colon \mathsf{C}(\mathcal{H}) \to \mathsf{C}(R)$ *preserves direct limits (and in particular coproducts). Therefore, also the induced functor* $\Psi \colon \mathsf{K}(\mathcal{H}) \to \mathsf{K}(R)$ *and its restriction* $\Psi \colon \mathsf{K}(\mathsf{fpInj}{-}\mathcal{H}) \to \mathsf{K}(R)$ *preserve coproducts.*

Proof. Coproducts in $\mathsf{K}(\mathcal{H})$ are computed termwise, as in $\mathsf{C}(\mathcal{H})$. Moreover, since $\mathsf{fpInj}{-}\mathcal{H}$ is closed under coproducts in $\mathcal{H}$, coproducts in $\mathsf{K}(\mathsf{fpInj}{-}\mathcal{H})$ are computed as in $\mathsf{K}(\mathcal{H})$. It is then enough to prove the claim for $\Psi \colon \mathsf{C}(\mathcal{H}) \to \mathsf{C}(R)$.

Now, let $X_\alpha := (\cdots \to X_\alpha^i \to X_\alpha^{i+1} \to \cdots) \in \mathsf{C}(\mathcal{H})$ be a direct system of objects, and consider their direct limit $\varinjlim X_\alpha = (\cdots \to \varinjlim X_\alpha^i \to \varinjlim X_\alpha^{i+1} \to$

$\cdots$). Ψ sends it to the totalisation of the bicomplex

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \mathsf{Hom}_{\mathcal{H}}(F_0, \varinjlim X^i_\alpha) & \longrightarrow & \mathsf{Hom}_{\mathcal{H}}(F_0, \varinjlim X^{i+1}_\alpha) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & \\
\cdots \longrightarrow & \mathsf{Hom}_{\mathcal{H}}(F_{-1}, \varinjlim X^i_\alpha) & \longrightarrow & \mathsf{Hom}_{\mathcal{H}}(F_{-1}, \varinjlim X^{i+1}_\alpha) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & \\
& \vdots & & \vdots & \\
& \downarrow & & \downarrow & \\
\cdots \longrightarrow & \mathsf{Hom}_{\mathcal{H}}(F_{-n}, \varinjlim X^i_\alpha) & \longrightarrow & \mathsf{Hom}_{\mathcal{H}}(F_{-n}, \varinjlim X^{i+1}_\alpha) & \longrightarrow \cdots
\end{array}$$

Since the F_i are finitely presented in $\mathcal{H}$, the functors $\mathsf{Hom}_{\mathcal{H}}(F_i, -)$ commute naturally with the direct limits, so $\mathcal{H}\mathrm{om}^{\bullet,\bullet}(T, \varinjlim X_\alpha)$ is isomorphic in $\mathsf{C}(\mathsf{C}(\mathcal{H}))$ to the direct limit of the bicomplexes $\mathcal{H}\mathrm{om}^{\bullet,\bullet}(T, X_\alpha)$. Totalisation also commutes with direct limits, and so Ψ preserves them. $\square$

In order to obtain a functor between the coderived categories, we want Ψ to preserve coacyclicity. Recall that a locally finitely presentable Grothendieck category $\mathcal{A}$ admits a natural notion of a pure exact sequence, and that a complex in $\mathsf{C}(\mathcal{A})$ is **pure-acyclic** if it is acyclic and in addition, each exact sequence $0 \to Z^i(X) \to X^i \to Z^{i+1}(X) \to 0$ induced by the cocycles is pure exact in $\mathcal{A}$.

We start by recalling the following fact.

Proposition 4.3.5 ([78])**.** *Over a locally coherent Grothendieck category, pure-acyclic complexes are coacyclic.*

Proof. This follows mainly from [78, §6.2]; we collect the argument for the convenience of the reader. Let $\mathcal{A}$ be a locally coherent Grothendieck category, and X a complex in $\mathsf{C}(\mathcal{A})$. Then X corresponds to a coacyclic object of $\mathsf{K}(\mathcal{A})$ if and only if it is $\mathsf{Ext}^1_{\mathsf{C}}$-orthogonal to $\mathsf{C}(\mathsf{Inj}(\mathcal{A}))$, i.e. if it belongs to the left class of the functorially complete cotorsion pair generated by disks of $\mathsf{fp}(\mathcal{A})$. Now, this left class is closed under retracts and transfinite extensions, and pure-acyclic complexes are (retracts of) transfinite extensions of disks of $\mathsf{fp}(\mathcal{A})$ in $\mathsf{C}(\mathcal{A})$ by [78, Lemma 5.6]. $\square$

Lemma 4.3.6. *The restriction* $\Psi\colon \mathsf{K}(\mathsf{fpInj}{-}\mathcal{H}) \to \mathsf{K}(R)$ *preserves coacyclic complexes.*

Proof. As a partial converse of Lemma 4.3.5, a complex $X \in \mathsf{C}(\mathcal{H})$ of fp-injectives is coacyclic in $\mathsf{K}(\mathcal{H})$ if and only if it is pure-acyclic [78, Proposition 6.11]. By [78, Lemma 4.14], a complex X in $\mathsf{C}(\mathcal{H})$ is pure-acyclic if and only if it is a direct limit of bounded contractible complexes. Since $\Psi\colon \mathsf{C}(\mathcal{H}) \to \mathsf{C}(R)$ preserves both direct limits (Lemma 4.3.4) and contractibility, $\Psi(X)$ will also be pure-acyclic by the same characterisation. Then we conclude by Lemma 4.3.5 that $\Psi(X)$ is also coacyclic. $\square$

In view of the equivalences

$$\mathsf{K}(\mathsf{Inj}(\mathcal{H})) \xrightarrow[\cong]{\subseteq} \mathsf{K}(\mathsf{fpInj}\text{–}\mathcal{H})/\{\text{pure acyclics}\} \xrightarrow[\cong]{} \mathsf{D}^{\mathsf{co}}(\mathcal{H})$$

by Lemma 4.3.6 we deduce that Ψ induces a functor

$$\mathbb{R}^{\mathsf{co}}\Psi \colon \mathsf{D}^{\mathsf{co}}(\mathcal{H}) \to \mathsf{D}^{\mathsf{co}}(R). \tag{4.3}$$

On an object $X \in \mathsf{D}^{\mathsf{co}}(\mathcal{H})$, $\mathbb{R}^{\mathsf{co}}\Psi$ is computed by first resolving X by a complex of fp-injectives (or even injectives), then applying Ψ and considering the resulting complex as an object of $\mathsf{D}^{\mathsf{co}}(R)$. When identifying $\mathsf{D}^{\mathsf{co}}(\mathcal{H}) \cong \mathsf{K}(\mathsf{Inj}(\mathcal{H}))$ and $\mathsf{D}^{\mathsf{co}}(R) \cong \mathsf{K}(\mathsf{Inj}(R))$, $\mathbb{R}^{\mathsf{co}}\Psi$ is then the composition

$$\mathbb{R}^{\mathsf{co}}\Psi\colon \quad \mathsf{K}(\mathsf{Inj}(\mathcal{H})) \xrightarrow{\subseteq} \mathsf{K}(\mathcal{H}) \xrightarrow{\Psi} \mathsf{K}(R) \xrightarrow{I_\lambda} \mathsf{K}(\mathsf{Inj}(R)) \tag{4.4}$$

where $I_\lambda \colon K(R) \to K(\mathsf{Inj}(R))$ is the reflection of Proposition 2.3.17.

Proposition 4.3.7. $\mathbb{R}^{\mathsf{co}}\Psi \colon \mathsf{D}^{\mathsf{co}}(\mathcal{H}) \to \mathsf{D}^{\mathsf{co}}(R)$ *is an equivalence.*

Proof. We want to argue by double *dévissage.*

First, $\mathbb{R}^{\mathsf{co}}\Psi \colon \mathsf{D}^{\mathsf{co}}(\mathcal{H}) \to \mathsf{D}^{\mathsf{co}}(R)$ preserves coproducts, since Ψ does, by Lemma 4.3.4. Now we show that it induces an equivalence between the subcategories of compact objects. In view of the identification $\mathsf{D}^{\mathsf{co}}(\mathcal{H}) \cong \mathsf{K}(\mathsf{Inj}(\mathcal{H}))$, a compact object of $\mathsf{D}^{\mathsf{co}}(\mathcal{H})$ is identified with the homotopically injective resolution X of an object in $\mathsf{D}^b(\mathsf{fp}(\mathcal{H}))$; in particular, this is a bounded below complex. When we apply Ψ and then I_λ, as in (4.4), we obtain again a bounded below complex, first in $\mathsf{K}(R)$ and then in $\mathsf{K}(\mathsf{Inj}(R))$. This last object $Y := I_\lambda\Psi(X)$, in particular, is a homotopically injective complex. Since we have $X \cong Q_rQX$ and $Y \cong Q_rQY$ in $\mathsf{K}(\mathsf{Inj}(\mathcal{H}))$ and $\mathsf{K}(\mathsf{Inj}(R))$, respectively, we can write

$$\mathbb{R}^{\mathsf{co}}\Psi(X) = Y \cong Q_rQY = Q_rQI_\lambda\Psi X = Q_rQ\Psi X \cong Q_rQ\Psi Q_rQX =: (*)$$

Now, by definition, $\mathbb{R}\mathsf{Hom}_{\mathcal{H}}(T,-) := Q\Psi Q_r$, so we can continue

$$(*) = Q_r\,\mathbb{R}\mathsf{Hom}_{\mathcal{H}}(T, QX)$$

It is therefore sufficient to show that $Q_r\,\mathbb{R}\mathsf{Hom}_{\mathcal{H}}(T, Q-)$ is an equivalence between $\mathsf{K}(\mathsf{Inj}(\mathcal{H}))^c$ and $\mathsf{K}(\mathsf{Inj}(R))^c$. Now, $Q\colon \mathsf{K}(\mathsf{Inj}(\mathcal{H}))^c \to \mathsf{D}^b(\mathsf{fp}(\mathcal{H}))$ and $Q_r\colon \mathsf{D}^b(\mathsf{mod}(R)) \to K(\mathsf{Inj}(R))^c$ are equivalences, and $\mathbb{R}\mathsf{Hom}_{\mathcal{H}}(T,-)\colon \mathsf{D}^b(\mathsf{fp}(\mathcal{H})) \to \mathsf{D}^b(\mathsf{mod}(R))$ is an equivalence by Lemma 4.3.3. ☐

Now that we have the equivalence between the coderived categories, we show that it preserves the recollements. First we need a technical lemma.

Lemma 4.3.8. $I_\lambda T \cong Q_lQT$ *in* $\mathsf{D}^{\mathsf{co}}(\mathcal{H}) \cong \mathsf{K}(\mathsf{Inj}(\mathcal{H}))$.

Proof. Let E be the homotopically injective resolution of T: we have a triangle in $\mathsf{K}(\mathcal{H})$

$$A \to T \to E \to A[1]$$

with A acyclic. Since T is bounded below, E and then A are also bounded below. A is therefore also coacyclic. This means that $E \cong I_\lambda T$ in $\mathsf{K}(\mathsf{Inj}(\mathcal{H}))$. Now, since E is homotopically injective we have $E \cong Q_r Q E \cong Q_r Q T$; but QT is compact in $\mathsf{D}(\mathcal{H})$, and therefore $Q_r Q T \cong Q_l Q T$ by Lemma 4.1.13. We conclude as wanted that $I_\lambda T \cong Q_l Q T$ in $\mathsf{K}(\mathsf{Inj}(\mathcal{H}))$. $\square$

Lemma 4.3.9. $\mathbb{R}^{\mathsf{co}}\Psi \colon \mathsf{D}^{\mathsf{co}}(\mathcal{H}) \to \mathsf{D}^{\mathsf{co}}(R)$ *preserves acyclics.*

Proof. Identifying $\mathsf{D}^{\mathsf{co}}(\mathcal{H}) \cong \mathsf{K}(\mathsf{Inj}(\mathcal{H}))$ and in view of (4.4), let $X \in \mathsf{K}(\mathsf{Inj}(\mathcal{H}))$ be acyclic. For every $n \in \mathbb{Z}$ we have

$$\begin{aligned} H^n I_\lambda \Psi X &\cong H^n \Psi X = H^n\, \mathcal{H}\mathrm{om}(T, X) \cong \\ &\cong \mathsf{Hom}_{\mathsf{K}(\mathcal{H})}(T, X[n]) \cong \mathsf{Hom}_{\mathsf{K}(\mathsf{Inj}(\mathcal{H}))}(I_\lambda T, X[n]) \overset{(1)}{\cong} \\ &\cong \mathsf{Hom}_{\mathsf{K}(\mathsf{Inj}(\mathcal{H}))}(Q_l Q T, X[n]) \cong \mathsf{Hom}_{\mathsf{D}(\mathcal{H})}(QT, QX[n]) \overset{(2)}{=} 0 \end{aligned}$$

where (1) is by Lemma 4.3.8 and (2) because $QX = 0$. $\square$

Theorem 4.3.10. $\mathbb{R}^{\mathsf{co}}\Psi \colon \mathsf{D}^{\mathsf{co}}(\mathcal{H}) \to \mathsf{D}^{\mathsf{co}}(R)$ *induces an equivalence of recollements, that is, there is a diagram*

$$\begin{array}{ccccc} \mathsf{S}(\mathcal{H}) & \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} & \mathsf{D}^{\mathsf{co}}(\mathcal{H}) & \substack{\leftarrow \\ \rightarrow \\ \leftarrow} & \mathsf{D}(()\mathcal{H}) \\ {\scriptstyle \mathbb{S}\Psi}\downarrow\cong & & {\scriptstyle \mathbb{R}^{\mathsf{co}}\Psi}\downarrow\cong & & {\scriptstyle \mathbb{R}\Psi}\downarrow\cong \\ \mathsf{S}(\mathsf{Mod}(R)) & \substack{\leftarrow \\ \rightarrow \\ \leftarrow} & \mathsf{D}^{\mathsf{co}}(R) & \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} & \mathsf{D}(R) \end{array}$$

in which the rows are the recollements from Remark 4.1.17 of $\mathcal{H}$ *and* $\mathsf{Mod}(R)$ *and such that all the six obvious squares commute.*

Proof. Identify $\mathsf{D}^{\mathsf{co}}(\mathcal{H}) \cong \mathsf{K}(\mathsf{Inj}(\mathcal{H}))$ and $\mathsf{D}^{\mathsf{co}}(R) \cong \mathsf{K}(\mathsf{Inj}(R))$. By Proposition 4.3.7, $\mathbb{R}^{\mathsf{co}}\Psi$ is an equivalence. By Lemma 4.3.9, it preserves acyclicity. In view of basic results on recollement equivalences (see §2.2), it is enough to show that the following square is commutative up to equivalence

$$\begin{array}{ccc} \mathsf{K}(\mathsf{Inj}(\mathcal{H})) & \xrightarrow{\quad Q \quad} & \mathsf{D}(\mathcal{H}) \\ {\scriptstyle \mathbb{R}^{\mathsf{co}}\Psi}\downarrow\cong & & {\scriptstyle \mathbb{R}\Psi}\downarrow\cong \\ \mathsf{K}(\mathsf{Inj}(R)) & \xrightarrow{\quad Q \quad} & \mathsf{D}(\mathsf{Mod}(R)) \end{array}$$

where $\mathbb{R}\Psi = \mathbb{R}\mathsf{Hom}_{\mathcal{H}}(T, -)$. Since $\mathbb{R}^{\mathsf{co}}\Psi$ preserves acyclics, the composition $Q\mathbb{R}^{\mathsf{co}}\Psi$ kills objects from $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(\mathcal{H}))$, and thus the approximation triangle with respect to the stable t-structure $(\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(\mathcal{H})), Q_r(\mathsf{D}(\mathcal{H}))$ in $\mathsf{K}(\mathsf{Inj}(\mathcal{H}))$ yields a natural equivalence $Q\mathbb{R}^{\mathsf{co}}\Psi \cong Q\mathbb{R}^{\mathsf{co}}\Psi Q_r Q$. Then we can compute similarly as in Proposition 4.3.7:

$$Q\mathbb{R}^{\mathsf{co}}\Psi Q_r Q = QI_\lambda \Psi Q_r Q \cong Q\Psi Q_r Q = \mathbb{R}\Psi Q.$$

The rest follows by denoting the induced triangle equivalence $\mathsf{S}(\mathcal{H}) \to \mathsf{S}(\mathsf{Mod}(R))$ by $\mathbb{S}\Psi$. $\square$

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