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INDIRIZZO MATEMATICA

CICLO XXVIII

On mass distribution and concentration phenomena for linear elliptic partial differential operators

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Riassunto

In questa tesi studiamo la dipendenza degli autovalori di operatori differenziali alle derivate parziali di tipo ellittico da perturbazioni della densità di massa su aperti dello spazio euclideo N-dimensionale. In particolare, proviamo risultati di dipendenza continua e analitica degli autovalori di operatori poliarmonici e li applichiamo ad alcuni problemi di ottimizzazione. Per provare i risultati di analiticità, adoperiamo una tecnica generale sviluppata da P.D. Lamberti e M. Lanza de Cristoforis, ottenendo formule per i differenziali di Frechét degli autovalori che ci permettono di caratterizzare le densità critiche sotto il vincolo di massa fissata. Inoltre, enunciamo un 'principio di massimo' per la classe di problemi di ottimizzazione considerata. In seguito, prendiamo in esame una famiglia particolare di densità di massa, ovvero densità che si concentrano al bordo degli aperti dove i problemi differenziali sono definiti. In questo caso, studiamo il comportamento asintotico degli autovalori e delle autofunzioni dei problemi di Neumann per l'operatore di Laplace e l'operatore biarmonico quando la massa si concentra al bordo. Proviamo in entrambi i casi, adattando una tecnica generale sviluppata da J.M. Arrieta, che gli autovalori e le autofunzioni del problema di Neumann convergono agli autovalori e alle autofunzioni di appropriati problemi limite di tipo Steklov. In particolare, il problema di tipo Steklov per l'operatore biarmonico così formulato viene introdotto per la prima volta in questa tesi, dove ne vengono poi studiate alcune proprietà. Nel caso dell'operatore di Laplace, proviamo la validità di un'espansione asintotica degli autovalori e delle autofunzioni del problema di Neumann fino al primo ordine ed otteniamo formule esplicite per i primi termini delle espansioni. Per ottenere questi risultati adattiamo al nostro problema delle tecniche di analisi asintotica utilizzate da M.E. Pérez e S.A. Nazarov per lo studio di sistemi vibranti con masse concentrate in punti o lungo certe curve. Per quanto riguarda il problema di Steklov per l'operatore biarmonico, consideriamo anche il problema della dipendenza degli autovalori dal dominio. Utilizzando sempre la tecnica generale sviluppata da P.D. Lamberti e M. Lanza de Cristoforis, proviamo che le palle sono domini critici per tutti gli autovalori. Inoltre, adattando l'argomento di F. Brock e R.

Weinstock per il problema di Steklov per l'operatore di Laplace, riusciamo a mostrare che la palla massimizza il primo autovalore positivo del problema di Steklov per l'operatore biarmonico tra tutti gli aperti limitati di misura fissata. Proviamo infine una versione quantitativa di questa disuguaglianza isoperimetrica, mostrando poi che l'esponente che compare nella disuguaglianza è ottimale.

La tesi è organizzata come segue. Il Capitolo 1 è dedicato ad alcuni preliminari. Nel Capitolo 2 consideriamo problemi di perturbazione di massa per operatori differenziali ellittici soggetti a diverse condizioni al bordo omogenee. Proviamo risultati di analiticità per gli autovalori e calcoliamo le formule per i differenziali di Frechét che poi verranno usate per caratterizzare le densità di massa critiche sotto il vincolo di massa fissata. Quindi proviamo che per un'ampia famiglia di operatori e condizioni al bordo non esistono densità di massa critiche sotto il solo vincolo di massa. Successivamente proviamo che gli autovalori sono debolmente* continui, il che permette di stabilire una sorta di 'principio di massimo' per la classe di problemi di ottimizzazione considerata nel capitolo. Nel Capitolo 3 consideriamo il problema agli autovalori per l'operatore di Laplace con condizioni al bordo di Neumann e densità di massa che si concentrano al bordo e mostriamo che gli autovalori e le autofunzioni convergono agli autovalori e alle autofunzioni di un opportuno problema di tipo Steklov per l'operatore di Laplace. Il risultato è ottenuto provando la convergenza in norma degli operatori risolventi. Inoltre, studiamo la dipendenza degli autovalori del problema di Steklov dalla densità di massa e mostriamo che sulla palla la densità costante è una densità di massa critica per un'opportuna famiglia di funzioni simmetriche degli autovalori. Nel Capitolo 4 discutiamo il comportamento asintotico degli autovalori del problema di Neumann per l'operatore di Laplace quando la massa si concentra al bordo. In particolare, nel caso della palla troviamo una formula esplicita per le derivate degli autovalori del problema di Neumann nel problema limite e deduciamo che localmente gli autovalori del problema di Steklov minimizzano gli autovalori del problema di Neumann. Inoltre, studiamo il comportamento asintotico degli autovalori del problema di Neumann su aperti limitati del piano e proviamo la validità di un'espansione asintotica per gli autovalori e le autofunzioni del problema di Neumann fino al primo ordine. Otteniamo formule esplicite per i primi termini delle espansioni in termini di soluzioni di opportuni problemi differenziali ausiliari. Nel Capitolo 5 formuliamo il problema agli autovalori per l'operatore biarmonico con condizioni di Neumann e di Steklov. Mostriamo che il problema di Steklov per l'operatore biarmonico può essere ottenuto a partire dal problema di Neumann con densità di massa che si concentra al bordo. In seguito, studiamo la dipendenza degli autovalori del problema di Steklov da perturbazioni del dominio provando formule di tipo Hadamard, e caratterizziamo i domini critici. Quindi, proviamo che per i problemi di Steklov e di Neumann le palle sono critiche. Per quanto riguarda il problema di Steklov, in aggiunta, proviamo che la palla massimizza il primo autovalore positivo tra tutti gli aperti limitati di misura fissata. Infine, all'interno del Capitolo 6 presentiamo alcuni risultati aggiuntivi su problemi di tipo Neumann. In particolare, studiamo la dipendenza degli autovalori dell'operatore biarmonico con condizioni di Neumann dal coefficiente di Poisson. Studiamo anche il comportamento degli autovalori del problema di Neumann per l'operatore di Laplace e per l'operatore biarmonico sulla corona quando la differenza dei due raggi tende a zero.

Abstract

In this thesis we study the dependence of the eigenvalues of elliptic partial differential operators upon mass density perturbations on open subsets of the N-dimensional euclidean space. We prove continuity and analyticity results for the eigenvalues of poly-harmonic operators and apply them to certain optimization problems. In order to prove analyticity, we use a general technique of P.D. Lamberti and M. Lanza de Cristoforis, and we obtain formulas for the Frechét differentials of the eigenvalues which are used to characterize critical mass densities under the constraint that the total mass is preserved. Then we state a sort of 'maximum principle' in spectral optimization problems for elliptic operators subject to mass density perturbations. Moreover, we consider a special class of densities, namely densities which concentrate near the boundary of open subsets of the N-dimensional euclidean space. We study the asymptotic behavior of the eigenvalues of Neumann-type problems for the Laplace and the biharmonic operator. By adapting a general technique of J.M. Arrieta, we prove that the Neumann eigenvalues converge to the appropriate limiting Steklov eigenvalues. In this way, we formulate a genuine Steklov eigenvalue problem for the biharmonic operator. In the case of the Laplace operator we prove the validity of an asymptotic expansion of the Neumann eigenvalues and eigenfunctions and provide formulas for the first terms in the expansions. We adapt to our case asymptotic analysis techniques used by M.E. Pérez and S.A. Nazarov to describe vibrating systems with masses concentrated at points or along curves. Moreover, we consider the problem of domain perturbations for the biharmonic Steklov problem obtained with this mass concentration procedure and prove that balls are critical domains for all the eigenvalues. Then we adapt the arguments of F. Brock and R. Weinstock to prove that the ball is actually a maximizer for the first positive eigenvalue among bounded domains of given measure. Moreover, we provide a quantitative version of such an isoperimetric inequality, showing also that it is sharp.

This thesis is organized as follows. Chapter 1 is dedicated to some preliminaries. In Chapter 2 we consider mass density perturbation problems for general elliptic operators of higher order subject to various homogeneous boundary conditions. We prove analyticity results for the eigenvalues and compute the Frechét differentials for the symmetric functions of the eigenvalues which are used to provide a characterization of critical mass densities under mass constraint. Then we prove that for a large class of operators and boundary conditions there are no critical mass densities under the constraint of preservation of the total mass. Moreover, we prove weak* continuity of the eigenvalues which allows to state a sort of 'maximum principle' for a class of spectral optimization problems. In Chapter 3 we consider the Neumann eigenvalue problem for the Laplace operator and mass densities which concentrate at the boundary. We prove that the Neumann eigenvalues converge to the appropriate limiting Steklov eigenvalues by proving strong convergence of the resolvent operators. Moreover, we consider the problem of mass density perturbations for the Steklov problem for the Laplace operator and show that in the case of the ball there exist critical mass densities under the sole constraint of preservation of the mass. In Chapter 4 we discuss the asymptotic behavior of the Neumann eigenvalues in the mass concentration phenomenon described in Chapter 3. In particular, in the case of the ball, we prove explicit formulas for the derivatives of the Neumann eigenvalues at the limiting Steklov problem and show that the Steklov eigenvalues locally minimize the Neumann eigenvalues. Moreover, we study the asymptotic behavior of the eigenvalues and the eigenfunctions of the Neumann problem in the case of bounded planar domains. We obtain explicit formulas for the first and second terms of the corresponding asymptotic expansions in terms of solutions to certain auxiliary boundary value problems. In Chapter 5 we introduce the Neumann eigenvalue problem and formulate the Steklov eigenvalue problem for the biharmonic operator. We show that the biharmonic Steklov problem which we introduce can be considered as a limiting Neumann problem for the biharmonic operator in a mass concentration phenomenon. Then we study the dependence of the symmetric functions of the eigenvalues of both Neumann and Steklov problems upon domain perturbations providing Hadamard type formulas, and we give a characterization of critical domains under volume constraint. Then we show that for Neumann and Steklov problems balls are critical domains. Regarding the Steklov problem, we also prove that the ball is a maximizer for the first positive eigenvalue among all bounded open sets of given measure. Finally, in Chapter 6 we include some additional results on Neumann-type problems. In particular, we study the dependence of the Neumann eigenvalues of the biharmonic operator upon the Poisson's ratio. Moreover, we study the behavior of the Neumann eigenvalues of the Laplace and the biharmonic operator on an annulus when the difference between the two radii goes to zero.

Introduction

Boundary value problems for linear elliptic partial differential equations arise in several models describing various physical phenomena and have been extensively studied for a long time. One of the most famous problems is perhaps the Poisson problem for the Laplace operator

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{0.0.1}$$

which arises for example in the study of the deformation of a fixed membrane of shape $\Omega \subset \mathbb{R}^2$ subject to an exterior force which is represented by the function f (see e.g., [31, 94, 105] for a detailed discussion and historical information). Problem (0.0.1) is a prototype of second order elliptic problems. The theory of linear second order elliptic equations is well developed and is considered nowadays classical.

However, many other phenomena in the applied sciences are modeled by higher order equations. It was already known at the beginning of the nineteenth century that the study of the bending of a clamped plate leads to the analysis of the following fourth order problem

$$\begin{cases} \Delta^2 u = f, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ represents the midplane of the plate and the function f represents the applied load. It was then natural to consider more general equations involving the polyharmonic operators $(-\Delta)^m$, $m \in \mathbb{N}$, subject to different types of homogeneous boundary conditions. It is worth mentioning the pioneeristic papers of Almansi [6, 7] and the book of Nicolesco [90] on this subject. These authors were among the first to study the properties of polyharmonic functions and higher order elliptic equations. Since then the interest for polyharmonic operators has grown but the theory of higher order elliptic equations is far less developed than the theory of analogous second order equations. As is well-known, this is also due to the fact that for higher order equations a maximum principle does not hold in general.

A general theory for boundary value problems for linear elliptic operators of order $2m$ has been developed by Agmon-Douglis-Nirenberg [2, 3, 4]. We also mention the recent book [47] which is devoted to an extensive study of boundary value problems for polyharmonic operators.

In this thesis we are mainly interested in eigenvalue problems for elliptic operators of second and higher order depending on a parameter ρ which plays the role of a mass density for the underlying physical system. In the case of the Laplace operator subject to Dirichlet boundary conditions such problem can be written as

$$\begin{cases} -\Delta u = \lambda \rho u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{0.0.2}$$

where ρ is a measurable positive and bounded function on Ω. Here and in the sequel Ω is an open subset of $\mathbb{R}^N$ with sufficiently smooth boundary. We also consider the eigenvalue problem for the Laplace operator subject to Neumann boundary conditions

$$\begin{cases} -\Delta u = \lambda \rho u, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases} \tag{0.0.3}$$

As for the biharmonic operator subject to Dirichlet boundary conditions, the eigenvalue problem reads

$$\begin{cases} \Delta^2 u = \lambda \rho u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases} \tag{0.0.4}$$

Also for the biharmonic operator, we consider different types of homogeneous boundary conditions such as Neumann boundary conditions

$$\begin{cases} \Delta^2 u = \lambda \rho u, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = \frac{\partial \Delta u}{\partial \nu} + \mathrm{div}_{\partial\Omega} \left(D^2 u \cdot \nu \right) = 0, & \text{on } \partial\Omega, \end{cases} \tag{0.0.5}$$

and intermediate boundary conditions

$$\begin{cases} \Delta^2 u = \lambda \rho u, & \text{in } \Omega, \\ u = \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega. \end{cases} \tag{0.0.6}$$

As we have said, keeping in mind important problems in linear elasticity (see e.g., [31]), we shall think of the weight ρ as a mass density. In fact, for $N = 2$ problems (0.0.2) and (0.0.3) are related to the study of the transverse vibrations of a thin elastic membrane which has a fixed or a free frame respectively, and the mass of which is displaced on $\Omega \subset \mathbb{R}^2$ with density ρ. On the other hand, problems (0.0.4), (0.0.5) and (0.0.6) are related to the study of the transverse vibrations of an elastic plate with density ρ which is clamped, free, and hinged, respectively. We shall refer to the quantity $\int_\Omega \rho dx$ as the total mass of the body.

As for higher order operators, we consider the following eigenvalue problem

$$\begin{cases} (-\Delta)^m u = \lambda\rho u, & \text{in } \Omega, \\ \frac{\partial^j u}{\partial \nu^j} = 0, \ \forall j = 0, ..., k-1 & \text{on } \partial\Omega, \\ \mathcal{B}_j u = 0, \ \forall j = 1, ..., m-k & \text{on } \partial\Omega, \end{cases} \tag{0.0.7}$$

where $\mathcal{B}_j$ are uniquely defined 'complementing' boundary operators (we refer to [88] for details). We observe that when $m = 1, k = 1$ problem (0.0.7) gives (0.0.2), while for $m = 1, k = 0$ we have problem (0.0.3). When $m = 2$ and $k = 2, 1$ and 0 we have problems (0.0.4), (0.0.6) and (0.0.5), respectively, for suitable operators $\mathcal{B}_j$. Problem (0.0.7) motivates the study of a more general class of eigenvalue problems for higher order operators of the form

$$\sum_{0\le|\alpha|,|\beta|\le m} (-1)^{|\alpha|} D^\alpha \left(A_{\alpha\beta} D^\beta u \right) = \lambda\rho u,$$

subject to homogeneous boundary conditions on an open subset Ω of $\mathbb{R}^N$. (We remark that the space dimension does not play any relevant role in our discussion and restriction to the case $N = 2$ will be done only in Subsection 4.2).

In this thesis we study the dependence of the solutions of the above mentioned problems upon perturbation of the density ρ, with special attention for the behavior of the eigenvalues.
We note that there is a vast literature on the dependence of the eigenvalues of the Laplace operator subject to Dirichlet and Neumann boundary conditions upon mass density perturbations.

One of the fundamental problems concerns the study of the qualitative behavior of the eigenvalues when the density is perturbed and the corresponding results concern continuity, differentiability and analyticity of the eigenvalues. A related fundamental problem concerns the optimization (maximization or minimization) of the eigenvalues (or suitable functionals of the eigenvalues) with respect to the variable ρ under suitable constraints, such as $\int_\Omega \rho dx = \text{const}$. We refer to the monographs [14, 31, 46, 59, 66, 92, 95] and to the papers [43, 44, 69] for an introduction to this subject. We also refer to the recent papers [27, 32, 33, 34, 73] for qualitative results on mass density perturbations problems and for information on the properties of maximizers and minimizers of certain functionals of the eigenvalues of composite membranes.

Another important problem concerns the study of particular classes of mass distributions ρ which are of order ε^{-1} in ε-neighborhoods of points or hypersurfaces contained in Ω and of order ε in the rest of Ω, as ε goes to zero. There is a vast literature concerning vibrating systems containing concentrated masses along curves or around certain points. We mention the extensive monographs [63, 82, 83, 87, 91, 97, 101] and the survey paper [81] for an introduction to the asymptotic analysis techniques for vibrating systems with concentrated masses. We also refer to the recent papers [49, 50, 51, 52, 53, 54, 55, 56, 80] for more information and results on the asymptotics of the eigenvalues and the eigenfunctions of vibrating systems containing stiff or heavy regions. We also mention the alternative approach of [9, 10, 11, 64, 65, 96], where the authors consider the asymptotic behavior of the solutions of elliptic or parabolic equations with terms concentrating at the boundary. In these cases, the results are obtained by means of functional analysis techniques and resolvent estimates for elliptic operators. It is also worth mentioning another approach for singularly perturbed problems based on potential theory and functional analysis proposed in [35, 79].

In this thesis we face three main problems. First, we study the qualitative behavior of the eigenvalues and, in particular, we prove continuity and analytiticy results for the dependence of suitable functions of the eigenvalues of elliptic operators upon mass density perturbations. These results are applied to certain optimization problems. Second, keeping in mind the above mentioned optimization problems, we consider some classes of mass densities concentrating at the boundary or at points, and study the asymptotic behavior of the eigenelements of elliptic operators under these singular mass density perturbations. Third, we define a genuine Steklov problem for the biharmonic operator by means of a family of Neumann-type problems with mass density concentrating at the boundary, and we study the dependence of its eigenvalues upon perturbation of Ω.

Concerning the first type of problems, our study is motivated by well-known results of Krein [69] and Cox and McLaughlin [32, 33, 34] concerning the description of optimal mass densities for the eigenvalues of the Dirichlet Laplacian under the assumption that the total mass is fixed and the additional condition $A \leq \rho \leq B$, where A and B are fixed positive constants. Complete solution to this problem for $N = 1$ was given by Krein in [69], where explicit formulas for the minimizers and the maximizers of all the eigenvalues were established. In particular, it turns out that optimal mass densities are bang-bang solutions, i.e., they satisfy the condition $(\rho - A)(B - \rho) = 0$ on Ω. The case $N > 1$ is discussed in [33, 34] and it is proved that maximizers and minimizers of the first eigenvalue of the Dirichlet Laplacian are bang-bang solutions. Moreover, we remark that Friedland [43] proves that the minimizers of suitable functionals of the eigenvalues are bang-bang as well.
Here, we prove the continuity of the eigenvalues not only with respect to the strong topology of $L^\infty(\Omega)$, but also with respect to the weak* topology, which is more relevant in optimization problems, see Theorem 2.1.5. In this sense we generalize a result of Cox and McLaughlin [32, 33, 34] for the Dirichlet Laplacian. Then we address the problem of the analyticity of the

eigenvalues. We use the general technique developed by Lamberti and Lanza de Cristoforis in [75] for compact self-adjoint operators in Hilbert spaces and prove that all simple eigenvalues and the elementary symmetric functions of multiple eigenvalues are real analytic functions of ρ, see Theorem 2.2.1. We remark that in general one cannot exepect to prove analyticity of the eigenvalues with respect to ρ. This is due to well-known bifurcation phenomena which prevent multiple eigenvalues from being differentiable functions of the parameters involved in the equation. In order to avoid such a situation, in the case of multiple eigenvalues we consider the elementary symmetric functions of the eigenvalues which have the effect of bypassing the splitting phenomenon. Then we compute the appropriate formulas for the Frechét differentials of the symmetric functions of the eigenvalues which are used to address extremum problems. We remark that the general technique of [75] has been used to study domain perturbation problems for different types of operators and boundary conditions, see e.g., [22, 23, 74, 76, 77]. As for mass density perturbation problems, we mention the paper of Lamberti [73] where it is considered the problem of the dependence of the eigenvalues of the Dirichlet Laplacian. In the spirit of [73], here we prove that for a large class of differential operators and boundary conditions, there are no critical mass densities under mass constraint, see Theorem 2.3.2. Moreover, we prove a sort of 'maximum principle' in optimization problems which can be stated as follows: if C is a weakly* compact set of mass densities with prescribed total mass, then '*all simple eigenvalues and the symmetric functions of multiple eigenvalues admit point of maximum and minimum in C with mass constraint and such points of maximum and minimul belong to ∂C*', see Corollaries 2.3.5 and 2.4.1 for the precise statement.

The above mentioned maximum principle and the corresponding absence of critical mass densities has led us to consider a slightly different kind of problems, where the mass density concentrates at the boundary. In this case, Steklov-type problems arise. Recall that the classical Steklov eigenvalue problem for the Laplace operator reads

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda \rho u, & \text{on } \partial\Omega, \end{cases} \tag{0.0.8}$$

where ρ is a measurable positive and bounded function defined on the boundary $\partial\Omega$. This problem has a rather different nature from the eigenvalue problems mentioned above. In fact, for this problem it is possible to find in some cases critical mass densities for the symmetric functions of the eigenvalues under the sole mass constraint $\int_{\partial\Omega} \rho d\sigma = 0$. For example, in the case of the ball in $\mathbb{R}^N$ it is possible to prove that the constant density is a critical point for suitable families of symmetric functions of the eigenvalues, see Corollary 3.2.6. This is not surprising. Indeed, it has been proved by Hersch, Payne and Schiffer in [61] for $N = 2$, that the constant density is a

maximizer for the product of the first two positive eigenvalues (in particular, it is a maximizer for the first positive eigenvalue). We also refer to the book of Bandle [14] for more results and further discussions on the dependence of the eigenvalues of the Steklov Laplacian upon density perturbations on planar domains.
Moreover, for what concerns the minimization problem, we prove that there there are no minimizers for the first positive eigenvalue of the Steklov Laplacian under mass density perturbations preserving the total mass, see Theorems 3.3.3, 3.3.10 and 3.3.19.
We refer to [99] for the physical derivation of problem (0.0.8). We also refer to the recent survey [48] for more information on the Steklov eigenvalues and to [74] for other related problems.

We note that for $N = 2$ problem (0.0.8) provides the vibration modes of a free elastic membrane the total mass of which is concentrated at the boundary with dentisy ρ. We provide an explanation of this known concentration phenomenon in terms of spectral convergence of operators. Namely, for any $\varepsilon > 0$ we define a suitable 'mass density' ρ_ε in the whole of Ω which is of order ε^{-1} in a ε-neighborhood of the boundary $\partial\Omega$, as ε goes to zero, while it is of order ε in the rest of Ω, and such that $\int_\Omega \rho_\varepsilon dx = \text{const}$. Then we consider the eigenvalues of problem (0.0.3) with density ρ_ε and show that such eigenvalues converge to the eigenvalues of problem (0.0.8) as ε goes to zero, see Theorem 3.1.21 and Corollary 3.1.42. This result can be proved by using the notion of compact convergence for the resolvent operators but can also be obtained as a consequence of the more general result proved in [10]. We refer to [9, 10, 11, 64, 65, 96] for a general approach to this kind of problems. Thus the Steklov problem can be considered as a limiting Neumann problem.

Then we address the problem of describing the asymptotic behavior of the Neumann eigenvalues in this mass concentration phenomenon. In particular we show that Neumann eigenvalues are differentiable with respect to ε in a small neighborhood of $\varepsilon = 0$ and provide explicit formulas for their derivatives. First, we consider the case of the unit ball centered at zero in $\mathbb{R}^N$ and next the case of smooth bounded domains in $\mathbb{R}^2$. In the case of the ball we prove that the Neumann eigenvalues have a monotone behavior near their limiting Steklov eigenvalues, and we can conclude that the Steklov eigenvalues locally minimize the Neumann eigenvalues for $\varepsilon > 0$ small enough, see Theorem 4.1.20 and Corollary 4.1.22. It is interesting to compare these results with those in [89] where authors consider the Neumann problem on the annulus $1 - \varepsilon < |x| < 1$ and prove that for $N = 2$ the first positive eigenvalue is a decreasing function of ε.
We note that the techniques that we use for the description of the asymptotic behavior of the Neumann eigenvalues in the case of the ball in $\mathbb{R}^N$ and in the case of general open subsets of $\mathbb{R}^2$ are completely different. In the case of the unit ball centered at zero we use Bessel functions to recast the

eigenvalue problem in the form of an equation $F(\lambda, \varepsilon) = 0$ in the unknowns λ, ε. Then after some preparatory work it is possible to apply the Implicit Function Theorem and conclude. We note that, despite the idea of the proof is rather simple and also used in other contexts (see e.g., [78]) this method requires lenghty computations, suitable Taylor's expansions and estimates on the corresponding remainders, as well as recursive formulas for the cross-products of Bessel functions and their derivatives. Importantly, we remark that the multiplicity of the eigenvalues which is often an obstruction in the application of standard asymptotic analysys, does not affect our method. On the other hand, in the case of a general open and bounded subset Ω of $\mathbb{R}^2$, we show the validity of an asymptotic expansion of the eigenvalues and of the eigenfunctions of problem (0.0.3) as ε goes to zero. In addition we provide explicit formulas for the first two coefficients in the expansions in terms of solutions to suitable auxiliary problems, see Theorems 4.2.10 and 4.2.14. In order to obtain our results, we follow the approach of [52, 53]. The results concerning the asymptotic behavior of Neumann eigenvalues on general domains in $\mathbb{R}^2$ have been obtained in collaboration with Dr. Matteo Dalla Riva (see also [36]).

The results on the behavior of the eigenvalues of the Neumann Laplacian when the mass concentrates at the boundary motivates the study of an analogous mass concentration problem for the biharmonic operator subject to Neumann-type boundary conditions. Namely, we consider the following Neumann-type problem

$$\begin{cases} \Delta^2 u - \tau \Delta u = \lambda \rho_\varepsilon u, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = \tau \frac{\partial u}{\partial \nu} - \frac{\partial \Delta u}{\partial \nu} - \mathrm{div}_{\partial\Omega}\left(D^2 u \cdot \nu\right) = 0, & \text{on } \partial\Omega, \end{cases} \tag{0.0.9}$$

where $\tau \geq 0$ is a fixed non-negative constant and represents the ratio of lateral tension due to flexural rigidity of the plate. Then the eigenvalues of (0.0.9) converge to the eigenvalues of the following Steklov-type problem for the biharmonic operator

$$\begin{cases} \Delta^2 u - \tau \Delta u = 0, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega, \\ \tau \frac{\partial u}{\partial \nu} - \frac{\partial \Delta u}{\partial \nu} - \mathrm{div}_{\partial\Omega}\left(D^2 u \cdot \nu\right) = \lambda u, & \text{on } \partial\Omega, \end{cases} \tag{0.0.10}$$

see Theorem 5.3.6 and Corollary 5.3.12. Thus problem (0.0.10) can be considered as a problem modeling the free vibration modes of a plate the mass of which is concentrated at the boundary, and therefore is a natural generalization to the biharmonic operator of the classical Steklov problem (0.0.8).

Moreover, we address the problem of the dependence of the eigenvalues of problem (0.0.10) upon domain perturbations. We note that domain perturbation problems have been widely studied in the case of the Laplace operator subject to different homogeneous boundary conditions (Dirichlet, Neumann, Steklov, etc.), in particular for shape optimization problems. We recall for instance the Faber-Krahn inequality proved in [41, 68], which says that the ball minimizes the first eigenvalue of the Dirichlet Laplacian among all domains with a fixed measure. Similar results have been shown also for other boundary conditions in [19, 104, 106]. As for the biharmonic operator much less is known. The well-known Rayleigh conjecture on the minimization of the first eigenvalue of the clamped plate has been solved by Nadirashvili for $N = 2$ in [86] and by Ashbaugh and Benguria for $N = 3$ in [12], while the general case remains an open problem (see also [85, 100]). Regarding Neumann boundary conditions, Chasman [28] proved that the ball is a maximizer for the first positive eigenvalue of problem (0.0.9). We refer to [57, 60] for a general approach to domain perturbation problems and to [59] for a comprehensive discussion on shape optimization problems for the eigenvalues of elliptic operators. We also refer to [22, 23] where the authors prove analiticity properties in the spirit of [75] for Dirichlet and intermediate boundary conditions respectively, and show that balls are critical domains for all elementary symmetric functions of the eigenvalues.

In this thesis, first, we prove analyicity results for the symmetric functions of the eigenvalues of problem (0.0.10) in the spirit of [75] (see Theorems 5.4.3 and 5.4.15) and show that balls are critical domains under volume coinstraint (see Theorem 5.4.21). Second, we prove that the ball is actually a maximizer for the first positive eigenvalue of problem (0.0.10) among all bounded open sets of given volume, for any constant $\tau > 0$, see Theorem 5.5.27. This is done by following the approach of [19, 28, 104]. We have also considered the problem of the stability of the optimal shape. In fact, we have provided a quantitative version of the isoperimetric inequality for the first positive Steklov eigenvalue and of the analogue inequality for the Neumann problem proved in [28]. Moreover, the two inequalities turn out to be sharp (see Theorems 5.6.22 and 5.6.59). The results concerning problem (0.0.10) have been obtained in collaboration with Dr. Davide Buoso (see also [24, 25]). The results concerning the sharpness of the isoperimetric inequalities have been obtained in collaboration with Dr. Davide Buoso and Dr. Laura M. Chasman (see also [21]).

Note that problem (0.0.10) should not be confused with another important Steklov-type problem already discussed in the literature, namely

$$\begin{cases} \Delta^2 u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \Delta u = \lambda \frac{\partial u}{\partial \nu}, & \text{on } \partial\Omega, \end{cases} \tag{0.0.11}$$

which has a rather different nature. In fact, for the first positive eigenvalue of problem (0.0.11) the minimization is an interesting open problem (rather than maximization), and explicit examples show that, surprisingly, the ball is not a minimizer.

At the end of this thesis we include some further results on Neumann-type problems. In particular, we consider the behavior of the eigenvalues of the biharmonic operator with Neumann boundary condition and non-zero Poisson's ratio σ, namely

$$\begin{cases} \Delta^2 u = \lambda u, & \text{in } \Omega, \\ (1-\sigma)\frac{\partial^2 u}{\partial \nu^2} + \sigma \Delta u = 0, & \text{on } \partial\Omega, \\ \frac{\partial \Delta u}{\partial \nu} + (1-\sigma)\mathrm{div}_{\partial\Omega}\left(D^2 u \cdot \nu\right)_{\partial\Omega} = 0, & \text{on } \partial\Omega, \end{cases} \tag{0.0.12}$$

where $\sigma \in [0,1[$. We refer to [39], where the author studies the behavior of the eigenvalues of the biharmonic operator on planar domains upon perturbations of the Poisson's ratio . We prove that all the eigenvalues of problem (0.0.12) go to zero as $\sigma \to 1^-$. Moreover, we show that problem (0.0.12) with $\sigma = 1$ admits an increasing sequence of positive eigenvalues of finite multiplicity diverging to $+\infty$ and which coincide with the eigenvalues of the Dirichlet problem for the biharmonic operator.

Moreover, we consider the eigenvalue problem for the Laplace and biharmonic operator with Neumann boundary conditions on an annulus of radii 1 and $1-\varepsilon$, with $\varepsilon \in]-\infty, 0[\cup]0,1[$. In particular, we are interested in the behavior of all the eigenvalues as $\varepsilon \to 0$. We mention the paper [89] where the authors consider the first positive eigenvalue of the Laplace operator with Neumann boundary conditions in the case $N = 2$, and prove that such eigenvalue is continuously differentiable with respect to ε in a suitable interval of $\mathbb{R}$ containing 0, and moreover, that it is an increasing function of ε. We prove an analogous result for all the Neumann eigenvalues of the Laplace and biharmonic operator for all $N \geq 2$. Namely, we provide an asymptotic expansion of all the Neumann eigenvalues at $\varepsilon = 0$. As a bypass product, we prove that all the positive eigenvalues are strictly increasing with respect to ε in a suitable neighborhood of 0.

The thesis is organized as follows. Chapter 1 is dedicated to some preliminaries. In Chapter 2 we consider mass density perturbation problems for general elliptic operators of higher order subject to various homogeneous boundary conditions. For all these cases we prove analyticity results for the eigenvalues in the spirit of [75] and compute the Frechét differentials for the symmetric functions of the eigenvalues which are used to provide a characterization of critical mass densities under mass constraint. Then we prove that for a large class of operators and boundary conditions there are no critical mass densities under the sole mass constraint. Moreover, we prove weak* continuity of the eigenvalues which combined with the results of non-existence of critical points for the eigenvalues, allow to state a sort of 'maximum principle' for a class of optimization problems. In Chapter 3 we consider the Neumann eigenvalue problem for the Laplace operator and mass densities which concentrate at the boundary. We prove that the Neumann eigenvalues converge to the appropriate limiting Steklov eigenvalues by means of strong convergence of the resolvent operators. Moreover, we consider the problem of mass density perturbations for the Steklov problem for the Laplace operator and show that this problem has a rather different nature than that of the problems considered in Chapter 2. In fact we show that in the case of the open unit ball there exist critical mass densities under the sole mass constraint. Moreover, in this chapter we consider other examples of mass concentrations for the Steklov and Dirichlet Laplacian. In Chapter 4 we discuss the asymptotic behavior of the Neumann eigenvalues in the mass concentration phenomenon described in Chapter 3. In particular, in the case of the ball, we prove explicit formulas for the derivatives of the Neumann eigenvalues at the limiting Steklov problem and show that the Steklov eigenvalue locally minimize the Neumann eigenvalues. Moreover, we study the asymptotic behavior of the eigenvalues and eigenfunctions of the Neumann problem in the case of bounded planar domains. We obtain explicit formulas for the first and second terms of the corresponding asymptotic expansions in terms of solutions to certain auxiliary boundary value problems, in the spirit of [52, 53]. In Chapter 5 we consider the biharmonic Neumann eigenvalue problem as described in [28] and problem (0.0.10). We show that problem (0.0.10) can be considered as a limiting Neumann problem for the biharmonic operator in a mass concentration phenomenon. Then we study the dependence of the symmetric functions of the eigenvalues of both Neumann and Steklov problems upon domain perturbations and provide Hadamard type formulas, which allow to give a characterization of critical domains under volume constraint. Then we show that for Neumann and Steklov problems balls are critical domains. Regarding the Steklov problem (0.0.10), we also prove that the ball is a maximizer of the first positive eigenvalue among all bounded open sets of given measure. Finally, in Chapter 6 we collect some results on the Neumann eigenvalues of the Laplace and the biharmonic operators. In particular, we study the dependence of the Neumann eigenvalues of the biharmonic operator upon the Poisson's ratio σ, with particular attention to the behavior of the eigenvalues as $\sigma \to 1^-$. Moreover, we study the asymptotic behavior of the Neumann eigenvalues both of the Laplace and the biharmonic operators on an annulus when the difference between the two radii goes to zero.

Part of the results in this thesis have been published or accepted for publication. The results in Chapter 2 on the dependence of the eigenvalues of general elliptic operators of higher order on the mass density and the corresponding 'maximum principle' have been published in [70]. The results in Chapter 3 on the Steklov eigenvalues as limiting Neumann eigenvalues have been partially published in [72]. The results in the first section of Chapter 4 on the asymptotic behavior of the Neumann eigenvalues on the ball are part of the paper [71], which has been accepted for publication. The results in the second section of Chapter 4 on the asymptotic behavior of the Neumann eigenvalues on planar domains are part of the paper [36] in preparation. The results on the fourth order Steklov problem in Chapter 5 have been published in [24, 25]. The results on the quantitative isoperimetric inequality for the Neumann problem and the sharpness of the Neumann and Steklov inequalities are part of the paper [21] in preparation. The results in the first section of Chapter 6 are part of the paper [93] in preparation.

List of principal symbols

$\mathbb{N}$	(natural numbers)
$\mathbb{N}_0$	(natural numbers including 0)
$f^{(-1)}$	(inverse function of an invertible function f)
r^{-1}, f^{-1}	(reciprocal of a real non-zero number and of a non-vanishing function)
$B, \partial B$	(open unit ball in $\mathbb{R}^N$ and unit sphere in $\mathbb{R}^N$, centered at zero)
ω_N	(Lebesgue measure of the unit ball)
ν	(outer unit normal to a smooth subset of $\mathbb{R}^N$)
$d\sigma$	(surface measure)
$\frac{\partial^k u}{\partial \nu^k}$	(normal k-th derivative of u)
$\lvert\alpha\rvert, \alpha!, D^\alpha$	(multi-index notation)
Df	(Jacobian matrix of f)
$D^2 f$	(Hessian matrix of f)
$\mathrm{div}_{\partial\Omega} F$	(tangential divergence of F defined by $\mathrm{div}_{\partial\Omega} F = \mathrm{div} F_{\vert\partial\Omega} - (DF \cdot \nu) \cdot \nu$)
$f(z) \in O(g(z))$ as $z \to 0$	(there exists $C > 0$ such that $\lvert f(z)\rvert \leq C\lvert g(z)\rvert$ for any z sufficiently close to zero)
(r, θ)	(standard spherical coordinates in $\mathbb{R}^N$, see (A))
Δ_S	(Laplace-Beltrami operator on the unit sphere ∂B of $\mathbb{R}^N$, see (B))

(A) Standard spherical coordinates (r, θ) in $\mathbb{R}^N$, where $\theta = (\theta_1, ...\theta_{N-1})$. The corresponding change of variables is

$$\begin{aligned} x_1 &= r\cos(\theta_1), \\ x_2 &= r\sin(\theta_1)\cos(\theta_2), \\ &\vdots \\ x_{N-1} &= r\sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{N-2})\cos(\theta_{N-1}), \\ x_N &= r\sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{N-2})\sin(\theta_{N-1}), \end{aligned}$$

with $\theta_1, ..., \theta_{N-2} \in [0, \pi]$, $\theta_{N-1} \in [0, 2\pi[$ (here it is understood that $\theta_1 \in [0, 2\pi[$ if $N = 2$).

(B) Laplace-Beltrami operator on the unit sphere ∂B in $\mathbb{R}^N$, defined by

$$\Delta_S = \sum_{j=1}^{N-1} \frac{1}{q_j (\sin\theta_j)^{N-j-1}} \frac{\partial}{\partial\theta_j} \left((\sin\theta_j)^{N-j-1} \frac{\partial}{\partial\theta_j} \right),$$

where

$$q_1 = 1, \quad q_j = (\sin\theta_1 \sin\theta_2 \cdots \sin_{\theta_{j-1}})^2, \quad j = 2, ..., N-1,$$

(see e.g., [67]).

Contents

Chapter 1

Preliminaries and notation

In this chapter we set the notation and introduce certain preliminary results which will be used in the sequel.

1.1 Sobolev Spaces

Let $N \in \mathbb{N}$. In the sequel we shall always consider $N \geq 2$ unless otherwise indicated. For any set V in $\mathbb{R}^N$ and $\rho > 0$ we denote by V_ρ the set $\{x \in V : d(x, \partial\Omega) > \rho\}$. Here $d(x, V)$ denotes the Euclidean distance from x to the set V. Moreover, by a cuboid we mean any roto-translation of a rectangular parallelepiped in $\mathbb{R}^N$.

Definition 1.1.1. *Let $\rho > 0$, $s, s' \in \mathbb{N}$, $s' \leq s$ and $\{V_j\}_{j=1}^s$ be a family of bounded open cuboids and $\{r_j\}_{j=1}^s$ be a family of isometries in $\mathbb{R}^N$. We say that $\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ is an atlas in $\mathbb{R}^N$ with the parameters $\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s$, briefly an atlas in $\mathbb{R}^N$. We denote by $C(\mathcal{A})$ the family of all open sets Ω in $\mathbb{R}^N$ satisfying the following properties:*

(i) $\Omega \subset \bigcup_{j=1}^{s} (V_j)_\rho$ *and* $(V_j)_\rho \cap \Omega \neq \emptyset$;

(ii) $V_j \cap \partial\Omega \neq \emptyset$ *for* $j = 1, \dots s'$, $V_j \cap \partial\Omega = \emptyset$ *for* $s' < j \leq s$;

(iii) for $j = 1, ..., s$

$$r_j(V_j) = \{\, x \in \mathbb{R}^N : \ a_{ij} < x_i < b_{ij},\, i = 1, ..., N\}$$

and

$$r_j(\Omega \cap V_j) = \{x \in \mathbb{R}^N : \ a_{Nj} < x_N < g_j(\bar{x}),\ \bar{x} \in W_j\},$$

where $\bar{x} = (x_1, ..., x_{N-1})$, $W_j = \{\bar{x} \in \mathbb{R}^{N-1} : \ a_{ij} < x_i < b_{ij},\, i = 1, ..., N-1\}$ *and* g_j *is a continuous function defined on* $\overline{W}_j$ *(it is meant that if* $s' < j \leq s$ *then* $g_j(\bar{x}) = b_{Nj}$ *for all* $\bar{x} \in \overline{W}_j$*); moreover for* $j = 1, \dots, s'$

$$a_{Nj} + \rho \leq g_j(\bar{x}) \leq b_{Nj} - \rho,$$

for all $\bar{x} \in \overline{W}_j$.

We say that an open set Ω *in* $\mathbb{R}^N$ *is an open set with a continuous boundary if* Ω *is of class* $C(\mathcal{A})$ *for some atlas* $\mathcal{A}$.

Let $m \in \mathbb{N}, M > 0$. *We say that an open set* Ω *is of class* $C^m_M(\mathcal{A})$ *if* Ω *is of class* $C(\mathcal{A})$ *and all the functions* g_j *in (iii) are of class* $C^m(\overline{W}_j)$ *with*

$$|g_j|_{c^m(\overline{W}_j)} = \sum_{1 \le |\alpha| \le m} \|D^\alpha g_j\|_{L^\infty(\overline{W}_j)} \le M.$$

We say that an open set Ω *in* $\mathbb{R}^N$ *is an open set of class* C^m *if* Ω *is of class* $C^m_M(\mathcal{A})$ *for some atlas* $\mathcal{A}$, $m \in \mathbb{N}$ *and* $M > 0$.

Let $N, m \in \mathbb{N}$, $1 \le p \le \infty$ and Ω be an open set in $\mathbb{R}^N$. Let $C^\infty_c(\Omega)$ be the space of those functions in $C^\infty(\Omega)$ which are compactly supported in Ω. We recall the following definitions.

Definition 1.1.2. *The Sobolev Space* $W^{m,p}(\Omega)$ *consists of all (real valued) functions* u *in* $L^p(\Omega)$ *with weak derivatives* $D^\alpha u$ *in* $L^p(\Omega)$ *for all* $|\alpha| \le m$.

We consider the space $W^{m,p}(\Omega)$ endowed with the norm defined by

$$\|u\|_{W^{m,p}(\Omega)} := \sum_{|\alpha| \le m} \|D^\alpha u\|_{L^p(\Omega)}, \quad \text{if p} \neq \infty,$$
$$\|u\|_{W^{m,\infty}(\Omega)} := \sum_{|\alpha| \le m} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

We recall the definition of $W^{m,p}_0(\Omega)$.

Definition 1.1.3. *Let* $1 \le p < \infty$. *We denote by* $W^{m,p}_0(\Omega)$ *the closure of* $C^\infty_c(\Omega)$ *in* $W^{m,p}(\Omega)$.

For $p = 2$, we write $H^m(\Omega) = W^{m,2}(\Omega)$, $H^m_0(\Omega) = W^{m,2}_0(\Omega)$. We recall the following results on the approximation of functions in $W^{m,p}(\Omega)$.

Theorem 1.1.4. (Global approximation by smooth functions). *Let* Ω *be an open set in* $\mathbb{R}^N$. *Let* $u \in W^{m,p}(\Omega)$ *for some* $1 \le p < \infty$. *Then there exists a sequence* $\{u_k\}_{k\in\mathbb{N}} \subset C^\infty(\Omega) \cap W^{m,p}(\Omega)$ *converging to* u *in* $W^{m,p}(\Omega)$.

Theorem 1.1.5. (Global approximation by smooth functions up to the boundary). *Let* Ω *be a bounded open set in* $\mathbb{R}^N$ *of class* C^1. *Let* $u \in W^{m,p}(\Omega)$ *for some* $1 \le p < \infty$. *Then there exists a sequence* $\{u_k\}_{k\in\mathbb{N}} \subset C^\infty(\overline{\Omega})$ *converging to* u *in* $W^{m,p}(\Omega)$.

As a consequence of Theorem 1.1.4 we have the following

Definition 1.1.6. *Let* Ω *be an open set in* $\mathbb{R}^N$, $1 \le p \le \infty$, $u \in L^p(\Omega)$. *Then* $u \in W^{1,p}(\Omega)$ *if and only if* u *coincides almost everywhere with a function* $\tilde{u}$ *such that for almost all lines* l *parallel to the coordinate axis,* $u_{|_l}$ *is locally absolutely continuous, and the classic derivatives* $\frac{\partial \tilde{u}}{\partial x_1}, \ldots, \frac{\partial \tilde{u}}{\partial x_N}$, *which exist almost everywhere, belong to* $L^p(\Omega)$.

Under suitable regularity assumptions on the open set Ω it makes sense to define the trace of a function $u \in W^{m,p}(\Omega)$ on $\partial\Omega$.

Theorem 1.1.7. (Trace). *Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1. Then there exists a bounded linear operator* Tr *from $W^{1,p}(\Omega)$ to $L^p(\partial\Omega)$ such that:*

i) $\mathrm{Tr}[u] = u_{|\partial\Omega}$ *if $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$;*

ii) $\|\mathrm{Tr}[u]\|_{L^p(\partial\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}$, $\forall u \in W^{1,p}(\Omega)$, *the constant C depending only on p and Ω.*

Theorem 1.1.8. *Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1. Then $u \in W_0^{1,p}(\Omega)$ if and only in $\mathrm{Tr}[u] = 0$.*

The next results concern the embeddings of Sobolev Spaces.

Theorem 1.1.9. (Gagliardo-Nirenberg-Sobolev inequality). *For $1 \leq p < N$ let the Sobolev exponent p^* be defined by $p^* := \frac{Np}{N-p}$. Then there exists $C > 0$, depending only on p and N, such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^N)},$$

for all $u \in W^{1,p}(\mathbb{R}^N)$.

Lemma 1.1.10. (Poincaré inequality). *Let Ω be an open set in $\mathbb{R}^N$ of finite measure, $1 \leq p < \infty$. Then there exists $C > 0$, depending only on p, N and Ω such that*

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)},$$

for all $u \in W_0^{1,p}(\Omega)$.

Theorem 1.1.11. (Rellich-Kondrakhov). *Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1, $1 \leq p < N$. Then $W^{1,p}(\Omega)$ is compactly embedded into $L^q(\Omega)$ for all $1 \leq q < p^*$. If $p \geq N$, then $W^{1,p}(\Omega)$ is compactly embedded into $L^q(\Omega)$ for all $q \geq 1$.*

Corollary 1.1.12. *If Ω is an open set of finite measure, then for all $1 \leq p < \infty$, $W_0^{1,p}(\Omega)$ is compactly embedded into $L^p(\Omega)$. If Ω is a bounded open set of class C^0, then for all $1 \leq p < \infty$, $W^{1,p}(\Omega)$ is compactly embedded into $L^p(\Omega)$.*

Theorem 1.1.13. (Poincaré-Wirtinger inequality). *Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1, $1 \leq p < \infty$. Then there exists $C > 0$, depending only on p, N and Ω such that*

$$\|u - (u)_\Omega\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)},$$

where $(u)_\Omega = \frac{\int_\Omega u}{|\Omega|}$.

From Theorem 1.1.11 it follows that the trace operator is compact.

Theorem 1.1.14. (Compact trace). *Let* Ω *be an open bounded set in* $\mathbb{R}^N$ *of class* C^1, $1 \leq p < \infty$. *Then the trace operator* $\mathrm{Tr} : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ *is compact.*

Remark 1.1.15. *We observe that Theorem 1.1.11 and Theorem 1.1.14 hold even under lower regularity assumptions on the boundary. In fact Theorem 1.1.11 holds e.g., for* Ω *of class* $C^{0,1}$ *(see [88, Theorem 6.1]).*

For all the proofs of the results contained in this subsection we refer to [26].

1.2 Compact operators and symmetric functions of the eigenvalues

In this section we recall some results from [75] which will be used in the sequel. For all the proofs of the results contained in this subsection we refer to [75].

Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be real Banach spaces. Let $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ be the Banach space of bounded linear maps from $\mathcal{X}$ to $\mathcal{Y}$ endowed with the usual norm $\|A\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} := \sup_{\|x\|_{\mathcal{X}}=1} \|Ax\|_{\mathcal{Y}}$. Let $\mathcal{B}(\mathcal{X} \times \mathcal{Y}, \mathcal{Z})$ be the space of bilinear continuous maps from $\mathcal{X} \times \mathcal{Y}$ to $\mathcal{Z}$, endowed with the usual norm of the uniform convergence on the product of the unit ball of $\mathcal{X}$ and the one of $\mathcal{Y}$. Let $(H, < \cdot, \cdot >)$ be a real Hilbert space, and $\|\cdot\|$ be the norm associated with a scalar product $< \cdot, \cdot >$ of H. We denote by H_Q the vector space H endowed with the scalar product $Q = Q(\cdot, \cdot)$, and by $\|\cdot\|_Q$ the associated norm. We denote by $\mathcal{K}(H, H)$ the subspace of $\mathcal{L}(H, H)$ of compact operators, which is closed in $\mathcal{L}(H, H)$. We denote by $\mathcal{K}_{\mathcal{S}}(H_Q, H_Q)$ the closed subspace of $\mathcal{K}(H_Q, H_Q)$ of those T such that $Q(Tu, u) = Q(u, Tu)$ for all $u, v \in H_Q$. Let T be a compact self-adjoint operator on H, and $\sigma(T)$ be the spectrum of T, which is well-known to be a finite or countable subset of $\mathbb{R}$. The elements of $\sigma(T) \setminus \{0\}$ are the eigenvalues of T, and 0 is the only possible accumulation point for $\sigma(T)$. For the characterization of the spectrum of a compact self-adjoint operator we refer to [18]. We denote by $j^+(T)$ the number of positive eigenvalues of T, each counted according to its multiplicity, and by $j^-(T)$ the number of negative eigenvalues of T, each counted according to its multiplicity. Following [75] we set

$$J^+(T) := \{j \in \mathbb{Z} : 1 \leq j \leq j^+(T)\},$$
$$J^-(T) := \{j \in \mathbb{Z} : -j^-(T) \leq j \leq -1\}.$$

Then there exists a unique function $j \to \mu_j(T)$ of $J(T) := J^+(T) \cup J^-(T)$ to $\mathbb{R}$, which is decreasing on $J^-(T)$ and on $J^+(T)$, with

$$\sigma(T) \setminus \{0\} = \{\mu_j(T) : j \in J(T)\},$$

and such that each eigenvalue is repeated according to its multiplicity. We set

$$\mathcal{B}_{\mathcal{S}}(H^2, \mathbb{R}) := \{B \in \mathcal{B}(H^2, \mathbb{R}) \,:\, B(u_1, u_2) = B(u_2, u_1) \text{ for all } u_1, u_2 \in H\},$$

which is a closed subspace of $\mathcal{B}(H^2, \mathbb{R})$, and

$$\mathcal{Q}(H^2, \mathbb{R}) := \{B \in \mathcal{B}_{\mathcal{S}}(H^2, \mathbb{R}) \,:\, \eta[B] > 0\},$$

where

$$\eta[B] := \inf \left\{ \frac{B(u, u)}{\|u\|^2} \,:\, u \in H \setminus \{0\} \right\}.$$

Note that the set $\mathcal{Q}(H)$ is the set of those scalar products on H which are coercive with respect to the fixed scalar product $< \cdot,\cdot >$. We observe that $Q \in \mathcal{B}_{\mathcal{S}}(H^2,\mathbb{R})$ is a coercive scalar product if and only if the embedding of H_Q in H is a homeomorphism. Now we set

$$\mathcal{M} := \{(Q,T) \in \mathcal{B}_{\mathcal{S}}(H^2,\mathbb{R}) \times \mathcal{K}(H,H) \, : \\ Q(Tu,v) = Q(u,Tv) \ \text{ for all } u,v \in H\}.$$

The set $\mathcal{M}$ is closed in $\mathcal{B}_{\mathcal{S}}(H^2,\mathbb{R}) \times \mathcal{K}(H,H)$. Moreover, we set

$$\mathcal{O} := \mathcal{M} \cap (\mathcal{Q}(H^2,\mathbb{R}) \times \mathcal{K}(H,H)) \\ = \{(Q,T) \in \mathcal{Q}(H^2,\mathbb{R}) \times \mathcal{K}(H,H) \, : \, T \in \mathcal{K}_{\mathcal{S}}(H_Q,H_Q)\}.$$

The set $\mathcal{O}$ is open in $\mathcal{M}$. We have the following theorem (see [75]).

Theorem 1.2.1. *Let H be a real Hilbert space, $j \in \mathbb{Z} \setminus \{0\}$. Then the set*

$$\mathcal{A}_j := \{(Q,T) \in \mathcal{O} \, : \, j \in J(T)\}$$

is open in $\mathcal{M}$, and the function $\mu_j[\cdot]$ which takes $(Q,T) \in \mathcal{A}_j$ to $\mu_j[T]$ is continuous.

We consider a fixed finite subset F of $\mathbb{Z} \setminus \{0\}$, and set

$$\mathcal{A}[F] := \{(Q,T) \in \mathcal{O} \, : \, j \in J(T) \, \forall j \in F \, , \ \mu_l[T] \notin \{\mu_j[T] \, : \, j \in F\} \\ \forall l \in J(T) \setminus F\}.$$

By Theorem 1.2.1 it follows that $\mathcal{A}[F]$ is open in $\mathcal{M}$ and $\mu_j[\cdot]$ are continuous on $\mathcal{A}[F]$. We consider the orthogonal projection $P_F[Q,T]$ of H_Q on the subsapce $E[T,F]$ generated by

$$\{u \in H_Q \, : \, Tu = \mu u, \ \exists \, \mu \in \{\mu_j[T] \, : \, j \in F\}\}.$$

We have the following lemma.

Lemma 1.2.2. *Let H be a real Hilbert space and F be a finite subset of $\mathbb{Z} \setminus \{0\}$. Then $E[T,F]$ has dimension equal to the cardinality of F, and it is an invariant subspace of H for T.*

We recall the following result (see [66]).

Theorem 1.2.3. *Let H be a real Hilbert space, F be a finite subset of $\mathbb{Z} \setminus \{0\}$. Then the map P_F which takes $(Q,T) \in \mathcal{A}[F]$ to $P_F[Q,T] \in \mathcal{L}(H,H)$ is continuous.*

The projection $P_F[Q,T]$ depends analytically on (Q,T), in the sense of the following theorem.

Theorem 1.2.4. *Let H be a real Hilbert space, F be a finite non-empty subset of $\mathbb{Z}\setminus\{0\}$ and $(\tilde{Q},\tilde{T}) \in \mathcal{A}[F]$. Then there exists an open neighbourhood $\widetilde{\mathcal{W}}$ of $(\tilde{Q},\tilde{T})$ in $\mathcal{Q}(H^2,\mathbb{R}) \times \mathcal{L}(H,H)$, and a real analytic operator $P_F^\sharp$ of $\widetilde{\mathcal{W}}$ to $\mathcal{L}(H,H)$ such that $P_F^\sharp[Q,T] = P_F[Q,T]$ for all $(Q,T) \in \widetilde{\mathcal{W}} \cap \mathcal{A}[F]$.*

It is possible to choose a orthonormal basis of $E[T,F]$ which depends analitically on (Q,T), as stated in the following lemma (see [75]).

Lemma 1.2.5. *Let H be a real Hilbert space, F be a finite subset of $\mathbb{Z}\setminus\{0\}$ and $(\tilde{Q},\tilde{T}) \in \mathcal{A}[F]$. Let $\{\tilde{u}_j : j \in F\}$ be an othonormal basis for $E[\tilde{T},F]$ in $H_{\tilde{Q}}$. Then there exists an open neighbourhood $\mathcal{W}_0$ of $(\tilde{Q},\tilde{T})$ in $\mathcal{Q}(H^2,\mathbb{R}) \times \mathcal{L}(H,H)$ which is contained in the neighbourhood $\tilde{\mathcal{W}}$ of Theorem 1.2.4 , and $|F|$ real analytic operators $u_j[\cdot,\cdot]$, $j \in F$, of $\mathcal{W}_0$ to H such that:*

i) $\{u_j[Q,T] : j \in F\}$ *is an orthonormal set in* H_Q*, for all* $(Q,T) \in \mathcal{W}_0$ *,*

ii) $\{u_j[Q,T] : j \in F\}$ *is an orthonormal basis for the range of* $P_F^\sharp[Q,T]$*, which coincide with* $E[T,F]$*, in* H_Q*, for all* $(Q,T) \in \mathcal{W}_0 \cap \mathcal{A}[F]$ *,*

iii) $u_j[\tilde{Q},\tilde{T}] = \tilde{u}_j$ *for all* $j \in F$*.*

We need also the following lemma.

Lemma 1.2.6. *Let H be a real Hilbert space, F be a finite subset of $\mathbb{Z}\setminus\{0\}$ and $(\tilde{Q},\tilde{T}) \in \mathcal{A}[F]$. Let $\{\tilde{u_1},...,\tilde{u_{|F|}}\}$ be an orthonormal basis of $E[\tilde{T},F]$ in $H_{\tilde{Q}}$, and $\{u_j[Q,T] : j = 1,...,|F|\}$ as in the previous lemma and $\mathcal{S}$ the map of $\mathcal{W}_0$ to the set $M_{|F|}(\mathbb{R})$ of $|F|\times|F|$ matrices with real coefficients, defined by*

$$\mathcal{S}[Q,T] = (\mathcal{S}_{hk}[Q,T])_{h,k=1,\ldots,|F|} := (Q(Tu_k[Q,T],u_h[Q,T]))_{h,k=1,\ldots,|F|}\,,$$

for all $(Q,T) \in \mathcal{W}_0$. Then $\mathcal{S}[\cdot,\cdot]$ is real analytic and $\mathcal{S}[Q,T]$ is symmetric for all $(Q,T) \in \mathcal{W}_0 \cap \mathcal{A}[F]$. Moreover, if $(Q,T) \in \mathcal{W}_0 \cap \mathcal{A}[F]$, then $\{\mu_j[T]\}_{j\in F}$ are the eigenvalues of $\mathcal{S}[Q,T]$ repeated according to their multiplicity. Finally, if we assume that $\mu_j[\tilde{T}]$ assume a common value $\tilde{\mu}_j$ for all $j \in F$, then the differential of $\mathcal{S}[\cdot,\cdot]$ in $(\tilde{Q},\tilde{T})$ is given by the formula

$$\begin{aligned} &d\mathcal{S}[\tilde{Q},\tilde{T}](\dot{Q},\dot{T}) \\ &\quad = \left(\tilde{Q}(\dot{T}\tilde{u}_k,\tilde{u}_h)\right)_{h,k=1,\ldots,|F|}, \quad \text{for all } (\dot{Q},\dot{T}) \in \mathcal{B}_{\mathcal{S}}(H^2,\mathbb{R}) \times \mathcal{L}(H,H)\,. \end{aligned}$$

Finally, as a consequence of Theorem 1.2.4, Lemma 1.2.5 and Lemma 1.2.6, we have the following theorem.

Theorem 1.2.7. *Let H be a real Hilbert space and F be a finite non-empty subset of $\mathbb{Z}\setminus\{0\}$. Let*

$$M_{F,s}[T] = \sum_{\substack{j_1,\ldots,j_s\in F\\ j_1<\cdots<j_s}} \mu_{j_1}[T]\cdots\mu_{j_s}[T], \quad \forall\, s \in \{1,\ldots,|F|\},$$

for all $(Q,T) \in \mathcal{A}[F]$*, be the elementary symmetric functions of the eigenvalues* $\mu_j[T]$ *with indexes* $j \in F$*. Let* $(\tilde{Q},\tilde{T}) \in \mathcal{A}[F]$*. Then there exists an open neighbourhood* $\widetilde{\mathcal{W}}$ *of* $(\tilde{Q},\tilde{T})$ *in* $\mathcal{Q}(H^2,\mathbb{R}) \times \mathcal{L}(H,H)$*, and real analytic functions* $M^\sharp_{F,s}[\cdot,\cdot]$*, for* $s = 1, ..., |F|$*, of* $\widetilde{\mathcal{W}}$ *in* $\mathbb{R}$ *such that*

$$M^\sharp_{F,s}[Q,T] = M_{F,s}[T],$$

for all $(Q,T) \in \widetilde{\mathcal{W}} \cap \mathcal{A}[F]$*, and for all* $s = 1, ..., |F|$*. If we further assume that there exists* $\tilde{\mu} \in \mathbb{R}$ *such that* $\tilde{\mu} = \mu_j[T]$ *for all* $j \in F$*, and if* $\{\tilde{u}_1, ..., \tilde{u}_{|F|}\}$ *is an orthonormal basis for* $E[\tilde{T},F]$ *in* $H_{\tilde{Q}}$*, then the partial derivative of* $M^\sharp_{F,s}$ *with respect to the variable* T *at* $(\tilde{Q},\tilde{T})$ *is given by the formula*

$$d_T M^\sharp_{F,s}[\tilde{Q},\tilde{T}](\dot{T}) = \binom{|F|-1}{s-1} \tilde{\mu}^{s-1} \sum_{l=1}^{|F|} \tilde{Q}(\dot{T}\tilde{u}_l, \tilde{u}_l),$$

for all $\dot{T} \in \mathcal{K}_{\mathcal{S}}(H_{\tilde{Q}}, H_{\tilde{Q}})$*, and for all* $s = 1, ..., |F|$*.*

1.3 Eigenvalues of elliptic operators

Let Ω be an open set in $\mathbb{R}^N$, $m \in \mathbb{N}$ and $V(\Omega)$ be a closed subspace of $H^m(\Omega)$ containing $H^m_0(\Omega)$ and such that the embedding $V(\Omega) \subset L^2(\Omega)$ is compact. We shall assume that $A_{\alpha\beta} \in L^\infty(\Omega)$ are fixed coefficients such that $A_{\alpha\beta} = A_{\beta\alpha}$ for all $\alpha, \beta \in \mathbb{N}^N_0$, with $|\alpha|, |\beta| \le m$. Let $\rho \in L^\infty(\Omega)$ be such that $\operatorname{ess\,inf}_\Omega \rho > 0$. We consider the following eigenvalue problem

$$\int_\Omega \sum_{0 \le |\alpha|,|\beta| \le m} A_{\alpha\beta} D^\alpha u D^\beta \varphi dx = \lambda \int_\Omega \rho u \varphi dx, \quad \forall \varphi \in V(\Omega), \tag{1.3.1}$$

in the unknowns $u \in V(\Omega)$ (the eigenfunction) and $\lambda \in \mathbb{R}$ (the eigenvalue). It is convenient to denote the left-hand side of equation (1.3.1) by $\mathcal{Q}[u,\varphi]$. It is also convenient to denote by $L^2_\rho(\Omega)$ the space $L^2(\Omega)$ endowed with the scalar product defined by

$$< u_1, u_2 >_\rho := \int_\Omega \rho u_1 u_2 dx, \ \forall u_1, u_2 \in L^2(\Omega).$$

Note that the corresponding norm $\|u\|_{L^2_\rho(\Omega)}$ is equivalent to the standard norm. We assume that the space $V(\Omega)$ and the coefficients $A_{\alpha\beta}$ are such that Gårding's inequality holds, i.e., we assume that there exist $a, b > 0$ such that

$$a\|u\|^2_{H^m(\Omega)} \le \mathcal{Q}[u,u] + b\|u\|^2_{L^2(\Omega)}, \tag{1.3.2}$$

for all $u \in V(\Omega)$. In many cases it will be convenient to normalize the constants a, b in such a way that

$$a\|u\|^2_{H^m(\Omega)} \le \mathcal{Q}[u,u] + b\|u\|^2_{L^2_\rho(\Omega)}. \tag{1.3.3}$$

For classical conditions on the coefficients $A_{\alpha\beta}$ ensuring the validity of (1.3.2) in the case $V(\Omega) = H^m_0(\Omega)$ we refer to [2, Theorem 7.6]. Moreover, we assume that there exists $c > 0$ such that

$$\mathcal{Q}[u,u] \leq c\|u\|^2_{H^m(\Omega)}, \tag{1.3.4}$$

for all $u \in V(\Omega)$. Note that since the coefficients $A_{\alpha\beta}$ are bounded, inequality (1.3.4) is always satisfied.

Under assumptions (1.3.3), (1.3.4), it is possible to prove that problem (1.3.1) has a divergent sequence of eigenvalues bounded from below by $-b$. For the sake of completeness, we recall here the standard procedure to recast problem (1.3.1) into an eigenvalue problem for a compact and self-adjoint operator on a Hilbert space. We consider the bounded linear operator L from $V(\Omega)$ to its dual $V(\Omega)'$ which takes any $u \in V(\Omega)$ to the functional $L[u]$ defined by $L[u][\varphi] := \mathcal{Q}[u,\varphi]$, for all $\varphi \in V(\Omega)$. Moreover, we consider the bounded linear operator I_ρ from $L^2_\rho(\Omega)$ to $V(\Omega)'$ which takes any $u \in L^2_\rho(\Omega)$ to the functional $I_\rho[u]$ defined by $I_\rho[u][\varphi] :=< u, \varphi >_\rho$, for all $\varphi \in V(\Omega)$. By inequalities (1.3.3), (1.3.4) and by the boundedness of the coefficients $A_{\alpha\beta}$, it follows that the quadratic form defined by the right-hand side of (1.3.3) induces in $V(\Omega)$ a norm equivalent to the standard norm of $H^m(\Omega)$. Hence by the Riesz Theorem, it follows that the operator $L + bI_\rho$ is a linear homeomorphism from $V(\Omega)$ onto $V(\Omega)'$, where b is as in (1.3.3). Thus, equation (1.3.1) is equivalent to the equation

$$(L + bI_\rho)^{(-1)} \circ I_\rho[u] = \mu u$$

where

$$\mu = (\lambda + b)^{-1}. \tag{1.3.5}$$

Thus, it is natural to consider the operator T_ρ from $L^2_\rho(\Omega)$ to itself defined by

$$T_\rho := i \circ (L + bI_\rho)^{(-1)} \circ I_\rho, \tag{1.3.6}$$

where i is the embedding of $V(\Omega)$ into $L^2_\rho(\Omega)$. In the sequel, we shall omit i and we shall simply write $T_\rho = (L + bI_\rho)^{(-1)} \circ I_\rho$. Note that

$$\begin{aligned} < T_\rho u_1, u_2 >_\rho &= I_\rho[u_2][(L + bI_\rho)^{(-1)} \circ I_\rho[u_1]] \\ &= (L + bI_\rho)[(L + bI_\rho)^{(-1)} \circ I_\rho[u_1]][(L + bI_\rho)^{(-1)} \circ I_\rho[u_2]], \end{aligned}$$

for all $u_1, u_2 \in L^2_\rho(\Omega)$. Thus, since the operator $L + bI_\rho$ is symmetric it follows that T_ρ is a self-adjoint operator in $L^2_\rho(\Omega)$. Moreover, if the embedding $V(\Omega) \subset L^2(\Omega)$ is compact then the operator T_ρ is compact. By inequality (1.3.3), T_ρ is injective. It follows that the spectrum of T_ρ is discrete and consists of a sequence of positive eigenvalues of finite multiplicity converging to zero. Then by (1.3.5) and standard spectral theory, we easily deduce the validity of the following theorem.

Theorem 1.3.7. *Let $\rho \in L^\infty(\Omega)$ be such that $\operatorname{ess\,inf}_\Omega \rho > 0$. Assume that inequalities (1.3.3) and (1.3.4) are satisfied for some $a, b, c > 0$. Then the eigenvalues of equation (1.3.1) have finite multiplicity and can be represented by means of a divergent sequence λ_j, $j \in \mathbb{N}$ as follows*

$$\lambda_j = \min_{\substack{E \subset V(\Omega) \\ \dim E = j}} \max_{\substack{u \in E \\ u \neq 0}} \frac{\int_\Omega \sum_{|\alpha|,|\beta| \le m} A_{\alpha\beta} D^\alpha u D^\beta u dx}{\int_\Omega \rho u^2 dx} . \qquad (1.3.8)$$

Each eigenvalue is repeated according to its multiplicity and

$$\lambda_j > -b + \frac{a}{\|\rho\|_{L^\infty(\Omega)}} ,$$

for all $j \in \mathbb{N}$. Moreover, the sequence $\mu_j = (b + \lambda_j)^{-1}$, $j \in \mathbb{N}$, represents all eigenvalues of the compact self-adjoint operator T_ρ.

1.4 Bessel functions, modified Bessel functions and ultraspherical Bessel functions

In this section we recall some facts from the theory of Bessel functions and of spherical harmonics, which will be used in the sequel. For all the proofs of the results concerning the Bessel functions contained in this section and for more information on Bessel functions we refer to [1]. For all the proofs of the results concerning the spherical harmonics and for an introduction to the theory of spherical harmonics we refer to [42].

1.4.1 Bessel functions

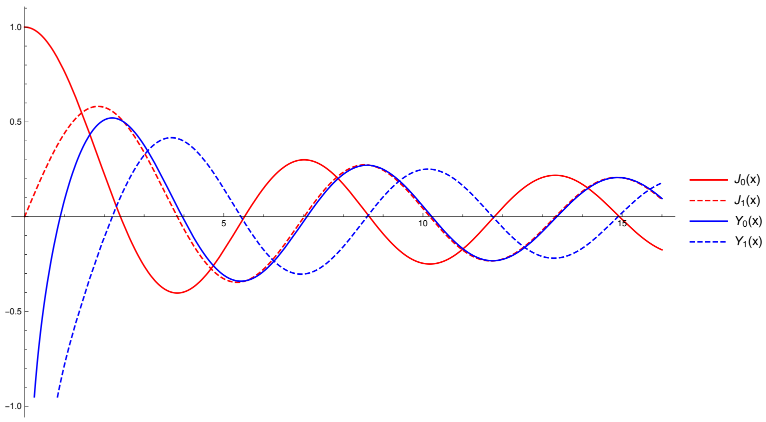
Consider the (complex) Bessel equation

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - \nu^2) w = 0, \ z \in \mathbb{C}, \qquad (1.4.1)$$

where $\nu \in \mathbb{C}$. As is well-known the solutions of this equation are given by the Bessel functions of the first kind $J_{\pm\nu}(z)$ and of the second kind $Y_\nu(z)$. Such functions are holomorphic in the variable $z \in \mathbb{C} \setminus \mathbb{R}_-$. Functions $J_\nu(z)$ and $J_{-\nu}(z)$ are linearly independent except when ν is an integer. $J_\nu(z)$ and $Y_\nu(z)$ are linearly independent for all ν. We recall some useful relations:

$$\begin{aligned} Y_\nu(z) &= \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\pi\nu)}, \qquad (1.4.2) \\ J_{-n}(z) &= (-1)^n J_n(z), \\ Y_{-n}(z) &= (-1)^n Y_n(z), \end{aligned}$$

where in the last two equations $n \in \mathbb{N}$. Note that if $\nu = 0$ or if ν is an integer, the right-hand side of the first equation is replaced by its limiting



value.
We recall some useful recurrence relations between solutions of (1.4.1):

$$\begin{aligned} \mathcal{C}_{\nu-1}(z)+\mathcal{C}_{\nu+1}(z) &= \frac{2\nu}{z}\mathcal{C}_\nu(z), \\ \mathcal{C}_{\nu-1}(z)-\mathcal{C}_{\nu+1}(z) &= 2\mathcal{C}'_\nu(z), \\ \mathcal{C}'_\nu(z) &= \mathcal{C}_{\nu-1}(z)-\frac{\nu}{z}\mathcal{C}_\nu(z), \\ \mathcal{C}'_\nu(z) &= -\mathcal{C}_{\nu+1}(z)+\frac{\nu}{z}\mathcal{C}_\nu(z), \end{aligned}$$

where $\mathcal{C}_\nu$ denotes J_ν, Y_ν or any linear combination of these functions. We also recall that

$$J'_0(z)=-J_1(z), \quad Y'_0(z)=-Y_1(z).$$

We have the following formulas for the derivatives:

$$\mathcal{C}^{(k)}_\nu(z)=\frac{1}{2^k}\Big(\mathcal{C}_{\nu-k}(z)-\binom{k}{1}\mathcal{C}_{\nu-k+2}(z) + \binom{k}{2}\mathcal{C}_{\nu-k+1}(z)-\ldots+(-1)^k\mathcal{C}_{\nu+k}(z)\Big),$$

for $k\in\mathbb{N}_0$. We also recall the following Taylor expansion of $J_\nu(z)$:

$$J_\nu(z)=\left(\frac{z}{2}\right)^\nu\sum_{k=0}^{\infty}\frac{\left(-\frac{z^2}{4}\right)^k}{k!\Gamma(\nu+k+1)}. \tag{1.4.3}$$

Note that from (1.4.2) and (1.4.3) it follows that $Y_\nu(z)$ has a singularity at $z=0$. In particular $Y_0(z)=\frac{2}{\pi}\ln(z)+o(\ln(z))$, $Y_\nu(z)=-\frac{\Gamma(\nu)}{\pi}\left(\frac{z}{2}\right)^{-\nu}+o(z^{-\nu})$ when $\mathrm{Re}(\nu)>0$, as z goes to zero. We have the following formulas for the Wronskians:

$$\begin{aligned} W(J_\nu(z),J_{-\nu}(z)) &:= J_{\nu+1}(z)J_{-\nu}(z)+J_\nu(z)J_{-(\nu+1)}(z)=-\frac{2\sin(\nu\pi)}{\pi z}, \\ W(J_\nu(z),Y_\nu(z)) &:= J_{\nu+1}(z)Y_\nu(z)-J_\nu(z)Y_{\nu+1}(z)=\frac{2}{\pi z}. \end{aligned}$$

We recall some useful facts on the zeros of cross products of Bessel functions. Let $\nu \in \mathbb{R}$ and $\lambda > 0$. The zeros of the function which takes $z \in \mathbb{C}$ to

$$J_\nu(z)Y_\nu(\lambda z) - J_\nu(\lambda z)Y_\nu(z)$$

are real and simple, in the sense that the derivative of the cross product with respect to z does not vanish at the zeroes. Moreover, there exist countably many zeroes of $J_\nu(z)Y_\nu(\lambda z) - J_\nu(\lambda z)Y_\nu(z)$. If $\lambda > 1$, the asymptotic expansion of the n-th zero with respect to λ near $\lambda = 1$ is given by

$$\beta + \frac{p}{\beta} + \frac{q - p^2}{\beta^3} + \frac{r - 4pq + 2p^3}{\beta^5} + O(\lambda - 1)^5, \tag{1.4.4}$$

where

$$\begin{aligned} \beta &= \frac{n\pi}{\lambda - 1}, \\ p &= \frac{\mu - 1}{8\lambda}, \\ q &= \frac{(\mu - 1)(\mu - 25)(\lambda^3 - 1)}{6(4\lambda)^3(\lambda - 1)}, \\ r &= \frac{(\mu - 1)(\mu^2 - 114\mu + 1073)(\lambda^5 - 1)}{5(4\lambda)^5(\lambda - 1)}, \end{aligned}$$

with $\mu = 4\nu^2$. The asymptotic expansion of the large positive zeros of the function which takes $z \in \mathbb{C}$ to

$$J'_\nu(z)Y'_\nu(\lambda z) - J'_\nu(\lambda z)Y'_\nu(z)$$

is given by (1.4.4) with the same β and

$$\begin{aligned} p &= \frac{\mu + 3}{8\lambda}, \\ q &= \frac{(\mu^2 + 46\mu - 63)(\lambda^3 - 1)}{6(4\lambda)^3(\lambda - 1)}, \\ r &= \frac{(\mu^3 + 185\mu^2 - 2053\mu + 1899)(\lambda^5 - 1)}{5(4\lambda)^5(\lambda - 1)}. \end{aligned}$$

The asymptotic expansion of the large positive zeros of the function which takes $z \in \mathbb{C}$ to

$$J'_\nu(z)Y_\nu(\lambda z) - J_\nu(\lambda z)Y'_\nu(z)$$

is given by (1.4.4) with

$$
\begin{aligned}
\beta &= \frac{\left(n-\frac{1}{2}\right)\pi}{\lambda-1}, \\
p &= \frac{(\mu+3)\lambda-(\mu-1)}{8\lambda(\lambda-1)}, \\
q &= \frac{(\mu^2+46\mu-63)\lambda^3-(\mu-1)(\mu-25)}{6(4\lambda)^3(\lambda-1)}, \\
r &= \frac{(\mu^3+185\mu^2-2053\mu+1899)\lambda^5-(\mu-1)(\mu^2-114\mu+1073)}{5(4\lambda)^5(\lambda-1)}.
\end{aligned}
$$

1.4.2 Modified Bessel function

Consider the modified Bessel equation

$$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}-(z^2+\nu^2)w=0,\ \ z\in\mathbb{C}.$$

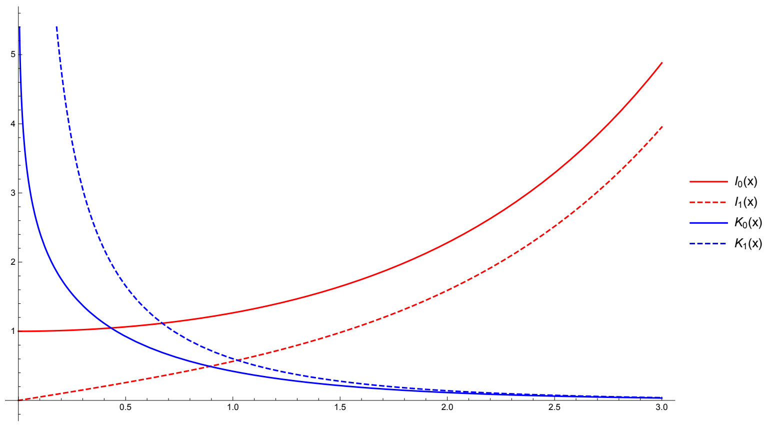
The solutions of this equation are given by the modified Bessel functions of the first kind $I_{\pm\nu}(z)$ and of the second kind $K_\nu(z)$. Such functions are holomorphic in the variable $z\in\mathbb{C}\setminus\mathbb{R}_-$. $I_\nu(z)$ and $I_{-\nu}(z)$ are linearly independent except when ν is an integer. $I_\nu(z)$ and $K_\nu(z)$ are linearly independent for all ν. The functions $I_\nu(z)$ and $K_\nu(z)$ are real and positive when $\nu>-1$ and $z>0$. We recall some useful relations:

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z)-I_\nu(z)}{\sin(\pi\nu)}, \tag{1.4.5}$$

$$I_{-n}(z) = I_n(z), \tag{1.4.6}$$

$$K_{-\nu}(z) = K_\nu(z),$$

where in (1.4.6) the index n is an integer number.



We recall some other useful recurrence relations:

$$\begin{aligned} \mathcal{C}_{\nu-1}(z) - \mathcal{C}_{\nu+1}(z) &= \frac{2\nu}{z}\mathcal{C}_\nu(z), \\ \mathcal{C}_{\nu-1}(z) + \mathcal{C}_{\nu+1}(z) &= 2\mathcal{C}'_\nu(z), \\ \mathcal{C}'_\nu(z) &= \mathcal{C}_{\nu-1}(z) - \frac{\nu}{z}\mathcal{C}_\nu(z), \\ \mathcal{C}'_\nu(z) &= -\mathcal{C}_{\nu+1}(z) + \frac{\nu}{z}\mathcal{C}_\nu(z), \end{aligned}$$

where $\mathcal{C}_\nu$ denotes here any of the functions I_ν, K_ν or any linear combination of them. We also recall that

$$I'_0(z) = I_1(z), \quad K'_0(z) = -K_1(z).$$

We have the following formulas for the derivatives:

$$\mathcal{C}_\nu^{(k)}(z) = \frac{1}{2^k}\Big(\mathcal{C}_{\nu-k}(z) + \binom{k}{1}\mathcal{C}_{\nu-k+2}(z) + \binom{k}{2}\mathcal{C}_{\nu-k+1}(z) + \ldots + \mathcal{C}_{\nu+k}(z)\Big),$$

for $k \in \mathbb{N}_0$. We have the following Taylor expansion of $I_\nu(z)$:

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^\infty \frac{\left(\frac{z^2}{4}\right)^k}{k!\Gamma(\nu+k+1)}. \tag{1.4.7}$$

Note that from (1.4.5) and (1.4.7) it follows that $K_\nu(z)$ has a singularity at $z = 0$, in particular $K_0(z) = -\ln(z) + o(\ln(z))$, $K_\nu(z) = \frac{\Gamma(\nu)}{2}\left(\frac{z}{2}\right)^{-\nu} + o(z^{-\nu})$ when $\mathrm{Re}(\nu) > 0$, as z goes to zero. We have the following formulas for the Wronskians:

$$\begin{aligned} W(I_\nu(z), I_{-\nu}(z)) &:= I_\nu(z)I_{-(\nu+1)}(z) - I_{\nu+1}(z)I_{-\nu}(z) = -\frac{2\sin(\nu\pi)}{\pi z}, \\ W(K_\nu(z), I_\nu(z)) &:= I_\nu(z)K_{\nu+1}(z) + I_{\nu+1}(z)K_\nu(z) = \frac{1}{z}. \end{aligned}$$

1.4.3 Ultraspherical Bessel and modified Bessel functions

We recall the definitions of ultraspherical Bessel functions and modified ultraspherical Bessel functions. Consider the ultraspherical Bessel equation

$$z^2\frac{d^2w}{dz^2} + (N-1)z\frac{dw}{dz} + \left(z^2 - l(l+N-2)\right)w = 0, \; z \in \mathbb{C}, \tag{1.4.8}$$

and the modified ultraspherical Bessel equation

$$z^2\frac{d^2w}{dz^2} + (N-1)z\frac{dw}{dz} - \left(z^2 + l(l+N-2)\right)w = 0, \; z \in \mathbb{C}, \tag{1.4.9}$$

for $l \in \mathbb{N}_0$. The ultraspherical Bessel functions of the first and second kind $j_l(z)$ and $y_l(z)$ are suitable linearly independent solutions of (1.4.8). The modified ultraspherical Bessel functions of the first and second kind $i_l(z)$ and $k_l(z)$ are suitable linearly independent solutions of (1.4.9). Namely, the functions $j_l(z), y_l(z), i_l(z)$ and $k_l(z)$ are defined in terms of the Bessel functions $J_\nu(z), Y_\nu(z), I_\nu(z), K_\nu(z)$, as follows:

$$\begin{aligned} j_l(z) &:= z^{-\frac{N-2}{2}} J_{\frac{N-2}{2}+l}(z), \\ y_l(z) &:= z^{-\frac{N-2}{2}} Y_{\frac{N-2}{2}+l}(z), \\ i_l(z) &:= z^{-\frac{N-2}{2}} I_{\frac{N-2}{2}+l}(z), \\ k_l(z) &:= z^{-\frac{N-2}{2}} K_{\frac{N-2}{2}+l}(z). \end{aligned}$$

The ultraspherical and modified ultraspherical Bessel functions have a number of recurrence relations which are inherited from those of the ordinary Bessel functions. We recall some of these relations for $j_l(z)$ and $i_l(z)$:

$$\begin{aligned} \frac{N-2+2l}{z} j_l(z) &= j_{l-1}(z) + j_{l+1}(z), \\ j_l'(z) &= \frac{l}{z} j_l(z) - j_{l+1}(z) = j_{l-1}(z) - \frac{l+N-2}{z} j_l(z), \\ \frac{N-2+2l}{z} i_l(z) &= i_{l-1}(z) - i_{l+1}(z), \\ i_l'(z) &= \frac{l}{z} i_l(z) + j_{l+1}(z). \end{aligned}$$

Note that for $N = 2$ the expressions above simplify to the corresponding relations for the ordinary Bessel functions. We recall some recurrence relations for the second derivatives:

$$\begin{aligned} j_l''(z) &= \left(\frac{l^2 - l}{z^2} - 1 \right) j_l(z) + \frac{N-1}{z} j_{l+1}(z), \\ i_l''(z) &= \left(\frac{l^2 - l}{z^2} + 1 \right) i_l(z) - \frac{N-1}{z} j_{l+1}(z). \end{aligned}$$

Again, when $N = 2$ each relation simplifies to the analogue for the ordinary Bessel functions. We have the following expansions of $j_l(z)$ and $i_l(z)$:

$$\begin{aligned} j_l(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k 2^{1-\frac{N}{2}}}{k!\Gamma(k+\frac{N}{2}+l)} \left(\frac{z}{2}\right)^{2k+l}, \\ i_l(z) &= \sum_{k=0}^{\infty} \frac{2^{1-\frac{N}{2}}}{k!\Gamma(k+\frac{N}{2}+l)} \left(\frac{z}{2}\right)^{2k+l}. \end{aligned} \tag{1.4.10}$$

From (1.4.10) it follows that $i_l(z)$ and its derivatives are all positive on $]0, +\infty[$.

1.4.4 Spherical harmonics

Let B be the unit ball in $\mathbb{R}^N$ centered at zero. We denote by ∂B the boundary of B, i.e., the unit sphere in $\mathbb{R}^N$. Let $k \in \mathbb{N}_0$ and $\mathcal{P}_k$ be the space of homogeneous polynomials of degree k on $\mathbb{R}^N$ and let

$$\begin{aligned} \mathscr{H}_k &:= \{P \in \mathcal{P}_k : \Delta P = 0\}, \\ \mathscr{S}_k &:= \left\{P_{|_{\partial B}} : P \in \mathscr{H}_k\right\}. \end{aligned}$$

The set $\mathscr{H}_k$ is the space of homogeneous harmonic polynomials of degree k and $\mathscr{S}_k$ is the space of their restriction on the unit sphere. The elements of $\mathscr{S}_k$ are the so-called spherical harmonics of degree k. We denote by r^2 the quantity $r^2 = |x|^2 = \sum_{j=1}^N x_j^2$, where $x = (x_1, x_2, ..., x_N)$ is a point in $\mathbb{R}^N$. We have the following lemma.

Lemma 1.4.11. *Let $k \in \mathbb{N}_0$. Then*

$$\mathcal{P}_k = \mathscr{H}_k \oplus r^2 \mathcal{P}_{k-2},$$

where $r^2\mathcal{P}_{k-2} = \left\{r^2 P : P \in \mathcal{P}_{k-2}\right\}$.

Corollary 1.4.12.

$$\mathcal{P}_k = \bigoplus_{j=0}^{k} r^j \mathscr{H}_{k-j}.$$

Corollary 1.4.13. *The restriction to the unit sphere of any element of $\mathcal{P}_k$ is a sum of spherical harmonics of degree at most k.*

Let $L^2(\partial B)$ the Hilbert space of the square integrable functions on ∂B with respect to the $N-1$ dimensional Haudorff measure of ∂B, endowed with the standard scalar product

$$< u, v >_{L^2(\partial B)} := \int_{\partial B} uv d\sigma,$$

for all $u, v \in L^2(\partial B)$. Here $d\sigma$ denotes the surface measure of ∂B. We have the following theorem on the representation of functions in $L^2(\partial B)$ in terms of spherical harmonics.

Theorem 1.4.14. *We have*

$$L^2(\partial B) = \bigoplus_{k=0}^{\infty} \mathscr{S}_k,$$

the expression on the right-hand side being an orthogonal direct sum with respect to the standard scalar product on $L^2(\partial B)$.

Let

$$d_k := \dim \mathscr{S}_k = \dim \mathscr{H}_k.$$

We have an explicit formula for d_k.

Lemma 1.4.15. *We have for* $\mathcal{P}_k$

$$\dim \mathcal{P}_k = \frac{(k+N-1)!}{k!(N-1)!}.$$

Corollary 1.4.16. *We have for* $\mathscr{S}_k$

$$d_k = (2k+N-2)\frac{(k+N-3)!}{k!(N-2)!}.$$

Let us denote by Δ_S the Laplace-Beltrami operator on the unit sphere ∂B in $\mathbb{R}^N$. We have the following lemma.

Lemma 1.4.17. *The solutions of*

$$-\Delta_S \phi = k(k+N-2)\phi, \text{ on } \partial B,$$

are the spherical harmonics of order k*.*

Chapter 2

Elliptic operators subject to mass density perturbations and maximum principles

In this chapter we discuss eigenvalue problems for general elliptic operators of arbitrary order subject to different homogeneous boundary conditions on open subsets of $\mathbb{R}^N$. The class of operators and boundary conditions which we consider is quite general and contains, for example, all the poly-harmonic operators subject to Dirichlet, intermediate, Neumann or mixed boundary conditions.

Let Ω be an open subset of $\mathbb{R}^N$ and $m \in \mathbb{N}$. We consider the elliptic partial differential operator $\mathcal{L}$ defined by

$$\mathcal{L}u := \sum_{0 \le |\alpha|, |\beta| \le m} (-1)^{\alpha} D^{\alpha} \left(A_{\alpha\beta} D^{\beta} u \right) \tag{2.0.1}$$

subject to homogeneous boundary conditions. By $\mathcal{R}$ we denote the subset of $L^{\infty}(\Omega)$ of those $\rho \in L^{\infty}(\Omega)$ such that $\operatorname{ess\,inf}_{\Omega} \rho > 0$. Let $\rho \in \mathcal{R}$ be fixed. We consider the eigenvalue problem

$$\mathcal{L}u = \lambda \rho u. \tag{2.0.2}$$

The weak formulation of problem (2.0.2) is

$$\int_{\Omega} \sum_{0 \le |\alpha|, |\beta| \le m} A_{\alpha\beta} D^{\alpha} u D^{\beta} \varphi dx = \lambda \int_{\Omega} \rho u \varphi dx, \quad \forall \varphi \in V(\Omega), \tag{2.0.3}$$

in the unknowns $u \in V(\Omega)$ (the eigenfunction) and $\lambda \in \mathbb{R}$ (the eigenvalue), where $V(\Omega) \subset H^m(\Omega)$ is the energy space associated with the boundary conditions imposed on $\mathcal{L}$. We assume that the coefficients $A_{\alpha\beta}$ are bounded real-valued functions such that $A_{\alpha\beta} = A_{\beta\alpha}$. Moreover, we assume that the space $V(\Omega)$ and the coefficients $A_{\alpha\beta}$ are such that inequalities (1.3.3) and (1.3.4) hold. If the embedding $V(\Omega) \subset L^2(\Omega)$ is compact, from Theorem 1.3.7 it follows that problem (2.0.3) admits a diverging sequence of eigenvalues of finite multiplicity

$$\lambda_1[\rho] \le \cdots \le \lambda_j[\rho] \le \cdots .$$

We prove a few results concerning the dependence of the eigenvalues $\lambda_j[\rho]$ upon variation of ρ.

2.1 Continuity of the eigenvalues

By the min-max principle (1.3.8) it follows that $\lambda_j[\rho]$ is a locally Lipschitz continuous functions of $\rho \in \mathcal{R}$. In fact, one can easily prove that

$$|\lambda_j[\rho_1] - \lambda_j[\rho_2]| \le \frac{\min\{\lambda_j[\rho_1], \lambda_j[\rho_2]\} + 2b}{\min\{\operatorname{ess\,inf}\rho_1, \operatorname{ess\,inf}\rho_2\}} \|\rho_1 - \rho_2\|_{L^\infty(\Omega)}\,, \tag{2.1.1}$$

for all $\rho_1, \rho_2 \in \mathcal{R}$ satisfying $\|\rho_1 - \rho_2\|_{L^\infty(\Omega)} < \min\{\operatorname{ess\,inf}\rho_1, \operatorname{ess\,inf}\rho_2\}$. Indeed, for $u \in V(\Omega)$ and $\rho_1, \rho_2 \in \mathcal{R}$ we have

$$\begin{aligned}
\left| \frac{\mathcal{Q}[u,u]}{\int_\Omega \rho_1 u^2 dx} - \frac{\mathcal{Q}[u,u]}{\int_\Omega \rho_2 u^2 dx} \right| &\le \frac{|\mathcal{Q}[u,u]|\left|\int_\Omega (\rho_2 - \rho_1) u^2 dx\right|}{\left(\int_\Omega \rho_1 u^2 dx\right)\left(\int_\Omega \rho_2 u^2 dx\right)} \\
&\le \frac{|\mathcal{Q}[u,u]| \int_\Omega u^2 dx \|\rho_2 - \rho_1\|_{L^\infty(\Omega)}}{\left(\int_\Omega \rho_1 u^2 dx\right)\left(\int_\Omega \rho_2 u^2 dx\right)} \le \frac{|\mathcal{Q}[u,u]| \|\rho_2 - \rho_1\|_{L^\infty(\Omega)}}{\left(\int_\Omega \rho_1 u^2 dx\right) \operatorname{ess\,inf}\rho_2} \\
&= \frac{\left|\mathcal{Q}[u,u] + b\int_\Omega \rho_1 u^2 dx - b\int_\Omega \rho_1 u^2 dx\right| \|\rho_2 - \rho_1\|_{L^\infty(\Omega)}}{\left(\int_\Omega \rho_1 u^2 dx\right) \operatorname{ess\,inf}\rho_2} \\
&\le \left(\frac{\mathcal{Q}[u,u]}{\int_\Omega \rho_1 u^2 dx} + 2b \right) \frac{\|\rho_2 - \rho_1\|_{L^\infty(\Omega)}}{\operatorname{ess\,inf}\rho_2}.
\end{aligned} \tag{2.1.2}$$

From (2.1.2) it follows that

$$\begin{aligned}
\frac{\mathcal{Q}[u,u]}{\int_\Omega \rho_1 u^2 dx} \left(1 - \frac{\|\rho_2 - \rho_1\|_{L^\infty(\Omega)}}{\operatorname{ess\,inf}\rho_2} \right) - \frac{2b\|\rho_2 - \rho_1\|_{L^\infty(\Omega)}}{\operatorname{ess\,inf}\rho_2} \\
\le \frac{\mathcal{Q}[u,u]}{\int_\Omega \rho_2 u^2 dx} \\
\le \frac{\mathcal{Q}[u,u]}{\int_\Omega \rho_1 u^2 dx} \left(1 + \frac{\|\rho_2 - \rho_1\|_{L^\infty(\Omega)}}{\operatorname{ess\,inf}\rho_2} \right) + \frac{2b\|\rho_2 - \rho_1\|_{L^\infty(\Omega)}}{\operatorname{ess\,inf}\rho_2}.
\end{aligned} \tag{2.1.3}$$

If ρ_1, ρ_2 satisfy $\|\rho_2 - \rho_1\|_{L^\infty(\Omega)} < \operatorname{ess\,inf}\rho_2$, then taking the infimum and the supremum in (2.1.3) yields

$$|\lambda_j[\rho_1] - \lambda_j[\rho_2]| \le \frac{\lambda_j[\rho_1] + 2b}{\operatorname{ess\,inf}\rho_2} \|\rho_1 - \rho_2\|_{L^\infty(\Omega)}. \tag{2.1.4}$$

It is easy to see that formula (2.1.4) still holds if in the right-hand side we replace $\lambda_j[\rho_1]$ with $\lambda_j[\rho_2]$ and $\operatorname{ess\,inf}\rho_2$ with $\operatorname{ess\,inf}\rho_1$. This proves formula (2.1.1).
Actually, $\lambda_j[\rho]$ depends with continuity on ρ not only with respect to the strong topology of $L^\infty(\Omega)$ but also with respect to the weak* topology, which is clearly more relevant in optimization problems. The following theorem was proved by Cox and McLaughlin [33] in the case of the Dirichlet Laplacian and mass densities uniformly bounded away from zero and infinity. The proof can be easily adapted to the general case. Moreover, it is possible to replace the uniform lower bound for ρ by a weaker assumption.

Theorem 2.1.5. *Let $C \subset \mathcal{R}$ be a bounded set. Assume that there exist $a, b, c > 0$ such that inequalities (1.3.3) and (1.3.4) are satisfied for all $\rho \in C$. Then the functions from C to $\mathbb{R}$ which take any $\rho \in C$ to $\lambda_j[\rho]$ are weakly* continuous for all $j \in \mathbb{N}$.*

Proof. Since C is bounded in $L^\infty(\Omega)$, it suffices to prove that given $\rho \in C$ and a sequence $\rho_k \in C$, $k \in \mathbb{N}$ such that $\rho_k \rightharpoonup^* \rho$ as $k \to \infty$ then $\lambda_j[\rho_k] \to \lambda_j[\rho]$. To do so, we first prove that for each $j \in \mathbb{N}$ there exists $L_j > 0$ such that $\lambda_j[\rho_k] \leq L_j$ for all $k \in \mathbb{N}$. This is clearly trivial if we assume that $0 < \alpha \leq \rho$ for all $\rho \in C$, in which case $\lambda_j[\rho] \leq \lambda_j[\alpha]$. Let $j \in \mathbb{N}$ be fixed and $u_1, \dots, u_j \in V(\Omega)$ be linearly independent eigenfunctions associated with the eigenvalues $\lambda_1[\rho], \dots, \lambda_j[\rho]$, normalized by $< u_r, u_s >_\rho = \delta_{rs}$ for all $r, s = 1, \dots, j$. Note that

$$\lim_{k\to\infty} \int_\Omega u_r u_s \rho_k dx = \int_\Omega u_r u_s \rho dx,$$

for all $r, s = 1, \dots, j$. Thus

$$\lim_{k\to\infty} \int_\Omega \left(\sum_{r=1}^{j} \gamma_r u_r \right)^2 \rho_k dx = \int_\Omega \left(\sum_{r=1}^{j} \gamma_r u_r \right)^2 \rho dx, \tag{2.1.6}$$

uniformly with respect to $\gamma = (\gamma_1, \dots, \gamma_j) \in \mathbb{R}^j$ with $|\gamma| \leq 1$. Let E be the linear space generated by $u_1, \dots, u_j$. By (2.1.6) it follows that for any $\varepsilon > 0$ there exists $k_\varepsilon \in \mathbb{N}$ such that

$$\begin{aligned}\frac{\int_\Omega \sum_{|\alpha|,|\beta|\leq m} A_{\alpha\beta} D^\alpha u D^\beta u dx}{\int_\Omega u^2 \rho_k dx} &\leq \frac{\int_\Omega \sum_{|\alpha|,|\beta|\leq m} A_{\alpha\beta} D^\alpha u D^\beta u dx}{\int_\Omega u^2 \rho dx} \\ &\quad + \varepsilon(\lambda_j[\rho] + 2b) \leq \lambda_j[\rho] + \varepsilon(\lambda_j[\rho] + 2b)\end{aligned} \tag{2.1.7}$$

for all $u \in E$, $k \geq k_\varepsilon$. By combining (1.3.8) and (2.1.7) we deduce that $\lambda_j[\rho_k] \leq \lambda_j[\rho] + \varepsilon(\lambda_j[\rho] + 2b)$ for all $k \geq k_\varepsilon$, which implies the existence of a uniform bound L_j as claimed above. The rest of the proof follows the lines of Cox [33]. Let $u_j[\rho_k]$, $j \in \mathbb{N}$ be a sequence of eigenfunctions associated with the eigenvalues $\lambda_j[\rho_k]$ normalized by $< u_j[\rho_k], u_l[\rho_k] >_{\rho_k} = \delta_{jl}$ for all $j, l \in \mathbb{N}$. Note that $\mathcal{Q}[u_j[\rho_k], u_j[\rho_k]] = \lambda_j[\rho_k]$ for all $k \in \mathbb{N}$, where we denoted by $\mathcal{Q}[u, \varphi]$ the left-hand side of (2.0.3). By inequality (1.3.3), the sequence $u_j[\rho_k]$, $k \in \mathbb{N}$ is bounded in the space $V(\Omega)$ equipped with the standard norm of $H^m(\Omega)$. It follows that possibly passing to subsequences, there exists $\bar{u}_j \in V(\Omega)$ such that $u_j[\rho_k]$ weakly converges to $\bar{u}_j$ as $k \to \infty$ in $V(\Omega)$, and there exists $\bar{\lambda}_j \in \mathbb{R}$ such $\lambda_j[\rho_k]$ converges to $\bar{\lambda}_j$ as $k \to \infty$. Moreover, since the embedding $V(\Omega) \subset L^2(\Omega)$ is compact we can directly assume that $u_j[\rho_k]$ converges to $\bar{u}_j$ strongly in $L^2(\Omega)$ as $k \to \infty$. By passing to the limit in the weak equation

$$\mathcal{Q}[u_j[\rho_k], \varphi] = \lambda_j[\rho_k] < u_j[\rho_k], \varphi >_{\rho_k}, \quad \forall\, \varphi \in V(\Omega)\,,$$

it follows that $\bar{\lambda}_j$ is an eigenvalue and of problem (2.0.3) and $\bar{u}_j$ a corresponding eigenfunction. Note that $< \bar{u}_j, \bar{u}_l >_{\rho} = \delta_{jl}$ for all $j, l \in \mathbb{N}$, hence λ_j, $j \in \mathbb{N}$ is a divergent sequence. It remains to prove that $\bar{\lambda}_j = \lambda_j[\rho]$ for all $j \in \mathbb{N}$. To do so, assume by contradiction that there exists an eigenfunction $\bar{u} \in V(\Omega)$ associated with an eigenvalue $\bar{\lambda}$ of the weak problem (2.0.3) such that $< \bar{u}, \bar{u}_j >_{\rho} = 0$ for all $j \in \mathbb{N}$. Assume that $\bar{u}$ is normalized by $\|\bar{u}\|_{\rho} = 1/(b + \bar{\lambda})$. By the Auchmuty principle [13] applied to the operator $L + bI_{\rho}$, where $L + bI_{\rho}$ has been defined in Section 1.3, we have

$$-\frac{1}{2(b + \lambda_j[\rho_k])} \le \frac{\mathcal{Q}[u, u] + b\|u\|^2_{L^2_{\rho_k}(\Omega)}}{2} - \|u - P_{n-1,\rho_k} u\|_{L^2_{\rho_k}(\Omega)}\;, \qquad (2.1.8)$$

for all $u \in V(\Omega)$ and $j, k \in \mathbb{N}$. Here $P_{j-1,\rho_k} u$ denotes the orthogonal projection in $L^2_{\rho_k}(\Omega)$ of u onto the space generated by $u_1[\rho_k], \ldots, u_{j-1}[\rho_k]$ for all $j \ge 2$ and $P_{0,\rho_k} u \equiv 0$. By setting $u = \bar{u}$ and passing to the limit in (2.1.8) as $k \to \infty$, we obtain

$$-\frac{1}{2(b + \bar{\lambda}_j)} \le \frac{\mathcal{Q}[\bar{u}, \bar{u}] + b\|\bar{u}\|^2_{L^2_{\rho}(\Omega)}}{2} - \|\bar{u}\|_{L^2_{\rho}(\Omega)} = -\frac{1}{2(b + \bar{\lambda})}$$

for all $j \in \mathbb{N}$, which contradicts the fact that $\bar{\lambda}_j \to \infty$ as $j \to \infty$. □

2.2 Analyticity of the eigenvalues

By classical results in perturbation theory, one can prove that $\lambda_j[\rho]$ depends real-analytically on ρ as long as $\lambda_j[\rho]$ is a simple eigenvalue. This is no longer true if the multiplicity of $\lambda_j[\rho]$ varies. In the case of multiple eigenvalues, analyticity can be proved for the symmetric functions of the eigenvalues. Namely, given a finite set of indexes $F \subset \mathbb{N}$, we set

$$\mathcal{R}[F] := \{\rho \in \mathcal{R} : \; \lambda_j[\rho] \ne \lambda_l[\rho], \; \forall\; j \in F,\, l \in \mathbb{N} \setminus F\}$$

and

$$\Lambda_{F,h}[\rho] := \sum_{\substack{j_1,\dots,j_h\in F\\ j_1<\dots<j_h}} \lambda_{j_1}[\rho]\cdots\lambda_{j_h}[\rho], \quad h=1,\dots,|F|.$$

Moreover, in order to compute formulas for the Frechét differentials, it is also convenient to set

$$\Theta[F] := \{\rho\in\mathcal{R}[F]:\ \lambda_{j_1}[\rho]=\lambda_{j_2}[\rho],\ \forall\ j_1,j_2\in F\}\,.$$

Then we have the following result:

Theorem 2.2.1. *Assume that there exist $a,b,c>0$ such that inequalities (1.3.3) and (1.3.4) are satisfied. Let F be a finite subset of $\mathbb{N}$. Then $\mathcal{R}[F]$ is an open set in $L^\infty(\Omega)$ and the functions $\Lambda_{F,h}$ are real-analytic in $\mathcal{R}[F]$. Moreover, if $F=\cup_{k=1}^n F_k$ and $\rho\in\cap_{k=1}^n\Theta[F_k]$ is such that for each $k=1,\dots,n$ the eigenvalues $\lambda_j[\rho]$ assume the common value $\lambda_{F_k}[\rho]$ for all $j\in F_k$, then the differentials of the functions $\Lambda_{F,h}$ at the point ρ are given by the formula*

$$d\Lambda_{F,h}[\rho][\dot\rho] = -\sum_{k=1}^n c_k\sum_{l\in F_k}\int_\Omega u_l^2\dot\rho dx\,, \tag{2.2.2}$$

for all $\dot\rho\in L^\infty(\Omega)$, where

$$c_k = \sum_{\substack{0\le h_1\le|F_1|\\ \dots\dots\\ 0\le h_n\le|F_n|\\ h_1+\dots+h_n=h}} \binom{|F_k|-1}{h_k-1}\lambda_{F_k}^{h_k}[\rho]\prod_{\substack{j=1\\ j\ne k}}^n\binom{|F_j|}{h_j}\lambda_{F_j}^{h_j}[\rho],$$

and for each $k=1,\dots,n$, $\{u_l\}_{l\in F_k}$ is an orthonormal basis in $L^2_\rho(\Omega)$ of the eigenspace associated with $\lambda_{F_k}[\rho]$.

Proof. We set

$$\tilde\Lambda_{F,h}[\rho] := \sum_{\substack{j_1,\dots,j_h\in F\\ j_1<\dots<j_h}} (\lambda_{j_1}[\rho]+b)\cdots(\lambda_{j_h}[\rho]+b)\,,$$

for all $\rho\in\mathcal{R}[F]$. Note that by elementary combinatorics, we have

$$\Lambda_{F,h}[\rho] = \sum_{k=0}^h(-b)^{h-k}\binom{|F|-k}{h-k}\tilde\Lambda_{F,k}[\rho]\,, \tag{2.2.3}$$

where we have set $\Lambda_{F,0}=\tilde\Lambda_{F,0}=1$.

By adapting to the operator $L+bI_\rho$ the same argument used in [76] for the Dirichlet Laplacian, thanks to Theorem 1.3.4 and Theorem 1.2.7, one can prove that $\mathcal{R}[F]$ is an open set in $L^\infty(\Omega)$ and that $\tilde{\Lambda}_{F,h}[\rho]$ depends real-analytically on $\rho \in \mathcal{R}[F]$. Thus, by (2.2.3) we deduce the real-analyticity of the functions $\Lambda_{F,h}$.

We now prove formula (2.2.2). First we assume that $n = 1$, hence $F = F_1$ and $\rho \in \Theta[F]$. For simplicity, we write $\lambda_F[\rho]$ rather than $\lambda_{F_1}[\rho]$. The same computations used in [76] yield the following formula for the Frechét differential $d\tilde{\Lambda}_{F,h}[\rho]$ of $\tilde{\Lambda}_{F,h}$ at the point $\rho \in \mathcal{R}[F]$:

$$d\tilde{\Lambda}_{F,h}[\rho][\dot{\rho}] = -(\lambda_F[\rho]+b)^{h+1}\binom{|F|-1}{h-1}\sum_{l\in F} < dT_\rho[\dot{\rho}][u_l], u_l >_\rho ,$$

for all $\dot{\rho} \in L^\infty(\Omega)$, where T_ρ is defined by (1.3.6) with $w = \rho$. By standard calculus and by recalling that $T_\rho u_l = (\lambda_F[\rho]+b)^{-1}u_l$ for all $l \in F$, we have

$$\begin{aligned} &< dT_\rho[\dot{\rho}][u_l], u_l >_\rho = -b < (L+bI_\rho)^{-1} dI_\rho[\dot{\rho}](L+bI_\rho)^{-1} I_\rho u_l, u_l >_\rho \\ &+ < (L+bI_\rho)^{-1} dI_\rho[\dot{\rho}]u_l, u_l >_\rho = \frac{\lambda_F[\rho]}{\lambda_F[\rho]+b} < (L+bI_\rho)^{-1} dI_\rho[\dot{\rho}]u_l, u_l >_\rho \\ &\qquad = \frac{\lambda_F[\rho]}{(\lambda_F[\rho]+b)^2}\int_\Omega u_l^2 \dot{\rho}\, dx \end{aligned}$$

hence

$$d\tilde{\Lambda}_{F,h}[\rho][\dot{\rho}] = -\lambda_F[\rho](\lambda_F[\rho]+b)^{h-1}\binom{|F|-1}{h-1}\sum_{l\in F}\int_\Omega u_l^2\dot{\rho}\, dx\,, \tag{2.2.4}$$

for all $\dot{\rho} \in L^\infty(\Omega)$. By (2.2.3) and (2.2.4) we get

$$\begin{aligned} &d\Lambda_{F,h}[\rho][\dot{\rho}] \\ &= -\sum_{k=1}^{h}\lambda_F[\rho](\lambda_F[\rho]+b)^{k-1}(-b)^{h-k}\binom{|F|-1}{k-1}\binom{|F|-k}{h-k}\sum_{l\in F}\int_\Omega u_l^2\dot{\rho}\, dx \\ &= -\lambda_F[\rho]\binom{|F|-1}{h-1}\sum_{k=0}^{h-1}\binom{h-1}{k}(\lambda_F[\rho]+b)^k(-b)^{h-1-k}\sum_{l\in F}\int_\Omega u_l^2\dot{\rho}\, dx, \end{aligned}$$

which immediately implies (2.2.2) for $n = 1$. We now consider the case $n > 1$. By means of a continuity argument, one can easily see that there exists an open neighborhood $\mathcal{W}$ of ρ in $\mathcal{R}[F]$ such that $\mathcal{W} \subset \cap_{k=1}^n \mathcal{R}[F_k]$. Thus,

$$\Lambda_{F,h} = \sum_{\substack{0\le h_1\le |F_1|,\dots,0\le h_n\le |F_n|\\ h_1+\cdots+h_n=h}}\prod_{k=1}^n \Lambda_{F_k,h_k} \tag{2.2.5}$$

on $\mathcal{W}$. By differentiating equality (2.2.5) at the point ρ and applying formula (2.2.2) for $n = 1$ to each function Λ_{F_k,h_k}, we deduce the validity of formula (2.2.2) for arbitrary values of $n \in \mathbb{N}$. □

2.3 Maximum principle

In this section we consider the case of general intermediate boundary conditions. This means that we assume that $V(\Omega)$ is a closed subspace of $H^m(\Omega)$ satisfying the inclusion

$$V(\Omega) \subset H^1_0(\Omega)\,. \tag{2.3.1}$$

Moreover, we assume that Ω has finite measure. For all $M > 0$ we set

$$L_M := \left\{ \rho \in L^\infty(\Omega) : \int_\Omega \rho dx = M \right\}.$$

The following theorem is a generalization of [76, Theorem 4.4] to the case of intermediate boundary conditions and can be thought as a kind of maximum principle.

Theorem 2.3.2. *Let all assumptions of Theorem 2.2.1 hold. Assume in addition that Ω has finite measure and inclusion (2.3.1) holds. Then for all $h = 1, \dots, |F|$ the map $\Lambda_{F,h}$ of $\mathcal{R}[F] \cap L_M$ to $\mathbb{R}$ which takes any $\rho \in \mathcal{R}[F] \cap L_M$ to $\Lambda_{F,h}[\rho]$ has no points of local maximum or minimum $\tilde{\rho}$ such that $\lambda_j[\tilde{\rho}]$ have the same sign and $\lambda_j[\tilde{\rho}] \neq 0$ for all $j \in F$.*

Proof. It is convenient to consider the real-valued function M defined on $L^\infty(\Omega)$ by $M[\rho] := \int_\Omega \rho dx$ for all $\rho \in L^\infty(\Omega)$. Assume by contradiction the existence of $\tilde{\rho}$ as in the statement. Then $\tilde{\rho}$ is a critical point for the function $\Lambda_{F,h}$ subject to the mass constraint $M[\rho] = M$. This implies the existence of a Lagrange multiplier, which means that there exists $c \in \mathbb{R}$ such that $d\Lambda_{F,h}[\tilde{\rho}] = cdM[\tilde{\rho}]$ (see e.g., [37, Theorem 26.1]). By formula (2.2.2), it follows that

$$\int_\Omega \left(\sum_{k=1}^n c_k \sum_{l \in F_k} u_l^2 \right) \dot{\rho} dx = c \int_\Omega \dot{\rho} dx,$$

for all $\dot{\rho} \in L^\infty(\Omega)$. Note that c_k are non-zero real numbers of the same sign. Since $\dot{\rho}$ is arbitrary, it follows that

$$\left(\sum_{k=1}^n c_k \sum_{l \in F_k} u_l^2 \right) = c, \quad \text{a.e. in } \Omega. \tag{2.3.3}$$

Since $u_l \in H^1_0(\Omega)$, then by a standard argument one can prove that the function $(\sum_{k=1}^n \sum_{l \in F_k} (\sqrt{|c_k|} u_l)^2)^{1/2}$ belongs to the space $H^1_0(\Omega)$ and equals $\sqrt{|c|}$ almost everywhere in Ω. As is well-known the space $H^1_0(\Omega)$ does not contain constant functions apart from the function identically equal to zero. Thus $c = 0$ and accordingly $u_l = 0$ for all $l \in F$, a contradiction. $\square$

Remark 2.3.4. *Theorem 2.3.2 concerns mass densities* $\tilde{\rho}$ *such that* $\lambda_j[\tilde{\rho}]$ *do not vanish and have the same sign for all* $j \in F$. *This assumption is clearly guaranteed for positively defined operators. Moreover, we note that the sign of the eigenvalues is preserved by small perturbations of* ρ. *Hence our assumption is not much restrictive in the analysis of bifurcation phenomena associated with multiple eigenvalues different from zero.*

Finally, by Theorems 2.1.5 and 2.3.2 we deduce the following:

Corollary 2.3.5. *Let all assumptions of Theorem 2.3.2 hold. Let* $C \subset \mathcal{R}[F]$ *be a weakly* compact set in* $L^\infty(\Omega)$. *Assume that there exist* $a, b > 0$ *such that inequality (1.3.3) is satisfied for all* $\rho \in C$. *Let* $M > 0$ *be such that* $C \cap L_M$ *is not empty. Assume that the eigenvalues* $\lambda_j[\rho]$ *have the same sign and do not vanish for all* $j \in F$, $\rho \in C$. *Then for all* $h = 1, \dots, |F|$, *the map* $\Lambda_{F,h}$ *from* $C \cap L_M$ *to* $\mathbb{R}$ *which takes* $\rho \in C \cap L_M$ *to* $\Lambda_{F,h}[\rho]$ *admits points of maximum and minimum and all such points belong to* $\partial C \cap L_M$.

Proof. Recall that weakly* compact sets are bounded. Thus, by Theorem 2.1.5 the functions $\Lambda_{F,h}$ are weakly* continuous on C hence they admit both maximum and minimum on the weakly* compact subset $C \cap L_M$ of C. By Corollary 2.3.2 the corresponding points of maximum and minimum cannot be interior points of C, hence they belong to $\partial C \cap L_M$. □

Condition (2.3.1) was used only to guarantee that $V(\Omega) \setminus \{0\}$ does not contain constant functions. Thus, one may replace condition (2.3.1) by slightly more general conditions. For example one may assume that $V(\Omega) \subset H^1_{0,\Gamma}(\Omega)$ where $H^1_{0,\Gamma}(\Omega)$ is the closure in $H^1(\Omega)$ of C^∞-functions vanishing in an open neighborhood of a suitable subset of Γ of $\partial\Omega$. In this case, one would talk about mixed-intermediate boundary conditions. We can argue as in the proof of Theorem 2.3.2 and get to condition (2.3.3). Then we note that the function $\left(\sum_{k=1}^n \sum_{l \in F_k} (\sqrt{|c_k|}u_l)^2\right)^{1/2}$ belongs to the space $H^1_{0,\Gamma}(\Omega)$ and equals $\sqrt{|c|}$ almost everywhere. The space $H^1_{0,\Gamma}(\Omega)$ does not contain non-zero constant functions and therefore $u_l = 0$ for all $l \in F$, a contradiction. Therefore Theorem 2.3.2 and Corollary 2.3.5 hold also in the case $V(\Omega) \subset H^1_{0,\Gamma}(\Omega)$.

Remark 2.3.6. *If* $V(\Omega)$ *is a closed subspace of* $H^m(\Omega)$ *containing constant functions different from zero, then we could argue as in the proof on Theorem 2.3.2 up to condition (2.3.3). Thus, in the general case one could simply characterize the critical mass densities of the functions* $\Lambda_{F,h}$ *as those mass densities for which condition (2.3.3) is satisfied. Clearly, in the case of simple eigenvalues condition (2.3.3) reduces to* $u = \text{const}$ *in* Ω *which implies that* $\lambda = 0$. *Thus, we conclude that the maximum principle stated in Theorem 2.3.2 holds for all simple eigenvalues and all homogeneous boundary conditions under consideration. As for multiple eigenvalues we note that* *the analysis of condition (2.3.3) is not straightforward as it may appear at a first glance.*

2.4 Poly-harmonic operators

In this sections we consider the case of poly-harmonic operators. Let $m \in \mathbb{N}$. Consider (2.0.1) and (2.0.2) with $A_{\alpha\beta} = \delta_{\alpha\beta} m!/\alpha!$ for all $\alpha, \beta \in {\mathbb{N}_0}^N$ with $|\alpha| = |\beta| = m$, where $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and $\delta_{\alpha\beta} = 0$ otherwise. Let $l \in \mathbb{N}_0$, $0 \le l \le m$ and $V(\Omega) = H^m(\Omega) \cap H^l_0(\Omega)$. Note that (1.3.3) and (1.3.4) are satisfied for any $b > 0$ where $a, c > 0$ are suitable constants possibly depending on b. Moreover, if $1 \le l \le m$ and the open set Ω has finite Lebesgue measure then the embedding $V(\Omega) \subset L^2(\Omega)$ is compact. If $l = 0$ and the open set Ω is bounded and has a Lipschitz continuous boundary then the embedding $V(\Omega) \subset L^2(\Omega)$ is compact (actually it is enough to assume that Ω is a bounded open set with a quasi-continuous boundary, see [26, Theorem 8]). Under these assumptions all corresponding eigenvalues λ_j are well-defined and non-negative.

Note that if $l = m$ then $V(\Omega) = H^m_0(\Omega)$ and by integrating by parts one can easily realize that the the bilinear form $\mathcal{Q}[u, \varphi]$ can be written in the more familiar form

$$\mathcal{Q}[u,\varphi] = \begin{cases} \int_\Omega \Delta^{\frac{m}{2}} u \Delta^{\frac{m}{2}} \varphi dx, & \text{if } m \text{ is even},\\ \int_\Omega \nabla\Delta^{\frac{m-1}{2}} u \nabla\Delta^{\frac{m-1}{2}} \varphi dx, & \text{if } m \text{ is odd}, \end{cases}$$

for all $u, \varphi \in H^m_0(\Omega)$. In this case we obtain the classical eigenvalue problem for poly-harmonic operators subject to the Dirichlet boundary conditions. The classical formulation of the Dirichlet problem problem is

$$\begin{cases} (-\Delta)^m u = \lambda\rho u, & \text{in } \Omega,\\ u = \frac{\partial u}{\partial\nu} = \cdots = \frac{\partial^{m-1} u}{\partial\nu^{m-1}} = 0, & \text{on } \partial\Omega, \end{cases}$$

where we denote by ν the outer unit normal to $\partial\Omega$. We recall that the Dirichlet problem arises in the study of vibrating strings for $N = 1$ and $m = 1$, membranes for $N = 2$ and $m = 1$, and clamped plates for $N = 2$ and $m = 2$. We refer to [31, 47, 94] for the physical derivation of the problem.

In the general case $l \le m$, the classical formulation of the eigenvalue problem is

$$\begin{cases} (-\Delta)^m u = \lambda\rho u, & \text{in } \Omega,\\ \frac{\partial^j u}{\partial\nu^j} = 0,\ \forall\, j = 0, \dots, l-1, & \text{on } \partial\Omega,\\ \mathcal{B}_j u = 0,\ \forall\, j = 1, \dots, m-l, & \text{on } \partial\Omega, \end{cases}$$

where $\mathcal{B}_j$ are uniquely defined 'complementing' boundary operators. See [88] for details. For $N \ge 2$, $m = 2$ and $l = 1$ we obtain the problem

$$\begin{cases} \Delta^2 u = \lambda\rho u, & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega,\\ \Delta u - (N-1)\kappa\frac{\partial u}{\partial\nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

where κ is the mean curvature of the boundary of Ω. This problem models a hinged vibrating rod for $N = 1$ and a hinged plate for $N = 2$. See [47] for further details.

Finally, we note that if $m = 2$ and $l = 0$ then $V(\Omega) = H^2(\Omega)$ and problem (1.3.1) is the weak formulation of a Neumann-type problem for the biharmonic operator

$$\begin{cases} \Delta^2 u = \lambda\rho u, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial^2 \nu} = 0, & \text{on } \partial\Omega, \\ \mathrm{div}_{\partial\Omega}\left(D^2 u \cdot \nu\right) + \frac{\partial \Delta u}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases}$$

Here $\mathrm{div}_{\partial\Omega}$ is the tangential divergence and $\left(D^2 u \cdot \nu\right)_{\partial\Omega}$ is the orthogonal projection of $D^2 u \cdot \nu$ onto the tangent hyperplane to $\partial\Omega$. This prolem models a free rod for $N = 1$ and a free plate for $N = 2$. See also [28].

We consider all the poly-harmonic operators subject to Dirichlet or intermediate boundary conditions, i.e., $1 \leq l \leq m$. From Theorem 2.1.5 and Theorem 2.3.2 we deduce the following:

Corollary 2.4.1. *Let $m, l \in \mathbb{N}$ with $1 \leq l \leq m$. Consider problem* (2.0.3) *with $A_{\alpha\beta} = \delta_{\alpha\beta} m!/\alpha!$ for all $\alpha, \beta \in \mathbb{N}_0^N$ with $|\alpha| = |\beta| = m$ and $V(\Omega) = H^m(\Omega) \cap H_0^l(\Omega)$. Assume that Ω has finite measure. Then for all $h = 1, ..., |F|$, the map $\Lambda_{F,h}$ of $\mathcal{R}[F] \cap L_M$ to $\mathbb{R}$ which takes any $\rho \in \mathcal{R}[F] \cap L_M$ to $\Lambda_{F,h}[\rho]$ has no points of local maximum or minimum $\tilde{\rho}$. Moreover, let $C \subset \mathcal{R}[F]$ be a weakly* compact set in $L^\infty(\Omega)$ such that $\inf_{\rho \in C} \operatorname{ess\,inf}_{x \in \Omega} \rho(x) > 0$. Let $M > 0$ be such that $C \cap L_M$ is not empty. Then for all $h = 1, ..., |F|$, the map $\Lambda_{F,h}$ admits points of maximum and minimum and all such points belong to $\partial C \cap L_M$.*

Proof. First we note that since $\inf_{\rho \in C} \operatorname{ess\,inf}_{x \in \Omega} \rho(x) > 0$, inequality (1.3.3) is satisfied for suitable constants $a, b > 0$ not depending on $\rho \in C$. Moreover, the embedding $H^m(\Omega) \cap H_0^l(\Omega) \subset L^2(\Omega)$ is compact. We also note that all the eigenvalues $\lambda_j[\rho]$ are non-negative. The proof now is the same as that of Theorem 2.3.2 and of Corollary 2.3.5. □

We consider now a particular class of weakly* compact sets. Let $A, B \in L^\infty(\Omega)$ be functions satisfying the condition

$$0 < \operatorname*{ess\,inf}_{x \in \Omega} A(x) < \operatorname*{ess\,sup}_{x \in \Omega} B(x) < \infty.$$

Let $C := \{\rho \in L^\infty(\Omega) : A \leq \rho \leq B\}$. Clearly, C is a weakly* compact set. Moreover, since all mass densities ρ are uniformly bounded away from zero and infinity, inequality (1.3.3) is satisfied for suitable constants $a, b > 0$ not depending on $\rho \in C$. Thus Corollary 2.4.1 is applicable to all non-zero eigenvalues. It turns out that point of maximum and minimum $\tilde{\rho}$ should coincide with $A(x)$ or $B(x)$ in a set of positive measure.

2.5 The Laplace operator with Neumann boundary conditions

As we have observed at the end of Section 2.3, the analysis of condition (2.3.3) in the case of multiple eigenvalues and energy spaces $V(\Omega)$ containing non-zero costant functions is not straightforward. We consider here the prototypical case of the Laplace operator with Neumann boundary conditions

$$\begin{cases} -\Delta u = \lambda \rho u, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.5.1}$$

The weak formulation of problem (2.5.1) is

$$\int_\Omega \nabla u \cdot \nabla \varphi dx = \lambda \int_\Omega \rho u \varphi dx, \quad \forall \varphi \in H^1(\Omega), \tag{2.5.2}$$

which correspond to the choice $A_{\alpha\beta} = \delta_{\alpha\beta}$, $V(\Omega) = H^1(\Omega)$ in (2.0.3). From Remark 2.3.6 it follows that in the case $V(\Omega) = H^1(\Omega)$ we have that $\tilde{\rho}$ is a critical point for $\Lambda_{F,h}$ provided condition (2.3.3) is satisfied. Under suitable regularity assumptions on the eigenfunctions associated with a double eigenvalue, we may prove that the validity of (2.3.3) implies that the eigenvalue must be zero. This is proved in the following theorem.

Theorem 2.5.3. *Let Ω be a bounded domain in $\mathbb{R}^N$ with a Lipschitz continuous boundary, $M > 0$ and $F = \{m, n\}$ with $m, n \in \mathbb{N}$, $m \neq n$. Let $\tilde{\rho} \in \mathcal{R}[F]$ be continuous and moreover, assume that the solutions to problem (2.5.2) are classical solutions and the nodal domains are sucht that the Divergence Theorem holds. Then for $h = 1, 2$, $\tilde{\rho}$ is not a critical mass density for the function which takes $\rho \in \mathcal{R}[F]$ to $\Lambda_{F,h}[\rho]$ under the constraint $\rho \in \mathcal{R} \cap L_M$.*

Proof. Let $\tilde{\rho} \in \mathcal{R}[F]$ be fixed. Then we have one of the following cases:

Case 1) $\tilde{\rho} \in \Theta[F]$. In this case $\lambda_F = \lambda_m = \lambda_n$ is an eigenvalue of multiplicity 2. Then by (2.2.4) it follows that

$$d\Lambda_{F,1}[\tilde{\rho}][\dot{\rho}] = -\lambda_F^2 \int_\Omega \dot{\rho}(u_m^2 + u_n^2) dx\,,$$
$$d\Lambda_{F,2}[\tilde{\rho}][\dot{\rho}] = -\lambda_F^3 \int_\Omega \dot{\rho}(u_m^2 + u_n^2) dx,$$

where $\{u_m, u_n\}$ is a orthogonal basis in $L^2_{\tilde{\rho}}(\Omega)$ of the eigenspace associated with λ_F.

Case 2) $\tilde{\rho} \in \bigcap_{k=1}^2 \Theta[F_k]$, where $F_1 = \{m\}$, $F_2 = \{n\}$. In this case $\lambda_{F_1} = \lambda_m, \lambda_{F_2} = \lambda_n$ are two simple eigenvalues. Then there exists an open neighborhood in $\mathcal{R}$ of $\tilde\rho$ such that $\mathcal{W} \subseteq \bigcap_{k=1}^{2} \mathcal{R}[F_k]$. Then

$$d\Lambda_{F,1}[\tilde\rho][\dot\rho] = d(\Lambda_{F_2,1} + \Lambda_{F_1,1})[\tilde\rho][\dot\rho] = -\int_\Omega \dot\rho(\lambda_{F_2}^2 u_n^2 + \lambda_{F_1}^2 u_m^2)dx\,,$$

$$d\Lambda_{F,2}[\tilde\rho][\dot\rho] = d(\Lambda_{F_1,1}\Lambda_{F_2,1})[\tilde\rho][\dot\rho] = -\int_\Omega \dot\rho(\lambda_{F_1}\lambda_{F_2}^2 u_n^2 + \lambda_{F_2}\lambda_{F_1}^2 u_m^2)dx,$$

where u_m is the eigenfunction associated with the eigenvalue λ_{F_1}, u_n is the eigenfunction associated with the eigenvalue λ_{F_2}, and u_m, u_n are such that $\int_\Omega \tilde\rho u_m^2 dx = 1$, $\int_\Omega \tilde\rho u_n^2 dx = 1$.

Suppose now that $\tilde\rho$ is a critical mass density for $\Lambda_{F,h}$, $h = 1,2$ in L_M. Then, in both cases, from condition (2.3.3) it follows that there exist $c_n, c_m > 0$, $c > 0$ such that

$$(c_n u_n^2 + c_m u_m^2) = c\,, \quad a.e. \text{ in } \Omega.$$

Let us consider separately the different cases:

i) $\tilde\rho \in \Theta[F]$, $d\Lambda_{F,1}[\tilde\rho][\dot\rho] = -\lambda_F^2 \int_\Omega \dot\rho(u_m^2 + u_n^2)dx$ (the case $d\Lambda_{F,2}[\tilde\rho][\dot\rho] = -\lambda_F^3 \int_\Omega \dot\rho(u_m^2 + u_n^2)dx$ is analogous). Then, by differentiating the equality

$$u_m^2 + u_n^2 = C \tag{2.5.4}$$

we obtain

$$u_m \nabla u_m + u_n \nabla u_n = 0 \tag{2.5.5}$$

which implies in particular

$$|\nabla u_m(x)|^2 = \frac{u_n^2(x)}{u_m^2(x)} |\nabla u_n(x)|^2\,,$$

for all $x \in \Omega$ such that $u_m(x) \neq 0$. Let us differentiate again in (2.5.5) and use the fact that $-\Delta u_m = \lambda_F \tilde\rho u_m$ and $-\Delta u_n = \lambda_F \tilde\rho u_n$. We obtain

$$|\nabla u_m(x)|^2 + |\nabla u_n(x)|^2 = \lambda_F \tilde\rho \left(u_m^2(x) + u_n^2(x)\right).$$

By combining (2.5.4), (2.5.5) and (2.5.6) we get

$$\left(\frac{u_n^2(x)}{u_m^2(x)} + 1\right) |\nabla u_n(x)|^2 = \lambda_F \tilde\rho C,$$

hence

$$|\nabla u_n(x)|^2 = \lambda_F \tilde\rho u_m(x)^2, \tag{2.5.6}$$

for all $x \in \Omega$ such that $u_m(x) \neq 0$. It is easy to see that (2.5.6) holds also if $x \in \Omega$ is such that $u_m(x) = 0$ because in this case u_n^2 has a maximum in x (see (2.5.4)), hence $\nabla u_n(x) = 0$. In the same way one can also show

$$|\nabla u_m(x)|^2 = \lambda_F \tilde{\rho} u_n^2(x).$$

ii) $\tilde{\rho} \in \bigcap_{k=1}^2 \Theta[F_k]$, $d\Lambda_{F,1}[\tilde{\rho}][\dot{\rho}] = -\int_\Omega \dot{\rho}(\lambda_{F_2}^2 u_n^2 + \lambda_{F_1}^2 u_m^2)dx$. In a similar way as for Case 1), since $\lambda_{F_2}^2 u_n^2 + \lambda_{F_1}^2 u_m^2 = C$ and $-\Delta u_m = \lambda_{F_1}\tilde{\rho} u_m$, $-\Delta u_n = \lambda_{F_2}\tilde{\rho} u_n$, we obtain the following relations:

$$|\nabla u_m(x)|^2 = \frac{\lambda_{F_2}^2}{C\lambda_{F_1}^2}\tilde{\rho}\left(\lambda_{F_1}^3 u_m^2(x) + \lambda_{F_2}^3 u_n^2(x)\right) u_n^2(x); \quad (2.5.7)$$
$$|\nabla u_n(x)|^2 = \frac{\lambda_{F_1}^2}{C\lambda_{F_2}^2}\tilde{\rho}\left(\lambda_{F_1}^3 u_m^2(x) + \lambda_{F_2}^3 u_n^2(x)\right) u_m^2(x).$$

iii) $\tilde{\rho} \in \bigcap_{k=1}^2 \Theta[F_k]$, $d\Lambda_{F,2}[\tilde{\rho}][\dot{\rho}] = -\int_\Omega \dot{\rho}(\lambda_{F_1}\lambda_{F_2}^2 u_n^2 + \lambda_{F_2}\lambda_{F_1}^2 u_m^2)dx$. By imposing $\lambda_{F_1}\lambda_{F_2}^2 u_n^2 + \lambda_{F_2}\lambda_{F_1}^2 u_m^2 = C$ we obtain

$$|\nabla u_m(x)|^2 = \frac{\lambda_{F_2}^2}{C}\tilde{\rho}\left(\lambda_{F_1}^2 u_m^2(x) + \lambda_{F_2}^2 u_n^2(x)\right) u_n^2(x); \quad (2.5.8)$$
$$|\nabla u_n(x)|^2 = \frac{\lambda_{F_1}^2}{C}\tilde{\rho}\left(\lambda_{F_1}^2 u_m^2(x) + \lambda_{F_2}^2 u_n^2(x)\right) u_m^2(x).$$

By (2.5.6), (2.5.7) and (2.5.8) we observe that in all cases, the nodal set of one of the two eigenfunctions coincides with the set where the gradient of the other vanishes. We also note that there are no points in Ω where both u_m and ∇u_m vanish (respectively u_n and ∇u_n). This implies that nodal sets of u_m are manifolds and coincide with the sets where ∇u_n vanishes. We observe that the nodal sets of the eigenfunctions u of problem (2.5.2) are not empty, since for such functions $\int_\Omega \tilde{\rho} u dx = 0$, hence u changes its sign on Ω.

Let us consider a nodal domain Ω_m of u_m. The function u_m doesn't change sign on Ω_m. The boundary $\partial\Omega_m$ of Ω_m can be written as $\partial\Omega_m = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \subset \partial\Omega$ and $\Gamma_2 \subset \Omega$. First we show that $\Gamma_1 \neq \emptyset$. Assume by contradiction that $\Gamma_1 = \emptyset$. The function $u_n|_{\Omega_m}$ is an eigenfunction of problem (2.5.1) with Ω replaced by Ω_m corresponding to the eigenvalue λ_{F_2}. Indeed the equation $-\Delta u_n = \lambda_{F_2} u_n$ is clearly satisfied on Ω_m and $\frac{\partial u_n}{\partial \nu} = 0$ on $\partial\Omega_m$, since ∇u_n is zero on $\partial\Omega_m$. Since $u_n|_{\Omega_m}$ is not identically zero, it must change sign. Thus, there exist at least two non-empty nodal domains for $u_n|_{\Omega_m}$ in Ω_m. We claim that al least one of them, say Ω_{m_n}, is relatively compact in Ω_m. If this were false, then there would exist at least a point x of $\partial\Omega_m$ such that $u_n(x) = 0$, hence $\nabla u_m(x) = 0$. But we since $\Gamma_1 = \emptyset$ we have $u_m(x) = 0$. Thus $u_n(x) = u_m(x) = 0$, hence $C = 0$, a contradiction.

Thus there exists a nodal domain Ω_{m_n} of $u_n|_{\Omega_m}$ such that $\overline{\Omega}_{m_n} \subset \Omega_m$. Now, $u_m|_{\Omega_{m_n}}$ solves problem (2.5.1) with λ_{F_1}, hence it must change sign Ω_{m_n}. But Ω_{m_n} is relatively compact in Ω_m, and on this set u_m has constant sign, a contradiction.

Thus we have proved that $\Gamma_1 \neq \emptyset$. Recall that u_m has constant sign on Ω_m. Moreover, $\frac{\partial u_n}{\partial \nu} = 0$ on Γ_1, while $\nabla u_n = 0$ on Γ_2, since here $u_m = 0$. Then $u_n|_{\Omega_m}$ is solution of problem (2.5.1) with Ω replaced by Ω_m corresponding to the eigenvalue λ_{F_2} and it changes sign on Ω_m. Let Ω_{m_n} be a nodal domain of $u_n|_{\Omega_m}$. By the arguments above we have that $\partial\Omega_{m_n} = \Gamma_{1,n} \cup \Gamma_{2,n}$, where $\emptyset \neq \Gamma_{1,n} \subset \partial\Omega_m$, and $\Gamma_{2,n} \subset \Omega_m$. We claim that there exists at least one nodal domain Ω_{m_n} such that $\Gamma_{1,n} \subset \partial\Omega$. If this were false, the boundary $\partial\Omega_{m_n}$ of each Ω_{m_n} would be of the type: $\partial\Omega_{m_n} = (\Gamma_{1,n} \cap \partial\Omega) \cup (\Gamma_{1,n} \cap (\partial\Omega_m \setminus (\partial\Omega \cap \partial\Omega_m))) \cup \Gamma_{2,n}$, and each of these partitions of $\partial\Omega_{m_n}$ would be non-empty. Since Ω_{m_n} is connected, $(\Gamma_{1,n} \cap (\partial\Omega_m \setminus (\partial\Omega \cap \partial\Omega_m))) \cap \Gamma_{2,n} \neq \emptyset$. On this set u_m and ∇u_m vanish, a contradiction. Thus there exists Ω_{m_n} such that $\Gamma_{1,n} \subset \partial\Omega$. Then $u_m|_{\Omega_{m_n}}$ is a nontrivial solution of problem (2.5.1) corresponding to the eigenvalue λ_{F_2} and changes its sign on Ω_{m_n}, a contradiction. This concludes the proof. □

As a consequence of Theorem 2.1.5 and Theorem 2.5.3 we have the following

Corollary 2.5.9. *Let Ω be a bounded domain in $\mathbb{R}^N$ with a Lipschitz continuous boundary, $F = \{m, n\}$ with $m, n \in \mathbb{N}$, $m \neq n$. Let $C \subseteq \mathcal{R}[F]$ be a weakly* compact subset of $L^\infty(\Omega)$ such that* $\inf_{\rho \in C} \operatorname{ess\,inf}_\Omega \rho > 0$. *Let $M > 0$ and $L_M = \{\rho \in L^\infty(\Omega) : \int_\Omega \rho = M\}$. Then for $h = 1, 2$, the function which takes $\rho \in C \cap L_M$ to $\Lambda_{F,h}[\rho]$ has maxima and minima, and if for such points the solutions of problem (2.5.2) are classic solutions, they must belong to $\partial C \cap L_M$.*

Proof. The proof is identical to that of Corollary 2.3.5. □

Chapter 3

Mass concentration phenomena for second order operators

In this chapter we consider the eigenvalue problem for the Laplace operator $-\Delta$ subject to Dirichlet and Neumann boundary conditions. In particular we shall consider eigenvalue problems of the type

$$-\Delta u = \lambda \rho_\varepsilon u$$

on a bounded open set Ω in $\mathbb{R}^N$, where ρ_ε is a measurable and positive function which depends on a small parameter $\varepsilon > 0$ and which is of order ε^{-1} in a ε-neighborhood of points or hypersurfaces contained in Ω as $\varepsilon \to 0$ and is of order ε in the rest of Ω, as $\varepsilon \to 0$.

For $N = 2$ this problem is related to the study of the vibration of a thin membrane which occupies at rest a planar region $\Omega \subset \mathbb{R}^2$ and the mass of which is displaced on the whole of Ω with density ρ_ε. Roughly speaking, we consider vibrating membranes the mass of which concentrates near points or hypersurfaces contained in Ω and we investigate the behavior of the eigenvalues, which represent the squares of the normal modes of vibration, as $\varepsilon \to 0$. Since the dimension does not play any relevant role in our discussion, we consider from now on open bounded sets in $\mathbb{R}^N$.

In the case of Neumann boundary conditions and mass densities which concentrate near the boundary of Ω, we obtain that the Neumann eigenvalues converge to the eigenvalues of the Steklov problem for the Laplace operator, which in this sense can be considered a limiting Neumann problem. Then we shall discuss the dependence of the eigenvalues of the Steklov problem upon mass density perturbations in the same spirit of Chapter 2.

3.1 Neumann to Steklov eigenvalues

Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. Let $\rho \in \mathcal{R}^{\mathcal{S}}$, where

$$\mathcal{R}^{\mathcal{S}} := \{\rho \in L^\infty(\partial\Omega) : \mathrm{ess\,inf}_{x\in\partial\Omega}\rho(x) > 0\}\,. \tag{3.1.1}$$

We consider the classical Steklov eigenvalue problem for the Laplace operator

$$\begin{cases} \Delta u = 0, & \text{in } \Omega\,, \\ \frac{\partial u}{\partial \nu} = \lambda\rho u, & \text{on } \partial\Omega\,, \end{cases} \tag{3.1.2}$$

in the unknowns u (the eigenfunction), λ (the eigenvalue). This problem models a free vibrating membrane whose mass is concentrated at the boundary with surface density ρ (see [99] for the derivation of the problem). We consider the weak formulation of (3.1.2)

$$\int_\Omega \nabla u \cdot \nabla\varphi dx = \lambda \int_{\partial\Omega} \rho u\varphi d\sigma\,, \quad \forall\varphi \in H^1(\Omega)\,, \tag{3.1.3}$$

in the unknowns $u \in H^1(\Omega)$, $\lambda \in \mathbb{R}$. Actually, we shall consider a problem in the space $H^1(\Omega)/\mathbb{R}$ since we need to get rid of the constants, which generate the eigenspace corresponding to the eigenvalue $\lambda = 0$.
We denote by Tr the trace operator acting from $H^1(\Omega)$ to $L^2(\partial\Omega)$. We denote by $\mathcal{J}_\rho^{\mathcal{S}}$ the continuous embedding of $L^2(\partial\Omega)$ into $H^1(\Omega)'$ defined by

$$\mathcal{J}_\rho^{\mathcal{S}}[u][\varphi] := \int_{\partial\Omega} \rho u\varphi d\sigma, \quad \forall u \in L^2(\partial\Omega), \varphi \in H^1(\Omega).$$

We set

$$H_\rho^{1,\mathcal{S}}(\Omega) := \left\{u \in H^1(\Omega) : \int_{\partial\Omega} \rho u d\sigma = 0\right\},$$

and we consider on $H^1(\Omega)$ the bilinear form

$$< u, v >:= \int_\Omega \nabla u \cdot \nabla v dx. \tag{3.1.4}$$

By the Poincaré-Wirtinger inequality, it turns out that the bilinear form (3.1.4) is a scalar product on $H_\rho^{1,\mathcal{S}}(\Omega)$ whose induced norm is equivalent to the standard one. In the sequel we will think of the space $H_\rho^{1,\mathcal{S}}$ as endowed with the scalar product (3.1.4). Let $F(\Omega)$ be defined by $F(\Omega) := \left\{G \in H^1(\Omega)' : G[1] = 0\right\}$. Then we consider the operator $\mathcal{M}_\rho^{\mathcal{S}}$ acting from $H_\rho^{1,\mathcal{S}}(\Omega)$ to $F(\Omega)$, defined by

$$\mathcal{M}_\rho^{\mathcal{S}}[u][\varphi] := \int_\Omega \nabla u \cdot \nabla\varphi dx, \quad \forall u \in H_\rho^{1,\mathcal{S}}(\Omega), \varphi \in H^1(\Omega).$$

It turns out that $\mathcal{M}_\rho^{\mathcal{S}}$ is a homeomorphism of $H_\rho^{1,\mathcal{S}}(\Omega)$ onto $F(\Omega)$. We define the operator $\pi_\rho^{\mathcal{S}}$ from $H^1(\Omega)$ to $H_\rho^{1,\mathcal{S}}(\Omega)$ by

$$\pi_\rho^{\mathcal{S}}[u] := u - \frac{\int_{\partial\Omega} \rho u d\sigma}{\int_{\partial\Omega} \rho d\sigma}, \tag{3.1.5}$$

for all $u \in H^1(\Omega)$. We consider the space $H^1(\Omega)/\mathbb{R}$ endowed with the bilinear form induced by (3.1.4). Such bilinear form renders $H^1(\Omega)/\mathbb{R}$ a Hilbert space. We denote by $\pi_\rho^{\sharp,\mathcal{S}}$ the map from $H^1(\Omega)/\mathbb{R}$ onto $H_\rho^{1,\mathcal{S}}(\Omega)$ defined by the equality $\pi_\rho^{\mathcal{S}} = \pi_\rho^{\sharp,\mathcal{S}} \circ p$, where p is the canonical projection of $H^1(\Omega)$ onto $H^1(\Omega)/\mathbb{R}$. The map $\pi_\rho^{\sharp,\mathcal{S}}$ turns out to be a homeomorphism. Finally, we define the operator $T_\rho^{\mathcal{S}}$ acting on $H^1(\Omega)/\mathbb{R}$ as follows

$$T_\rho^{\mathcal{S}} := (\pi_\rho^{\sharp,\mathcal{S}})^{-1} \circ (\mathcal{M}_\rho^{\mathcal{S}})^{-1} \circ \mathcal{J}_\rho^{\mathcal{S}} \circ \mathrm{Tr} \circ \pi_\rho^{\sharp,\mathcal{S}}. \tag{3.1.6}$$

Lemma 3.1.7. *The pair (λ, u) of the set $\mathbb{R} \times (H_\rho^{1,\mathcal{S}}(\Omega) \setminus \{0\})$ satisfies (3.1.3) if and only if $\lambda > 0$ and the pair $(\lambda^{-1}, p[u])$ of the set $\mathbb{R} \times ((H^1(\Omega)/\mathbb{R}) \setminus \{0\})$ satisfies the equation*

$$\lambda^{-1} p[u] = T_\rho^{\mathcal{S}} p[u].$$

We have the following theorem.

Theorem 3.1.8. *The operator $T_\rho^{\mathcal{S}}$ is a compact self-adjoint operator in $H^1(\Omega)/\mathbb{R}$, whose eigenvalues coincide with the reciprocals of the positive eigenvalues of problem (3.1.3). In particular, the set of eigenvalues of problem (3.1.3) is contained in $[0, +\infty[$ and consists of the image of a sequence increasing to $+\infty$. Each eigenvalue has finite multiplicity.*

Proof. For the self-adjointness, it suffices to observe that

$$\begin{aligned} < T_\rho^{\mathcal{S}} u, v >_{H^1(\Omega)/\mathbb{R}} &= < (\pi_\rho^{\sharp,\mathcal{S}})^{-1} \circ (\mathcal{M}_\rho^{\mathcal{S}})^{-1} \circ \mathcal{J}_\rho^{\mathcal{S}} \circ \mathrm{Tr} \circ \pi_\rho^{\sharp,\mathcal{S}} u, v >_{H^1(\Omega)/\mathbb{R}} \\ &= \mathcal{M}_\rho^{\mathcal{S}}[(\mathcal{M}_\rho^{\mathcal{S}})^{-1} \circ \mathcal{J}_\rho^{\mathcal{S}} \circ \mathrm{Tr} \circ \pi_\rho^{\sharp,\mathcal{S}} u][\pi_\rho^{\sharp,\mathcal{S}} v] \\ &= \mathcal{J}_\rho^{\mathcal{S}}[\mathrm{Tr} \circ \pi_\rho^{\sharp,\mathcal{S}} u][\pi_\rho^{\sharp,\mathcal{S}} v], \quad \forall u, v \in H^1(\Omega)/\mathbb{R}, \end{aligned}$$

and that $\mathcal{J}_\rho^{\mathcal{S}}[\mathrm{Tr} \circ \pi_\rho^{\sharp,\mathcal{S}} u][\pi_\rho^{\sharp,\mathcal{S}} v]$ is symmetric. As for compactness, just observe that the trace operator acting from $H^1(\Omega)$ to $L^2(\partial\Omega)$ is compact. The remaining statements follow by standard spectral theory. □

Therefore the eigenvalues of (3.1.3) can be represented by means of an increasing sequence

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots .$$

The first positive eigenvalue is λ_2 as proved in the following theorem.

Theorem 3.1.9. *The first eigenvalue* λ_1 *of* (3.1.3) *is zero and the corresponding eigenfunctions are the constant functions on* Ω. *Moreover,* $\lambda_2 > 0$.

Proof. It is straightforward to prove that the constant functions on Ω are eigenfunctions of (3.1.3) with eigenvalue $\lambda = 0$. Suppose now that u is an eigenfunction corresponding to the eigenvalue $\lambda = 0$. Then we have

$$\int_\Omega |\nabla u|^2 dx = 0.$$

Therefore, since Ω is connected, u is constant. This concludes the proof. □

We can characterize the eigenvalues of (3.1.3) by means of the Rayleigh Min-Max Principle:

$$\lambda_j = \min_{\substack{E \subset H^1(\Omega) \\ \dim E = j}} \max_{\substack{u \in E \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_{\partial\Omega} \rho u^2 d\sigma}. \tag{3.1.10}$$

Now we turn our attention to the Neumann eigenvalue problem. Let $\rho \in \mathcal{R}$, where

$$\mathcal{R} := \left\{ \rho \in L^\infty(\Omega) : \operatorname{ess}\inf_{x \in \Omega} \rho(x) > 0 \right\}. \tag{3.1.11}$$

We consider the classical Neumann eigenvalue problem for the Laplace operator

$$\begin{cases} -\Delta u = \lambda \rho u, & \text{in } \Omega\,, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega\,, \end{cases} \tag{3.1.12}$$

in the unknowns u (the eigenfunction), λ (the eigenvalue). This problem models a free vibrating membrane of mass density ρ (see e.g., [31] for the derivation of the problem). We consider the weak formulation of problem (3.1.12)

$$\int_\Omega \nabla u \cdot \nabla \varphi dx = \lambda \int_\Omega \rho u \varphi dx\,, \quad \forall \varphi \in H^1(\Omega)\,, \tag{3.1.13}$$

in the unknowns $u \in H^1(\Omega)$, $\lambda \in \mathbb{R}$. In the sequel we shall recast this problem in $H^1(\Omega)/\mathbb{R}$ since we need to get rid of the constants, which generate the eigenspace corresponding to the eigenvalue $\lambda = 0$. We denote by i the canonical embedding of $H^1(\Omega)$ into $L^2(\Omega)$. We denote by $\mathcal{J}_\rho^{\mathcal{N}}$ the continuous embedding of $L^2(\Omega)$ into $H^1(\Omega)'$, defined by

$$\mathcal{J}_\rho^{\mathcal{N}}[u][\varphi] := \int_\Omega \rho u \varphi dx \quad \forall u \in L^2(\Omega), \varphi \in H^1(\Omega).$$

We set

$$H^{1,\mathcal{N}}_\rho(\Omega) := \left\{ u \in H^1(\Omega) : \int_\Omega u \rho dx = 0 \right\}. \tag{3.1.14}$$

In the sequel we shall consider the space $H^{1,\mathcal{N}}_{\rho}(\Omega)$ as endowed with the scalar product (3.1.4). This scalar product induces on $H^{1,\mathcal{N}}_{\rho}(\Omega)$ a norm which is equivalent to the standard one. Then we consider the operator $\mathcal{M}^{\mathcal{N}}_{\rho}$ acting from $H^{1,\mathcal{N}}_{\rho}(\Omega)$ to $F(\Omega)$ defined by

$$\mathcal{M}^{\mathcal{N}}_{\rho}[u][\varphi] := \int_{\Omega} \nabla u \cdot \nabla \varphi dx, \quad \forall u \in H^{1,\mathcal{N}}_{\rho}(\Omega), \varphi \in H^{1}(\Omega).$$

The operator $\mathcal{M}^{\mathcal{N}}_{\rho}$ turns out to be a linear homeomorphism of $H^{1,\mathcal{N}}_{\rho}(\Omega)$ onto $F(\Omega)$. We define the operator $\pi^{\mathcal{N}}_{\rho}$ from $H^{1}(\Omega)$ to $H^{1,\mathcal{N}}_{\rho}(\Omega)$ as

$$\pi^{\mathcal{N}}_{\rho}[u] := u - \frac{\int_{\Omega} u\rho dx}{\int_{\Omega} \rho dx},$$

for all $u \in H^{1}(\Omega)$. We consider the space $H^{1}(\Omega)/\mathbb{R}$ endowed with de bilinear form induced by (3.1.4). Such form renders $H^{1}(\Omega)/\mathbb{R}$ a Hilbert space. We denote by $\pi^{\sharp,\mathcal{N}}_{\rho}$ the map from $H^{1}(\Omega)/\mathbb{R}$ onto $H^{1,\mathcal{N}}_{\rho}(\Omega)$ defined by the equality $\pi^{\mathcal{N}}_{\rho} = \pi^{\sharp,\mathcal{N}}_{\rho} \circ p$, where p is the canonical projection of $H^{1}(\Omega)$ onto $H^{1}(\Omega)/\mathbb{R}$.

We define the operator $T^{\mathcal{N}}_{\rho}$ acting on $H^{1}(\Omega)/\mathbb{R}$ as follows

$$T^{\mathcal{N}}_{\rho} := (\pi^{\sharp,\mathcal{N}}_{\rho})^{-1} \circ (\mathcal{M}^{\mathcal{N}}_{\rho})^{-1} \circ \mathcal{J}^{\mathcal{N}}_{\rho} \circ i \circ \pi^{\sharp,\mathcal{N}}_{\rho}. \tag{3.1.15}$$

Lemma 3.1.16. *The pair* (λ, u) *of the set* $\mathbb{R} \times (H^{1,\mathcal{N}}_{\rho}(\Omega) \setminus \{0\})$ *satisfies (3.1.13) if and only if* $\lambda > 0$ *and the pair* $(\lambda^{-1}, p[u])$ *of the set* $\mathbb{R} \times ((H^{1}(\Omega)/\mathbb{R}) \setminus \{0\})$ *satisfies the equation*

$$\lambda^{-1} p[u] = T^{\mathcal{N}}_{\rho} p[u].$$

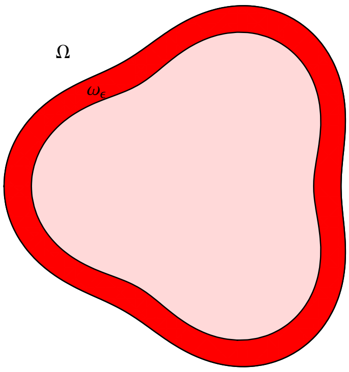
We have the following theorem.

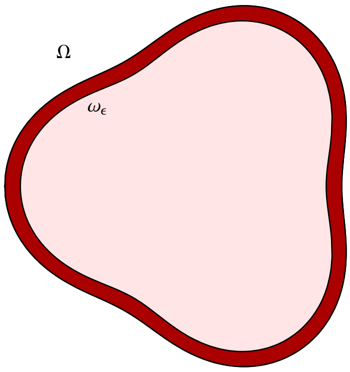
Theorem 3.1.17. *The operator* $T^{\mathcal{N}}_{\rho}$ *is a compact self-adjoint operator in* $H^{1}(\Omega)/\mathbb{R}$ *and its eigenvalues coincide with the reciprocals of the positive eigenvalues of problem (3.1.13). In particular, the set of eigenvalues of problem (3.1.13) is contained in* $[0, +\infty[$ *and consists of the image of a sequence increasing to* $+\infty$. *Each eigenvalue has finite multiplicity.*

Proof. The proof is similar to that of Theorem 3.1.8. Just note that the embedding i from $H^{1}(\Omega)$ to $L^{2}(\Omega)$ is compact. □

We have the following theorem on the spectrum of problem (3.1.13).

Theorem 3.1.18. *The first eigenvalue* λ_1 *of* (3.1.13) *is zero and the corresponding eigenfunctions are the constants. Moreover, the second eigenvalue* λ_2 *of* (3.1.13) *is positive.*





Now we consider problem (3.1.13) with densities which concentrate in a neighborhood of the boundary of Ω. Let us denote by ω_ε the set defined by

$$\omega_\varepsilon := \{x \in \Omega : d(x, \partial\Omega) < \varepsilon\} . \tag{3.1.19}$$

Let us fix a positive number $M > 0$ and define the family of densities $\{\rho_\varepsilon\}_{\varepsilon\in]0,\varepsilon_0[} \subset \mathcal{R}$ as follows:

$$\rho_\varepsilon(x) := \begin{cases} \varepsilon, & \text{if } x \in \Omega \setminus \overline{\omega}_\varepsilon, \\ \frac{M-\varepsilon|\Omega\setminus\overline{\omega}_\varepsilon|}{|\omega_\varepsilon|}, & \text{if } x \in \omega_\varepsilon, \end{cases} . \tag{3.1.20}$$

for $\varepsilon \in]0, \varepsilon_0[$, where ε_0 is sufficiently small. We note that $\int_\Omega \rho_\varepsilon dx = M$ for all $\varepsilon \in]0, \varepsilon_0[$. We refer to the quantity M as the total mass of the body.

Problem (3.1.3), and problem (3.1.13) with density ρ_ε are strictly related. In fact, under the assumption that Ω is of class C^2, it is possible to prove that the eigenvalues of problem (3.1.13) with density ρ_ε converge to the eigenvalues of problem (3.1.3) with $\rho \equiv \frac{M}{|\partial\Omega|}$. This is a consequence of the following theorem.

Theorem 3.1.21. *Let Ω be a bounded domain of class C^2 in $\mathbb{R}^N$. Let the operator $T^{\mathcal{S}}_{\frac{M}{|\partial\Omega|}}$ and $T^{\mathcal{N}}_{\rho_\varepsilon}$ be defined as in* (3.1.6) *and* (3.1.15) *respectively. Then $T^{\mathcal{N}}_{\rho_\varepsilon}$ converges in norm to $T^{\mathcal{S}}_{\frac{M}{|\partial\Omega|}}$ as $\varepsilon \to 0$.*

We need some preliminary results in order to prove Theorem 3.1.21. First of all we note that $\pi_c^{\sharp,\mathcal{S}} = \pi_1^{\sharp,\mathcal{S}}$ for all $c \in \mathbb{R}$, with $c \neq 0$. This can be deduced from (3.1.5).

Now we recall some facts from standard calculus which will be used in the sequel. Let M be a parametric hypersurface in $\mathbb{R}^N$ of class C^2, i.e., there exists a function $\phi \in C^2(\overline{D})$ from D to $\mathbb{R}^N$, where D is a bounded open subset of $\mathbb{R}^{N-1}$ such that $\phi(D) = M$. We assume that $\text{rank} D\phi(y_1, ..., y_{N-1}) = N-1$ for all $(y_1, ..., y_{N-1}) \in D$, where we denoted by $D\phi$ the Jacobian matrix of

ϕ. We set

$$M(\varepsilon) := \{\phi(y_1, ..., y_{N-1}) + t\nu(y_1, ..., y_{N-1}) : (y_1, ..., y_{N-1}) \in D, t \in]0, \varepsilon[\},$$

where $\nu(y_1, ..., y_{N-1})$ is the normal vector to $\phi(D)$ at the point $\phi(y_1, ..., y_{N-1})$ given by

$$\nu(y_1, ..., y_{N-1}) := \frac{\frac{\partial \phi}{\partial y_1} \wedge \cdots \wedge \frac{\partial \phi}{\partial y_{N-1}}}{\left|\frac{\partial \phi}{\partial y_1} \wedge \cdots \wedge \frac{\partial \phi}{\partial y_{N-1}}\right|}.$$

We consider the map ψ from $D \times]0, \varepsilon[$ onto $M(\varepsilon)$ defined by

$$\psi(y_1, ..., y_{N-1}, t) := \phi(y_1, ..., y_{N-1}) + t\nu(y_1, ..., y_{N-1})$$

for all $(y_1, ..., y_{N-1}) \in D$, $t \in]0, \varepsilon[$. We need to compute $\det D\psi$. We have

$$\begin{aligned}\det D\psi = \det &\left[\frac{\partial \phi}{\partial y_1} \quad \cdots \quad \frac{\partial \phi}{\partial y_{N-1}} \quad \nu(y_1, ..., y_{N-1})\right] \\ &+ t g_1\left(\frac{\partial \phi}{\partial y_1}, \frac{\partial \nu}{\partial y_1}, ..., \frac{\partial \phi}{\partial y_{N-1}}, \frac{\partial \nu}{\partial y_{N-1}}\right) + \cdots \\ &+ t^{N-1} g_{N-1}\left(\frac{\partial \phi}{\partial y_1}, \frac{\partial \nu}{\partial y_1}, ..., \frac{\partial \phi}{\partial y_{N-1}}, \frac{\partial \nu}{\partial y_{N-1}}\right),\end{aligned}$$

where g_i are suitable compositions of sums and products of the first partial derivatives of ϕ and ν. As is known, the first term in the sum is equal to

$$\left|\frac{\partial \phi}{\partial y_1} \wedge \cdots \wedge \frac{\partial \phi}{\partial y_{N-1}}\right|,$$

which is the $(N-1)$-dimensional measure of the hypersurface. We are ready to prove the following lemma. For the sake of completeness we include also statement *ii*).

Lemma 3.1.22. *Let M be a parametric hypersurface and (D, ϕ) a parametrization of M. Assume that $\inf_D \left|\frac{\partial \phi}{\partial y_1} \wedge \cdots \wedge \frac{\partial \phi}{\partial y_{N-1}}\right| > 0$. Assume also that $\varepsilon_0 > 0$ is such that ψ is a diffeomorphism for all $\varepsilon \in]0, \varepsilon_0[$. Let $f_\varepsilon, f \in H^1(M(\varepsilon_0))$ for all $\varepsilon > 0$ be such that $f_\varepsilon \to f$ in $H^1(M(\varepsilon_0))$ as $\varepsilon \to 0$. Then we have*

i)

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{M(\varepsilon)} f dx = \int_M f d\sigma; \tag{3.1.23}$$

ii)

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{M(\varepsilon)} (f_\varepsilon - f)\, dx = 0.$$

Proof. We write the proof in the case $N = 3$ for simplicity. The proof for $N > 3$ is analogous and can be carried out by using the same arguments. We consider

$$\frac{1}{\varepsilon}\int_{M(\varepsilon)} f(x)dx = \frac{1}{\varepsilon}\int_0^\varepsilon \int_D (f\circ\psi)(y_1,y_2,t)\left|\det D\psi\right| dy_1dy_2dt,$$

and compute the limit as $\varepsilon \to 0$. We observe that

$$\begin{aligned}\det D\psi &= \det\left[\tfrac{\partial\phi}{\partial y_1},\quad \tfrac{\partial\phi}{\partial y_2},\quad \nu(y_1,y_2)\right] + t\det\left[\tfrac{\partial\nu}{\partial y_1},\quad \tfrac{\partial\phi}{\partial y_2},\quad \nu(y_1,y_2)\right]\\ &- t\det\left[\tfrac{\partial\nu}{\partial y_2},\quad \tfrac{\partial\phi}{\partial y_1},\quad \nu(y_1,y_2)\right] + t^2\det\left[\tfrac{\partial\nu}{\partial y_1},\quad \tfrac{\partial\nu}{\partial y_2},\quad \nu(y_1,y_2)\right]. \quad (3.1.24)\end{aligned}$$

Moreover

$$\begin{aligned}&\frac{1}{\varepsilon}\int_0^\varepsilon \int_D (f\circ\psi)(y_1,y_2,t)\left|\det D\psi\right| dy_1dy_2dt\\ &= \frac{1}{\varepsilon}\int_0^\varepsilon \int_D ((f\circ\psi)(y_1,y_2,t)-(f\circ\psi)(y_1,y_2,0))\left|\det D\psi\right| dy_1dy_2dt\\ &\qquad + \frac{1}{\varepsilon}\int_0^\varepsilon \int_D (f\circ\psi)(y_1,y_2,0)\left|\det D\psi\right| dy_1dy_2dt. \quad (3.1.25)\end{aligned}$$

For the first summand in the right-hand side of (3.1.25), we observe that for a.e. $(y_1,y_2)\in D$, we have $|(f\circ\psi)(y_1,y_2,t)-(f\circ\psi)(y_1,y_2,0)| \le \int_0^\varepsilon \left|\frac{\partial(f\circ\psi)}{\partial t'}(y_1,y_2,t')\right| dt'$. Then, since $f\in H^1(M(\varepsilon))$, we have

$$\begin{aligned}&\frac{1}{\varepsilon}\int_0^\varepsilon \int_D |(f\circ\psi)(y_1,y_2,t)-(f\circ\psi)(y_1,y_2,0)|\left|\det D\psi\right| dy_1dy_2dt\\ &\qquad \le \int_D\int_0^\varepsilon \left|\frac{\partial(f\circ\psi)}{\partial t}(y_1,y_2,t)\right| \left|\det D\psi\right| dtdy_1dy_2\\ &\qquad\qquad \le |M(\varepsilon)|^{\frac{1}{2}}\left\|\nabla f\right\|_{L^2(M(\varepsilon))}.\end{aligned}$$

Thus the first summand in the right-hand side of (3.1.25) vanishes as $\varepsilon\to 0$. For the second summand, observe that for $(y_1,y_2)\in D$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\varepsilon \left|\det D\psi(y_1,y_2,t)\right| dt = \left|\det\left[\tfrac{\partial\phi}{\partial y_1},\quad \tfrac{\partial\phi}{\partial y_2},\quad \nu(y_1,y_2)\right]\right| = \left|\frac{\partial\phi}{\partial y_1}\wedge\frac{\partial\phi}{\partial y_2}\right|,$$

since the terms in (3.1.24) containing t vanish as $\varepsilon\to 0$. The last quantity is exactly the area element of the surface. Then we get

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{M(\varepsilon)} f dx = \int_M f d\sigma.$$

Now we prove statement *ii*). We consider

$$
\begin{aligned}
&\frac{1}{\varepsilon}\int_{M(\varepsilon)}(f_\varepsilon - f)\,dx\\
&\qquad = \frac{1}{\varepsilon}\int_0^\varepsilon\int_D ((f_\varepsilon\circ\psi)(y_1,y_2,t)-(f\circ\psi)(y_1,y_2,t))\,|\mathrm{det}D\psi|\,dy_1dy_2dt\\
&\qquad = \frac{1}{\varepsilon}\int_0^\varepsilon\int_D ((f_\varepsilon\circ\psi)(y_1,y_2,0)-(f\circ\psi)(y_1,y_2,0))\,|\mathrm{det}D\psi|\,dy_1dy_2dt\\
&+\frac{1}{\varepsilon}\int_0^\varepsilon\int_D\int_0^t\left(\frac{\partial(f_\varepsilon\circ\psi)}{\partial t'}(y_1,y_2,t')-\frac{\partial(f\circ\psi)}{\partial t'}(y_1,y_2,t')\right)dt'\,|\mathrm{det}D\psi|\,dy_1dy_2dt\\
&\qquad \le \frac{1}{\varepsilon}\int_0^\varepsilon\int_D |(f_\varepsilon\circ\psi)(y_1,y_2,0)-(f\circ\psi)(y_1,y_2,0)|\,|\mathrm{det}D\psi|\,dy_1dy_2dt\\
&+\frac{1}{\varepsilon}\int_0^\varepsilon\int_D\int_0^t\left|\frac{\partial(f_\varepsilon\circ\psi)}{\partial t'}(y_1,y_2,t')-\frac{\partial(f\circ\psi)}{\partial t'}(y_1,y_2,t')\right|dt'\,|\mathrm{det}D\psi|\,dy_1dy_2dt.
\end{aligned}
\tag{3.1.26}
$$

We set

$$
\begin{aligned}
G_1(y_1,y_2) &:= \left|\det\left[\tfrac{\partial\nu}{\partial y_1},\quad \tfrac{\partial\phi}{\partial y_2},\quad \nu(y_1,y_2)\right]-\det\left[\tfrac{\partial\nu}{\partial y_2},\quad \tfrac{\partial\phi}{\partial y_1},\quad \nu(y_1,y_2)\right]\right|,\\
G_2(y_1,y_2) &:= \left|\det\left[\tfrac{\partial\nu}{\partial y_1,}\quad \tfrac{\partial\nu}{\partial y_2},\quad \nu(y_1,y_2)\right]\right|.
\end{aligned}
$$

We have for the first summand of (3.1.26)

$$
\begin{aligned}
&\frac{1}{\varepsilon}\int_0^\varepsilon\int_D |(f_\varepsilon\circ\psi)(y_1,y_2,0)-(f\circ\psi)(y_1,y_2,0)|\,|\mathrm{det}D\psi|\,dy_1dy_2dt\\
&\quad\le \frac{1}{\varepsilon}\int_0^\varepsilon\int_D |(f_\varepsilon\circ\psi)(y_1,y_2,0)-(f\circ\psi)(y_1,y_2,0)|\left|\frac{\partial\phi}{\partial y_1}\wedge\frac{\partial\phi}{\partial v}\right|dy_1dy_2dt\\
&\quad +\frac{1}{\varepsilon}\int_0^\varepsilon\int_D |(f_\varepsilon\circ\psi)(y_1,y_2,0)-(f\circ\psi)(y_1,y_2,0)|\,tG_1(y_1,y_2)dy_1dy_2dt\\
&\quad +\frac{1}{\varepsilon}\int_0^\varepsilon\int_D |(f_\varepsilon\circ\psi)(y_1,y_2,0)-(f\circ\psi)(y_1,y_2,0)|\,t^2G_2(y_1,y_2)dy_1dy_2dt\\
&\qquad\qquad = \int_M |f_\varepsilon - f|\,d\sigma\\
&\qquad +\frac{\varepsilon}{2}\int_D |(f_\varepsilon\circ\psi)(y_1,y_2,0)-(f\circ\psi)(y_1,y_2,0)|\,G_1(y_1,y_2)dy_1dy_2\\
&\qquad +\frac{\varepsilon^2}{3}\int_D |(f_\varepsilon\circ\psi)(y_1,y_2,0)-(f\circ\psi)(y_1,y_2,0)|\,G_2(y_1,y_2)dy_1dy_2\\
&\qquad\qquad\qquad \le C\int_M |f_\varepsilon - f|\,d\sigma,
\end{aligned}
$$

where C is a positive constant which is bounded, uniformly in $\varepsilon > 0$. Thus the first summand in (3.1.26) vanishes as $\varepsilon \to 0$ because $f_\varepsilon \to f$ in $L^2(M)$ hence in $L^1(M)$. Now we consider the second summand in (3.1.26). We have

$$\frac{1}{\varepsilon}\int_0^\varepsilon\int_D\int_0^t\left|\frac{\partial(f_\varepsilon\circ\psi)}{\partial t'}(y_1,y_2,t')-\frac{\partial(f\circ\psi)}{\partial t'}(y_1,y_2,t')\right|dt'\,|\mathrm{det}D\psi|\,dy_1dy_2dt$$
$$\le\frac{1}{\varepsilon}\int_0^\varepsilon\int_D t^{\frac12}\left(\int_0^t\left|\frac{\partial(f_\varepsilon\circ\psi)}{\partial t'}(y_1,y_2,t')-\frac{\partial(f\circ\psi)}{\partial t'}(y_1,y_2,t')\right|^2dt'\right)^{\frac12}|\mathrm{det}D\psi|\,dy_1dy_2dt$$
$$\le\frac{C}{\varepsilon}\int_0^\varepsilon t^{\frac12}\left(\int_D\int_0^t\left|\frac{\partial(f_\varepsilon\circ\psi)}{\partial t'}(y_1,y_2,t')-\frac{\partial(f\circ\psi)}{\partial t'}(y_1,y_2,t')\right|^2\cdot|\mathrm{det}D\psi|\,dt'dy_1dy_2\right)^{\frac12}dt$$
$$\le C\varepsilon^{\frac12}\left\|\nabla\left(f_\varepsilon-f\right)\right\|_{L^2(M(\varepsilon))}.$$

This concludes the proof. □

We define the set $(\partial\Omega)^{\varepsilon_0}$ by

$$(\partial\Omega)^{\varepsilon_0}:=\left\{x\in\mathbb{R}^N: d(x,\partial\Omega)<\varepsilon_0\right\}.$$

Moreover, we denote by $\nu(\bar{x})$ the outer unit normal to $\partial\Omega$ at a point $\bar{x}\in\partial\Omega$. We recall the following theorem.

Theorem 3.1.27. (Tubular Neighborhood Theorem). *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^2. Then there exists $\varepsilon_0>0$ such that for each $x\in(\partial\Omega)^{\varepsilon_0}$ there exists a unique couple $(\bar{x},s)\in\partial\Omega\times]-\varepsilon_0,\varepsilon_0[$ such that $x=\bar{x}+s\nu(\bar{x})$, where $\nu(\bar{x})$ is the outer unit normal to $\partial\Omega$ at the point $\bar{x}$. Moreover, $\bar{x}$ is the (unique) nearest to x point of the boundary and $s=d(x,\partial\Omega)$. Finally, possibly reducing the value of ε_0, the map $x\mapsto(\bar{x},s)$ is a diffeomorphism of class C^1 from $(\partial\Omega)^{\varepsilon_0}$ onto $\partial\Omega\times]-\varepsilon_0,\varepsilon_0[$.*

We are ready to prove the following lemma.

Lemma 3.1.28. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^2. Let $\varepsilon>0$ and $\rho_\varepsilon\in\mathcal{R}$ be as in* (3.1.20)*. Then the following statements hold:*

i) For all $\varphi\in H^1(\Omega)/\mathbb{R}$, $\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[\varphi]\to\pi^{\sharp,\mathcal{S}}_1[\varphi]$ in $L^2(\Omega)$ as $\varepsilon\to0$ (hence also in $H^1(\Omega)$);

ii) if $u_\varepsilon\rightharpoonup u$ in $H^1(\Omega)/\mathbb{R}$ as $\varepsilon\to0$ then possibly passing to a subsequence, $\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[u_\varepsilon]\to\pi^{\sharp,\mathcal{S}}_1[u]$ in $L^2(\Omega)$ as $\varepsilon\to0$;

iii) assume that $u_\varepsilon,w_\varepsilon,u,w\in H^1(\Omega)$ are such that $u_\varepsilon\to u$, $w_\varepsilon\to w$ in $L^2(\Omega)$ as $\varepsilon\to0$, and that $\|\nabla u_\varepsilon\|_{L^2(\Omega)},\|\nabla u\|_{L^2(\Omega)}\le C$, $\|\nabla w_\varepsilon\|_{L^2(\Omega)}$, $\|\nabla w\|_{L^2(\Omega)}\le C$, uniformly in $\varepsilon>0$. Then

$$\int_\Omega\rho_\varepsilon\left(u_\varepsilon-u\right)w_\varepsilon dx\to0,$$

and

$$\int_\Omega\rho_\varepsilon\left(w_\varepsilon-w\right)udx\to0,$$

as $\varepsilon\to0$.

Proof. We start by proving statement i). It is sufficient to show that

$$\lim_{\varepsilon\to 0}\left\|\frac{\int_\Omega \rho_\varepsilon\tilde{\varphi}dx}{M}-\frac{\int_{\partial\Omega}\tilde{\varphi}d\sigma}{|\partial\Omega|}\right\|_{L^2(\Omega)}=0,$$

where $\tilde{\varphi}\in H^1(\Omega)$ is such that $\varphi=p[\tilde{\varphi}]$. Since the equality

$$\lim_{\varepsilon\to 0}\int_\Omega \rho_\varepsilon\tilde{\varphi}dx=\frac{M}{|\partial\Omega|}\int_{\partial\Omega}\tilde{\varphi}d\sigma$$

holds by Lemma 3.1.22, i), we have the desired result.
Now we prove statement iii). We note that since $u_\varepsilon\to u$, $w_\varepsilon\to w$ as $\varepsilon\to 0$ in $L^2(\Omega)$ and $u_\varepsilon, w_\varepsilon$ are uniformly bounded in $H^1(\Omega)$, then $u_\varepsilon\rightharpoonup u$, $w_\varepsilon\rightharpoonup w$, $\mathrm{Tr}[u_\varepsilon]\to\mathrm{Tr}[u]$ and $\mathrm{Tr}[w_\varepsilon]\to\mathrm{Tr}[w]$ as $\varepsilon\to 0$. Then, in order to prove iii) it is sufficient to prove that

$$\lim_{\varepsilon\to 0}\int_\Omega \rho_\varepsilon u_\varepsilon w_\varepsilon dx=0$$

whenever $u_\varepsilon\to 0$ in $L^2(\Omega)$ and $\mathrm{Tr}[u_\varepsilon]\to 0$ in $L^2(\partial\Omega)$. We have that

$$\int_\Omega \rho_\varepsilon u_\varepsilon w_\varepsilon dx=\varepsilon\int_{\Omega\setminus\overline{\omega}_\varepsilon}u_\varepsilon w_\varepsilon dx+C(\varepsilon)\int_{\omega_\varepsilon}u_\varepsilon w_\varepsilon dx, \tag{3.1.29}$$

where $C(\varepsilon)=\frac{M-\varepsilon|\Omega\setminus\overline{\omega}_\varepsilon|}{|\omega_\varepsilon|}$. The first summand clearly is $O(\varepsilon)$ as $\varepsilon\to 0$. By multiplying and dividing the second summand by ε and observing that $\varepsilon C(\varepsilon)\le C'<+\infty$ for $\varepsilon>0$ small enough, we obtain that the second summand in the right-hand side of (3.1.29) is less than or equal to

$$C'\cdot\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|u_\varepsilon w_\varepsilon|\,dx.$$

We now apply Theorem 3.1.27. Let $\varepsilon_0>0$ be as in Theorem 3.1.27 and let $\varepsilon\in]0,\varepsilon_0[$. Let $x_0\in\partial\Omega$ and U_0 be a neighborhood of x_0 in $\mathbb{R}^N$ such that there exists $V_0\subset\mathbb{R}^{N-1}$ and a parametrization $\phi\in C^2(V_0)$ such that the map ψ from $V_0\times]0,\varepsilon[$ onto $M(\varepsilon)=\{x\in\Omega: d(x,\partial\Omega\cap U_0)<\varepsilon\}$ defined by

$$\psi(p,t):=\phi(p)+t\nu(p)\,,\quad\forall(p,t)\in V_0\times]0,\varepsilon[$$

is a diffeomorphism from $V_0\times]0,\varepsilon[$ onto $M(\varepsilon)$. Here $p=(p_1,...,p_{N-1})\in\mathbb{R}^{N-1}$ and $\nu(p)$ denotes the unit inner normal to $\partial\Omega$ at $\phi(p)$. Now we consider

$$\int_{M(\varepsilon)}\frac{1}{\varepsilon}|u_\varepsilon w_\varepsilon|\,dx=\int_{V_0}\int_0^\varepsilon\frac{|\mathrm{det}D\psi|}{\varepsilon}|(u_\varepsilon\circ\psi)(p,t)|\,|(w_\varepsilon\circ\psi)(p,t)|\,dtdp. \tag{3.1.30}$$

For almost every $p \in V_0$, $(u_\varepsilon \circ \psi)(p,t)$, $(w_\varepsilon \circ \psi)(p,t)$ are absolutely continuous on $[0,\varepsilon]$ and since this set is compact, also their product is absolutely continuous. Let $p \in V_0$ be fixed. We have

$$\begin{aligned}
&(u_\varepsilon \circ \psi)(p,t)(w_\varepsilon \circ \psi)(p,t) \\
&\qquad\qquad = (u_\varepsilon \circ \psi)(p,0)(w_\varepsilon \circ \psi)(p,0) \\
&+ \int_0^t \frac{\partial(u_\varepsilon \circ \psi)}{\partial t'}(p,t')(w_\varepsilon \circ \psi)(p,t') + (u_\varepsilon \circ \psi)(p,t')\frac{\partial(w_\varepsilon \circ \psi)}{\partial t'}(p,t')dt', \\
&(u_\varepsilon \circ \psi)(p,t') = (u_\varepsilon \circ \psi)(p,0) + \int_0^{t'} \frac{\partial(u_\varepsilon \circ \psi)}{\partial s}(p,s)ds, \\
&(w_\varepsilon \circ \psi)(p,t') = (w_\varepsilon \circ \psi)(p,0) + \int_0^{t'} \frac{\partial(w_\varepsilon \circ \psi)}{\partial s}(p,s)ds. \qquad (3.1.31)
\end{aligned}$$

We observe that, for fixed ε and for a.e. $p \in V_0$, the quantity $C_1(t,p) = \left(\int_0^t \left|\frac{\partial(u_\varepsilon \circ\psi)}{\partial t'}(t',p)\right|^2 dt'\right)^{\frac{1}{2}}$ is increasing in $0 \le t \le \varepsilon$ hence $C_1(t,p) \le C_1(\varepsilon,p)$ for all $0 \le t \le \varepsilon$.

The same result holds for $C_2(t,p) = \left(\int_0^t \left|\frac{\partial(w_\varepsilon \circ\psi)}{\partial t'}(t',p)\right|^2 dt'\right)^{\frac{1}{2}}$. Then, for $0 \le t' \le t \le \varepsilon$ we have

$$\begin{aligned}
\left|(u_\varepsilon \circ \psi)(p,t')\right| &\le |(u_\varepsilon \circ \psi)(p,0)| + t'^{\frac{1}{2}} C_1(t',p) \\
&\le |(u_\varepsilon \circ \psi)(p,0)| + t^{\frac{1}{2}} C_1(\varepsilon,p), \qquad (3.1.32)
\end{aligned}$$

$$\begin{aligned}
\left|(w_\varepsilon \circ \psi)(p,t')\right| &\le |(w_\varepsilon \circ \psi)(p,0)| + t'^{\frac{1}{2}} C_2(t',p) \\
&\le |(w_\varepsilon \circ \psi)(p,0)| + t^{\frac{1}{2}} C_2(\varepsilon,p). \qquad (3.1.33)
\end{aligned}$$

Now, let us consider the right hand side in (3.1.30). By using (3.1.31):

$$\begin{aligned}
&\int_{V_0}\int_0^\varepsilon \frac{|\det D\psi|}{\varepsilon} |(u_\varepsilon \circ \psi)(p,t)|\,|(w_\varepsilon \circ \psi)(p,t)|\,dtdp \\
&\qquad \le \int_{V_0}\int_0^\varepsilon \frac{1}{\varepsilon} |(u_\varepsilon \circ \psi)(p,0)|\,|(w_\varepsilon \circ \psi)(p,0)|\,|\det D\psi|\,dtdp \\
&+ \|\det D\psi\|_{L^\infty(V_0\times[0,\varepsilon])} \int_{V_0}\int_0^\varepsilon \frac{1}{\varepsilon}\int_0^t \left|\frac{\partial(u_\varepsilon \circ \psi)}{\partial t'}(p,t')\right| |(w_\varepsilon \circ \psi)(p,t')|\,dt'dtdp \\
&+ \|\det D\psi\|_{L^\infty(V_0\times[0,\varepsilon])} \int_{V_0}\int_0^\varepsilon \frac{1}{\varepsilon}\int_0^t |(u_\varepsilon \circ \psi)(p,t')| \left|\frac{\partial(w_\varepsilon \circ \psi)}{\partial t'}(p,t')\right| dt'dtdp. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.1.34)
\end{aligned}$$

Now using (3.1.32) and (3.1.33) and the fact that $\|\nabla u_\varepsilon\|_{L^2(\Omega)}, \|\nabla w_\varepsilon\|_{L^2(\Omega)} \le C$, it is easy to prove that the second and third summand in the right-hand side of (3.1.34) vanish as $\varepsilon \to 0$. For the first summand in the right-hand side of (3.1.34), we observe that

$$
\begin{aligned}
\mathrm{det} D\psi = \det \left[\tfrac{\partial \phi}{\partial p_1} \quad \cdots \quad \tfrac{\partial \phi}{\partial p_{N-1}} \quad \nu(p_1,...,p_{N-1}) \right] \\
+ t g_1 \left(\frac{\partial \phi}{\partial p_1}, \frac{\partial \nu}{\partial p_1}, ..., \frac{\partial \phi}{\partial p_{N-1}}, \frac{\partial \nu}{\partial p_{N-1}} \right) + \cdots \\
+ t^{N-1} g_{N-1} \left(\frac{\partial \phi}{\partial p_1}, \frac{\partial \nu}{\partial p_1}, ..., \frac{\partial \phi}{\partial p_{N-1}}, \frac{\partial \nu}{\partial p_{N-1}} \right),
\end{aligned}
$$

where g_i are suitable compositions of sums and products of the first partial derivatives of ϕ and ν. It is not restrictive to assume that

$$
\inf_{V_0} \left| \frac{\partial \phi}{\partial p_1} \wedge \cdots \wedge \frac{\partial \phi}{\partial p_{N-1}} \right| > 0.
$$

Now, using the same argument as in the proof of statement *ii*) of Lemma 3.1.22, we obtain

$$
\begin{aligned}
&\frac{1}{\varepsilon} \int_{V_0} \int_0^{\varepsilon} |(u_\varepsilon \circ \psi)(p,0)| \, |(w_\varepsilon \circ \psi)(p,0)| \, |\mathrm{det} D\psi| \, dt dp \\
&\quad \leq \frac{1}{\varepsilon} \int_0^{\varepsilon} \int_{V_0} |(u_\varepsilon \circ \psi)(p,0)| \, |(w_\varepsilon \circ \psi)(p,0)| \left| \frac{\partial \phi}{\partial p_1} \wedge \cdots \wedge \frac{\partial \phi}{\partial p_{N-1}} \right| dp dt \\
&\qquad + \sum_{i=1}^{N-1} \frac{1}{\varepsilon} \int_0^{\varepsilon} \int_{V_0} |(u_\varepsilon \circ \psi)(p,0)| \, |(w_\varepsilon \circ \psi)(p,0)| \, t^i \, |g_i(p)| \, dp dt \\
&\qquad \qquad \leq \tilde{C} \int_{\partial\Omega \cap U_0} |u_\varepsilon| \, |w_\varepsilon| \, d\sigma,
\end{aligned}
$$

where $\tilde{C}$ is uniformly bounded in $\varepsilon \in]0, \varepsilon_0[$. Since $\mathrm{Tr}[u_\varepsilon] \to 0$ in $L^2(\partial\Omega)$ as $\varepsilon \to 0$, it follows that also the first summand in the right-hand side of (3.1.34) vanishes as $\varepsilon \to 0$.

Since ω_ε can be covered by a finite number of open sets of the type $M(\varepsilon)$, say $\omega_\varepsilon \subset \bigcup_{i=1}^m M_i(\varepsilon)$, we have that

$$
\frac{1}{\varepsilon} \int_{\omega_\varepsilon} |u_\varepsilon w_\varepsilon| \, dx \leq \sum_{i=1}^m \frac{1}{\varepsilon} \int_{M_i(\varepsilon)} |u_\varepsilon w_\varepsilon| \, dx.
$$

This concludes the proof of statement *iii*).

We now prove statement *ii*). Let $\tilde{u}_\varepsilon, \tilde{u} \in H_1^{1,\mathcal{N}}(\Omega)$ (see (3.1.14)) be such that $u_\varepsilon = p[\tilde{u}_\varepsilon]$, $u = p[\tilde{u}]$. We have

$$
\begin{aligned}
&\left\| \pi_{\rho_\varepsilon}^{\sharp,\mathcal{N}}[u_\varepsilon] - \pi_1^{\sharp,\mathcal{S}}[u] \right\|_{L^2(\Omega)} \\
&\qquad \leq \left\| \pi_{\rho_\varepsilon}^{\sharp,\mathcal{N}}[u_\varepsilon] - \pi_{\rho_\varepsilon}^{\sharp,\mathcal{N}}[u] \right\|_{L^2(\Omega)} + \left\| \pi_{\rho_\varepsilon}^{\sharp,\mathcal{N}}[u] - \pi_1^{\sharp,\mathcal{S}}[u] \right\|_{L^2(\Omega)}. \quad (3.1.35)
\end{aligned}
$$

By statement *i*) it follows that the second summand in the right hand side of (3.1.35) goes to zero as $\varepsilon \to 0$. For the first summand, we have

$$\begin{aligned}\left\|\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[u_\varepsilon]-\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[u]\right\|_{L^2(\Omega)} &= \left\|\tilde{u}_\varepsilon - \frac{\int_\Omega \rho_\varepsilon \tilde{u}_\varepsilon dx}{M} - \tilde{u} + \frac{\int_\Omega \rho_\varepsilon \tilde{u} dx}{M}\right\|_{L^2(\Omega)} \\ &\leq \|\tilde{u}_\varepsilon - \tilde{u}\|_{L^2(\Omega)} + \frac{\left\|\int_\Omega \rho_\varepsilon\left(\tilde{u}_\varepsilon-\tilde{u}\right)dx\right\|_{L^2(\Omega)}}{M} \\ &\leq \|\tilde{u}_\varepsilon - \tilde{u}\|_{L^2(\Omega)} + \left(\frac{|\Omega|^{\frac{1}{2}}}{M}\right)\cdot\left|\int_\Omega \rho_\varepsilon\left(\tilde{u}_\varepsilon-\tilde{u}\right)dx\right|.\end{aligned}$$

Now, if we prove that $\tilde{u}_\varepsilon \to \tilde{u}$ in $L^2(\Omega)$ we are done, since the result follows by statement *iii*) with $w_\varepsilon \equiv 1$. Since $u_\varepsilon \rightharpoonup u$ in $H^1(\Omega)/\mathbb{R}$, then $\tilde{u}_\varepsilon \rightharpoonup \tilde{u}$ in $H^{1,\mathcal{N}}_1(\Omega)$ as $\varepsilon \to 0$. From the compactness of the embedding of $H^{1,\mathcal{N}}_1(\Omega)$ into $L^2(\Omega)$ it follows that $\tilde{u}_\varepsilon \to \tilde{u}$ in $L^2(\Omega)$, as $\varepsilon \to 0$. This concludes the proof of statement *ii*) and of the theorem. □

We recall the following definition.

Definition 3.1.36. *Let H be a real Hilbert space, $\mathcal{K}(H,H)$ be the Banach subspace of $\mathcal{L}(H,H)$ of those $T \in \mathcal{L}(H,H)$ which are compact. A set $\mathcal{K} \subset \mathcal{K}(H,H)$ is said to be collectively compact if and only if the set*

$$\{K[x] : K \in \mathcal{K}, x \in B\},$$

where B is the open unit ball in H, has compact closure. We say that a sequence of compact operators $\{K_n\}_{n\in\mathbb{N}}$ compactly converges to the operator K if $\{K_n\}_{n\in\mathbb{N}}$ is collectively compact and $K_n[x_n] \to K[x]$ whenever $x_n \to x$ in H.

We refer to [8, 103] for details. We are now ready to prove Theorem 3.1.21.

Proof of Theorem 3.1.21. We prove that $T^{\mathcal{N}}_{\rho_\varepsilon}$ compactly converges to the compact operator $T^{\mathcal{S}}_{\frac{M}{|\partial\Omega|}}$. This implies, in fact, that

$$\lim_{\varepsilon\to 0}\left\|\left(T^{\mathcal{N}}_{\rho_\varepsilon} - T^{\mathcal{S}}_{\frac{M}{|\partial\Omega|}}\right)^2\right\|_{\mathcal{L}(H^1(\Omega)/\mathbb{R},H^1(\Omega)/\mathbb{R})} = 0. \tag{3.1.37}$$

Then, since the operators $\left\{T^{\mathcal{N}}_{\rho_\varepsilon}\right\}_{\varepsilon\in]0,\varepsilon_0[}$ and $T^{\mathcal{S}}_{\frac{M}{|\partial\Omega|}}$ are self-adjoint, property (3.1.37) is equivalent to convergence in norm. We refer to [8, 103] for a proof of (3.1.37) and for a more detailed discussion on compact convergence of compact operators on Hilbert spaces. We recall that, by definition, $T^{\mathcal{N}}_{\rho_\varepsilon}$ compactly converges to $T^{\mathcal{S}}_{\frac{M}{|\partial\Omega|}}$ if the following requirements are fulfilled:

i) if $\|u_\varepsilon\|_{H^1(\Omega)/\mathbb{R}} \leq C$ for all $\varepsilon \in]0,\varepsilon_0[$, then the family $\{T^{\mathcal{N}}_{\rho_\varepsilon} u_\varepsilon\}_{\varepsilon\in]0,\varepsilon_0[}$ has compact closure in $H^1(\Omega)/\mathbb{R}$;

ii) if $u_\varepsilon \to u$ in $H^1(\Omega)/\mathbb{R}$, then $T^{\mathcal{N}}_{\rho_\varepsilon} u_\varepsilon \to T^{\mathcal{S}}_{\frac{M}{|\partial\Omega|}} u$ in $H^1(\Omega)/\mathbb{R}$.

We prove i) first. Let $u, \varphi \in H^1(\Omega)/\mathbb{R}$. We have

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\Omega \rho_\varepsilon \pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[u]\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[\varphi]dx &= \lim_{\varepsilon\to 0}\int_\Omega \rho_\varepsilon\left(\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[u] - \pi^{\sharp,\mathcal{S}}_1[u]\right)\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[\varphi]dx \\ &+ \lim_{\varepsilon\to 0}\int_\Omega \rho_\varepsilon \pi^{\sharp,\mathcal{S}}_1[u]\left(\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[\varphi] - \pi^{\sharp,\mathcal{S}}_1[\varphi]\right)dx \\ &+ \left(\lim_{\varepsilon\to 0}\int_\Omega \rho_\varepsilon \pi^{\sharp,\mathcal{S}}_1[u]\pi^{\sharp,\mathcal{S}}_1[\varphi]dx - \frac{M}{|\partial\Omega|}\int_{\partial\Omega}\pi^{\sharp,\mathcal{S}}_1[u]\pi^{\sharp,\mathcal{S}}_1[\varphi]d\sigma\right) \\ &+ \frac{M}{|\partial\Omega|}\int_{\partial\Omega}\pi^{\sharp,\mathcal{S}}_1[u]\pi^{\sharp,\mathcal{S}}_1[\varphi]d\sigma. \qquad (3.1.38)\end{aligned}$$

By Lemma 3.1.28, *iii)* we have that the first and second summands in the right-hand side of (3.1.38) go to zero as $\varepsilon \to 0$. As for the third summand, from Lemma 3.1.22, *i)* applied to the function $f = 1$ we have that $|\omega_\varepsilon| = \varepsilon|\partial\Omega| + o(\varepsilon)$ as $\varepsilon \to 0$. Therefore $\rho_\varepsilon = \frac{M}{\varepsilon|\partial\Omega|} + o(1)$ as $\varepsilon \to 0$. Thus, from Lemma 3.1.22 and formula (3.1.23) it follows that also the third summand of (5.3.9) goes to zero as $\varepsilon \to 0$. Moreover, the equality $(\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon})^{-1} \circ (\mathcal{M}^{\mathcal{N}}_{\rho_\varepsilon})^{-1} = (\pi^{\sharp,\mathcal{S}}_1)^{-1} \circ (\mathcal{M}^{\mathcal{S}}_1)^{-1}$ holds. Therefore, from (3.1.38) it follows that $T^{\mathcal{N}}_{\rho_\varepsilon} u$ is bounded for each $u \in H^1(\Omega)/\mathbb{R}$. Thus, by Banach-Steinhaus Theorem, there exists C' such that $\|T^{\mathcal{N}}_{\rho_\varepsilon}\|_{\mathcal{L}(H^1(\Omega)/\mathbb{R}, H^1(\Omega)/\mathbb{R})} \leq C'$ for all $\varepsilon \in]0,\varepsilon_0[$. Moreover, since $\|u_\varepsilon\|_{H^1(\Omega)/\mathbb{R}} \leq C$ for all $\varepsilon \in]0,\varepsilon_0[$, possibly passing to a subsequence, we have that $u_\varepsilon \rightharpoonup u$ in $H^1(\Omega)/\mathbb{R}$, for some $u \in H^1(\Omega)/\mathbb{R}$. This implies that, possibly passing to a subsequence, $T^{\mathcal{N}}_{\rho_\varepsilon} u_\varepsilon \rightharpoonup w$ in $H^1(\Omega)/\mathbb{R}$ as $\varepsilon \to 0$, for some $w \in H^1(\Omega)/\mathbb{R}$. We show that $w = T^{\mathcal{S}}_{\frac{M}{|\partial\Omega|}} u$. To shorten our notation we set $w_\varepsilon := T^{\mathcal{N}}_{\rho_\varepsilon} u_\varepsilon$. By Lemma 3.1.28, *i)* we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_\Omega \nabla(\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[w_\varepsilon]) \cdot \nabla(\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[\varphi])dx \\ &\quad = \int_\Omega \nabla(\pi^{\sharp,\mathcal{S}}_1[w]) \cdot \nabla(\pi^{\sharp,\mathcal{S}}_1[\varphi])dx, \qquad (3.1.39)\end{aligned}$$

for all $\varphi \in H^1(\Omega)/\mathbb{R}$.

On the other hand, since $\left(\mathcal{M}^{\mathcal{N}}_{\rho_\varepsilon} \circ \pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}\right) w_\varepsilon = \left(\mathcal{J}^{\mathcal{N}}_{\rho_\varepsilon} \circ i \circ \pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}\right) u_\varepsilon$, we have that

$$\int_\Omega \nabla(\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[w_\varepsilon]) \cdot \nabla(\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[\varphi])dx = \int_\Omega \rho_\varepsilon \pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[u_\varepsilon]\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[\varphi]dx \qquad (3.1.40)$$

Then, by Lemma 3.1.28, *iii*), (3.1.39) and (3.1.40) we have

$$
\begin{aligned}
&< w, \varphi >_{H^1(\Omega)/\mathbb{R}} \\
&\qquad = \lim_{\varepsilon\to 0} < w_\varepsilon, \varphi >_{H^1(\Omega)/\mathbb{R}} = \lim_{\varepsilon\to 0} \int_\Omega \rho_\varepsilon \pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[u_\varepsilon] \pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[\varphi] dx \\
&\qquad = \lim_{\varepsilon\to 0} \int_\Omega \rho_\varepsilon \left(\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[u_\varepsilon] - \pi^{\sharp,\mathcal{S}}_1[u] \right) \pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[\varphi] dx \\
&\qquad + \lim_{\varepsilon\to 0} \int_\Omega \rho_\varepsilon \pi^{\sharp,\mathcal{S}}_1[u] \left(\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[\varphi] - \pi^{\sharp,\mathcal{S}}_1[\varphi] \right) dx \\
&\qquad + \lim_{\varepsilon\to 0} \int_\Omega \rho_\varepsilon \pi^{\sharp,\mathcal{S}}_1[u] \pi^{\sharp,\mathcal{S}}_1[\varphi] dx \\
&\qquad = \frac{M}{|\partial\Omega|} \int_{\partial\Omega} \pi^{\sharp,\mathcal{S}}_1[u] \pi^{\sharp,\mathcal{S}}_1[\varphi] d\sigma \\
&\qquad = < T^{\mathcal{S}}_{\frac{M}{|\partial\Omega|}} u, \varphi >_{H^2(\Omega)/\mathbb{R}},
\end{aligned}
$$

hence $w = T^{\mathcal{S}}_{\frac{M}{|\partial\Omega|}} u$. In a similar way one can prove that $\|w_\varepsilon\|_{H^1(\Omega)/\mathbb{R}} \to \|w\|_{H^1(\Omega)/\mathbb{R}}$. In fact

$$
\begin{aligned}
\lim_{\varepsilon\to 0} \|w_\varepsilon\|^2_{H^1(\Omega)/\mathbb{R}} &= \lim_{\varepsilon\to 0} \int_\Omega \rho_\varepsilon \left(\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[u_\varepsilon] - \pi^{\sharp,\mathcal{S}}_1[u] \right) \pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[w_\varepsilon] dx \\
&+ \lim_{\varepsilon\to 0} \int_\Omega \rho_\varepsilon \pi^{\sharp,\mathcal{S}}_1[u] \left(\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[w_\varepsilon] - \pi^{\sharp,\mathcal{S}}_1[w_\varepsilon] \right) dx \\
&+ \lim_{\varepsilon\to 0} \int_\Omega \rho_\varepsilon \pi^{\sharp,\mathcal{S}}_1[u] \left(\pi^{\sharp,\mathcal{S}}_1[w_\varepsilon] - \pi^{\sharp,\mathcal{S}}_1[w] \right) dx \\
&+ \lim_{\varepsilon\to 0} \int_\Omega \rho_\varepsilon \pi^{\sharp,\mathcal{S}}_1[u] \pi^{\sharp,\mathcal{S}}_1[w] dx \\
&= \frac{M}{|\partial\Omega|} \int_{\partial\Omega} \pi^{\sharp,\mathcal{S}}_1[u] \pi^{\sharp,\mathcal{S}}_1[w] d\sigma = \|w\|^2_{H^1(\Omega)/\mathbb{R}}.
\end{aligned}
$$

This proves *i*). As for point *ii*), let $u_\varepsilon \to u$ in $H^1(\Omega)/\mathbb{R}$. Then there exists C'' such that $\|u_\varepsilon\|_{H^1(\Omega)/\mathbb{R}} \leq C''$ for all $\varepsilon \in]0, \varepsilon_0[$. Then, by the same argument used for point *i*), for each sequence $\varepsilon_j \to 0$, possibly passing to a subsequence, we have $T^{\mathcal{N}}_{\rho_{\varepsilon_j}} u_{\varepsilon_j} \to T^{\mathcal{S}}_{\frac{M}{|\partial\Omega|}} u$. Since this is true for each $\{\varepsilon_j\}_{j\in\mathbb{N}}$, we have the convergence for the whole family, i.e., $T^{\mathcal{N}}_{\rho_\varepsilon} u_\varepsilon \to T^{\mathcal{S}}_{\frac{M}{|\partial\Omega|}} u$. This concludes the proof. □

We need the following well-known result.

Theorem 3.1.41. *Let H be a real Hilbert space and $\{A_\varepsilon\}_{\varepsilon\in]0,\varepsilon_0[}$ be a family of bounded self-adjoint operators converging in norm to the bounded self-adjoint operator A, i.e., $\lim_{\varepsilon\to 0} \|A_\varepsilon - A\|_{\mathcal{L}(H,H)} = 0$. Then isolated eigenvalues λ of A of finite multiplicity are exactly the limits of the eigenvalues of*

A_ε, counting the multiplicity. Moreover, the corresponding eigenprojections converge in norm.

Thanks to Theorem 3.1.41, as an immediate corollary of Theorem 3.1.21 we have

Corollary 3.1.42. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^2. Let $\lambda_j[\rho_\varepsilon]$ denote the eigenvalues of problem (3.1.13) with density ρ_ε on Ω for all $j \in \mathbb{N}$. Let λ_j, $j \in \mathbb{N}$ denote the eigenvalues of problem (3.1.3) corresponding to the constant surface density $\frac{M}{|\partial\Omega|}$. Then $\lim_{\varepsilon\to 0} \lambda_j[\rho_\varepsilon] = \lambda_j$ for all $j \in \mathbb{N}$.*

3.2 The Steklov eigenvalue problem. Mass density perturbations

In this section we discuss the dependence of the eigenvalues of problem (3.1.2) on the weight ρ. We shall obtain results of continuity and real analiticity of the eigenvalues in the spirit of Theorem 2.1.5 and Theorem 2.2.1. We note that this problem has a rather different behavior under mass density perturbations with respect to the operators considered in Chapter 2. In fact, in some particular cases we are able to find mass densities which are critical for the symmetric functions of the eigenvalues under mass constraint. Through all this section Ω is a bounded domain of class C^1. Moreover, we shall denote the eigenvalues of problem (3.1.2) by $\lambda_j[\rho]$ for all $j \in \mathbb{N}$.

3.2.1 Continuity and analyticity of the eigenvalues

By the min-max principle (3.1.10) it is possible to prove that $\lambda_j[\rho]$ is a locally Lipschitz continuous function of $\rho \in \mathcal{R}^{\mathcal{S}}$. In fact as in Section 2.1 it is possible to prove that

$$|\lambda_j[\rho_1] - \lambda_j[\rho_2]| \leq \frac{\min\{\lambda_j[\rho_1], \lambda_j[\rho_2]\}}{\min\{\operatorname{ess\,inf}_{\partial\Omega} \rho_1, \operatorname{ess\,inf}_{\partial\Omega} \rho_2\}} \|\rho_1 - \rho_2\|_{L^\infty(\partial\Omega)},$$

for all $\rho_1, \rho_2 \in \mathcal{R}^{\mathcal{S}}$ satisfying $\|\rho_1 - \rho_2\|_{L^\infty(\partial\Omega)} < \min\{\operatorname{ess\,inf}_{\partial\Omega} \rho_1, \operatorname{ess\,inf}_{\partial\Omega} \rho_2\}$. The eigenvalues $\lambda_j[\rho]$ depend with continuity on ρ also with respect the weak* topology of $L^\infty(\partial\Omega)$. We have the following theorem.

Theorem 3.2.1. *Let $C \subset \mathcal{R}^{\mathcal{S}}$ be a bounded set. Then the function which takes $\rho \in C$ to $\lambda_j[\rho]$ is continuous in the weak* topology of $L^\infty(\partial\Omega)$.*

Proof. The proof is analogous to that of Theorem 2.1.5 and accordingly is omitted. □

We prove now that all simple eigenvalues and the symmetric functions of the eigenvalues of problem (3.1.2) depend real analitically on ρ and provide

Hadamard-type formulas for the corresponding Frechét differentials. Let F be a finite nonempty subset of $\mathbb{N}$. We set

$$\begin{aligned}\mathcal{R}^{\mathcal{S}}[F] &:= \{\rho \in \mathcal{R}^{\mathcal{S}} \,:\, \lambda_j[\rho] \neq \lambda_l[\rho]\,,\ \forall j \in F, l \in \mathbb{N} \setminus F\}\ ,\\ \Theta^{\mathcal{S}}[F] &:= \{\rho \in \mathcal{R}^{\mathcal{S}}[F] \,:\, \lambda_{j_1}[\rho] = \lambda_{j_2}[\rho]\,,\ \forall j_1, j_2 \in F\}.\end{aligned}$$

Given $\rho \in \mathcal{R}^{\mathcal{S}}$, we denote by $L^2_\rho(\partial\Omega)$ the space $L^2(\partial\Omega)$ endowed with the bilinear form

$$< u, v >_{\rho,\partial\Omega} := \int_{\partial\Omega} \rho u v d\sigma.$$

Such bilinear form is a scalar product on $L^2(\partial\Omega)$ which induces on $L^2(\partial\Omega)$ a norm equivalent to the standard one. Then we have the following result.

Theorem 3.2.2. *Let F be a nonempty finite subset of $\mathbb{N}$. Then $\mathcal{R}^{\mathcal{S}}[F]$ is open in $L^\infty(\partial\Omega)$ and the symmetric functions of the eigenvalues*

$$\Lambda_{F,h}[\rho] = \sum_{\substack{j_1,\dots,j_h \in F\\ j_1 < \dots j_h}} \lambda_{j_1}[\rho] \cdots \lambda_{j_h}[\rho]\,, \quad h = 1, \dots, |F|\,,$$

are real analytic in $\mathcal{R}^{\mathcal{S}}[F]$. Moreover, if $F = \cup_{k=1}^n F_k$ and $\rho \in \cap_{k=1}^n \Theta^{\mathcal{S}}[F_k]$ is such that for each $k = 1, \dots, n$ the eigenvalues $\lambda_j[\rho]$ assume the common value $\lambda_{F_k}[\rho]$ for all $j \in F_k$, then the differentials of the functions $\Lambda_{F,h}$ at the point ρ are given by the formula

$$d\Lambda_{F,h}[\rho][\dot\rho] = -\sum_{k=1}^n c_k \sum_{l \in F_k} \int_{\partial\Omega} u_l^2 \dot\rho d\sigma\,, \tag{3.2.3}$$

for all $\dot\rho \in L^\infty(\partial\Omega)$, where

$$c_k = \sum_{\substack{0 \le h_1 \le |F_1|\\ \dots\dots\\ 0 \le h_n \le |F_n|\\ h_1 + \dots + h_n = h}} \binom{|F_k| - 1}{h_k - 1} \lambda_{F_k}^{h_k}[\rho] \prod_{\substack{j=1\\ j \neq k}}^n \binom{|F_j|}{h_j} \lambda_{F_j}^{h_j}[\rho],$$

and for each $k = 1, \dots, n$, $\{u_l\}_{l \in F_k}$ is an orthonormal basis in $L^2_\rho(\partial\Omega)$ of the eigenspace associated with $\lambda_{F_k}[\rho]$.

Proof. The proof follows the same lines as that of Theorem 2.2.1 and is accordingly omitted. □

3.2.2 Critical mass densities

We consider now the problem of finding critical mass densities for the symmetric functions of the eigenvalues under mass constraint, i.e., mass densities ρ which satisfy $\mathrm{Ker} dM_\partial[\rho] \subset \mathrm{Ker} d\Lambda_{F,h}[\rho]$, where $M_\partial[\rho] := \int_{\partial\Omega} \rho d\sigma$. As a consequence of Theorem 3.2.2 and formula (3.2.3) combined with the Lagrange Multipliers Theorem, we can give a characterization of such critical mass densities.

Corollary 3.2.4. *Let all assumptions of Theorem 3.2.2 hold. Then $\rho \in \mathcal{R}^{\mathcal{S}}$ is a critical mass density for $\Lambda_{F,h}$ for some $h = 1, ..., |F|$, subject to mass constraint if and only if there exists $c \geq 0$ such that*

$$\left(\sum_{k=1}^{n} \sum_{l \in F_k} (\mathrm{Tr} u_l)^2 \right) = c, \quad \text{a.e. on } \partial\Omega. \tag{3.2.5}$$

The analysis carried out in Chapter 2 has pointed out that for a large class of non-negative elliptic operators subject to homogeneous boundary conditions there are no critical mass densities for simple eigenvalues and the symmetric functions of multiple eigenvalues. In the case of Steklov boundary conditions the situation is much different. Indeed, if Ω is a ball, then a critical mass density exists.

Corollary 3.2.6. *Let B be the unit ball in $\mathbb{R}^N$. Let $k \in \mathbb{N}_0$. Let us denote by n_k the number of linearly independent spherical harmonics of degree strictly less than k in $\mathbb{R}^N$ and by d_k the number of linearly independent spherical harmonics of degree k (see Corollary 1.4.16). Let $F = \{n_k + 1, ..., n_k + d_k\}$ and $M > 0$. Then the constant mass density $\rho \equiv \frac{M}{|\partial B|}$ is a critical mass density for $\Lambda_{F,h}$ for $h = 1, ..., d_k$ under the constraint $\int_{\partial\Omega} \rho d\sigma = M$.*

Proof. It is well-known that the eigenvalues of problem (3.1.2) on the unit ball with constant density $\rho \equiv \frac{M}{|\partial B|}$ are of the form $\lambda_k = \frac{|\partial B| k}{M}$, $k \in \mathbb{N}_0$. Each eigenvalue λ_k has multiplicity d_k and the eigenfunctions associated with λ_k are exactly the homogeneous harmonic polynomials of degree k in $\mathbb{R}^N$. Therefore, the set $\{u_{k,j} := |x|^k H_{k,j}\}_{j=1}^{d_k}$, where $\{H_{k,j}\}_{j=1}^{d_k}$ is a basis for the spherical harmonics of degree k in $\mathbb{R}^N$ and $\{H_{k,j}\}_{j=1}^{d_k}$ are normalized such that $\frac{M}{|\partial B|} \int_{\partial B} H_{k,j} H_{k,i} d\sigma = \delta_{i,j}$ for all $i, j = 1, ..., d_k$, is a basis for the eigenspace associated with the eigenvalue λ_k. Thus condition (3.2.5) is satisfied since it is well-known (see e.g., [40]) that

$$\sum_{j=1}^{d_k} u_{k,j}^2 = c_k \quad \text{on } \partial B,$$

for a suitable constant $c_k > 0$. Then the constant density $\frac{M}{|\partial B|}$ is a critical mass density for $\Lambda_{F,h}$. □

Remark 3.2.7. *In the same hypothesis of Corollary 3.2.6, consider the particular case of $F = \{2, ..., N + 1\}$. We note that $d_1 = N$ and the set $\{u_{1,j} := c_j x_j\}_{j=1}^{N}$, where $c_j = \left(\frac{M}{|\partial B|} \int_{\partial B} x_j^2 d\sigma \right)^{-1}$ for all $j = 1, ..., N$ is a orthonormal basis in $L^2(\partial B)$ of the eigenspace associated with the first positive eigenvalue $\frac{M}{|\partial B|}$ of problem (3.1.2) on the unit ball with constant density, which has multiplicity N. Then the constant density $\frac{M}{|\partial B|}$ is a critical mass density for $\Lambda_{F,h}$ for $h = 1, ..., N$ under the constraint $\int_{\partial B} \rho d\sigma = M$.*

It is interesting to compare Corollary 3.2.6 and Remark 3.2.7 with a classical result proved by Hersch, Payne and Schiffer [61] in the case of a class of planar domains. We recall that in the case of the unit ball in $\mathbb{R}^2$ and constant density $\frac{M}{2\pi}$, we have that $\lambda_2[\frac{M}{2\pi}] = \lambda_3[\frac{M}{2\pi}] = \frac{2\pi}{M}$. We have the following theorem.

Theorem 3.2.8 (Hersch,Payne,Schiffer)**.** *Let Ω be the unit disk in $\mathbb{R}^2$ centered at zero and $M > 0$ be fixed. Then*

$$\lambda_2[\rho]\lambda_3[\rho] \leq \frac{M^2}{4\pi^2}.$$

The equality is attained only at $\rho \equiv \frac{M}{2\pi}$.

Thus in the case of a ball in $\mathbb{R}^2$ the constant mass density is in fact the unique maximizer for the first positive eigenvalue $\lambda_2[\rho]$ among all mass densities preserving the total mass. We refer to [14] for further discussions on the problem of maximization of Steklov eigenvalues subject to mass density perturbations.

In the next section we consider the problem of minimizing $\lambda_2[\rho]$ among all mass densities preserving the total mass.

3.3 Minimization of the first positive Steklov eigenvalue

Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1 and $M > 0$ be a fixed number. Let $\lambda_2[\rho]$ be the first positive eigenvalue of problem (3.1.2) on Ω. We shall prove that there exists a sequence ρ_ε of densities such that $\int_{\partial\Omega} \rho_\varepsilon d\sigma = M$ and $\lambda_2[\rho_\varepsilon] \to 0$ as $\varepsilon \to 0$. Therefore, the problem

$$\min_{\substack{\rho \in \mathcal{R}^{\mathcal{S}} \\ \int_{\partial\Omega} \rho d\sigma = M}} \lambda_2[\rho]$$

has no solutions. In Subsection 3.3.1 we prove the result for the unit ball in $\mathbb{R}^2$. In Subsection 3.3.2 we extend the result to the case of the unit ball in $\mathbb{R}^N$ for $N \geq 3$. Finally, in Subsection 3.3.3 we consider the case of general bounded domains of class C^1 in $\mathbb{R}^2$.

3.3.1 The case of the ball in $\mathbb{R}^2$

Through all this subsection, we consider problem (3.1.2) when $\Omega = B$ is the unit ball in $\mathbb{R}^2$ centered at zero. We denote by $\mathcal{R}_\rho[u]$ the Rayleigh quotient of a function $u \in H^1(B)$:

$$\mathcal{R}_\rho[u] = \frac{\int_B |\nabla u|^2 dxdy}{\int_{\partial B} \rho u^2 d\sigma},$$

where $\rho \in \mathcal{R}^{\mathcal{S}}$. From (3.1.10), we have the following variational representation of the first positive eigenvalue of (3.1.2)

$$\lambda_2[\rho] = \min_{\substack{0 \neq u \in H^1(B) \\ \int_{\partial B} \rho u d\sigma = 0}} \mathcal{R}_\rho[u]. \tag{3.3.1}$$

It is convenient to use polar coordinates $(r, \theta) \in \mathbb{R}_+ \times \partial B$ in $\mathbb{R}^2$ and the corresponding change of variables $x = r\cos(\theta), y = r\sin(\theta)$. Given $M > 0$, we define the family of densities $\{\rho_\varepsilon\}_{\varepsilon \in]0,\varepsilon_0[} \subset L^2(\partial B)$ written in polar coordinates as follows:

$$\rho_\varepsilon(\theta) := \begin{cases} \varepsilon, & \text{if } \theta \in [\varepsilon, \pi - \varepsilon] \cup [\pi + \varepsilon, 2\pi - \varepsilon], \\ \frac{M - \varepsilon(2\pi - 4\varepsilon)}{4\varepsilon}, & \text{if } \theta \in [0, \varepsilon[\cup]\pi - \varepsilon, \pi + \varepsilon[\cup]2\pi - \varepsilon, 2\pi[, \end{cases} \tag{3.3.2}$$

for all $\varepsilon \in]0, \varepsilon_0[$ with ε_0 sufficiently small. Note that $\int_{\partial B} \rho_\varepsilon d\sigma = M$ for all $\varepsilon \in]0, \varepsilon_0[$. The densities ρ_ε are piecewise constant and concentrate in a neighborhood of two antipodal points, while they vanish in all the other points, as $\varepsilon \to 0$. We have the following theorem.

Theorem 3.3.3. *Let ρ_ε be defined by* (3.3.2)*. Let $\lambda_2[\rho_\varepsilon]$ be the first positive eigenvalue of problem* (3.1.2) *with $\rho = \rho_\varepsilon$ on the unit ball B in $\mathbb{R}^2$. Then*

$$\lim_{\varepsilon \to 0} \lambda_2[\rho_\varepsilon] = 0. \tag{3.3.4}$$

Proof. Let B^+ be the ball of radius 1 centered at $(1, 0)$, B^- the ball of radius 1 centered at $(-1, 0)$, B_ε^+ the ball of radius $\sqrt{2 - 2\cos(\varepsilon)}$ centered at $(1, 0)$ and B_ε^- the ball of radius $\sqrt{2 - 2\cos(\varepsilon)}$ centered at $(-1, 0)$. We introduce the family of trial functions u_ε (see Figure 3.2) given by

$$u_\varepsilon(x, y) := \begin{cases} -\ln(\sqrt{(1 - x)^2 + y^2}), & \text{if } (x, y) \in (B \cap B^+) \setminus B_\varepsilon^+, \\ -\ln(\varepsilon), & \text{if } (x, y) \in B \cap B_\varepsilon^+, \\ \ln(\sqrt{(1 + x)^2 + y^2}), & \text{if } (x, y) \in (B \cap B^-) \setminus B_\varepsilon^-, \\ \ln(\varepsilon), & \text{if } (x, y) \in B \cap B_\varepsilon^-, \\ 0, & \text{if } (x, y) \in B \setminus (B^+ \cup B^-). \end{cases}$$

By construction $u_\varepsilon \in H^1(B)$ and by symmetry $\int_{\partial B} \rho_\varepsilon u_\varepsilon d\sigma = 0$ for all $\varepsilon \in]0, \varepsilon_0[$. Hence, u_ε is a suitable trial function for (3.3.1).

We have

$$\lambda_1[\rho_\varepsilon] \leq \mathcal{R}_{\rho_\varepsilon}[u_\varepsilon], \tag{3.3.5}$$

for all $\varepsilon \in]0, \varepsilon_0[$. Note that $|u_\varepsilon|$, $|\nabla u_\varepsilon|$ and ρ_ε are symmetric with respect to the x and the y axes. Then we compute the integrals appearing in $\mathcal{R}_{\rho_\varepsilon}[u_\varepsilon]$ restricted to $B \cap \{(x, y) : x > 0, y > 0\}$. We consider the numerator of $\mathcal{R}_{\rho_\varepsilon}[u_\varepsilon]$

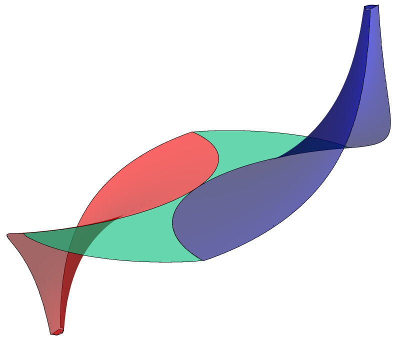


Figure 3.2: Trial function u_ε with $\varepsilon = 0.05$.

first. We have

$$\int_{B\cap\{x>0,y>0\}} |\nabla u_\varepsilon|^2 dxdy = \int_{B\cap B^+\cap\{y>0\}} |\nabla u_\varepsilon|^2 dxdy$$
$$\leq \int_{B^+\cap\{x<1,y>0\}} |\nabla \tilde{u}_\varepsilon|^2 dxdy, \quad (3.3.6)$$

where the first equality holds since $u_\varepsilon = 0$ in $B \setminus (B^+ \cup B^-)$, and the last inequality follows from the fact that we are integrating a positive function on a larger set. Here we have denoted by $\tilde{u}_\varepsilon$ the function on B^+ defined by $\tilde{u}_\varepsilon(x,y) = -\ln(\sqrt{(1-x)^2+y^2})$ for $2-2\cos(\varepsilon) \leq (1-x)^2+y^2 \leq 1$ and $\tilde{u}_\varepsilon(x,y) = -\ln(\varepsilon)$ for $(1-x)^2+y^2 < 2-2\cos(\varepsilon)$. Now we use the polar coordinates (r,θ) with respect to the point $(1,0)$ and the corresponding change of variables $(x,y) = \psi(r,\theta) = (1+r\cos(\theta), r\sin(\theta))$, with $r \in [0,+\infty[$, $\theta \in [0,2\pi[$. In this new coordinates $\tilde{u}_\varepsilon(\psi(r,\theta)) = -\ln(r)$ for $\sqrt{2-2\cos(\varepsilon)} < r < 1$ and $\tilde{u}_\varepsilon(\psi(r,\theta)) = -\ln(\varepsilon)$, for $0 < r\sqrt{2-2\cos(\varepsilon)}$. In this new coordinates the right-hand side of (3.3.6) is written as

$$\int_{B^+\cap\{x<1,y>0\}} |\nabla \tilde{u}_\varepsilon|^2 dxdy$$
$$= \int_{\frac{\pi}{2}}^{\pi} \int_{\sqrt{2-2\cos(\varepsilon)}}^{1} \frac{1}{r} drd\theta = -\frac{\pi}{2}\ln(\sqrt{2-2\cos(\varepsilon)}).$$

We note that

$$\ln(\sqrt{2-2\cos(\varepsilon)}) = \ln(\varepsilon) + O(\varepsilon^2), \quad (3.3.7)$$

as $\varepsilon \to 0$.
Now we consider the denominator of $\mathcal{R}_{\rho_\varepsilon}[u_\varepsilon]$. We have

$$\begin{aligned}
\int_{\partial B\cap\{x>0,y>0\}} \rho_\varepsilon u^2 d\sigma = & \int_0^\varepsilon \frac{M-\varepsilon(2\pi-4\varepsilon)}{4\varepsilon}(\ln\varepsilon)^2 d\theta \\
& + \int_\varepsilon^{\frac{\pi}{3}} \varepsilon \left[\ln\left(2\sin\left(\frac{\theta}{2}\right)\right)\right]^2 d\theta \\
= & \left(\frac{M-\varepsilon(2\pi-4\varepsilon)}{4}\right)(\ln\varepsilon)^2 + \varepsilon\int_{2\sin\left(\frac{\varepsilon}{2}\right)}^1 \frac{(\ln s)^2}{\sqrt{1-\frac{s^2}{4}}}ds \\
\geq & \left(\frac{M-\varepsilon(2\pi-4\varepsilon)}{4}\right)(\ln\varepsilon)^2 + \frac{\varepsilon}{\cos\left(\frac{\varepsilon}{2}\right)}\int_{2\sin\left(\frac{\varepsilon}{2}\right)}^1 (\ln s)^2 ds \\
= & \left(\frac{M-\varepsilon(2\pi-4\varepsilon)}{4}\right)(\ln\varepsilon)^2 \\
& + \frac{\varepsilon}{\cos\left(\frac{\varepsilon}{2}\right)}\left[2-4\sin\left(\frac{\varepsilon}{2}\right)+4\sin\left(\frac{\varepsilon}{2}\right)\ln\left(2\sin\left(\frac{\varepsilon}{2}\right)\right)\right. \\
& \left. -2\sin\left(\frac{\varepsilon}{2}\right)\left[\ln\left(2\sin\left(\frac{\varepsilon}{2}\right)\right)\right]^2\right].
\end{aligned}$$

We note that

$$\begin{aligned}
\frac{\varepsilon}{\cos\left(\frac{\varepsilon}{2}\right)}&\left[2-4\sin\left(\frac{\varepsilon}{2}\right)+4\sin\left(\frac{\varepsilon}{2}\right)\ln\left(2\sin\left(\frac{\varepsilon}{2}\right)\right)\right. \\
&\left. -2\sin\left(\frac{\varepsilon}{2}\right)\left[\ln\left(2\sin\left(\frac{\varepsilon}{2}\right)\right)\right]^2\right] \\
&= 2\varepsilon + O(\varepsilon^2(\ln\varepsilon)^2),
\end{aligned}$$

as $\varepsilon \to 0$. From (3.3.7) and (3.3.7), it follows that

$$\frac{\int_{B\cap\{x>0,y>0\}} |\nabla u_\varepsilon|^2 dxdy}{\int_{\partial B\cap\{x>0,y>0\}} \rho_\varepsilon u^2 d\sigma} \leq -\frac{\pi\left(\ln(\varepsilon)+O(\varepsilon^2)\right)}{2\left(\frac{(M-\varepsilon(2\pi-4\varepsilon))}{4}(\ln\varepsilon)^2+2\varepsilon+O(\varepsilon^2(\ln\varepsilon)^2)\right)},$$

as $\varepsilon \to 0$. Then, from (3.3.5) and using a Taylor expansion, we have

$$\lambda_1[\rho_\varepsilon] \leq \frac{2\pi}{M|\ln(\varepsilon)|} + \frac{4\pi\varepsilon\left(4-\pi(\ln\varepsilon)^2\right)}{M^2(\ln\varepsilon)^3} + O\left(\frac{\varepsilon^2}{\ln(\varepsilon)}\right),$$

as $\varepsilon \to 0$, which yields (3.3.4). Moreover we have an upper bound for the rate of convergence of $\lambda_1[\rho_\varepsilon]$ to zero, as $\varepsilon \to 0$ (see Figure 3.3). □

Remark 3.3.8. *We note that a basis of the harmonic functions on the unit ball B in $\mathbb{R}^2$ is given (in polar coordinates) by*

$$1, \ln(r), r^l\cos(l\theta), r^l\sin(l\theta), r^{-l}\cos(l\theta), r^{-l}\sin(l\theta),$$

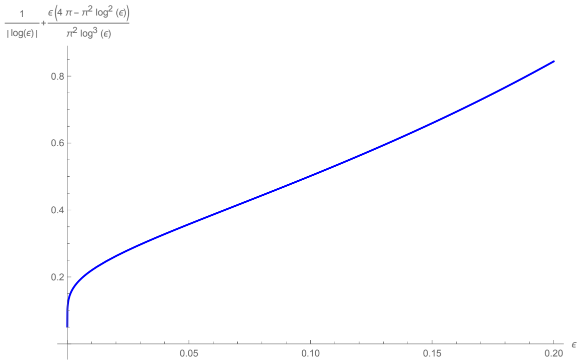


Figure 3.3: Plot of $\frac{2\pi}{M|\ln(\varepsilon)|} + \frac{4\pi\varepsilon\left(4-\pi(\ln\varepsilon)^2\right)}{M^2(\ln\varepsilon)^3}$ with $M = 2\pi$ and $\varepsilon \in]0, 0.2[$.

for $l \in \mathbb{N}, l \geq 1$. *When we consider problem* (3.1.2)*, we require that the solutions are regular in the interior of the domain. Therefore we consider only the functions* $1, r^l\cos(l\theta), r^l\sin(l\theta)$, $l \in \mathbb{N}, l \geq 1$. *In the case of problem* (3.1.2) *with density given by* (3.3.2)*, the coefficient* ρ_ε *vanishes everywhere but in two points, say* p^+, p^-*, where it blows up, as* $\varepsilon \to 0$. *Therefore, intuitively the harmonic functions which better mimic the behavior of an eigenfunction near these two points are exactly* $\ln(r), r^{-l}\cos(l\theta), r^{-l}\sin(l\theta)$*, translated in such a way that the singularity occurs at the points where the coefficient blows up. In particular, we note that the harmonic function* $\ln(r)$ *corresponds to the value* $l = 0$*, i.e., it is candidate to be the "second eigenfunction" of the zero eigenvalue of the limiting problem. Intuitively, the limiting problem (whatever it means), has a zero eigenvalue of multiplicity* 2*, and the eigenspace is spanned by the constant and by a suitable translations of* $\ln(r)$. *By following this heuristic intuition, one could guess what are the test functions to be used in Rayleigh quotient for the first positive eigenvalue in the case of the unit ball* B *in* $\mathbb{R}^N$*, for* $N \geq 3$.

3.3.2 The case of the ball in $\mathbb{R}^N$ with $N \geq 3$

Thorugh this subsection we denote by B be the unit ball in $\mathbb{R}^N$ centered at zero, with $N \geq 3$. We introduce the spherical coordinates $(r,\theta) = (r,\theta_1,...,\theta_{N-1}) \in \mathbb{R}_+ \times \partial B$ in $\mathbb{R}^N$ centered at $x_0 = (x_{1,0}, x_{2,0}, ..., x_{N,0}) \in \mathbb{R}^N$, and the corresponding change of variables given by

$$
\begin{aligned}
x_1 &= x_{1,0} + r\cos(\theta_1),\\
x_2 &= x_{2,0} + r\sin(\theta_1)\cos(\theta_2),\\
x_3 &= x_{3,0} + r\sin(\theta_1)\sin(\theta_2)\cos(\theta_3),\\
&\vdots\\
x_{N-1} &= x_{N-1,0} + r\sin(\theta_1)\cdots\sin(\theta_{N-2})\cos(\theta_{N-1}),
\end{aligned}
$$

$$x_N \quad = \quad x_{N,0} + r\sin(\theta_1)\cdots\sin(\theta_{N-2})\sin(\theta_{N-1}).$$

The spherical volume element in these coordinates is given by

$$d\sigma = r^{N-1}\sin(\theta_1)^{N-2}\sin(\theta_2)^{N-3}\cdots\sin(\theta_{N-2})drd\theta_1 d\theta_2\cdots d\theta_{N-1}.$$

We define the subsets of $\mathbb{R}^N$, $B^+, B^-, B_{\varepsilon^+}, B_\varepsilon^-$ as follows

$$\begin{cases} B^+ := B((1,0,...,0),1), \\ B^- := B((-1,0,...,0),1), \\ B_\varepsilon^+ := B((1,0,...,0),\varepsilon), \\ B_\varepsilon^- := B((-1,0,...,0),\varepsilon). \end{cases}$$

We introduce the family of densities $\{\rho_\varepsilon\}_{\varepsilon\in]0,\varepsilon_0[} \subset L^2(\partial\Omega)$ defined by

$$\rho_\varepsilon(x) := \begin{cases} \varepsilon, & \text{if } x\in\partial B\setminus(B_\varepsilon^+\cup B_\varepsilon^-), \\ \frac{M-\varepsilon(N\omega_N-2|C_\varepsilon|)}{2|C_\varepsilon|}, & \text{if } x\in\partial B\cap(B_\varepsilon^+\cup B_\varepsilon^-), \end{cases} \tag{3.3.9}$$

where $M>0$ is fixed, $N\omega_N$ is the $N-1$-dimensional measure of ∂B and $C_\varepsilon := \partial B\cap B_\varepsilon^+$. Note that $|\partial B\cap B_\varepsilon^+| = |\partial B\cap B_\varepsilon^-| = |C_\varepsilon|$. By construction $\int_{\partial B}\rho_\varepsilon d\sigma = M$ for all $\varepsilon\in]0,\varepsilon_0[$, for a suitable $\varepsilon_0>0$.

We have the following theorem.

Theorem 3.3.10. *Let ρ_ε be defined by* (3.3.9). *Let $\lambda_2[\rho_\varepsilon]$ be the first positive eigenvalue of problem* (3.1.2) *with $\rho=\rho_\varepsilon$ on the unit ball B in $\mathbb{R}^N$. Then*

$$\lim_{\varepsilon\to 0}\lambda_2[\rho_\varepsilon] = 0. \tag{3.3.11}$$

Proof. Let u_ε be the function on B defined by

$$u_\varepsilon(x) := \begin{cases} \frac{1}{\left(\sqrt{(x_1-1)^2+x_2^2+\cdots+x_N^2}\right)^{N-2}} - 1, & \text{if } x\in B\cap(B^+\setminus B_\varepsilon^+), \\ \frac{1}{\varepsilon^{N-2}} - 1, \text{if } x\in B\cap B_\varepsilon^+, & \\ 1-\frac{1}{\left(\sqrt{(x_1+1)^2+x_2^2+\cdots+x_N^2}\right)^{N-2}}, & \text{if } x\in B\cap(B^-\setminus B_\varepsilon^-), \\ 1-\frac{1}{\varepsilon^{N-2}}, & \text{if } x\in B\cap B_\varepsilon^-, \\ 0, & \text{if } x\in B\setminus(B^+\cup B^-). \end{cases}$$

By construction, $u_\varepsilon\in H^1(B)$ and $\int_{\partial B}\rho_\varepsilon u_\varepsilon d\sigma = 0$ for all $\varepsilon\in]0,\varepsilon_0[$. Hence u_ε is a suitable trial function for the Rayleigh quotient of $\lambda_2[\varepsilon]$. Indeed we have

$$\lambda_2[\rho_\varepsilon] \le \mathcal{R}_{\rho_\varepsilon}[u_\varepsilon], \tag{3.3.12}$$

for all $\varepsilon\in]0,\varepsilon_0[$. By the symmetry of $|u_\varepsilon|$, $|\nabla u_\varepsilon|$ and ρ_ε we can consider the integrals appearing in the Rayleigh quotients restricted on $B\cap\{x_1>0\}$.

We consider the numerator of $\mathcal{R}_{\rho_\varepsilon}[u_\varepsilon]$ first and use the spherical coordinates centered in $(1,0,...,0)$. We have

$$\int_{B\cap\{x_1>0\}}|\nabla u_\varepsilon|^2dx \leq \frac{N(N-2)^2\omega_N}{2}\int_\varepsilon^1 r^{1-N}dr = \frac{N(N-2)\omega_N}{2}\left(\frac{1}{\varepsilon^{N-2}}-1\right), \quad (3.3.13)$$

where we extended the function u_ε to the whole of B^+ as we did in the proof of Theorem 3.3.3. Now we consider the denominator of $\mathcal{R}_{\rho_\varepsilon}[u_\varepsilon]$. We have

$$\int_{\partial B\cap\{x_1>0\}}\rho_\varepsilon u_\varepsilon^2 d\sigma = \frac{M-\varepsilon(N\omega_N-2|C_\varepsilon|)}{2}\left(\frac{1}{\varepsilon^{N-2}}-1\right)^2 + \varepsilon\int_{\partial B\cap(B^+\setminus B_\varepsilon^+)}\left(\frac{1}{\left(\sqrt{(x_1-1)^2+x_2^2+\cdots+x_N^2}\right)^{N-2}}-1\right)^2 d\sigma. \quad (3.3.14)$$

We need to estimate the second term in the right-hand side of (3.3.14). It is straightforward to see that

$$\varepsilon\int_{\partial B\cap(B^+\setminus B_\varepsilon^+)}\left(\frac{1}{\left(\sqrt{(x_1-1)^2+x_2^2+\cdots+x_N^2}\right)^{N-2}}-1\right)^2 d\sigma \geq \varepsilon\left(\frac{1}{\varepsilon^{N-2}}-1\right)^2|\partial B\cap(B^+\setminus B_\varepsilon^+)| = |\partial B\cap B^+|\left(\frac{1}{\varepsilon^{N-2}}-1\right)^2\varepsilon - |C_\varepsilon|.\left(\frac{1}{\varepsilon^{N-2}}-1\right)^2\varepsilon \quad (3.3.15)$$

We note that $|C_\varepsilon|\in O(\varepsilon^{N-1})$ as $\varepsilon\to 0$. From (3.3.12), (3.3.13), (3.3.14) and (3.3.15) we have

$$\lambda_2[\rho_\varepsilon] \leq \frac{N(N-2)\omega_N}{2\left(\frac{1}{\varepsilon^{N-2}}-1\right)\left(\frac{M-\varepsilon(N\omega_N-2|C_\varepsilon|)}{2}+\varepsilon|\partial B\cap B^+|-\varepsilon|C_\varepsilon|\right)}. \quad (3.3.16)$$

We perform a Taylor expansion of the right-hand side of (3.3.16) and obtain

$$\lambda_2[\rho_\varepsilon] \leq \varepsilon^{N-2}\left(\frac{N(N-2)\omega_N}{M} - \frac{N(N-2)\omega_N(2|\partial B\cap B^+|-N\omega_N)\varepsilon}{M^2}+O(\varepsilon^2)\right),$$

which yields formula (3.3.11). Moreover, we have an upper bound for the rate of convergence of $\lambda_2[\varepsilon]$ which depends on M and N (see Figure 3.4). □

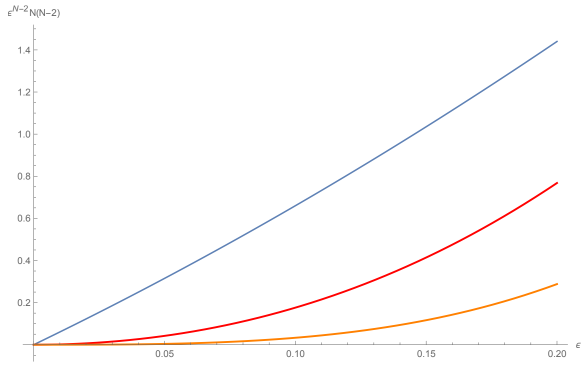


Figure 3.4: Plot of $\varepsilon^{N-2}\left(\frac{N(N-2)\omega_N}{M}\right)$ with $M=\omega_N$, $\varepsilon\in]0,0.2[$ and $N=3$ (Blue), $N=4$ (Red), $N=5$ (Orange).

3.3.3 The case of an arbitrary $\Omega\subset\mathbb{R}^2$

In this subsection we consider a bounded domain Ω in $\mathbb{R}^2$ of class C^1. We shall consider only the case of $\mathbb{R}^2$ since the computations are less involved. The result for $N>2$ can be obtained by following the same scheme (see also Subection 3.3.2). Through this subsection we denote by $x=(x_1,x_2)$ an element of $\mathbb{R}^2$.

Let $x_+,x_-\in\partial\Omega$ be such that

$$d(x_+,x_-)\geq d(x_1,x_2),\quad \forall x_1,x_2\in\partial\Omega.$$

The distance function d from $\mathbb{R}^2$ to $\mathbb{R}$ is continuous and $\partial\Omega\times\partial\Omega$ is a compact set, therefore there it has a maximum. We write $x_+=(x_{+,1},x_{+,2})$ and $x_-=(x_{-,1},x_{-,2})$. We set

$$d^*:=\frac{d(x_+,x_-)}{2}.$$

We introduce the subsets $B_*^+,B_*^-,B_\varepsilon^+,B_\varepsilon^-$ of $\mathbb{R}^2$ defined in the following way (see Figure 3.5):

$$\begin{cases} B_*^+:=B(x_+,d^*),\\ B_*^-:=B(x_-,d^*),\\ B_\varepsilon^+:=B(x_+,\varepsilon),\\ B_\varepsilon^-:=B(x_-,\varepsilon).\end{cases}$$

Let $M>0$ be fixed. For all $\varepsilon\in]0,\varepsilon_0[$ with ε_0 small enough, we introduce the family $\{\rho_\varepsilon\}_{\varepsilon\in]0,\varepsilon_0[}\subset L^2(\partial\Omega)$ defined by

$$\rho_\varepsilon(x):=\begin{cases}\varepsilon, & \text{if } x\in\partial\Omega\setminus(B_*^+\cup B_*^-),\\ \rho_+, & \text{if } x\in\partial\Omega\cap B_\varepsilon^+,\\ \rho_-, & \text{if } x\in\partial\Omega\cap B_\varepsilon^-,\end{cases}\tag{3.3.17}$$

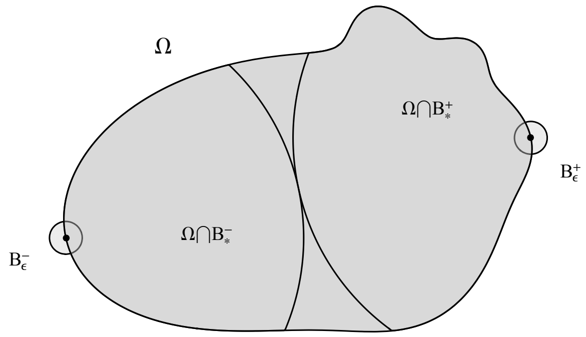


Figure 3.5: The regions $\Omega, B_\varepsilon^+, B_\varepsilon^-, \Omega\cap B_*^+, \Omega\cap B_*^-$.

where ρ_+, ρ_- are suitable constants depending on M and ε which solve the following linear system

$$\begin{cases} |\partial\Omega\cap B_\varepsilon^+|\rho_+ + |\partial\Omega\cap B_\varepsilon^-|\rho_- + |\partial\Omega\setminus(B_\varepsilon^+\cup B_\varepsilon^-)|\varepsilon = M, \\ |\partial\Omega\cap B_\varepsilon^+|\rho_+ - |\partial\Omega\cap B_\varepsilon^-|\rho_- = 0. \end{cases} \tag{3.3.18}$$

Note that the determinant of the matrix associated with system (3.3.18) is given by $-(|\partial\Omega\cap B_\varepsilon^+||\partial\Omega\cap B_\varepsilon^-|)^2$ and is different from zero for all $\varepsilon\in]0,\varepsilon_0[$. Moreover, $M-|\partial\Omega\setminus(B_\varepsilon^+\cup B_\varepsilon^-)|\varepsilon\neq 0$ for all $\varepsilon\in]0,\varepsilon_0[$ for ε_0 small enough. We solve system (3.3.3) and obtain

$$\begin{aligned} \rho_+ &= \frac{M-|\partial\Omega\setminus(B_\varepsilon^+\cup B_\varepsilon^-)|\varepsilon}{2|\partial\Omega\cap B_\varepsilon^+|} \\ \rho_- &= \frac{M-|\partial\Omega\setminus(B_\varepsilon^+\cup B_\varepsilon^-)|\varepsilon}{2|\partial\Omega\cap B_\varepsilon^-|}. \end{aligned}$$

By construction, $\int_{\partial\Omega}\rho_\varepsilon d\sigma = M$ for all $\varepsilon\in]0,\varepsilon_0[$.

We have the following theorem.

Theorem 3.3.19. *Let Ω be a bounded domain in $\mathbb{R}^2$ of class C^1 and let ρ_ε be defined by* (3.3.17)*. Let $\lambda_2[\rho_\varepsilon]$ be the first positive eigenvalue of problem* (3.1.2) *with $\rho=\rho_\varepsilon$ on Ω. Then*

$$\lim_{\varepsilon\to 0}\lambda_2[\rho_\varepsilon] = 0. \tag{3.3.20}$$

Proof. Let the function u_ε^+ be defined by

$$u_\varepsilon^+(x) := \begin{cases} -\ln\left(\frac{\varepsilon}{d^*}\right), & \text{if } x\in\Omega\cap B_\varepsilon^+, \\ -\ln\left(\frac{|x-x_+|}{d^*}\right), & \text{if } x\in\Omega\cap(B_*^+\setminus B_\varepsilon^+). \end{cases}$$

Let $\alpha(\varepsilon) \in \mathbb{R}$ be defined by

$$\alpha(\varepsilon) := \frac{|\partial\Omega \cap B_\varepsilon^+|\rho_+ \ln\left(\frac{\varepsilon}{d^*}\right) + \varepsilon \int_{\partial\Omega\cap(B_*^+\setminus B_\varepsilon^+)} \ln\left(\frac{|x-x_+|}{d^*}\right) d\sigma}{|\partial\Omega \cap B_\varepsilon^-|\rho_- \ln(\frac{\varepsilon}{d^*}) + \varepsilon \int_{\partial\Omega\cap(B_*^-\setminus B_\varepsilon^-)} \ln\left(\frac{|x-x_-|}{d^*}\right) d\sigma},$$

for $\varepsilon \in]0, \varepsilon_0[$. We define the function u_ε^- by

$$u_\varepsilon^-(x) := \begin{cases} \alpha(\varepsilon) \ln\left(\frac{\varepsilon}{d^*}\right), & \text{if } x \in \Omega \cap B_\varepsilon^-, \\ \alpha(\varepsilon) \ln\left(\frac{|x-x_-|}{d^*}\right), & \text{if } x \in \Omega \cap (B_*^- \setminus B_\varepsilon^-). \end{cases}$$

Let the function u_ε from Ω to $\mathbb{R}$ be defined by

$$u_\varepsilon(x) := \begin{cases} u_\varepsilon^+(x), & \text{if } x \in \Omega \cap (B_*^+), \\ u_\varepsilon^-(x), & \text{if } x \in \Omega \cap (B_*^-), \\ 0, & \text{if } x \in \Omega \setminus (B_*^+ \cup B_*^-). \end{cases}$$

By definition $\int_{\partial\Omega} \rho_\varepsilon u_\varepsilon = 0$ for all $\varepsilon \in]0, \varepsilon_0[$, where ε_0 is small enough.

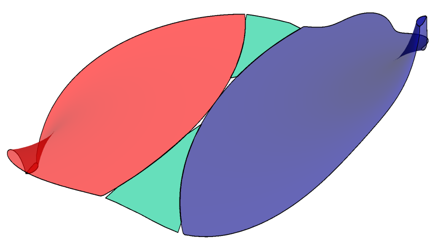


Figure 3.6: Trial function u_ε with $\varepsilon = 0.05$.

We need more information on the coefficient $\alpha(\varepsilon)$. We note that $\alpha(\varepsilon) \geq 0$ for all $\varepsilon \in]0, \varepsilon_0[$. Moreover, we note that

$$\begin{aligned} \alpha(\varepsilon) &\leq \frac{|\partial\Omega \cap B_\varepsilon^+|\rho_+ \ln\left(\frac{\varepsilon}{d^*}\right)}{|\partial\Omega \cap B_\varepsilon^-|\rho_- \ln(\frac{\varepsilon}{d^*}) + |\partial\Omega \cap (B_*^- \setminus B_\varepsilon^-)|\varepsilon \ln\left(\frac{\varepsilon}{d^*}\right)} \\ &\leq \frac{|\partial\Omega \cap B_\varepsilon^+|\rho_+ \ln\left(\frac{\varepsilon}{d^*}\right)}{|\partial\Omega \cap B_\varepsilon^-|\rho_- \ln(\frac{\varepsilon}{d^*})} + O(\varepsilon \ln(\varepsilon)) = 1 + O(\varepsilon \ln(\varepsilon)), \end{aligned}$$

as $\varepsilon \to 0$. Hence, if ε_0 is small enough, $\alpha(\varepsilon)$ is strictly positive and bounded away from zero and infinity, uniformly in $\varepsilon \in]0, \varepsilon_0[$. Moreover $\lim_{\varepsilon\to 0} \alpha(\varepsilon) = 1$, see the second equation in (3.3.18). Then u_ε is well-defined and belongs to $H^1(\Omega)$. Therefore u_ε is a suitable test function. We have

$$\begin{aligned}
&\lambda_2[\rho_\varepsilon] \\
&\quad\leq \frac{\int_{\Omega\cap B_*^+} |\nabla u_\varepsilon|^2 dx + \int_{\Omega\cap B_*^-} |\nabla u_\varepsilon|^2 dx}{\int_{\partial\Omega\cap B_\varepsilon^+} \rho_+ u_\varepsilon^2 d\sigma + \int_{\partial\Omega\cap B_\varepsilon^-} \rho_- u_\varepsilon^2 d\sigma + \varepsilon \int_{\partial\Omega\cap(B_*^+\setminus B_\varepsilon^+)} u_\varepsilon^2 d\sigma + \varepsilon \int_{\partial\Omega\cap(B_*^-\setminus B_\varepsilon^-)} u_\varepsilon^2 d\sigma} \\
&\leq \frac{\int_{B_*^+} |\nabla u_\varepsilon|^2 dx + \int_{B_*^-} |\nabla u_\varepsilon|^2 dx}{\left(\rho_+|\partial\Omega\cap B_\varepsilon^+| + \alpha(\varepsilon)^2\rho_-|\partial\Omega\cap B_\varepsilon^-|\right) \ln\left(\frac{\varepsilon}{d^*}\right)^2 + \varepsilon \int_{\partial\Omega\cap(B_*^+\setminus B_\varepsilon^+)} u_\varepsilon^2 d\sigma + \varepsilon \int_{\partial\Omega\cap(B_*^-\setminus B_\varepsilon^-)} u_\varepsilon^2 d\sigma} \\
&\qquad\qquad \leq \frac{(\alpha(\varepsilon)^2+1)\int_{B_*^+} |\nabla u_\varepsilon|^2 dx}{\left(\rho_+|\partial\Omega\cap B_\varepsilon^+| + \rho_-\alpha(\varepsilon)^2|\partial\Omega\cap B_\varepsilon^-|\right) \ln\left(\frac{\varepsilon}{d^*}\right)^2} \\
&\qquad\qquad = -\frac{2\pi\left(\alpha(\varepsilon)^2+1\right)\ln\left(\frac{\varepsilon}{d^*}\right)}{\left(\frac{M-|\partial\Omega\setminus(B_\varepsilon^+\cup B_\varepsilon^-)|\varepsilon}{2}\right)(\alpha(\varepsilon)^2+1)\ln\left(\frac{\varepsilon}{d^*}\right)^2} \\
&\qquad\qquad\qquad = \frac{4\pi}{|\ln\left(\frac{\varepsilon}{d^*}\right)|\left(M - |\partial\Omega\setminus(B_\varepsilon^+\cup B_\varepsilon^-)|\right)}. \qquad (3.3.21)
\end{aligned}$$

We use a suitable Taylor's expansion in (3.3.21) and obtain

$$\lambda_2[\rho_\varepsilon] \leq \frac{4\pi}{M|\log\left(\frac{\varepsilon}{d^*}\right)|} + \frac{4\pi\varepsilon|\partial\Omega\setminus(B_\varepsilon^+\cup B_\varepsilon^-)|}{M^2|\ln\left(\frac{\varepsilon}{d^*}\right)|} + O\left(\frac{\varepsilon^2}{\ln(\varepsilon)}\right),$$

as $\varepsilon \to 0$ (see Figure 3.7). This yields formula (3.3.20). □

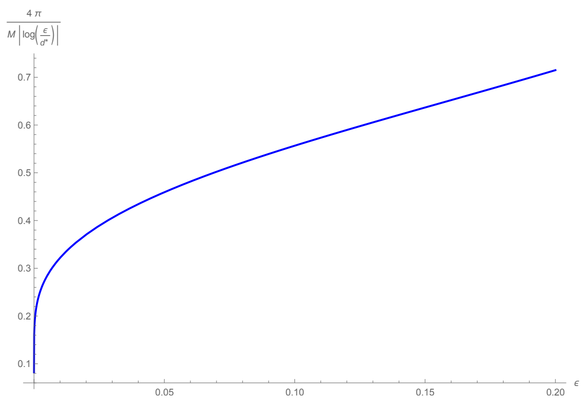


Figure 3.7: Plot of $\frac{4\pi}{M|\log(\frac{\varepsilon}{d^*})|}$ with $M = 2.9$, $d^* = 1$, $\varepsilon \in]0, 0.2[$.

3.4 On the optimization of the first positive Dirichlet and Neumann eigenvalues

In this section we consider the eigenvalue problem for the Laplace operator subject to Dirichlet and Neumann boundary conditions and with density

$\rho \in \mathcal{R}$, where $\mathcal{R}$ is given by (3.1.11). We show that there exist densities preserving the total mass such that the first positive Dirichlet eigenvalue is arbitrarily close to zero or arbitrary large. Also, we prove that there exist densities preserving the total mass such that the first positive Neumann eigenvalue is arbitrarily close to zero. The maximization of the first Neumann eigenvalue among all densities preserving the total mass seems to be a more complicated issue. Actually, we have been informed that this problem has been solved by means of techniques from differential geometry. The answer is that there exists a uniform upper bound for the Neumann eigenvalues under the constraint that the mass is fixed. It is still unclear if such an upper bound is attained or not. We refer to the paper [30] for a geometric approach to the study of the spectrum of the Laplacian.

3.4.1 Optimization of the first Dirichlet eigenvalue

Let Ω be a domain in $\mathbb{R}^N$ with finite measure, $\rho \in \mathcal{R}$ and $M > 0$ be a fixed number. We consider the Dirichlet eigenvalue problem for the Laplace operator

$$\begin{cases} -\Delta u = \lambda \rho u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.4.1}$$

in the unknowns u (the eigenfunction) and λ (the eigenvalue). It is well-known that for all $\rho \in \mathcal{R}$, problem (3.4.1) admits a diverging sequence of positive eigenvalues of finite multiplicity

$$0 < \lambda_1[\rho] < \lambda_2[\rho] \leq \cdots \leq \lambda_j[\rho] \leq \cdots .$$

The first eigenvalue $\lambda_1[\rho]$ is positive and simple, and an eigenfunction associated with $\lambda_1[\rho]$ does not change sign in Ω.

The aim of this subsection is to prove that there exist sequences $\hat{\rho}_\varepsilon$ and $\check{\rho}_\varepsilon$ in $\mathcal{R}$ such that $\int_\Omega \hat{\rho}_\varepsilon dx = \int_\Omega \check{\rho}_\varepsilon dx = M$ for all $\varepsilon \in]0, \varepsilon_0[$, and $\lambda_1[\hat{\rho}_\varepsilon] \to +\infty$, $\lambda_1[\check{\rho}_\varepsilon] \to 0$, as $\varepsilon \to 0$. Thus, the problems

$$\max_{\substack{\rho \in \mathcal{R} \\ \int_\Omega \rho dx = M}} \lambda_1[\rho]$$

and

$$\min_{\substack{\rho \in \mathcal{R} \\ \int_\Omega \rho dx = M}} \lambda_1[\rho]$$

have no solutions.

We start with the problem of the maximization. In this case we need the additional assumption that Ω is a bounded domain of class C^2. Let $\hat{\rho}_\varepsilon = \rho_\varepsilon$, where ρ_ε is defined by (3.1.20). We note that $\int_\Omega \hat{\rho}_\varepsilon dx = M$ for all $\varepsilon \in]0, \varepsilon_0[$, for a suitable $\varepsilon_0 > 0$ small. We have the following theorem.

Theorem 3.4.2. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^2. Let $\hat{\rho}_\varepsilon$ be defined by* (3.1.20)*. Let $\lambda_1[\hat{\rho}_\varepsilon]$ be the first eigenvalue of problem* (3.4.1) *with $\rho = \hat{\rho}_\varepsilon$. Then*

$$\lim_{\varepsilon\to 0} \lambda_1[\hat{\rho}_\varepsilon] = +\infty.$$

Proof. For all $\varepsilon \in]0, \varepsilon_0[$, with $\varepsilon_0 > 0$ small enough, let $u_\varepsilon \in H^1_0(\Omega)$ be the unique eigenfunction associated with $\lambda_1[\hat{\rho}_\varepsilon]$ such that $\int_\Omega \hat{\rho}_\varepsilon u_\varepsilon^2 dx = 1$. Then it holds

$$\lambda_1[\hat{\rho}_\varepsilon] = \int_\Omega |\nabla u_\varepsilon|^2 dx,$$

for all $\varepsilon \in]0, \varepsilon_0[$. Assume by contradiction that there exists a constant $C > 0$ which does not depend on $\varepsilon > 0$, such that $\lambda_1[\hat{\rho}_\varepsilon] \leq C$, for all $\varepsilon \in]0, \varepsilon_0[$. Thus, $\int_\Omega |\nabla u_\varepsilon|^2 dx \leq C$ and from the Poincaré inequality, $\|u_\varepsilon\|_{L^2(\Omega)} \leq C'$, for a constant $C' > 0$ which does not depend on $\varepsilon > 0$. Then there exists $u \in H^1_0(\Omega)$ such that, possibly passing to a subsequence, $u_\varepsilon \rightharpoonup u$ in $H^1_0(\Omega)$, as $\varepsilon \to 0$, and $u_\varepsilon \to u$ in $L^2(\Omega)$, as $\varepsilon \to 0$. From Lemma 3.1.22 it follows that $\lim_{\varepsilon\to 0} \int_\Omega \hat{\rho}_\varepsilon u_\varepsilon^2 dx = \frac{M}{|\partial\Omega|} \int_{\partial\Omega} u^2 d\sigma = 0$, which is a contradiction with the fact that $\int_\Omega \hat{\rho}_\varepsilon u_\varepsilon^2 dx = 1$ for all $\varepsilon \in]0, \varepsilon_0[$. Thus $\lambda_1[\hat{\rho}_\varepsilon] \to +\infty$ as $\varepsilon \to 0$. This concludes the proof of the theorem. $\square$

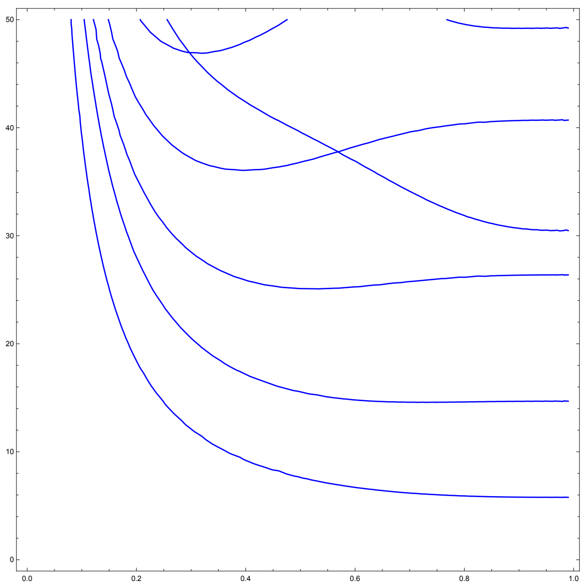


Figure 3.8: Eigenvalues of problem (3.4.1) on the unit ball in $\mathbb{R}^2$ with $\rho = \hat{\rho}_\varepsilon$, $M = \pi$, $\varepsilon \in]0, 1[$.

Now we consider the problem of the minimization of the first Dirichlet eigenvalue. Let Ω be a domain in $\mathbb{R}^N$ with finite measure. Let $x_0 \in \Omega$ and B_0 be a ball centered at x_0 such that $\overline{B}_0 \subset \Omega$. We denote by r_0 the radius of B_0. For all $\varepsilon \in]0, r_0/2[$, let B_ε the ball of radius ε centered at x_0. Let the function $\check{\rho}_\varepsilon \in L^\infty(\Omega)$ be defined by

$$\check{\rho}_\varepsilon(x) := \begin{cases} \frac{M-\varepsilon(|\Omega|-\varepsilon^N\omega_N)}{\varepsilon^N\omega_N}, & \text{if } x \in B_\varepsilon, \\ \varepsilon, & \text{if } x \in \Omega \setminus \overline{B}_\varepsilon. \end{cases} \tag{3.4.3}$$

By definition, $\int_\Omega \check{\rho}_\varepsilon dx = M$ for all $\varepsilon \in]0, r_0/2[$.

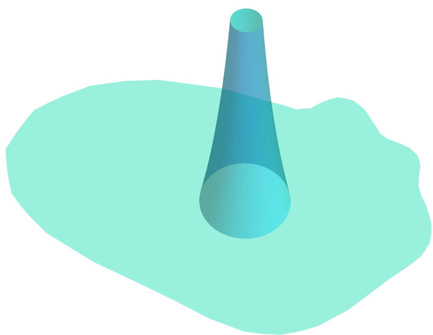


Figure 3.9: Test function u_ε on $\Omega \subset \mathbb{R}^2$.

We have the following theorem.

Theorem 3.4.4. *Let Ω be a domain in $\mathbb{R}^N$ with finite measure. Let $\check{\rho}_\varepsilon$ be defined by* (3.4.3)*. Let $\lambda_1[\check{\rho}_\varepsilon]$ be the first eigenvalue of problem* (3.4.1) *with $\rho = \check{\rho}_\varepsilon$. Then*

$$\lim_{\varepsilon \to 0} \lambda_1[\check{\rho}_\varepsilon] = 0. \tag{3.4.5}$$

Proof. We distinguish the case $N = 2$ and $N \geq 3$. We start by proving (3.4.5) in the case $N = 2$. Let $u_\varepsilon \in H_0^1(\Omega)$ be defined by

$$u_\varepsilon(x) := \begin{cases} -\ln\left(\frac{\varepsilon}{r_0}\right), & \text{if } x \in B_\varepsilon, \\ -\ln\left(\frac{|x-x_0|}{r_0}\right), & \text{if } x \in B_0 \setminus \overline{B}_\varepsilon, \\ 0, & \text{if } x \in \Omega \setminus \overline{B}_0. \end{cases}$$

By definition, $u_\varepsilon \in H^1_0(\Omega)$ for all $\varepsilon \in]0, r_0/2[$. From the Min-Max principle we have that

$$\lambda_1[\check\rho_\varepsilon] \le \frac{\int_\Omega |\nabla u_\varepsilon|^2 dx}{\int_\Omega \check\rho_\varepsilon u_\varepsilon^2 dx}. \tag{3.4.6}$$

It is convenient to use standard polar coordinates $(r,\theta) \in \mathbb{R}_+ \times \partial B$ in $\mathbb{R}^2$ with respect to the point x_0. We have for the numerator in (3.4.6)

$$\int_\Omega |\nabla u_\varepsilon|^2 dx = \int_0^{2\pi} \int_\varepsilon^{r_0} \frac{1}{r} dr d\theta = 2\pi \ln\left(\frac{r_0}{\varepsilon}\right). \tag{3.4.7}$$

For the denominator we have

$$\begin{aligned}
&\int_\Omega \check\rho_\varepsilon u_\varepsilon^2 dx \\
&= \varepsilon \int_0^{2\pi} \int_\varepsilon^{r_0} \left(\ln\left(\frac{r}{r_0}\right)\right)^2 r dr d\theta + \frac{M - \varepsilon(|\Omega| - \pi\varepsilon^2)}{\pi\varepsilon^2} \int_0^{2\pi} \int_0^{\varepsilon} \left(\ln\left(\frac{\varepsilon}{r_0}\right)\right)^2 r dr d\theta \\
&= \frac{\pi}{2}\left(r_0^2 - \varepsilon^2 - 2\varepsilon^2 \ln\left(\frac{r_0}{\varepsilon}\right)\left(1 + \ln\left(\frac{r_0}{\varepsilon}\right)\right)\right)\varepsilon + \left(M - \varepsilon(|\Omega| - \pi\varepsilon^2)\right)\left(\ln\left(\frac{\varepsilon}{r_0}\right)\right)^2.
\end{aligned} \tag{3.4.8}$$

Therefore, from (3.4.6), (3.4.7) and (3.4.8) we have

$$\begin{aligned}
&\lambda_1[\check\rho_\varepsilon] \\
&\quad \le \frac{2\pi \ln\left(\frac{r_0}{\varepsilon}\right)}{\frac{\pi}{2}\left(r_0^2 - \varepsilon^2 - 2\varepsilon^2 \ln\left(\frac{r_0}{\varepsilon}\right)\left(1 + \ln\left(\frac{r_0}{\varepsilon}\right)\right)\right)\varepsilon + \left(M - \varepsilon(|\Omega| - \pi\varepsilon^2)\right)\left(\ln\left(\frac{\varepsilon}{r_0}\right)\right)^2}.
\end{aligned} \tag{3.4.9}$$

The right-hand side of (3.4.9) clearly goes to zero as $\varepsilon \to 0$. In fact, we can make a Taylor expansion and obtain that the right-hand side of (3.4.9) equals

$$\frac{2\pi}{M\,|\ln(\varepsilon/r_0)|} + O\left(\frac{\varepsilon}{|\ln(\varepsilon)|}\right),$$

as $\varepsilon \to 0$. This concludes the proof in the case $N = 2$.

Consider now the case $N \ge 3$. Let $u_\varepsilon \in H^1_0(\Omega)$ be defined by

$$u_\varepsilon(x) := \begin{cases} \frac{1}{\varepsilon^{N-2}} - \frac{1}{r_0^{N-2}}, & \text{if } x \in B_\varepsilon, \\ \frac{1}{|x-x_0|^{N-2}} - \frac{1}{r_0^{N-2}}, & \text{if } x \in B_0 \setminus \overline{B}_\varepsilon, \\ 0, & \text{if } x \in \Omega \setminus \overline{B}_0. \end{cases}$$

By definition, $u_\varepsilon \in H^1_0(\Omega)$ for all $\varepsilon \in]0, r_0/2[$. From the Min-Max principle, $\lambda_1[\check\rho_\varepsilon]$ satisfies the inequality (3.4.6).

It is convenient to use standard spherical coordinates $(r,\theta) \in \mathbb{R}_+ \times \partial B$ in $\mathbb{R}^N$ with respect to the point x_0. We have for the numerator in (3.4.6)

$$\begin{aligned}\int_\Omega |\nabla u_\varepsilon|^2 dx \\ &= \int_{\partial B}\int_\varepsilon^{r_0} ((2-N)r^{1-N})^2 r^{N-1} dr d\sigma = N\omega_N(N-2)(\varepsilon^{2-N} - r_0^{2-N}).\end{aligned} \tag{3.4.10}$$

For the denominator we have

$$\begin{aligned}\int_\Omega \check{\rho}_\varepsilon u_\varepsilon^2 dx &= \varepsilon \int_{\partial B}\int_\varepsilon^{r_0} \left(\frac{1}{r^{N-2}} - \frac{1}{r_0^{N-2}}\right)^2 r^{N-1} dr d\sigma \\ &\quad + \frac{M - \varepsilon(|\Omega| - \omega_N\varepsilon^N)}{\omega_N \varepsilon^N}\int_{\partial B}\int_0^\varepsilon \left(\frac{1}{\varepsilon^{N-2}} - \frac{1}{r_0^{N-2}}\right)^2 r^{N-1} dr d\sigma \\ &= \omega_N \left(N r_0^{2-N}\varepsilon^2 + \frac{N}{N-4}\varepsilon^{4-N} - r_0^{4-N}\varepsilon^N - \frac{(N-2)^2}{N-4} r_0^{4-N}\right)\varepsilon \\ &\qquad + \left(M - \varepsilon(|\Omega| - \varepsilon^N\omega_N)\right)(r_0^{2-N} - \varepsilon^{2-N}),\end{aligned} \tag{3.4.11}$$

if $N \geq 3$ and $N \neq 4$, while

$$\begin{aligned}\int_\Omega \check{\rho}_\varepsilon u_\varepsilon^2 dx &= \varepsilon \int_{\partial B}\int_\varepsilon^{r_0} \left(\frac{1}{r^2} - \frac{1}{r_0^2}\right)^2 r^3 dr d\sigma \\ &\quad + \frac{M - \varepsilon(|\Omega| - \omega_4\varepsilon^4)}{\omega_4\varepsilon^4}\int_{\partial B}\int_0^\varepsilon \left(\frac{1}{\varepsilon^2} - \frac{1}{r_0^2}\right)^2 r^3 dr d\sigma \\ &= \frac{(r_0^2 - \varepsilon^2)((r_0^2-\varepsilon^2)(M - \varepsilon|\Omega|) - 2r_0^2\varepsilon^5\omega_4)}{r_0^4\varepsilon^4} - 4\varepsilon\omega_4 \ln\left(\frac{\varepsilon}{r_0}\right),\end{aligned} \tag{3.4.12}$$

if $N = 4$. Therefore, from (3.4.6), (3.4.10), (3.4.11) and (3.4.12), we have

$$\begin{aligned}&\lambda_1[\check{\rho}_\varepsilon] \\ &\leq \frac{N\omega_N(N-2)(\varepsilon^{2-N} - r_0^{2-N})}{\omega_N\left(N r_0^{2-N}\varepsilon^2 + \frac{N}{N-4}\varepsilon^{4-N} - r_0^{4-N}\varepsilon^N - \frac{(N-2)^2}{N-4}r_0^{4-N}\right)\varepsilon + \left(M - \varepsilon(|\Omega| - \varepsilon^N\omega_N)\right)(r_0^{2-N} - \varepsilon^{2-N})}\end{aligned} \tag{3.4.13}$$

if $N \geq 3$, $N \neq 4$, and

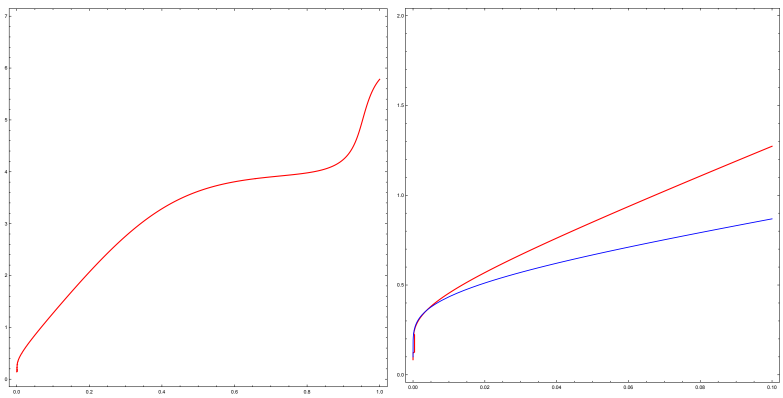
$$\lambda_1[\check{\rho}_\varepsilon] \leq \frac{8\omega_4(\varepsilon^{-2} - r_0^{-2})}{\frac{(r_0^2-\varepsilon^2)((r_0^2-\varepsilon^2)(M-\varepsilon|\Omega|)-2r_0^2\varepsilon^5\omega_4)}{r_0^4\varepsilon^4} - 4\varepsilon\omega_4\ln\left(\frac{\varepsilon}{r_0}\right)}, \tag{3.4.14}$$

if $N = 4$.
The right-hand side of both (3.4.13) and (3.4.14) clearly goes to zero as

$\varepsilon \to 0$. In fact, we can make a Taylor expansion and obtain that the right-hand side of both (3.4.13) and (3.4.14) equals

$$\frac{N(N-2)}{M}\varepsilon^{N-2} + O(\varepsilon^{N-1}),$$

as $\varepsilon \to 0$. This concludes the proof in the case $N \geq 3$. $\square$



(a) $\lambda_1[\check{\rho}_\varepsilon]$ on the unit ball in $\mathbb{R}^2$, $M = \pi$, $\varepsilon \in]0,1[$.

(b) $\lambda_1[\check{\rho}_\varepsilon]$ (red) and $\frac{2}{|\ln(\varepsilon)|}$ (blue), $\varepsilon \in]0, 0.1[$.

3.4.2 Minimization of the first positive Neumann eigenvalue

Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1 and $M>0$ be a fixed number. We consider the eigenvalue problem (3.1.12). We recall that the first eigenvalue $\lambda_1[\rho]$ of problem (3.1.12) is zero, and the corresponding eigenfunctions are the constant functions on Ω. The second eigenvalue $\lambda_2[\rho]$ is positive. The aim of this subsection is to prove that there exists a sequence $\rho_\varepsilon\in\mathcal{R}$ such that $\int_\Omega \rho_\varepsilon dx = M$ for all $\varepsilon\in]0,\varepsilon_0[$, for a suitable $\varepsilon_0>0$, and such that $\lambda_2[\rho_\varepsilon]\to 0$ as $\varepsilon\to 0$. Thus the problem

$$\min_{\substack{\rho\in\mathcal{R}\\ \int_\Omega \rho dx = M}} \lambda_2[\rho]$$

has no solutions.

Let $x_1,x_2\in\Omega$, $x_1\neq x_2$, and let $r_0>0$ be such that $\overline{B}_1\subset\Omega$, $\overline{B}_2\subset\Omega$ and $B_1\cap B_2=\emptyset$, where B_1 and B_2 are balls of radius r_0 centered at x_1 and x_2, respectively. Let $\varepsilon\in]0,r_0/2[$ and let $B_{\varepsilon,1}$, $B_{\varepsilon,2}$ be the balls of radius ε centered at x_1 and x_2, respectively. We introduce the function $\rho_\varepsilon\in\mathcal{R}$ defined by

$$\rho_\varepsilon(x) := \begin{cases} \varepsilon, & \text{if } x\in\Omega\setminus\overline{(B_{\varepsilon,1}\cup B_{\varepsilon_2})},\\ \frac{M-\varepsilon(|\Omega|-2\varepsilon^N\omega_N)}{2\varepsilon^N\omega_N}, & \text{if } x\in B_{\varepsilon,1}\cup B_{\varepsilon,2}, \end{cases} \tag{3.4.15}$$

for all $\varepsilon\in]0,r_0/2[$. We note that $\int_\Omega\rho_\varepsilon dx = M$ for all $\varepsilon\in]0,r_0/2[$.
We consider problem (3.1.12) with $\rho=\rho_\varepsilon$. We have the following theorem.

Theorem 3.4.16. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. Let ρ_ε be defined by* (3.4.15)*. Let $\lambda_2[\rho_\varepsilon]$ be the second eigenvalue of problem* (3.1.12) *with $\rho=\rho_\varepsilon$. Then*

$$\lim_{\varepsilon\to 0}\lambda_2[\rho_\varepsilon]=0. \tag{3.4.17}$$

Proof. Let $u_\varepsilon\in H^1(\Omega)$ be defined by

$$u_\varepsilon(x) := \begin{cases} -\ln\left(\frac{\varepsilon}{r_0}\right), & \text{if } x\in B_{\varepsilon,1},\\ \ln\left(\frac{\varepsilon}{r_0}\right), & \text{if } x\in B_{\varepsilon,2},\\ -\ln\left(\frac{|x-x_1|}{r_0}\right), & \text{if } x\in B_1\setminus\overline{B}_{\varepsilon,1},\\ \ln\left(\frac{|x-x_2|}{r_0}\right), & \text{if } x\in B_2\setminus\overline{B}_{\varepsilon,2},\\ 0, & \text{if } x\in\Omega\setminus\overline{(B_1\cup B_2)}, \end{cases}$$

if $N = 2$, and by

$$u_\varepsilon(x) := \begin{cases} \frac{1}{\varepsilon^{N-2}} - \frac{1}{r_0^{N-2}}, & \text{if } x \in B_{\varepsilon,1}, \\ -\frac{1}{\varepsilon^{N-2}} + \frac{1}{r_0^{N-2}}, & \text{if } x \in B_{\varepsilon,2}, \\ \frac{1}{|x-x_1|^{N-2}} - \frac{1}{r_0^{N-2}}, & \text{if } x \in B_1 \setminus \overline{B}_{\varepsilon,1}, \\ -\frac{1}{|x-x_2|^{N-2}} + \frac{1}{r_0^{N-2}}, & \text{if } x \in B_2 \setminus \overline{B}_{\varepsilon,2}, \\ 0, & \text{if } x \in \Omega \setminus \overline{(B_1 \cup B_2)}, \end{cases}$$

if $N \geq 3$. Clearly $u_\varepsilon \in H^1(\Omega)$ for all $\varepsilon \in]0, r_0/2[$. Moreover, by definition, $\int_\Omega \rho_\varepsilon u_\varepsilon dx = 0$ for all $\varepsilon \in]0, r_0/2[$.

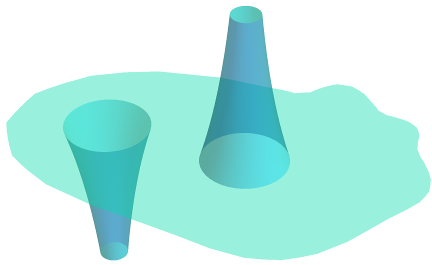


Figure 3.11: Test function u_ε on $\Omega \subset \mathbb{R}^2$.

We recall that by the Min-Max principle we have

$$\lambda_2[\rho_\varepsilon] = \min_{\substack{u \in H^1(\Omega) \\ \int_\Omega \rho_\varepsilon dx = 0}} \frac{\int_\Omega |\nabla u_\varepsilon|^2 dx}{\int_\Omega \rho_\varepsilon u_\varepsilon^2 dx}.$$

Then it follows that

$$\lambda_2[\rho_\varepsilon] \leq \frac{\int_\Omega |\nabla u_\varepsilon|^2}{\int_\Omega \rho_\varepsilon u_\varepsilon^2 dx},$$

for all $\varepsilon \in]0, r_0/2[$. Now the proof of (3.4.17) follows the same lines as the proof of formula (3.4.5) in Theorem (3.4.4), and is accordingly omitted. In particular, we prove that

$$\lim_{\varepsilon \to 0} \frac{\int_\Omega |\nabla u_\varepsilon|^2 dx}{\int_\Omega \rho_\varepsilon u_\varepsilon^2 dx} = 0,$$

which implies the validity of (3.4.17). □

Chapter 4

Neumann and Steklov problems: an asymptotic analysis

As we have highlighted in Chapter 3 (Theorem 3.1.21 and Corollary 3.1.42), we can consider the Steklov eigenvalues of the Laplace operator as limiting Neumann eigenvalues in a problem of mass concentration at the boundary of a bounded domain of class C^2 in $\mathbb{R}^N$. Through this chapter we shall use a slightly different notation for the eigenvalues and the eigenfunctions of problems (3.1.2) and (3.1.12) with respect to that used in Chapter 3. The notation used in this chapter is more convenient in view of the analysis which we shall carry out. We consider problem (3.1.2) with constant density $\rho \equiv \frac{M}{|\partial\Omega|}$, namely

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{M}{|\partial\Omega|}\mu u, & \text{on } \partial\Omega, \end{cases} \tag{4.0.1}$$

in the unknowns μ (the eigenvalue) and u (the eigenfunction), where $M > 0$ is a fixed constant. We consider also problem (3.1.12) with density $\rho = \rho_\varepsilon$, where ρ_ε is given by (3.1.20), namely

$$\begin{cases} -\Delta u = \lambda \rho_\varepsilon u, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.0.2}$$

in the unknowns λ (the eigenvalue) and u (the eigenfunction).

In this chapter we discuss the asymptotic behavior of the Neumann eigenvalues and find explicit formulas for their derivatives at the limiting problem in the case of the open unit ball in $\mathbb{R}^N$ (Section 4.1) and of general bounded domain of class C^2 in $\mathbb{R}^2$ (Section 4.2). In particular, in the case of the ball in $\mathbb{R}^N$ we deduce that the Neumann eigenvalues have a monotone behavior in the limit and that Steklov eigenvalues locally minimize the Neumann eigenvalues.

The techniques used in Section 4.1 and Section 4.2 are completely different. In Section 4.1 we use the explicit form of the solutions of problem (4.0.2) on the unit ball in terms of Bessel functions. In this way problem (4.0.2) is recasted in the form of an equation $F(\lambda, \varepsilon) = 0$. Then we carry out the analysis of this implicit equation and find a formula for the derivative of the eigenvalues at $\varepsilon = 0$. In Section 4.2 we use classical techniques of asymptotic analysis in the spirit of [50, 52, 53]. In particular, we postulate an asymptotic expansion of the eigenelements of problem (4.0.2) and justify the expansion up to the first order.

We remark that the techniques used in Section 4.1 allow to overcome the problem of the multiplicity of the eigenvalues, which is usually an obstruction in the application of the techniques of asymptotic analysis. In fact, the analysis carried out in Section 4.1 involves all multiple eigenvalues. On the other hand, in Section 4.2 the techniques of asymptotic analysis which we used allow to deal only with simple eigenvalues.

4.1 The case of the ball in $\mathbb{R}^N$

Let B be the open unit ball centered at zero in $\mathbb{R}^N$, $N \geq 2$. We denote by ω_N the measure of B. Therefore, the $(N-1)$-dimensional measure of ∂B is given by $N\omega_N$. In this section we consider problem (4.0.1) with $\Omega = B$. As is well-known (see also Corollary 3.2.6) the eigenvalues of problem (4.0.1) on the unit ball with constant density $\frac{M}{N\omega_N}$ are given explicitly by the sequence

$$\mu_l = \frac{N\omega_N}{M} l, \qquad l \in \mathbb{N}_0, \tag{4.1.1}$$

and the eigenfunctions corresponding to μ_l are the homogeneous harmonic polynomials of degree l. In particular, the multiplicity of μ_l is $(2l+N-2)(l+N-3)!/(l!(N-2)!)$, and only μ_0 is simple, the corresponding eigenfunctions being the constant functions.

As in Chapter 3, for any $\varepsilon \in]0,1[$, we consider a mass density ρ_ε in the whole of B defined by (3.1.20). We recall that for any $x \in B$ we have $\rho_\varepsilon(x) \to 0$ as $\varepsilon \to 0$, and $\int_B \rho_\varepsilon dx = M$ for all $\varepsilon > 0$, which means that the total mass M is fixed and concentrates at the boundary of B as $\varepsilon \to 0$. Then we consider problem (4.0.2) on B with $\rho = \rho_\varepsilon$.

The eigenvalues of (4.0.2) on B with density ρ_ε have finite multiplicity and form a sequence

$$\lambda_0(\varepsilon) < \lambda_1(\varepsilon) \leq \lambda_2(\varepsilon) \leq \cdots,$$

depending on ε, with $\lambda_0(\varepsilon) = 0$. Recall that by Corollary 3.1.42 we have that for any $l \in \mathbb{N}_0$

$$\lambda_l(\varepsilon) \to \mu_l, \quad \text{as } \varepsilon \to 0, \tag{4.1.2}$$

(see also [10], [72]).

In this section we study the asymptotic behavior of $\lambda_l(\varepsilon)$ as $\varepsilon \to 0$. Namely, we prove that such eigenvalues are continuously differentiable with respect to ε for $\varepsilon \geq 0$ small enough, and that the following formula holds

$$\lambda_l'(0) = \frac{2l\mu_l}{3} + \frac{2\mu_l^2}{N(2l+N)}. \tag{4.1.3}$$

In particular, for $l \neq 0$, $\lambda_l'(0) > 0$ hence $\lambda_l(\varepsilon)$ is strictly increasing and the Steklov eigenvalues μ_l minimize the Neumann eigenvalues $\lambda_l(\varepsilon)$ for ε small enough. We note that our analysis concerns all eigenvalues μ_l with arbitrary indexes and multiplicity, and that we do not prove global monotonocity of $\lambda_l(\varepsilon)$, which in fact does not hold for any l; see Figures 4.1, 4.2.

In Subsection 4.1.1 we prove formula (4.1.3). The proof of the results relies on the use of Bessel functions which allows to recast problem (4.0.2) in the form of an equation $F(\lambda, \varepsilon) = 0$ in the unknowns λ, ε. Then, after some preparatory work, it is possible to apply the Implicit Function Theorem and conclude. The application of this method requires suitable Taylor's expansions and estimates for the corresponding remainders, as well as recursive formulas for the cross-products of Bessel functions and their derivatives. The estimates for the remainders are contained in Subsection 4.1.2. In Subsection 4.1.3 we consider the case $N = 1$. In Subsection 4.1.4 we study the behavior of the eigenvalues and their derivatives under dilations of the domain. In Subsection 4.1.5 we establish recursive formulas for the cross-products of Bessel functions and their derivatives used in Subsection 4.1.1.

4.1.1 Asymptotic behavior of Neumann eigenvalues

In the sequel we shall use the standard spherical coordinates $(r, \theta) \in \mathbb{R}_+ \times \partial B$ in $\mathbb{R}^N$.

We note that, when $d(x, \partial B) < \varepsilon$, we have $\rho_\varepsilon(x) = \frac{M - \varepsilon\omega_N(1-\varepsilon)^N}{\omega_N(1-(1-\varepsilon)^N)}$. To shorten notation, in what follows we will denote by a and b the quantities defined by

$$a := \sqrt{\lambda}\varepsilon(1-\varepsilon), \quad \text{and} \quad b := \sqrt{\lambda\tilde{\rho}_\varepsilon}(1-\varepsilon),$$

where

$$\tilde{\rho}_\varepsilon := \frac{M - \varepsilon\omega_N(1-\varepsilon)^N}{\omega_N\left(1-(1-\varepsilon)^N\right)}.$$

We have the following lemma.

Lemma 4.1.4. *Given an eigenvalue λ of problem (4.0.2), a corresponding eigenfunction u is of the form $u(r,\theta) = S_l(r)H_l(\theta)$ where $H_l(\theta)$ is a spherical harmonic of some order $l \in \mathbb{N}_0$ and*

$$S_l(r) = \begin{cases} r^{1-\frac{N}{2}} J_{\nu_l}(\sqrt{\lambda\varepsilon} r), & \text{if } r < 1-\varepsilon, \\ r^{1-\frac{N}{2}} \left(\alpha J_{\nu_l}(\sqrt{\lambda\tilde{\rho}_\varepsilon} r) + \beta Y_{\nu_l}(\sqrt{\lambda\tilde{\rho}_\varepsilon} r)\right), & \text{if } 1-\varepsilon < r < 1, \end{cases} \tag{4.1.5}$$

where $\nu_l := \frac{(N+2l-2)}{2}$ *and* α, β *are given by*

$$\alpha = \frac{\pi b}{2}\left(J_{\nu_l}(a)Y'_{\nu_l}(b) - \frac{a}{b}J'_{\nu_l}(a)Y_{\nu_l}(b)\right)$$
$$\beta = \frac{\pi b}{2}\left(\frac{a}{b}J_{\nu_l}(b)J'_{\nu_l}(a) - J'_{\nu_l}(b)J_{\nu_l}(a)\right).$$

Proof. Recall that the Laplace operator can be written in spherical coordinates as

$$\Delta = \partial_{rr} + \frac{N-1}{r}\partial_r + \frac{1}{r^2}\Delta_S,$$

where Δ_S is the Laplace-Beltrami operator on the unit sphere ∂B of $\mathbb{R}^N$. In order to solve the equation $-\Delta u = \lambda\rho_\varepsilon u$, we separate variables so that $u(r,\theta) = S(r)H(\theta)$. Then using $l(l+N-2)$, $l \in \mathbb{N}_0$, as separation constant, we obtain the equations

$$r^2S'' + r(N-1)S' + r^2\lambda\rho_\varepsilon S - l(l+N-2)S = 0 \tag{4.1.6}$$

and

$$-\Delta_S H = l(l+N-2)H. \tag{4.1.7}$$

By setting $S(r) = r^{1-\frac{N}{2}}\tilde{S}(r)$ into (4.1.6), it follows that $\tilde{S}(r)$ satisfies the Bessel equation

$$\tilde{S}'' + \frac{1}{r}\tilde{S}' + \left(\lambda\rho_\varepsilon - \frac{\nu_l^2}{r^2}\right)\tilde{S} = 0.$$

Since the solutions u of problem (4.0.2) are bounded on Ω and $Y_{\nu_l}(z)$ blows up at $z=0$, it follows that for $r < 1-\varepsilon$, $S(r)$ is a multiple of the function $r^{1-\frac{N}{2}}J_{\nu_l}(\sqrt{\lambda\varepsilon}r)$. For $1-\varepsilon < r < 1$, $S(r)$ is a linear combination of the functions $r^{1-\frac{N}{2}}J_{\nu_l}(\sqrt{\lambda\tilde{\rho}_\varepsilon}r)$ and $r^{1-\frac{N}{2}}Y_{\nu_l}(\sqrt{\lambda\tilde{\rho}_\varepsilon}r)$. On the other hand, the solutions of equation (4.1.7) are the spherical harmonics of order l. Thus u can be written as in (4.1.5) for suitable values of $\alpha, \beta \in \mathbb{R}$.

Now it remains to compute the coefficient α and β in (4.1.5). Since the right-hand side of the equation in (4.0.2) is a function in $L^2(\Omega)$ then by standard regularity theory a solution u of (4.0.2) belongs to the standard Sobolev space $H^2(\Omega)$, hence α and β must be chosen in such a way that u and $\partial_r u$ are continuous at $r = 1-\varepsilon$, that is

$$\begin{cases} \alpha J_{\nu_l}(\sqrt{\lambda\tilde{\rho}_\varepsilon}(1-\varepsilon)) + \beta Y_{\nu_l}(\sqrt{\lambda\tilde{\rho}_\varepsilon}(1-\varepsilon)) = J_{\nu_l}(\sqrt{\lambda\varepsilon}(1-\varepsilon)), \\ \alpha J'_{\nu_l}(\sqrt{\lambda\tilde{\rho}_\varepsilon}(1-\varepsilon)) + \beta Y'_{\nu_l}(\sqrt{\lambda\tilde{\rho}_\varepsilon}(1-\varepsilon)) = \sqrt{\frac{\varepsilon}{\tilde{\rho}_\varepsilon}}J'_{\nu_l}(\sqrt{\lambda\varepsilon}(1-\varepsilon)). \end{cases}$$

Solving the system we obtain

$$\alpha = \frac{J_{\nu_l}(a)Y'_{\nu_l}(b) - \frac{a}{b}J'_{\nu_l}(a)Y_{\nu_l}(b)}{J_{\nu_l}(b)Y'_{\nu_l}(b) - J'_{\nu_l}(b)Y_{\nu_l}(b)}, \quad \beta = \frac{\frac{a}{b}J_{\nu_l}(b)J'_{\nu_l}(a) - J'_{\nu_l}(b)J_{\nu_l}(a)}{J_{\nu_l}(b)Y'_{\nu_l}(b) - J'_{\nu_l}(b)Y_{\nu_l}(b)}.$$

Note that $J_{\nu_l}(b)Y'_{\nu_l}(b) - J'_{\nu_l}(b)Y_{\nu_l}(b)$ is the Wronskian in b, which is known to be $\frac{2}{\pi b}$ (see Section 1.4, see also [1, §9]). This concludes the proof. □

We are ready to establish an implicit characterization of the eigenvalues of (4.0.2).

Lemma 4.1.8. *The nonzero eigenvalues λ of problem (4.0.2) are given implicitly as zeros of the equation*

$$\left(1-\frac{N}{2}\right)P_1(a,b)+\frac{b}{(1-\varepsilon)}P_2(a,b)=0, \tag{4.1.9}$$

where

$$P_1(a,b)=J_{\nu_l}(a)\left(Y'_{\nu_l}(b)J_{\nu_l}(\frac{b}{1-\varepsilon})-J'_{\nu_l}(b)Y_{\nu_l}(\frac{b}{1-\varepsilon})\right) + \frac{a}{b}J'_{\nu_l}(a)\left(J_{\nu_l}(b)Y_{\nu_l}(\frac{b}{1-\varepsilon})-Y_{\nu_l}(b)J_{\nu_l}(\frac{b}{1-\varepsilon})\right),$$

$$P_2(a,b)=J_{\nu_l}(a)\left(Y'_{\nu_l}(b)J'_{\nu_l}(\frac{b}{1-\varepsilon})-J'_{\nu_l}(b)Y'_{\nu_l}(\frac{b}{1-\varepsilon})\right) + \frac{a}{b}J'_{\nu_l}(a)\left(J_{\nu_l}(b)Y'_{\nu_l}(\frac{b}{1-\varepsilon})-Y_{\nu_l}(b)J'_{\nu_l}(\frac{b}{1-\varepsilon})\right).$$

Proof. By Lemma 4.1.4, an eigenfunction u associated with an eigenvalue λ is of the form $u(r,\theta)=S_l(r)H_l(\theta)$, where for $r>1-\varepsilon$

$$S_l(r)=\frac{\pi b}{2}r^{1-\frac{N}{2}}\left[\left(J_{\nu_l}(a)Y'_{\nu_l}(b)-\frac{a}{b}J'_{\nu_l}(a)Y_{\nu_l}(b)\right)J_{\nu_l}(\frac{br}{1-\varepsilon}) + \left(\frac{a}{b}J_{\nu_l}(b)J'_{\nu_l}(a)-J'_{\nu_l}(b)J_{\nu_l}(a)\right)Y_{\nu_l}(\frac{br}{1-\varepsilon})\right].$$

We require that $\frac{\partial u}{\partial \nu}=\frac{\partial u}{\partial r}_{|r=1}=0$, which is true if and only if

$$\begin{aligned}\frac{\pi b}{2}\left(1-\frac{N}{2}\right)&\left[\left(J_{\nu_l}(a)Y'_{\nu_l}(b)-\frac{a}{b}J'_{\nu_l}(a)Y_{\nu_l}(b)\right)J_{\nu_l}(\frac{b}{1-\varepsilon}) + \left(\frac{a}{b}J_{\nu_l}(b)J'_{\nu_l}(a)-J'_{\nu_l}(b)J_{\nu_l}(a)\right)Y_{\nu_l}(\frac{b}{1-\varepsilon})\right]\\ &+\frac{\pi b^2}{2(1-\varepsilon)}\left[\left(J_{\nu_l}(a)Y'_{\nu_l}(b)-\frac{a}{b}J'_{\nu_l}(a)Y_{\nu_l}(b)\right)J'_{\nu_l}(\frac{b}{1-\varepsilon}) + \left(\frac{a}{b}J_{\nu_l}(b)J'_{\nu_l}(a)-J'_{\nu_l}(b)J_{\nu_l}(a)\right)Y'_{\nu_l}(\frac{b}{1-\varepsilon})\right]=0.\end{aligned}$$

The previous equation can be clearly rewritten in the form (4.1.9). $\square$

We now prove the following lemma.

Lemma 4.1.10. *Equation (4.1.9) can be written in the form*

$$\lambda^2\varepsilon\left(\frac{M}{3N\omega_N}-\frac{1}{\nu_l(1+\nu_l)}\right)+\lambda\varepsilon\left(\frac{N}{2}-\nu_l+\frac{(2-N)N\omega_N}{2\nu_l(1+\nu_l)M}\right)$$
$$-2\lambda+\frac{2N\omega_N l}{M}-\frac{2N\omega_N l}{M}\left(\frac{N-1}{2}-\frac{\omega_N}{M}-\nu_l\right)\varepsilon+\mathcal{R}(\lambda,\varepsilon)=0 \quad (4.1.11)$$

where $\mathcal{R}(\lambda,\varepsilon)\in O(\varepsilon\sqrt{\varepsilon})$ *as* $\varepsilon\to 0$.

Proof. We plan to divide the left hand-side of (4.1.9) by $J'_{\nu_l}(a)$ and to analyze the resulting terms using the known Taylor's series for Bessel functions. Note that $J'_{\nu_l}(a)>0$ for all ε small enough. We split our analysis into three steps.

Step 1. We consider the term $\frac{P_2(a,b)}{J'_{\nu_l}(a)}$, that is

$$\frac{J_{\nu_l}(a)}{J'_{\nu_l}(a)}\left[Y'_{\nu_l}(b)J'_{\nu_l}(\frac{b}{1-\varepsilon})-Y'_{\nu_l}(\frac{b}{1-\varepsilon})J'_{\nu_l}(b)\right]$$
$$+\frac{a}{b}\left[Y'_{\nu_l}(\frac{b}{1-\varepsilon})J_{\nu_l}(b)-Y_{\nu_l}(b)J'_{\nu_l}(\frac{b}{1-\varepsilon})\right]. \quad (4.1.12)$$

Using Taylor's formula, we write the derivatives of the Bessel functions in (4.1.12), call them $\mathcal{C}'_{\nu_l}$, as follows

$$\mathcal{C}'_{\nu_l}\left(\frac{b}{1-\varepsilon}\right)$$
$$=\mathcal{C}'_{\nu_l}(b)+\mathcal{C}''_{\nu_l}(b)\frac{\varepsilon b}{1-\varepsilon}+\cdots+\frac{\mathcal{C}^{(n)}_{\nu_l}(b)}{(n-1)!}\left(\frac{\varepsilon b}{1-\varepsilon}\right)^{n-1}+o\left(\frac{\varepsilon b}{1-\varepsilon}\right)^{n-1}. \quad (4.1.13)$$

Then, using (4.1.13) with $n=4$ for J'_{ν_l} and Y'_{ν_l} we get

$$\frac{J_{\nu_l}(a)}{J'_{\nu_l}(a)}\left[\frac{\varepsilon b}{1-\varepsilon}\left(Y'_{\nu_l}(b)J''_{\nu_l}(b)-J'_{\nu_l}(b)Y''_{\nu_l}(b)\right)\right.$$
$$+\frac{\varepsilon^2b^2}{2(1-\varepsilon)^2}\left(Y'_{\nu_l}(b)J'''_{\nu_l}(b)-J'_{\nu_l}(b)Y'''_{\nu_l}(b)\right)$$
$$\left.+\frac{\varepsilon^3b^3}{6(1-\varepsilon)^3}\left(Y'_{\nu_l}(b)J''''_{\nu_l}(b)-J'_{\nu_l}(b)Y''''_{\nu_l}(b)\right)+R_1(b)\right]$$
$$+\frac{a}{b}\left[\left(J_{\nu_l}(b)Y'_{\nu_l}(b)-Y_{\nu_l}(b)J'_{\nu_l}(b)\right)+\frac{\varepsilon b}{1-\varepsilon}\left(J_{\nu_l}(b)Y''_{\nu_l}(b)-Y_{\nu_l}(b)J''_{\nu_l}(b)\right)\right.$$
$$\left.+\frac{\varepsilon^2b^2}{2(1-\varepsilon)^2}\left(J_{\nu_l}(b)Y'''_{\nu_l}(b)-Y_{\nu_l}(b)J'''_{\nu_l}(b)\right)+R_2(b)\right], \quad (4.1.14)$$

where $R_1(b)$, $R_2(b)$ are the appropriate remainders in the Taylor's formulas.

Let R_3 be the remainder defined in Lemma 4.1.23. We set

$$R(\lambda,\varepsilon) := R_3(a)\left[\frac{\varepsilon b}{1-\varepsilon}\left(Y'_{\nu_l}(b)J''_{\nu_l}(b) - J'_{\nu_l}(b)Y''_{\nu_l}(b)\right)\right.$$
$$+\frac{\varepsilon^2 b^2}{2(1-\varepsilon)^2}\left(Y'_{\nu_l}(b)J'''_{\nu_l}(b) - J'_{\nu_l}(b)Y'''_{\nu_l}(b)\right)$$
$$\left.+\frac{\varepsilon^3 b^3}{6(1-\varepsilon)^3}\left(Y'_{\nu_l}(b)J''''_{\nu_l}(b) - J'_{\nu_l}(b)Y''''_{\nu_l}(b)\right)\right]$$
$$+R_1(b)\left[\frac{a}{\nu_l} + \frac{a^3}{2{\nu_l}^2(1+\nu_l)}\right] + R_2(b)\frac{a}{b} + R_3(a)R_1(b).$$

By Lemma 4.1.28, it turns out that $R(\lambda,\varepsilon) \in O(\varepsilon^3)$ as $\varepsilon \to 0$.

We also set

$$f(\varepsilon) := b_1^2(\varepsilon)a_1^3(\varepsilon)f_1(\varepsilon);$$
$$g(\varepsilon) := b_1^2(\varepsilon)a_1(\varepsilon)g_1(\varepsilon) + a_1^3(\varepsilon)g_2(\varepsilon);$$
$$h(\varepsilon) := a_1(\varepsilon)h_1(\varepsilon) + \varepsilon^2\frac{a_1^3(\varepsilon)}{b_1^2(\varepsilon)}h_2(\varepsilon);$$
$$k(\varepsilon) := \frac{a_1(\varepsilon)}{b_1^2(\varepsilon)}k_1(\varepsilon),$$

where

$$a_1(\varepsilon) := \frac{a}{\sqrt{\lambda}\varepsilon} = (1-\varepsilon);$$
$$b_1(\varepsilon) := b\sqrt{\frac{\varepsilon}{\lambda}};$$
$$f_1(\varepsilon) := \frac{1}{6{\nu_l}^2(1+\nu_l)(1-\varepsilon)^3};$$
$$g_1(\varepsilon) := \frac{1}{3\nu_l(1-\varepsilon)^3};$$
$$g_2(\varepsilon) := -\frac{1}{{\nu_l}^2(1+\nu_l)(1-\varepsilon)} + \frac{\varepsilon}{2{\nu_l}^2(1+\nu_l)(1-\varepsilon)^2} - \frac{\varepsilon^2(3+2{\nu_l}^2)}{6{\nu_l}^2(1+\nu_l)(1-\varepsilon)^3};$$
$$h_1(\varepsilon) := -\frac{2}{\nu_l(1-\varepsilon)} + \frac{\varepsilon}{\nu_l(1-\varepsilon)^2} - \frac{\varepsilon^2(3+2{\nu_l}^2)}{3\nu_l(1-\varepsilon)^3} - \frac{\varepsilon}{(1-\varepsilon)^2};$$
$$h_2(\varepsilon) := \frac{1}{(1+\nu_l)(1-\varepsilon)} - \frac{3\varepsilon}{2(1+\nu_l)(1-\varepsilon)^2} + \frac{\varepsilon^2({\nu_l}^4+11{\nu_l}^2)}{6{\nu_l}^2(1+\nu_l)(1-\varepsilon)^3};$$
$$k_1(\varepsilon) := 2 + \frac{2\varepsilon\nu_l}{(1-\varepsilon)} - \frac{3\varepsilon^2\nu_l}{(1-\varepsilon)^2} + \frac{\varepsilon^3({\nu_l}^4+11{\nu_l}^2)}{3\nu_l(1-\varepsilon)^3} - \frac{2\varepsilon}{(1-\varepsilon)} + \frac{\varepsilon^2(2+{\nu_l}^2)}{(1-\varepsilon)^2}.$$

Note that functions f, g, h, k are continuous at $\varepsilon = 0$ and $f(0)$, $g(0)$, $h(0)$, $k(0) \neq 0$.

Using in (4.1.14) formula (4.1.24) for $J_{\nu_l}(a)/J'_{\nu_l}(a)$ and the explicit formulas for the cross products of Bessel functions given by Lemma 4.1.41 and

Corollary 4.1.46, (4.1.12) can be written as

$$\frac{1}{\sqrt{\lambda}\pi}\varepsilon\sqrt{\varepsilon}k(\varepsilon)+\frac{\sqrt{\lambda}}{\pi}\varepsilon\sqrt{\varepsilon}h(\varepsilon)+\frac{\lambda\sqrt{\lambda}}{\pi}\varepsilon^2\sqrt{\varepsilon}g(\varepsilon)+\frac{\lambda^2\sqrt{\lambda}}{\pi}\varepsilon^3\sqrt{\varepsilon}f(\varepsilon)+R(\lambda,\varepsilon). \tag{4.1.15}$$

Step 2. We consider the quantity $\frac{P_1(a,b)}{J'_{\nu_l}(a)}$, that is

$$\frac{J_{\nu_l}(a)}{J'_{\nu_l}(a)}\left[Y'_{\nu_l}(b)J_{\nu_l}(\frac{b}{1-\varepsilon})-J'_{\nu_l}(b)Y_{\nu_l}(\frac{b}{1-\varepsilon})\right] + \frac{a}{b}\left[J_{\nu_l}(b)Y_{\nu_l}(\frac{b}{1-\varepsilon})-Y_{\nu_l}(b)J_{\nu_l}(\frac{b}{1-\varepsilon})\right]. \tag{4.1.16}$$

Proceeding as in Step 1 and setting

$$\begin{aligned}
\tilde{f}(\varepsilon) &:= -\frac{a_1^3(\varepsilon)b_1(\varepsilon)}{2\pi{\nu_l}^2(1+\nu_l)(1-\varepsilon)^2};\\
\tilde{g}(\varepsilon) &:= \frac{a_1^3(\varepsilon)}{b_1(\varepsilon)}\left(\frac{1}{\pi{\nu_l}^2(1+\nu_l)}+\frac{\varepsilon^2}{2\pi(1+\nu_l)(1-\varepsilon)^2}\right)-\frac{a_1(\varepsilon)b_1(\varepsilon)}{\nu_l\pi(1-\varepsilon)^2};\\
\tilde{h}(\varepsilon) &:= \frac{a_1(\varepsilon)}{b_1(\varepsilon)}\left(\frac{2}{\nu_l\pi}+\frac{2\varepsilon}{\pi(1-\varepsilon)}+\frac{(\nu_l-1)}{\pi(1-\varepsilon)^2}\varepsilon^2\right),
\end{aligned}$$

one can prove that (4.1.16) can be written as

$$\varepsilon\tilde{h}(\varepsilon)+\lambda\varepsilon^2\tilde{g}(\varepsilon)+\lambda^2\varepsilon^3\tilde{f}(\varepsilon)+\hat{R}(\lambda,\varepsilon), \tag{4.1.17}$$

where $\hat{R}(\lambda,\varepsilon)\in O(\varepsilon^2\sqrt{\varepsilon})$ as $\varepsilon\to 0$, see Lemma 4.1.28.

Step 3. We combine (4.1.15) and (4.1.17) and rewrite equation (4.1.9) in the form

$$\begin{aligned}
&\varepsilon(1-\frac{N}{2})\tilde{h}(\varepsilon)+\varepsilon\frac{b_1(\varepsilon)k(\varepsilon)}{\pi(1-\varepsilon)}+\lambda\varepsilon^2(1-\frac{N}{2})\tilde{g}(\varepsilon)+\lambda\varepsilon\frac{b_1(\varepsilon)h(\varepsilon)}{\pi(1-\varepsilon)}\\
&+\lambda^2\varepsilon^3(1-\frac{N}{2})\tilde{f}(\varepsilon)+\lambda^2\varepsilon^2\frac{b_1(\varepsilon)g(\varepsilon)}{\pi(1-\varepsilon)}+\lambda^3\varepsilon^3\frac{b_1(\varepsilon)f(\varepsilon)}{\pi(1-\varepsilon)}+\mathcal{R}_0(\lambda,\varepsilon)=0,
\end{aligned} \tag{4.1.18}$$

where

$$\mathcal{R}_0(\lambda,\varepsilon)=\frac{\sqrt{\lambda}b_1(\varepsilon)}{(1-\varepsilon)\sqrt{\varepsilon}}R(\lambda,\varepsilon)+\left(1-\frac{N}{2}\right)\hat{R}(\lambda,\varepsilon).$$

Note that $\mathcal{R}_0(\lambda,\varepsilon)\in O(\varepsilon^2\sqrt{\varepsilon})$ as $\varepsilon\to 0$. Dividing by ε in (4.1.18) and setting $\mathcal{R}_1(\lambda,\varepsilon):=\frac{\mathcal{R}_0(\lambda,\varepsilon)}{\varepsilon}$, we obtain

$$\begin{aligned}
&(1-\frac{N}{2})\tilde{h}(\varepsilon)+\frac{b_1(\varepsilon)k(\varepsilon)}{\pi(1-\varepsilon)}+\lambda\varepsilon(1-\frac{N}{2})\tilde{g}(\varepsilon)+\lambda\frac{b_1(\varepsilon)h(\varepsilon)}{\pi(1-\varepsilon)}\\
&+\lambda^2\varepsilon^2(1-\frac{N}{2})\tilde{f}(\varepsilon)+\lambda^2\varepsilon\frac{b_1(\varepsilon)g(\varepsilon)}{\pi(1-\varepsilon)}+\lambda^3\varepsilon^2\frac{b_1(\varepsilon)f(\varepsilon)}{\pi(1-\varepsilon)}+\mathcal{R}_1(\lambda,\varepsilon)=0.
\end{aligned} \tag{4.1.19}$$

We now multiply in (4.1.19) by $\frac{\pi\nu_l(1-\varepsilon)}{b_1(\varepsilon)}$ which is a positive quantity for all $\varepsilon \in]0,1[$. Taking into account the definitions of functions $g, h, k, \tilde{g}, \tilde{h}$, we can finally rewrite (4.1.19) in the form

$$\lambda^2\varepsilon\left(\frac{\hat{\rho}(\varepsilon)}{3} - \frac{1}{\nu_l(1+\nu_l)}\right) + \lambda\varepsilon\left(\frac{N}{2} - \nu_l + \frac{2-N}{2\nu_l(1+\nu_l)\hat{\rho}(\varepsilon)}\right) - 2\lambda + \frac{2l\,(1+\varepsilon\nu_l)}{\hat{\rho}(\varepsilon)} + \mathcal{R}(\lambda,\varepsilon) = 0,$$

where

$$\hat{\rho}(\varepsilon) := \varepsilon\tilde{\rho}(\varepsilon) = \frac{M - \omega_N\varepsilon(1-\varepsilon)^N}{\omega_N\left(N - \frac{N(N-1)}{2}\varepsilon - \sum_{k=3}^{N}\binom{N}{k}(-1)^k\varepsilon^{k-1}\right)},$$

and $\mathcal{R}(\lambda,\varepsilon) \in O(\varepsilon\sqrt{\varepsilon})$ as $\varepsilon \to 0$. The formulation in (4.1.11) can be easily deduced by observing that

$$\hat{\rho}_\varepsilon = \frac{M}{N\omega_N} + 2\frac{M}{N\omega_N}\left(\frac{N-1}{4} - \frac{\omega_N}{2M}\right)\varepsilon + O(\varepsilon^2), \quad \text{as } \varepsilon \to 0.$$

□

We are now ready to prove the main result of this section.

Theorem 4.1.20. *All eigenvalues of problem (4.0.2) have the following asymptotic behavior*

$$\lambda_l(\varepsilon) = \mu_l + \left(\frac{2l\mu_l}{3} + \frac{2\mu_l^2}{N(2l+N)}\right)\varepsilon + o(\varepsilon), \quad \text{as } \varepsilon \to 0, \tag{4.1.21}$$

where μ_l are the eigenvalues of problem (4.0.1).

Moreover, for all $l \in \mathbb{N}_0$ the functions defined by $\lambda_l(\varepsilon)$ for $\varepsilon > 0$ and $\lambda_l(0) = \mu_l$, are continuous in the whole of $[0,1[$ and of class C^1 in a neighborhood of $\varepsilon = 0$.

Proof. By using the Min-Max Principle and related standard arguments, one can easily prove that $\lambda_l(\varepsilon)$ depends with continuity on $\varepsilon > 0$ (cfr. formula (2.1.1), see also [70, 75]). Moreover, by using (4.1.2) the maps $\varepsilon \mapsto \lambda_l(\varepsilon)$ can be extended by continuity at the point $\varepsilon = 0$ by setting $\lambda_l(0) = \mu_l$.

In order to prove differentiability of $\lambda_l(\varepsilon)$ around zero and the validity of (4.1.21), we consider equation (4.1.11) and apply the Implicit Function Theorem. Note that equation (4.1.11) can be written in the form $F(\lambda,\varepsilon) = 0$

where F is a function of class C^1 in the variables $(\lambda, \varepsilon) \in]0, \infty[\times[0, 1[$, with

$$\begin{aligned} F(\lambda, 0) &= -2\lambda + \frac{2N\omega_N l}{M}, \\ F'_\lambda(\lambda, 0) &= -2, \\ F'_\varepsilon(\lambda, 0) &= \lambda^2\left(\frac{M}{3N\omega_N} - \frac{1}{\nu_l(1+\nu_l)}\right) + \lambda\left(\frac{N}{2} - \nu_l + \frac{(2-N)N\omega_N}{2\nu_l(1+\nu_l)M}\right) \\ &\quad -\frac{2N\omega_N l}{M}\left(\frac{N-1}{2} - \frac{\omega_N}{M} - \nu_l\right) \end{aligned}$$

By (4.1.1), $\mu_l = N\omega_N l/M$ hence $F(\mu_l, 0) = 0$. Since $F'_\lambda(\mu_l, 0) \neq 0$, the Implicit Function Theorem combined with the continuity of the functions $\lambda_l(\cdot)$ allows to conclude that functions $\lambda_l(\cdot)$ are of class C^1 around zero.

We now compute the derivative of $\lambda_l(\cdot)$ at zero. Using the equality $N\omega_N/M = \mu_l/l$ and recalling that $\nu_l = l + N/2 - 1$ we get

$$\begin{aligned} F'_\varepsilon(\mu_l, 0) &= \mu_l^2\left(\frac{l}{3\mu_l} - \frac{1}{\nu_l(1+\nu_l)}\right) \\ &\quad + \mu_l\left(1 - l + \frac{\mu_l(2-N)}{2l\nu_l(1+\nu_l)}\right) - 2\mu_l\left(\frac{1}{2} - l - \frac{\mu_l}{Nl}\right) \\ &= \mu_l^2\left(\frac{1}{\nu_l(1+\nu_l)}\left(\frac{2-N}{2l} - 1\right) + \frac{2}{Nl}\right) + \frac{4}{3}\mu_l l \\ &= \frac{4\mu_l^2}{N^2 + 2Nl} + \frac{4}{3}\mu_l l. \end{aligned}$$

Finally, formula $\lambda'_l(0) = -F'_\varepsilon(\mu_l, 0)/F'_\lambda(\mu_l, 0)$ yields (4.1.3) and the validity of (4.1.21). $\square$

Corollary 4.1.22. *For any $l \in \mathbb{N}$ there exists $\delta_l > 0$ such that the function $\lambda_l(\cdot)$ is strictly increasing in the interval $[0, \delta_l[$. In particular, $\mu_l < \lambda_l(\varepsilon)$ for all $\varepsilon \in]0, \delta_l[$.*

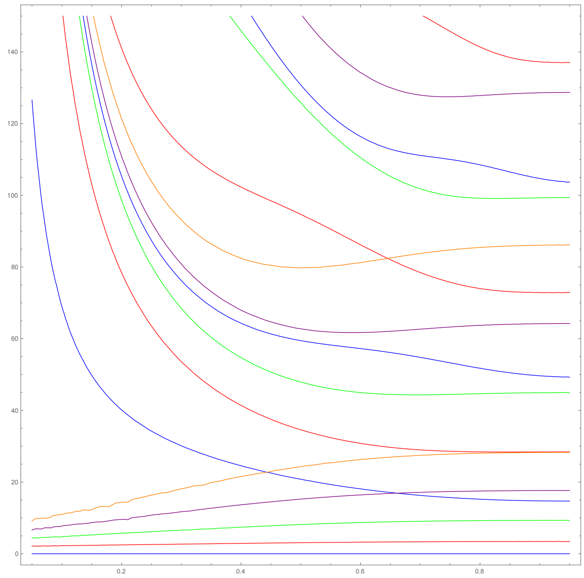


Figure 4.1: Solution branches of equation (4.1.9) with $N = 2$, $M = \pi$ in the region $(\varepsilon, \lambda) \in]0, 1[\times]0, 150[$. The colors refer to the choice of l in (4.1.9), in particular blue ($l = 0$), red ($l = 1$), green ($l = 2$), purple ($l = 3$), orange ($l = 4$).

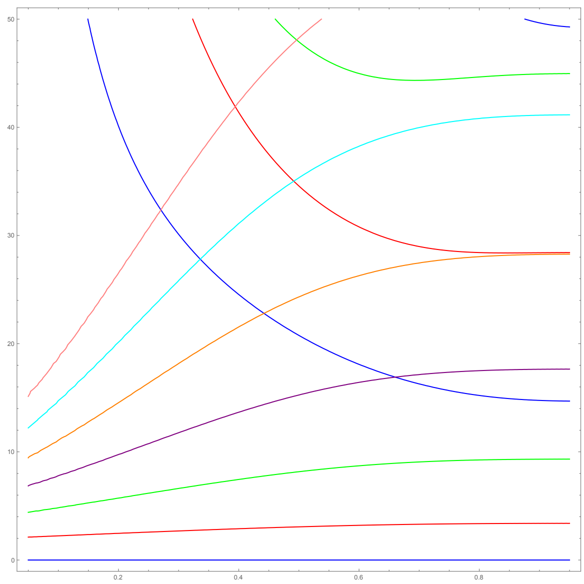


Figure 4.2: Solution branches of equation (4.1.9) with $N = 2$, $M = \pi$ in the region $(\varepsilon, \lambda) \in]0, 1[\times]0, 50[$. The colors refer to the choice of l in (4.1.9), in particular blue ($l = 0$), red ($l = 1$), green ($l = 2$), purple ($l = 3$), orange ($l = 4$), cyan ($l = 5$), pink ($l = 6$) .

4.1.2 Estimates for the remainders

This subsection is devoted to the proof of a few technical estimates used in the proof of Lemma 4.1.10.

Lemma 4.1.23. *The function R_3 defined by*

$$\frac{J_\nu(z)}{J'_\nu(z)} = \frac{z}{\nu} + \frac{z^3}{2\nu^2(1+\nu)} + R_3(z), \tag{4.1.24}$$

is $O(z^5)$ as $z \to 0$.

Proof. Recall the following well-known representation of the Bessel functions of the first species

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{j=0}^{+\infty} \frac{(-1)^j}{j!\Gamma(j+\nu+1)} \left(\frac{z}{2}\right)^{2j}. \tag{4.1.25}$$

For clarity, we simply write

$$J_\nu(z) = z^\nu(a_0 + a_2 z^2 + a_4 z^4 + O(z^5)), \tag{4.1.26}$$

hence

$$J'_\nu(z) = z^{\nu-1}(\nu a_0 + (\nu+2)a_2 z^2 + (\nu+4)a_4 z^4 + O(z^5)), \tag{4.1.27}$$

where the coefficients a_0, a_2, a_4 are defined by (4.1.25). By (4.1.26), (4.1.27) and standard computations it follows that

$$\frac{J_\nu(z)}{J'_\nu(z)} = \frac{z}{\nu} - \frac{2a_2}{\nu^2 a_0} z^3 + O(z^5),$$

which gives exactly (4.1.24). ☐

Lemma 4.1.28. *For any $\lambda > 0$ the remainders $R(\lambda,\varepsilon)$ and $\hat{R}(\lambda,\varepsilon)$ defined in the proof of Lemma 4.1.10 are $O(\varepsilon^3)$, $O(\varepsilon^2\sqrt{\varepsilon})$, respectively, as $\varepsilon \to 0$. Moreover, the same holds true for the corresponding partial derivatives $\partial_\lambda R(\lambda,\varepsilon)$, $\partial_\lambda \hat{R}(\lambda,\varepsilon)$.*

Proof. First, we consider $R_3(a) = R_3(\sqrt{\lambda\varepsilon}(1-\varepsilon))$ where R_3 is defined in Lemma 4.1.23 and we differentiate it with respect to λ. We obtain

$$\frac{\partial R_3(a)}{\partial\lambda} = \frac{aR'_3(a)}{2\lambda},$$

hence by Lemma 4.1.23 we can conclude that $R_3(a)$ and $\frac{\partial R_3(a)}{\partial\lambda}$ are $O(\varepsilon^2\sqrt{\varepsilon})$ as $\varepsilon \to 0$.

Now consider $R_1(b)$ and $R_2(b)$ defined in the proof of Lemma 4.1.10. Since $\lambda > 0$, we have that $b > 0$ hence the Bessel functions are analytic in b and we can write

$$R_1(b) = \sum_{k=4}^{+\infty} \frac{\varepsilon^k b^k}{k!(1-\varepsilon)^k} \left(Y_\nu'(b) J_\nu^{(k+1)}(b) - J_\nu'(b) Y_\nu^{(k+1)}(b) \right),$$

and

$$\begin{aligned} 2\sqrt{\lambda}\frac{\partial R_1(b)}{\partial \lambda} \\ = \frac{\varepsilon b_1(\varepsilon)}{\sqrt{\varepsilon}(1-\varepsilon)} \sum_{k=4}^{+\infty} \frac{b^{k-1}\varepsilon^{k-1}}{(k-1)!(1-\varepsilon)^{k-1}} \left(Y_\nu'(b) J_\nu^{(k+1)}(b) - J_\nu'(b) Y_\nu^{(k+1)}(b) \right) \\ + \frac{b_1(\varepsilon)}{\sqrt{\varepsilon}} \sum_{k=4}^{+\infty} \frac{\varepsilon^k b^k}{k!(1-\varepsilon)^k} \left(Y_\nu'(b) J_\nu^{(k+1)}(b) - J_\nu'(b) Y_\nu^{(k+1)}(b) \right)'. \end{aligned}$$

Using the fact that $b = \sqrt{\lambda/\varepsilon} b_1(\varepsilon)$ and Lemma 4.1.41 we conclude that all the cross products of the form $Y_\nu'(b) J_\nu^{(k+1)}(b) - J_\nu'(b) Y_\nu^{(k+1)}(b)$ and their derivatives $(Y_\nu'(b) J_\nu^{(k+1)}(b) - J_\nu'(b) Y_\nu^{(k+1)}(b))'$ are $O(\sqrt{\varepsilon})$ and $O(\varepsilon)$ respectively, as $\varepsilon \to 0$. It follows that $R_1(b)$ and $\partial_\lambda R_1(b)$ are $O(\varepsilon^2\sqrt{\varepsilon})$ as $\varepsilon \to 0$.

Similarly,

$$R_2(b) = \sum_{k=3}^{+\infty} \frac{\varepsilon^k b^k}{k!(1-\varepsilon)^k} \left(J_\nu(b) Y_\nu^{(k+1)}(b) - Y_\nu(b) J_\nu^{(k+1)}(b) \right),$$

and

$$\begin{aligned} 2\sqrt{\lambda}\frac{\partial R_2(b)}{\partial \lambda} \\ = \frac{\varepsilon b_1(\varepsilon)}{\sqrt{\varepsilon}(1-\varepsilon)} \sum_{k=3}^{+\infty} \frac{b^{k-1}\varepsilon^{k-1}}{(k-1)!(1-\varepsilon)^{k-1}} \left(J_\nu(b) Y_\nu^{(k+1)}(b) - Y_\nu(b) J_\nu^{(k+1)}(b) \right) \\ + \frac{b_1(\varepsilon)}{\sqrt{\varepsilon}} \sum_{k=3}^{+\infty} \frac{\varepsilon^k b^k}{k!(1-\varepsilon)^k} \left(J_\nu(b) Y_\nu^{(k+1)}(b) - Y_\nu(b) J_\nu^{(k+1)}(b) \right)', \end{aligned}$$

hence $R_2(b)$ and $\partial_\lambda R_2(b)$ are $O(\varepsilon^2)$ as $\varepsilon \to 0$.

Summing up all the terms, using Lemma 4.1.40 and Corollary 4.1.46, we obtain

$$\begin{aligned} R(\lambda, \varepsilon) = R_3(a) \left[\frac{2\varepsilon}{\pi(1-\varepsilon)} \left(\frac{\nu^2}{b^2} - 1 \right) + \frac{\varepsilon^2}{\pi(1-\varepsilon)^2} \left(1 - \frac{3\nu^2}{b^2} \right) \right. \\ \left. + \frac{\varepsilon^3 b^2}{3\pi(1-\varepsilon)^3} \left(\frac{\nu^4 + 11\nu^2}{b^4} - \frac{3 + 2\nu^2}{b^2} + 1 \right) \right] \\ + R_1(b) \left[\frac{a}{\nu} + \frac{a^3}{2\nu^2(1+\nu)} \right] + R_2(b)\frac{a}{b} + R_3(a) R_1(b). \end{aligned}$$

We conclude that $R(\lambda, \varepsilon)$ is $O(\varepsilon^3)$ as $\varepsilon \to 0$. Moreover, it easily follows that $\frac{\partial R(\lambda,\varepsilon)}{\partial \lambda}$ is also $O(\varepsilon^3)$ as $\varepsilon \to 0$.

The proof of the estimates for $\hat{R}$ and its derivatives is similar and we omit it. □

Remark 4.1.29. *According to standard Landau's notation, saying that a function $f(z)$ is $O(g(z))$ as $z \to 0$ means that there exists $C > 0$ such that $|f(z)| \leq C|g(z)|$ for any z sufficiently close to zero. Thus, using Landau's notation in the statements of Lemmas 4.1.10, 4.1.28 understands the existence of such constants C, which in principle may depend on $\lambda > 0$. However, a careful analysis of the proofs reveals that given a bounded interval of the type $[A, B]$ with $0 < A < B$ then the appropriate constants C in the estimates can be taken independent of $\lambda \in [A, B]$.*

4.1.3 The case $N = 1$

We include here a description of the case $N = 1$ for the sake of completeness. Let Ω be the open interval $]-1, 1[$. Problem (4.0.1) reads

$$\begin{cases} u''(x) = 0, & \text{if } x \in]-1, 1[, \\ u'(\pm 1) = \pm \frac{M}{2} \mu u(\pm 1), \end{cases} \tag{4.1.30}$$

in the unknowns μ and u. It is easy to see that the only eigenvalues are $\mu_0 = 0$ and $\mu_1 = \frac{2}{M}$ and they are associated with the constant functions and the function $u(x) = x$, respectively. As in (3.1.20), we define a mass density ρ_ε on the whole of $]-1, 1[$ by

$$\rho_\varepsilon(x) := \begin{cases} \frac{M}{2\varepsilon} - 1 + \varepsilon & \text{if } x \in]-1, -1+\varepsilon[\cup]1-\varepsilon, 1[, \\ \varepsilon & \text{if } x \in]-1+\varepsilon, 1-\varepsilon[. \end{cases}$$

Note that for any $x \in]-1, 1[$ we have $\rho_\varepsilon(x) \to 0$ as $\varepsilon \to 0$, and $\int_{-1}^{1} \rho_\varepsilon dx = M$ for all $\varepsilon > 0$. Problem (4.0.2) for $N = 1$ reads

$$\begin{cases} -u''(x) = \lambda \rho_\varepsilon(x) u(x), & \text{if } x \in]-1, 1[, \\ u'(-1) = u'(1) = 0. \end{cases} \tag{4.1.31}$$

It is well-known from Sturm-Liouville theory that problem (4.1.31) has an increasing sequence of non-negative eigenvalues of multiplicity one. We denote the eigenvalues of (4.1.31) by $\lambda_l(\varepsilon)$ with $l \in \mathbb{N}_0$. For any $\varepsilon \in]0, 1[$, the only zero eigenvalue is $\lambda_0(\varepsilon)$ and the corresponding eigenfunctions are the constant functions.

We establish an implicit characterization of the eigenvalues of (4.1.31).

Lemma 4.1.32. *The eigenvalues λ of problem* (4.1.31) *are given implicitly* *as zeros of the equation*

$$2\sqrt{\varepsilon\left(\frac{M}{2\varepsilon}-1+\varepsilon\right)}\cos\left(2\sqrt{\lambda\varepsilon}(1-\varepsilon)\right)\sin\left(2\varepsilon\sqrt{\lambda\left(\frac{M}{2\varepsilon}-1+\varepsilon\right)}\right)$$
$$+\left[-\frac{M}{2\varepsilon}+1+\left(\frac{M}{2\varepsilon}-1+2\varepsilon\right)\cos\left(2\varepsilon\sqrt{\lambda\left(\frac{M}{2\varepsilon}-1+\varepsilon\right)}\right)\right]\sin\left(2\sqrt{\lambda\varepsilon}(1-\varepsilon)\right)$$
$$=0. \quad (4.1.33)$$

Proof. Given an eigenvalue $\lambda > 0$ of problem (4.1.31), a corresponding solution u of (4.1.31) is of the form

$$u(x)=\begin{cases} A\cos(\sqrt{\lambda\rho_e}x)+B\sin(\sqrt{\lambda\rho_e}x), & \text{if } x\in]-1,-1+\varepsilon[,\\ C\cos(\sqrt{\lambda\rho_i}x)+D\sin(\sqrt{\lambda\rho_i}x), & \text{if } x\in]-1+\varepsilon,1-\varepsilon[,\\ E\cos(\sqrt{\lambda\rho_e}x)+F\sin(\sqrt{\lambda\rho_e}x), & \text{if } x\in]1-\varepsilon,1[,\end{cases}$$

where $\rho_i=\varepsilon, \rho_e=\frac{M}{2\varepsilon}-1+\varepsilon$ and A,B,C,D,E,F are suitable real numbers. By imposing the continuity condition for u and u' at the points $x=-1+\varepsilon$ and $x=1-\varepsilon$ and the boundary conditions, we obtain a homogeneous system of six linear equations in six unknowns of the form $\mathcal{M}v=0$, where $v=(A,B,C,D,E,F)$ and $\mathcal{M}$ is the matrix associated with the system given by

$$\mathcal{M}=\begin{bmatrix} \alpha & -\beta & -\gamma & \delta & 0 & 0\\ 0 & 0 & \gamma & \delta & -\alpha & -\beta\\ \sqrt{\rho_e}\beta & \sqrt{\rho_e}\alpha & -\sqrt{\rho_i}\delta & -\sqrt{\rho_i}\gamma & 0 & 0\\ 0 & 0 & -\sqrt{\rho_i}\delta & \sqrt{\rho_i}\gamma & \sqrt{\rho_e}\beta & -\sqrt{\rho_e}\alpha\\ \eta & \zeta & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \eta & -\zeta \end{bmatrix},$$

where

$$\begin{aligned}\alpha &:= \cos(\sqrt{\lambda\rho_e}(1-\varepsilon))\\ \beta &:= \sin(\sqrt{\lambda\rho_e}(1-\varepsilon))\\ \gamma &:= \cos(\sqrt{\lambda\rho_i}(1-\varepsilon))\\ \delta &:= \sin(\sqrt{\lambda\rho_i}(1-\varepsilon))\\ \zeta &:= \cos(\sqrt{\lambda\rho_e})\\ \eta &:= \sin(\sqrt{\lambda\rho_e}).\end{aligned}$$

We impose the condition $\det\mathcal{M}=0$. Easy computations give

$$2\Big(\beta\delta\eta\sqrt{\rho_i}+\alpha\delta\eta\sqrt{\rho_e}+\alpha\delta\zeta\sqrt{\rho_i}-\beta\gamma\zeta\sqrt{\rho_e}\Big)\Big(\beta\gamma\eta\sqrt{\rho_i}-\alpha\delta\eta\sqrt{\rho_e}+\alpha\gamma\zeta\sqrt{\rho_i}+\beta\delta\zeta\sqrt{\rho_e}\Big)$$
$$=0.$$

This yields formula (4.1.33). □

Note that $\lambda = 0$ is a solution of (4.1.33) for all $\varepsilon > 0$, then we consider only the case of nonzero eigenvalues. Using standard Taylor's formulas, we easily prove the following lemma.

Lemma 4.1.34. *Equation* (4.1.33) *can be rewritten in the form*

$$M - \frac{\lambda M^2}{2} + \frac{\lambda M^2}{6}\left(1 + \lambda\left(2 + \frac{M}{2}\right)\right)\varepsilon + R(\lambda, \varepsilon) = 0, \qquad (4.1.35)$$

where $R(\lambda, \varepsilon) \in O(\varepsilon^2)$ *as* $\varepsilon \to 0$.

Proof. It is sufficient to consider a suitable Taylor's expansion in formula (4.1.33). In particular we expand the functions $\sin\left(2\varepsilon\sqrt{\lambda\left(\frac{M}{2\varepsilon} - 1 + \varepsilon\right)}\right)$ and $\sin\left(2\sqrt{\lambda\varepsilon}(1-\varepsilon)\right)$ up to the third order, and the functions $\cos\left(2\sqrt{\lambda\varepsilon}(1-\varepsilon)\right)$ and $\cos\left(2\varepsilon\sqrt{\lambda\left(\frac{M}{2\varepsilon} - 1 + \varepsilon\right)}\right)$ up to the second order. We obtain

$$\begin{aligned} F(\lambda,\varepsilon) = {} & 4\sqrt{\lambda\varepsilon}\Big(\frac{M}{2} - \varepsilon + \varepsilon^2\Big)\Big(1 - 2\lambda\varepsilon(1-\varepsilon)^2\Big)\Big(1 - \frac{2}{3}\lambda\varepsilon\big(\frac{M}{2} - \varepsilon + \varepsilon^2\big)\Big) \\ & + 2\sqrt{\lambda\varepsilon}\Big(1 - \varepsilon - \frac{2\lambda\varepsilon(1-\varepsilon)^3}{3}\Big)\Big[1 - \frac{M}{2\varepsilon} + \Big(2\varepsilon - 1 + \frac{M}{2\varepsilon}\Big)\Big(1 - 2\lambda\varepsilon\big(\frac{M}{2} - \varepsilon + \varepsilon^2\big)\Big)\Big] \\ & + R_0(\varepsilon) = 0, \qquad (4.1.36) \end{aligned}$$

where $R_0(\varepsilon) \in O(\varepsilon^2\sqrt{\varepsilon})$. We divide the right-hand side of (4.1.36) by $2\sqrt{\lambda\varepsilon}$ and set $R(\varepsilon) = \frac{R_0(\varepsilon)}{2\sqrt{\lambda\varepsilon}}$. From standard computations formula (4.1.35) easily follows. □

Finally, we can prove the following theorem. Note that formula (4.1.38) is the same as (4.1.21) with $N = 1, l = 1$.

Theorem 4.1.37. *The first nonzero eigenvalue of problem* (4.1.31) *has the following asymptotic behavior*

$$\lambda_1(\varepsilon) = \mu_1 + \frac{2}{3}(\mu_1 + \mu_1^2)\varepsilon + o(\varepsilon) \quad \text{as } \varepsilon \to 0, \qquad (4.1.38)$$

where $\mu_1 = 2/M$ *is the only nonzero eigenvalue of problem* (4.1.30). *Moreover, for* $l > 1$ *we have that* $\lambda_l(\varepsilon) \to +\infty$ *as* $\varepsilon \to 0$.

Proof. The proof is similar to that of Theorem 4.1.20. It is possible to prove that the eigenvalues $\lambda_l(\varepsilon)$ of (4.1.31) depend with continuity on $\varepsilon > 0$. We consider equation (4.1.35) and apply the Implicit Function Theorem. Equation (4.1.35) can be written in the form $F(\lambda, \varepsilon) = 0$, with F of class C^1 in $]0, +\infty[\times[0, 1[$ with $F(\lambda, 0) = M - \frac{\lambda M^2}{2}$, $F'_\lambda(\lambda, 0) = -\frac{M^2}{2}$ and $F'_\varepsilon(\lambda, 0) = \frac{\lambda M^2}{6}(1 + \lambda(2 + \frac{M}{2}))$.

Since $\mu_1 = \frac{2}{M}$, $F(\mu_1, 0) = 0$ and $F'_\lambda(\mu_1, 0) \neq 0$, the zeros of equation (4.1.38) in a neighborhood of $(\lambda, 0)$ are given by the graph of a C^1-function $\varepsilon \mapsto \lambda(\varepsilon)$ with $\lambda(0) = \mu_1$. We note that $\lambda(\varepsilon) = \lambda_1(\varepsilon)$ for all ε small enough. Indeed, assuming by contradiction that $\lambda(\varepsilon) = \lambda_l(\varepsilon)$ with $l \geq 2$, we would obtain that, possibly passing to a subsequence, $\lambda_1(\varepsilon) \to \bar{\lambda}$ as $\varepsilon \to 0$, for some $\bar{\lambda} \in [0, \lambda_1[$. Then passing to the limit in (4.1.35) as $\varepsilon \to 0$ we would obtain a contradiction. Thus, $\lambda_1(\cdot)$ is of class C^1 in a neighborhood of zero and $\lambda_1'(0) = -F'_\varepsilon(\lambda_1, 0)/F'_\lambda(\lambda_1, 0)$ which yields formula (4.1.38).

The divergence of the higher eigenvalues $\lambda_l(\varepsilon)$ with $l > 1$, as $\varepsilon \to 0$ is deduced by the fact that the existence of a converging subsequence of the form $\lambda_l(\varepsilon_n)$, $n \in \mathbb{N}$ would provide the existence of an eigenvalue for the limiting problem (4.1.30) different from μ_0 and μ_1, which is not admissible. □

4.1.4 Behavior of the eigenvalues under dilations

We consider problems (4.0.1) and (4.0.2) when $\Omega = B_R$ is the ball centered in zero and of radius R in $\mathbb{R}^N$, $N \geq 2$. Let $M > 0$ be fixed. Let the function $\rho_{\varepsilon,M,R} \in L^\infty(B_R)$ be defined by

$$\rho_{\varepsilon,M,R}(x) := \begin{cases} \varepsilon, & \text{if } x \in B_{R(1-\varepsilon)}, \\ \frac{M - \varepsilon\omega_N R^N (1-\varepsilon)^N}{\omega_N R^N (1-(1-\varepsilon)^N)}, & \text{if } x \in B_R \setminus \overline{B_{R(1-\varepsilon)}}. \end{cases}$$

We consider now the dependence of $\lambda_l(\varepsilon)$ and μ_l and their derivatives on the radius R. We denote by $\lambda_l(\varepsilon, M, R)$ the eigenvalues of problem (4.0.2) on B_R with density $\rho_{\varepsilon,M,R}$, while we denote by $\mu_l(M, R)$ the eigenvalues of problem (4.0.1) on B_R. We introduce the following quantity:

$$\mathcal{R}(u, \varepsilon, M, R) := \frac{\int_{B_R} |\nabla u|^2 \, dx}{\int_{B_R} u^2 \rho_{\varepsilon,M,R} \, dx}. \tag{4.1.39}$$

From the Min-Max Principle (see (1.3.8)) we have

$$\lambda_l(\varepsilon, M, R) = \min_{\substack{E \subset H^1(B_R) \\ \dim E = l+1}} \max_{0 \neq u \in E} \mathcal{R}(u, \varepsilon, M, R), \quad \forall l \in \mathbb{N},$$

We perform the change of variables $x = Ry$ in (4.1.39). We obtain

$$\mathcal{R}(u, \varepsilon, M, R) = R^{-2} \mathcal{R}\left(u(R\cdot), \varepsilon, \frac{M}{R^N}, 1 \right).$$

It follows that

$$\lambda_l(\varepsilon, M, R) = R^{-2} \lambda_l(\varepsilon, \frac{M}{R^N}, 1).$$

Moreover, we have

$$\mu_l(M,R) = \frac{lN\omega_N}{M}R^{N-2},$$
$$\mu_l(\frac{M}{R^N},1) = \frac{lN\omega_N}{M}R^N,$$

therefore $\mu_l\left(\frac{M}{R^N},1\right) = R^{-2}\mu_l(M,R)$. We rewrite formula (4.1.3) as

$$\partial_\varepsilon\lambda_l(\varepsilon,M,1)|_{\varepsilon=0} = \frac{2l\mu_l(M,1)}{3} + \frac{2\mu_l^2(M,1)\omega_N}{N(2l+N)}.$$

We obtain

$$\begin{aligned}\partial_\varepsilon\lambda_l(\varepsilon,M,R)|_{\varepsilon=0} &= R^{-2}\partial_\varepsilon\lambda_l(\varepsilon,\frac{M}{R^N},1)|_{\varepsilon=0}\\ &= R^{-2}\left(\frac{2l\mu_l(\frac{M}{R^N},1)}{3} + \frac{2\mu_l^2(\frac{M}{R^N},1)}{N(2l+N)}\right)\\ &= R^{-4}\left(\frac{2l\mu_l(M,R)}{3} + \frac{2R^{-2}\mu_l^2(M,R)}{N(2l+N)}\right).\end{aligned}$$

4.1.5 Cross products of Bessel functions

We provide here explicit formulas for the cross products of Bessel functions used in this section.

Lemma 4.1.40. *The following identities hold:*

$$\begin{aligned}Y_\nu(z)J_\nu'(z) - J_\nu(z)Y_\nu'(z) &= -\frac{2}{\pi z};\\ Y_\nu(z)J_\nu''(z) - J_\nu(z)Y_\nu''(z) &= \frac{2}{\pi z^2};\\ Y_\nu'(z)J_\nu''(z) - J_\nu'(z)Y_\nu''(z) &= \frac{2}{\pi z}\left(\frac{\nu^2}{z^2}-1\right).\end{aligned}$$

Proof. It is well-known (see Section 1.4, see also [1, §9]) that

$$J_\nu(z)Y_\nu'(z) - Y_\nu(z)J_\nu'(z) = J_{\nu+1}(z)Y_\nu(z) - J_\nu(z)Y_{\nu+1}(z) = \frac{2}{\pi z},$$

which gives the first identity in the statement. The second identity holds since

$$J_\nu(z)Y_\nu''(z) - Y_\nu(z)J_\nu''(z) = (J_\nu(z)Y_\nu'(z) - Y_\nu(z)J_\nu'(z))' = \left(\frac{2}{\pi z}\right)' = -\frac{2}{\pi z^2}.$$

The third identity holds since

$$
\begin{aligned}
&Y'_\nu(z)J''_\nu(z) - J'_\nu(z)Y''_\nu(z)\\
&\quad = Y'_\nu(z)\left(J_{\nu-1}(z) - \frac{\nu}{z}J_\nu(z)\right)' - J'_\nu(z)\left(Y_{\nu-1}(z) - \frac{\nu}{z}Y_\nu(z)\right)'\\
&\quad = Y'_\nu(z)J'_{\nu-1}(z) - J'_\nu(z)Y'_{\nu-1}(z) + \frac{\nu}{z^2}\left(Y'_\nu(z)J_\nu(z) - J'_\nu(z)Y_\nu(z)\right)\\
&\quad = \left(Y'_\nu(z)\frac{1}{2}\left(J_{\nu-2}(z) - J_\nu(z)\right) - J'_\nu(z)\frac{1}{2}\left(Y_{\nu-2}(z) - Y_\nu(z)\right)\right) + \frac{2\nu}{\pi z^3}\\
&\quad = \frac{1}{2}\left(Y'_\nu(z)J_{\nu-2}(z) - J'_\nu(z)Y_{\nu-2}(z)\right)\\
&\qquad - \frac{1}{2}\left(Y'_\nu(z)J_\nu(z) - J'_\nu(z)Y_\nu(z)\right) + \frac{2\nu}{\pi z^3}\\
&\quad = \frac{1}{2}\left(J'_\nu(z)Y_\nu(z) - Y'_\nu(z)J_\nu(z)\right)\\
&\qquad + \frac{\nu-1}{z}\left(Y'_\nu(z)J_{\nu-1}(z) - J'_\nu(z)Y_{\nu-1}(z)\right) - \frac{1}{\pi z} + \frac{2\nu}{\pi z^3}\\
&\quad = \frac{\nu-1}{z}\left(J_{\nu-1}(z)\left(Y_{\nu-1}(z) - \frac{\nu}{z}Y_\nu(z)\right) - Y_{\nu-1}(z)\left(J_{\nu-1}(z) - \frac{\nu}{z}J_\nu(z)\right)\right)\\
&\qquad - \frac{2}{\pi z} + \frac{2\nu}{\pi z^3}\\
&\quad = -\frac{\nu(\nu-1)}{z^2}\left(Y_\nu(z)J_{\nu-1}(z) - J_\nu(z)Y_{\nu-1}(z)\right) - \frac{2}{\pi z} + \frac{2\nu}{\pi z^3}\\
&\quad = \frac{2}{\pi z}\left(-1 + \frac{\nu^2}{z^2}\right),
\end{aligned}
$$

where the first, second and fourth equalities follow respectively from the well-known formulas $\mathcal{C}'_\nu(z) = \mathcal{C}_{\nu-1}(z) - \frac{\nu}{z}\mathcal{C}_\nu(z)$, $2\mathcal{C}'_\nu(z) = \mathcal{C}_{\nu-1}(z) - \mathcal{C}_{\nu+1}(z)$ and $\mathcal{C}_{\nu-2}(z) + \mathcal{C}_\nu(z) = \frac{2(\nu-1)}{z}\mathcal{C}_{\nu-1}(z)$, where $\mathcal{C}_\nu(z)$ stands both for $J_\nu(z)$ and $Y_\nu(z)$ (see Section 1.4, see also [1, §9]). This proves the lemma. □

Lemma 4.1.41. *The following identities hold:*

$$
\begin{aligned}
Y_\nu(z)J_\nu^{(k)}(z) - J_\nu(z)Y_\nu^{(k)}(z) &= \frac{2}{\pi z}\left(r_k + R_{\nu,k}(z)\right); && (4.1.42)\\
Y'_\nu(z)J_\nu^{(k)}(z) - J'_\nu(z)Y_\nu^{(k)}(z) &= \frac{2}{\pi z}\left(q_k + Q_{\nu,k}(z)\right), && (4.1.43)
\end{aligned}
$$

for all $k > 2$ and $\nu \geq 0$, where $r_k, q_k \in \{0, 1, -1\}$, and $Q_{\nu,k}(z)$, $R_{\nu,k}(z)$ are finite sums of quotients of the form $\frac{c_{\nu,k}}{z^m}$, with $m \geq 1$ and $c_{\nu,k}$ a suitable constant, depending on ν, k.

Proof. We will prove (4.1.42) and (4.1.43) by induction. Identities (4.1.42) and (4.1.43) hold for $k = 1$ and $k = 2$ by Lemma 4.1.40. Suppose now that

$$
\begin{aligned}
Y_\nu(z)J_\nu^{(k)}(z) - J_\nu(z)Y_\nu^{(k)}(z) &= \frac{2}{\pi z}\left(r_k + R_{\nu,k}(z)\right),\\
Y'_\nu(z)J_\nu^{(k)}(z) - J'_\nu(z)Y^{(k)}(z) &= \frac{2}{\pi z}\left(q_k + Q_{\nu,k}(z)\right),
\end{aligned}
$$

hold for all $\nu \geq 0$. First consider

$$Y_\nu'(z)J_\nu^{(k+1)}(z) - J_\nu'(z)Y_\nu^{(k+1)}(z).$$

We use the recurrence relations $\mathcal{C}_{\nu+1}(z) + \mathcal{C}_{\nu-1}(z) = \frac{2\nu}{z}\mathcal{C}_\nu(z)$ and $2\mathcal{C}'(z) = \mathcal{C}_{\nu-1}(z) - \mathcal{C}_{\nu+1}(z)$, where $\mathcal{C}_\nu(z)$ stands both for $J_\nu(z)$ and $Y_\nu(z)$ (see [1, §9]). We have

$$\begin{aligned}
Y_\nu'(z)J_\nu^{(k+1)}(z) - J_\nu'(z)Y_\nu^{(k+1)}(z) &= Y_\nu'(z)(J_\nu')^{(k)}(z) - J_\nu'(z)(Y_\nu')^{(k)}(z)\\
&= \frac{1}{4}\Big[(Y_{\nu-1}(z) - Y_{\nu+1}(z))\,(J_{\nu-1}(z) - J_{\nu+1}(z))^{(k)}\\
&\quad - (J_{\nu-1}(z) - J_{\nu+1}(z))\,(Y_{\nu-1}(z) - Y_{\nu+1}(z))^{(k)}\Big]\\
= \frac{1}{4}\Big[\Big(Y_{\nu-1}(z)J_{\nu-1}^{(k)}(z) - J_{\nu-1}(z)Y_{\nu-1}^{(k)}(z)\Big) &+ \Big(Y_{\nu+1}(z)J_{\nu+1}^{(k)}(z) - J_{\nu+1}(z)Y_{\nu+1}^{(k)}(z)\Big)\\
+ \Big(J_{\nu+1}(z)Y_{\nu-1}^{(k)}(z) - Y_{\nu-1}(z)J_{\nu+1}^{(k)}(z)\Big) &+ \Big(J_{\nu-1}(z)Y_{\nu+1}^{(k)}(z) - Y_{\nu+1}(z)J_{\nu-1}^{(k)}(z)\Big)\Big]\\
&= \frac{1}{4}\Big[\frac{2}{\pi z}\,(r_k + R_{\nu-1,k}(z) + r_k + R_{\nu+1,k}(z))\\
&+ \frac{2\nu}{z}\Big(J_\nu(z)Y_{\nu-1}^{(k)} - Y_\nu(z)J_{\nu-1}^{(k)}(z) + J_\nu(z)Y_{\nu+1}^{(k)}(z) - Y_\nu(z)J_{\nu+1}^{(k)}(z)\Big)\\
&- \Big(J_{\nu-1}(z)Y_{\nu-1}^{(k)}(z) - Y_{\nu-1}(z)J_{\nu-1}^{(k)}(z) + J_{\nu+1}(z)Y_{\nu+1}^{(k)}(z) - Y_{\nu+1}J_{\nu+1}^{(k)}(z)\Big)\Big]\\
&= \frac{1}{4}\Big[\frac{4}{\pi z}\,(2r_k + R_{\nu-1,k}(z) + R_{\nu+1,k}(z))\\
&+ \frac{2\nu}{z}\Big(J_\nu(z)\,(Y_{\nu-1}(z) + Y_{\nu+1}(z))^{(k)} - Y_\nu(z)\,(J_{\nu-1}(z) + J_{\nu+1}(z))^{(k)}\Big)\Big]\\
&= \frac{1}{\pi z}\,(2r_k + R_{\nu-1,k}(z) + R_{\nu+1,k}(z))\\
&+ \frac{\nu^2}{z}\left(J_\nu(z)\left(\frac{1}{z}Y_\nu(z)\right)^{(k)} - Y_\nu(z)\left(\frac{1}{z}J_\nu(z)\right)^{(k)}\right)\\
&= \frac{2}{\pi z}\Big[r_k + \frac{1}{2}\,(R_{\nu-1,k}(z) + R_{\nu+1,k}(z))\\
&\quad - \frac{\nu^2}{z}\sum_{j=0}^{k}\frac{k!(-1)^{k-j}}{j!z^{k-j+1}}\,(r_j + R_{\nu,j}(z))\Big]. \qquad (4.1.44)
\end{aligned}$$

We prove now (4.1.43)

$$\begin{aligned}
Y_\nu(z)J_\nu^{(k+1)}(z) - J_\nu(z)Y_\nu^{(k+1)}(z) &= \Big(Y_\nu(z)J_\nu^{(k)}(z) - J_\nu(z)Y_\nu^{(k)}(z)\Big)'\\
&\quad - \Big(Y_\nu'(z)J_\nu^{(k)}(z) - J_\nu'(z)Y_\nu^{(k)}(z)\Big)\\
&= \frac{2}{\pi z}\left(-q_k - Q_{\nu,k}(z) - \frac{r_k}{z} - \frac{R_{\nu,k}(z)}{z} + R_{\nu,k}'(z)\right). \qquad (4.1.45)
\end{aligned}$$

This concludes the proof. □

Corollary 4.1.46. *The following formulas hold*

$$\begin{aligned} J_\nu(z)Y_\nu'''(z) - Y_\nu(z)J_\nu'''(z) &= \frac{2}{\pi z}\left(\frac{2+\nu^2}{z^2} - 1\right); \\ Y_\nu'(z)J_\nu'''(z) - J_\nu'(z)Y_\nu'''(z) &= \frac{2}{\pi z^2}\left(1 - \frac{3\nu^2}{z^2}\right); \\ Y_\nu'(z)J_\nu''''(z) - J_\nu'(z)Y_\nu''''(z) &= \frac{2}{\pi z}\left(1 - \frac{3+2\nu^2}{z^2} + \frac{\nu^4+11\nu^2}{z^4}\right). \end{aligned}$$

Proof. From Lemma 4.1.41 (see in particular (4.1.45)) it follows

$$\begin{aligned} J_\nu(z)Y_\nu'''(z) - Y_\nu(z)J_\nu'''(z) &= -\frac{2}{\pi z}\left[-q_2 - Q_{\nu,2}(z) - \frac{r_2}{z} - \frac{R_{\nu,2}(z)}{z} + R_{\nu,2}'(z)\right] \\ &= \frac{2}{\pi z}\left(\frac{2+\nu^2}{z^2} - 1\right). \end{aligned}$$

Next we compute

$$\begin{aligned} &Y_\nu'(z)J_\nu'''(z) - J_\nu'(z)Y_\nu'''(z) \\ &\quad = \frac{2}{\pi z}\left[r_2 + R_{\nu,2}(z) - \frac{\nu^2}{z}\sum_{j=0}^{2}\frac{2(-1)^{2-j}}{j!z^{2-j+1}}\left(r_j + R_{\nu,j}(z)\right)\right] \\ &\quad = \frac{2}{\pi z^2}\left(1 - \frac{3\nu^2}{z^2}\right). \end{aligned}$$

Finally, by (4.1.44) with $k = 3$, we have

$$\begin{aligned} Y_\nu'(z)J_\nu''''(z) - J_\nu'(z)Y_\nu''''(z) &= \frac{2}{\pi z}\left[r_3 + \frac{1}{2}\left(R_{\nu-1,3}(z) + R_{\nu+1,3}(z)\right)\right. \\ &\quad \left. - \frac{\nu^2}{z}\sum_{j=0}^{3}\frac{6(-1)^{3-j}}{j!z^{3-j+1}}\left(r_j + R_{\nu,j}(z)\right)\right] \\ &= \frac{2}{\pi z}\left(1 - \frac{3+2\nu^2}{z^2} + \frac{\nu^4+11\nu^2}{z^4}\right). \end{aligned}$$

□

4.2 Bounded domains of class C^2 in $\mathbb{R}^2$

In this section we consider problems (4.0.1) and (4.0.2) on bounded domains of class C^2 in $\mathbb{R}^2$. The aim of this section is to study the asymptotic behavior of the eigenvalues $\lambda_j(\varepsilon)$ of problem (4.0.2) and of the corresponding eigenfunctions which we shall denote here by $u_{j,\varepsilon}$, as ε goes to zero. To do so, we show the validity of an asymptotic expansion for $\lambda_j(\varepsilon)$ and $u_{j,\varepsilon}$ as ε goes to zero. In addition, we provide explicit formulas for the first two coefficients in the expansions in terms of solutions to suitable auxiliary problems. In particular we establish a closed formula for the derivatives of $\lambda_j(\varepsilon)$ at $\varepsilon = 0$. We

shall see that such formula in the case of the unit ball centered at zero in $\mathbb{R}^2$ agrees with formula (4.1.3) when $N = 2$ (see Subsection 4.2.5). We remark that asymptotic for vibrating systems containing masses concentrated along curves or around certain points have been considered by several authors in the last decades (see e.g., [50, 52, 53, 80, 98]). In particular we shall follow the approach of [50, 52, 53].
For the sake of completeness, in Subsection 4.2.6 we shall also present the formal heuristic computations which allow to postulate the expansions proved in this section.

We denote by $\mathcal{H}_\varepsilon(\Omega)$ the Hilbert space consisting of the functions in the standard Sobolev Space $H^1(\Omega)$ endowed with the bilinear form

$$\langle u, v\rangle_\varepsilon := \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega \rho_\varepsilon u v dx, \quad \forall u, v \in \mathcal{H}_\varepsilon(\Omega). \tag{4.2.1}$$

The bilinear form (4.2.1) induces on $H^1(\Omega)$ a norm which is equivalent to the standard one. We denote such a norm by $\|\cdot\|_\varepsilon$. We recall that the weak formulation of problem (4.0.2) can be stated as follows: a pair $(\lambda(\varepsilon), u_\varepsilon) \in \mathbb{R} \times H^1(\Omega)$ is a solution of (4.0.2) in the weak sense if and only if

$$\int_\Omega \nabla u_\varepsilon \cdot \nabla \varphi dx = \lambda(\varepsilon) \int_\Omega \rho_\varepsilon u_\varepsilon \varphi dx, \quad \forall \varphi \in H^1(\Omega).$$

We introduce the operator $\mathcal{A}_\varepsilon$ from $\mathcal{H}_\varepsilon(\Omega)$ to itself which maps a function $f \in \mathcal{H}_\varepsilon(\Omega)$ to the uniquely determined $u \in \mathcal{H}_\varepsilon$ satisfying the equation

$$\int_\Omega \nabla u \cdot \nabla \varphi dx + \int_\Omega \rho_\varepsilon u \varphi dx = \int_\Omega \rho_\varepsilon f \varphi dx, \quad \forall \varphi \in \mathcal{H}_\varepsilon(\Omega). \tag{4.2.2}$$

In the sequel we shall exploit the following lemma. We refer to [91, §III.1] for its proof.

Lemma 4.2.3. *Let $A : H \to H$ be a linear, self-adjoint, positive and compact operator from a separable Hilbert space H to itself. Let $u \in H$, with $\|u\|_H = 1$. Let $\eta, r > 0$ such that $\|Au - \eta u\|_H \le r$. Then, there exists an eigenvalue η^* of the operator A which satisfy the inequality $|\eta - \eta^*| \le r$. Moreover, for any $r^* > r$ there exists $u^* \in H$ with $\|u^*\|_H = 1$, u^* belonging to the space generated by all the eigenspaces associated with an eigenvalue of the operator A lying on the segment $[\eta - r^*, \eta + r^*]$ and such that*

$$\|u - u^*\|_H \le \frac{2r}{r^*}.$$

We also need the following lemma.

Lemma 4.2.4. *Let $u \in H^1(\Omega)$ and ρ_ε be defined as in* (3.1.20)*. Then there exists $C_\Omega > 0$ which does not depend on $\varepsilon > 0$ and u, such that*

$$\left\| u - \frac{1}{M}\int_\Omega \rho_\varepsilon u dx \right\|_{L^2(\Omega)} \le C_\Omega \|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H^1(\Omega).$$

Proof. Suppose by contradiction that for any $k \in \mathbb{N}$ there exists ε_k with $\varepsilon_k \to 0$ as $k \to +\infty$, and $u_k \in H^1(\Omega)$ such that

$$\left\| u_k - \frac{1}{M}\int_\Omega \rho_{\varepsilon_k} u_k dx \right\|_{L^2(\Omega)} > k\|\nabla u_k\|_{L^2(\Omega)}. \tag{4.2.5}$$

Set

$$v_k := \left\| u_k - \frac{1}{M}\int_\Omega \rho_{\varepsilon_k} u_k dx \right\|_{L^2(\Omega)}^{-1} \left(u_k - \frac{1}{M}\int_\Omega \rho_{\varepsilon_k} u_k dx \right).$$

Then $\int_\Omega \rho_{\varepsilon_k} v_k dx = 0$, $\|v_k\|_{L^2(\Omega)} = 1$ for all $k \in \mathbb{N}$, and from (4.2.5), $\|\nabla v_k\| < \frac{1}{k}$. Then v_k is bounded in $H^1(\Omega)$, hence, possibly passing to a subsequence, $v_k \rightharpoonup v$ weakly in $H^1(\Omega)$ and $v_k \to v$ strongly in $L^2(\Omega)$, for some $v \in H^1(\Omega)$. Moreover, since $\|\nabla v_k\|_{L^2(\Omega)} < \frac{1}{k}$ it follows that $\nabla v = 0$ a.e. in Ω. We have that for all $k \in \mathbb{N}$

$$0 = \int_\Omega \rho_{\varepsilon_k} v_k dx = \int_\Omega \rho_{\varepsilon_k}(v_k - v)dx + \int_\Omega \rho_{\varepsilon_k} v dx.$$

It is standard to prove that

$$\lim_{k\to+\infty} \int_\Omega \rho_{\varepsilon_k} v dx = \frac{M}{|\partial\Omega|}\int_{\partial\Omega} v d\sigma,$$

see Lemma 3.1.22. Moreover, we have that

$$\int_\Omega \rho_{\varepsilon_k}(v_k - v)dx = \int_{\omega_{\varepsilon_k}} \frac{\tilde{\rho}_{\varepsilon_k}}{\varepsilon_k}(v_k - v)dx + \varepsilon \int_{\Omega\setminus\overline{\omega}_{\varepsilon_k}} (v_k - v)dx. \tag{4.2.6}$$

Since the sequence $\{v_k\}$ is bounded in $H^1(\Omega)$, the second term in the right-hand side of (4.2.6) goes to zero as k tends to $+\infty$. Also, from the fact that $v_k \to v$ strongly in $L^2(\Omega)$, and $\|\nabla v_k\|_{L^2(\Omega)}$ and $\|\nabla v\|_{L^2(\Omega)}$ are uniformly bounded in $k \in \mathbb{N}$, it is possible to prove that the first term in the right-hand side of (4.2.6) goes to zero as k tends to $+\infty$ (see Lemma 3.1.28). Then we have that $\int_{\partial\Omega} v d\sigma = 0$ and $\|v\|_{L^2(\Omega)} = 1$. Hence, by the standard Poincaré-Wirtinger inequality for functions in $H^1(\Omega)$ having zero integral mean on the boundary, we have

$$0 \le \|v\|_{L^2(\Omega)} = \left\| v - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} v d\sigma \right\|_{L^2(\Omega)} \le \tilde{C}_\Omega \|\nabla v\|_{L^2(\Omega)} = 0,$$

where the constant $\tilde{C}_\Omega$ depends only on the open set Ω. Therefore $v \equiv 0$ in $H^1(\Omega)$, which is a contradiction since $\|v\|_{L^2(\Omega)} = 1$. □

We also recall that from Corollary 3.1.42 we have

$$\lim_{\varepsilon\to 0} \lambda_j(\varepsilon) = \mu_j,$$

for all $j \in \mathbb{N}_0$. Therefore, by Corollary 3.1.42, it follows that the function $\lambda_j(\cdot)$ which takes $\varepsilon > 0$ to $\lambda_j(\varepsilon)$ can be extended with continuity at $\varepsilon = 0$ by setting $\lambda_j(0) := \mu_j$ for all $j \in \mathbb{N}_0$. Suppose that $\mu_{j-1} \lneqq \mu_j \le \mu_{j+1} \le \dots \le \mu_{j+m} \lneqq \mu_{j+m+1}$ and $\lambda_{j-1}(\varepsilon) \lneqq \lambda_j(\varepsilon) \le \lambda_{j+1}(\varepsilon) \le \dots \le \lambda_{j+m}(\varepsilon) \lneqq \lambda_{j+m+1}(\varepsilon)$ for all $\varepsilon > 0$ in a suitable right neighborhood of zero. Let us denote by $P_{j,m}(\varepsilon)$ the operator from $L^2(\Omega)$ to itself which maps a function $f \in L^2(\Omega)$ to its projection on the space generated by all the eigenfunctions associated with the eigenvalues $\lambda_j(\varepsilon), \dots, \lambda_{j+m}(\varepsilon)$, and by $Q_{j,m}$ the operator from $L^2(\Omega)$ to itself which maps a function $f \in L^2(\Omega)$ to its projection on the space generated by all the eigenfunctions associated with the eigenvalues $\mu_j, \dots, \mu_{j+m}$. Then, from Corollary 3.1.42 it follows that

$$\lim_{\varepsilon \to 0} \|P_{j,m}(\varepsilon) - Q_{j,m}\|_{\mathcal{L}(L^2(\Omega), L^2(\Omega))} = 0.$$

In particular, if μ_j is a simple eigenvalue, then there exist $\varepsilon_j > 0$ such that $\lambda_j(\varepsilon)$ is simple for all $\varepsilon \in]0, \varepsilon_j[$.

It is well-known that there exists $\varepsilon_0 > 0$ such that the map $x \mapsto x - \varepsilon\nu(x)$ is a diffeomorphism from $\partial\Omega$ to $\partial\omega_\varepsilon \cap \Omega$ for all $\varepsilon \in]0, \varepsilon_0[$, where ω_ε is defined by (3.1.19) (see Theorem 3.1.27).

We shall use curvilinear coordinates in the strip ω_ε. Let $\gamma : [0, |\partial\Omega|[\to \partial\Omega$ be the arc length parametrization of the boundary $\partial\Omega$. Then we consider the map $\psi : [0, |\partial\Omega|[\times]0, \varepsilon[\to \omega_\varepsilon$ defined by $\psi(s,t) := \gamma(s) - t\nu(\gamma(s))$ for all $(s,t) \in [0, |\partial\Omega|[\times]0, \varepsilon[$, where $\nu(\gamma(s))$ denotes the outer unit normal to $\partial\Omega$ at $\gamma(s)$. We denote by $\kappa(s)$ the signed curvature of $\partial\Omega$, namely $\kappa(s) := \gamma_1'(s)\gamma_2''(s) - \gamma_2'(s)\gamma_1''(s)$ for all $s \in [0, |\partial\Omega|[$.

In order to study problem (4.0.2) it is also convenient to introduce a change of variables by setting $\xi = \frac{t}{\varepsilon}$. Accordingly, we denote by ψ_ε the function from $[0, |\partial\Omega|[\times]0, 1[$ to ω_ε defined by $\psi_\varepsilon(s,\xi) := \gamma(s) - \varepsilon\xi\nu(\gamma(s))$ for all $(s,\xi) \in [0, |\partial\Omega|[\times]0, 1[$. The variable ξ is usually called rapid variable. Note that in this new system of coordinates (s,ξ), the strip ω_ε is transformed into a band of length $|\partial\Omega|$ and width 1 (see Figures 4.3, 4.4). Moreover, we note that if $\varepsilon < (\sup_{s\in[0,|\partial\Omega|[} |\kappa(s)|)^{-1}$, we have $1 - \varepsilon\xi\kappa(s) > 0$ for all $\xi \in]0, 1[$, so that $|\det D\psi_\varepsilon| = \varepsilon(1 - \varepsilon\xi\kappa(s))$. From now on we consider $\varepsilon \in]0, \varepsilon_0[$ where ε_0 is sufficiently small and is such that $\varepsilon_0 < (\sup_{s\in[0,|\partial\Omega|[} |\kappa(s)|)^{-1}$.

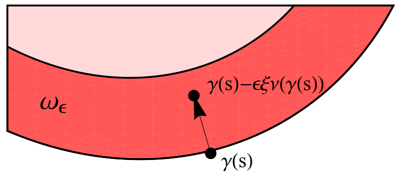


Figure 4.3

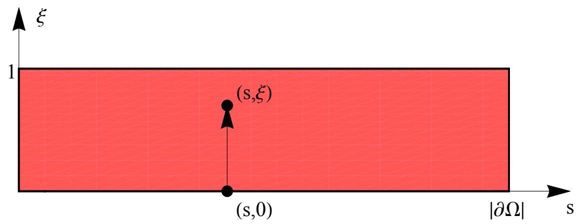


Figure 4.4

We also need to write the gradient of a function u with respect to the coordinates (s,ξ). We have

$$(\nabla u\circ\psi_\varepsilon)(s,\xi)=\begin{pmatrix}\frac{\gamma_1'(s)}{1-\varepsilon\xi\kappa(s)}\partial_s(u\circ\psi_\varepsilon(s,\xi))-\frac{\gamma_2'(s)+\varepsilon\xi\gamma_1''(s)}{\varepsilon(1-\varepsilon\xi\kappa(s))}\partial_\xi(u\circ\psi_\varepsilon(s,\xi))\\ \frac{\gamma_2'(s)}{1-\varepsilon\xi\kappa(s)}\partial_s(u\circ\psi_\varepsilon(s,\xi))+\frac{\gamma_1'(s)-\varepsilon\xi\gamma_2''(s)}{\varepsilon(1-\varepsilon\xi\kappa(s))}\partial_\xi(u\circ\psi_\varepsilon)(s,\xi)\end{pmatrix},$$

and therefore

$$\begin{aligned}&(\nabla u\circ\psi_\varepsilon\cdot\nabla v\circ\psi_\varepsilon)(s,\xi)\\ &\quad=\frac{1}{\varepsilon^2}\partial_\xi(u\circ\psi_\varepsilon(s,\xi))\partial_\xi(v\circ\psi_\varepsilon(s,\xi))+\frac{\partial_s(u\circ\psi_\varepsilon(s,\xi))\partial_s(v\circ\psi_\varepsilon(s,\xi))}{(1-\varepsilon\xi\kappa(s))^2}\\ &\quad=\frac{1}{\varepsilon^2}\partial_\xi(u\circ\psi_\varepsilon(s,\xi))\partial_\xi(v\circ\psi_\varepsilon(s,\xi))+\partial_s(u\circ\psi_\varepsilon(s,\xi))\partial_s(v\circ\psi_\varepsilon(s,\xi))\\ &\qquad+\varepsilon\xi\kappa(s)\sum_{j=1}^{+\infty}(j+1)(\varepsilon\xi\kappa(s))^{j-1}\partial_s(u\circ\psi_\varepsilon(s,\xi))\partial_s(v\circ\psi_\varepsilon(s,\xi)),\end{aligned}\tag{4.2.7}$$

for all $(s,\xi)\in[0,|\partial\Omega|[\times]0,1[$.
We observe that for all $\varepsilon<(\sup_{s\in[0,|\partial\Omega|[}|\kappa(s)|)^{-1}$ the series in the last line of (4.2.7) is convergent.

Finally, we can write $\rho_\varepsilon=\varepsilon+\frac{1}{\varepsilon}\tilde\rho_\varepsilon\chi_{\omega_\varepsilon}$, where $\chi_{\omega_\varepsilon}$ is the characteristic function of ω_ε and

$$\tilde\rho_\varepsilon:=\varepsilon\left(\frac{M-\varepsilon|\Omega\setminus\overline{\omega}_\varepsilon|}{|\omega_\varepsilon|}\right)-\varepsilon^2,\tag{4.2.8}$$

for all $\varepsilon \in]0, \varepsilon_0[$. We note that $\frac{M-\varepsilon|\Omega\setminus\overline{\omega}_\varepsilon|}{|\omega_\varepsilon|} \in O(\varepsilon^{-1})$ as $\varepsilon \to 0$. We note also that there exist $\varepsilon_0 > 0$ and $r_2 > r_1 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$ it holds

$$r_2 < \tilde{\rho}_\varepsilon < r_1. \tag{4.2.9}$$

4.2.1 Asymptotic expansions and derivatives of the eigenvalues

Let μ_j be a simple eigenvalue of problem (4.0.1). It is not restrictive to consider simple eigenvalues since the Steklov eigenvalues are generically simple (see e.g., [5, 102]).
We recall that from Corollary 3.1.42 it follows that there exists $\varepsilon_j \in]0, \varepsilon_0[$ such that for all $\varepsilon \in]0, \varepsilon_j[$ the eigenvalue $\lambda_j(\varepsilon)$ of problem (4.0.2) is simple and $\lambda_j(\varepsilon) \to \mu_j$ as $\varepsilon \to 0$. We shall prove the following theorem concerning an asymptotic expansion of $\lambda_j(\varepsilon)$.

Theorem 4.2.10. *Let $j \in \mathbb{N}_0$. Assume that μ_j is a simple eigenvalue of problem* (4.0.1)*. Then*

$$\lambda_j(\varepsilon) = \mu_j + \varepsilon\mu_j^1 + O(\varepsilon^2) \tag{4.2.11}$$

as $\varepsilon \to 0$, where

$$\mu_j^1 = \frac{|\Omega|\mu_j}{M} - \frac{|\partial\Omega|\mu_j}{M}\int_\Omega u_j^2 dx + \frac{2M\mu_j^2}{3|\partial\Omega|} + \frac{\mu_j}{2}\int_{\partial\Omega} u_j^2 \kappa d\sigma - \frac{K\mu_j}{2|\partial\Omega|}, \tag{4.2.12}$$

and $u_j \in H^1(\Omega)$ is the unique eigenfunction of problem (4.0.1) *associated with the eigenvalue μ_j such that $\int_{\partial\Omega} u_j^2 d\sigma = 1$ and $K \in \mathbb{R}$ is given by*

$$K := \int_0^{|\partial\Omega|} \kappa(s) ds. \tag{4.2.13}$$

We also prove an asymptotic expansion for the eigenfunction $u_{j,\varepsilon}$ associated with $\lambda_j(\varepsilon)$. This is contained in the following theorem.

Theorem 4.2.14. *Let $j \in \mathbb{N}_0$. Assume that μ_j is a simple eigenvalue of problem* (4.0.1) *and $\varepsilon_j > 0$ is such that $\lambda_j(\varepsilon)$ is a simple eigenvalue of problem* (4.0.2) *for all $\varepsilon \in]0, \varepsilon_j[$. Let u_j be the unique eigenfunction of problem* (4.0.1) *associated with μ_j such that $\int_{\partial\Omega} u_j^2 d\sigma = 1$. Let $u_{j,\varepsilon}$ be the unique eigenfunction of problem* (4.0.2) *corresponding to $\lambda_j(\varepsilon)$ such that $\frac{|\partial\Omega|}{M}\int_\Omega \rho_\varepsilon u_{j,\varepsilon}^2 dx = 1$ for all $\varepsilon \in]0, \varepsilon_j[$. Then there exist $u_j^1 \in H^1(\Omega)$ and $w_j, w_j^1 \in H^1([0, |\partial\Omega|[\times]0, 1[)$ such that*

$$u_{j,\varepsilon} = u_j + \varepsilon u_j^1 + \varepsilon v_j + \varepsilon^2 v_j^1 + O(\varepsilon^2) \quad \text{in } L^2(\Omega), \tag{4.2.15}$$

as $\varepsilon \to 0$, where the functions $v_j, v_j^1 \in H^1(\Omega)$ are the extensions by 0 of $w_j \circ \psi_\varepsilon^{(-1)}$ and $w_j^1 \circ \psi_\varepsilon^{(-1)}$ respectively to Ω.

We shall present explicit formulas for w_j and w_j^1 (see (4.2.16) and (4.2.32)) and we shall identify u_j^1 as the solution of a certain boundary value problem (see problem (4.2.31)).

The proof of Theorems 4.2.10 and 4.2.14 consists of two steps. In the first step (Subsection 4.2.2) we show that the quantity $\lambda_j(\varepsilon)-\mu_j$ is of order ε as ε tends to zero. Moreover, in this step we introduce the function w_j. We also show that $\|u_{j,\varepsilon}-u_j-\varepsilon v_j\|_{L^2(\Omega)}$ is of order ε as ε tends to zero. In the second step (Subsection 4.2.3) we complete the proof of Theorems 4.2.10 and 4.2.14. Moreover, we introduce the boundary value problem solved by u_j^1 and the function w_j^1. In Subsection 4.2.4 we recall some technical results on the well-posedness of the auxiliary boundary value problem solved by u_j^1.

4.2.2 First step of the proof of Theorems 4.2.10 and 4.2.14

The aim of this subsection is to prove formulas (4.2.28) and (4.2.30), that is, justify a part of the expansions (4.2.11) and (4.2.15).
Let $j\in\mathbb{N}_0$. Assume that μ_j is a simple eigenvalue of problem (4.0.1). Let u_j be the unique eigenfunction of (4.0.1) associated with μ_j such that $\int_{\partial\Omega}u_j^2d\sigma=1$. Let $\varepsilon_j\in]0,\varepsilon_0[$ be such that $\lambda_j(\varepsilon)$ is a simple eigenvalue of (4.0.2) for all $\varepsilon\in]0,\varepsilon_j[$. Let the function $w_j(s,\xi)$ from $[0,|\partial\Omega|[\times]0,1[$ to $\mathbb{R}$ be defined by

$$w_j(s,\xi):=-\frac{M\mu_j}{2|\partial\Omega|}(u_j\circ\psi_\varepsilon(s,0))\,(\xi-1)^2. \tag{4.2.16}$$

The function w_j solves the following problem

$$\begin{cases}-\partial_\xi^2 w_j(s,\xi)=\frac{M\mu_j}{|\partial\Omega|}(u_j\circ\psi_\varepsilon(s,0)), & (s,\xi)\in[0,|\partial\Omega|[\times]0,1[,\\ \partial_\xi w_j(s,0)=\frac{M\mu_j}{|\partial\Omega|}(u_j\circ\psi_\varepsilon(s,0)), & s\in[0,|\partial\Omega|[,\\ w_j(s,1)=\partial_\xi w_j(s,1)=0, & s\in[0,|\partial\Omega|[.\end{cases} \tag{4.2.17}$$

Now let $v_j\in H^1(\Omega)$ be defined as in Theorem 4.2.14. We note that by construction $v_j\in H^1(\Omega)$. We plan to apply Lemma 4.2.3 to the compact and self-adjoint operator $\mathcal{A}_\varepsilon$ acting on the Hilbert space $\mathcal{H}_\varepsilon(\Omega)$. We note that $\lambda_j(\varepsilon)$ is an eigenvalue of (4.0.2) if and only if $\frac{1}{1+\lambda_j(\varepsilon)}$ is an eigenvalue of $\mathcal{A}_\varepsilon$. Moreover, we note that $|\mu_j-\mu_{j-1}|>0$ and $|\mu_j-\mu_{j+1}|>0$, and that $\lambda_j(\varepsilon)\neq\lambda_{j-1}(\varepsilon)$ and $\lambda_j(\varepsilon)\neq\lambda_{j+1}(\varepsilon)$ for all $\varepsilon\in]0,\varepsilon_j[$. Then, by the continuity of the eigenvalues, it follows that there exists a constant $\delta_j>0$ which does not depend on $\varepsilon>0$ and such that $\left|\frac{1}{1+\mu_j}-\frac{1}{1+\lambda_{j-1}(\varepsilon)}\right|,\left|\frac{1}{1+\mu_j}-\frac{1}{1+\lambda_{j+1}(\varepsilon)}\right|>\delta_j$, and $\left|\frac{1}{1+\mu_j}-\frac{1}{1+\lambda_j(\varepsilon)}\right|\leq\delta_j$, for all $\varepsilon\in]0,\varepsilon_j[$ (possibly choosing $\varepsilon_j>0$ smaller). We plan to apply Lemma 4.2.3 with $\eta=\frac{1}{1+\mu_j}$, $u=\frac{u_j+\varepsilon v_j}{\|u_j+\varepsilon v_j\|_\varepsilon}$,

$r^* = \delta_j$ and a suitable $r > 0$. First we prove that

$$\left| \left\langle \mathcal{A}_\varepsilon(u_j + \varepsilon v_j) - \frac{1}{1+\mu_j}(u_j + \varepsilon v_j), \varphi \right\rangle_\varepsilon \right| \leq C\varepsilon \|\varphi\|_\varepsilon, \quad \forall \varphi \in H^1(\Omega), \tag{4.2.18}$$

where $C > 0$ is a constant which does not depend on ε and φ. Indeed, by (4.2.1) and (4.2.2) we have

$$\begin{aligned}
&\left| \left\langle \mathcal{A}_\varepsilon(u_j + \varepsilon v_j) - \frac{1}{1+\mu_j}(u_j + \varepsilon v_j), \varphi \right\rangle_\varepsilon \right| \\
&= \left| \int_\Omega \rho_\varepsilon u_j \varphi dx + \int_{\omega_\varepsilon} \varepsilon \rho_\varepsilon v_j \varphi dx - \frac{1}{1+\mu_j} \left(\int_\Omega \nabla u_j \cdot \nabla \varphi dx + \int_\Omega \rho_\varepsilon u_j \varphi dx \right.\right. \\
&\qquad \left.\left. + \int_{\omega_\varepsilon} \varepsilon \nabla v_j \cdot \nabla \varphi dx + \int_{\omega_\varepsilon} \varepsilon \rho_\varepsilon v_j \varphi dx \right) \right| \\
&= \frac{\mu_j}{1+\mu_j} \left| \varepsilon \int_\Omega u_j \varphi dx + \int_{\omega_\varepsilon} \frac{1}{\varepsilon} \tilde{\rho}_\varepsilon u_j \varphi dx - \frac{M}{|\partial\Omega|} \int_{\partial\Omega} u_j \varphi d\sigma \right. \\
&\qquad \left. + \varepsilon \int_{\omega_\varepsilon} \rho_\varepsilon v_j \varphi dx - \frac{\varepsilon}{\mu_j} \int_{\omega_\varepsilon} \nabla v_j \cdot \nabla \varphi dx \right|.
\end{aligned} \tag{4.2.19}$$

We introduce the following quantities:

$$\begin{aligned}
J_{1,\varepsilon} &:= \varepsilon \int_\Omega u_j \varphi dx, \\
J_{2,\varepsilon} &:= \int_{\omega_\varepsilon} \frac{1}{\varepsilon} \tilde{\rho}_\varepsilon u_j \varphi dx, \\
J_{3,\varepsilon} &:= \frac{M}{|\partial\Omega|} \int_{\partial\Omega} u_j \varphi d\sigma, \\
J_{4,\varepsilon} &:= \varepsilon \int_{\omega_\varepsilon} \rho_\varepsilon v_j \varphi dx, \\
J_{5,\varepsilon} &:= \varepsilon \int_{\omega_\varepsilon} \nabla v_j \cdot \nabla \varphi dx.
\end{aligned}$$

We have that the expression inside the absolute value which appears in the last term of (4.2.19) equals

$$J_{1,\varepsilon} + J_{2,\varepsilon} - J_{3,\varepsilon} + J_{4,\varepsilon} - \frac{1}{\mu_j} J_{5,\varepsilon}.$$

We study the quantities $J_{1,\varepsilon}$, $J_{2,\varepsilon}$, $J_{3,\varepsilon}$, $J_{4,\varepsilon}$ and $J_{5,\varepsilon}$ separately. Through all the rest of the section we will denote by C a positive constant which does not depend on ε and φ (and which can eventually be re-defined line by line). We consider $J_{1,\varepsilon}$ first. We have

$$J_{1,\varepsilon} = \varepsilon \int_\Omega u_j \varphi dx \leq C\varepsilon \|u_j\|_\varepsilon \|\varphi\|_\varepsilon. \tag{4.2.20}$$

This follows from the Hölder inequality $\int_\Omega u_j\varphi dx \le \|u_j\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)}$ and from Lemma 4.2.4. In fact, given $\varphi \in H^1(\Omega)$ we have

$$\|\varphi\|_{L^2(\Omega)} = \left\|\varphi - \frac{1}{M}\int_\Omega \rho_\varepsilon \varphi dx + \frac{1}{M}\int_\Omega \rho_\varepsilon\varphi dx\right\|_{L^2(\Omega)}$$
$$\le \left\|\varphi - \frac{1}{M}\int_\Omega \rho_\varepsilon\varphi dx\right\|_{L^2(\Omega)} + \left\|\frac{1}{M}\int_\Omega \rho_\varepsilon\varphi dx\right\|_{L^2(\Omega)}$$
$$\le C_\Omega\|\nabla\varphi\|_{L^2(\Omega)} + \frac{|\Omega|^{\frac12}}{M^{\frac12}}\left(\int_\Omega \rho_\varepsilon\varphi^2 dx\right)^{\frac12} \le \max\left\{C_\Omega, \frac{|\Omega|^{\frac12}}{M^{\frac12}}\right\}\|\varphi\|_\varepsilon, \quad (4.2.21)$$

where in the second inequality we have used the fact that $\rho_\varepsilon > 0$ in the following way:

$$\int_\Omega \rho_\varepsilon\varphi dx = \int_\Omega \rho_\varepsilon^{\frac12}\rho_\varepsilon^{\frac12}\varphi dx \le \left(\int_\Omega \rho_\varepsilon dx\right)^{\frac12}\left(\int_\Omega \rho_\varepsilon\varphi^2 dx\right)^{\frac12}.$$

Moreover, we observe that $\|u_j\|_\varepsilon \le C$. This follows from the fact that $\lim_{\varepsilon\to0}\int_\Omega \rho_\varepsilon u_j^2 dx = \frac{M}{|\partial\Omega|}\int_{\partial\Omega} u_j^2 d\sigma = \frac{M}{|\partial\Omega|}$.
We now consider $J_{2,\varepsilon}$.

$$J_{2,\varepsilon} = \int_{\omega_\varepsilon}\frac{1}{\varepsilon}\tilde\rho_\varepsilon u_j\varphi dx$$
$$= \int_0^{|\partial\Omega|}\int_0^1 \tilde\rho_\varepsilon(u_j\circ\psi_\varepsilon(s,\xi))(\varphi\circ\psi_\varepsilon(s,\xi))(1-\varepsilon\xi\kappa(s))d\xi ds$$
$$= \int_0^{|\partial\Omega|}\int_0^1 \tilde\rho_\varepsilon(u_j\circ\psi_\varepsilon)(\varphi\circ\psi_\varepsilon)d\xi ds$$
$$- \int_0^{|\partial\Omega|}\int_0^1 \tilde\rho_\varepsilon(u_j\circ\psi_\varepsilon(s,\xi))(\varphi\circ\psi_\varepsilon(s,\xi))\frac{\xi\kappa(s)}{1-\varepsilon\xi\kappa(s)}\varepsilon(1-\varepsilon\xi\kappa(s))d\xi ds$$
$$\le \int_0^{|\partial\Omega|}\int_0^1 \tilde\rho_\varepsilon(u_j\circ\psi_\varepsilon)(\varphi\circ\psi_\varepsilon)d\xi ds + C\varepsilon\int_{\omega_\varepsilon}\frac{\tilde\rho_\varepsilon}{\varepsilon}|u_j\varphi|dx$$
$$\le \int_0^{|\partial\Omega|}\int_0^1 \tilde\rho_\varepsilon(u_j\circ\psi_\varepsilon)(\varphi\circ\psi_\varepsilon)d\xi ds + C\varepsilon\left\|u_j\right\|_\varepsilon\left\|\varphi\right\|_\varepsilon.$$

Then

$$J_{1,\varepsilon} + J_{2,\varepsilon} - J_{3,\varepsilon} + \frac{1}{\mu_j}J_{5,\varepsilon}$$
$$= \varepsilon\int_\Omega u_j\varphi dx + \int_{\omega_\varepsilon}\frac{1}{\varepsilon}\tilde\rho_\varepsilon u_j\varphi dx$$
$$- \frac{M}{|\partial\Omega|}\int_0^{|\partial\Omega|}\int_0^1 (u_j\circ\psi_\varepsilon(s,0))(\varphi\circ\psi_\varepsilon(s,\xi))d\xi ds$$

$$\leq \int_0^{|\partial\Omega|}\int_0^1 \tilde{\rho}_\varepsilon (u_j\circ\psi_\varepsilon)(\varphi\circ\psi_\varepsilon)d\xi ds$$
$$-\frac{M}{|\partial\Omega|}\int_0^{|\partial\Omega|}\int_0^1 (u_j\circ\psi_\varepsilon(s,0))(\varphi\circ\psi_\varepsilon(s,\xi))d\xi ds + C\varepsilon\left\|u_j\right\|_\varepsilon\left\|\varphi\right\|_\varepsilon .$$

Consider now

$$\int_0^{|\partial\Omega|}\int_0^1\left(\tilde{\rho}_\varepsilon(u_j\circ\psi_\varepsilon(s,\xi))-\frac{M}{|\partial\Omega|}(u_j\circ\psi_\varepsilon(s,0))\right)(\varphi\circ\psi_\varepsilon(s,\xi))d\xi ds$$
$$=\int_0^{|\partial\Omega|}\int_0^1 \tilde{\rho}_\varepsilon\left(u_j\circ\psi_\varepsilon(s,\xi))-(u_j\circ\psi_\varepsilon(s,0))\right)(\varphi\circ\psi_\varepsilon(s,\xi))d\xi ds$$
$$+\int_0^{|\partial\Omega|}\int_0^1 (u_j\circ\psi_\varepsilon(s,0))\left(\tilde{\rho}_\varepsilon-\frac{M}{|\partial\Omega|}\right)(\varphi\circ\psi_\varepsilon(s,\xi))d\xi ds.$$

We note that $|\omega_\varepsilon| = \varepsilon|\partial\Omega| - \frac{\varepsilon^2}{2}\int_0^{|\partial\Omega|}\kappa(s)ds = \varepsilon|\partial\Omega| - \frac{\varepsilon^2}{2}K$, where K is defined by (4.2.13). From standard Taylor's expansions of the right-hand side of (4.2.8) it follows that

$$\tilde{\rho}_\varepsilon = \frac{M}{|\partial\Omega|} + \frac{\frac{1}{2}KM - |\Omega||\partial\Omega|}{|\partial\Omega|^2}\varepsilon + F(\varepsilon), \tag{4.2.22}$$

where $F(\varepsilon) \in O(\varepsilon^2)$ as $\varepsilon \to 0$. It follows that

$$\int_0^{|\partial\Omega|}\int_0^1 (u_j\circ\psi_\varepsilon(s,0))\left(\tilde{\rho}_\varepsilon-\frac{M}{|\partial\Omega|}\right)(\varphi\circ\psi_\varepsilon(s,\xi))d\xi ds$$
$$\leq C\varepsilon\int_0^{|\partial\Omega|}\int_0^1 |(u_j\circ\psi_\varepsilon(s,0))(\varphi\circ\psi_\varepsilon(s,\xi))|d\xi ds$$
$$\leq C\varepsilon\|u_j\|_{L^2(\partial\Omega)}\left(\int_0^{|\partial\Omega|}\int_0^1(\varphi\circ\psi_\varepsilon(s,\xi))^2\frac{\varepsilon(1-\varepsilon\xi\kappa(s))}{\varepsilon(1-\varepsilon\xi\kappa(s))}d\xi ds\right)^{\frac{1}{2}}$$
$$\leq C\varepsilon\|u_j\|_\varepsilon\left(\int_{\omega_\varepsilon}\frac{1}{\varepsilon}\varphi^2 dx\right)^{\frac{1}{2}} \leq C\varepsilon\left\|u_j\right\|_\varepsilon\left\|\varphi\right\|_\varepsilon ,$$

where in the last line we have used the fact that $\|u_j\|_{L^2(\partial\Omega)} \leq C_\Omega\|\nabla u_j\|_{L^2(\Omega)}$ and (4.2.9). Moreover, since u_j is a solution of (4.0.1), by standard elliptic regularity (see e.g., [2]), it follows that $u_j \in C^2(\Omega)\cap C^{1,\alpha}(\overline{\Omega})$ for all $\alpha \in]0,1[$ (note that we only assume Ω of class C^2). Since ψ_ε is a diffeomorphism of class C^2, for all $(s,\xi) \in [0,|\partial\Omega|[\times]0,1[$ we have

$$(u_j\circ\psi_\varepsilon(s,\xi)) - (u_j\circ\psi_\varepsilon(s,0)) = \xi\partial_\xi(u_j\circ\psi_\varepsilon)(s,\xi^*),$$

for some $\xi^* \in (0,\xi)$. Then, by recalling the definition of ψ and setting $t^* := \varepsilon\xi^*$, it follows that

$$
\begin{aligned}
\int_0^{|\partial\Omega|}\int_0^1 &\tilde{\rho}_\varepsilon\left((u_j\circ\psi_\varepsilon(s,\xi)) - (u_j\circ\psi_\varepsilon(s,0))\right)\varphi d\xi ds \\
&= \tilde{\rho}_\varepsilon \int_0^{|\partial\Omega|}\int_0^1 \xi\partial_\xi(u_j\circ\psi_\varepsilon)(s,\xi^*)(\varphi\circ\psi_\varepsilon(s,\xi))d\xi ds \\
&= \tilde{\rho}_\varepsilon \int_0^{|\partial\Omega|}\int_0^\varepsilon t\partial_t(u_j\circ\psi)(s,t^*)(\varphi\circ\psi(s,t))\frac{dt}{\varepsilon}ds \\
&\le C\|u_j\|_{C^1(\overline{\Omega})}\int_0^{|\partial\Omega|}\int_0^\varepsilon \frac{t}{\varepsilon^{\frac12}}\frac{(\varphi\circ\psi)}{\varepsilon^{\frac12}}dtds \\
&\le \frac{C\|u_j\|_{C^1(\overline{\Omega})}\varepsilon}{\sqrt{3}}\int_0^{|\partial\Omega|}\left(\int_0^\varepsilon \frac{(\varphi\circ\psi)^2}{\varepsilon}dt\right)^{\frac12} ds \le C\varepsilon\|\varphi\|_\varepsilon, \quad (4.2.23)
\end{aligned}
$$

where we have exploited Hölder inequality to prove the second inequality. We consider now $J_{4,\varepsilon}$.

$$
J_{4,\varepsilon} = \varepsilon\int_{\omega_\varepsilon}\rho_\varepsilon v_j\varphi dx \le \varepsilon\|v_j\|_\varepsilon\|\varphi\|_\varepsilon.
$$

From the definition of v_j (see (4.2.16)) it is standard to prove that $\|v_j\|_\varepsilon \le C$.

We consider $J_{5,\varepsilon}$ and pass to the coordinates (s,ξ). We use formula (4.2.7) and we obtain

$$
\begin{aligned}
J_{5,\varepsilon} &= \varepsilon\int_{\omega_\varepsilon}\nabla v_j\nabla\varphi dx \\
&= \varepsilon^2\int_0^{|\partial\Omega|}\int_0^1\Bigg(\frac{1}{\varepsilon^2}\partial_\xi w_j(s,\xi)\partial_\xi(\varphi\circ\psi_\varepsilon)(s,\xi) + \partial_s w_j(s,\xi)\partial_s(\varphi\circ\psi_\varepsilon)(s,\xi) \\
&+\varepsilon\xi\kappa(s)\xi\sum_{j=1}^{+\infty}(j+1)(\varepsilon\xi\kappa(s))^{j-1}\partial_s w_j(s,\xi)\partial_s(\varphi\circ\psi_\varepsilon)(s,\xi)\Bigg)(1-\varepsilon\xi\kappa(s))d\xi ds \\
&\le \int_0^{|\partial\Omega|}\int_0^1 \partial_\xi w_j(s,\xi)\partial_\xi(\varphi\circ\psi_\varepsilon)(s,\xi)d\xi ds + C\varepsilon\|\nabla\varphi\|_{L^2(\Omega)} \\
&= -\frac{M\mu_j}{|\partial\Omega|}\int_{\partial\Omega}u_j\varphi d\sigma + \frac{M\mu_j}{|\partial\Omega|}\int_0^{|\partial\Omega|}\int_0^1(u_j\circ\psi_\varepsilon(s,0))(\varphi\circ\psi_\varepsilon(s,\xi))d\xi ds \\
&\qquad + C\varepsilon\|\nabla\varphi\|_{L^2(\Omega)}.
\end{aligned}
$$

Thus inequality (4.2.18) is proved.

We prove now that $\|u_j+\varepsilon v_j\|_\varepsilon$ is uniformly bounded for $\varepsilon\in]0,\varepsilon_0[$. This follows by the following lemma.

Lemma 4.2.24. *Let u_j be the unique eigenfunction of* (4.0.1) *associated with the eigenvalue μ_j such that $\int_{\partial\Omega} u_j^2 d\sigma = 1$. Then there exists a constant C which does not depend on ε such that for all $\varepsilon \in]0, \varepsilon_0[$ it holds*

$$\|u_j + \varepsilon v_j\|_\varepsilon^2 = \frac{M}{|\partial\Omega|}(1 + \mu_j) + O(\varepsilon), \tag{4.2.25}$$

as $\varepsilon \to 0$.

Proof. We use the explicit formula for $\|\cdot\|_\varepsilon$ and we write

$$\|u_j + \varepsilon v_j\|_\varepsilon^2 = L_{1,\varepsilon} + L_{2,\varepsilon} + L_{3,\varepsilon},$$

where

$$\begin{aligned} L_{1,\varepsilon} &:= \int_\Omega \rho_\varepsilon u_j^2 dx, \\ L_{2,\varepsilon} &:= \int_\Omega |\nabla u_j|^2 dx, \\ L_{3,\varepsilon} &:= \varepsilon^2 \int_{\omega_\varepsilon} v_j^2 dx - 2\varepsilon \int_{\omega_\varepsilon} u_j v_j dx + \varepsilon^2 \int_{\omega_\varepsilon} |\nabla v_j|^2 dx - 2\varepsilon \int_{\omega_\varepsilon} \nabla u_j \cdot \nabla v_j dx. \end{aligned}$$

We consider $L_{1,\varepsilon} - \frac{M}{|\partial\Omega|}$ first. We have

$$\begin{aligned} L_{1,\varepsilon} - \frac{M}{|\partial\Omega|} &= L_{1,\varepsilon} - \frac{M}{|\partial\Omega|} \int_{\partial\Omega} u_j^2 d\sigma \\ &= \frac{1}{\varepsilon} \int_{\omega_\varepsilon} \tilde{\rho}_\varepsilon u_j^2 dx - \frac{M}{|\partial\Omega|} \int_{\partial\Omega} u_j^2 d\sigma + \varepsilon \int_\Omega u_j^2 dx \\ &= \frac{M}{|\partial\Omega|} \frac{1}{\varepsilon} \int_{\omega_\varepsilon} u_j^2 dx - \frac{M}{|\partial\Omega|} \int_{\partial\Omega} u_j^2 d\sigma \\ &+ \int_{\omega_\varepsilon} \left(\frac{\frac{1}{2}KM - |\Omega||\partial\Omega|}{|\partial\Omega|^2} + \frac{F(\varepsilon)}{\varepsilon} \right) u_j^2 dx + \varepsilon \int_\Omega u_j^2 dx \\ &\leq \frac{M}{|\partial\Omega|} \frac{1}{\varepsilon} \int_{\omega_\varepsilon} u_j^2 dx - \frac{M}{|\partial\Omega|} \int_{\partial\Omega} u_j^2 d\sigma + C\|u_j\|_{C(\overline{\Omega})}^2 \varepsilon. \end{aligned}$$

We consider $\frac{1}{\varepsilon}\int_{\omega_\varepsilon} u_j^2 dx - \int_{\partial\Omega} u_j^2 d\sigma$. We have

$$\begin{aligned} &\frac{1}{\varepsilon} \int_{\omega_\varepsilon} u_j^2 dx - \int_{\partial\Omega} u_j^2 d\sigma \\ &= \int_0^{|\partial\Omega|} \frac{1}{\varepsilon} \left(\int_0^\varepsilon (u_j \circ \psi(s,t))^2 (1 - t\kappa(s)) - (u_j \circ \psi(s,0))^2 dt \right) ds \\ &\leq \int_0^{|\partial\Omega|} \frac{1}{\varepsilon} \left(C\|u_j\|_{C^1(\overline{\Omega})}^2 \int_0^\varepsilon t dt \right) ds \leq C\varepsilon. \end{aligned}$$

We consider now $L_{2,\varepsilon}$. Since u_j is an eigenfunction of (4.0.1) and is normalized by $\int_{\partial\Omega} u_j^2 d\sigma = 1$, we have

$$L_{2,\varepsilon} = \int_{\Omega} |\nabla u_j|^2 dx = \frac{M\mu_j}{|\partial\Omega|} \int_{\partial\Omega} u_j^2 d\sigma = \frac{M\mu_j}{|\partial\Omega|}.$$

Finally, it is standard to prove that $|L_{3,\varepsilon}| \leq C\varepsilon$. We prove the result for the fourth summand in $L_{3,\varepsilon}$. The result for the other summands is obtained in a similar way. It is convenient to use the coordinates (s,t) on ω_ε. We have

$$\begin{aligned}\varepsilon \int_{\omega_\varepsilon} \nabla u_j \cdot \nabla v_j dx \\ &= \varepsilon \int_0^{|\partial\Omega|} \int_0^{\varepsilon} \Big(\partial_t (u_j \circ \psi)(s,t) \partial_t w_j (s, \frac{t}{\varepsilon}) \\ &+ \frac{\partial_s (u_j \circ \psi)(s,t) \partial_s w_j (s, \frac{t}{\varepsilon})}{(1 - t\kappa(s))^2} \Big) (1 - t\kappa(s)) dt ds \\ &\leq C\varepsilon \int_0^{|\partial\Omega|} \int_0^{\varepsilon} \frac{\|u_j\|^2_{C^1(\overline{\Omega})}}{\varepsilon} dt ds \leq C\varepsilon.\end{aligned}$$

This concludes the proof of Lemma 4.2.24. □

In order to apply Lemma 4.2.3 it is sufficient to multiply both sides of (4.2.18) by $\|u_j + \varepsilon v_j\|_\varepsilon^{-1}$. Thanks to Lemma 4.2.24 we have

$$\left| \left\langle \mathcal{A}_\varepsilon \left(\frac{u_j + \varepsilon v_j}{\|u_j + \varepsilon v_j\|_\varepsilon} \right) - \frac{1}{1+\mu_j} \left(\frac{u_j + \varepsilon v_j}{\|u_j + \varepsilon v_j\|_\varepsilon} \right), \varphi \right\rangle_\varepsilon \right| \leq C \sqrt{\frac{|\partial\Omega|}{M}} (1+\mu_j)^{-\frac{1}{2}} \varepsilon \|\varphi\|_\varepsilon, \quad \forall \varphi \in H^1(\Omega). \quad (4.2.26)$$

Now we set $\varphi := \mathcal{A}_\varepsilon \left(\frac{u_j + \varepsilon v_j}{\|u_j + \varepsilon v_j\|_\varepsilon} \right) - \frac{1}{1+\mu_j} \left(\frac{u_j + \varepsilon v_j}{\|u_j + \varepsilon v_j\|_\varepsilon} \right)$ in (4.2.26) and we obtain

$$\left\| \mathcal{A}_\varepsilon \left(\frac{u_j + \varepsilon v_j}{\|u_j + \varepsilon v_j\|_\varepsilon} \right) - \frac{1}{1+\mu_j} \left(\frac{u_j + \varepsilon v_j}{\|u_j + \varepsilon v_j\|_\varepsilon} \right) \right\|_\varepsilon \leq C \sqrt{\frac{|\partial\Omega|}{M}} (1+\mu_j)^{-\frac{1}{2}} \varepsilon.$$

Then we apply Lemma 4.2.3 with $\eta = \frac{1}{1+\mu_j}$, $u = \frac{u_j + \varepsilon v_j}{\|u_j + \varepsilon v_j\|_\varepsilon}$, $r^* = \delta_j$ and $r = C\sqrt{\frac{|\partial\Omega|}{M}} (1+\mu_j)^{-\frac{1}{2}} \varepsilon$. It follows that there exists an eigenvalue η^* of $\mathcal{A}_\varepsilon$ such that

$$\left| \frac{1}{1+\mu_j} - \eta^* \right| \leq C \sqrt{\frac{|\partial\Omega|}{M}} (1+\mu_j)^{-\frac{1}{2}} \varepsilon. \quad (4.2.27)$$

When $\varepsilon \in]0, \varepsilon_j[$ (possibly choosing ε_j smaller), the only eigenvalue of $\mathcal{A}_\varepsilon$ which satisfies (4.2.27) is $\frac{1}{1+\lambda_j(\varepsilon)}$. Thus we have that

$$\lambda_j(\varepsilon) = \mu_j + O(\varepsilon), \quad (4.2.28)$$

as $\varepsilon \to 0$. Let $u_{j,\varepsilon}$ be the unique eigenfunction of (4.0.2) associated with the eigenvalue $\lambda_j(\varepsilon)$ such that $\frac{|\partial\Omega|}{M}\int_\Omega \rho_\varepsilon u_{j,\varepsilon}^2 dx = 1$. We observe that

$$\|u_{j,\varepsilon}\|_\varepsilon^2 = \int_\Omega \rho_\varepsilon u_{j,\varepsilon}^2 dx + \int_\Omega |\nabla u_{j,\varepsilon}|^2 dx = \frac{M}{|\partial\Omega|}(1+\lambda_j(\varepsilon)).$$

From (4.2.25) and (4.2.28) it follows that there exists a constant $c > 0$ which does not depend on $\varepsilon \in]0, \varepsilon_j[$ such that

$$\left|(1+\lambda_j(\varepsilon))^{\frac{1}{2}} - (1+\mu_j)^{\frac{1}{2}}\right| \le c\varepsilon$$

and

$$\left|\|u_j + \varepsilon v_j\|_\varepsilon - \frac{M^{\frac{1}{2}}}{|\partial\Omega|^{\frac{1}{2}}}(1+\mu_j)^{\frac{1}{2}}\right| \le c\varepsilon.$$

From this fact it follows that

$$\begin{aligned}
&\|u_{j,\varepsilon} - u_j - \varepsilon v_j\|_{L^2(\Omega)} \\
&\qquad = \frac{M^{\frac{1}{2}}}{|\partial\Omega|^{\frac{1}{2}}}(1+\mu_j)^{\frac{1}{2}} \left\| \left(\frac{\|u_{j,\varepsilon}\|_\varepsilon}{\frac{M^{\frac{1}{2}}}{|\partial\Omega|^{\frac{1}{2}}}(1+\mu_j)^{\frac{1}{2}}} \right) \frac{u_{j,\varepsilon}}{\|u_{j,\varepsilon}\|_\varepsilon} \right. \\
&\qquad\qquad \left. - \left(\frac{\|u_j+\varepsilon v_j\|_\varepsilon}{\frac{M^{\frac{1}{2}}}{|\partial\Omega|^{\frac{1}{2}}}(1+\mu_j)^{\frac{1}{2}}} \right) \frac{u_j+\varepsilon v_j}{\|u_j+\varepsilon v_j\|_\varepsilon} \right\|_{L^2(\Omega)} \\
&\le \frac{M^{\frac{1}{2}}}{|\partial\Omega|^{\frac{1}{2}}}(1+\mu_j)^{\frac{1}{2}} \left\| \frac{u_{j,\varepsilon}}{\|u_{j,\varepsilon}\|_\varepsilon} + c'\varepsilon \frac{u_{j,\varepsilon}}{\|u_{j,\varepsilon}\|_\varepsilon} - \frac{u_j+\varepsilon v_j}{\|u_j+\varepsilon v_j\|_\varepsilon} - c'\varepsilon \frac{u_j+\varepsilon v_j}{\|u_j+\varepsilon v_j\|_\varepsilon} \right\|_{L^2(\Omega)} \\
&\qquad \le \frac{M^{\frac{1}{2}}}{|\partial\Omega|^{\frac{1}{2}}}(1+\mu_j)^{\frac{1}{2}} \left\| \frac{u_{j,\varepsilon}}{\|u_{j,\varepsilon}\|_\varepsilon} - \frac{u_j+\varepsilon v_j}{\|u_j+\varepsilon v_j\|_\varepsilon} \right\|_{L^2(\Omega)} \\
&\qquad\qquad + \frac{M^{\frac{1}{2}}}{|\partial\Omega|^{\frac{1}{2}}}(1+\mu_j)^{\frac{1}{2}} \frac{c'\varepsilon}{\|u_{j,\varepsilon}\|_\varepsilon} \|u_{j,\varepsilon}\|_{L^2(\Omega)} \\
&\qquad + \frac{M^{\frac{1}{2}}}{|\partial\Omega|^{\frac{1}{2}}}(1+\mu_j)^{\frac{1}{2}} \frac{c'\varepsilon}{\|u_j+\varepsilon v_j\|_\varepsilon} \|u_j+\varepsilon v_j\|_{L^2(\Omega)} \\
&\qquad \le \max\left\{ \frac{C_\Omega M^{\frac{1}{2}}}{|\partial\Omega|^{\frac{1}{2}}}, \frac{|\Omega|^{\frac{1}{2}}}{|\partial\Omega|^{\frac{1}{2}}} \right\} (1+\mu_j)^{\frac{1}{2}} \left\| \frac{u_{j,\varepsilon}}{\|u_{j,\varepsilon}\|_\varepsilon} - \frac{u_j+\varepsilon v_j}{\|u_j+\varepsilon v_j\|_\varepsilon} \right\|_\varepsilon \\
&\qquad\qquad + \max\left\{ \frac{C_\Omega M^{\frac{1}{2}}}{|\partial\Omega|^{\frac{1}{2}}}, \frac{|\Omega|^{\frac{1}{2}}}{|\partial\Omega|^{\frac{1}{2}}} \right\} (1+\mu_j)^{\frac{1}{2}} 2c'\varepsilon, \quad (4.2.29)
\end{aligned}$$

for a suitable $c' > 0$ which does not depend on $\varepsilon \in]0, \varepsilon_j[$.
In the last inequality we have used the fact that for all $f \in H^1(\Omega)$, $\|f\|_{L^2(\Omega)} \le \max\left\{C_\Omega, \frac{|\Omega|^{\frac{1}{2}}}{M^{\frac{1}{2}}}\right\} \|f\|_\varepsilon$ (see also formula (4.2.20)). From Lemma 4.2.3, it follows that there exists a function $u^* \in \mathcal{H}_\varepsilon(\Omega)$ with $\|u^*\|_\varepsilon = 1$ belonging to

the space generated by all the eigenfunctions associated with the eigenvalues lying in the segment $\left[\frac{1}{1+\mu_j}-\delta_j, \frac{1}{1+\mu_j}+\delta_j\right]$ such that

$$\left\|u^* - \frac{u_j+\varepsilon v_j}{\|u_j+\varepsilon v_j\|_\varepsilon}\right\|_\varepsilon \leq \frac{2C}{\delta_j}\sqrt{\frac{|\partial\Omega|}{M}}(1+\mu_j)^{\frac{1}{2}}\varepsilon.$$

Since for all $\varepsilon \in]0, \varepsilon_j[$, $\frac{1}{1+\lambda_j(\varepsilon)}$ is the only eigenvalue of $\mathcal{A}_\varepsilon$ lying in the segment $\left[\frac{1}{1+\mu_j}-\delta_j, \frac{1}{1+\mu_j}+\delta_j\right]$, and $\lambda_j(\varepsilon)$ is simple, it follows that necessarily $u^* = \frac{u_{j,\varepsilon}}{\|u_{j,\varepsilon}\|_\varepsilon}$. Thus, there exists $c_j > 0$ (possibly depending on j) and which does not depend on ε, such that

$$\left\|\frac{u_{j,\varepsilon}}{\|u_{j,\varepsilon}\|_\varepsilon} - \frac{u_j+\varepsilon v_j}{\|u_j+\varepsilon v_j\|_\varepsilon}\right\|_\varepsilon \leq c_j\varepsilon,$$

and therefore

$$\|u_{j,\varepsilon}-u_j-\varepsilon v_j\|_{L^2(\Omega)} \leq \max\left\{\frac{C_\Omega M^{\frac{1}{2}}}{|\partial\Omega|^{\frac{1}{2}}}, \frac{|\Omega|^{\frac{1}{2}}}{|\partial\Omega|^{\frac{1}{2}}}\right\}(1+\mu_j)^{\frac{1}{2}}(2c'+c_j)\varepsilon. \quad (4.2.30)$$

This concludes the first step of the proof of Theorems (4.2.10) and (4.2.14).

4.2.3 Second Step of the proof of Theorems 4.2.10 and 4.2.14

The aim of this subsection is to complete the justification of (4.2.11) and (4.2.15) and therefore to complete the proof of Theorems 4.2.10 and 4.2.14. Let $\mu_j^1 \in \mathbb{R}$ be defined as in (4.2.12). Let u_j be the unique solution of problem (4.0.1) such that $\int_{\partial\Omega} u_j^2 d\sigma = 1$. Then by using Theorem 4.2.66 in Subsection 4.2.4, there exists a unique weak solution u_j^1 of the boundary value problem

$$\begin{cases} -\Delta u_j^1 = \mu_j u_j & \text{in } \Omega, \\ \partial_\nu u_j^1 - \frac{M\mu_j}{|\partial\Omega|}u_j^1 = \left(\frac{M\mu_j}{2|\partial\Omega|^2}(K-|\partial\Omega|\kappa) - \frac{2M^2\mu_j^2}{3|\partial\Omega|^2} - \frac{|\Omega|\mu_j}{|\partial\Omega|}\right) + \frac{M\mu_j^1}{|\partial\Omega|}u_j & \text{on } \partial\Omega, \end{cases} \quad (4.2.31)$$

which satisfies the following conditions

$$\int_{\partial\Omega} u_j^1 u_j d\sigma = \left(\frac{\mu_j^1}{2\mu_j} + \frac{M\mu_j}{3|\partial\Omega|}\right)$$

and

$$\int_{\partial\Omega} \left(u_j^1\right)^2 d\sigma = 1 + \left(\frac{\mu_j^1}{2\mu_j} + \frac{M\mu_j}{3|\partial\Omega|}\right)^2.$$

In fact, from Theorem 4.2.66 it follows that the solution of problem (4.2.31) is unique up to multiples of u_j. Moreover, by standard elliptic regularity (see e.g., [2]), $u_j^1 \in C^2(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, for all $\alpha \in]0, 1[$.

Next, we introduce the function $w_j^1(s,\xi)$ from $[0,|\partial\Omega|[\times]0,1[$ to $\mathbb{R}$ defined by

$$\begin{aligned} w_j^1(s,\xi) :=& -\frac{\kappa(s)M\mu_j}{6|\partial\Omega|}(u_j\circ\psi_\varepsilon(s,0))(\xi-1)^3 \\ &+\frac{M^2\mu_j^2}{24|\partial\Omega|^2}(u_j\circ\psi_\varepsilon(s,0))(\xi^2+2\xi+9)(\xi-1)^2 \\ &+\Bigg(\frac{|\Omega|\mu_j}{2|\partial\Omega|}(u_j\circ\psi_\varepsilon(s,0)) - \frac{M}{2|\partial\Omega|}(\mu_j(u_j^1\circ\psi_\varepsilon(s,0)) \\ &+\mu_j^1(u_j\circ\psi_\varepsilon(s,0))) - \frac{KM\mu_j}{4|\partial\Omega|^2}(u_j\circ\psi_\varepsilon(s,0))\Bigg)(\xi-1)^2, \quad (4.2.32) \end{aligned}$$

for all $(s,\xi)\in[0,|\partial\Omega|[\times]0,1[$. (See also (4.2.13) for the definition of K.) We note that the function w_j^1 solves the following differential equation

$$\begin{aligned} -\partial_\xi^2 w_j^1(s,\xi) =& -\kappa(s)\partial_\xi w_j(s,\xi) + \frac{M}{|\partial\Omega|}\Bigg(\mu_j(u_j^1\circ\psi_\varepsilon(s,0)) + \mu_j w_j(s,\xi) \\ &+\mu_j^1(u_j\circ\psi_\varepsilon(s,0)) - \xi\frac{M\mu_j^2}{|\partial\Omega|}(u_j\circ\psi_\varepsilon(s,0)) \\ &-\frac{|\Omega|\mu_j}{M}(u_j\circ\psi_\varepsilon(s,0)) + \frac{K\mu_j}{2|\partial\Omega|}(u_j\circ\psi_\varepsilon(s,0))\Bigg) \quad (4.2.33) \end{aligned}$$

for all $(s,\xi)\in[0,\partial\Omega)\times]0,1[$. Moreover, on the boundary we have

$$w_j^1(s,1) = \partial_\xi w_j^1(s,1) = 0 \quad (4.2.34)$$

for all $s\in[0,|\partial\Omega|[$. Now let $v_j^1\in H^1(\Omega)$ be defined as in Theorem 4.2.14. We note that by construction $v_j^1\in H^1(\Omega)$. We plan to apply again Lemma 4.2.3. To do so, we prove that there exists a constant $C>0$ which does not depend on ε such that

$$\begin{aligned} \left|\left\langle \mathcal{A}_\varepsilon(u_j+\varepsilon v_j+\varepsilon u_j^1+\varepsilon^2 v_j^1) - \frac{1}{1+\mu_j+\varepsilon\mu_j^1}(u_j+\varepsilon v_j+\varepsilon u_j^1+\varepsilon^2 v_j^1),\varphi\right\rangle_\varepsilon\right| \\ \le C\varepsilon^2\|\varphi\|_\varepsilon, \quad (4.2.35) \end{aligned}$$

for all $\varphi\in H^1(\Omega)$ and all $\varepsilon\in]0,\varepsilon_0[$. As usual, through this subsection we denote by C a positive constant which does not depend on ε and φ. The constant C may eventually be re-defined line by line.

Consider the left-hand side of (4.2.35). We have

$$\left\langle \mathcal{A}_\varepsilon(u_j + \varepsilon v_j + \varepsilon u_j^1 + \varepsilon^2 v_j^1) - \frac{1}{1+\mu_j+\varepsilon\mu_j^1}(u_j + \varepsilon v_j + \varepsilon u_j^1 + \varepsilon^2 v_j^1), \varphi \right\rangle_\varepsilon$$
$$= \frac{1}{1+\mu_j+\varepsilon\mu_j^1}\left((\mu_j+\varepsilon\mu_j^1)\int_\Omega \rho_\varepsilon(u_j + \varepsilon v_j + \varepsilon u_j^1 + \varepsilon^2 v_j^1)\varphi dx \right.$$
$$\left. - \int_\Omega \nabla(u_j + \varepsilon v_j + \varepsilon u_j^1 + \varepsilon^2 v_j^1)\cdot\nabla\varphi dx \right) \quad (4.2.36)$$

We introduce the following quantities:

$$\begin{aligned}
I_{1,\varepsilon} &:= (\mu_j+\varepsilon\mu_j^1)\varepsilon\int_\Omega (u_j+\varepsilon u_j^1)\varphi dx,\\
I_{2,\varepsilon} &:= (\mu_j+\varepsilon\mu_j^1)\varepsilon^2\int_{\omega_\varepsilon} (v_j+\varepsilon v_j^1)\varphi dx,\\
I_{3,\varepsilon} &:= (\mu_j+\varepsilon\mu_j^1)\int_{\omega_\varepsilon} \frac{\tilde{\rho}_\varepsilon}{\varepsilon}(u_j + \varepsilon v_j + \varepsilon u_j^1 + \varepsilon^2 v_j^1)\varphi dx,\\
I_{4,\varepsilon} &:= \int_\Omega \nabla u_j\cdot\nabla\varphi dx,\\
I_{5,\varepsilon} &:= \varepsilon\int_\Omega \nabla u_j^1\cdot\nabla\varphi dx,\\
I_{6,\varepsilon} &:= \int_{\omega_\varepsilon} \nabla(\varepsilon v_j+\varepsilon^2 v_j^1)\cdot\nabla\varphi dx.
\end{aligned}$$

We observe that the expression in brackets in the right-hand side of (4.2.36) equals

$$I_{1,\varepsilon}+I_{2,\varepsilon}+I_{3,\varepsilon}-I_{4,\varepsilon}-I_{5,\varepsilon}-I_{6,\varepsilon}. \quad (4.2.37)$$

In what follows we shall recall the definition (4.2.8) of $\tilde{\rho}_\varepsilon$. We consider each term separately. We start from $I_{1,\varepsilon}$. We have

$$I_{1,\varepsilon} = I_{1,1,\varepsilon} + I_{1,2,\varepsilon},$$

where

$$\begin{aligned}
I_{1,1,\varepsilon} &:= \varepsilon\mu_j\int_\Omega u_j\varphi dx, \quad (4.2.38)\\
I_{1,2,\varepsilon} &:= \varepsilon^2\mu_j^1\int_\Omega u_j\varphi dx + \varepsilon^2\mu_j\int_\Omega u_j^1\varphi dx + \varepsilon^3\mu_j^1\int_\Omega u_j^1\varphi dx.
\end{aligned}$$

We note that $I_{1,2,\varepsilon} \le C\varepsilon^2\|\varphi\|_\varepsilon$. Indeed, for the first term of $I_{1,2,\varepsilon}$ we have

$$\varepsilon^2\mu_j^1\int_\Omega u_j\varphi dx \le C\varepsilon^2\|u_j\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)} \le C\varepsilon^2\|\varphi\|_\varepsilon.$$

The other terms can be treated in a similar way (see also (4.2.21)).

We consider now $I_{2,\varepsilon}$. Thanks to (4.2.9) we have

$$I_{2,\varepsilon} = (\mu_j + \varepsilon\mu_j^1)\varepsilon^2 \int_{\omega_\varepsilon} (v_j + \varepsilon v_j^1)\varphi dx = (\mu_j + \varepsilon\mu_j^1)\frac{\varepsilon^3}{\tilde\rho_\varepsilon} \int_{\omega_\varepsilon} \frac{\tilde\rho_\varepsilon}{\varepsilon}(v_j + \varepsilon v_j^1)\varphi dx$$
$$\leq (\mu_j + \varepsilon\mu_j^1)\frac{\varepsilon^3}{\tilde\rho_\varepsilon}\left(\int_{\omega_\varepsilon} \frac{\tilde\rho_\varepsilon}{\varepsilon}(v_j + \varepsilon v_j^1)^2 dx\right)^{\frac{1}{2}} \|\varphi\|_\varepsilon \leq C\varepsilon^{\frac{5}{2}}\|\varphi\|_\varepsilon. \quad (4.2.39)$$

We consider $I_{3,\varepsilon}$. We have

$$I_{3,\varepsilon} = I_{3,1,\varepsilon} + I_{3,2,\varepsilon} + I_{3,3,\varepsilon} + I_{3,4,\varepsilon} + I_{3,5,\varepsilon},$$

where

$$\begin{aligned}
I_{3,1,\varepsilon} &:= \mu_j \int_{\omega_\varepsilon} \frac{\tilde\rho_\varepsilon}{\varepsilon} u_j\varphi dx,\\
I_{3,2,\varepsilon} &:= \mu_j \int_{\omega_\varepsilon} \tilde\rho_\varepsilon u_j^1\varphi dx,\\
I_{3,3,\varepsilon} &:= \mu_j \int_{\omega_\varepsilon} \tilde\rho_\varepsilon v_j\varphi dx,\\
I_{3,4,\varepsilon} &:= \mu_j^1 \int_{\omega_\varepsilon} \tilde\rho_\varepsilon u_j\varphi dx,\\
I_{3,5\varepsilon} &:= \mu_j\varepsilon \int_{\omega_\varepsilon} \tilde\rho_\varepsilon v_j^1\varphi dx + \mu_j^1\varepsilon \int_{\omega_\varepsilon} \tilde\rho_\varepsilon u_j^1\varphi dx + \mu_j^1\varepsilon \int_{\omega_\varepsilon} \tilde\rho_\varepsilon v_j\varphi dx\\
&\quad + \mu_j^1\varepsilon^2 \int_{\omega_\varepsilon} \tilde\rho_\varepsilon v_j^1\varphi dx.
\end{aligned}$$

We consider first $I^1_{3,1,\varepsilon}$ and use formula (4.2.22). We have

$$I_{3,1,\varepsilon} = \mu_j \int_{\omega_\varepsilon} \frac{M}{\varepsilon|\partial\Omega|} u_j\varphi dx + \mu_j \int_{\omega_\varepsilon} \frac{KM}{2|\partial\Omega|^2} u_j\varphi dx$$
$$- \mu_j \int_{\omega_\varepsilon} \frac{|\Omega|}{|\partial\Omega|} u_j\varphi dx + \mu_j\frac{F(\varepsilon)}{\varepsilon} \int_{\omega_\varepsilon} u_j\varphi dx. \quad (4.2.40)$$

The last term in (4.2.40) can be bounded from above by $C\varepsilon^2\|\varphi\|_\varepsilon$. In fact we have

$$\mu_j F(\varepsilon) \int_{\omega_\varepsilon} \frac{1}{\varepsilon} u_j\varphi dx = \frac{\mu_j F(\varepsilon)}{\tilde\rho_\varepsilon} \int_{\omega_\varepsilon} \frac{\tilde\rho_\varepsilon}{\varepsilon} u_j\varphi dx \leq \frac{\mu_j F(\varepsilon)}{\tilde\rho_\varepsilon}\|u_j\|_\varepsilon\|\varphi\|_\varepsilon \leq C\varepsilon^2\|\varphi\|_\varepsilon.$$

For the first term in (4.2.40) we have

$$\mu_j \int_{\omega_\varepsilon} \frac{M}{\varepsilon|\partial\Omega|} u_j\varphi dx$$
$$= \mu_j \int_0^{|\partial\Omega|}\int_0^1 \frac{M}{|\partial\Omega|}(u_j\circ\psi_\varepsilon)(\varphi\circ\psi_\varepsilon) d\xi ds$$
$$- \mu_j\varepsilon \int_0^{|\partial\Omega|}\int_0^1 \frac{M}{|\partial\Omega|}\xi\kappa(s)(u_j\circ\psi_\varepsilon(s,\xi))(\varphi\circ\psi_\varepsilon(s,\xi)) d\xi ds, \quad (4.2.41)$$

while for the second term in (4.2.40) we have

$$\begin{aligned}
&\mu_j \int_{\omega_\varepsilon} \frac{KM}{2|\partial\Omega|^2} u_j \varphi dx \\
&\qquad = \mu_j \varepsilon \int_0^{|\partial\Omega|} \int_0^1 \frac{KM}{2|\partial\Omega|^2} (u_j \circ \psi_\varepsilon)(\varphi \circ \psi_\varepsilon) d\xi ds \\
&\qquad - \mu_j \varepsilon^2 \int_0^{|\partial\Omega|} \int_0^1 \frac{KM}{2|\partial\Omega|^2} \xi \kappa(s) (u_j \circ \psi_\varepsilon(s,\xi))(\varphi \circ \psi_\varepsilon(s,\xi)) d\xi ds \\
&\qquad \leq \mu_j \varepsilon \int_0^{|\partial\Omega|} \int_0^1 \frac{KM}{2|\partial\Omega|^2} (u_j \circ \psi_\varepsilon)(\varphi \circ \psi_\varepsilon) d\xi ds + C\varepsilon^2 \|\varphi\|_\varepsilon. \qquad (4.2.42)
\end{aligned}$$

The last inequality can be proved by observing that

$$\begin{aligned}
&- \mu_j \varepsilon^2 \int_0^{|\partial\Omega|} \int_0^1 \frac{KM}{2|\partial\Omega|^2} \xi \kappa(s) (u_j \circ \psi_\varepsilon(s,\xi))(\varphi \circ \psi_\varepsilon(s,\xi)) d\xi ds \\
&\qquad \leq \varepsilon^2 \frac{C}{\tilde{\rho}_\varepsilon} \int_{\omega_\varepsilon} \frac{\tilde{\rho}_\varepsilon}{\varepsilon} u_j \varphi dx \leq C\varepsilon^2 \|u_j\|_\varepsilon \|\varphi\|_\varepsilon.
\end{aligned}$$

In a similar way, for the third term in (4.2.40), we have

$$\begin{aligned}
&- \mu_j \int_{\omega_\varepsilon} \frac{|\Omega|}{|\partial\Omega|} u_j \varphi dx \\
&\qquad \leq -\mu_j \varepsilon \int_0^{|\partial\Omega|} \int_0^1 \frac{|\Omega|}{|\partial\Omega|} (u_j \circ \psi_\varepsilon)(\varphi \circ \psi_\varepsilon) d\xi ds + C\varepsilon^2 \|\varphi\|_\varepsilon. \qquad (4.2.43)
\end{aligned}$$

We collect (4.2.41), (4.2.42) and (4.2.43) and we obtain that

$$\begin{aligned}
I_{3,1,\varepsilon} \leq & \mu_j \int_0^{|\partial\Omega|} \int_0^1 \frac{M}{|\partial\Omega|} (u_j \circ \psi_\varepsilon)(\varphi \circ \psi_\varepsilon) d\xi ds \\
& - \mu_j \varepsilon \int_0^{|\partial\Omega|} \int_0^1 \frac{M}{|\partial\Omega|} \xi \kappa(s) (u_j \circ \psi_\varepsilon(s,\xi))(\varphi \circ \psi_\varepsilon(s,\xi)) d\xi ds \\
& + \mu_j \varepsilon \int_0^{|\partial\Omega|} \int_0^1 \frac{KM}{2|\partial\Omega|^2} (u_j \circ \psi_\varepsilon)(\varphi \circ \psi_\varepsilon) d\xi ds \\
& - \mu_j \varepsilon \int_0^{|\partial\Omega|} \int_0^1 \frac{|\Omega|}{|\partial\Omega|} (u_j \circ \psi_\varepsilon)(\varphi \circ \psi_\varepsilon) d\xi ds + C\varepsilon^2 \|\varphi\|_\varepsilon.
\end{aligned}$$

We consider now $I_{3,2,\varepsilon}$. Thanks to formula (4.2.22), we have

$$\begin{aligned}
I_{3,2,\varepsilon} = & \mu_j \int_{\omega_\varepsilon} \frac{M}{|\partial\Omega|} u_j^1 \varphi dx + \mu_j \varepsilon \int_{\omega_\varepsilon} \frac{KM}{2|\partial\Omega|^2} u_j^1 \varphi dx \\
& - \mu_j \varepsilon \int_{\omega_\varepsilon} \frac{|\Omega|}{|\partial\Omega|} u_j^1 \varphi dx + F(\varepsilon)\varepsilon^2 \int_{\omega_\varepsilon} u_j^1 \varphi dx \\
& \leq \mu_j \varepsilon \int_0^{|\partial\Omega|} \int_0^1 \frac{M}{|\partial\Omega|} (u_j^1 \circ \psi_\varepsilon)(\varphi \circ \psi_\varepsilon) d\xi ds + C\varepsilon^2 \|\varphi\|_\varepsilon.
\end{aligned}$$

Similarly, for $I_{3,3,\varepsilon}$ and $I_{3,4,\varepsilon}$ we obtain

$$I_{3,3,\varepsilon} = \mu_j \int_{\omega_\varepsilon} \tilde{\rho}_\varepsilon v_j \varphi dx \leq \varepsilon \mu_j \int_0^{|\partial\Omega|} \int_0^1 \frac{M}{|\partial\Omega|} w_j (\varphi \circ \psi_\varepsilon) d\xi ds + C\varepsilon^2 \|\varphi\|_\varepsilon$$
$$= -\varepsilon \int_0^{|\partial\Omega|} \int_0^1 \frac{M^2 \mu_j^2}{2|\partial\Omega|^2} (u_j \circ \psi_\varepsilon(s,0))(\xi - 1)^2 (\varphi \circ \psi_\varepsilon(s,\xi)) d\xi ds + C\varepsilon^2 \|\varphi\|_\varepsilon.$$

and

$$I_{3,4,\varepsilon} = \mu_j^1 \int_{\omega_\varepsilon} \tilde{\rho}_\varepsilon u_j \varphi dx \leq \mu_j^1 \varepsilon \int_0^{|\partial\Omega|} \int_0^1 \frac{M}{|\partial\Omega|} (u_j \circ \psi_\varepsilon)(\varphi \circ \psi_\varepsilon) d\xi ds + C\varepsilon^2 \|\varphi\|_\varepsilon.$$

We note that that by the definitions of u_j^1, v_j and v_j^1 (see (4.2.31), (4.2.16) and (4.2.32)) one can prove that the norms $\|u_j^1\|_\varepsilon$, $\|v_j\|_\varepsilon$ and $\|v_j^1\|_\varepsilon$ are bounded from above, uniformly in $\varepsilon \in]0, \varepsilon_0[$. Then we verify that $I_{3,5,\varepsilon} \leq C\varepsilon^2$. We show this inequality only for the first summand in $I_{3,5,\varepsilon}$. The proof for the other summands is similar and is accordingly omitted. By the definition of $\tilde{\rho}_\varepsilon$ and by the Cauchy-Shwartz inequality we have

$$\mu_j \varepsilon \int_{\omega_\varepsilon} \tilde{\rho}_\varepsilon v_j^1 \varphi dx = \mu_j \varepsilon^2 \int_{\omega_\varepsilon} \frac{\tilde{\rho}_\varepsilon}{\varepsilon} v_j^1 \varphi dx \leq \mu_j \varepsilon^2 C \|v_j^1\|_\varepsilon \|\varphi\|_\varepsilon \leq C\varepsilon^2 \|\varphi\|_\varepsilon.$$

We have proved that

$$\begin{aligned}
I_{3,\varepsilon} & \\
\leq & \mu_j \int_0^{|\partial\Omega|} \int_0^1 \frac{M}{|\partial\Omega|} (u_j \circ \psi_\varepsilon)(\varphi \circ \psi_\varepsilon) d\xi ds \\
& - \mu_j \varepsilon \int_0^{|\partial\Omega|} \int_0^1 \frac{M}{|\partial\Omega|} \xi \kappa(s) (u_j \circ \psi_\varepsilon(s,\xi))(\varphi \circ \psi_\varepsilon(s,\xi)) d\xi ds \\
& + \mu_j \varepsilon \int_0^{|\partial\Omega|} \int_0^1 \frac{KM}{2|\partial\Omega|^2} (u_j \circ \psi_\varepsilon)(\varphi \circ \psi_\varepsilon) d\xi ds \\
& - \mu_j \varepsilon \int_0^{|\partial\Omega|} \int_0^1 \frac{|\Omega|}{|\partial\Omega|} (u_j \circ \psi_\varepsilon)(\varphi \circ \psi_\varepsilon) d\xi ds \\
& + \mu_j \varepsilon \int_0^{|\partial\Omega|} \int_0^1 \frac{M}{|\partial\Omega|} (u_j^1 \circ \psi_\varepsilon)(\varphi \circ \psi_\varepsilon) d\xi ds \\
& - \varepsilon \int_0^{|\partial\Omega|} \int_0^1 \frac{M^2 \mu_j^2}{2|\partial\Omega|^2} (u_j \circ \psi_\varepsilon(s,0))(\xi - 1)^2 (\varphi \circ \psi_\varepsilon(s,\xi)) d\xi ds \\
& + \mu_j^1 \varepsilon \int_0^{|\partial\Omega|} \int_0^1 \frac{M}{|\partial\Omega|} (u_j \circ \psi_\varepsilon)(\varphi \circ \psi_\varepsilon) d\xi ds + C\varepsilon^2 \|\varphi\|_\varepsilon. \quad (4.2.44)
\end{aligned}$$

We consider now $I_{4,\varepsilon}$. We have

$$I_{4,\varepsilon} = \int_\Omega \nabla u_j \cdot \nabla \varphi dx = \int_{\partial\Omega} \frac{M\mu_j}{|\partial\Omega|} u_j \varphi d\sigma. \quad (4.2.45)$$

Next we consider $I_{5,\varepsilon}$

$$I_{5,\varepsilon} = \varepsilon \int_\Omega \nabla u_j^1 \cdot \nabla \varphi dx = \varepsilon \int_\Omega \mu_j u_j \varphi dx$$
$$+ \varepsilon \int_{\partial\Omega} \left(\frac{M\mu_j}{2|\partial\Omega|^2}(K - |\partial\Omega|\kappa(s)) - \frac{2M^2\mu_j^2}{3|\partial\Omega|^2} + \frac{M\mu_j^1 - |\Omega|\mu_j}{|\partial\Omega|} \right) u_j \varphi d\sigma$$
$$+ \varepsilon \int_{\partial\Omega} \frac{M\mu_j}{|\partial\Omega|} u_j^1 \varphi d\sigma. \quad (4.2.46)$$

Now we consider the quantity $I_{6,\varepsilon}$.

$$I_{6,\varepsilon} = I_{6,1,\varepsilon} + I_{6,2,\varepsilon},$$

where

$$I_{6,1,\varepsilon} := \varepsilon \int_{\omega_\varepsilon} \nabla v_j \cdot \nabla \varphi dx,$$
$$I_{6,2,\varepsilon} := \varepsilon^2 \int_{\omega_\varepsilon} \nabla v_j^1 \cdot \nabla \varphi dx.$$

Thanks to formula (4.2.7) and integrating by parts with respect to the variable ξ in the interval $]0,1[$, we have

$$I_{6,1,\varepsilon} = \varepsilon \int_{\omega_\varepsilon} \nabla v_j \cdot \nabla \varphi dx$$
$$= \varepsilon^2 \int_0^{|\partial\Omega|} \int_0^1 \left(\frac{1}{\varepsilon^2} \partial_\xi w_j(s,\xi) \partial_\xi (\varphi \circ \psi_\varepsilon)(s,\xi) + \frac{\partial_s w_j(s,\xi) \partial_s (\varphi \circ \psi_\varepsilon)(s,\xi)}{(1 - \varepsilon\xi\kappa(s))^2} \right) (1 - \varepsilon\xi\kappa(s)) d\xi ds$$
$$\le \int_0^{|\partial\Omega|} \int_0^1 \frac{M\mu_j}{|\partial\Omega|} (u_j \circ \psi_\varepsilon(s,0))(\varphi \circ \psi_\varepsilon(s,\xi)) d\xi ds - \int_{\partial\Omega} \frac{M\mu_j}{|\partial\Omega|} u_j \varphi d\sigma$$
$$-\varepsilon \int_0^{|\partial\Omega|} \int_0^1 \frac{\mu_j M}{|\partial\Omega|} \kappa(s)\,(2\xi - 1)\,(u_j \circ \psi_\varepsilon(s,0))(\varphi \circ \psi_\varepsilon(s,\xi)) d\xi ds + C\varepsilon^2 \|\varphi\|_\varepsilon. \quad (4.2.47)$$

Moreover, by an analogous argument, we have

$$I_{6,2,\varepsilon} = \varepsilon^2 \int_{\omega_\varepsilon} \nabla v_j^1 \cdot \nabla \varphi dx$$
$$\le -\varepsilon \int_{\partial\Omega} \left(\frac{M\mu_j(K - |\partial\Omega|\kappa)}{2|\partial\Omega|^2} - \frac{2M^2\mu_j^2}{3|\partial\Omega|^2} + \frac{M\mu_j^1 - |\Omega|\mu_j}{|\partial\Omega|} \right) u_j \varphi d\sigma$$
$$- \varepsilon \int_{\partial\Omega} \frac{M\mu_j}{|\partial\Omega|} u_j^1 \varphi d\sigma$$

$$
\begin{aligned}
&+\varepsilon\int_0^{|\partial\Omega|}\int_0^1 \frac{M\mu_j\kappa(s)}{|\partial\Omega|}(\xi-1)(u_j\circ\psi_\varepsilon(s,0))(\varphi\circ\psi_\varepsilon(s,\xi))d\xi ds\\
&+\varepsilon\int_0^{|\partial\Omega|}\int_0^1 \frac{M}{|\partial\Omega|}\Big(\mu_j(u_j^1\circ\psi_\varepsilon(s,0))-\frac{M\mu_j^2}{2|\partial\Omega|}(\xi-1)^2(u_j\circ\psi_\varepsilon(s,0))\\
&\qquad+\mu_j^1(u_j\circ\psi_\varepsilon(s,0))-\frac{M\mu_j^2}{|\partial\Omega|}\xi(u_j\circ\psi_\varepsilon(s,0))\\
&\qquad-\frac{|\Omega|\mu_j}{M}(u_j\circ\psi_\varepsilon(s,0))+\frac{K\mu_j}{2|\partial\Omega|}(u_j\circ\psi_\varepsilon(s,0))\Big)(\varphi\circ\psi_\varepsilon(s,\xi))d\xi ds\\
&\qquad+C\varepsilon^2\|\varphi\|_\varepsilon. \qquad (4.2.48)
\end{aligned}
$$

We now recall (4.2.37). By (4.2.38), (4.2.39), (4.2.44), (4.2.45), (4.2.46), (4.2.47) and (4.2.48) we compute the following inequality

$$
\begin{aligned}
&I_{1,\varepsilon}+I_{2,\varepsilon}+I_{3,\varepsilon}-I_{4,\varepsilon}-I_{5,\varepsilon}-I_{6,\varepsilon}\\
&\le\frac{M\mu_j}{|\partial\Omega|}\int_0^{|\partial\Omega|}\int_0^1\Big((u_j\circ\psi_\varepsilon(s,\xi))-(u_j\circ\psi_\varepsilon(s,0))\\
&\qquad+\varepsilon\frac{M\mu_j}{|\partial\Omega|}\xi(u_j\circ\psi_\varepsilon(s,0))\Big)(\varphi\circ\psi_\varepsilon(s,\xi))d\xi ds \qquad (4.2.49)\\
&-\varepsilon\frac{M\mu_j}{|\partial\Omega|}\int_0^{|\partial\Omega|}\int_0^1((u_j\circ\psi_\varepsilon(s,\xi))\\
&\qquad-(u_j\circ\psi_\varepsilon(s,0)))(\varphi\circ\psi_\varepsilon(s,\xi))\xi\kappa(s)d\xi ds \qquad (4.2.50)\\
&+\varepsilon\frac{\mu_j^1 M}{|\partial\Omega|}\int_0^{|\partial\Omega|}\int_0^1((u_j\circ\psi_\varepsilon(s,\xi))\\
&\qquad-(u_j\circ\psi_\varepsilon(s,0)))(\varphi\circ\psi_\varepsilon(s,\xi))d\xi ds \qquad (4.2.51)\\
&+\varepsilon\frac{M\mu_j}{|\partial\Omega|}\int_0^{|\partial\Omega|}\int_0^1\big((u_j^1\circ\psi_\varepsilon(s,\xi))\\
&\qquad-(u_j^1\circ\psi_\varepsilon(s,0))\big)(\varphi\circ\psi_\varepsilon(s,\xi))d\xi ds \qquad (4.2.52)\\
&-\varepsilon\frac{\mu_j|\Omega|}{|\partial\Omega|}\int_0^{|\partial\Omega|}\int_0^1((u_j\circ\psi_\varepsilon(s,\xi))\\
&\qquad-(u_j\circ\psi_\varepsilon(s,0)))(\varphi\circ\psi_\varepsilon(s,\xi))d\xi ds \qquad (4.2.53)\\
&+\varepsilon\frac{\mu_j KM}{2|\partial\Omega|^2}\int_0^{|\partial\Omega|}\int_0^1((u_j\circ\psi_\varepsilon(s,\xi))\\
&\qquad-(u_j\circ\psi_\varepsilon(s,0)))(\varphi\circ\psi_\varepsilon(s,\xi))d\xi ds. \qquad (4.2.54)\\
&+C\varepsilon^2\|\varphi\|_\varepsilon.
\end{aligned}
$$

Since both u_j and u_j^1 are of class $C^1(\Omega)$, we can conclude that the terms (4.2.50)-(4.2.54) can be bounded from above by $C\varepsilon^2\|\varphi\|_\varepsilon$ (see (4.2.23)). It remains to estimate (4.2.49). We recall that $u_j\in C^2(\Omega)$. It is more convenient to consider coordinates (s,t) and the corresponding change of variable $x=\psi(s,t)$. We note that $\partial_t u_j(\psi(s,0))=-\partial_\nu u_j(\gamma(s))=-\frac{\mu_j M}{|\partial\Omega|}u_j(\psi(s,0))$.

Therefore, for each $(s,t) \in [0,\partial\Omega)\times]0,1[$, there exist $t^* \in (0,t)$ and $t^{**} \in (0,t^*)$ such that

$$(u_j \circ \psi)(s,t) - (u_j \circ \psi)(s,0) = t\partial_t(u_j \circ \psi)(s,t^*)$$

and

$$\partial_t(u_j \circ \psi)(s,t^*) - \partial_t(u_j \circ \psi)(s,0) = t\partial_t^2(u_j \circ \psi)(s,t^{**}).$$

Then for (4.2.49) we have

$$\begin{aligned}
&\frac{M\mu_j}{|\partial\Omega|}\int_0^{|\partial\Omega|}\int_0^1 \left((u_j \circ \psi_\varepsilon(s,\xi)) - (u_j \circ \psi_\varepsilon(s,0)) + \varepsilon\frac{M\mu_j}{|\partial\Omega|}\xi(u_j \circ \psi_\varepsilon(s,0))\right) \\
&\qquad\qquad \cdot (\varphi \circ \psi_\varepsilon(s,\xi))d\xi ds \\
&= \frac{M\mu_j}{|\partial\Omega|}\int_0^{|\partial\Omega|}\frac{1}{\varepsilon}\int_0^\varepsilon t(\partial_t\,(u_j \circ \psi)(s,t^*) - \partial_t(u_j \circ \psi)(s,0))\,(\varphi \circ \psi(s,t))dtds \\
&\qquad \le \frac{M\mu_j C}{|\partial\Omega|}\frac{1}{\varepsilon}\int_0^{|\partial\Omega|}\int_0^\varepsilon t^2\partial_t^2(u_j \circ \psi)(s,t^{**})(\varphi \circ \psi(s,t))dtds \\
&\qquad\quad \le \frac{M\mu_j C}{|\partial\Omega|\varepsilon^{\frac{1}{2}}}\|u_j\|_{C^2(\Omega)}\int_0^{|\partial\Omega|}\int_0^\varepsilon t^2\frac{(\varphi \circ \psi(s,t))}{\varepsilon^{\frac{1}{2}}}dtds \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \le \frac{M\mu_j C}{|\partial\Omega|\sqrt{5}}\varepsilon^2\|\varphi\|_\varepsilon.
\end{aligned}$$

This proves (4.2.35).

We note that $\|u_j+\varepsilon v_j+\varepsilon u_j^1+\varepsilon^2 v_j^1\|_\varepsilon^{-1}$ is uniformly bounded for $\varepsilon \in]0,\varepsilon_0[$. We shall need a more precise estimate of $\|u_j + \varepsilon v_j + \varepsilon u_j^1 + \varepsilon^2 v_j^1\|_\varepsilon^{-1}$, which is the aim of the following lemma.

Lemma 4.2.55. *Let u_j be the unique eigenfunction of* (4.0.1) *associated with the eigenvalue μ_j such that $\int_{\partial\Omega} u_j^2 d\sigma = 1$. Let u_j^1 be the solution of* (4.2.64) *which satisfies*

$$\int_{\partial\Omega} u_j^1 u_j d\sigma = \left(\frac{\mu_j^1}{2\mu_j} + \frac{M\mu}{3|\partial\Omega|}\right).$$

Then there exists a constant $C > 0$ which does not depend on ε such that for all $\varepsilon \in]0,\varepsilon_0[$ it holds

$$\|u_j + \varepsilon v_j + \varepsilon u_j^1 + \varepsilon^2 v_j^1\|_\varepsilon^2 = \frac{M}{|\partial\Omega|}\left(1 + \mu_j + \varepsilon\mu_j^1\right) + O(\varepsilon^2), \tag{4.2.56}$$

as $\varepsilon \to 0$.

Proof. We have

$$\|u_j + \varepsilon v_j + \varepsilon u_j^1 + \varepsilon^2 v_j^1\|_\varepsilon^2 = N_{1,\varepsilon} + N_{2,\varepsilon} + N_{3,\varepsilon} + N_{4,\varepsilon} + N_{5,\varepsilon},$$

where

$$\begin{aligned}
N_{1,\varepsilon} &:= \varepsilon\int_{\Omega}\left(u_j^2+2\varepsilon u_j u_j^1+\varepsilon^2\left(u_j^1\right)^2\right)dx,\\
N_{2,\varepsilon} &:= \varepsilon\int_{\omega_\varepsilon}\left(\varepsilon^2 v_j^2+\varepsilon^4\left(v_j^1\right)^2+2\varepsilon u_j v_j+2\varepsilon^2 u_j^1 v_j+2\varepsilon^3 u_j^1 v_j^1+2\varepsilon^2 u_j v_j^1+2\varepsilon^3 v_j v_j^1\right)dx,\\
N_{3,\varepsilon} &:= \int_{\omega_\varepsilon}\frac{\tilde\rho_\varepsilon}{\varepsilon}\Big(u_j^2+2\varepsilon u_j u_j^1+\varepsilon^2\left(u_j^1\right)^2+\varepsilon^2 v_j^2+\varepsilon^4\left(v_j^1\right)^2+2\varepsilon u_j v_j+2\varepsilon^2 u_j^1 v_j\\
&\qquad +2\varepsilon^3 u_j^1 v_j^1+2\varepsilon^2 u_j v_j^1+2\varepsilon^3 v_j v_j^1\Big)dx,\\
N_{4,\varepsilon} &:= \int_{\Omega}|\nabla u_j|^2+2\varepsilon\nabla u_j\cdot\nabla u_j^1+\varepsilon^2|\nabla u_j^1|^2 dx,\\
N_{5,\varepsilon} &:= \int_{\omega_\varepsilon}\varepsilon^2|\nabla v_j|^2+\varepsilon^4|\nabla v_j^1|^2+2\varepsilon\nabla u_j\cdot\nabla v_j+2\varepsilon^2\nabla u_j\cdot\nabla v_j^1\\
&\qquad +2\varepsilon^2\nabla v_j\nabla u_j^1+2\varepsilon^3\nabla v_j\cdot\nabla v_j^1+2\varepsilon^3\nabla u_j^1\cdot\nabla v_j^1 dx.
\end{aligned}$$

We start from $N_{1,\varepsilon}$. We note that from standard elliptic regularity we have that $\|u_j\|_{C(\overline{\Omega})}, \|u_j^1\|_{C(\overline{\Omega})}\leq C$. Therefore it holds

$$N_{1,\varepsilon}\leq\varepsilon\int_{\Omega}u_j^2dx+C\varepsilon^2.$$

Consider now $N_{2,\varepsilon}$. From the definition of v_j and v_j^1 we have that $\|v_j\|_{C(\overline{\omega}_\varepsilon)}$, $\|v_j^1\|_{C(\overline{\omega}_\varepsilon)}\leq C$. Moreover, from the fact that $|\omega_\varepsilon|\leq C\varepsilon$, it immediately follows that

$$N_{2,\varepsilon}\leq C\varepsilon^2.$$

Now we consider $N_{3,\varepsilon}$. By the same arguments above, we have that the third, fourth, fifth, seventh, eighth, ninth and tenth summands of $N_{3,\varepsilon}$ can be bounded from above by $C\varepsilon^2$. Therefore

$$N_{3,\varepsilon}\leq N_{3,\varepsilon}^1+N_{3,\varepsilon}^2+N_{3,\varepsilon}^3+C\varepsilon^2,$$

where

$$\begin{aligned}
N_{3,1,\varepsilon} &:= \int_{\omega_\varepsilon}\frac{\tilde\rho_\varepsilon}{\varepsilon}u_j^2dx,\\
N_{3,2,\varepsilon} &:= 2\int_{\omega_\varepsilon}\tilde\rho_\varepsilon u_j u_j^1dx,\\
N_{3,3,\varepsilon} &:= 2\int_{\omega_\varepsilon}\tilde\rho_\varepsilon u_j v_j dx.
\end{aligned}$$

Consider $N_{3,1,\varepsilon}$ first. We use the expansion (4.2.22) for $\tilde\rho_\varepsilon$. Moreover, since $u_j\in C^2(\Omega)$ we have that $(u_j\circ\psi(s,\xi))^2=(u_j\circ\psi(s,0))^2+2tu_j(\psi(s,0))\partial_t(u_j\circ\psi)(s,0)+Ct^2$, which implies

$$(u_j\circ\psi(s,\xi))^2=(u_j\circ\psi(s,0))^2+2t\frac{M\mu_j}{|\partial\Omega|}(u_j\circ\psi(s,0))^2+Ct^2,$$

since u_j is a solution of (4.0.1). Then we have

$$N_{3,1,\varepsilon} = \int_0^{|\partial\Omega|}\int_0^{\varepsilon}\left(\frac{M}{\varepsilon|\partial\Omega|} + \frac{\frac{1}{2}KM - |\Omega||\partial\Omega|}{|\partial\Omega|^2} + \frac{F(\varepsilon)}{\varepsilon}\right)$$
$$\cdot\left((u_j\circ\psi(s,0))^2 + 2t\frac{M\mu_j}{|\partial\Omega|}(u_j\circ\psi(s,0))^2 + Ct^2\right)(1-t\kappa(s))dtds$$

From standard computations and recalling that $\int_{\partial\Omega} u_j^2 d\sigma = 1$, it follows that

$$N_{3,1,\varepsilon} \leq \frac{M}{|\partial\Omega|} + \varepsilon\left(\frac{\frac{1}{2}KM - |\Omega||\partial\Omega|}{|\partial\Omega|^2}\right) - \varepsilon\frac{M^2\mu}{|\partial\Omega|^2} - \varepsilon\frac{M}{2|\partial\Omega|}\int_{\partial\Omega} u_j^2\kappa d\sigma + C\varepsilon^2.$$

Very similar computations for $N_{3,2,\varepsilon}$ and $N_{3,3,\varepsilon}$ yield

$$N_{3,2,\varepsilon} \leq \varepsilon\frac{2M}{|\partial\Omega|}\left(\frac{\mu_j^1}{2\mu_j} + \frac{M\mu_j}{3|\partial\Omega|}\right) + C\varepsilon^2$$

and

$$N_{3,3,\varepsilon} \leq -\varepsilon\frac{M^2\mu_j}{3|\partial\Omega|^2} + C\varepsilon^2.$$

Now we pass to the terms involving the gradients. First we recall that from standard elliptic regularity $\|u_j\|_{C^1(\Omega)}, \|u_j^1\|_{C^1(\Omega)} \leq C$. Moreover, from the definition of v and v_j, it follows that $\|v_j\|_{C^1(\Omega)}, \|v_j^1\|_{C^1(\Omega)} \leq \frac{C}{\varepsilon}$. We consider $N_{4,\varepsilon}$. We have

$$N_{4,\varepsilon} \leq \frac{M\mu_j}{|\partial\Omega|}\int_{\partial\Omega} u_j^2 d\sigma + 2\varepsilon\frac{M\mu_j}{|\partial\Omega|}\int_{\partial\Omega} u_j u_j^1 d\sigma + C\varepsilon^2$$
$$\leq \frac{M\mu_j}{|\partial\Omega|} + \varepsilon\left(\frac{M\mu_j^1}{|\partial\Omega|} + \frac{2M^2\mu_j^2}{3|\partial\Omega|^2}\right) + C\varepsilon^2.$$

Next we consider $N_{5,\varepsilon}$. We note that all the summands but the first and the third can be bounded from above by $C\varepsilon^2$. Therefore we have

$$N_{5,\varepsilon} = N_{5,1,\varepsilon} + N_{5,2,\varepsilon} + C\varepsilon^2,$$

where

$$N_{5,1,\varepsilon} := \varepsilon^2\int_{\omega_\varepsilon}|\nabla v_j|^2 dx,$$
$$N_{5,2,\varepsilon} := 2\varepsilon\int_{\omega_\varepsilon}\nabla u_j\cdot\nabla v_j dx.$$

Consider $N_{5,\varepsilon}^1$. Passing to coordinates (s,ξ) and using formula (4.2.7), from standard computations it follows that

$$N_{5,1,\varepsilon} \leq \varepsilon\frac{M^2\mu_j^2}{3|\partial\Omega|^2} + C\varepsilon^2.$$

In a very similar way one can prove that for $N_{5,2,\varepsilon}$ it holds

$$N_{5,2,\varepsilon} \leq -\varepsilon \frac{M^2\mu_j^2}{|\partial\Omega|^2} + C\varepsilon^2.$$

We collect all the expression appearing into $N_{1,\varepsilon}, N_{2,\varepsilon}, N_{3,\varepsilon}, N_{4,\varepsilon}$ and $N_{5,\varepsilon}$ according to the powers of ε appearing. We have

$$\begin{aligned}
&\|u_j + \varepsilon v_j + \varepsilon u_j^1 + \varepsilon^2 v_j^1\|_\varepsilon^2 \\
&\qquad \leq \frac{M}{|\partial\Omega|}\Bigg[1 + \mu_j \\
&+ \varepsilon\left(\frac{|\partial\Omega|}{M}\int_\Omega u_j^2 dx - \frac{|\Omega|}{M} - \frac{2M\mu_j}{3|\partial\Omega|} + \frac{K}{2|\partial\Omega|} - \frac{1}{2}\int_{\partial\Omega} u_j^2\kappa d\sigma + \frac{\mu_j^1}{\mu_j} + \mu_j^1\right)\Bigg] \\
&\qquad + C\varepsilon^2.
\end{aligned}$$

We note that

$$\frac{\mu_j^1}{\mu_j} = -\frac{|\partial\Omega|}{M}\int_\Omega u_j^2 dx + \frac{|\Omega|}{M} - \frac{2M\mu_j}{3|\partial\Omega|} - \frac{K}{2|\partial\Omega|} + \frac{1}{2}\int_{\partial\Omega} u_j^2\kappa d\sigma,$$

therefore

$$\|u_j + \varepsilon v_j + \varepsilon u_j^1 + \varepsilon^2 v_j^1\|_\varepsilon^2 \leq \frac{M}{|\partial\Omega|}\left(1 + \mu_j + \varepsilon\mu_j^1\right) + C\varepsilon^2.$$

This concludes the proof. □

As we did in Section 4.2.2, in order to apply Lemma 4.2.3 it is sufficient to multiply both sides of (4.2.35) by $\|u_j+\varepsilon v_j+\varepsilon u_j^1+\varepsilon^2 v_j^1\|_\varepsilon^{-1}$. From Lemma 4.2.55 it follows that

$$\begin{aligned}
&\left|\left\langle \mathcal{A}_\varepsilon\left(\frac{u_j + \varepsilon v_j + \varepsilon u_j^1 + \varepsilon^2 v_j^1}{\|u_j + \varepsilon v_j + \varepsilon u_j^1 + \varepsilon^2 v_j^1\|_\varepsilon}\right)\right.\right. \\
&\qquad \left.\left. - \frac{1}{1 + \mu_j + \varepsilon\mu_j^1}\left(\frac{u_j + \varepsilon v_j + \varepsilon u_j^1 + \varepsilon^2 v_j^1}{\|u_j + \varepsilon v_j + \varepsilon u_j^1 + \varepsilon^2 v_j^1\|_\varepsilon}\right), \varphi\right\rangle_\varepsilon\right| \\
&\qquad\qquad \leq C\sqrt{\frac{|\partial\Omega|}{M}}(1+\mu_j)^{-\frac{1}{2}}\left(1 - \frac{\mu_j^1}{2(1+\mu_j)}\varepsilon\right)\varepsilon^2\|\varphi\|_\varepsilon, \quad (4.2.57)
\end{aligned}$$

for all $\varphi \in H^1(\Omega)$. Now we set $\varphi := \frac{u_j+\varepsilon v_j+\varepsilon u_j^1+\varepsilon^2 v_j^1}{\|u_j+\varepsilon v_j+\varepsilon u_j^1+\varepsilon^2 v_j^1\|_\varepsilon}$ in (4.2.57), which gives

$$\left\| \mathcal{A}_\varepsilon \left(\frac{u_j + \varepsilon v_j + \varepsilon u_j^1 + \varepsilon^2 v_j^1}{\|u_j + \varepsilon v_j + \varepsilon u_j^1 + \varepsilon^2 v_j^1\|_\varepsilon} \right) \right.$$
$$\left. - \frac{1}{1+\mu_j+\varepsilon\mu_j^1} \left(\frac{u_j + \varepsilon v_j + \varepsilon u_j^1 + \varepsilon^2 v_j^1}{\|u_j + \varepsilon v_j + \varepsilon u_j^1 + \varepsilon^2 v_j^1\|_\varepsilon} \right) \right\|_\varepsilon$$
$$\leq C\sqrt{\frac{|\partial\Omega|}{M}}(1+\mu_j)^{-\frac{1}{2}} \left(1 - \frac{\mu_j^1}{2(1+\mu_j)}\varepsilon \right) \varepsilon^2.$$

From Lemma 4.2.3 applied with $\eta = \frac{1}{1+\mu_j+\varepsilon\mu_j^1}$, $u = \frac{u_j+\varepsilon v_j+\varepsilon u_j^1+\varepsilon^2 v_j^1}{\|u_j+\varepsilon v_j+\varepsilon u_j^1+\varepsilon^2 v_j^1\|_\varepsilon}$ and $r = C\sqrt{\frac{|\partial\Omega|}{M}}(1+\mu_j)^{-\frac{1}{2}}\left(1 - \frac{\mu_j^1}{2(1+\mu_j)}\varepsilon\right)\varepsilon^2$, it follows that there exists an eigenvalue η^* of $\mathcal{A}_\varepsilon$ such that

$$\left| \frac{1}{1+\mu_j+\varepsilon\mu_j^1} - \eta^* \right| \leq C\sqrt{\frac{|\partial\Omega|}{M}}(1+\mu_j)^{-\frac{1}{2}} \left(1 - \frac{\mu_j^1}{2(1+\mu_j)}\varepsilon \right) \varepsilon^2. \quad (4.2.58)$$

When $\varepsilon \in]0, \varepsilon_j[$ (possibly choosing ε_j smaller), the only eigenvalue of $\mathcal{A}_\varepsilon$ which satisfies (4.2.58) is $\frac{1}{1+\lambda_j(\varepsilon)}$. Thus we have that

$$\lambda_j(\varepsilon) = \mu_j + \varepsilon\mu_j^1 + O(\varepsilon^2), \quad (4.2.59)$$

as $\varepsilon \to 0$. Let $u_{j,\varepsilon}$ be the unique eigenfunction of (4.0.2) associated with the eigenvalue $\lambda_j(\varepsilon)$ such that $\frac{M}{|\partial\Omega|}\int_\Omega \rho_\varepsilon u_{j,\varepsilon}^2 dx = 1$. We recall that

$$\|u_{j,\varepsilon}\|_\varepsilon^2 = \frac{M}{|\partial\Omega|}\left(1+\lambda_j(\varepsilon)\right).$$

From (4.2.56) and (4.2.59) it follows that there exists a constant $c > 0$ which does not depend on ε such that

$$|(1+\lambda_j(\varepsilon))^{\frac{1}{2}} - (1+\mu_j+\varepsilon\mu_j^1)^{\frac{1}{2}}| \leq c\varepsilon^2 \quad (4.2.60)$$

and

$$\left| \|u_j + \varepsilon v_j + \varepsilon u_j^1 + \varepsilon^2 v_j^1\|_\varepsilon - \frac{M^{\frac{1}{2}}}{|\partial\Omega|^{\frac{1}{2}}}(1+\mu_j+\varepsilon\mu_j^1)^{\frac{1}{2}} \right| \leq c\varepsilon^2. \quad (4.2.61)$$

Moreover, from Lemma 4.2.3, it follows that for all $\delta > 0$, there exists a function $u^* \in \mathcal{H}_\varepsilon(\Omega)$ with $\|u^*\|_\varepsilon = 1$ belonging to the space generated by all the eigenfunctions associated with the eigenvalues of $\mathcal{A}_\varepsilon$ lying in the segment

$\left[\frac{1}{1+\mu_j+\varepsilon\mu_j^1}-\delta, \frac{1}{1+\mu_j+\varepsilon\mu_j^1}+\delta\right]$ such that

$$\left\| u^* - \frac{u_j+\varepsilon v_j+\varepsilon u_j^1+\varepsilon^2 v_j^1}{\|u_j+\varepsilon v_j+\varepsilon u_j^1+\varepsilon^2 v_j^1\|_\varepsilon}\right\|_\varepsilon \leq \frac{2C}{\delta}\sqrt{\frac{|\partial\Omega|}{M}}(1+\mu_j)^{-\frac{1}{2}}\left(1-\frac{\mu_j^1}{2(1+\mu_j)}\varepsilon\right)\varepsilon^2. \quad (4.2.62)$$

By the continuity of the eigenvalues $\lambda_j(\varepsilon)$ and the simplicity of $\lambda_j(\varepsilon)$ for all $\varepsilon \in]0,\varepsilon_j[$, it follows that (possibly choosing $\varepsilon_j > 0$ smaller) there exists $\delta_j > 0$ such that $\left|\frac{1}{1+\mu_j+\varepsilon\mu_j^1}-\frac{1}{1+\lambda_{j-1}(\varepsilon)}\right| > \delta_j$, $\left|\frac{1}{1+\mu_j+\varepsilon\mu_j^1}-\frac{1}{1+\lambda_{j+1}(\varepsilon)}\right| > \delta_j$ and $\left|\frac{1}{1+\mu_j+\varepsilon\mu_j^1}-\frac{1}{1+\lambda_j(\varepsilon)}\right| \leq \delta_j$, for all $\varepsilon \in]0,\varepsilon_j[$. Then we choose $\delta = \delta_j$ in (4.2.62). Since for all $\varepsilon \in]0,\varepsilon_j[$, $\frac{1}{1+\lambda_j(\varepsilon)}$ is the only eigenvalue lying in the segment $\left[\frac{1}{1+\mu_j+\varepsilon_j^1}-\delta_j, \frac{1}{1+\mu_j+\varepsilon\mu_j^1}+\delta_j\right]$ and $\lambda_j(\varepsilon)$ is simple, it follows that necessarily $u^* = \frac{u_{j,\varepsilon}}{\|u_{j,\varepsilon}\|_\varepsilon}$. Thus there exists $c_j > 0$ (possibly depending on j) which does not depend on ε, such that

$$\left\| \frac{u_{j,\varepsilon}}{\|u_{j,\varepsilon}\|_\varepsilon} - \frac{u_j+\varepsilon v_j+\varepsilon u_j^1+\varepsilon^2 v_j^1}{\|u_j+\varepsilon v_j+\varepsilon u_j^1+\varepsilon^2 v_j^1\|_\varepsilon}\right\|_\varepsilon \leq c_j\varepsilon^2. \quad (4.2.63)$$

Finally, by computations similar to those in (4.2.29), using (4.2.63), (4.2.60), (4.2.61), and following the same lines as in the proof of (4.2.30), it is possible to prove that

$$\|u_{j,\varepsilon} - u_j - \varepsilon v_j - \varepsilon u_j^1 - \varepsilon^2 v_j^1\|_{L^2(\Omega)} \leq C(1+\mu_j+\varepsilon\mu_j^1)\varepsilon^2.$$

This concludes the proof of Theorems 4.2.10 and 4.2.14.

4.2.4 Well-posedness of problem (4.2.31)

Let u_j be the unique eigenfunction associated with a simple eigenvalue μ_j of problem (4.0.1) such that $\int_{\partial\Omega} u_j^2 d\sigma = 1$. Then we consider the following problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \partial_\nu u - \frac{M\mu_j}{|\partial\Omega|}u = g_1 + \lambda g_2 & \text{on } \partial\Omega, \end{cases} \quad (4.2.64)$$

where $f \in L^2(\Omega)$ and $g_1, g_2 \in L^2(\partial\Omega)$ are given data which satisfy the condition $\int_{\partial\Omega} g_2 u_j d\sigma \neq 0$, while the unknowns are the constant λ and the function u. The weak formulation of problem (4.2.64) reads: find $(u,\lambda) \in H^1(\Omega)\times\mathbb{R}$ such that

$$\int_\Omega \nabla u\cdot\nabla\varphi dx - \frac{M\mu_j}{|\partial\Omega|}\int_{\partial\Omega} u\varphi d\sigma = \int_\Omega f\varphi dx + \int_{\partial\Omega} g_1\varphi d\sigma + \lambda\int_{\partial\Omega} g_2\varphi d\sigma, \quad (4.2.65)$$

for all $\varphi \in H^1(\Omega)$. We have the following theorem.

Theorem 4.2.66. *Problem* (4.2.64) *admits a weak solution* $(u, \lambda) \in H^1(\Omega) \times \mathbb{R}$ *if and only if*

$$\lambda = -\left(\int_\Omega f u_j dx + \int_{\partial\Omega} g_1 u_j d\sigma\right)\left(\int_{\partial\Omega} g_2 u_j d\sigma\right)^{-1}. \tag{4.2.67}$$

Moreover, given a solution u *of* (4.2.64), *all the solutions of* (4.2.64) *are given by* $u + Au_j$ *with* $A \in \mathbb{R}$.

Proof. Consider the operator $\mathcal{A}_1$ from $H^1(\Omega)$ to $H^1(\Omega)'$ which takes $u \in H^1(\Omega)$ to the functional $\mathcal{A}_1[u]$ defined by

$$\mathcal{A}_1[u][\varphi] := \int_\Omega \nabla u \cdot \nabla \varphi dx + \int_{\partial\Omega} u\varphi d\sigma\,, \quad \varphi \in H^1(\Omega).$$

As is well-known, the operator $\mathcal{A}_1$ is a homeomorphism from $H^1(\Omega)$ to $H^1(\Omega)'$. Then we consider the trace operator Tr from $H^1(\Omega)$ to $L^2(\partial\Omega)$ which is compact and the operator $\mathcal{J}$ from $L^2(\partial\Omega)$ to $H^1(\Omega)'$ defined by

$$\mathcal{J}[u][\varphi] := \int_{\partial\Omega} u\varphi d\sigma\,, \quad \forall \varphi \in H^1(\Omega).$$

We define the operator $\mathcal{A}_2$ from $H^1(\Omega)$ to $H^1(\Omega)'$ as

$$\mathcal{A}_2 := -\left(1 + \frac{M\mu_j}{|\partial\Omega|}\right)\mathcal{J} \circ \mathrm{Tr}.$$

We define the operator $\mathcal{A}$ from $H^1(\Omega)$ to $H^1(\Omega)'$ as $\mathcal{A} := \mathcal{A}_1 + \mathcal{A}_2$. This is the sum of an invertible operator and a compact operator, therefore $\mathcal{A}$ is a Fredholm operator. Finally we denote by $B(\lambda)$ the element $B(\lambda) \in H^1(\Omega)'$ defined by

$$B(\lambda)[\varphi] := \int_\Omega f\varphi dx + \int_{\partial\Omega} g_1\varphi d\sigma + \lambda \int_{\partial\Omega} g_2 \varphi d\sigma\,, \quad \varphi \in H^1(\Omega).$$

Problem (4.2.65) is recasted as follows: find $(\lambda, u) \in \mathbb{R} \times H^1(\Omega)$ such that

$$\mathcal{A}[u] = B(\lambda).$$

The kernel of $\mathcal{A}$ is finite dimensional and is the space of those u^* such that

$$\int_\Omega \nabla u^* \cdot \nabla \varphi dx - \frac{M\mu_j}{|\partial\Omega|}\int_{\partial\Omega} u^* \varphi d\sigma = 0\,, \quad \forall \varphi \in H^1(\Omega).$$

Since we have assumed that μ_j is a simple eigenvalue associated with the eigenfunction u_j, it follows that the kernel of $\mathcal{A}$ coincides with the one dimensional subspace of $H^1(\Omega)$ generated by u_j. Therefore, problem (4.2.64) has solution if and only if $B(\lambda)$ satisfies the equality

$$\int_\Omega f u_j dx + \int_{\partial\Omega} g_1 u_j d\sigma + \lambda \int_{\partial\Omega} g_2 u_j d\sigma = 0.$$

Accordingly, problem (4.2.65) has solution if and only if λ is given by (4.2.67). Finally we note that the solution u of problem (4.2.65) is defined up to multiples of u_j. □

4.2.5 The case of the unit ball in $\mathbb{R}^2$

In this subsection we consider the case $\Omega = B$, where B is the open unit ball centered at zero in $\mathbb{R}^2$. We have already established a formula for the derivatives of all the eigenvalues $\lambda_j(\varepsilon)$ at $\varepsilon = 0$ in Section 4.1 (see formula (4.1.3)). We show now that formula (4.2.12) in the case of the unit ball in $\mathbb{R}^2$ formally is the same as formula (4.1.3). We recall that the eigenvalues of problem (4.0.1) on B are given by

$$\mu_{2j-1} = \mu_{2j} = \frac{2\pi j}{M}\ , \ j \in \mathbb{N},$$

while $\mu_0 = 0$. We note that all the positive eigenvalues have multiplicity two. It is convenient to use polar coordinates $(r, \theta) \in \mathbb{R}_+ \times \partial B$ in $\mathbb{R}^2$ and the corresponding change of variables $x = \phi_s(r, \theta) = (r\cos(\theta), r\sin(\theta))$. The eigenfunctions associated with the eigenvalue $\mu_{2j-1} = \mu_{2j}$ are the two-dimensional harmonic polynomials $u_{j,1}, u_{j,2}$ of degree j, which can be written in polar coordinates as

$$\begin{aligned} u_{j,1}(r,\theta) &= r^j \cos(j\theta), \\ u_{j,2}(r,\theta) &= r^j \sin(j\theta). \end{aligned}$$

Then we consider problem (4.0.2) when $\Omega = B$. It is standard to show (see e.g., Lemma 4.1.4, see also [71, 72]) that all the eigenvalues of problem (4.0.2) on B have multiplicity which is an integer multiple of two, except the first one which is zero and has multiplicity one. Moreover, for a fixed $j \in \mathbb{N}$, there exists $\varepsilon_j > 0$ such that $\lambda_j(\varepsilon)$ has multiplicity two for all $\varepsilon \in]0, \varepsilon_j[$ (cfr. Corollary 3.1.42 and [71]). Therefore, from Corollary 3.1.42 (see also [71]) it follows that the positive eigenvalues of (4.0.2) on B can be labelled with two indexes k and l and denoted by $\lambda_{2k-1,l}(\varepsilon) = \lambda_{2k,l}(\varepsilon)$, for $k, l \in \mathbb{N}$. The corresponding eigenfunctions, which we denote by $u_{0,l,\varepsilon}, u_{k,l,\varepsilon,1}$ and $u_{k,l,\varepsilon,2}$ can be written in the following form

$$\begin{aligned} u_{0,l,\varepsilon} &= R_{0,l}(r), \\ u_{k,l,\varepsilon,1} &= R_{k,l}(r)\cos(k\theta), \\ u_{k,l,\varepsilon,2} &= R_{k,l}(r)\sin(k\theta), \end{aligned}$$

where $R_{k,l}(r)$ are suitable linear combinations of Bessel Functions of the first and second species and order k (see Lemma 4.1.4). Moreover, it is possible to prove that $\lambda_{2k-1,1}(\varepsilon) \to \mu_{2k-1}$, $\lambda_{2k,1}(\varepsilon) \to \mu_{2k}$, $\lambda_{2k-1,l}(\varepsilon) \to +\infty$, $\lambda_{2k,l}(\varepsilon) \to +\infty$ for $l \geq 2$, $u_{k,1,\varepsilon,1} \to u_{k,1}$ and $u_{\varepsilon,k,1,2} \to u_{k,2}$ in the $L^2(\Omega)$ norm, as $\varepsilon \to 0$ (see Theorem 4.1.37).

We note that, in principle, Theorem 4.2.10 could not be applied in this case since all the eigenvalues are multiple. From Theorem 4.1.20 we have that

$$\begin{aligned}\lambda_{2j-1,1}(\varepsilon) = \mu_{2j-1} + \left(\frac{2j\mu_{2j-1}}{3} + \frac{\mu_{2j-1}^2}{2(j+1)}\right)\varepsilon + o(\varepsilon) \\ = \frac{2\pi j}{M} + \frac{2j^2\pi}{M}\left(\frac{2}{3} + \frac{\pi}{M(1+j)}\right)\varepsilon + o(\varepsilon), \quad (4.2.68)\end{aligned}$$

as $\varepsilon \to 0$. The same formula holds if we substitute $\lambda_{2j-1,1}(\varepsilon)$ and μ_{2j-1} with $\lambda_{2j,1}(\varepsilon)$ and μ_{2j} respectively.

Now, let us consider formally formula (4.2.12) with $u_j = \pi^{-\frac{1}{2}}(r^j \cos(j\theta)) \circ \phi_s^{(-1)}$ and observe that the mean curvature on ∂B is constant end equals 1. Standard computations yield formula (4.2.68). This suggests that in some sense in the case of the ball, Theorem 4.2.10 still holds, despite the multiplicity of the eigenvalues is greater than one. This is not surprising. In fact we could have replaced through all this section the space $H^1(\Omega)$ with the space $H^1_j(\Omega)$ of those functions u in $H^1(\Omega)$ which are orthogonal to $(r^j \cos(j\theta)) \circ \phi_s^{(-1)}$ with respect to the $H^1(\Omega)$ scalar product. In this way the eigenvalue μ_{2j-1} becomes a simple eigenvalue and Theorem 4.2.10 can be applied. However, this is not straightforward.

The method used in this section is more general than the method used in Section 4.1 and allows to find a formula for the derivative of the eigenvalues $\lambda(\varepsilon)$ of problem (4.0.2) for a quite wide class of domains in $\mathbb{R}^2$.

4.2.6 Heuristic determination of the expansions

In this subsection we show how to guess asymptotic expansions for the eigenvalues and eigenfunctions of problem (4.0.2) of the type (4.2.11) and (4.2.15). Let μ_j be a simple eigenvalue of problem (4.0.1) and for $\varepsilon > 0$ small enough, let $\lambda_j(\varepsilon)$ be a simple eigenvalue of problem (4.0.2) such that $\lambda_j(\varepsilon) \to \mu_j$ as ε goes to zero (see 3.1.42). Then

$$\lambda_j(\varepsilon) = \mu_j + o(1),$$

as $\varepsilon \to 0$. The first step in order to postulate an asymptotic expansion is to guess the powers of ε of the lower order terms. In analogy with formula

(4.1.21) for the unit ball in $\mathbb{R}^N$ we postulate an asymptotic expansion for $\lambda_j(\varepsilon)$ of the following form

$$\lambda_j(\varepsilon) = \mu_j + \varepsilon\mu_j^1 + o(\varepsilon), \tag{4.2.69}$$

as $\varepsilon \to 0$, i.e., we guess that the second term in the asymptotic expansion of $\lambda_j(\varepsilon)$ is of order ε as $\varepsilon \to 0$. Since we want to find a formula for the derivative of the eigenvalues at $\varepsilon = 0$, we shall consider the expansion (4.2.69) and neglect further terms.
Now we consider an asymptotic expansion for the eigenfunction $u_{j,\varepsilon}$ associated with the eigenvalue $\lambda_j(\varepsilon)$ of the form

$$u_{j,\varepsilon} = u_j + o(1), \tag{4.2.70}$$

as $\varepsilon \to 0$. The equality in (4.2.70) is understood in the sense of the $L^2(\Omega)$ norm. In formula (4.2.70) we have considered the eigenfunction $u_{j,\varepsilon}$ normalized by $\int_\Omega \rho_\varepsilon u_{j,\varepsilon}^2 dx = M/|\partial\Omega|$. We denoted by u_j the unique eigenfunction of (4.0.1) associated with the eigenvalue μ_j normalized by $\int_{\partial\Omega} u_j^2 d\sigma = 1$. By looking at (4.2.69) we can argue that the second term in the expansion of $u_{\varepsilon,j}$ is of order ε as $\varepsilon \to 0$, therefore we consider an expansion of the following type

$$u_{j,\varepsilon} = u_j + \varepsilon U_{\varepsilon,j}^1 + o(\varepsilon), \tag{4.2.71}$$

as $\varepsilon \to 0$, for some function $U_{\varepsilon,j}^1$ which possibly depends explicitly on $\varepsilon > 0$. Since the coefficient ρ_ε is piecewise constant and is of order ε^{-1} in ω_ε, we need to introduce in the expansion of the eigenfunctions some correcting terms which are supported on ω_ε and which are usually called 'boundary layers' (see [52, 53]). Therefore we consider the function $U_{\varepsilon,j}^1$ of the following form

$$U_{\varepsilon,j}^1 = u_j^1 + v_j + \varepsilon v_j^1,$$

where $u_j^1 \in H^1(\Omega)$ is supported on the whole of Ω and $v_j, v_j^1 \in H^1(\Omega)$ are extension by zero of $w_j \circ \psi_\varepsilon^{(-1)}$ and $w_j^1 \circ \psi_\varepsilon^{(-1)}$, where w_j and w_j^1 are functions in $H^1([0, |\partial\Omega|[\times]0,1[)$. In particular, we will look for functions v_j and v_j^1 which are uniformly bounded in $\varepsilon > 0$, while their gradients are of order ε^{-1}. Therefore the postulated expansion (4.2.71) can be rewritten as

$$u_{j,\varepsilon} = u_j + \varepsilon u_j^1 + \varepsilon v_j + \varepsilon^2 v_j^1 + o(\varepsilon), \tag{4.2.72}$$

as $\varepsilon \to 0$. We note that in formula (4.2.72) a term of order ε^2 appears. This is not surprising. In order to characterize u_j^1, v_j and v_j^1, we shall plug the asymptotic expansion (4.2.72) into (4.0.2), and therefore we shall take derivatives of v_j and v_j^1, which are of order ε^{-1}. Rouhgly speaking, εv_j is of order $O(1)$ and $\varepsilon^2 v_j^1$ is of order $O(\varepsilon)$ as $\varepsilon \to 0$, in the sense of the $H^1(\omega_\varepsilon)$ norm.

We compute now $|\omega_\varepsilon|$. We pass to the variables (s,ξ) and obtain

$$|\omega_\varepsilon| = \int_{\omega_\varepsilon} dx = \int_0^{|\partial\Omega|}\int_0^1 \varepsilon(1-\varepsilon\xi\kappa(s))d\xi ds = \varepsilon|\partial\Omega| - \frac{\varepsilon^2}{2}\int_0^{|\partial\Omega|}\kappa(s)ds.$$

We denoted by K the quantity $\int_0^{|\partial\Omega|}\kappa(s)ds$ (see (4.2.13)). Now we expand in Taylor series the quantity $\frac{M-\varepsilon|\Omega\setminus\overline{\omega}_\varepsilon|}{|\omega_\varepsilon|}$ around $\varepsilon=0$. We have

$$\frac{M-\varepsilon|\Omega\setminus\overline{\omega}_\varepsilon|}{|\omega_\varepsilon|} = \frac{M}{\varepsilon|\partial\Omega|} + \frac{\frac{1}{2}KM - |\Omega||\partial\Omega|}{|\partial\Omega|^2} + O(\varepsilon), \tag{4.2.73}$$

as $\varepsilon\to 0$. Note that ρ_ε can be written in the following equivalent form

$$\rho_\varepsilon = \varepsilon + \frac{1}{\varepsilon}\tilde{\rho}_\varepsilon\chi_{\omega_\varepsilon},$$

where $\chi_{\omega_\varepsilon}$ is the characteristic function of the set ω_ε and $\tilde{\rho}_\varepsilon$ is given by formula (4.2.8). Thanks to formula (4.2.73), $\tilde{\rho}_\varepsilon$ can be written in the form (4.2.22).

We also need to write the Laplace operator in the variables (s,ξ). From standard calculus it follows that

$$\Delta = \frac{1}{\varepsilon^2}\partial_\xi^2 - \frac{1}{\varepsilon}\kappa(s)\partial_\xi - \kappa(s)^2\xi\partial_\xi + \partial_s^2 + \cdots,$$

where the remaining terms are of order $O(\varepsilon)$ as $\varepsilon\to 0$. We denoted by dots further asymptotic terms which are not of use in order to postulate the asymptotic expansions (4.2.69) and (4.2.72).

Suppose for the moment that the functions u_j and u_j^1 are regular enough. For example, suppose that they are of class C^2. We have

$$\begin{aligned}(u_j\circ\psi_\varepsilon)(s,\xi) &= (u_j\circ\psi_\varepsilon)(s,0) + \xi\partial_\xi(u_j\circ\psi)(s,0) + O(\varepsilon^2), &(4.2.74)\\ (u_j^1\circ\psi_\varepsilon)(s,\xi) &= (u_j^1\circ\psi_\varepsilon)(s,0) + O(\varepsilon), &(4.2.75)\end{aligned}$$

as $\varepsilon\to 0$. Note that $\partial_\xi(u_j\circ\psi)(s,0) = -\varepsilon\partial_\nu u_j(\gamma(s))$. We also recall (see Theorem 4.2.14) that

$$\begin{aligned}w_j(s,\xi) &= (v_j\circ\psi_\varepsilon)(s,\xi),\\ w_j^1(s,\xi) &= (v_j^1\circ\psi_\varepsilon)(s,\xi).\end{aligned}$$

Now we plug (4.2.22), (4.2.69), (4.2.70), (4.2.73), (4.2.74) and (4.2.75) in (4.0.2). We have for the left-hand side of the differential equation in (4.0.2)

$$\begin{aligned}&-\Delta(u_j+\varepsilon u_j^1+\varepsilon v_j+\varepsilon^2 v_j^1)\\ &\quad= -\left(\Delta u_j+\varepsilon\Delta u_j^1\right) - \left(\frac{1}{\varepsilon}\partial_\xi^2 w_j - \kappa(s)\partial_\xi w_j + \partial_\xi^2 w_j^1 + O(\varepsilon)\right),\end{aligned} \tag{4.2.76}$$

as $\varepsilon \to 0$, where the first summand of (4.2.76) is defined in the whole of Ω, while the second summand is supported in the strip ω_ε and is written in the coordinates (s,ξ). For the right-hand side of the differential equation in (4.0.2) we have

$$\rho_\varepsilon\lambda_j(\varepsilon)(u_j+\varepsilon u_j^1+\varepsilon v_j+\varepsilon^2 v_j^1)$$
$$=\left(\varepsilon+\frac{1}{\varepsilon}\tilde\rho_\varepsilon\chi_{\omega_\varepsilon}\right)\left(\mu_j+\varepsilon\mu_j^1+O(\varepsilon^2)\right)\left(u_j+\varepsilon u_j^1+\varepsilon v_j+\varepsilon^2 v_j^1\right)$$
$$=P_1+P_2. \quad (4.2.77)$$

For the reader's convenience, we split formula (4.2.77) into two parts. The first is supported on the whole of Ω, the second is supported on ω_ε and is expressed in the variables (s,ξ). We denote these two parts by P_1 and P_2 respectively. We have

$$P_1=\varepsilon\mu_j u_j+O(\varepsilon^2) \quad (4.2.78)$$

as $\varepsilon \to 0$, and

$$P_2=\frac{1}{\varepsilon}\frac{M\mu_j}{|\partial\Omega|}(u_j\circ\psi_\varepsilon)(s,0)+\left(\frac{\frac{1}{2}KM-|\Omega||\partial\Omega|}{|\partial\Omega|^2}\mu_j+\frac{M\mu_j^1}{|\partial\Omega|}\right)(u_j\circ\psi_\varepsilon)(s,0)$$
$$-\frac{M\mu_j\xi\partial_\nu u_j(\gamma(s))}{|\partial\Omega|}+\frac{M\mu_j}{|\partial\Omega|}(u_j^1\circ\psi_\varepsilon)(s,0)+\frac{M\mu_j}{|\partial\Omega|}w_j(s,\xi)+O(\varepsilon), \quad (4.2.79)$$

as $\varepsilon \to 0$. Note that in (4.2.79) and in the second term of (4.2.76) it appears a reminder which is of order $O(\varepsilon)$ as $\varepsilon \to 0$. Moreover, we note that these terms are supported in the strip ω_ε which has measure of order $O(\varepsilon)$ as $\varepsilon \to 0$. Therefore, roughly speaking, we can think of the terms of order $O(\varepsilon)$ supported on ω_ε as terms of order $O(\varepsilon^2)$ as $\varepsilon \to 0$.

As for boundary conditions we have

$$\partial_\nu u_j(\gamma(s))-\partial_\xi w_j(s,0)+\varepsilon\left(\partial_\nu u_j^1(\gamma(s))-\partial_\xi w_j^1(s,0)\right)+O(\varepsilon^2)=0. \quad (4.2.80)$$

The next step is to match the quantities in (4.2.76), (4.2.78), (4.2.79) and (4.2.80) according to the powers of ε which appear. We obtain

$$\begin{cases}\Delta u_j=0, & \text{on } \Omega,\\ -\Delta u_j^1=\mu_j u_j, & \text{on } \Omega,\\ -\partial_\xi^2 w_j(s,\xi)=\frac{M\mu_j}{|\partial\Omega|}(u_j\circ\psi_\varepsilon)(s,0), & \text{on } [0,|\partial\Omega|[\times]0,1[,\end{cases}$$

and

$$-\partial_\xi^2 w_j^1(s,\xi)=-\kappa(s)\partial_\xi w_j(s,\xi)+\frac{M}{|\partial\Omega|}\Big(\mu_j(u_j^1\circ\psi_\varepsilon)(s,0)+\mu_j w_j(s,\xi)$$
$$+\mu_j^1(u_j\circ\psi_\varepsilon)(s,0)-\mu_j\xi\partial_\nu(u_j(\gamma(s)))$$
$$-\frac{|\Omega|\mu_j}{M}(u_j\circ\psi_\varepsilon)(s,0)+\frac{K\mu_j}{2|\partial\Omega|}(u_j\circ\psi_\varepsilon)(s,0)\Big) \text{ on } [0,|\partial\Omega|[\times]0,1[, \quad (4.2.81)$$

while for boundary conditions we have

$$\begin{aligned} \partial_\nu u_j(\gamma(s)) &= \partial_\xi w_j(s,0), \quad s \in [0, |\partial\Omega|[\,; \\ \partial_\nu u_j^1(\gamma(s)) &= \partial_\xi w_j^1(s,0), \quad s \in [0, |\partial\Omega|[\,. \end{aligned} \tag{4.2.82}$$

Now we write the compatibility conditions which must be satisfied by u_j, u_j^1, v_j, v_j^1. First, note that

$$\partial_\xi w_j(s,1) = \partial_\xi w_j^1(s,0) = 0.$$

In fact, we integrate $-\partial_\xi^2 w_j(s,\xi)$ with respect to $\xi \in]0,1[$

$$-\int_0^1 \partial_\xi^2 w_j(s,\xi) d\xi = \frac{M\mu_j}{|\partial\Omega|}(u_j \circ \psi_\varepsilon)(s,0),$$

therefore

$$\partial_\xi w_j(s,0) - \partial_\xi w_j(s,1) = \frac{M\mu_j}{|\partial\Omega|}(u_j \circ \psi_\varepsilon)(s,0),$$

which in view of (4.2.82) reads

$$\partial_\nu u_j(\gamma(s)) - \partial_\xi w_j(s,1) = \frac{M\mu_j}{|\partial\Omega|}(u_j \circ \psi_\varepsilon)(s,0).$$

This last fact yields

$$\begin{cases} \partial_\nu u_j(\gamma(s)) = \frac{M\mu_j}{|\partial\Omega|}(u_j \circ \psi_\varepsilon)(s,0), \\ \partial_\xi w_j(s,1) = 0. \end{cases} \tag{4.2.83}$$

Note that for each $s \in [0, |\partial\Omega|[$, the function $\xi \mapsto w_j(s,\xi)$ is defined up to constants. We choose w_j such that $w_j(s,1) = 0$ for all $s \in [0, |\partial\Omega|[$. With this choice w_j is uniquely determined and solves problem (4.2.17). Moreover, from the compatibility condition (4.2.83) it follows that u_j solves problem (4.0.1).

Now we repeat the same procedure for w_j^1. We integrate (4.2.81) with respect to $\xi \in]0,1[$ and from the compatibility conditions, as we did for w_j, it follows that w_j^1 and u_j^1 solve problems (4.2.33)-(4.2.34) and (4.2.64) respectively.

Remark 4.2.84. *We have chosen w_j and w_j^1 to satisfy $w_j(s,1) = w_j^1(s,1) = 0$, so that $u_j + \varepsilon u_j^1 + \varepsilon v_j + \varepsilon v_j^1 \in H^1(\Omega)$.*

Remark 4.2.85. *We have chosen the particular powers of ε in* (4.2.69) *and* (4.2.72) *since a posteriori the matching of all the terms and the compatibility conditions produce auxiliary problems which are well-posed. If we try to postulate an asymptotic expansion with different powers of ε (e.g., $\mu_j + \varepsilon^{\frac{1}{2}}\mu_j^1 + O(\varepsilon^{\frac{3}{2}})$ and the analogue for $u_{j,\varepsilon}$), this would lead to define problems the solutions of which are trivial, i.e., identically zero).*

Chapter 5

Mass concentration phenomena for fourth order operators. A new biharmonic Steklov problem

In this chapter we discuss the eigenvalue problem for the biharmonic operator Δ^2 subject to Steklov-type and Neumann-type boundary conditions. This operator is related do the study of the bending of a plate via the Kirchoff-Love model (see e.g., [31]).

Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1, $N \geq 2$. We introduce the following Steklov-type problem for the biharmonic operator

$$\begin{cases} \Delta^2 u - \tau\Delta u = 0, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega, \\ \tau\frac{\partial u}{\partial \nu} - \mathrm{div}_{\partial\Omega}\big(D^2 u \cdot \nu\big) - \frac{\partial \Delta u}{\partial \nu} = \lambda\rho u, & \text{on } \partial\Omega, \end{cases} \tag{5.0.1}$$

in the unknowns u (the eigenfunction), λ (the eigenvalue), where $\tau > 0$ is a fixed positive constant, $\rho \in \mathcal{R}^{\mathcal{S}}$, where $\mathcal{R}^{\mathcal{S}}$ is defined by (3.1.1). Here $\mathrm{div}_{\partial\Omega}$ denotes the tangential divergence operator and D^2u the Hessian matrix of u. We recall that the tangential divergence $\mathrm{div}_{\partial\Omega}F$ of a vector field F is defined as $\mathrm{div}_{\partial\Omega}F = \mathrm{div}F_{|\partial\Omega} - (DF.\nu)\cdot\nu$, where DF is the Jacobiam matrix of F. For $N = 2$, this problem is related to the study of the vibrations of a thin elastic plate with a free frame and mass displaced on the boundary with density ρ. The spectrum consists of a diverging sequence of eigenvalues of finite multiplicity

$$0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \tag{5.0.2}$$

where we agree to repeat the eigenvalues according to their multiplicity. We note that problem (5.0.1) is the analogue for the biharmonic operator of the classical Steklov problem (3.1.2) for the Laplace operator considered in Chapter 3 and 4. Then we consider the following Neumann-type problem for the biharmonic operator

$$\begin{cases} \Delta^2 u - \tau\Delta u = \lambda\rho u, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega, \\ \tau\frac{\partial u}{\partial \nu} - \operatorname{div}_{\partial\Omega}\big(D^2 u \cdot \nu\big) - \frac{\partial \Delta u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{5.0.3}$$

where $\rho \in \mathcal{R}$ and $\mathcal{R}$ is defined by (3.1.11) (we refer to [28] for the derivation of the boundary conditions). It is well known that this problem arises in the study of a free vibrating plate whose mass is displaced on the whole of Ω with density ρ. Also in this case the spectrum consist of a diverging sequence of eigenvalues of finite multiplicity as in (5.0.2).

In this chapter we consider the dependence of the eigenvalues λ_j of problem (5.0.1) both on the density ρ and the domain Ω. Moreover we shall understand problem (5.0.1) as the limit of Neumann problems (5.0.3) when the mass ρ is concentrating at the boundary of Ω. This behavior is analogous to that of the Laplace operator with Neumann boundary conditions and mass concentrated in a neighborhood of the boundary considered in Section 3.1.

We remark that problem (5.0.1) should not be confused with other Steklov-type problems already discussed in the literature. For example, in [20] the authors consider the following problem

$$\begin{cases} \Delta^2 u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \Delta u = \lambda\frac{\partial u}{\partial \nu}, & \text{on } \partial\Omega, \end{cases}$$

which has a rather different nature.
We shall think of the Steklov problem (5.0.1) as the natural fourth order version of problem (3.1.2). For this purpose we briefly derive problem (5.0.1) starting from a physical model (see also [99, 105]).

5.1 Formulating the problem

In this section we provide a physical interpretation of problem (5.0.1) for $N = 2$ as the equation of a thin vibrating plate. As usual, we assume that the mass of the plate is displaced in the middle plane of the plate parallel to its faces. When the body is at its equilibrium it covers a planar domain Ω with boundary $\partial\Omega$ in $\mathbb{R}^2$. We describe the vertical deviation from the equilibrium of each point (x, y) of Ω at time t by means of a function $v = v(x, y, t)$. We suppose that the whole mass of the plate is concentrated at the boundary with a density which we denote by $\rho(x, y)$. Moreover, we assume that $\rho(x, y)$ is bounded and positive on $\partial\Omega$. Under these assumptions, the total kinetic energy of the plate is given by

$$T = \frac{1}{2}\int_{\partial\Omega} \rho \dot{v}^2 d\sigma,$$

where we denote by $\dot{v}$ the derivative of v with respect to the time t, and by $d\sigma$ the surface measure on $\partial\Omega$. Now we obtain an expression for the potential energy of the plate. By following [105, §10.8], under the assumption that the strain potential energy at each point depends only on the strain configuration at that point and that the Poisson ratio of the material is zero, we have that the strain potential energy is given by

$$V_s = \frac{1}{2}\int_{\Omega} \big(v_{xx}^2 + v_{yy}^2 + 2v_{xy}^2\big) dxdy.$$

Besides V_s, we have another term of the potential energy due to the lateral tension

$$V_\tau = \frac{\tau}{2}\int_{\Omega} \big(v_x^2 + v_y^2\big) dxdy,$$

where $\tau > 0$ is the ratio of lateral tension due to flexural rigidity. The Hamilton's integral in the time interval $[t_1, t_2]$ of the system is given by

$$\begin{aligned} \mathcal{H} &= \int_{t_1}^{t_2} T - V_s - V_\tau \, dt \\ &= \frac{1}{2}\int_{t_1}^{t_2}\int_{\partial\Omega} \rho\dot{v}^2 d\sigma dt - \frac{1}{2}\int_{t_1}^{t_2}\int_{\Omega} \big(v_{xx}^2 + v_{yy}^2 + 2v_{xy}^2\big) + \tau\big(v_x^2 + v_y^2\big) dxdydt. \end{aligned}$$

According to Hamilton's Variational Principle, the actual motion of the system minimizes such integral. Let $v(x, y, t)$ be a minimizer for $\mathcal{H}$. Consider the one-parameter family

$$v(x, y, t) + \varepsilon\eta(x, y, t),$$

where we η is a twice continuously differentiable and such that $\eta(x, y, t_1) = \eta(x, y, t_2) = 0$. We consider the Hamiltion'n integral $\mathcal{H}(\varepsilon)$ given by

$$\begin{aligned} \mathcal{H}(\varepsilon) = \frac{1}{2}\int_{t_1}^{t_2}\int_{\partial\Omega} \rho(\dot{v} + \varepsilon\dot{\eta})^2 d\sigma dt \\ - \frac{\tau}{2}\int_{t_1}^{t_2}\int_{\Omega} (v_x + \varepsilon\eta_x)^2 + (v_y + \varepsilon\eta_y)^2 dxdydt \\ - \frac{1}{2}\int_{t_1}^{t_2}\int_{\Omega} (v_{xx} + \varepsilon\eta_{xx})^2 + (v_{yy} + \varepsilon\eta_{yy})^2 + 2(v_{xy} + \varepsilon\eta_{xy})^2 dxdydt. \end{aligned}$$

We have that $\mathcal{H}(\varepsilon)$ has a minimum at $\varepsilon = 0$, therefore $\frac{d\mathcal{H}}{d\varepsilon}_{|\varepsilon=0} = 0$. We compute $\frac{d\mathcal{H}}{d\varepsilon}_{|\varepsilon=0}$.

$$\begin{aligned}\frac{d\mathcal{H}}{d\varepsilon}_{|\varepsilon=0} &= \int_{t_1}^{t_2}\int_{\partial\Omega} \rho\dot{v}\dot{\eta}d\sigma dt \\ &- \int_{t_1}^{t_2}\int_{\Omega}\Big(v_{xx}\eta_{xx} + v_{yy}\eta_{yy} + 2v_{xy}\eta_{xy} + \tau(v_x\eta_x + v_y\eta_y)\Big)dxdydt = 0.\end{aligned} \tag{5.1.1}$$

We integrate by parts with respect to the variable t the first summand of (5.1.1). We have

$$\int_{t_1}^{t_2}\int_{\partial\Omega}\rho\dot{v}\dot{\eta}d\sigma dt = -\int_{t_1}^{t_2}\int_{\partial\Omega}\rho\ddot{v}\eta d\sigma dt, \tag{5.1.2}$$

since $\eta = 0$ at $t = t_1, t_2$. Now we integrate by parts separately all the terms in the second summand of (5.1.1). We have

$$\begin{aligned}\int_\Omega v_{xx}\eta_{xx}dxdy &= \int_\Omega v_{xxxx}\eta dxdy + \int_{\partial\Omega}\Big[v_{xx}\eta_x - v_{xxx}\eta\Big]\nu_{(x)}d\sigma, \\ \int_\Omega v_{yy}\eta_{yy}dxdy &= \int_\Omega u_{yyyy}\eta dxdy + \int_{\partial\Omega}\Big[v_{yy}\eta_y - v_{yyy}\eta\Big]\nu_{(y)}d\sigma, \\ 2\int_\Omega u_{xy}\eta_{xy}dxdy &= 2\int_\Omega v_{xxyy}\eta dxdy + \int_{\partial\Omega}\Big[v_{xy}\eta_y - u_{xyy}\eta\Big]\nu_{(x)}d\sigma \\ &+ \int_{\partial\Omega}\Big[v_{xy}\eta_x - v_{xxy}\eta\Big]\nu_{(y)}d\sigma, \\ \tau\int_\Omega v_x\eta_x &= -\tau\int_\Omega v_{xx}\eta dxdy + \tau\int_{\partial\Omega}v_x\eta\nu_{(x)}d\sigma, \\ \tau\int_\Omega v_y\eta_y &= -\tau\int_\Omega v_{yy}\eta dxdy + \tau\int_{\partial\Omega}v_y\eta\nu_{(y)}d\sigma,\end{aligned}$$

where we have denoted by $\nu_{(x)}$ and $\nu_{(y)}$ the components of the unit outer normal ν to $\partial\Omega$. The terms involving the integrals over the whole of Ω sum up to

$$\int_\Omega \Big(\Delta^2 v - \tau\Delta v\Big)\eta dxdy, \tag{5.1.3}$$

while the boundary terms equal

$$\int_{\partial\Omega}\Big(\tau\frac{\partial v}{\partial\nu} - \frac{\partial\Delta v}{\partial\nu}\partial\nu\Big)\eta + \big(D^2v\cdot\nu\big)\cdot\nabla\eta d\sigma. \tag{5.1.4}$$

We use the Divergence Theorem to find a more suitable expression for the boundary integral $\int_{\partial\Omega}\big(D^2v\cdot\nu\big)\cdot\nabla\eta d\sigma$. We have (see [28, 38] for the details)

$$\int_{\partial\Omega}\big(D^2v\cdot\nu\big)\cdot\nabla\eta d\sigma = \int_{\partial\Omega}\frac{\partial^2 v}{\partial\nu^2}\frac{\partial\eta}{\partial\nu} - \mathrm{div}_{\partial\Omega}\big(D^2v\cdot\nu\big)_{\partial\Omega}\eta d\sigma, \tag{5.1.5}$$

where $\left(D^2v\cdot\nu\right)_{\partial\Omega}=D^2v\cdot\nu-\frac{\partial^2 v}{\partial\nu^2}\nu$ is the tangential part of $D^2v\cdot\nu$. From (5.1.2), (5.1.3), (5.1.4) and (5.1.5) it follows that v satisfies

$$-\int_{t_1}^{t_2}\int_{\partial\Omega}\rho\ddot{v}\eta d\sigma dt-\int_{t_1}^{t_2}\int_{\Omega}\eta\left(\Delta^2v-\tau\Delta v\right)dxdydt$$
$$-\int_{t_1}^{t_2}\int_{\partial\Omega}\frac{\partial\eta}{\partial\nu}\frac{\partial^2 v}{\partial\nu^2}-\eta\left(\tau\frac{\partial v}{\partial\nu}-\mathrm{div}_{\partial\Omega}\left(D^2v.\nu\right)_{\partial\Omega}-\frac{\partial\Delta v}{\partial\nu}\right)d\sigma dt=0,$$

for all $\eta\in C^2(\Omega\times[t_1,t_2])$ such that $\eta(x,y,t_1)=\eta(x,y,t_2)=0$ and $(x,y)\in\Omega$. Since η is arbitrary we obtain

$$\begin{cases}\Delta^2v-\tau\Delta v=0, & \text{in }\Omega,\\ \frac{\partial^2 v}{\partial\nu^2}=0, & \text{on }\partial\Omega,\\ \rho\ddot{v}+\tau\frac{\partial v}{\partial\nu}-\mathrm{div}_{\partial\Omega}\left(D^2v.\nu\right)-\frac{\partial\Delta v}{\partial\nu}=0, & \text{on }\partial\Omega,\end{cases}\tag{5.1.6}$$

for all $t\in[t_1,t_2]$. We remark that we have written $\mathrm{div}_{\partial\Omega}\left(D^2v.\nu\right)$ instead of $\mathrm{div}_{\partial\Omega}\left(D^2v.\nu\right)_{\partial\Omega}$ since $\left(D^2v.\nu\right)_{\partial\Omega}=D^2v.\nu-\frac{\partial^2 v}{\partial\nu^2}\nu$ and $\frac{\partial^2 v}{\partial\nu^2}=0$ on $\partial\Omega$.

We separate the variables and, as is customary, we look for solutions to problem (5.1.6) of the form $v(x,y,t)=u(x,y)w(t)$. We find that the temporal component $w(t)$ solves the ordinary differential equation $-\ddot{w}(t)=\lambda w(t)$ for all $t\in[t_1,t_2]$, while the spatial component u solves problem (5.0.1).

Note that in the sequel we shall not put any restriction on the space dimension. Thus Ω will always denote a bounded domain in $\mathbb{R}^N$ of class C^1, with $N\geq 2$.

5.2 The Steklov spectrum

We consider the weak formulation of problem (5.0.1),

$$\int_\Omega D^2u:D^2\varphi+\tau\nabla u\cdot\nabla\varphi dx=\lambda\int_{\partial\Omega}\rho u\varphi d\sigma\,,\quad\forall\varphi\in H^2(\Omega)\,,\tag{5.2.1}$$

in the unknowns $u\in H^2(\Omega)$, $\lambda\in\mathbb{R}$, where

$$D^2u:D^2\varphi:=\sum_{i,j=1}^N\frac{\partial^2 u}{\partial x_i\partial x_j}\frac{\partial^2\varphi}{\partial x_i\partial x_j}$$

denotes the Frobenius product. Actually, we will consider a problem in the space $H^2(\Omega)/\mathbb{R}$ since we need to get rid of the constants, which generate the eigenspace corresponding to the eigenvalue $\lambda=0$. We denote by $\mathcal{J}_\rho^{\mathcal{S}}$ the continuous embedding of $L^2(\partial\Omega)$ into $H^2(\Omega)'$ defined by

$$\mathcal{J}_\rho^{\mathcal{S}}[u][\varphi]:=\int_{\partial\Omega}\rho u\varphi d\sigma,\quad\forall u\in L^2(\partial\Omega),\varphi\in H^2(\Omega).$$

We set

$$H^{2,\mathcal{S}}_\rho(\Omega) := \left\{ u \in H^2(\Omega) : \int_{\partial\Omega} \rho u d\sigma = 0 \right\},$$

and we consider in $H^2(\Omega)$ the bilinear form

$$< u, v >:= \int_\Omega D^2u : D^2v + \tau\nabla u \cdot \nabla v dx. \tag{5.2.2}$$

By the Poincaré-Wirtinger Inequality, it turns out that this bilinear form is indeed a scalar product on $H^{2,\mathcal{S}}_\rho(\Omega)$ whose induced norm is equivalent to the standard one. In the sequel we will think of the space $H^{2,\mathcal{S}}_\rho(\Omega)$ as endowed with the scalar product (5.2.2). Let $F(\Omega)$ be defined by $F(\Omega) := \{G \in H^2(\Omega)' : G[1] = 0\}$. Then, we consider the operator $\mathcal{P}^{\mathcal{S}}_\rho$ as an operator from $H^{2,\mathcal{S}}_\rho(\Omega)$ to $F(\Omega)$, defined by

$$\mathcal{P}^{\mathcal{S}}_\rho[u][\varphi] := \int_\Omega D^2u : D^2\varphi + \tau\nabla u\cdot\nabla\varphi dx,\ \forall u \in H^{2,\mathcal{S}}_\rho(\Omega), \varphi \in H^2(\Omega). \tag{5.2.3}$$

It turns out that $\mathcal{P}^{\mathcal{S}}_\rho$ is a homeomorphism of $H^{2,\mathcal{S}}_\rho(\Omega)$ onto $F(\Omega)$. We define the operator $\pi^{\mathcal{S}}_\rho$ from $H^2(\Omega)$ to $H^{2,\mathcal{S}}_\rho(\Omega)$ by

$$\pi^{\mathcal{S}}_\rho[u] := u - \frac{\int_{\partial\Omega} \rho u d\sigma}{\int_{\partial\Omega} \rho d\sigma}. \tag{5.2.4}$$

We consider the space $H^2(\Omega)/\mathbb{R}$ endowed with the bilinear form induced by (5.2.2). Such bilinear form renders $H^2(\Omega)/\mathbb{R}$ a Hilbert space. We denote by $\pi^{\sharp,\mathcal{S}}_\rho$ the map from $H^2(\Omega)/\mathbb{R}$ onto $H^{2,\mathcal{S}}_\rho(\Omega)$ defined by the equality $\pi^{\mathcal{S}}_\rho = \pi^{\sharp,\mathcal{S}}_\rho \circ p$, where p is the canonical projection of $H^2(\Omega)$ onto $H^2(\Omega)/\mathbb{R}$. The map $\pi^{\sharp,\mathcal{S}}_\rho$ turns out to be a homeomorphism. Finally, we define the operator $T^{\mathcal{S}}_\rho$ acting on $H^2(\Omega)/\mathbb{R}$ as follows

$$T^{\mathcal{S}}_\rho := (\pi^{\sharp,\mathcal{S}}_\rho)^{-1} \circ (\mathcal{P}^{\mathcal{S}}_\rho)^{-1} \circ \mathcal{J}^{\mathcal{S}}_\rho \circ \mathrm{Tr} \circ \pi^{\sharp,\mathcal{S}}_\rho, \tag{5.2.5}$$

where Tr denotes the trace operator acting from $H^2(\Omega)$ to $L^2(\partial\Omega)$.

Lemma 5.2.6. *The pair (λ, u) of the set $(\mathbb{R} \setminus \{0\}) \times (H^{2,\mathcal{S}}_\rho(\Omega) \setminus \{0\})$ satisfies (5.2.1) if and only if $\lambda > 0$ and the pair $(\lambda^{-1}, p[u])$ of the set $\mathbb{R} \times ((H^2(\Omega)/\mathbb{R}) \setminus \{0\})$ satisfies the equation*

$$\lambda^{-1}p[u] = T^{\mathcal{S}}_\rho p[u].$$

We have the following theorem.

Theorem 5.2.7. *The operator $T^{\mathcal{S}}_\rho$ is a non-negative compact self-adjoint operator in $H^2(\Omega)/\mathbb{R}$, whose eigenvalues coincide with the reciprocals of the positive eigenvalues of problem (5.2.1). In particular, the set of eigenvalues of problem (5.2.1) is contained in $[0, +\infty[$ and consists of the image of a sequence increasing to $+\infty$. Each eigenvalue has finite multiplicity.*

Proof. For the self-adjointness, it suffices to observe that

$$
\begin{aligned}
< T_\rho^{\mathcal{S}} u, v >_{H^2(\Omega)/\mathbb{R}} = &< (\pi_\rho^{\sharp,\mathcal{S}})^{-1} \circ (\mathcal{P}_\rho^{\mathcal{S}})^{-1} \circ \mathcal{J}_\rho^{\mathcal{S}} \circ \mathrm{Tr} \circ \pi_\rho^{\sharp,\mathcal{S}} u, v >_{H^2(\Omega)/\mathbb{R}} \\
&= \mathcal{P}_\rho^{\mathcal{S}}[(\mathcal{P}_\rho^{\mathcal{S}})^{-1} \circ \mathcal{J}_\rho^{\mathcal{S}} \circ \mathrm{Tr} \circ \pi_\rho^{\sharp,\mathcal{S}} u][\pi_\rho^{\sharp,\mathcal{S}} v] \\
&= \mathcal{J}_\rho^{\mathcal{S}}[\mathrm{Tr} \circ \pi_\rho^{\sharp,\mathcal{S}} u][\pi_\rho^{\sharp,\mathcal{S}} v], \quad \forall u, v \in H^2(\Omega)/\mathbb{R},
\end{aligned}
$$

and that $\mathcal{J}_\rho^{\mathcal{S}}[\mathrm{Tr}\circ\pi_\rho^{\sharp,\mathcal{S}} u][\pi_\rho^{\sharp,\mathcal{S}} v]$ is symmetric. As for compactness, just observe that the trace operator Tr acting from $H^1(\Omega)$ to $L^2(\partial\Omega)$ is compact. The remaining statements follow by standard spectral theory. □

As a consequence we have that the spectrum of (5.2.1) consists of an increasing sequence of non-negative eigenvalues of finite multiplicity. Note that the first positive eigenvalue is λ_2 as proved by the following theorem.

Theorem 5.2.8. *The first eigenvalue λ_1 of (5.2.1) is zero and the corresponding eigenfunctions are the constants. Moreover, $\lambda_2 > 0$.*

Proof. It is straightforward to see that constant functions are eigenfunctions of (5.2.1) with eigenvalue $\lambda = 0$. Suppose now that u is an eigenfunction corresponding to the eigenvalue $\lambda = 0$. Then we have

$$
\int_\Omega |D^2 u|^2 + \tau |\nabla u|^2 dx = 0,
$$

where $|D^2 u|^2 = \sum_{i,j=1}^N \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2$. Since $\nabla u = 0$, it follows that u is constant. Then the eigenvalue $\lambda = 0$ has multiplicity one. □

Thus λ_2 is the first positive eigenvalue of (5.2.1) which is usually called the fundamental tone. Note that we can charactrize λ_2 by means of the Rayleigh principle

$$
\lambda_2 = \min_{\substack{0 \neq u \in H^2(\Omega) \\ \int_{\partial\Omega} \rho u d\sigma = 0}} \frac{\int_\Omega |D^2 u|^2 + \tau |\nabla u|^2 dx}{\int_{\partial\Omega} \rho u^2 d\sigma}. \tag{5.2.9}
$$

5.3 Neumann problem and behavior of Neumann eigenvalues under mass concentration at the boundary

We consider now problem (5.0.3). Let ω_ε be the set defined by (3.1.19). We fix a positive number $M > 0$ and choose in (5.0.3) $\rho = \rho_\varepsilon$, where ρ_ε is defined by (3.1.20). If in addition we assume that Ω is of class C^2, ε_0 can be chosen in such a way that the map $x \mapsto x - \nu\varepsilon$ is a diffeomorphism between $\partial\Omega$ and $\partial\omega_\varepsilon \cap \Omega$ for all $\varepsilon \in]0, \varepsilon_0[$ (see Theorem 3.1.27). We note that $\int_\Omega \rho_\varepsilon dx = M$ for all $\varepsilon \in]0, \varepsilon_0[$. We refer to the quantity M as the total mass of the body.

We prove, under the hypothesis that Ω is of class C^2, convergence of the eigenvalues and eigenfunctions of problem (5.0.3) with density ρ_ε to the eigenvalues and eigenfunctions of problem (5.2.1) with constant surface density $\frac{M}{|\partial\Omega|}$ when the parameter ε goes to zero (see Corollary 5.3.12). This provides a further interpretation of problem (5.2.1) as the equation of a free vibrating plate whose mass is concentrated at the boundary in the case of domains of class C^2.

We consider the weak formulation of problem (5.0.3) with density ρ_ε,

$$\int_\Omega D^2u : D^2\varphi + \tau\nabla u \cdot \nabla\varphi dx = \lambda \int_\Omega \rho_\varepsilon u\varphi dx, \quad \forall \varphi \in H^2(\Omega)\,, \tag{5.3.1}$$

in the unknowns $u \in H^2(\Omega)$, $\lambda \in \mathbb{R}$. In the sequel we shall recast this problem in $H^2(\Omega)/\mathbb{R}$ since we need to get rid of the constants, which generate the eigenspace corresponding to the eigenvalue $\lambda = 0$. We denote by i the canonical embedding of $H^2(\Omega)$ into $L^2(\Omega)$. We denote by $\mathcal{J}_{\rho_\varepsilon}^{\mathcal{N}}$ the continuous embedding of $L^2(\Omega)$ into $H^2(\Omega)'$, defined by

$$\mathcal{J}_{\rho_\varepsilon}^{\mathcal{N}}[u][\varphi] := \int_\Omega \rho_\varepsilon u\varphi dx, \quad \forall u \in L^2(\Omega), \varphi \in H^2(\Omega).$$

We set

$$H_{\rho_\varepsilon}^{2,\mathcal{N}}(\Omega) := \left\{ u \in H^2(\Omega) : \int_\Omega u\rho_\varepsilon dx = 0 \right\}.$$

In the sequel we will think of the space $H_{\rho_\varepsilon}^{2,\mathcal{N}}(\Omega)$ as endowed with the scalar product (5.2.2). This scalar product induces on $H_{\rho_\varepsilon}^{2,\mathcal{N}}(\Omega)$ a norm which is equivalent to the standard one. We denote by $\pi_{\rho_\varepsilon}^{\mathcal{N}}$ the map from $H^2(\Omega)$ to $H_{\rho_\varepsilon}^{2,\mathcal{N}}(\Omega)$ defined by

$$\pi_{\rho_\varepsilon}^{\mathcal{N}}[u] := u - \frac{\int_\Omega u\rho_\varepsilon dx}{\int_\Omega \rho_\varepsilon dx},$$

for all $u \in H^2(\Omega)$. We denote by $\pi_{\rho_\varepsilon}^{\sharp,\mathcal{N}}$ the map from $H^2(\Omega)/\mathbb{R}$ onto $H_{\rho_\varepsilon}^{2,\mathcal{N}}(\Omega)$ defined by the equality $\pi_{\rho_\varepsilon}^{\mathcal{N}} = \pi_{\rho_\varepsilon}^{\sharp,\mathcal{N}} \circ p$, where p is the canonical projection of $H^2(\Omega)$ onto $H^2(\Omega)/\mathbb{R}$. As in (5.2.3), we consider the operator $\mathcal{P}_{\rho_\varepsilon}^{\mathcal{N}}$ as a map from $H_{\rho_\varepsilon}^{2,\mathcal{N}}(\Omega)$ to $F(\Omega)$ defined by

$$\mathcal{P}_{\rho_\varepsilon}^{\mathcal{N}}[u][\varphi] := \int_\Omega D^2u : D^2\varphi + \tau\nabla u \cdot \nabla\varphi dx,\ \forall u \in H_{\rho_\varepsilon}^{2,\mathcal{N}}(\Omega), \varphi \in H^2(\Omega).$$

It turns out that $\mathcal{P}_{\rho_\varepsilon}^{\mathcal{N}}$ is a linear homeomorphism of $H_{\rho_\varepsilon}^{2,\mathcal{N}}(\Omega)$ onto $F(\Omega)$. Finally, let the operator $T_{\rho_\varepsilon}^{\mathcal{N}}$ from $H^2(\Omega)/\mathbb{R}$ to itself be defined by

$$T_{\rho_\varepsilon}^{\mathcal{N}} := (\pi_{\rho_\varepsilon}^{\sharp,\mathcal{N}})^{-1} \circ (\mathcal{P}_{\rho_\varepsilon}^{\mathcal{N}})^{-1} \circ \mathcal{J}_{\rho_\varepsilon}^{\mathcal{N}} \circ i \circ \pi_{\rho_\varepsilon}^{\sharp,\mathcal{N}}. \tag{5.3.2}$$

Lemma 5.3.3. *The pair (λ, u) of the set $(\mathbb{R} \setminus \{0\}) \times (H^{2,\mathcal{N}}_{\rho_\varepsilon}(\Omega) \setminus \{0\})$ satisfies (5.3.1) if and only if $\lambda > 0$ and the pair $(\lambda^{-1}, p[u])$ of the set $\mathbb{R} \times ((H^2(\Omega)/\mathbb{R}) \setminus \{0\})$ satisfies the equation*

$$\lambda^{-1} p[u] = T^{\mathcal{N}}_{\rho_\varepsilon} p[u].$$

As in Theorem 5.2.7 it is easy to prove the following theorem.

Theorem 5.3.4. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1 and $\varepsilon \in]0, \varepsilon_0[$. The operator $T^{\mathcal{N}}_{\rho_\varepsilon}$ is a compact self-adjoint operator in $H^2(\Omega)/\mathbb{R}$ and its eigenvalues coincide with the reciprocals of the positive eigenvalues $\lambda_j(\rho_\varepsilon)$ of problem (5.3.1) for all $j \in \mathbb{N}$. Moreover, the set of eigenvalues of problem (5.3.1) is contained in $[0, +\infty[$ and consists of the image of a sequence increasing to $+\infty$. Each eigenvalue has finite multiplicity.*

We have the following theorem on the spectrum of problem (5.3.1) (see also Theorem 5.2.8).

Theorem 5.3.5. *The first eigenvalue λ_1 of (5.3.1) is zero and the corresponding eigenfunctions are the constants. Moreover, $\lambda_2 > 0$.*

Now we highlight the relations between problems (5.2.1) and (5.3.1) when Ω is of class C^2. In particular we plan to prove the following theorem.

Theorem 5.3.6. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^2. Let the operators $T^{\mathcal{S}}_{\frac{M}{|\partial\Omega|}}$ and $T^{\mathcal{N}}_{\rho_\varepsilon}$ from $H^2(\Omega)/\mathbb{R}$ to itself be defined as in (5.2.5) and (5.3.2) respectively. Then $T^{\mathcal{N}}_{\rho_\varepsilon}$ converges in norm to $T^{\mathcal{S}}_{\frac{M}{|\partial\Omega|}}$ as $\varepsilon \to 0$.*

In order to prove Theorem 5.3.6 we need the following lemma. We remark that by (5.2.4), $\pi^{\sharp,\mathcal{S}}_c = \pi^{\sharp,\mathcal{S}}_1$ for all $c \in \mathbb{R}$, with $c \neq 0$.

Lemma 5.3.7. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^2. Let $\rho_\varepsilon \in \mathcal{R}$ be as in (3.1.20). Then the following statements hold.*

i) *For all $\varphi \in H^2(\Omega)/\mathbb{R}$, $\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[\varphi] \to \pi^{\sharp,\mathcal{S}}_1[\varphi]$ in $L^2(\Omega)$ (hence also in $H^2(\Omega)$) as $\varepsilon \to 0$;*

ii) *If $u_\varepsilon \rightharpoonup u$ in $H^2(\Omega)/\mathbb{R}$, then possibly passing to a subsequence $\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[u_\varepsilon] \to \pi^{\sharp,\mathcal{S}}_1[u]$ in $L^2(\Omega)$ as $\varepsilon \to 0$;*

iii) *Assume that $u_\varepsilon, u, w_\varepsilon, w \in H^2(\Omega)$ are such that $u_\varepsilon \to u$, $w_\varepsilon \to w$ in $L^2(\Omega)$ and $\mathrm{Tr}[u_\varepsilon] \to \mathrm{Tr}[u]$, $\mathrm{Tr}[w_\varepsilon] \to \mathrm{Tr}[w]$ in $L^2(\partial\Omega)$ as $\varepsilon \to 0$. Moreover assume that there exists a constant $C > 0$ such that $\|\nabla u_\varepsilon\|_{L^2(\Omega)} \leq C$, $\|\nabla w_\varepsilon\|_{L^2(\Omega)} \leq C$ for all $\varepsilon \in]0, \varepsilon_0[$. Then*

$$\int_\Omega \rho_\varepsilon \left(u_\varepsilon - u\right) w_\varepsilon dx \to 0,$$

and

$$\int_{\Omega} \rho_\varepsilon \left(w_\varepsilon - w\right) u dx \to 0,$$

as $\varepsilon \to 0$

Proof. The proof is analogous to that of Lemma 3.1.28 and accordingly is omitted (see also [72] and references therein for details). □

Proof of Theorem 5.3.6. The proof is similar to that of Theorem 3.1.21. We repeat it here for completeness. It is sufficient to prove that the family $\left\{T_{\rho_\varepsilon}^{\mathcal{N}}\right\}_{\varepsilon\in]0,\varepsilon_0[}$ of compact operators, compactly converges to the compact operator $T_{\frac{M}{|\partial\Omega|}}^{\mathcal{S}}$ (see Definition 3.1.36). This implies, in fact, that

$$\lim_{\varepsilon\to 0}\left\|\left(T_{\rho_\varepsilon}^{\mathcal{N}} - T_{\frac{M}{|\partial\Omega|}}^{\mathcal{S}}\right)^2\right\|_{\mathcal{L}(H^2(\Omega)/\mathbb{R},H^2(\Omega)/\mathbb{R})} = 0. \tag{5.3.8}$$

Then, since the operators $\left\{T_{\rho_\varepsilon}^{\mathcal{N}}\right\}_{\varepsilon\in]0,\varepsilon_0[}$ and $T_{\frac{M}{|\partial\Omega|}}^{\mathcal{S}}$ are self-adjoint, property (5.3.8) is equivalent to convergence in norm (see [8, 103] for a more detailed discussion on compact convergence of compact operators on Hilbert spaces). We recall that, by definition, $T_{\rho_\varepsilon}^{\mathcal{N}}$ compactly converges to $T_{\frac{M}{|\partial\Omega|}}^{\mathcal{S}}$ if the following requirements are fulfilled:

i) if $\|u_\varepsilon\|_{H^2(\Omega)/\mathbb{R}} \leq C$ for all $\varepsilon \in]0,\varepsilon_0[$, then the family $\{T_{\rho_\varepsilon}^{\mathcal{N}}u_\varepsilon\}_{\varepsilon\in]0,\varepsilon_0[}$ is relatively compact in $H^2(\Omega)/\mathbb{R}$;

ii) if $u_\varepsilon \to u$ in $H^2(\Omega)/\mathbb{R}$, then $T_{\rho_\varepsilon}^{\mathcal{N}}u_\varepsilon \to T_{\frac{M}{|\partial\Omega|}}^{\mathcal{S}} u$ in $H^2(\Omega)/\mathbb{R}$.

We prove i) first. Let $u,\varphi \in H^2(\Omega)/\mathbb{R}$ we have

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\Omega}\rho_\varepsilon\pi_{\rho_\varepsilon}^{\sharp,\mathcal{N}}[u]\pi_{\rho_\varepsilon}^{\sharp,\mathcal{N}}[\varphi]dx &= \lim_{\varepsilon\to 0}\int_{\Omega}\rho_\varepsilon\left(\pi_{\rho_\varepsilon}^{\sharp,\mathcal{N}}[u]-\pi_1^{\sharp,\mathcal{S}}[u]\right)\pi_{\rho_\varepsilon}^{\sharp,\mathcal{N}}[\varphi]dx \\
&+ \lim_{\varepsilon\to 0}\int_{\Omega}\rho_\varepsilon\pi_1^{\sharp,\mathcal{S}}[u]\left(\pi_{\rho_\varepsilon}^{\sharp,\mathcal{N}}[\varphi]-\pi_1^{\sharp,\mathcal{S}}[\varphi]\right)dx \\
&+\left(\lim_{\varepsilon\to 0}\int_{\Omega}\rho_\varepsilon\pi_1^{\sharp,\mathcal{S}}[u]\pi_1^{\sharp,\mathcal{S}}[\varphi]dx - \frac{M}{|\partial\Omega|}\int_{\partial\Omega}\pi_1^{\sharp,\mathcal{S}}[u]\pi_1^{\sharp,\mathcal{S}}[\varphi]d\sigma\right) \\
&+\frac{M}{|\partial\Omega|}\int_{\partial\Omega}\pi_1^{\sharp,\mathcal{S}}[u]\pi_1^{\sharp,\mathcal{S}}[\varphi]d\sigma.
\end{aligned} \tag{5.3.9}$$

By Lemma 5.3.7, *ii)* we have that the first and second summands in the right-hand side of (5.3.9) go to zero as $\varepsilon \to 0$. As for the third summand, from Lemma 3.1.22 i) applied to the function $f = 1$ we have that $|\omega_\varepsilon| = \varepsilon|\partial\Omega| + o(\varepsilon)$ as $\varepsilon \to 0$. Therefore $\rho_\varepsilon = \frac{M}{\varepsilon|\partial\Omega|} + o(1)$ as $\varepsilon \to 0$. Thus, from

Lemma 3.1.22 and formula (3.1.23) it follows that also the third summand of (5.3.9) goes to zero as $\varepsilon \to 0$. Moreover, the equality $(\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon})^{-1} \circ (\mathcal{P}^{\mathcal{N}}_{\rho_\varepsilon})^{-1} = (\pi^{\sharp,\mathcal{S}}_1)^{-1} \circ (\mathcal{P}^{\mathcal{S}}_1)^{-1}$ holds. Therefore, from (5.3.9) it follows that $T^{\mathcal{N}}_{\rho_\varepsilon} u$ is bounded for each $u \in H^2(\Omega)/\mathbb{R}$. Thus, by Banach-Steinhaus Theorem, there exists C' such that $\|T^{\mathcal{N}}_{\rho_\varepsilon}\|_{\mathcal{L}(H^2(\Omega)/\mathbb{R}, H^2(\Omega)/\mathbb{R})} \le C'$ for all $\varepsilon \in]0, \varepsilon_0[$. Moreover, since $\|u_\varepsilon\|_{H^2(\Omega)/\mathbb{R}} \le C$ for all $\varepsilon \in]0, \varepsilon_0[$, possibly passing to a subsequence, we have that $u_\varepsilon \rightharpoonup u$ in $H^2(\Omega)/\mathbb{R}$, for some $u \in H^2(\Omega)/\mathbb{R}$. Then, possibly passing to a subsequence, $T^{\mathcal{N}}_{\rho_\varepsilon} u_\varepsilon \rightharpoonup w$ in $H^2(\Omega)/\mathbb{R}$ as $\varepsilon \to 0$, for some $w \in H^2(\Omega)/\mathbb{R}$. We show that $w = T^{\mathcal{S}}_{\frac{M}{|\partial\Omega|}} u$. To shorten our notation we set $w_\varepsilon := T^{\mathcal{N}}_{\rho_\varepsilon} u_\varepsilon$. By Lemma 5.3.7, *i)* we have

$$\lim_{\varepsilon\to 0} \int_\Omega D^2(\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[w_\varepsilon]) : D^2(\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[\varphi]) + \tau \nabla(\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[w_\varepsilon]) \cdot \nabla(\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[\varphi]) dx$$
$$= \int_\Omega D^2(\pi^{\sharp,\mathcal{S}}_1[w]) : D^2(\pi^{\sharp,\mathcal{S}}_1[\varphi]) + \tau \nabla(\pi^{\sharp,\mathcal{S}}_1[w]) \cdot \nabla(\pi^{\sharp,\mathcal{S}}_1[\varphi]) dx, \quad (5.3.10)$$

for all $\varphi \in H^2(\Omega)/\mathbb{R}$. On the other hand, since by (5.3.2) $\left(\mathcal{P}^{\mathcal{N}}_{\rho_\varepsilon} \circ \pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}\right) w_\varepsilon = \left(\mathcal{J}^{\mathcal{N}}_{\rho_\varepsilon} \circ i \circ \pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}\right) u_\varepsilon$, we have that

$$\int_\Omega D^2(\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[w_\varepsilon]) : D^2(\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[\varphi]) + \tau \nabla(\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[w_\varepsilon]) \cdot \nabla(\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[\varphi]) dx$$
$$= \int_\Omega \rho_\varepsilon \pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[u_\varepsilon] \pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[\varphi] dx \quad (5.3.11)$$

Then, by Lemma 5.3.7, *iii)*, (5.3.10) and (5.3.11) we have

$$< w, \varphi >_{H^2(\Omega)/\mathbb{R}} = \lim_{\varepsilon\to 0} < w_\varepsilon, \varphi >_{H^2(\Omega)/\mathbb{R}} = \lim_{\varepsilon\to 0} \int_\Omega \rho_\varepsilon \pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[u_\varepsilon] \pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[\varphi] dx$$
$$= \lim_{\varepsilon\to 0} \int_\Omega \rho_\varepsilon \left(\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[u_\varepsilon] - \pi^{\sharp,\mathcal{S}}_1[u]\right) \pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[\varphi] dx$$
$$+ \lim_{\varepsilon\to 0} \int_\Omega \rho_\varepsilon \pi^{\sharp,\mathcal{S}}_1[u] \left(\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[\varphi] - \pi^{\sharp,\mathcal{S}}_1[\varphi]\right) dx$$
$$+ \lim_{\varepsilon\to 0} \int_\Omega \rho_\varepsilon \pi^{\sharp,\mathcal{S}}_1[u] \pi^{\sharp,\mathcal{S}}_1[\varphi] dx = \frac{M}{|\partial\Omega|} \int_{\partial\Omega} \pi^{\sharp,\mathcal{S}}_1[u] \pi^{\sharp,\mathcal{S}}_1[\varphi] d\sigma$$
$$= < T^{\mathcal{S}}_{\frac{M}{|\partial\Omega|}} u, \varphi >_{H^2(\Omega)/\mathbb{R}},$$

hence $w = T^{\mathcal{S}}_{\frac{M}{|\partial\Omega|}} u$. In a similar way one can prove that $\|w_\varepsilon\|_{H^2(\Omega)/\mathbb{R}} \to$

$\|w\|_{H^2(\Omega)/\mathbb{R}}$. In fact

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\|w_\varepsilon\|^2_{H^2(\Omega)/\mathbb{R}} &= \lim_{\varepsilon\to 0}\int_\Omega \rho_\varepsilon\left(\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[u_\varepsilon]-\pi^{\sharp,\mathcal{S}}_1[u]\right)\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[w_\varepsilon]dx\\
&+\lim_{\varepsilon\to 0}\int_\Omega \rho_\varepsilon\pi^{\sharp,\mathcal{S}}_1[u]\left(\pi^{\sharp,\mathcal{N}}_{\rho_\varepsilon}[w_\varepsilon]-\pi^{\sharp,\mathcal{S}}_1[w_\varepsilon]\right)dx\\
&+\lim_{\varepsilon\to 0}\int_\Omega \rho_\varepsilon\pi^{\sharp,\mathcal{S}}_1[u]\left(\pi^{\sharp,\mathcal{S}}_1[w_\varepsilon]-\pi^{\sharp,\mathcal{S}}_1[w]\right)dx\\
&+\lim_{\varepsilon\to 0}\int_\Omega \rho_\varepsilon\pi^{\sharp,\mathcal{S}}_1[u]\pi^{\sharp,\mathcal{S}}_1[w]dx\\
&=\frac{M}{|\partial\Omega|}\int_{\partial\Omega}\pi^{\sharp,\mathcal{S}}_1[u]\pi^{\sharp,\mathcal{S}}_1[w]d\sigma=\|w\|^2_{H^2(\Omega)/\mathbb{R}}\,.
\end{aligned}
$$

This proves *i*). As for point *ii*), let $u_\varepsilon\to u$ in $H^2(\Omega)/\mathbb{R}$. Then there exists C'' such that $\|u_\varepsilon\|_{H^2(\Omega)/\mathbb{R}}\le C''$ for all $\varepsilon\in]0,\varepsilon_0[$. Then, by the same argument used for point *i*), for each sequence $\varepsilon_j\to 0$, possibly passing to a subsequence, we have $T^{\mathcal{N}}_{\rho_{\varepsilon_j}}u_{\varepsilon_j}\to T^{\mathcal{S}}_{\frac{M}{|\partial\Omega|}}u$. Since this is true for each $\{\varepsilon_j\}_{j\in\mathbb{N}}$, we have the convergence for the whole family, i.e., $T^{\mathcal{N}}_{\rho_\varepsilon}u_\varepsilon\to T^{\mathcal{S}}_{\frac{M}{|\partial\Omega|}}u$. This concludes the proof. □

Thanks to Theorem 3.1.41, as an immediate corollary of Theorem 5.3.6 we have

Corollary 5.3.12. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^2. Let $\lambda_j[\rho_\varepsilon]$ denote the eigenvalues of problem (5.3.1) on Ω for all $j\in\mathbb{N}$. Let λ_j, $j\in\mathbb{N}$ denote the eigenvalues of problem (5.2.1) corresponding to the constant surface density $\frac{M}{|\partial\Omega|}$. Then $\lim_{\varepsilon\to 0}\lambda_j[\rho_\varepsilon]=\lambda_j$ for all $j\in\mathbb{N}$.*

5.4 Symmetric functions of the eigenvalues. Isovolumetric perturbations

In this section we address the problem of the dependece of the eigenvalues of problems (5.0.1) and (5.0.3) upon perturbations of the domain Ω.

We shall compute Hadamard-type formulas for both the Steklov and the Neumann problems, which will be used to investigate the behavior of the eigenvalues subject to isovolumetric perturbations. To do so, we use the so called transplantation method, see [60] for a general introduction to this approach. We will study problems (5.0.1) and (5.0.3) on $\phi(\Omega)$, for a suitable homeomorphism ϕ, where Ω has to be thought as a fixed bounded domain of class C^1. Therefore, we introduce the following class of functions

$$
\Phi(\Omega):=\left\{\phi\in\left(C^2\left(\overline{\Omega}\right)\right)^N:\phi\text{ injective and }\inf_\Omega|\det D\phi|>0\right\}.
$$

We observe that if Ω is of class C^1 and $\phi \in \Phi(\Omega)$, then also $\phi(\Omega)$ is of class C^1 and $\phi^{(-1)} \in \Phi(\phi(\Omega))$. Therefore, it makes sense to study both problems (5.0.1) and problem (5.0.3) on $\phi(\Omega)$. Moreover, we endow the space $C^2(\overline{\Omega})$ with the standard norm

$$||f||_{C^2(\overline{\Omega})} = \sup_{|\alpha|\leq 2,\ x\in\overline{\Omega}} |D^\alpha f(x)|.$$

Note that $\Phi(\Omega)$ is open in $\left(C^2(\overline{\Omega})\right)^N$ (see e.g., [75, Lemma 3.11]).

It is known that balls play a relevant role in the study of isovolumetric perturbations of the domain Ω for all the eigenvalues of the Dirichlet and Neumann Laplacian. We refer to [75, 77], where the authors prove that the elementary symmetric functions of the eigenvalues depend real analytically on the domain, providing also Hadamard-type formulas for the corresponding derivatives. Then, in [76] they show that balls are critical points for such functions under volume constraint.

From now on we will consider problems (5.0.1) and (5.0.3) with constant mass density $\rho \equiv 1$.

5.4.1 The Steklov problem

We plan to study the Steklov problem on the domain $\phi(\Omega)$ for $\phi \in \Phi(\Omega)$, i.e.,

$$\begin{cases} \Delta^2 u - \tau\Delta u = 0, & \text{in } \phi(\Omega), \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\phi(\Omega), \\ \tau\frac{\partial u}{\partial \nu} - \mathrm{div}_{\partial\phi(\Omega)}(D^2 u\cdot\nu) - \frac{\partial \Delta u}{\partial \nu} = \lambda u, & \text{on } \partial\phi(\Omega). \end{cases} \tag{5.4.1}$$

To do so, we pull back the resolvent operator of (5.4.1) to Ω. Therefore, we are interested in the operator $\mathcal{P}_\phi^{\mathcal{S}}$ from $H_\phi^{2,\mathcal{S}}(\Omega)$ to $F(\Omega)$, defined by

$$\begin{aligned} \mathcal{P}_\phi^{\mathcal{S}}[u][\varphi] := &\int_\Omega (D^2(u\circ\phi^{-1})\circ\phi) : (D^2(\varphi\circ\phi^{-1})\circ\phi)|\det D\phi| dx \\ &+ \tau\int_\Omega (\nabla(u\circ\phi^{-1})\circ\phi)\cdot(\nabla(\varphi\circ\phi^{-1})\circ\phi)|\det D\phi| dx, \end{aligned} \tag{5.4.2}$$

for all $u \in H_\phi^{2,\mathcal{S}}(\Omega)$, $\varphi \in H^2(\Omega)$, where

$$H_\phi^{2,\mathcal{S}}(\Omega) := \left\{ u \in H^2(\Omega) : \int_{\partial\Omega} u|\nu(\nabla\phi)^{-1}||\det D\phi| d\sigma = 0 \right\}.$$

Moreover, for every $\phi \in \Phi(\Omega)$, we consider the map $\mathcal{J}_\phi^{\mathcal{S}}$ from $L^2(\partial\Omega)$ to $H^2(\Omega)'$ defined by

$$\mathcal{J}_\phi^{\mathcal{S}}[u][\varphi] := \int_{\partial\Omega} u\varphi|\nu(\nabla\phi)^{-1}||\det D\phi| d\sigma, \quad \forall u \in L^2(\partial\Omega), \varphi \in H^2(\Omega).$$

It is easily seen that the form (5.4.2) is a scalar product on $H^{2,\mathcal{S}}_\phi(\Omega)$. We will think of the space $H^{2,\mathcal{S}}_\phi(\Omega)$ as endowed with the scalar product (5.4.2). We denote by $\pi^{\mathcal{S}}_\phi$ the map from $H^2(\Omega)$ to $H^{2,\mathcal{S}}_\phi(\Omega)$ defined by

$$\pi^{\mathcal{S}}_\phi(u) := u - \frac{\int_{\partial\Omega} u|\nu(\nabla\phi)^{-1}||\det D\phi|dx}{\int_{\partial\Omega} |\nu(\nabla\phi)^{-1}||\det D\phi|dx},$$

and by $\pi^{\sharp,\mathcal{S}}_\phi$ the map from $H^2(\Omega)/\mathbb{R}$ onto $H^{2,\mathcal{S}}_\phi(\Omega)$ defined by the equality $\pi^{\mathcal{S}}_\phi = \pi^{\sharp,\mathcal{S}}_\phi \circ p$. Clearly, $\pi^{\sharp,\mathcal{S}}_\phi$ is a homeomorphism, and we can recast problem (5.4.1) as

$$\lambda^{-1}u = W^{\mathcal{S}}_\phi u,$$

where

$$W^{\mathcal{S}}_\phi := (\pi^{\sharp,\mathcal{S}}_\phi)^{-1} \circ (\mathcal{P}^{\mathcal{S}}_\phi)^{-1} \circ \mathcal{J}^{\mathcal{S}}_\phi \circ \mathrm{Tr} \circ \pi^{\sharp,\mathcal{S}}_\phi.$$

The operator $W^{\mathcal{S}}_\phi$ can be shown to be compact and self-adjoint, as we have done for the operator $T^{\mathcal{S}}_\rho$ defined by (5.2.5) in Theorem 5.2.7 (see also [77, Theorem 2.1]).

In order to avoid bifurcation phenomena, which usually occur when dealing with multiple eigenvalues, we focus our attention on the elementary symmetric functions of the eigenvalues. This is the aim of the following theorem.

Theorem 5.4.3. *Let Ω be a bounded domain of $\mathbb{R}^N$ of class C^1. Let F be a finite non-empty subset of $\mathbb{N}$. Let*

$$\mathcal{A}_\Omega[F] := \{\phi \in \Phi(\Omega) : \lambda_l[\phi] \notin \{\lambda_j[\phi] : j \in F\}\ \forall l \in \mathbb{N} \setminus F\}.$$

Then the following statements hold.

i) The set $\mathcal{A}_\Omega[F]$ is open in $\Phi(\Omega)$. The map P_F of $\mathcal{A}_\Omega[F]$ to the space $\mathcal{L}\left(H^2(\Omega), H^2(\Omega)\right)$ which takes $\phi \in \mathcal{A}_\Omega[F]$ to the orthogonal projection of $H^{2,\mathcal{S}}_\phi(\Omega)$ onto its (finite dimensional) subspace generated by

$$\left\{u \in H^{2,\mathcal{S}}_\phi(\Omega) : \mathcal{P}^{\mathcal{S}}_{\mathrm{Id}}[u\circ\phi^{-1}] = \lambda_j[\phi]\mathcal{J}^{\mathcal{S}}_{\mathrm{Id}} \circ \mathrm{Tr}[u\circ\phi^{-1}] \text{ for some } j \in F\right\}$$

is real analytic.

ii) Let $s \in \{1, \dots, |F|\}$. The function $\Lambda_{F,s}$ from $\mathcal{A}_\Omega[F]$ to $\mathbb{R}$ defined by

$$\Lambda_{F,s}[\phi] := \sum_{j_1<\cdots<j_s\in F} \lambda_{j_1}[\phi]\cdots\lambda_{j_s}[\phi]$$

is real analytic.

iii) Let

$$\Theta_\Omega[F] := \{\phi \in \mathcal{A}_\Omega[F] : \lambda_j[\phi] \text{ have a common value } \lambda_F[\phi]\ \forall j \in F\}.$$

Then the real analytic functions

$$\left(\binom{|F|}{1}^{-1}\Lambda_{F,1}\right)^{\frac{1}{1}}, \dots, \left(\binom{|F|}{|F|}^{-1}\Lambda_{F,|F|}\right)^{\frac{1}{|F|}},$$

of $\mathcal{A}_\Omega[F]$ to $\mathbb{R}$ coincide on $\Theta_\Omega[F]$ with the function which takes ϕ to $\lambda_F[\phi]$.

Proof. The proof can be done adapting that of [77, Theorem 2.2 and Corollary 2.3] (see also [75]). □

In order to compute explicit formulas for the differentials of the functions $\Lambda_{F,s}$, we need the following technical lemma.

Lemma 5.4.4. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1, and let $\tilde\phi \in \Phi(\Omega)$ be such that $\tilde\phi(\Omega)$ is of class C^2. Let $u_1, u_2 \in H^2(\Omega)$ be such that $v_i = u_i \circ \tilde\phi^{(-1)} \in H^4(\tilde\phi(\Omega))$ for $i = 1, 2$ and*

$$\frac{\partial^2 v_1}{\partial \nu^2} = \frac{\partial^2 v_2}{\partial \nu^2} = 0 \quad \text{on} \quad \partial\tilde\phi(\Omega).$$

Then we have

$$\begin{aligned} d|_{\phi=\tilde\phi}\mathcal{P}^{\mathcal{S}}_\phi[\psi][u_1][u_2] &= \int_{\partial\tilde\phi(\Omega)} (D^2v_1 : D^2v_2 + \tau\nabla v_1\cdot\nabla v_2)\mu\cdot\nu d\sigma \\ &+ \int_{\partial\tilde\phi(\Omega)} \Big(\mathrm{div}_{\partial\tilde\phi(\Omega)}(D^2v_1.\nu)\nabla v_2 + \mathrm{div}_{\partial\tilde\phi(\Omega)}(D^2v_2.\nu)\nabla v_1\Big)\cdot\mu d\sigma \\ +\int_{\partial\tilde\phi(\Omega)} &\left(\frac{\partial\Delta v_1}{\partial\nu}\nabla v_2 + \frac{\partial\Delta v_2}{\partial\nu}\nabla v_1\right)\cdot\mu d\sigma - \tau\int_{\partial\tilde\phi(\Omega)}\left(\frac{\partial v_1}{\partial\nu}\nabla v_2 + \frac{\partial v_2}{\partial\nu}\nabla v_1\right)\cdot\mu d\sigma \\ &- \int_{\tilde\phi(\Omega)} \big((\Delta^2 v_1 - \tau\Delta v_1)\nabla v_2 + (\Delta^2 v_2 - \tau\Delta v_2)\nabla v_1\big)\cdot\mu d\sigma, \end{aligned} \tag{5.4.5}$$

for all $\psi \in (C^2(\overline{\Omega}))^N$, where $\mu = \psi\circ\tilde\phi^{-1}$.

Proof. We have

$$\begin{aligned}
& d|_{\phi=\tilde\phi}\mathcal{P}^{\mathcal{S}}_{\phi}[\psi][u_1][u_2] \\
&\quad = \int_\Omega (d|_{\phi=\tilde\phi} D^2(u_1\circ\phi^{-1})\circ\phi)[\psi] : (D^2(u_2\circ\tilde\phi^{-1})\circ\tilde\phi)|\det D\tilde\phi| dx \\
&\quad + \tau\int_\Omega (d|_{\phi=\tilde\phi}\nabla(u_1\circ\phi^{-1})\circ\phi)[\psi]\cdot(\nabla(u_2\circ\tilde\phi^{-1})\circ\tilde\phi)|\det D\tilde\phi| dx \\
&\quad + \int_\Omega (D^2(u_1\circ\tilde\phi^{-1})\circ\tilde\phi) : (d|_{\phi=\tilde\phi} D^2(u_2\circ\phi^{-1})\circ\phi)[\psi]|\det D\tilde\phi| dx \\
&\quad + \tau\int_\Omega (\nabla(u_1\circ\tilde\phi^{-1})\circ\tilde\phi)\cdot(d|_{\phi=\tilde\phi}\nabla(u_2\circ\phi^{-1})\circ\phi)[\psi]|\det D\tilde\phi| dx \\
&\quad + \int_\Omega (D^2(u_1\circ\tilde\phi^{-1})\circ\tilde\phi) : (D^2(u_2\circ\tilde\phi^{-1})\circ\tilde\phi) d|_{\phi=\tilde\phi}|\det D\phi|[\psi] dx \\
& + \tau\int_\Omega (\nabla(u_1\circ\tilde\phi^{-1})\circ\tilde\phi)\cdot(\nabla(u_2\circ\tilde\phi^{-1})\circ\tilde\phi) d|_{\phi=\tilde\phi}|\det D\phi|[\psi] dx,
\end{aligned} \tag{5.4.6}$$

and we note that the last two summands in (5.4.6) equals

$$\int_{\tilde\phi(\Omega)} \big(D^2v_1 : D^2v_2 + \tau\nabla v_1\cdot\nabla v_2\big)\,\mathrm{div}\mu\, dy.$$

(See also Proprosition 5.4.18). By standard calculus we have (see [23, formula (2.15)])

$$D^2(u\circ\phi^{-1})\circ\phi = (\nabla\phi)^{-t}D^2u(\nabla\phi)^{-1} + \left(\sum_{k,l=1}^N \frac{\partial u}{\partial x_k}\frac{\partial\sigma_{k,i}}{\partial x_l}\sigma_{l,j}\right)_{i,j},$$

where $\sigma = (\nabla\phi)^{-1}$. This yields the following formula

$$d|_{\phi=\tilde\phi}(D^2(u\circ\phi^{-1})\circ\phi)[\psi]\circ\tilde\phi^{-1} = -D^2v\nabla\mu - \nabla\mu^t D^2v - \sum_{r=1}^N \frac{\partial v}{\partial y_r}D^2\mu_r, \tag{5.4.7}$$

where $\mu = \psi\circ\tilde\phi^{-1}$ and $v = u\circ\tilde\phi^{-1}$. We rewrite formula (5.4.7) componentwise getting

$$\begin{aligned}
&\Big(d|_{\phi=\tilde\phi}(D^2(u\circ\phi^{-1})\circ\phi)[\psi]\circ\tilde\phi^{-1}\Big)_{i,j} \\
&\qquad\qquad = -\sum_{r=1}^N\left(\frac{\partial^2 v}{\partial y_i\partial y_r}\frac{\partial\mu_r}{\partial y_j} + \frac{\partial^2 v}{\partial y_j\partial y_r}\frac{\partial\mu_r}{\partial y_i} + \frac{\partial^2\mu_r}{\partial y_i\partial y_j}\frac{\partial v}{\partial y_r}\right).
\end{aligned}$$

Moreover (see [75, Lemma 3.26])

$$\Big(d|_{\phi=\tilde\phi}(\nabla(u\circ\phi^{-1})\circ\phi)[\psi]\circ\tilde\phi^{-1}\Big)_i = -\sum_{r=1}^N\frac{\partial v}{\partial y_r}\frac{\partial\mu_r}{\partial y_i}.$$

Now we use Einstein notation, dropping all the summation symbols. The first summand of the right hand side of (5.4.6) equals

$$-\int_{\tilde{\phi}(\Omega)} \left(\frac{\partial^2 v_1}{\partial y_i \partial y_r} \frac{\partial \mu_r}{\partial y_j} + \frac{\partial^2 v_1}{\partial y_j \partial y_r} \frac{\partial \mu_r}{\partial y_i} + \frac{\partial^2 \mu_r}{\partial y_i \partial y_j} \frac{\partial v_1}{\partial y_r} \right) \frac{\partial^2 v_2}{\partial y_i \partial y_j} dy. \qquad (5.4.8)$$

In order to compute (5.4.8), integrating by parts, we have

$$\begin{aligned}
\int_{\tilde{\phi}(\Omega)} \frac{\partial^2 v_1}{\partial y_i \partial y_r} \frac{\partial \mu_r}{\partial y_j} \frac{\partial^2 v_2}{\partial y_i \partial y_j} dy &= \int_{\partial \tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_i} \frac{\partial \mu_r}{\partial y_j} \nu_r \frac{\partial^2 v_2}{\partial y_i \partial y_j} d\sigma \\
&- \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_i} \frac{\partial \mathrm{div}\mu}{\partial y_j} \frac{\partial^2 v_2}{\partial y_i \partial y_j} dy - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_i} \frac{\partial \mu_r}{\partial y_j} \frac{\partial^3 v_2}{\partial y_i \partial y_j \partial y_r} dy \\
&= \int_{\partial \tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_i} \frac{\partial \mu_r}{\partial y_j} \nu_r \frac{\partial^2 v_2}{\partial y_i \partial y_j} d\sigma - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_i} \frac{\partial \mu_r}{\partial y_j} \frac{\partial^3 v_2}{\partial y_i \partial y_j \partial y_r} dy \\
&- \int_{\partial \tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_i} \mathrm{div}\mu \frac{\partial^2 v_2}{\partial y_i \partial y_j} \nu_j d\sigma + \int_{\tilde{\phi}(\Omega)} D^2 v_1 : D^2 v_2 \mathrm{div}\mu dy \\
&+ \int_{\tilde{\phi}(\Omega)} \mathrm{div}\mu \nabla v_1 \cdot \nabla \Delta v_2 dy,
\end{aligned}$$

and

$$\begin{aligned}
\int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_r} \frac{\partial^2 \mu_r}{\partial y_i \partial y_j} \frac{\partial^2 v_2}{\partial y_i \partial y_j} dy &= \int_{\partial \tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_r} \frac{\partial \mu_r}{\partial y_i} \nu_j \frac{\partial^2 v_2}{\partial y_i \partial y_j} d\sigma \\
&- \int_{\tilde{\phi}(\Omega)} \frac{\partial^2 v_1}{\partial y_r \partial y_j} \frac{\partial \mu_r}{\partial y_i} \frac{\partial^2 v_2}{\partial y_i \partial y_j} dy - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_r} \frac{\partial \mu_r}{\partial y_i} \frac{\partial \Delta v_2}{\partial y_i} dy \\
&= \int_{\partial \tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_r} \frac{\partial \mu_r}{\partial y_i} \nu_j \frac{\partial^2 v_2}{\partial y_i \partial y_j} d\sigma - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_r} \frac{\partial \mu_r}{\partial y_i} \frac{\partial \Delta v_2}{\partial y_i} dy \\
&- \int_{\partial \tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_j} \frac{\partial \mu_r}{\partial y_i} \nu_r \frac{\partial^2 v_2}{\partial y_i \partial y_j} d\sigma + \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_j} \frac{\partial \mathrm{div}\mu}{\partial y_i} \frac{\partial^2 v_2}{\partial y_i \partial y_j} dy \\
&+ \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_j} \frac{\partial \mu_r}{\partial y_i} \frac{\partial^3 v_2}{\partial y_i \partial y_j \partial y_r} dy \\
&= \int_{\partial \tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_r} \frac{\partial \mu_r}{\partial y_i} \nu_j \frac{\partial^2 v_2}{\partial y_i \partial y_j} d\sigma - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_r} \frac{\partial \mu_r}{\partial y_i} \frac{\partial \Delta v_2}{\partial y_i} dy \\
&- \int_{\partial \tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_j} \frac{\partial \mu_r}{\partial y_i} \nu_r \frac{\partial^2 v_2}{\partial y_i \partial y_j} d\sigma + \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_j} \frac{\partial \mu_r}{\partial y_i} \frac{\partial^3 v_2}{\partial y_i \partial y_j \partial y_r} dy \\
&+ \int_{\partial \tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_j} \mathrm{div}\mu \frac{\partial^2 v_2}{\partial y_i \partial y_j} \nu_i d\sigma - \int_{\tilde{\phi}(\Omega)} D^2 v_1 : D^2 v_2 \mathrm{div}\mu dy \\
&- \int_{\tilde{\phi}(\Omega)} \nabla v_1 \cdot \nabla \Delta v_2 \mathrm{div}\mu dy.
\end{aligned}$$

We also have

$$\int_{\tilde\phi(\Omega)}\frac{\partial v_1}{\partial y_r}\frac{\partial \mu_r}{\partial y_i}\frac{\partial v_2}{\partial y_i}dy=\int_{\partial\tilde\phi(\Omega)}\frac{\partial v_2}{\partial\nu}\nabla v_1\cdot\mu d\sigma-\int_{\tilde\phi(\Omega)}\Delta v_2\nabla v_1\cdot\mu dy$$
$$-\int_{\tilde\phi(\Omega)}\frac{\partial v_2}{\partial y_i}\frac{\partial^2 v_1}{\partial y_i\partial y_r}\mu_r dy=\int_{\partial\tilde\phi(\Omega)}\frac{\partial v_2}{\partial\nu}\nabla v_1\cdot\mu d\sigma-\int_{\tilde\phi(\Omega)}\Delta v_2\nabla v_1\cdot\mu dy$$
$$-\int_{\partial\tilde\phi(\Omega)}\nabla v_1\cdot\nabla v_2\mu\cdot\nu d\sigma+\int_{\tilde\phi(\Omega)}\frac{\partial v_1}{\partial y_i}\frac{\partial^2 v_2}{\partial y_i\partial y_r}\mu_r dy+\int_{\tilde\phi(\Omega)}\nabla v_1\cdot\nabla v_2\mathrm{div}\mu dy.$$

It follows that

$$\begin{aligned}
&d|_{\phi=\tilde\phi}\mathcal{P}^{\mathcal{S}}_{\phi}[\psi][u_1][u_2]\\
&=-\int_{\tilde\phi(\Omega)}D^2v_1:D^2v_2\mathrm{div}\mu dy-\int_{\partial\tilde\phi(\Omega)}\left(\frac{\partial v_1}{\partial y_i}\frac{\partial^2 v_2}{\partial y_i\partial y_j}+\frac{\partial v_2}{\partial y_i}\frac{\partial^2 v_1}{\partial y_i\partial y_j}\right)\frac{\partial\mu_r}{\partial y_j}\nu_r d\sigma\\
&\quad+\int_{\tilde\phi(\Omega)}\left(\frac{\partial v_1}{\partial y_i}\frac{\partial^3 v_2}{\partial y_i\partial y_j\partial y_r}+\frac{\partial v_2}{\partial y_i}\frac{\partial^3 v_1}{\partial y_i\partial y_j\partial y_r}\right)\frac{\partial\mu_r}{\partial y_j}dy\\
&\quad+\int_{\partial\tilde\phi(\Omega)}\left(\frac{\partial v_1}{\partial y_i}\frac{\partial^2 v_2}{\partial y_i\partial y_j}+\frac{\partial v_2}{\partial y_i}\frac{\partial^2 v_1}{\partial y_i\partial y_j}\right)\nu_j\mathrm{div}\mu d\sigma\\
&\quad-\int_{\tilde\phi(\Omega)}\left(\nabla v_1\cdot\nabla\Delta v_2+\nabla v_2\cdot\nabla\Delta v_1\right)\mathrm{div}\mu dy\\
&\quad-\int_{\partial\tilde\phi(\Omega)}\left(\frac{\partial v_1}{\partial y_r}\frac{\partial^2 v_2}{\partial y_i\partial y_j}+\frac{\partial v_2}{\partial y_r}\frac{\partial^2 v_1}{\partial y_i\partial y_j}\right)\nu_j\frac{\partial\mu_r}{\partial y_i}d\sigma\\
&\quad+\int_{\tilde\phi(\Omega)}\left(\frac{\partial v_1}{\partial y_r}\frac{\partial\Delta v_2}{\partial y_i}+\frac{\partial v_2}{\partial y_r}\frac{\partial\Delta v_1}{\partial y_i}\right)\frac{\partial\mu_r}{\partial y_i}dy\\
&\quad-\tau\int_{\partial\tilde\phi(\Omega)}\left(\frac{\partial v_1}{\partial\nu}\nabla v_2+\frac{\partial v_2}{\partial\nu}\nabla v_1\right)\cdot\mu d\sigma\\
&\quad+\tau\int_{\tilde\phi(\Omega)}\left(\Delta v_1\nabla v_2+\Delta v_2\nabla v_1\right)\cdot\mu dy+\tau\int_{\partial\tilde\phi(\Omega)}\nabla v_1\cdot\nabla v_2\mu\cdot\nu d\sigma. \qquad (5.4.9)
\end{aligned}$$

Now we recall that

$$\mathrm{div}\mu=\mathrm{div}_{\partial\tilde\phi(\Omega)}\mu+\frac{\partial\mu}{\partial\nu}\cdot\nu \ \text{ on } \partial\tilde\phi(\Omega),$$

(see also [38, §8.5]) and that, since $\nu=\nabla d$, where d is the distance from the boundary defined in an appropriate tubular neighborhood of the boundary, then $\nabla\nu=(\nabla\nu)^t$ and $\frac{\partial\nu}{\partial\nu}=0$, from which it follows that

$$\nabla_{\partial\tilde\phi(\Omega)}\nu=(\nabla_{\partial\tilde\phi(\Omega)}\nu)^t \ \text{ on } \partial\tilde\phi(\Omega).$$

We will use these identities throughout all the following computations.

Using the fact that

$$\frac{\partial^2 v_1}{\partial\nu^2}=\frac{\partial^2 v_2}{\partial\nu^2}=0 \ \text{ on } \partial\tilde\phi(\Omega),$$

we get that the sixth summand in (5.4.9) equals

$$
\begin{aligned}
&-\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial y_r}(D^2v_2.\nu)_{\partial\tilde{\phi}(\Omega)}+\frac{\partial v_2}{\partial y_r}(D^2v_1.\nu)_{\partial\tilde{\phi}(\Omega)}\right)\cdot\nabla_{\partial\tilde{\phi}(\Omega)}\mu_r d\sigma\\
&\qquad-\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial y_r}\frac{\partial^2 v_2}{\partial\nu^2}+\frac{\partial v_2}{\partial y_r}\frac{\partial^2 v_1}{\partial\nu^2}\right)\frac{\partial\mu_r}{\partial\nu}d\sigma\\
&\qquad=\int_{\partial\tilde{\phi}(\Omega)}\left(\nabla_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial y_r}\right)(D^2v_2.\nu)_{\partial\tilde{\phi}(\Omega)}\right.\\
&\qquad\quad\left.+\nabla_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_2}{\partial y_r}\right)(D^2v_1.\nu)_{\partial\tilde{\phi}(\Omega)}\right)\mu_r d\sigma\\
&+\int_{\partial\tilde{\phi}(\Omega)}\left(\mathrm{div}_{\partial\tilde{\phi}(\Omega)}(D^2v_1.\nu)_{\partial\tilde{\phi}(\Omega)}\nabla v_2+\mathrm{div}_{\partial\tilde{\phi}(\Omega)}(D^2v_2.\nu)_{\partial\tilde{\phi}(\Omega)}\nabla v_1\right)\cdot\mu d\sigma\\
&\qquad=\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial^2 v_1}{\partial y_i\partial y_r}\frac{\partial^2 v_2}{\partial y_i\partial y_j}+\frac{\partial^2 v_2}{\partial y_i\partial y_r}\frac{\partial^2 v_1}{\partial y_i\partial y_j}\right)\nu_j\mu_r d\sigma\\
&+\int_{\partial\tilde{\phi}(\Omega)}\left(\mathrm{div}_{\partial\tilde{\phi}(\Omega)}(D^2v_1.\nu)_{\partial\tilde{\phi}(\Omega)}\nabla v_2+\mathrm{div}_{\partial\tilde{\phi}(\Omega)}(D^2v_2.\nu)_{\partial\tilde{\phi}(\Omega)}\nabla v_1\right)\cdot\mu d\sigma.
\end{aligned}
\tag{5.4.10}
$$

The seventh summand in (5.4.9) equals

$$
\begin{aligned}
&\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial\Delta v_1}{\partial\nu}\nabla v_2+\frac{\partial\Delta v_2}{\partial\nu}\nabla v_1\right)\cdot\mu d\sigma-\int_{\tilde{\phi}(\Omega)}\left(\Delta^2v_1\nabla v_2+\Delta^2v_2\nabla v_1\right)\cdot\mu d\sigma\\
&\qquad-\int_{\tilde{\phi}(\Omega)}\left(\frac{\partial^2 v_1}{\partial y_i\partial y_r}\frac{\partial\Delta v_2}{\partial y_i}+\frac{\partial^2 v_2}{\partial y_i\partial y_r}\frac{\partial\Delta v_1}{\partial y_i}\right)\mu_r dy\\
&=\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial\Delta v_1}{\partial\nu}\nabla v_2+\frac{\partial\Delta v_2}{\partial\nu}\nabla v_1\right)\cdot\mu d\sigma-\int_{\tilde{\phi}(\Omega)}\left(\Delta^2v_1\nabla v_2+\Delta^2v_2\nabla v_1\right)\cdot\mu d\sigma\\
&\qquad-\int_{\partial\tilde{\phi}(\Omega)}\left(\nabla v_1\cdot\nabla\Delta v_2+\nabla v_2\cdot\nabla\Delta v_1\right)\mu\cdot\nu d\sigma\\
&\qquad+\int_{\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial y_i}\frac{\partial^2\Delta v_2}{\partial y_i\partial y_r}+\frac{\partial v_2}{\partial y_i}\frac{\partial^2\Delta v_1}{\partial y_i\partial y_r}\right)\mu_r dy\\
&\qquad\quad+\int_{\tilde{\phi}(\Omega)}\left(\nabla v_1\cdot\nabla\Delta v_2+\nabla v_2\cdot\nabla\Delta v_1\right)\mathrm{div}\mu dy.
\end{aligned}
\tag{5.4.11}
$$

The second summand in (5.4.9) equals

$$
\begin{aligned}
&-\int_{\partial\tilde{\phi}(\Omega)}\nabla(\nabla v_1\cdot\nabla v_2)\nabla(\mu_r)\nu_r d\sigma\\
&\qquad=-\int_{\partial\tilde{\phi}(\Omega)}\nabla_{\partial\tilde{\phi}(\Omega)}(\nabla v_1\cdot\nabla v_2)\nabla_{\partial\tilde{\phi}(\Omega)}(\mu_r)\nu_r d\sigma\\
&\qquad\qquad-\int_{\partial\tilde{\phi}(\Omega)}\frac{\partial}{\partial\nu}(\nabla v_1\cdot\nabla v_2)\frac{\partial\mu_r}{\partial\nu}\nu_r d\sigma.
\end{aligned}
\tag{5.4.12}
$$

The third summand in (5.4.9) equals

$$\int_{\partial\tilde\phi(\Omega)}\left(\frac{\partial v_1}{\partial y_i}\frac{\partial^3 v_2}{\partial y_i\partial y_j\partial y_r}+\frac{\partial v_2}{\partial y_i}\frac{\partial^3 v_1}{\partial y_i\partial y_j\partial y_r}\right)\nu_j\mu_r d\sigma$$
$$-\int_{\tilde\phi(\Omega)}\left(\frac{\partial v_1}{\partial y_i}\frac{\partial^2\Delta v_2}{\partial y_i\partial y_r}+\frac{\partial v_2}{\partial y_i}\frac{\partial^2\Delta v_1}{\partial y_i\partial y_r}\right)\mu_r dy$$
$$-\int_{\tilde\phi(\Omega)}\left(\frac{\partial^2 v_1}{\partial y_i\partial y_j}\frac{\partial^3 v_2}{\partial y_i\partial y_j\partial y_r}+\frac{\partial^2 v_2}{\partial y_i\partial y_j}\frac{\partial^3 v_1}{\partial y_i\partial y_j\partial y_r}\right)\mu_r dy$$
$$=\int_{\partial\tilde\phi(\Omega)}\left(\frac{\partial v_1}{\partial y_i}\frac{\partial^3 v_2}{\partial y_i\partial y_j\partial y_r}+\frac{\partial v_2}{\partial y_i}\frac{\partial^3 v_1}{\partial y_i\partial y_j\partial y_r}\right)\nu_j\mu_r d\sigma$$
$$-\int_{\tilde\phi(\Omega)}\left(\frac{\partial v_1}{\partial y_i}\frac{\partial^2\Delta v_2}{\partial y_i\partial y_r}+\frac{\partial v_2}{\partial y_i}\frac{\partial^2\Delta v_1}{\partial y_i\partial y_r}\right)\mu_r dy$$
$$-\int_{\partial\tilde\phi(\Omega)}D^2v_1:D^2v_2\mu\cdot\nu d\sigma+\int_{\tilde\phi(\Omega)}D^2v_1:D^2v_2\mathrm{div}\mu dy. \quad (5.4.13)$$

From (5.4.9)-(5.4.13), it follows that

$$d|_{\phi=\tilde\phi}\mathcal{P}_\phi^{\mathcal{S}}[\psi][u_1][u_2]=-\int_{\partial\tilde\phi(\Omega)}\nabla_{\partial\tilde\phi(\Omega)}(\nabla v_1\cdot\nabla v_2)\nabla_{\partial\tilde\phi(\Omega)}(\mu_r)\nu_r d\sigma$$
$$-\int_{\partial\tilde\phi(\Omega)}\frac{\partial}{\partial\nu}(\nabla v_1\cdot\nabla v_2)\frac{\partial\mu_r}{\partial\nu}\nu_r d\sigma+\int_{\partial\tilde\phi(\Omega)}\frac{\partial}{\partial\nu}(\nabla v_1\cdot\nabla v_2)\mathrm{div}\mu d\sigma$$
$$+\int_{\partial\tilde\phi(\Omega)}\left(\mathrm{div}_{\partial\tilde\phi(\Omega)}(D^2v_1.\nu)_{\partial\tilde\phi(\Omega)}\nabla v_2+\mathrm{div}_{\partial\tilde\phi(\Omega)}(D^2v_2.\nu)_{\partial\tilde\phi(\Omega)}\nabla v_1\right)\cdot\mu d\sigma$$
$$+\int_{\partial\tilde\phi(\Omega)}\left(\frac{\partial^2 v_1}{\partial y_i\partial y_r}\frac{\partial^2 v_2}{\partial y_i\partial y_j}+\frac{\partial^2 v_2}{\partial y_i\partial y_r}\frac{\partial^2 v_1}{\partial y_i\partial y_j}\right)\nu_j\mu_r d\sigma$$
$$+\int_{\partial\tilde\phi(\Omega)}\left(\frac{\partial v_1}{\partial y_i}\frac{\partial^3 v_2}{\partial y_i\partial y_j\partial y_r}+\frac{\partial v_2}{\partial y_i}\frac{\partial^3 v_1}{\partial y_i\partial y_j\partial y_r}\right)\nu_j\mu_r d\sigma$$
$$-\int_{\partial\tilde\phi(\Omega)}D^2v_1:D^2v_2\mu\cdot\nu d\sigma-\int_{\partial\tilde\phi(\Omega)}\left(\nabla v_1\cdot\nabla\Delta v_2+\nabla v_2\cdot\nabla\Delta v_1\right)\mu\cdot\nu d\sigma$$
$$+\int_{\partial\tilde\phi(\Omega)}\left(\frac{\partial\Delta v_1}{\partial\nu}\nabla v_2+\frac{\partial\Delta v_2}{\partial\nu}\nabla v_1\right)\cdot\mu d\sigma-\int_{\tilde\phi(\Omega)}\left(\Delta^2v_1\nabla v_2+\Delta^2v_2\nabla v_1\right)\cdot\mu d\sigma$$
$$-\tau\int_{\partial\tilde\phi(\Omega)}\left(\frac{\partial v_1}{\partial\nu}\nabla v_2+\frac{\partial v_2}{\partial\nu}\nabla v_1\right)\cdot\mu d\sigma$$
$$+\tau\int_{\tilde\phi(\Omega)}\left(\Delta v_1\nabla v_2+\Delta v_2\nabla v_1\right)\cdot\mu dy+\tau\int_{\partial\tilde\phi(\Omega)}\nabla v_1\cdot\nabla v_2\mu\cdot\nu d\sigma$$
$$=-\int_{\partial\tilde\phi(\Omega)}\nabla_{\partial\tilde\phi(\Omega)}(\nabla v_1\cdot\nabla v_2)\nabla_{\partial\tilde\phi(\Omega)}(\mu_r)\nu_r d\sigma$$
$$+\int_{\partial\tilde\phi(\Omega)}\frac{\partial}{\partial\nu}(\nabla v_1\cdot\nabla v_2)\mathrm{div}_{\partial\tilde\phi(\Omega)}\mu d\sigma$$

$$+\int_{\partial\tilde\phi(\Omega)}\Big(\mathrm{div}_{\partial\tilde\phi(\Omega)}(D^2v_1.\nu)_{\partial\tilde\phi(\Omega)}\nabla v_2+\mathrm{div}_{\partial\tilde\phi(\Omega)}(D^2v_2.\nu)_{\partial\tilde\phi(\Omega)}\nabla v_1\Big)\cdot\mu d\sigma$$
$$+\int_{\partial\tilde\phi(\Omega)}\left(\frac{\partial\Delta v_1}{\partial\nu}\nabla v_2+\frac{\partial\Delta v_2}{\partial\nu}\nabla v_1\right)\cdot\mu d\sigma-\tau\int_{\partial\tilde\phi(\Omega)}\left(\frac{\partial v_1}{\partial\nu}\nabla v_2+\frac{\partial v_2}{\partial\nu}\nabla v_1\right)\cdot\mu d\sigma$$
$$+\int_{\partial\tilde\phi(\Omega)}\frac{\partial}{\partial\nu}\left(\frac{\partial}{\partial y_r}(\nabla v_1\cdot\nabla v_2)\right)\mu_r d\sigma$$
$$-\int_{\partial\tilde\phi(\Omega)}D^2v_1:D^2v_2\mu\cdot\nu d\sigma-\int_{\partial\tilde\phi(\Omega)}(\nabla v_1\cdot\nabla\Delta v_2+\nabla v_2\cdot\nabla\Delta v_1)\mu\cdot\nu d\sigma$$
$$-\int_{\tilde\phi(\Omega)}\big(\Delta^2v_1\nabla v_2+\Delta^2v_2\nabla v_1\big)\cdot\mu d\sigma$$
$$+\tau\int_{\tilde\phi(\Omega)}(\Delta v_1\nabla v_2+\Delta v_2\nabla v_1)\cdot\mu dy+\tau\int_{\partial\tilde\phi(\Omega)}\nabla v_1\cdot\nabla v_2\mu\cdot\nu d\sigma.\quad(5.4.14)$$

The first summand on the right hand side of (5.4.14) equals

$$\int_{\partial\tilde\phi(\Omega)}\Delta_{\partial\tilde\phi(\Omega)}(\nabla v_1\cdot\nabla v_2)\mu\cdot\nu d\sigma+\int_{\partial\tilde\phi(\Omega)}\nabla_{\partial\tilde\phi(\Omega)}(\nabla v_1\cdot\nabla v_2)\cdot(\nabla_{\partial\tilde\phi(\Omega)}\nu_r)\mu_r d\sigma,$$

while the sixth one equals

$$\int_{\partial\tilde\phi(\Omega)}\frac{\partial^2}{\partial\nu^2}(\nabla v_1\cdot\nabla v_2)\mu\cdot\nu d\sigma+\int_{\partial\tilde\phi(\Omega)}\nabla_{\partial\tilde\phi(\Omega)}\left(\frac{\partial}{\partial\nu}(\nabla v_1\cdot\nabla v_2)\right)\cdot\mu d\sigma$$
$$-\int_{\partial\tilde\phi(\Omega)}\nabla_{\partial\tilde\phi(\Omega)}(\nabla v_1\cdot\nabla v_2)\cdot(\nabla_{\partial\tilde\phi(\Omega)}\nu_r)\mu_r d\sigma.$$

Using the fact that

$$\int_{\partial\tilde\phi(\Omega)}\mathrm{div}_{\partial\tilde\phi(\Omega)}\left(\frac{\partial}{\partial\nu}(\nabla v_1\cdot\nabla v_2)\cdot\mu\right)d\sigma=\int_{\partial\tilde\phi(\Omega)}K\frac{\partial}{\partial\nu}(\nabla v_1\cdot\nabla v_2)\mu\cdot\nu d\sigma,$$

where K denotes the mean curvature of $\partial\tilde\phi(\Omega)$ (see [38, §8.5]), we obtain

$$d|_{\phi=\tilde\phi}\mathcal{P}^{\mathcal{S}}_\phi[\psi][u_1][u_2]=\int_{\partial\tilde\phi(\Omega)}\Delta_{\partial\tilde\phi(\Omega)}(\nabla v_1\cdot\nabla v_2)\mu\cdot\nu d\sigma$$
$$+\int_{\partial\tilde\phi(\Omega)}K\frac{\partial}{\partial\nu}(\nabla v_1\cdot\nabla v_2)\mu\cdot\nu d\sigma+\int_{\partial\tilde\phi(\Omega)}\frac{\partial^2}{\partial\nu^2}(\nabla v_1\cdot\nabla v_2)\mu\cdot\nu d\sigma$$
$$-\int_{\partial\tilde\phi(\Omega)}D^2v_1:D^2v_2\mu\cdot\nu d\sigma-\int_{\partial\tilde\phi(\Omega)}(\nabla v_1\cdot\nabla\Delta v_2+\nabla v_2\cdot\nabla\Delta v_1)\mu\cdot\nu d\sigma$$
$$+\int_{\partial\tilde\phi(\Omega)}\Big(\mathrm{div}_{\partial\tilde\phi(\Omega)}(D^2v_1.\nu)_{\partial\tilde\phi(\Omega)}\nabla v_2+\mathrm{div}_{\partial\tilde\phi(\Omega)}(D^2v_2.\nu)_{\partial\tilde\phi(\Omega)}\nabla v_1\Big)\cdot\mu d\sigma$$
$$+\int_{\partial\tilde\phi(\Omega)}\left(\frac{\partial\Delta v_1}{\partial\nu}\nabla v_2+\frac{\partial\Delta v_2}{\partial\nu}\nabla v_1\right)\cdot\mu d\sigma-\tau\int_{\partial\tilde\phi(\Omega)}\left(\frac{\partial v_1}{\partial\nu}\nabla v_2+\frac{\partial v_2}{\partial\nu}\nabla v_1\right)\cdot\mu d\sigma$$

$$
\begin{aligned}
&- \int_{\tilde\phi(\Omega)} \left(\Delta^2 v_1 \nabla v_2 + \Delta^2 v_2 \nabla v_1\right) \cdot \mu d\sigma \\
&+ \tau \int_{\tilde\phi(\Omega)} (\Delta v_1 \nabla v_2 + \Delta v_2 \nabla v_1) \cdot \mu dy + \tau \int_{\partial\tilde\phi(\Omega)} \nabla v_1 \cdot \nabla v_2 \mu \cdot \nu d\sigma \\
&= \int_{\partial\tilde\phi(\Omega)} \Delta(\nabla v_1 \cdot \nabla v_2)\mu \cdot \nu d\sigma - \int_{\partial\tilde\phi(\Omega)} D^2 v_1 : D^2 v_2 \mu \cdot \nu d\sigma \\
&- \int_{\partial\tilde\phi(\Omega)} (\nabla v_1 \cdot \nabla\Delta v_2 + \nabla v_2 \cdot \nabla\Delta v_1)\mu \cdot \nu d\sigma \\
&+ \int_{\partial\tilde\phi(\Omega)} \left(\mathrm{div}_{\partial\tilde\phi(\Omega)} (D^2 v_1.\nu)_{\partial\tilde\phi(\Omega)} \nabla v_2 + \mathrm{div}_{\partial\tilde\phi(\Omega)} (D^2 v_2.\nu)_{\partial\tilde\phi(\Omega)} \nabla v_1\right) \cdot \mu d\sigma \\
&+ \int_{\partial\tilde\phi(\Omega)} \left(\frac{\partial\Delta v_1}{\partial\nu}\nabla v_2 + \frac{\partial\Delta v_2}{\partial\nu}\nabla v_1\right) \cdot \mu d\sigma - \tau \int_{\partial\tilde\phi(\Omega)} \left(\frac{\partial v_1}{\partial\nu}\nabla v_2 + \frac{\partial v_2}{\partial\nu}\nabla v_1\right) \cdot \mu d\sigma \\
&- \int_{\tilde\phi(\Omega)} \left((\Delta^2 v_1 - \tau\Delta v_1)\nabla v_2 + (\Delta^2 v_2 - \tau\Delta v_2)\nabla v_1\right) \cdot \mu d\sigma \\
&+ \tau \int_{\partial\tilde\phi(\Omega)} \nabla v_1 \cdot \nabla v_2 \mu \cdot \nu d\sigma.
\end{aligned}
$$

Using the equality

$$
\Delta(\nabla v_1 \cdot \nabla v_2) = \nabla\Delta v_1 \cdot \nabla v_2 + \nabla v_1 \cdot \nabla\Delta v_2 + 2D^2 v_1 : D^2 v_2
$$

we finally get formula (5.4.5). □

Now we can compute Hadamard-type formulas for the eigenvalues of problem (5.4.1).

Theorem 5.4.15. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. Let F be a finite non-empty subset of $\mathbb{N}$. Let $\tilde\phi \in \Theta_\Omega[F]$ be such that $\partial\tilde\phi(\Omega) \in C^4$. Let $v_1, \dots, v_{|F|}$ be an orthonormal basis of the eigenspace associated with the eigenvalue $\lambda_F[\tilde\phi]$ of problem (5.4.1) in $L^2(\partial\tilde\phi(\Omega))$. Then*

$$
\begin{aligned}
d|_{\phi=\tilde\phi}(\Lambda_{F,s})[\psi] = -\lambda_F^{s-1}[\tilde\phi] \binom{|F|-1}{s-1} \sum_{l=1}^{|F|} \int_{\partial\tilde\phi(\Omega)} \Big(\lambda_F K v_l^2 \\
+ \lambda_F \frac{\partial(v_l^2)}{\partial\nu} - \tau|\nabla v_l|^2 - |D^2 v_l|^2\Big)\mu \cdot \nu d\sigma,
\end{aligned}
$$

for all $\psi \in (C^2(\overline{\Omega}))^N$, where $\mu = \psi \circ \tilde\phi^{(-1)}$, and K denotes the mean curvature of $\partial\tilde\phi(\Omega)$.

Proof. First of all we note that $v_1, ..., v_{|F|} \in H^4(\tilde\phi(\Omega))$ (see e.g., [47, §2.5]). We set $u_l = v_l \circ \tilde\phi$ for $l = 1, \dots, |F|$. For $|F| > 1$ (case $|F| = 1$ is similar),

$s \leq |F|$, we have

$$d|_{\phi=\tilde{\phi}}(\Lambda_{F,s})[\psi] = -\lambda_F^{s+2}[\tilde{\phi}]\binom{|F|-1}{s-1}\sum_{l=1}^{|F|}\mathcal{P}_{\tilde{\phi}}^{\mathcal{S}}\left[d|_{\phi=\tilde{\phi}}W_{\phi}^{\mathcal{S}}[\psi][p(u_l)]\right][p(u_l)]\,. \tag{5.4.16}$$

We refer to [75, Theorem 3.38] for a proof of formula (5.4.16).

By standard calculus in normed spaces we have:

$$\begin{aligned}
&\mathcal{P}_{\tilde{\phi}}^{\mathcal{S}}\left[\mathrm{d}|_{\phi=\tilde{\phi}}\left((\pi_{\phi}^{\sharp,\mathcal{S}})^{-1}\circ\left(\mathcal{P}_{\phi}^{\mathcal{S}}\right)^{-1}\circ\mathcal{J}_{\phi}^{\mathcal{S}}\circ\mathrm{Tr}\circ\pi_{\phi}^{\sharp,\mathcal{S}}\right)[\psi][p(u_l)]\right][p(u_l)]\\
&\quad=\mathcal{P}_{\tilde{\phi}}^{\mathcal{S}}\left[(\pi_{\tilde{\phi}}^{\sharp,\mathcal{S}})^{-1}\circ\left(\mathcal{P}_{\tilde{\phi}}^{\mathcal{S}}\right)^{-1}\circ\mathrm{d}|_{\phi=\tilde{\phi}}\left(\mathcal{J}_{\phi}^{\mathcal{S}}\circ\mathrm{Tr}\circ\pi_{\phi}^{\sharp,\mathcal{S}}\right)[\psi][p(u_l)]\right][p(u_l)]\\
&\qquad+\mathcal{P}_{\tilde{\phi}}^{\mathcal{S}}\left[\mathrm{d}|_{\phi=\tilde{\phi}}\left((\pi_{\phi}^{\sharp,\mathcal{S}})^{-1}\circ\left(\mathcal{P}_{\phi}^{\mathcal{S}}\right)^{-1}\right)[\psi]\circ\mathcal{J}_{\tilde{\phi}}^{\mathcal{S}}\circ\mathrm{Tr}\circ\pi_{\tilde{\phi}}^{\sharp,\mathcal{S}}[p(u_l)]\right][p(u_l)]\,.
\end{aligned}$$

Now note that:

$$\begin{aligned}
&\mathcal{P}_{\tilde{\phi}}^{\mathcal{S}}\left[(\pi_{\tilde{\phi}}^{\sharp,\mathcal{S}})^{-1}\circ\left(\mathcal{P}_{\tilde{\phi}}^{\mathcal{S}}\right)^{-1}\circ\mathrm{d}|_{\phi=\tilde{\phi}}\left(\mathcal{J}_{\phi}^{\mathcal{S}}\circ\mathrm{Tr}\circ\pi_{\phi}^{\sharp,\mathcal{S}}\right)[\psi][p(u_l)]\right][p(u_l)]\\
&\qquad=\int_{\partial\tilde{\phi}(\Omega)}\left(Kv_l^2+\frac{\partial(v_l^2)}{\partial\nu}\right)\mu\cdot\nu d\sigma-\int_{\partial\tilde{\phi}(\Omega)}\nabla(v_l^2)\cdot\mu d\sigma,
\end{aligned}$$

(see also [74, Lemma 3.3]) and

$$\begin{aligned}
&\mathcal{P}_{\tilde{\phi}}^{\mathcal{S}}\left[\mathrm{d}|_{\phi=\tilde{\phi}}\left((\pi_{\phi}^{\sharp,\mathcal{S}})^{-1}\circ\left(\mathcal{P}_{\phi}^{\mathcal{S}}\right)^{-1}\right)[\psi]\circ\mathcal{J}_{\tilde{\phi}}^{\mathcal{S}}\circ\mathrm{Tr}\circ\pi_{\tilde{\phi}}^{\sharp,\mathcal{S}}[p(u_l)]\right][p(u_l)]\\
&\qquad=-\lambda_F^{-1}d|_{\phi=\tilde{\phi}}\left(\mathcal{P}_{\phi}^{\mathcal{S}}\circ\pi_{\phi}^{\mathcal{S}}\right)[\psi][u_l][\pi_{\tilde{\phi}}^{\mathcal{S}}(u_l)].
\end{aligned}$$

(We refer to [77, Lemma 2.4] for more explicit computations). Using formula (5.4.5) we obtain

$$\begin{aligned}
&\mathcal{P}_{\tilde{\phi}}^{\mathcal{S}}\left[\mathrm{d}|_{\phi=\tilde{\phi}}\left((\pi_{\phi}^{\sharp,\mathcal{S}})^{-1}\circ\left(\mathcal{P}_{\phi}^{\mathcal{S}}\right)^{-1}\right)[\psi]\circ\mathcal{J}_{\tilde{\phi}}^{\mathcal{S}}\circ\mathrm{Tr}\circ\pi_{\tilde{\phi}}^{\sharp,\mathcal{S}}[p(u_l)]\right][p(u_l)]\\
&\qquad=-\lambda_F^{-1}\int_{\partial\tilde{\phi}(\Omega)}\left(|D^2v_l|^2+\tau|\nabla v_l|^2\right)\mu\cdot\nu d\sigma+\int_{\partial\tilde{\phi}(\Omega)}\nabla(v_l^2)\cdot\mu d\sigma.
\end{aligned}$$

This concludes the proof. □

5.4.2 Isovolumetric perturbatons

Now we turn our attention to extremum problems of the type

$$\min_{\mathcal{V}(\phi)=\text{const.}}\Lambda_{F,s}[\phi]\ \text{or}\ \max_{\mathcal{V}(\phi)=\text{const.}}\Lambda_{F,s}[\phi],$$

where $\mathcal{V}(\phi)$ denotes the Lebesgue measure of $\phi(\Omega)$, i.e.,

$$\mathcal{V}(\phi):=\int_{\phi(\Omega)}dx=\int_{\Omega}|\det D\phi|dx. \tag{5.4.17}$$

Note that all ϕ's realizing one of the extrema are critical points under measure constraint, i.e., $\mathrm{Ker}\, d\mathcal{V}(\phi) \subseteq \mathrm{Ker}\, d\Lambda_{F,s}[\phi]$. We have the following result (see [76, Proposition 2.10]).

Proposition 5.4.18. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. Then the following statements hold.*

i) The map $\mathcal{V}$ from $\Phi(\Omega)$ to $\mathbb{R}$ defined in (5.4.17) is real analytic. Moreover, the differential of $\mathcal{V}$ at $\tilde{\phi} \in \Phi(\Omega)$ is given by the formula

$$d|_{\phi=\tilde{\phi}}\mathcal{V}(\phi)[\psi] = \int_{\tilde{\phi}(\Omega)} \mathrm{div}(\psi \circ \tilde{\phi}^{-1}) dy = \int_{\partial\tilde{\phi}(\Omega)} (\psi \circ \tilde{\phi}^{-1}) \cdot \nu d\sigma.$$

ii) For $\mathcal{V}_0 \in]0, +\infty[$, let

$$V(\mathcal{V}_0) := \{\phi \in \Phi(\Omega) : \mathcal{V}(\phi) = \mathcal{V}_0\}.$$

If $V(\mathcal{V}_0) \neq \emptyset$, then $V(\mathcal{V}_0)$ is a real analytic manifold of $(C^2(\overline{\Omega}))^N$ of codimension 1.

Using Lagrange Multipliers Theorem, it is easy to prove the following theorem.

Theorem 5.4.19. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. Let F be a non-empty finite subset of $\mathbb{N}$. Let $\mathcal{V}_0 \in]0, +\infty[$. Let $\tilde{\phi} \in V(\mathcal{V}_0)$ be such that $\partial\tilde{\phi}(\Omega) \in C^4$ and $\lambda_j[\tilde{\phi}]$ have a common value $\lambda_F[\tilde{\phi}]$ for all $j \in F$ and $\lambda_l[\tilde{\phi}] \neq \lambda_F[\tilde{\phi}]$ for all $l \in \mathbb{N} \setminus F$. For $s = 1, \dots, |F|$, the function $\tilde{\phi}$ is a critical point for $\Lambda_{F,s}$ on $V(\mathcal{V}_0)$ if and only if there exists an orthonormal basis $v_1, \dots, v_{|F|}$ of the eigenspace corresponding to the eigenvalue $\lambda_F[\tilde{\phi}]$ of problem (5.4.1) in $L^2(\partial\tilde{\phi}(\Omega))$, and a constant $c \in \mathbb{R}$ such that*

$$\sum_{l=1}^{|F|} \left(\lambda_F[\tilde{\phi}] \left(K v_l^2 + \frac{\partial \left(v_l^2\right)}{\partial \nu} \right) - \tau |\nabla v_l|^2 - |D^2 v_l|^2 \right) = c, \text{ a.e. on } \partial\tilde{\phi}(\Omega). \tag{5.4.20}$$

Now that we have a characterization for the criticality of $\tilde{\phi}$, we may wonder whether balls are critical domains. It turns out that indeed balls are criticals for the symmetric functions of the eigenvalues, as proved in the following theorem.

Theorem 5.4.21. *Let Ω be a bounded domain of $\mathbb{R}^N$ of class C^1. Let $\tilde{\phi} \in \Phi(\Omega)$ be such that $\tilde{\phi}(\Omega)$ is a ball. Let $\tilde{\lambda}$ be an eigenvalue of problem (5.4.1) in $\tilde{\phi}(\Omega)$, and let F be the set of $j \in \mathbb{N}$ such that $\lambda_j[\tilde{\phi}] = \tilde{\lambda}$. Then $\Lambda_{F,s}$ has a critical point at $\tilde{\phi}$ on $V(\mathcal{V}(\tilde{\phi}))$, for all $s = 1, \dots, |F|$.*

Proof. Using Lemma 5.4.22 below and the fact that the mean curvature is constant for a ball, condition (5.4.20) is immediately seen to be satisfied. □

Lemma 5.4.22. *Let B be the unit ball in $\mathbb{R}^N$ centered at zero, and let λ be an eigenvalue of problem (5.4.1) in B. Let F be the subset of $\mathbb{N}$ of all indexes j such that the j-th eigenvalue of problem (5.4.1) in B coincides with λ. Let $v_1, \dots, v_{|F|}$ be an orthonormal basis of the eigenspace associated with the eigenvalue λ, where the orthonormality is taken with respect to the scalar product in $L^2(\partial B)$. Then*

$$\sum_{j=1}^{|F|} v_j^2, \ \sum_{j=1}^{|F|} |\nabla v_j|^2, \ \sum_{j=1}^{|F|} |D^2 v_j|^2$$

are radial functions.

Proof. Let $O_N(\mathbb{R})$ denote the group of orthogonal linear transformations in $\mathbb{R}^N$. Since the Laplace operator is invariant under rotations, then $v_k \circ A$, where $A \in O_N(\mathbb{R})$, is an eigenfunction with eigenvalue λ; moreover, $\{v_j \circ A : j = 1, \dots, |F|\}$ is a orthonormal basis for the eigenspace associated with λ. Since both $\{v_j : j = 1, \dots, |F|\}$ and $\{v_j \circ A : j = 1, \dots, |F|\}$ are orthonormal bases, then there exists $R[A] \in O_N(\mathbb{R})$ with matrix $(R_{ij}[A])_{i,j=1,\dots,|F|}$ such that

$$v_j = \sum_{l=1}^{|F|} R_{jl}[A] v_l \circ A.$$

This implies that

$$\sum_{j=1}^{|F|} v_j^2 = \sum_{j=1}^{|F|} (v_j \circ A)^2,$$

from which we get that $\sum_{j=1}^{|F|} v_j^2$ is radial. Moreover, using standard calculus, we get

$$\sum_{j=1}^{|F|} |\nabla v_j|^2 = \sum_{l_1,l_2=1}^{|F|} R_{jl_1}[A] R_{jl_2}[A] \, (\nabla v_{l_1} \circ A) \cdot (\nabla v_{l_2} \circ A) = \sum_{l=1}^{|F|} |\nabla v_l \circ A|^2,$$

and

$$\begin{aligned} D^2 v_j \cdot D^2 v_j &= \sum_{l_1,l_2=1}^{|F|} R_{jl_1}[A] R_{jl_2}[A] A^t \cdot (D^2 v_{l_1} \circ A) \cdot A \cdot A^t \cdot (D^2 v_{l_2} \circ A) \cdot A \\ &= \sum_{l_1,l_2=1}^{|F|} R_{jl_1}[A] R_{jl_2}[A] A^t \cdot (D^2 v_{l_1} \circ A) \cdot (D^2 v_{l_2} \circ A) \cdot A, \end{aligned}$$

hence

$$|D^2 v_j|^2 = \mathrm{tr}(D^2 v_j \cdot D^2 v_j) = \sum_{l_1,l_2=1}^{|F|} R_{jl_1}[A] R_{jl_2}[A] (D^2 v_{l_1} \circ A) : (D^2 v_{l_2} \circ A),$$

from which we get

$$\sum_{j=1}^{|F|} |D^2 v_j|^2 = \sum_{j=1}^{|F|} |D^2 v_j \circ A|^2.$$

□

5.4.3 The Neumann problem

As we have done for the Steklov problem, we study the Neumann problem for the biharmonic operator on $\phi(\Omega)$, i.e.,

$$\begin{cases} \Delta^2 u - \tau \Delta u = \lambda u, & \text{in } \phi(\Omega), \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\phi(\Omega), \\ \tau \frac{\partial u}{\partial \nu} - \mathrm{div}_{\partial\phi(\Omega)}(D^2 u \cdot \nu) - \frac{\partial \Delta u}{\partial \nu} = 0, & \text{on } \partial\phi(\Omega). \end{cases} \quad (5.4.23)$$

We consider the operator $\mathcal{P}_\phi^{\mathcal{N}}$ from $H_\phi^{2,\mathcal{N}}(\Omega)$ to $F(\Omega)$, defined by

$$\begin{aligned} \mathcal{P}_\phi^{\mathcal{N}}[u][\varphi] := & \int_\Omega (D^2(u \circ \phi^{-1}) \circ \phi) : (D^2(\varphi \circ \phi^{-1}) \circ \phi) |\det D\phi| dx \\ & + \tau \int_\Omega (\nabla(u \circ \phi^{-1}) \circ \phi) \cdot (\nabla(\varphi \circ \phi^{-1}) \circ \phi) |\det D\phi| dx, \quad (5.4.24) \end{aligned}$$

for all $u \in H_\phi^{2,\mathcal{N}}(\Omega)$, $\varphi \in H^2(\Omega)$, where

$$H_\phi^{2,\mathcal{N}}(\Omega) := \left\{ u \in H^2(\Omega) : \int_\Omega u |\det D\phi| dx = 0 \right\},$$

Moreover, for every $\phi \in \Phi(\Omega)$, we consider the map $\mathcal{J}_\phi^{\mathcal{N}}$ from $L^2(\Omega)$ to $H^2(\Omega)'$ defined by

$$\mathcal{J}_\phi^{\mathcal{N}}[u][\varphi] := \int_\Omega u\varphi |\det D\phi| d\sigma, \quad \forall u \in L^2(\Omega), \varphi \in H^2(\Omega).$$

We will think of the space $H_\phi^{2,\mathcal{N}}(\Omega)$ as endowed with the scalar product induced by (5.4.24). We denote by $\pi_\phi^{\mathcal{N}}$ the map from $H^2(\Omega)$ to $H_\phi^{2,\mathcal{N}}(\Omega)$ defined by

$$\pi_\phi^{\mathcal{N}}(u) := u - \frac{\int_\Omega u |\det D\phi| dx}{\int_\Omega |\det D\phi| dx},$$

and by $\pi_\phi^{\sharp,\mathcal{N}}$ the map from $H^2(\Omega)/\mathbb{R}$ onto $H_\phi^{2,\mathcal{N}}(\Omega)$ defined by the equality $\pi_\phi^{\mathcal{N}} = \pi_\phi^{\sharp,\mathcal{N}} \circ p$. Clearly, $\pi_\phi^{\sharp,\mathcal{N}}$ is a homeomorphism, and we can recast problem (5.4.23) as

$$\lambda^{-1} u = W_\phi^{\mathcal{N}} u,$$

where $W^{\mathcal{N}}_{\phi} := (\pi^{\sharp,\mathcal{N}}_{\phi})^{-1} \circ (\mathcal{P}^{\mathcal{N}}_{\phi})^{-1} \circ \mathcal{J}^{\mathcal{N}}_{\phi} \circ i \circ \pi^{\sharp,\mathcal{N}}_{\phi}$ and i is the canonical embedding of $H^2(\Omega)$ into $L^2(\Omega)$. An analogue of Theorem 5.4.3 can be stated also in this case. Therefore, we can compute Hadamard-type formulas for the Neumann eigenvalues. This is contained in the following theorem.

Theorem 5.4.25. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. Let F be a finite non-empty subset of $\mathbb{N}$. Let $\tilde{\phi} \in \Theta_\Omega[F]$ be such that $\partial\tilde{\phi}(\Omega) \in C^4$. Let $v_1, \dots, v_{|F|}$ be an orthonormal basis of the eigenspace associated with the eigenvalue $\lambda_F[\tilde{\phi}]$ of problem (5.4.23) in $L^2(\tilde{\phi}(\Omega))$. Then*

$$\begin{aligned} & d|_{\phi=\tilde{\phi}}(\Lambda_{F,s})[\psi] \\ & \qquad = -\lambda_F^{s-1}[\tilde{\phi}] \binom{|F|-1}{s-1} \sum_{l=1}^{|F|} \int_{\partial\tilde{\phi}(\Omega)} \left(\lambda_F v_l^2 - \tau|\nabla v_l|^2 - |D^2 v_l|^2\right) \mu \cdot \nu d\sigma, \end{aligned}$$

for all $\psi \in (C^2(\overline{\Omega}))^N$, where $\mu = \psi \circ \tilde{\phi}^{-1}$.

Proof. The proof is similar to that of Theorem 5.4.15.

First of all we note that, by elliptic regularity theory, $v_1, \dots, v_{|F|} \in H^4(\tilde{\phi}(\Omega))$ (see [47, §2.5]). We set $u_l = v_l \circ \tilde{\phi}$ for $l = 1, \dots, |F|$. For $|F| > 1$ (case $|F| = 1$ is similar), $s \leq |F|$, we have

$$d|_{\phi=\tilde{\phi}}(\Lambda_{F,s})[\psi] = -\lambda_F^{s+2}[\tilde{\phi}] \binom{|F|-1}{s-1} \sum_{l=1}^{|F|} \mathcal{P}^{\mathcal{N}}_{\tilde{\phi}} \left[d|_{\phi=\tilde{\phi}} W^{\mathcal{N}}_{\phi}[\psi][p(u_l)] \right] [p(u_l)] \, .$$

By standard calculus in normed spaces we have:

$$\begin{aligned} & \mathcal{P}^{\mathcal{N}}_{\tilde{\phi}} \left[\mathrm{d}|_{\phi=\tilde{\phi}} \left((\pi^{\sharp,\mathcal{N}}_{\phi})^{-1} \circ \left(\mathcal{P}^{\mathcal{N}}_{\phi}\right)^{-1} \circ \mathcal{J}^{\mathcal{N}}_{\phi} \circ i \circ \pi^{\sharp,\mathcal{N}}_{\phi} \right) [\psi][p(u_l)] \right] [p(u_l)] \\ & \quad = \mathcal{P}^{\mathcal{N}}_{\tilde{\phi}} \left[(\pi^{\sharp,\mathcal{N}}_{\tilde{\phi}})^{-1} \circ \left(\mathcal{P}^{\mathcal{N}}_{\tilde{\phi}}\right)^{-1} \circ \mathrm{d}|_{\phi=\tilde{\phi}} \left(\mathcal{J}^{\mathcal{N}}_{\phi} \circ i \circ \pi^{\sharp,\mathcal{N}}_{\phi} \right) [\psi][p(u_l)] \right] [p(u_l)] \\ & \qquad + \mathcal{P}^{\mathcal{N}}_{\tilde{\phi}} \left[\mathrm{d}|_{\phi=\tilde{\phi}} \left((\pi^{\sharp,\mathcal{N}}_{\phi})^{-1} \circ \left(\mathcal{P}^{\mathcal{N}}_{\phi}\right)^{-1} \right) [\psi] \circ \mathcal{J}^{\mathcal{N}}_{\tilde{\phi}} \circ i \circ \pi^{\sharp,\mathcal{N}}_{\tilde{\phi}}[p(u_l)] \right] [p(u_l)] \, . \end{aligned}$$

Now note that

$$\begin{aligned} & \mathcal{P}^{\mathcal{N}}_{\tilde{\phi}} \left[(\pi^{\sharp,\mathcal{N}}_{\tilde{\phi}})^{-1} \circ \left(\mathcal{P}^{\mathcal{N}}_{\tilde{\phi}}\right)^{-1} \circ \mathrm{d}|_{\phi=\tilde{\phi}} \left(\mathcal{J}^{\mathcal{N}}_{\phi} \circ i \circ \pi^{\sharp,\mathcal{N}}_{\phi} \right) [\psi][p(u_l)] \right] [p(u_l)] \\ & \qquad\qquad = \int_{\tilde{\phi}(\Omega)} v_l^2 \mathrm{div} \mu dy, \end{aligned}$$

(see also Proposition 5.4.18) and

$$\begin{aligned} & \mathcal{P}^{\mathcal{N}}_{\tilde{\phi}} \left[\mathrm{d}|_{\phi=\tilde{\phi}} \left((\pi^{\sharp,\mathcal{N}}_{\phi})^{-1} \circ \left(\mathcal{P}^{\mathcal{N}}_{\phi}\right)^{-1} \right) [\psi] \circ \mathcal{J}^{\mathcal{N}}_{\tilde{\phi}} \circ i \circ \pi^{\sharp,\mathcal{N}}_{\tilde{\phi}}[p(u_l)] \right] [p(u_l)] \\ & \qquad\qquad = -\lambda_F^{-1} d|_{\phi=\tilde{\phi}} \left(\mathcal{P}^{\mathcal{N}}_{\phi} \circ \pi^{\mathcal{N}}_{\phi} \right) [\psi][u_l][\pi^{\mathcal{N}}_{\tilde{\phi}}(u_l)]. \end{aligned}$$

Using formula (5.4.5) we obtain

$$\mathcal{P}^{\mathcal{N}}_{\tilde{\phi}}\left[\mathrm{d}|_{\phi=\tilde{\phi}}\left((\pi^{\sharp,\mathcal{N}}_{\phi})^{-1}\circ\left(\mathcal{P}^{\mathcal{N}}_{\phi}\right)^{-1}\right)[\psi]\circ\mathcal{J}^{\mathcal{N}}_{\tilde{\phi}}\circ i\circ\pi^{\sharp,\mathcal{N}}_{\tilde{\phi}}[p(u_l)]\right][p(u_l)]$$
$$=-\lambda_F^{-1}\int_{\partial\tilde{\phi}(\Omega)}\left(|D^2v_l|^2+\tau|\nabla v_l|^2\right)\mu\cdot\nu d\sigma+\int_{\tilde{\phi}(\Omega)}\nabla(v_l^2)\cdot\mu dy.$$

To conclude, just observe that

$$\int_{\tilde{\phi}(\Omega)}\nabla(v_l^2)\cdot\mu dy=\int_{\partial\tilde{\phi}(\Omega)}v_l^2\mu\cdot\nu d\sigma-\int_{\tilde{\phi}(\Omega)}(v_l^2)\mathrm{div}\mu dy.$$

□

Now we can state the analogue of Theorem 5.4.19 for problem (5.4.23).

Theorem 5.4.26. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. Let F be a non-empty finite subset of $\mathbb{N}$. Let $\mathcal{V}_0\in]0,+\infty[$. Let $\tilde{\phi}\in V(\mathcal{V}_0)$ be such that $\partial\tilde{\phi}(\Omega)\in C^4$ and $\lambda_j[\tilde{\phi}]$ have a common value $\lambda_F[\tilde{\phi}]$ for all $j\in F$ and $\lambda_l[\tilde{\phi}]\neq\lambda_F[\tilde{\phi}]$ for all $l\in\mathbb{N}\setminus F$. For $s=1,\dots,|F|$, the function $\tilde{\phi}$ is a critical point for $\Lambda_{F,s}$ on $V(\mathcal{V}_0)$ if and only if there exists an orthonormal basis $v_1,\dots,v_{|F|}$ of the eigenspace corresponding to the eigenvalue $\lambda_F[\tilde{\phi}]$ of problem (5.4.23) in $L^2(\tilde{\phi}(\Omega))$, and a constant $c\in\mathbb{R}$ such that*

$$\sum_{l=1}^{|F|}\left(\lambda_F v_l^2-\tau|\nabla v_l|^2-|D^2v_l|^2\right)=c,\ \text{a.e. on }\partial\tilde{\phi}(\Omega).$$

We observe that Lemma 5.4.22 holds for problem (5.4.23) as well, since in the proof we have only used the rotation invariance of the Laplace operator. Then, we are led to the following theorem.

Theorem 5.4.27. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. Let $\tilde{\phi}\in\Phi(\Omega)$ be such that $\tilde{\phi}(\Omega)$ is a ball. Let $\tilde{\lambda}$ be an eigenvalue of problem (5.4.23) on $\tilde{\phi}(\Omega)$, and let F be the set of $j\in\mathbb{N}$ such that $\lambda_j[\tilde{\phi}]=\tilde{\lambda}$. Then $\Lambda_{F,s}$ has a critical point at $\tilde{\phi}$ on $V(\mathcal{V}(\tilde{\phi}))$, for all $s=1,\dots,|F|$.*

5.5 The fundamental tone of the ball. An isoperimetric inequality

In the previous section we have shown that the ball is a critical point for all the elementary symmetric functions of the eigenvalues of problem (5.0.1) when $\rho\equiv1$. In this section we prove that the ball is actually a maximizer for the fundamental tone, that is

$$\lambda_2(\Omega)\leq\lambda_2(\Omega^*),\tag{5.5.1}$$

where Ω^* is a ball such that $|\Omega|=|\Omega^*|$. Through all this section we consider problem (5.0.1) whith $\rho\equiv1$.

5.5.1 Eigenvalues and eigenfunctions on the ball

We compute the eigenvalues and the eigenfunctions of (5.0.1) when $\Omega = B$ is the unit ball in $\mathbb{R}^N$ centered at the origin. It is convenient to use spherical coordinates $(r,\theta) = (r,\theta_1,...,\theta_{N-1}) \in \mathbb{R}_+ \times \partial B$ in $\mathbb{R}^N$.
The boundary conditions of (5.0.1) in this case are written as

$$\frac{\partial^2 u}{\partial r^2}_{|r=1} = 0, \tag{5.5.2}$$

$$\tau\frac{\partial u}{\partial r} - \frac{1}{r^2}\Delta_S\Big(\frac{\partial u}{\partial r} - \frac{u}{r}\Big) - \frac{\partial \Delta u}{\partial r}_{|r=1} = \lambda u_{|r=1}, \tag{5.5.3}$$

where Δ_S is the angular part of the Laplacian.
It is well known that the eigenfunctions can be written as a product of a radial part and an angular part (see [28] for details). The radial part is given in terms of ultraspherical modified Bessel functions and powertype functions and the angular part is given in terms of spherical harmonics. We have the following theorem.

Theorem 5.5.4. *Let Ω be the unit ball in $\mathbb{R}^N$ centered at the origin. Any eigenfunction u_l of problem (5.0.1) is of the form $u_l(r,\theta) = R_l(r)H_l(\theta)$ where $H_l(\theta)$ is a spherical harmonic of some order $l \in \mathbb{N}$ and*

$$R_l(r) = A_l r^l + B_l i_l(\sqrt{\tau} r),$$

where A_l and B_l are suitable constants such that

$$B_l = \frac{l(1-l)}{\tau i_l''(\sqrt{\tau})} A_l.$$

Moreover, the eigenvalue $\lambda_{(l)}$ associated with the eigenfunction u_l is delivered by the formula

$$\begin{aligned}\lambda_{(l)} = l\Big((1-l)l i_l(\sqrt{\tau}) + \tau i_l''(\sqrt{\tau})\Big)^{-1}\Big[&3(l-1)l(l+N-2)i_l(\sqrt{\tau})\\ &- (l-1)\sqrt{\tau}\big(N-1+2Nl+2l(l-2)l+\tau\big)i_l'(\sqrt{\tau})\\ &+ \tau\big((l-1)(l+2N-3)+\tau\big)i_l''(\sqrt{\tau})\\ &+ (l-1)\tau\sqrt{\tau}i_l'''(\sqrt{\tau})\Big],\end{aligned} \tag{5.5.5}$$

for any $l \in \mathbb{N}_0$.

Proof. Solutions to problem (5.0.1) in the unit ball are smooth (see e.g., [47, Theorem 2.20]). We consider two cases: $\Delta u = 0$ and $\Delta u \neq 0$.

Let u be such that $\Delta u = 0$. The Laplacian can be written in spherical coordinates as

$$\Delta = \partial_{rr} + \frac{N-1}{r}\partial_r + \frac{1}{r^2}\Delta_S.$$

Separating variables so that $u = R(r)Y(\theta)$ we obtain the equations

$$R'' + \frac{N-1}{r}R' - \frac{l(l+N-2)}{r^2}R = 0 \tag{5.5.6}$$

and

$$\Delta_S Y = -l(l+N-2)Y. \tag{5.5.7}$$

The solutions to equation (5.5.6) are given by $R(r) = ar^l + br^{2-N-l}$ if $l > 0, N \geq 2$, and by $R(r) = a + b\log(r)$ if $l = 0, N = 2$. Since the solutions cannot blow up at $r = 0$, we must impose $b = 0$. The solutions of the second equation are the spherical harmonics of order l. Then u can be written as

$$u(r,\theta) = a_l r^l Y_l(\theta)$$

for some $l \in \mathbb{N}_0$.

Let us consider now the case $\Delta u \neq 0$. We set $v = \Delta u$ and solve the equation

$$\Delta v = \tau v.$$

By writing $v = R(r)Y(\theta)$ we obtain that R solves the equation

$$R'' + \frac{N-1}{r}R' - \frac{l(l+N-2)}{r^2}R = \tau R, \tag{5.5.8}$$

while Y solves equation (5.5.7). Equation (5.5.8) is the modified ultraspherical Bessel equation that is solved by the modified ultraspherical Bessel functions of first and second kind $i_l(\sqrt{\tau}r)$ and $k_l(\sqrt{\tau}r)$. Since the solutions cannot blow up at $r = 0$, we must choose only $i_l(z)$ since $k_l(z)$ has a singularity at $z = 0$. Then

$$v(r,\theta) = b_{l_1} i_{l_1}(\sqrt{\tau}r)Y_{l_1}(\theta)$$

for some $l_1 \in \mathbb{N}_0$. Now $v = \frac{\Delta v}{\tau} = \Delta u$, that is $\Delta(v/\tau - u) = 0$. This means that

$$u(r,\theta) = \frac{b_{l_1}}{\tau} i_{l_1}(\sqrt{\tau}r)Y_{l_1}(\theta) - c_{l_2} r^{l_2} Y_{l_2}(\theta) \tag{5.5.9}$$

for some $l_2 \in \mathbb{N}_0$.

Now we prove that the indexes l_1 and l_2 in (5.5.9) must coincide. This can be shown by imposing the boundary condition (5.5.2), which can be written as

$$b_{l_1} i''_{l_1}(\sqrt{\tau})Y_{l_1}(\theta) - c_{l_2} l_2(l_2 - 1)Y_{l_2}(\theta) = 0.$$

If the two indexes do not agree, the coefficients of $Y_{l_i}, i = 1,2$ must vanish since spherical harmonics with different indexes are linearly independent on $\partial\Omega$. Since $i''_{l_1}(\sqrt{\tau}) > 0$, this implies $b_{l_1} = 0$ and therefore $l_2 = 0$ or $l_2 = 1$. Thus we have

$$u_l(r,\theta) = \Big(A_l r^l + B_l i_l(\sqrt{\tau}r)\Big)H_l(\theta), \tag{5.5.10}$$

with suitable constants A_l, B_l. In the case $l \neq 0, 1$, again from the boundary condition (5.5.2) we have

$$l(l-1)A_l + \tau i_l''(\sqrt{\tau})B_l = 0, \tag{5.5.11}$$

hence $B_l = \frac{l(1-l)}{\tau i_l''(\sqrt{\tau})}A_l$. Note that the formula holds also in the case $l = 0, 1$ since these indexes correspond to $B_l = 0$.

Finally, let us consider the boundary condition (5.5.3). Using in (5.5.3) the representation of u_l provided by formula (5.5.10), we get

$$\Bigg[\Big(-\lambda + l\big((l-1)(l+N-2)+\tau\big)\Big)A_l + \Big(-\big(3l(l+N-2)+\lambda\big)i_l(\sqrt{\tau}) - \sqrt{\tau}\big((N-1-2Nl-2(l-2)l-\tau)i_l'(\sqrt{\tau}) + (N-1)\sqrt{\tau}i_l''(\sqrt{\tau}) + \tau i_l'''(\sqrt{\tau})\big)\Big)B_l\Bigg] H_l(\theta) = \lambda\left(A_l + B_l i_l(\sqrt{\tau})\right)H_l(\theta).$$

Using equality (5.5.11) we get that the function u_l given by (5.5.10) is an eigenfunction of (5.0.1) on the unit ball. Moreover, as a consequence, we also get formula (5.5.5) for the associated eigenvalue. This concludes the proof. □

We are ready to state and prove the following theorem concerning the first positive eigenvalue.

Theorem 5.5.12. *Let Ω be the unit ball in $\mathbb{R}^N$ centered at the origin. The first positive eigenvalue of (5.0.1) is $\lambda_2 = \lambda_{(1)} = \tau$. The corresponding eigenspace is generated by $\{x_1, x_2, ..., x_N\}$.*

Proof. By Theorem 5.5.4, $0 = \lambda_{(0)} < \tau = \lambda_{(1)}$. We consider formula (5.5.5) with $l = 2$. We have

$$\lambda_{(2)} = 2\Big(\tau i_2''(\sqrt{\tau}) - 2i_2(\sqrt{\tau})\Big)^{-1}\Big[6Ni_2(\sqrt{\tau}) - \sqrt{\tau}(5N-1+\tau)i_2'(\sqrt{\tau}) + \tau(2N-1+\tau)i_2''(\sqrt{\tau}) + \tau\sqrt{\tau}i_2'''(\sqrt{\tau})\Big]. \tag{5.5.13}$$

In order to prove that $\lambda_{(2)} > \tau$, we use some well-known recurrence relations between ultraspherical Bessel functions (see [1, p. 376]),

$$\begin{aligned}
i_l'(\sqrt{\tau}) &= \frac{l}{\sqrt{\tau}}i_l(\sqrt{\tau}) + i_{l+1}(\sqrt{\tau}),\\
i_l''(\sqrt{\tau}) &= \frac{l(l-1)}{\tau}i_l(\sqrt{\tau}) + \frac{l+2}{\tau}i_{l+1}(\sqrt{\tau}) + i_{l+2}(\sqrt{\tau}),\\
i_l'''(\sqrt{\tau}) &= \frac{l(l-1)(l-2)}{\tau\sqrt{\tau}}i_l(\sqrt{\tau}) + \frac{l(2l+1)}{\tau}i_{l+1}(\sqrt{\tau})\\
&\quad + \frac{2(l+2)}{\sqrt{\tau}}i_{l+2}(\sqrt{\tau}) + i_{l+3}(\sqrt{\tau}).
\end{aligned}$$

Using these relations in (5.5.13), we obtain an equivalent formula for $\lambda_{(2)}$,

$$\lambda_{(2)} = 2\Big(5\sqrt{\tau}i_3(\sqrt{\tau}) + \tau i_4(\sqrt{\tau})\Big)^{-1}\Big[(10N-2+2\tau)i_2(\sqrt{\tau}) + \big(2-10N+(7+10N)\sqrt{\tau}-2\tau+5\tau\sqrt{\tau}\big)i_3(\sqrt{\tau}) + \tau(8+2N+\tau)i_4(\sqrt{\tau}) + \tau\sqrt{\tau}i_5(\sqrt{\tau})\Big].$$

By well-known properties of the functions I_ν (see [1, §9]), it follows that $i_l \geq i_{l+1}$ for all $l \in \mathbb{N}_0$. This implies

$$(10N-2+2\tau)i_2(\sqrt{\tau}) + \big(2-10N+(7+10N)\sqrt{\tau}-2\tau+5\tau\sqrt{\tau}\big)i_3(\sqrt{\tau}) + \tau(8+2N+\tau)i_4(\sqrt{\tau}) + \tau\sqrt{\tau}i_5(\sqrt{\tau}) \geq \Big(5\tau\sqrt{\tau}i_3(\sqrt{\tau}) + \tau^2 i_4(\sqrt{\tau})\Big),$$

then

$$\lambda_{(2)} \geq 2\tau > \tau = \lambda_{(1)}.$$

Now it remains to prove that $\lambda_{(l)}$ is an increasing function of l for $l \geq 2$. We adapt the method used in [28, Theorem 3]. We claim that for any smooth radial function $R(r)$ the Rayleigh quotient

$$\mathcal{Q}(R(r)H_l(\theta)) = \frac{\int_{\partial B}\int_0^1 \left(|D^2(R(r)H_l(\theta))|^2 + \tau|\nabla(R(r)H_l(\theta))|^2\right) r^{N-1} dr d\sigma(\theta)}{\int_{\partial B} R(r)^2 H_l(\theta)^2 d\sigma(\theta)}$$

is an increasing function of l for $l \geq 2$. Here and in the sequel we shall denote by $\sigma(\theta)$ the $(N-1)$-dimensional measure element of the unit sphere, which is given by

$$d\sigma(\theta) = \sin(\theta_1)^{N-2}\sin(\theta_2)^{N-3}\cdots\sin(\theta_{N-2})d\theta_1\cdots d\theta_{N-1}$$

We consider the spherical harmonics to be normalized with respect to the $L^2(\partial B)$ scalar product. In particular, we have that the denominator of $\mathcal{Q}(R(r)H_l(\theta))$, $D[R(r)H_l(\theta)]$ is $R^2(1)$. Now we need to write the numerator of the Rayleigh quotient in a suitable way. We recall that the numerator of the Rayleigh quotient of a function $u \in H^2(B)$ is given by

$$N[u] = \int_B |D^2u|^2 + \tau|\nabla u|^2 dx,$$

We use the following pointwise identity to re-write the Hessian term:

$$|D^2u|^2 = \frac{1}{2}\Delta\left(|\nabla u|^2\right) - \nabla u \cdot \nabla(\Delta u). \tag{5.5.14}$$

We also need to write the gradient in spherical coordinates

$$\nabla u = \frac{\partial u}{\partial r}\vec{r} + \frac{1}{r}\nabla_S u, \tag{5.5.15}$$

where $\frac{1}{r}\nabla_S u$ is the tangential gradient of u on ∂B and $\vec{r} = \frac{x}{r}$ is the unit normal. We note that the two vectors in the right-hand side of (5.5.15) are orthogonal. We recall that, since we considered the spherical harmonics to be normalized with respect to the $L^2(\partial B)$ scalar product, we have

$$\int_{\partial B} |\nabla_S H_l(\theta)|^2 d\sigma(\theta) = l(l+N-2). \tag{5.5.16}$$

To simplify the notation, we set $k := l(l+N-2)$. Moreover, from the Tangential Divergence Theorem, it follows that

$$\int_{\partial B} \frac{1}{r^2}\Delta_S u d\sigma(\theta) = \int_{\partial B} \operatorname{div}_{\partial B}\left(\frac{1}{r}\nabla_S u\right) d\sigma(\theta) = 0 \tag{5.5.17}$$

We use (5.5.14) to re-write the integral of the Hessian as follows:

$$\int_B |D^2u|^2 dx = \int_B \frac{1}{2}\Delta\left(|\nabla u|^2\right) - \nabla u \cdot \nabla(\Delta u) dx. \tag{5.5.18}$$

Now we choose in (5.5.18) $u = R(r)H_l(\theta)$. We have

$$\begin{aligned}
&\int_B |D^2u|^2 dx \\
&\quad = \frac{1}{2}\int_{\partial B}\int_0^1 \left(\frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_S\right) \\
&\quad \left((R'(r))^2|H_l(\theta)|^2 + \frac{R(r)^2}{r^2}|\nabla_S H_l(\theta)|^2\right) r^{N-1} dr d\sigma(\theta) \\
&\quad - \int_{\partial B}\int_0^1 \left(R'(r)H_l(\theta)\vec{r} + \frac{R(r)}{r}\nabla_S H_l(\theta)\right) \\
&\quad \cdot \nabla\left((R''(r) + \frac{N-1}{r}R'(r) - \frac{k}{r^2}R(r))H_l(\theta)\right) r^{N-1} dr d\sigma(\theta) \\
&= \int_{\partial B}\int_0^1 \left((R''(r))^2 + R'(r)R'''(r) + \frac{N-1}{r}R'(r)R''(r)\right)|H_l(\theta)|^2 r^{N-1} dr d\sigma(\theta) \\
&\quad + \int_{\partial B}\int_0^1 \left(\frac{(R'(r))^2}{r^2} + \frac{R(r)R''(r)}{r^2}\right. \\
&\quad \left. + \frac{N-5}{r^3}R(r)R'(r) + \frac{4-N}{r^4}R(r)^2\right)|\nabla_S H_l(\theta)|^2 r^{N-1} dr d\sigma(\theta) \\
&\quad - \int_{\partial B}\int_0^1 \left(R'(r)R'''(r) + \frac{N-1}{r}R'(r)R''(r)\right. \\
&\quad \left. - \frac{N-1}{r^2}(R'(r))^2 + \frac{2k}{r^3}R(r)R'(r)\right)|H_l(\theta)|^2 r^{N-1} dr d\sigma(\theta) \\
&- \int_{\partial B}\int_0^1 \left(\frac{R(r)R''(r)}{r^2} + \frac{N-1}{r^3}R(r)R'(r) - \frac{k}{r^4}R(r)^2\right)|\nabla_S H_l(\theta)|^2 r^{N-1} dr d\sigma(\theta) \\
&= \int_{\partial_B}\int_0^1 \left((R''(r))^2 + \frac{N-1}{r^2}(R'(r))^2 - \frac{2k}{r^3}R(r)R'(r)\right)|H_l(\theta)|^2 r^{N-1} dr d\sigma(\theta)
\end{aligned}$$

$$+\int_{\partial B}\int_0^1\left(\frac{(R'(r))^2}{r^2}-\frac{4}{r^3}R(r)R'(r)+\frac{4-N+k}{r^4}R(r)^2\right)|\nabla H_l(\theta)|^2r^{N-1}drd\sigma(\theta), \tag{5.5.19}$$

where in the first equality we have used the fact that $-\Delta_S H_l = kH_l$ and in the second equality we have used the fact that

$$\int_{\partial B}\int_0^1\frac{1}{r^2}\Delta_S\left((R'(r))^2|H_l(\theta)|^2+\frac{R(r)^2}{r^2}|\nabla_S H_l(\theta)|^2\right)r^{N-1}drd\sigma(\theta)=0$$

which is a consequence of (5.5.17). Now, expanding the integrands in (5.5.19) and using (5.5.16) and the orthonormality of $H_l(\theta)$ with respect to the $L^2(\partial B)$ scalar product, we obtain that

$$\int_B|D^2u|^2dx=\int_0^1\left((R''(r))^2+\frac{N-1}{r^2}(R'(r))^2\right.$$
$$\left.+\frac{2k}{r^4}\left(rR'(r)-\frac{3}{2}R(r)\right)^2+\frac{k(k+N-1/2)}{r^4}R(r)^2\right)r^{N-1}dr. \tag{5.5.20}$$

As for the gradient term we have

$$\int_B|\nabla u|^2dx$$
$$=\int_{\partial B}\int_0^1\left((R'(r))^2|H_l(\theta)|^2+\frac{R(r)^2}{r^2}|\nabla_S H_l(\theta)|^2\right)r^{N-1}drd\sigma(\theta)$$
$$=\int_0^1\left((R'(r))^2+\frac{k}{r^2}R(r)^2\right)r^{N-1}dr \tag{5.5.21}$$

Combining (5.5.20) and (5.5.21) we have that the numerator of the Rayleigh quotient $N[R(r)H_l(\theta)]$ can be written in the following form

$$N[R(r)H_l(\theta)]$$
$$=\int_0^1\left(\frac{2k}{r^4}\left(rR'-\frac{3}{2}R\right)^2+\frac{k(k-N-1/2)}{r^4}R^2+\tau\frac{kR^2}{r^2}\right)r^{N-1}dr$$
$$+\int_0^1\left((R''^2)+\frac{N-1}{r^2}(R')^2+\tau(R')^2\right)r^{N-1}dr.$$

The above expression is increasing in k for $k\geq N+1/2$ and since k is an increasing function of l, we easily get that each term involving l is an increasing function of l for $l\geq 2$. Thus the claim above is proved.

For each $l\in\mathbb{N}_0$,

$$\lambda_{(l)}=\inf\mathcal{Q}(u)=\inf\frac{\int_B|D^2u|^2+\tau|\nabla u|^2dx}{\int_{\partial B}u^2d\sigma}, \tag{5.5.22}$$

where the infimum is taken among all functions u that are $L^2(\partial B)$-orthogonal to the first $m-1$ eigenfunctions u_i and $m \in \mathbb{N}$ is such that $\lambda_{(l)} = \lambda_m$ is the $m-$th eigenvalue of problem (5.0.1). The eigenfunctions u_l are of the form $u_l = R_l(r)H_l(\theta)$, and u_l realizes the infimum in (5.5.22). Then

$$\lambda_{(l)} = \mathcal{Q}(R_l(r)H_l(\theta)) \leq \mathcal{Q}(R_{l+1}(r)H_l(\theta)) \leq \mathcal{Q}(R_{l+1}(r)Y_{l+1}(\theta)) = \lambda_{(l+1)},$$

where the first inequality follows from the fact that $R_{l+1}(r)H_l(\theta)$ is also orthogonal with respect to the $L^2(\partial B)$ scalar product to the first $m-1$ eigenfunctions $R_i(r)Y_i(\theta)$ for $i = 1, ...m-1$, and then it is a suitable trial function in (5.5.22). The second inequality follows from the fact that the quotient $\mathcal{Q}(R(r)H_l(\theta))$ is an increasing function of l, for $l \geq 2$. This concludes the proof. $\square$

5.5.2 The isoperimetric inequality

In this subsection we prove the isoperimetric inequality (5.5.1). Actually, we prove a stronger result, that is a quantitative version of (5.5.1). We adapt to our case a result of [16], where the authors prove a quantitative version of the Brock-Weinstock inequality for the Steklov Laplacian. We also refer to [58, 84] where these kind of questions have been considered for the first time (see also [17, 45]).

Throughout this section Ω is a bounded domain of class C^1. We recall the following lemma from [16].

Lemma 5.5.23. *Let Ω be an open set with Lipschitz boundary and $p > 1$. Then*

$$\int_{\partial\Omega} |x|^p d\sigma \geq \int_{\partial\Omega^*} |x|^p d\sigma \left(1 + c_{N,p}\left(\frac{|\Omega \triangle \Omega^*|}{|\Omega|}\right)^2\right),$$

where Ω^* is the ball centered at zero with the same measure as Ω, $\Omega\triangle\Omega^*$ is the symmetric difference of Ω and Ω^*, and $c_{N,p}$ is a constant depending only on N and p given by

$$c_{N,p} := \frac{(N+p-1)(p-1)}{4} \frac{\sqrt[N]{2}-1}{N} \left(\min_{t \in [1, \sqrt[N]{2}]} t^{p-2}\right).$$

We also recall the following characterization of the inverses of the eigenvalues of (5.0.1) from [62] (see also [14]).

Lemma 5.5.24. *Let Ω be a bounded domain of class C^1 in $\mathbb{R}^N$. Then the eigenvalues of problem (5.0.1) on Ω satisfy,*

$$\sum_{l=k+1}^{k+N} \frac{1}{\lambda_l(\Omega)} = \max\left\{\sum_{l=k+1}^{k+N} \int_{\partial\Omega} v_l^2 d\sigma\right\}, \tag{5.5.25}$$

where the maximum is taken over the families $\{v_l\}_{l=k+1}^{k+N}$ *in* $H^2(\Omega)$ *satisfying* $\int_\Omega D^2 v_i : D^2 v_j + \tau \nabla v_i \cdot \nabla v_j dx = \delta_{ij}$*, and* $\int_{\partial\Omega} v_i u_j d\sigma = 0$ *for all* $i = k+1, ..., k+N$ *and* $j = 1, 2, ..., k$*, where* $u_1, u_2, ..., u_k$ *are the first* k *eigenfunctions of problem (5.0.1).*

For every open set $\Omega \in \mathbb{R}^N$ with finite measure, we recall the definition of Fraenkel asymmetry

$$\mathcal{A}(\Omega) := \inf \left\{ \frac{\|\chi_\Omega - \chi_B\|_{L^1(\mathbb{R}^N)}}{|\Omega|} : B \text{ ball with } |B| = |\Omega| \right\}. \tag{5.5.26}$$

The quantity $\mathcal{A}(\Omega)$ is the distance in the $L^1(\mathbb{R}^N)$ norm of a set Ω from the set of all balls of the same measure as Ω. This quantity turns out to be a suitable distance between sets for the purposes of stability estimates of eigenvalues. Note that $\mathcal{A}(\Omega)$ is scaling invariant and $0 \leq \mathcal{A}(\Omega) < 2$.

We are ready to prove the following theorem.

Theorem 5.5.27. *For every domain* Ω *in* $\mathbb{R}^N$ *of class* C^1 *the following estimate holds*

$$\lambda_2(\Omega) \leq \lambda_2(\Omega^*) \left(1 - \delta_N \mathcal{A}(\Omega)^2\right), \tag{5.5.28}$$

where δ_N *is given by*

$$\delta_N := \frac{c_{N,2}}{2} = \frac{N+1}{8N} \left(\sqrt[N]{2} - 1 \right),$$

and Ω^* *is a ball with the same measure as* Ω*.*

Proof. Let Ω be a bounded domain of class C^1 in $\mathbb{R}^N$ with the same measure as the unit ball B. We consider in (5.5.25) $l = 2, ..., N+1$ and $v_l = (\tau|\Omega|)^{-1/2} x_l$ as trial functions. The trial functions must have zero integral mean over $\partial\Omega$. This can be obtained by a change of coordinates $x = y - \frac{1}{|\partial\Omega|} \int_{\partial\Omega} y d\sigma$. Moreover, the functions v_l satisfy the normalization condition of Lemma 5.5.24. Then v_l can be used as test functions in (5.5.25). We get

$$\sum_{l=2}^{N+1} \frac{1}{\lambda_l(\Omega)} \geq \frac{1}{\tau|\Omega|} \int_{\partial\Omega} |x|^2 d\sigma.$$

We use Lemma 5.5.23 with $p = 2$. This yields

$$\begin{aligned} \sum_{l=2}^{N+1} \frac{1}{\lambda_l(\Omega)} &\geq \frac{1}{\tau|\Omega|} \int_{\partial B} |x|^2 d\sigma \left(1 + c_{N,2} \left(\frac{|\Omega \triangle B|}{|\Omega|}\right)^2\right) \\ &= \frac{N|B|}{\tau|B|} \left(1 + c_{N,2} \left(\frac{|\Omega \triangle B|}{|\Omega|}\right)^2\right) = \frac{N}{\tau} \left(1 + c_{N,2} \left(\frac{|\Omega \triangle B|}{|\Omega|}\right)^2\right) \\ &= \sum_{l=2}^{N+1} \frac{1}{\lambda_l(B)} \left(1 + c_{N,2} \left(\frac{|\Omega \triangle B|}{|\Omega|}\right)^2\right). \end{aligned} \tag{5.5.29}$$

Suppose now that $\lambda_2(\Omega) \geq \frac{\tau}{2}$, otherwise estimate (5.5.28) is trivially true, since $0 \leq \mathcal{A}(\Omega) < 2$. Since $\lambda_2(\Omega) \leq \lambda_l(\Omega)$ for all $l \geq 3$, inequality (5.5.29) and the definition of $\mathcal{A}(\Omega)$ yield

$$\lambda_2(\Omega)\left(1 + c_{N,2}\mathcal{A}(\Omega)^2\right) \leq \lambda_2(B). \tag{5.5.30}$$

Therefore, since $\lambda_2(\Omega) \geq \frac{\tau}{2}$ and $\lambda_2(B) = \tau$, from (5.5.30) we have

$$\lambda_2(\Omega) \leq \tau - \lambda_2(\Omega)c_{N,2}\mathcal{A}(\Omega)^2 \leq \tau\left(1 - \frac{c_{N,2}\mathcal{A}(\Omega)^2}{2},\right)$$

which implies (5.5.28) with $\delta_N = \frac{1}{8}\min\{1, \frac{N+1}{N}(\sqrt[N]{2} - 1)\}$. We note that $\min\{1, \frac{N+1}{N}(\sqrt[N]{2}-1)\} = \frac{N+1}{N}(\sqrt[N]{2}-1)$. This concludes the proof in the case Ω has the same measure as the unit ball.

The proof for general finite values of $|\Omega|$ relies on the well-known scaling properties of the eigenvalues. Namely, for all $s > 0$, if we write an eigenvalue of problem (5.0.1) as $\lambda(\tau, \Omega)$, we have

$$\lambda(\tau, \Omega) = s^3\lambda(s^{-2}\tau, s\Omega).$$

This is easy to prove by looking at the variational characterization of $\lambda(\tau, \Omega)$ and $\lambda(s^{-2}\tau, s\Omega)$ and performing a change of variable $x \mapsto x/s$ in the Rayleigh quotient (5.2.9). This last observation concludes the proof of the theorem. $\square$

The isoperimetric inequality (5.5.1) is an immediate consequence of Theorem 5.5.27.

Corollary 5.5.31. *Among all bounded domains of class C^1 with fixed measure, the ball maximizes the first non-negative eigenvalue of problem (5.0.1), that is $\lambda_2(\Omega) \leq \lambda_2(\Omega^*)$, where $\lambda_2(\Omega)$ has been defined in (5.2.9) and Ω^* is a ball with the same measure as Ω.*

5.5.3 Some remarks on the case $\tau = 0$

Throughout this chapter we have only considered problems (5.0.1) and (5.0.3) with $\tau > 0$. If we set $\tau = 0$, problem (5.0.1) reads

$$\begin{cases} \Delta^2 u = 0, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega, \\ -\mathrm{div}_{\partial\Omega}\left(D^2u \cdot \nu\right) - \frac{\partial \Delta u}{\partial \nu} = \lambda u, & \text{on } \partial\Omega, \end{cases} \tag{5.5.32}$$

while problem (5.0.3) reads

$$\begin{cases} \Delta^2 u = \lambda u, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega, \\ \mathrm{div}_{\partial\Omega}\left(D^2u \cdot \nu\right) + \frac{\partial \Delta u}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases} \tag{5.5.33}$$

Problems (5.5.32) and (5.5.33) model free vibrating plates which are not subject to lateral tension. These problems have a sequence of non-negative eigenvalues of finite multiplicity and the corresponding eigenfunctions form a orthonormal basis of $H^2(\Omega)$. The coordinate functions $x_1, ..., x_N$ and the constant functions are eigenfunctions of both problems (5.5.32) and (5.5.33) corresponding to the eigenvalue $\lambda = 0$, which has multiplicity $N+1$. Therefore, the first non-zero eigenvalue is the $(N+2)$-th eigenvalue.

As we did in Theorem 5.3.6, we can define the family of problems

$$\begin{cases} \Delta^2 u = \lambda \rho_\varepsilon u, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega, \\ \mathrm{div}_{\partial\Omega}\big(D^2 u \cdot \nu\big) + \frac{\partial \Delta u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{5.5.34}$$

where ρ_ε is defined as in (3.1.20). We have the following theorem, whose proof can be easily done adapting that of Theorem 5.3.6.

Theorem 5.5.35. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^2. Let ρ_ε be defined by (3.1.20). Let $\lambda_j[\rho_\varepsilon]$ be the eigenvalues of problem (5.5.34) on Ω for all $j \in \mathbb{N}$. Let λ_j, $j \in \mathbb{N}$ denote the eigenvalues of problem (5.5.32) corresponding to the constant surface density $\frac{M}{|\partial\Omega|}$. Then we have $\lim_{\varepsilon\to 0} \lambda_j[\rho_\varepsilon] = \lambda_j$ for all $j \in \mathbb{N}$.*

It is clear that a discussion similar to that of Section 5.4 can be carried out for problems (5.5.32) and (5.5.33) as well, by means of a change of the projections $\pi_\phi^{\mathcal{S}}, \pi_\phi^{\mathcal{N}}$ according to the kernel. In particular, all the formulas in Section 5.4 remain true, by setting $\tau = 0$. Then we have the following theorem.

Theorem 5.5.36. *Let Ω be a bounded domain in $\mathbb{R}^N$. Let $\tilde{\phi} \in \Phi(\Omega)$ be such that $\tilde{\phi}(\Omega)$ is a ball. Let $\tilde{\lambda}$ be an eigenvalue of problem (5.5.32) (problem (5.5.33) respectively) in $\tilde{\phi}(\Omega)$, and let F be the set of $j \in \mathbb{N}$ such that $\lambda_j[\tilde{\phi}] = \tilde{\lambda}$. Then $\Lambda_{F,s}$ has a critical point at $\tilde{\phi}$ on $V(\mathcal{V}(\tilde{\phi}))$, for all $s = 1, \dots, |F|$.*

Moreover, for problem (5.5.32), it is possible to identify the fundamental modes and the fundamental tone on the ball. We have the following

Theorem 5.5.37. *Let $\Omega = B$ be the unit ball in $\mathbb{R}^N$. The eigenfunctions of problem (5.5.32) are of the form*

$$u_l(r,\theta) = \left(A_l r^l + B_l r^{2+l}\right) H_l(\theta),$$

for $l \in \mathbb{N}_0$, where A_l and B_l are suitable constants such that

$$B_l = -\frac{l(l-1)}{(l+2)(l+1)} A_l.$$

The eigenvalues $\lambda_{(l)}$ of problem (5.5.32) corresponding to the eigenfunctions $u_l(r,\theta)$ are delivered by the formula

$$\lambda_{(l)} = \frac{l(l-1)\big(N + 2Nl + (l-1)(2+3l)\big)}{1+2l}. \tag{5.5.38}$$

The first positive eigenvalue is

$$\lambda_{N+2} = \lambda_{(2)} = 2\left(N + \frac{8}{5}\right), \tag{5.5.39}$$

and the corresponding eigenfunctions are

$$u_2(r,\theta) = \big(6r^2 - r^4\big)H_2(\theta). \tag{5.5.40}$$

Proof. The proof is similar to that of Theorem 5.5.4, from which it differs only for the use of biharmonic functions on the ball as solutions of the differential equation $\Delta^2 u = 0$. Any smooth and bounded function u on B which satisfies $\Delta^2 u = 0$ can be written in spherical coordinates (r,θ) as a linear combination of functions of the form $\left(ar^l + br^{2+l}\right)H_l(\theta)$ with $l \in \mathbb{N}_0$ (see also [6, 7, 90] for a complete characterization of biharmonic and poly-harmonic functions on the unit ball). The rest of the proof follows the same lines as that of Theorem 5.5.4. Then, once formula (5.5.38) has been established, it is straightforward to identify the fundamental tone (5.5.39) and the corresponding modes (5.5.40). □

Now we have an explicit form for the fundamental tone and for the corresponding eigenfunctions in the case of the unit ball which suggests how to construct trial functions for the Rayleigh quotient of λ_{N+2}. If we want to use a function of the form $R(r)H_2(\theta)$ as a test function as we did in Theorem 5.5.27 we must impose that $R(r)H_2(\theta)$ is othogonal to the constants and to the coordinate functions with respect to the $L^2(\partial\Omega)$ scalar product and we can no more obtain this just by translating the domain Ω.

We note that functions of the form $R(r)H_2(\theta)$ are suitable trial functions for the annuli centered at zero. We recall from formula (5.5.40) that the radial part of an eigenfunction associated with the first positive eigenvalue λ_{N+2} of th unit ball is of the form $6r^2 - r^4$. We want to construct a test function for the annulus centered at zero of the form $R(r)H_2(\theta)$ in such a way that the radial part $R(r)$ equals $6r^2 - r^4$ whenever $r \leq 1$. Moreover we want that the radial part is an increasing function of r, for $r \in [0, +\infty[$. We note that $6r^2 - r^4$ is increasing for $r \in [0,1]$. Note that test functions must belong to $H^2(\Omega)$. Therefore we choose $R(r) = 8r - 3$ for $r > 1$ which is increasing and moreover, $R(r)$ and $R'(r)$ are continuous at $r = 1$. With this choice of the test function it possible to compare the Rayleigh quotients of the annulus and of the ball and prove that the first positive eigenvalue of the ball centered at zero is bigger than the first positive eigenvalue of an annulus centered at zero with the same measure. We need the following preliminary lemma.

Lemma 5.5.41. *Let Ω be an open set in $\mathbb{R}^N$ and Ω^* be the ball centered at zero with the same measure as Ω. Let F be a measurable and radial function such that*

$$F(x) > F(y) \quad \forall x \in \Omega^*, y \notin \Omega^*. \tag{5.5.42}$$

Then

$$\int_\Omega F dx \leq \int_{\Omega^*} F dx,$$

with equality if and only if $\Omega = \Omega^*$.

Proof. Since $|\Omega| = |\Omega^*|$ we have that $|\Omega \setminus \Omega^*| = |\Omega^* \setminus \Omega|$. Let F be a measurable function satisfying condition (5.5.42). We have

$$\begin{aligned}
\int_\Omega F dx = \int_{\Omega\cap\Omega^*} F dx + \int_{\Omega\setminus\Omega^*} F dx \\
\leq \int_{\Omega\cap\Omega^*} F dx + \sup_{x\in\Omega\cap\Omega^*} |F(x)||\Omega \setminus \Omega^*| \\
\leq \int_{\Omega\cap\Omega^*} F dx + \inf_{x\in\Omega^*\setminus\Omega} |F(x)||\Omega^* \setminus \Omega| \\
\leq \int_{\Omega\cap\Omega^*} F dx + \int_{\Omega^*\setminus\Omega} F dx = \int_{\Omega^*} F dx.
\end{aligned}$$

The second inequality follows from (5.5.42). Note that if $|\Omega \setminus \Omega^*| > 0$, either the second inequality or the third is strict by the strict inequality in (5.5.42). □

We have the following theorem (see also Figure 5.1).

Theorem 5.5.43. *Let B be the unit ball in $\mathbb{R}^N$ centered at zero. Let $\rho > 0$ and A_ρ be the subset of $\mathbb{R}^N$ be defined by*

$$A_\rho := \left\{ x \in \mathbb{R}^N : \rho < |x| < \left(1 + \rho^N\right)^{\frac{1}{N}} \right\}.$$

Then

$$\lambda_{N+2}(B) \geq \lambda_{N+2}(A_\rho)$$

for all $\rho \in]0, +\infty[$.

Proof. Let the function $R(r)$ from $[0, +\infty[$ to $\mathbb{R}$ be defined by

$$R(r) = \begin{cases} 6r^2 - r^4, & \text{if } r \in [0, 1], \\ 8r - 3, & \text{if } r \in]1, +\infty[. \end{cases}$$

We note that by construction, $R \in C^2([0, +\infty[)$. We set $v(r, \theta) := R(r)H_2(\theta)$, where $H_2(\theta)$ is a spherical harmonic of degree 2 normalized by $\int_{\partial B} H_2^2 d\sigma =$

1. With this choice we also have that $\int_{\partial B} |\nabla_S H_2|^2 d\sigma = 2N$. Moreover, from well-known properties of spherical harmonics, we have that it holds

$$\int_{\partial A_\rho} H_2 d\sigma = 0$$

and

$$\int_{\partial A_\rho} \frac{x_l}{|x_l|} H_2 d\sigma = 0 , \quad \forall l = 1, ...N.$$

We recall the following characterization of $\lambda_{N+2}(A_\rho)$

$$\lambda_{N+2}(A_\rho) = \min_{\substack{0\neq u\in H^2(\Omega) \\ \int_{\partial A_\rho} u d\sigma = \int_{\partial A_\rho} x_i u d\sigma = 0,\ \forall i=1,\dots,N}} \frac{\int_{A_\rho} |D^2 u|^2 dx}{\int_{\partial A_\rho} u^2 d\sigma}. \tag{5.5.44}$$

Therefore, since v is a suitable test function for the Rayleigh quotient (5.5.44) we have

$$\lambda_{N+2}(A_\rho) \leq \frac{\int_{A_\rho} |D^2 v|^2 dx}{\int_{\partial A_\rho} v^2 d\sigma}. \tag{5.5.45}$$

By following the same lines of the proof of formula (5.5.20) we compute the integral of $|D^2 v|^2$ on A_ρ. We have

$$\begin{aligned}
\int_{A_\rho} &|D^2 v|^2 dx \\
&= \int_\rho^{(1+\rho^N)^{\frac{1}{N}}} \Big(R''(r)^2 + \frac{5N-1}{r^2} R'(r)^2 \\
&\quad - \frac{12N}{r^3} R'(r)R(r) + \frac{2N(3N+4)}{r^4} R(r)^2 \Big) r^{N-1} dr \\
&= \frac{1}{N\omega_N} \int_{A_\rho} F(|x|) dx,
\end{aligned} \tag{5.5.46}$$

where in the last equality we have multiplied and divided the integral by $\int_{\partial B} d\sigma$ which equals $N\omega_N$, and we have re-written the integral in cartesian coordinates. The function $F(r)$ is given by

$$F(r) := R''(r)^2 + \frac{5N-1}{r^2} R'(r)^2 - \frac{12N}{r^3} R'(r)R(r) + \frac{2N(3N+4)}{r^4} R(r)^2.$$

As for the denominator of the Rayleigh quotient (5.5.45), from the normalization of $H_2(\theta)$ we have

$$\int_{\partial A_\rho} v^2 d\sigma = R(\rho)^2 + R\left((1+\rho^N)^{\frac{1}{N}} \right)^2. \tag{5.5.47}$$

From (5.5.46) and (5.5.47) we have

$$\lambda_{N+2}(A_\rho) \leq \frac{\int_{A_\rho} F(|x|)dx}{N\omega_N\left(R(\rho)^2 + R\left(\left(1+\rho^N\right)^{\frac{1}{N}}\right)^2\right)},$$

From standard computations it follows that $R(\rho)^2 + R\left(\left(1+\rho^N\right)^{\frac{1}{N}}\right)^2$ is an increasing function of ρ, for $\rho \in [0, +\infty[$. Therefore it attains its minimum at $\rho = 0$ and such a minimum is 25. As for the numerator, it is possible to prove that the function $F(r)$ is a decreasing function of r, for $r \in [0,1]$. In fact, when $r \in [0,1]$ we have

$$F(r) = 72N(3N+2) - 24(8+3N(N+2))r^2 + 2(64+N(3N+20)),$$

and

$$F'(r) = 8(64+N(3N+20))r^3 - 48(8+3N(N+2))r.$$

Since $r \in [0,1]$, $r^3 \leq r$. Then

$$F'(r) \leq 8(64+N(3N+20))r - 48(8+3N(N+2)) = -8r(16(N-1)+15N^2) < 0.$$

Now, let $r \in]1, +\infty[$. We set $\tilde{F}(r) := r^4(F(1) - F(r))$ for $r \geq 1$. We have

$$\tilde{F}(r) \\ = 2(5N+4)(15N-8)r^4 - 64(6N^2+N-1)r^2 + 96N(3N+1)r - 18N(3N+4).$$

We show that $\tilde{F}(r) > 0$ for all $r \in]1, +\infty[$. Clearly $\tilde{F}(1) = 0$. We compute $\tilde{F}'(1) = 8(16(N-1)+15N^2) > 0$. Moreover we have $\tilde{F}''(r) = 24(5N+4)(15N-8)r^2 - 128(6N^2+N-1) > 8(N(129N+44)-80) > 0$ since $r > 1$. We have shown that $F(|x_1|) > F(|x_2|)$ for all x_1, x_2 such that $x_1 \in B$, $x_2 \notin B$. Therefore, from Lemma 5.5.41 it follows that

$$\int_{A_\rho} F(|x|)dx \leq \int_B F(|x|)dx = 10(8+5N)N\omega_N.$$

From this it follows that

$$\lambda_{N+2}(A_\rho) \leq \frac{10(8+5N)N\omega_N}{25N\omega_N} = 2\left(N + \frac{8}{5}\right) = \lambda_{N+2}(B).$$

$\square$

The results contained in this subsection suggest that the ball should be a maximizer also for problems (5.5.32) and (5.5.33). For what concerns problem (5.5.33), unfortunately a characterization of the fundamental tone is still unavaiable.

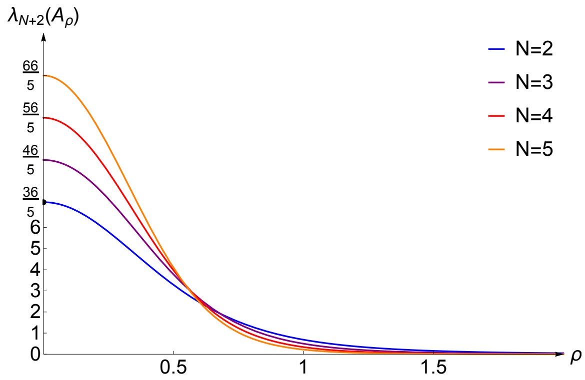


Figure 5.1

5.6 Neumann isoperimetric inequality in quantitative form. Sharpness of Neumann and Steklov inequalities

In this section, we consider problem (5.0.3) with $\rho \equiv 1$, namely problem

$$\begin{cases} \Delta^2 u - \tau \Delta u = \lambda u, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega, \\ \tau \frac{\partial u}{\partial \nu} - \mathrm{div}_{\partial\Omega}\big(D^2 u \cdot \nu\big) - \frac{\partial \Delta u}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases} \tag{5.6.1}$$

We recall the following isoperimetric inequality for the fundamental tone of problem (5.6.1) proved in [28]

$$\lambda_2(\Omega) \le \lambda_2(\Omega^*), \tag{5.6.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ of class C^1, $\lambda_2(\Omega)$ is the first positive eigenvalue of problem (5.6.1) on Ω, and Ω^* is a ball such that $|\Omega| = |\Omega^*|$. The aim of this section is to improve inequality (5.6.2) in the following quantitative form

$$\lambda_2(\Omega) \le \lambda_2(\Omega^*) \left(1 - C\mathcal{A}(\Omega)^2\right), \tag{5.6.3}$$

where $\mathcal{A}(\Omega)$ is the Fraenkel asymmetry, defined in (5.5.26).
In this section, we also prove the sharpness of both inequality (5.6.3) and inequality (5.5.28) corresponding to the biharmonic Steklov problem (5.5.28).

We introduce now some preliminaries which are used throughout the section, and we recall some results proved in [28]. Let B be the unit ball in $\mathbb{R}^N$ centered at zero. For a fixed $\tau > 0$, we take positive constants a, b satisfying $a^2 b^2 = \lambda_2(B)$ and $b^2 - a^2 = \tau$. We then define the function

$$R(r) = j_1(ar) + \gamma i_1(br),$$

where

$$\gamma = -\frac{a^2 j_1''(a)}{b^2 i_1''(b)},$$

and $j_1(z)$ and $i_1(z)$ are the ultraspherical and modified ultraspherical Bessel functions of the first species and order 1 respectively. We then define the function $\rho : [0,+\infty[\to [0,+\infty[$ by

$$\rho(r) := \begin{cases} R(r), & \text{if } r \in [0,1[\\ R(1) + (r-1)R'(1), & \text{if } r \in [1,+\infty[. \end{cases}$$

Let $u_k : \mathbb{R}^N \to \mathbb{R}$ be the functions defined by

$$u_k(x) := \rho(|x|)\frac{x_k}{|x_k|}, \tag{5.6.4}$$

for $k = 1, \dots, N$. The functions $u_{k|_B}$ are in fact the eigenfunctions associated with the eigenvalue $\lambda_2(B)$ of the Neumann problem (5.6.1) on the unit ball B, which has multiplicity N (see [29, Theorem 3]). Moreover, we have (see [28, p. 437])

$$\begin{aligned} \sum_{k=1}^N |u_k|^2 &= \rho(|x|)^2; \\ \sum_{k=1}^N |Du_k|^2 &= \frac{N-1}{r^2}\rho(|x|)^2 + (\rho'(|x|))^2; \\ \sum_{k=1}^N |D^2u_k|^2 &= (\rho''(|x|))^2 + \frac{3(N-1)}{|x|^4}(\rho(|x|) - |x|\rho'(|x|))^2. \end{aligned}$$

We denote by $N[\rho]$ the quantity

$$N[\rho] := \sum_{k=1}^N |D^2u_k|^2 + \tau|Du_k|^2.$$

Finally, we recall some properties enjoyed by the functions ρ and $N[\rho]$ which were proved in [28].

Lemma 5.6.5. *The function ρ satisfies the following properties.*

i) $\rho''(r) \le 0$ for all $r \ge 0$, therefore ρ' is non-increasing.

ii) $\rho(r) - r\rho'(r) \ge 0$ for all $r \ge 0$, equality holding only for $r = 0$.

iii) The function $\rho(r)^2$ is strictly increasing.

iv) The function $\rho(r)^2/r^2$ is decreasing.

v) The function $3(\rho(r) - r\rho'(r))^2/r^4 + \tau\rho^2(r)/r^2$ *is decreasing.*

vi) $N[\rho(r_1)] > N[\rho(r_2)]$ *for all* $r_1 \in [0, 1[$, $r_2 \in [1, +\infty[$.

vii) For all $r \geq 0$ *we have*

$$N[\rho] = (\rho'')^2 + \frac{3(N-1)(\rho(r) - r\rho'(r))^2}{r^4} + \tau\frac{(N-1)\rho^2(r)}{r^2} + \tau(\rho'(r))^2.$$

viii) For all $r \geq 1$, $N[\rho]$ *is decreasing.*

5.6.1 Quantitative isoperimetric inequality for the Neumann problem

In this subsection we state and prove the Neumann quantitative inequality:

Theorem 5.6.6. *For every bounded domain* Ω *in* $\mathbb{R}^N$ *of class* C^1 *the following estimate holds*

$$\lambda_2(\Omega) \leq \lambda_2(\Omega^*)\left(1 - \eta_{N,\tau,|\Omega|}\mathcal{A}(\Omega)^2\right), \tag{5.6.7}$$

where

$$\eta_{N,\tau,|\Omega|} > 0$$

is a positive constant, and Ω^* *is a ball such that* $|\Omega^*| = |\Omega|$.

Proof. Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1 with the same measure as the unit ball B. We recall the variational characterization of the second eigenvalue $\lambda_2(\Omega)$ of (5.6.1) on Ω:

$$\lambda_2(\Omega) = \inf_{\substack{0 \neq u \in H^2(\Omega) \\ \int_\Omega u dx = 0}} \frac{\int_\Omega |D^2u|^2 + \tau|Du|^2 dx}{\int_\Omega u^2 dx}. \tag{5.6.8}$$

Let $u_k(x)$, for $k = 1, \dots, N$ be the eigenfunctions corresponding to $\lambda_2(B)$ defined in (5.6.4). Clearly $u_{k|_\Omega} \in H^2(\Omega)$ by construction. It is possible to choose the origin of the coordinate axes in $\mathbb{R}^N$ in such a way that $\int_\Omega u_k dx = 0$ for all $k = 1, \dots, N$. With this choice, the functions u_k are suitable trial functions for the Rayleigh quotient (5.6.8). Once we have fixed the origin, let

$$\alpha := \frac{|\Omega \triangle B|}{|\Omega|}.$$

By definition of Fraenkel asymmetry we have

$$\mathcal{A}(\Omega) \leq \alpha. \tag{5.6.9}$$

From the variational characterization (5.6.8), it follows that for each $k = 1, \dots, N$,

$$\lambda_2(\Omega) \leq \frac{\int_\Omega |D^2u_k|^2 + \tau|Du_k|^2 dx}{\int_\Omega u_k^2 dx}. \tag{5.6.10}$$

We multiply both sides of (5.6.10) by $\int_\Omega u_k^2 dx$ and sum over $k = 1, \dots, N$, obtaining

$$\lambda_2(\Omega) \le \frac{\int_\Omega N[\rho]dx}{\int_\Omega \rho^2 dx}. \tag{5.6.11}$$

The same procedure for $\lambda_2(B)$ clearly yields

$$\lambda_2(B) = \frac{\int_B N[\rho]dx}{\int_B \rho^2 dx}. \tag{5.6.12}$$

From (5.6.11) and (5.6.12) it follows that

$$\lambda_2(B)\int_B \rho^2 dx - \lambda_2(\Omega)\int_\Omega \rho^2 dx \ge \int_B N[\rho]dx - \int_\Omega N[\rho]dx \ge 0, \tag{5.6.13}$$

where the last inequality follows from Lemma 5.6.5, vi) and [28, Lemma 14].

Now we consider the two balls B_1 and B_2 centered at the origin with radii r_1, r_2 taken such that $|\Omega \cap B| = |B_1| = \omega_N r_1^N$ and $|\Omega \setminus B| = |B_2 \setminus B| = \omega_N(r_2^N - 1)$. Then $|B_2| = \omega_N r_2^N$, and by construction

$$1 - r_1^N = \frac{\alpha}{2} = r_2^N - 1.$$

This follows since $|\Omega| + |B| = |\Omega \triangle B| + 2|\Omega \cap B|$ and then $1 - r_1^N = \alpha/2$. Similarly, $|\Omega \setminus B| + |\Omega \cap B| = |\Omega|$, hence $r_1^N = 2 - r_2^N$ and then $r_2^N - 1 = \alpha/2$. Now we observe, again by Lemma 5.6.5, vi) and viii) that

$$\int_\Omega N[\rho]dx \le \int_{B_1} N[\rho]dx + \int_{B_2 \setminus B} N[\rho]dx.$$

From this and (5.6.13), we obtain

$$\begin{aligned}\lambda_2(B)\int_B \rho^2 dx - \lambda_2(\Omega)\int_\Omega \rho^2 dx &\ge \int_B N[\rho]dx - \int_\Omega N[\rho]dx \\ &\ge \int_{B\setminus B_1} N[\rho]dx - \int_{B_2\setminus B} N[\rho]dx.\end{aligned} \tag{5.6.14}$$

Since the function $\rho(r)^2$ is strictly increasing by Lemma 5.6.5, iii), we have

$$\int_\Omega \rho^2 dx \ge \int_B \rho^2 dx = N\omega_N \int_0^1 \rho^2(r) r^{N-1} dr =: C^{(1)}_{N,\tau}.$$

Therefore

$$\begin{aligned}&\lambda_2(B)\int_B \rho^2 dx - \lambda_2(\Omega)\int_\Omega \rho^2 dx \\ &\quad = (\lambda_2(B) - \lambda_2(\Omega))\int_B \rho^2 dx + \lambda_2(\Omega)\left(\int_B \rho^2 dx - \int_\Omega \rho^2 dx\right) \\ &\quad \le C^{(1)}_{N,\tau}\left(\lambda_2(B) - \lambda_2(\Omega)\right).\end{aligned} \tag{5.6.15}$$

Now we consider the right-hand side of (5.6.14). We write $N[\rho]$ more explicitly in terms of ρ, obtaining:

$$\begin{aligned}\int_{B\setminus B_1} N[\rho]dx &= N\omega_N \int_{r_1}^1 \Big((\rho''(r))^2 + \frac{3(N-1)(\rho(r)-r\rho'(r))^2}{r^4} \\ &\quad + \tau(\rho'(r))^2 + \frac{\tau(N-1)}{r^2}\rho(r)^2\Big) r^{N-1}dr \\ &\geq N\omega_N \int_{r_1}^1 \Bigg(\frac{3(N-1)(\rho(r)-r\rho'(r))^2}{r^4} \\ &\quad +\tau(\rho'(r))^2 + \frac{\tau(N-1)}{r^2}\rho(r)^2\Bigg) r^{N-1}dr \\ &\geq \omega_N \left(3(N-1)(R(1)-R'(1))^2 + \tau R'(1)^2 + \tau(N-1)R(1)^2\right)(1-r_1^N),\end{aligned} \tag{5.6.16}$$

where in the last inequality, we use the fact that the integrand is non-increasing in r by Lemma 5.6.5, i) and v). Moreover,

$$\begin{aligned}&\int_{B_2\setminus B} N[\rho]dx \\ &= N\omega_N \int_1^{r_2} \Bigg(\frac{3(N-1)}{r^4}(R(1)-R'(1))^2 + \tau R'(1)^2 \\ &\quad + \frac{\tau(N-1)}{r^2}\Big((R(1)-R'(1))^2 + 2rR'(1)(R(1)-R'(1))\Big) \\ &\quad + \frac{\tau(N-1)}{r^2}\Big(r^2R'(1)^2\Big)\Bigg) r^{N-1}dr \\ &\leq N\omega_N \int_1^{r_2} \Bigg(N\tau R'(1)^2 + \frac{N-1}{r}\left((3+\tau)(R(1)-R'(1))^2\right. \\ &\quad \left. +2\tau R'(1)(R(1)-R'(1))\right)\Bigg) r^{N-1}dr \\ &= N\omega_N \tau R'(1)^2(r_2^N-1) + N\omega_N\left((3+\tau)(R(1)-R'(1))^2\right. \\ &\quad \left. +2\tau R'(1)(R(1)-R'(1))\right)(r_2^{N-1}-1),\end{aligned} \tag{5.6.17}$$

where we have estimated the quantities $1/r^2$ and $1/r^4$ by $1/r$. We note that $r_2 = (1+\alpha/2)^{1/N}$ and $0 \leq \alpha \leq 2$. Using the Taylor expansion up to order 1 and remainder in Lagrange form, we obtain

$$\begin{aligned} r_2^{N-1} &= 1 + \frac{N-1}{N}\frac{\alpha}{2} - \frac{(N-1)\left(1+\frac{\xi}{2}\right)^{\frac{N-1}{N}-2}}{8N^2}\alpha^2 \\ &\leq 1 + \frac{N-1}{N}\frac{\alpha}{2} - \frac{(N-1)2^{\frac{N-1}{N}-2}}{8N^2}\alpha^2 = 1 + \frac{N-1}{N}\frac{\alpha}{2} - c_N\alpha^2,\end{aligned} \tag{5.6.18}$$

for some $\xi \in (0,\alpha)$, where c_N is a positive constant which depends only on N. Using (5.6.16), (5.6.17), (5.6.18) and the fact that $1-r_1^N = r_2^N - 1 = \alpha/2$

in the right-hand side of (5.6.14), we obtain:

$$\begin{aligned}\int_{B\setminus B_1} N[\rho]dx - \int_{B_2\setminus B} N[\rho]dx \\ \geq -N\omega_N\Big((3+\tau)(R(1)-R'(1))^2 \\ + 2\tau R'(1)(R(1)-R'(1))\Big)\left(\frac{N-1}{N}\frac{\alpha}{2} - c_N\alpha^2\right) \\ + \omega_N\left(3(N-1)(R(1)-R'(1))^2 + \tau R'(1)^2 + \tau(N-1)R(1)^2\right)\frac{\alpha}{2} \\ - N\omega_N\tau R'(1)^2\frac{\alpha}{2} \\ =: C^{(2)}_{N,\tau}\alpha^2,\end{aligned} \tag{5.6.19}$$

where the constant $C^{(2)}_{N,\tau} > 0$ is given by

$$C^{(2)}_{N,\tau} = N\omega_N\left((3+\tau)(R(1)-R'(1))^2 + 2\tau R'(1)(R(1)-R'(1))\right)c_N.$$

From (5.6.9), (5.6.15) and (5.6.19) it follows

$$\lambda_2(B) - \lambda_2(\Omega) \geq \frac{C^{(2)}_{N,\tau}}{C^{(1)}_{N,\tau}}\mathcal{A}(\Omega)^2,$$

and therefore,

$$\lambda_2(\Omega) \leq \lambda_2(B)\left(1 - \frac{C^{(2)}_{N,\tau}}{\lambda_2(B)C^{(1)}_{N,\tau}}\mathcal{A}(\Omega)^2\right). \tag{5.6.20}$$

The isoperimetric inequality is thus proved in the case of Ω with the same measure as the unit ball. The inequality for a generic domain Ω follows from the well-known scaling properties of the eigenvalues of problem (5.6.1). Writing our eigenvalues as $\lambda_2(\tau,\Omega)$ to make explicit the dependence on the parameter τ, we have

$$\lambda_2(\tau,\Omega) = s^4\lambda_2(s^{-2}\tau, s\Omega), \tag{5.6.21}$$

for all $s > 0$. From (5.6.20) and taking $s = (\omega_N/|\Omega|)^{1/N}$ in (5.6.21), it follows that for every Ω in $\mathbb{R}^N$ of class C^1 we have

$$\begin{aligned}\lambda_2(\tau,\Omega) &= s^4\lambda_2(s^{-2}\tau, s\Omega) \\ &\leq s^4\lambda_2(s^{-2}\tau, B)\left(1 - \frac{C^{(2)}_{N,s^{-2}\tau}}{\lambda_2(s^{-2}\tau,B)C^{(1)}_{N,s^{-2}\tau}}\mathcal{A}(s\Omega)\right) \\ &= \lambda_2(\tau,\Omega^*)\left(1 - \frac{C^{(2)}_{N,s^{-2}\tau}}{\lambda_2(s^{-2}\tau,B)C^{(1)}_{N,s^{-2}\tau}}\mathcal{A}(\Omega)\right).\end{aligned}$$

We set $\eta_{N,\tau,|\Omega|} := \frac{C^{(2)}_{N,s^{-2}\tau}}{\lambda_2(s^{-2}\tau,B)C^{(1)}_{N,s^{-2}\tau}}$. This concludes the proof of the theorem.

□

5.6.2 Sharpness of the Neumann inequality

In this subsection we prove that inequality (5.6.7) is sharp, that is, the exponent 2 for the Fraenkel asymmetry is optimal in the decay rate of $\lambda_2(\Omega^*) - \lambda_2(\Omega)$. To do so, we exhibit a family of sets $\{\Omega_\varepsilon\}_{\varepsilon>0}$ approaching the unit ball centered at zero such that $\mathcal{A}(\Omega_\varepsilon) \in O(\varepsilon)$ and $\lambda_2(\Omega^*) - \lambda_2(\Omega) \in O(\varepsilon^2)$, as $\varepsilon \to 0$. We have the following.

Theorem 5.6.22. *Let B be the unit ball in $\mathbb{R}^N$ centered at zero. There exist a family $\{\Omega_\varepsilon\}_{\varepsilon>0}$ of smooth domains and positive constants c_1, c_2, c_3, c_4 and r_1, r_2, r_3, r_4 which do not depend on $\varepsilon > 0$ such that*

$$r_1\varepsilon^2 \leq \Big||\Omega_\varepsilon| - |B|\Big| \leq r_2\varepsilon^2, \tag{5.6.23}$$

$$c_1\varepsilon \leq c_2\mathcal{A}(\Omega_\varepsilon) \leq \frac{|\Omega_\varepsilon \triangle B|}{|\Omega_\varepsilon|} \leq c_3\mathcal{A}(\Omega_\varepsilon) \leq c_4\varepsilon, \tag{5.6.24}$$

and

$$r_3\varepsilon^2 \leq |\lambda_2(\Omega_\varepsilon) - \lambda_2(B)| \leq r_4\varepsilon^2, \tag{5.6.25}$$

for all $\varepsilon \in]0, \varepsilon_0[$, where $\varepsilon_0 > 0$ is sufficiently small.

In order to prove Theorem 5.6.22, we follow the same lines of [16], where the authors consider the same issue in the case of the Steklov eigenvalue problem for the Laplace operator.
First, we consider a class of domains satisfying suitable geometrical assumptions (see (5.6.26) and (5.6.27)). Under such assumptions, it is standard to prove that (5.6.23) and (5.6.24) hold. In order to prove (5.6.25), we construct suitable test functions for the Rayleigh quotient of $\lambda_2(\Omega_\varepsilon)$ starting from the eigenfunctions associated with $\lambda_2(B)$. We obtain an estimate for $\lambda_2(B) - \lambda_2(\Omega)$ in terms of some error functions $R_1(\varepsilon), R_2(\varepsilon)$ (see (5.6.38)). Then we prove that a better estimates of $R_1(\varepsilon)$ and $R_2(\varepsilon)$ produces a better estimates on $\lambda_2(B) - \lambda_2(\Omega)$, see Lemma 5.6.39. Then we prove Lemmas 5.6.40 and 5.6.52, which give us suitable estimates on $R_1(\varepsilon)$ and $R_2(\varepsilon)$, finally providing (5.6.25).

Let us define the family of domains $\{\Omega_\varepsilon\}_{\varepsilon>0}$ as follows:

$$\Omega_\varepsilon := \left\{ x \in \mathbb{R}^N : x = 0 \text{ or } |x| < 1 + \varepsilon\psi\left(\frac{x}{|x|}\right)\right\}, \tag{5.6.26}$$

where ψ is a function belonging to the following class

$$\mathcal{P} := \left\{ \psi \in C^\infty(\partial B) : \int_{\partial B} \psi d\sigma = \int_{\partial B} (a \cdot x)\psi d\sigma = \int_{\partial B} (a \cdot x)^2 \psi d\sigma = 0 \right\}, \tag{5.6.27}$$

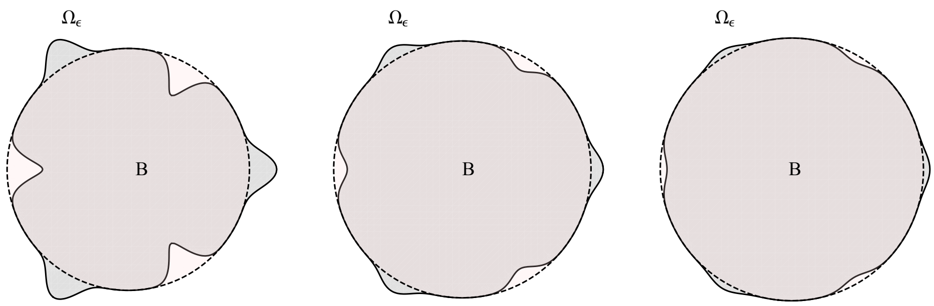


Figure 5.2: Domains Ω_ε defined by (5.6.26) with $\psi \in \mathcal{P}$.

for all $a \in \mathbb{R}^N$. (Note that the class $\mathcal{P}$ is non-empty, in fact all spherical harmonics H_l of degree $l \geq 3$ belong to this class, see Figure 5.2).

Under this choice of Ω_ε, the existence of constants $r_1, r_2, c_1, \dots, c_4$ satisfying inequalities (5.6.23) and (5.6.24) follow immediately from [16, Lemma 6.2]. Thus, we need only to prove (5.6.25).

Let $\lambda_2(\Omega_\varepsilon)$ be the first positive eigenvalue of the Neumann problem (5.6.1) on Ω_ε, and let u_ε be an associated eigenfunction normalized by $\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} = 1$, so that

$$\int_{\Omega_\varepsilon} |D^2 u_\varepsilon|^2 + \tau |Du_\varepsilon|^2 \, dx = \lambda_2(\Omega_\varepsilon). \tag{5.6.28}$$

By standard elliptic regularity (see e.g., [47, §2.4.3]), since Ω_ε is of class C^∞ by construction, we may take a sufficiently small $\varepsilon_0 > 0$ so that $u_\varepsilon \in C^\infty(\overline{\Omega}_\varepsilon)$ for all $\varepsilon \in]0, \varepsilon_0[$. Moreover, for all $k \in \mathbb{N}$, the sets Ω_ε are of class C^k uniformly in $\varepsilon \in]0, \varepsilon_0[$, which means that there exist constants $H_k > 0$ which do not depend on ε such that

$$\|u_\varepsilon\|_{C^k(\overline{\Omega}_\varepsilon)} \leq H_k. \tag{5.6.29}$$

Now let $\tilde{u}_\varepsilon$ be a C^5 extension of u_ε to some open neighborhood A of $B \cup \Omega_\varepsilon$. Then, there exists $K_A > 0$ which does not depend on $\varepsilon > 0$ such that

$$\|\tilde{u}_\varepsilon\|_{C^5(\overline{A})} \leq K_A \|u_\varepsilon\|_{C^5(\overline{\Omega}_\varepsilon)} \leq K_A H_5. \tag{5.6.30}$$

From the fact that $\int_{\Omega_\varepsilon} u_\varepsilon \, dx = 0$ and $|B \setminus \Omega_\varepsilon|, |\Omega_\varepsilon \setminus B| \in O(\varepsilon)$ as $\varepsilon \to 0$, it follows that the quantity $\delta := \frac{1}{|B|} \int_B \tilde{u}_\varepsilon \, dx$ satisfies

$$\delta = \frac{1}{|B|} \int_B \tilde{u}_\varepsilon \, dx = \frac{1}{|B|} \left(\int_{B \setminus \Omega_\varepsilon} \tilde{u}_\varepsilon \, dx - \int_{\Omega_\varepsilon \setminus B} u_\varepsilon \, dx \right) \leq c\varepsilon, \tag{5.6.31}$$

where $c > 0$ does not depend on $\varepsilon \in]0, \varepsilon_0[$. Now we set

$$v_\varepsilon := \tilde{u}_{\varepsilon|_B} - \delta. \tag{5.6.32}$$

The function v_ε is of class $C^5(\overline{B})$, $\int_B v_\varepsilon\, dx = 0$ and satisfies

$$\|v_\varepsilon\|_{C^5(\overline{B})} \le K_1 \tag{5.6.33}$$

for some constant $K_1 > 0$ independent of $\varepsilon \in]0, \varepsilon_0[$. Therefore, v_ε is a suitable trial function for the Rayleigh quotient of $\lambda_2(B)$ (see formula (6.1.13)). Thus,

$$\lambda_2(B) \le \frac{\int_B |D^2 v_\varepsilon|^2 + \tau |D v_\varepsilon|^2\, dx}{\int_B {v_\varepsilon}^2\, dx}. \tag{5.6.34}$$

We now consider the quantity $\left|\int_B v_\varepsilon^2 - \tilde{u}_\varepsilon^2\, dx\right|$. We have

$$\begin{aligned} \left|\int_B v_\varepsilon^2 - \tilde{u}_\varepsilon^2\, dx\right| &= \left|\int_B \delta^2 - 2\delta\tilde{u}_\varepsilon\, dx\right| = \left|\int_B \delta(-v_\varepsilon - \tilde{u}_\varepsilon)\, dx\right| \\ &= \frac{1}{|B|}\left(\int_B \tilde{u}_\varepsilon\, dx\right)^2 \le K_2\varepsilon^2, \end{aligned} \tag{5.6.35}$$

where $K_2 > 0$ is a positive constant independent of $\varepsilon \in]0, \varepsilon_0[$. Moreover, we have that by (5.6.30) and (5.6.33),

$$\begin{aligned} \left|\int_{B\setminus\Omega_\varepsilon} v_\varepsilon^2 - \tilde{u}_\varepsilon^2 dx\right| &\le \int_{B\setminus\Omega_\varepsilon} |v_\varepsilon^2 - \tilde{u}_\varepsilon^2| dx \le K_3 \int_{B\setminus\Omega_\varepsilon} |v_\varepsilon - \tilde{u}_\varepsilon| dx \\ &= K_3 \frac{|B\setminus\Omega_\varepsilon|}{|B|}\left|\int_B \tilde{u}_\varepsilon\, dx\right| \le k\varepsilon^2, \end{aligned} \tag{5.6.36}$$

where $K_3, k > 0$ are positive constants independent of $\varepsilon \in]0, \varepsilon_0[$. Therefore, from (5.6.28), (5.6.34), (5.6.35), and (5.6.36), it follows that

$$\begin{aligned} \lambda_2(B) &\le \frac{\int_{B\cap\Omega_\varepsilon} |D^2 u_\varepsilon|^2 + \tau|Du_\varepsilon|^2 dx + \int_{B\setminus\Omega_\varepsilon} |D^2 v_\varepsilon|^2 + \tau|Dv_\varepsilon|^2\, dx}{\int_B \tilde{u}_\varepsilon^2 dx - K_2\varepsilon^2} \\ &\le \frac{\lambda_2(\Omega_\varepsilon) + \int_{B\setminus\Omega_\varepsilon} |D^2 v_\varepsilon|^2 + \tau|Dv_\varepsilon|^2 dx - \int_{\Omega_\varepsilon\setminus B} |D^2 u_\varepsilon|^2 + \tau|Du_\varepsilon|^2\, dx}{1 + \int_{B\setminus\Omega_\varepsilon} v_\varepsilon^2 dx - \int_{\Omega_\varepsilon\setminus B} u_\varepsilon^2 dx - (k + K_2)\varepsilon^2}. \end{aligned} \tag{5.6.37}$$

We introduce now the two error terms $R_1(\varepsilon)$ and $R_2(\varepsilon)$ defined by

$$R_1(\varepsilon) := \int_{B\setminus\Omega_\varepsilon} |D^2 v_\varepsilon|^2 + \tau|Dv_\varepsilon|^2 dx - \int_{\Omega_\varepsilon\setminus B} |D^2 u_\varepsilon|^2 + \tau|Du_\varepsilon|^2\, dx$$

and

$$R_2(\varepsilon) := \int_{B\setminus\Omega_\varepsilon} v_\varepsilon^2 dx - \int_{\Omega_\varepsilon\setminus B} u_\varepsilon^2 dx.$$

Then inequality (5.6.37) can be rewritten as

$$\lambda_2(B) \le \frac{\lambda_2(\Omega_\varepsilon) + R_1(\varepsilon)}{1 + R_2(\varepsilon) - K_4\varepsilon^2}, \tag{5.6.38}$$

where $K_4 = k + K_2$. From the uniform estimates (5.6.29) and (5.6.33) on u_ε and v_ε, it easily follows that $R_1, R_2 \in O(\varepsilon)$ as $\varepsilon \to 0$, which together with (5.6.38) immediately yields $\lambda_2(B) \le \lambda_2(\Omega_\varepsilon) + C\varepsilon$ for some constant $C > 0$ which does not depend on $\varepsilon \in]0, \varepsilon_0[$ (taking $\varepsilon_0 > 0$ smaller if necessary).

We observe that, due to the close relation of $R_1(\varepsilon)$ and $R_2(\varepsilon)$ with the difference $\lambda_2(B) - \lambda_2(\Omega_\varepsilon)$, a better estimate for $R_1(\varepsilon)$ and $R_2(\varepsilon)$ provides a better estimate for $\lambda_2(B) - \lambda_2(\Omega_\varepsilon)$. More precisely, we have the following lemma.

Lemma 5.6.39. *Let K_4 be as in* (5.6.38)*. Let $\omega : [0,1] \to [0, +\infty[$ be a continuous function such that $\frac{t^2}{K_4} \le \omega(t) \le K_4 t$. If there exists a constant $C > 0$ such that $|R_1(\varepsilon)| \le C\omega(\varepsilon)$ and $|R_2(\varepsilon)| \le C\omega(\varepsilon)$, then there exists a constant $C' > 0$ such that*

$$\lambda_2(B) \le \lambda_2(\Omega_\varepsilon) + C'\omega(\varepsilon)$$

for every sufficient small $\varepsilon > 0$.

Proof. We refer to [17, Lemma 6.2] for the proof (see also [16, Lemma 6.7]). □

We also need the following:

Lemma 5.6.40. *Let ω be a function as in Lemma 5.6.39, and let v_ε be as in* (5.6.32)*. Suppose that there exists $C > 0$ such that for all $\varepsilon > 0$ sufficiently small we have $|R_1(\varepsilon)| \le C\omega(\varepsilon)$ and $|R_2(\varepsilon)| \le C\omega(\varepsilon)$. Then there exists an eigenfunction ξ_ε associated with $\lambda_2(B)$ such that*

$$\|v_\varepsilon - \xi_\varepsilon\|_{C^3(\overline{B})} \le \tilde{C}\sqrt{\omega(\varepsilon)},$$

for some $\tilde{C} > 0$ which does not depend on $\varepsilon > 0$.

Proof. Let $\{\xi_n\}_{n \ge 1}$ be an orthonormal basis of $L^2(B)$ consisting of eigenfunctions of the Neumann biharmonic problem (5.6.1) for $\Omega = B$, with ξ_1 constant. Note then that by our normalization, we have

$$\int_B |D^2\xi_n|^2 + \tau|D\xi_n|^2\, dx = \lambda_n(B) \quad \forall n \in \mathbb{N}.$$

We may now write $v_\varepsilon = \sum_{n=1}^{+\infty} a_n(\varepsilon)\xi_n$. Note that $a_1(\varepsilon) \equiv 0$ since v_ε has zero integral mean over B and ξ_1 is a constant. We have

$$\begin{aligned}\sum_{n=2}^{+\infty} a_n(\varepsilon)^2 - 1 = \|v_\varepsilon\|^2_{L^2(B)} - 1 &= \int_B v_\varepsilon^2 dx - \int_{\Omega_\varepsilon} u_\varepsilon^2 dx \\ &= \int_B (v_\varepsilon^2 - \tilde{u}_\varepsilon^2)dx - \int_{B\setminus\Omega_\varepsilon} (v_\varepsilon^2 - \tilde{u}_\varepsilon^2)dx + R_2(\varepsilon),\end{aligned}$$

and then by using (5.6.35), (5.6.36) we obtain

$$\left|\sum_{n=2}^{+\infty} a_n(\varepsilon)^2 - 1\right| \le K_4\varepsilon^2 + C\omega(\varepsilon) \le C_1\omega(\varepsilon). \tag{5.6.41}$$

We may now write

$$\begin{aligned}
\lambda_2(\Omega_\varepsilon) &= \int_{\Omega_\varepsilon} |D^2u_\varepsilon|^2 + \tau|Du_\varepsilon|^2 dx \\
&= \int_B |D^2v_\varepsilon|^2 + \tau|Dv_\varepsilon|^2\, dx + \int_{\Omega_\varepsilon\setminus B} |D^2u_\varepsilon|^2 + \tau|Du_\varepsilon|^2 dx \\
&\quad - \int_{B\setminus\Omega_\varepsilon} |D^2v_\varepsilon|^2 + \tau|Dv_\varepsilon|^2\, dx \\
&= \sum_{n=2}^{+\infty} a_n(\varepsilon)^2\lambda_n(B) - R_1(\varepsilon).
\end{aligned}$$

From Lemma 5.6.39 it follows that

$$|\lambda_2(B) - \lambda_2(\Omega_\varepsilon)| \le C'\omega(\varepsilon),$$

therefore

$$\left|\sum_{n=2}^{+\infty} a_n(\varepsilon)^2\lambda_n(B) - \lambda_2(B)\right| = |\lambda_2(\Omega_\varepsilon) + R_1(\varepsilon) - \lambda_2(B)| \le C_2\omega(\varepsilon). \tag{5.6.42}$$

By the symmetry of the ball, the first nonzero eigenvalue $\lambda_2(B)$ has multiplicity N, and so $\lambda_2(B) = \lambda_3(B) = \cdots = \lambda_{N+1}(B) < \lambda_{N+2}(B)$. Therefore

$$\begin{aligned}
C_2\omega(\varepsilon) &\ge \left|\sum_{n=2}^{N+1} a_n(\varepsilon)^2\lambda_2(B) + \sum_{n=N+2}^{+\infty} a_n(\varepsilon)^2\lambda_n(B) - \lambda_2(B)\right| \\
&= \left|\lambda_2(B)\left(\sum_{n=2}^{+\infty} a_n(\varepsilon)^2 - 1\right) + \sum_{n=N+2}^{+\infty} a_n(\varepsilon)^2\left(\lambda_n(B) - \lambda_2(B)\right)\right| \\
&\ge (\lambda_{N+2}(B) - \lambda_2(B)) \sum_{n=N+2}^{+\infty} a_n(\varepsilon)^2 - \lambda_2(B)C_1\omega(\varepsilon),
\end{aligned}$$

which yields

$$\sum_{n=N+2}^{+\infty} a_n(\varepsilon)^2 \le C_3\omega(\varepsilon), \tag{5.6.43}$$

hence by (5.6.41)

$$\left|\sum_{n=2}^{N+1} a_n(\varepsilon)^2 - 1\right| \le C_4\omega(\varepsilon). \tag{5.6.44}$$

Revisiting (5.6.42), we see

$$
\begin{aligned}
C_2\omega(\varepsilon) &\geq \left|\sum_{n=2}^{N+1} a_n(\varepsilon)^2\lambda_2(B) + \sum_{n=N+2}^{+\infty} a_n(\varepsilon)^2\lambda_n(B) - \lambda_2(B)\right| \\
&= \left|\lambda_2(B)\left(\sum_{n=2}^{N+1} a_n(\varepsilon)^2 - 1\right) + \sum_{n=N+2}^{+\infty} a_n(\varepsilon)^2\lambda_n(B)\right| \\
&\geq \lambda_2(B)\left(\sum_{n=2}^{N+1} a_n(\varepsilon)^2 - 1\right) + \sum_{n=N+2}^{+\infty} a_n(\varepsilon)^2\lambda_n(B),
\end{aligned}
$$

which, together with (5.6.43) and (5.6.44) yields

$$
\sum_{n=N+2}^{+\infty} a_n(\varepsilon)^2\lambda_n(B) \leq C_2\omega(\varepsilon) - \lambda_2(B)\left(\sum_{n=2}^{N+1} a_n(\varepsilon)^2 - 1\right) \leq C_5\omega(\varepsilon). \tag{5.6.45}
$$

Now set $\varphi := \sum_{n=2}^{N+1} a_n(\varepsilon)\xi_n = v_\varepsilon - \sum_{n=N+2}^{\infty} a_n(\varepsilon)\xi_n$ and define the norm $\|\cdot\|_{H^2_\tau(B)}$ by

$$
\|h\|^2_{H^2_\tau(B)} := \int_B |D^2h|^2 + \tau|Dh|^2 + h^2\,dx, \qquad h \in H^2(B).
$$

This norm is equivalent to the standard $H^2(B)$ norm by coercivity of the bilinear form.

We now estimate the quantity $\|v_\varepsilon - \varphi\|_{H^2_\tau(B)}$. We have

$$
\begin{aligned}
\|v_\varepsilon - \varphi\|^2_{H^2_\tau(B)} &= \int_B |D^2(v_\varepsilon - \varphi)|^2 + \tau|D(v_\varepsilon - \varphi)|^2 + (v_\varepsilon - \varphi)^2 dx \\
&= \int_B \sum_{n=N+2}^{+\infty} a_n(\varepsilon)^2(|D^2\xi_n|^2 + \tau|D\xi_n|^2 + \xi_n^2)dx \\
&= \sum_{n=N+2}^{+\infty} a_n(\varepsilon)^2(1 + \lambda_n(B)) \leq C_6\omega(\varepsilon),
\end{aligned} \tag{5.6.46}
$$

where the last inequality follows from (5.6.43) and (5.6.45). Thus the function v_ε is $\sqrt{\omega(\varepsilon)}$-close to φ in the $H^2_\tau(B)$ norm.

Now we pass from the bound on the $H^2_\tau(B)$ norm to the bound on the $C^3(\overline{B})$ norm. To do so, we use standard elliptic regularity estimates for the biharmonic operator. We have that, in $B \cap \Omega_\varepsilon$,

$$
\Delta^2 v_\varepsilon - \tau\Delta v_\varepsilon = \Delta^2 u_\varepsilon - \tau\Delta u_\varepsilon = \lambda_2(\Omega_\varepsilon)u_\varepsilon = \lambda_2(\Omega_\varepsilon)(v_\varepsilon + \delta).
$$

Recall that $\delta \in O(\varepsilon)$ as $\varepsilon \to 0$ by (5.6.31). We set

$$
f_\varepsilon := \Delta^2 v_\varepsilon - \tau\Delta v_\varepsilon. \tag{5.6.47}
$$

Note that in particular, $f_\varepsilon = \lambda_2(\Omega_\varepsilon)(v_\varepsilon + \delta)$ on $B \cap \Omega_\varepsilon$. Then defining the functions $g_\varepsilon^{(1)}$ and $g_\varepsilon^{(2)}$ on ∂B by $g_\varepsilon^{(1)} := \frac{\partial^2 v_\varepsilon}{\partial \nu^2}$ and $g_\varepsilon^{(2)} := \tau \frac{\partial v_\varepsilon}{\partial \nu} - \mathrm{div}_{\partial B}(D^2 v_\varepsilon \cdot \nu) - \frac{\partial \Delta v_\varepsilon}{\partial \nu}$, we see that the function v_ε uniquely solves the following problem:

$$\begin{cases} \Delta^2 u - \tau \Delta u = f_\varepsilon, & \text{in } B,\\ \frac{\partial^2 u}{\partial \nu^2} = g_\varepsilon^{(1)}, & \text{on } \partial B,\\ \tau \frac{\partial u}{\partial \nu} - \mathrm{div}_{\partial B}(D^2 u \cdot \nu) - \frac{\partial \Delta u}{\partial \nu} = g_\varepsilon^{(2)}, & \text{on } \partial B,\\ \int_B u dx = 0. \end{cases}$$

Now let $f := \lambda_2(B)\varphi$. Then by definition the function φ is the unique solution of

$$\begin{cases} \Delta^2 u - \tau \Delta u = f, & \text{in } B,\\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial B,\\ \tau \frac{\partial u}{\partial \nu} - \mathrm{div}_{\partial B}(D^2 u \cdot \nu) - \frac{\partial \Delta u}{\partial \nu} = 0, & \text{on } \partial B,\\ \int_B u\, dx = 0. \end{cases}$$

Finally, define the function $w := v_\varepsilon - \varphi = \sum_{n=N+2}^{\infty} a_n(\varepsilon)\xi_n$, which is the unique solution of

$$\begin{cases} \Delta^2 w - \tau \Delta w = f_\varepsilon - f, & \text{in } B,\\ \frac{\partial^2 w}{\partial \nu^2} = g_\varepsilon^{(1)}, & \text{on } \partial B,\\ \tau \frac{\partial w}{\partial \nu} - \mathrm{div}_{\partial B}(D^2 w \cdot \nu) - \frac{\partial \Delta w}{\partial \nu} = g_\varepsilon^{(2)}, & \text{on } \partial B,\\ \int_B w\, dx = 0. \end{cases}$$

For every $p > N$ we have (see e.g., [47, Theorem 2.20])

$$\|w\|_{W^{4,p}(B)} \le C \left(\|f_\varepsilon - f\|_{L^p(B)} + \|g_\varepsilon^{(1)}\|_{W^{2-\frac{1}{p},p}(\partial B)} + \|g_\varepsilon^{(2)}\|_{W^{1-\frac{1}{p},p}(\partial B)} \right). \tag{5.6.48}$$

We consider separately the three summands in the right-hand side of (5.6.48). We start from the first summand. Recall that for any $x \in B \cap \Omega_\varepsilon$ we have (see (5.6.47)) $f_\varepsilon(x) = \lambda_2(\Omega_\varepsilon)(v_\varepsilon(x) + \delta)$. Moreover, for any $x \in B$ there exists $\tilde{x} \in B \cap \Omega_\varepsilon$ such that $|x - \tilde{x}| \le C\varepsilon$. Therefore, from the regularity of v_ε (see (5.6.33)), for $x \in B$ we have, as $\varepsilon \to 0$,

$$\begin{aligned} f_\varepsilon(x) &= \Delta^2 v_\varepsilon(x) - \tau \Delta v_\varepsilon(x) = \Delta^2 v_\varepsilon(\tilde{x}) - \tau \Delta v_\varepsilon(\tilde{x}) + O(\varepsilon)\\ &= \lambda_2(\Omega_\varepsilon)(v_\varepsilon(\tilde{x}) + \delta) + O(\varepsilon) = \lambda_2(\Omega_\varepsilon) v_\varepsilon(x) + O(\varepsilon). \end{aligned}$$

Thus, as $\varepsilon \to 0$, for all $p > N$, we have by Lemma 5.6.39 and by (5.6.46)

$$\begin{aligned} \|f_\varepsilon - f\|_{L^p(B)} &= \|\lambda_2(\Omega_\varepsilon) v_\varepsilon - \lambda_2(B)\varphi\|_{L^p(B)} + O(\varepsilon) \qquad (5.6.49)\\ &\le |\lambda_2(\Omega_\varepsilon) - \lambda_2(B)| \|v_\varepsilon\|_{L^p(B)} + |\lambda_2(B)| \|v_\varepsilon - \varphi\|_{L^p(B)} + O(\varepsilon)\\ &\le C_7 \omega(\varepsilon) + C_8 \sqrt{\omega(\varepsilon)} + O(\varepsilon) \le C_9 \sqrt{\omega(\varepsilon)}. \end{aligned}$$

Now we consider the second summand in the right-hand side of (5.6.48). From the fact that $g_\varepsilon^{(1)} = \frac{\partial^2 v_\varepsilon}{\partial \nu^2}$, the fact that v_ε is an extension of u_ε, the regularity of both u_ε and v_ε (5.6.29), (5.6.33) and from the fact that $\frac{\partial^2 u_\varepsilon}{\partial \nu^2} = 0$ on $\partial\Omega_\varepsilon$, we may conclude

$$\|g_\varepsilon^{(1)}\|_{W^{2-\frac{1}{p},p}(\partial B)} \leq C\varepsilon. \tag{5.6.50}$$

For the same reason, for the third summand in the right-hand side of (5.6.48) we have

$$\|g_\varepsilon^2\|_{W^{2-\frac{1}{p},p}(\partial B)} \leq C\varepsilon. \tag{5.6.51}$$

From (5.6.48) and the bounds (5.6.49), (5.6.50), and (5.6.51) on each summand, it follows that for all $p > N$,

$$\|v_\varepsilon - \varphi\|_{W^{4,p}(B)} \leq C_{10}\sqrt{\omega(\varepsilon)},$$

and thus, from the Sobolev embedding theorem,

$$\|v_\varepsilon - \varphi\|_{C^3(\overline{B})} \leq \tilde{C}\sqrt{\omega(\varepsilon)}.$$

This concludes the proof of the lemma. □

The next lemma gives us refined bounds on $|R_1(\varepsilon)|$ and $|R_2(\varepsilon)|$.

Lemma 5.6.52. *Let $\omega(t), v_\varepsilon$ be as in Lemma 5.6.39. Suppose that for all $\varepsilon > 0$ small enough there exists an eigenfunction ξ_ε associated with $\lambda_2(B)$ such that*

$$\|v_\varepsilon - \xi_\varepsilon\|_{C^3(\overline{B})} \leq C\sqrt{\omega(\varepsilon)}, \tag{5.6.53}$$

for some $C > 0$ which does not depend on $\varepsilon > 0$. Then there exists $\tilde{C} > 0$ which does not depend on ε such that $|R_1(\varepsilon)|, |R_2(\varepsilon)| \leq \tilde{C}\varepsilon\sqrt{\omega(\varepsilon)}$.

Proof. It is convenient to use spherical coordinates $(r,\theta) \in \mathbb{R}_+ \times \partial B$ in $\mathbb{R}^N$ and the corresponding change of variables $x = \phi(r,\theta)$. We denote by $\mathcal{D}$ and $\tilde{\mathcal{D}}$ the sets $\mathcal{D} := \partial(\Omega_\varepsilon \setminus B) \cap \partial B$ and $\tilde{\mathcal{D}} = \partial(B \setminus \Omega_\varepsilon) \cap \partial B$. Observe $\psi \geq 0$ on $\mathcal{D}$ and $\psi \leq 0$ on $\tilde{\mathcal{D}}$.

Thanks to the regularity of u_ε and $\tilde{u}_\varepsilon$ (see (5.6.30)), on $\Omega_\varepsilon \setminus B$ we have

$$\begin{aligned} D^2u_\varepsilon \circ \phi(1+\varepsilon\psi,\theta) &= D^2u_\varepsilon \circ \phi(1,\theta) + O(\varepsilon), \\ Du_\varepsilon \circ \phi(1+\varepsilon\psi,\theta) &= Du_\varepsilon \circ \phi(1,\theta) + O(\varepsilon), \end{aligned}$$

as $\varepsilon \to 0$. Therefore, integrating with respect to the radius r and applying our definition of v_ε (5.6.32), we see

$$\begin{aligned} \int_{\Omega_\varepsilon\setminus B} |D^2u_\varepsilon|^2 + \tau|Du_\varepsilon|^2 dx &= \varepsilon\int_{\mathcal{D}} \left(\left|D^2u_\varepsilon\right|^2 + \tau\left|Du_\varepsilon\right|^2\right)\psi d\sigma + O(\varepsilon^2) \\ &= \varepsilon\int_{\mathcal{D}} \left(\left|D^2v_\varepsilon\right|^2 + \tau\left|Dv_\varepsilon\right|^2\right)\psi d\sigma + O(\varepsilon^2), \end{aligned}$$

as $\varepsilon \to 0$. Similarly,

$$\int_{B\setminus\Omega_\varepsilon} |D^2 v_\varepsilon|^2 + \tau |Dv_\varepsilon|^2 dx = -\varepsilon \int_{\tilde{\mathcal{D}}} \left(\left|D^2 v_\varepsilon\right|^2 + \tau \left|Dv_\varepsilon\right|^2 \right) \psi d\sigma + O(\varepsilon^2),$$

as $\varepsilon \to 0$. From these and from hypothesis (5.6.53), we see

$$\begin{aligned} |R_1(\varepsilon)| &\le \varepsilon \left| \int_{\partial B} \psi \left(\left|D^2 v_\varepsilon\right|^2 + \tau \left|Dv_\varepsilon\right|^2 \right) d\sigma \right| + O(\varepsilon^2) \\ &\le \varepsilon \left| \int_{\partial B} \psi \left(\left|D^2 \xi_\varepsilon\right|^2 + \tau \left|D\xi_\varepsilon\right|^2 \right) d\sigma \right| + C\varepsilon\sqrt{\omega(\varepsilon)} + O(\varepsilon^2) \\ &\le \tilde{C}\varepsilon\sqrt{\omega(\varepsilon)}, \quad (5.6.54) \end{aligned}$$

as $\varepsilon \to 0$, where in the last inequality we have used the following fact of eigenfunctions of $\lambda_2(B)$:

$$\left(\left|D^2 \xi_\varepsilon\right|^2 + \tau \left|D\xi_\varepsilon\right|^2 \right)\Big|_{\partial B} = (a \cdot x)^2, \quad (5.6.55)$$

for some $a \in \mathbb{R}^N$ (cf. (5.6.4)).

By following the very same scheme we can prove the analogue of (5.6.54) for $R_2(\varepsilon)$. This concludes the proof of the lemma. □

We can now proceed to complete the proof of Theorem 5.6.2.

Let $\omega_0(\varepsilon) := |R_1(\varepsilon)| + |R_2(\varepsilon)|$. This function is continuous in ε and, moreover, has the property

$$\frac{\varepsilon^2}{K} \le \omega_0(\varepsilon) \le K\varepsilon.$$

The first inequality follows from Theorem 5.6.6, while the second follows from the fact that $R_1, R_2 \in O(\varepsilon)$. By Lemma 5.6.40, it follows that there exists an eigenfunction ξ_ε of the Neumann problem (5.6.1) on B associated with eigenvalue $\lambda_2(B)$ such that

$$\|v_\varepsilon - \xi_\varepsilon\|_{C^3(\overline{B})} \le C\sqrt{\omega_0(\varepsilon)}.$$

Now we apply Lemma 5.6.52, obtaining

$$\omega_0(\varepsilon) \le 2\tilde{C}\varepsilon\sqrt{\omega_0(\varepsilon)},$$

and therefore

$$\sqrt{\omega_0(\varepsilon)} = \frac{|R_1(\varepsilon)| + |R_2(\varepsilon)|}{\sqrt{\omega_0(\varepsilon)}} \le 2\tilde{C}\varepsilon.$$

From this it follows that $\omega_0(\varepsilon) \le 4\tilde{C}^2\varepsilon^2$, and hence both $|R_1(\varepsilon)|, |R_2(\varepsilon)| \le 4\tilde{C}^2\varepsilon^2$. Finally, we apply Lemma 5.6.39 and obtain that

$$\lambda_2(B) \le \lambda_2(\Omega_\varepsilon) + \mathcal{C}\varepsilon^2,$$

for a constant $\mathcal{C} > 0$ which does not depend on $\varepsilon \in]0, \varepsilon_0[$. This concludes the proof of Theorem 5.6.22.

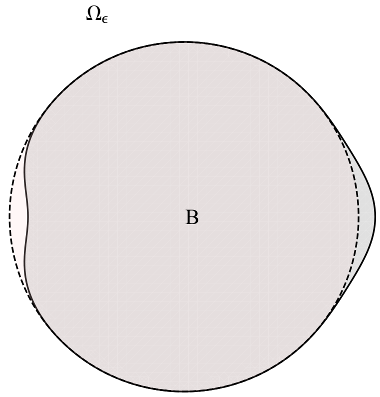


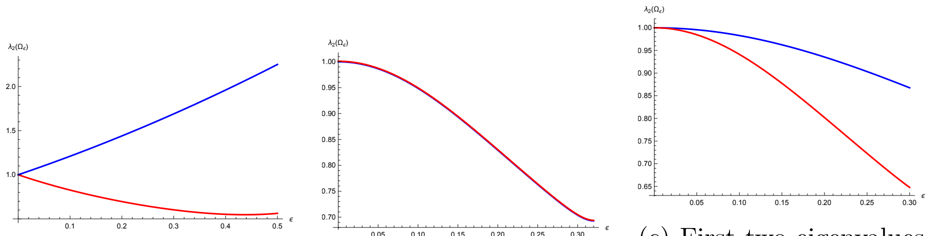
Figure 5.3: The domain Ω_ε introduced in [17].

Remark 5.6.56. *In [17], the authors provided an explicit construction of a family* $\{\Omega_\varepsilon\}_\varepsilon$ *in* $\mathbb{R}^2$ *suitable to show the sharpness of their inequality (see Figure 5.3). This family turns out to be suitable to show also the sharpness of inequalities* (5.5.28) *and* (5.6.7)*. On the other hand, in [16] the authors gave only sufficient conditions to generate a suitable family* $\{\Omega_\varepsilon\}_\varepsilon$*, which are exactly those we apply in* (5.6.27)*. We observe that the first two conditions, namely*

$$\int_{\partial B} \psi d\sigma = \int_{\partial B} a \cdot x\psi d\sigma = 0, \tag{5.6.57}$$

have a purely geometrical meaning, and are used to prove inequalities (5.6.23) *and* (5.6.24) *(cf. [16, Lemma 6.2]). In particular, the first one says that the measure of* Ω_ε *is the same as the measure of* B *up to an error of order* ε^2*. The second says that the baricenter of* Ω_ε *is the origin up to an error of order* ε^2*. This implies in particular that* $\mathcal{A}(\Omega_\varepsilon)$ *is of order* ε*. The third condition has instead a stricter relation with the problem, since any function* ξ *belonging to the eigenspace associated with* $\lambda_2(B)$ *satisfies equality* (5.6.55)*, which is crucial in the proof of* (5.6.25)*. This is due to the fact that it can be factorized as a radial part times a spherical harmonic polynomial of degree* 1*. This also tells us that the correct conditions to impose are still* (5.6.27) *when considering the Steklov problem. In particular, as pointed out in [16, Remark 6.9], ellipsoids satisfy conditions* (5.6.57)*, and hence inequalities* (5.6.23) *and* (5.6.24) *hold, but miss the final condition, and therefore are not a suitable family for this problem.*

Remark 5.6.58. *We note that* $\lambda_2(B)$ *has multiplicity* N*. Thus* $\lambda_2(\Omega)$ *is not differentiable at* $\Omega = B$*. This implies that along some directions* $\lambda_2(\Omega)$ *could have a non-trivial super-differential. This is exactly what happens when we consider the eigenvalues of nearly spherical ellipsoids (see Figure 5.4a). In order to show that this does not happen for every direction, we build a*



(a) First two eigenvalues of the ellipsoids $\frac{x^2}{1+\varepsilon^2} + (1+\varepsilon^2)y^2 = 1$ in $\mathbb{R}^2$.

(b) First (double) eigenvalue of the domains $\Omega_\varepsilon \subset \mathbb{R}^2$ in Figure 5.2.

(c) First two eigenvalues of the domains $\Omega_\varepsilon \subset \mathbb{R}^2$ introduced in [17] (see also Figure 5.3).

family Ω_ε apporaching the ball such that the multiplicity of the eigenvalue is preserved in a neighborhood of the ball. This is sufficient to ensure that along such direction the eigenvalue is differentiable and therefore it converges with the sharp exponent 2 *(see Figure 5.4b). As already discussed, the conditions given in* (5.6.27) *are sufficient to achieve the exponent* 2 *for the Fraenkel asymmetry. The family of domains Ω_ε introduced in [17] (see Figure 5.3) does not satisfy the second condition in* (5.6.27) *for all $a \in \mathbb{R}^N$, but the eigenvalues converge with the sharp exponent* 2*. We also note that along this direction the eigenvalue $\lambda_2(\Omega)$ is not differentiable (see Figure 5.4c).*

5.6.3 Sharpness of the Steklov inequality

In this subsection we prove the following result, that tells us that inequality (5.5.28) is sharp. Due to the strong similarities between the two problems, we shall maintain the same notation as in the previous subsection.

Theorem 5.6.59. *Let B be the unit ball in $\mathbb{R}^N$ centered at zero. There exist a family $\{\Omega_\varepsilon\}_{\varepsilon>0}$ of smooth domains and positive constants c_1, c_2, c_3, c_4 and r_1, r_2, r_3, r_4, which do not depend on $\varepsilon > 0$ such that* (5.6.23), (5.6.24) *and* (5.6.25) *hold for all $\varepsilon \in]0, \varepsilon_0[$, where $\lambda_2(\Omega)$ denotes the first positive eigenvalue of problem* (5.0.1) *with $\rho \equiv 1$.*

To prove this theorem, we begin by defining the family $\{\Omega_\varepsilon\}_{\varepsilon>0}$ as in (5.6.26). Thus it remains only to prove (5.6.25).

We recall the variational characterization of the first positive eigenvalue of the Steklov problem (5.0.1) with $\rho \equiv 1$ on a domain Ω:

$$\lambda_2(\Omega) = \inf_{\substack{0 \neq u \in H^2(\Omega) \\ \int_{\partial\Omega} u \, d\sigma = 0}} \frac{\int_\Omega |D^2 u|^2 + \tau |Du|^2 \, dx}{\int_{\partial\Omega} u^2 \, d\sigma}. \tag{5.6.60}$$

Let $\lambda_2(\Omega_\varepsilon)$ be the first positive eigenvalue of the Steklov problem (5.0.1) on Ω_ε, and let u_ε be an associated eigenfunction, normalized by

$$\int_{\partial\Omega_\varepsilon} u_\varepsilon^2 dx = 1.$$

Then by (5.6.60),

$$\int_{\Omega_\varepsilon} |D^2 u_\varepsilon|^2 + \tau |\nabla u_\varepsilon|^2 dx = \lambda_2(\Omega_\varepsilon).$$

By standard elliptic regularity (see e.g., [47, §2.4.3]), since Ω_ε is of class C^∞ by construction, we have that $u_\varepsilon \in C^\infty(\overline{\Omega_\varepsilon})$ for all $\varepsilon \in]0, \varepsilon_0[$. Moreover, for all $k \in \mathbb{N}$, the sets Ω_ε are of class C^k uniformly in $\varepsilon \in]0, \varepsilon_0[$, which means that there exists a constant $H_k > 0$ independent of ε such that

$$\|u_\varepsilon\|_{C^k(\overline{\Omega_\varepsilon})} \leq H_k.$$

Let now $\tilde{u}_\varepsilon$ be a C^5 extension of u_ε to an open neighborhood A of $B \cup \Omega_\varepsilon$. Then, there exists $K_A > 0$ which does not depend on $\varepsilon > 0$ such that

$$\|\tilde{u}_\varepsilon\|_{C^5(\overline{A})} \leq K_A \|u_\varepsilon\|_{C^5(\overline{\Omega_\varepsilon})} \leq K_A H_5.$$

As in the Neumann case, take $\delta := \frac{1}{|\partial B|} \int_{\partial B} \tilde{u}_\varepsilon \, d\sigma$ to be the mean of $\tilde{u}_\varepsilon$ over ∂B. From the fact that $\int_{\partial \Omega_\varepsilon} u_\varepsilon dx = 0$ and $|B \setminus \Omega_\varepsilon|, |\Omega_\varepsilon \setminus B| \in O(\varepsilon)$ as $\varepsilon \to 0$, it follows that as as $\varepsilon \to 0$,

$$\delta = \frac{1}{|\partial B|} \int_{\partial B} \tilde{u}_\varepsilon \, d\sigma \in O(\varepsilon).$$

(See also [16, formula (6.15)]). Now let us set

$$v_\varepsilon := \tilde{u}_{\varepsilon|_B} - \delta \tag{5.6.61}$$

This function is of class $C^5(\overline{B})$ and satisfies $\int_{\partial B} v_\varepsilon \, d\sigma = 0$ and

$$\|v_\varepsilon\|_{C^5(B)} \leq K_1$$

for a constant $K_1 > 0$ independent of $\varepsilon \in]0, \varepsilon_0[$. Therefore, v_ε is a suitable trial function for the Rayleigh quotient of $\lambda_2(B)$, and so

$$\lambda_2(B) \leq \frac{\int_B |D^2 v_\varepsilon|^2 + \tau |\nabla v_\varepsilon|^2 dx}{\int_{\partial B} {v_\varepsilon}^2 \, d\sigma}.$$

On the other hand,

$$\left| \int_{\partial B} v_\varepsilon^2 - \tilde{u}_\varepsilon^2 \, d\sigma \right| = \left| \int_{\partial B} \delta^2 - 2\delta \tilde{u}_\varepsilon \, d\sigma \right| \leq K_2 \varepsilon^2,$$

where $K_2 > 0$ is a positive constant independent of $\varepsilon \in]0, \varepsilon_0[$. Therefore, we may write

$$\lambda_2(B) \leq \frac{\lambda_2(\Omega_\varepsilon) + R_1(\varepsilon)}{1 + R_2(\varepsilon) - K_3 \varepsilon^2},$$

where we have once again defined the error terms

$$R_1(\varepsilon) := \int_{B\setminus\Omega_\varepsilon} |D^2 v_\varepsilon|^2 + \tau|\nabla v_\varepsilon|^2 dx - \int_{\Omega_\varepsilon\setminus B} |D^2 u_\varepsilon|^2 + \tau|\nabla u_\varepsilon|^2 dx,$$

and

$$R_2(\varepsilon) := \int_{\partial B} v_\varepsilon^2 \, d\sigma - \int_{\partial\Omega_\varepsilon} u_\varepsilon^2 \, d\sigma.$$

At this point, we note that the observations made in Subsection 5.6.2 remain valid here. Therefore, in order to conclude the proof of (5.6.25), we need only the following lemmas, which are the analogues of Lemmas 5.6.39, 5.6.40 and 5.6.52 and which we recall here for the reader's convenience.

Lemma 5.6.62. *Let K_3 be as in* (5.6.3)*. Let $\omega : [0,1] \to [0,+\infty[$ be a continuous function such that $\frac{t^2}{K_3} \le \omega(t) \le K_3 t$. If there exists a constant $C > 0$ such that $|R_1(\varepsilon)| \le C\omega(\varepsilon)$ and $|R_2(\varepsilon)| \le C\omega(\varepsilon)$, then there exists a constant $C' > 0$ such that*

$$\lambda_2(B) \le \lambda_2(\Omega_\varepsilon) + C'\omega(\varepsilon)$$

for every sufficient small $\varepsilon > 0$.

Proof. See [16, Lemma 6.7]. □

Lemma 5.6.63. *Let ω be a function as in Lemma 5.6.62, and let v_ε be as in* (5.6.61)*. Suppose that there exists $C > 0$ such that for all $\varepsilon > 0$ sufficiently small we have $|R_1(\varepsilon)| \le C\omega(\varepsilon)$ and $|R_2(\varepsilon)| \le C\omega(\varepsilon)$. Then there exists an eigenfunction ξ_ε associated with $\lambda_2(B)$ such that*

$$\|v_\varepsilon - \xi_\varepsilon\|_{C^3(\overline{B})} \le \tilde{C}\sqrt{\omega(\varepsilon)},$$

for some $\tilde{C} > 0$ which does not depend on $\varepsilon > 0$.

Proof. The proof is essentially identical to that of Lemma 5.6.40 and hence the details are omitted. □

Lemma 5.6.64. *Let $\omega(t), v_\varepsilon$ be as in Lemma 5.6.62. Suppose that for all $\varepsilon > 0$ small enough there exists an eigenfunction ξ_ε associated with $\lambda_2(B)$ such that*

$$\|v_\varepsilon - \xi_\varepsilon\|_{C^3(\overline{B})} \le C\sqrt{\omega(\varepsilon)},$$

for some $C > 0$ which does not depend on $\varepsilon > 0$. Then there exists $\tilde{C} > 0$ which does not depend on ε such that $|R_1(\varepsilon)|, |R_2(\varepsilon)| \le \tilde{C}\varepsilon\sqrt{\omega(\varepsilon)}$.

Proof. Regarding the bound on R_1, we refer to the proof of Lemma 5.6.52. For R_2, we refer to [16, Lemma 6.8], observing that if ξ_ε is an eigenfunction associated with $\lambda_2(B)$, then

$$\mathrm{div}_{\partial B}(D^2\xi_\varepsilon \cdot \nu) + \frac{\partial \Delta \xi_\varepsilon}{\partial \nu} = 0$$

on ∂B, and therefore the second boundary condition in (5.0.1) reads $\tau \frac{\partial \xi_\varepsilon}{\partial \nu} = \lambda_2(B)\xi_\varepsilon$. □

Chapter 6

A few properties of the eigenvalues of Neumann-type problems

In this chapter we collect a number of results concerning the eigenvalues of Neumann-type problems for the Laplace and biharmonic operators. This chapter is divided in two sections where we consider two different problems. In Section 6.1 we discuss the eigenvalue problem for the biharmonic operator subject to Neumann-type boundary conditions, with particular attention to the dependence of the eigenvalues upon variations of the Poisson's ratio. In Section 6.2 we consider the eigenvalue problem for the Laplace and biharmonic operators subject to Neumann boundary conditions on annuli, and describe the behavior of the eigenvalues as the difference of the two radii goes to zero. In particular, we obtain a counterexample to the domain monotonicity of all the Neumann eigenvalues.

6.1 Neumann eigenvalues of the biharmonic operator

In this section we study the dependence of the eigenvalues of the biharmonic operator subject to Neumann and Dirichlet boundary conditions upon perturbations of the Poisson's ratio σ. In particular, we prove that the Neumann eigenvalues are Lipschitz continuous with respect to $\sigma \in [0,1[$ and that all the Neumann eigenvalues go to zero as $\sigma \to 1^-$. Moreover, we show that the Neumann problem defined by setting $\sigma = 1$ admits a sequence of positive eigenvalues of finite multiplicity which are not limiting points of Neumann eigenvalues with $\sigma \in [0,1[$ as $\sigma \to 1^-$, and which coincide with the Dirichlet eigenvalues of the Biharmonic operator.

Let Ω be a bounded domain in $\mathbb{R}^N$ of class $C^{4,\alpha}$ for some $\alpha \in]0,1[$. Let

$\sigma \in [0, 1[$. We consider the following Neumann problem for the biharmonic operator with Poisson's ratio $\sigma \in [0, 1[$

$$\begin{cases} \Delta^2 u = \lambda u, & \text{in } \Omega, \\ (1-\sigma)\frac{\partial^2 u}{\partial \nu^2} + \sigma \Delta u = 0, & \text{on } \partial\Omega, \\ \frac{\partial \Delta u}{\partial \nu} + (1-\sigma)\mathrm{div}_{\partial\Omega}\left(D^2 u \cdot \nu\right)_{\partial\Omega} = 0, & \text{on } \partial\Omega, \end{cases} \tag{6.1.1}$$

in the unknowns u (the eigenfunction) and λ (the eigenvalue). For $N = 2$ this problem is related to the study of the transverse vibrations of a thin plate with a free edge and which occupies at rest a planar region of shape $\Omega \subset \mathbb{R}^2$. The coefficient σ represents the Poisson's ratio of the material the plate is made of. We refer e.g., to [31] for more details on the physical interpretation of problem (6.1.1) and on the Poisson's ratio σ. We also mention the paper [39], where the author studies the dependence of the vibrational modes of a plate subject to homogeneous boundary conditions upon the Poisson's ratio $\sigma \in \left]0, \frac{1}{2}\right[$, providing also a perturbation formula for the frequencies as functions of the Poisson's coefficient.

We recall that problem (6.1.1) admits an infinite sequence of non-negative eigenvalues of finite multiplicity which depend on $\sigma \in [0, 1[$ and which we denote here by

$$0 = \lambda_1(\sigma) = \lambda_2(\sigma) = \cdots = \lambda_{N+1}(\sigma) < \lambda_{N+2}(\sigma) \le \cdots \le \lambda_j(\sigma) \le \cdots .$$

We note that $\lambda = 0$ is an eigenvalue of (6.1.1) of multiplicity $N + 1$, and a set of linearly independent eigenfunctions associated with $\lambda = 0$ is given by $\{1, x_1, ..., x_N\}$.

If we set $\sigma = 1$, problem (6.1.1) reads

$$\begin{cases} \Delta^2 u = \lambda u, & \text{in } \Omega, \\ \Delta u = 0, & \text{on } \partial\Omega, \\ \frac{\partial \Delta u}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases} \tag{6.1.2}$$

We note that the differential operator associated with problem (6.1.2) is not a Fredholm operator. Indeed all the harmonic functions in Ω are eigenfunctions corresponding to the eigenvalue $\lambda = 0$, hence the kernel of the associated operator is infinite dimensional. We also note that the boundary conditions in (6.1.2) do not satisfy the so-called 'complementing conditions' (see [2, §10] and [47] for details), which are necessary conditions for the well-posedness of a differential problem. Nevertheless, problem (6.1.2) admits a countable number of positive eigenvalues of finite multiplicity diverging to $+\infty$, which we denote here by

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots ,$$

see Theorem 6.1.11 below.
We show that $\lambda_j(\sigma) \to 0$ as $\sigma \to 1^-$ for all $j \in \mathbb{N}$. Thus, the positive

eigenvalues of problem (6.1.2) are not limiting points for the eigenvalues of problem (6.1.1) as $\sigma \to 1^-$. Moreover, we show that the positive eigenvalues λ_j of problem (6.1.2) coincide with the eigenvalues of the Dirichlet problem for the biharmonic operator, namely problem

$$\begin{cases} \Delta^2 w = \mu w, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega, \\ \frac{\partial w}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases} \tag{6.1.3}$$

We recall that the eigenvalues of (6.1.3) form an increasing sequence of positive eigenvalues of finite multiplicity diverging to $+\infty$, which we denote here by

$$0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_j \leq \cdots . \tag{6.1.4}$$

6.1.1 Eigenvalues of Neumann and Dirichlet problems

We consider problems (6.1.1), (6.1.2) and (6.1.3) in their weak formulation. The weak formulation of problem (6.1.1) when $\sigma \in [0,1[$ is

$$\int_\Omega (1-\sigma) D^2 u : D^2 \varphi + \sigma \Delta u \Delta \varphi dx = \lambda \int_\Omega u \varphi dx \,, \tag{6.1.5}$$

for all $\varphi \in H^2(\Omega)$, in the unknowns $u \in H^2(\Omega)$, $\lambda \in \mathbb{R}$, where we recall that $D^2 u : D^2 \varphi = \sum_{i,j=1}^N \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 \varphi}{\partial x_i \partial x_j}$ denotes the Frobenius product.
Actually we will recast problem (6.1.5) in $H^2(\Omega)/N$, where $N \subset H^2(\Omega)$ is the subspace of $H^2(\Omega)$ generated by the functions $\{1, x_1, ..., x_N\}$. To do so, we set

$$H^2_N(\Omega) := \left\{ u \in H^2(\Omega) \ : \ \int_\Omega u dx = \int_\Omega \frac{\partial u}{\partial x_i} dx = 0 \,, \forall i = 1, ..., N \right\}.$$

In the sequel we will think of the space $H^2_N(\Omega)$ as endowed with the bilinear form given by the left-hand side of (6.1.5). From the fact that $|D^2 u|^2 \geq \frac{1}{N} (\Delta u)^2$ for all $u \in H^2(\Omega)$ and from the Poincaré-Wirtinger inequality, it follows that such bilinear form defines on $H^2_N(\Omega)$ a scalar product whose induced norm is equivalent to the standard one. We denote by π_N the map from $H^2(\Omega)$ to $H^2_N(\Omega)$ defined by

$$\pi_N[u] := u - \frac{1}{|\Omega|} \int_\Omega u + \frac{1}{|\Omega|^2} \sum_{i=1}^N \left(\int_\Omega \frac{\partial u}{\partial x_i} dx \right) \left(\int_\Omega x_i dx \right) - \frac{1}{|\Omega|} \sum_{i=1}^N \left(\int_\Omega \frac{\partial u}{\partial x_i} dx \right) x_i,$$

for all $u \in H^2(\Omega)$. We denote by $\pi_N^\sharp$ the map from $H^2(\Omega)/N$ onto $H^2_N(\Omega)$ defined by the equality $\pi_N = \pi_N^\sharp \circ p$, where p is the canonical projection of

$H^2(\Omega)$ onto $H^2(\Omega)/N$. The map $\pi_N^\sharp$ turns out to be a homeomorphism. Let $F(\Omega)$ be defined by $F(\Omega) := \{G \in H^2(\Omega)' : G[1] = G[x_i] = 0\,,\ \forall i = 1,..,N\}$. Then we consider the operator $\mathcal{P}_\sigma$ as an operator from $H^2_N(\Omega)$ to $F(\Omega)$ defined by

$$\mathcal{P}_\sigma[u][\varphi] := \int_\Omega (1-\sigma)D^2u : D^2\varphi + \sigma\Delta u\Delta\varphi dx\,, \quad \forall u \in H^2_N(\Omega), \varphi \in H^2(\Omega).$$

It turns out that $\mathcal{P}_\sigma$ is a homeomorphism of $H^2_N(\Omega)$ onto $F(\Omega)$. We denote by $\mathcal{J}$ the continuous embedding of $L^2(\Omega)$ into $H^2(\Omega)'$ defined by

$$\mathcal{J}[u][\varphi] := \int_\Omega u\varphi dx\,, \quad \forall u \in L^2(\Omega), \varphi \in H^2(\Omega).$$

Finally, we define the operator T_σ acting on $H^2(\Omega)/\mathrm{N}$ as follows:

$$T_\sigma = (\pi_N^\sharp)^{(-1)} \circ \mathcal{P}_\sigma^{(-1)} \circ \mathcal{J} \circ i \circ \pi_N^\sharp,$$

where i denotes the embedding of $H^2(\Omega)$ into $L^2(\Omega)$.

Lemma 6.1.6. *The pair (λ, u) of the set $(\mathbb{R} \setminus \{0\}) \times (H^2_N(\Omega) \setminus \{0\})$ satisfies (6.1.5) if and only if $\lambda > 0$ and the pair $(\lambda^{-1}, p[u])$ of the set $\mathbb{R} \times (H^2(\Omega)/N \setminus \{0\})$ satisfies the equation*

$$\lambda^{-1}p[u] = T_\sigma p[u].$$

We have the following theorem.

Theorem 6.1.7. *The operator T_σ is a non-negative compact self-adjoint operator in $H^2(\Omega)/N$, whose eigenvalues coincide with the reciprocals of the positive eigenvalues of problem (6.1.5). In particular, the set of eigenvalues of problem (6.1.5) is contained in $[0, +\infty[$ and consists of the image of a sequence increasing to $+\infty$ and each eigenvalue has finite multiplicity. Moreover, the first eigenvalue is $\lambda = 0$ and has multiplicity $N + 1$, and a set of linearly independent eigenfunctions corresponding to $\lambda = 0$ is given by $\{1, x_1, ..., x_N\}$.*

Proof. It is easy to prove that the operator T_σ is self-adjoint. The compactness of the operator T_σ follows from the compactness of the embedding i. The last statement is straightforward. □

In an analogous way it is possible to show that the eigenvalues of (6.1.3) are positive and of finite multiplicity. In fact, the weak formulation of problem (6.1.3) reads: find $(u, \lambda) \in H^2_0(\Omega) \times \mathbb{R}$ such that u solves equation $\int_\Omega \Delta u\Delta\varphi dx = \lambda \int_\Omega u\varphi dx$ for all $\varphi \in H^2_0(\Omega)$. We note that this this is equivalent to find $(u, \lambda) \in H^2_0(\Omega) \times \mathbb{R}$ such that equation (6.1.5) holds for all $\varphi \in H^2_0(\Omega)$. From the Poincaré inequality it follows that the bilinear form given by the right-hand side of (6.1.5) defines on $H_0^2(\Omega)$ a scalar product whose induced norm is equivalent to the standard one. Therefore the analogue of Theorem 6.1.7 holds. Thus, the eigenvalues of problem (6.1.3) are positive and can be represented by means of an infinite sequence diverging to $+\infty$ of the form (6.1.4), and the corresponding eigenfunctions form a orthonormal basis of $H_0^2(\Omega)$.

Finally, we show that problem (6.1.2) admits an infinite sequence of positive eigenvalues. We have already observed that all harmonic functions in $H^2(\Omega)$ are eigenfunctions corresponding to the eigenvalue $\lambda = 0$. We start by recalling the following direct decomposition of the space $H^2(\Omega)$ via the biharmonic operator (see [15, Theorem 4.7] for details):

$$H^2(\Omega) = H_h^2(\Omega) \oplus \Delta(H^4(\Omega) \cap H_0^2(\Omega)), \tag{6.1.8}$$

where $H_h^2(\Omega) := \{u \in H^2(\Omega) : \Delta u = 0\}$ is the space of harmonic functions in $H^2(\Omega)$. The fact that the sum is direct follows since if $f \in H_h^2(\Omega)$ and $u \in H^4(\Omega) \cap H_0^2(\Omega)$ with $\Delta u = f$, then

$$\begin{aligned}\int_\Omega (\Delta u)^2 dx &= \int_\Omega \Delta u f dx \\ &= \int_\Omega \Delta f u dx + \int_{\partial\Omega} f \frac{\partial u}{\partial \nu} d\sigma - \int_{\partial\Omega} \frac{\partial f}{\partial \nu} u d\sigma \\ &= \int_\Omega \Delta f u dx = 0,\end{aligned}$$

thus $\Delta u = 0$ and then $f = 0$.

In order to characterize the positive eigenvalues of problem (6.1.2) and to get rid of the harmonic functions which are the eigenfunctions associated with the eigenvalue $\lambda = 0$, we will obtain a problem in $\Delta(H^4(\Omega) \cap H_0^2(\Omega))$. Thus, we consider the following weak formulation of problem (6.1.2) for $\lambda \neq 0$. (In the case $\lambda = 0$, the solution of (6.1.2) are exactly the harmonic functions in $H^2(\Omega)$.)

$$\int_\Omega \Delta^2 u \Delta^2 \varphi dx = \lambda \int_\Omega \Delta u \Delta \varphi, \quad \forall u, \varphi \in H^4(\Omega) \cap H_0^2(\Omega), \tag{6.1.9}$$

in the unknowns $u \in H^4(\Omega) \cap H_0^2(\Omega)$, $\lambda \in \mathbb{R}$. In fact, assume we have a solution $u \in H^4(\Omega)$ to (6.1.2). For any $\varphi \in H^2(\Omega)$ we have

$$\begin{aligned}\lambda \int_\Omega u\varphi dx &= \int_\Omega \Delta^2 u \varphi dx \\ &= \int_\Omega \Delta u \Delta \varphi dx + \int_{\partial\Omega} \frac{\partial \Delta u}{\partial \nu} \varphi d\sigma - \int_{\partial\Omega} \Delta u \frac{\partial \varphi}{\partial \nu} d\sigma \\ &= \int_\Omega \Delta u \Delta \varphi dx.\end{aligned}$$

Thus

$$\int_\Omega \Delta u \Delta \varphi dx = \lambda \int_\Omega u\varphi dx. \tag{6.1.10}$$

By (6.1.8) we have that

$$\varphi = \varphi_h + \Delta\psi,$$

where $\varphi_h \in H^2(\Omega)$ is harmonic and $\psi \in H^4(\Omega) \cap H^2_0(\Omega)$. Thus

$$\int_\Omega \Delta u \left(\Delta\varphi_h + \Delta^2\psi\right) dx = \lambda \int_\Omega u \left(\varphi_h + \Delta\psi\right).$$

Now if $\lambda \neq 0$, then by (6.1.10) we have

$$0 = \int_\Omega \Delta u \Delta\varphi_h dx = \lambda \int_\Omega u\varphi_h,$$

then $\int_\Omega u\varphi_h dx = 0$. Hence

$$\int_\Omega \Delta u \Delta^2\psi dx = \lambda \int_\Omega u\Delta\psi dx.$$

Similarly, $u = u_h + \Delta v$, where $u_h \in H^2(\Omega)$ is harmonic and $v \in H^4(\Omega) \cap H^2_0(\Omega)$. Thus

$$\begin{aligned}\int_\Omega \Delta\left(u_h + \Delta v\right)\Delta^2\psi dx &= \lambda \int_\Omega \left(u_h + \Delta v\right)\Delta\psi dx \\ &= \lambda\int_\Omega \Delta v\Delta\psi dx + \lambda\int_\Omega \Delta u_h \psi dx + \int_{\partial\Omega} u_h \frac{\partial\psi}{\partial\nu} d\sigma - \int_{\partial\Omega} \frac{\partial u_h}{\partial\nu}\psi d\sigma \\ &= \lambda\int_\Omega \Delta v\Delta\psi dx.\end{aligned}$$

We note that there exists a constant $C > 0$ such that $\int_\Omega \Delta^2 u \Delta^2\varphi dx \leq C\|u\|_{H^4(\Omega)}\|\varphi\|_{H^4(\Omega)}$ and $\|u\|_{H^4(\Omega)} \leq C\|\Delta^2 u\|_{L^2(\Omega)}$ for all $u, \varphi \in H^4(\Omega) \cap H^2_0(\Omega)$ (the second inequality follows from standard elliptic regularity for the Dirichlet problem for the biharmonic operator and from the regularity assumptions of Ω, see [47, Theorem 2.20] for details). Therefore, the bilinear form given by the left-hand side of (6.1.9) defines on $H^4(\Omega)\cap H^2_0(\Omega)$ a scalar product whose induced norm is equivalent to the standard norm of $H^4(\Omega)$. Thus, the analogue of Theorem 6.1.7 holds.

Theorem 6.1.11. *The set of eigenvalues of problem* (6.1.2) *is contained in* $[0, +\infty[$. *The eigenspace corresponding to the eigenvalue* $\lambda = 0$ *has infinite dimension and all harmonic functions in* $H^2(\Omega)$ *are eigenfunctions associated with* $\lambda = 0$. *Moreover, the set of positive eigenvalues consists of the image of a sequence increasing to* $+\infty$. *Each positive eigenvalue has finite multiplicity and the corresponding eigenfunctions form a orthonormal basis of* $H^4(\Omega) \cap H^2_0(\Omega)$.

6.1.2 Dependence of the Neumann eigenvalues upon the Poisson's ratio

In the first part of this subsection we consider the behavior of the eigenvalues of problem (6.1.1) as $\sigma \to 1^-$. In the second part, we show that the positive eigenvalues of problem (6.1.2) and the eigenvalues of problem (6.1.3) coincide. We start by proving the following lemma.

Lemma 6.1.12. *The function which maps* $\sigma \in [0,1[$ *to* $\lambda_j(\sigma)$ *is Lipschitz continuous.*

Proof. We recall that for each $\sigma \in [0,1[$ we have the following formula for $\lambda_j(\sigma)$

$$\lambda_j(\sigma) = \inf_{\substack{E \leq H^2(\Omega) \\ \dim E = j}} \sup_{0 \neq u \in E} \frac{\int_\Omega (1-\sigma)|D^2u|^2 + \sigma(\Delta u)^2 dx}{\int_\Omega u^2 dx}. \tag{6.1.13}$$

Therefore, for each $\sigma_1, \sigma_2 \in [0,1[$ and $u \in H^2(\Omega)$ we have

$$\begin{aligned} \left| \frac{\int_\Omega (1-\sigma_1)|D^2u|^2 + \sigma_1(\Delta u)^2 dx}{\int_\Omega u^2 dx} - \frac{\int_\Omega (1-\sigma_2)|D^2u|^2 + \sigma_2(\Delta u)^2 dx}{\int_\Omega u^2 dx} \right| \\ \leq |\sigma_1 - \sigma_2| \frac{\int_\Omega |D^2u|^2 + (\Delta u)^2 dx}{\int_\Omega u^2 dx} \\ \leq (1+N)|\sigma_1 - \sigma_2| \frac{\int_\Omega |D^2u|^2 dx}{\int_\Omega u^2 dx}, \end{aligned} \tag{6.1.14}$$

then taking the infimum and the supremum in (6.1.14) yields

$$|\lambda_j(\sigma_1) - \lambda_j(\sigma_2)| \leq (1+N)\lambda_j(0)|\sigma_1 - \sigma_2|.$$

Then the function $\lambda_j(\sigma)$ is Lipschitz in $[0,1[$. $\square$

Remark 6.1.15. *From Lemma 6.1.12 it follows that, since* $\lambda_j(\sigma)$ *is Lipschitz on* $[0,1[$*, then it is uniformly continuous in the whole of* $[0,1]$*. Hence, for all* $j \in \mathbb{N}$*, there exist* $\overline{\lambda}_j \in \mathbb{R}$ *such that* $\lim_{\sigma \to 1^-} \lambda_j(\sigma) = \overline{\lambda}_j$*.*

In the next theorem we show that $\lim_{\sigma \to 1^-} \lambda_j(\sigma) = 0$ for all $j \in \mathbb{N}$.

Theorem 6.1.16. *For all* $j \in \mathbb{N}$ *it holds* $\lim_{\sigma \to 1^-} \lambda_j(\sigma) = 0$*.*

Proof. We recall that the space $H^2_h(\Omega)$ is closed in $H^2(\Omega)$ and therefore it is a Hilbert space, endowed with the standard scalar product of $H^2(\Omega)$. Let $\{u_i\}_{i=1}^\infty$ be a set of linearly independent functions in $H^2_h(\Omega)$ such that $\int_\Omega u_i u_k = \delta_{ik}$ for all $i,k \in \mathbb{N}$. Then, from (6.1.13) we have that for all $j \in \mathbb{N}$ it holds

$$\lambda_j(\sigma) \leq \max_{c_1,\dots,c_j \in \mathbb{R}} \frac{(1-\sigma)\int_\Omega \left|\sum_{i=1}^j c_i D^2 u_i\right|^2 dx}{\int_\Omega \left(\sum_{i=1}^j c_i u_i\right)^2 dx},$$

where we have chosen as j-dimensional space E in (6.1.13) the space generated by $\{u_1, ..., u_j\}$. Then we have

$$\max_{c_1,\dots,c_j\in\mathbb{R}} \frac{(1-\sigma)\int_\Omega \left|\sum_{i=1}^j c_i D^2 u_i\right|^2 dx}{\int_\Omega \left(\sum_{i=1}^j c_i u_i\right)^2 dx} \leq \max_{c_1,\dots,c_j\in\mathbb{R}} j(1-\sigma)\frac{\sum_{i=1}^j c_i^2 \int_\Omega |D^2u_i|^2 dx}{\sum_{i=1}^j c_i^2} \leq j(1-\sigma)\max_{i=1,\dots,j}\int_\Omega |D^2u_i|^2 dx,$$

and therefore $\lim_{\sigma\to 1^-}\lambda_j(\sigma) = 0$ for all $j \in \mathbb{N}$. □

Thus, the positive eigenvalues of problem (6.1.2) are not limiting points for the eigenvalues of problem (6.1.1) as $\sigma \to 1^-$.

Now we consider problems (6.1.2) and (6.1.3). We note that, under the assumptions that Ω is of class $C^{4,\alpha}$, we have that the eigenfunctions w of problem (6.1.3) are of class $C^{4,\alpha}(\overline{\Omega})$ (see [47, Theorem 2.20]). We have the following theorem.

Theorem 6.1.17. *All the positive eigenvalues of problem* (6.1.2) *coincide with the eigenvalues of problem* (6.1.3).

Proof. Let μ be an eigenvalue of problem (6.1.3) and let $w \in H_0^2(\Omega)$ be an eigenfunction associated with μ . Let $v_0 \in H^2(\Omega)\cap H_0^1(\Omega)$ be the unique solution of

$$\begin{cases} \Delta v_0 = w, & \text{in } \Omega,\\ v_0 = 0, & \text{on } \partial\Omega. \end{cases}$$

We set $v_h = v_0 + h$ for some harmonic function $h \in H^2(\Omega)$. Now we consider the following problem: find a harmonic function h such that

$$\begin{cases} \Delta^2 v_h = \mu v_h, & \text{in } \Omega,\\ \Delta v_h = 0, & \text{on } \partial\Omega,\\ \frac{\partial \Delta v_h}{\partial\nu} = 0, & \text{on } \partial\Omega. \end{cases}$$

Clearly $\Delta v_{h|\partial\Omega} = \frac{\partial\Delta v_h}{\partial\nu}_{|\partial\Omega} = 0$ for all harmonic functions h. As for the differential equation, we have $\Delta^2(v_0+h) = \mu(v_0+h)$ if and only if $\Delta(\Delta v_0 + \Delta h) = \mu(v_0+h)$, that is $\Delta w = \mu(v_0+h)$ and therefore $h = \frac{\Delta w}{\mu} - v_0$, which is clearly harmonic and belongs to $H^2(\Omega)$. Therefore each eigenvalue μ of problem (6.1.3) is an eigenvalue of problem (6.1.2) and a corresponding eigenfunction is given by $v = \frac{\Delta w}{\mu}$. On the other hand, suppose that $\lambda > 0$ is an eigenvalue of problem (6.1.2) and let $u \in H^4(\Omega)\cap H_0^2(\Omega)$ be a corresponding eigenfunction. Then, the function $w = \Delta u$ is in $H_0^2(\Omega)$ and solves

$$\begin{cases} \Delta^2 w = \lambda w, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega, \\ \frac{\partial w}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

therefore, λ is an eigenvalue of problem (6.1.3) with corresponding eigenfunction Δv. □

6.1.3 Neumann and Dirichlet eigenvalues in the case of the unit ball

In this section we consider problems (6.1.1), (6.1.2) and (6.1.3) when $\Omega = B$ is the unit ball in $\mathbb{R}^N$ centered at zero. In this case it is possible to perform explicit computations which allow to recast the eigenvalue problems (6.1.1), (6.1.2) and (6.1.3) into suitable equations of the form $F(\lambda) = 0$ and then gather informations on the behavior of the eigenvalues.

It is convenient to use the standard spherical coordinates $(r, \theta) \in \mathbb{R}_+ \times \partial B$. Recall that by $H_l(\theta)$ we denote a spherical harmonic of order $l \in \mathbb{N}_0$. We also recall that for each $l \in \mathbb{N}_0$, H_l is a solution of the equation $-\Delta_S H_l = l(l + N - 2)H_l$.
As customary, for $l \in \mathbb{N}_0$, we denote by j_l and i_l the ultraspherical and modified ultraspherical Bessel functions of the first species and order l respectively.

We consider first problem (6.1.3) on B. We have the following lemma.

Lemma 6.1.18. *Given an eigenvalue μ of problem* (6.1.3) *on B, a corresponding eigenfunction w is of the form $w(r, \theta) = W_l(r)H_l(\theta)$, for some $l \in \mathbb{N}_0$, where*

$$W_l(r) = \alpha j_l(\sqrt[4]{\mu} r) + \beta i_l(\sqrt[4]{\mu} r), \tag{6.1.19}$$

for suitable $\alpha, \beta \in \mathbb{R}$.

We refer e.g., to [29] for the proof of Lemma 6.1.18. We establish now an implicit characterization of the eigenvalues of problem (6.1.3) on B.

Lemma 6.1.20. *The eigenvalues μ of problem* (6.1.3) *on B are given implicitly as zeroes of the equation*

$$j_l(\sqrt[4]{\mu}) i_l'(\sqrt[4]{\mu}) - i_l(\sqrt[4]{\mu}) j_l'(\sqrt[4]{\mu}) = 0. \tag{6.1.21}$$

Proof. By Lemma 6.1.18, an eigenfunction w associated with an eigenvalue μ is of the form $w(r, \theta) = W_l(r)H_l(\theta)$, where $W_l(r)$ is given by (6.1.19). We recall that in spherical coordinates the Dirichlet boundary conditions are written as

$$w_{|_{r=1}} = \partial_r w_{|_{r=1}} = 0.$$

By imposing such boundary conditions to $w(r,\theta)$ we obtain the following homogeneous system of two equations in two unknowns α and β

$$\begin{cases} \alpha j_l(\sqrt[4]{\mu}) + \beta i_l(\sqrt[4]{\mu}) = 0, \\ \alpha j_l'(\sqrt[4]{\mu}) + \beta i_l'(\sqrt[4]{\mu}) = 0, \end{cases}$$

which has solutions if and only if its determinant vanishes. This yields formula (6.1.21). □

Now we consider problem (6.1.1) on B. We have the following lemma (see e.g., [28] for the proof).

Lemma 6.1.22. *Given an eigenvalue λ of problem* (6.1.1) *with $\sigma \in [0,1]$ on B, a corresponding eigenfunction u is of the form $u(r,\theta) = U_l(r)H_l(\theta)$, for some $l \in \mathbb{N}_0$, where*

$$U_l(r) = \alpha j_l(\sqrt[4]{\lambda} r) + \beta i_l(\sqrt[4]{\lambda} r), \tag{6.1.23}$$

for suitable $\alpha, \beta \in \mathbb{R}$.

We have the following lemma on the eigenvalues of problem (6.1.1) on B.

Lemma 6.1.24. *The eigenvalues λ of problem* (6.1.1) *with $\sigma \in [0,1]$ on B are given implicitly as zeroes of the equation*

$$\det M(\lambda,\sigma) = 0, \tag{6.1.25}$$

where $M(\lambda,\sigma)$ is the 2×2 matrix defined by

$$\begin{bmatrix} \begin{matrix} \sqrt{\lambda} j_l''(\sqrt[4]{\lambda}) \\ +(N-1)\sqrt[4]{\lambda}\sigma j_l'(\sqrt[4]{\lambda}) \\ -l(l+N-2)\sigma j_l(\sqrt[4]{\lambda}) \end{matrix} & \begin{matrix} \sqrt{\lambda} i_l''(\sqrt[4]{\lambda}) \\ +(N-1)\sqrt[4]{\lambda}\sigma i_l'(\sqrt[4]{\lambda}) \\ -l(l+N-2)\sigma i_l(\sqrt[4]{\lambda}) \end{matrix} \\ \\ \begin{matrix} \sqrt[4]{\lambda^3} j_l'''(\sqrt[4]{\lambda}) + (N-1)\sqrt{\lambda} j_l''(\sqrt[4]{\lambda}) \\ +\sqrt[4]{\lambda}\left(1-N+l(\sigma-2)(N+l-2)\right) j_l'(\sqrt[4]{\lambda}) \\ -l(l+N-2)(\sigma-3) j_l(\sqrt[4]{\lambda}) \end{matrix} & \begin{matrix} \sqrt[4]{\lambda^3} i_l'''(\sqrt[4]{\lambda}) + (N-1)\sqrt{\lambda} i_l''(\sqrt[4]{\lambda}) \\ +\sqrt[4]{\lambda}\left(1-N+l(\sigma-2)(N+l-2)\right) i_l'(\sqrt[4]{\lambda}) \\ -l(l+N-2)(\sigma-3) i_l(\sqrt[4]{\lambda}) \end{matrix} \end{bmatrix} \tag{6.1.26}$$

Proof. By Lemma 6.1.22, an eigenfunction u associated with an eigenvalue λ is of the form $u(r,\theta) = U_l(r)H_l(\theta)$, where $U_l(r)$ is given by (6.1.23). We recall that in spherical coordinates the Neumann boundary conditions are written as

$$\begin{cases} (1-\sigma)\partial_{rr}^2 u + \sigma \Delta u_{|r=1} = 0, \\ \partial_r(\Delta u) + (1-\sigma)\frac{1}{r^2}\Delta_S\left(\partial_r u - \frac{u}{r}\right)_{|r=1} = 0, \end{cases}$$

see [28] for details. By imposing boundary conditions to the function u we obtain a system of two equations in two unknowns α and β, and the associated matrix is given by (6.1.26). Thus the eigenvalues must solve equation (6.1.25). □

We give now an alternative proof of Theorem 6.1.17 when $\Omega = B$ is the unit ball in $\mathbb{R}^N$ centered at zero based on the explicit representations of the eigenvalues discussed in this section. We have the following theorem.

Theorem 6.1.27. *Equations* $\det M(\lambda, 1) = 0$ *and* (6.1.21) *admit the same non-zero solutions.*

Proof. We consider (6.1.26) with $\sigma = 1$. Let $\lambda > 0$ be a solution of $\det M(\lambda, 1) = 0$. We compute $F(\lambda) = \det M(\lambda, 1)$. We have

$$\begin{aligned}
F(\lambda) = &-\sqrt[4]{\lambda}l(l-1)(N+l-2)(N+l-1)\left(j_l(\sqrt[4]{\lambda})i_l'(\sqrt[4]{\lambda}) - i_l(\sqrt[4]{\lambda})j_l'(\sqrt[4]{\lambda})\right)\\
&+\sqrt{\lambda}l(N+1)(l+N-2)\left(j_l(\sqrt[4]{\lambda})i_l''(\sqrt[4]{\lambda}) - i_l(\sqrt[4]{\lambda})j_l''(\sqrt[4]{\lambda})\right)\\
&-\lambda^{3/4}(N(N-1)+l(N+l-2))\left(j_l'(\sqrt[4]{\lambda})i_l''(\sqrt[4]{\lambda}) - i_l'(\sqrt[4]{\lambda})j_l''(\sqrt[4]{\lambda})\right)\\
&+\lambda^{3/4}l(l+N-2)(j_l(\sqrt[4]{\lambda})i_l'''(\sqrt[4]{\lambda}) - i_l(\sqrt[4]{\lambda})j_l'''(\sqrt[4]{\lambda}))\\
&-\lambda(N-1)\left(j_l'(\sqrt[4]{\lambda})i_l'''(\sqrt[4]{\lambda}) - i_l'(\sqrt[4]{\lambda})j_l'''(\sqrt[4]{\lambda})\right)\\
&+\lambda^{5/4}\left(j_l''(\sqrt[4]{\lambda})i_l'''(\sqrt[4]{\lambda}) - i_l''(\sqrt[4]{\lambda})j_l'''(\sqrt[4]{\lambda})\right).
\end{aligned} \tag{6.1.28}$$

We recall the well-known recurrence formulas for Bessel functions and their derivatives (see [1, 9.1.27 and 9.6.26])

$$\begin{array}{ll}
J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z}J_\nu(z), & I_{\nu-1}(z) - I_{\nu+1}(z) = \frac{2\nu}{z}J_\nu(z),\\
J_{\nu-1}(z) - J_{\nu+1}(z) = 2J_\nu'(z), & I_{\nu-1}(z) + I_{\nu+1}(z) = 2I_\nu'(z),\\
J_\nu'(z) = J_{\nu-1}(z) - \frac{\nu}{z}J_\nu(z), & I_\nu'(z) = I_{\nu-1}(z) - \frac{\nu}{z}I_\nu(z),\\
J_\nu'(z) = -J_{\nu+1}(z) + \frac{\nu}{z}J_\nu(z), & I_\nu'(z) = I_{\nu+1}(z) + \frac{\nu}{z}I_\nu(z).
\end{array} \tag{6.1.29}$$

We set $C_l^{\pm}(z) = I_{\frac{N}{2}+l}(z)J_{\frac{N}{2}-1+l}(z) \pm I_{\frac{N}{2}-1+l}(z)J_{\frac{N}{2}+l}(z)$. We use (6.1.29) to get

$$j_l(z)i_l'(z) - i_l(z)j_l'(z) = z^{2-N}C_l^+(z), \tag{6.1.30}$$

$$\begin{aligned}
j_l(z)i_l''(z) - i_l(z)j_l''(z) = {}& z^{1-N}\left(2zI_{\frac{N}{2}-1+l}(z)J_{\frac{N}{2}-1+l}(z)\right.\\
&\left.-(N-1)C_l^+(z)\right),
\end{aligned} \tag{6.1.31}$$

$$\begin{aligned}
j_l'(z)i_l''(z) - i_l'(z)j_l''(z) = {}& z^{-N}\left(z^2C_l^-(z) + 2lzI_{\frac{N}{2}-1+l}(z)J_{\frac{N}{2}-1+l}(z)\right.\\
&\left.-l(l+N-2)C_l^+(z)\right),
\end{aligned} \tag{6.1.32}$$

$$\begin{aligned}
j_l(z)i_l'''(z) - i_l(z)j_l'''(z) = {}& z^{-N}\left(z^2C_l^-(z) + 2(1-N+l)zI_{\frac{N}{2}-1+l}(z)J_{\frac{N}{2}-1+l}(z)\right.\\
&\left.+(N(N-1)+l(l+N-2))C_l^+(z)\right),
\end{aligned} \tag{6.1.33}$$

$$\begin{aligned}
j_l'(z)i_l'''(z) - i_l'(z)j_l'''(z) = {}& z^{-1-N}\left(-2z^3I_{\frac{N}{2}+l}(z)J_{\frac{N}{2}+l}(z) + (1-N+2l)z^2C_l^-(z)\right.\\
&+2l(1-N+l)zI_{\frac{N}{2}-1+l}(z)J_{\frac{N}{2}-1+l}(z)\\
&\left.+l(l+N-2)(N+1)C_l^+(z)\right),
\end{aligned} \tag{6.1.34}$$

$$\begin{aligned}
j_l''(z)i_l'''(z) - i_l''(z)j_l'''(z) = {}& z^{-2-N}\left(-z^4C_l^+(z) + 2(N-1)z^3I_{\frac{N}{2}+l}(z)J_{\frac{N}{2}+l}(z)\right.\\
&-(N+1)(2l+1)z^2C_l^-(z)\\
&-2(N-3)(l-1)lzI_{\frac{N}{2}-1+l}(z)J_{\frac{N}{2}-1+l}(z)\\
&\left.+l(l-1)(l+N-2)(l+N-1)C_l^+(z)\right).
\end{aligned} \tag{6.1.35}$$

Thanks to (6.1.30)-(6.1.35), expression (6.1.28) simplifies to

$$F(\lambda) = \lambda^{5/4} \left(j_l(\sqrt[4]{\lambda}) i_l'(\sqrt[4]{\lambda}) - i_l(\sqrt[4]{\lambda}) j_l'(\sqrt[4]{\lambda}) \right). \tag{6.1.36}$$

Therefore by comparing (6.1.36) with (6.1.21) we see that the non-zero eigenvalues of problem (6.1.2) and the eigenvalues of problem (6.1.3) on the unit ball coincide. □

Remark 6.1.37. *From Theorem 6.1.16 it follows that all the eigenvalues* $\lambda_j(\sigma) \to 0$ *as* $\sigma \to 1^-$. *This means that there are infinitely many branches of solutions* $\sigma \mapsto \lambda(\sigma)$ *of equation* (6.1.25) *such that* $\lambda(\sigma) \to 0$ *as* $\sigma \to 1^-$. *Theorem 6.1.27 shows that there are also infinitely many branches* $\sigma \mapsto \lambda(\sigma)$ *such that* $\lambda(\sigma) \to \mu$ *as* $\sigma \to 1^-$, *for some solution* $\mu > 0$ *to equation* (6.1.21) *(see Figure 6.1).*

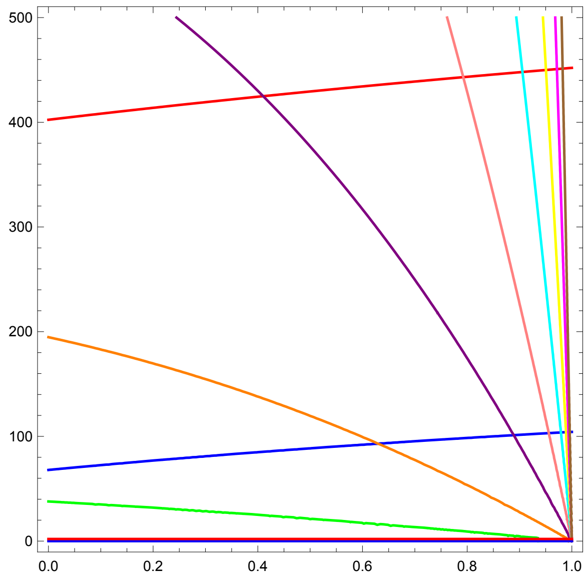


Figure 6.1: Solution branches of equation (6.1.25) with $N = 2$ for $(\sigma, \lambda) \in]0,1[\times]0,500[$. The color refers to the choice of l in (6.1.25): blue ($l = 0$), red ($l = 1$), green ($l = 2$), orange ($l = 3$), purple ($l = 4$), pink ($l = 5$), cyan ($l = 6$), yellow ($l = 7$), magenta ($l = 8$), brown ($l = 9$).

6.2 Neumann eigenvalues on annuli

In this section we consider the behavior of the Neumann eigenvalues both for the Laplace and the biharmonic operator on the annulus of radii $1 - \varepsilon$ and 1, for $\varepsilon \in] - \varepsilon_0, 0[\cup]0, \varepsilon_0[$, for a suitable $\varepsilon_0 > 0$ small.

We mention the paper [89], where the authors consider the Neumann Laplacian on the annulus of radii $1-\varepsilon$ and 1 and prove that for $N=2$ the first positive eigenvalue converges to 1 as $\varepsilon \to 0$ and it is an increasing function of ε for $\varepsilon \in]-\varepsilon_0, \varepsilon_0[$, for a suitable $\varepsilon_0 > 0$. As a bypass product, they obtain a counterexaple of the domain monotonicity for the first Neumann eigenvalue of the Laplace operator. The proof of the results in [89] relies on the use of the variational characterization of the first Neumann eigenvalue and on estimates for the derivatives of the eigenvalues of suitable Sturm-Liouville problems. In this section we consider the explicit characterization of all the eigenvalues in terms of zeros of suitable combinations of Bessel's functions, as well as suitable Taylor's expansions of such functions and estimates for the correspondig remainders. This technique allows to consider all eigenvalues with arbitrary indexes and multiplicity, and can be used also for the corresponding problem for the biharmonic operator.

Let $\varepsilon \in]-\infty, 0[\cup]0, 1[$ and let $A_\varepsilon \subset \mathbb{R}^N$ be the set defined by

$$A_\varepsilon = \begin{cases} x \in \mathbb{R}^N : 1-\varepsilon < |x| < 1, & \text{if } \varepsilon \in]0,1[, \\ x \in \mathbb{R}^N : 1 < |x| < 1-\varepsilon, & \text{if } \varepsilon \in]-\infty, 0[. \end{cases} \tag{6.2.1}$$

We consider the Neumann eigenvalue problem for the Laplace operator in A_ε, namely

$$\begin{cases} -\Delta u = \lambda u, & \text{in } A_\varepsilon, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial A_\varepsilon. \end{cases} \tag{6.2.2}$$

It is convenient to use standard spherical coordinates $(r,\theta) \in \mathbb{R}_+ \times \partial B$ in $\mathbb{R}^N$. We recall that the eigenvalues of problem (6.2.2) have finite multiplicity and can be represented by means of an increasing sequence diverging to $+\infty$ of the form

$$0 = \lambda_0(\varepsilon) < \lambda_1(\varepsilon) \le \cdots \le \lambda_l(\varepsilon) \le \cdots .$$

We are interested in the behavior of the map $\varepsilon \mapsto \lambda_j(\varepsilon)$ as $\varepsilon \to 0$.

6.2.1 Eigenvalues of the Laplace operator on the annulus of $\mathbb{R}^2$

In this subsection we consider the case $N = 2$. We have the following theorem on the asymptotic behavior of the eigenvalues of (6.2.2) as $\varepsilon \to 0$.

Theorem 6.2.3. *For $N=2$, all the eigenvalues of problem* (6.2.2) *have the following asymptotic behavior*

$$\lambda_l(\varepsilon) = l^2 + \varepsilon l^2 + O(\varepsilon^2), \tag{6.2.4}$$

as $\varepsilon \to 0$. In particular, $\lambda_l'(0) > 0$ for all $l \ge 1$.

Proof. In polar coordinates, the first equation in (6.2.2) reads

$$-\partial^2_{rr}u - \frac{1}{r}\partial_r u - \frac{1}{r^2}\partial^2_{\theta\theta}u = \lambda u, \tag{6.2.5}$$

for $\theta \in [0, 2\pi[$ and $r \in]1-\varepsilon, 1[$ (if $\varepsilon > 0$), or $r \in]1, 1-\varepsilon[$ (if $\varepsilon < 0$). The boundary conditions read

$$\partial_r u(1,\theta) = \partial_r u(1-\varepsilon,\theta) = 0, \quad \forall \theta \in [0, 2\pi[. \tag{6.2.6}$$

Moreover $u(r,0) = u(r,2\pi)$ for all $r \in]1-\varepsilon, 1[$ (if $\varepsilon > 0$) or $r \in]1, 1-\varepsilon[$ (if $\varepsilon < 0$). As customary, by separating variables so that $u(r,\theta) = w(r)v(\theta)$ and using l^2 as separation constant, with $l \in \mathbb{N}_0$, we have that $v(\theta)$ is a solution of

$$-v''(\theta) = l^2 v(\theta), \tag{6.2.7}$$

that is $v(\theta) = A\cos(l\theta) + B\sin(l\theta)$ for $l \in \mathbb{N}_0$ and $A, B \in \mathbb{R}$. Then $u(r,\theta) = w(r)\left(A\cos(l\theta) + B\sin(l\theta)\right)$, for some $A, B \in \mathbb{R}$. By using formula (6.2.7) into (6.2.5), we obtain that w solves

$$w''(r) + \frac{1}{r}w'(r) + \left(\lambda - \frac{l^2}{r^2}\right)w(r) = 0. \tag{6.2.8}$$

This implies that u is of the form $u(r,\theta) = w_l(r)\left(A\cos(l\theta) + B\sin(l\theta)\right)$, where

$$w_l(r) = \alpha J_l(\sqrt{\lambda}r) + \beta Y_l(\sqrt{\lambda}r), \tag{6.2.9}$$

where J_l and Y_l are the Bessel functions of the first and second species and order l respectively, and $\alpha, \beta \in \mathbb{R}$ are suitable constants. By imposing boundary conditions (6.2.6) we obtain that λ must satisfy the condition $\det \mathcal{B} = 0$, where

$$\mathcal{B} = \begin{bmatrix} J_l'(\sqrt{\lambda}) & Y_l'(\sqrt{\lambda}) \\ J_l'(\sqrt{\lambda}(1-\varepsilon)) & Y_l'(\sqrt{\lambda}(1-\varepsilon)) \end{bmatrix}.$$

Then λ is such that

$$J_l'(\sqrt{\lambda})Y_l'(\sqrt{\lambda}(1-\varepsilon)) - J_l'(\sqrt{\lambda}(1-\varepsilon))Y_l'(\sqrt{\lambda}) = 0 \tag{6.2.10}$$

Now we expand the left-hand side of (6.2.10) in Taylor series around $\varepsilon = 0$ up to the second order, obtaining

$$\begin{aligned} &-\sqrt{\lambda}\varepsilon\left[J_l'(\sqrt{\lambda})Y_l''(\sqrt{\lambda}) - J_l''(\sqrt{\lambda})Y_l'(\sqrt{\lambda})\right] \\ &\qquad + \frac{\lambda\varepsilon^2}{2}\left[J_l'(\sqrt{\lambda})Y_l'''(\sqrt{\lambda}) - J_l'''(\sqrt{\lambda})Y_l'(\sqrt{\lambda})\right] + R(\lambda,\varepsilon) = 0, \end{aligned} \tag{6.2.11}$$

where $R(\lambda,\varepsilon)$ is given by

$$R(\lambda,\varepsilon) = \sum_{k=3}^{\infty} \frac{(-1)^k \varepsilon^k \sqrt{\lambda}^k}{k!}\left(Y_l'(\sqrt{\lambda})J_l^{(k+1)}(\sqrt{\lambda}) - J_l'(\sqrt{\lambda})Y_l^{(k+1)}(\sqrt{\lambda})\right),$$

and is $O(\varepsilon^3)$ as $\varepsilon \to 0$. The fact that the remainder $R(\lambda, \varepsilon)$ in (6.2.11) is of order ε^3 follows from the fact that for all $m \in \mathbb{N}_0$, the eigenvalues of (6.2.2) which we denote here by $\lambda_m(\varepsilon)$, are bounded away from zero and infinity uniformly in ε, as $\varepsilon \to 0$. Indeed, by the Rayleigh Min-Max Principle, we have that

$$\lambda_m(\varepsilon) \leq \max_{c_0,\dots,c_m \in \mathbb{R}^{m+1}\setminus\{0\}} \frac{\int_{A_\varepsilon} |\sum_{k=0}^m c_k \nabla \varphi_k|^2 dx}{\int_{A_\varepsilon} \left(\sum_{k=0}^m c_k \varphi_k\right)^2 dx},$$

where $\varphi_k(r,\theta) = r^k \cos(k\theta)$. We consider the case $\varepsilon > 0$ (the case $\varepsilon < 0$ is analogous). From standard computations it follows that

$$\begin{aligned}
&\lambda_m(\varepsilon) \\
&\quad \leq \max_{c_0,\dots,c_m \in \mathbb{R}^{m+1}\setminus\{0\}} \frac{\int_0^{2\pi}\int_{1-\varepsilon}^1 \left(\sum_{k=0}^m c_k k r^{k-1}\cos(k\theta)\right)^2 r dr d\theta}{\pi \sum_{k=0}^m c_k^2 \frac{\left(1-(1-\varepsilon)^{2(1+k)}\right)}{2(1+k)}} \\
&\qquad + \frac{\int_0^{2\pi}\int_{1-\varepsilon}^1 \frac{1}{r}\left(\sum_{k=0}^m c_k k r^k \sin(k\theta)\right)^2 dr d\theta}{\pi \sum_{k=0}^m c_k^2 \frac{\left(1-(1-\varepsilon)^{2(1+k)}\right)}{2(1+k)}} \\
&\leq (m+1) \max_{c_0,\dots,c_m \in \mathbb{R}^{m+1}\setminus\{0\}} \frac{\int_0^{2\pi}\int_{1-\varepsilon}^1 \left(\sum_{k=0}^m c_k^2 k^2 r^{2(k-1)}\cos(k\theta)^2\right) r dr d\theta}{\pi \sum_{k=0}^m c_k^2 \frac{\left(1-(1-\varepsilon)^{2(1+k)}\right)}{2(1+k)}} \\
&\qquad + \frac{\int_0^{2\pi}\int_{1-\varepsilon}^1 \left(\sum_{k=0}^m c_k^2 k^2 r^{2(k-1)}\sin(k\theta)^2\right) r dr d\theta}{\pi \sum_{k=0}^m c_k^2 \frac{\left(1-(1-\varepsilon)^{2(1+k)}\right)}{2(1+k)}} \\
&\qquad = (m+1) \max_{c_0,\dots,c_m \in \mathbb{R}^{m+1}\setminus\{0\}} \frac{\sum_{k=0}^m c_k^2 k \left(1-(1-\varepsilon)^{2k}\right)}{\sum_{k=0}^m c_k^2 \frac{\left(1-(1-\varepsilon)^{2(1+k)}\right)}{2(1+k)}}.
\end{aligned}$$

We note that, for all choices of $(c_0, ..., c_m) \in \mathbb{R}^{m+1} \setminus \{0\}$,

$$\lim_{\varepsilon \to 0} \frac{\sum_{k=0}^m c_k^2 k \left(1-(1-\varepsilon)^{2k}\right)}{\sum_{k=0}^m c_k^2 \frac{\left(1-(1-\varepsilon)^{2(1+k)}\right)}{2(1+k)}} = 2\frac{\sum_{k=0}^m k^2 c_k^2}{\sum_{k=1}^m c_k^2} \leq 2(m+1)^2.$$

Thus, for $\varepsilon > 0$ sufficiently small, $\lambda_m(\varepsilon)$ is uniformly bounded from above, as $\varepsilon \to 0$. We now prove that all the positive eigenvalues are bounded away from zero. In fact, given a positive eigenvalue λ of problem (6.2.2), a corresponding eigenfunction is given by $w_l(r)(A\cos(l\theta) + B\sin(l\theta))$, for some $l \in \mathbb{N}_0$, and $w_l(r)$ is as in (6.2.9). Then we have

$$\begin{aligned}\lambda = &\frac{\int_0^{2\pi}\int_{1-\varepsilon}^1 r w_l'(r)^2 (A\cos(l\theta)+B\sin(l\theta))^2 drd\theta}{\int_0^{2\pi}\int_{1-\varepsilon}^1 w_l(r)^2(A\cos(l\theta)+B\sin(l\theta))^2 rdrd\theta}\\ &+\frac{\int_0^{2\pi}\int_{1-\varepsilon}^1 \frac{1}{r}\left(w_l(r)^2 l^2(-A\sin(l\theta)+B\cos(l\theta))^2\right)drd\theta}{\int_0^{2\pi}\int_{1-\varepsilon}^1 w_l(r)^2(A\cos(l\theta)+B\sin(l\theta))^2 rdrd\theta}\\ &\geq \frac{\int_0^{2\pi}\int_{1-\varepsilon}^1 \frac{1}{r}\left(w_l(r)^2 l^2(-A\sin(l\theta)+B\cos(l\theta))^2\right)drd\theta}{\int_0^{2\pi}\int_{1-\varepsilon}^1 w_l(r)^2(A\cos(l\theta)+B\sin(l\theta))^2 rdrd\theta}\\ &= l^2\frac{\int_{1-\varepsilon}^1 \frac{w_l(r)^2}{r}dr}{\int_{1-\varepsilon}^1 w_l(r)^2 rdr}\geq 1, \end{aligned}\tag{6.2.12}$$

if $\varepsilon \in]0,1[$, while

$$\begin{aligned}\lambda = &\frac{\int_0^{2\pi}\int_{1}^{1-\varepsilon} r w_l'(r)^2 (A\cos(l\theta)+B\sin(l\theta))^2 drd\theta}{\int_0^{2\pi}\int_{1}^{1-\varepsilon} w_l(r)^2(A\cos(l\theta)+B\sin(l\theta))^2 rdrd\theta}\\ &+\frac{\int_0^{2\pi}\int_{1}^{1-\varepsilon} \frac{1}{r}\left(w_l(r)^2 l^2(-A\sin(l\theta)+B\cos(l\theta))^2\right)drd\theta}{\int_0^{2\pi}\int_{1}^{1-\varepsilon} w_l(r)^2(A\cos(l\theta)+B\sin(l\theta))^2 rdrd\theta}\\ &\geq \frac{\int_0^{2\pi}\int_{1}^{1-\varepsilon} \frac{1}{r}\left(w_l(r)^2 l^2(-A\sin(l\theta)+B\cos(l\theta))^2\right)drd\theta}{\int_0^{2\pi}\int_{1}^{1-\varepsilon} w_l(r)^2(A\cos(l\theta)+B\sin(l\theta))^2 rdrd\theta}\\ &= l^2\frac{\int_{1}^{1-\varepsilon} \frac{w_l(r)^2}{r}dr}{\int_{1}^{1-\varepsilon} w_l(r)^2 rdr}\geq \frac{1}{4}, \end{aligned}\tag{6.2.13}$$

if $\varepsilon \in]-1,0[$.

We note now that $\lambda = 0$ is an eigenvalue of problem (6.2.2) and the associated eigenfunctions are the constant functions on A_ε. In particular, $\lambda = 0$ is the first eigenvalue of (6.2.8) with $l = 0$ and with Neumann boundary conditions (6.2.6). Moreover, all the non-zero eigenvalues of problem (6.2.2) are bounded away from zero. Then in equation (6.2.11), we can assume $\lambda \neq 0$. Dividing by ε and multiplying by $\pi\lambda$ in (6.2.11), and recalling that (see Lemmas 4.1.40 and 4.1.46)

$$J_l'(\sqrt{\lambda})Y_l''(\sqrt{\lambda}) - J_l''(\sqrt{\lambda})Y_l'(\sqrt{\lambda}) = -\frac{2}{\pi\sqrt{\lambda}}\left(\frac{l^2}{\lambda}-1\right) \tag{6.2.14}$$

$$J_l'(\sqrt{\lambda})Y_l'''(\sqrt{\lambda}) - J_l'''(\sqrt{\lambda})Y_l'(\sqrt{\lambda}) = -\frac{2}{\pi\lambda}\left(1-\frac{3l^2}{\lambda}\right), \tag{6.2.15}$$

equation (6.2.11) can be rewritten as

$$2\left(l^2-\lambda\right) - \left(\lambda - 3l^2\right)\varepsilon + \tilde{R}(\lambda,\varepsilon), \tag{6.2.16}$$

where $\tilde{R}(\lambda,\varepsilon) = \pi\lambda R(\lambda,\varepsilon)/\varepsilon \in O(\varepsilon^2)$ as $\varepsilon \to 0$.

We consider equation (6.2.16) and apply the Implicit Function Theorem. Equation (6.2.16) can be written in the form $F(\lambda,\varepsilon) = 0$, with F of class C^1 in $]0,+\infty[\times[0,1[$, and $F(\lambda,0) = 2\left(l^2-\lambda\right)$, $\partial_\lambda F(\lambda,0) = -2$, and $\partial_\varepsilon F(\lambda,0) = 3l^2 - \lambda$. Since $F(l^2,0) = 0$ and $\partial_\lambda F(l^2,0) = -2 \neq 0$ for all $l \in \mathbb{N}_0$, the zeros of equation (6.2.11) in a neighborhood of $(\lambda,0)$ are given by the graph of a C^1-function $\varepsilon \mapsto \lambda(\varepsilon)$ with $\lambda(0) = l^2$. Moreover $\lambda'(0) = -\partial_\varepsilon F(l^2,0)/\partial_\lambda F(l^2,0) = l^2$ for $l \in \mathbb{N}_0$. Then $\lambda(\varepsilon) = l^2 + \varepsilon l^2 + O(\varepsilon^2)$ as $\varepsilon \to 0$, for all $l \in \mathbb{N}$. Moreover $\lambda_0(\varepsilon) = 0$ for all $\varepsilon \in]-\infty,0[\cup]0,1[$, thus formula (6.2.4) holds also for $l = 0$. Moreover $\lambda_l'(0) > 0$ for all $l \geq 1$. This concludes the proof of the theorem. □

Remark 6.2.17. *The limiting eigenvalues are the eigenvalues μ of $-v''(s) = \mu v(s)$ on $]0,2\pi[$ with periodic conditions $v(0) = v(2\pi)$.*

Remark 6.2.18. *We note that, by standard Sturm-Liouville theory, for all $l \in \mathbb{N}_0$, equation* (6.2.8) *with Neumann boundary conditions* (6.2.6) *admits a diverging sequence of non-negative and simple eigenvalues, which we denote by $\lambda_{l,k}(\varepsilon)$, $k \in \mathbb{N}_0$, with $\lambda_{l,k_1}(\varepsilon) < \lambda_{l,k_2}(\varepsilon)$ if $k_1 < k_2$. In particular, $\lambda_{0,0}(\varepsilon) = 0$ for all $\varepsilon \in]-\infty,0[\cup]0,1[$, and $\lambda_{l,k}(\varepsilon) \geq \frac{1}{4}$ for all $\varepsilon \in]-1,0[\cup]0,1[$ (this follows from* (6.2.12) *and* (6.2.13)*) and for all $l,k \in \mathbb{N}_0$, $(l,k) \neq (0,0)$. Moreover, as a consequence of Theorem 6.2.3, we have that $\lambda_{l,0} \to l^2$ for all $l \in \mathbb{N}_0$ and $\lambda_{l,k} \to +\infty$ for all $l \in \mathbb{N}_0$ and for all $k \geq 1$, as $\varepsilon \to 0$.*

Such behavior of the eigenvalues can be also explained as follows. Let us recall that the k-th zero of the cross product

$$J_l'(z)Y_l'(\alpha z) - Y_l'(z)J_l'(\alpha z) \tag{6.2.19}$$

has a well-known asymptotic expansion. We set $\eta = 4l^2$, $\beta = \frac{k\pi}{\alpha-1}$, $p = \frac{\eta+3}{8\alpha}$, $q = \frac{(\eta^2+46\eta-63)(\alpha^3-1)}{6(4\alpha)^3(\alpha-1)}$, $r = \frac{(\eta^3+185\eta^2-2053\eta+1899)}{5(4\eta)^5(\alpha-1)}$. Then the $k-$th zero of (6.2.19) *is given by*

$$\beta + \frac{p}{\beta} + \frac{q-p^2}{\beta^3} + \frac{r-4pq+2p^3}{\beta^5} + \cdots$$

We exploit all the computations by setting $\alpha = 1-\varepsilon$ and $z = \sqrt{\lambda}$ in (6.2.19) *and collect the terms according to the powers of ε, obtaining*

$$\lambda_{l,k}(\varepsilon) = \frac{\pi^2 k^2}{\varepsilon^2} + l^2 + \varepsilon l^2 + O(\varepsilon^2). \tag{6.2.20}$$

Note that $\lambda_{l,0} = l^2 + \varepsilon l^2 + O(\varepsilon^2)$ which converges to $\lambda_l(0) = l^2$ as $\varepsilon \to 0$, while for $k > 1$, $\lambda_{l,k}(\varepsilon) \to +\infty$ as $\varepsilon \to 0$. See also Figure 6.2.

Remark 6.2.21. *Problem* (6.2.2) *presents some analogies with the eigenvalue problem for the Laplace operator $-\Delta u = \lambda u$ on $]0,2\pi[\times]0,\varepsilon[$ with*

Neumann conditions at $x_2 = 0$ *and* $x_2 = \varepsilon$ *and periodic condition at* $x_1 = 0$ *and* $x_1 = 2\pi$*. In this case, for each fixed* ε *the eigenvalues are given by* $\frac{\pi^2 k^2}{\varepsilon^2} + l^2$*, with* $l, k \in \mathbb{N}_0$*, which are exactly the first two terms appearing in formula* (6.2.20).

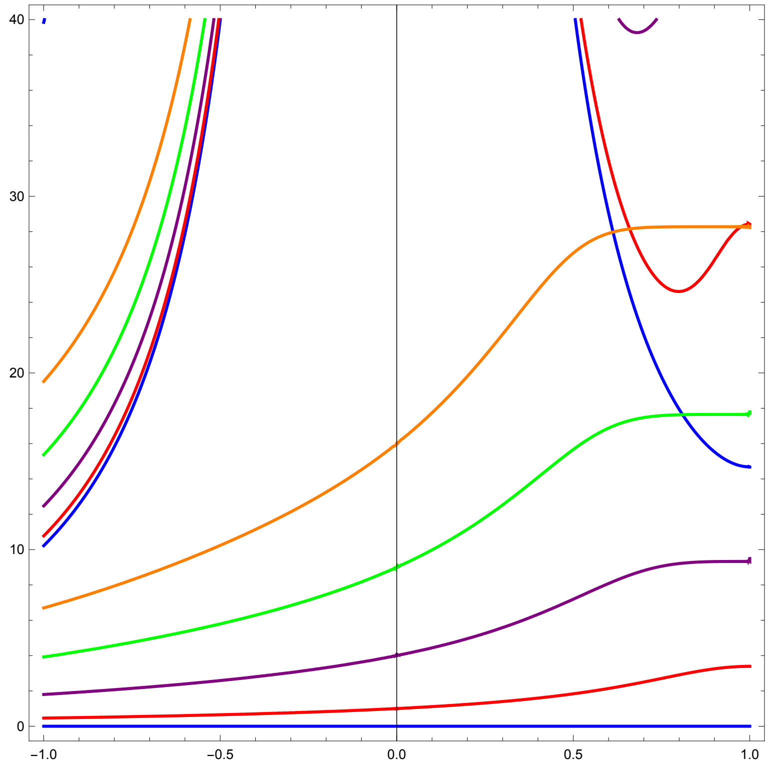


Figure 6.2: Solution branches of equation (6.2.10) for $(\varepsilon, \lambda) \in]-1, 1[\times]0, 40[$. The color refers to the choice of l in (6.2.10). In particular $l = 0$ (blue), $l = 1$ (red), $l = 2$ (purple), $l = 3$ (green), $l = 4$ (orange).

6.2.2 Eigenvalues of the Laplace operator on the annulus of $\mathbb{R}^N$

In this subsection we consider the case $N \geq 3$. We have the following theorem on the behavior of the eigenvalues of (6.2.2) as $\varepsilon \to 0$.

Theorem 6.2.22. *All the eigenvalues of problem* (6.2.2) *have the following asymptotic behavior*

$$\lambda_l(\varepsilon) = l(l + N - 2) + \varepsilon l(l + N - 2) + O(\varepsilon^2),$$

as $\varepsilon \to 0$. *In particular,* $\lambda_l'(0) > 0$ *for all* $l \geq 1$.

Proof. The proof follows the same lines as that of Theorem 6.2.3. We repeat briefly it here for the reader's convenience. As is customary, we separate variables and look for solutions to problem (6.2.2) of the form $u(r, \theta) = w(r)H(\theta)$. We use $l(l + N - 2)$ with $l \in \mathbb{N}_0$ as separation constant. It turns out that H solves the equation

$$-\Delta_S H = l(l + N - 2)H, \quad \text{on } \partial B,$$

for some $l \in \mathbb{N}_0$. Therefore $H(\theta) = H_l(\theta)$, where $H_l(\theta)$ is a spherical harmonic of some order $l \in \mathbb{N}_0$. The radial part $w(r)$ solves the equation

$$w''(r) + \frac{N-1}{r} w'(r) + \left(\lambda - \frac{l(l+N-2)}{r^2}\right) w(r) = 0.$$

This implies that, given an eigenvalue λ of problem (6.2.2) on A_ε, a corresponding eigenfunction is of the form

$$u(r,\theta) = (\alpha j_l(\sqrt{\lambda} r) + \beta y_l(\sqrt{\lambda} r)) H_l(\theta),$$

where $\alpha, \beta \in \mathbb{R}$ are suitable constants and j_l and y_l are ultraspherical Bessel functions of the first and second species and order l respectively. Now we impose Neumann boundary conditions. We obtain

$$\begin{aligned} &\alpha j_l'(\lambda) + \beta y_l'(\lambda) = 0, \\ &\alpha j_l'(\lambda(1-\varepsilon)) + \beta y_l'(\lambda(1-\varepsilon)) = 0. \end{aligned}$$

Thus, the eigenvalues are implicitly given by the equation $\det \mathcal{B} = 0$, where

$$\mathcal{B} = \begin{bmatrix} j_l'(\lambda) & y_l'(\lambda) \\ j_l'(\lambda(1-\varepsilon)) & y_l'(\lambda(1-\varepsilon))0 \end{bmatrix}.$$

From standard computations it follows that the eigenvalues of problem (6.2.2) are solutions to the following implicit equation

$$\begin{aligned} &(N-2)^2 \left(J_{\frac{N}{2}-1+l}(\sqrt{\lambda}) Y_{\frac{N}{2}-1+l}(\sqrt{\lambda}(1-\varepsilon)) - Y_{\frac{N}{2}-1+l}(\sqrt{\lambda}) J_{\frac{N}{2}-1+l}(\sqrt{\lambda}(1-\varepsilon)) \right) \\ &+ 2(N-2)\sqrt{\lambda} \left(Y'_{\frac{N}{2}-1+l}(\sqrt{\lambda}) J_{\frac{N}{2}-1+l}(\sqrt{\lambda}(1-\varepsilon)) - J'_{\frac{N}{2}-1+l}(\sqrt{\lambda}) Y_{\frac{N}{2}-1+l}(\sqrt{\lambda}(1-\varepsilon)) \right) \\ &+2(N-2)(1-\varepsilon)\sqrt{\lambda} \left(Y_{\frac{N}{2}-1+l}(\sqrt{\lambda}) J'_{\frac{N}{2}-1+l}(\sqrt{\lambda}(1-\varepsilon)) - J_{\frac{N}{2}-1+l}(\sqrt{\lambda}) Y'_{\frac{N}{2}-1+l}(\sqrt{\lambda}(1-\varepsilon)) \right) \\ &+4(1-\varepsilon)\lambda \left(J'_{\frac{N}{2}-1+l}(\sqrt{\lambda}) Y'_{\frac{N}{2}-1+l}(\sqrt{\lambda}(1-\varepsilon)) - Y'_{\frac{N}{2}-1+l}(\sqrt{\lambda}) J_{\frac{N}{2}-1+l}(\sqrt{\lambda}(1-\varepsilon)) \right) = 0. \end{aligned} \tag{6.2.23}$$

We expand the left-hand side of (6.2.23) in ε near $\varepsilon = 0$ up to the second order. Then we divide both sides of the expansion of (6.2.23) by ε and use formulas (6.2.14) and (6.2.15). From standard computations we obtain that equation (6.2.23) can be re-written in the following equivalent form

$$8(l(l+N-2) - \lambda) + 4(l(l+N-2) + \lambda)\varepsilon + R(\lambda, \varepsilon) = 0,$$

where $R(\lambda, \varepsilon) \in O(\varepsilon^2)$, uniformly in $\lambda > 0$, as $\varepsilon \to 0$. Thus, by letting $\varepsilon \to 0$ we obtain that the following asymptotic expansion of the eigenvalues $\lambda_l(\varepsilon)$ of (6.2.2) holds

$$\lambda(\varepsilon) = l(l+N-2) + l(l+N-2)\varepsilon + O(\varepsilon^2), \tag{6.2.24}$$

as $\varepsilon \to 0$. This concludes the proof of the theorem. □

Remark 6.2.25. *We note that in a neighborhood of $\varepsilon = 0$, the eigenvalues of problem* (6.2.2) *behave like* (6.2.24)*. In particular, we deduce that all the eigenvalues of* (6.2.2) *are not monotonic with respect to the inclusion of sets (this is shown in the case $N = 2$ for the first eigenvalue of problem* (6.2.2) *in [89]). In fact, if $0 < \varepsilon_1 < \varepsilon_2$ are sufficiently small (possibly depending on l), then $A_{\varepsilon_2} \supseteq A_{\varepsilon_1}$ and $\lambda_l(\varepsilon_2) > \lambda_l(\varepsilon_1)$. Conversely, if $\varepsilon_2 < \varepsilon_1 < 0$, then $A_{\varepsilon_2} \supseteq A_{\varepsilon_1}$ and $\lambda_l(\varepsilon_2) < \lambda_l(\varepsilon_1)$.*

6.2.3 Eigenvalues of the biharmonic operator on the annulus of $\mathbb{R}^N$

Let $\varepsilon \in]-1, 0[\cup]0, 1[$ and let $A_\varepsilon \subset \mathbb{R}^N$ be the set defined by (6.2.1). We consider the following eigenvalue problem for the biharmonic operator subject to Neumann boundary conditions on A_ε

$$\begin{cases} \Delta^2 u = \lambda u, & \text{in } A_\varepsilon, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial A_\varepsilon, \\ \mathrm{div}_{\partial\Omega}\left(D^2 u.\nu\right) + \frac{\partial \Delta u}{\partial \nu} = 0, & \text{on } \partial A_\varepsilon. \end{cases} \tag{6.2.26}$$

We recall that in spherical coordinates we have $\Delta u = \partial^2_{rr} u + \frac{N-1}{r}\partial_r u + \frac{1}{r^2}\Delta_S u$. The boundary conditions in (6.2.26) are written in spherical coordinates as

$$\begin{cases} \partial_{rr} u_{|r=1} = 0, \\ \partial_{rr} u_{|r=1-\varepsilon} = 0, \\ \frac{1}{r^2}\Delta_S\left(\partial_r u - \frac{u}{r}\right) + \partial_r\left(\Delta u\right)_{|r=1} = 0, \\ \frac{1}{r^2}\Delta_S\left(\partial_r u - \frac{u}{r}\right) + \partial_r\left(\Delta u\right)_{|r=1-\varepsilon} = 0. \end{cases} \tag{6.2.27}$$

By following [28], in order to find a solution to the differential equation $\Delta^2 u = \lambda u$, we factor the eigenvalue equation equation as

$$(\Delta + a^2)(\Delta - a^2)u = 0,$$

where $a > 0$ is such that $\lambda = a^4$. The eigenfunctions turn out to be linear combinations of v_1 and v_2 where $-\Delta v_1 = a^2 v_1$ and $\Delta v_2 = a^2 v_2$. It follows that

$$v_1 = \left(j_l(ar) + y_l(ar)\right) H_l(\theta), \qquad v_2 = \left(i_l(ar) + k_l(ar)\right) H_l(\theta),$$

where j_l, y_l are the ultraspherical Bessel functions of the first and second kind respectively, and i_l, k_l are the ultraspherical modified Bessel functions of the first and second kind respectively, and H_l is a spherical harmonic of some order $l \in \mathbb{N}_0$. Then, it is standard to prove that, given an eigenvalue λ of problem (6.2.26), a corresponding eigenfunction u has the following form (see [28] for more details)

$$u(r,\theta) = \left(A j_l(ar) + B y_l(ar) + C i_l(ar) + D k_l(ar)\right) H_l(\theta),$$

for $l \in \mathbb{N}_0$, where $a > 0$ is such that $a^4 = \lambda$ and $A, B, C, D \in \mathbb{R}$ are suitable constants. Now we impose the boundary conditions (6.2.27). We obtain i)

$$a^2 \left(Aj_l''(a) + By_l''(a) + Ci_l''(a) + Dk_l''(a)\right) = 0,$$

ii)

$$a^2 \left(Aj_l''(a-a\varepsilon) + By_l''(a-a\varepsilon) + Ci_l''(a-a\varepsilon) + Dk_l''(a-a\varepsilon)\right) = 0,$$

iii)

$$\begin{aligned}
&A\big(3l(l+N-2)j_l(a) + a(1-N)j_l'(a) \\
&\quad -2al(l+N-2)j_l'(a) + a^2(N-1)j_l''(a) + a^3 j_l'''(a)\big) \\
&\quad + B\big(3l(l+N-2)y_l(a) + a(1-N)y_l'(a) \\
&\quad -2al(l+N-2)y_l'(a) + a^2(N-1)y_l''(a) + a^3 y_l'''(a)\big) \\
&\quad + C\big(3l(l+N-2)i_l(a) + a(1-N)i_l'(a) \\
&\quad -2al(l+N-2)i_l'(a) + a^2(N-1)i_l''(a) + a^3 i_l'''(a)\big) \\
&\quad + D\big(3l(l+N-2)k_l(a) + a(1-N)k_l'(a) \\
&\quad -2al(l+N-2)k_l'(a) + a^2(N-1)k_l''(a) + a^3 k_l'''(a)\big) = 0,
\end{aligned}$$

iv)

$$\begin{aligned}
&A\left(\frac{3l(l+N-2)}{(1-\varepsilon)^3}j_l(a-a\varepsilon) + \frac{a(1-N)}{(1-\varepsilon)^2}j_l'(a-a\varepsilon)\right. \\
&\left. -\frac{2al(l+N-2)}{(1-\varepsilon)^2}j_l'(a-a\varepsilon) + \frac{a^2(N-1)}{1-\varepsilon}j_l''(a-a\varepsilon) + a^3 j_l'''(a-a\varepsilon)\right) \\
&\quad + B\left(\frac{3l(l+N-2)}{(1-\varepsilon)^3}y_l(a-a\varepsilon) + \frac{a(1-N)}{(1-\varepsilon)^2}y_l'(a-a\varepsilon)\right. \\
&\left. -\frac{2al(l+N-2)}{(1-\varepsilon)^2}y_l'(a-a\varepsilon) + \frac{a^2(N-1)}{1-\varepsilon}y_l''(a-a\varepsilon) + a^3 y_l'''(a-a\varepsilon)\right) \\
&\quad + C\left(\frac{3l(l+N-2)}{(1-\varepsilon)^3}i_l(a-a\varepsilon) + \frac{a(1-N)}{(1-\varepsilon)^2}i_l'(a-a\varepsilon)\right. \\
&\left. -\frac{2al(l+N-2)}{(1-\varepsilon)^2}i_l'(a-a\varepsilon) + \frac{a^2(N-1)}{1-\varepsilon}i_l''(a-a\varepsilon) + a^3 i_l'''(a-a\varepsilon)\right) \\
&\quad + D\left(\frac{3l(l+N-2)}{(1-\varepsilon)^3}k_l(a-a\varepsilon) + \frac{a(1-N)}{(1-\varepsilon)^2}k_l'(a-a\varepsilon)\right. \\
&\left. -\frac{2al(l+N-2)}{(1-\varepsilon)^2}k_l'(a-a\varepsilon) + \frac{a^2(N-1)}{1-\varepsilon}k_l''(a-a\varepsilon) + a^3 k_l'''(a-a\varepsilon)\right) \\
&= 0.
\end{aligned}$$

This is a system of four equations in four unknowns, $A, B, C, D \in \mathbb{R}$, which has solution if and only if the determinant of the associated matrix is zero.

For a fixed $l \in \mathbb{N}_0$, let $\mathcal{B}_l(\lambda, \varepsilon)$ be the matrix associated with the linear system. The eigenvalues are implicitly characterized by the equation $\det\mathcal{B}_l(\lambda, \varepsilon) = 0$. In principle, the equation can be handled in the same way as for the second order case, by using a suitable Taylor's expansion of $\det\mathcal{B}_l(\lambda, \varepsilon)$ in $\varepsilon = 0$ and then simplifying by using suitable recurrence formulas for cross products of Bessel functions and their derivatives. Actually, the computations become very long and involved, and it seems quite difficult to obtain a simple closed formulas for the limiting eigenvalues and their derivatives. We have used the software *Mathematica* to perform symbolic computations in the case $N = 2$. The software was not able to handle the case $N \geq 3$, which has a larger number of terms and an additional symbolic quantity N.

6.2.4 Symbolic computations for the eigenvalues of the biharmonic operator on the annulus of $\mathbb{R}^2$

We have used the software *Mathematica* for symbolic computations in order to expand and simplify the expression $\det\mathcal{B}_l(\lambda, \varepsilon)$ of Subsection 6.2.3.

As in the case of the Laplace operator, we have expanded in Taylor series the function $\det\mathcal{B}_l(\lambda, \varepsilon)$ up to the third order around $\varepsilon = 0$. We considered only the case $N = 2$. After a long computation time we obtained the following expression

$$\frac{1}{\pi}\left(-8\left(\lambda + 2(\lambda - 1)l^2 + 4l^4 - 2l^6\right)\varepsilon^2 + 8\left(\lambda + 2(1 + \lambda)l^2 - 4l^4 + 2l^6\right)\varepsilon^3\right) + O(\varepsilon^4) = 0.$$

Using the same arguments as in the case of the eigenvalues of the Laplace operator on the annulus, we find that near $\varepsilon = 0$ the behavior of the eigenvalue $\lambda_l(\varepsilon)$ is

$$\lambda_l(\varepsilon) = \frac{2l^2(l^2 - 1)^2}{1 + 2l^2} + \frac{4l^2(l^2 - 1)^2}{1 + 2l^2}\varepsilon + O(\varepsilon^2),$$

as $\varepsilon \to 0$, for all $l \in \mathbb{N}_0$. See Figure 6.3

This shows that also in the case of the biharmonic operator and $N = 2$ there is no monotonicity of all the eigenvalues under inclusion of domains.

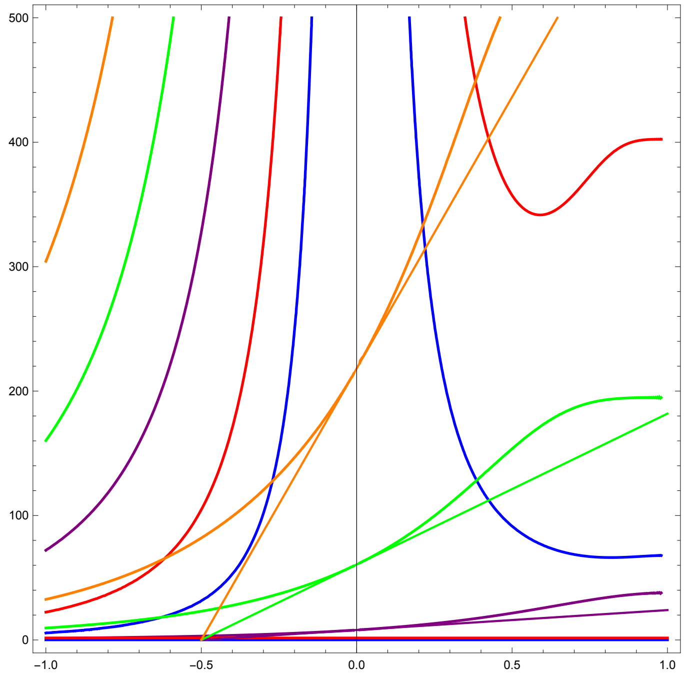


Figure 6.3: Solution branches of equation $\det \mathcal{B}(\lambda, \varepsilon)$ with $N = 2$ and $(\varepsilon, \lambda) \in]-1, 1[\times]0, 500[$. The colors refers to the choice of l in the equation. In particular, $l = 0$ (blue), $l = 1$ (red), $l = 2$ (purple), $l = 3$ (green), $l = 4$ (orange). We also plotted the line $\frac{2l^2(l^2-1)^2}{1+2l^2} + \frac{4l^2(l^2-1)^2}{1+2l^2}\varepsilon$ with $l = 2$ (purple), $l = 3$ (green) and $l = 4$ (orange).

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