SIAM J. CONTROL OPTIM.
Vol. 60, No. 1, pp. 280–309

NONDEGENERATE ABNORMALITY, CONTROLLABILITY, AND GAP PHENOMENA IN OPTIMAL CONTROL WITH STATE CONSTRAINTS*

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Abstract. In optimal control theory, *infimum gap* means that there is a gap between the infimum values of a given minimum problem and an extended problem, obtained by enlarging the set of original solutions and controls. The gap phenomenon is somewhat "dual" to the problem of the *controllability* of the original control system to an extended solution. In this paper we present sufficient conditions for the absence of an infimum gap and for controllability for a wide class of optimal control problems subject to endpoint and state constraints. These conditions are based on a nondegenerate version of the nonsmooth constrained maximum principle, expressed in terms of subdifferentials. In particular, under some new constraint qualification conditions, we prove that (i) if an extended minimizer is a *nondegenerate normal* extremal, then no gap shows up; (ii) given an extended solution verifying the constraints, either it is a *nondegenerate abnormal* extremal or the original system is controllable to it. An application to the impulsive extension of a free end-time, nonconvex optimization problem with control-polynomial dynamics illustrates the results.

Key words. optimal control problems, maximum principle, state constraints, gap phenomena, controllability, nondegeneracy

AMS subject classifications. 49K15, 34K45, 49N25

DOI. 10.1137/20M1382465

1. Introduction.

In the calculus of variations and in the theory of optimal control it is a rather common procedure to enlarge the space of solutions for those problems that do not admit a solution in a, say, ordinary space. Of course, a fundamental requirement for a good extension is that there is no gap between the infimum of the original problem and that of the extended problem. However, even if the set of ordinary solutions is C^0-dense in the set of extended trajectories, in the presence of constraints an infimum gap does in general occur, whenever all ordinary solutions in a C^0-neighborhood of a feasible extended trajectory (a local extended minimizer, for instance) violate the constraints. In this case, we will refer to the extended trajectory as *isolated*. By defining the original control system *controllable* to an extended trajectory whenever the trajectory is not isolated, we see that gap avoidance and controllability are strictly related issues. Since Warga's early works [32, 33], it has emerged that the existence of an infimum gap, or better, following our terminology, the fact that an extended trajectory is isolated, is related to the validity of a maximum principle in *abnormal* form (as customary, abnormality means that the scalar multiplier associated with the cost is zero). In particular, results of this kind have been obtained for the classical extension by relaxation (convex [25, 26] or in measure [33, 15]) and, more recently, for the impulsive extension of control-affine systems with unbounded controls, with or without state constraints (see [13], [23], respectively).

*Received by the editors November 24, 2020; accepted for publication (in revised form) October 4, 2021; published electronically January 25, 2022.
https://doi.org/10.1137/20M1382465
Funding: This research is partially supported by INdAM-GNAMPA Project 2020, "Extended control problems: gap, higher order conditions and Lyapunov functions," and by Padua University grant SID 2018, "Controllability, stabilizability and infimum gaps for control systems," project BIRD 187147.

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Let us also mention [24], where a general extension is considered, but in the absence of state constraints and for smooth data. These results are obtained by different techniques, which essentially reflect two different approaches to the maximum principle: approach (a), based on the construction of approximating cones to reachable sets and on set separation arguments [32, 33, 15, 24], and approach (b), which makes use of perturbation and penalization techniques and of the Ekeland's variational principle [25, 26, 23, 13]. For nonsmooth optimal control problems, methods (a), (b) are not easily comparable, as they require different assumptions on dynamics and target but, above all, lead to different abnormality conditions, which involve the "derivative containers" introduced in [33] or the "quasi-differential quotients" defined in [24] in case (a), while in case (b) one uses a by now standard form of the nonsmooth constrained maximum principle due to Clarke, expressed in terms of subdifferentials (see [7]).

The main purpose of this paper is to extend approach (b), applied so far only to particular cases, to identify under which general assumptions for the extension of an optimal control problem the following statement is valid: *an isolated extended trajectory is an abnormal extremal.* Furthermore, we give sufficient conditions for which we prove the stronger result: *an isolated extended trajectory is an abnormal extremal of a nondegenerate version of the maximum principle.*

Precisely, we consider the optimization problem

$$(P)\begin{cases} \qquad\text{minimize } \Psi(y(S)) \\ \text{over } (\omega,\alpha,y)\in\mathscr{V}(S)\times\mathscr{A}(S)\times W^{1,1}([0,S];\mathbb{R}^n), \text{ verifying} \\ \dot y(s)=\mathscr{F}(s,y(s),\omega(s),\alpha(s)) \text{ a.e.}, \quad y(0)=\check x_0, \\ h(s,y(s))\le 0 \quad \forall s\in[0,S], \qquad y(S)\in\mathscr{T}, \end{cases}$$

where $\mathscr{V}(S):=L^1([0,S];V)$, $\mathscr{A}(S):=L^1([0,S];A)$, and the *extended* optimization problem, say, (P_e), which is obtained by (P) replacing in the minimization the control set $\mathscr{V}(S)$ with the larger set $\mathscr{W}(S):=L^1([0,S];W)$, where $W=\overline{V}$. The data comprise the functions $\Psi:\mathbb{R}^n\to\mathbb{R}$, $\mathscr{F}:\mathbb{R}\times\mathbb{R}^n\times W\times A\to\mathbb{R}^n$, $h:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, the bounded set $V\subset\mathbb{R}^m$, the compact set $A\subset\mathbb{R}^q$, and the closed set $\mathscr{T}\subset\mathbb{R}^n$. We refer to any triple (ω,α,y), where $(\omega,\alpha)\in\mathscr{W}(S)\times\mathscr{A}(S)$ and y solves the Cauchy problem

$$\dot y(s)=\mathscr{F}(s,y(s),\omega(s),\alpha(s)) \quad \text{a.e.}, \qquad y(0)=\check x_0, \tag{1.1}$$

as an *extended process* or simply a *process*. A process (ω,α,y) is *feasible* if $h(s,y(s))\le 0$ for all $s\in[0,S]$ and $y(S)\in\mathscr{T}$. The processes (ω,α,y) of (P), where $\omega\in\mathscr{V}(S)$, will be called *strict sense processes*.

As a further extension, we consider the convex relaxation of (P_e):

$$(P_r)\begin{cases} \qquad\text{minimize } \Psi(y(S)) \\ \text{over } (\underline{\omega},\underline{\alpha},\lambda,y)\in\mathscr{W}^{1+n}(S)\times\mathscr{A}^{1+n}(S)\times\Lambda_n(S)\times W^{1,1}([0,S];\mathbb{R}^n), \text{ verifying} \\ \dot y(s)=\displaystyle\sum_{k=0}^n\lambda^k(s)\mathscr{F}(s,y(s),\omega^k(s),\alpha^k(s)) \text{ a.e.}, \quad y(0)=\check x_0, \\ h(s,y(s))\le 0 \quad \forall s\in[0,S], \qquad y(S)\in\mathscr{T}, \end{cases}$$

where $\Lambda_n(S):=L^1([0,S];\Delta_n)$ and Δ_n is the n-dimensional simplex:

$$\Delta_n:=\left\{\lambda=(\lambda^0,\dots,\lambda^n):\quad \lambda^k\ge 0,\ k=0,\dots,n,\quad \sum_{k=0}^n\lambda^k=1\right\}.$$

Problem (P_r) is briefly referred to as the *relaxed* problem and a process $(\underline{\omega}, \underline{\alpha}, \lambda, y)$ for (P_r) is referred to as *relaxed* process. We will identify a process (ω, α, y) with the relaxed process $(\underline{\omega}, \underline{\alpha}, \lambda, y)$, where $\underline{\omega} := (\omega, \dots, \omega)$, $\underline{\alpha} := (\alpha, \dots, \alpha)$, and, for instance, $\lambda := (1+n)^{-1}(1, \dots, 1)$.

The controls α and ω play a different role, as only the control set $\mathscr{V}(S)$ to which ω belongs is extended. This distinction is reflected by the hypotheses on the dynamics $\mathscr{F}$. Referring to section 2 for details, we observe that while continuity of $\mathscr{F}$ in a will be enough, with respect to w a form of uniform continuity will be needed, both of $\mathscr{F}$ and of its Clarke generalized Jacobian $D_x\mathscr{F}$. Moreover, not only $\overline{V} = W$, but there must also exist an increasing sequence of closed subsets $V_i \subseteq V$ such that $\cup_i V_i = V$ (in Remark 2.2 below we will discuss possible extensions of this hypothesis). This formulation of the problem includes as special cases both the extension by convex relaxation considered in [26] (if $\mathscr{F}$ does not depend on w) and the impulsive, in general nonconvex, extension investigated in [23, 13].

In Theorem 2.1, we state our first main result, that any isolated feasible relaxed process is an abnormal extremal. The relevance of this result lies, in fact, in its consequences, which are (i) a "normality test" for no gap, namely, if for a (local) minimizer $\bar{z} := (\underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{y})$ of (P_r) or (P_e) the cost multiplier is $\neq 0$ for any set of multipliers in the maximum principle, at $\bar{z}$ there is no (local) infimum gap; (ii) the original control system (1.1) is controllable to any feasible relaxed process which is not an abnormal extremal (see Theorems 2.2 and 2.3 below).

When the state constraint is active at the initial point $(0, \check{x}_0)$, it is well-known that sets of degenerate multipliers such that *any* feasible relaxed process is abnormal do exist, making the above results (i), (ii) in fact useless. This "degeneracy question" seems to have been disregarded in the literature on the relationship between gap and normality, apart from [13], where, however, conditions are introduced which are never met in the case of a fixed initial point.

Based on the above considerations, in section 3 we provide a condition inspired by the nondegeneracy conditions proposed in [9, 10] ((H4) below), under which we refine the results of section 2. In particular, we establish that any feasible relaxed process which is isolated is an abnormal extremal for a *nondegenerate* maximum principle, and derive as corollaries a "nondegenerate normality test" for no gap and a "nondegenerate controllability condition" (see Theorems 3.1, 3.2, and 3.3 below).

The "normality" and the "nondegenerate normality" tests are useful especially because in certain situations they allow us to deduce the absence of gap from easily verifiable conditions, in the form of constraint and endpoint qualification conditions for normality, on which there is a wide literature (see, e.g., [11, 12, 16, 2] and references therein). As shown in [23, 22, 13], where some explicit normality sufficient conditions for the control-affine impulsive extension are provided, these conditions are in general weaker than those previously obtained to get the absence of gap directly, as in [1, 18].

In section 4 we extend the previous results to free end-time optimal control problems. We limit ourselves to considering the case of Lipschitz continuous time dependence, leaving the case of measurable time dependence to future investigations. Actually, Lipschitz continuous time dependence always arises in the impulsive extension of nonlinear problems with unbounded controls under the graph-completion approach, to which we apply our results in section 5. Impulsive optimal control problems have been extensively studied together with their applications, mostly in the case of control-affine systems, starting from [28, 31, 5, 17, 19]. We focus instead on the less investigated case of control-polynomial dynamics [27, 21]. Among applications for which the polynomial dependence is relevant let us mention Lagrangian mechanical systems, possibly with friction forces, where inputs are identified with the derivatives of some coordinates. In this case, the degree of the polynomial is 2, as a consequence of the fact that the kinetic energy is a quadratic form of the velocity (see, e.g., [4, 6]).

1.1. Notation and preliminaries. Given an interval $I \subseteq \mathbb{R}$ and a set $X \subseteq \mathbb{R}^k$, we write $W^{1,1}(I;X)$, $L^1(I;X)$, $L^\infty(I;X)$, for the space of absolutely continuous functions, Lebesgue integrable functions, and essentially bounded functions defined on I and with values in X, respectively. For all the classes of functions introduced so far, we will not specify domain and codomain when the meaning is clear and we will use $\|\cdot\|_{L^1(I)}$, $\|\cdot\|_{L^\infty(I)}$, or $\|\cdot\|_{L^1}$, $\|\cdot\|_{L^\infty}$ to denote the L^1 and the ess-sup norm, respectively. Furthermore, we denote by $\ell(X)$, $\mathrm{co}(X)$, $\mathrm{Int}(X)$, $\overline{X}$, and ∂X the Lebesgue measure, the convex hull, the interior, the closure, and the boundary of X, respectively. As customary, χ_X is the characteristic function of X, namely $\chi_X(x) = 1$ if $x \in X$ and $\chi_X(x) = 0$ if $x \in \mathbb{R}^k \setminus X$. For any subset $Y \subset X$, $\mathrm{proj}_Y X$ will denote the projection of X on Y. We denote the closed unit ball in $\mathbb{R}^k$ by $\mathbb{B}_k$, omitting the dimension when it is clear from the context. Given a closed set $\mathscr{O} \subseteq \mathbb{R}^k$ and a point $z \in \mathbb{R}^k$, we define the distance of z from $\mathscr{O}$ as $d_{\mathscr{O}}(z) := \min_{y\in\mathscr{O}} |z-y|$. We set $\mathbb{R}_{\geq 0} := [0,+\infty[$. For any $a, b \in \mathbb{R}$, we write $a \vee b := \max\{a,b\}$. We use $NBV^+([0,S];\mathbb{R})$ to denote the space of increasing, real valued functions μ on $[0,S]$ of bounded variation, vanishing at the point 0 and right continuous on $]0,S[$. Each $\mu \in NBV^+([0,S];\mathbb{R})$ defines a Borel measure on $[0,S]$, still denoted by μ, its total variation function is indicated by $\|\mu\|_{TV}$ or by $\mu([0,S])$, and its support is $\mathrm{spt}\{\mu\}$.

Some standard constructs from nonsmooth analysis are employed in this paper. For background material we refer the reader, for instance, to [7, 29]. A set $K \subseteq \mathbb{R}^k$ is a *cone* if $\alpha k \in K$ for any $\alpha > 0$, whenever $k \in K$. Take a closed set $D \subseteq \mathbb{R}^k$ and a point $\bar{x} \in D$, the *limiting normal cone* $N_D(\bar{x})$ of D at $\bar{x}$ is given by

$$N_D(\bar{x}) := \left\{ \eta \in \mathbb{R}^k : \exists x_i \overset{D}{\to} \bar{x},\, \eta_i \to \eta \ \text{ such that } \ \limsup_{x\to x_i} \frac{\eta_i \cdot (x - x_i)}{|x - x_i|} \leq 0 \ \ \forall i \right\},$$

in which the notation $x_i \xrightarrow{D} \bar{x}$ is used to indicate that all points in the converging sequence $(x_i)_i$ lay in D. Taking a lower semicontinuous function $G : \mathbb{R}^k \to \mathbb{R}$ and a point $\bar{x} \in \mathbb{R}^k$, the *limiting subdifferential* of G at $\bar{x}$ is

$$\partial G(\bar{x}) := \left\{ \xi\colon\ \exists \xi_i \to \xi,\, x_i \to \bar{x} \text{ s.t. } \limsup_{x\to x_i} \frac{\xi_i \cdot (x - x_i) - G(x) + G(x_i)}{|x - x_i|} \leq 0 \ \ \forall i \right\}.$$

If $G : \mathbb{R}^k \times \mathbb{R}^h \to \mathbb{R}$ is a lower semicontinuous function and $(\bar{x},\bar{y}) \in \mathbb{R}^k \times \mathbb{R}^h$, we write $\partial_x G(\bar{x},\bar{y})$, $\partial_y G(\bar{x},\bar{y})$ to denote the *partial limiting subdifferential of G at $(\bar{x},\bar{y})$ w.r.t. x, y*, respectively. When G is differentiable, ∇G is the usual gradient operator and $\nabla_x G$, $\nabla_y G$ denote the partial derivatives of G. Given a locally Lipschitz continuous function $G : \mathbb{R}^k \to \mathbb{R}$ and $\bar{x} \in \mathbb{R}^k$, the *reachable hybrid subdifferential* of G at $\bar{x}$ is

$$\partial^{*>} G(\bar{x}) := \{\xi\colon\ \exists (x_i)_i \subset \mathrm{diff}(G) \setminus \{\bar{x}\} \text{ s.t. } x_i \to \bar{x},\, G(x_i) > 0\ \forall i,\ \nabla G(x_i) \to \xi\}$$

while the *reachable gradient* of G at $\bar{x}$ is

$$\partial^* G(\bar{x}) := \{\xi\colon\ \exists (x_i)_i \subset \mathrm{diff}(G) \setminus \{\bar{x}\} \text{ s.t. } x_i \to \bar{x} \text{ and } \nabla G(x_i) \to \xi\},$$

where $\mathrm{diff}(G)$ denotes the set of differentiability points of G. We define the *hybrid subdifferential* as $\partial^> G(\bar{x}) := \mathrm{co}\, \partial^{*>} G(\bar{x})$. The set $\partial^* G(\bar{x})$ is nonempty, closed, and in general nonconvex, and its convex hull coincides with the Clarke subdifferential

$\partial^c G(\bar{x})$, that is, $\partial^c G(\bar{x}) = \operatorname{co} \partial^* G(\bar{x})$. Finally, when G is locally Lipschitz continuous, $\partial^c G(\bar{x}) = \operatorname{co} \partial G(\bar{x})$. Given a locally Lipschitz continuous function $G : \mathbb{R}^k \to \mathbb{R}^l$ and $\bar{x} \in \mathbb{R}^k$, we write $DG(\bar{x})$ to denote the *Clarke generalized Jacobian*, defined as

$$DG(\bar{x}) := \operatorname{co} \{\xi\colon \exists (x_i)_i \subset \operatorname{diff}(G) \setminus \{\bar{x}\} \text{ s.t. } x_i \to \bar{x} \text{ and } \nabla G(x_i) \to \xi\},$$

where now ∇G denotes the classical Jacobian matrix of G. If $G : \mathbb{R}^k \times \mathbb{R}^h \to \mathbb{R}^l$ and $(\bar{x}, \bar{y}) \in \mathbb{R}^k \times \mathbb{R}^h$, $D_x G(\bar{x}, \bar{y})$, $D_y G(\bar{x}, \bar{y})$ denote the *Clarke generalized Jacobian of G at* $(\bar{x}, \bar{y})$ *w.r.t.* x, y, respectively.

2. Infimum gap, isolated processes, and abnormality. In the following, when the final time $S > 0$ is clear from the context, we simply write $\mathscr{V}$, $\mathscr{W}$, $\mathscr{A}$, Λ_n, instead of $\mathscr{V}(S)$, $\mathscr{W}(S)$, $\mathscr{A}(S)$, $\Lambda_n(S)$, respectively.

2.1. Basic assumptions. We shall consider the following hypotheses, in which $(\bar{\underline{\omega}}, \bar{\underline{\alpha}}, \bar{\lambda}, \bar{y})$ is a feasible relaxed process, which we call the *reference process* and, for some $\theta > 0$, we set

$$\Sigma_\theta := \{(s, x) \in \mathbb{R} \times \mathbb{R}^n : \quad s \in [0, S],\ x \in \bar{y}(s) + \theta\, \mathbb{B}\}.$$

(H1) *There exists a sequence* $(V_i)_i$ *of closed subsets of* V *such that* $V_i \subseteq V_{i+1}$ *for every* i *and* $\bigcup_{i=1}^{+\infty} V_i = V$.

(H2) *The constraint function* h *is upper semicontinuous and* K_h*-Lipschitz continuous in* x, *uniformly w.r.t.* s *in* $[0, S]$.

(H3) (i) *For all* $(x, w, a) \in \{x \in \mathbb{R}^n : \ (s, x) \in \Sigma_\theta \ \text{ for some } s \in [0, S]\} \times W \times A$, $\mathscr{F}(\cdot, x, w, a)$ *is Lebesgue measurable on* $[0, S]$ *and for any* $(s, x) \in \Sigma_\theta$, $\mathscr{F}(s, x, \cdot, \cdot)$ *is continuous on* $W \times A$. *Moreover, there exists* $k \in L^1([0, S]; \mathbb{R}_{\geq 0})$ *such that, for all* (s, x, w, a), $(s, x', w, a) \in \Sigma_\theta \times W \times A$, *we have*

$$|\mathscr{F}(s, x, w, a)| \leq k(s), \qquad |\mathscr{F}(s, x', w, a) - \mathscr{F}(s, x, w, a)| \leq k(s)|x' - x|.$$

(ii) *There exists some continuous increasing function* $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *with* $\varphi(0) = 0$ *such that for any* $(s, x, a) \in \Sigma_\theta \times A$, *we have*

$$\begin{aligned} &|\mathscr{F}(s, x, w', a) - \mathscr{F}(s, x, w, a)| \leq k(s)\varphi(|w' - w|) \qquad \forall w',\ w \in W, \\ &D_x \mathscr{F}(s, x, w', a) \subseteq D_x \mathscr{F}(s, x, w, a) + k(s)\varphi(|w' - w|)\, \mathbb{B} \qquad \forall w',\ w \in W. \end{aligned}$$

When hypothesis (H3) *is valid for* $k \equiv K_{\mathscr{F}}$ *for some constant* $K_{\mathscr{F}} > 0$, *we will refer to* (H3) *as* (H3)$'$.

Remark 2.1. Condition (H1), which is always satisfied when the set V is relatively open, implies (and is in general stronger than) the density of $\mathscr{V}$ in $\mathscr{W}$ in the L^1-norm. Indeed, given $\omega \in \mathscr{W}$, from (H1) it follows that for any $\delta > 0$ there exists some index i_δ, such that the Hausdorff distance $d_H(V_i, W) < \delta/S$ for every $i \geq i_\delta$. Hence, by the selection theorem [3, Thm. 2, p. 91] for such i there is some measurable function $\omega_i(s) \in \operatorname{proj}_{V_i}(\omega(s))$ for a.e. s, which thus verifies $\|\omega_i - \omega\|_{L^1} \leq S\|\omega_i - \omega\|_{L^\infty} \leq S d_H(V_i, W) < \delta$. In particular, when $\mathscr{V} = L^1([0, S]; V)$ for some subset $V \subset W$ such that $\operatorname{int}(W) \subseteq V \subseteq W$ and $W = \overline{\operatorname{int}(W)}$, the validity of (H1) follows by elementary properties of closed and open subsets of $\mathbb{R}^n$.

Remark 2.2. As one can easily deduce from the proofs in section 6 below, condition (H1) could be replaced by the hypothesis that there exists a subset $\mathscr{V} \subset \mathscr{W} := L^1([0, S]; W)$ which is closed by finite concatenation and verifies the following:
(i) *there exists an increasing sequence of closed subsets* $(\mathscr{V}_i)_i \subseteq \mathscr{V}$ *such that* $\cup_i \mathscr{V}_i = \mathscr{V}$ *and, for any* $\omega \in \mathscr{W}$ *and* $\delta > 0$, *there are* i_δ *and* $\omega_\delta \in \mathscr{V}_{i_\delta}$, *such that* $\|\omega_\delta - \omega\|_{L^1} \leq \delta$;

(ii) *for every i, for the optimization problem obtained from (P) by replacing $\mathscr{V}$ with $\mathscr{V}_i$, a nonsmooth constrained maximum principle is valid.*

For example, from [14] a condition sufficient for (ii) to hold true is the C^0-closure of the set of the solutions to (1.1) as $(\omega,\alpha)\in\mathscr{V}_i\times\mathscr{A}$ for every i.

Remark 2.3. Condition (H3)(ii) is satisfied, for instance, when $\mathscr{F}(s,x,w,a)=\mathscr{F}_1(s,x,a)+\mathscr{F}_2(s,x,w,a)$, where $\mathscr{F}_1$, $\mathscr{F}_2$ verify hypothesis (H3)(i) and, in addition, the function $\mathscr{F}_2(s,\cdot,w,a)$ is C^1 and $\nabla_x\mathscr{F}_2$ is continuous on the compact set $\Sigma_\theta\times W\times A$. Another situation where condition (H3) is verified is when the dynamic function has a polynomial dependence on the control variable w, with locally Lipschitz continuous coefficients in the state variable, as we will see in detail in section 5 (see (5.1) and hypothesis (H5) below).

2.2. Infimum gap and isolated processes. Let us write Γ, Γ_e, Γ_r, to denote the sets of strict sense, extended, and relaxed processes which are feasible, respectively. As observed in the introduction, Γ, Γ_e can be identified with subsets of Γ_r.

DEFINITION 2.1 (minimizer). *A process $\bar z:=(\underline{\bar\omega},\underline{\bar\alpha},\bar\lambda,\bar y)\in\Gamma_g$, $g\in\{e,r\}$, is called a local Ψ-minimizer for problem (P_g) if, for some $\delta>0$, one has*

$$\Psi(\bar y(S))=\inf\left\{\Psi(y(S)):\quad (\underline{\omega},\underline{\alpha},\lambda,y)\in\Gamma_g,\quad \|y-\bar y\|_{L^\infty}<\delta\right\}.$$

The process $\bar z$ is a Ψ-minimizer for problem (P_g) if $\Psi(\bar y(S))=\inf_{\Gamma_g}\Psi(y(S))$.

DEFINITION 2.2 (infimum gap). *Fix $\bar z:=(\underline{\bar\omega},\underline{\bar\alpha},\bar\lambda,\bar y)\in\Gamma_r$.*
(i) *Let $\Psi:\mathbb{R}^n\to\mathbb{R}$ be a continuous function. When there is some $\delta>0$ such that*

$$\Psi(\bar y(S))<\ \inf\left\{\Psi(y(S)):\quad (\omega,\alpha,y)\in\Gamma,\quad \|y-\bar y\|_{L^\infty}<\delta\right\}$$

(as customary, when the set is empty we set the infimum $=+\infty$), we say that at $\bar z$ there is a local Ψ-infimum gap. *We say that there is a* Ψ-infimum gap *with problem (P_g), $g\in\{r,e\}$, if $\inf_{\Gamma_g}\Psi(y(S))<\ \inf_\Gamma\Psi(y(S))$.*
(ii) *When at $\bar z$ there is a local Ψ-infimum gap or if there is a Ψ-infimum gap (with (P_e) or (P_r)) for some Ψ as above, we say that* at $\bar z$ there is a local infimum gap *or that* there is an infimum gap *with (P_e) or (P_r), respectively.*

Obviously, a Ψ-infimum gap with (P_e) implies a Ψ-infimum gap with (P_r), and it may happen that $\inf_{\Gamma_r}\Psi(y(S))<\ \inf_{\Gamma_e}\Psi(y(S))<\ \inf_\Gamma\Psi(y(S))$.

The notion of local infimum gap at $\bar z$ is equivalent to the notion of *isolated process*, first introduced in [23], which is independent of any optimization problem.

DEFINITION 2.3 (isolated process and controllability). *We say that a process $\bar z:=(\underline{\bar\omega},\underline{\bar\alpha},\bar\lambda,\bar y)\in\Gamma_r$ is* isolated *(in Γ) if, for some $\delta>0$, one has*

$$\{(\omega,\alpha,y)\in\Gamma:\quad \|y-\bar y\|_{L^\infty}<\delta\}=\emptyset.$$

The control system (1.1) *is said to be* controllable to $\bar z$ *if $\bar z$ is not isolated.*

PROPOSITION 2.1. *Let $\bar z:=(\underline{\bar\omega},\underline{\bar\alpha},\bar\lambda,\bar y)\in\Gamma_r$. Then*

(i) *if $\bar z$ is isolated, then at $\bar z$ there is a local infimum gap and, for some $\delta>0$, one has $\inf\left\{\Psi(y(S)):\quad (\omega,\alpha,y)\in\Gamma,\quad \|y-\bar y\|_{L^\infty}<\delta\right\}=+\infty$ for every continuous Ψ;*
(ii) *if at $\bar z$ there is a local infimum gap, then $\bar z$ is isolated.*

As a consequence, $\bar z$ is isolated if and only if at $\bar z$ there is a local infimum gap.

Proof. The proof of (i) is trivial, hence we limit ourselves to proving (ii). Suppose that at $\bar{z}$ there is a Ψ-local infimum gap for some continuous Ψ and some $\delta > 0$, but $\bar{z}$ is not isolated. Then, for every $i \in \mathbb{N}$, $i \geq \frac{1}{\delta}$, there exists a feasible strict sense process $z_i = (\omega_i, \alpha_i, y_i) \in \Gamma$ such that $\|y_i - \bar{y}\|_{L^\infty} < \frac{1}{i}$, and, by the continuity of Ψ,

$$\Psi(\bar{y}(S)) < \inf \{\Psi(y(S)) : (\omega, \alpha, y) \in \Gamma, \|y - \bar{y}\|_{L^\infty} < \delta\} \leq \lim_i \Psi(y_i(S)), = \Psi(\bar{y}(S)),$$

which gives the desired contradiction. ☐

Incidentally, if at $\bar{z}$ there is a local Ψ-infimum gap for some Ψ, for some $\delta > 0$ there is in fact a local $\tilde{\Psi}$-infimum gap and $\inf \{\tilde{\Psi}(y(S)) : (\omega, \alpha, y) \in \Gamma, \|y - \bar{y}\|_{L^\infty} < \delta\} = +\infty$ for *every* continuous function $\tilde{\Psi}$.

2.3. Abnormality and infimum gap. We introduce a notion of normal and abnormal extremal for the relaxed optimization problem.

Definition 2.4 (normal and abnormal extremal). *Let $\bar{z} := (\underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{y})$ be a feasible relaxed process. Given a function $\Psi : \mathbb{R}^n \to \mathbb{R}$ which is Lipschitz continuous on a neighborhood of $\bar{y}(S)$, we say that $\bar{z}$ is a* Ψ-extremal *if there exist a path $p \in W^{1,1}([0, S]; \mathbb{R}^n)$, $\gamma \geq 0$, $\mu \in NBV^+([0, S]; \mathbb{R})$, $m : [0, S] \to \mathbb{R}^n$ Borel measurable and μ-integrable function, verifying the following conditions:*

$$\|p\|_{L^\infty} + \|\mu\|_{TV} + \gamma \neq 0; \tag{2.1}$$

$$-\dot{p}(s) \in \sum_{k=0}^{n} \bar{\lambda}^k(s) \operatorname{co} \partial_x \left\{q(s) \cdot \mathscr{F}(s, \bar{y}(s), \bar{\omega}^k(s), \bar{\alpha}^k(s))\right\} \quad a.e.; \tag{2.2}$$

$$-q(S) \in \gamma \partial \Psi(\bar{y}(S)) + N_{\mathscr{T}}(\bar{y}(S)); \tag{2.3}$$

for every $k = 0, \ldots, n$, for a.e. $s \in [0, S]$, one has

$$q(s) \cdot \mathscr{F}\big(s, \bar{y}(s), \bar{\omega}^k(s), \bar{\alpha}^k(s)\big) = \max_{(w,a) \in W \times A} q(s) \cdot \mathscr{F}\big(s, \bar{y}(s), w, a\big); \tag{2.4}$$

$$m(s) \in \partial_x^{>} h(s, \bar{y}(s)), \qquad \mu\text{-a.e.}; \tag{2.5}$$

$$spt(\mu) \subseteq \{s \in [0, S] : h(s, \bar{y}(s)) = 0\}, \tag{2.6}$$

where

$$q(s) := \begin{cases} p(s) + \int_{[0,s[} m(\sigma)\mu(d\sigma), & s \in [0, S[, \\ p(S) + \int_{[0,S]} m(\sigma)\mu(d\sigma), & s = S. \end{cases}$$

We will call a Ψ-extremal normal *if all possible choices of (p, γ, μ, m) as above have $\gamma > 0$ and* abnormal *when it is not normal. Since the notion of abnormal Ψ-extremal is actually independent of Ψ, in the following abnormal Ψ-extremals will be simply called* abnormal extremals.

Theorem 2.1. *Let $\bar{z} := (\underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{y})$ be a feasible relaxed process. Assume that hypotheses* (H1)–(H3) *are verified. If at $\bar{z}$ there is a local infimum gap, then $\bar{z}$ is an abnormal extremal.*

Remark 2.4. Since an extended process is a special case of a relaxed process, implicit in the definition of relaxed extremal is the definition of extended extremal. In particular, if $\bar{z} = (\bar{\omega}, \bar{\alpha}, \bar{y}) \in \Gamma_e$, clearly the costate differential inclusion (2.2) and the maximality condition (2.4) in the above definition read, for a.e. $s \in [0, S]$,

$$-\dot p(t) \in \operatorname{co}\partial_x\Big\{q(s)\cdot\mathscr{F}(s,\bar y(s),\bar\omega(s),\bar\alpha(s))\Big\}, \tag{2.7}$$

$$q(s)\cdot\mathscr{F}\big(s,\bar y(s),\bar\omega(s),\bar\alpha(s)\big) = \max_{(w,a)\in W\times A} q(s)\cdot\mathscr{F}\big(s,\bar y(s),w,a\big), \tag{2.8}$$

respectively.

A first noteworthy direct consequence of Theorem 2.1 is the following sufficient condition for the absence of an infimum gap.

THEOREM 2.2. *Suppose there exists a local Ψ-minimizer $\bar z := (\underline{\bar\omega},\underline{\bar\alpha},\bar\lambda,\bar y)$ for (P_e) or (P_r), for which hypotheses* (H1)–(H3) *are verified and Ψ is Lipschitz continuous in a neighborhood of $\bar y(S)$, and which is a normal Ψ-extremal. Then, at $\bar z$ there is no local Ψ-infimum gap for (P_e) or (P_r), respectively. If $\bar z$ is a Ψ-minimizer, then there is no infimum gap for (P_e) or (P_r), respectively.*

Remark 2.5. By a well-known constrained maximum principle (see [29, Chap. 9]), local Ψ-minimizers of (P_r) are Ψ-extremal in a stronger form than in Definition 2.4, in which the costate differential inclusion (2.2) is replaced by

$$-\dot p(s) \in \operatorname{co}\partial_x\left\{\sum_{k=0}^{n}\bar\lambda^k(s)\,q(s)\cdot\mathscr{F}(s,\bar y(s),\bar\omega^k(s),\bar\alpha^k(s))\right\} \quad \text{for a.e. } s\in[0,S]. \tag{2.9}$$

The need to consider (2.2) derives from the perturbation technique used in the proof of Theorem 2.1 (see also [25]). In fact, (2.2) may differ from (2.9) only in case of nonsmooth dynamics. Precisely, if $\mathscr{F}(s,\cdot,\bar\omega^k(s),\bar\alpha^k(s))$ is continuously differentiable at $\bar y(s)$, for all $k=0,\dots,n$ and a.e. $s\in[0,S]$, then both differential inclusions reduce to the adjoint equation

$$-\dot p(s) = \sum_{k=0}^{n}\bar\lambda^k(s)\,q(s)\cdot\nabla_x\mathscr{F}(s,\bar y(s),\bar\omega^k(s),\bar\alpha^k(s)) \quad \text{for a.e. } s\in[0,S].$$

Thanks to Proposition 2.1, from Theorem 2.1 we can also deduce the following sufficient condition for controllability to the reference trajectory.

THEOREM 2.3. *Let $\bar z := (\underline{\bar\omega},\underline{\bar\alpha},\bar\lambda,\bar y)$ be a feasible relaxed process and assume that hypotheses* (H1)–(H3) *are verified. Then, either*

(i) *$\bar z$ is not isolated in Γ, namely, there exists a sequence of feasible processes $(\omega_i,\alpha_i,y_i)\in\Gamma$ such that $\|y_i-\bar y\|_{L^\infty}\to 0$ as $i\to+\infty$, or*

(ii) *$\bar z$ is an abnormal extremal.*

Proof. Suppose by contradiction statements (i) and (ii) are false, namely, let $\bar z$ be an isolated process which is not an abnormal extremal. Then, by Proposition 2.1 at $\bar z$ there is a local infimum gap and $\bar z$ should be an abnormal extremal by Theorem 2.1. ∎

The proof of Theorem 2.1 is given in section 6.

3. Infimum gap and nondegenerate abnormality. In the particular case of fixed initial point, when the state constraint is active at the initial time there always exist sets of degenerate multipliers, as, for instance, $\gamma=0$, $\mu=\delta_{\{0\}}$, $p(s)=-m(0)\in\partial_x^{>}h(0,\check x_0)$ for all $s\in[0,S]$.[1] With the degenerate multipliers, the maximum principle is clearly useless, not only to select minimizers but also to identify no-gap conditions in the form of a "normality test."

[1] For any $r\in\mathbb{R}$, $\delta_{\{r\}}$ is the Dirac unit measure concentrated at r.

DEFINITION 3.1 (nondegenerate normal and abnormal extremal). *Assume that* $\bar{z} := (\underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{y})$ *is a feasible relaxed process. Given a function* $\Psi : \mathbb{R}^n \to \mathbb{R}$ *which is Lipschitz continuous on a neighborhood of* $\bar{y}(S)$*, let* $\bar{z}$ *be a* Ψ*-extremal. We call a* nondegenerate multiplier *any set of multipliers* (p, γ, μ, m) *that meets the conditions of Definition* 2.4 *and also verifies the following strengthened nontriviality condition*

$$\mu(]0, S]) + \|q\|_{L^\infty} + \gamma \neq 0, \tag{3.1}$$

where q *is as in Definition* 2.4*. We call* $\bar{z}$ *a* nondegenerate normal Ψ-extremal *if all possible choices of nondegenerate multipliers have* $\gamma > 0$ *and a* nondegenerate abnormal Ψ-extremal *when there exists at least one nondegenerate multiplier with* $\gamma = 0$*. Since nondegenerate abnormal* Ψ*-extremals do not depend on* Ψ*, in the following we will call them simply* nondegenerate abnormal extremals.

As is easy to see, a nondegenerate abnormal extremal is always an abnormal extremal, and, conversely, any normal Ψ-extremal is also nondegenerate normal. However, we may have situations where a nondegenerate normal Ψ-extremal is not a normal Ψ-extremal, as illustrated in Example 5.1 below.

To introduce our constraint qualification conditions, we define $\Lambda^1_n \equiv \Lambda^1_n(S)$ as

$$\Lambda^1_n := L^1([0, S]; \Delta^1_n), \ \Delta^1_n := \cup_{k=0}^n \{e_k\} \ \ (e_0, \dots, e_n \text{ canonical basis of } \mathbb{R}^{1+n}), \tag{3.2}$$

and extend the relaxed control system by introducing a new variable, ξ. Precisely, with a small abuse of notation, in the following we call a relaxed process any element $(\underline{\omega}, \underline{\alpha}, \lambda, \xi, y)$ with $(\underline{\omega}, \underline{\alpha}, \lambda) \in \mathscr{W}^{1+n} \times \mathscr{A}^{1+n} \times \Lambda_n$ and (ξ, y) which satisfies

$$\begin{cases} (\dot{\xi}, \dot{y})(s) = \Big(\lambda(s), \sum_{k=0}^n \lambda^k(s) \mathscr{F}(s, y(s), \omega^k(s), \alpha^k(s))\Big) \text{ a.e.}, \\ (\xi, y)(0) = (0, \check{x}_0). \end{cases} \tag{3.3}$$

Observe that a relaxed process $(\underline{\omega}, \underline{\alpha}, \lambda, \xi, y)$ with $\lambda \in \Lambda^1_n$ corresponds to the extended process (ω, α, ξ, y), where $(\omega, \alpha) := \sum_{k=0}^n (\omega_k, \alpha_k) \chi_{\{s \in [0,S]:\ \lambda(s) = e_k\}}$. In particular, the function y is a solution of (1.1) associated with the control (ω, α).

Let $\bar{z} := (\underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{\xi}, \bar{y})$ be a feasible relaxed process. Define the constraint set

$$\Omega := \{(s, x) \in \mathbb{R}^{1+n} : \quad h(s, x) \leq 0\}.$$

We shall consider the following hypothesis.

(H4) *If* $(0, \check{x}_0) \in \partial\Omega$*, there exist some* $\tilde{\delta} > 0$*,* $\bar{s} \in]0, S]$*, a sequence of extended processes* $(\tilde{\omega}_i, \tilde{\alpha}_i, \tilde{\lambda}_i, \tilde{\xi}_i, \tilde{y}_i)_i$ *with* $(\tilde{\omega}_i, \tilde{\alpha}_i, \tilde{\lambda}_i)_i \subset (\mathscr{W}(S) \cap \mathscr{V}(\bar{s})) \times \mathscr{A} \times \Lambda^1_n$*, and sequences* $(\hat{\omega}_i, \hat{\alpha}_i)_i \subset \mathscr{V}(\bar{s}) \times \mathscr{A}(\bar{s})$*, and* $(\tilde{r}_i)_i \subset L^1([0, S]; \mathbb{R}_{\geq 0})$ *with* $\lim_{i \to +\infty} \|\tilde{r}_i\|_{L^1} = 0$*, such that the following properties* (i)–(iv) *are verified.*

(i) *One has*

$$\lim_{i \to +\infty} \|(\tilde{\xi}_i, \tilde{y}_i) - (\bar{\xi}, \bar{y})\|_{L^\infty} = 0; \tag{3.4}$$

(ii) *for every* i*, one has*

$$h(s, \tilde{y}_i(s)) \leq 0 \qquad \forall s \in [0, \bar{s}]; \tag{3.5}$$

(iii) *for every* i*, there is a Lebesgue measurable subset* $\tilde{E}_i \subset [0, S]$ *such that, for a.e.* $s \in \tilde{E}_i$*, one has*

$$(\tilde{\omega}_i, \tilde{\alpha}_i, \tilde{\lambda}_i)(s) \in \bigcup_{k=0}^n \{(\bar{\omega}^k(s), \bar{\alpha}^k(s), e^k)\} + (\tilde{r}_i(s)\mathbb{B}_m) \times \{0\} \times \{0\}, \tag{3.6}$$

and $\lim_{i \to +\infty} \ell(\tilde{E}_i) = S$*;*

(iv) *for every i and for all $(\zeta_0, \zeta) \in \partial^* h(0, \check{x}_0)$, for a.e. $s \in [0, \bar{s}]$ one has*

$$\zeta \cdot \big[\mathscr{F}(s, \check{x}_0, (\hat{\omega}_i, \hat{\alpha}_i)(s)) - \mathscr{F}(s, \check{x}_0, (\tilde{\omega}_i, \tilde{\alpha}_i)(s))\big] \le -\tilde{\delta}. \tag{3.7}$$

Remark 3.1. Some comments on hypothesis (H4) are in order.

(1) It prescribes additional conditions to assumptions (H1)–(H3) only when the initial point $(0, \check{x}_0)$ lies on the boundary of the constraint set Ω. Incidentally, this is not equivalent to having $h(0, \check{x}_0) = 0$, as it may clearly happen that $h(0, \check{x}_0) = 0$ but $(0, \check{x}_0) \in \mathrm{Int}(\Omega)$.

(2) When $(0, \check{x}_0) \in \partial\Omega$, the first part of hypothesis (H4) substantially requires the existence of strict sense processes that approximate the reference process and satisfy the state constraint on some (small) interval $[0, \bar{s}]$, with controls which are close to controls $(\bar{\omega}_i, \bar{\alpha}_i, \bar{\lambda}_i)$ belonging to $\bigcup_{k=0}^n \{(\bar{\omega}^k(s), \bar{\alpha}^k(s), e^k)\}$ for a.e. $s \in [0, S]$. Let us point out that, disregarding the state constraint (3.5), the existence of approximating controls that satisfy the remaining conditions (3.4), (3.6) follows by the relaxation theorem together with hypothesis (H1), as we will see in the proof of Theorem 2.1 below, in subsection 6.2. Relation (3.7), on the other hand, is an adaptation of known constraint qualification conditions (see, e.g., [9, 10]), in which the reference (relaxed) control is replaced by approximating strict sense controls $(\tilde{\omega}_i, \tilde{\alpha}_i) \in \mathscr{V}(\bar{s}) \times \mathscr{A}(\bar{s})$.

(3) If hypotheses (H1), (H2), and (H3)$'$ with reference to $\bar{z}$ are verified, then in hypothesis (H4) one can assume that the control sequence $(\hat{\omega}_i, \hat{\alpha}_i)_i$ belongs to the extended control set $\mathscr{W}(\bar{s}) \times \mathscr{A}(\bar{s})$ rather than $\mathscr{V}(\bar{s}) \times \mathscr{A}(\bar{s})$. Indeed, using the notation of (H1)–(H3)$'$, let us choose some $\rho > 0$ such that $K_h\, K_{\mathscr{F}} \varphi(\rho) \le \frac{\tilde{\delta}}{2}$, and let $j \in \mathbb{N}$ verify $d_H(V_j, W) \le \rho$. Hence, for every $i \in \mathbb{N}$ there exists a measurable selection $\hat{\omega}_i^*(s) \in \mathrm{proj}_{V_j}(\hat{\omega}_i(s))$ for a.e. $s \in [0, S]$, such that $\|\hat{\omega}_i^* - \hat{\omega}_i\|_{L^\infty} \le \rho$ (see also Remark 2.1), and, for all $(\zeta_0, \zeta) \in \partial^* h(0, \check{x}_0)$ (by adding and subtracting $\zeta \cdot \mathscr{F}(s, \check{x}_0, (\hat{\omega}_i^*, \hat{\alpha}_i)(s))$), one has

$$\zeta \cdot \big[\mathscr{F}(s, \check{x}_0, (\hat{\omega}_i^*, \hat{\alpha}_i)(s)) - \mathscr{F}(s, \check{x}_0, (\tilde{\omega}_i, \tilde{\alpha}_i)(s))\big] \le -\frac{\tilde{\delta}}{2}, \quad \text{a.e. } s \in [0, \bar{s}],$$

as soon as $(\hat{\omega}_i, \hat{\alpha}_i)$ satisfies (3.7).

(4) When hypothesis (H3)$'$ is verified, then the upper semicontinuity of the set valued map $\partial^* h(\cdot, \cdot)$ and (3.7) in (H4) imply that there exist δ, $\varepsilon > 0$ such that for any $(\zeta_0, \zeta) \in \partial^* h(\sigma, x)$ with $\sigma \in [0, \varepsilon]$ and $x \in \{\check{x}_0\} + \varepsilon\mathbb{B}$, for any $s \le \bar{s}$, for any continuous path $y : [0, s] \to \{\check{x}_0\} + \varepsilon\mathbb{B}$, and for any measurable map $\eta : [0, s] \to \{0, 1\}$, the following integral condition holds:

$$\int_0^s \eta(\sigma)\, \zeta \cdot \big[\mathscr{F}(\sigma, y, \hat{\omega}_i, \hat{\alpha}_i)(\sigma) - \mathscr{F}(\sigma, y, \tilde{\omega}_i, \tilde{\alpha}_i)(\sigma)\big] d\sigma \le -\delta\, \ell(s, \eta(\cdot)), \tag{3.8}$$

where

$$\ell(s, \eta(\cdot)) := \ell(\{\sigma \in [0, s] \,:\, \eta(\sigma) = 1\}). \tag{3.9}$$

In particular, relation (3.8) holds for any $(\zeta_0, \zeta) \in \partial^c h(\sigma, x)$, as the scalar product is bilinear, and for all $\zeta \in \partial_x^{>} h(\sigma, x)$, since (see, e.g., [29, Thm. 5.3.1])

$$\partial_x^{>} h(\sigma, x) \subseteq \partial_x^c h(\sigma, x) \subseteq \{\zeta \,:\, \exists \zeta_0 \text{ s.t. } (\zeta_0, \zeta) \in \partial^c h(\sigma, x)\} \qquad \forall (\sigma, x) \in \mathbb{R}^{1+n}.$$

Relation (3.8) is in fact the condition used in the proof of Theorem 3.1 below.

(5) When hypothesis (H3)′ is verified, it is not difficult to verify that condition (3.8) still holds if we replace (H4), (iv), with the following assumption:

(iv)′ *there exists* $\tilde{\varepsilon} > 0$ *such that, for every* i, *for all* $\zeta \in \partial^c_x h(\sigma, x)$ *with* $\sigma \in [0, \tilde{\varepsilon}]$ *and* $x \in \{\check{x}_0\} + \tilde{\varepsilon}\mathbb{B}$, *for a.e.* $s \in [0, \bar{s}]$ *one has*

$$\zeta \cdot \left[\mathscr{F}(s, \check{x}_0, (\hat{\omega}_i, \hat{\alpha}_i)(s)) - \mathscr{F}(s, \check{x}_0, (\tilde{\omega}_i, \tilde{\alpha}_i)(s))\right] \leq -\tilde{\delta}. \tag{3.10}$$

Theorem 2.1 can be refined as follows.

Theorem 3.1. *Let* $\bar{z} := (\underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{\xi}, \bar{y})$ *be a feasible relaxed process. Assume that hypotheses* (H1)–(H2)–(H3)′–(H4) *are verified. If at* $\bar{z}$ *there is a local infimum gap, then* $\bar{z}$ *is a nondegenerate abnormal extremal.*

As in the previous section, from Theorem 3.1 one can derive the following results.

Theorem 3.2. *Suppose there exists a local* Ψ*-minimizer* $\bar{z} := (\underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{\xi}, \bar{y})$ *for* (P_e) *or* (P_r) *for which* (H1)–(H2)–(H3)′–(H4) *are verified, where* Ψ *is Lipschitz continuous in a neighborhood of* $\bar{y}(S)$, *and which is a nondegenerate normal* Ψ*-extremal. Then, at* $\bar{z}$ *there is no local infimum gap for* (P_e) *or* (P_r), *respectively. If* $\bar{z}$ *is a* Ψ*-minimizer, then there is no infimum gap for* (P_e) *or* (P_r), *respectively.*

Theorem 3.3. *Let* $\bar{z} := (\underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{\xi}, \bar{y})$ *be a feasible relaxed process and assume that hypotheses* (H1)–(H2)–(H3)′–(H4) *are verified. Then, either*

(i) $\bar{z}$ *is not isolated in* Γ, *namely, there exists a sequence of feasible processes* $(\omega_i, \alpha_i, y_i) \in \Gamma$ *such that* $\|y_i - \bar{y}\|_{L^\infty} \to 0$ *as* $i \to +\infty$, *or*

(ii) $\bar{z}$ *is a nondegenerate abnormal extremal.*

The proof of Theorem 3.1 is given in section 6.

Remark 3.2. As will be clear from the proofs in section 6, Theorems 3.1, 3.2, and 3.3—as well as Theorems 2.1, 2.2, and 2.3—remain true if we replace the fixed set of control values A with a compact Borel measurable multifunction $A : [0, S] \rightsquigarrow \mathbb{R}^q$.

Moreover, we could substitute the endpoint constraint $(y(0), y(S)) \in \{\check{x}_0\} \times \mathscr{T}$ with the more general constraint

$$(y(0), y(S)) \in \tilde{\mathscr{T}}$$

for some closed subset $\tilde{\mathscr{T}} \subset \mathbb{R}^{n+n}$. In fact, in this case the proof of Theorem 2.1 can be easily adapted, while, arguing as in [13], one can deduce that Theorem 3.1 remains true if hypothesis (H4) is replaced, for instance, by the following condition:

$$\partial^>_x h(0, y(0)) \cap (-\Pi_{x_1} N_{\mathscr{T}}(y(0), y(S))) = \emptyset.^2 \tag{3.11}$$

Nevertheless, condition (3.11) is never satisfied when $\tilde{\mathscr{T}} = \{\check{x}_0\} \times \mathscr{T}$ and $(0, \check{x}_0)$ lies on the boundary of the state constraint. This is the main reason why in this article we prefer to focus on this special case.

4. Free end-time problems with Lipschitz time dependence.

We consider the optimization problem

$$(P^*) \begin{cases} \text{minimize } \Psi(S, y(S)) \\ \text{over the set of } S > 0 \text{ and } (\omega, \alpha, y) \in \mathscr{V}(S) \times \mathscr{A}(S) \times W^{1,1}([0,S]; \mathbb{R}^n), \\ \text{verifying the Cauchy problem (1.1) on } [0, S] \text{ and the constraints} \\ h(s, y(s)) \leq 0 \quad \forall s \in [0, S], \qquad (S, y(S)) \in \mathscr{T}^*, \end{cases}$$

[2]Let $X \subseteq \mathbb{R}^{k_1+k_2}$ for some natural numbers k_1, k_2, and write $x = (x_1, x_2) \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$ for any $x \in X$. Then, $\Pi_{x_i} X$ will denote the projection of X on $\mathbb{R}^{k_i}$ for $i = 1, 2$.

where $\mathscr{T}^*$ is a closed subset of $\mathbb{R}^{1+n}$ and $\Psi : \mathbb{R}^{1+n} \to \mathbb{R}$. An *extended process* or *process* is an element (S, ω, α, y), where $S > 0$, $(\omega, \alpha) \in \mathscr{W}(S) \times \mathscr{A}(S)$, and y solves the Cauchy problem (1.1) on $[0, S]$. When $\omega \in \mathscr{V}(S)$, the process is called a *strict sense process*. A process (S, ω, α, y) is *feasible* if $h(s, y(s)) \leq 0$ for all $s \in [0, S]$ and $(S, y(S)) \in \mathscr{T}^*$. We call an *extended problem*, and write (P_e^*), the problem of minimizing $\Psi(S, y(S))$ over the set of feasible extended processes.

The associated *relaxed problem* is

$$(P_r^*) \begin{cases} \qquad\qquad \text{minimize } \Psi(S, y(S)) \\ \text{over } S > 0,\ (\underline{\omega}, \underline{\alpha}, \lambda, y) \in \mathscr{W}^{1+n}(S) \times \mathscr{A}^{1+n}(S) \times \Lambda_n(S) \times W^{1,1}([0,S];\mathbb{R}^n) \text{ s.t.} \\ \dot{y}(s) = \displaystyle\sum_{k=0}^{n} \lambda^k(s) \mathscr{F}(s, y(s), \omega^k(s), \alpha^k(s)) \text{ a.e. } s \in [0, S], \quad y(0) = \check{x}_0, \\ h(s, y(s)) \leq 0 \quad \forall s \in [0, S], \qquad (S, y(S)) \in \mathscr{T}^*, \end{cases}$$

A process $(S, \underline{\omega}, \underline{\alpha}, \lambda, y)$ for (P_r^*) is referred to as a *relaxed* process. As in the previous sections, we can identify the set of strict sense processes (extended processes) with the subset of relaxed processes $(S, \underline{\omega}, \underline{\alpha}, \lambda, y)$ with $(\underline{\omega}, \underline{\alpha}, \lambda) \in \mathscr{V}^{1+n}(S) \times \mathscr{A}^{1+n}(S) \times \Lambda_n^1(S)$ $[(\underline{\omega}, \underline{\alpha}, \lambda) \in \mathscr{W}^{1+n}(S) \times \mathscr{A}^{1+n}(S) \times \Lambda_n^1(S)]$. We will use Γ^*, Γ_e^*, Γ_r^* to denote the sets of feasible strict sense, feasible extended, and feasible relaxed processes, respectively.

Throughout this section, we strengthen hypotheses (H2)-(H3) treating time as a state variable. As in section 3, we add to (P_r^*) the variable $\xi(s) = \int_0^s \lambda(s')\, ds'$, $s \in [0, S]$, and call a relaxed process any element $(S, \underline{\omega}, \underline{\alpha}, \lambda, \xi, y)$, where $(\underline{\omega}, \underline{\alpha}, \lambda) \in \mathscr{W}^{1+n}(S) \times \mathscr{A}^{1+n}(S) \times \Lambda_n(S)$ and (ξ, y) solves (3.3) on $[0, S]$.

We shall consider the following hypotheses, in which $(\bar{S}, \underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{\xi}, \bar{y})$ is a given feasible relaxed process for (P_r^*) and, for some $\theta > 0$, we set

$$\Sigma_\theta^* := \left\{ (t, x) \in \mathbb{R} \times \mathbb{R}^n : \quad (t, x) \in (s, \bar{y}(s)) + \theta\, \mathbb{B}_{1+n}, \quad s \in [0, \bar{S}] \right\}.$$

(H2)* *The constraint function h is K_h-Lipschitz continuous in Σ_θ^*.*

(H3)* (i) *The function $\mathscr{F}$ is continuous on $\Sigma_\theta^* \times W \times A$. Furthermore, there is some constant $K_{\mathscr{F}} > 0$ such that, for all (s, x, w, a), $(s', x', w, a) \in \Sigma_\theta^* \times W \times A$,*

$$|\mathscr{F}(s', x', w, a) - \mathscr{F}(s, x, w, a)| \leq K_{\mathscr{F}} |(s', x') - (s, x)|.$$

(ii) *There exists some continuous increasing function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\varphi(0) = 0$ such that for any $(s, x, a) \in \Sigma_\theta^* \times A$, we have*

$$D_{t,x} \mathscr{F}(s, x, w', a) \subseteq D_{t,x} \mathscr{F}(s, x, w, a) + \varphi(|w' - w|)\, \mathbb{B} \qquad \forall w',\ w \in W.$$

Let us identify a continuous function $\tilde{y} : [0, \tau] \to \mathbb{R}^k$ with its extension to $\mathbb{R}$ by constant extrapolation of the left and right endpoint values. Then, for all τ_1, $\tau_2 > 0$, and $(\tilde{y}_1, \tilde{y}_2) \in C^0([0, \tau_1]; \mathbb{R}^k) \times C^0([0, \tau_2]; \mathbb{R}^k)$, we define the distance

$$(4.1) \qquad d_\infty\big((\tau_1, \tilde{y}_1), (\tau_2, \tilde{y}_2)\big) := |\tau_2 - \tau_1| + \|\tilde{y}_2 - \tilde{y}_1\|_\infty, \qquad (\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\mathbb{R})}).$$

We can now extend the concepts of *local minimizer*, *local infimum gap*, and *isolated process* to free end-time problems, by formally replacing trajectories y and L^∞-norm over trajectories with pairs (S, y) endowed with the distance d_∞. For instance, if $\bar{z} := (\bar{S}, \underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{y})$ is a feasible relaxed process, *at $\bar{z}$ there is a local infimum* gap if there is some $\delta > 0$ such that

$$\Psi(\bar{S}, \bar{y}(S)) < \inf \left\{ \Psi(S, y(S)) : \quad (S, \omega, \alpha, y) \in \Gamma^*, \quad d_\infty\big((S, y), (\bar{S}, \bar{y})\big) < \delta \right\}$$

for some continuous function $\Psi : \mathbb{R}^{1+n} \to \mathbb{R}$, while $\bar{z}$ is an *isolated process* if

$$\big\{(S,\omega,\alpha,y) \in \Gamma^* : \quad d_\infty\big((S,y),(\bar{S},\bar{y})\big) < \delta\big\} = \emptyset$$

for some $\delta > 0$. As in the case with fixed end-time, *at $\bar{z}$ there is a local infimum gap if and only if $\bar{z}$ is isolated.*

Definition 4.1 (extremal and nondegenerate extremal). *Let $\bar{z} := (\bar{S}, \underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{y})$ be a feasible relaxed process and assume that hypotheses* (H2)*, (H3)* *are verified. Given a function $\Psi : \mathbb{R}^{1+n} \to \mathbb{R}$ which is Lipschitz continuous on a neighborhood of $(\bar{S}, \bar{y}(\bar{S}))$, we say that $\bar{z}$ is a* Ψ-extremal *if there exist a pair of paths $(p_*, p) \in W^{1,1}([0,\bar{S}];\mathbb{R}^{1+n})$, $\gamma \geq 0$, $\mu \in NBV^+([0,\bar{S}];\mathbb{R})$, $(m_*, m) : [0,\bar{S}] \to \mathbb{R}^{1+n}$ Borel measurable and μ-integrable functions, verifying the following conditions:*

$$\|p\|_{L^\infty} + \|\mu\|_{TV} + \gamma \neq 0; \tag{4.2}$$

$$\big(\dot{p}_*, -\dot{p}\big)(s) \in \sum_{k=0}^{n} \bar{\lambda}^k(s) \operatorname{co} \partial_{t,x} \left\{q(s) \cdot \mathscr{F}(s, \bar{y}(s), \bar{\omega}^k(s), \bar{\alpha}^k(s))\right\} \ \ a.e.\ \ s \in [0,\bar{S}];$$

$$\big(q_*(\bar{S}), -q(\bar{S})\big) \in \gamma \partial \Psi\left(S, \bar{y}(S)\right) + N_{\mathscr{T}}(\bar{S}, \bar{y}(\bar{S}));$$

for every $k = 0, \dots, n$, for a.e. $s \in [0,\bar{S}]$, one has

$$q(s) \cdot \mathscr{F}\Big(s, \bar{y}(s), \bar{\omega}^k(s), \bar{\alpha}^k(s)\Big) = \max_{(w,a) \in W \times A} q(s) \cdot \mathscr{F}\Big(s, \bar{y}(s), w, a\Big); \tag{4.3}$$

$$\sum_{k=0}^{n} \bar{\lambda}^k(s) q(s) \cdot \mathscr{F}(s, \bar{y}(s), \bar{\omega}^k(s), \bar{\alpha}^k(s)) = q_*(s) \ \ a.e.\ \ s \in [0,\bar{S}]; \tag{4.4}$$

$$(m_*, m)(s) \in \partial^{>}_{t,x} h\left(s, \bar{y}(s)\right) \ \ \mu\text{-}a.e.\ \ s \in [0,\bar{S}];$$

$$spt(\mu) \subseteq \{s \in [0,\bar{S}] \,:\, h\left(s, \bar{y}(s)\right) = 0\},$$

where $$(q_*, q)(s) := \begin{cases} (p_*, p)(s) + \int_{[0,s[} (m_*, m)(\sigma)\mu(d\sigma), & s \in [0,\bar{S}[, \\ (p_*, p)(\bar{S}) + \int_{[0,\bar{S}]} (m_*, m)(\sigma)\mu(d\sigma), & s = \bar{S}. \end{cases}$$

A Ψ-extremal is normal *if all possible choices of $(p_*, p, \gamma, \mu, m_*, m)$ as above have $\gamma > 0$ and* abnormal *when it is not normal. Given a Ψ-extremal $\bar{z}$, we call a* nondegenerate multiplier *any set of multipliers $(p_*, p, \gamma, \mu, m_*, m)$ and (q_*, q) as above, that also verify*

$$\mu(]0,S]) + \|q\|_{L^\infty} + \gamma \neq 0. \tag{4.5}$$

A Ψ-extremal is nondegenerate normal *if all the choices of nondegenerate multipliers have $\gamma > 0$, and it is* nondegenerate abnormal *when there exists a nondegenerate multiplier with $\gamma = 0$. In the following, abnormal (nondegenerate abnormal) Ψ-extremals will be simply called* abnormal (nondegenerate abnormal) extremals.

Theorems 2.1 and 3.1 extend to free end-time optimization problems as follows.

Theorem 4.1. *Let $\bar{z} := (\bar{S}, \underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{\xi}, \bar{y})$ be a feasible relaxed process for (P_r^*), and suppose that at $\bar{z}$ there is a local infimum gap.*

(i) *If hypotheses* (H1)–(H2)*–(H3)* *hold, then $\bar{z}$ is an abnormal extremal.*

(ii) *If, in addition, also hypothesis* (H4) *for $S = \bar{S}$ is verified, then $\bar{z}$ is a nondegenerate abnormal extremal.*

Proof. Let $\bar{z} := (\bar{S}, \underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{\xi}, \bar{y})$ be a feasible relaxed process at which there is a local infimum gap. By the above considerations, this is equivalent to supposing that $\bar{z}$ is an isolated process. Adapting a standard time-rescaling procedure (see, e.g., [29, Thm. 8.7.1]), we transform problems (P^*), (P^*_e), and (P^*_r) into fixed end-time problems and show that $\bar z$ is an isolated process also with respect to feasible strict sense processes of a *rescaled, fixed end-time problem.* At this point, the thesis follows by applying Theorems 2.1 and 3.1 to the rescaled problem.

From the fact that $\bar z$ is isolated, it follows that there exists some $\delta > 0$ such that

$$\big\{(S,\omega,\alpha,y)\in\Gamma^*:\quad d_\infty\big((S,y),(\bar S,\bar y)\big)<3\delta\big\}=\emptyset. \tag{4.6}$$

Set $\bar\delta := \min\big\{\frac{\delta}{3\bar S\,K_{\mathscr F}},\frac12\big\}$.

We define the *rescaled optimization problem* $(\hat P^*)$ as

$$(\hat P^*)\begin{cases} \qquad\qquad\text{minimize } \Psi(y^*(\bar S),y(\bar S))\\ \text{over } (\omega,\alpha,\zeta,y^*,y)\in\mathscr V(\bar S)\times\mathscr A(\bar S)\times L^1([0,\bar S];[-\bar\delta,\bar\delta])\times W^{1,1}([0,\bar S];\mathbb R^{1+n}) \text{ s.t.}\\ (\dot y^*,\dot y)(s)=(1+\zeta(s))\big(1,\mathscr F(y^*(s),y(s),\omega(s),\alpha(s))\big) \text{ a.e.},\quad (y^*,y)(0)=(0,\check x_0),\\ h(y^*(s),y(s))\le 0\quad\forall s\in[0,\bar S],\qquad (y^*(\bar S),y(\bar S))\in\mathscr T^*,\end{cases}$$

A process $(\omega,\alpha,\zeta,y^*,y)$ for the fixed end-time problem $(\hat P^*)$ is referred to as a *rescaled strict sense process.* Let $\hat\Gamma^*$ denote the set of the *feasible* rescaled strict sense processes, that is, the set rescaled strict sense processes that verify $h(y^*(s),y(s))\le 0$ for all $s\in[0,\bar S]$ and $(y^*(\bar S),y(\bar S))\in\mathscr T^*$. We call *rescaled extended processes* the processes $(\omega,\alpha,\zeta,y^*,y)$ with $\omega\in\mathscr W(\bar S)$ and write $(\hat P^*_e)$ to denote the rescaled extended problem associated with $(\hat P^*)$.

We can identify $\bar z = (\bar S,\underline{\bar\omega},\underline{\bar\alpha},\bar\lambda,\bar\xi,\bar y)$ with a process $\check z := (\underline{\check\omega},\underline{\check\alpha},\underline{\check\zeta},\check\lambda,\check\xi,\check y^*,\check y)\in \mathscr W^{2+n}(\bar S)\times\mathscr A^{2+n}(\bar S)\times(L^1[0,\bar S];[-\bar\delta,\bar\delta]))^{2+n}\times\Lambda_{n+1}(\bar S)\times W^{1,1}([0,\bar S];\mathbb R^{2+n}\times\mathbb R^{1+n})$ of the rescaled relaxed problem associated with $(\hat P^*_e)$, by setting

$$\underline{\check\omega}:=(w,\underline{\bar\omega}),\quad \underline{\check\alpha}:=(a,\underline{\bar\alpha}),\quad \underline{\check\zeta}:=0,\ \ \check\lambda:=(0,\bar\lambda),\quad \check\xi=(0,\bar\xi),\quad \check y^*:=id,\quad \check y:=\bar y,$$

for arbitrary $w\in W$ and $a\in A$. Since $\bar z$ is an isolated feasible relaxed process for the free end-time problem, $\check z$ is feasible and isolated for the relaxed rescaled problem. In particular, we claim that

$$\Big\{(\omega,\alpha,\zeta,y^*,y)\in\hat\Gamma^*:\quad \|(y^*,y)-(\check y^*,\check y)\|_{L^\infty([0,\bar S])}<\delta\Big\}=\emptyset. \tag{4.7}$$

Indeed, let $(\omega,\alpha,\zeta,y^*,y)$ be an arbitrary feasible, rescaled strict sense process verifying $\|(y^*,y)-(\check y^*,\check y)\|_{L^\infty([0,\bar S])}<\delta$. Consider the time-transformation $y^*:[0,\bar S]\to[0,S]$, where $S:=y^*(\bar S)$. Observe that y^* is a strictly increasing, Lipschitz continuous function, with Lipschitz continuous inverse, $(y^*)^{-1}$. It can be deduced that the process $(S,\hat\omega,\hat\alpha,\hat y)$, where

$$(\hat\omega,\hat\alpha,\hat y):=(\omega,\alpha,y)\circ(y^*)^{-1}\quad\text{in } [0,S],$$

is a feasible strict sense process for the free end-time problem (P^*), i.e., $(S,\hat\omega,\hat\alpha,\hat y)\in\Gamma^*$. Indeed, recalling the definitions of $\bar\delta$ and d_∞,[3] after some calculations, we get

$$\begin{aligned} d_\infty\big((S,\hat y),(\bar S,\bar y)\big) &= |S-\bar S|+\|\hat y-\bar y\|_\infty \le |y^*(\bar S)-\check y^*(\bar S)|\\ &+\sup_{s\in[0,S\vee\bar S]}\big[|y((y^*)^{-1}(s\wedge S))-\check y((y^*)^{-1}(s\wedge S))|+|\check y((y^*)^{-1}(s\wedge S))-\check y(s\wedge\bar S)|\big]\\ &\le \|y^*-\check y^*\|_{L^\infty([0,\bar S])}+\|y-\check y\|_{L^\infty([0,\bar S])}+\delta<3\delta, \end{aligned}$$

[3]By definition, $\hat y$ and $\check y$ are replaced with their constant, continuous extensions to $\mathbb R$.

since, in particular,

$$\sup_{s\in[0,S\vee\bar S]}|\check y((y^*)^{-1}(s\wedge S))-\check y(s\wedge\bar S)|\le 3K_{\mathscr F}\bar S\,\bar\delta\le\delta.$$

Therefore, (4.6) yields (4.7), and the feasible rescaled relaxed process $\check z$ is isolated in $\hat\Gamma^*$, as claimed. In order to apply the results of Theorems 2.1 and 3.1 to $(\hat P^*)$ with reference to the process $\check z$, it remains to show that if we consider (w,a,λ,ζ) as control variables and $\tilde x:=(s,x)$ as the state variable for the (now time-independent) problem $(\hat P^*)$, all the hypotheses assumed in their statements are fulfilled. To this aim, observe that (H2) trivially follows from (H2)*, while (H3)* easily implies (H3)$'$. In particular, the compactness of $\Sigma^*_\theta\times W\times A\times[-\bar\delta,\bar\delta]$ and the continuity of $(1+\zeta)\mathscr F$ on it guarantee the validity of (H3)$'$ for the rescaled dynamics function $(1+\zeta)\mathscr F$. Finally, recalling that $\check\zeta\equiv 0$ and $\check y^*(s)=s$ for all $s\in[0,\bar S]$, hypothesis (H4) can be trivially reformulated as a hypothesis on the rescaled process $\check z$. At this point, from Theorems 2.1 and 3.1 we can derive that $\bar z$ is an abnormal extremal, nondegenerate when hypothesis (H4) is verified. In particular, the only nontrivial results, namely conditions (4.3), (4.4), and the nontriviality conditions (4.2), (4.5) can be obtained through routine arguments (see, e.g., the proof of [29, Thm. 8.7.1]). ∎

Again, from Theorem 4.1 we can get normality tests for gap avoidance and sufficient controllability conditions for the free end-time problem completely analogous to Theorems 2.2–2.3 and 3.2–3.3, respectively.

5. An application to nonconvex, control-polynomial impulsive problems. We consider the free end-time optimal control problem:

$$(\mathscr P^*)\begin{cases}\qquad\text{minimize }\ \Psi(T,y(T),v(T))\\ \text{over } T>0,\ (u,a,y,v)\in L^d([0,T];U)\times L^1([0,T];A)\times W^{1,1}([0,T];\mathbb R^{n+1})\ \text{ s.t.}\\ (\dot y,\dot v)(t)=\left(f(t,y,a)+\displaystyle\sum_{k=1}^d\left(\sum_{1\le j_1\le\cdots\le j_k\le m}g^k_{j_1,\dots,j_k}(t,y)\ u^{j_1}\cdots u^{j_k}\right),|u|^d\right)\ \text{a.e.,}\\ (y,v)(0)=(\check x_0,0),\\ h(t,y(t))\le 0\qquad\forall\,t\in[0,T],\quad v(T)\le K,\quad (T,y(T))\in\mathscr T^*.\end{cases}$$

Here, $U\subseteq\mathbb R^m$ is a closed cone, $A\subseteq\mathbb R^q$ is a compact subset, $K>0$ is a fixed constant, possibly equal to $+\infty$, and the target set $\mathscr T^*\subseteq\mathbb R^{1+n}$ is closed. Notice that $v(t)$ is simply the L^d-norm to the power d of the control function u on $[0,t]$. The variable v is sometimes called *fuel* or *energy* and $v\mapsto\Psi(t,x,v)$ is usually assumed monotone nondecreasing for every (t,x) (see, e.g., [20, 21]). The integer $d\ge1$ will be called the *degree* of the control system. Problem $(\mathscr P^*)$ is referred to as the *original problem* and we call a process (T,u,a,y,v) for $(\mathscr P^*)$ an *original process.* We say that (T,u,a,y,v) is *feasible* if $h(t,y(t))\le0$ for all $t\in[0,T]$, $v(T)\le K$, and $(T,y(T))\in\mathscr T^*$.

Throughout this section, we shall consider the following structural hypothesis:

(H5) *The functions $f:\mathbb R^{1+n}\times A\to\mathbb R^n$, $g^k_{j_1\dots j_k}:\mathbb R^{1+n}\to\mathbb R^n$ are continuous, all $g^k_{j_1\dots j_k}$ are locally Lipschitz continuous, and $f(\cdot,\cdot,a)$ is locally Lipschitz continuous uniformly w.r.t. $a\in A$. Furthermore, the constraint function $h:\mathbb R^{1+n}\to\mathbb R$ is locally Lipschitz continuous.*

5.1. The impulsive extension. In order to apply the theory developed in the previous sections, we reformulate problem $(\mathscr P^*)$ and embed it into a free end-time extended problem with bounded controls. To do this, we use a compactification procedure based on a reparameterization technique, commonly adopted to obtain an impulsive extension of unbounded control problems, as generalized to polynomial systems (see, e.g., [27, 21]). Let us choose

$$W := \left\{(w^0, w) \in \mathbb{R}_{\geq 0} \times U : \quad (w^0)^d + |w|^d = 1\right\}, \quad V := \left\{(w^0, w) \in W : \quad w^0 > 0\right\}.$$

For every $S > 0$, we set $\mathscr{W}(S) := L^1([0,S]; W)$,[4] $\mathscr{V}(S) := L^1([0,S]; V)$, and $\mathscr{A}(S) := L^1([0,S]; A)$, and introduce the *space-time* or *extended problem*:[5]

$$(P_e^*) \begin{cases} \qquad\qquad \text{minimize } \ \Psi(y^0(S), y(S), \nu(S)) \\ \text{over } S > 0,\ (\omega^0, \omega, \alpha, y^0, y, \nu) \in \mathscr{W}(S) \times \mathscr{A}(S) \times W^{1,1}([0,S]; \mathbb{R}^{1+n+1}) \text{ s.t.} \\ (\dot{y}^0, \dot{y}, \dot{\nu})(s) = \Big((\omega^0)^d(s), \mathscr{F}(y^0(s), y(s), \omega^0(s), \omega(s), \alpha(s)), |\omega(s)|^d\Big) \text{ a.e.}, \\ (y^0, y, \nu)(0) = (0, \check{x}_0, 0), \\ h(y^0(s), y(s)) \leq 0 \ \ \forall\ s \in [0,S], \quad \big(y^0(S), y(S), \nu(S)\big) \in \mathscr{T}^* \times]-\infty, K], \end{cases}$$

where, for any $(t, x, w^0, w, a) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_{\geq 0} \times U \times A$, we have set

$$\mathscr{F}(t, x, w^0, w, a) := f(t, x, a)(w^0)^d + \sum_{k=1}^{d} \left(\sum_{1 \leq j_1 \leq \cdots \leq j_k \leq m} g^k_{j_1, \ldots, j_k}(t, x)\ w^{j_1} \cdots w^{j_k}\, (w^0)^{d-k} \right). \tag{5.1}$$

Adopting notation and terminology of section 4, a process $(S, \omega^0, \omega, \alpha, y^0, y, \nu)$ of problem (P_e^*) is referred to as an *extended* process and it is *feasible* if $h(y^0(s), y(s)) \leq 0$ for all $s \in [0,S]$ and $\big(y^0(S), y(S), \nu(S)\big) \in \mathscr{T}^* \times]-\infty, K]$. When $\omega^0 > 0$ almost everywhere, namely $(\omega^0, \omega) \in \mathscr{V}(S)$, $(S, \omega^0, \omega, \alpha, y^0, y, \nu)$ is called a *strict sense* process. The problem of minimizing $\Psi(y^0(S), y(S))$ over feasible strict sense processes is still denoted by (P^*).

The associated *relaxed problem* is

$$(P_r^*) \begin{cases} \qquad\qquad \text{minimize } \ \Psi(y^0(S), y(S), \nu(S)) \\ \text{over } S > 0, \text{ controls } (\underline{\omega}^0, \underline{\omega}, \underline{\alpha}, \lambda) \in \mathscr{W}^{3+n}(S) \times \mathscr{A}^{3+n}(S) \times \Lambda_{2+n}(S), \text{ and} \\ \text{ trajectories } (y^0, y, \nu) \in W^{1,1}([0,S]; \mathbb{R}^{1+n+1}) \text{ s.t.} \\ (\dot{y}^0, \dot{y}, \dot{\nu})(s) = \displaystyle\sum_{k=0}^{2+n} \lambda^k(s) \Big((\omega^{0,k})^d(s), \mathscr{F}(y^0(s), y(s), \omega^{0,k}(s), \omega^k(s), \alpha^k(s)), \left|\omega^k(s)\right|^d\Big), \\ (y^0, y, \nu)(0) = (0, \check{x}_0, 0), \\ h(s, y(s)) \leq 0 \quad \forall s \in [0,S], \qquad (S, y(S)) \in \mathscr{T}^*. \end{cases}$$

A process $(S, \underline{\omega}^0, \underline{\omega}, \underline{\alpha}, \lambda, y^0, y, \nu)$ for (P_r^*) is referred to as a *relaxed* process. We will use Γ^*, Γ_e^*, Γ_r^* to denote the sets of feasible strict sense, feasible extended, and feasible relaxed processes, respectively.

The original problem $(\mathscr{P}^*)$ can be identified with problem (P^*), as established by the following lemma, an immediate consequence of the chain rule.

Lemma 5.1 (embedding). *Assume hypothesis* (H5). Then the map

$$\mathscr{I} : \{(T, u, \alpha, y, v), \text{ original processes}\} \to \{(S, \omega^0, \omega, \alpha, y^0, y, \nu), \text{ extended processes}\}$$

[4] The controls $(\omega^0, \omega) \in \mathscr{W}(S)$ actually belong to $L^\infty \cap L^1$, since W is compact.

[5] The original time t coincides now with the state component y^0, while s is the new "pseudotime" variable.

defined as

$$\mathscr{I}(T, u, \underset{\sim}{a}, \underset{\sim}{y}, v) := (S, \omega^0, \omega, \alpha, \zeta, y^0, y, \nu),$$

where, setting $\sigma(t) := t + v(t)$ for all $t \in [0, T]$,

$$S := \sigma(T), \quad (y^0, y, \nu)(s) := (id, \underset{\sim}{y}, v) \circ \sigma^{-1}(s) \ \ \forall s \in [0, S],$$
$$(\omega^0, \omega)(s) := (1 + |u|^d)^{-\frac{1}{d}} \ (1, u) \circ \sigma^{-1}(s), \quad \alpha(s) := \underset{\sim}{a} \circ \sigma^{-1}(s) \ \text{ a.e. } s \in [0, S],^6$$

is injective and has as image the subset of strict sense processes. Moreover, $\mathscr{I}$ maps any *feasible* original process into a *feasible* strict sense process, with the same cost.

The extended problem (P_e^*) consists thus in considering processes $(S, \omega^0, \omega, \alpha, y^0, y, \nu)$, where ω^0 may be zero on nondegenerate subintervals of $[0, S]$. On these intervals, the time variable $t = y^0$ is constant—i.e., the time stops—while the state variable y evolves, according to $\mathscr{F}(t, y, 0, \omega, \alpha) = \sum_{1 \le j_1 \le \cdots \le j_d \le m} g^d_{j_1,\ldots,j_d}(t, y)\ \omega^{j_1} \cdots \omega^{j_d}$, which can be called *fast dynamics*. For this reason, problem (P_e^*) is often referred to as the *impulsive extension* of the original problem $(\mathscr{P}^*)$ (more details on polynomial impulsive problems can be found in [27, 21] and references therein).

Let us introduce the *unmaximized Hamiltonian* H, defined by

$$H(t, x, p_0, p, \pi, \omega^0, \omega, a) := p_0(\omega^0)^d + p \cdot \mathscr{F}(t, x, w^0, w, a) + \pi|\omega|^d$$

for all $(t, x, p_0, p, \pi, \omega^0, \omega, a) \in \mathbb{R}^{1+n+1+n+1} \times W \times A$. The concepts of *extremal* and *nondegenerate extremal* read now as follows.

Definition 5.2 (extremal and nondegenerate extremal). *Assume* (H5) *and let* $\bar{z} := (\bar{S}, \underline{\bar{\omega}}^0, \underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{y}^0, \bar{y}, \bar{\nu})$ *be a feasible relaxed process. Given a cost function* Ψ *which is Lipschitz continuous on a neighborhood of* $(\bar{y}^0(\bar{S}), \bar{y}(S), \bar{\nu}(\bar{S}))$*, we say that* $\bar{z}$ *is a* Ψ-extremal *if there exist a path* $(p_0, p) \in W^{1,1}([0, \bar{S}]; \mathbb{R} \times \mathbb{R}^n)$*,* $\gamma \ge 0$*,* $\pi \le 0$*,* $\mu \in NBV^+([0, \bar{S}]; \mathbb{R})$*,* $(m_0, m) : [0, \bar{S}] \to \mathbb{R}^{1+n}$ *Borel measurable and* μ*-integrable functions, verifying the following conditions:*

(5.2) $\|p_0\|_{L^\infty} + \|p\|_{L^\infty} + \|\mu\|_{TV} + \gamma \neq 0;$

for a.e. $s \in [0, S]$*, one has*

(5.3) $$(-\dot{p}_0, -\dot{p})(s) \in \sum_{k=0}^{2+n} \bar{\lambda}^k(s) \text{co}\ \partial_{t,x}\ H\Big(\bar{y}^0(s), \bar{y}(s), q_0(s), q(s), \pi, \bar{\omega}^{0,k}(s), \bar{\omega}^k(s), \bar{\alpha}^k(s)\Big);$$

(5.4) $$\big(-q_0(\bar{S}), -q(\bar{S}), -\pi\big) \in \gamma\partial\Psi\left(\bar{y}^0(\bar{S}), \bar{y}(\bar{S}), \bar{\nu}(\bar{S})\right) + N_{\mathscr{T}^* \times]-\infty, K]}\left(\bar{y}^0(\bar{S}), \bar{y}(\bar{S}), \bar{\nu}(\bar{S})\right);$$

for every $k = 0, \ldots 2 + n$*, for a.e.* $s \in [0, S]$*, one has*

(5.5) $$H\Big(\bar{y}^0(s), \bar{y}(s), q_0(s), q(s), \pi, \bar{\omega}^{0,k}(s), \bar{\omega}^k(s), \bar{\alpha}^k(s)\Big) = \max_{(\omega^0, \omega, a) \in W \times A} H\Big(\bar{y}^0(s), \bar{y}(s), q_0(s), q(s), \pi, \omega^0, \omega, a\Big) = 0;$$

$$(m_0, m)(s) \in \partial^>_{t,x}\, h\left(\bar{y}^0(s), \bar{y}(s)\right)\ \mu\text{-a.e.}; \quad spt(\mu) \subseteq \{s \in [0, \bar{S}] \,:\, h\left(\bar{y}^0(s), \bar{y}(s)\right) = 0\},$$

where $$(q_0, q)(s) := \begin{cases} (p_0, p)(s) + \int_{[0,s[} (m_0, m)(\tau)\mu(d\tau), & s \in [0, \bar{S}[, \\ (p_0, p)(\bar{S}) + \int_{[0,\bar{S}]} (m_0, m)(\tau)\mu(d\tau), & s = \bar{S}. \end{cases}$$

We will call a Ψ*-extremal* normal *if all* $(p_0, p, \pi, \gamma, \mu, m_0, m)$ *as above have* $\gamma > 0$*, and* abnormal *when it is not normal.*

[6] Since every L^d-equivalence class contains Borel measurable representatives, we are tacitly assuming that all L^d-maps we are considering are Borel measurable.

Given a Ψ*-extremal* $\bar{z}$ *we call* nondegenerate multipliers *all* $(p_0, p, \pi, \gamma, \mu, m_0, m)$ *and* (q_0, q) *as above, that also verify*

$$\mu(]0, S]) + \|q_0\|_{L^\infty} + \|q\|_{L^\infty} + \gamma \neq 0. \tag{5.6}$$

If $\gamma\partial_v\Psi\left(\bar{y}^0(\bar{S}), \bar{y}(\bar{S}), \bar{\nu}(\bar{S})\right) = 0$ *and* $\bar{\nu}(\bar{S}) < K$*, then* $\pi = 0$*. Furthermore, if* $\bar{y}^0(0) < \bar{y}^0(\bar{S})$*,* (5.2) *(resp.,* (5.6)*) can be strengthened to*

$$\|p\|_{L^\infty} + \|\mu\|_{TV} + \gamma \neq 0 \quad [\mu(]0, S]) + \|q\|_{L^\infty} + \gamma \neq 0]. \tag{5.7}$$

We will call a Ψ*-extremal* nondegenerate normal *if all* $(p_0, p, \pi, \gamma, \mu, m_0, m)$ *and* (q_0, q) *as above and verifying* (5.6) *have* $\gamma > 0$*, and* nondegenerate abnormal *when it is not nondegenerate normal.*

5.2. Main results. Given a feasible relaxed process $\bar{z} := (\bar{S}, \underline{\bar{\omega}}^0, \underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{y}^0, \bar{y}, \bar{\nu})$, let us consider the obvious corresponding version of hypothesis (H4) (on $[0, \bar{S}]$), which we still denote (H4) for simplicity. From Theorem 4.1 follow gap-abnormality relations for the original problem with respect to its relaxed impulsive extension (P_r^*).

THEOREM 5.1. *Let* $\bar{z} := (\bar{S}, \underline{\bar{\omega}}^0, \underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{y}^0, \bar{y}, \bar{\nu})$ *be a feasible process for the relaxed impulsive extension* (P_r^*)*, and suppose that at* $\bar{z}$ *there is a local infimum gap. If hypothesis* (H5) *is verified, then* $\bar{z}$ *is an abnormal extremal. If hypothesis* (H4) *is also satisfied, then* $\bar{z}$ *is a nondegenerate abnormal extremal.*

Proof. It is sufficient to show that hypothesis (H5) allows the application of Theorem 4.1(i). In fact, with regard to the conditions on the multipliers which are peculiar of the impulsive problem (the absence of the multiplier π in the nontriviality conditions (5.2), (5.6), and the strengthened versions in (5.7)), these can be proved exactly as in the control-affine case (see [23, Thm. 3.1], [13, Thm. 1.1]). To this aim, we observe that hypothesis (H1) is trivially verified, by choosing, e.g., $V_i := \{(w^0, w) \in V : \quad w^0 \geq \frac{1}{i+1}\}$ for every $i \in \mathbb{N}$, while (H5) yields (H2)* directly. Condition (H3)* easily follows from (H5), taking into account the control-polynomial structure of the dynamics as regards point (ii). ∎

As corollaries, we have the following.

THEOREM 5.2. *Assume hypothesis* (H5) *and let* Ψ *be locally Lipschitz continuous.*
(i) *Let* $\bar{z}$ *be a local* Ψ*-minimizer for* (P_e^*) *or* (P_r^*) *which is a normal* Ψ*-extremal. Then, at* $\bar{z}$ *there is no local infimum gap. If* $\bar{z}$ *is a* Ψ*-minimizer, then there is no infimum gap.*
(ii) *Let* $\bar{z}$ *be a local* Ψ*-minimizer for* (P_e^*) *or* (P_r^*)*, at which condition* (H4) *is verified and which is a nondegenerate normal* Ψ*-extremal. Then, at* $\bar{z}$ *there is no local infimum gap. If* $\bar{z}$ *is a* Ψ*-minimizer, then there is no infimum gap.*

THEOREM 5.3. *Assume hypothesis* (H5)*. Then, either*
(i) $\bar{z}$ *is not isolated in* Γ^* *or*
(ii) $\bar{z}$ *is an abnormal extremal, in fact, a nondegenerate abnormal extremal, if condition* (H4) *is verified.*

Remark 5.1. The above results extend and complement the results previously obtained in [13]. First of all, in this paper we consider control systems with polynomial dependence on the unbounded control, whereas in [13] we have only dealt with the case of control-affine systems. This is a substantial difference, because this extension allows us to treat higher order, not only first order, impulse inputs, which occur, for instance, in some applications to Lagrangian mechanics (see, e.g., [4, 6]).

Furthermore, in [13] convex relaxation of the extended problem is not considered. Notice that in case $d = 1$ the set of velocities of the extended system is in general not convex, unless the drift f is independent of a and the cone U is convex (see, e.g., [19]). Hence, generally the extended problem does not admit a minimum.

Finally, in the present paper we prove a nondegenerate normality test for no gap in the case of fixed initial point, which had been left as an open question in [13].

Remark 5.2. When the feasible reference process $\bar z$ belongs to the subclass of extended processes, namely $\bar z := (\bar S, \bar\omega^0, \bar\omega, \bar\alpha, \bar y^0, \bar y, \bar\nu) \in \Gamma_e^*$, in the previous theorems condition (H4) can be replaced by the following, simpler condition:[7]

(H6) If $(0, \check x_0) \in \partial\Omega$, there are some $\tilde\delta > 0$, $\bar s \in]0, \bar S]$, some sequence of strict sense processes $(\tilde\omega_i^0, \tilde\omega_i, \tilde\alpha_i, \tilde y_i^0, \tilde y_i, \tilde\nu_i)_i \subset \mathscr{V}(\bar s) \times \mathscr{A}(\bar s) \times W^{1,1}([0,\bar s]; \mathbb{R}^{1+n+1})$, some sequences $(\hat\omega_i^0, \hat\omega_i, \hat\alpha_i)_i \subset \mathscr{W}(\bar s) \times \mathscr{A}(\bar s)$, and $(\tilde r_i)_i \subset L^1([0,\bar s]; \mathbb{R}_{\ge 0})$ with $\lim_{i\to+\infty} \|\tilde r_i\|_{L^1([0,\bar s])} = 0$, such that the following properties (i)–(iii) are verified.

(i) For every i, one has

$$h(\tilde y_i^0(s), \tilde y_i(s)) \le 0 \qquad \forall s \in [0, \bar s];$$

(ii) for every i, there is a Lebesgue measurable subset $\tilde E_i \subset [0, \bar s]$ such that

$$(\tilde\omega_i^0, \tilde\omega_i)(s) \in (\bar\omega^0, \bar\omega)(s) + \tilde r_i(s)\mathbb{B}, \quad \tilde\alpha_i(s) = \bar\alpha(s), \quad \text{a.e. } s \in \tilde E_i;$$
$$\lim_{i\to+\infty} \ell(\tilde E_i) = \bar s;$$

(iii) for every i, for all $(\zeta_0, \zeta) \in \partial^* h(0, \check x_0)$, and for a.e. $s \in [0, \bar s]$, one has

$$\begin{aligned}\zeta_0 \cdot &[(\hat\omega_i^0(s))^d - (\tilde\omega_i^0(s))^d] \\ &+\zeta \cdot \left[\mathscr{F}(0, \check x_0, (\hat\omega_i^0, \hat\omega_i, \hat\alpha_i)(s)) - \mathscr{F}(0, \check x_0, (\tilde\omega_i^0, \tilde\omega_i, \tilde\alpha_i)(s))\right] \le -\tilde\delta.\end{aligned}$$

LEMMA 5.3. *Assume hypothesis* (H5) *and let* $\bar z := (\bar S, \bar\omega^0, \bar\omega, \bar\alpha, \bar y^0, \bar y, \bar\nu) \in \Gamma_e^*$. *Then, condition* (H6) *implies condition* (H4).

Proof. To prove that (H5)–(H6) imply condition (H4), let us consider a sequence $\delta_i \downarrow 0$ and for every i define the strict sense control

$$(\check\omega_i^0, \check\omega_i, \check\alpha_i)(s) := \begin{cases} (\tilde\omega_i^0, \tilde\omega_i, \tilde\alpha_i)(s) & \text{if } s \in [0, \bar s], \\ (\bar\omega^0, \bar\omega, \bar\alpha)(s) & \text{if } s \in]\bar s, \bar S] \text{ and } \bar\omega^0(s) > 0, \\ (\delta_i, \sqrt[d]{1-\delta_i^d}\,\bar\omega(s), \bar\alpha(s)) & \text{if } s \in]\bar s, \bar S] \text{ and } \bar\omega^0(s) = 0, \end{cases}$$

where $(\tilde\omega_i^0, \tilde\omega_i, \tilde\alpha_i)$ is as in (H6). By identifying, for every i, the strict sense process, say, $(\check\omega_i^0, \check\omega_i, \check\alpha_i, \check y_i^0, \check y_i, \check\nu_i)$, of (P^*) corresponding to $(\check\omega_i^0, \check\omega_i, \check\alpha_i)$ with a relaxed process—as we have been doing since the introduction—we derive that conditions (H4)(ii) and (H4)(iii) on $\check E_i := \tilde E_i \cup]\bar s, \bar S] \subset [0, \bar S]$ are verified. Condition (H4)(i) follows from well-known continuity properties of the input-output map, associated with the control system in (P^*). Finally, in view of Remark 3.1(3), (H6)(iii) implies (H4)(iv), although the controls $(\hat\omega_i^0, \hat\omega_i, \hat\alpha_i)$ are extended, not necessarily strict sense, controls. ∎

In some situations, hypothesis (H6) simplifies considerably.

[7] We recall that $\Omega = \{(t,x) : \ h(t,x) \le 0\}$.

LEMMA 5.4. *Assume* (H5). *Let* $(0,\check{x}_0)\in\partial\Omega$ *and let* $\bar{z}:=(\bar{S},\bar{\omega}^0,\bar{\omega},\bar{\alpha},\bar{y}^0,\bar{y},\bar{\nu})$ *be a feasible extended process. If there are some* $\tilde{\delta}>0$, $\bar{s}\in]0,\bar{S}]$, *and an extended control* $(\hat{\omega}^0,\hat{\omega},\hat{\alpha})\in\mathscr{W}(\bar{s})\times\mathscr{A}(\bar{s})$ *such that, for all* $(\zeta_0,\zeta)\in\partial^*h(0,\check{x}_0)$ *and for a.e.* $s\in[0,\bar{s}]$,

$$\begin{aligned}\zeta_0\cdot[(\hat{\omega}^0(s))^d-(\bar{\omega}^0(s))^d]\\ +\zeta\cdot\left[\mathscr{F}(0,\check{x}_0,(\hat{\omega}^0,\hat{\omega},\hat{\alpha})(s))-\mathscr{F}(0,\check{x}_0,(\bar{\omega}^0,\bar{\omega},\bar{\alpha})(s))\right]\le-\tilde{\delta},\end{aligned}\tag{5.8}$$

and either $\bar{\omega}^0>0$ *a.e. in* $[0,\bar{s}]$, *or there is some* $\tilde{\delta}_1>0$ *such that, for a.e.* $s\in[0,\bar{s}]$,

$$\sup_{(\zeta_0,\zeta)\in\partial^*h(0,\check{x}_0)}\left[\zeta_0\cdot(\bar{\omega}^0(s))^d+\zeta\cdot\mathscr{F}(0,\check{x}_0,((\bar{\omega}^0,\bar{\omega}),\bar{\alpha})(s))\right]\le-\tilde{\delta}_1,\tag{5.9}$$

then condition (H6) *is satisfied.*

Proof of Lemma 5.4. Let us first suppose that $\bar{\omega}^0>0$ a.e. in $[0,\bar{s}]$. Then, conditions (H6)(i), (ii) are verified by choosing, for every i, $(\tilde{\omega}^0_i,\tilde{\omega}_i,\tilde{\alpha}_i)=(\bar{\omega}^0,\bar{\omega},\bar{\alpha})$, while (H6)(iii) follows directly from (5.8), by taking $(\hat{\omega}^0_i,\hat{\omega}_i,\hat{\alpha}_i)\equiv(\hat{\omega}^0,\hat{\omega},\hat{\alpha})$ for every i. If instead (5.9) is assumed, let us consider a sequence $\delta_i\downarrow 0$ and for every i, let us set

$$(\tilde{\omega}^0_i,\tilde{\omega}_i,\tilde{\alpha}_i)(s):=\begin{cases}(\bar{\omega}^0,\bar{\omega},\bar{\alpha})(s) & \text{if } \bar{\omega}^0(s)>0,\\ (\delta_i,\,\sqrt[d]{1-\delta_i^d}\,\bar{\omega}(s),\bar{\alpha}(s)) & \text{if } \bar{\omega}^0(s)=0,\end{cases}$$

for a.e. $s\in[0,\bar{s}]$. Then, $(\tilde{\omega}^0_i,\tilde{\omega}_i,\tilde{\alpha}_i)\in\mathscr{V}(\bar{s})\times\mathscr{A}(\bar{s})$ verifies (H6)(ii) with $\tilde{E}_i=[0,\bar{s}]$ and $\tilde{r}_i\equiv\delta_i$. Let $(\tilde{y}^0_i,\tilde{y}_i,\tilde{\nu}_i)$ be the corresponding solution of the extended control system in (P_e^*) with initial condition $(\tilde{y}^0_i,\tilde{y}_i,\tilde{\nu}_i)(0)=(0,\check{x}_0,0)$. From condition (5.9), using the Lebourg mean value theorem [29, Thm. 4.5.3] to estimate $h(\tilde{y}^0_i(s),\tilde{y}_i(s))-h(0,\check{x}_0)$, one can derive that $h(\tilde{y}^0_i(s),\tilde{y}_i(s))\le 0$ for all $s\in[0,\bar{s}]$, for every i large enough, so proving the validity of (H6)(i). Finally, from condition (5.8) (by adding and subtracting $\zeta_0\cdot(\tilde{\omega}^0_i(s))^d+\zeta\cdot\mathscr{F}(0,\check{x}_0,(\tilde{\omega}^0_i,\tilde{\omega}_i,\tilde{\alpha}_i)(s))$ and by taking $(\hat{\omega}^0_i,\hat{\omega}_i,\hat{\alpha}_i)=(\hat{\omega}^0,\hat{\omega},\hat{\alpha})$), we get condition (H6)(iii), possibly reducing $\tilde{\delta}$, for all i large enough. ∎

5.3. An example. The following example illustrates how Theorem 5.2(ii), in which the normality hypothesis is understood in its nondegenerate form, can be used to exclude the occurence of an infimum gap. Of course, Theorem 5.2(i) is of no use here, because for problems of this nature, in which the initial state lies in the boundary of the state constraint set, extremals are never normal in the sense of Definition 5.2.

Example 5.1. Consider the problem

$$\left\{\begin{array}{l}\qquad\qquad\text{minimize}\quad -y(1)\\ \text{over } (u,y,v)\in L^1([0,1];\mathbb{R}^2)\times W^{1,1}([0,1];\mathbb{R}^3\times\mathbb{R}) \text{ satisfying}\\ (\dot{y},\dot{v})(t)=\Big(f(y(t))+g_1(y(t))\,u^1(t)+g_2(y(t))\,u^2(t),\ |u(t)|\Big),\\ (y,v)(0)=((1,0,0),0),\\ y(t)\in\Omega\ \ \forall t\in[0,1],\ v(1)\le 2,\ y(1)\in\mathscr{T},\end{array}\right.\tag{5.10}$$

in which $\Omega:=[-1,1]^3$, $\mathscr{T}:=[-1,0]\times[0,1]^2$, and

$$g_1(x):=\begin{pmatrix}1\\0\\0\end{pmatrix},\quad g_2(x):=\begin{pmatrix}0\\-1\\-x^1\end{pmatrix},\quad f(x):=\begin{pmatrix}0\\x^2x^3\\0\end{pmatrix}\quad\forall x\in\mathbb{R}^3\,.$$

Here, $W = \{(\omega^0, w) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^2 : \quad w^0 + |w| = 1\}$, $V = \{(\omega^0, w) \in W : \quad w^0 > 0\}$, and the associated extended problem is

$$\begin{cases} \qquad\qquad \text{minimize} \quad -y^1(S) \\ \text{over } S > 0,\ (\omega^0, \omega^1, \omega^2, y^0, y, \nu) \in \mathscr{W}(S) \times W^{1,1}([0, S]; \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}) \quad \text{satisfying} \\ (\dot{y}^0, \dot{y}, \dot{\nu})(s) = \Big(\omega^0(s),\ f(y(s))\omega^0(s) + g_1(y(s))\,\omega^1(s) + g_2(y(s))\,\omega^2(s),\ |\omega(s)|\Big), \\ (y^0, y, \nu)(0) = (0, (1, 0, 0), 0), \\ y(s) \in \Omega \ \ \forall s \in [0, S], \quad (y^0(S), y(S), \nu(S)) \in \{1\} \times \mathscr{T} \times]-\infty, 2]. \end{cases}$$

As it is easy to see, an extended minimizer is given by the following feasible extended process $\bar{z} := (\bar{S}, \bar{\omega}^0, \bar{\omega}, \bar{y}^0, \bar{y}, \bar{\nu})$, where

$$\bar{S} = 2, \qquad (\bar{\omega}^0, \bar{\omega}) = (\bar{\omega}^0, \bar{\omega}^1, \bar{\omega}^2) = (1, 0, 0)\chi_{[0,1]} + (0, -1, 0)\chi_{]1,2]},$$
$$(\bar{y}^0, \bar{y}, \bar{\nu}) = (\bar{y}^0, \bar{y}^1, \bar{y}^2, \bar{y}^3, \bar{\nu}) = (s, 1, 0, 0, 0)\chi_{[0,1]} + (1, 2 - s, 0, 0, s - 1)\chi_{[1,2]}.$$

From the maximum principle [13, Thm. 1.1], $\bar{z}$ is a Ψ-extremal accordingly to Definition 5.2. Hence, there exist a set of multipliers $(p_0, p, \pi, \gamma, \mu)$ and functions (m_0, m) with $\pi = 0$, since $\nabla_v \Psi \equiv 0$ and $\bar{\nu}(2) = 1 < 2$, $m_0 \equiv 0$, as the state constraint does not depend on time, and $\mu([0, 2]) = \mu([0, 1])$. Moreover, for every $s \in [0, 1]$ the fact that $\bar{y}(s) \in \Omega$ is equivalent to $h(\bar{y}(s)) \leq 0$, with $h(x^1, x^2, x^3) := x^1 - 1$, so that the condition $m(s) \in \partial_x^> h(\bar{y}(0))$ μ-a.e. yields $m(s) = (1, 0, 0)$ μ-a.e. in $[0, 1]$. By the adjoint equation, it follows that the path $(p_0, p) = (p_0, p_1, p_2, p_3) \equiv (\bar{p}_0, \bar{p}_1, \bar{p}_2, \bar{p}_3)$ is constant. From the transversality condition

$$-(q_0, q_1, q_2, q_3)(2) \in \gamma\{(0, -1, 0, 0)\} + \mathbb{R} \times N_{\mathscr{T}}(0, 0, 0),$$

where $q_0 \equiv \bar{p}_0$, and $q(s) = (\bar{p}_1 + \mu([0, 1]), \bar{p}_2, \bar{p}_3)$ for all $s \in]1, 2]$, we derive that $\bar{p}_0$, $\bar{p}_1 \in \mathbb{R}$, $\bar{p}_2$, $\bar{p}_3 \geq 0$, and $q_1(2) = \bar{p}_1 + \mu([0, 1]) = \gamma - \alpha_1$ with $\alpha_1 \geq 0$. The maximality condition in $]1, 2]$ implies that $\bar{p}_2 = \bar{p}_3 = 0$. In particular, from the relations

$$\max_{w^1 \in [-1,1]} \left\{q_1(s)w^1\right\} \chi_{[0,1]}(s) = \bar{p}_0 \chi_{[0,1]}(s) = 0, \quad -q_1(s)\chi_{]1,2]}(s) = 0,$$

we also deduce that $\bar{p}_0 = 0$, $q_1(s) = \bar{p}_1 + \mu([0, s[) = 0$ for a.e. $s \in [0, 1[$, and $q_1(s) = \bar{p}_1 + \mu([0, 1]) = \gamma - \alpha_1 = 0$ for every $s \in]1, 2]$. In particular, $q(s) = 0$ for a.e. $s \in [0, 2]$, $\mu([0, s[) = -\bar{p}_1$ for a.e. $s \in [0, 1]$ implies that ($\bar{p}_1 \leq 0$ and) $\mu = -\bar{p}_1 \mu(\{0\})$, while the last relation yields that $\gamma = \alpha_1$.

It is immediate to see that the set of degenerate multipliers (p_0, p, γ, μ) with $p_0 = p_2 = p_3 = 0$, $p_1 = -1$, $\mu = \delta_{\{0\}}$, and $\gamma = 0$ meets all the conditions of the maximum principle. So, $\bar{z}$ is an abnormal extremal. However, since $\bar{w}^0 > 0$ for a.e. $s \in [0, 1]$ and the control $(\hat{\omega}^0, \hat{\omega}) = (\hat{\omega}^0, \hat{\omega}^1, \hat{\omega}^2) \equiv (0, -1, 0)$ verifies (5.8), from Lemma 5.4 it follows that condition (H6) is satisfied. Therefore, in view of Theorem 5.2(ii), to deduce that there is no infimum gap it is enough to observe that $\bar{z}$ is nondegenerate normal, namely, that $\gamma \neq 0$ for all sets of multipliers as above, which in addition verify

$$\mu(]0, 2]) + \|q\|_{L^\infty} + \gamma \neq 0.$$

This is true, since the previous calculations imply that $\|q\|_{L^\infty} = 0$ and $\mu(]0, 2]) = 0$.

6. Proofs of Theorems 2.1 and 3.1. Preliminarily, let us observe that, since the proofs involve only relaxed and extended processes with trajectories close to the reference trajectory $(\bar{\xi}, \bar{y})$ and the controls assume values in compact sets, using standard cut-off techniques we can assume that all hypotheses (H2)–(H3)$'$ are satisfied not only in Σ_θ, but in the whole space $\mathbb{R}^{1+n}$. Hence, for any $(\underline{\omega}, \underline{\alpha}, \lambda) \in \mathscr{W}^{1+n} \times \mathscr{A}^{1+n} \times \Lambda_n$ there is a unique solution $(\xi, y)[\underline{\omega}, \underline{\alpha}, \lambda]$ to (3.3) defined on $[0, S]$. Similarly, for any $(\omega, \alpha) \in \mathscr{W} \times \mathscr{A}$, we will write $y[\omega, \alpha]$ to denote the corresponding solution to (1.1).

6.1. Proof of Theorem 3.1. The proof is divided into several steps in which successive sequences of optimization problems are introduced that have as admissible controls only strict sense controls, and costs that measure how much a process violates the constraints. Using the Ekeland principle, minimizers are then built for these problems, which converge to the initial isolated process. Furthermore, applying a maximum principle to these approximate problems with reference to the abovementioned minimizers, we obtain in the limit a set of multipliers with $\gamma = 0$ *and verifying the strengthened nontriviality conditions* (3.1) for the relaxed problem with reference to the isolated process $(\bar{\underline{\omega}}, \bar{\underline{\alpha}}, \bar{\lambda}, \bar{y})$.

Step 1. Define the function $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}$, given by

$$\Phi\,(x, z) := d_{\mathscr{T}}(x) \vee z$$

and for any $y \in W^{1,1}([0, S]; \mathbb{R}^n)$, introduce the payoff

$$\mathscr{J}(y) := \Phi\Big(y(S),\ \max_{s \in [0,S]} h(s, y(s))\Big).$$

Fix a sequence $(\varepsilon_i)_i$ such that $\varepsilon_i \downarrow 0$. Let $(\tilde{\omega}_i, \tilde{\alpha}_i, \tilde{\lambda}_i)_i$ be a control sequence as in hypothesis (H4), such that, eventually passing to a subsequence, for every i, the corresponding trajectory $(\tilde{\xi}_i, \tilde{y}_i)$ of (3.3) verifies

$$\|(\tilde{\xi}_i, \tilde{y}_i) - (\bar{\xi}, \bar{y})\|_{L^\infty} \le \varepsilon_i. \tag{6.1}$$

For every i, let $\rho_i \ge 0$ verify

$$\rho_i^4 = \sup\left\{\ \mathscr{J}(y) :\quad (\omega, \alpha, y) \in \Gamma, \quad \|y - \bar{y}\|_{L^\infty} \le 2\varepsilon_i\ \right\}.$$

By the Lipschitz continuity of Φ, it follows that $\lim_{i\to+\infty} \rho_i^4 = 0$. Moreover, $\rho_i > 0$ for every i large enough, since $\bar{z}$ is an isolated process by Proposition 2.1.

In the following, as it is clearly not restrictive, we will always assume that the properties valid from a certain index onward apply to each index $i \in \mathbb{N}$. By well-known continuity properties of the input-output map $(\omega, \alpha) \mapsto y[\omega, \alpha]$, for every ε_i there exists $\delta_i > 0$ such that if $\|\omega - \tilde{\omega}_i\|_{L^1} \le \delta_i$, then $\|y[\omega, \tilde{\alpha}_i] - \tilde{y}_i\|_{L^\infty} \le \varepsilon_i$. According to hypothesis (H1) and Remark 2.1, for any i there exists an element of the sequence $(V_j)_j$, which we denote by V_{δ_i}, and some $\mathring{\omega}_i \in \mathscr{V}_{\delta_i} := L^1([0, S]; V_{\delta_i})$ such that $\|\mathring{\omega}_i - \tilde{\omega}_i\|_{L^1} \le \delta_i$. In particular, if we define

$$\check{\omega}_i(s) := \begin{cases} \tilde{\omega}_i(s) & \text{a.e. } s \in [0, \rho_i], \\ \mathring{\omega}_i(s) & \text{a.e. } s \in]\rho_i, S], \end{cases} \qquad (\check{\alpha}_i(s), \check{\lambda}_i(s)) := (\tilde{\alpha}_i(s), \tilde{\lambda}_i(s)) \ \ \text{a.e. } s \in [0, S],$$

and $(\check{\xi}_i, \check{y}_i) := (\xi, y)[\check{\omega}_i, \dots, \check{\omega}_i, \check{\alpha}_i, \dots, \check{\alpha}_i, \check{\lambda}_i]$, then $\|\check{\omega}_i - \tilde{\omega}_i\|_{L^1} \le \delta_i$ and $\check{y}_i$ is a strict sense trajectory such that $\|(\check{\xi}_i, \check{y}_i) - (\tilde{\xi}_i, \tilde{y}_i)\|_{L^\infty} \le \varepsilon_i$. From (6.1) it follows that

$$\|(\check{\xi}_i, \check{y}_i) - (\bar{\xi}, \bar{y})\|_{L^\infty} \le 2\varepsilon_i. \tag{6.2}$$

Hence, by the very definition of ρ_i we deduce that for any i the process $\check{z}_i := (\check{\omega}_i, \check{\alpha}_i, \check{\lambda}_i, \check{\eta}_i, \check{\xi}_i, \check{y}_i)$, where $\check{\eta}_i \equiv 0$, is a ρ_i^4-minimizer for the optimal control problem:

$$(\hat{P}_i)\begin{cases} \qquad\qquad\qquad\qquad \text{Minimize} \quad \mathscr{J}(y) \\ \text{over the set of control } (\omega,\alpha,\lambda,\eta)\in\mathscr{V}_{\delta_i}\times\mathscr{A}\times\Lambda^1_n\times L^1([0,S];\{0,1\}), \\ \text{and trajectories } \ (\xi,y)\in W^{1,1}([0,S];\mathbb{R}^{1+n}\times\mathbb{R}^n), \text{ satisfying} \\ \dot{\xi}(s)=\lambda(s) \qquad \text{a.e. } s\in[0,S], \\ \dot{y}(s)=\mathscr{F}(s,y,\tilde{\omega}_i,\tilde{\alpha}_i)+\eta(s)[\mathscr{F}(s,y,\hat{\omega}_i,\hat{\alpha}_i)-\mathscr{F}(s,y,\tilde{\omega}_i,\tilde{\alpha}_i)] \text{ a.e. } s\in[0,\rho_i], \\ \dot{y}(s)=\mathscr{F}(s,y(s),\omega(s),\alpha(s)) \qquad \text{a.e. } s\in]\rho_i,S], \\ (\xi,y)(0)=(0,\check{x}_0), \end{cases}$$

where $(\hat{\omega}_i,\hat{\alpha}_i)$ is as in hypothesis (H4). We call an element $(\omega,\alpha,\lambda,\eta,\xi,y)$ verifying the constraints in $(\hat{P}_i)$ a *process for problem* $(\hat{P}_i)$ and use Γ_i to denote the set of such processes. By introducing, for every $(\omega',\alpha',\lambda',\eta',\xi',y')$, $(\omega,\alpha,\lambda,\eta,\xi,y)\in\Gamma_i$, the distance

$$\begin{aligned}(6.3)\qquad &\mathbf{d}((\omega',\alpha',\lambda',\eta',\xi',y'),(\omega,\alpha,\lambda,\eta,\xi,y)) \\ &\qquad := \|\omega'-\omega\|_{L^1([0,S])}+\ell\{s\in[0,S]:\ (\alpha',\lambda',\eta')(s)\neq(\alpha,\lambda,\eta)(s)\},\end{aligned}$$

we can make $(\Gamma_i,\mathbf{d})$ a complete metric space. Then, from Ekeland's principle it follows that there exists a process $z_i:=(\omega_i,\alpha_i,\lambda_i,\eta_i,\xi_i,y_i)\in\Gamma_i$, which is a minimizer of the optimization problem

$$(P_i)\begin{cases}\text{Minimize } \ \mathscr{J}(y)+\rho_i^2\int_0^S[|\omega(s)-\omega_i(s)|+\ell_i(s,\alpha(s),\lambda(s),\eta(s))]\,ds \\ \text{over } \ (\omega,\alpha,\lambda,\eta,\xi,y)\in\Gamma_i,\end{cases}$$

where $\ell_i(s,a,\lambda,\eta):=\chi_{\{(a,\lambda,\eta)\neq(\alpha_i(s),\lambda_i(s),\eta_i(s))\}}$ for any $(s,a,\lambda,\eta)\in[0,S]\times A\times\Lambda^1_n\times\{0,1\}$, and verifies

$$(6.4)\qquad \mathbf{d}\big((\omega_i,\alpha_i,\lambda_i,\eta_i,\xi_i,y_i),(\check{\omega}_i,\check{\alpha}_i,\check{\lambda}_i,\check{\eta}_i,\check{\xi}_i,\check{y}_i)\big)\le\rho_i^2.$$

Thus, by (6.2) and the continuity of the input-output map associated with the control system (3.3), it follows that, eventually passing to a subsequence, as $i\to+\infty$,

$$(6.5)\qquad \big\|(\xi_i,y_i)-(\bar{\xi},\bar{y})\big\|_{L^\infty}\to 0, \quad (\dot{\xi}_i,\dot{y}_i)\rightharpoonup(\dot{\bar{\xi}},\dot{\bar{y}}) \quad \text{weakly in } L^1.$$

Furthermore, hypothesis (H4) and (6.4) imply that, for every i, there exist some nonempty subset $E_i\subseteq\tilde{E}_i\subseteq[0,S]$ and some $r_i\in L^1([0,S];\mathbb{R}_{\ge 0})$ with $r_i\ge\tilde{r}_i$ ($\tilde{E}_i$, $\tilde{r}_i$ as in (H4)) such that, as $i\to+\infty$, $\ell(E_i)\to S$, $\|r_i\|_{L^1}\to 0$, and

$$(6.6)\qquad (\omega_i,\alpha_i,\lambda_i)(s)\in\bigcup_{k=0}^n\{(\bar{\omega}^k(s),\bar{\alpha}^k(s),e^k)\}+(r_i(s),0,0)\mathbb{B} \quad \text{for a.e. } s\in E_i.$$

From (6.5) and the fact that $\bar{z}$ is isolated, it follows that $\mathscr{J}(y_i)>0$ for all i, namely, at least one of the following inequalities holds true:[8]

$$(6.7)\qquad d_{\mathscr{T}}(y_i(S))>0, \qquad c_i:=\max_{s\in[0,S]}h(s,y_i(s))>0.$$

Step 2. For each $i\in\mathbb{N}$, set

$$\tilde{h}(s,x,c):=h(s,x)-c \qquad \forall(s,x,c)\in\mathbb{R}^{1+n+1}.$$

[8]Notice that, to any process $(\omega,\alpha,\lambda,\eta,\xi,y)\in\Gamma_i$ corresponds a strict sense process $(\check{\omega},\check{\alpha},\check{y})$, where $\check{y}\equiv y$ and $(\check{\omega},\check{\alpha})(s)=(\tilde{\omega}_i,\tilde{\alpha}_i)(s)+\eta(s)(\hat{\omega}_i-\tilde{\omega}_i,\hat{\alpha}_i-\tilde{\alpha}_i)(s)$ a.e. $s\in[0,\rho_i]$, $(\check{\omega},\check{\alpha})(s)=(\omega,\alpha)(s)$ a.e. $s\in]\rho_i,S]$.

The process $(z_i, c_i) = (\omega_i, \alpha_i, \lambda_i, \eta_i, \xi_i, y_i, c_i)$ turns out to be a minimizer for

$$(Q_i)\begin{cases}\text{Minimize } \Phi(y(S), c(S)) + \rho_i^2 \int_0^S [|\omega(s) - \omega_i(s)| + \ell_i(s, \alpha(s), \lambda(s), \eta(s))]\, ds\\ \text{over } (\omega, \alpha, \lambda, \eta, \xi, y) \in \Gamma_i,\ c \in W^{1,1}([0,S];\mathbb{R}),\ \text{verifying}\\ \dot c(s) = 0, \qquad \tilde h(s, y(s), c(s)) \le 0 \quad \forall s \in [0,S].\end{cases}$$

Passing eventually to a subsequence, we may suppose that

either (a) $c_i > 0$ *for each* $i \in \mathbb{N}$ or (b) $c_i \le 0$ *for each* $i \in \mathbb{N}$.

Case (a). Preliminarily, we show that, for every i, one has $h(s, y_i(s)) < c_i$ for all $s \in [0, \rho_i]$. This result is a straightforward consequence of the following lemma.

LEMMA 6.1. *For every* $i \in \mathbb{N}$*, one has* $h(s, y_i(s)) \le 0$ *for all* $s \in [0, \rho_i]$.

Proof. From a standard application of the Gronwall's lemma one can deduce that there is $\bar C > 0$ such that, for every i, one has

$$|y_i(s) - \tilde y_i(s)| \le \bar C\, \ell(s, \eta_i(\cdot)) \qquad \forall s \in]0, \rho_i], \tag{6.8}$$

where the nondecreasing map $s \mapsto \ell(s, \eta_i(\cdot))$ is as in (3.9). Fix now $i \in \mathbb{N}$. By the Lebourg mean value theorem [29, Thm. 4.5.3], for every $s \in [0, \rho_i]$ there exists $(\zeta^s_{0_i}, \zeta^s_i) \in \partial^c h(s, x_i(s))$ for some $x_i(s)$ in the segment $[y_i(s)\,,\, \tilde y_i(s)] \subseteq \mathbb{R}^n$, such that[9]

$$\begin{aligned}&h(s, y_i(s)) - h(s, \tilde y_i(s)) = \zeta^s_i \cdot (y_i(s) - \tilde y_i(s))\\ &\quad = \int_0^s \zeta^s_i \cdot [\mathscr{F}(\sigma, y_i, \tilde\omega_i, \tilde\alpha_i) - \mathscr{F}(\sigma, \tilde y_i, \tilde\omega_i, \tilde\alpha_i)]\, d\sigma\\ &\qquad\qquad + \int_0^s \eta(\sigma)\zeta^s_i \cdot [\mathscr{F}(\sigma, y_i, \hat\omega_i, \hat\alpha_i) - \mathscr{F}(\sigma, y_i, \tilde\omega_i, \tilde\alpha_i)]\, d\sigma\\ &\quad \le \int_0^s \bar C K_h K_{\mathscr{F}}\, \ell(\sigma, \eta_i(\cdot)) d\sigma - \delta\, \ell(s, \eta_i(\cdot)) \le \ell(s, \eta_i(\cdot))\left(-\delta + \bar C K_h K_{\mathscr{F}}\, s\right) \le 0,\end{aligned}$$

where the last relations follow from (3.8), (6.8), and the fact that $s \le \rho_i \downarrow 0$. Finally, condition (3.5) implies the thesis. $\square$

Our aim is now to apply the Pontryagin maximum principle to problem (Q_i) with reference to the minimizer (z_i, c_i), for which, thanks to Lemma 6.1, the constraint is inactive on $[0, \rho_i]$. By standard arguments (see the proof of [13, Thm. 2.2]) we deduce that $\partial^{>}_{t,x,c} \tilde h(s, y_i(s), c_i) = \partial^{>}_{t,x} h(s, y_i(s)) \times \{-1\}$ and that, if $(\beta_{y_i}, \beta_{c_i}) \in \partial\Phi(y_i(S), c_i(S))$, then there is some $\sigma_i^1, \sigma_i^2 \ge 0$ with $\sigma_i^1 + \sigma_i^2 = 1$, such that $\beta_{y_i} \in \sigma_i^1\ (\partial d_{\mathscr{T}}(y_i(S)) \cap \partial\mathbb{B}_n)$, $\beta_{c_i} = \sigma_i^2$, and $\sigma_i^k = 0$, $k \in \{1,2\}$, when the maximum in $d_{\mathscr{T}}(y_i(S)) \vee c_i(S)$ is strictly greater than the kth term in the maximization. Thus, the maximum principle [29, Thm. 9.3.1] yields the existence of some multipliers $(p_i, \pi_i) \in W^{1,1}([0,S];\mathbb{R}^{n+1})$ associated with (y_i, c_i), $\mu_i \in NBV^+([0,S];\mathbb{R})$, $\gamma_i \ge 0$, σ_i^1, $\sigma_i^2 \ge 0$ with $\sum_{k=1}^2 \sigma_i^k = 1$, and Borel measurable, μ_i-integrable functions $m_i : [0,S] \to \mathbb{R}^n$, such that

(i)′ $\|p_i\|_{L^\infty} + \|\mu_i\|_{TV} + \gamma_i + \|\pi_i\|_{L^\infty} = 1$;

(ii)′ $-\dot p_i(s) \in \mathrm{co}\ \partial_x\{q_i(s) \cdot \mathscr{F}(s, y_i, \omega_i, \alpha_i)(s)\}$ for a.e. $s \in [\rho_i, S]$, and $\dot\pi_i(s) = 0$ for a.e. $s \in [0,S]$;

(iii)′ $-q_i(S) \in \gamma_i\, \sigma_i^1\ (\partial d_{\mathscr{T}}(y_i(S)) \cap \partial\mathbb{B}_n)$, $\pi_i(0) = 0$, $-\pi_i(S) + \int_{[0,S]} \mu_i(d\sigma) = \gamma_i \sigma_i^2$;

(iv)′ $m_i(s) \in \partial^{>}_x h\,(s, y_i(s)) \qquad \mu_i$-a.e. $s \in [0,S]$,

(v)′ $spt(\mu_i) \subseteq \{s :\ h\,(s, y_i(s)) - c_i = 0\} \subset [\rho_i, S]$,

[9]Notice that by the boundedness of the dynamics, both $\tilde y_i(s)$ and $y_i(s)$ lay on $\check x_0 + sK_{\mathscr{F}}\mathbb{B}$. Hence, for i sufficiently large, $s \in [0, \varepsilon]$ and $x_i(s) \in \check x_0 + \varepsilon\mathbb{B}$, where $\varepsilon > 0$ is as in Remark 3.1(4).

(vi)$'_1$ $\int_0^{\rho_i} \eta_i\, p_i \cdot [\mathscr{F}(s, y_i, \hat{\omega}_i, \hat{\alpha}_i) - \mathscr{F}(s, y_i, \tilde{\omega}_i, \tilde{\alpha}_i)]\, ds \geq \int_0^{\rho_i} \{(1-\eta_i)\, p_i \cdot [\mathscr{F}(s, y_i, \hat{\omega}_i, \hat{\alpha}_i) - \mathscr{F}(s, y_i, \tilde{\omega}_i, \tilde{\alpha}_i)] - 2\gamma_i \rho_i^2\}\, ds$; [10]

(vi)$'_2$ $\int_{\rho_i}^S q_i \cdot \mathscr{F}(s, y_i, \omega_i, \alpha_i) ds \geq \int_{\rho_i}^S \{q_i \cdot \mathscr{F}(s, y_i, \omega, \alpha) - 2\gamma_i \rho_i^2\}\, ds$ for all $(\omega, \alpha, \lambda, \eta) \in \mathscr{V}_{\delta_i} \times \mathscr{A} \times \Lambda_n^1 \times L^1([0, S]; \{0, 1\})$,

where

$$q_i(s) := \begin{cases} p_i(s) + \int_{[0,s[} m_i(\sigma)\mu_i(d\sigma), & s \in [0, S[, \\ p_i(S) + \int_{[0,S]} m_i(\sigma)\mu_i(d\sigma), & s = S. \end{cases}$$

Observe that, for each i, by (ii)$'$ and (iii)$'$ we derive $\|\mu_i\|_{TV} = \int_{[0,S]} \mu_i(ds) = \gamma_i \sigma_i^2$ and $\pi_i \equiv 0$. Furthermore, since $\|m_i\|_{L^\infty} \leq K_h$, then by (iii)$'$ we get

$$\gamma_i \sigma_i^1 = |q_i(S)| \leq \|p_i\|_{L^\infty} + K_h \|\mu_i\|_{TV}.$$

By summing up these estimates and the nontriviality condition (i)$'$, we get

$$2\|p_i\|_{L^\infty} + (2 + K_h)\|\mu_i\|_{TV} + \geq \gamma_i(\sigma_i^1 + \sigma_i^2 - 1) + 1 = 1.$$

Hence, scaling the multipliers, we obtain $\|p_i\|_{L^\infty} + \|\mu_i\|_{TV} = 1$ and $\gamma_i \leq \tilde{L} := 2 + K_h$.

Case (b). Now, $c_i \leq 0$ for each i, so that (6.7) implies $d_{\mathscr{T}}(y_i(S)) > 0$. Thus, the process $(\omega_i, \alpha_i, \lambda_i, \eta_i, y_i, \xi_i, \hat{c}_i)$ with $\hat{c}_i := c_i + \hat{\varepsilon}$ for $\hat{\varepsilon} > 0$ suitably small is still a minimizer of problem (Q_i) and, in addition, it verifies $h(s, y_i(s)) - \hat{c}_i < 0$ for all $s \in [0, S]$ (namely, the state constraint is inactive on $[0, S]$). Hence, by applying the maximum principle for problem (Q_i) with reference to this minimizer we deduce the existence of multipliers $(p_i, \pi_i) \in W^{1,1}([0, S]; \mathbb{R}^{n+1})$, which satisfy conditions (i)$'$–(vi)$'$ with $\pi_i \equiv 0$, $\mu_i = 0$, $\sigma_i^2 = 0$, and $\gamma_i > 0$. In this case, from (iii)$'$ we get $0 < \gamma_i = |q_i(S)| = |p_i(S)| \leq \|p_i\|_{L^\infty}$, and, scaling the multipliers appropriately, we obtain $\|p_i\|_{L^\infty} = 1$ and $\gamma_i \leq 2$ ($\leq \tilde{L}$ as above).

Step 3. For either the case where $c_i > 0$ for each i or the case where $c_i \leq 0$ for each i, passing to the limit as $i \to +\infty$ for suitable subsequences and arguing as in the proof of [13, Thm. 2.2, Step 4], we can deduce the existence of a set of multipliers $p \in W^{1,1}([0, S]; \mathbb{R}^n)$, $\mu \in NBV^+([0, S]; \mathbb{R})$ and a Borel measurable, μ-integrable function $m : [0, S] \to \mathbb{R}^n$, such that

$$\begin{aligned} &\|p\|_{L^\infty} + \|\mu\|_{TV} = 1, && spt(\mu) \subseteq \{s \in [0, S] : h(s, \bar{y}(s)) = 0\}, \\ &-q(S) \in N_{\mathscr{T}}(\bar{y}(S)), && m(s) \in \partial_x^> h(s, \bar{y}(s))\ \mu\text{-a.e. } s \in [0, S], \end{aligned} \tag{6.9}$$

where

$$q(s) := \begin{cases} p(s) + \int_{[0,s[} m(\sigma)\mu(d\sigma), & s \in [0, S[, \\ p(S) + \int_{[0,S]} m(\sigma)\mu(d\sigma), & s = S. \end{cases}$$

Furthermore,

$$q_i \to q \text{ in } L^1, \qquad p_i \to p \text{ in } L^\infty, \qquad \dot{p}_i \rightharpoonup \dot{p} \text{ weakly in } L^1. \tag{6.10}$$

[10] By (v)$'$ it follows that $q_i \equiv p_i$ on $[0, \rho_i]$. Notice also that (vi)$'_1$ holds in a more general form; in fact we can replace $1 - \eta_i(\cdot)$ in the right-hand side with any measurable function $\eta : [0, \rho_i] \to \{0, 1\}$. Furthermore, we assume without loss of generality $\operatorname{diam}(W) = 1$, since W is supposed to be compact, so that $|\omega(s) - \omega_i(s)| + \ell_i(s, \alpha(s), \lambda(s), \eta(s)) \leq 2$ for any $s \in [0, S]$.

Now, let $\Omega_i := [\rho_i, S] \cap E_i$, where E_i is as in (6.6), so that $\ell(\Omega_i) \to S$. Recalling $\partial_x(q \cdot \mathscr{F}) = q \cdot D_x\mathscr{F}$, by (ii)′, (6.6), and (H3)(ii) we deduce, for a.e. $s \in \Omega_i$,

$$\begin{aligned}&\big(-\dot{p}_i, \dot{\xi}_i, \dot{y}_i\big)(s) \in \bigcup_{k=0}^{n} \Big[\big(\text{co}\ \partial_x\{q_i(s)\cdot\mathscr{F}(s, y_i, \bar{\omega}^k, \bar{\alpha}^k)(s)\}\big) \\ &\qquad\qquad \times\{(e^k, \mathscr{F}(s, y_i, \bar{\omega}^k, \bar{\alpha}^k)(s))\}\Big] + \big((1+K_h)\varphi(r_i(s))\mathbb{B}_n\big) \times \{0\} \times \big(\varphi(r_i(s))\mathbb{B}_n\big) \\ &\subseteq \bigcup_{k=0}^{n} \Big[\big(\text{co}\ \partial_x\{q(s)\cdot\mathscr{F}(s, y_i, \bar{\omega}^k, \bar{\alpha}^k)(s)\}\big) \times \{(e^k, \mathscr{F}(s, y_i, \bar{\omega}^k, \bar{\alpha}^k)(s))\}\Big] \\ &\qquad\qquad +\Big(\big((1+K_h)\varphi(r_i(s)) + K_{\mathscr{F}}|q_i(s)-q(s)|\big)\mathbb{B}_n\Big) \times \{0\} \times \big(\varphi(r_i(s))\mathbb{B}_n\big).\end{aligned}$$

By the properties of $\varphi(\cdot)$, the compactness of W, and the dominated convergence theorem we have $\varphi(r_i) \to 0$ in L^1 as $i \to \infty$. Hence, all the hypotheses of the compactness of trajectories theorem [29, Thm. 2.5.3] are satisfied, so that we can pass to the limit and get

$$\begin{aligned}\Big(-\dot{p}, \dot{\bar{\xi}}, \dot{\bar{y}}\Big)(s) \in \text{co}\Big(\bigcup_{k=0}^{n} \big[\big(\text{co}\ \partial_x\{q(s)\cdot\mathscr{F}(s, \bar{y}, \bar{\omega}^k, \bar{\alpha}^k)(s)\}\big) \\ \times\{(e^k, \mathscr{F}(s, \bar{y}, \bar{\omega}^k, \bar{\alpha}^k)(s))\}\big]\Big) \quad \text{for a.e. } s \in [0, S].\end{aligned}$$

By the Caratheodory representation theorem, there exists a measurable function $\lambda = (\lambda^0, \dots, \lambda^n) \in \Lambda_n$ such that

$$\begin{aligned}\Big(-\dot{p}, \dot{\bar{\xi}}, \dot{\bar{y}}\Big)(s) \in \sum_{k=0}^{n} \lambda^k(s)\Big(\text{co}\ \partial_x\left\{q(s)\cdot\mathscr{F}(s, \bar{y}, \bar{\omega}^k, \bar{\alpha}^k)(s)\right\} \\ \times\left\{(e^k, \mathscr{F}(s, \bar{y}, \bar{\omega}^k, \bar{\alpha}^k)(s))\right\}\Big) \quad \text{for a.e. } s \in [0, S].\end{aligned} \tag{6.11}$$

But now

$$\dot{\bar{\xi}}(s) = \sum_{k=0}^{n} \lambda^k(s)e^k = \sum_{k=0}^{n} \bar{\lambda}^k(s)e^k \qquad \text{a.e. } s \in [0, S].$$

Therefore, for every $k = 0, \dots, n+1$, $\lambda^k(s) = \bar{\lambda}^k(s)$ a.e. $s \in [0, S]$ and (2.2) is proved.

Let us prove (2.4). Taking $(\omega, \alpha) \in \mathscr{W} \times \mathscr{A}$, by (H1) and Remark 2.1 there exists a sequence $(v_i)_i \in \mathscr{V}$ such that $v_i \in \mathscr{V}_{\delta_i}$ for any i and $\|\omega - v_i\|_{L^1} \le \delta_i \downarrow 0$. By $(\text{vi})'_2$, we deduce that, for any i, one has

$$\int_0^S q_i(s)\cdot\dot{y}_i(s)\chi_{[\rho_i,S]}(s)ds \ge \int_0^S \left\{q_i(s)\cdot\mathscr{F}(s, y_i, v_i, \alpha)(s) - 2\gamma_i\rho_i^2\right\}\chi_{[\rho_i,S]}(s)ds.$$

Passing to the limit and using (6.5), (6.10) in the left-hand side and the dominated convergence theorem in the right-hand side, we obtain

$$\int_0^S q(s)\cdot\dot{\bar{y}}(s)\,ds \ge \int_0^S q(s)\cdot\mathscr{F}(s, \bar{y}(s), \omega(s), \alpha(s))\,ds.$$

Since this last relation holds for any selector $(\omega, \alpha) \in \mathscr{W} \times \mathscr{A}$, by a measurable selection theorem we can conclude that

$$q(s)\cdot\dot{\bar{y}}(s) = \max_{(w,a)\in W\times A} q(s)\cdot\mathscr{F}(s, \bar{y}(s), w, a) \qquad \text{a.e. } s \in [0, S]. \tag{6.12}$$

Finally, (6.12) trivially implies (2.4). Thus $\bar{z}$ is an abnormal extremal. To prove that it is in fact a nondegenerate abnormal extremal, it remains to show that the above multipliers verify the strengthened nontriviality condition

$$\|q\|_{L^\infty} + \mu(]0, S]) \neq 0. \tag{6.13}$$

To this aim, assume by contradiction that $\|q\|_{L^\infty} + \mu(]0, S]) = 0$. Then, the nontriviality condition (6.9) yields that $\mu(\{0\}) \neq 0$ and $p \equiv -\mu(\{0\})\zeta$ for some $\zeta \in \partial_x^> h(0, \check{x}_0)$. For every i, by the maximality condition (vi)$'_1$ and condition (3.8), it follows that

$$\begin{aligned}
0 \geq & \int_0^{\rho_i} (1-2\eta_i)\, p \cdot [\mathscr{F}(s, y_i, \hat{\omega}_i, \hat{\alpha}_i) - \mathscr{F}(s, y_i, \tilde{\omega}_i, \tilde{\alpha}_i)]\, ds \\
& + \int_0^{\rho_i} \left\{(1-2\eta_i)\,(p_i - p) \cdot [\mathscr{F}(s, y_i, \hat{\omega}_i, \hat{\alpha}_i) - \mathscr{F}(s, y_i, \tilde{\omega}_i, \tilde{\alpha}_i)] - 2\gamma_i \rho_i^2\right\} ds \\
\geq & \int_0^{\rho_i} p \cdot [\mathscr{F}(s, y_i, \hat{\omega}_i, \hat{\alpha}_i) - \mathscr{F}(s, y_i, \tilde{\omega}_i, \tilde{\alpha}_i)]\, \chi_{\{\sigma:\ \eta_i(\sigma)=0\}}(s)\, ds \\
& - \int_0^{\rho_i} p \cdot [\mathscr{F}(s, y_i, \hat{\omega}_i, \hat{\alpha}_i) - \mathscr{F}(s, y_i, \tilde{\omega}_i, \tilde{\alpha}_i)]\, \chi_{\{\sigma:\ \eta_i(\sigma)=1\}}(s)\, ds \\
& \quad -2\rho_i(K_{\mathscr{F}} \|p_i - p\|_{L^\infty} + \tilde{L}\rho_i^2) \\
\geq & \ \mu(\{0\})\, \delta\, \ell(\rho_i, 1-\eta_i(\cdot)) - 2K_{\mathscr{F}} K_h\, \ell(\rho_i, \eta_i(\cdot)) - 2\rho_i(K_{\mathscr{F}} \|p_i - p\|_{L^\infty} + \tilde{L}\rho_i^2) \\
\geq & \ \rho_i \left[\mu(\{0\})\, \delta - \mu(\{0\})\, \delta\, \rho_i - 2K_{\mathscr{F}} K_h \rho_i - 2K_{\mathscr{F}} \|p_i - p\|_{L^\infty} - 2\tilde{L}\rho_i^2\right] > 0,
\end{aligned}$$

where we use the facts that $\ell(\rho_i, \eta_i(\cdot)) \leq \rho_i^2$ and consequently $\ell(\rho_i, 1-\eta_i(\cdot)) \geq \rho_i - \rho_i^2$, which follow from (6.4). Thus, we obtain the desired contradiction.

6.2. Proof of Theorem 2.1. Preliminarily, observe that hypothesis (H3) can be reduced to (H3)$'$. We can clearly take $k \geq 1$ in assumption (H3), but actually we may (and we do) assume without loss of generality $k \equiv 1$. Indeed, reasoning as in [8, sect. 2], we can introduce the time change $t = \sigma(s) := \int_0^s k(\tau) d\tau$, so that $(\hat{\underline{\omega}}, \hat{\underline{\alpha}}, \hat{\lambda}, \hat{y}) := (\underline{\omega}, \underline{\alpha}, \lambda, y) \circ \sigma^{-1}$ is a process for the *transformed* problem, with dynamics $\hat{F} = \frac{1}{k} \sum_{j=0}^n \lambda^j\, \mathscr{F}(s, y, \omega^j, \alpha^j)$, verifying (H3) for $k \equiv 1$, and interval $[0, \sigma(S)]$, if and only if $(\underline{\omega}, \underline{\alpha}, \lambda, y)$ is a process for the relaxed problem. Furthermore, the transformed process, say, $\hat{z}$, corresponding to $\bar{z} := (\underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{\xi}, \bar{y})$ is isolated for the transformed problem, and if $\hat{z}$ is an abnormal extremal for the transformed problem for some $(\hat{p}, 0, \hat{\mu}, \hat{m})$ as in Definition 2.4, then $\bar{z}$ is an abnormal extremal with (p, γ, μ, m) verifying $p = \hat{p} \circ \sigma$, $\gamma = 0$, $d\mu = k\, d\hat{\mu}$, and $m = \hat{m} \circ \sigma$.

First of all, we notice that $(\bar{\xi}, \bar{y})$ is a solution of the differential inclusion

$$(\dot{\xi}, \dot{y})(s) \in \mathrm{co} \bigcup_{k=0}^n \{(e^k, \mathscr{F}(s, y(s), \bar{\omega}^k(s), \bar{\alpha}^k(s)))\} \qquad \text{a.e. } s \in [0, S].$$

Let us fix a sequence $\varepsilon_i \downarrow 0$. By the relaxation theorem [29, Thm. 2.7.2], there exists a sequence of extended processes $(\bar{\omega}_i, \bar{\alpha}_i, \bar{\lambda}_i)(s) \in \bigcup_{k=0}^n \{(\bar{\omega}^k(s), \bar{\alpha}^k(s), e^k)\}$ for a.e. $s \in [0, S]$ such that, for any i, the corresponding trajectory $(\bar{\xi}_i, \bar{y}_i) := (\xi, y)[\bar{\omega}_i, \dots, \bar{\omega}_i, \bar{\alpha}_i, \dots, \bar{\alpha}_i, \bar{\lambda}_i]$ satisfies

$$\|(\bar{\xi}_i, \bar{y}_i) - (\bar{\xi}, \bar{y})\|_{L^\infty} \leq \varepsilon_i.$$

Let $\mathscr{J}(\cdot)$ and $(\rho_i)_i$ be as in the proof of Theorem 3.1 and $(\delta_i)_i$ such that for every $\omega \in \mathscr{W}$ with $\|\omega - \bar{\omega}_i\|_{L^1} \leq \delta_i$, then $\|y[\omega, \bar{\alpha}_i] - \bar{y}_i\|_{L^\infty} \leq \varepsilon_i$. Then, thanks to hypothesis

(H1), for any i there exists $\check{\omega}_i \in \mathscr{V}_{\delta_i}$ such that $\|\check{\omega}_i - \bar{\omega}_i\|_{L^1} \le \delta_i$. As a consequence, if we define $\check{\alpha}_i \equiv \bar{\alpha}_i$, $\check{\lambda}_i \equiv \bar{\lambda}_i$, and $\check{y}_i = y[\check{\omega}_i, \check{\alpha}_i]$, then $\check{y}_i$ is a strict sense trajectory that satisfies $\|\check{y}_i - \bar{y}_i\|_{L^\infty} \le \varepsilon_i$ and

$$(\check{\omega}_i, \check{\alpha}_i, \check{\lambda}_i)(s) \in \bigcup_{k=0}^{n} \{(\bar{\omega}^k(s), \bar{\alpha}^k(s), e^k)\} + (\check{r}_i(s)\mathbb{B}_m) \times \{0\} \times \{0\} \qquad \text{a.e. } s \in [0, S]$$

for some measurable sequence $\check{r}_i \to 0$ in L^1. Therefore, the process $(\check{\omega}_i, \check{\alpha}_i, \check{\lambda}_i, \check{\xi}_i, \check{y}_i)$ is a ρ_i^4-minimizer for the optimal control problem

$$(\check{P}_i) \begin{cases} \text{minimize} \quad \mathscr{J}(y) \\ \text{over } (\omega, \alpha, \lambda, y, \xi) \in \mathscr{V}_{\delta_i} \times \mathscr{A} \times \Lambda_n^1 \times W^{1,1}([0, S]; \mathbb{R}^{n+n}), \text{ satisfying} \\ (\dot{\xi}, \dot{y})(s) = \big(\lambda(s),\ \mathscr{F}(s, y(s), \omega(s), \alpha(s))\big) \ \text{ a.e. } s \in [0, S], \\ (\xi, y)(0) = (0, \check{x}_0). \end{cases}$$

From now on, except for minor obvious changes, the proof proceeds as the proof of Theorem 3.1 and is actually simpler, since we disregard the nondegeneracy issue.

7. Conclusions. In this article we provide sufficient conditions in the form of a normality test for the absence of gap phenomena when we pass from a quite general optimal control problem with nonsmooth data, endpoint and state constraints, to an extended version of it, in a new unified framework, that embraces both the extension of the class of ordinary controls to include impulse controls and convex relaxation. When the initial point lies on the boundary of the constraint set, we also introduce some nondegeneracy conditions under which we obtain a *nondegenerate* normality test, which may detect the absence of a gap in a situation where the usual normality test is of no use.

The uniform framework introduced in this paper may have implications for future infimum gap research in several directions. On the one hand, it may be the starting point for some generalizations, such as, for instance, (i) determine sufficient conditions to avoid gap phenomena in free end-time optimal control problems with measurable time dependence, in the presence of endpoint and state constraints. This situation is completely different from the case of Lipschitz continuous time dependence investigated in section 4, since the time-rescaling procedure used there is no longer applicable. To our knowledge, there is only one result of this kind, which concerns the classical extension by convex relaxation without state constraints (see [30]). (ii) Establish under which general assumptions a strict sense local minimizer which is not also an extended or a relaxed local minimizer satisfies the (extended) Pontryagin maximum principle in abnormal form. Even in this case, results are known only for the relaxed extension (see [25, 26, 30]).

On the other hand, by considering different extension/relaxation procedures for classes of control systems not considered in this paper (such as distributed parameters systems or multistage problems), it should be possible to adapt the analysis developed in this work to weaken the regularity assumptions on the dynamics function $\mathscr{F}$ in the control variable w. This would be a first step toward unifying the results on the infimum gap phenomenon achieved by following the two different approaches (a) and (b) described in the introduction, which is a long-standing issue. In fact, approach (a) generally requires less regularity of $\mathscr{F}$ than (b).

Acknowledgment. We wish to thank the referees, whose valuable inquiries and remarks have considerably helped us to improve the presentation of the results.

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