On the explicit representation of the trace space $H^{\frac{3}{2}}$ and of the solutions to biharmonic Dirichlet problems on Lipschitz domains via multi-parameter Steklov problems

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Abstract

We consider the problem of describing the traces of functions in $H^2(\Omega)$ on the boundary of a Lipschitz domain Ω of $\mathbb{R}^N$, $N \geq 2$. We provide a definition of those spaces, in particular of $H^{\frac{3}{2}}(\partial\Omega)$, by means of Fourier series associated with the eigenfunctions of new multi-parameter biharmonic Steklov problems which we introduce with this specific purpose. These definitions coincide with the classical ones when the domain is smooth. Our spaces allow to represent in series the solutions to the biharmonic Dirichlet problem. Moreover, a few spectral properties of the multi-parameter biharmonic Steklov problems are considered, as well as explicit examples. Our approach is similar to that developed by G. Auchmuty for the space $H^1(\Omega)$, based on the classical second order Steklov problem.

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1 Introduction

We consider the trace spaces of functions in $H^2(\Omega)$ when Ω is a bounded Lipschitz domain in $\mathbb{R}^N$, briefly Ω is of class $C^{0,1}$, for $N \geq 2$. It is well known that there exists a linear and continuous operator Γ called the *total trace*, from $H^2(\Omega)$ to $L^2(\partial\Omega) \times L^2(\partial\Omega)$ defined by $\Gamma(u) = (\gamma_0(u), \gamma_1(u))$, where $\gamma_0(u)$ is the *trace* of u on $\partial\Omega$ and $\gamma_1(u)$ is the *normal derivative* of u. In particular, for $u \in C^2(\overline{\Omega})$, $\gamma_0(u) = u_{|\partial\Omega}$ and $\gamma_1(u) = \frac{\partial u}{\partial \nu} = \nabla u_{|\partial\Omega} \cdot \nu$, where ν denotes the outer unit normal to $\partial\Omega$.

A relevant problem in the theory of Sobolev Spaces consists in describing the *trace spaces* $\gamma_0(H^2(\Omega))$, $\gamma_1(H^2(\Omega))$, and the *total trace space* $\Gamma(H^2(\Omega))$. This problem has important implications in the study of solutions to fourth order elliptic partial differential equations.

From a historical point of view, this issue finds its origins in [23] where J. Hadamard proposed his famous counterexample pointing out the importance to understand which conditions on the datum g guarantee that the solution v to the Dirichlet problem

$$\begin{cases} \Delta v = 0, \text{ in } \Omega, \\ v = g, \quad \text{on } \partial\Omega, \end{cases}$$

has square summable gradient. In modern terms, this problem can be reformulated as the problem of finding necessary and sufficient conditions on g such that $g = \gamma_0(u)$ for some $u \in H^1(\Omega)$.

If the domain Ω is of class $C^{2,1}$, then it is known that $\gamma_0(H^2(\Omega)) = H^{\frac{3}{2}}(\partial\Omega)$, $\gamma_1(H^2(\Omega)) = H^{\frac{1}{2}}(\partial\Omega)$, and $\Gamma(H^2(\Omega)) = H^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, where $H^{\frac{3}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$ are the classical Sobolev spaces of fractional order (see e.g., [22,31] for their definitions). However, if Ω is an arbitrary bounded domain of class $C^{0,1}$ there is no such a simple description and not many results are available in the literature.

We note that a complete description of the traces of all derivatives up to the order $m - 1$ of a function $u \in H^m(\Omega)$ is due to O. Besov who provided an explicit but quite technical representation theorem, see [6,7], see also [8]. Simpler descriptions are not available with the exception of a few special cases. For example, when Ω is a polygon in $\mathbb{R}^2$ the trace spaces are described by using the classical trace theorem applied to each side of the polygon, complemented with suitable compatibility conditions at the vertexes, see [22] also for higher dimensional polyhedra. For more general planar domains another simple description is given in [20].

Our list of references cannot be exhaustive and we refer to the recent monograph [30] which treats the trace problem in presence of corner or conical singularities in $\mathbb{R}^3$, as well as further results on N-dimensional polyhedra. We also quote the fundamental paper [24] by V. Kondrat'ev for a pioneering work in this type of problems. Interested readers can find more information in our recent survey paper [26].

Thus, the definition of the space $H^{\frac{3}{2}}(\partial\Omega)$ turns out to be problematic and for this reason sometimes the space $H^{\frac{3}{2}}(\partial\Omega)$ is simply defined by setting $H^{\frac{3}{2}}(\partial\Omega) := \gamma_0(H^2(\Omega))$ without providing an explicit representation. Note that standard definitions of $H^s(\partial\Omega)$ when $s \in (1, 2]$ require that Ω is of class at least C^2.

In the present paper we provide decompositions of the space $H^2(\Omega)$ of the form $H^2(\Omega) = H^2_{\mu,D}(\Omega) + \mathcal{H}^2_{0,N}(\Omega)$ and $H^2(\Omega) = H^2_{\lambda,N}(\Omega) + \mathcal{H}^2_{0,D}(\Omega)$. The spaces $\mathcal{H}^2_{0,N}(\Omega)$ and $\mathcal{H}^2_{0,D}(\Omega)$ are the subspaces of $H^2(\Omega)$ of those functions u such that $\gamma_1(u) = 0$ and $\gamma_0(u) = 0$, respectively. The spaces $H^2_{\mu,D}(\Omega)$ and $H^2_{\lambda,N}(\Omega)$ are associated with suitable Steklov problems of biharmonic type (namely, problems (BS_μ) and (BS_λ) described here below), depending on real parameters μ, λ, and admit Fourier bases of Steklov eigenfunctions, see (3.5) and (3.15). Under the sole assumptions that Ω is of class $C^{0,1}$, we use those bases to define in a natural way two spaces at the boundary which we denote by $\mathcal{S}^{\frac{3}{2}}(\partial\Omega)$ and $\mathcal{S}^{\frac{1}{2}}(\partial\Omega)$ and we prove that

$$\gamma_0(H^2(\Omega)) = \gamma_0(H^2_{\lambda,N}(\Omega)) = \mathcal{S}^{\frac{3}{2}}(\partial\Omega)$$

and

$$\gamma_1(H^2(\Omega)) = \gamma_1(H^2_{\mu,D}(\Omega)) = \mathcal{S}^{\frac{1}{2}}(\partial\Omega),$$

see Theorem 4.1. Thus, if one would like to define the space $H^{\frac{3}{2}}(\partial\Omega)$ as $\gamma_0(H^2(\Omega))$, our result gives an explicit description of $H^{\frac{3}{2}}(\partial\Omega)$.

It turns out that the analysis of problems (BS_μ)-(BS_λ) provides further information on the total trace $\Gamma(H^2(\Omega))$. In particular, we prove the inclusion $\Gamma(H^2(\Omega)) \subseteq \mathcal{S}^{\frac{3}{2}}(\partial\Omega) \times \mathcal{S}^{\frac{1}{2}}(\partial\Omega)$ and show that in general this inclusion is strict if Ω is assumed to be only of class $C^{0,1}$. Moreover, we show that any couple $(f, g) \in \mathcal{S}^{\frac{3}{2}}(\partial\Omega) \times \mathcal{S}^{\frac{1}{2}}(\partial\Omega)$ belongs to $\Gamma(H^2(\Omega))$ if and only if it satisfies a certain compatibility condition, see Theorem 4.4.

If Ω is of class $C^{2,1}$, we recover the classical result, namely $\Gamma(H^2(\Omega)) = \mathcal{S}^{\frac{3}{2}}(\partial\Omega) \times \mathcal{S}^{\frac{1}{2}}(\partial\Omega)$, which implies that $\mathcal{S}^{\frac{3}{2}}(\partial\Omega) = H^{\frac{3}{2}}(\partial\Omega)$ and $\mathcal{S}^{\frac{1}{2}}(\partial\Omega) = H^{\frac{1}{2}}(\partial\Omega)$.

The two families of problems which we are going to introduce depend on a parameter $\sigma \in \left(-\frac{1}{N-1}, 1\right)$, which in applications to linear elasticity represents the Poisson coefficient of the elastic material of the underlying system for $N = 2$.

The first family of BS_μ - 'Biharmonic Steklov μ' problems is defined as follows:

$$\begin{cases} \Delta^2 v = 0, & \text{in } \Omega, \\ (1-\sigma)\frac{\partial^2 v}{\partial\nu^2} + \sigma\Delta v = \lambda(\mu)\frac{\partial v}{\partial\nu}, & \text{on } \partial\Omega, \\ -(1-\sigma)\text{div}_{\partial\Omega}(D^2v \cdot \nu)_{\partial\Omega} - \frac{\partial\Delta v}{\partial\nu} = \mu v, & \text{on } \partial\Omega, \end{cases} \quad (BS_\mu)$$

in the unknowns $v, \lambda(\mu)$, where $\mu \in \mathbb{R}$ is fixed. Here D^2u denotes the Hessian matrix of u, $\text{div}_{\partial\Omega}F := \text{div}F - (\nabla F \cdot \nu)\nu$ denotes the tangential divergence of a vector field F and $F_{\partial\Omega} := F - (F \cdot \nu)\nu$ denotes the tangential component of F.

The second family of BS_λ - 'Biharmonic Steklov λ' problems is defined as follows:

$$\begin{cases} \Delta^2 u = 0, & \text{in } \Omega, \\ (1-\sigma)\frac{\partial^2 u}{\partial\nu^2} + \sigma\Delta u = \lambda\frac{\partial u}{\partial\nu}, & \text{on } \partial\Omega, \\ -(1-\sigma)\text{div}_{\partial\Omega}(D^2u \cdot \nu)_{\partial\Omega} - \frac{\partial\Delta u}{\partial\nu} = \mu(\lambda)u, & \text{on } \partial\Omega, \end{cases} \quad (BS_\lambda)$$

in the unknowns $u, \mu(\lambda)$, where $\lambda \in \mathbb{R}$ is fixed.

Note that since Ω is assumed to be of class $C^{0,1}$, problems (BS$_\mu$) and (BS$_\lambda$) have to be considered in the weak sense, see (3.4) and (3.14) for the appropriate formulations.

Up to our knowledge, the Steklov problems (BS$_\mu$) and (BS$_\lambda$) are new in the literature. Other Steklov-type problems for the biharmonic operator have been discussed in the literature. We mention the DBS - 'Dirichlet Biharmonic Steklov' problem

$$\begin{cases} \Delta^2 v = 0, & \text{in } \Omega, \\ (1-\sigma)\frac{\partial^2 v}{\partial \nu^2} + \sigma \Delta v = \eta \frac{\partial v}{\partial \nu}, & \text{on } \partial\Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases} \qquad \text{(DBS)}$$

in the unknowns v, η, and the NBS - 'Neumann Biharmonic Steklov' problem

$$\begin{cases} \Delta^2 u = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \\ -(1-\sigma)\text{div}_{\partial\Omega}(D^2 u \cdot \nu)_{\partial\Omega} - \frac{\partial \Delta u}{\partial \nu} = \xi u, & \text{on } \partial\Omega, \end{cases} \qquad \text{(NBS)}$$

in the unknowns u, ξ. Problem (DBS) for $\sigma = 1$ has been studied by many authors (see e.g., [3,9,16–18,25,29]); for the case $\sigma \neq 1$ we refer to [10], see also [4,17] for $\sigma = 0$. Problem (NBS) has been discussed in [25,28,29] for $\sigma = 1$. We point out that problem (BS$_\lambda$) with $\sigma = \lambda = 0$ has been introduced in [12] as the natural fourth order generalization of the classical Steklov problem for the Laplacian (see also [11]). As we shall see, problem (BS$_\lambda$) shares much more analogies with the classical Steklov problem than those already presented in [12], in particular it plays a role in describing the space $\gamma_0(H^2(\Omega))$ similar to that played by the Steklov problem for the Laplacian in describing $\gamma_0(H^1(\Omega))$ (cf. [2]).

If $\mu < 0$, problem (BS$_\mu$) has a discrete spectrum which consists of a divergent sequence $\{\lambda_j(\mu)\}_{j=1}^{\infty}$ of non-negative eigenvalues of finite multiplicity. Similarly, if $\lambda < \eta_1$, where $\eta_1 > 0$ is the first eigenvalue of (DBS), problem (BS$_\lambda$) has a discrete spectrum which consists of a divergent sequence $\{\mu_j(\lambda)\}_{j=1}^{\infty}$ of eigenvalues of finite multiplicity and bounded from below. (For other values of μ and λ the description of the spectra of (BS$_\mu$) and (BS$_\lambda$) is more involved, see Appendix C.)

The eigenfunctions associated with the eigenvalues $\lambda_j(\mu)$ define a Hilbert basis of the above mentioned space $H^2_{\mu,D}(\Omega)$ which is the orthogonal complement in $H^2(\Omega)$ of $\mathcal{H}^2_{0,N}(\Omega)$ with respect to a suitable scalar product. Moreover, the normal derivatives of those eigenfunctions allow to define the above mentioned space $\mathcal{S}^{\frac{1}{2}}(\partial\Omega)$, see (4.2). Similarly, the eigenfunctions associated with the eigenvalues $\mu_j(\lambda)$ define a Hilbert basis of the space $H^2_{\lambda,N}(\Omega)$ which is the orthogonal complement in $H^2(\Omega)$ of $\mathcal{H}^2_{0,D}(\Omega)$ with respect to a suitable scalar product. Moreover, the traces of those eigenfunctions allow to define the space $\mathcal{S}^{\frac{3}{2}}(\partial\Omega)$, see (4.1).

The definitions in (4.1) and (4.2) are given by means of Fourier series and the coefficients in such expansions need to satisfy certain summability conditions, which are strictly related to the asymptotic behavior of the eigenvalues of (BS$_\lambda$) and (BS$_\mu$).

Note that

$$\mu_j(\lambda) \sim C_N \left(\frac{j}{|\partial\Omega|}\right)^{\frac{3}{N-1}} \quad \text{and} \quad \lambda_j(\mu) \sim C_N' \left(\frac{j}{|\partial\Omega|}\right)^{\frac{1}{N-1}}, \quad \text{as } j \to +\infty, \quad (1.1)$$

where C_N, C_N' depend only on N, see Appendix B. In view of (1.1) and (4.1)–(4.2), we can identify the space $\mathcal{S}^{\frac{3}{2}}(\partial\Omega)$ with the space of sequences

$$\left\{(s_j)_{j=1}^\infty \in \mathbb{R}^\infty : (j^{\frac{3}{2(N-1)}} s_j)_{j=1}^\infty \in l^2\right\} \quad (1.2)$$

and the space $\mathcal{S}^{\frac{1}{2}}(\partial\Omega)$ with the space

$$\left\{(s_j)_{j=1}^\infty \in \mathbb{R}^\infty : (j^{\frac{1}{2(N-1)}} s_j)_{j=1}^\infty \in l^2\right\}. \quad (1.3)$$

Observe the natural appearance of the exponents $\frac{3}{2}$ and $\frac{1}{2}$ in (1.2) and (1.3). It is remarkable that, in essence, a summability condition analogous to that in (1.3) is already present in [23, Formula (3)] for the case of the unit disk D of the plane and the space $H^{\frac{1}{2}}(\partial D) = \gamma_0(H^1(D))$.

Using the representations (4.1) and (4.2) we are able to provide necessary and sufficient conditions for the solvability in $H^2(\Omega)$ of the Dirichlet problem

$$\begin{cases} \Delta^2 u = 0, & \text{in } \Omega, \\ u = f, & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = g, & \text{on } \partial\Omega, \end{cases} \quad (1.4)$$

under the sole assumption that Ω is of class $C^{0,1}$, and to represent in Fourier series the solutions. We note that different necessary and sufficient conditions for the solvability of problem (1.4) in the larger space $H(\Delta, \Omega) = \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\}$ have been found in [4] by using the (DBS) problem with $\sigma = 1$ and the classical Dirichlet-to-Neumann map. We also refer to [5,15,34,35] for a different approach to the solvability of higher order problems on Lipschitz domains. Note that in [5,15,34,35] the authors consider notions of weak solutions which differ substantially from the standard variational one used in this paper, and the solutions to problem (1.4) are allowed to have infinite Dirichlet energy. For instance, in [15, Thm. 3.1] the boundary data f, g belong to $L_1^2(\partial\Omega)$, $L^2(\partial\Omega)$ respectively, and are assumed by the solution u as non-tangential limits; accordingly, u is not expected to belong to $H^2(\Omega)$ but just to $H^{3/2}(\Omega)$, see [15, p. 110].

Since we have not been able to find problems (BS_μ) and (BS_λ) in the literature, we believe that it is worth including in the present paper also some information on their spectral behavior, which may have a certain interest on its own. In particular, we prove Lipschitz continuity results for the functions $\mu \mapsto \lambda_j(\mu)$ and $\lambda \mapsto \mu_j(\lambda)$ and we show that problems (DBS) and (NBS) can be seen as limiting problems for (BS_μ) and (BS_λ) as $\mu \to -\infty$ and $\lambda \to -\infty$, respectively. We also perform a complete study

of the eigenvalues in the unit ball in $\mathbb{R}^N$ for $\sigma = 0$, and we discuss the asymptotic behavior of $\lambda_j(\mu)$ and $\mu_j(\lambda)$ on smooth domains when $j \to +\infty$. Finally, we briefly discuss problems (BS$_\mu$) and (BS$_\lambda$) also when $\mu > 0$ and $\lambda > \eta_1$.

Our approach is similar to that developed by G. Auchmuty in [2] for the trace space of $H^1(\Omega)$, based on the classical second order Steklov problem

$$
\begin{cases}
\Delta u = 0, & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = \lambda u, & \text{on } \partial\Omega.
\end{cases}
$$

We also refer to [32] for related results.

This paper is organized as follows. In Section 2 we introduce some notation and discuss a few preliminary results. In Section 3 we discuss problems (BS$_\mu$) and (BS$_\lambda$) when $\mu < 0$ and $\lambda < \eta_1$. In Section 4 we define the spaces $\mathcal{S}^{\frac{3}{2}}(\partial\Omega)$ and $\mathcal{S}^{\frac{1}{2}}(\partial\Omega)$ and the representation theorems for the trace spaces of $H^2(\Omega)$. In Subsect. 4.1 we prove a representation result for the solutions of the biharmonic Dirichlet problem. In Appendix A we provide a complete description of problems (BS$_\mu$) and (BS$_\lambda$) on the unit ball for $\sigma = 0$. In Appendix B we briefly discuss asymptotic laws for the eigenvalues. In Appendix C we discuss problems (BS$_\mu$) and (BS$_\lambda$) when $\mu > 0$ and $\lambda > \eta_1$.

2 Preliminaries and notation

For a bounded domain (i.e., a bounded open connected set) Ω in $\mathbb{R}^N$, we denote by $H^1(\Omega)$ the standard Sobolev space of functions in $L^2(\Omega)$ with all weak derivatives of the first order in $L^2(\Omega)$ endowed with its standard norm $\|u\|_{H^1(\Omega)} := \left(\|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}$ for all $u \in H^1(\Omega)$. Note that in this paper we consider $L^2(\Omega)$ as a space of real-valued functions and we always assume $N \geq 2$.

By $H^2(\Omega)$ we denote the standard Sobolev space of functions in $L^2(\Omega)$ with all weak derivatives of the first and second order in $L^2(\Omega)$ endowed with the norm $\|u\|_{H^2(\Omega)} := \left(\|D^2 u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}$ for all $u \in H^2(\Omega)$. We denote by $H_0^1(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$ and by $H_0^2(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $H^2(\Omega)$. The space $C_c^\infty(\Omega)$ is the space of all functions in $C^\infty(\Omega)$ with compact support in Ω. If the boundary is sufficiently regular (e.g., if Ω is of class $C^{0,1}$), the norm defined by $\sum_{|\alpha| \leq 2} \|D^\alpha u\|_{L^2(\Omega)}$ is a norm on $H^2(\Omega)$ equivalent to the standard one.

By definition, a domain of class $C^{0,1}$ is such that locally around each point of its boundary it can be described as the sub-graph of a Lipschitz continuous function. Also, we shall say that Ω is of class $C^{2,1}$ if locally around each point of its boundary the domain can be described as the sub-graph of a function of class $C^{2,1}$.

By $(\cdot, \cdot)_{\partial\Omega}$ we denote the standard scalar product of $L^2(\partial\Omega)$, namely

$$
(u, v)_{\partial\Omega} := \int_{\partial\Omega} uv\, d\sigma, \quad \forall u, v \in L^2(\partial\Omega).
$$

We denote by $\gamma_0(u) \in L^2(\partial\Omega)$ the trace of u and by $\gamma_1(u) \in L^2(\partial\Omega)$ the normal derivative of u, that is, $\gamma_1(u) = \nabla u \cdot v$. By Γ we denote the total trace operator from $H^2(\Omega)$ to $L^2(\partial\Omega) \times L^2(\partial\Omega)$ defined by

$$\Gamma_2(u) = (\gamma_0(u), \gamma_1(u)),$$

for all $u \in H^2(\Omega)$. The operator Γ is compact. If Ω is of class $C^{2,1}$ then Γ is a linear and continuous operator from $H^2(\Omega)$ onto $H^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ admitting a right continuous inverse. We refer to e.g., [31] for more details.

Here $H^{\frac{3}{2}}(\partial\Omega)$, $H^{\frac{1}{2}}(\partial\Omega)$ denote the standard Sobolev spaces of fractional order (see e.g., [22,31] for more details). For any $\sigma \in \left(-\frac{1}{N-1}, 1\right)$, $\mu, \lambda \in \mathbb{R}$ and $u, v \in H^2(\Omega)$ we set

$$\mathcal{Q}_\sigma(u, v) = (1 - \sigma) \int_\Omega D^2u : D^2v dx + \sigma \int_\Omega \Delta u \Delta v dx,$$

$$\mathcal{Q}_{\mu, D}(u, v) = \mathcal{Q}_\sigma(u, v) - \mu(\gamma_0(u), \gamma_0(v))_{\partial\Omega},$$

and

$$\mathcal{Q}_{\lambda, N}(u, v) = \mathcal{Q}_\sigma(u, v) - \lambda(\gamma_1(u), \gamma_1(v))_{\partial\Omega},$$

where $D^2u : D^2v = \sum_{i,j=1}^N \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j}$ denotes the Frobenius product of the Hessians matrices. Note that if $\sigma \in \left(-\frac{1}{N-1}, 1\right)$, then the quadratic form $\mathcal{Q}_\sigma$ is coercive in $H^2(\Omega)$ and the norm $\left(\mathcal{Q}_\sigma(u, u) + \|u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}$ is equivalent to the standard norm of $H^2(\Omega)$, see e.g., [14].

It is easy to see that if Ω is a bounded domain of class $C^{0,1}$ then the space $H^2(\Omega)$ can be endowed with the equivalent norm

$$\left(\|D^2u\|^2_{L^2(\Omega)} + \|\gamma_0(u)\|^2_{L^2(\partial\Omega)}\right)^{\frac{1}{2}}.$$

We set

$$\mathcal{H}^2_{0, D}(\Omega) = \left\{u \in H^2(\Omega) : \gamma_0(u) = 0\right\}$$

and

$$\mathcal{H}^2_{0, N}(\Omega) = \left\{u \in H^2(\Omega) : \gamma_1(u) = 0\right\}.$$

The spaces $\mathcal{H}^2_{0, D}(\Omega)$ and $\mathcal{H}^2_{0, N}(\Omega)$ are closed subspaces of $H^2(\Omega)$ and $\mathcal{H}^2_{0, N}(\Omega) \cap \mathcal{H}^2_{0, D}(\Omega) = H^2_0(\Omega)$. We also note that $\mathcal{H}^2_{0, D} = H^2(\Omega) \cap H^1_0(\Omega)$.

It is useful to recall the so-called biharmonic Green formula

$$\int_\Omega D^2\psi : D^2\varphi dx = \int_\Omega (\Delta^2\psi)\varphi dx + \int_{\partial\Omega} \frac{\partial^2\psi}{\partial v^2} \frac{\partial\varphi}{\partial v} d\sigma$$
$$- \int_{\partial\Omega} \left(\text{div}_{\partial\Omega}(D^2\psi \cdot v)_{\partial\Omega} + \frac{\partial\Delta\psi}{\partial v}\right)\varphi d\sigma, \tag{2.1}$$

valid for all sufficiently smooth ψ, φ, see [1].

The biharmonic functions in $H^2(\Omega)$ are defined as those functions $u \in H^2(\Omega)$ such that $\int_\Omega D^2 u : D^2 \varphi \, dx = 0$ for all $\varphi \in H_0^2(\Omega)$, or equivalently, thanks to (2.1), those functions $u \in H^2(\Omega)$ such that $\int_\Omega \Delta u \Delta \varphi \, dx = 0$ for all $\varphi \in H_0^2(\Omega)$. We denote by $\mathcal{B}_N(\Omega)$ the space of biharmonic functions with zero normal derivative, that is the orthogonal complement of $H_0^2(\Omega)$ in $\mathcal{H}_{0,N}^2(\Omega)$ with respect to $\mathcal{Q}_\sigma$:

$$\mathcal{B}_N(\Omega) := \left\{ u \in \mathcal{H}_{0,N}^2(\Omega) : \mathcal{Q}_\sigma(u, \varphi) = 0 \,, \forall \varphi \in H_0^2(\Omega) \right\}. \tag{2.2}$$

By formula (2.1) and a standard approximation we deduce that

$$\mathcal{B}_N(\Omega) := \left\{ u \in \mathcal{H}_{0,N}^2(\Omega) : \int_\Omega \Delta u \Delta \varphi \, dx = 0 \,, \forall \varphi \in H_0^2(\Omega) \right\}. \tag{2.3}$$

We note that $\mathcal{B}_N(\Omega)$ is the space of the biharmonic functions in $\mathcal{H}_{0,N}^2(\Omega)$. Thus

$$\mathcal{H}_{0,N}^2(\Omega) = H_0^2(\Omega) \oplus \mathcal{B}_N(\Omega).$$

Analogously, we denote by $\mathcal{B}_D$ the space of biharmonic functions with zero boundary trace, that is the orthogonal complement of $H_0^2(\Omega)$ in $\mathcal{H}_{0,D}^2(\Omega)$ with respect to $\mathcal{Q}_\sigma$:

$$\mathcal{B}_D(\Omega) := \left\{ u \in \mathcal{H}_{0,D}^2(\Omega) : \mathcal{Q}_\sigma(u, \varphi) = 0 \,, \forall \varphi \in H_0^2(\Omega) \right\}. \tag{2.4}$$

By formula (2.1) and standard approximation we deduce that

$$\mathcal{B}_D(\Omega) := \left\{ u \in \mathcal{H}_{0,D}^2(\Omega) : \int_\Omega \Delta u \Delta \varphi \, dx = 0 \,, \forall \varphi \in H_0^2(\Omega) \right\}. \tag{2.5}$$

We note that $\mathcal{B}_D(\Omega)$ is the space of biharmonic functions in $\mathcal{H}_{0,D}^2(\Omega)$. Thus

$$\mathcal{H}_{0,D}^2(\Omega) = H_0^2(\Omega) \oplus \mathcal{B}_D(\Omega).$$

Finally, by $\mathbb{N}$ we denote the set of positive natural numbers and by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$.

3 Multi-parameter Steklov problems

In this section we provide the appropriate weak formulations of problems (BS$_\mu$) and (BS$_\lambda$). In particular we prove that both problems have discrete spectrum provided $\mu < 0$ and $\lambda < \eta_1$, respectively. Here η_1 is the first eigenvalue of problem (3.1) below, which is the weak formulation of (DBS). We remark that $\eta_1 > 0$ and that $\xi_1 = 0$ is the first eigenvalue of problem (NBS), hence the condition $\mu < 0$ reads $\mu < \xi_1$. We also provide a variational characterization of the eigenvalues.

Through all this section Ω will be a bounded domain of class $C^{0,1}$ and $\sigma \in \left(-\frac{1}{N-1}, 1 \right)$ will be fixed.

3.1 The (DBS) and (NBS) problems

Before analyzing problems (BS_μ) and (BS_λ) we need to recall a few facts about problems (DBS) and (NBS).

Problem (DBS) is understood in the weak sense as follows:

$$\int_\Omega (1-\sigma)D^2v : D^2\varphi + \sigma\,\Delta v \Delta\varphi dx = \eta \int_{\partial\Omega} \frac{\partial v}{\partial\nu}\frac{\partial\varphi}{\partial\nu}d\sigma\,, \quad \forall\varphi \in \mathcal{H}_{0,D}^2(\Omega), \quad (3.1)$$

in the unknowns $v \in \mathcal{H}_{0,D}^2(\Omega)$, $\eta \in \mathbb{R}$. Note that formulation (3.1) is justified by formula (2.1). Indeed, by applying formula (2.1), one can easily see that if v is a smooth solution to problem (3.1), then v is a solution to the classical problem (DBS) (the same considerations can be done for all other problems discussed in this paper).

We have the following theorem.

Theorem 3.1 *Let Ω be a bounded domain in $\mathbb{R}^N$ of class $C^{0,1}$ and let $\sigma \in \left(-\frac{1}{N-1}, 1\right)$. The eigenvalues of problem (3.1) have finite multiplicity and are given by a non-decreasing sequence of positive real numbers η_j defined by*

$$\eta_j = \min_{\substack{V \subset \mathcal{H}_{0,D}^2(\Omega) \\ \dim V = j}} \max_{\substack{v \in V \\ u \neq 0}} \frac{\mathcal{Q}_\sigma(v, v)}{\int_{\partial\Omega} \left(\frac{\partial v}{\partial\nu}\right)^2 d\sigma}, \tag{3.2}$$

where each eigenvalue is repeated according to its multiplicity. Moreover, there exists a Hilbert basis $\{v_j\}_{j=1}^\infty$ of $\mathcal{B}_D(\Omega)$ of eigenfunctions v_j associated with the eigenvalues η_j. Finally, by normalizing the eigenfunctions v_j with respect to $\mathcal{Q}_\sigma$ for all $j \geq 1$, the functions $\hat{v}_j := \sqrt{\eta_j}\gamma_1(v_j)$ define a Hilbert basis of $L^2(\partial\Omega)$ with respect to its standard scalar product.

Problem (NBS) is understood in the weak sense as follows:

$$\int_\Omega (1-\sigma)D^2u : D^2\varphi + \sigma\,\Delta u \Delta\varphi dx = \xi \int_{\partial\Omega} u\varphi d\sigma\,, \quad \forall\varphi \in \mathcal{H}_{0,N}^2(\Omega), \tag{3.3}$$

in the unknowns $u \in \mathcal{H}_{0,N}^2(\Omega)$, $\xi \in \mathbb{R}$. We have the following theorem.

Theorem 3.2 *Let Ω be a bounded domain in $\mathbb{R}^N$ of class $C^{0,1}$ and let $\sigma \in \left(-\frac{1}{N-1}, 1\right)$. The eigenvalues of problem (3.3) have finite multiplicity and are given by a non-decreasing sequence of non-negative real numbers ξ_j defined by*

$$\xi_j = \min_{\substack{U \subset \mathcal{H}_{0,N}^2(\Omega) \\ \dim U = j}} \max_{\substack{u \in U \\ u \neq 0}} \frac{\mathcal{Q}_\sigma(u, u)}{\int_{\partial\Omega} u^2 d\sigma},$$

where each eigenvalue is repeated according to its multiplicity. The first eigenvalue $\xi_1 = 0$ has multiplicity one and the corresponding eigenfunctions are the constant functions on Ω. Moreover, there exists a Hilbert basis $\{u_j\}_{j=1}^\infty$ of $\mathcal{B}_N(\Omega)$ of eigenfunctions u_j associated with the eigenvalues ξ_j. Finally, by normalizing the eigenfunctions

u_j with respect to $\mathcal{Q}_\sigma$ for all $j \geq 2$, the functions $\hat{u}_j := \sqrt{\xi_j}\gamma_0(u_j)$, $j \geq 2$, and $\hat{u}_1 = |\partial\Omega|^{-1/2}$ define a Hilbert basis of $L^2(\partial\Omega)$ with respect to its standard scalar product.

The proofs of Theorems 3.1 and 3.2 can be carried out exactly as those of Theorems 3.3 and 3.10 presented in Subsects. 3.2 and 3.3. Note that the condition $\sigma \in \left(-\frac{1}{N-1}, 1\right)$ is used to guarantee the coercivity of the form $\mathcal{Q}_\sigma$ discussed in the previous section.

3.2 The BS_μ eigenvalue problem

For any $\mu \in \mathbb{R}$, the weak formulation of problem (BS_μ) reads

$$\int_\Omega (1-\sigma)D^2 v : D^2\varphi + \sigma\Delta v\Delta\varphi\, dx - \mu\int_{\partial\Omega} v\varphi\, d\sigma$$
$$= \lambda(\mu)\int_{\partial\Omega}\frac{\partial v}{\partial\nu}\frac{\partial\varphi}{\partial\nu}d\sigma\,, \quad \forall\varphi \in H^2(\Omega), \tag{3.4}$$

in the unknowns $v \in H^2(\Omega)$, $\lambda(\mu) \in \mathbb{R}$, and can be re-written as

$$\mathcal{Q}_{\mu,D}(v,\varphi) = \lambda(\mu)\,(\gamma_1(v), \gamma_1(\varphi))_{\partial\Omega}\,, \quad \forall\varphi \in H^2(\Omega).$$

We prove that for all $\mu < 0$, problem (BS_μ) admits an increasing sequence of eigenvalues of finite multiplicity diverging to $+\infty$ and the corresponding eigenfunctions form a basis of $H^2_{\mu,D}(\Omega)$, where $H^2_{\mu,D}(\Omega)$ denotes the orthogonal complement of $\mathcal{H}^2_{0,N}(\Omega)$ in $H^2(\Omega)$ with respect to the scalar product $\mathcal{Q}_{\mu,D}$, namely

$$H^2_{\mu,D}(\Omega) = \left\{v \in H^2(\Omega) : \mathcal{Q}_{\mu,D}(v,\varphi) = 0\,, \forall\varphi \in \mathcal{H}^2_{0,N}(\Omega)\right\}. \tag{3.5}$$

To do so, we recast problem (3.4) in the form of an eigenvalue problem for a compact self-adjoint operator acting on a Hilbert space. We consider on $H^2(\Omega)$ the equivalent norm

$$\|v\|^2_{\mu,D} = \mathcal{Q}_{\mu,D}(v,v)$$

which is associated with the scalar product defined by

$$\langle v,\varphi\rangle_{\mu,D} = \mathcal{Q}_{\mu,D}(v,\varphi),$$

for all $v, \varphi \in H^2(\Omega)$. Then we define the operator $B_{\mu,D}$ from $H^2(\Omega)$ to its dual $(H^2(\Omega))'$ by setting

$$B_{\mu,D}(v)[\varphi] = \langle v,\varphi\rangle_{\mu,D}\,, \quad \forall v, \varphi \in H^2(\Omega).$$

By the Riesz Theorem it follows that $B_{\mu,D}$ is a surjective isometry. Then we consider the operator J_1 from $H^2(\Omega)$ to $(H^2(\Omega))'$ defined by

$$J_1(v)[\varphi] = (\gamma_1(v), \gamma_1(\varphi))_{\partial\Omega}, \quad \forall v, \varphi \in H^2(\Omega). \tag{3.6}$$

The operator J_1 is compact since γ_1 is a compact operator from $H^2(\Omega)$ to $L^2(\partial\Omega)$. Finally, we set

$$T_{\mu,D} = B_{\mu,D}^{(-1)} \circ J_1. \tag{3.7}$$

From the compactness of J_1 and the boundedness of $B_{\mu,D}$ it follows that $T_{\mu,D}$ is a compact operator from $H^2(\Omega)$ to itself. Moreover, $\langle T_{\mu,D}(v), \varphi \rangle_{\mu,D} = (\gamma_1(v), \gamma_1(\varphi))_{\partial\Omega}$, for all $u, \varphi \in H^2(\Omega)$, hence $T_{\mu,D}$ is self-adjoint. Note that $\mathrm{Ker}\, T_{\mu,D} = \mathrm{Ker}\, J_1 = \mathcal{H}_{0,N}^2(\Omega)$ and the non-zero eigenvalues of $T_{\mu,D}$ coincide with the reciprocals of the eigenvalues of (3.4), the eigenfunctions being the same.

We are now ready to prove the following theorem.

Theorem 3.3 *Let Ω be a bounded domain in $\mathbb{R}^N$ of class $C^{0,1}$ and let $\sigma \in \left(-\frac{1}{N-1}, 1\right)$. Let $\mu < 0$. Then the eigenvalues of (3.4) have finite multiplicity and are given by a non-decreasing sequence of positive real numbers $\{\lambda_j(\mu)\}_{j=1}^{\infty}$ defined by*

$$\lambda_j(\mu) = \min_{\substack{V \subset H^2(\Omega) \\ \dim V = j}} \max_{\substack{v \in V \\ \frac{\partial v}{\partial \nu} \neq 0}} \frac{\mathcal{Q}_{\mu,D}(v, v)}{\int_{\partial\Omega} \left(\frac{\partial v}{\partial \nu}\right)^2 d\sigma}, \tag{3.8}$$

where each eigenvalue is repeated according to its multiplicity.

Moreover there exists a basis $\{v_{j,\mu}\}_{j=1}^{\infty}$ of $H_{\mu,D}^2(\Omega)$ of eigenfunctions $v_{j,\mu}$ associated with the eigenvalues $\lambda_j(\mu)$.

By normalizing the eigenfunctions $v_{j,\mu}$ with respect to $\mathcal{Q}_{\mu,D}$, the functions defined by $\{\hat{v}_{j,\mu}\}_{j=1}^{\infty} := \{\sqrt{\lambda_j(\mu)}\gamma_1(v_{j,\mu})\}_{j=1}^{\infty}$ form a Hilbert basis of $L^2(\partial\Omega)$ with respect to its standard scalar product.

Proof Since $\mathrm{Ker}\, T_{\mu,D} = \mathcal{H}_{0,N}^2$, by the Hilbert-Schmidt Theorem applied to the compact self-adjoint operator $T_{\mu,D}$ it follows that $T_{\mu,D}$ admits an increasing sequence of positive eigenvalues $\{q_j\}_{j=1}^{\infty}$ bounded from above, converging to zero and a corresponding Hilbert basis $\{v_{j,\mu}\}_{j=1}^{\infty}$ of eigenfunctions of $H_{\mu,D}^2(\Omega)$. Since $q \neq 0$ is an eigenvalue of $T_{\mu,D}$ if and only if $\lambda = \frac{1}{q}$ is an eigenvalue of (3.4) with the same eigenfunctions, we deduce the validity of the first part of the statement. In particular, formula (3.8) follows from the standard min-max formula for the eigenvalues of compact self-adjoint operators. Note that $\lambda_1(\mu) > 0$, since $\mathcal{Q}_{\mu,D}(v, v) = 0$ if and only if $v = 0$.

To prove the final part of the theorem, we recast problem (3.4) into an eigenvalue problem for the compact self-adjoint operator $T'_{\mu,D} = \gamma_1 \circ B_{\mu,D}^{(-1)} \circ J'_1$, where J'_1 denotes the map from $L^2(\partial\Omega)$ to the dual of $H^2(\Omega)$ defined by

$$J'_1(v)[\varphi] = (v, \gamma_1(\varphi))_{\partial\Omega}, \quad \forall v \in L^2(\partial\Omega), \varphi \in H^2(\Omega).$$

We apply again the Hilbert-Schmidt Theorem and observe that $T_{\mu,D}$ and $T'_{\mu,D}$ admit the same non-zero eigenvalues and that the eigenfunctions of $T'_{\mu,D}$ are exactly the normal derivatives of the eigenfunctions of $T_{\mu,D}$. From (3.4) we deduce that if the eigenfunctions $v_{j,\mu}$ of $T_{\mu,D}$ are normalized by $\mathcal{Q}_{\mu,D}(v_{j,\mu}, v_{k,\mu}) = \delta_{jk}$, where δ_{jk} is the Kronecker symbol, then the normalization of the traces of their normal derivatives in $L^2(\partial\Omega)$ are obtained by multiplying $\gamma_1(v_{j,\mu})$ by $\sqrt{\lambda_j(\mu)}$. This concludes the proof. $\qquad\square$

We present now a few results on the behavior of the eigenvalues of (3.4) for $\mu \in (-\infty, 0)$, in particular we prove a Lipschitz continuity result for the eigenvalues $\lambda_j(\mu)$ with respect to μ and find their limits as $\mu \to -\infty$.

Theorem 3.4 *For any $j \in \mathbb{N}$ and $\delta > 0$, the function $\lambda_j : (-\infty, -\delta] \to (0, +\infty)$ which takes $\mu \in (-\infty, -\delta]$ to $\lambda_j(\mu) \in (0, +\infty)$ is Lipschitz continuous on $(-\infty, -\delta]$.*

Proof Without loss of generality we assume that $\mu_1, \mu_2 \in (-\infty, -\delta]$ and that $\mu_1 < \mu_2$. Let $v \in H^2(\Omega)$. Then

$$
\begin{aligned}
0 &\leq \frac{\mathcal{Q}_{\mu_1,D}(v, v)}{\int_{\partial\Omega}\left(\frac{\partial v}{\partial \nu}\right)^2 d\sigma} - \frac{\mathcal{Q}_{\mu_2,D}(v, v)}{\int_{\partial\Omega}\left(\frac{\partial v}{\partial \nu}\right)^2 d\sigma} = (\mu_2 - \mu_1)\frac{\int_{\partial\Omega} v^2 d\sigma}{\int_{\partial\Omega}\left(\frac{\partial v}{\partial \nu}\right)^2 d\sigma} \\
&\leq -\frac{(\mu_2 - \mu_1)}{\mu_1}\frac{\mathcal{Q}_{\mu_1,D}(v, v)}{\int_{\partial\Omega}\left(\frac{\partial v}{\partial \nu}\right)^2 d\sigma}.
\end{aligned}
$$

Hence

$$
\frac{\mathcal{Q}_{\mu_1,D}(v, v)}{\int_{\partial\Omega}\left(\frac{\partial v}{\partial \nu}\right)^2 d\sigma} \geq \frac{\mathcal{Q}_{\mu_2,D}(v, v)}{\int_{\partial\Omega}\left(\frac{\partial v}{\partial \nu}\right)^2 d\sigma} \tag{3.9}
$$

and

$$
\frac{\mathcal{Q}_{\mu_2,D}(v, v)}{\int_{\partial\Omega}\left(\frac{\partial v}{\partial \nu}\right)^2 d\sigma} \geq \frac{\mathcal{Q}_{\mu_1,D}(v, v)}{\int_{\partial\Omega}\left(\frac{\partial v}{\partial \nu}\right)^2 d\sigma}\left(1 + \frac{(\mu_2 - \mu_1)}{\mu_1}\right) \tag{3.10}
$$

By taking the infimum and the supremum over j dimensional subspaces of $H^2(\Omega)$ into (3.9) and (3.10), and by (3.8), we get

$$
|\lambda_j(\mu_1) - \lambda_j(\mu_2)| \leq \frac{\lambda_j(\mu_1)}{|\mu_1|}|\mu_2 - \mu_1| \leq \lambda_j(\mu_1)\frac{|\mu_2 - \mu_1|}{\delta}.
$$

This concludes the proof. $\qquad\square$

We now investigate the behavior of the eigenvalues $\lambda_j(\mu)$ as $\mu \to -\infty$. First, we need to recall a few facts about the convergence of operators defined on variable spaces. As customary, we consider families of spaces and operators depending on a small parameter $\varepsilon \geq 0$ with $\varepsilon \to 0$. This will be applied later with $\varepsilon = -\frac{1}{\mu}$ and $\mu \to -\infty$.

Let us denote by $\mathcal{H}_\varepsilon$ a family of Hilbert spaces for all $\varepsilon \in [0, \varepsilon_0)$ and assume that there exists a corresponding family of linear operators $E_\varepsilon : \mathcal{H}_0 \to \mathcal{H}_\varepsilon$ such that, for any $u \in \mathcal{H}_0$

$$\|E_\varepsilon(u)\|_{\mathcal{H}_\varepsilon} \to \|u\|_{\mathcal{H}_0}, \quad \text{as } \varepsilon \to 0.$$

We recall the definition of compact convergence of operators in the sense of [33].

Definition 3.5 We say that a family $\{K_\varepsilon\}_{\varepsilon \in [0, \varepsilon_0)}$ of compact operators $K_\varepsilon \in \mathcal{L}(\mathcal{H}_\varepsilon)$ converges compactly to K_0 if

   i) for any $\{u_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0)}$ with $\|u_\varepsilon - E_\varepsilon(u)\|_{\mathcal{H}_\varepsilon} \to 0$ as $\varepsilon \to 0$, then $\|K_\varepsilon(u_\varepsilon) - E_\varepsilon(K_0(u))\|_{\mathcal{H}_\varepsilon} \to 0$ as $\varepsilon \to 0$;

   ii) for any $\{u_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0)}$ with $u_\varepsilon \in \mathcal{H}_\varepsilon$, $\|u_\varepsilon\|_{\mathcal{H}_\varepsilon} = 1$, then $\{K_\varepsilon(u_\varepsilon)\}_{\varepsilon \in (0, \varepsilon_0)}$ is precompact in the sense that for all sequences $\varepsilon_n \to 0$ there exist a sub-sequence $\varepsilon_{n_k} \to 0$ and $w \in \mathcal{H}_0$ such that $\|K_{\varepsilon_{n_k}}(u_{\varepsilon_{n_k}}) - E_{\varepsilon_{n_k}}(w)\|_{\mathcal{H}_{\varepsilon_{n_k}}} \to 0$ as $k \to +\infty$.

We also recall the following theorem, where by spectral convergence of a family of operators we mean the convergence of the eigenvalues and the convergence of the eigenfunctions in the sense of [33], see also [17, §2].

Theorem 3.6 *Let $\{K_\varepsilon\}_{\varepsilon \in [0, \varepsilon_0)}$ be non-negative, compact self-adjoint operators in the Hilbert spaces $\mathcal{H}_\varepsilon$. Assume that their eigenvalues are given by $\{\sigma_j(\varepsilon)\}_{j=1}^\infty$. If K_ε compactly converge to K_0, then there is spectral convergence of K_ε to K_0 as $\varepsilon \to 0$. In particular, for every $j \in \mathbb{N}$ $\sigma_j(\varepsilon) \to \sigma_j(0)$, as $\varepsilon \to 0$.*

Let $T_D : \mathcal{H}_{0,D}^2(\Omega) \to \mathcal{H}_{0,D}^2(\Omega)$ be defined by $T_D = B_D^{(-1)} \circ J_1$, where B_D is the operator from $\mathcal{H}_{0,D}^2(\Omega)$ to its dual $(\mathcal{H}_{0,D}^2(\Omega))'$ given by

$$B_D(v)[\varphi] = \mathcal{Q}_\sigma(v, \varphi), \quad \forall v, \varphi \in \mathcal{H}_{0,D}^2(\Omega),$$

and J_1 is defined in (3.6). By the Riesz Theorem it follows that B_D is a surjective isometry. The operator T_D is the resolvent operator associated with problem (3.1) and plays the same role of $T_{\mu,D}$ defined in (3.7). In fact, as in the proof of Theorem 3.3 it is possible to show that T_D admits an increasing sequence of non-zero eigenvalues $\{q_j\}_{j=1}^\infty$ bounded from above and converging to 0. Moreover, a number $q \neq 0$ is an eigenvalue of T_D if and only if $\eta = \frac{1}{q}$ is an eigenvalue of (3.1), with the same eigenfunctions.

We have now a family of compact self-adjoint operators $T_{\mu,D}$ each defined on the Hilbert space $H^2(\Omega)$ endowed with the scalar product $\mathcal{Q}_{\mu,D}$, and the compact self-adjoint operator T_D defined on $\mathcal{H}_{0,D}^2(\Omega)$ endowed with the scalar product $\mathcal{Q}_\sigma$. We are ready to state and prove the following theorem.

Theorem 3.7 *The family of operators $\{T_{\mu,D}\}_{\mu \in (-\infty, 0)}$ compactly converges to T_D as $\mu \to -\infty$. In particular,*

$$\lim_{\mu \to -\infty} \lambda_j(\mu) = \eta_j, \tag{3.11}$$

for all $j \in \mathbb{N}$, where η_j are the eigenvalues of (3.1).

Proof For each $\mu \in (-\infty, 0)$ we define the map $E_\mu \equiv E : \mathcal{H}^2_{0,D}(\Omega) \to H^2(\Omega)$ simply by setting $E(u) = u$, for all $u \in \mathcal{H}^2_{0,D}(\Omega)$.

In view of Definition 3.5, we have to prove that

i) if $\{u_\mu\}_{\mu<0} \subset H^2(\Omega)$ and $u \in \mathcal{H}^2_{0,D}(\Omega)$ are such that $\mathcal{Q}_{\mu,D}(u_\mu - u, u_\mu - u) \to 0$ as $\mu \to -\infty$, then

$$\mathcal{Q}_{\mu,D}(T_{\mu,D}(u_\mu) - T_D(u), T_{\mu,D}(u_\mu) - T_D(u)) \to 0, \quad \text{as } \mu \to -\infty;$$

ii) if $\{u_\mu\}_{\mu<0} \subset H^2(\Omega)$ is such that $\mathcal{Q}_{\mu,D}(u_\mu, u_\mu) = 1$ for all $\mu < 0$, then for every sequence $\mu_n \to -\infty$ there exists a sub-sequence $\mu_{n_k} \to -\infty$ and $v \in \mathcal{H}^2_{0,D}(\Omega)$ such that

$$\mathcal{Q}_{\mu_{n_k},D}(T_{\mu_{n_k},D}(u_{\mu_{n_k}}) - v, T_{\mu_{n_k},D}(u_{\mu_{n_k}}) - v) \to 0, \quad \text{as } k \to +\infty. \quad (3.12)$$

We start by proving i). By the assumptions in i), it follows that u_μ is uniformly bounded in $H^2(\Omega)$ for μ in a neighborhood of $-\infty$. Indeed, by definition

$$\mathcal{Q}_{\mu,D}(T_{\mu,D}(u_\mu), \varphi) = \int_{\partial\Omega} \frac{\partial u_\mu}{\partial \nu} \frac{\partial \varphi}{\partial \nu} d\sigma, \quad \forall \varphi \in H^2(\Omega), \quad (3.13)$$

hence, by choosing $\varphi = T_{\mu,D}(u_\mu)$, we find that the family $\{T_{\mu,D}(u_\mu)\}_{\mu<0}$ is bounded in $H^2(\Omega)$. Thus, possibly passing to a sub-sequence, $T_{\mu,D}(u_\mu) \rightharpoonup v$ in $H^2(\Omega)$, and $\gamma_0(T_{\mu,D}(u_\mu)) \to \gamma_0(v)$ in $L^2(\partial\Omega)$, as $\mu \to -\infty$, which implies that $\gamma_0(v) = 0$ since the term $-\mu \int_{\partial\Omega} T_{\mu,D}(u_\mu)^2 d\sigma$ is bounded in μ. Thus $v \in \mathcal{H}^2_{0,D}(\Omega)$.

Choosing $\varphi \in \mathcal{H}^2_{0,D}(\Omega)$ and passing to the limit in (3.13) we have that

$$\mathcal{Q}_\sigma(v, \varphi) = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \frac{\partial \varphi}{\partial \nu} d\sigma, \quad \forall \varphi \in \mathcal{H}^2_{0,D}(\Omega),$$

hence $v = T_D(u)$. Thus $T_{\mu,D}(u_\mu) \rightharpoonup T_D(u)$ in $H^2(\Omega)$. Moreover, the convergence is stronger because

$$\lim_{\mu \to -\infty} \mathcal{Q}_{\mu,D}(T_{\mu,D}(u_\mu) - T_D(u), T_{\mu,D}(u_\mu) - T_D(u))$$

$$= \lim_{\mu \to -\infty} \left(\mathcal{Q}_{\mu,D}(T_{\mu,D}(u_\mu), T_{\mu,D}(u_\mu)) \right.$$

$$\left. -2\mathcal{Q}_{\mu,D}(T_{\mu,D}(u_\mu), T_D(u)) + \mathcal{Q}_{\mu,D}(T_D(u), T_D(u)) \right)$$

$$= \mathcal{Q}_\sigma(T_D(u), T_D(u)) - 2\mathcal{Q}_\sigma(T_D(u), T_D(u)) + \mathcal{Q}_\sigma(T_D(u), T_D(u)) = 0,$$

which proves point i).

Note that the equality $\lim_{\mu \to -\infty} \mathcal{Q}_{\mu,D}(T_{\mu,D}(u_\mu), T_{\mu,D}(u_\mu)) = \mathcal{Q}_\sigma(T_D(u), T_D(u))$ is a consequence of

$$\lim_{\mu \to -\infty} \mathcal{Q}_{\mu,D}(T_{\mu,D}(u_\mu), T_{\mu,D}(u_\mu)) = \lim_{\mu \to -\infty} \int_{\partial\Omega} \frac{\partial u_\mu}{\partial \nu} \frac{\partial T_{\mu,D}(u_\mu)}{\partial \nu} d\sigma$$

$$= \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \frac{\partial T_D(u)}{\partial \nu} d\sigma = \mathcal{Q}_\sigma(T_D(u), T_D(u)).$$

The proof of point ii) is similar. If $\mathcal{Q}_{\mu,D}(u_\mu, u_\mu) = 1$, up to sub-sequences $u_\mu \rightharpoonup u \in H^2(\Omega)$, $\gamma_0(u_\mu) \to \gamma_0(u)$, and $\gamma_1(u_\mu) \to \gamma_1(u)$ as $\mu \to -\infty$. Moreover, $\|\gamma_0(u_\mu)\|^2_{L^2(\partial\Omega)} \le -\frac{1}{\mu}$, hence $\|\gamma_0(u_\mu)\|^2_{L^2(\partial\Omega)} \to 0$ as $\mu \to -\infty$. This implies that $\gamma_0(u) = 0$ and that $u \in \mathcal{H}^2_{0,D}(\Omega)$. Then it is possible to repeat the same arguments above to conclude the validity of (3.12) with $v = T_D(u)$.

Thus $T_{\mu,D}$ compactly converges to T_D and (3.11) follows by Theorem 3.6. $\qquad \square$

Remark 3.8 We also note that each eigenvalue $\lambda_j(\mu)$ is non-increasing with respect to μ, for $\mu \in (-\infty, 0)$. In fact from the Min-Max Principle (3.8) it immediately follows that for all $j \in \mathbb{N}$, $\lambda_j(\mu_1) \le \lambda_j(\mu_2)$ if $\mu_1 > \mu_2$.

Now we consider the behavior of the first eigenvalue as $\mu \to 0^-$.

Lemma 3.9 *We have*

$$\lim_{\mu \to 0^-} \lambda_1(\mu) = 0$$

Proof Let $p \in \mathbb{R}^N$ be fixed. From (3.8) we get

$$0 < \lambda_1(\mu) = \min_{\substack{v \in H^2(\Omega) \\ v \ne 0}} \frac{\mathcal{Q}_{\mu,D}(v,v)}{\int_{\partial\Omega} \left(\frac{\partial v}{\partial \nu}\right)^2 d\sigma} \le \frac{\mathcal{Q}_{\mu,D}(p \cdot x, p \cdot x)}{\int_{\partial\Omega}(p \cdot v)^2 d\sigma} = -\mu \frac{\int_{\partial\Omega}(p \cdot x)^2 d\sigma}{\int_{\partial\Omega}(p \cdot v)^2 d\sigma},$$

for all $\mu \in (-\infty, 0)$. By letting $\mu \to 0^-$ we obtain the result. $\qquad \square$

3.3 The BS_λ eigenvalue problem

The weak formulation of problem (BS$_\lambda$) reads

$$\int_\Omega (1-\sigma)D^2 u : D^2\varphi + \sigma \Delta u \Delta \varphi dx - \lambda \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \frac{\partial \varphi}{\partial \nu} d\sigma$$

$$= \mu(\lambda) \int_{\partial\Omega} u\varphi d\sigma, \quad \forall \varphi \in H^2(\Omega), \tag{3.14}$$

in the unknowns $u \in H^2(\Omega)$, $\mu(\lambda) \in \mathbb{R}$, and can be re-written as

$$\mathcal{Q}_{\lambda,N}(u, \varphi) = \mu(\lambda)(\gamma_0(u), \gamma_0(\varphi))_{\partial\Omega}, \quad \forall \varphi \in H^2(\Omega).$$

We prove that for all $\lambda < \eta_1$, where η_1 is the first eigenvalue of (DBS), problem (BS$_\lambda$) admits an increasing sequence of eigenvalues of finite multiplicity diverging to $+\infty$ and the corresponding eigenfunctions form a basis of $H^2_{\lambda,N}(\Omega)$, where $H^2_{\lambda,N}(\Omega)$

denotes the orthogonal complement of $\mathcal{H}^2_{0,D}(\Omega)$ in $H^2(\Omega)$ with respect to $\mathcal{Q}_{\lambda,N}$, that is

$$H^2_{\lambda,N}(\Omega) = \left\{ u \in H^2(\Omega) : \mathcal{Q}_{\lambda,N}(u,\varphi) = 0, \ \forall \varphi \in \mathcal{H}^2_{0,D}(\Omega) \right\}. \tag{3.15}$$

Since in general $\mathcal{Q}_{\lambda,N}$ is not a scalar product, we find it convenient to consider on $H^2(\Omega)$ the norm

$$\|u\|^2_{\lambda,N} = \mathcal{Q}_{\lambda,N}(u,u) + b\|\gamma_0(u)\|^2_{L^2(\Omega)}, \tag{3.16}$$

where $b > 0$ is a fixed number which is chosen as follows. If $\lambda < 0$, no restrictions are required on $b > 0$, since the norm $\|\cdot\|_{\lambda,N}$ is equivalent to the standard norm of $H^2(\Omega)$ for all $b > 0$. Assume now that $0 \le \lambda < \eta_1$. From Theorem 3.7 and Lemma 3.9 we have that $(0, \eta_1) \subseteq \lambda_1((-\infty, 0))$, hence there exists $\mu \in (-\infty, 0)$ and $\varepsilon \in (0, 1)$ such that $\lambda_1(\mu) = \frac{\lambda+\varepsilon}{1-\varepsilon} < \eta_1$. Then

$$\begin{aligned} \mathcal{Q}_{\lambda,N}(u,u) &= \varepsilon \mathcal{Q}_{-1,N}(u,u) + (1-\varepsilon)\mathcal{Q}_{\lambda_1(\mu),N}(u,u) \\ &\ge \varepsilon \mathcal{Q}_{-1,N}(u,u) + (1-\varepsilon)\mu\|\gamma_0(u)\|^2_{L^2(\partial\Omega)}. \end{aligned} \tag{3.17}$$

Thus, by choosing any b satisfying

$$b > -(1-\varepsilon)\mu, \tag{3.18}$$

it follows by (3.17) and (3.18) that $\|\cdot\|_{\lambda,N}$ is a norm equivalent to the standard norm of $H^2(\Omega)$.

The norm $\|\cdot\|_{\lambda,N}$ is associated with the scalar product defined by

$$\langle u, \varphi \rangle_{\lambda,N} = \mathcal{Q}_{\lambda,N}(u,\varphi) + b(\gamma_0(u), \gamma_0(\varphi))_{\partial\Omega}, \tag{3.19}$$

for all $u, \varphi \in H^2(\Omega)$.

We now recast problem (3.14) in the form of an eigenvalue problem for a compact self-adjoint operator acting on a Hilbert space. To do so, we define the operator $B_{\lambda,N}$ from $H^2(\Omega)$ to its dual $(H^2(\Omega))'$ by setting

$$B_{\lambda,N}(u)[\varphi] = \langle u, \varphi \rangle_{\lambda,N}, \quad \forall u, \varphi \in H^2(\Omega).$$

By the Riesz Theorem it follows that $B_{\lambda,N}$ is a surjective isometry. Then we consider the operator J_0 from $H^2(\Omega)$ to $(H^2(\Omega))'$ defined by

$$J_0(u)[\varphi] = (\gamma_0(u), \gamma_0(\varphi))_{\partial\Omega}, \quad \forall u, \varphi \in H^2(\Omega). \tag{3.20}$$

The operator J_0 is compact since γ_0 is a compact operator from $H^2(\Omega)$ to $L^2(\partial\Omega)$. Finally, we set

$$T_{\lambda,N} = B^{(-1)}_{\lambda,N} \circ J_0. \tag{3.21}$$

From the compactness of J_0 and the boundedness of $B_{\lambda,N}$ it follows that $T_{\lambda,N}$ is a compact operator from $H^2(\Omega)$ to itself. Moreover, $\langle B_{\lambda,N}(u), \varphi \rangle_{\lambda,N} = (\gamma_0(u), \gamma_0(\varphi))_{\partial\Omega}$, for all $u, \varphi \in H^2(\Omega)$, hence $T_{\lambda,N}$ is self-adjoint.

Note that $\text{Ker } T_{\lambda,N} = \text{Ker } J_0 = \mathcal{H}^2_{0,D}(\Omega)$ and the non-zero eigenvalues of $T_{\lambda,N}$ coincide with the reciprocals of the eigenvalues of (3.14) shifted by b, the eigenfunctions being the same.

We are now ready to prove the following theorem.

Theorem 3.10 *Let Ω be a bounded domain in $\mathbb{R}^N$ of class $C^{0,1}$, $\sigma \in \left(-\frac{1}{N-1}, 1\right)$, and $\lambda < \eta_1$. Then the eigenvalues of (3.14) have finite multiplicity and are given by a non-decreasing sequence of real numbers $\{\mu_j(\lambda)\}_{j=1}^{\infty}$ defined by*

$$\mu_j(\lambda) = \min_{\substack{U \subset H^2(\Omega) \\ \dim U = j}} \max_{\substack{u \in U \\ \gamma_0(u) \neq 0}} \frac{\mathcal{Q}_{\lambda,N}(u,u)}{\int_{\partial\Omega} u^2 d\sigma}, \tag{3.22}$$

where each eigenvalue is repeated according to its multiplicity. Moreover, there exists a Hilbert basis $\{u_{j,\lambda}\}_{j=1}^{\infty}$ of $H^2_{\lambda,N}(\Omega)$ (endowed with the scalar product (3.19)) of eigenfunctions $u_{j,\lambda}$ associated with the eigenvalues $\mu_j(\lambda)$ and the following statements hold:

i) *If $\lambda < 0$ then $\mu_1(\lambda) = 0$ is an eigenvalue of multiplicity one and the corresponding eigenfunctions are the constant functions. Moreover, if $\tilde{u}_{j,\lambda}$ denote the normalizations of $u_{j,\lambda}$ with respect to $\mathcal{Q}_{\lambda,N}$ for all $j \geq 2$, the functions $\hat{u}_{j,\lambda} := \sqrt{\mu_j(\lambda)}\gamma_0(\tilde{u}_{j,\lambda})$, $j \geq 2$, and $\hat{u}_{1,\lambda} := |\partial\Omega|^{-1/2}$ define a Hilbert basis of $L^2(\partial\Omega)$ with respect to its standard scalar product.*

ii) *If $0 \leq \lambda < \eta_1$, then $\mu(\lambda) = 0$ is an eigenvalue. Moreover, if $\mu_{j_0}(\lambda)$ is the first positive eigenvalue, and $\tilde{u}_{j,\lambda}$ denote the normalizations of $u_{j,\lambda}$ with respect to $\mathcal{Q}_{\lambda,N}$ for all $j \geq j_0$, and $\{\hat{u}_{j,\lambda}\}_{j=1}^{j_0-1}$ denotes a orthonormal basis with respect to the $L^2(\partial\Omega)$ scalar product of the eigenspace associated to $\mu_1(\lambda), ..., \mu_{j_0-1}(\lambda)$ restricted to $\partial\Omega$, then the functions $\hat{u}_{j,\lambda} := \sqrt{\mu_j(\lambda)}\gamma_0(\tilde{u}_{j,\lambda})$, $j \geq j_0$, and $\{\hat{u}_{j,0}\}_{j=1}^{j_0-1}$, define a Hilbert basis of $L^2(\partial\Omega)$ with respect to its standard scalar product. Finally, if $\lambda = 0$, then $j_0 = N + 2$ and the eigenspace corresponding to $\mu_1(0) = \cdots = \mu_{N+1}(0) = 0$ is generated by $\{1, x_1, ..., x_N\}$; if $\lambda > 0$, then $\mu_1(\lambda) < 0$.*

Proof Since $\text{Ker } J_0 = \mathcal{H}^2_{0,D}(\Omega)$, by the Hilbert-Schmidt Theorem applied to $T_{\lambda,N}$ it follows that $T_{\lambda,N}$ admits a non-increasing sequence of positive eigenvalues $\{p_j\}_{j=1}^{\infty}$ bounded from above, converging to zero and a corresponding Hilbert basis $\{u_{j,\lambda}\}$ of eigenfunctions of $H^2_{\lambda,N}(\Omega)$. We note that $p \neq 0$ is an eigenvalue of $T_{\lambda,N}$ if and only if $\mu = \frac{1}{p} - b$ is an eigenvalue of (3.14), the eigenfunction being the same.

Formula (3.22) follows from the standard min-max formula for the eigenvalues of compact self-adjoint operators.

If $\lambda < 0$, then $\mu_1(\lambda) = 0$ and a corresponding eigenfunction $u_{1,\lambda}$ satisfies $D^2 u_{1,\lambda} = 0$ in Ω, hence it is a linear function; moreover, since $\frac{\partial u_{1,\lambda}}{\partial \nu} = 0$ on $\partial\Omega$, $u_{1,\lambda}$ has to be constant.

If $\lambda = 0$, then $\mu = 0$ is an eigenvalue and a corresponding eigenfunction is a linear function. Hence $\mu_1(0) = \cdots = \mu_{N+1}(0) = 0$ and the associated eigenspace is spanned by $\{1, x_1, ..., x_N\}$.

If $0 < \lambda < \eta_1$, then by (3.11) and Lemma 3.9, there exists $\mu < 0$ such that $\lambda_1(\mu) = \lambda$, hence μ is an eigenvalue of (3.14). Moreover, by definition we have that for all $u \in H^2(\Omega)$ with $\gamma_0(u) \neq 0$

$$\frac{Q_{\lambda,N}(u,u)}{\int_{\partial\Omega} u^2 d\sigma} = \frac{Q_{\lambda_1(\mu),N}(u,u)}{\int_{\partial\Omega} u^2 d\sigma} \geq \mu,$$

hence $\mu_1(\lambda) = \mu < 0$.

To prove the final part of the theorem, we recast problem (3.14) into an eigenvalue problem for the compact self-adjoint operator $T'_{\lambda,N} = \gamma_0 \circ B^{(-1)}_{\lambda,N} \circ J'_0$, where J'_0 denotes the map from $L^2(\partial\Omega)$ to the dual of $H^2(\Omega)$ defined by

$$J'_0(u)[\varphi] = (u, \gamma_0(\varphi))_{\partial\Omega}, \quad \forall u \in L^2(\partial\Omega), \varphi \in H^2(\Omega).$$

We apply again the Hilbert-Schmidt Theorem and observe that $T_{\lambda,N}$ and $T'_{\lambda,N}$ admit the same non-zero eigenvalues and that the eigenfunctions of $T'_{\lambda,N}$ are exactly the traces of the eigenfunctions of $T_{\lambda,N}$. From (3.14) we deduce that if we normalize the eigenfunction $u_{j,\lambda}$ of $T_{\lambda,N}$ associated with positive eigenvalues and we denote them by $\tilde{u}_{j,\lambda}$, then the normalization of their traces in $L^2(\partial\Omega)$ are obtained by multiplying $\gamma_0(\tilde{u}_{j,\lambda})$ by $\sqrt{\mu_j(\lambda)}$. The rest of the proof easily follows.   □

As we have done for problem (3.4), we present now a few results on the behavior of the eigenvalues of (3.14) for $\lambda \in (-\infty, \eta_1)$. We have the following theorem on the Lipschitz continuity of eigenvalues, the proof of which is similar to that of Theorem 3.4 and is accordingly omitted.

Theorem 3.11 *For any $j \in \mathbb{N}$ and $\delta > 0$, the functions $\mu_j : (-\infty, \eta_1 - \delta] \to [0, +\infty)$ which takes $\lambda \in (-\infty, \eta_1 - \delta]$ to $\mu_j(\lambda) \in \mathbb{R}$ are Lipschitz continuous on $(-\infty, \eta_1 - \delta]$.*

We now investigate the behavior of the eigenvalues $\mu_j(\lambda)$ as $\lambda \to -\infty$. In order state the analogue of Theorem 3.7, we consider the operator $T_N : \mathcal{H}^2_{0,N}(\Omega) \to \mathcal{H}^2_{0,N}(\Omega)$ defined by $T_N = B^{(-1)}_N \circ J_0$, where B_N is the operator from $\mathcal{H}^2_{0,N}(\Omega)$ to its dual $(\mathcal{H}^2_{0,N}(\Omega))'$ given by

$$B_N(v)[\varphi] = \mathcal{Q}_\sigma(v, \varphi) + b(\gamma_0(v), \gamma_0(\varphi))_{\partial\Omega}, \quad \forall v, \varphi \in \mathcal{H}^2_{0,N}(\Omega), \qquad (3.23)$$

and b has the same value as in the definition of the operator $T_{\lambda,N}$, see (3.18), and J_0 is defined in (3.20). Note that the constant b can be chosen to be independent of λ for $\lambda < 0$. By the Riesz Theorem it follows that B_N is a surjective isometry. The operator T_N is the resolvent operator associated with problem (3.3) and plays the same role of $T_{\lambda,N}$ defined in (3.21). In fact, as in the proof of Theorem 3.10 it is possible to show that T_N admits an increasing sequence of non-zero eigenvalues $\{p_j\}_{j=1}^\infty$ bounded from above and converging to 0. Moreover, a number $p \neq 0$ is an eigenvalue of T_N if and only if $\xi = \frac{1}{p} - b$ is an eigenvalue of (3.3), with the same eigenfunctions.

We have now a family of compact self-adjoint operators $T_{\lambda,N}$ each defined on the Hilbert space $H^2(\Omega)$ endowed with the scalar product (3.19), and the compact self-adjoint operator T_N defined on $\mathcal{H}^2_{0,N}(\Omega)$ endowed with the scalar product defined by the right-hand side of (3.23). We have the following theorem, the proof of which is similar to that of Theorem 3.7 and is accordingly omitted.

Theorem 3.12 *The family of operators* $\{T_{\lambda,N}\}_{\lambda \in (-\infty, \eta_1)}$ *compactly converges to* T_N *as* $\lambda \to -\infty$. *In particular,*

$$\lim_{\lambda \to -\infty} \mu_j(\lambda) = \xi_j, \tag{3.24}$$

for all $j \in \mathbb{N}$, *where* ξ_j *are the eigenvalues of* (3.3).

Remark 3.13 We also note that each eigenvalue $\mu_j(\lambda)$ is non-increasing with respect to λ, for $\lambda \in (-\infty, \eta_1)$. In fact from the Min-Max Principle (3.22) it immediately follows that for all $j \in \mathbb{N}$, $\mu_j(\lambda_1) \le \mu_j(\lambda_2)$ if $\lambda_1 > \lambda_2$.

4 Characterization of trace spaces of $H^2(\Omega)$ via biharmonic Steklov eigenvalues

In this section we shall use the Hilbert basis of eigenfunctions $v_{j,\mu}$ and $\hat{v}_{j,\mu}$ given by Theorem 3.3 and the Hilbert basis of eigenfunctions $u_{j,\lambda}$, $\hat{u}_{j,\lambda}$ given by Theorem 3.10, for all $\mu \in (-\infty, 0)$ and $\lambda \in (-\infty, \eta_1)$. We recall that by definition, the functions $v_{j,\mu}$ and $u_{j,\lambda}$ are normalized with respect to $Q_{\mu,D}(\cdot, \cdot)$ and $Q_{\lambda,N}(\cdot, \cdot) + b(\gamma_0(\cdot), \gamma_0(\cdot))_{\partial\Omega}$ respectively, while $\hat{v}_{j,\mu}$ and $\hat{u}_{j,\lambda}$ are normalized with respect to the standard scalar product of $L^2(\partial\Omega)$.

We will also denote by l^2 the space of sequences $s = (s_j)_{j=1}^{\infty}$ of real numbers satisfying $\|s\|^2_{l^2} = \sum_{j=1}^{\infty} s_j^2 < \infty$.

We define the spaces

$$\mathcal{S}^{\frac{3}{2}}(\partial\Omega) = \mathcal{S}^{\frac{3}{2}}_{\lambda}(\partial\Omega) := \left\{ f \in L^2(\partial\Omega) : f = \sum_{j=1}^{\infty} \hat{a}_j \hat{u}_{j,\lambda} \text{ with } \left(\sqrt{|\mu_j(\lambda)|} \hat{a}_j \right)_{j=1}^{\infty} \in l^2 \right\}, \tag{4.1}$$

and

$$\mathcal{S}^{\frac{1}{2}}(\partial\Omega) = \mathcal{S}^{\frac{1}{2}}_{\mu}(\partial\Omega) := \left\{ f \in L^2(\partial\Omega) : f = \sum_{j=1}^{\infty} \hat{b}_j \hat{v}_{j,\mu} \text{ with } \left(\sqrt{\lambda_j(\mu)} \hat{b}_j \right)_{j=1}^{\infty} \in l^2 \right\}. \tag{4.2}$$

These spaces are endowed with the natural norms defined by

$$\|f\|^2_{\mathcal{S}^{\frac{3}{2}}_{\lambda}(\partial\Omega)} = \sum_{j=1}^{j_0-1} \hat{a}_j^2 + \sum_{j=j_0}^{\infty} \mu_j(\lambda) \hat{a}_j^2,$$

where j_0 is as in Theorem 3.10, and

$$\|f\|^2_{\mathcal{S}^{\frac{1}{2}}_\mu(\partial\Omega)} = \sum_{j=1}^\infty \lambda_j(\mu)\hat{b}_j^2.$$

Recall that if $\lambda = 0$, $j_0 = N + 2$ and if $\lambda < 0$, then $j_0 = 2$.

These spaces allow to describe the trace spaces for $H^2(\Omega)$. In particular, $\mathcal{S}^{\frac{3}{2}}_\lambda(\partial\Omega)$ and $\mathcal{S}^{\frac{1}{2}}_\mu(\partial\Omega)$ turn out to be independent of λ and μ. Namely, we have the following.

Theorem 4.1 *Let Ω be a bounded domain in $\mathbb{R}^N$ of class $C^{0,1}$. Then*

$$\gamma_0(H^2(\Omega)) = \gamma_0(H^2_{\lambda,N}(\Omega)) = \mathcal{S}^{\frac{3}{2}}(\partial\Omega) \ (= \mathcal{S}^{\frac{3}{2}}_\lambda(\partial\Omega)) \tag{4.3}$$

and

$$\gamma_1(H^2(\Omega)) = \gamma_1(H^2_{\mu,D}(\Omega)) = \mathcal{S}^{\frac{1}{2}}(\partial\Omega) \ (= \mathcal{S}^{\frac{1}{2}}_\mu(\partial\Omega)). \tag{4.4}$$

In particular, the spaces $\mathcal{S}^{\frac{3}{2}}_\lambda(\partial\Omega)$ and $\mathcal{S}^{\frac{1}{2}}_\mu(\partial\Omega)$ do not depend on $\lambda \in (-\infty, \eta_1)$ and $\mu \in (-\infty, 0)$.

Moreover, if Ω is of class $C^{2,1}$ then

$$\Gamma(H^2(\Omega)) = \mathcal{S}^{\frac{3}{2}}(\partial\Omega) \times \mathcal{S}^{\frac{1}{2}}(\partial\Omega),$$

hence

$$\mathcal{S}^{\frac{3}{2}}(\partial\Omega) = H^{\frac{3}{2}}(\partial\Omega)$$

and

$$\mathcal{S}^{\frac{1}{2}}(\partial\Omega) = H^{\frac{1}{2}}(\partial\Omega).$$

Proof Let us begin by proving (4.3). By the definition of $H^2_{\lambda,N}(\Omega)$ given in (3.15) and by Theorem 3.10 we have that any $u \in H^2(\Omega)$ can be written as

$$u = u_\lambda + v_D$$

where $v_D \in \mathcal{H}^2_{0,D}(\Omega)$ and

$$u_\lambda = \sum_{j=1}^\infty a_j u_{j,\lambda}$$

for some coefficients a_j satisfying $\sum_{j=1}^\infty a_j^2 < \infty$. Here $\{u_{j,\lambda}\}_{j=1}^\infty$ is a orthonormal basis of $H^2_{\lambda,N}(\Omega)$ with respect to the scalar product (3.19) with b satisfying (3.18). Let j_0 be as in Theorem 3.10. Hence we can write

$$u_\lambda = \sum_{j=1}^{j_0-1} a_j u_{j,\lambda} + \sum_{j=j_0}^{\infty} a_j u_{j,\lambda}$$

$$= \sum_{j=1}^{j_0-1} a_j u_{j,\lambda} + \sum_{j=j_0}^{\infty} \left(a_j \sqrt{\mathcal{Q}_{\lambda,N}(u_{j,\lambda}, u_{j,\lambda})} \right) \cdot \frac{u_{j,\lambda}}{\sqrt{\mathcal{Q}_{\lambda,N}(u_{j,\lambda}, u_{j,\lambda})}}$$

$$= \sum_{j=1}^{j_0-1} a_j u_{j,\lambda} + \sum_{j=j_0}^{\infty} \tilde{a}_j \tilde{u}_{j,\lambda},$$

where $\tilde{u}_{j,\lambda} = \frac{u_{j,\lambda}}{\sqrt{\mathcal{Q}_{\lambda,N}(u_{j,\lambda}, u_{j,\lambda})}}$ are the eigenfunctions normalized with respect to $\mathcal{Q}_{\lambda,N}$ and $\tilde{a}_j$ still satisfy $\sum_{j=j_0+1}^{\infty} \tilde{a}_j^2 < \infty$ (in fact $0 < \mathcal{Q}_{\lambda,N}(u_{j,\lambda}, u_{j,\lambda}) \le 1$ for all $j \ge j_0$). Clearly $\gamma_0(u) = \gamma_0(u_\lambda)$, hence by the continuity of the trace operator we have that

$$\gamma_0(u_\lambda) = \sum_{j=1}^{j_0-1} a_j \gamma_0(u_{j,\lambda}) + \sum_{j=j_0}^{\infty} \tilde{a}_j \gamma_0(\tilde{u}_{j,\lambda})$$

$$= \sum_{j=1}^{j_0-1} \frac{a_j}{\sqrt{\mu_j(\lambda)+b}} \cdot \left(\sqrt{\mu_j(\lambda)+b} \cdot \gamma_0(u_{j,\lambda}) \right) + \sum_{j=j_0}^{\infty} \frac{\tilde{a}_j}{\sqrt{\mu_j(\lambda)}} \cdot \gamma_0 \left(\sqrt{\mu_j(\lambda)} \tilde{u}_{j,\lambda} \right)$$

$$= \sum_{j=1}^{j_0-1} \hat{a}_j \hat{u}_{j,\lambda} + \sum_{j=j_0}^{\infty} \hat{a}_j \hat{u}_{j,\lambda} = \sum_{j=1}^{\infty} \hat{a}_j \hat{u}_{j,\lambda},$$

where we have set

$$\hat{a}_j = \frac{a_j}{\sqrt{\mu_j(\lambda)+b}}, \quad \hat{u}_{j,\lambda} = \sqrt{\mu_j(\lambda)+b} \cdot \gamma_0(u_{j,\lambda})$$

for $j = 1, ..., j_0 - 1$ and

$$\hat{a}_j = \frac{\tilde{a}_j}{\sqrt{\mu_j(\lambda)}}, \quad \hat{u}_{j,\lambda} = \sqrt{\mu_j(\lambda)} \cdot \gamma_0(\tilde{u}_{j,\lambda})$$

for $j \ge j_0$. This proves that $\gamma_0(H^2_{\lambda,N}(\Omega)) \subseteq \mathcal{S}_\lambda^{\frac{3}{2}}(\partial\Omega)$.

We prove now the opposite inclusion. Let $f \in \mathcal{S}_\lambda^{\frac{3}{2}}(\partial\Omega)$. Then $f = \sum_{j=1}^{\infty} \hat{a}_j \hat{u}_{j,\lambda}$ with $\sum_{j=1}^{\infty} |\mu_j(\lambda)| \hat{a}_j^2 < \infty$. Let $u := \sum_{j=1}^{\infty} a_j u_{j,\lambda}$ where

$$a_j = \sqrt{\mu_j(\lambda)+b} \cdot \hat{a}_j \tag{4.5}$$

By definition, $u \in H^2(\Omega)$ since $\sum_{j=1}^{\infty} a_j^2 < \infty$. Moreover, we note that

$$f = \sum_{j=1}^{\infty} \hat{a}_j \hat{u}_{j,\lambda} = \sum_{j=1}^{\infty} \hat{a}_j \sqrt{\mu_j(\lambda) + b} \cdot \frac{\hat{u}_{j,\lambda}}{\sqrt{\mu_j(\lambda) + b}} = \sum_{j=1}^{\infty} \hat{a}_j \sqrt{\mu_j(\lambda) + b} \cdot \gamma_0(u_{j,\lambda}),$$

(4.6)

hence $f = \gamma_0(u) \in \gamma_0(H^2(\Omega))$.

The proof of (4.4) follows the same lines as that of (4.3) and is accordingly omitted.

We deduce then that the spaces $\mathcal{S}_\lambda^{\frac{3}{2}}(\partial\Omega)$ and $\mathcal{S}_\mu^{\frac{1}{2}}(\partial\Omega)$ do not depend on the particular choice of $\lambda \in (-\infty, \eta_1)$ and $\mu \in (-\infty, 0)$. In particular, we have proved that $\Gamma(H^2(\Omega)) \subseteq \mathcal{S}_\lambda^{\frac{3}{2}}(\partial\Omega) \times \mathcal{S}_\mu^{\frac{1}{2}}(\partial\Omega)$.

Assume now that Ω is of class $C^{2,1}$. We prove that $\mathcal{S}_\lambda^{\frac{3}{2}}(\partial\Omega) \times \mathcal{S}_\mu^{\frac{1}{2}}(\partial\Omega) \subseteq \Gamma(H^2(\Omega))$. This will imply $\Gamma(H^2(\Omega)) = \mathcal{S}_\lambda^{\frac{3}{2}}(\partial\Omega) \times \mathcal{S}_\mu^{\frac{1}{2}}(\partial\Omega)$.

Let $(f, g) \in \mathcal{S}_\lambda^{\frac{3}{2}}(\partial\Omega) \times \mathcal{S}_\mu^{\frac{1}{2}}(\partial\Omega)$. This means that $f = \gamma_0(u_\lambda)$, $g = \gamma_1(v_\mu)$ for some $u_\lambda \in H_{\lambda,N}^2(\Omega)$, $v_\mu \in H_{\mu,D}^2(\Omega)$. We claim that there exist $v_D \in \mathcal{H}_{0,D}^2(\Omega)$ and $u_N \in \mathcal{H}_{0,N}^2(\Omega)$ such that $u_\lambda + v_D = v_\mu + u_N$. To do so, it suffices to prove the existence of $v_D \in \mathcal{H}_{0,D}^2(\Omega)$ and $u_N \in \mathcal{H}_{0,N}^2(\Omega)$ such that $u_\lambda - v_\mu = u_N - v_D$. We claim that

$$H^2(\Omega) = \mathcal{H}_{0,D}^2(\Omega) + \mathcal{H}_{0,N}^2(\Omega).$$

(4.7)

Indeed, given $u \in H^2(\Omega)$, one can find by the classical Total Trace Theorem a function $u_1 \in H^2(\Omega)$ such that $\gamma_0(u_1) = 0$ and $\gamma_1(u_1) = \gamma_1(u)$. Thus $u = u_1 + (u - u_1)$ with $\gamma_1(u - u_1) = 0$ and the claim is proved. Thus the existence of functions v_D and u_N follows by (4.7) and the function $u = u_\lambda + v_D = v_\mu + u_N$ is such that $f = \gamma_0(u)$ and $g = \gamma_1(u)$.   □

Remark 4.2 Theorem 4.1 gives an explicit spectral characterization of the space $\gamma_0(H^2(\Omega))$ of traces of functions in $H^2(\Omega)$ when Ω is a bounded domain of class $C^{0,1}$ in $\mathbb{R}^N$. This space corresponds to $H^{\frac{3}{2}}(\partial\Omega)$ when Ω is of class $C^{2,1}$. In this case explicit descriptions of the space $H^{\frac{3}{2}}(\partial\Omega)$ are available in the literature and typically are given by means local charts and explicit representation of derivatives, see e.g., [22,31].

For domains of class $C^{0,1}$, it is not clear what is the appropriate definition of $H^{\frac{3}{2}}(\partial\Omega)$. Sometimes $H^{\frac{3}{2}}(\partial\Omega)$ is defined just by setting

$$H^{\frac{3}{2}}(\partial\Omega) := \gamma_0(H^2(\Omega)).$$

According to this definition, Theorem 4.1 implies that $H^{\frac{3}{2}}(\partial\Omega) = \mathcal{S}^{\frac{3}{2}}(\partial\Omega)$ also for domains of class $C^{0,1}$.

From Theorem 4.1 it follows that if Ω is a domain of class $C^{0,1}$, then

$$\Gamma(H^2(\Omega)) \subseteq \mathcal{S}^{\frac{3}{2}}(\partial\Omega) \times \mathcal{S}^{\frac{1}{2}}(\partial\Omega),$$

(4.8)

and equality holds if Ω is of class $C^{2,1}$. We observe that if Ω is not of class $C^{2,1}$, then in general equality does not hold in (4.8). Indeed, we have the following counterexample.

Counterexample 4.3 *Let* $\Omega = (0,1) \times (0,1)$ *be unit square in* $\mathbb{R}^2$. *We prove that*

$$\Gamma(H^2(\Omega)) \subsetneqq \mathcal{S}^{\frac{3}{2}}(\partial\Omega) \times \mathcal{S}^{\frac{1}{2}}(\partial\Omega).$$

To do so, we consider the real-valued function φ *defined in* Ω *by* $\varphi(x_1, x_2) = x_1$ *for all* $(x_1, x_2) \in \Omega$ *and we prove that the couple* $(\gamma_0(\varphi), 0) \in (\mathcal{S}^{\frac{3}{2}}(\partial\Omega) \times \mathcal{S}^{\frac{1}{2}}(\partial\Omega)) \setminus \Gamma(H^2(\Omega))$. *It is obvious that* $\gamma_0(\varphi) \in \mathcal{S}^{\frac{3}{2}}(\partial\Omega)$ *since* $\varphi \in H^2(\Omega)$. *Assume now by contradiction that* $(\gamma_0(\varphi), 0) \in \Gamma(H^2(\Omega))$, *that is, there exists* $u \in H^2(\Omega)$ *such that* $\gamma_0(u) = \gamma_0(\varphi)$ *and* $\gamma_1(u) = 0$. *Clearly, since* $\gamma_0(u) = \gamma_0(\varphi)$, *there exists* $v_D \in \mathcal{H}^2_{0,D}$ *such that*

$$u = \varphi + v_D$$

and hence

$$\gamma_1(v_D) = \gamma_1(u) - \gamma_1(\varphi) = -\nabla x_1 \cdot \nu_{|\partial\Omega} = -\nu_1.$$

It follows that v_D *is a function in* $H^2(\Omega)$ *such that* $\gamma_0(v_D) = 0$ *and* $\gamma_1(v_D) = -\nu_1$, *but this opposes a well-known necessary (and sufficient) condition for a couple* $(f, g) \in H^1(\partial\Omega) \times L^2(\partial\Omega)$ *to belong to* $\Gamma(H^2(\Omega))$, *namely*

$$\frac{\partial f}{\partial \tau} \tau + g\nu \in H^{\frac{1}{2}}(\partial\Omega), \tag{4.9}$$

where τ *is the unit tangent vector (positively oriented with respect to the outer unit* ν *to* Ω), *see* [21,22]. *Indeed, the couple* $(0, -\nu_1)$ *does not satisfy condition* (4.9).

In order to characterize those couples $(f, g) \in \mathcal{S}^{\frac{3}{2}}(\partial\Omega) \times \mathcal{S}^{\frac{1}{2}}(\partial\Omega)$ which belong to $\Gamma(H^2(\Omega))$ when Ω is of class $C^{0,1}$, we need the spaces $\mathscr{S}^{\frac{3}{2}}(\partial\Omega)$ and $\mathscr{S}^{\frac{1}{2}}(\partial\Omega)$ defined by

$$\mathscr{S}^{\frac{3}{2}}(\partial\Omega) := \gamma_0(\mathcal{H}^2_{0,N}) = \gamma_0(\mathcal{B}_N(\Omega))$$

and

$$\mathscr{S}^{\frac{1}{2}}(\partial\Omega) := \gamma_1(\mathcal{H}^2_{0,D}) = \gamma_1(\mathcal{B}_D(\Omega)).$$

The spaces $\mathscr{S}^{\frac{3}{2}}(\partial\Omega)$ and $\mathscr{S}^{\frac{1}{2}}(\partial\Omega)$ have explicit descriptions similar to those of $\mathcal{S}^{\frac{3}{2}}_\lambda(\partial\Omega)$ and $\mathcal{S}^{\frac{1}{2}}_\mu(\partial\Omega)$, namely

$$\mathscr{S}^{\frac{3}{2}}(\partial\Omega) = \left\{ f \in L^2(\partial\Omega) : f = \sum_{j=1}^{\infty} \hat{c}_j \hat{u}_j \text{ with } (\sqrt{\xi_j}\hat{c}_j)_{j=1}^{\infty} \in l^2 \right\}. \tag{4.10}$$

and

$$\mathscr{S}^{\frac{1}{2}}(\partial\Omega) = \left\{ g \in L^2(\partial\Omega) : \ g = \sum_{j=1}^{\infty} \hat{d}_j \hat{v}_j \text{ with } (\sqrt{\eta_j}\hat{d}_j)_{j=1}^{\infty} \in l^2 \right\}. \qquad (4.11)$$

Here $\hat{u}_j = \sqrt{\xi_j}\gamma_0(u_j)$, $j \geq 2$, where $\{u_j\}_{j=1}^{\infty}$ is a Hilbert basis of eigenfunctions of problem (3.3), normalized with respect to $\mathcal{Q}_\sigma$, with the understanding that u_1 and $\hat{u}_1$ equal the constant $|\partial\Omega|^{-1/2}$, and $\hat{v}_j = \sqrt{\eta_j}\gamma_1(v_j)$, where $\{v_j\}_{j=1}^{\infty}$ is a Hilbert basis of eigenfunctions of problem (3.1) normalized with respect to $\mathcal{Q}_\sigma$.

Note that

$$\mathscr{S}^{\frac{3}{2}}(\partial\Omega) \times \mathscr{S}^{\frac{1}{2}}(\partial\Omega) = \Gamma(\mathcal{H}^2_{0,N}(\Omega) + \mathcal{H}^2_{0,D}(\Omega)) \subseteq \Gamma(H^2(\Omega)).$$

One can see by similar arguments as in Counterexample 4.3 that in general $\mathscr{S}^{\frac{3}{2}}(\partial\Omega) \times \mathscr{S}^{\frac{1}{2}}(\partial\Omega) \subsetneqq \Gamma(H^2(\Omega))$ if Ω is not of class $C^{2,1}$, while equality occurs if Ω is of class $C^{2,1}$ by (4.7).

We are now ready to characterize the trace space $\Gamma(H^2(\Omega))$ for domains Ω of class $C^{0,1}$.

Theorem 4.4 *Let Ω be a bounded domain in $\mathbb{R}^N$ of class $C^{0,1}$. Let $(f, g) \in \mathcal{S}^{\frac{3}{2}}(\partial\Omega) \times \mathcal{S}^{\frac{1}{2}}(\partial\Omega) = \mathcal{S}^{\frac{3}{2}}_\lambda(\partial\Omega) \times \mathcal{S}^{\frac{1}{2}}_\mu(\partial\Omega)$ be given by*

$$f = \sum_{j=1}^{\infty} \hat{a}_j \hat{u}_{j,\lambda}, \quad g = \sum_{j=1}^{\infty} \hat{b}_j \hat{v}_{j,\mu} \qquad (4.12)$$

for some $\lambda \in (-\infty, \eta_1)$, $\mu \in (-\infty, 0)$, with $\left(\sqrt{|\mu_j(\lambda)|}\hat{a}_j\right)_{j=1}^{\infty}, \left(\sqrt{\lambda_j(\mu)}\hat{b}_j\right)_{j=1}^{\infty} \in l^2$. Then (f, g) belongs to $\Gamma(H^2(\Omega))$ if and only if

$$\sum_{j=1}^{\infty} a_j \gamma_1(u_{j,\lambda}) - g \in \mathscr{S}^{\frac{1}{2}}(\partial\Omega), \qquad (4.13)$$

where a_j are given by (4.5).

Equivalently, (f, g) belongs to $\Gamma(H^2(\Omega))$ if and only if

$$\sum_{j=1}^{\infty} b_j \gamma_0(v_{j,\mu}) - f \in \mathscr{S}^{\frac{3}{2}}(\partial\Omega), \qquad (4.14)$$

where $b_j = \sqrt{\lambda_j(\mu)}\hat{b}_j$.

Proof Assume that $(f, g) \in \Gamma(H^2(\Omega))$. Then $f = \gamma_0(u_\lambda + v_D)$ where $v_D \in \mathcal{H}^2_{0,D}(\Omega)$ and $u_\lambda = \sum_{j=1}^{\infty} a_j u_{j,\lambda}$ with the coefficients a_j given by (4.5). Moreover, $g = \gamma_1(u_\lambda + v_D)$ by the continuity of the trace operator. We deduce that

$$\gamma_1(u_\lambda) - g = -\gamma_1(v_D) \in \mathscr{S}^{\frac{1}{2}}(\partial\Omega).$$

This proves (4.13). Vice versa, assume that (4.13) holds. Then there exist $v_D \in \mathcal{H}^2_{0,D}(\Omega)$ such that $\gamma_1(v_D) = \sum_{j=1}^\infty a_j\gamma_1(u_{j,\lambda}) - g$. Thus

$$\gamma_1\left(\sum_{j=1}^\infty a_j u_{j,\lambda} - v_D\right) = g$$

and

$$\gamma_0\left(\sum_{j=1}^\infty a_j u_{j,\lambda} - v_D\right) = f$$

by (4.6). The proof of the second part of the statement follows the same lines as that of the first part and is accordingly omitted. □

4.1 Representation of the solutions to the Dirichlet problem

Using the Steklov expansions in (4.1) and (4.2) and the characterization of the total trace space $\Gamma(H^2(\Omega))$ given by Theorem 4.4 we are able to describe the solutions to the Dirichlet problem (1.4).

Corollary 4.5 *Let Ω be a bounded domain in $\mathbb{R}^N$ of class $C^{0,1}$, $(f, g) \in L^2(\partial\Omega) \times L^2(\partial\Omega)$. Then, there exists a solution $u \in H^2(\Omega)$ to problem (1.4) if and only if the couple (f, g) belongs to $\mathscr{S}^{\frac{3}{2}}(\partial\Omega) \times \mathscr{S}^{\frac{1}{2}}(\partial\Omega)$ and satisfies condition (4.13) or, equivalently, condition (4.14). In this case, if f, g are represented as in (4.12), then the solution u can be represented in each of the following two forms:*

i) *if $u_\lambda := \sum_{j=1}^\infty a_j u_{j,\lambda}$ where a_j are given by (4.5) and $g - \gamma_1(u_\lambda)$ is represented by $\sum_{j=1}^\infty \hat{d}_j \hat{v}_j \in \mathscr{S}^{\frac{1}{2}}(\partial\Omega)$, then*

$$u = u_\lambda + v_D$$

with $u_\lambda \in H^2_{\lambda,N}(\Omega)$ and $v_D = \sum_{j=1}^\infty d_j v_j \in \mathcal{B}_D(\Omega)$, $d_j = \sqrt{\eta_j}\hat{d}_j$ for all $j \in \mathbb{N}$.

ii) *if $v_\mu := \sum_{j=1}^\infty b_j v_{j,\mu}$ where $b_j = \sqrt{\lambda_j(\mu)}\hat{b}_j$ and $f - \gamma_0(v_\mu)$ is represented by $\sum_{j=1}^\infty \hat{c}_j \hat{u}_j \in \mathscr{S}^{\frac{3}{2}}(\partial\Omega)$, then*

$$u = v_\mu + u_N$$

with $v_\mu \in H^2_{\mu,D}(\Omega)$ and $u_N = \sum_{j=1}^\infty c_j u_j \in \mathcal{B}_N(\Omega)$, $c_j = \sqrt{\xi_j}\hat{c}_j$ for all $j \in \mathbb{N}$, $j \geq 2$, $c_1 = \hat{c}_1$.

Moreover the solution u is unique

Proof The first part of the statement is an immediate consequence of Theorem 4.4. Indeed, if there exists a solution $u \in H^2(\Omega)$ then (f, g) belongs to $\Gamma(H^2(\Omega))$, hence it satisfies (4.13) and (4.14).

Vice versa, if (f, g) satisfies (4.13), then $u_\lambda \in H^2_{\lambda,N}(\Omega)$, $v_D \in \mathcal{B}_D(\Omega)$ are well-defined and $u = u_\lambda + v_D$ is a biharmonic function in $H^2(\Omega)$ such that $\gamma_0(u) = \gamma_0(u_\lambda) = f$ and $\gamma_1(v_D) = g - \gamma_1(u_\lambda)$, hence $\gamma_1(u) = \gamma_1(u_\lambda + v_D) = g$.

Similarly, if (f, g) satisfies (4.14), then $v_\mu \in H^2_{\mu,D}(\Omega)$, $u_N \in \mathcal{B}_N(\Omega)$ are well-defined and $u = v_\mu + u_N$ is a biharmonic function in $H^2(\Omega)$ such that $\gamma_1(u) = \gamma_1(v_\mu) = g$ and $\gamma_0(u_N) = f - \gamma_0(v_\mu)$, hence $\gamma_0(u) = \gamma_0(v_\mu + u_N) = f$.

The uniqueness of the solution in $H^2(\Omega)$ follows from the fact that a solution u in $H^2(\Omega)$ of (1.4) with $f = g = 0$ must belong to $H^2_0(\Omega)$ and, since it is biharmonic, it must also belong to the orthogonal of $H^2_0(\Omega)$, hence $u = 0$.   □

Appendix A. Eigenvalues and eigenfunctions on the ball

In this section we compute the eigenvalues and the eigenfunctions of (BS_μ) and (BS_λ) when $\Omega = B$ is the unit ball in $\mathbb{R}^N$ centered at the origin and $\sigma = 0$. It is convenient to use spherical coordinates (r, θ), where $\theta = (\theta_1, ..., \theta_{N-1})$. The corresponding transformation of coordinates is

$$
\begin{aligned}
x_1 &= r\cos(\theta_1), \\
x_2 &= r\sin(\theta_1)\cos(\theta_2), \\
&\vdots \\
x_{N-1} &= r\sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{N-2})\cos(\theta_{N-1}), \\
x_N &= r\sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{N-2})\sin(\theta_{N-1}),
\end{aligned}
$$

with $\theta_1, ..., \theta_{N-2} \in [0, \pi]$, $\theta_{N-1} \in [0, 2\pi)$ (here it is understood that $\theta_1 \in [0, 2\pi)$ if $N = 2$).

The boundary conditions for fixed parameters $\lambda, \mu \in \mathbb{R}$

$$
\begin{cases}
\frac{\partial^2 u}{\partial \nu^2} = \lambda \frac{\partial u}{\partial \nu}, & \text{on } \partial B, \\
-\mathrm{div}_{\partial\Omega}(D^2 u \cdot \nu)_{\partial\Omega} - \frac{\partial \Delta u}{\partial \nu} = \mu u, & \text{on } \partial B,
\end{cases}
\tag{A.1}
$$

are written in spherical coordinates as

$$
\begin{cases}
\frac{\partial^2 u}{\partial r^2}\big|_{r=1} = \lambda \frac{\partial u}{\partial r}\big|_{r=1}, \\
-\frac{1}{r^2} \Delta_S \left(\frac{\partial u}{\partial r} - \frac{u}{r} \right) - \frac{\partial \Delta u}{\partial r}\big|_{r=1} = \mu u|_{r=1},
\end{cases}
\tag{A.2}
$$

where Δ_S is the Laplace-Beltrami operator on the unit sphere. It is well known that the eigenfunctions can be written as a product of a radial part and an angular part (see e.g., [13] for details). In particular, the radial part is given in terms of power-type functions, while the angular part is written in terms of spherical harmonics. We have the following theorem.

Theorem A.1 *Let $\Omega = B$ be the unit ball in $\mathbb{R}^N$ and $(\lambda, \mu) \in \mathbb{R}^2$. Then there exists a non-trivial solution to problem $\Delta^2 u = 0$ on B with boundary conditions (A.1) if and only if there exists $l \in \mathbb{N}_0$ such that $\det M_l(\lambda, \mu) = 0$, where $M_l(\lambda, \mu)$ is the matrix defined by*

$$
M_l(\lambda, \mu) = \begin{pmatrix} l(l - 1 - \lambda) & (l + 2)(l + 1 - \lambda) \\ l(l + N - 2)(l - 1) - \mu & l(l(l - 5) + N(l - 1) - 2) - \mu, \end{pmatrix}
\tag{A.3}
$$

for all $l \in \mathbb{N}_0$. If $(\lambda, \mu) \in \mathbb{R}^2$ solves the equation $\det M_l(\lambda, \mu) = 0$ for some $l \in \mathbb{N}_0$, then the associated solutions can be written in the form

$$
u_l(r, \theta) = \left(A_l r^l + B_l r^{l+2} \right) H_l(\theta),
$$

where $(A_l, B_l) \in \mathbb{R}^2$ solves the linear system $M_l(\lambda, \mu) \cdot (A_l, B_l) = 0$ and $H_l(\theta)$ is a spherical harmonic of degree l in $\mathbb{R}^N$.

Proof It is well-known that the weak solutions of $\Delta^2 u = 0$ in the unit ball complemented with (A.1) are smooth (see e.g., [19, §2]). Moreover, we recall that any function u satisfying $\Delta^2 u = 0$ on the unit ball B along with the two homogeneous boundary conditions (A.2) can be written in spherical coordinates in the form

$$
u_l(r, \theta) = (A r^l + B r^{l+2}) H_l(\theta),
\tag{A.4}
$$

for $l \in \mathbb{N}_0$, where $A, B \in \mathbb{R}$ are arbitrary constants and $H_l(\theta)$ is a spherical harmonic of degree l in $\mathbb{R}^N$ (see e.g., [12, §5-6] for details). By using the explicit form (A.4) in (A.2), we find that the constants A, B need to satisfy a homogeneous system of two linear equations, whose associated matrix is given by (A.3). Hence a non-trivial solution exists if and only if the determinant of (A.3) is zero. The rest of the statement is now a straightforward consequence. $\square$

By Theorem A.1 we immediately deduce the following characterization of the eigenvalues of (BS_μ) and (BS_λ). Here by m_l we denote the dimension of the space of the spherical harmonics of degree $l \in \mathbb{N}_0$ in $\mathbb{R}^N$, that is

$$
m_l = \frac{(2l + N - 2)(l + N - 3)!}{l!(N - 2)!}.
$$

Corollary A.2 *Any eigenvalue $\lambda(\mu)$ of (BS_μ) on B satisfies the equation*

$$\lambda(\mu)\left(2l^3 + (N-1)l^2 - (N-2)l - \mu\right)$$
$$= \left(3l^4 + 2(N-2)l^3 - (N+1)l^2 - (N-2)l - (1+2l)\mu\right), \quad (A.5)$$

for some $l \in \mathbb{N}_0$. Any eigenvalue $\mu(\lambda)$ of (BS_λ) on B satisfies the equation

$$\mu(\lambda)\,(2l+1-\lambda)$$
$$= \left(3l^4 + 2(N-2)l^3 - (N+1)l^2 - (N-2)l - \left(2l^3 + (N-1)l^2 - (N-2)l\right)\lambda\right),$$
$$(A.6)$$

for some $l \in \mathbb{N}_0$. The multiplicity of $\mu(\lambda)$ and $\lambda(\mu)$ corresponding to an index $l \in \mathbb{N}_0$ equals the dimension m_l of the space of the spherical harmonics of degree l in $\mathbb{R}^N$.

By using similar arguments, one can easily prove that the eigenvalues of problems (DBS) and (NBS) on the unit ball B can also be determined explicitly.

Theorem A.3 *Any eigenvalue η of problem (DBS) on B is of the form*

$$\eta = 2l + 1 \quad (A.7)$$

for some $l \in \mathbb{N}_0$ and its multiplicity equals the dimension m_l of the space of spherical harmonics of degree l in $\mathbb{R}^N$.

Any eigenvalue ξ of problem (NBS) on B is of the form

$$\xi = l(2l^2 + (N-1)l - N + 2), \quad (A.8)$$

for some $l \in \mathbb{N}_0$ and its multiplicity equals the dimension m_l of the space of spherical harmonics of degree l in $\mathbb{R}^N$.

By combining Corollary A.2 and Theorem A.3 we can prove the following statement, where $\lambda_{(l)}(\mu)$, $\mu_{(l)}(\lambda)$, $\eta_{(l)}$ and $\xi_{(l)}$ denote the eigenvalues of (BS_μ), (BS_λ), (DBS) and (NBS), respectively, associated with spherical harmonics of order $l \in \mathbb{N}_0$.

Theorem A.4 *For all $l \in \mathbb{N}$ and $\mu \neq \xi_{(l)}$*

$$\lambda_{(l)}(\mu) = \frac{\left(3l^4 + 2(N-2)l^3 - (N+1)l^2 - (N-2)l - \eta_{(l)}\mu\right)}{\left(\xi_{(l)} - \mu\right)}. \quad (A.9)$$

For all $l \in \mathbb{N}$ and $\lambda \neq \eta_{(l)}$

$$\mu_{(l)}(\lambda) = \frac{\left(3l^4 + 2(N-2)l^3 - (N+1)l^2 - (N-2)l - \xi_{(l)}\lambda\right)}{\left(\eta_{(l)} - \lambda\right)}. \quad (A.10)$$

Moreover, $\lambda_{(0)}(\mu) = \eta_{(0)} = 1$ for all $\mu \in \mathbb{R}$ and $\mu_{(0)}(\lambda) = \xi_{(0)} = 0$ for all $\lambda \in \mathbb{R}$.

Proof According to the change of notation for eigenvalues, we denote by $\lambda_{(l)}(\mu)$, $\mu_{(l)}(\lambda)$, $\eta_{(l)}$ and $\xi_{(l)}$ the eigenvalues of problems (BS_μ), (BS_λ), (DBS) and (NBS) corresponding to the choice of $l \in \mathbb{N}_0$ in (A.5), (A.6), (A.7) and (A.8). Each of such eigenvalues has multiplicity m_l.

We note that (A.5) and (A.6) can be rewritten as

$$\lambda_{(l)}(\mu)\left(\xi_{(l)} - \mu\right) = \left(3l^4 + 2(N-2)l^3 - (N+1)l^2 - (N-2)l - \eta_{(l)}\mu\right) \quad \text{(A.11)}$$

and

$$\mu_{(l)}(\lambda)\left(\eta_{(l)} - \lambda\right) = \left(3l^4 + 2(N-2)l^3 - (N+1)l^2 - (N-2)l - \xi_{(l)}\lambda\right). \quad \text{(A.12)}$$

If $l \in \mathbb{N}$, then from (A.11) and (A.12) we deduce the validity of (A.9) when $\mu \neq \xi_{(l)}$ and of (A.10) when $\lambda \neq \eta_{(l)}$. If $l = 0$, the condition $\det M_0(\lambda, \mu) = 0$ can be written in the form $\mu(\lambda - 1) = 0$ which allows to conclude the proof. $\qquad\square$

We note that

$$\lim_{\mu \to -\infty} \lambda_{(l)}(\mu) = \eta_{(l)}$$

and

$$\lim_{\lambda \to -\infty} \mu_{(l)}(\lambda) = \xi_{(l)}$$

for all $l \in \mathbb{N}_0$. This is coherent with Theorems 3.7 and 3.12. Moreover,

$$\lim_{\mu \to \xi_{(l)}^\pm} \lambda_{(l)}(\mu) = \pm\infty$$

and

$$\lim_{\lambda \to \eta_{(l)}^\pm} \mu_{(l)}(\lambda) = \pm\infty$$

for all $l \in \mathbb{N}$. We have shown that the branches of eigenvalues $\lambda_{(l)}(\mu)$ and $\mu_{(l)}(\lambda)$ are analytic functions of their parameters on $\mathbb{R}\setminus\{\xi_{(l)}\}_{l\in\mathbb{N}_0}$ and $\mathbb{R}\setminus\{\eta_{(l)}\}_{l\in\mathbb{N}_0}$ respectively. In particular, the branch $\lambda_{(l)}(\mu)$ is a equilateral hyperbole with $\eta_{(l)}$ as horizontal asymptote and $\xi_{(l)}$ as vertical asymptote, if $l \geq 1$, while it is coincides with $\{(\mu, 1) : \mu \in \mathbb{R}\}$ for $l = 0$. The branch $\mu_{(l)}(\lambda)$ is a equilateral hyperbole with $\xi_{(l)}$ as horizontal asymptote and $\eta_{(l)}$ as vertical asymptote, if $l \geq 1$, while it is coincides with $\{(\lambda, 0) : \mu \in \mathbb{R}\}$ for $l = 0$. The situation is illustrated in Fig. 1.
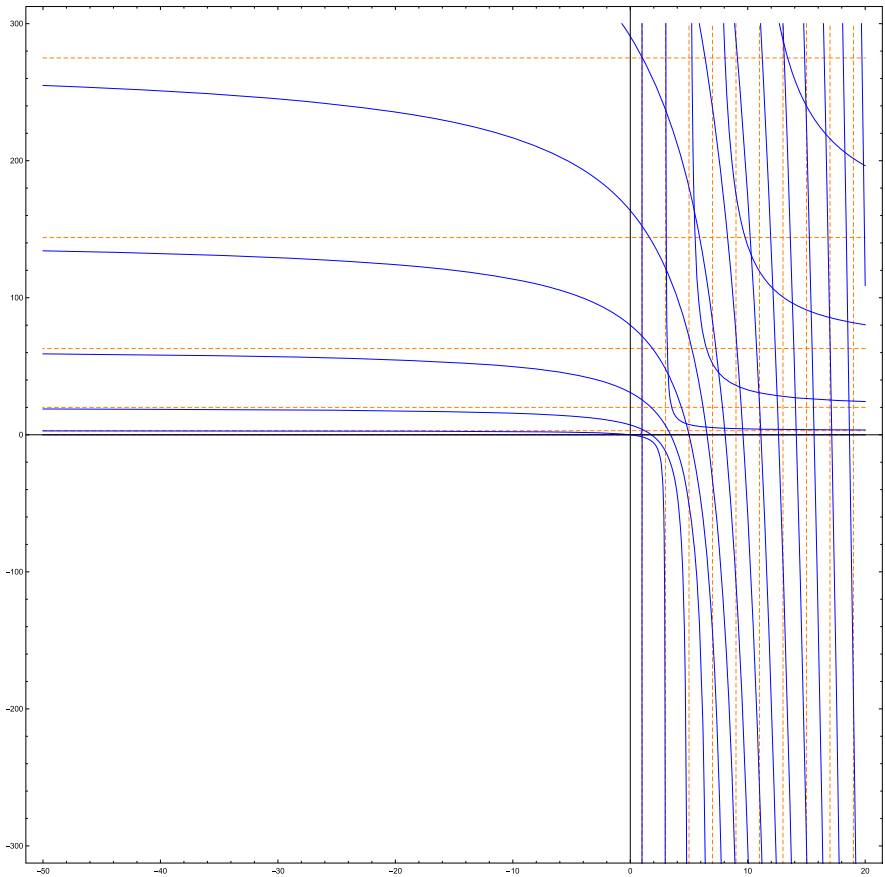
Fig. 1 Eigenvalues $\mu_{(l)}(\lambda)$ of (BS$_\lambda$) as functions of λ (the parameter λ correspond to the abscissa). Vertical asymptotes are the eigenvalues $\eta_{(l)}$ of (DBS). Horizontal asymptotes are the eigenvalues $\xi_{(l)}$ of (NBS). A reflection along the angle bisector of the first and third quadrant gives the eigenvalues $\lambda_{(l)}(\mu)$ of (BS$_\mu$) as functions of μ

Appendix B. Asymptotic formulas

It is proved in [28,29] that if Ω is a bounded domain in $\mathbb{R}^N$ with C^∞ boundary, then the eigenvalues of problems (DBS) and (NBS) with $\sigma = 1$ satisfy the following asymptotic laws

$$\eta_j \sim \frac{4\pi}{\omega_{N-1}^{\frac{1}{N-1}}} \left(\frac{j}{|\partial\Omega|} \right)^{\frac{1}{N-1}}, \tag{B.1}$$

and

$$\xi_j \sim \frac{16\pi^3}{\omega_{N-1}^{\frac{3}{N-1}}} \left(\frac{j}{|\partial\Omega|} \right)^{\frac{3}{N-1}}, \tag{B.2}$$

as $j \to \infty$. Here ω_{N-1} denotes the volume of the unit ball in $\mathbb{R}^{N-1}$.

We state the following theorem, whose proof is omitted since it follows exactly the same lines as those of Theorems 1.1 and 1.2 of [29].

Theorem B.1 *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^∞. Then formulas* (B.1) *and* (B.2) *hold for all $\sigma \in \left(-\frac{1}{N-1}, 1 \right)$. Moreover, the two following asymptotic formulas hold for the eigenvalues of problems* (BS$_\mu$) *and* (BS$_\lambda$)

$$\lambda_j(\mu) \sim \frac{3\pi}{\omega_{N-1}^{\frac{1}{N-1}}} \left(\frac{j}{|\partial\Omega|} \right)^{\frac{1}{N-1}},$$ (B.3)

and

$$\mu_j(\lambda) \sim \frac{12\pi^3}{\omega_{N-1}^{\frac{3}{N-1}}} \left(\frac{j}{|\partial\Omega|} \right)^{\frac{3}{N-1}}$$ (B.4)

as $j \to \infty$.

We note that the principal term in the asymptotic expansions of the eigenvalues depends neither on the Poisson's ratio σ nor on μ or λ. However, lower order terms have to depend on μ and λ, since, as $\mu, \lambda \to -\infty$, $\lambda_j(\mu) \to \eta_j$ and $\mu_j(\lambda) \to \xi_j$, and asymptotic formulas of $\lambda_j(\mu)$ and η_j, and of $\mu_j(\lambda)$ and ξ_j, differ from a factor $\frac{3}{4}$.

Remark B.2 We remark that the approach used in [29] requires that the boundary of Ω is of class C^∞. However, as proved by another technique in [28], the asymptotic formulas for the eigenvalues of (DBS) and (NBS) when $\sigma = 1$ hold when Ω is of class C^2.

We now show that in the case of the unit ball B in $\mathbb{R}^N$ it is possible to recover formulas (B.1), (B.2), (B.3) and (B.4) by using the explicit computations in Appendix A.

Note that for a fixed $l \in \mathbb{N}_0$, the dimension of the space of spherical harmonics of degree less or equal than l is $\frac{(2l+N-1)(N+l-2)!}{l!(N-1)!}$. By (A.8) we deduce that

$$\xi_j = l(2l^2 + (N-1)l - N + 2)$$ (B.5)

whenever $j \in \mathbb{N}$ is such that

$$\frac{(2l+N-3)(N+l-3)!}{(l-1)!(N-1)!} < j \leq \frac{(2l+N-1)(N+l-2)!}{l!(N-1)!}$$ (B.6)

Moreover,

$$\lim_{l \to +\infty} \frac{(2l+N-3)(N+l-3)!}{(l-1)!(N-1)!} \div \frac{2l^{N-1}}{(N-1)!}$$
$$= \lim_{l \to +\infty} \frac{(2l+N-1)(N+l-2)!}{l!(N-1)!} \div \frac{2l^{N-1}}{(N-1)!} = 1.$$

From (B.5) and (B.6) we deduce that

$$\xi_j \sim 2^{\frac{N-4}{N-1}}(N-1)!^{\frac{3}{N-1}} j^{\frac{3}{N-1}}, \quad \text{as } j \to +\infty.$$

We note that this is exactly (B.2). Indeed, recalling that $|\partial B| = N\omega_N$, a standard computation shows that

$$2^{\frac{N-4}{N-1}}(N-1)!^{\frac{3}{N-1}} = \frac{16\pi^3}{\omega_{N-1}^{\frac{3}{N-1}}} \cdot \frac{1}{|\partial\Omega|^{\frac{3}{N-1}}},$$

for which it is useful to note that

$$\omega_N \omega_{N-1} = \frac{2^N \pi^{N-1}}{N!}.$$

In the same way we verify that

$$\eta_j \sim 2^{\frac{N-2}{N-1}}(N-1)!^{\frac{1}{N-1}} j^{\frac{1}{N-1}},$$

$$\lambda_j(\mu) \sim \frac{3}{4} 2^{\frac{N-2}{N-1}}(N-1)!^{\frac{1}{N-1}} j^{\frac{1}{N-1}},$$

and

$$\mu_j(\lambda) \sim \frac{3}{4} 2^{\frac{N-4}{N-1}}(N-1)!^{\frac{3}{N-1}} j^{\frac{3}{N-1}},$$

as $j \to +\infty$, and these asymptotic formulas correspond to formulas (A.7), (A.5) and (A.6).

Appendix C. The (BS$_\mu$) and (BS$_\lambda$) problems for $\mu > 0$ and $\lambda > \eta_1$

In this section we briefly discuss problem (BS$_\mu$) for $\mu > 0$ and problem (BS$_\lambda$) for $\lambda > \eta_1$.

We begin with problem (BS$_\mu$). Assume that $\mu \in \mathbb{R}$ is such that $\xi_j < \mu < \xi_{j+1}$ for some $j \in \mathbb{N}$. Recall that ξ_j denote the eigenvalues of problem (NBS). We denote by U_j the subspace of $H^2(\Omega)$ generated by all eigenfunctions u_i associated with the eigenvalues ξ_i with $i \leq j$ and we set

$$U_j^\perp = \left\{ u \in H^2(\Omega) : \mathcal{Q}_{\mu,D}(u, \varphi) = 0, \quad \forall \varphi \in U_j \right\}.$$

The space $U_j^\perp$ is a closed subspace of $H^2(\Omega)$. We have the following result.

Theorem C.1 *Let Ω be a bounded domain in $\mathbb{R}^N$ of class $C^{0,1}$ and assume that $\xi_j < \mu < \xi_{j+1}$ for some $j \in \mathbb{N}$. Then*

$$H^2(\Omega) = U_j \oplus U_j^\perp. \tag{C.1}$$

Moreover, there exists $b \geq 0$ such that the quadratic form $Q_{\mu,D}(u,v) + b(\gamma_1(u), \gamma_1(v))_{\partial\Omega}$, $u, v \in H^2(\Omega)$, is coercive on $U_j^\perp$.

Note that the decomposition (C.1) is not straightforward since $Q_{\mu,D}$ does not define a scalar product for $\mu > 0$. However, in order to prove Theorem C.1 one can easily adapt the analogous proof in [27, Theorem 5.1] but we omit the details.

We observe that, whenever $u \in U_j^\perp$, one can consider only test functions $\varphi \in U_j^\perp$ in the weak formulation of (3.4). Indeed, adding to φ a test function $\tilde{\varphi} \in U_j$ leaves both sides of the equation unchanged. Hence one can perform the same analysis as in the case $\mu < 0$ and state an analogous version of Theorem 3.3 with the space $H^2(\Omega)$ replaced by $U_j^\perp$. We leave this to the reader.

Remark C.2 If $\mu = \xi_j$ for some $j \in \mathbb{N}$ the situation is more involved and is not analyzed here. However, if we assume that ξ_j is an eigenvalue of (3.3) of multiplicity m such that

$$Q_\sigma(u,\varphi) = \xi_j(\gamma_0(u), \gamma_0(\varphi))_{\partial\Omega}, \quad \forall\varphi \in H^2(\Omega), u \in U_{\xi_j}, \tag{C.2}$$

where U_{ξ_j} is the eigenspace generated by all the eigenfunctions $\{u_j^1, ..., u_j^m\}$ in $\mathcal{H}_{0,N}^2(\Omega)$ associated with ξ_j, the problem becomes simpler. Indeed, any function $w \in H^2(\Omega)$ can be written in the form $w = u + v$, where $u \in U_{\xi_j}$ and $v \in H_{\xi_j}^2(\Omega)$, where

$$H_{\xi_j}^2(\Omega) := \left\{ v \in H^2(\Omega) : (\gamma_0(v), \gamma_0(u_j^i))_{\partial\Omega} = 0 \text{ for all } i = 1, ..., m \right\}.$$

Hence, whenever $u \in H_{\xi_j}^2(\Omega)$, one can consider only test functions in $\varphi \in H_{\xi_j}^2(\Omega)$ in the weak formulation (3.4) with $\mu = \xi_j$. In fact, adding to φ a function $\tilde{\varphi} \in U_{\xi_j}$ leaves both sides of the equation unchanged. Thus we can perform the same analysis of Theorem C.1 with $H^2(\Omega)$ replaced by $H_{\xi_j}^2(\Omega)$.

Note that equation (C.2) is satisfied with $\xi_1 = 0$ and u a constant function.

We also have an analogous result for problem (BS_λ) with $\lambda > \eta_1$. Assume that $\lambda \in \mathbb{R}$ is such that $\eta_j < \lambda < \eta_{j+1}$ for some $j \in \mathbb{N}$. Recall that η_j denote the eigenvalues of problem (DBS). We denote by V_j the subspace of $H^2(\Omega)$ generated by all eigenfunctions v_i associated with the eigenvalues η_i with $i \leq j$ and we set

$$V_j^\perp = \left\{ v \in H^2(\Omega) : Q_{\lambda,N}(v,\varphi) = 0, \quad \forall\varphi \in V_j \right\}.$$

The space $V_j^\perp$ is a closed subspace of $H^2(\Omega)$. We have the following result.

Theorem C.3 *Let Ω be a bounded domain in $\mathbb{R}^N$ of class $C^{0,1}$ and assume that $\eta_j < \lambda < \eta_{j+1}$. Then*

$$H^2(\Omega) = V_j \oplus V_j^\perp. \tag{C.3}$$

Moreover, there exists $b \geq 0$ such that the quadratic form $Q_{\lambda,N}(u,v) + b(\gamma_0(u), \gamma_0(v))_{\partial\Omega}$, $u, v \in H^2(\Omega)$, is coercive on $V_j^\perp$.

We observe that, whenever $v \in V_j^\perp$, one can consider only test functions $\varphi \in V_j^\perp$ in the weak formulation of (3.14). In fact, adding to φ a test function $\tilde{\varphi} \in V_j$ leaves both sides of the equation unchanged. Hence one can perform the same analysis as in the case $\lambda < \eta_1$ and state an analogous version of Theorem 3.10 with the space $H^2(\Omega)$ replaced by $V_j^\perp$. We leave this to the reader.

Remark C.4 As in Remark C.2, one can treat the case $\lambda = \eta_j$ for some j in the special situation when η_j is an eigenvalue of (3.1) of multiplicity m such that

$$\mathcal{Q}_\sigma(v, \varphi) = \eta_j(\gamma_1(v), \gamma_1(\varphi)), \quad \forall \varphi \in H^2(\Omega), v \in V_{\eta_j},$$

where V_{η_j} is the eigenspace generated by all the eigenfunctions $\{v_j^1, ..., v_j^m\}$ in $\mathcal{H}_{0,D}^2(\Omega)$ associated with η_j. This happens in the case of the unit ball with $\eta_1 = 1$. We observe that any function $w \in H^2(\Omega)$ can be written in the form $w = v + u$, where $v \in V_{\eta_j}$ and $u \in H_{\eta_j}^2(\Omega)$, where

$$H_{\eta_j}^2(\Omega) := \left\{ u \in H^2(\Omega) : (\gamma_1(u), \gamma_1(v_j^i))_{\partial\Omega} = 0 \text{ for all } i = 1, ..., m \right\}.$$

Hence, whenever $v \in H_{\eta_j}^2(\Omega)$, one can consider in the weak formulation (3.14) with $\lambda = \eta_j$ only test functions in $\varphi \in H_{\eta_j}^2(\Omega)$. In fact, adding to φ a function $\tilde{\varphi} \in V_{\eta_j}$ leaves both sides of the equation unchanged. Thus we can perform the same analysis of Theorem C.3 with $H^2(\Omega)$ replaced by $H_{\eta_j}^2(\Omega)$.

We conclude this section with a few more remarks. We have observed that the eigenvalue $\mu = 0$ is always an eigenvalue of (3.14) when $\lambda < \eta_1$, and that the constant functions belong to the eigenspace associated with $\mu = 0$, and in particular belong to the space $\mathcal{H}_{0,N}^2(\Omega)$ and are eigenfunctions associated with the first eigenvalue $\xi_1 = 0$ of problem (NBS). Such a situation may occur also for other eigenvalues, as well as for problem (3.4) (as we have seen in the case of the ball in $\mathbb{R}^N$ with the eigenvalue $\lambda(\mu) = \eta_1 = 1$ for all $\mu \in \mathbb{R}$). The following lemma clarifies this phenomenon.

Lemma C.5 *Let $j \in \mathbb{N}$. Then one of the following two alternatives occurs for problem* (3.4):

i) $\lambda_j(\mu) < \eta_j$ *for all $\mu \in (-\infty, 0)$;*

ii) *there exists $\mu_0 \in (-\infty, 0)$ such that $\lambda_j(\mu_0) = \eta_j$. In this case, $\eta_j = \lambda_j(\mu)$ for all $\mu \in (-\infty, \mu_0]$ and η_j is an eigenvalue of problem* (3.4) *for any $\mu \in \mathbb{R}$.*

Proof From the Min-Max Principles (3.2) and (3.8) we have that $\lambda_j(\mu) \le \eta_j$ for all $\mu \in (-\infty, 0)$, $j \in \mathbb{N}$. Assume now that there exists $\mu_0 \in (-\infty, 0)$ such that $\lambda_j(\mu_0) = \eta_j$. Again from (3.2) and (3.8) we deduce that we can choose an eigenfunction v_{j,μ_0} associated with $\lambda_j(\mu_0)$ which belong to $\mathcal{H}_{0,D}^2(\Omega)$ and which coincide with an eigenfunction u_j of problem (3.1) associated with η_j. In fact the eigenfunctions associated with $\lambda_j(\mu)$ are exactly the functions realizing the equality in (3.8). We also note that

$$\mathcal{Q}_\sigma(u_j, \varphi) = \mathcal{Q}_{\mu,D}(u_j, \varphi), \quad \forall \varphi \in H^2(\Omega), \mu \in \mathbb{R},$$

hence

$$\mathcal{Q}_{\mu,D}(u_j, \varphi) = \eta_j(u_j, \varphi)_{\partial\Omega}, \quad \forall \varphi \in H^2(\Omega), \mu \in \mathbb{R},$$

hence η_j is an eigenvalue of (3.4) for all $\mu \in \mathbb{R}$ and in particular $\eta_j = \lambda_j(\mu)$ for all $\mu \in (-\infty, \mu_0]$. This concludes the proof. $\qquad\square$

In the same way one can prove the following.

Lemma C.6 *Let* $j \in \mathbb{N}$. *Then one of the following two alternatives occurs for problem* (3.14):

i) $\mu_j(\lambda) < \xi_j$ *for all* $\lambda \in (-\infty, \eta_1)$;

ii) *there exists* $\lambda_0 \in (-\infty, \eta_1)$ *such that* $\mu_j(\lambda_0) = \xi_j$. *In this case,* $\xi_j = \mu_j(\lambda)$ *for all* $\lambda \in (-\infty, \lambda_0]$ *and* ξ_j *is an eigenvalue of problem* (3.14), *for any* $\lambda \in \mathbb{R}$.

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