

ATOMIC DECOMPOSITION OF FINITE SIGNED MEASURES ON COMPACTS OF \mathbb{R}^n

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ABSTRACT. Recently there has been interest in pairs of Banach spaces (E_0, E) in an $o - O$ relation and with $E_0^{**} = E$. It is known that this can be done for Lipschitz spaces on suitable metric spaces. In this paper we consider the case of a compact subset K of \mathbb{R}^n with the euclidean metric, which does not give an $o - O$ structure, but we use part of the theory concerning these pairs to find an atomic decomposition of the predual of $Lip(K)$. In particular, since the space $\mathfrak{M}(K)$ of finite signed measures on K , when endowed with the Kantorovich-Rubinstein norm, has as dual space $Lip(K)$, we can give an atomic decomposition for this space.

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1. INTRODUCTION

L. Hanin has dedicated some papers [8, 9] to the description of spaces in duality with Lipschitz spaces, namely spaces of finite signed Borel measures on compact metric spaces K . In what follows we will consider K a compact domain in \mathbb{R}^n equipped with the euclidean norm, which we denote here by $|\cdot|$. More precisely, when we endow the space $\mathfrak{M}(K)$ of such measures on K with the so-called Kantorovich-Rubinstein norm and consider its completion, we obtain a space that is isometric to the predual of the space of Lipschitz functions of K .

The Kantorovich-Rubinstein norm (see section 2) was introduced in the context of optimal transport theory. As a matter of fact, the distance, induced by the norm, between two measures μ and ν with same total mass, i.e. $\mu(K) = \nu(K)$, is simply the cost of the optimal transport from one to the other (see next section for definitions).

Other than identifying $\mathfrak{M}(K)^*$ as $Lip(K)$, passing to duals, one can also investigate embedding properties of $\mathfrak{M}(K)^c$, or of $\mathfrak{M}_0(K)^c$, in its bidual $Lip(K)^*$, where $\mathfrak{M}_0(K)$ is the subspace of $\mathfrak{M}(K)$ containing only measures with null total mass.

An interesting consequence of this approach is that it inspires the introduction of the dual problem in optimal transport theory. As a matter of fact, by thinking

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of elements in $\mathfrak{M}(K)$ as functionals on $\text{Lip}(K)$ we obtain that the Kantorovich-Rubinstein norm on $\mathfrak{M}(K)$ is equivalent to the operator norm

$$\|\mu\|_{KR} = \sup \left\{ \int_K f d\mu, \quad f \in \text{Lip}_1(K) \right\}$$

where $\text{Lip}_1(K)$ is the set of Lipschitz functions with Lipschitz constant $L \leq 1$. The fundamental problem of optimal transport theory, i.e. finding, if it exists, a minimizer to the minimization problem occurring in the definition of $\|\mu - \nu\|_{KR}$ with μ and ν measures with same total mass, is then equivalently formulated as a maximization problem. It was proven in [3] that elements in $\mathfrak{M}_0(K)^c$ are precisely those for which the dual problem admits maximizers. Moreover, in the same paper, the space $\mathfrak{M}_0(K)^c$ is also characterized as the space of the distributional divergences of $L^1(K; \mathbb{R}^n)$ functions.

In this paper we will give an atomic decomposition of the spaces $\mathfrak{M}(K)$ and $\mathfrak{M}_0(K)$ by restriction of the decomposition of their completions, seen as preduals of Lipschitz spaces. We recall that the description of atomic decompositions of Hölder spaces on compact spaces was given in [10] and [1], following different approaches; in particular, in [10] the atomic decomposition is closer to other "classical" examples [4, 6], while in [1] a more abstract atomic decomposition is obtained. We decided to follow this second approach, based on techniques from [6], which are inspired by the o - O construction in [14].

In particular, we will see in the third section that elements of the embedded copy of $\mathfrak{M}_0(K)^c$ in $\text{Lip}(K)^*$ can be thought of as all the infinite sums of the type

$$\mu = \sum_{j=1}^{+\infty} \frac{\delta_{x_j} - \delta_{y_j}}{|x_j - y_j|} \alpha_j \quad \text{with } \alpha_j \text{ satisfying } \sum_{j=1}^{+\infty} |\alpha_j| < +\infty$$

and where $\{x_j\}_{j \in \mathbb{N}}$ and $\{y_j\}_{j \in \mathbb{N}}$ are two disjoint countable dense subsets of K . These infinite sums are intended as bounded linear functionals on Lipschitz functions f in the following way

$$\langle \mu, f \rangle = \sum_{j=1}^{+\infty} \frac{f(x_j) - f(y_j)}{|x_j - y_j|} \alpha_j$$

where the right hand side is finite because $\frac{|f(x_j) - f(y_j)|}{|x_j - y_j|}$ is bounded by the Lipschitz constant of f and α_j is a sequence in ℓ^1 .

On the other hand for some choices of α_j , μ is not a finite signed Borel measure on K , even if the sequence of partial sums is a Cauchy sequence in the Kantorovich norm, showing that $\mathfrak{M}_0(K)$ is not complete. A fourth section of this paper is dedicated to obtain a similar result for $\mathfrak{M}(K)^c$. In such a case, since we are not identifying functions that differ from each other by a constant, the atomic decomposition will be not only expressed as an infinite linear combination of dipoles, but a correction term in form of an atom (i.e. δ_{x_j}) has to be added to each summand.

2. LIPSCHITZ SPACES, SPACES OF BOREL MEASURES AND THEIR COMPLETIONS

In this section we will introduce the notation concerning the spaces and the norms we will work with. Let us fix a bounded open set $\Omega \subset \mathbb{R}^n$ and let us denote $K = \bar{\Omega}$.

2.1. Lipschitz spaces and fractional Sobolev spaces.

Definition 2.1. We define the **Lipschitz spaces**

$$\text{Lip}(K) = \left\{ f : K \rightarrow \mathbb{R} : \sup_{\substack{(x,y) \in K^2 \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|} < +\infty \right\}$$

and

$$\text{Lip}_0(K) = \text{Lip}(K) / \mathbb{R},$$

i.e. the Lipschitz space $\text{Lip}(K)$ modulo constant functions.

In $\text{Lip}_0(K)$, to simplify the notation, we will identify any function $f : K \rightarrow \mathbb{R}$ with its equivalence class. If we endow $\text{Lip}_0(K)$ with the norm

$$\|f\|_{\text{Lip}_0(K)} = \sup_{\substack{(x,y) \in K^2 \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|},$$

then this normed space is a Banach space, while on $\text{Lip}(K)$ the functional $\|\cdot\|_{\text{Lip}_0(K)}$ would only work as a seminorm.

Furthermore, $\text{Lip}(K)$ would be a Banach space if endowed with the norm

$$\|f\|_{\text{Lip}(K)} = \max\{\|f\|_{\text{Lip}_0(K)}, \|f\|_{L^\infty(K)}\}.$$

In the following we will need to embed the spaces $\text{Lip}(K)$ and $\text{Lip}_0(K)$ in suitable reflexive Banach spaces. For our purposes, the natural candidates are fractional Sobolev spaces. An almost complete survey on such spaces is given in [5].

Definition 2.2. Let us denote by $W^{s,p}(\Omega)$ for $s \in (0, 1)$ and $p > 1$ the fractional Sobolev space consisting of the functions $f \in L^p(\Omega)$ such that

$$\|f\|_{W^{s,p}(\Omega)}^p := \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{ps+n}} dx dy < +\infty.$$

If we endow $W^{s,p}(\Omega)$ with the norm

$$\|f\|_{W^{s,p}(\Omega)} = \|f\|_{\dot{W}^{s,p}(\Omega)} + \|f\|_{L^p(\Omega)}$$

it is a reflexive separable Banach space. The homogeneous fractional Sobolev space $\dot{W}^{s,p}(\Omega)$ is defined as $\dot{W}^{s,p}(\Omega) = W^{s,p}(\Omega) / \mathbb{R}$ and if we endow this space with the norm $\|f\|_{\dot{W}^{s,p}(\Omega)}$ it is a reflexive separable Banach space.

Remark 2.1. Let us recall that if $ps > n$, by a fractional Morrey-type embedding theorem, we have that $W^{s,p}(\Omega) \hookrightarrow C(K)$. In this case we will always consider the continuous version of a function in $W^{s,p}(\Omega)$.

Another characterization of $\dot{W}^{s,p}(\Omega)$ for $sp > n$ is given as the space of functions $f \in W^{s,p}(\Omega)$ such that $f(z) = 0$, for an a priori fixed point $z \in K$ (here we are implicitly using the embedding $W^{s,p}(\Omega) \hookrightarrow C(K)$). In particular we have (by using the same idea adopted for $\text{Lip}(K)$) that the norm

$$\|f\|_{W^{s,p}(\Omega),z} = \|f\|_{\dot{W}^{s,p}(\Omega)} + |f(z)|$$

is equivalent to $\|\cdot\|_{W^{s,p}(\Omega)}$. By identifying $C(K) / \mathbb{R}$ in the same way we have $\dot{W}^{s,p}(\Omega) \hookrightarrow C(K) / \mathbb{R}$.

2.2. Spaces of Borel measures and Lipschitz-free spaces.

Definition 2.3. We denote the space of finite signed Borel measures on K by $\mathfrak{M}(K)$, the subspace of finite positive measures on K by $\mathfrak{M}_+(K)$, and the subspace of $\mathfrak{M}(K)$ consisting only of measures μ such that $\mu(K) = 0$ by $\mathfrak{M}_0(K)$.

Via the Hahn-Jordan decomposition, a signed measure μ can be seen as the difference of two positive Borel measures μ^+ and μ^- , i.e. $\mu = \mu^+ - \mu^-$; the total variation of μ is defined as the sum of the two, i.e. $|\mu| = \mu^+ + \mu^-$.

The total variation $\mu \in \mathfrak{M}(K) \mapsto |\mu|(K) \in \mathbb{R}$ is a norm on $\mathfrak{M}(K)$ that gives to the space the structure of Banach space. However, it does not take into account the metric structure of the domain K (for instance $|\delta_x - \delta_y|(K) = 2$, for any $(x, y) \in K^2$). On the other hand, even in the more general setting of a compact metric space K , Kantorovich and Rubinstein introduced a norm $\|\cdot\|_{KR}$ on $\mathfrak{M}(K)$ inducing a distance that is a natural extension of the distance on K .

As a matter of fact, K naturally embeds in $\mathfrak{M}(K)$ by associating to each point x in K the Dirac measure δ_x concentrated in x . We will introduce a norm $\|\cdot\|_{KR}$ that will have the interesting property that $\|\delta_x - \delta_y\|_{KR} = |x - y|$, in some sense extending the metric on K to $\mathfrak{M}(K)$.

To define the Kantorovich-Rubinstein norm on $\mathfrak{M}(K)$, we first start by doing so on the space $\mathfrak{M}_0(K) \subset \mathfrak{M}(K)$ of **balanced measures** μ , i.e. such that $\mu(K) = 0$ and hence $\mu^+(K) = \mu^-(K)$.

Definition 2.4 ([11, 12, 13]). Consider any $\mu \in \mathfrak{M}_0(K)$ and define a family $\Psi_\mu \subset \mathfrak{M}_+(K \times K)$ of positive Borel measures on the Cartesian square $K \times K$ of K in the following way: $\Psi \in \Psi_\mu$ if and only if, for any Borel set $E \subset K$, $\Psi(K, E) - \Psi(E, K) = \mu(E)$ (called **balance condition**)

The Kantorovich-Rubinstein norm of μ is defined as

$$\|\mu\|_{KR_0} := \inf \left\{ \int_{K \times K} |x - y| d\Psi(x, y) : \Psi \in \Psi_\mu \right\}.$$

Definition 2.5. For $\mu \in \mathfrak{M}(K)$ we define the “extended” Kantorovich-Rubinstein norm of μ as

$$\|\mu\|_{KR} := \inf \{ \|\nu\|_{KR_0} + |\mu - \nu|(K) : \nu \in \mathfrak{M}_0(K) \}.$$

An important thing to notice is that $(\mathfrak{M}_0(K), \|\cdot\|_{KR_0})$ and $(\mathfrak{M}(K), \|\cdot\|_{KR})$ are not Banach spaces.

Remark 2.2. Given $(x, y) \in K$ we have $\|\delta_x - \delta_y\|_{KR_0} = |x - y|$ while $\|\delta_x\|_{KR} = 1$, showing that the Kantorovich-Rubinstein norm satisfies the desired property of concordance with the metric on K .

The completion of the space of finite Borel measure on K with respect to the Kantorovich-Rubinstein norm is denoted by $\mathfrak{M}(K)^c$, while we denote by $\mathfrak{M}_0(K)^c$ the completion of $\mathfrak{M}_0(K)$ with respect to the norm $\|\cdot\|_{KR_0}$.

It has been shown (see for instance [7]) that $\mathfrak{M}(K)^*$ is isometric to $\text{Lip}(K)$ while $\mathfrak{M}_0(K)^*$ is isometric to $\text{Lip}_0(K)$. Moreover, it is interesting to recall another characterization of $\mathfrak{M}_0(K)^c$. Indeed, in [3] it is shown that if K is a compact subset of \mathbb{R}^n then for any distribution $\mu \in \mathfrak{M}_0(K)^c$ there exists a function $f \in L^1(K; \mathbb{R}^n)$ such that

$$\mu = \text{div } f.$$

Moreover (see [3]), for any distribution $\mu \in \mathfrak{M}_0(K)^c$ there exists a function $f \in B_{\text{Lip}_0(K)}$ such that

$$\|\mu\|_{KR} = \int_K f d\mu,$$

so that the norm is attained.

3. ATOMIC DECOMPOSITION OF $\mathfrak{M}_0(K)^c$

Our aim is to give an atomic decomposition of elements μ of $\mathfrak{M}_0(K)^c$, and so in particular of measures that are null on K , as an infinite sum of simpler elements that we will call *atoms*.

Definition 3.1. We will call δ -**atom** any measure $\mu \in \mathfrak{M}(K)$ whose support is finite. Moreover, we call **dipoles** the measure $\mu \in \mathfrak{M}_0(K)$ of the form $\mu = \alpha(\delta_x - \delta_y)$ for some $\alpha \in \mathbb{R}$ and $(x, y) \in K^2$.

To obtain a decomposition of elements of $\mathfrak{M}_0(K)^c$ - which will induce a decomposition of elements of $\mathfrak{M}_0(K)$ - we generalize the approach of [2], which relies on the o - O structure of $(C^{0,\alpha}, C^{0,\alpha})$, by using results contained in [6], which allow us to remove the dependence on the "little o" space, because for Lip and Lip_0 it is trivial. We start by writing Lip_0 in a suitable way.

Lemma 3.1. *There exists a sequence of operators $(L_j)_{j \in \mathbb{N}} : X = (\dot{W}^{s,p}(\Omega)) \rightarrow Y = \mathbb{R}$ such that*

$$\text{Lip}_0(K) = \{f \in \dot{W}^{s,p}(\Omega) : \sup_{j \in \mathbb{N}} |L_j f| < +\infty\}$$

and

$$\|f\|_{\text{Lip}_0(K)} = \sup_{j \in \mathbb{N}} |L_j f|.$$

Proof. First of all, let us fix $s \in (0, 1)$ and $ps > n$, so that $\dot{W}^{s,p}(\Omega) \hookrightarrow C(K)/\mathbb{R}$. Let us consider $D_1 \subset K$ a numerable set such that $K = \overline{D_1}$ and $K_1 = K \setminus D_1$. Now let us consider $D_2 \subset K_1$ a numerable set such that $K_1 = \overline{D_2}$. Finally, let us define $D = D_1 \times D_2$. Observe that $D_1 \cap D_2 = \emptyset$ so, for any $(x, y) \in D$, $x \neq y$. Moreover, D is numerable, hence we can enumerate $D = \{(x_j, y_j)\}_{j \in \mathbb{N}}$. Finally $\overline{D} = K \times K$. Let us define

$$L_j : f \in \dot{W}^{s,p}(\Omega) \rightarrow \frac{f(x_j) - f(y_j)}{|x_j - y_j|} \in \mathbb{R}.$$

L_j is obviously linear. Moreover, since $\dot{W}^{s,p}(\Omega) \hookrightarrow C(K)/\mathbb{R}$ we have

$$\frac{f(x_j) - f(y_j)}{|x_j - y_j|} \leq \frac{2}{|x_j - y_j|} \|f\|_{L^\infty(K)} \leq C_j \|f\|_{\dot{W}^{s,p}(\Omega)},$$

hence $L_j \in (\dot{W}^{s,p}(\Omega))^*$ for any $j \in \mathbb{N}$.

Finally, let us observe that by density of D in $K \times K$ and continuity of $f \in \dot{W}^{s,p}(\Omega)$ it holds

$$\|f\|_{\text{Lip}_0(K)} = \sup_{j \in \mathbb{N}} |L_j f|$$

concluding the proof. \square

Now that we have this rewriting of the definition of $\text{Lip}_0(K)$ we can use the techniques exploited in [6] to obtain the desired atomic decomposition.

Theorem 3.2. *There exists a constant $C \in (0, 1)$ such that for any distribution $\mu \in \mathfrak{M}_0(K)^c$ there exists a sequence $(\alpha_j)_{j \in \mathbb{N}} \in \ell^1(\mathbb{R})$ such that*

$$\mu = \sum_{j=1}^{+\infty} \frac{\delta_{x_j} - \delta_{y_j}}{|x_j - y_j|} \alpha_j$$

and

$$C \sum_{j=1}^{+\infty} |\alpha_j| \leq \|\mu\|_{KR_0} \leq \sum_{j=1}^{+\infty} |\alpha_j|,$$

where the sequences $(x_j)_{j \in \mathbb{N}}$ and $(y_j)_{j \in \mathbb{N}}$ are defined in Lemma 3.1. Moreover, the sequence of δ -atoms $(\mu_j)_{j \in \mathbb{N}} \subset \mathfrak{M}_0(K)$ defined as

$$\mu_j = \frac{\delta_{x_j} - \delta_{y_j}}{|x_j - y_j|}$$

spans $\mathfrak{M}_0(K)^c$, with $\|\mu_j\|_{KR_0} = 1$ for any $j \in \mathbb{N}$. In particular the δ -atoms μ_j are dipoles, hence admit support of cardinality exactly 2.

Proof. By [6, Theorem 3] we know that there exists $C \in (0, 1)$ such that for any $\mu \in \mathfrak{M}_0(K)^c$

$$\mu = \sum_{j=1}^{+\infty} L_j^* \alpha_j,$$

where L_j^* is the adjoint operator of L_j , and

$$C \sum_{j=1}^{+\infty} \|L_j^* \alpha_j\|_{KR_0} \leq \|\mu\|_{KR_0} \leq \sum_{j=1}^{+\infty} \|L_j^* \alpha_j\|_{KR_0}.$$

Now let us recall that

$$\langle f, L_j^* \alpha_j \rangle = \langle L_j f, \alpha_j \rangle = \frac{f(x_j) - f(y_j)}{|x_j - y_j|} \alpha_j$$

but, since $L_j^* : \mathbb{R} \rightarrow (\dot{W}^{s,p}(\Omega))^*$ we also have

$$\int_K f dL_j^* \alpha_j = \langle f, L_j^* \alpha_j \rangle$$

hence

$$L_j^* \alpha_j = \frac{\delta_{x_j} - \delta_{y_j}}{|x_j - y_j|} \alpha_j$$

concluding the proof. \square

The problem of characterizing the space $\mathfrak{M}_0(K)^c$ has been faced in several ways. In particular it is interesting to remember that in [3], such a space is shown to be isometric to the space $L^1(K; \mathbb{R}^d)/V_0$ where $V_0 = \{\sigma \in L^1(K; \mathbb{R}^d) : \operatorname{div} \sigma = 0\}$, given by $\sigma \in L^1(K; \mathbb{R}^d)/V_0 \mapsto -\operatorname{div} \sigma \in \mathfrak{M}_0(K)^c$. The motivation of such research towards a characterization of $\mathfrak{M}_0(K)^c$ is linked (as the authors state in the introduction of their paper) to the convergence of infinite sums of dipoles to distributions that are not balanced measures. Here we have shown that such infinite sums of dipoles are indeed the main component of $\mathfrak{M}_0(K)^c$ and the dipoles represent an atomic part of such a space. Let us finally recall that the infinite sums of dipoles are shown to have a characterization as $-\operatorname{div} \sigma$ for some $\sigma \in L^1(K; \mathbb{R}^d)$ by using the theory of tangential measures (see [3, Example 3.7]).

4. ATOMIC DECOMPOSITION OF $\mathfrak{M}(K)^c$

This section is devoted to a similar atomic decomposition in the larger space $\mathfrak{M}(K)^c$, with the help of the space $\text{Lip}(K)$. This time we cannot use the same operators as in Lemma 3.1 since they define a seminorm on $\text{Lip}(K)$. The following rewriting of $\text{Lip}(K)$ relies on the fact that we can consider on \mathbb{R}^2 the ℓ^∞ norm.

Lemma 4.1. *There exists a sequence of operators $(L_j)_{j \in \mathbb{N}} \in \mathcal{L}(W^{s,p}(\Omega), \mathbb{R}^2)$, where we equip \mathbb{R}^2 with the norm $\|(x, y)\|_{\ell^\infty} = \max\{|x|, |y|\}$, such that*

$$\text{Lip}(K) = \{f \in W^{s,p}(\Omega) : \sup_{j \in \mathbb{N}} \|L_j f\|_{\ell^\infty} < +\infty\}$$

and

$$\|f\|_{\text{Lip}(K)} = \sup_{j \in \mathbb{N}} \|L_j f\|_{\ell^\infty}.$$

Proof. First of all, let us fix $s \in (0, 1)$ and $ps > n$, so that $W^{s,p}(\Omega) \hookrightarrow C(K)$. Let us consider $D_1 \subset K$ a numerable set such that $K = \overline{D_1}$ and $K_1 = K \setminus D_1$. Now let us consider $D_2 \subset K_1$ a numerable set such that $K_1 = \overline{D_2}$. Finally, let us define $D = D_1 \times D_2$. Observe that $D_1 \cap D_2 = \emptyset$ so, for any $(x, y) \in D$, $x \neq y$. Moreover, D is numerable, hence we can enumerate $D = \{(x_j, y_j)\}_{j \in \mathbb{N}}$. Finally $\overline{D} = K \times K$. Let us define

$$L_j : f \in \dot{W}^{s,p}(\Omega) \rightarrow \left(\frac{f(x_j) - f(y_j)}{|x_j - y_j|}, f(x_j) \right) \in \mathbb{R}^2.$$

L_j is obviously linear. Moreover, since $W^{s,p}(\Omega) \hookrightarrow C(K)$ we have

$$\max \left\{ \frac{|f(x_j) - f(y_j)|}{|x_j - y_j|}, |f(x_j)| \right\} \leq \max \left\{ \frac{2}{|x_j - y_j|}, 1 \right\} \|f\|_{L^\infty(K)} \leq C_j \|f\|_{W^{s,p}(\Omega)},$$

hence $L_j \in \mathcal{L}(W^{s,p}(\Omega), \mathbb{R}^2)$ for any $j \in \mathbb{N}$.

Finally, let us observe that by density of D in $K \times K$, D_1 in K , and continuity of $f \in \dot{W}^{s,p}(\Omega)$ it holds

$$\|f\|_{\text{Lip}(K)} = \sup_{j \in \mathbb{N}} \|L_j f\|_{\ell^\infty}$$

concluding the proof. \square

As we did in the previous section, we can now use the techniques of [6] to obtain the atomic decomposition of $\mathfrak{M}(K)^c$.

Theorem 4.2. *There exists a constant $C \in (0, 1)$ such that for any distribution $\mu \in \mathfrak{M}(K)^c$ there exists a sequence $((\alpha_j^1, \alpha_j^2))_{j \in \mathbb{N}} \in \ell^1(\mathbb{R}^2)$ such that*

$$\mu = \sum_{j=1}^{+\infty} \left(\frac{\delta_{x_j} - \delta_{y_j}}{|x_j - y_j|} \alpha_j^1 + \delta_{x_j} \alpha_j^2 \right)$$

and

$$(4.1) \quad C \sum_{j=1}^{+\infty} \left| |\alpha_j^1| - |\alpha_j^2| \right| \leq \|\mu\|_{KR_0} \leq \sum_{j=1}^{+\infty} |\alpha_j^1| + |\alpha_j^2|,$$

where the sequences $(x_j)_{j \in \mathbb{N}}$ and $(y_j)_{j \in \mathbb{N}}$ are defined in Lemma 4.1. In particular, the sequence of δ -atoms $(\mu_j)_{j \in \mathbb{N}} \subset \mathfrak{M}(K)$ defined as

$$(4.2) \quad \mu_j = \begin{cases} \frac{\delta_{x_k} - \delta_{y_k}}{|x_k - y_k|} & j = 2k - 1 \\ \delta_{x_k} & j = 2k \end{cases}$$

spans $\mathfrak{M}(K)^c$, and $\|\mu_j\|_{KR} = 1$ for any $j \in \mathbb{N}$.

Proof. By [6, Theorem 3] we know that there exists $C \in (0, 1)$ such that for any $\mu \in \mathfrak{M}(K)^c$

$$\mu = \sum_{j=1}^{+\infty} L_j^* \alpha_j,$$

where L_j^* is the adjoint operator of L_j , $\alpha_j = (\alpha_j^1, \alpha_j^2) \in \mathbb{R}^2$, and

$$(4.3) \quad C \sum_{j=1}^{+\infty} \|L_j^* \alpha_j\|_{KR} \leq \|\mu\|_{KR_0} \leq \sum_{j=1}^{+\infty} \|L_j^* \alpha_j\|_{KR}.$$

Now let us recall that

$$\langle f, L_j^* \alpha_j \rangle = \langle L_j f, \alpha_j \rangle = \frac{f(x_j) - f(y_j)}{|x_j - y_j|} \alpha_j^1 + f(x_j) \alpha_j^2$$

but, since $L_j^* : \mathbb{R}^2 \rightarrow \mathcal{L}(\dot{W}^{s,p}(\Omega), \mathbb{R}^2)$ we also have

$$\int_K f dL_j^* \alpha_j = \langle f, L_j^* \alpha_j \rangle$$

hence

$$L_j^* \alpha_j = \frac{\delta_{x_j} - \delta_{y_j}}{|x_j - y_j|} \alpha_j^1 + \delta_{x_j} \alpha_j^2.$$

Now let us determine some upper and lower bounds for $\|L_j^* \alpha_j\|_{KR}$. To do this, let us recall that

$$\|\delta_x - \delta_y\|_{KR} = |x - y| \quad \forall x, y \in K, \quad \|\delta_x\|_{KR} = 1 \quad \forall x \in K.$$

Hence we have for the upper bound

$$(4.4) \quad \|L_j^* \alpha_j\|_{KR} \leq \frac{\|\delta_{x_j} - \delta_{y_j}\|_{KR}}{|x_j - y_j|} |\alpha_j^1| + \|\delta_{x_j}\|_{KR} |\alpha_j^2| = |\alpha_j^1| + |\alpha_j^2|$$

while for the lower bound

$$(4.5) \quad \|L_j^* \alpha_j\|_{KR} \geq \left| \frac{\|\delta_{x_j} - \delta_{y_j}\|_{KR}}{|x_j - y_j|} |\alpha_j^1| - \|\delta_{x_j}\|_{KR} |\alpha_j^2| \right| = \left| |\alpha_j^1| - |\alpha_j^2| \right|.$$

Using Equations (4.4) and (4.5) in Equation (4.3) we finally achieve Equation (4.1). \square

Remark 4.3. Let us observe that the sequence of δ -atoms $(\mu_j)_{j \in \mathbb{N}}$ is composed by delta measures and dipoles. In particular if j is even, then μ_j is a delta measure and then the cardinality of its support is exactly 1. On the other hand, if j is odd, then μ_j is a dipole and then the cardinality of its support is exactly 2. Thus we have that for any distribution $\mu \in \mathfrak{M}(K)^c$ there exists a sequence $(\alpha_j)_{j \in \mathbb{N}} \in \ell^1(\mathbb{R})$ such that $\mu = \sum_{j=1}^{+\infty} \alpha_j \mu_j$ where μ_j are δ -atoms with support of cardinality at most 2.

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