Università degli Studi di Padova

Sede amministrativa: Università degli Studi di Padova

Dipartimento di Matematica "Tullio Levi-Civita"

CORSO DI DOTTORATO DI RICERCA IN SCIENZE MATEMATICHE

CURRICOLO MATEMATICA

CICLO XXXI

On some examples in higher Auslander–Reiten theory

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Riassunto

Questa tesi è dedicata allo studio di alcuni esempi di algebre d-representation finite e infinite. Queste algebre sono la generalizzazione delle classiche algebre di tipo di rappresentazione finito e infinito nel contesto della teoria di Auslander-Reiten in dimensione superiore, introdotta da Iyama negli anni 2000. Nella prima parte della tesi introduciamo nuovi esempi di algebre d-representation infinite sfruttando la loro relazione con le algebre bimodulo $(d+1)$-Calabi-Yau di parametro di Gorenstein 1. Per prima cosa richiameremo alcuni risultati di Bocklandt, Schedler e Wemyss, e di Herschend, Iyama e Oppermann, che mostrano alcuni esempi dati da skew group algebre di sottogruppi finiti di gruppi lineari generali. Come generalizzazione delle classiche algebre ereditarie tame di tipo $\tilde{\mathrm{A}}$, Herschend, Iyama e Oppermann hanno introdotto le algebre d-representation infinite di tipo $\tilde{\mathrm{A}}$ considerando sottogruppi abeliani di $\mathrm{SL}(d+1,\mathbb{C})$. Nel nostro lavoro otteniamo una costruzione simile per alcuni sottogruppi non abeliani. Più precisamente, studiamo la skew group algebra di alcuni sottogruppi metaciclici immersi in $\mathrm{SL}(s,\mathbb{C})$ e $\mathrm{SL}(s+1,\mathbb{C})$, quando s è un numero primo. A tale scopo diamo una descrizione dei quiver di McKay con superpotenziale di tali gruppi, e ne consideriamo dei grading ottenuti da tagli. Inoltre, mostriamo che per $s=2$ i nostri esempi corrispondono alle classiche algebre ereditarie tame di tipo $\tilde{\mathrm{D}}$. La seconda parte della tesi tratta di skew group algebre di algebre Jacobiane. Dimostriamo che, se un gruppo ciclico finito agisce sull'algebra Jacobiana $\mathcal{P}(Q,W)$ di un quiver con potenziale (Q,W) soddisfando alcune assunzioni, allora la skew group algebra $\mathcal{P}(Q,W)*G$ è Morita equivalente all'algebra Jacobiana di un altro quiver con potenziale $Q_G, W_G)$, che è descritto esplicitamente. Il quiver con potenziale originale può essere quindi recuperato tramite una costruzione di skew group algebra rispetto a un'azione naturale del gruppo duale di G. Una delle motivazioni per questo lavoro risiede in un teorema di Herschend e Iyama, che mette in relazione i quiver con potenziale auto-iniettivi alle algebre 2-representation finite. Alla luce di ciò, studiamo il comportamento dei tagli su tali quiver sotto la nostra costruzione, allo scopo di ottenere nuovi esempi di algebre 2-representation finite da quelli già conosciuti. Presentiamo alcuni esempi dove la nostra costruzione può essere applicata, come i quiver con potenziale planari dove il gruppo agisce per rotazioni, e in particolare quelli auto-iniettivi che sono ottenuti da diagrammi di Postnikov.

Abstract

This thesis is devoted to the study of some examples of d-representation finite and infinite algebras. These algebras are the generalization of the classical representation finite and infinite ones in the context of higher dimensional Auslander-Reiten theory, which was introduced by Iyama in the 2000's. In the first part of the thesis we introduce new examples of d-representation infinite algebras exploiting their relation with bimodule $(d+1)$-Calabi Yau algebras of Gorenstein parameter 1. Firstly we review some results by Bocklandt, Schedler and Wemyss, and by Herschend, Iyama and Oppermann, which show some examples given by skew group algebras of finite subgroups of general linear groups. As a generalization of the classical tame hereditary algebras of type $\tilde{\mathrm{A}}$, Herschend, Iyama and Oppermann introduced d-representation infinite algebras of type $\tilde{\mathrm{A}}$ by considering abelian subgroups of $\mathrm{SL}(d+1, \mathbb{C})$. In our work we achieve a similar construction for some non-abelian groups. More precisely, we study the skew group algebra of some metacyclic groups embedded in $\mathrm{SL}(s, \mathbb{C})$ and $\mathrm{SL}(s+1, \mathbb{C})$, when s is a prime number. To this aim we give a description of the McKay quivers with superpotential of such groups, and we consider gradings on them obtained from cuts. Moreover, we show that for $s = 2$ our examples correspond to the classical tame hereditary algebras of type $\tilde{\mathrm{D}}$. The second part of the thesis deals with skew group algebras of Jacobian algebras. We prove that, if a finite cyclic group G acts on the Jacobian algebra $\mathcal{P}(Q, W)$ of a quiver with potential (Q, W) satisfying some assumptions, then the skew group algebra $\mathcal{P}(Q, W) * G$ is Morita equivalent to the Jacobian algebra of another quiver with potential (Q_G, W_G), which is explicitly described. The original quiver with potential can then be recovered by a skew group algebra construction with a natural action of the dual group of G. One of the motivations for this work lies in a theorem of Herschend and Iyama, which relates self-injective quivers with potential to 2-representation finite algebras. In view of this, we study the behaviour of cuts on such quivers under our construction, in order to obtain new examples of 2-representation finite algebras from old ones. We present some examples where our construction can be applied, such as planar quivers with potentials where the group acts by rotations, and in particular the self-injective ones which are obtained by Postnikov diagrams.

Introduction

Auslander-Reiten theory was developed in a series of papers during the seventies [3, 4, 5, 6, 7] and it has since represented one of the most important tools in the representation theory of associative algebras. At the beginning of the 2000's, this theory has been generalized to "higher dimensions" by Iyama and his coauthors [37, 36, 38, 39, 40, 41, 28, 29, 31, 30, 43], providing new insights not only into representation theory but also into other areas such as algebraic geometry and commutative algebra.

The classical Auslander-Reiten theory aims to describe the category of finitely generated modules $\operatorname{mod} A$ over an Artin algebra A. In its higher dimensional analogue the object of study is not the whole category $\operatorname{mod} A$ anymore, but some special subcategories called d-cluster tilting subcategories. Namely, a full subcategory $\mathcal{C}$ of $\operatorname{mod} A$ is called d-cluster tilting if it is functorially finite (i.e., every module in $\operatorname{mod} A$ has both a left and a right $\mathcal{C}$-approximation) and

$$\begin{aligned}\mathcal{C} &= \{X \in \operatorname{mod} A \mid \operatorname{Ext}^i_A(\mathcal{C}, X) = 0 \text{ for all } i = 1, \dots, d-1\} \\ &= \{X \in \operatorname{mod} A \mid \operatorname{Ext}^i_A(X, \mathcal{C}) = 0 \text{ for all } i = 1, \dots, d-1\}.\end{aligned}$$

In particular, the only 1-cluster tilting subcategory of $\operatorname{mod} A$ is $\operatorname{mod} A$ itself.

Indecomposable modules play a central role in Auslander-Reiten theory, since each module has an essentially unique direct sum decomposition into indecomposables. Thus it comes natural to classify algebras into two families: the representation finite ones are the ones which have finitely many isomorphism classes of indecomposables, while all the others are called representation infinite. The latter can be further subdivided into two families: the tame and the wild algebras. Hereditary representation finite algebras over an algebraically closed field were classified by Gabriel [21]: they are given by path algebras of simply laced Dynkin quivers. By taking the extended analogues of these quivers one gets all hereditary representation tame algebras [19, 51].

Both representation finite and infinite algebras have a generalization in higher Auslander-Reiten theory. The d-representation finite algebras were introduced by Iyama and Oppermann in [40]: they are the algebras of global dimension at most d which have a d-cluster tilting module (i.e., a module M such that $\operatorname{add} M$ is a d-cluster tilting subcategory of $\operatorname{mod} A$). The definition of a d-representation infinite algebra, introduced by Herschend, Iyama and Oppermann in [31], is a little more complicated and we will give it in §1.4.2.

The idea is to generalize the following property of representation infinite algebras: if we apply any power of the inverse Auslander-Reiten translation to an indecomposable projective module, then we always get a non-zero module.

The definitions of hereditary representation finite and infinite algebras can be recovered by putting $d = 1$ in the above ones. While we have a complete classification of these algebras in this case, the same cannot be said for $d \geq 2$. A family of d-representation finite algebras, which generalize the classical representation finite algebras of type A, was defined in [40]. Apart from them, the only explicit examples of d-representation finite algebras known so far were constructed for $d = 2$ by Herschend and Iyama [29] and Pasquali [54].

Regarding d-representation infinite algebras, there is a family of examples, introduced in [31], which generalizes the classical representation infinite algebras of type $\tilde{\mathrm{A}}$. Moreover, additional examples can be obtained from tensor products [31, §2.1]. Apart from the aforementioned ones, the only other examples arise in connection with non-commutative algebraic geometry [49, 30, 14].

The aim of this thesis is to construct explicitly more examples of d-representation finite and infinite algebras. A key role will be played by higher preprojective algebras. Given an algebra A of global dimension at most d, its $(d+1)$-preprojective algebra is defined as the tensor algebra $\Pi_{d+1}(A) := T_A \operatorname{Ext}^d_A(A^*, A)$. There is a natural grading on it such that its degree 0 part is A. When $d = 1$, $\Pi_{d+1}(A)$ is isomorphic to the classical preprojective algebra which was first defined by Gel'fand and Ponomarev in [22]. It is shown in [41] and [31] that the property of being d-representation finite, respectively infinite, can be translated into properties of the corresponding higher preprojective algebra. The situation is particularly nice in the case of 2-representation finite algebras. In fact, it was shown in [29] that their higher preprojective algebras are exactly the self-injective Jacobian algebras of quivers with potential. The latter have been widely studied in representation theoretical contexts in recent years, mainly because of their importance in the theory of cluster algebras [17, 18, 13].

The main tool we will exploit for constructing new examples is given by skew group algebras. These constructions allow us to obtain new algebras from old ones by means of group actions. In Chapter 2 we will consider skew group algebras of polynomial rings by actions of finite subgroups of general linear groups. When considering subgroups of $\mathrm{SL}(2, \mathbb{C})$, such algebras are exactly the preprojective algebras of hereditary representation tame algebras [58]. As a generalization of this result, in our work we will use results of [11] and [31] in order to obtain d-representation infinite algebras from subgroups of $\mathrm{SL}(d+1, \mathbb{C})$. In Chapter 3 we will instead consider skew group algebras of Jacobian algebras. In particular, we will take advantage of the description given by Reiten and Riedtmann [57] of the skew group algebra of the path algebra of a quiver bound by relations, where the group acting on it is finite and cyclic. We will prove that, under some conditions, we obtain in this way another Jacobian algebra.

The thesis is divided into three chapters. In Chapter 1 we will set up some notation and introduce some preliminary notions. In Chapter 2 we will construct examples of d-representation infinite algebras coming from skew group algebras of a family of subgroups of $\mathrm{SL}(d+1)$: this construction works only when either d or $d+1$ is a prime number. This part of the thesis is based on paper [24], which was written by the author during a visiting period at Uppsala University. Chapter 3 is dedicated to the study of skew group algebras of Jacobian algebras. We will prove that, under some assumptions, the property of being the Jacobian algebra of a quiver with potential is preserved under a skew group algebra construction. This in turn will allow us to find new examples of 2-representation finite algebras, thanks to the results of [29] which relate them with quivers with potential. These results were obtained by the author in the joint work [25] with Andrea Pasquali.

Next we give a more detailed description of each chapter of the thesis.

Chapter 1: Preliminaries

This chapter is devoted to introducing some basic facts and notations, and to reviewing some preliminary known results from the literature which will be needed in the rest of the thesis.

In Section 1.1 we fix some notations about quivers and path algebras, which are key tools in our work. In Section 1.2 we review some basic facts about bimodules and gradings which will be used when studying properties of higher preprojective algebras. In Section 1.3 we introduce skew group algebras, which will play a key role in Chapters 2 and 3. In Section 1.4 we will give a brief review of some basic constructions in higher dimensional Auslander-Reiten theory and in particular we will introduce d-representation finite and infinite algebras.

Chapter 2: Higher representation infinite algebras from metacyclic groups

In this chapter we will introduce examples of $(s-1)$- and s-representation infinite algebras for each prime number s. To this aim we will exploit the following result.

Theorem 1 ([1],[45],[49],[31]). *There is a bijection between isomorphism classes of d-representation infinite algebras and isomorphism classes of bimodule $(d+1)$-Calabi-Yau algebras of Gorenstein parameter 1 with finite dimensional degree 0 part. This bijection is realized by sending a d-representation infinite algebra A to its higher preprojective algebra $\Pi_{d+1}(A)$ and a bimodule $(d+1)$-Calabi-Yau algebra B of Gorenstein parameter 1 to its degree 0 part B_0.*

Bimodule $(d+1)$-Calabi-Yau algebras of Gorenstein parameter 1 are a special class of positively graded algebras. Their definition will be given in Section 2.2: here we just mention that being bimodule $(d+1)$-Calabi-Yau is a property of the algebra itself and does not involve the grading, while the Gorenstein parameter is an attribute of the grading.

A source of bimodule $(d+1)$-Calabi-Yau algebras is provided by the following construction. Let V be a $\mathbb{C}$-vector space of dimension $d+1$ and let $\mathbb{C}[V]$ be the algebra of polynomial functions in V. Any finite subgroup G of $\mathrm{SL}(V)$ has a natural action on $\mathbb{C}[V]$, and we can construct the skew group algebra $\mathbb{C}[V] * G$. The latter is Morita equivalent to the path algebra of a quiver Q_G (called the McKay quiver of G) modulo some relations: we will denote this algebra by Π_G. It is proved in [11] that the relations in Π_G are induced by a superpotential ω_G and that both $\mathbb{C}[V] * G$ and Π_G are bimodule $(d+1)$-Calabi-Yau algebras.

According to Theorem 1, in order to obtain d-representation infinite algebras we need to find suitable gradings on Π_G. If $d = 1$ this is always possible by the following theorem.

Theorem 2 ([58]). *Let G be a non-trivial finite subgroup of $\mathrm{SL}(V, \mathbb{C})$, where $\dim_{\mathbb{C}} V = 2$. Then the skew group algebra $\mathbb{C}[V] * G$ is Morita equivalent to the preprojective algebra of a quiver whose underlying graph is an extended simply laced Dynkin diagram.*

Note that assigning to the preprojective algebra of an extended Dynkin diagram a grading which satisfies Theorem 1 is equivalent to giving an orientation of such diagram. Hence, by Theorem 2, we can obtain all hereditary tame algebras, since the latter are classified by extended Dynkin diagrams.

The correspondence between finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ and simply laced Dynkin diagrams is given in the following way:

{Finite subgroups of $\mathrm{SL}(2, \mathbb{C})$}	$\longleftrightarrow$	{Simply laced Dynkin diagrams}
Cyclic		Type A_m
Binary dihedral		Type D_m
Binary tetrahedral		Type E_6
Binary octahedral		Type E_7
Binary icosahedral		Type E_8

This bijection was already known as "McKay correspondence" [48], and it has the following geometrical interpretation. Consider the quotient space $\mathbb{C}^2/G$, where G is a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$ acting naturally on $\mathbb{C}^2$. This space has a singularity at the origin 0 and we can take a minimal resolution $Y \to \mathbb{C}^2/G$: then the preimage of 0 is a union of irreducible components isomorphic to $\mathbb{P}^1_{\mathbb{C}}$, and the corresponding intersection graph is exactly the Dynkin diagram associated to G.

In view of Theorem 2, we can ask the following question.

Question 1. *If G is a finite subgroup of $\mathrm{SL}(d+1, \mathbb{C})$, then is $\mathbb{C}[V] * G$ Morita equivalent to the $(d+1)$-preprojective algebra of some d-representation infinite algebra?*

In other words, can we always find a grading on Π_G which gives it Gorenstein parameter 1 and such that the degree 0 part is finite dimensional?

Unfortunately, the answer to this question can be negative if $d \geq 2$. For example, Thibault proved in [60] that if G is conjugate to a subgroup of $\mathrm{SL}(d_1, \mathbb{C}) \times \mathrm{SL}(d_2, \mathbb{C})$ for some $d_1, d_2 \geq 1$ such that $d_1 + d_2 = d + 1$, then $\mathbb{C}[V] * G$ cannot be Morita equivalent to a higher preprojective algebra.

On the other end, there are also situations where the answer to Question 1 is affirmative. For example, this is the case for the d-representation infinite algebras of type $\tilde{\mathrm{A}}$ defined in [31], which are obtained from abelian subgroups of $\mathrm{SL}(d+1, \mathbb{C})$.

Our aim is to find other examples of groups for which the answer to Question 1 is affirmative. Given a prime number s and positive integers m, r, t satisfying certain conditions, we will consider the finite subgroup G of $\mathrm{GL}(s,\mathbb{C})$ generated by the following matrices:

$$\alpha = \begin{pmatrix} \varepsilon_m & 0 & \cdots & 0 \\ 0 & \varepsilon_m^r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon_m^{r^{s-1}} \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & 0 & \cdot & 0 & \varepsilon_m^t \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

where ε_m is a primitive m-th root of unity. The conditions we impose imply that G satisfies the following properties:

- the subgroup A generated by α is normal in G, and it is cyclic of order m;
- the quotient G/A is cyclic of order s and so it is simple, since s is a prime number;
- the conjugation by β induces an action of G/A on A, where the generator of G/A sends α to α^r.

In particular we have that G is a metacyclic group, i.e., it is an extension of cyclic groups. Moreover, we will assume that α is not a scalar multiple of the identity: this implies that G is not abelian, and so our examples are different from the ones studied in [31].

For such a group G the following two cases can occur:

(SL) G is contained in $\mathrm{SL}(s,\mathbb{C})$;

(GL) G is not contained in $\mathrm{SL}(s,\mathbb{C})$.

In case (GL), Π_G is the path algebra of the McKay quiver of G modulo some relations which are induced by a *twisted* superpotential ω_G. In particular, Π_G is not bimodule s-Calabi-Yau: however, there is a natural embedding of $\mathrm{GL}(s,\mathbb{C})$ in $\mathrm{SL}(s+1,\mathbb{C})$ and we can consider the image of G under it, which we denote by G'. Then $\Pi_{G'}$ is $(s+1)$-Calabi-Yau, since $G' \leq \mathrm{SL}(s+1,\mathbb{C})$, and $\Pi_{G'}$ and the superpotential $\omega_{G'}$ can be easily obtained from Π_G and ω_G.

In Section 2.5 we will give a description of the McKay quiver Q_G of a metacyclic group G and of a (twisted) superpotential ω_G. For this purpose, we will rely on the already known description of the McKay quiver Q_A of the abelian subgroup $A \leq G$, and we will exploit a G/A-action on Q_A. Moreover, we will prove in Theorem 2.5.19 that every path in the superpotential ω_G of G in a certain sense "comes from a path in the superpotential of A": in this way we obtain only a partial description of ω_G, but it will suffice for our purposes. In fact, we will use this result to prove, in Proposition 2.5.23, that if we have a grading on Q_A such that ω_A is homogeneous of degree a and which satisfies an invariance hypothesis, then there exists a grading on Q_G such that ω_G is homogeneous of degree a. An analogous result is proven in Proposition 2.5.24 for the embedded group $G' \leq \mathrm{SL}(s+1,\mathbb{C})$.

In Section 2.6 we will describe explicitly some gradings satisfying the above assumptions. Following [31], we will give a geometric picture of the McKay quiver of Q_A and we will show that we can obtain gradings on this quiver by considering particular subsets of arrows called *cuts*. Moreover, we will show that gradings satisfying Proposition 2.5.23 can be obtained from cuts which are invariant under the G/A-action.

In Section 2.7 we will give some examples where the previous results can be applied. In particular, we will define two families of metacyclic groups $M(s,b) \leq \mathrm{SL}(s,\mathbb{C})$ and $\hat{M}(s,b) \leq \mathrm{GL}(s,\mathbb{C})$, for all prime numbers s and for all integers $b \geq 1$. We will show that both this families of groups give a positive answer to Question 1: more precisely, we have the following result.

Theorem 3 (Corollary 2.7.5)**.** *Let s be a prime number.*

(a) For each integer $b \geq 1$, there exists an $(s-1)$-representation infinite algebra which is the degree 0 part of Π_G, where $G = M(s,b)$.

(b) For each integer $b \geq 2$ such that $(b,s) = 1$, there exists an s-representation infinite algebra which is the degree 0 part of $\Pi_{G'}$, where $G = \hat{M}(s,b)$ and G' is its embedding in $\mathrm{SL}(s+1,\mathbb{C})$.

Finally, we will compute some examples for $s = 2, 3$. In the (SL) case, for $s = 2$, we will show that with our construction we obtain all tame hereditary algebras of type $\tilde{\mathrm{D}}$. For $s = 3$, the groups considered belong to the family of *trihedral groups*, which have already been studied from a geometric point of view (see for example [34, 35, 47]).

Other groups which also raised some interest in geometry are binary dihedral groups in $\mathrm{GL}(2,\mathbb{C})$ (see [53, 47]): they can be obtained from our construction in the (GL) case with $s = 2$.

Chapter 3: Skew group algebras of Jacobian algebras and 2-representation finite algebras

The aim of this chapter is to study the skew group algebra of a Jacobian algebra coming from a quiver with potential (QP for short). As we mentioned before, one of our main motivations is given by the following result (see Section 3.1 for the definitions).

Theorem 4 ([29, Theorem 3.11])**.** *If (Q,W) is a self-injective QP with a cut C, then the corresponding truncated Jacobian algebra is 2-representation finite. Moreover, every basic 2-representation finite algebra is obtained in this way.*

Let Λ be a finite-dimensional algebra of the form $\Bbbk Q/I$, where Q is a quiver and I is an admissible ideal. If G is a finite group acting on Λ by automorphisms, then Reiten and Riedtmann [57] describe the quiver Q_G of (a basic version of) the skew group algebra $\Lambda * G$. This description is complete if G is cyclic, and Demonet extended it to a complete description for arbitrary finite groups if Λ is hereditary [16]. However, describing the relations on this quiver is quite difficult in general.

As we already mentioned, we are interested in the case where I is induced by a potential W in Q. It turns out that the skew group algebra of Λ is Morita equivalent to a Jacobian algebra. This result was proven in [46] in greater generality: the action of G on the QP induces an action on the corresponding Ginzburg dg algebra defined in [23], and the skew group dg algebra is Morita equivalent to the Ginzburg dg algebra associated to another QP. The quiver obtained is Q_G, and the potential is the image of W under a natural map. The result about Jacobian algebras can then be recovered by taking the 0-th homology of the corresponding Ginzburg dg algebras. However, the new potential is not explicitly described; it is expressed as a linear combination of cycles of Q_G only in some examples [46, §4.5].

In this thesis we eschew the dg setting and focus on Jacobian algebras of QPs. Our aim is, under some assumptions on the group action, to construct explicitly a potential for the skew group algebra we mentioned above. More precisely, we will prove the following result.

Theorem 5 (Theorem 3.3.7)**.** *Let (Q, W) be a QP, and let $\Lambda = \mathcal{P}(Q, W)$ be its Jacobian algebra. Let G be a finite cyclic group acting on (Q, W) as per the assumptions (A1)-(A7) of §3.3.1. Let Q_G be the quiver constructed in §3.3.2, W_G the potential on Q_G defined in §3.3.3, and $\eta \in \Lambda * G$ the idempotent defined in §3.3.1. Then*

$$\mathcal{P}(Q_G, W_G) \cong \eta(\mathcal{P}(Q, W) * G)\eta.$$

Note that in general Jacobian algebras are defined as quotients of the complete path algebra of a quiver. In our results, however, we only consider the case where taking the completion is superfluous and we can actually work with the usual path algebra.

As observed in [57, §5], there is a natural action of the dual group $\hat{G}$ on $\Lambda * G$, which restricts to an action on the basic algebra $\eta(\Lambda * G)\eta$. Reiten and Riedtmann prove that $(\Lambda * G) * \hat{G}$ is Morita equivalent to Λ if G is abelian, so it is natural to ask whether one gets back the original QP by applying this second skew group algebra construction. To do so, one needs to find assumptions which guarantee that W_G is fixed by $\hat{G}$ as an element of $\Bbbk Q_G \cong \eta((\Bbbk Q) * G)\eta$, and which are preserved under taking skew group algebras. If $G = \mathbb{Z}/2\mathbb{Z}$, it was shown in [2] that indeed we get (Q, W) back (and in fact the Ginzburg dg algebra of (Q, W)). We extend Amiot and Plamondon's result to our setting (assumptions (A1)-(A7) of §3.3.1), via a direct check using our formula for W_G:

Theorem 6 (Proposition 3.5.3 and Corollary 3.5.4)**.** *There is an isomorphism of quivers $\phi : (Q_G)_{\hat{G}} \cong Q$ such that, if we extend it to an isomorphism between the corresponding path algebras, we have $\phi((W_G)_{\hat{G}}) = W$. This induces an algebra isomorphism*

$$\theta\left((\eta\,(\Lambda * G)\,\eta) * \hat{G}\right)\theta \cong \Lambda,$$

where $\Lambda = \mathcal{P}(Q, W)$ and θ is the idempotent defined in Section 3.5.

A simple example of the above construction which is good to have in mind is illustrated in Example 3.8.1, and specifically in the quivers of Figure 3.4 and Figure 3.5. Here we take Q to be the QP of the 3-preprojective algebra of type A_4, so the potential is given by the sum of all 3-cycles with alternating signs. The group $G = \mathbb{Z}/3\mathbb{Z}$ acts by rotations in the plane, and the quiver Q_G is given in Figure 3.5. Here the action of $\hat{G}$ permutes the vertices 4^i and multiplies the arrow $\tilde{\delta}$ by a third root of unity, and one can check that by performing the same construction on Q_G one gets Q back.

In view of Theorem 4, we look with special interest at the case where $\Lambda = \mathcal{P}(Q, W)$ is self-injective. In [57] it is proved that the skew group algebra construction preserves self-injectivity. We show that it also preserves the property of being Frobenius, and compute a Nakayama automorphism of $\Lambda * G$ if the bilinear form on Λ is G-equivariant. As a consequence, we prove that if Λ is the Jacobian algebra of a planar self-injective QP and we take G generated by a Nakayama automorphism, then $\Lambda * G$ is symmetric. We show that G-invariant cuts on (Q, W) induce cuts on (Q_G, W_G), and the corresponding truncated Jacobian algebras are obtained from each other by a skew group algebra construction. Thus we have that, under some hypotheses, 2-representation finiteness is preserved under taking skew group algebras. Moreover we give some sufficient conditions on (Q, W) which imply that all the truncated Jacobian algebras of (Q_G, W_G) are derived equivalent. It was recently shown in [46], by different methods, that in fact the property of being d-representation (in)finite is always preserved under taking skew group algebras. An example where the 2-representation finite algebra is constructed from tensor product of Dynkin quivers is illustrated in Example 3.8.6. We also look at a case where Λ is not self-injective in Example 3.8.7. Here we realise an Auslander algebra as a truncated Jacobian algebra, thus checking directly a special case of [57, Theorem 1.3(c)(iv)].

There is a natural class of QPs with a group action satisfying our assumptions, namely rotation-invariant planar QPs. Planar QPs were introduced in [29] as they behave particularly nicely when they have self-injective Jacobian algebras. It turns out that in all known examples of self-injective planar QPs a Nakayama automorphism acts by a rotation, hence they fit nicely in our setting. Recently it has been shown that Postnikov diagrams have connections with planar self-injective QPs: in [54] it is proved that the QP coming from an (a, n)-Postnikov diagram on a disk (as in [9]) is self-injective if and only if the diagram is rotation invariant. Thus, our construction produces many examples of symmetric Jacobian algebras, one for every such Postnikov diagram. An example is given as Example 3.8.3.

Contents

Chapter 1

Preliminaries

In this chapter we will set up notations state some preliminary results which will be used in the following chapters.

Throughout the thesis we will use the following notations and conventions. We fix a base field $\Bbbk$. Algebras are assumed to be associative unital $\Bbbk$-algebras. We denote by $\operatorname{rad} A$ the Jacobson radical of an algebra A.

Unless stated otherwise, modules are considered right modules. If A is an algebra, we denote by $\operatorname{mod} A$ the category of finitely generated A-modules. For an A-module M, we denote by $\operatorname{add} M$ the full subcategory of $\operatorname{mod} A$ which consists of direct summands of finite direct sums of copies of M.

For an algebra A, we say that $\operatorname{gl.dim} A \leq d$ if every A-module has a projective resolution of length at most d.

For an A-module M, we denote by $M^* = \operatorname{Hom}_{\Bbbk}(M, \Bbbk)$ its standard dual, which has a natural structure of left A-module.

Let A be an algebra and $\varphi\colon A \to A$ be an algebra endomorphism. For an A-module M, we define M_φ to be the right A-module which is equal to M as a vector space but whose action is given by $m \cdot a = m\varphi(a)$, for all $m \in M$ and $a \in A$.

If X is a subset of a ring A, we denote by $\langle X \rangle$ the two-sided ideal of A generated by X.

1.1 Quivers and path algebras

A *quiver* $Q = (Q_0, Q_1, s, t)$ is the data of a set of vertices Q_0, a set of arrows Q_1 and two maps $\mathfrak{s}, \mathfrak{t}\colon Q_1 \to Q_0$ which assign to an arrow respectively its source and its target. The *path algebra* $\Bbbk Q$ of Q is defined as the free $\Bbbk$-vector space generated by all paths in Q, with product given by concatenation. We use the convention that the product $\alpha\beta$ of two arrows is intended as "first do β, then α". If p is a path in a quiver and α is an arrow, we use the notation $\alpha \in p$ to indicate that α appears as one of the arrows in p. A *relation* of a quiver is a linear combination of paths with the same start and end.

Let Q be a quiver. An ideal $I \subseteq \Bbbk Q$ is called *admissible* if there exists an $N \geq 2$ such that $\langle Q_1 \rangle^N \subseteq I \subseteq \langle Q_1 \rangle^2$. It is well known that any finite dimensional algebra is Morita equivalent to $\Bbbk Q/I$ for some quiver Q and some admissible ideal I.

1.2 Bimodules and gradings

In this section we will set up some notation about bimodules and graded algebras and modules. Gradings will play a central role throughout this thesis, since higher preprojective algebras have a natural graded structure. Moreover, bimodules will be relevant in the characterization of d-representation infinite algebras in Section 2.2.

Let A be an algebra. An A-*bimodule* M is a left A-module and a right A-module such that $(am)b = a(mb)$ and $\lambda m = m\lambda$ for all $a, b \in A$, $m \in M$, $\lambda \in \Bbbk$. Let $A^e := A^{\mathrm{op}} \otimes_{\Bbbk} A$ be the enveloping algebra of A. Then an A^e-module M can be considered as an A-bimodule by putting $amb = m(a \otimes b)$ for all $m \in M$, $a, b \in A$.

Given an A^e-module M, we define the bimodule dual $M^\vee := \mathrm{Hom}_{A^e}(M, A^e)$: we will regard it as an A^e-module with action given by $(\psi(a \otimes b))(m) = (b \otimes a)\psi(m)$ for all $\psi \in M^*$, $m \in M$, $a, b \in A$. We can also give an A^e-module structure to the standard dual M^* by setting $(\psi(a \otimes b))(m) = \psi(m(b \otimes a))$.

All the gradings we will consider are over $\mathbb{Z}$, unless stated otherwise. For a homogeneous element x in a graded algebra or module, we will often denote its degree by $|x|$ when no confusion arises from what grading we are considering. If A is a graded algebra and $M = \bigoplus_{d \in \mathbb{Z}} M_d$ is a finite dimensional graded A-module, then we can give the standard dual M^* a structure of graded A-module by setting $(M^*)_d = M^*_{-d}$. If $M = \bigoplus_{d \in \mathbb{Z}} M_d$ is a finitely generated graded projective A-module, then we have a natural grading on the dual $\mathrm{Hom}_A(M, A)$, where the degree d part $\mathrm{Hom}_A(M, A)_d$ is given by morphisms $M \to A$ which are homogeneous of degree d.

If A is a graded algebra, then A^e inherits naturally a graded structure. The functor $(-)^\vee$ preserves the finitely generated graded projective A^e-modules, where the graded structure on $M^\vee$ is given as above.

An example of a graded algebra which will be important in our work is the tensor algebra of an A-bimodule M, which is defined as $T_A M = \bigoplus_{i \geq 0} M^{\otimes_A i}$, where $M^{\otimes_A i} := M \otimes_A \overset{(n \text{ times})}{\cdots} \otimes_A M$.

1.3 Skew group algebras

Skew group algebras will be widely used throughout this thesis. Here we recall their definition.

Let G be a finite group acting on an algebra Λ by algebra automorphisms.

Definition 1.3.1. The *skew group algebra* $\Lambda * G$ is the algebra defined by:

- its underlying vector space is $\Lambda \otimes_{\Bbbk} \Bbbk G$;

- multiplication is given by

$$(\lambda \otimes g)(\mu \otimes h) = \lambda g(\mu) \otimes gh$$

for $\lambda, \mu \in \Lambda$ and $g, h \in G$, extended by linearity and distributivity.

There is a natural algebra monomorphism $\Lambda \to \Lambda * G$ given by $\lambda \mapsto \lambda \otimes 1$. Notice that the algebra $\Lambda * G$ is not basic in general.

It is shown in [57] that many representation theoretical properties of Λ are inherited by $\Lambda * G$. In particular, this holds for the property of being self-injective: this fact will be of particular interest in Chapter 3.

1.4 Higher Auslander-Reiten theory

In this section we introduce d-representation finite and infinite algebras, which will play a central role in the rest of the thesis. They both have finite global dimension, so we will start by reviewing some general notions about algebras satisfying this property. For the results of this sections we mainly follow [31] and [40].

Let $d \geq 1$ be an integer and let A be a finite dimensional algebra of global dimension at most d. There are functors

$$\nu := \operatorname{Hom}_A(-, A_A)^* \colon \operatorname{mod} A \to \operatorname{mod} A, \quad \nu^{-1} := \operatorname{Hom}_A((-)^*, {}_AA) \colon \operatorname{mod} A \to \operatorname{mod} A$$

which induce quasi-inverse equivalences between the subcategories $\operatorname{proj} A$ and $\operatorname{inj} A$ of, respectively, finitely generated projective and injective A-modules. So we get induced quasi-inverse triangle equivalences between the respective bounded homotopy categories $\mathbf{K}^{\mathrm{b}}(\operatorname{proj} A)$ and $\mathbf{K}^{\mathrm{b}}(\operatorname{inj} A)$. Since $\operatorname{gl.dim} A \leq d$, both these categories are triangle equivalent to the bounded derived category $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$, hence we obtain quasi-inverse triangle equivalences

$$\mathbf{R}\operatorname{Hom}_A(-, A_A)^* \colon \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A) \to \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A),$$
$$\mathbf{R}\operatorname{Hom}_A((-)^*, {}_AA) \colon \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A) \to \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$$

which we will call again, respectively, ν and ν^{-1}. Moreover, by [27], we have that ν is a Serre functor [12] of $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$.

To an algebra of finite global dimension we can associate an higher preprojective algebra.

Definition 1.4.1. Let A be a finite dimensional algebra of global dimension at most d. Its $(d+1)$-*preprojective algebra* $\Pi_{n+1}(A)$ is defined as the tensor algebra

$$\Pi_{d+1}(A) := T_A \operatorname{Ext}_A^d(A^*, A)$$

of the A-bimodule $\operatorname{Ext}_A^d(A^*, A)$.

For an algebra A, having global dimension at most 1 is equivalent to being hereditary. If A is basic and finite dimensional, this means that $A \cong \Bbbk Q$ for a finite acyclic quiver Q. In this case $\Pi_2(A)$ is isomorphic to the classical preprojective algebra [22] of Q (see [59] for a proof).

1.4.1 d-representation finite algebras

We will now review the definition of d-representation finite algebras, which were defined in [40] in order to generalize the hereditary representation finite algebras. In Chapter 3 we will construct some examples of them for $d = 2$.

We recall that an A-module M is called *d-cluster tilting* if

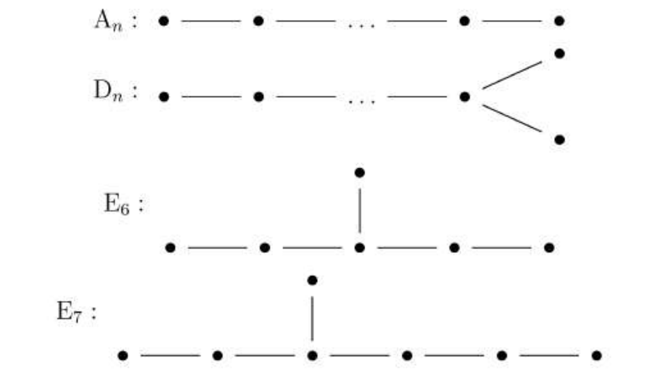
$$\begin{aligned} \operatorname{add} M &= \{X \in \operatorname{mod} A \mid \operatorname{Ext}^i_A(M, X) = 0 \text{ for all } i = 1, \ldots, d-1\} \\ &= \{X \in \operatorname{mod} A \mid \operatorname{Ext}^i_A(X, M) = 0 \text{ for all } i = 1, \ldots, d-1\}. \end{aligned}$$

Definition 1.4.2. A finite dimensional algebra A is *d-representation finite* if $\operatorname{gl.dim} A \leq d$ and there exists a d-cluster tilting module.

Note that a 1-cluster tilting module is simply a representation generator of $\operatorname{mod} A$. Hence A is 1-representation finite if and only if $\operatorname{gl.dim} A \leq 1$ (i.e., it is hereditary) and it is representation finite. In fact, in this case a 1-cluster tilting module is the direct sum of all (representatives of isoclasses of) indecomposable A-modules. A classification of hereditary representation finite algebras over an algebraically closed field is given by the following theorem.

Theorem 1.4.3 ([21])**.** *Suppose that $\Bbbk$ is algebraically closed. An hereditary algebra A is representation finite if and only if $A \cong \Bbbk Q$, where Q is a disjoint union of simply laced Dynkin quivers.*

Simply laced Dynkin quivers are the ones whose underlying graph is one of the following:



We have the following characterization of d-representation finite algebras.

Theorem 1.4.4 ([41])**.** *Let A be an algebra of global dimension at most d. Then A is d-representation finite if and only if its $(d+1)$-preprojective algebra $\Pi_{d+1}(A)$ is self-injective and satisfies the "vosnex" property (see [41, §3.1]).*

If $d = 2$ then the vosnex property is always satisfied: this case will be treated in Section 3.1.

1.4.2 d-representation infinite algebras

Chapter 2 is devoted to construct examples of d-representation infinite algebras. Following [31], we now recall their definition and illustrate a characterization of them. In Section 2.2 we will see that d-representation infinite algebras can be characterized by a property of their $(d+1)$-preprojective algebra.

Let A be an algebra of global dimension at most d. Recall from Section 1.4 that we have a Serre functor ν of $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$; we define $\nu_d := \nu \circ [-d] \colon \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A) \to \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$. Note that we can embed $\operatorname{mod} A$ in $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$ by considering modules as complexes concentrated in degree 0.

Definition 1.4.5. A finite dimensional algebra A is *d-representation infinite* if $\operatorname{gl.dim} A \leq d$ and $\nu_d^{-i}(A) \in \operatorname{mod} A$ for all $i \geq 0$.

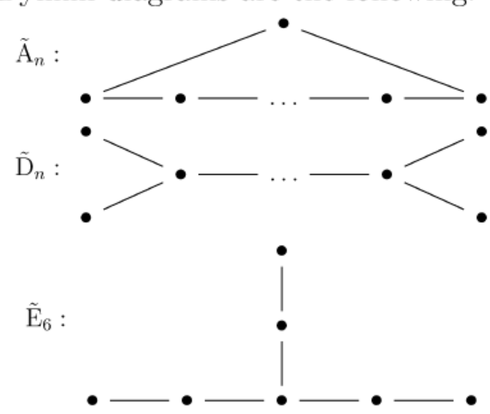
These algebras are a generalization of hereditary representation infinite algebras. In fact, when $\operatorname{gl.dim} A \leq 1$, the 0-th cohomology of the functor ν_1^{-1} coincides with the inverse Auslander-Reiten translation τ^-. Hence we have that an hereditary algebra A is 1-representation infinite if and only if $\tau^{-i}(A) \neq 0$ for all $i \geq 0$, and it is known by classical Auslander-Reiten theory [8] that the latter condition is equivalent to A being representation infinite.

Representation infinite algebras can be subdivided into two families: the tame and the wild algebras. In imprecise words, a representation infinite algebra A is tame if all indecomposable modules of a fixed dimension appear in finitely many one-parameter families; it is wild if, for any finite dimensional algebra B, the category $\operatorname{mod} B$ can be embedded in $\operatorname{mod} A$. A theorem by Drozd [20] says that every finite dimensional algebra is either representation finite, tame or wild.

We have a classification of hereditary representation tame algebras similar to the one of Theorem 1.4.3.

Theorem 1.4.6 ([19, 51])**.** *Suppose that $\Bbbk$ is algebraically closed. An hereditary algebra A is representation tame if and only if $A \cong \Bbbk Q$, where Q is a disjoint union of simply laced extended Dynkin quivers.*

The extended versions of Dynkin diagrams are the following:



$\tilde{E}_7$:

$\tilde{E}_8$:

In [31] the following generalization of the definition of a tame algebra is given.

Definition 1.4.7. A d-representation infinite algebra A is d-tame if its $(d+1)$-preprojective algebra is a Noetherian R-algebra, i.e., it is an R-algebra for some commutative Noetherian ring R and it is finitely generated as an R-module.

We will see in Chapter 2 that all hereditary tame algebras are 1-tame. Moreover, all d-representation infinite algebras we will construct in that chapter will be d-tame (cf. Proposition 2.3.7).

Chapter 2

Higher representation infinite algebras from metacyclic groups

In this chapter we will construct examples of d-representation infinite algebras from McKay quivers of metacyclic groups. In Section 2.1 we will define derivation quotient algebras in the sense of [11], and we will see that, by putting suitable gradings on them, we can obtain examples of d-representation infinite algebras. We will show in Section 2.3 that some examples of derivation quotient algebras are provided by skew group algebras of finite subgroups of a special linear group. Then we will consider the case where these groups are metacyclic: in Section 2.4 we summarize their representation theory and in Section 2.5 we describe their McKay quivers. In Section 2.6 we will study gradings on these quivers by means of cuts. Finally, we will illustrate some examples in Section 2.7.

Throughout this chapter we will assume that the base field $\Bbbk$ is the field $\mathbb{C}$ of complex numbers.

2.1 Derivation quotient algebras and gradings

In this section we describe superpotentials and derivation quotient algebras following [11], and we consider gradings on them.

2.1.1 Superpotentials

Let Q be a quiver. We will denote by $\mathbb{C}Q_k$ the subspace of $\mathbb{C}Q$ generated by paths of length k and set $S := \mathbb{C}Q_0$, $V := \mathbb{C}Q_1$. Then S is a finite dimensional semisimple $\mathbb{C}$-algebra. We will give V the unique S-bimodule structure where $\mathfrak{t}(a)a\mathfrak{s}(a) = a$ for all $a \in Q_1$: in this way $\mathbb{C}Q$ is identified with the tensor algebra $T_S V$.

Given a path $p \in \mathbb{C}Q_m$, we can define for any $k \leq m$ the left and right partial derivatives with respect to p as the maps $\partial_p, \delta_p \colon \mathbb{C}Q_k \to \mathbb{C}Q_{m-k}$ given by

$$\partial_p q := \begin{cases} r & \text{if } q = pr, \\ 0 & \text{otherwise,} \end{cases} \qquad q\delta_p := \begin{cases} r & \text{if } q = rp, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the set of arrows Q_1 is a basis of V, and we have a dual basis $\{a^*\,,\, a \in Q_1\}$ of V^*. These yield bases for the vector spaces $\mathbb{C}Q_k$ and $\mathbb{C}Q_k^*$. For a path $p = a_1 \cdots a_k \in \mathbb{C}Q_k$, we will denote by p^* the element $a_k^* \cdots a_1^* \in \mathbb{C}Q_k^*$.

Definition 2.1.1 ([11])**.** An element $\omega \in \mathbb{C}Q_n$ is called a *superpotential of degree* n if it satisfies the following two conditions:

(1) ω is a linear combination of cyclic paths (equivalently, $\omega s = s\omega$ for any $s \in S$);

(2) $\sigma(\omega) = (-1)^{n-1}\omega$, where σ is the map defined on paths as $\sigma(a_1 \cdots a_n) = a_n a_1 \cdots a_{n-1}$.

For later purposes, it will be convenient to consider the twisted analogue of this definition. We call *twist* a $\mathbb{C}$-algebra automorphism τ of $\mathbb{C}Q$ which satisfies $\tau(\mathbb{C}Q_k) \subseteq \mathbb{C}Q_k$ and permutes the primitive idempotents. If M is an S^e-module, we define its (right) twist M_τ as the vector space M with S^e-action given by $m.(s \otimes t) := m(s \otimes \tau(t))$.

Definition 2.1.2. An element $\omega \in \mathbb{C}Q_n$ is called a *twisted superpotential of degree* n if it satisfies the following two conditions:

(1) ω is a linear combination of paths p satisfying $t(p) = \tau(s(p))$ (equivalently, $\omega s = \tau(s)\omega$ for any $s \in S$);

(2) $\sigma^\tau(\omega) = (-1)^{n-1}\omega$, where σ^τ is the map defined on paths as

$$\sigma^\tau(a_1 \cdots a_n) = \tau(a_n) a_1 \cdots a_{n-1}$$

.

Note that if the twist is trivial we recover the definition of superpotential.

Given a twisted superpotential $\omega \in \mathbb{C}Q_n$ and an integer $k \leq n$, we can define an S^e-module morphism

$$\Delta_k^\omega \colon \mathbb{C}Q_k^* \otimes_S S_\tau \to \mathbb{C}Q_{n-k}$$

by $\Delta_k^\omega(p^* \otimes s) := \partial_p \omega s$. We denote by W_{n-k} the image of Δ_k^ω.

Definition 2.1.3. Let $\omega \in \mathbb{C}Q_n$ be a (twisted) superpotential. The *derivation quotient algebra* of ω of order k is defined as

$$\mathcal{D}(\omega, k) := \mathbb{C}Q/\langle W_{n-k} \rangle = \mathbb{C}Q/\langle \partial_p \omega\,,\, p \text{ path of length } k \rangle,$$

where $\langle W_{n-k} \rangle$ denotes the smallest two-sided ideal of $\mathbb{C}Q$ which contains W_{n-k}.

For later use we give the following definition.

Definition 2.1.4. Write a superpotential $\omega \in \mathbb{C}Q_n$ as

$$\omega = \sum_{|p|=n} c_p p,$$

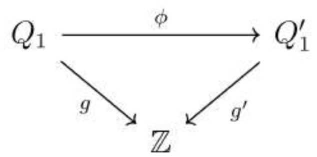
for some scalars $c_p \in \mathbb{C}$. Then we define the *support of* ω to be the set

$$\operatorname{supp}(\omega) := \{p \,|\, p \text{ path of length } n,\ c_p \neq 0\}.$$

2.1.2 Graded quivers

We recall that a morphism of quivers $\phi\colon Q \to Q'$ consists in two maps $Q_0 \to Q'_0$, $Q_1 \to Q'_1$ (which, abusing of notation, we will both denote again by ϕ) which are compatible with the source and target maps.

Definition 2.1.5.
- A ($\mathbb{Z}-$)*graded quiver* is a couple (Q, g) consisting of a quiver Q and a map $g\colon Q_1 \to \mathbb{Z}$. A *morphism of graded quivers* $\phi\colon (Q, g) \to (Q', g')$ is given by a morphism of quivers $\phi\colon Q \to Q'$ such that the following diagram commutes:



- Let $\phi\colon Q \to Q'$ be a morphism of quivers and suppose that Q is graded by g. We will say that ϕ is g-*gradable* if, for every arrow $a \in Q'_1$, $\phi^{-1}(a)$ is either empty or a homogeneous subset of Q_1 (i.e., all its elements have the same degree).

Let $\phi\colon Q \to Q'$ be a morphism of quivers. There is a natural way to induce a grading on Q from a grading on Q', and vice versa, which we now illustrate.

Definition 2.1.6. (1) Suppose that g is a grading on Q'. Then we define a grading $\phi^* g$ by putting $(\phi^* g)(a) = g(\phi(a))$ for all $a \in Q_1$. Note that this is the unique grading on Q which makes ϕ a morphism of graded quivers.

(2) Suppose that g is a grading on Q and that ϕ is g-gradable. It is clear that there always exists a grading g' on Q' which makes ϕ a morphism of graded quivers: indeed, it is enough to put $g'(a)$ equal to k if the elements of $\phi^{-1}(a)$ have all degree k, and to any integer if $\phi^{-1}(a) = \varnothing$. If, in addition, we assume that $\phi\colon Q_1 \to Q'_1$ is surjective, then such a grading is unique and we will denote it by $\phi_* g$.

Remark 2.1.7. A grading on Q induces in a natural way a grading on the path algebra $\mathbb{C}Q$. Note that in general this grading does not coincide with the natural one on $\mathbb{C}Q$ given by path length (i.e., the one obtained by putting all arrows in degree 1).

From now on we will fix a grading on $\mathbb{C}Q$ which comes from a grading on Q. Note that in this case all elements of S have degree 0. We will now show that if ω is a superpotential which is homogeneous with respect to this grading, then we get a graded structure on $\mathcal{D}(\omega, k)$.

Lemma 2.1.8. *Suppose that $\omega \in \mathbb{C}Q_n$ is a superpotential which is homogeneous of degree a. Then W_i is a graded S^e-submodule of $\mathbb{C}Q$ for all $i = 0, \ldots, n$. In particular the derivation quotient algebra $\mathcal{D}(\omega, k)$ inherits in a natural way a structure of graded $\mathbb{C}$-algebra.*

Proof. Firstly we show that the morphism Δ_k^ω is homogeneous of degree a. Indeed, if $p^* \in \mathbb{C}Q_k^*$ has degree d, then p has degree $-d$ and by definition of partial derivative we have $|\Delta_k^\omega(p^*)| = |\partial_p \omega| = a - d$.

Hence each W_i is a graded S^e-submodule of $\mathbb{C}Q$ because it is the image of an homogeneous morphism. This implies that the ideal generated by it is homogeneous and so we have a well defined grading on the quotient $D(\omega, k) = \mathbb{C}Q/\langle W_{n-k}\rangle$. $\square$

2.2 Graded Calabi-Yau algebras and the Gorenstein parameter

We will now show how we can obtain n-representation infinite algebras from derivation quotient algebras.

Definition 2.2.1. *Let a be an integer and $n \geq 2$. A positively graded $\mathbb{C}$-algebra $A = \bigoplus_{i\geq 0} A_i$ is called* bimodule n-Calabi-Yau of Gorenstein parameter a *if the A^e-module A has a bounded resolution $P_\bullet$ of finitely generated graded projective A^e-modules such that we have an isomorphism of complexes*

$$\Phi\colon P_\bullet \xrightarrow{\sim} P_\bullet^\vee[n](-a).$$

Here $[n]$ denotes the shift of complexes, while $(-a)$ denotes the shift of the grading.

In other words, we want a commutative diagram

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & P_n & \xrightarrow{d_n} & \cdots & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \Phi_n} & & & & \downarrow{\scriptstyle \Phi_1} & & \downarrow{\scriptstyle \Phi_0} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & P_0^\vee & \xrightarrow{d_1^\vee} & \cdots & \xrightarrow{d_{n-1}^\vee} & P_{n-1}^\vee & \xrightarrow{d_n^\vee} & P_n^\vee & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

where the maps Φ_i are isomorphisms and homogeneous of degree $-a$.

The above definition is motivated by the following theorem.

Theorem 2.2.1 ([1],[45],[49],[31]). *If A is a bimodule n-Calabi-Yau of Gorenstein parameter 1 such that $\dim_{\mathbb{C}} A_0 < \infty$, then A_0 is $(n-1)$-representation infinite. Vice versa, if B is $(n-1)$-representation infinite, then its higher preprojective algebra $\Pi_n(B)$, equipped with the tensor algebra grading, is bimodule n-Calabi-Yau of Gorenstein parameter 1 and A is the degree 0 part of $\Pi_n(B)$.*

In view of this, we will be interested in finding examples of bimodule n-Calabi-Yau algebras of Gorenstein parameter 1. In the next subsection, following [11], we will describe a way to obtain such examples from derivation quotient algebras.

2.2.1 Graded Calabi-Yau algebras from derivation quotient algebras

Let Q be a quiver and retain the notation of Section 2.1. We put $A := \mathcal{D}(\omega, n-2)$ and fix a positive grading on $\mathbb{C}Q$ such that ω is homogeneous of degree a. This induces, by Lemma 2.1.8, a grading on A which is again positive.

Set $P_i := A \otimes_S W_i \otimes_S A$ if $0 \leq i \leq n$ and $P_i := 0$ otherwise. Then we have a complex of projective A^e-modules

$$P_\bullet = (\ldots \to 0 \to P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \ldots \xrightarrow{d_1} P_0 \to 0 \to \ldots), \tag{2.2.2}$$

with differentials $d_i \colon P_i \to P_{i-1}$ defined by

$$d_i = \varepsilon_i(d_i^l + (-1)^i d_i^r),$$

where

$$d_i^l(1 \otimes \partial_p \omega \otimes 1) := \sum_{b \in Q_1} b \otimes \partial_b \partial_p \omega \otimes 1,$$

$$d_i^r(1 \otimes \partial_p \omega \otimes 1) := \sum_{b \in Q_1} 1 \otimes \partial_p \omega \delta_b \otimes b,$$

$$\varepsilon_i := \begin{cases} (-1)^{i(n-i)} & \text{if } i < (n+1)/2, \\ 1 & \text{otherwise.} \end{cases}$$

Clearly each P_i is a graded A^e-module, because W_i is a graded S^e-submodule of $\mathbb{C}Q$. Moreover it is easy to see that the differentials d_i have degree zero, so $P_\bullet$ is a complex of graded projective A^e-modules. The following is a graded version of [11, Theorem 6.2].

Theorem 2.2.2. *Suppose that the complex $P_\bullet$ defined above is a resolution of A (i.e., it is exact in positive degrees and $H^0(P_\bullet) = A$). Then A is n-bimodule Calabi-Yau of Gorenstein parameter a.*

Proof. It is proved in [11] that if $P_\bullet$ is a projective resolution of A, then it is self dual, i.e., $P_\bullet \cong P_\bullet^\vee[n]$. The duality isomorphism given in loc. cit. is described as follows (see also [44] for more details). For each $i = 0, \ldots, n$ we have a perfect pairing

$$\langle , \rangle \colon W_i \otimes W_{n-i} \to \mathbb{C}$$

given by $\langle \partial_p \omega, \partial_q \omega \rangle := c_{qp}$. The isomorphism of A^e-modules

$$\Phi_i \colon P_i = A \otimes_S W_i \otimes_S A \to \mathrm{Hom}_{A^e}(A \otimes_S W_{n-i} \otimes_S A, A^e) = P_{n-i}^\vee$$

is given by $\Phi_i(\alpha \otimes \partial_p \omega \otimes \alpha')(\beta \otimes \partial_q \omega \otimes \beta') = \alpha' \beta \otimes \langle \partial_p \omega, \partial_q \omega \rangle \otimes \beta' \alpha$. These maps commute with the differentials and thus yield an isomorphism of complexes $\Phi_\bullet \colon P_\bullet \to P_\bullet^\vee[n]$. These facts can be proved using the same calculations of the proof of [44, Theorem 3.21].

Now we are only left to show that Φ_i is homogeneous of degree $-a$. It is enough if we prove that if $\partial_p\omega \in W_i$ has degree d, then $\Phi_i(1\otimes\partial_p\omega\otimes 1)$ has degree $d-a$. For this it suffices to show that $\Phi_i(1\otimes\partial_p\omega\otimes 1)(1\otimes\partial_q\omega\otimes 1)$ has degree $e+d-a$ whenever $|\partial_q\omega| = e$. Note that we have $|p| = |\omega| - |\partial_p\omega| = a-d$, $|q| = a-e$ and so $|\partial_{qp}\omega| = a - |p| - |q| = d+e-a$. Suppose now that $d+e \neq a$: then $|\partial_{qp}\omega| \neq 0$, but this implies that $\partial_{qp}\omega = 0$ because $\partial_{qp}\omega \in S$, so $\Phi_i(1\otimes\partial_p\omega\otimes 1)(1\otimes\partial_q\omega\otimes 1) = 0$. If instead $d+e=a$, then it is clear that $\Phi_i(1\otimes\partial_p\omega\otimes 1)(1\otimes\partial_q\omega\otimes 1)$ has degree $e+d-a=0$. □

2.3 Skew group algebras and McKay quivers

In [11] it is shown that a source of derivation quotient algebras is provided by a family of skew group algebras. In this section we will summarize this construction and consider the graded case in order to apply Theorem 2.2.2. We start by recalling some basic notions about representation theory of finite groups.

2.3.1 Representations of finite groups

Let G be a finite group. We will sometimes identify representations of G with modules over the group algebra $\mathbb{C}G$. Given two representation M and N of G, their tensor product $M\otimes_{\mathbb{C}} N$ will be regarded as a representation of G via the action $g(m\otimes n) = gm\otimes gn$.

If H is a subgroup of G and M is a representation of H, we call $\mathrm{Ind}_H^G(M)$ the representation of G induced by M, which, as left modules, is defined by $\mathbb{C}G\otimes_{\mathbb{C}H} M$. If we fix a set $\{g_1,\dots,g_n\}$ of left coset representatives of G/H, then $\mathrm{Ind}_H^G(M)$ is isomorphic to $\bigoplus_{i=1}^n g_i\otimes M$ with G-action given by $g(g_i\otimes m) = g_j\otimes hm$ for all $g\in G$, where g_j is the coset representative which satisfy $gg_i = g_jh$ for $h\in H$. In order to simplify the notation we will write g_im for $g_i\otimes m$.

If N is a representation of G, we call $\mathrm{Res}_H^G(N)$ the restriction of M to H. It is well known (see for example [10]) that the restriction functor is a right adjoint to the induction functor. More precisely we have an isomorphism of vector spaces

$$\begin{aligned}\mathrm{Hom}_H(N, \mathrm{Res}_H^G(M)) &\longrightarrow \mathrm{Hom}_G(\mathrm{Ind}_H^G(N), M)\\ \phi &\longmapsto (gx\mapsto g\phi(x))\end{aligned}$$

which is functorial in M and N.

We also have an isomorphism

$$\begin{aligned}\mathrm{Ind}_H^G(\mathrm{Res}_H^G(M)\otimes_{\mathbb{C}} N) &\longrightarrow M\otimes_{\mathbb{C}}\mathrm{Ind}_H^G(N)\\ g(m\otimes n) &\longmapsto gm\otimes gn\end{aligned}$$

of representations of G.

2.3.2 McKay quivers

Let V be a $\mathbb{C}$-vector space of dimension n and let G be a finite subgroup of $\mathrm{GL}(V)$. Call $\mathbb{C}[V]$ the algebra of polynomial functions on V: then G acts on it in a natural way and we can form the skew group algebra $\mathbb{C}[V] * G$. We have the following description of the latter.

Theorem 2.3.1 ([11]). *The skew group algebra $\mathbb{C}[V]*G$ is Morita equivalent to a basic algebra Π_G which is a derivation quotient algebra of order $n-2$ with a (twisted) superpotential ω of degree n. More explicitly, we have*

$$\Pi_G = \mathbb{C}Q/\langle \partial_p \omega \,,\, |p| = n-2 \rangle,$$

where Q is the McKay quiver of G (see Definition 2.3.2).

Definition 2.3.2. Let $\mathrm{Irr}(G)$ be a complete set of representatives for the irreducible representations of G. The *McKay quiver Q of G* relative to V is described as follows. Its set of vertices is $\mathrm{Irr}(G)$ and, for any $S, T \in \mathrm{Irr}(G)$, the set of arrows going from S to T is given by a basis of the vector space $\mathrm{Hom}_G(S, V \otimes_{\mathbb{C}} T)$.

We now give an explicit description, following [11], of the superpotential ω of the algebra Π_G.

We denote by $\det_V$ the 1-dimensional representation of G where each $g \in G$ acts as the multiplication by $\det(g)$. Clearly $\det_V$ is isomorphic to the exterior product $\bigwedge^n V$. Now consider the functor $\tau := \det_V \otimes_{\mathbb{C}} \_\_$ and note that it sends irreducible representations to irreducible representations. Hence we get a bijection $\tau\colon \mathrm{Irr}(G) \to \mathrm{Irr}(G)$: it is easy to see that its inverse, which we denote by τ^-, is given by tensoring by $\det_{V^*}$. Clearly we have that $\tau(V \otimes_{\mathbb{C}} S) = V \otimes_{\mathbb{C}} \tau(S)$, so τ extends to an automorphism of the path algebra $\tau\colon \mathbb{C}Q \to \mathbb{C}Q$ which preserves the path length: this will be our twist.

Consider now a path

$$p\colon v_1 \xrightarrow{a_1} v_2 \xrightarrow{a_2} \dots \to v_n \xrightarrow{a_n} v_{n+1}$$

of length n in the McKay quiver Q. If we call S_i the representation corresponding to the vertex v_i, then we can view the arrows as morphisms $a_i\colon S_i \to V \otimes S_{i+1}$. This induces a composition

$$S_1 \xrightarrow{a_1} V \otimes S_2 \xrightarrow{\mathrm{id}_V \otimes a_2} \dots \xrightarrow{\mathrm{id}_{V^{\otimes n-1}} \otimes a_n} V^{\otimes n} \otimes S_{n+1} \xrightarrow{\alpha_V^n \otimes \mathrm{id}_{S_{n+1}}} \det_V \otimes S_{n+1} = \tau(S_{n+1}), \tag{2.3.1}$$

where, for each $m \leq n$, we denote by α_V^m the antisymmetrizer $V^{\otimes m} \to \bigwedge^m V$, $x_1 \otimes \cdots \otimes x_m \mapsto x_1 \wedge \cdots \wedge x_m$. By Schur's Lemma, the composition morphism (2.3.1) is the multiplication by a scalar, which will be denoted by c_p. Note in particular that $c_p \neq 0$ only when $\tau(t(p)) = s(p)$.

Theorem 2.3.3 ([11]). *The superpotential ω of the algebra Π_G is given by*

$$\omega = \sum_{|p|=n} (c_p \dim t(p)) p.$$

Remark 2.3.4. It is worth pointing out that the superpotential described above depends on the choice of the basis for the arrows in Q.

Suppose that the basis we choose for $\mathrm{Hom}_G(S, V \otimes_{\mathbb{C}} T)$ is invariant under the twist. Then the automorphism τ of $\mathbb{C}Q$ is actually induced by an automorphism of the quiver Q and, by [11, Lemma 4.3], the coefficients of the superpotential have the property that

$$c_{a_1 \cdots a_n} = (-1)^{n-1} c_{\tau(a_n) a_1 \cdots a_{n-1}}$$

for all paths $a_1 \cdots a_n$ in Q. In case our basis is invariant only up to multiplication by a non-zero scalar, we can still define an automorphism of Q, which we call τ', by putting $\tau'(v) := \tau(v)$ if $v \in Q_0$ and $\tau'(a) := a'$ if $a \in Q_1$, where a' is the only arrow such that $\tau(a)$ is a multiple of a'. It is easy to see that in this case we have the following weaker version of [11, Lemma 4.3].

Lemma 2.3.5. *If $p = a_1 \cdots a_n$ is a path of length n in Q, then the coefficient c_p satisfies*

$$c_{a_1 \cdots a_n} = \mu c_{\tau'(a_n) a_1 \cdots a_{n-1}}$$

for a non-zero scalar $\mu \in \mathbb{C}^\times$. In particular, $a_1 \cdots a_s$ lies in $\mathrm{supp}(\omega)$ if and only if $\tau'(a_n) a_1 \cdots a_{n-1}$ lies in $\mathrm{supp}(\omega)$.

The main reason we are interested in skew group algebras comes from the following corollary to Theorem 2.2.2.

Corollary 2.3.6. *Suppose that G is contained in $\mathrm{SL}(V)$. Put a grading on the path algebra $\mathbb{C}Q$ of the McKay quiver of G in such a way that the superpotential ω described in Theorem 2.3.3 is homogeneous of degree* 1.

Then the algebra $\Pi_G = \mathbb{C}Q/\langle \partial_p \omega \,,\, |p| = n-2 \rangle$, equipped with the grading induced by $\mathbb{C}Q$, is n-bimodule Calabi-Yau of Gorenstein parameter 1. *In particular, its degree zero part, if finite dimensional, is an $(n-1)$-representation infinite algebra.*

Proof. In [11] the authors prove that any skew group algebra $\mathbb{C}[V] * G$, for a finite subgroup G of $\mathrm{SL}(V)$, has the property of being n-Calabi-Yau and Koszul. Moreover they show that, for any derivation quotient algebra A satisfying these two properties, the complex (2.2.2) is a resolution of A. This applies in particular to $\mathbb{C}[V] * G$, and also to Π_G since being n-Calabi-Yau and Koszul is invariant under Morita equivalence. Hence we can conclude by Theorem 2.2.2 that Π_G is n-bimodule Calabi-Yau of Gorenstein parameter 1.

The last assertion then follows directly from Theorem 2.2.1. □

We end this section with the following remark (cf. [31, Example 6.11]).

Proposition 2.3.7. *All $(n-1)$-representation infinite algebras constructed as in Corollary 2.3.6 are $(n-1)$-tame in the sense of Definition 1.4.7. In particular, all hereditary representation tame algebras are 1-tame.*

Proof. Recall that for such an $(n-1)$-representation infinite algebra A, the higher preprojective algebra $\Pi_n(A)$ is Morita equivalent to $\mathbb{C}[V]*G$ for some finite subgroup G in $\mathrm{SL}(V)$. By a theorem of Auslander (see for example [42] for a proof) we have an isomorphism of $\mathbb{C}[V]^G$-algebras $\mathbb{C}[V]*G \cong \mathrm{End}_{\mathbb{C}[V]^G}\mathbb{C}[V]$, where $\mathbb{C}[V]^G$ denotes the invariant subring. A theorem of E. Noether [52] implies that $\mathbb{C}[V]^G$ is a finitely generated $\mathbb{C}$-algebra (and so it is Noetherian) and $\mathbb{C}[V]$ is a finitely generated $\mathbb{C}[V]^G$-module. Hence we have that also $\mathbb{C}[V]*G$ is finitely generated over $\mathbb{C}[V]^G$, so A is $(n-1)$-tame. The last assertion follows from Theorem 2 of the Introduction, which says that all hereditary representation tame algebras arise as in Corollary 2.3.6. □

2.3.3 Subgroups of $\mathrm{GL}(d,\mathbb{C})$ embedded in $\mathrm{SL}(d+1,\mathbb{C})$

We saw previously that we can obtain examples of $(n-1)$-representation infinite algebras from skew group algebras of finite subgroups of $\mathrm{SL}(n,\mathbb{C})$. If, instead, we start from a subgroup of $\mathrm{GL}(n,\mathbb{C})$ not contained in $\mathrm{SL}(n,\mathbb{C})$, then we cannot apply Corollary 2.3.6 anymore. However, every subgroup of $\mathrm{GL}(n,\mathbb{C})$ can be regarded as a subgroup of $\mathrm{SL}(n+1,\mathbb{C})$ by means of the natural embedding $\mathrm{GL}(n,\mathbb{C}) \hookrightarrow \mathrm{SL}(n+1,\mathbb{C})$ given by

$$X \mapsto \left(\begin{array}{c|c} X & 0 \\ \hline 0 & \frac{1}{\det(X)} \end{array}\right).$$

For a finite subgroup G of $\mathrm{GL}(V)$, where V is a $\mathbb{C}$-vector space of dimension n, we denote by G' its image under the above embedding, so that G' is a subgroup of $\mathrm{SL}(W)$ for a vector space $W \supseteq V$ of dimension $n+1$. Let Q be the McKay quiver of G relative to V and ω be the associated twisted superpotential: if Q' denotes the McKay quiver of G' relative to W, then it is known (see for example [26]) that a superpotential ω' for $\Pi_{G'}$ can be obtained from ω by adding some arrows. More precisely we have the following.

Proposition 2.3.8. *Fix a basis for the arrows of Q which is invariant, up to multiplication by a non-zero scalar, under the twist. Then Q can be viewed as a subquiver of Q', and the latter can be obtained from the former by adding an arrow $i \to \tau(i)$ for each vertex $i \in Q_0$. Moreover, the support of the superpotential ω' is obtained from the one of ω by adding these arrows. More precisely, we have that a path*

$$\tau(i_n) \to i_1 \to \ldots \to i_n$$

is in $\mathrm{supp}(\omega)$ *if and only if the path*

$$\tau(i_n) \to i_1 \to \ldots \to i_n \to \tau(i_n)$$

is in $\mathrm{supp}(\omega')$*, and all the paths in* $\mathrm{supp}(\omega')$*, up to cyclic permutation, are obtained in this way.*

2.4 Metacyclic groups

We now introduce a family of groups to which we will later apply the results of the previous sections. All groups in this family will satisfy the following property.

Definition 2.4.1. A group G is metacyclic if it has a normal cyclic subgroup A such that G/A is cyclic.

Some generalities about metacyclic groups can be found in [15, §47]. In particular, it is shown in loc. cit. that one can associate to a metacyclic group some integers which must satisfy certain conditions. On the contrary, in the following we will start with integers satisfying such conditions and associate to them a metacyclic group embedded in a general linear group.

Definition 2.4.2. Let m, r, s, t be positive integers satisfying the following conditions:

(M1) $(m, r) = 1$, where (m, r) indicates the greatest common divisor of m and r;

(M2) $r^s \equiv 1 \pmod{m}$;

(M3) $(r-1)t \equiv 0 \pmod{m}$.

Define G to be the finite subgroup of $\mathrm{GL}(s, \mathbb{C})$ generated by the following matrices:

$$\alpha = \begin{pmatrix} \varepsilon_m & 0 & \cdots & 0 \\ 0 & \varepsilon_m^r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon_m^{r^{s-1}} \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & 0 & \cdots & 0 & \varepsilon_m^t \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

where ε_m is a fixed primitive m-th root of unity. We will refer to G as the *metacyclic group associated to* m, r, s, t and we will denote by A the subgroup of G generated by α.

We now make some comments on the above definition. By (M1) we have that A is cyclic of order m, and it is normal in G because, by (M2), $\beta^{-1}\alpha\beta = \alpha^r$. Condition (M3) implies that $\beta^s = \alpha^t$, hence it easy to see that G/A is a cyclic group of order s generated by the class of β. This shows that G is metacyclic; in particular, the order of G is sm.

At a later stage we will need to consider additional conditions on the integers m, r, s, t, in order to apply the results of the previous sections or to simplify some calculations. For convenience we will list all of them now and refer to them in the following whenever they will be needed:

(M4) s is a prime number;

(M5) $r \not\equiv 1 \pmod{m}$;

(M6) $m = sn$ for an integer n;

(M7) $r - 1 = sb$ for an integer b.

Let us make a brief comment on these additional conditions. We will need (M4) because it implies that G/A has prime order, and so it is a simple group: this will simplify a lot the description of the irreducibles representation of G, allowing us to use [15, Corollary 47.14] in the next subsection. Condition (M5) will be used from Section 2.5: in particular it implies that G is not abelian, thus ensuring that we are not in the case already studied in [31]. Another consequence of (M5) is Proposition 2.5.4, thanks to which we will have fewer cases to analyse in the study of the superpotential associated to G in Section 2.5. Conditions (M6) and (M7) will be introduced in Section 2.6 in order to prove the existence of gradings which satisfy the hypotheses of Corollary 2.3.6.

2.4.1 Irreducible representations

The representation theory of metacyclic groups is well known. Here we summarize, following [15], the description of their irreducible representations, in order to fix some notation for the following sections.

Let G be the metacyclic group associated to integers m, r, s, t. From now on we will suppose that G satisfies also condition (M4). Note that this implies that the quotient G/A is simple, because it is cyclic of prime order.

Let $\mathrm{Irr}(A) = \{S_i\}_{i=0}^{m-1}$ be a complete set of non-isomorphic irreducible representations of A. Since A is abelian, the S_i's are all 1-dimensional: we put $S_i = \mathbb{C}v_i$ and assume that the action of A on S_i is given by $\alpha v_i = \varepsilon_m^i v_i$. We will often consider the indices i as integers modulo m and identify $\mathrm{Irr}(A)$ with $\mathbb{Z}/m\mathbb{Z}$ via $S_i \leftrightarrow i$. Note in particular that this gives $S_i \otimes S_j \cong S_{i+j}$.

Consider now the induced representations $\mathrm{Ind}_A^G(S_i) =: T_i$. We choose as a basis of T_i the set $\{v_i, \beta v_i, \ldots, \beta^{s-1} v_i\}$, so G acts on T_i in the following way:

$$\alpha(\beta^k v_i) = \varepsilon_m^{r^k i} \beta^k v_i, \qquad k = 0, \ldots, s-1,$$
$$\beta(\beta^k v_i) = \beta^{k+1} v_i, \qquad k = 0, \ldots, s-2,$$
$$\beta(\beta^{s-1} v_i) = \beta^s v_i = \alpha^t v_i = \varepsilon_m^{ti} v_i.$$

Thus the matrices of α and β with respect to the action on T_i are

$$\alpha \mapsto \begin{pmatrix} \varepsilon_m^i & 0 & \cdots & 0 \\ 0 & \varepsilon_m^{ri} & \cdot & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon_m^{r^{s-1}i} \end{pmatrix}, \qquad \beta \mapsto \begin{pmatrix} 0 & 0 & \cdots & 0 & \varepsilon_m^{ti} \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

The following proposition tells us when the representation T_i is irreducible.

Proposition 2.4.3 ([15]). *(a) T_i is irreducible if and only if $r^k i \not\equiv i \pmod m$ for all $k = 1, \ldots, s-1$;*

(b) $T_i \cong T_j$ *if and only if there exist a* k *such that* $r^k i = j$.

Remark 2.4.4. Note that, since s is prime, we can replace the condition (a) above by:

(a') T_i is irreducible if and only if $ri \not\equiv i \pmod{m}$.

Suppose now that $ri \equiv i \pmod{m}$, so that T_i is not irreducible. The matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & \varepsilon_m^{ti} \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

which represents the action of β, has characteristic polynomial $p(\lambda) = (-1)^s(\lambda^s - \varepsilon_m^{ti})$ and so it is diagonalisable. Its eigenvalues are $\{\lambda_{i,\ell} \,|\, \ell = 0, \dots, s-1\}$, where $\lambda_{i,\ell} := \eta_i \varepsilon_s^{\ell}$ and $\eta_i := \varepsilon_m^{\frac{t}{s}i}$ is a fixed s-th root of ε_m^{ti}. The elements

$$w_i^{(\ell)} := \sum_{k=0}^{s-1} \lambda_{i,\ell}^{s-1-k} \beta^k v_i, \qquad \ell = 0, \dots, s-1,$$

provide a basis of eigenvectors for β. Indeed,

$$\beta w_i^{(\ell)} = \sum_{k=0}^{s-1} \lambda_{i,\ell}^{s-k-1} \beta(\beta^k v_i) = \sum_{h=1}^{s-1} \lambda_{i,\ell}^{s-h} \beta^h v_i + \varepsilon_m^{ti} v_i = \lambda_{i,\ell} \sum_{h=1}^{s-1} \lambda_{i,\ell}^{s-h-1} \beta^h v_i + \lambda_{i,\ell}^s v_i = \lambda_{i,\ell}\, w_i^{(\ell)}$$

and

$$\alpha w_i^{(\ell)} = \sum_{k=0}^{s-1} \lambda_{i,\ell}^{s-k-1} \alpha(\beta^k v_i) = \sum_{k=0}^{s-1} \lambda_{i,\ell}^{s-k-1} \varepsilon_m^{r^k i} \beta^k v_i = \sum_{k=0}^{s-1} \lambda_{i,\ell}^{s-k-1} \varepsilon_m^{i} \beta^k v_i = \varepsilon_m^i w_i^{(\ell)},$$

where, in the second equality, we used the fact that $r^k i \equiv i \pmod{m}$.

Hence the 1-dimensional subspace $T_i^{(\ell)} := \mathbb{C} w_i^{(\ell)}$ is a subrepresentation of T_i, and we have a decomposition

$$T_i \cong \bigoplus_{\ell=0}^{s-1} T_i^{(\ell)}.$$

2.5 The skew group algebra of a metacyclic group

In this section, G will be the metacyclic group associated to some integers m, r, s, t satisfying conditions (M1),...,(M5). Note that (M5) implies that G is not abelian: we are assuming it because the case of skew group algebras of abelian groups has already been studied in [31]. Moreover, this assumption will simplify a lot some calculations at a later stage (see Proposition 2.5.4).

The action of G/A on G by conjugation induces an automorphism φ of G given by $\varphi(g) = \beta^{-1}g\beta$. This in turn induces an action of G/A on $\mathrm{Irr}(A)$ given by $\varphi(i) = ri$. From now on we will fix a set $\mathcal{D}$ of representatives in $\mathbb{Z}/m\mathbb{Z}$ of this action and we will denote by $\mathcal{F}$ the set of fixed points. Note that the orbits of the fixed points have cardinality 1, so we have $\mathcal{F} \subseteq \mathcal{D}$ regardless of what choice for $\mathcal{D}$ we made. By Proposition 2.4.3 we have the following result.

Proposition 2.5.1. *The set* $\mathrm{Irr}(G) = \{T_i \mid i \in \mathcal{D} \smallsetminus \mathcal{F}\} \cup \{T_i^{(\ell)} \mid i \in \mathcal{F}\,, \ell = 0, \ldots, s-1\}$ *is a complete set of nonisomorphic irreducible representations of* G*. Moreover we have that* $\dim_{\mathbb{C}} T_i = s$ *and* $\dim_{\mathbb{C}} T_i^{(\ell)} = 1$.

Notation. For each $i \in \mathbb{Z}/m\mathbb{Z}$ we call $\underline{i}$ its representative in $\mathcal{D}$, i.e., the only element of the G/A-orbit of i which is contained in $\mathcal{D}$ (note that $\underline{i} = i$ if $i \in \mathcal{D}$). We fix an integer $\kappa_i \in \{0, \ldots, s-1\}$ such that $r^{\kappa_i}\underline{i} \equiv i$. If $i \in \mathcal{D} \smallsetminus \mathcal{F}$ it is clear, by Proposition 2.4.3(a), that we can choose κ_i in a unique way; otherwise, for a fixed point $i \in \mathcal{F}$, we set $\kappa_i := 0$.

Example 2.5.2. Let $m = 21$, $r = 4$, $s = 3$, $t = 0$, so the action of G/A on $\mathbb{Z}/21\mathbb{Z}$ is given by the multiplication by 4. A possible choice of representatives is

$$\mathcal{D} = \{0, 4, 7, 8, 9, 12, 13, 14, 17\} \subseteq \mathbb{Z}/21\mathbb{Z}.$$

In this case we have, for example, $\underline{1} = \underline{4} = \underline{16} = 4$ and $\kappa_1 = 2$, $\kappa_4 = 0$, $\kappa_{16} = 1$. We will analyse this example in detail in Example 2.7.7.

Call V the s-dimensional natural representation of G, which coincides with T_1. We may note that, by Remark 2.4.4, condition (M5) is equivalent to say that V is irreducible. It is easy to see that, for all i, we have an isomorphism $\mathrm{Res}_A^G(T_i) \cong \bigoplus_{k=0}^{s-1} S_{r^k i}$, $\beta^k v_i \mapsto v_{r^k i}$. In the case where i is a fixed point, we have an isomorphism $\mathrm{Res}_A^G(T_i^{(\ell)}) \cong S_i$, $w_i^{(\ell)} \mapsto v_i$. We will call again V the restriction of V to A, which clearly coincides with the natural representation of A and is isomorphic to $\bigoplus_{k=0}^{s-1} S_{r^k}$.

Our strategy for describing the skew group algebra of G will exploit the action of G/A on the McKay quiver of A. The structure of the latter, and also the relations which give an isomorphism with the skew group algebra of A, are well known. The following description is a particular case of [11, Corollary 4.1].

Proposition 2.5.3 ([11])**.** *Let* Q_A *be the McKay quiver of* A*. Then the following holds.*

- Q_A *has vertices* $\mathbb{Z}/m\mathbb{Z}$ *and an arrow* $x_k^i \colon i \to i - r^k$ *for each* $i \in \mathbb{Z}/m\mathbb{Z}$ *and* $k = 0, \ldots, s-1$ *(sometimes, if this does not cause confusion, we will omit the superscript and write* x_k *in place of* x_k^i*).*
- *Let* $\mathfrak{S}_s$ *be the symmetric group on* $\{0, \ldots, s-1\}$*. For each permutation* $\sigma \in \mathfrak{S}_s$ *and each vertex* i*, define a path*

$$\mathbf{p}_\sigma^i := x_{\sigma(s-1)} \cdots x_{\sigma(0)} \colon i \to i - \textstyle\sum_{k=0}^{s-1} r^{\sigma(k)}.$$

Then the superpotential of Q_A is given by

$$\omega_A = \sum_{i \in (Q_A)_0} \sum_{\sigma \in \mathfrak{S}_s} (-1)^\sigma \mathbf{p}^i_\sigma,$$

where $(-1)^\sigma$ is the sign of the permutation σ.

In particular, the skew group algebra $\mathbb{C}[V] * A$ is isomorphic to the path algebra of Q_A modulo the relations

$$\{x_h^{i-r^k} x_k^i = x_k^{i-r^h} x_h^i \,|\, i \in \mathbb{Z}/m\mathbb{Z},\, 0 \le k, h \le s-1\}.$$

Proof. The proof is given in [11, Corollary 4.1]. Here we only mention, since it will be needed later, which basis of arrows we choose.

For each $i \in \mathbb{Z}/m\mathbb{Z}$ and $k = 0, \dots, s-1$, we will take x_k^i to be the morphism in $\operatorname{Hom}_A(S_i, V \otimes S_{i-r^k}) \cong \operatorname{Hom}_A(S_{r^k} \otimes S_{i-r^k}, V \otimes S_{i-r^k})$ defined by $v_{r^k} \otimes v_{i-r^k} \mapsto v_{r^k} \otimes v_{i-r^k}$. □

2.5.1 The vertices and arrows of Q_G

Now we will consider the McKay quiver of G, which we denote by Q_G. Proposition 2.5.1 gives us a description of its vertices: we will call i the vertex corresponding to T_i for $i \in \mathcal{D} \smallsetminus \mathcal{F}$, and $i^{(\ell)}$ the vertex corresponding to $T_i^{(\ell)}$ for $i \in \mathcal{F}$, $\ell = 0, \dots, s-1$. By an abuse of terminology, for $i \in \mathcal{F}$, we will call "fixed points" both the vertex $i \in (Q_A)_0$ and each of the vertices $i^{(\ell)} \in (Q_G)_0$, for $\ell = 0, \dots, s-1$.

We will now describe the arrows of Q_G. Recall that in order to do this we must choose a basis of the vector space $\operatorname{Hom}_G(S, V \otimes T)$ for all $S, T \in \operatorname{Irr}(G)$.

First note that for $i, j \in \mathbb{Z}/m\mathbb{Z}$ we have, by the isomorphisms discussed in §2.3.1,

$$\begin{aligned}\operatorname{Hom}_G(T_i, V \otimes T_j) &= \operatorname{Hom}_G(\operatorname{Ind}_A^G(S_i), V \otimes T_j) \cong \operatorname{Hom}_A(S_i, \operatorname{Res}_A^G(V \otimes T_j)) \\ &\cong \operatorname{Hom}_A(S_i, V \otimes \operatorname{Res}_A^G(T_j)) \cong \operatorname{Hom}_A(S_i, V \otimes (\textstyle\bigoplus_{k=0}^{s-1} S_{r^k j})). \qquad (2.5.1)\end{aligned}$$

By Proposition 2.5.3 $\operatorname{Hom}_A(S_i, V \otimes (\bigoplus_{k=0}^{s-1} S_{r^k j}))$ is generated by elements of the form x_p^i, for some $a, p \in \{0, \dots, s-1\}$ which satisfy $r^a j \equiv i - r^p \pmod m$. Now take such an element $x_p^i \in \operatorname{Hom}_A(S_i, V \otimes (\bigoplus_{k=0}^{s-1} S_{r^k j}))$ and call $x_{p,a}^i$ its image in $\operatorname{Hom}_G(T_i, V \otimes T_j)$ under the inverses of the isomorphisms (2.5.1). Then, recalling that we identify $\beta^a v_j \in \operatorname{Res}_A^G(T_j)$ with $v_{r^a j} \in \bigoplus_{k=0}^{s-1} S_{r^k j}$, it is easy to see that this homomorphism is explicitly given by

$$\begin{aligned} x_{p,a}^i \colon T_i &\longrightarrow V \otimes T_j \\ v_i &\longmapsto \beta^p v_1 \otimes \beta^a v_j. \end{aligned}$$

Before going on we make the following observation.

Proposition 2.5.4. *In the quivers Q_G and Q_A there are no arrows between two fixed points.*

Proof. Suppose that we have an arrow $i^{(\ell)} \to j^{(\ell')}$ in Q_G, where $i, j \in \mathcal{F}$ and $0 \leq \ell, \ell' \leq s-1$. Then the space $\mathrm{Hom}_G(T_i^{(\ell)}, V \otimes T_j^{(\ell')}) \cong \mathrm{Hom}_G(T_i^{(\ell)}, T_{1+j})$ is different from zero. By (M5), V is irreducible and thus so is $V \otimes T_j^{(\ell')} \cong T_{1+j}$. Hence, by Schur's Lemma, we must have $T_i^{(\ell)} \cong T_{1+j}$, which is a contradiction because these representations have different dimensions as vector spaces.

Now take two fixed points $i, j \in (Q_A)_0$. The arrows $i \to j$ in Q_A are given by a basis of $\mathrm{Hom}_A(S_i, V \otimes S_j) \cong \mathrm{Hom}_A(S_i, \bigoplus_{k=0}^{s-1} S_{r^k+j})$, so if there exists such an arrow then we must have $i \equiv r^a + j \pmod{m}$ for some a. This implies that r^a is a fixed point, because so is $i - j$. Hence we have $r \equiv 1$, which is again a contradiction by (M5). $\square$

So for choosing a basis of the arrows we only have to consider the following three possibilities.

(1) $i, j \in \mathcal{D} \setminus \mathcal{F}$. Then we choose $\{x^i_{p,a} \mid r^a j \equiv i - r^p \pmod{m}\}$ as a basis of $\mathrm{Hom}_G(T_i, V \otimes T_j)$. These elements form indeed a basis because they are the image under an isomorphism of a basis of $\mathrm{Hom}_A(S_i, V \otimes (\bigoplus_{k=0}^{s-1} S_{r^k j}))$.

(2) $i \in \mathcal{D} \setminus \mathcal{F}$ and $j \in \mathcal{F}$. For any $\ell = 0, \ldots, s-1$, we call $\pi_j^{(\ell)} \colon T_j \to T_j^{(\ell)}$ the projection morphism and, given a, p which satisfy $r^a j \equiv i - r^p \pmod{m}$, we define

$$x^{i(\ell)}_{p,a} := (\mathrm{id}_V \otimes \pi_j^{(\ell)}) \circ x^i_{p,a} \colon T_i \to V \otimes T_j^{(\ell)}.$$

Note that, since j is a fixed point, $r^a j \equiv j$ and so every choice of a satisfies the condition above. We choose $\{x^{i(\ell)}_{p,0} \mid j \equiv i - r^p \pmod{m}\}$ to be our basis for $\mathrm{Hom}_G(T_i, V \otimes T_j^{(\ell)})$. This set is indeed a basis, because it has cardinality at most 1 and $\mathrm{Hom}_G(T_i, V \otimes T_j^{(\ell)})$ has dimension at most 1.

(3) $i \in \mathcal{F}$ and $j \in \mathcal{D} \setminus \mathcal{F}$. For any $\ell = 0, \ldots, s-1$, we call $\iota_i^{(\ell)} \colon T_i^{(\ell)} \to T_i$ the inclusion morphism and, given a, p which satisfy $r^a j \equiv i - r^p \pmod{m}$, we define

$$x^{(\ell)i}_{p,a} := x^i_{p,a} \circ \iota_i^{(\ell)} \colon T_i^{(\ell)} \to V \otimes T_j.$$

Note that, since i is a fixed point, we have that $r^{a+b} j \equiv i - r^{p+b}$ for any b, so we can assume that the exponent of r is zero in the condition above. We choose $\{x^{(\ell)i}_{p,0} \mid j \equiv i - r^p \pmod{m}\}$ to be our basis for $\mathrm{Hom}_G(T_i^{(\ell)}, V \otimes T_j)$. This set is indeed a basis, because it has cardinality at most 1 and $\mathrm{Hom}_G(T_i^{(\ell)}, V \otimes T_j)$ has dimension at most 1.

The following lemma consists in some calculations which will be used in the next subsections.

Lemma 2.5.5. *(a) We have* $\pi_i^{(\ell)}(v_i) = \frac{1}{s} \varepsilon_m^{-ti} \lambda_{i,\ell} w_i^{(\ell)}$ *and* $\iota_i^{(\ell)}(w_i^{(\ell)}) = \sum_{k=0}^{s-1} \lambda_{i,\ell}^{s-1-k} \beta^k v_i$.

(b) $x^{i(\ell)}_{p,0}(v_i) = \frac{\varepsilon_m^{-tj}}{s} \lambda_{j,\ell} \beta^p v_1 \otimes w_j^{(\ell)}$.

(c) $x_{p,0}^{(\ell)i}(w_i^{(\ell)}) = \sum_{k=0}^{s-1} \lambda_{i,\ell}^{s-1-k} \beta^{k+p} v_1 \otimes \beta^k v_j$.

(d) If j is a fixed point, the composition

$$T_i \xrightarrow{x_{p,0}^{i(\ell)}} V \otimes T_j^{(\ell)} \xrightarrow{\mathrm{id}_V \otimes x_{q,0}^{(\ell)j}} V \otimes V \otimes T_h$$

sends v_i to

$$\frac{1}{s} \sum_{k=0}^{s-1} \lambda_{j,\ell}^{s-k} \beta^p v_1 \otimes \beta^{k+q} v_1 \otimes \beta^k v_h.$$

Proof. (a) The second claim is immediate from the definition of $w_i^{(\ell)}$. For the first claim, it is enough to show that $v_i = \frac{1}{s}\sum_{\ell=0}^{s-1} \varepsilon_m^{-ti} \lambda_{i,\ell} w_i^{(\ell)}$. Indeed, we have

$$\frac{1}{s}\sum_{\ell=0}^{s-1} \varepsilon_m^{-ti} \lambda_{i,\ell} w_i^{(\ell)} = \frac{1}{s}\sum_{\ell=0}^{s-1} \varepsilon_m^{-ti} \lambda_{i,\ell} \sum_{k=0}^{s-1} \lambda_{i,\ell}^{s-1-k} \beta^k v_i = \frac{1}{s}\sum_{k=0}^{s-1} \varepsilon_m^{-ti} \left(\sum_{\ell=0}^{s-1} \lambda_{i,\ell}^{s-k} \right) \beta^k v_i.$$

Recalling that $\lambda_{i,\ell} = \eta_i \varepsilon_s^{\ell}$, where $\eta_i^s = \varepsilon_m^{ti}$, we get

$$\sum_{\ell=0}^{s-1} \lambda_{i,\ell}^{s-k} = \eta_i^{s-k} \sum_{\ell=0}^{s-1} \varepsilon_s^{l(s-k)} = \eta_i^{s-k} \sum_{\ell=0}^{s-1} \varepsilon_s^{-kl} = \begin{cases} s\eta_i^s = s\varepsilon_m^{ti} & \text{if } k = 0, \\ \frac{1-\varepsilon_s^{-ks}}{1-\varepsilon_s^{-k}} = 0 & \text{if } k \neq 0, \end{cases}$$

and the result follows.

(b), (c), (d) follow immediately from (a). □

Remark 2.5.6. We may note that in the previous lemma we made an abuse of notation. Indeed, the basis of T_i is made of the elements $\beta^k v_1$ for $k \in \{0, \dots, s-1\}$, so when we write $\beta^k v_i$ for some $k \notin \{0, \dots, s-1\}$ this should be intended as $\beta^{\bar{k}} v_i$ (where $\bar{k}$ denotes the smallest non-negative integer in the equivalence class of k modulo s) multiplied by a constant, more precisely by a power of ε_m^{ti}. The exact value of this constant will not be important for us in what follows, the only thing which should be pointed out is that it is different from zero. So, keeping this in mind, we will tacitly carry on this abuse of notation and continue to write $\beta^k v_i$ even if $k \notin \{0, \dots, s-1\}$.

Our next aim will be to describe the superpotential of Q_G. Actually, this superpotential will be twisted, where the twist is induced by the tensor product with the representation $\det_V$. Hence we will start by describing explicitly this automorphism.

2.5.2 The twist in Q_G

Set $c := \sum_{k=0}^{s-1} r^k$ and note that it is a fixed point. We may observe that the elements α and β act on $\det_V$ respectively as the multiplication by ε_m^c and by $(-1)^{s-1}\varepsilon_m^t$. Since t is a fixed point, we have that $r^k t \equiv t \pmod m$ for all k, thus $tc \equiv ts \pmod m$. Hence, putting

$d_s := \begin{cases} 0 & \text{if } s > 2, \\ 1 & \text{if } s = 2 \end{cases}$, we get that $\lambda_{c,d_s} = \varepsilon_m^{\frac{t}{s}c} \varepsilon_s^{d_s} = \varepsilon_m^t (-1)^{s-1}$ and so $\det_V = T_c^{(d_s)}$. It is easily checked that the maps

$$\begin{aligned} T_i \otimes \det_V &\to T_{i+c} \\ v_i \otimes w_c^{(d_s)} &\mapsto v_{i+c} \end{aligned} \tag{2.5.2}$$

and, if i is a fixed point,

$$\begin{aligned} T_i^{(\ell)} \otimes \det_V &\to T_{i+c}^{(\ell+d_s)} \\ w_i^{(\ell)} \otimes w_c^{(d_s)} &\mapsto w_{i+c}^{(\ell+d_s)}, \end{aligned} \tag{2.5.3}$$

are isomorphisms (for the second map, note that $\lambda_{i,\ell}\lambda_{c,d_s} = \lambda_{i+c,\ell+d_s}$).

From now on, we will need to make the following assumption.

Assumption 2.5.7. The set of representatives $\mathcal{D}$ is closed under the sum by c.

Hence we have that the twist acts on vertices by $\tau(i) = i + c$ if $i \in \mathcal{D} \setminus \mathcal{F}$, and $\tau(i^{(\ell)}) = (i+c)^{(\ell+d_s)}$ if $i \in \mathcal{F}$, $0 \leq \ell \leq s-1$.

The following lemma can be easily proved using the isomorphisms (2.5.2) and (2.5.3).

Lemma 2.5.8. *The twist τ sends each arrow in Q_G to a non-zero scalar multiple of another arrow. More precisely, we have:*

(1) if $i, j \in \mathcal{D} \setminus \mathcal{F}$, then $\tau(x_{p,a}^i) = \lambda_{c,d_s}^{-a} x_{p,a}^{i+c}$;

(2) if $i \in \mathcal{D} \setminus \mathcal{F}$ and $j \in \mathcal{F}$, then $\tau(x_{p,0}^{i(\ell)}) = \lambda_{c,d_s}^{s-1} x_{p,0}^{i(\ell+d_s)}$;

(3) if $i \in \mathcal{F}$ and $j \in \mathcal{D} \setminus \mathcal{F}$, then $\tau(x_{p,0}^{(\ell)i}) = \lambda_{c,d_s}^{1-s} x_{p,0}^{(\ell+d_s)i}$.

In view of the above lemma, we have that τ induces in a natural way an automorphism τ' of Q_G, as we observed in the discussion above Lemma 2.3.5.

2.5.3 The superpotential of Q_G

Now we are ready to begin the study of the superpotential ω_G. Our aim will be to prove Theorem 2.5.19, which says that every path in $\operatorname{supp}(\omega_G)$ is induced from a path in $\operatorname{supp}(\omega_A)$ (we will make this statement more precise by introducing a new quiver $\tilde{Q}_G$, see Definition 2.5.15). We start by proving two lemmas which describe explicitly the map $T_i \to V^{\otimes u} \otimes T_j$ associated to a path of length u in Q_G. This will be done only when such a path satisfies some technical assumptions, since, as we will see in the proof of Theorem 2.5.19, the general case can be traced back to that of paths of this kind. Finally, in Theorem 2.5.20 we will see that a converse of Theorem 2.5.19 holds for paths which contain at most one fixed point.

Lemma 2.5.9. *For $i_1, i_2, \dots, i_{u+1} \in \mathbb{Z}/m\mathbb{Z}$, let $a_1, \dots, a_u, p_1, \dots, p_u$ be integers which satisfy $r^{a_h} i_{h+1} \equiv i_h - r^{p_h} \pmod m$ for each $h = 1, \dots, u$. Then:*

(1) the composition

$$T_{i_1} \xrightarrow{x^{i_1}_{p_1,a_1}} V \otimes T_{i_2} \xrightarrow{1_V \otimes x^{i_2}_{p_2,a_2}} \dots \xrightarrow{1_{V^{\otimes u-1}} \otimes x^{i_u}_{p_u,a_u}} V^{\otimes u} \otimes T_{i_{u+1}}$$

sends v_{i_1} to the element

$$\beta^{p_1} v_1 \otimes \beta^{a_1+p_2} v_1 \otimes \beta^{a_1+a_2+p_3} v_1 \otimes \cdots \otimes \beta^{a_1+\cdots+a_{u-1}+p_u} v_1 \otimes \beta^{a_1+\cdots+a_u} v_{i_{u+1}};$$

(2) we have

$$r^{a_1+\cdots+a_u} i_{u+1} \equiv i_1 - r^{p_1} - r^{a_1+p_2} - \cdots - r^{a_1+\cdots+a_{u-1}+p_u} \pmod m. \tag{2.5.4}$$

Proof. (1) We proceed by induction on u. The case $u = 1$ is clear by the definition of the $x^i_{p,a}$'s. Now suppose $u > 1$. By the induction hypothesis we have

$$\begin{aligned}
&(1_{V^{\otimes u-1}} \otimes x^{i_u}_{p_u,a_u}) \circ \cdots \circ (x^{i_1}_{p_1,a_1})(v_{i_1}) = \\
&= (1_{V^{\otimes u-1}} \otimes x^{i_u}_{p_u,a_u})(\beta^{p_1} v_1 \otimes \beta^{a_1+p_2} v_1 \otimes \cdots \otimes \beta^{a_1+\cdots+a_{u-2}+p_{u-1}} v_1 \otimes \beta^{a_1+\cdots+a_{u-1}} v_{i_u}) \\
&= \beta^{p_1} v_1 \otimes \beta^{a_1+p_2} v_1 \otimes \cdots \otimes \beta^{a_1+\cdots+a_{u-2}+p_{u-1}} v_1 \otimes \beta^{a_1+\cdots+a_{u-1}} x^{i_u}_{p_u,a_u}(v_{i_u}) \\
&= \beta^{p_1} v_1 \otimes \beta^{a_1+p_2} v_1 \otimes \cdots \otimes \beta^{a_1+\cdots+a_{u-2}+p_{u-1}} v_1 \otimes \beta^{a_1+\cdots+a_{u-1}} (\beta_1^{p_u} v_1 \otimes \beta^{a_u} v_{i_{u+1}}) \\
&= \beta^{p_1} v_1 \otimes \beta^{a_1+p_2} v_1 \otimes \beta^{a_1+a_2+p_3} v_1 \otimes \cdots \otimes \beta^{a_1+\cdots+a_{u-1}+p_u} v_1 \otimes \beta^{a_1+\cdots+a_u} v_{i_{u+1}}.
\end{aligned}$$

(2) We proceed again by induction on u. The case $u = 1$ is clear since $r^{a_1} i_2 \equiv i_1 - r^{p_1}$. Now suppose that $r^{a_1+\cdots+a_{u-1}} i_u \equiv i_1 - r^{p_1} - r^{a_1+p_2} - \cdots - r^{a_1+\cdots+a_{u-2}+p_{u-1}}$. By adding on both sides $-r^{a_1+\cdots+a_{u-1}+p_u}$ we see that is enough to prove that $r^{a_1+\cdots+a_{u-1}} i_u - r^{a_1+\cdots+a_{u-1}+p_u} \equiv r^{a_1+\cdots+a_u} i_{u+1}$, but this follows immediately from the fact that $r^{a_u} i_{u+1} \equiv i_u - r^{p_u}$. □

Lemma 2.5.10. *Let $i_1, i_2, \dots, i_{u+1} \in \mathcal{D}$ and let $a_1, \dots, a_u, p_1, \dots, p_u$ be integers which satisfy $r^{a_j} i_{j+1} \equiv i_j - r^{p_j} \pmod m$ for each $j = 1, \dots, u$. Suppose that the only fixed points among the i_j's are $i_{f_1}, \dots, i_{f_h}$ for some integers $f_1, \dots, f_h \in \{2, \dots, u\}$, and that $a_{f_j-1} = a_{f_j} = 0$ for any j. We also assume that $f_{j+1} - f_j \geq 2$ for each $j = 2, \dots, h-1$.*

Fix integers $\ell_1, \dots, \ell_h \in \{0, \dots, s-1\}$. We introduce the following simplified notation: for each $j = 1, \dots, u$, we set $i'_j := i_{f_j-1}$, $p'_j := p_{f_j-1}$, $i''_j := i_{f_j}$ and $p''_j := p_{f_j}$.

Consider the following path of length u:

$$i_1 \xrightarrow{x^{i_1}_{p_1,a_1}} \dots \to i'_j \xrightarrow{x^{i'_j(\ell_j)}_{p'_j,0}} i''^{(\ell_j)}_j \xrightarrow{x^{(\ell_j)i''_j}_{p''_j,0}} i_{f_j+1} \to \dots \xrightarrow{x^{i_u}_{p_u,a_u}} i_{u+1}. \tag{2.5.5}$$

Then the corresponding composition $T_{i_1} \to \dots \to V^{\otimes u} \otimes T_{i_{u+1}}$ sends v_{i_1} to

$$\frac{1}{s^h} \sum_{k_1,\dots,k_h=0}^{s-1} \left(\prod_{j=1}^{h} \varepsilon_s^{-ti_{f_j}} \lambda^{s-k_j}_{i_{f_j},\ell_j} \right) \beta^{q_1} v_1 \otimes \cdots \otimes \beta^{q_u} v_1 \otimes \beta^{q_{u+1}} v_{i_{u+1}}, \tag{2.5.6}$$

where $f_0 := 0$ *and*

$$\begin{aligned}
q_1 &:= p_1, \\
&\dots \\
q_{f_1-1} &:= a_1 + \cdots + a_{f_1-2} + p_{f_1-1}, \\
q_{f_1} &:= k_1 + a_1 + \cdots + a_{f_1-1} + p_{f_1}, \\
&\dots \\
q_{f_2-1} &:= k_1 + a_1 + \cdots + a_{f_2-2} + p_{f_2-1}, \\
q_{f_2} &:= k_1 + k_2 + a_1 + \cdots + a_{f_2-1} + p_{f_2}, \\
&\dots \\
q_{f_h-1} &:= k_1 + \cdots + k_{h-1} + a_1 + \cdots + a_{f_h-2} + p_{f_h-1}, \\
q_{f_h} &:= k_1 + \cdots + k_h + a_1 + \cdots + a_{f_h-1} + p_{f_h}, \\
&\dots \\
q_u &:= k_1 + \cdots + k_h + a_1 + \cdots + a_{u-1} + p_u \\
q_{u+1} &:= k_1 + \cdots + k_h + a_1 + \cdots + a_u.
\end{aligned}$$

Remark 2.5.11. Note that the assertion of the lemma makes sense also for $h = 0$. In this case the element (2.5.6) is equal to

$$\beta^{p_1} v_1 \otimes \beta^{a_1+p_2} v_1 \otimes \cdots \otimes \beta^{a_1+\cdots+a_{u-1}+p_u} v_1 \otimes \beta^{a_1+\cdots+a_u} v_{i_{u+1}},$$

and so the result follows from Lemma 2.5.9.

Proof of Lemma 2.5.10. We proceed by induction on the number h of fixed points. The case $h = 0$ is clear by the previous remark.

Now fix an $h \geq 1$ and suppose that the statement is valid for every path of any length which contains strictly less than h fixed points. So let $\mathbf{p}$ be the path (2.5.5) and consider the subpath

$$\mathbf{p}' \colon i_1 \to \ldots \to i_{f_h-1}.$$

Clearly $\mathbf{p}'$ contains $h-1$ fixed points, and neither its starting nor its ending points are among them. So $\mathbf{p}'$ satisfies the induction hypothesis and we have that the composition $T_{i_1} \to \ldots \to V^{\otimes f_h-2} \otimes T_{f_h-1}$ sends v_{i_1} to

$$\frac{1}{s^{h-1}} \sum_{k_1,\ldots,k_{h-1}=0}^{s-1} \left(\prod_{j=1}^{h-1} \varepsilon_s^{-t i_{f_j}} \lambda_{i_{f_j},\ell_j}^{s-k_j} \right) \beta^{q_1} v_1 \otimes \cdots \otimes \beta^{q_{f_h-2}} v_1 \otimes \beta^{q'_{f_h-1}} v_{i_{f_h-1}},$$

where $q'_{f_h-1} := k_1 + \cdots + k_{h-1} + a_1 + \cdots + a_{f_h-2}$. If we apply to this element the map

$$V^{\otimes(f_h-2)} \otimes T_{f_h-1} \xrightarrow{\left(\mathrm{id}_{V^{\otimes(f_h-2)}}\right) \otimes x_{p'_h,0}^{i'_h(\ell_h)}} V^{\otimes(f_h-1)} \otimes T_{f_h}^{(\ell_h)} \xrightarrow{\left(\mathrm{id}_{V^{\otimes(f_h-1)}}\right) \otimes x_{p''_h,0}^{(\ell_h)i''_h}} V^{\otimes f_h} \otimes T_{f_h+1}$$

we get

$$\frac{1}{s^{h-1}}\sum_{k_1,\dots,k_{h-1}=0}^{s-1}\left(\prod_{j=1}^{h-1}\varepsilon_s^{-ti_{f_j}}\lambda_{i_{f_j},\ell_j}^{s-k_j}\right)\beta^{q_1}v_1\otimes\cdots\otimes\beta^{q_{f_h-2}}v_1\otimes\beta^{q'_{f_h-1}}\theta(v_{i_{f_h-1}}), \qquad (2.5.7)$$

where $\theta := \left(\mathrm{id}_V\otimes x_{p''_h,0}^{(\ell_h)i''_h}\right)\circ x_{p'_h,0}^{i'_h(\ell_h)}$. By Lemma 2.5.5 we have that

$$\theta(v_{i_{f_h-1}}) = \frac{\varepsilon_s^{-ti_{f_h}}}{s}\sum_{k_h=0}^{s-1}\lambda_{i_{f_h},\ell_h}^{s-k_h}\beta^{p_{f_h-1}}v_1\otimes\beta^{k_h+p_{f_h}}v_1\otimes\beta^{k_h}v_{i_{f_h+1}},$$

so the (2.5.7) becomes

$$\frac{1}{s^{h}}\sum_{k_1,\dots,k_{h}=0}^{s-1}\left(\prod_{j=1}^{h-1}\varepsilon_s^{-ti_{f_j}}\lambda_{i_{f_j},\ell_j}^{s-k_j}\right)\beta^{q_1}v_1\otimes\cdots\otimes\beta^{q_{f_h-1}}v_1\otimes\beta^{q_{f_h}}v_1\otimes\beta^{q'_{f_h-1}+k_h}v_{i_{f_h+1}},$$

since $q_{f_h-1} = k_1+\cdots+k_{h-1}+a_1+\cdots+a_{f_h-2}+p_{f_h-1} = q'_{f_h-1}+p_{f_h-1}$, $q_{f_h} = k_1+\cdots+k_h+a_1+\cdots+a_{f_h-1}+p_{f_h} = q'_{f_h-1}+a_{f_h-1}+k_h+p_{f_h} = q'_{f_h-1}+k_h+p_{f_h}$. Hence, if we go on calculating the exponents in the same way as in Lemma 2.5.9, we obtain the result we wanted to prove. □

Let us illustrate in an example how one can compute the element (2.5.6) of Lemma 2.5.10.

Example 2.5.12. Let G be the metacyclic group associated to $m = 21$, $r = 4$, $s = 3$, $t = 0$ (see Example 2.7.7 for further details about this example). Choose

$$\mathcal{D} = \{0,4,7,8,9,12,13,14,17\}\subseteq\mathbb{Z}/21\mathbb{Z}$$

as a set of representatives for the G/A-action. For an $\ell\in\{0,1,2\}$, consider the path

$$\mathbf{p}\colon 12 \xrightarrow{x_{0,1}^{12}} 8 \xrightarrow{x_{0,0}^{8(\ell)}} 7^{(\ell)} \xrightarrow{x_{2,0}^{(\ell)7}} 12$$

in Q_G, so, according to the notation of Lemma 2.5.10, we have $i_1 = 12$, $i_2 = 8$, $i_3 = 7$, $i_4 = 12$, $a_1 = 1$, $a_2 = a_3 = 0$, $p_1 = p_2 = 0$, $p_3 = 2$, $h = 1$, $f_1 = 3$, $\lambda_{i_3,\ell} = \varepsilon_3^{\ell+1}$. In this case the element (2.5.6) becomes

$$\begin{aligned}&\frac{1}{3}\sum_{k=0}^{2}\lambda_{i_3,\ell}\beta^{p_1}v_1\otimes\beta^{a_1+p_2}v_1\otimes\beta^{k+a_1+a_2+p_3}v_1\otimes\beta^{k+a_1+a_2+a_3}v_1\\ =&\frac{1}{3}\left(\varepsilon_3^{3(\ell+1)}v_1\otimes\beta v_1\otimes v_1\otimes\beta v_1+\varepsilon_3^{2(\ell+1)}v_1\otimes\beta v_1\otimes\beta v_1\otimes\beta^2 v_1+\varepsilon_3^{\ell+1}v_1\otimes\beta v_1\otimes\beta^2 v_1\otimes v_1\right).\end{aligned}$$

Note that if we apply the antisymmetrizer to it we get

$$\frac{1}{3}\varepsilon_3^{\ell+1}v_1\wedge\beta v_1\wedge\beta^2 v_1\otimes v_1.$$

This implies that the coefficient $c_{\mathbf{p}}$ of $\mathbf{p}$ in ω_G is equal to $\frac{1}{3}\varepsilon_3^{\ell+1}$, and in particular we have that $\mathbf{p}\in\operatorname{supp}(\omega_G)$.

The following proposition, which will be used in the proof of Theorem 2.5.20, gives a necessary and sufficient condition for a path without fixed points to lie in $\operatorname{supp}(\omega_G)$.

Proposition 2.5.13. *Let*

$$\mathbf{p}\colon i_1 \xrightarrow{x^{i_1}_{p_1,a_1}} \dots \xrightarrow{x^{i_s}_{p_s,a_s}} i_{s+1}$$

be a path in Q_G which contains no fixed points. Then $\mathbf{p}$ lies in the support of ω_G if and only if $\{p_1, a_1+p_2, \dots, a_1+\dots+a_{s-1}+p_s\}$ is a complete set of representatives of the integers modulo s.

Proof. The path $\mathbf{p}$ can be identified with the composition

$$T_{i_1} \to \dots \to V^{\otimes s} \otimes T_{i_{s+1}}.$$

Composing this map with the antisymmetrizer we obtain a morphism

$$T_{i_1} \to \det_V \otimes T_{i_{s+1}},$$

which, by Lemma 2.5.9, sends v_{i_1} to $\beta^{p_1}v_1 \wedge \beta^{a_1+p_2}v_1 \wedge \dots \wedge \beta^{a_1+\dots+a_{s-1}+p_s}v_1 \otimes \beta^{a_1+\dots+a_s}v_{i_{s+1}}$. Then it is clear that this element is different from zero if and only if the integers $p_1, a_1+p_2, \dots, a_1+\dots+a_{s-1}+p_s$ are pairwise different modulo s. $\square$

Example 2.5.14. Let us consider again the case where $m = 21$, $r = 4$, $s = 3$, $t = 0$. The path

$$12 \xrightarrow{x^{12}_{2,0}} 17 \xrightarrow{x^{17}_{0,1}} 4 \xrightarrow{x^{4}_{0,2}} 12$$

is in $\operatorname{supp}(\omega_G)$ because $\{p_1, a_1+p_2, a_1+a_2+p_3\} = \{2, 0, 1\}$.

On the other end, we have that

$$12 \xrightarrow{x^{12}_{2,0}} 17 \xrightarrow{x^{17}_{2,2}} 4 \xrightarrow{x^{4}_{0,2}} 12$$

is not in $\operatorname{supp}(\omega_G)$, since $\{p_1, a_1+p_2, a_1+a_2+p_3\} = \{2, 2, 2\}$.

Proposition 2.5.13 tells us exactly when a path containing no fixed points is in the support of ω_G. However, in view of Lemma 2.5.10, obtaining a similar statement for paths containing fixed points seems more difficult and we will not prove such a result. Nevertheless, we will be able to show that every path in the support of ω_G comes, in a certain sense, from a path in the support of ω_A, and this will be sufficient for our purposes. In order to make this more precise, we give the following definition.

Definition 2.5.15. Let $\tilde{Q}_G$ be the quiver defined in the following way. Its set of vertices is $\mathcal{D}$, while for the arrows $i \to j$ we have the following three possibilities:

(1) $i, j \in \mathcal{D} \setminus \mathcal{F}$: in this case the arrows between i and j in $\tilde{Q}_G$ are the same as in Q_G;

(2) $i \in \mathcal{D} \setminus \mathcal{F}$ and $j \in \mathcal{F}$: we put an arrow $x^i_{p,0}$ in $\tilde{Q}_G$ whenever $j \equiv i - r^p$ for some p;

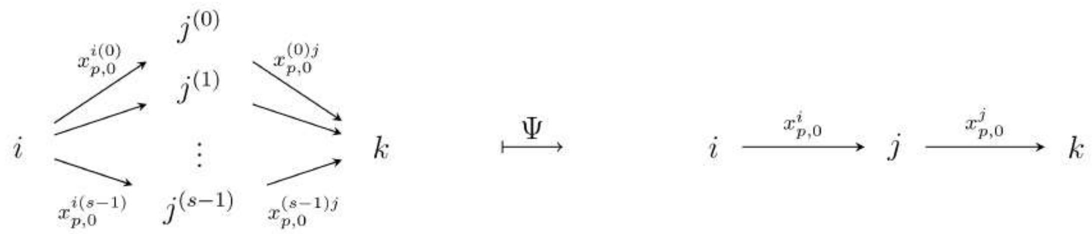


Figure 2.1. The local behaviour of Ψ at fixed points.

(3) $i \in \mathcal{F}$ and $j \in \mathcal{D} \setminus \mathcal{F}$: we put an arrow $x^{i}_{p,0}$ in $\tilde{Q}_G$ whenever $j \equiv i - r^p$ for some p.

We will see later that $\tilde{Q}_G$ can be seen as the quotient of Q_A by an action of G/A.

We now define two morphisms of quivers

$$\Phi\colon Q_A \to \tilde{Q}_G, \qquad \Psi\colon Q_G \to \tilde{Q}_G.$$

For $i \in (Q_A)_0$ we put $\Phi(i) = \underline{i}$. Given an arrow $x^i_q\colon i \to i - r^q$ in Q_A, we set $\Phi(x^i_q) = x^{\underline{i}}_{p,a}\colon \underline{i} \to \underline{i - r^q}$, where

$$(p,a) = \begin{cases} (q - \kappa_i, \kappa_{i-r^q} - \kappa_i) & \text{if } \underline{i}, \underline{i-r^q} \in \mathcal{D} \setminus \mathcal{F}; \\ (q - \kappa_i, 0) & \text{if } \underline{i} \in \mathcal{D} \setminus \mathcal{F},\ \underline{i - r^q} \in \mathcal{F}; \\ (q - \kappa_{i-r^q}, 0) & \text{if } \underline{i} \in \mathcal{F},\ \underline{i - r^q} \in \mathcal{D} \setminus \mathcal{F}. \end{cases}$$

We define $\Psi(i) = i$ if $i \in \mathcal{D} \setminus \mathcal{F}$ and $\Psi(i^{(\ell)}) = i$ if $i \in \mathcal{F}$, $0 \le \ell \le s-1$. Moreover we put $\Psi(x^i_{p,a}) = x^i_{p,a}$, $\Psi(x^{i(\ell)}_{p,0}) = x^i_{p,0}$, $\Psi(x^{(\ell)i}_{p,0}) = x^i_{p,0}$ whenever the notation makes sense. So basically we can consider Ψ simply as the map which "forgets" about the splitting of fixed points (see Figure 2.1).

Consider the subquivers $Q_G \setminus \mathcal{F}$ and $\tilde{Q}_G \setminus \mathcal{F}$ of, respectively, Q_G and $\tilde{Q}_G$, which are obtained by removing the fixed points and the arrows adjacent to them. Then it is clear that $\Psi\,|_{Q_G \setminus \mathcal{F}}\colon Q_G \setminus \mathcal{F} \to \tilde{Q}_G \setminus \mathcal{F}$ is an isomorphism. So the fact that we used the same names to indicate vertices and arrows in these subquivers will cause no confusion: actually, we will often treat them as if they were the same quiver.

Recall that G/A acts on the vertices of Q_A via the automorphism φ given by the multiplication by r. We can extend this to an automorphism of Q_A by setting $\varphi(x^i_q) = x^{ri}_{q+1}$.

Consider now the orbit quiver $Q_A/(G/A)$. We will denote by $[i]$ the orbit of $i \in (Q_A)_0$ and by $[x^i_q]$ the orbit of $x^i_q \in (Q_A)_1$.

Proposition 2.5.16. *The morphism Φ induces an isomorphism of quivers*

$$\tilde{\Phi}\colon Q_A/(G/A) \to \tilde{Q}_G$$

.

Proof. For each $i \in (Q_A)_0$ we have $\Phi(\varphi(i)) = \Phi(ri) = \underline{ri} = \underline{i}$, so Φ, as a map between vertices, factors through the action of G/A. Hence we get a map $(Q_A/(G/A))_0 \to (\tilde{Q}_G)_0$, $[i] \mapsto \Phi(i) = \underline{i}$, which is obviously a bijection because $(\tilde{Q}_G)_0 = \mathcal{D}$ is a set of representatives of the G/A-orbits.

Now we consider the arrows. For each $x_q^i \in (Q_A)_1$ we have $\Phi(\varphi(x_q^i)) = \Phi(x_{q+1}^{ri}) = x_{p,a}^{\underline{ri}} = x_{p,a}^{\underline{i}}$, where

$$(p,a) = \begin{cases} (q+1-\kappa_{ri}, \kappa_{r(i-r^q)} - \kappa_{ri}) & \text{if } \underline{ri}, \underline{r(i-r^q)} \in \mathcal{D} \smallsetminus \mathcal{F}; \\ (q+1-\kappa_{ri}, 0) & \text{if } \underline{ri} \in \mathcal{D} \smallsetminus \mathcal{F},\ \underline{r(i-r^q)} \in \mathcal{F}; \\ (q+1-\kappa_{r(i-r^q)}, 0) & \text{if } \underline{ri} \in \mathcal{F},\ \underline{r(i-r^q)} \in \mathcal{D} \smallsetminus \mathcal{F}. \end{cases}$$

It is easy to check that $\underline{ri} = \underline{i}$, $\underline{r(i-r^q)} = \underline{i-r^q}$, $\kappa_{ri} = \kappa_i + 1$, $\kappa_{r(i-r^q)} = \kappa_{i-r^q} + 1$, so it becomes clear from the definition of Φ that $\Phi(\varphi(x_q^i)) = \Phi(x_q^i)$. Hence Φ factors through the action of G/A and we get a map $\tilde{\Phi}\colon (Q_A/(G/A))_1 \to (\tilde{Q}_G)_1$, $[x_q^i] \mapsto \Phi(x_q^i)$.

To show that this map is surjective, take an arrow $x_{q,b}^i\colon i \to j$ in $(\tilde{Q}_G)_1$, so $r^b j \equiv i - r^q$ holds. Note that $\kappa_i = 0$ and $\kappa_{i-r^q} = b$. By definition, $\Phi(x_q^i) = x_{p,a}^i$, where

$$(p,a) = \begin{cases} (q-\kappa_i, \kappa_{i-r^q} - \kappa_i) = (q,b) & \text{if } i,j \in \mathcal{D} \smallsetminus \mathcal{F}; \\ (q-\kappa_i, 0) = (q,b) & \text{if } i \in \mathcal{D} \smallsetminus \mathcal{F},\ j \in \mathcal{F}; \\ (q-\kappa_{i-r^q}, 0) = (q,b) & \text{if } i \in \mathcal{F},\ j \in \mathcal{D} \smallsetminus \mathcal{F}. \end{cases}$$

This means that $\Phi(x_q^i) = x_{q,b}^i$, so we showed that Φ (and consequently $\tilde{\Phi}$) is surjective on arrows.

Now we prove that $\tilde{\Phi}$ is injective. Let x_p^i, x_q^j be two arrows in Q_A and suppose that $\Phi(x_p^i) = \Phi(x_q^j)$: we want to show that x_p^i and x_q^j lie in the same G/A-orbit. Clearly we have that $\underline{i} = \underline{j}$ and $\underline{i-r^p} = \underline{j-r^q}$, so there exist $a, b \in \{0, \dots, s-1\}$ such that $j \equiv r^a i \pmod m$ and $j - r^q \equiv r^b(i - r^p) \pmod m$ (we take $a = 0$ and $b = 0$ respectively when i and $i - r^p$ are fixed points). Note that $r^{\kappa_i}\underline{j} \equiv r^{\kappa_i}\underline{i} \equiv i \equiv r^{-a} j$, so $\kappa_j = \kappa_i - a$. Similarly we have $\kappa_{j-r^q} = \kappa_{i-r^p} - b$. By definition of Φ, we write $\Phi(x_p^i) = x_{p',a'}^{\underline{i}}$ and $\Phi(x_q^j) = x_{q',b'}^{\underline{j}}$, where

$$(p',a') = \begin{cases} (p-\kappa_i, \kappa_{i-r^p} - \kappa_i) & \text{if } \underline{i}, \underline{i-r^p} \in \mathcal{D} \smallsetminus \mathcal{F}; \\ (p-\kappa_i, 0) & \text{if } \underline{i} \in \mathcal{D} \smallsetminus \mathcal{F},\ \underline{i-r^p} \in \mathcal{F}; \\ (p-\kappa_{i-r^p}, 0) & \text{if } \underline{i} \in \mathcal{F},\ \underline{i-r^p} \in \mathcal{D} \smallsetminus \mathcal{F} \end{cases}$$

and

$$(q',b') = \begin{cases} (q-\kappa_j, \kappa_{j-r^q} - \kappa_j) & \text{if } \underline{j}, \underline{j-r^q} \in \mathcal{D} \smallsetminus \mathcal{F}; \\ (q-\kappa_j, 0) & \text{if } \underline{j} \in \mathcal{D} \smallsetminus \mathcal{F},\ \underline{j-r^q} \in \mathcal{F}; \\ (q-\kappa_{j-r^q}, 0) & \text{if } \underline{j} \in \mathcal{F},\ \underline{j-r^q} \in \mathcal{D} \smallsetminus \mathcal{F}. \end{cases}$$

By hypothesis $(p',a') = (q',b')$. Hence, in the first two cases we have $p - \kappa_i = q - \kappa_j = q - \kappa_i + a$, so $q = p + a$. This means that $x_q^j = x_{p+a}^{r^a i} = \varphi^a(x_p^i)$ and so $[x_p^i] = [x_q^j]$. In the last case we have $p - \kappa_{i-r^p} = q - \kappa_{j-r^q} = q - \kappa_{i-r^p} + b$, so $q = p + b$. Moreover, both i and j are fixed points, so we can write $j \equiv r^b i \pmod m$. Hence we have that $x_q^j = x_{p+b}^{r^b i} = \varphi^b(x_p^i)$ and so $[x_p^i] = [x_q^j]$. ∎

Example 2.5.17. We shall illustrate the behaviour of Φ in the case of Example 2.5.12. The quivers Q_A, Q_G and $\tilde{Q}_G$ are depicted, respectively, in Figures 2.2, 2.4 and 2.5. Recall that their set of vertices are

$$(Q_A)_0 = \mathbb{Z}/21\mathbb{Z}, \quad (Q_G)_0 = (\tilde{Q}_G)_0 = \{0, 4, 7, 8, 9, 12, 13, 14, 17\}.$$

The map Φ is given on vertices by:

$$\begin{aligned} \Phi(0) = 0, \quad & \Phi(1) = \Phi(4) = \Phi(16) = 4, \quad & \Phi(2) = \Phi(8) = \Phi(11) = 8, \\ \Phi(7) = 7, \quad & \Phi(9) = \Phi(15) = \Phi(18) = 9, \quad & \Phi(3) = \Phi(6) = \Phi(12) = 12, \\ \Phi(14) = 14, \quad & \Phi(10) = \Phi(13) = \Phi(19) = 13, \quad & \Phi(5) = \Phi(17) = \Phi(20) = 17. \end{aligned}$$

We now describe how Φ behaves only on some of the arrows of Q_G:

$$\begin{aligned} \Phi(x_0^{17}\colon 17 \to 16) = \Phi(x_1^5\colon 5 \to 1) = \Phi(x_2^{20}\colon 20 \to 4) = x_{0,1}^{17}\colon 17 \to 4, \\ \Phi(x_0^5\colon 5 \to 4) = \Phi(x_1^{20}\colon 20 \to 16) = \Phi(x_2^{17}\colon 17 \to 1) = x_{2,2}^{17}\colon 17 \to 4, \\ \Phi(x_0^0\colon 0 \to 20) = \Phi(x_1^0\colon 0 \to 17) = \Phi(x_2^0\colon 0 \to 5) = x_{1,0}^0\colon 0 \to 17. \end{aligned}$$

There is a natural way to define an automorphism of $\tilde{Q}_G$ which is compatible with both the twists of Q_A and Q_G via the morphisms Φ and Ψ. More precisely, we have the following proposition/definition.

Proposition 2.5.18. *For each $i \in (Q_A)_0$ and each $x_q^i \in (Q_A)_1$, set $\tau([i]) := [\tau(i)] = [i+c]$ and $\tau([x_q^i]) := [\tau(x_q^i)] = [x_q^{i+c}]$. Then this assignments induce a well defined automorphism of $Q_A/(G/A)$, which in turn induces an automorphism of $\tilde{Q}_G$ under the isomorphism of Proposition 2.5.16: by an abuse of notation, we will again denote both these maps by τ. Moreover these maps are compatible with the other twists, in the sense that the following diagram commutes:*

$$\begin{array}{ccccccc} Q_A & \longrightarrow & Q_A/(G/A) & \xrightarrow{\tilde{\Phi}} & \tilde{Q}_G & \xleftarrow{\Psi} & Q_G \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau'} \\ Q_A & \longrightarrow & Q_A/(G/A) & \xrightarrow{\tilde{\Phi}} & \tilde{Q}_G & \xleftarrow{\Psi} & Q_G \end{array}$$

Proof. For each $i \in (Q_A)_0$ and each $x_q^i \in (Q_A)_1$, we have $\tau([ri]) = [ri + c] = [ri + rc] = [i+c] = \tau([i])$ and similarly $\tau([x_{q+1}^{ri}]) = [x_{q+1}^{ri+rc}] = \tau([x_q^i])$, so τ is well defined on $Q_A/(G/A)$.

For the commutativity of the diagram, note that the two squares on the left commute by definition, so we only have to deal with the square on the right. For $i \in \mathcal{D} \setminus \mathcal{F}$ we have $\tau(\Psi(i)) = \tau(i) = i + c = \Psi(i + c) = \Psi(\tau'(i))$, while for $i \in \mathcal{F}$ we have $\tau(\Psi(i^{(\ell)})) = \tau(i) = i + c = \Psi((i + c)^{(\ell+d_s)}) = \Psi(\tau'(i^{(\ell)}))$. Hence the result follows. $\square$

Theorem 2.5.19. *Let $\mathbf{p}$ be a path in Q_G which lies in* $\operatorname{supp}(\omega_G)$. *Then there exists a path $\tilde{\mathbf{p}} \in \operatorname{supp}(\omega_A)$ such that $\Psi(\mathbf{p}) = \Phi(\tilde{\mathbf{p}})$.*

Proof. We will first consider a particular case and then the general one.

Case 1. Suppose that $\mathbf{p}$ is in the form described in Lemma 2.5.10, so we retain all the notation from there. The fact that $\mathbf{p}$ is in $\operatorname{supp}(\omega_G)$ implies that there exist $k_1, \ldots, k_h$ such that the integers $q_1, \ldots, q_s$ are pairwise different modulo s. Otherwise, there would be two linearly dependent tensor factors for each element of the sum (2.5.6), so we would get zero after applying the antisymmetrizer. To simplify the notation, for the rest of the proof we will set $N_k := r^{q_1} + \cdots + r^{q_k}$ for each $k = 1, \ldots, s$ and $N_0 := 0$. We also set $f_0 := 0$ and $f_{h+1} := s + 1$. Note that what follows will make sense also for $h = 0$.

Define $\tilde{\mathbf{p}}$ to be the path

$$\tilde{\mathbf{p}}\colon i_1 \to i_1 - N_1 \to \ldots \to i_1 - N_s$$

in Q_A. Since we assumed the q_j's to be pairwise different, by Proposition 2.5.3 we deduce that $\tilde{\mathbf{p}} \in \operatorname{supp}(\omega_A)$. So it remains to show that $\Psi(\mathbf{p}) = \Phi(\tilde{\mathbf{p}})$.

Firstly, we claim that

$$i_1 - N_k \equiv r^{q_{k+1} - p_{k+1}} i_{k+1} \tag{2.5.8}$$

for all $k = 0, \ldots, s$. We will prove this by induction on k, the case $k = 0$ being clear. Suppose that $k \geq 1$, then there exists a $j \in \{1, \ldots, h+1\}$ such that $f_{j-1} \leq k \leq f_j - 1$. If $k \neq f_j - 1$, we have that $q_{k+1} - p_{k+1} = k_1 + \cdots + k_{j-1} + a_1 + \cdots + a_k$, so by induction hypothesis we obtain

$$\begin{aligned} r^{q_{k+1} - p_{k+1}} i_{k+1} &= r^{k_1 + \cdots + k_{j-1} + a_1 + \cdots + a_{k-1}} (r^{a_k} i_{k+1}) \equiv r^{q_k - p_k} (i_k - r^{p_k}) \\ &\equiv i_1 - N_{k-1} - r^{q_k} = i_1 - N_k. \end{aligned}$$

If $k = f_j - 1$, using the fact that i_{k+1} is a fixed point, we have

$$\begin{aligned} r^{q_{k+1} - p_{k+1}} i_{k+1} &\equiv r^{q_k - p_k} i_{k+1} \equiv r^{q_k - p_k} (i_k - r^{p_k}) \\ &\equiv i_1 - N_{k-1} - r^{q_k} = i_1 - N_k. \end{aligned}$$

Hence the claim follows.

Now if we apply Φ to $\tilde{\mathbf{p}}$ we obtain the path $\Phi(\tilde{\mathbf{p}})\colon i_1 \to \ldots \to i_{s+1}$, because $\underline{i_1 - N_k} = i_{k+1}$ by the equation (2.5.8). By definition of Φ, the arrows in this path are given by $x^{i_k}_{p'_k, a'_k} : \underline{i_1 - N_{k-1}} \to \underline{i_1 - N_k}$, where

$$(p'_k, a'_k) = \begin{cases} (q_k - \kappa_{i_1 - N_{k-1}}, \kappa_{i_1 - N_k} - \kappa_{i_1 - N_{k-1}}) & \text{if } \underline{i_1 - N_{k-1}},\ \underline{i_1 - N_k} \in \mathcal{D} \smallsetminus \mathcal{F}; \\ (q_k - \kappa_{i_1 - N_{k-1}}, 0) & \text{if } \underline{i_1 - N_{k-1}} \in \mathcal{D} \smallsetminus \mathcal{F},\ \underline{i_1 - N_k} \in \mathcal{F}; \\ (q_k - \kappa_{i_1 - N_k}, 0) & \text{if } \underline{i_1 - N_{k-1}} \in \mathcal{F},\ \underline{i_1 - N_k} \in \mathcal{D} \smallsetminus \mathcal{F}. \end{cases}$$

We may observe that in all cases we have $(p'_k, a'_k) = (p_k, a_k)$, because the (2.5.8) implies that, for all k,

$$\kappa_{i_1 - N_k} = \begin{cases} q_{k+1} - p_{k+1} & \text{if } i_1 - N_k \text{ is not a fixed point,} \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\Psi(\mathbf{p}) = \Phi(\tilde{\mathbf{p}})$.

Case 2. We now consider the general case. Let $\mathbf{p} \in \operatorname{supp}(\omega_G)$ and write $\mathbf{p} = \mathbf{p}_1 \cdots \mathbf{p}_s$. By Proposition 2.5.4 we can assume that no arrow $\mathbf{p}_i$ both starts and ends with a fixed point. Note that $s(\mathbf{p}) = \tau(t(\mathbf{p}))$, so $s(\mathbf{p})$ is a fixed point if and only if $t(\mathbf{p})$ is. If $s(\mathbf{p})$ is not a fixed point, then $\mathbf{p}$ is in the form discussed in Case 1. Otherwise, suppose that $s(\mathbf{p}) = s(\mathbf{p}_s)$ is a fixed point and consider the path $\mathbf{q} := \tau'(\mathbf{p}_s)\mathbf{p}_1 \cdots \mathbf{p}_{s-1}$. By Lemma 2.3.5, we have that $\mathbf{q}$ lies in $\operatorname{supp}(\omega_G)$; moreover neither $s(\mathbf{q}) = s(\mathbf{p}_{s-1}) = t(\mathbf{p}_s)$ nor $t(\mathbf{q}) = t(\tau'(\mathbf{p}_s)) = \tau(t(\mathbf{p}_s))$ are fixed points, because otherwise $\mathbf{p}_s$ would connect two fixed points. So $\mathbf{q}$ is as in case 1, and we can write $\Psi(\mathbf{q}) = \Phi(\tilde{\mathbf{q}})$ for some path $\tilde{\mathbf{q}} = \tilde{\mathbf{q}}_1 \cdots \tilde{\mathbf{q}}_s \in \operatorname{supp}(\omega_A)$. Now note that, by Proposition 2.5.18, we have $\Psi(\mathbf{q}) = \Psi(\tau'(\mathbf{p}_s))\Psi(\mathbf{p}_1)\cdots\Psi(\mathbf{p}_{s-1}) = \tau(\Psi(\mathbf{p}_s))\Psi(\mathbf{p}_1)\cdots\Psi(\mathbf{p}_{s-1})$, so $\tau(\Psi(\mathbf{p}_s)) = \Phi(\tilde{\mathbf{q}}_1)$, $\Psi(\mathbf{p}_i) = \Phi(\tilde{\mathbf{q}}_{i+1})$ for all $i = 1, \dots, s-1$ and in particular $\Psi(\mathbf{p}_s) = \tau^-(\Phi(\tilde{\mathbf{q}}_1)) = \Phi(\tau^-(\tilde{\mathbf{q}}_1))$. Now set $\tilde{\mathbf{p}} := \tilde{\mathbf{q}}_2 \cdots \tilde{\mathbf{q}}_s \tau^-(\tilde{\mathbf{q}}_1)$: then $\Psi(\mathbf{p}) = \Phi(\tilde{\mathbf{p}})$ and, by Lemma 2.3.5, $\tilde{\mathbf{p}} \in \operatorname{supp}(\omega_A)$, hence the result follows. □

The previous theorem tells us that $\Psi(\operatorname{supp}(\omega_G)) \subseteq \Phi(\operatorname{supp}(\omega_A))$, but we do not know if the equality holds. The following proposition shows that this happens at least when we restrict the support to paths containing at most one fixed point.

Theorem 2.5.20. *Let*

$$\mathbf{p}\colon i \to i - r^{q_1} \to \ldots \to i - r^{q_1} - \cdots - r^{q_s}$$

be a path in Q_A which passes through at most one fixed point and suppose that $\mathbf{p} \in \operatorname{supp}(\omega_A)$. Then there exists a path $\tilde{\mathbf{p}}$ in Q_G such that $\tilde{\mathbf{p}} \in \operatorname{supp}(\omega_G)$ and $\Phi(\mathbf{p}) = \Psi(\tilde{\mathbf{p}})$.

Proof. We consider separately the cases where $\mathbf{p}$ contains zero or one fixed point.

Case 1. Suppose that $\mathbf{p}$ has no fixed points and consider the path $\Phi(\mathbf{p})$ in $\tilde{Q}_G$. It is clear that we can lift it to a path

$$\tilde{\mathbf{p}}\colon \underline{i} \to \underline{i - r^{q_1}} \to \ldots \to \underline{i - r^{q_1} - \cdots - r^{q_s}}$$

in Q_G, since Ψ acts as the identity outside the fixed points. Thus $\Phi(\mathbf{p}) = \Psi(\tilde{\mathbf{p}})$.

We are left to show that $\tilde{\mathbf{p}} \in \operatorname{supp}(\omega_G)$. In the following we will use the notation $i_h := i - r^{q_1} - \cdots - r^{q_{h-1}}$. By definition of Φ, the arrows of $\tilde{\mathbf{p}}$ are given by $x^{i_h}_{p_h, a_h} \colon \underline{i_h} \to \underline{i_{h+1}}$ for each $h = 1, \dots, s$, where $p_h := q_h - \kappa_{i_h}$ and $a_h := \kappa_{i_{h+1}} - \kappa_{i_h}$. Hence we have that

$$\{p_1, a_1 + p_2, \dots, a_1 + \cdots + a_{s-1} + p_s\} = \{q_1 - \kappa_i, q_2 - \kappa_i, \dots, q_s - \kappa_i\}.$$

Note that $\mathbf{p} \in \operatorname{supp}(\omega_A)$ implies that $q_1, \dots, q_s$ are pairwise different modulo s, so the same is true for $p_1, a_1 + p_2, \dots, a_1 + \cdots + a_{s-1} + p_s$. Hence it follows from Proposition 2.5.13 that the path $\tilde{\mathbf{p}}$ lies in the support of ω_G.

Case 2. Now we consider the case where $\mathbf{p}$ has exactly one fixed point. We keep the notation $i_h := i - r^{q_1} - \cdots - r^{q_{h-1}}$ as in the previous case.

By an argument similar to the one in Case 2 of the proof of Theorem 2.5.19, we can assume that the only fixed point in $\mathbf{p}$ is i_s. It is clear that $\Phi(\mathbf{p})$ can be lifted to a path

$$\tilde{\mathbf{p}}\colon \underline{i_1} \to \underline{i_2} \to \ldots \to \underline{i_{s-1}} \to \underline{i_s}^{(\ell)} \to \underline{i_{s+1}}$$

in Q_G, for an integer $0 \le \ell \le s-1$. Thus $\Phi(\mathbf{p}) = \Psi(\tilde{\mathbf{p}})$ and the arrows in $\tilde{\mathbf{p}}$ are given by $x^{\underline{i_j}}_{p_j,a_j}\colon \underline{i_j} \to \underline{i_{j+1}}$ for $j = 1, \ldots, s-2$, $x^{\underline{i_{s-1}}(\ell)}_{p_{s-1},a_{s-1}}\colon \underline{i_{s-1}} \to \underline{i_s}^{(\ell)}$, $x^{(\ell)\underline{i_s}}_{p_s,a_s}\colon \underline{i_s}^{(\ell)} \to \underline{i_{s+1}}$, where

$$(p_j, a_j) = \begin{cases} (q_j - \kappa_{i_j}, \kappa_{i_{j+1}} - \kappa_{i_j}) & \text{if } 1 \le j \le s-2, \\ (q_j - \kappa_{i_j}, 0) & \text{if } j = s-1, \\ (q_j - \kappa_{i_{j+1}}, 0) & \text{if } j = s. \end{cases}$$

The path $\tilde{\mathbf{p}}$ induces a morphism $T_{\underline{i_1}} \to \bigwedge^s V \otimes T_{\underline{i_{s+1}}}$ which, by Lemma 2.5.10, sends $v_{\underline{i_1}}$ to

$$\frac{1}{s}\sum_{k=0}^{s-1} \varepsilon_s^{-t\underline{i_s}} \lambda^{s-k}_{\underline{i_s},\ell} \beta^{p_1} v_1 \wedge \beta^{p_1+a_2} v_1 \wedge \cdots \wedge \beta^{a_1+\cdots+a_{s-2}+p_{s-1}} v_1 \wedge \beta^{k+a_1+\cdots+a_{s-1}+p_s} v_1 \otimes \beta^{k+a_1+\cdots+a_s} v_{\underline{i_{s+1}}}.$$

For all $j = 0, \ldots, s-2$ we have $a_1 + \cdots + a_j + p_{j+1} = q_{j+1} - \kappa_{i_1}$, while $k + a_1 + \cdots + a_{s-1} + p_s = k - \kappa_{i_1} + \kappa_{i_{s-1}} + q_s - \kappa_{i_{s+1}}$ and $k + a_1 + \cdots + a_s = k - \kappa_{i_1} + \kappa_{i_{s-1}}$. So the previous sum becomes

$$\frac{1}{s}\sum_{k=0}^{s-1} \varepsilon_s^{-t\underline{i_s}} \lambda^{s-k}_{\underline{i_s},\ell} \beta^{q_1-\kappa_{i_1}} v_1 \wedge \beta^{q_2-\kappa_{i_1}} v_1 \wedge \cdots \wedge \beta^{q_{s-1}-\kappa_{i_1}} v_1 \wedge \beta^{k-\kappa_{i_1}+\kappa_{i_{s-1}}+q_s-\kappa_{i_{s+1}}} v_1 \otimes \beta^{k-\kappa_{i_1}+\kappa_{i_{s-1}}} v_{i_1}.$$

Note that since $\mathbf{p} \in \operatorname{supp}(\omega_A)$ we must have $i_{s+1} \equiv i_1 + c \pmod{m}$, and moreover $\underline{i_{s+1}} \equiv \underline{i_1} + c$ because $\mathcal{D}$ is closed under the twist, by Assumption 2.5.7: this implies that $\kappa_{i_{s+1}} = \kappa_{i_1}$. Hence in the above sum all terms are zero except the one where $k = \kappa_{i_1} - \kappa_{i_{s-1}}$, and thus we obtain

$$\frac{1}{s}\varepsilon_s^{-t\underline{i_s}} \lambda^{s-\kappa_{i_1}+\kappa_{i_{s-1}}}_{\underline{i_s},\ell} \beta^{q_1-\kappa_{i_1}} v_1 \wedge \beta^{q_2-\kappa_{i_1}} v_1 \wedge \cdots \wedge \beta^{q_{s-1}-\kappa_{i_1}} v_1 \wedge \beta^{q_s-\kappa_{i_1}} v_1 \otimes v_{i_1}.$$

This is clearly a non-zero element, because $q_1, \ldots, q_s$ are pairwise different modulo s, and so the coefficient in ω_G corresponding to the path $\tilde{\mathbf{p}}$ is non-zero. Hence the result follows. ☐

Corollary 2.5.21. *For $s = 2, 3$ we have $\Psi(\operatorname{supp}(\omega_G)) = \Phi(\operatorname{supp}(\omega_A))$.*

Proof. Clearly in these cases every path of length s can contain at most one fixed point, hence the result follows immediately from Theorem 2.5.20. ☐

Example 2.5.22. Let us retain the case of Example 2.5.12. By Corollary 2.5.21 we can describe explicitly the paths in $\operatorname{supp}(\omega_G)$. We know that $\operatorname{supp}(\omega_A)$ consists in all the cyclic permutations of paths of type $i \xrightarrow{x_0^i} i-1 \xrightarrow{x_1^{i-1}} i-5 \xrightarrow{x_2^{i-5}} i$ and $i \xrightarrow{x_0^i} i-1 \xrightarrow{x_2^{i-1}} i-17 \xrightarrow{x_1^{i-17}} i$.

Hence $\operatorname{supp}(\omega_G)$ is made of the paths which are induced by these ones via the procedure described in the proof of Theorem 2.5.20.

For example, given an $\ell \in \{0,1,2\}$, the path $12 \xrightarrow{x_0^{12}} 11 \xrightarrow{x_1^{11}} 7 \xrightarrow{x_2^7} 12$ in $\operatorname{supp}(\omega_A)$ induces a path

$$\mathbf{p}\colon 12 \xrightarrow{x_{0,1}^{12}} 8 \xrightarrow{x_{0,0}^{8(\ell)}} 7^{(\ell)} \xrightarrow{x_{2,0}^{(\ell)7}} 12$$

in $\operatorname{supp}(\omega_G)$ (note that this was already shown by a direct computation in Example 2.5.12). The reader should be careful that this is not the same path which is induced by $12 \xrightarrow{x_1^{12}} 8 \xrightarrow{x_0^8} 7 \xrightarrow{x_2^7} 12$, since the arrows $x_0^{12}\colon 12 \to 11$ and $x_1^{12}\colon 12 \to 8$ in Q_A yield two different arrows from 12 to 8 in Q_G.

2.5.4 Gradings of Q_G

We will now illustrate a way to obtain gradings on Q_G which make ω_G homogeneous.

Proposition 2.5.23. *Let d_A be a grading on Q_A such that ω_A is homogeneous of degree a and the morphism of quivers Φ is d_A-gradable. Then there exists a grading d_G on Q_G such that ω_G is homogeneous of degree a with respect to it.*

Proof. By Proposition 2.5.16 we have that Φ is surjective on arrows: this, together with the fact that Φ is d_A-gradable, implies that we can define the grading $\Phi_* d_A$ on $\tilde{Q}_G$ (see Definition 2.1.6). We now define a grading on Q_G by $d_G := \Psi^*\Phi_* d_A$. Note that with these definitions both Φ and Ψ become morphisms of graded quivers.

Now we must show that ω_G is homogeneous of degree a with respect to d_G. Let $\mathbf{p} \in \operatorname{supp}(\omega_G)$, then it is enough to prove that $d_G(\mathbf{p}) = a$. By Theorem 2.5.19 there exists $\tilde{\mathbf{p}} \in \operatorname{supp}(\omega_A)$ such that $\Phi(\tilde{\mathbf{p}}) = \Psi(\mathbf{p})$, hence, since $d_A(\omega_A) = a$, we must have that

$$d_G(\mathbf{p}) = (\Psi^*\Phi_* d_A)(\mathbf{p}) = (\Phi_* d_A)(\Psi(\mathbf{p})) = (\Phi_* d_A)(\Phi(\tilde{\mathbf{p}})) = d_A(\tilde{\mathbf{p}}) = a.$$

□

We will see in Section 2.6 how one can find in practice gradings which fit the setting of Proposition 2.5.23.

2.5.5 Metacyclic groups embedded in $\mathrm{SL}(s+1,\mathbb{C})$

Our final aim will be to obtain $(s-1)$-representation infinite algebras from the McKay quiver Q_G: however, by Corollary 2.3.6, this can be done when G is contained in $\mathrm{SL}(s,\mathbb{C})$. Nevertheless, if this condition is not satisfied, we can still use the results in §2.3.3 and get examples of s-representation infinite algebras by embedding G in $\mathrm{SL}(s+1,\mathbb{C})$.

Denote by G' and A' the images in $\mathrm{SL}(s+1,\mathbb{C})$ of, respectively, G and A under this embedding, and call $Q_{G'}$, $Q_{A'}$, $\omega_{G'}$, $\omega_{A'}$ the corresponding McKay quivers and superpotentials. We now want to show that an analogue of Proposition 2.5.23 holds in this setting.

Recall that by Proposition 2.3.8 Q_G (resp. Q_A) is a subquiver of $Q_{G'}$ (resp. $Q_{A'}$), and the latter is obtained from the former by adding all the arrows $i \to \tau(i)$. Now consider the automorphism τ of $\tilde{Q}_G$ defined in Proposition 2.5.18. We define $\tilde{Q}_{G'}$ as the quiver obtained from $\tilde{Q}_G$ by adding an arrow $i \to \tau(i)$ for each vertex $i \in (\tilde{Q}_G)_0$.

Since the morphisms Φ and Ψ are compatible with τ, we can naturally extend them to morphisms Φ' and Ψ', so that the following diagram commutes:

$$\begin{array}{ccccc} Q_{A'} & \xrightarrow{\Phi'} & \tilde{Q}_{G'} & \xleftarrow{\Psi'} & Q_{G'} \\ \uparrow & & \uparrow & & \uparrow \\ Q_A & \xrightarrow{\Phi} & \tilde{Q}_G & \xleftarrow{\Psi} & Q_G \end{array}$$

Note also that the G/A-action on Q_A can be extended to a G/A-action on $Q_{A'}$, and $\tilde{Q}_{G'}$ can be thought as the quotient of $Q_{A'}$ by this action.

Proposition 2.5.24. *Let d_A be a grading on $Q_{A'}$ such that $\omega_{A'}$ is homogeneous of degree a and the morphism Φ' is d_A-gradable. Then there exists a grading d_G on $Q_{G'}$ such that $\omega_{G'}$ is homogeneous of degree a.*

Proof. Note that $\Psi'(\mathrm{supp}(\omega_{G'})) \subseteq \Phi'(\mathrm{supp}(\omega_{A'}))$: indeed, every path in the support of $\omega_{G'}$ is, up to cyclic permutation, in the form $a_{\mathbf{p}} \cdot \mathbf{p}$ for a path $\mathbf{p} \in \mathrm{supp}(\omega_G)$, where $a_{\mathbf{p}}$ is the arrow $t(\mathbf{p}) \to \tau(t(\mathbf{p}))$. Hence it is enough to show that $a_{\mathbf{p}}$ is in the image of Φ', but this is true because both Φ and Ψ are compatible with the twists.

Now it is easy to check that the same proof of Proposition 2.5.23 carries over if we just replace Q_A, Q_G, ω_A, ω_G, Φ, Ψ respectively by $Q_{A'}$, $Q_{G'}$, $\omega_{A'}$, $\omega_{G'}$, Φ', Ψ'. □

2.6 Cuts

In this section we will illustrate a method to define explicitly some gradings on the McKay quivers we studied so far. For this purpose, we will first describe Q_A and $Q_{A'}$ using a construction of [31]. In order to encompass both the cases of A and A', we will first do this in a more general setting.

Fix an integer $N \geq 2$. Let $\{e_0, \ldots, e_{N-1}\}$ be the canonical basis of $\mathbf{R}^N$ and let

$$E = \{(x_0, \ldots, x_{N-1}) \in \mathbf{R}^N \mid \sum_{i=0}^{N-1} x_i = 0\}.$$

The set $\{e_i - e_j \in E \,|\, 0 \leq i \neq j \leq N-1\}$ is a root system of type A_{N-1} (see for example [33]). We take as simple roots $\alpha_i := e_i - e_{i-1}$ for $i = 1, \ldots, N-1$, and we set $\alpha_0 = \alpha_N := e_0 - e_{N-1} = -\sum_{i=1}^{N-1} \alpha_i$. Define the root lattice L as the lattice in E generated by the simple roots.

We define a quiver $Q = Q^{(N)}$ as follows. Its vertices are $Q_0 := L$; for each vertex $v \in Q_0$ and each $k = 0, \ldots, N-1$, we have an arrow $a_k^v \colon v \to v + \alpha_k$. Sometimes, when this does not cause any confusion, we will drop the superscript and write a_k in place of a_k^v.

Let $m, r_1, \ldots, r_N$ be positive integers such that $(r_i, m) = 1$ for all i. Let H be the subgroup of $\mathrm{SL}(N, \mathbb{C})$ generated by the matrix

$$\begin{pmatrix} \varepsilon_m^{r_1} & 0 & \cdots & 0 \\ 0 & \varepsilon_m^{r_2} & \cdot & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon_m^{r_N} \end{pmatrix}.$$

It is easy to see that H is cyclic of order m. Let $\eta\colon L \to \mathbb{Z}/m\mathbb{Z}$ be the homomorphism of abelian groups defined by $\eta(\alpha_j) = -r_j$ for all $j = 1, \ldots, N$. It is clearly surjective because each r_j is invertible modulo m: hence it induces an isomorphism (which we call again η)

$$\eta\colon L/B \xrightarrow{\sim} \mathbb{Z}/m\mathbb{Z},$$

where $B := \ker(\eta)$.

The subgroup $B \leq L$ acts on L by translations and this extends naturally to an action on the quiver Q, so we can form the orbit quiver Q/B. We will denote by $\overline{a}_k^{\overline{v}}\colon \overline{v} \to \overline{v} + \overline{\alpha}_k$ the arrow in Q/B corresponding to the orbit of a_k^v.

Now consider the McKay quiver Q_H. It has vertices $(Q_H)_0 := \mathbb{Z}/m\mathbb{Z}$ and an arrow $x_k^i\colon i \to i - r_k$ for all $i \in (Q_H)_0$, $k = 1, \ldots, N$. This follows from [11, Corollary 4.1]; alternatively, it can be proved in the same way of Proposition 2.5.3.

Proposition 2.6.1. *The map η extends to an isomorphism of quivers $\eta\colon Q/B \to Q_H$, which is given on arrows by $\eta(\overline{a}_k^{\overline{v}}) = x_k^{\eta(\overline{v})}$. Moreover the McKay relations in Q_H described in Proposition 2.5.3 correspond to the relations*

$$\{\overline{a}_h^{\overline{v}+\overline{\alpha}_k}\overline{a}_k^{\overline{v}} = \overline{a}_k^{\overline{v}+\overline{\alpha}_h}\overline{a}_h^{\overline{v}} \mid \overline{v} \in L/B,\ 0 \leq k, h \leq N-1\}$$

in Q/B.

Proof. This follows immediately from Proposition 2.5.3. □

Definition 2.6.2. A subset C of the arrows of Q is called a *cut* if every path of the form

$$a_{\sigma(0)} \cdots a_{\sigma(N-1)}\colon v \to v,$$

for a permutation $\sigma \in \mathfrak{S}_N$, contains exactly one arrow of C.

Given a cut C, we define a grading d_C on Q by setting

$$d_C(a) = \begin{cases} 1 & \text{if } a \in C, \\ 0 & \text{otherwise.} \end{cases}$$

We have the following result (cf. [31, Theorem 5.6]).

Proposition 2.6.3. *Let C be a cut on Q which is invariant under the action of B. Then d_C induces a grading on Q_H such that the skew group algebra of H becomes N-bimodule Calabi-Yau of Gorenstein parameter 1.*

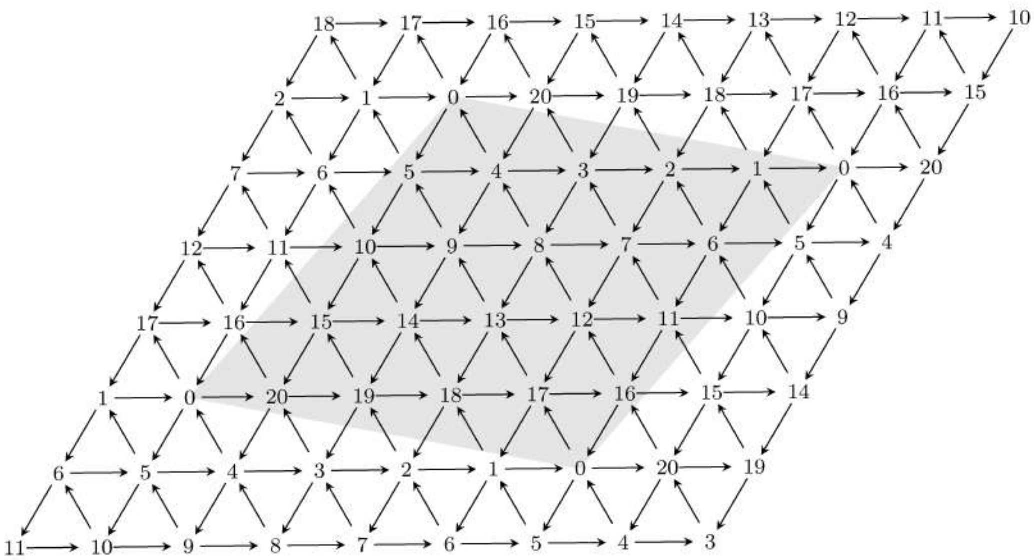


Figure 2.2. A part of the infinite quiver $Q^{(3)}$. Each vertex is labelled with its image under η, where we set $m = 21$, $r_1 = 1$, $r_2 = 4$, $r_3 = 16$. The McKay quiver Q_H is obtained by taking the vertices in the shadowed parallelogram and identifying the upper side with the lower one and the left side with the right one (note that in this way Q_H can be naturally embedded in a real 2-dimensional torus).

Proof. The projection morphism $Q \to Q/B$ is clearly surjective on arrows, and it is d_C-gradable because C is B-invariant. Hence it induces a grading on Q/B, and in turn one on Q_H via η: we denote the latter by d_C^H. It is clear, by Propositions 2.5.3 and 2.6.1, that the fact that C is a cut implies that the superpotential ω_H of Q_H is homogeneous of degree 1. Hence the statement follows by Corollary 2.3.6. □

We will now apply this construction to the setting of Section 2.5, in order to get an analogue of Proposition 2.6.3 for metacyclic groups.

Let G be the metacyclic group associated to some integers m, r, s, t. From now on we will assume that all the conditions (M1),. . . ,(M7) hold.

In the following we will treat separately the cases where $G \subseteq \mathrm{SL}(s, \mathbb{C})$ and $G \not\subseteq \mathrm{SL}(s, \mathbb{C})$, and we will refer to them respectively as (SL) and (GL). Depending on the case, the objects we introduced at the beginning of this section will assume the following values:

(SL) $H = A$, $N = s$, $r_i = r^{i-1}$ for all $i = 1, \dots, s$;

(GL) $H = A'$ (according to the notation of Section 2.5.5), $N = s + 1$, $r_i = r^{i-1}$ for all $i = 1, \dots, s$, $r_{s+1} = -c$.

Before going on we first give an example of a cut in Q which will be important in the following in both the above cases.

2.6.1 An example of cut

We keep the notation introduced previously, but from now on we will consider it only in the cases (SL) and (GL).

Let l be a positive integer. For $k \in \{1, \dots, l\}$, define $\gamma = \gamma_k \colon L \to \mathbb{Z}/sl\mathbb{Z}$ as the group homomorphism given by $\gamma(\alpha_i) = k$ for all $i = 1, \dots, N-1$. Note that in this way we have $\gamma(\alpha_0) = -(N-1)k$.

If $x \in \mathbb{Z}/sl\mathbb{Z}$, we denote by $\overline{x}$ the unique representative of x in $\{0, \dots, sl-1\}$.

Definition 2.6.4. Given $k \in \{1, \dots, l\}$, we define the following subset of Q_1:

$$C_k^{(l)} = C_k := \left\{ a_i \colon v \to v + \alpha_i \,|\, \overline{\gamma(v)} \geq \overline{\gamma(v+\alpha_i)},\ 0 \leq i \leq N-1 \right\}.$$

Proposition 2.6.5. *Let $l \geq 1$, $1 \leq k \leq l$.*

(a) In case (SL), C_k is a cut in Q for all k. In case (GL), C_k is a cut in Q for all $k < l$.

(b) Every path in Q of length greater or equal than sl contains at least an arrow of C_k.

(c) Suppose that l divides both the integers n and b defined in conditions (M6) and (M7). Then, for all $k = 1, \dots, l$, C_k is invariant under the action of B.

Proof. (a) Suppose we have a path $a_{i_0} \cdots a_{i_{N-1}} \colon v \to v$ in Q with $\{i_0, \dots, i_{N-1}\} = \{0, \dots, N-1\}$: we have to show that there exist exactly one j such that $a_{i_j} \in C_k$. Up to a cyclic permutation, we can assume that $i_0 = 0$. If $N = s$, we have $(N-1)k = sk - k < sl$; if $N = s+1$ then by hypothesis $k < l$, and so $(N-1)k = sk < sl$. In both cases $(N-1)k < sl$, so, since $\gamma(v+\alpha_0) = \gamma(v) - (N-1)k$, we have that $a_{i_0} \in C_k$ if and only if $\overline{\gamma(v)} - (N-1)k \geq 0$. Now put $v_j := v + \sum_{h=0}^{j} \alpha_{i_h}$ and note that, for $j = 1, \dots, N-1$, we have $a_{i_j} \colon v_{j-1} \to v_j$ and $\gamma(v_j) = \gamma(v_{j-1}) + k$. Hence $a_{i_j} \notin C_k$ if and only if $\overline{\gamma(v_j)} = \overline{\gamma(v_{j-1})} + k$, and $a_{i_j} \in C_k$ if and only if $\overline{\gamma(v_{j-1})} + k \geq sl$.

Now suppose that $a_{i_0} \in C_k$. By the above discussion we have that $\overline{\gamma(v_0)} = \overline{\gamma(v)} - (N-1)k \geq 0$, so, if $1 \leq j \leq N-1$, then $0 < \overline{\gamma(v_0)} + jk = \overline{\gamma(v)} - (N-1-j)k \leq \overline{\gamma(v)} < sl$ and thus $\overline{\gamma(v_j)} = \overline{\gamma(v_0) + jk} = \overline{\overline{\gamma(v_0)} + jk} = \overline{\gamma(v_0)} + jk$. This means that, for all $j = 1, \dots, N-1$, $\overline{\gamma(v_j)} = \overline{\gamma(v_{j-1})} + k$ and hence $a_{i_j} \notin C_k$.

Assume now that $a_{i_0} \notin C_k$, so that we have $\overline{\gamma(v)} - (N-1)k < 0$. Suppose by absurd that no a_{i_j} is in C_k. Then we must have $\overline{\gamma(v_{N-1})} = \overline{\gamma(v_0)} + (N-1)k$, but $v_{N-1} = v$ and so $\overline{\gamma(v_0)} + (N-1)k = \overline{\gamma(v_{N-1})} = \overline{\gamma(v)} < (N-1)k$, which is a contradiction because $\overline{\gamma(v_0)} \geq 0$. Hence there exist a $j \geq 1$ such that a_{i_j} is in C_k: by an argument similar to above, it can be easily proved that such a j must be unique.

(b) Consider a path $p = a_{i_1} \cdots a_{i_h}$ in Q. If no arrow in p is contained in C_k, then we must have a chain of inequalities $\overline{\gamma(v)} < \overline{\gamma(v+\alpha_{i_1})} < \cdots < \overline{\gamma(v + \sum_{j=1}^{h} \alpha_{i_j})}$ in $\{0, \dots, sl-1\}$, and this is clearly impossible if $h \geq sl$.

(c) Suppose that $v = \sum_{j=1}^{N-1} \mu_j \alpha_j$ is in B: this means that $\eta(v) = -\sum_{j=1}^{N-1} \mu_j r^{j-1} \equiv 0 \pmod{m}$. Since $l|n$, we have $sl|sn = m$ and so we get $\sum_{j=1}^{N-1} \mu_j r^{j-1} \equiv 0 \pmod{sl}$. Moreover we have that $sl|sb = r-1$, so $r \equiv 1 \pmod{sl}$ and thus $\sum_{j=1}^{N-1} \mu_j r^{j-1} \equiv \sum_{j=1}^{N-1} \mu_j \equiv 0 \pmod{sl}$. Hence we get $\gamma(v) = k \sum_{j=1}^{N-1} \mu_j = 0$, which clearly implies that C_k is B-invariant. □

2.6.2 The case $G \subseteq \mathrm{SL}(s, \mathbb{C})$

Recall from Section 2.5.2 that the determinants of the generators of G are $\det(\alpha) = \varepsilon_m^c$, where $c = \sum_{i=0}^{s-1} r^i$, and $\det(\beta) = (-1)^{s-1}\varepsilon_m^t$. Hence, if $G \subseteq \mathrm{SL}(s, \mathbb{C})$, we have that

$$c \equiv 0 \pmod{m}, \qquad t = \begin{cases} n & \text{if } s = 2, \\ 0 & \text{if } s > 2. \end{cases}$$

Clearly we have that A is contained in $\mathrm{SL}(s, \mathbb{C})$, too. The vertices of the quiver $Q = Q^{(s)}$ are the points of a root lattice of type A_{s-1}, with basis given by $\alpha_1, \ldots, \alpha_{s-1}$. We have an isomorphism of quivers $\eta\colon Q/B \to Q_A$, where B is the kernel of the group homomorphism $\eta\colon L \to \mathbb{Z}/m\mathbb{Z}$ given $\eta(\alpha_j) = -r^{j-1}$ for all $j = 1, \ldots, s-1$. Note that, since $c \equiv 0 \pmod{m}$, we have $\eta(\alpha_0) = \eta(\alpha_s) = -r^{s-1}$.

Recall that the action of G/A on A induces an automorphism φ of Q_A, which is given on vertices by $\varphi(i) = ri$. Identifying Q/B with Q_A via η, this induces an automorphism of Q/B given by $\varphi(\overline{\alpha}_j) = \overline{\alpha}_{j+1}$, $0 \le j \le s-1$. Moreover it is easy to check that this automorphism lifts to an action on Q defined by $\varphi(\alpha_j) = \alpha_{j+1}$.

Theorem 2.6.6. *Let C be a cut in Q which is invariant under both the actions of B and G/A, and let d_C be the grading on Q induced by C. Then there exists a grading on Q_G such that Π_G is s-Calabi-Yau of Gorenstein parameter 1 with the grading induced by it.*

In particular, if the degree 0 part $(\Pi_G)_0$ of Π_G is finite dimensional, then $(\Pi_G)_0$ is $(s-1)$-representation infinite.

Proof. Since C is B-invariant, by Proposition 2.6.3 it induces a grading d_C^A on Q_A such that the superpotential ω_A becomes homogeneous of degree 1. Moreover, the fact that C is G/A invariant implies that the morphism $\Phi\colon Q_A \to \tilde{Q}_G$ is d_C^A-gradable: hence, by Proposition 2.5.23, there exists a grading d_C^G on Q_G such that ω_G is homogeneous of degree 1. So, applying Corollary 2.3.6, the result follows. □

Corollary 2.6.7. *Let l be a positive integer which divides both n and b and let $k \in \{1, \ldots, l\}$. Then the cut $C_k^{(l)}$ of Definition 2.6.4 induces a grading on Π_G such that $(\Pi_G)_0$ is $(s-1)$-representation infinite.*

Proof. By Proposition 2.6.5(c) we have that $C_k^{(l)}$ is B-invariant. It is also G/A-invariant, since the action of G/A permutes the set $\{\alpha_0, \ldots, \alpha_{s-1}\}$. So the result follows from Theorem 2.6.6 if we prove that $(\Pi_G)_0$ is finite dimensional. To achieve this, is enough to show that there exists an integer M such that every path in Q_G of length greater or equal than M

has degree 1. Clearly it is enough to prove this for $\tilde{Q}_G$. Let p be a path in $\tilde{Q}_G$ of length h. Then it is easy to see that p lifts to a path of length h in Q. Hence, by Proposition 2.6.5(b), it is enough to take $M = sl$. □

2.6.3 The case $G \not\subseteq \mathrm{SL}(s, \mathbb{C})$

We retain all the notation of §2.5.5. So we embed G and A in $\mathrm{SL}(s+1, \mathbb{C})$ and we denote their images by G' and A' respectively. In this case, the vertices of the quiver $Q = Q^{(s+1)}$ are the points of a root lattice of type A_s, with basis $\alpha_1, \ldots, \alpha_s$.

We have an isomorphism $\eta\colon Q/B \to Q_{A'}$, where B is the kernel of the group homomorphism $\eta\colon L \to \mathbb{Z}/m\mathbb{Z}$ given by $\eta(\alpha_j) = -r^{j-1}$ for $j = 1, \ldots, s$. Note that in this case the element $\alpha_0 = \alpha_{s+1}$ is sent to c.

The action of G/A on Q_A, which is given on vertices by $\varphi(i) = ri$, extends naturally to $Q_{A'}$. Identifying the latter with Q/B via η, this induces an automorphism of Q/B given by $\varphi(\overline{\alpha}_j) = \overline{\alpha}_{j+1}$ for $j = 1, \ldots, s-1$, $\varphi(\overline{\alpha}_s) = \overline{\alpha}_1$. Moreover, it is easy to check that this automorphism lifts to an action on Q defined by $\varphi(\alpha_j) = \alpha_{j+1}$, $j = 1, \ldots, s-1$, $\varphi(\alpha_s) = \alpha_1$. It is also worth to point out that α_0 is now fixed by this action.

Theorem 2.6.8. *Let C be a cut in Q which is invariant under both the actions of B and G/A, and let d_C be the grading on Q induced by C. Then there exists a grading on $Q_{G'}$ such that the induced grading on $\Pi_{G'}$ makes it $(s+1)$-Calabi-Yau of Gorenstein parameter 1.*

In particular, if the degree 0 part $(\Pi_{G'})_0$ of $\Pi_{G'}$ is finite dimensional, then $(\Pi_{G'})_0$ is s-representation infinite.

Proof. Since C is B-invariant, by Proposition 2.6.3 it induces a grading d_C^A on $Q_{A'}$ such that the superpotential $\omega_{A'}$ becomes homogeneous of degree 1. Moreover, the fact that C is G/A invariant implies that the morphism $\Phi'\colon Q_{A'} \to \tilde{Q}_{G'}$ is d_C^A-gradable. Hence we can apply Proposition 2.5.24 to get a grading d_C^G on $Q_{G'}$ such that $\omega_{G'}$ is homogeneous of degree 1. So the result follows from Corollary 2.3.6. □

Corollary 2.6.9. *Let l be a positive integer which divides both n and b and let $1 \leq k \leq l-1$. Then the cut $C_k^{(l)}$ of Definition 2.6.4 induces a grading on $\Pi_{G'}$ such that $(\Pi_{G'})_0$ is s-representation infinite.*

Proof. Note that $C_k^{(l)}$ is G/A-invariant, since the action of G/A permutes the set $\{\alpha_0, \ldots, \alpha_s\}$. The rest of the proof is analogue to the one of Corollary 2.6.7. □

We end this section with the following easy observation, which gives a method of obtaining new gradings in both the cases (SL) and (GL).

Remark 2.6.10. If a cut in Q contains an arrow which starts or ends with a fixed point j, then all the corresponding splitting arrows in Q_G (respectively $Q_{G'}$) will have degree 1 with respect to the grading defined in Theorem 2.6.6 (respectively Theorem 2.6.8). In this case all the paths in the superpotential ω_G (respectively $\omega_{G'}$) passing through $j^{(\ell)}$ will

Figure 2.3. On the left we have an example of a grading obtained as in Theorem 2.6.6 in the local neighbourhood of a fixed point. Here the thick arrows have degree 1, while the others have degree 0. Applying the procedure described in Remark 2.6.10 for $\ell = 1$ we obtain the grading illustrated on the right.

contain a subpath of the form $i \xrightarrow{a} j^{(\ell)} \xrightarrow{b} k$, where one arrow among a and b has degree 1 and the other has degree 0. Hence it is easy to see that if we define a new grading in the quiver by swapping the degrees of a and b (see Figure 2.3), then the degree of the superpotential remains unchanged and so the algebra Π_G (respectively $\Pi_{G'}$) has again Gorenstein parameter 1. Moreover, if the degree 0 part of Π_G with the old grading is finite dimensional, then the same is true if we consider the new grading.

2.7 Examples

We have seen so far that we can obtain examples of higher representation infinite algebras from skew group algebras of some finite groups which satisfy certain conditions, but we still don't know how rich the class of such groups is. The aim of this section is to show that we have indeed many examples of them.

We start by defining two families of metacyclic groups.

Definition 2.7.1. Let s be a prime number.

(a) For each $b \geq 1$ we define $M(s,b)$ to be the metacyclic group in $\mathrm{GL}(s,\mathbb{C})$ associated to integers m, r, s, t, where we set

$$r := sb + 1, \quad m := \sum_{j=0}^{s-1} r^j, \quad t := \begin{cases} m/s & \text{if } s = 2, \\ 0 & \text{if } s > 2, \end{cases}$$

(b) For each $b \geq 2$ we define $\hat{M}(s,b)$ to be the metacyclic group in $\mathrm{GL}(s,\mathbb{C})$ associated to integers m, r, s, t, where we set

$$r := sb + 1, \quad m := b\sum_{j=0}^{s-1} r^j, \quad t := \begin{cases} m/s & \text{if } s = 2, \\ 0 & \text{if } s > 2, \end{cases}$$

Proposition 2.7.2. *The groups we defined in Definition 2.7.1 satisfy all the conditions (M1),...,(M7). Moreover, we have that* $M(s,b) \subseteq \mathrm{SL}(s,\mathbb{C})$ *and* $\hat{M}(s,b) \not\subseteq \mathrm{SL}(s,\mathbb{C})$.

Proof. (a) We consider first the case of $M(s,b)$, where s is prime and $b \geq 1$. Condition (M7) is clear by definition of r. It is easy to see that $r^s - 1 = (r-1)m = sbm$, so (M2) holds. For (M1), we note that $m - 1 = \sum_{j=1}^{s-1} r^j$ is a multiple of r, so clearly $(m,r) = 1$. By (M7) we have $m = \sum_{j=0}^{s-1} r^j \equiv \sum_{j=0}^{s-1} 1 \equiv 0 \pmod s$, so we have (M6). Now we can write $m = sn$, so $t = \begin{cases} n & \text{if } s = 2, \\ 0 & \text{if } s > 2, \end{cases}$. Hence (M3) is clear if $s > 2$; for $s = 2$ note that we have $r = 2b+1$, $m = 1 + r = 2(b+1)$, $n = b+1$, so $(r-1)t = 2bn = 2b(b+1) \equiv 0 \pmod m$. Condition (M5) is clear because $1 < r < m$. Finally, since $c = \sum_{j=0}^{s-1} r^j = m$, by the discussion at the beginning of Subsection 2.6.2 we have that $M(b,s) \subseteq \mathrm{SL}(s,\mathbb{C})$.

(b) Now we consider $\hat{M}(s,b)$, so in this case s is prime and $b \geq 2$. Again, (M7) is clear. It is easy to see that $r^s - 1 = sm$, so (M2) holds. Now note that $m - b = b\sum_{j=1}^{s-1} r^j$ is a multiple of r, so $(m,r) = (b,r) = (b, sb+1) = 1$ and (M1) holds. Condition (M6) is clear, so we can write $m = sn$. Hence we have again $t = \begin{cases} n & \text{if } s = 2, \\ 0 & \text{if } s > 2, \end{cases}$, so (M3) is clear except for $s = 2$: in this case $m = 2b(b+1)$ and $n = b(b+1)$, thus $(r-1)t = 2bn = 2b^2(b+1) \equiv 0 \pmod m$. Condition (M5) holds because $1 < r < m$. Finally, since $b \geq 2$ it is clear that $1 < c < m$: this implies that $\det(\alpha) = \varepsilon_m^c \neq 1$ and thus $\hat{M}(s,b) \not\subseteq \mathrm{SL}(s,\mathbb{C})$. □

In case (GL), in order to apply the results of the previous section, we need G to satisfy Assumption 2.5.7. This will not be always the case: in the following we will exhibit a sufficient condition for this to happen, and we will show that the groups of the form $\hat{M}(s,b)$ satisfy it. We will also see in Example 2.7.9 that this condition is not necessary.

Proposition 2.7.3. *Let* r, m, s, t *be integers which satisfy conditions (M1),...,(M7), and let* G *be the associated metacyclic group. Consider the automorphism* τ *of* $\mathbb{Z}/m\mathbb{Z}$ *given by the sum by* c*, and call* u *its order. Suppose that* $(u,s) = 1$*, then there exists a complete set* $\mathcal{D}$ *of representatives for the action of* G/A *on* $\mathbb{Z}/m\mathbb{Z}$ *which is closed under* τ.

Remark 2.7.4. The integer u we just defined is the smallest positive integer which is a solution to the equation $cx \equiv 0 \pmod m$, and hence it is equal to $\frac{m}{(c,m)}$.

Proof of Proposition 2.7.3. Clearly any set $\mathcal{D}$ of representatives contains the set $\mathcal{F}$ of fixed points. Moreover, $\mathcal{F}$ is closed under τ, so we only have to show a suitable way to choose the elements in $\mathcal{D} \smallsetminus \mathcal{F}$.

We have already seen in Proposition 2.5.18 that τ induces an action on the G/A-orbits of $\mathbb{Z}/m\mathbb{Z}$ given by $\tau([i]) = [\tau(i)] = [i+c]$, where $[i]$ denotes the G/A-orbit of i. So take an $i_1 \in \mathbb{Z}/m\mathbb{Z}$ and call k_1 the smallest positive integer such that $\tau^{k_1}([i_1]) = [i_1]$. Let $\mathcal{D}_1 := \{i_1, \tau(i_1), \ldots, \tau^{k_1-1}(i_1)\}$: the elements of this set clearly provide representatives of different G/A-orbits. Now we want to show that $\mathcal{D}_1$ is invariant under τ, which is equivalent to say that $i_1 + k_1 c \equiv i_1 \pmod m$.

The fact that $[i_1 + k_1 c] = \tau^{k_1}([i_1]) = [i_1]$ implies that there exists an $h \in \{0, \ldots, s-1\}$ such that $i_1 + k_1 c \equiv r^h i_1 \pmod m$. Now an easy induction shows that $r^{lh} i_1 \equiv i_1 + lk_1 c$

(mod m) for all $l \geq 1$. In particular we have that $r^{uh} i_1 \equiv i_1 + uk_1 c \pmod{m}$, and the latter is equivalent to i_1 modulo m because $cu \equiv 0 \pmod{m}$. Since i_1 is not in $\mathcal{F}$, we must have that $uh \equiv 0 \pmod{s}$, but this implies that $h \equiv 0 \pmod{s}$ because we have assumed that $(u, s) = 1$. Hence $h = 0$ and so $i_1 + k_1 c \equiv i_1 \pmod{m}$, which shows that $\mathcal{D}_1$ is invariant under τ.

Now we can choose an element $i_2 \in \mathbb{Z}/m\mathbb{Z}$ which does not belong to any of the orbits of the elements in $\mathcal{D}_1$, and applying the same argument as above we can construct a set $\mathcal{D}_2 = \{i_2, \tau(i_2), \ldots, \tau^{k_2-1}(i_2)\}$ which is invariant under τ and such that the orbits of elements in $\mathcal{D}_1$ are disjoint from the ones of the elements in $\mathcal{D}_2$. Repeating this procedure until we can, we obtain a sequence of sets $\mathcal{D}_1, \ldots, \mathcal{D}_N$, and $\mathcal{D} := \bigcup_{j=1}^{N} \mathcal{D}_j \cup \mathcal{F}$ provides a complete set of representatives for the G/A-action which is invariant under τ. □

We now show that we can always obtain a higher representation infinite algebra from the examples we discussed above.

Corollary 2.7.5. *Let s be a prime number.*

(a) For each integer $b \geq 1$, there exists an $(s-1)$-representation infinite algebra which is the degree 0 part of Π_G, where $G = M(s, b)$.

(b) For each integer $b \geq 2$ such that $(b, s) = 1$, there exists an s-representation infinite algebra which is the degree 0 part of $\Pi_{G'}$, where $G = \hat{M}(s, b)$ and G' is its embedding in $\mathrm{SL}(s+1, \mathbb{C})$.

Proof. (a) Take a positive integer l which divides both n and b and an integer $1 \leq k \leq l$ (for example, we could choose $l = k = 1$). Then the result follows by applying Corollary 2.6.7.

(b) Recall that in this case we have $m = bc$, so $u = b$. Hence we have $(u, s) = 1$, so by Proposition 2.7.3 there exists a set of representatives $\mathcal{D}$ which is invariant under τ'. Now take an integer $l \geq 2$ which divides both n and b and an integer $1 \leq k < l$. Note that $m = bc$ and c is a multiple of s, so $n = b\frac{c}{s}$: hence we can choose, for example, $l = b \geq 2$ and $k = 1$, and so it is always possible to find l, k which satisfy the above properties. Then the result follows by applying Corollary 2.6.9. □

In the following we will give some examples for $s = 2, 3$.

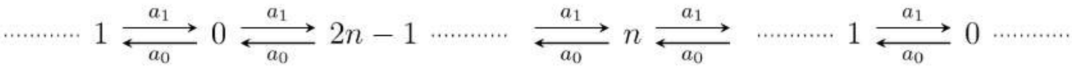
2.7.1 Examples for $G \subseteq \mathrm{SL}(s, \mathbb{C})$

Example 2.7.6. Let $s = 2$. We now want to describe all metacyclic groups in $\mathrm{SL}(2, \mathbb{C})$ which satisfy conditions (M1),...,(M7).

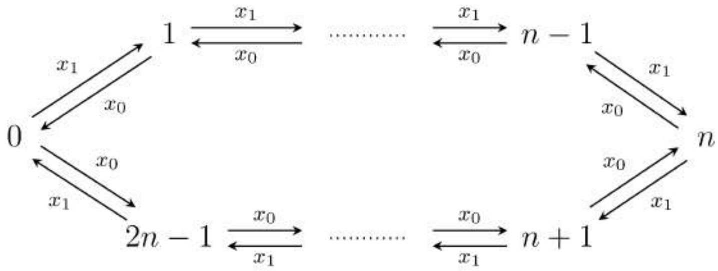
By (M6) we can take $m = 2n$ for an integer $n \geq 2$. Since we want our group to be in $\mathrm{SL}(2, \mathbb{C})$, it is clear that $r = m - 1$ is the unique (modulo m) possible choice of r. Also, as we already observed previously, we must take $t = n$. Hence the corresponding metacyclic group G is generated by the matrices

$$\alpha = \begin{pmatrix} \varepsilon_{2n} & 0 \\ 0 & \varepsilon_{2n}^{-1} \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The quiver Q is the preprojective quiver of type $\tilde{\mathrm{A}}^{\infty}_{\infty}$ and can be drawn on a line:

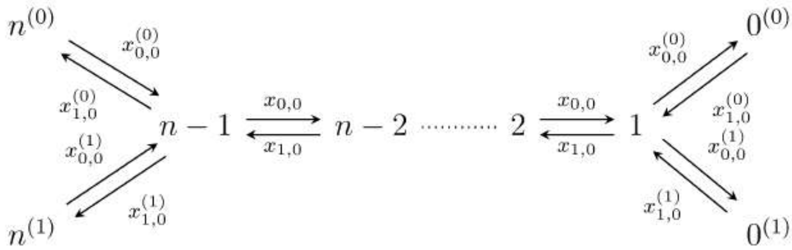


Here we have labelled every vertex by its image in $\mathbb{Z}/m\mathbb{Z}$ under the morphism η described in Section 2.6. Then Q_A, which we recall being isomorphic to the orbit quiver Q/A, is obtained by taking the subquiver of the above quiver given by some $m+1$ consecutive vertices $0, 1, \ldots, 2n-1, 0$ and identifying the two vertices labelled by 0. Note that in this way we obtain the preprojective quiver of type $\tilde{\mathrm{A}}_{2n-1}$:



By Proposition 2.5.3, its path algebra modulo the preprojective relations is isomorphic to the skew group algebra $\mathbb{C}[V] * A$.

Now consider the action of $G/A \cong \mathbb{Z}/2\mathbb{Z}$, which sends i to $-i$. On Q it is given by rotating the quiver of 180° around the vertex 0, while on Q_A it is given by reflecting with respect to the horizontal line passing through 0 and n. It is easy to see that $\mathcal{D} = \{0, \ldots, n\}$ is a complete set of representatives in $(Q_A)_0$ of this action, and that the only fixed points are 0 and n. Hence the quiver Q_G is given by



which is the preprojective quiver of type $\tilde{\mathrm{D}}_{n+2}$.

Using Corollary 2.5.21 and the description of the superpotential of Q_A, we can see that the support of ω_G is given by all paths of the form $x_{0,0}x_{1,0}$, $x_{1,0}x_{0,0}$, $x^{(\ell)}_{0,0}x^{(\ell)}_{1,0}$, $x^{(\ell)}_{1,0}x^{(\ell)}_{0,0}$, $\ell = 0, 1$. Moreover, any cut in Q which is invariant under the actions of B and G/A induces an orientation of the graph underlying Q_G. So if we consider the corresponding induced grading on Π_G and take the degree 0 part, we obtain an hereditary representation infinite algebra of type $\tilde{\mathrm{D}}_{n+2}$. Using Remark 2.6.10 it is easy to see that all these algebras are obtainable in this way.

We may note that the relations we get from the superpotential in our case are different from the classical preprojective relations. However, we still have that Π_G is isomorphic to a preprojective algebra of type $\tilde{\mathrm{D}}_{n+2}$, because we know, by Theorem 2.2.1, that it is isomorphic to the preprojective algebra of $(\Pi_G)_0$, and we saw that $(\Pi_G)_0$ is the path algebra of a quiver of type $\tilde{\mathrm{D}}_{n+2}$.

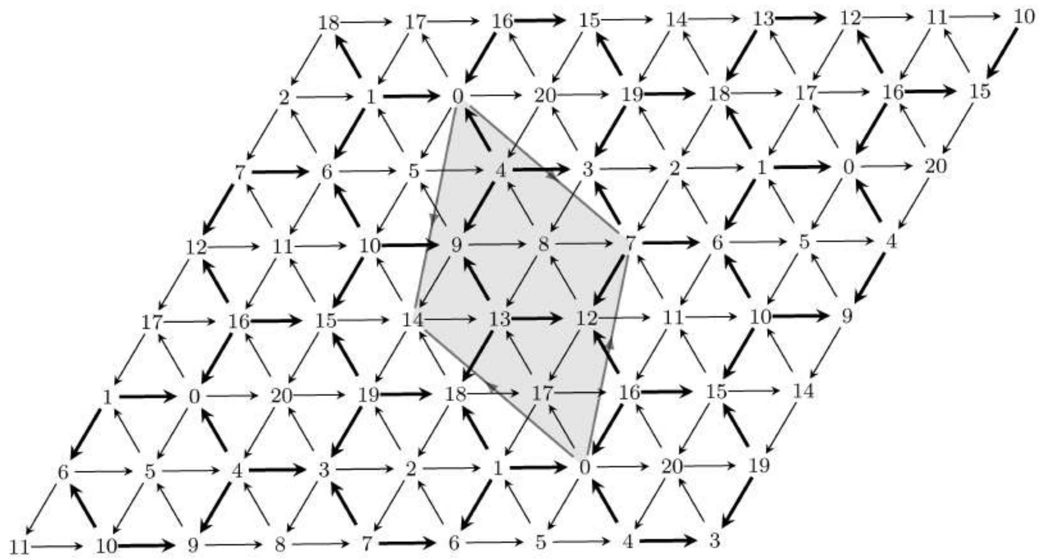


Figure 2.4. The cut $C_1^{(1)}$ for $m = 21$, $r = 4$, $s = 3$ (the arrows in the cut are represented by thick lines). A complete set of representatives $\mathcal{D}$ for the G/A-action is given by the vertices contained in the shadowed rhombus (the two 0's are identified), and the fixed points (i.e., 0, 7 and 14) are the vertices of the latter. The quiver $\tilde{Q}_G$ is obtained by identifying the edges of the rhombus whose adjacent vertices have the same name, according to the orientation depicted. For a picture of Q_G see Figure 2.5.

Example 2.7.7. Now we exhibit an example where $s = 3$. Let $G = M(3, 1)$, so we have $m = 21$, $r = 4$, $t = 0$. Then G is generated by the matrices

$$\alpha = \begin{pmatrix} \varepsilon_{21} & 0 & 0 \\ 0 & \varepsilon_{21}^4 & 0 \\ 0 & 0 & \varepsilon_{21}^{16} \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We have already depicted the quivers Q and Q_A in Figure 2.2. In this case, the G/A-action on Q is given by an anticlockwise rotation of 120° around the origin 0, and it is easy to see that the induced action on Q_A can be realized as an anticlockwise rotation of 120° around each fixed point.

In Figure 2.4 we show a way to realize the quiver $\tilde{Q}_G$, together with the cut $C_1^{(1)}$. The quiver Q_G and the grading induced by this cut are illustrated in Figure 2.5. Note that in this case Corollary 2.5.21 holds, so the paths in the support of the superpotential ω_G are exactly the ones induced by paths in $\mathrm{supp}(\omega_A)$.

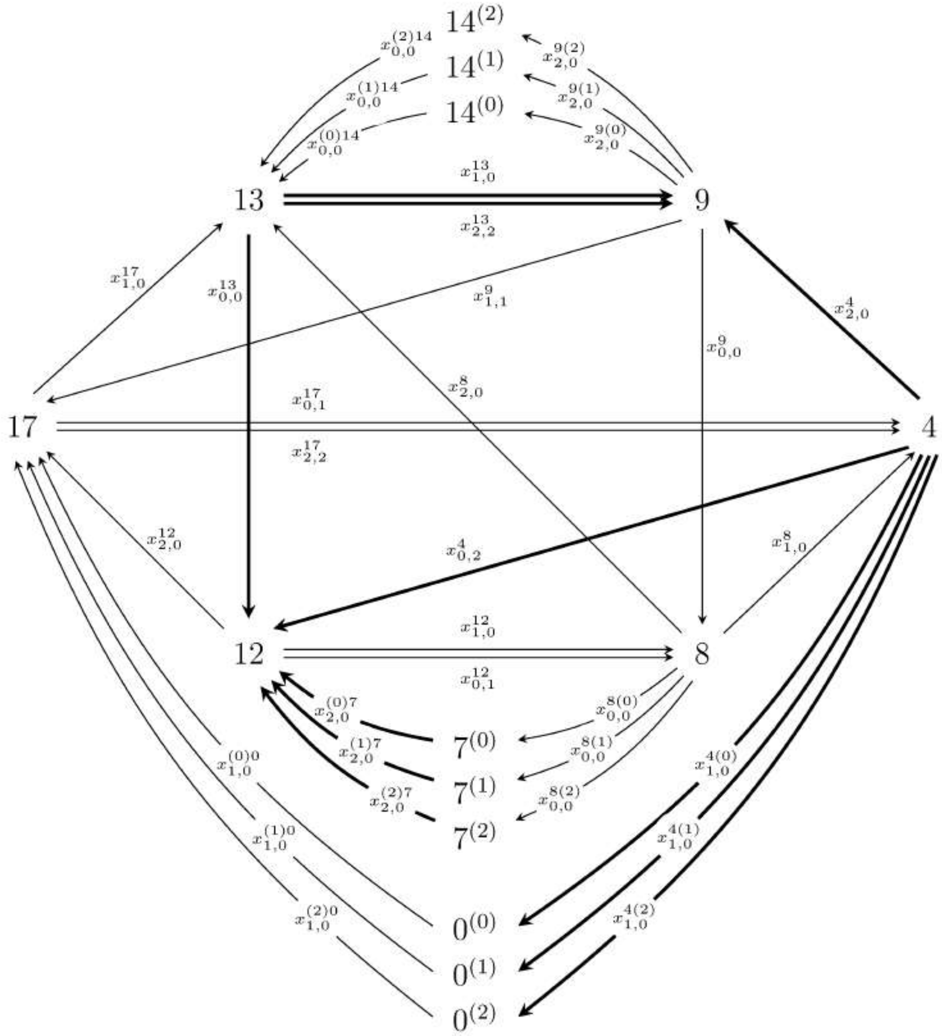


Figure 2.5. The quiver Q_G for $m = 21$, $r = 4$, $s = 3$. The thick arrows have degree 1 with respect to the grading associated to the cut of Figure 2.4, so the quiver of the corresponding 2-representation infinite algebra $(\Pi_G)_0$ is obtained by deleting these arrows.

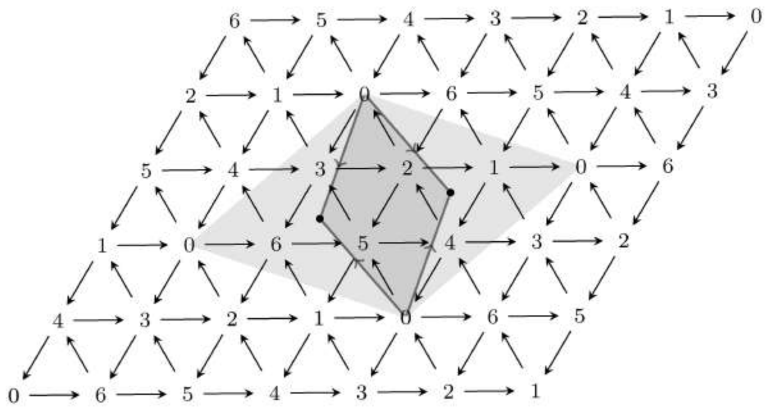


Figure 2.6. The quiver $Q^{(3)}$ with $m = 7$, $r = 2$. Fundamental domains for the quivers Q_A and $\tilde{Q}_G$ are given by, respectively, the light and the dark shadowed regions. The vertices of the latter are the fixed points for the G/A action on the torus.

Now we will give an example where the conditions (M6) and (M7) do not hold, and we will show that in this case we have no invariant cuts.

Example 2.7.8. Set $m = 7$, $r = 2$, $s = 3$, $t = 0$, and let G be the corresponding metacyclic group. The quivers Q, Q_A and $\tilde{Q}_G$ are described in Figure 2.6. For a picture of Q_G, see [11, Example 5.5]. Note that the quiver Q_A can be embedded in a torus which carries a G/A-action with three fixed points. However, among these fixed points only one corresponds to a vertex of the quiver: the others are located in the "barycentre of a triangle", as we can see in the figure. Now consider the cyclic path $2 \to 1 \to 4 \to 2$: the action of G/A sends each arrow in it to the arrow which precedes it in the path. Since a cut must contain exactly one arrow from this path, it is clear that we cannot have cuts which are G/A-invariant.

2.7.2 Examples for $G \not\subseteq \mathrm{SL}(s, \mathbb{C})$

Example 2.7.9. Let $G = M(2, 2)$, so we have $s = 2$, $r = 5$, $m = 12$ and $t = 6$. The group G is generated by the matrices

$$\alpha = \begin{pmatrix} \varepsilon_{12} & 0 \\ 0 & \varepsilon_{12}^5 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and is not contained in $\mathrm{SL}(2, \mathbb{C})$, because $c = 1 + r = 6$. Thus its image G' under the embedding in $\mathrm{SL}(3, \mathbb{C})$ is generated by

$$\alpha = \begin{pmatrix} \varepsilon_{12} & 0 & \\ 0 & \varepsilon_{12}^5 & 0 \\ 0 & 0 & \varepsilon_{12}^6 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

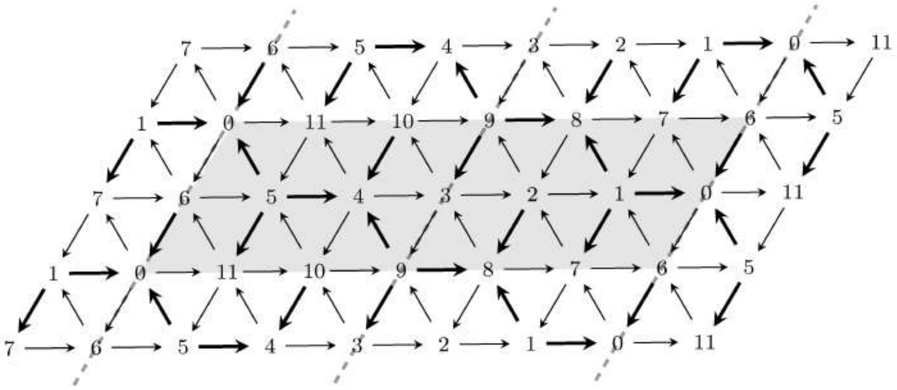


Figure 2.7. The quiver $Q^{(3)}$, where the vertices are labelled with their image under the isomorphism η according to the setting of Example 2.7.9. The quiver $Q_{A'}$ is given by the vertices contained in the shaded parallelogram, where the vertices on opposite sides with the same label are identified. Hence $Q_{A'}$ can be embedded on a torus. The action of G/A on $Q_{A'}$ is given by reflecting along the dashed lines, and the vertices contained in the latter are the fixed points. The thick arrows represent the cut $C_1^{(2)}$.

The quivers $Q_{A'}$ and $Q_{G'}$, together with some cuts, are represented in Figures 2.7, 2.8 and 2.9. Note that the arrows of type $i \to \tau(i)$ are the ones which point in south-west direction.

We end this example by remarking that in this case $u = s = 2$, and so the condition $(u, s) = 1$ in Proposition 2.7.3 is not satisfied. However, it is still possible to choose a set of representatives $\mathcal{D}$ which is invariant under τ' (see Figure 2.9), so such condition is not necessary. Moreover, in Figure 2.8 we exhibit a cut which is not induced by one of the form $C_k^{(l)}$, but it is easily checked that it still yields a grading on Π_G such that $(\Pi_G)_0$ is 2-representation infinite.

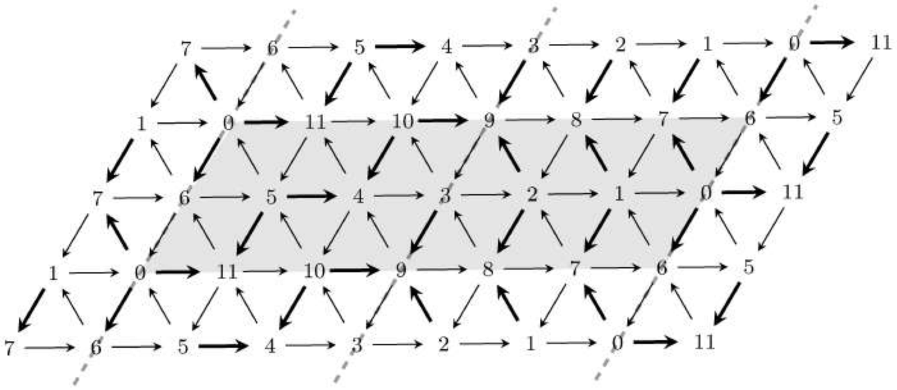


Figure 2.8. A cut in the setting of Example 2.7.9 which is not of the form $C_k^{(l)}$ for any k, l.

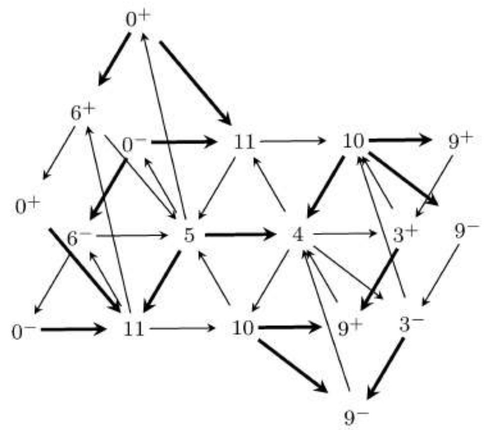


Figure 2.9. The quiver $Q_{G'}$ obtained from Figure 2.8 by choosing as set of representatives $\mathcal{D}$ the vertices contained in the left-side half of the parallelogram. Note that now only the upper and lower sides are identified. The thick arrows are induced by an invariant cut in Q.

Chapter 3

Skew group algebras of Jacobian algebras and 2-representation finite algebras

In this chapter we study skew group algebras of Jacobian algebras of quivers with potential. Sections 3.1 and 3.2 consist in some preliminaries about quivers with potential and self-injective algebras. In Section 3.3 we will set up our assumptions and state our main result, whose proof will be given in Section 3.4. In Section 3.5 we will explain how we can get our original algebra back by doing another skew group algebra construction using the dual group. In Section 3.6 we apply our results to planar rotation-invariant QPs. In Section 3.7 we consider how cuts behave with respect to taking skew group algebras, and the consequences for truncated Jacobian algebras. Section 3.8 consists of some examples which illustrate our construction.

3.1 Quivers with potential and 2-representation finite algebras

Let Q be a quiver. Denote by $\widehat{\Bbbk Q}$ the completion of $\Bbbk Q$ with respect to the $\langle Q_1 \rangle$-adic topology. Define

$$\operatorname{com}_Q = \overline{\left[\widehat{\Bbbk Q}, \widehat{\Bbbk Q}\right]} \subseteq \widehat{\Bbbk Q},$$

where $\overline{}$ denotes closure. Thus $\widehat{\Bbbk Q}/\operatorname{com}_Q$ has a topological basis consisting of cycles in Q. In particular there is a unique continuous linear map

$$\sigma : \widehat{\Bbbk Q}/\operatorname{com}_Q \to \widehat{\Bbbk Q}$$

induced by

$$\alpha_1 \cdots \alpha_n \mapsto \sum_{m=1}^{n} \alpha_m \cdots \alpha_n \alpha_1 \cdots \alpha_{m-1}.$$

For each $\alpha \in Q_1$ define $d_\alpha : \langle Q_1 \rangle \to \widehat{\Bbbk Q}$ to be the continuous linear map given by $d_\alpha(\alpha p) = p$ and $d_\alpha(q) = 0$ if q does not end with α. Define the *cyclic derivative* with respect to an arrow α to be $\partial_\alpha = d_\alpha \circ \sigma : \langle Q_1 \rangle / \operatorname{com}_Q \to \widehat{\Bbbk Q}$. It will be convenient to take derivatives with respect to multiples of arrows. For $\lambda \in \Bbbk^*$, define $\partial_{\lambda\alpha}(c) = \lambda^{-1}\partial_\alpha(c)$. A *potential* is an element $W \in \langle Q_1 \rangle^3 / (\langle Q_1 \rangle^3 \cap \operatorname{com}_Q)$, i.e., a (possibly infinite) linear combination of cycles of length at least 3. A potential is called *finite* if it can be written as a finite linear combination of cycles. By an abuse of notation, if c is a cycle in Q we will denote again by c the corresponding element of $\langle Q_1 \rangle^3 / (\langle Q_1 \rangle^3 \cap \operatorname{com}_Q)$ and consider it up to cyclic permutation of its arrows. We call the pair (Q, W) a *quiver with potential (QP)* and define its Jacobian algebra to be

$$\mathcal{P}(Q, W) = \widehat{\Bbbk Q} \Big/ \overline{\langle \partial_\alpha W \mid \alpha \in Q_1 \rangle}.$$

In our setting, the completion will not play any role, due the following proposition.

Proposition 3.1.1 ([54, Proposition 2.3]). *If W is a finite potential and the ideal*

$$\langle \partial_\alpha W \mid \alpha \in Q_1 \rangle \subseteq \Bbbk Q$$

is admissible, then

$$\mathcal{P}(Q, W) \cong \Bbbk Q \Big/ \langle \partial_\alpha W \mid \alpha \in Q_1 \rangle.$$

In the following we will state a result by Herschend and Iyama which relates QPs with 2-representation finite algebras.

Let (Q, W) be a QP. For a subset $C \subseteq Q_1$ we can define a grading d_C on Q by setting

$$d_C(\alpha) = \begin{cases} 1, & \text{if } \alpha \in C; \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3.1.2. A subset $C \subseteq Q_1$ is called a *cut* if W is homogeneous of degree 1 with respect to d_C.

Note that a cut induces a grading on the Jacobian algebra $\mathcal{P}(Q, W)$. We call its degree 0 part a *truncated Jacobian algebra* and denote it by $\mathcal{P}(Q, W)_C$.

Now let A be a basic algebra of global dimension at most 2 and recall that, by Theorem 1.4.4, its 3-preprojective algebra $\Pi_3(A)$ is self-injective. By [45] we have that $\Pi_3(A) \cong \mathcal{P}(Q_A, W_A)$ for some QP (Q_A, W_A). Moreover, there exists a cut C_A such that $A \cong \mathcal{P}(Q_A, W_A)_{C_A}$. The following theorem tells us that 2-representation finite algebras are exactly the truncated Jacobian algebras of QPs which satisfy the above properties.

Theorem 3.1.3 ([29, Theorem 3.11]). *If (Q, W) is a self-injective QP (i.e., a QP whose Jacobian algebra is self-injective) and C is a cut, then $\mathcal{P}(Q, W)_C$ is 2-representation finite. Moreover, every basic 2-representation finite algebra is obtained in this way.*

3.2 Self-injective algebras

Motivated by the situation of Theorem 3.1.3, in this section we will recall some general facts about self-injective (and Frobenius) algebras. Some references are for instance [50] or [32]. In §3.2.1 we will show that the property of being Frobenius is preserved under taking skew group algebras.

An algebra Λ is *self-injective* if it is injective as a right Λ-module. It is *Frobenius* if there is a bilinear form $(-,-)$ on Λ which is nondegenerate and multiplicative (i.e., $(a,bc)=(ab,c)$ for all $a,b,c\in\Lambda$). It is *symmetric* if this form can be taken to be symmetric. Frobenius algebras are self-injective, and the converse is true if and only if $\dim \mathrm{Hom}_\Lambda(S,\Lambda)=\dim S$ for all simples S. In particular, self-injective basic algebras are exactly the Frobenius basic algebras.

If Λ is Frobenius, then from the nondegenerate bilinear form we get an isomorphism $f:\Lambda\to D\Lambda$ of vector spaces, given by $f(v)=(-,v)$. Moreover f is an isomorphism of left Λ-modules since

$$f(\lambda v)=(-,\lambda v)=(-\lambda,v)=\lambda(f(v)).$$

Nondegeneracy of the form implies that there exists a unique $\Bbbk$-linear map $\varphi:\Lambda\to\Lambda$ satisfying

$$(a,b)=(b,\varphi(a))$$

for all $a,b\in\Lambda$. In fact such a φ is an algebra automorphism, and f becomes a right module isomorphism $f:\Lambda_\varphi\to D\Lambda$. If we choose a different bilinear form and hence a different isomorphism $g:\Lambda\to D\Lambda$ of vector spaces, then $g(a)=f(au)$ for some unit $u\in\Lambda$. Then the corresponding automorphism ψ is given by $\psi(a)=u\varphi(a)u^{-1}$, so φ is unique as an outer automorphism of Λ. The automorphism φ is called a *Nakayama automorphism* of Λ. In particular, Λ is symmetric if and only if $\varphi=\mathrm{id}_{\mathrm{Out}(\Lambda)}$.

3.2.1 Skew group algebras of Frobenius algebras

We are interested in studying skew group algebras of Frobenius algebras, and in particular the case where G is generated by a Nakayama automorphism.

Remark 3.2.1. In [57, Theorem 1.3(c)(iii)] it is proved that skew group algebras of self-injective algebras are always self-injective. In the discussion that follows we show that the property of being Frobenius is also preserved under taking skew group algebras.

Let G be a finite group acting on a Frobenius algebra Λ by automorphisms. The algebra $\Bbbk G$ is always Frobenius and in fact symmetric. We denote by $(-,-)$ the corresponding symmetric nondegenerate bilinear form on $\Bbbk G$ as well. This form can be taken to be $(h,l)=\delta_{hl^{-1}}$ for $h,l\in G$, extended bilinearly. Then we can define a bilinear form $\langle -,-\rangle$ on the skew group algebra $\Lambda * G$ by setting

$$\langle \lambda\otimes l,\mu\otimes m\rangle=(\lambda,l(\mu))(l,m)$$

for $\lambda,\mu\in\Lambda$ and $l,m\in G$, extended bilinearly.

Lemma 3.2.2. *The form $\langle -, - \rangle$ is multiplicative and nondegenerate. In particular, $\Lambda * G$ is Frobenius.*

Proof. We have

$$\begin{aligned}\langle (\lambda \otimes l)(\mu \otimes m), \nu \otimes n \rangle &= \langle \lambda l(\mu) \otimes lm, \nu \otimes n \rangle = \\ &= (\lambda l(\mu), (lm)(\nu))(lm, n) = \\ &= (\lambda, l(\mu) l(m(\nu)))(l, mn) = \\ &= (\lambda, l(\mu m(\nu)))(l, mn) = \\ &= \langle \lambda \otimes l, \mu m(\nu) \otimes mn \rangle = \\ &= \langle \lambda \otimes l, (\mu \otimes m)(\nu \otimes n) \rangle\end{aligned}$$

for all $\lambda, \mu, \nu \in \Lambda$ and $l, m, n \in G$. This proves multiplicativity.

Assume now that there exists $\sum_i \xi_i \otimes z_i \in \Lambda * G$ such that $\langle \sum_i \xi_i \otimes z_i, x \rangle = 0$ for all $x \in \Lambda * G$. Without loss of generality we can take every z_i to be an element of G. Take $x = \lambda \otimes l$ with $\lambda \in \Lambda$ and $l \in G$. Then

$$0 = \sum_i \langle \xi_i \otimes z_i, \lambda \otimes l \rangle = \sum_i (\xi_i, z_i(\lambda))(z_i, l) = \sum_{z_i = l^{-1}} (\xi_i, l^{-1}(\lambda)) = \left(\sum_{z_i = l^{-1}} \xi_i, l^{-1}(\lambda) \right).$$

Since l acts by an automorphism and $(-, -)$ is nondegenerate, it follows that $\sum_{z_i = l^{-1}} \xi_i = 0$. By iterating this argument for all possible values of l, we get that

$$\sum_i \xi_i \otimes z_i = \sum_{l \in G} \left(\sum_{z_i = l^{-1}} \xi_i \right) \otimes l^{-1} = 0.$$

Assume instead that $\langle x, \sum_i \xi_i \otimes z_i \rangle = 0$ for all $x \in \Lambda * G$. Again we suppose that $z_i \in G$ and we take $x = \lambda \otimes l$ with $\lambda \in \Lambda$ and $l \in G$. Then

$$0 = \sum_i \langle \lambda \otimes l, \xi_i \otimes z_i \rangle = \sum_i (\lambda, l(\xi_i))(l, z_i) = \sum_{z_i = l^{-1}} (\lambda, l(\xi_i)) = \left(\lambda, l \left(\sum_{z_i = l^{-1}} \xi_i \right) \right)$$

so that $\sum_{z_i = l^{-1}} \xi_i = 0$ and we can argue as above. This proves nondegeneracy. □

If the bilinear form on Λ is G-equivariant, we can find a Nakayama automorphism of $\Lambda * G$. Let us choose a Nakayama automorphism φ of Λ.

Proposition 3.2.3. *If $(g(\lambda), g(\mu)) = (\lambda, \mu)$ for all $g \in G, \lambda, \mu \in \Lambda$, then $\varphi \otimes 1$ is a Nakayama automorphism of $\Lambda * G$.*

Proof. Let $\lambda, \mu \in \Lambda$ and $l, m \in G$. Then

$$\begin{aligned}\langle \lambda \otimes l, \mu \otimes m \rangle &= \delta_{lm^{-1}}(\lambda, l(\mu)) = \\ &= \delta_{lm^{-1}}(l(\mu), \varphi(\lambda)) = \\ &= \delta_{lm^{-1}}(\mu, l^{-1}\varphi(\lambda)) = \\ &= \delta_{lm^{-1}}(\mu, m\varphi(\lambda)) = \\ &= \langle \mu \otimes m, \varphi(\lambda) \otimes l \rangle.\end{aligned}$$

□

Corollary 3.2.4. *If φ generates the image $\mathrm{im}(G) \subseteq \mathrm{Aut}(\Lambda)$, then $\Lambda * G$ is symmetric.*

Proof. Since φ is an element in $\mathrm{im}(G)$, we know that there is an $h \in G$ which acts on Λ as φ. Now let $g \in G$. By assumption, there exists an integer j such that g acts on Λ as φ^j. Then we have

$$(\lambda, \mu) = (\mu, \varphi(\lambda)) = (\varphi(\lambda), \varphi(\mu)) = (\varphi^j(\lambda), \varphi^j(\mu)) = (g(\lambda), g(\mu)),$$

so we can apply Proposition 3.2.3 and get that

$$\varphi \otimes 1 : \lambda \otimes l \mapsto h(\lambda) \otimes l$$

is a Nakayama automorphism of $\Lambda * G$. Notice now that $h(\lambda) \otimes l = (1 \otimes h)(\lambda \otimes l)(1 \otimes h)^{-1}$, so that $\varphi \otimes 1$ is the identity as an outer automorphism of $\Lambda * G$, which means that $\Lambda * G$ is symmetric. □

We include the following lemma, which we will use in Section 3.6.

Lemma 3.2.5. *Let Λ be a symmetric algebra, and $e \in \Lambda$ an idempotent. Then $e\Lambda e$ is symmetric.*

Proof. Let $\langle -, - \rangle$ be a symmetric multiplicative nondegenerate bilinear form on Λ. Then the restricted form on $e\Lambda e$ is a symmetric multiplicative bilinear form on $e\Lambda e$. Let now $u \in e\Lambda e$ such that $\langle u, - \rangle_{|e\Lambda e} = 0$. Let $v \in \Lambda$ and observe that

$$\langle u, v \rangle = \langle eue, v \rangle = \langle eu, ev \rangle = \langle ev, eu \rangle = \langle eve, u \rangle = 0$$

so that $u = 0$ since the form is nondegenerate on Λ. □

3.3 Setup and result

Let (Q, W) be a quiver with potential and let $\Lambda = \mathcal{P}(Q, W)$ be its Jacobian algebra. Write $W = \sum_c a(c)c$, and recall that we consider cycles up to cyclic permutation. We assume that W is finite and that the cyclic derivatives of W generate an admissible ideal of $\Bbbk Q$. In what follows we will freely use integers as indices for convenience, even when they should be seen as elements of $\mathbb{Z}/n\mathbb{Z}$.

3.3.1 Assumptions

Let G be a cyclic group of order n with generator g, acting on $\Bbbk Q$. We make the following assumptions.

(A1) The field $\Bbbk$ contains a primitive n-th root of unity ζ. In particular, $n \neq 0$ in $\Bbbk$.

(A2) The action of G permutes the vertices of Q and maps every arrow to a multiple of an arrow.

(A3) If α is an arrow between two fixed vertices, then $g(\alpha) = \zeta^{b(\alpha)}\alpha$ for an integer $b(\alpha)$.

(A4) Every vertex of Q which is not fixed by G has an orbit of cardinality n.

(A5) We have $GW = W$.

Since G preserves the potential, we get an induced action of G on Λ. We define a second "forgetful" action $*$ of G on Q by $g * v = g(v)$ for $v \in Q_0$ and $g * \alpha = \beta$ whenever β is an arrow and $g(\alpha)$ is a scalar multiple of β.

Remark 3.3.1. Suppose that an arrow α is such that $g(\alpha) = \zeta^i\beta$ for some arrow $\beta \neq \alpha$. Then, by assumption (A4), one of $\mathfrak{s}(\alpha)$ and $\mathfrak{t}(\alpha)$ has an orbit of size n, so $|G * \alpha| = n$. We can replace β with $\zeta^{-i}\beta$ as the element in $\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda$ representing the corresponding arrow. By doing this for all n distinct arrows in the orbit of α, we get that on this orbit the action of G coincides with the $*$ action of G. The potential W is not affected by this procedure, if we see it as an element of $\widehat{\Bbbk Q}/\operatorname{com}_Q$, so it is still invariant under G. However, note that the expression of W as a linear combination of cycles in Q is possibly changed.

In view of the above observation, we can without loss of generality make the additional assumption:

(A6) Arrows with at least one end which is not fixed are sent to arrows by the action of G.

So for an arrow α between two fixed vertices we have $g * \alpha = \alpha = \zeta^{-b(\alpha)}g(\alpha)$, while for all other arrows we have $g * \alpha = g(\alpha) = \beta$ for some arrow $\beta \neq \alpha$.

We need to make some further assumptions about the relationship between G and W. It turns out that it is convenient to impose conditions on the number of fixed vertices appearing in cycles of W. We make the following assumption.

(A7) Every cycle c appearing in W is of one of the following types:

  (i) the cycle c goes through no vertices fixed by G;

  (ii) the cycle c goes through exactly one (counted with multiplicity) vertex fixed by G;

  (iii) the cycle c goes through exactly one (counted with multiplicity) vertex not fixed by G;

(iv) the cycle c goes only through vertices which are fixed by G.

Remark 3.3.2. These assumptions are strong. We need them to construct a QP (Q_G, W_G) such that the skew group algebra of $\mathcal{P}(Q, W)$ is Morita equivalent to $\mathcal{P}(Q_G, W_G)$. However, the assumptions are satisfied in many examples, and they are weak enough to still hold for (Q_G, W_G). This in turn allows us to come back to (Q, W) via a skew group algebra construction with a natural action of the dual group $\hat{G}$ (see Section 3.5).

Remark 3.3.3. From our assumptions, it follows that cycles of a given type are mapped by G to multiples of cycles of the same type. By assumption (A6), cycles of type (i) and (ii) contain only arrows that are mapped to arrows, so those cycles are mapped to cycles. If $c = \alpha_1 \dots \alpha_l$ is of type (iv), then $g(c) = \zeta^{\sum_i b(\alpha_i)} c$, so from $GW = W$ we obtain that $\sum_i b(\alpha_i) = 0 \pmod{n}$ and $Gc = c$. In particular, $g * c = g(c)$ for all c of type (i), (ii), (iv).

The reader wishing to have examples of QPs with group actions satisfying these assumptions is advised to have in mind the QPs of Example 3.8.1. In particular, the two QPs of Figure 3.4 and Figure 3.5 both have an action of $\mathbb{Z}/3\mathbb{Z}$, one sending arrows to arrows and the other multiplying $\tilde{\delta}$ by a third root of unity. They are the quivers with potential corresponding to each other's skew group algebra under these actions. All cycles of the first one are of type (i) or (ii), while all cycles of the second one are of type (iii) or (iv).

3.3.2 The quiver of $\Lambda * G$

We now describe the quiver Q_G of the skew group algebra $\Lambda * G$ following [57]. We first define an idempotent $\eta \in \Lambda * G$ such that $\eta(\Lambda * G)\eta$ is basic and Morita equivalent to $\Lambda * G$. We decompose η as a sum of primitive orthogonal idempotents, and use those to label the vertices of Q_G. Then we choose elements in $\eta(\Lambda * G)\eta$ to be the arrows.

A complete list of primitive orthogonal idempotents for the group algebra $\Bbbk G$ is given by

$$e_\mu = \frac{1}{n} \sum_{i=0}^{n-1} \zeta^{i\mu} g^i,$$

for $\mu = 0, \dots, n-1$.

Fix a set $\mathcal{E}$ of representatives of vertices of Q under the action of G. We write $\mathcal{E} = \mathcal{E}' \sqcup \mathcal{E}''$, where $\mathcal{E}'$ and $\mathcal{E}''$ consist of the vertices in $\mathcal{E}$ whose orbits have cardinality n and 1 respectively. We define the following idempotents in $\Lambda * G$:

- for each vertex $\varepsilon \in \mathcal{E}'$ we put $\eta^\varepsilon = \varepsilon \otimes 1$;
- for each vertex $\varepsilon \in \mathcal{E}''$ and $\mu = 0, \dots, n-1$ we put $\eta^\varepsilon_\mu = \varepsilon \otimes e_\mu$.

Set

$$\eta = \sum_{\varepsilon \in \mathcal{E}'} \eta^\varepsilon + \sum_{\varepsilon \in \mathcal{E}''} \sum_{\mu=0}^{n-1} \eta^\varepsilon_\mu.$$

Note in particular that $\eta = \hat{\varepsilon} \otimes 1$, where $\hat{\varepsilon}$ is the idempotent of Λ corresponding to $\mathcal{E}$. By [57, §2.3] the algebra $\eta(\Lambda * G)\eta$ is Morita equivalent to $\Lambda * G$. A complete list of primitive orthogonal idempotents for $\eta(\Lambda * G)\eta$ is

$$\{\eta^{\varepsilon} \mid \varepsilon \in \mathcal{E}'\} \cup \left\{\eta^{\varepsilon}_{\mu} \mid \varepsilon \in \mathcal{E}'', \mu = 0, \dots, n-1\right\}.$$

Remark 3.3.4. The idempotent η is not canonical, in that it depends on choosing some vertices of Q. However, it is convenient to define it in this way to get a natural action of the dual group $\hat{G}$ on $\eta(\Lambda * G)\eta$. By contrast, the authors of [2] choose a canonically defined basic algebra for their Morita equivalence, but in exchange they have to choose vertices of Q in order to be able to define such an action.

Now we will fix a basis for the arrows of the quiver Q_G of $\eta(\Lambda * G)\eta$. There are four different cases to consider.

(1) Let β be an arrow between two non-fixed vertices of Q. Then there is exactly one arrow α in the G-orbit of β such that $\mathfrak{t}(\alpha) \in \mathcal{E}'$. Thus α is of the form $\alpha\colon g^t\varepsilon \to \varepsilon'$, with $\varepsilon, \varepsilon' \in \mathcal{E}'$ and $0 \leq t \leq n-1$. We call α an arrow of type (1), and define an element $\tilde{\alpha} \in \eta(\Lambda * G)\eta$ by

$$\tilde{\alpha} = \alpha \otimes g^t.$$

This will be an arrow in Q_G from η^{ε} to $\eta^{\varepsilon'}$.

(2) Let β be an arrow in Q from a non-fixed vertex to a fixed vertex. Then there is exactly one arrow α in the G-orbit of β such that $\mathfrak{s}(\alpha) \in \mathcal{E}'$. Thus α is of the form $\alpha\colon \varepsilon \to \varepsilon'$, with $\varepsilon \in \mathcal{E}'$, $\varepsilon' \in \mathcal{E}''$. We call α an arrow of type (2), and define elements $\tilde{\alpha}^{\mu} \in \eta(\Lambda * G)\eta$ by

$$\tilde{\alpha}^{\mu} = (1 \otimes e_{\mu})(\alpha \otimes 1)$$

for $\mu = 0, \dots, n-1$. These will be arrows in Q_G from η^{ε} to $\eta^{\varepsilon'}_{\mu}$ respectively.

(3) Let β be an arrow in Q from a fixed vertex to a non-fixed vertex. Then there is exactly one arrow α in the G-orbit of β such that $\mathfrak{t}(\alpha) \in \mathcal{E}'$. Thus α is of the form $\alpha\colon \varepsilon \to \varepsilon'$, with $\varepsilon \in \mathcal{E}''$, $\varepsilon' \in \mathcal{E}'$. We call α an arrow of type (3), and define elements $\tilde{\alpha}^{\mu} \in \eta(\Lambda * G)\eta$ by

$$\tilde{\alpha}^{\mu} = \alpha \otimes e_{\mu}$$

for $\mu = 0, \dots, n-1$. These will be arrows in Q_G from η^{ε}_{μ} to $\eta^{\varepsilon'}$ respectively.

(4) Let α be an arrow between two fixed vertices, i.e., $\alpha\colon \varepsilon \to \varepsilon'$ with $\varepsilon, \varepsilon' \in \mathcal{E}''$. Recall that by assumption $g(\alpha) = \zeta^{b(\alpha)}\alpha$. We call α an arrow of type (4), and define elements $\tilde{\alpha}^{\mu} \in \eta(\Lambda * G)\eta$ by

$$\tilde{\alpha}^{\mu} = \alpha \otimes e_{\mu}$$

for $\mu = 0, \dots, n-1$. These will be arrows in Q_G from η^{ε}_{μ} to $\eta^{\varepsilon'}_{\mu - b(\alpha)}$ respectively.

For an arrow $\alpha : g^t(\varepsilon) \to \varepsilon'$ of type (1), we define $t(\alpha) = t$. Note that this integer is well defined modulo n, since the orbit of ε has cardinality n. If instead α is an arrow of type (2), (3), or (4), we put $t(\alpha) = 0$.

Proposition 3.3.5. *This choice gives a basis of* $\operatorname{rad}\eta(\Lambda * G)\eta/\operatorname{rad}^2\eta(\Lambda * G)\eta$, *and the start and target of arrows in* Q_G *are as claimed above.*

Proof. The vector space spanned by the arrows of Q decomposes as a direct sum of $\Bbbk G$-modules into the spans of the G-orbits of the arrows. Therefore it is enough to look at one G-orbit of an arrow at a time, and we can assume that there are no multiple arrows in Q.

Let us now look at the four cases. If $\alpha : g^t\varepsilon \to \varepsilon'$ is of type (1), then the n arrows in $G\alpha$ give rise to a unique arrow $\tilde{\alpha} : \eta^\varepsilon \to \eta^{\varepsilon'}$. By [57, Theorem 1.3(d)(i)] we have that $\operatorname{rad}^i \Lambda * G = (\operatorname{rad}^i \Lambda)\Lambda * G$, so that a basis of the space of arrows from η^ε to $\eta^{\varepsilon'}$ is given by $\{\varepsilon'\beta h(\varepsilon) \otimes h\}$ with $\beta \in Q_1$. So the only β contributing is the only arrow in $G\alpha$ ending in ε', and this basis is $\{\tilde{\alpha} = \alpha \otimes g^{t(\alpha)}\}$.

Let now $\alpha : \varepsilon \to \varepsilon'$ be of type (2). Then the n arrows in $G\alpha$ give rise to n arrows of Q_G. By the above argument, we get that a basis of $(\varepsilon' \otimes 1)(\operatorname{rad}\Lambda * G/\operatorname{rad}^2 \Lambda * G)(\varepsilon \otimes 1)$ is given by $\{g^i(\alpha) \otimes g^i \mid i = 0, \dots, n-1\}$. Then the set $\{\tilde{\alpha}^\mu = (1 \otimes e_\mu)(\alpha \otimes 1) \mid \mu = 0, \dots, n-1\}$ is also a basis, since

$$(1 \otimes e_\mu)(\alpha \otimes 1) = \frac{1}{n}\sum_{i=0}^{n-1} \zeta^{i\mu} g^i(\alpha) \otimes g^i.$$

Now $\eta^{\varepsilon'}_\nu \tilde{\alpha}^\mu = \tilde{\alpha}^\mu$ if $\nu = \mu$, and 0 otherwise, so each $\tilde{\alpha}^\mu$ is indeed an arrow of Q_G from η^ε to $\eta^{\varepsilon'}_\mu$.

If $\alpha : \varepsilon \to \varepsilon'$ is an arrow of type (3) or (4), by similar arguments we get that $\{\alpha \otimes g^i\}$ is a basis of $(\varepsilon' \otimes 1)(\operatorname{rad}\Lambda * G/\operatorname{rad}^2 \Lambda * G)(\varepsilon \otimes 1)$. Then $\{\tilde{\alpha}^\mu = \alpha \otimes e_\mu\}$ is also a basis, and it consists of arrows. □

The choice of vertices and arrows we have made defines an isomorphism $J : \Bbbk Q_G \to \eta((\Bbbk Q) * G)\eta$ by [57, §2.3].

3.3.3 Cycles in Q_G and the potential W_G

We want to define a potential W_G on Q_G, so we need to construct cycles in Q_G depending on those appearing in W. Recall that we write $W = \sum_c a(c)c$, and that we consider cycles up to cyclic permutation. We will define, for every cycle c appearing in W, a cycle $\hat{c}$ in $G * c$ depending on our choice of representatives of the vertices. Moreover, to every $\hat{c}$ we will associate a cycle $\tilde{c}$ in Q_G.

(i) Let c be a cycle of type (i) in W. Then choose $\hat{c}$ in $G * c$ such that

$$\hat{c}: \qquad \varepsilon_0 = \varepsilon_l \xrightarrow{g^{t_1+\cdots+t_{l-1}}(\alpha_l)} g^{t_1+\cdots+t_{l-1}}(\varepsilon_{l-1}) \longrightarrow \cdots \xrightarrow{g^{t_1}(\alpha_2)} g^{t_1}(\varepsilon_1) \xrightarrow{\alpha_1} \varepsilon_0$$

with $\varepsilon_i \in \mathcal{E}'$ for all i. Notice that this is indeed (in general) a choice, the only requirement is that $\hat{c}$ should go through at least one vertex in $\mathcal{E}'$. Set moreover $\hat{d} = \hat{c}$ for all the other $d \in G * c$. Note that each α_i is an arrow of type (1) and $t_i = t(\alpha_i)$. Define a cycle $\tilde{c}$ in Q_G by

$$\tilde{c} = \tilde{\alpha}_1 \cdots \tilde{\alpha}_l.$$

(ii) Let c be a cycle of type (ii) in W. There is a unique $\hat{c} \in G * c$ that can be written as above, with $\varepsilon_i \in \mathcal{E}'$ for $i \neq 1$, $\varepsilon_1 \in \mathcal{E}''$ and $t_1 = 0$. Note that for $i \geq 3$, α_i is of type (1) and $t_i = t(\alpha_i)$, while $g^{-t_2}(\alpha_2)$ is of type (2) and α_1 is of type (3). Define cycles $\tilde{c}^\mu$ in Q_G by

$$\tilde{c}^\mu = \tilde{\alpha}_1^\mu \widetilde{g^{-t_2}(\alpha_2)}^\mu \tilde{\alpha}_3 \cdots \tilde{\alpha}_l$$

for $\mu = 0, \ldots, n-1$, and call $p(c) = t_2$.

(iii) Let c be a cycle of type (iii) in W. There is a unique $\hat{c} \in G * c$ that can be written as above, with $\varepsilon_i \in \mathcal{E}''$ for $i \neq 1$, $\varepsilon_1 \in \mathcal{E}'$ and $t_i = 0$ for all i. Notice that for $i \geq 3$, α_i is of type (4), while α_2 is of type (3) and α_1 is of type (2). Put $b_i = b(\alpha_i) + \cdots + b(\alpha_l)$ for all $i \geq 3$ and define cycles $\tilde{c}^\mu$ in Q_G by

$$\tilde{c}^\mu = \tilde{\alpha}_1^\mu \tilde{\alpha}_2^{\mu - b_3} \tilde{\alpha}_3^{\mu - b_4} \cdots \tilde{\alpha}_{l-1}^{\mu - b_l} \tilde{\alpha}_l^\mu$$

for $\mu = 0, \ldots, n-1$. Call $q(c) = b_3$, and notice that $g(\hat{c}) = \zeta^{q(c)} g * \hat{c}$ (and in fact $g(c) = \zeta^{q(c)} g * c$).

(iv) Let c be a cycle of type (iv) in W. Thus $Gc = c$ in $\Bbbk Q$ and we can write $\hat{c} = c$ as above, with $\varepsilon_i \in \mathcal{E}'$ and $t_i = 0$ for all i. Notice that each α_i is an arrow of type (4). Put $b_i = b(\alpha_i) + \cdots + b(\alpha_l)$ for all i and define cycles $\tilde{c}^\mu$ in Q_G by

$$\tilde{c}^\mu = \tilde{\alpha}_1^{\mu - b_2} \tilde{\alpha}_2^{\mu - b_3} \cdots \tilde{\alpha}_{l-1}^{\mu - b_l} \tilde{\alpha}_l^\mu$$

for $\mu = 0, \ldots, n-1$.

Now define $\mathcal{C}(x) = \{\hat{c} \mid c \text{ cycle of } W \text{ of type } x\}$ for $x =$ (i), (ii), (iii), (iv). Then $\mathcal{C} = \bigsqcup \mathcal{C}(x)$ is a cross-section of cycles of W under the $*$ action of G.

We can now define a (finite) potential W_G on Q_G by setting

$$W_G = \sum_{c \in \mathcal{C}(\mathrm{i})} a(c) \frac{|Gc|}{n} \tilde{c} + \sum_{c \in \mathcal{C}(\mathrm{ii})} a(c) \sum_{\mu=0}^{n-1} \zeta^{-p(c)\mu} \tilde{c}^\mu + \sum_{c \in \mathcal{C}(\mathrm{iii}) \cup \mathcal{C}(\mathrm{iv})} a(c) \sum_{\mu=0}^{n-1} \tilde{c}^\mu.$$

Remark 3.3.6. Note that all cycles in W_G have length at least 3, since each of them has the same length of a cycle in W. Moreover the sums in W_G are made over subsets of cycles which appear in W, hence they are all finite. This means that W_G is indeed a finite potential in Q_G.

3.3.4 Main result

We are ready to state our main result. Recall that we assume that (Q, W) is a QP with finite potential such that the cyclic derivatives of W generate an admissible ideal of $\Bbbk Q$. Call $\Lambda = \mathcal{P}(Q, W)$ the Jacobian algebra of (Q, W).

Theorem 3.3.7. *Let G be a finite cyclic group acting on (Q, W) as per the assumptions (A1)-(A7). Then*

$$\mathcal{P}(Q_G, W_G) \cong \eta(\Lambda * G)\eta.$$

We give a proof of this result in §3.4.3, and outline here the strategy we will use. By [57, §2.3], the algebra $\eta(\Lambda * G)\eta$ is isomorphic to $\Bbbk Q_G$ modulo a certain ideal. Our first step, carried out in §3.4.1, is to give explicit generators for this ideal in our setting. However, these generators will not be relations of Q_G (i.e., linear combinations of paths in Q_G with common start and end). In §3.4.2, we express them in terms of the derivatives of the potential W_G, which will allow us to conclude.

Remark 3.3.8. The statement that there exists a potential W' such that $\mathcal{P}(Q_G, W') \cong \eta(\Lambda * G)\eta$ follows, by taking the 0-th cohomology of the corresponding dg algebras, from a much more general result proved in [46, Corollary 1.3]. Moreover, [46, Lemma 4.4.1] expresses a suitable W' as an element of $\eta(\Lambda * G)\eta$, and W' is written as a linear combination of paths in Q_G in the examples of [46, §4.5]. Our Theorem 3.3.7 states that the potential W_G, which we constructed under our assumptions (A1)-(A7), has the same property.

3.4 Proof of main result

3.4.1 Ideals of skew group algebras

In order to prove Theorem 3.3.7, we need some observations about ideals of skew group algebras.

Proposition 3.4.1. *Let A be a ring and let η be an idempotent of A. Let $I = AXA$ for some subset $X \subseteq A$, such that $\eta x \eta = x$ for all $x \in X$. Then*

$$\eta \frac{A}{I} \eta = \frac{\eta A \eta}{\langle X \rangle}.$$

Proof. It is enough to prove that $\eta I \eta = \langle X \rangle$. Let $\kappa = 1 - \eta$. Then $\eta A = \eta A \eta \oplus \eta A \kappa$ and $A\eta = \eta A \eta \oplus \kappa A \eta$. Observe that $\eta x \eta = x$ implies $\kappa x = x \kappa = 0$. Then

$$\eta I \eta = \eta AXA\eta = \eta A \eta X \eta A \eta \oplus \eta A \eta X \kappa A \eta \oplus \eta A \kappa X \eta A \eta \oplus \eta A \kappa X \kappa A \eta = \eta A \eta X \eta A \eta = \langle X \rangle.$$

□

Now retain the notation of Section 3.3. So $\Lambda = \Bbbk Q/\mathcal{R}$, where $\mathcal{R} = \langle R \rangle$ and $R = \{\partial_\alpha W \mid \alpha \in Q_1\}$, and the action of G on Λ leaves R stable. Then we know by [57, §2.2] that

$$\Lambda * G \cong \frac{(\Bbbk Q) * G}{\langle R \otimes 1 \rangle}.$$

Recall that we have an idempotent $\eta = \hat{\varepsilon} \otimes 1$, for an idempotent $\hat{\varepsilon}$ in $\Bbbk Q$, such that $\eta((\Bbbk Q) * G)\eta \cong \Bbbk Q_G$. We have the following lemmas.

Lemma 3.4.2. *Suppose that $\langle R\rangle$ is an admissible ideal of $\Bbbk Q$. Then the ideal $\eta\langle R\otimes 1\rangle\eta$ of $\eta((\Bbbk Q)*G)\eta$ is admissible.*

Proof. Let $A=\Bbbk Q$. Since $\mathcal{R}=\langle R\rangle$ is admissible, we have $(\operatorname{rad} A)^N\subseteq\mathcal{R}\subseteq(\operatorname{rad} A)^2$ for some $N\geq 2$. Consider $\mathcal{R}$ as a subset of $A*G$ under the natural inclusion $A\to A*G$, so $\langle R\otimes 1\rangle=(A*G)\mathcal{R}(A*G)$. By [57, Theorem 1.3(d)(ii)] we have $(A*G)(\operatorname{rad} A)^i=(\operatorname{rad} A)^i(A*G)=(\operatorname{rad} A*G)^i$ for all $i\geq 1$, so

$$(A*G)(\operatorname{rad} A)^N(A*G)\subseteq(A*G)\mathcal{R}(A*G)\subseteq(A*G)(\operatorname{rad} A)^2(A*G)$$

becomes

$$(\operatorname{rad} A*G)^N\subseteq\langle R\otimes 1\rangle\subseteq(\operatorname{rad} A*G)^2.$$

Then the claim follows from the fact that $\eta(\operatorname{rad} A*G)\eta=\operatorname{rad}(\eta(A*G)\eta)$. □

Lemma 3.4.3. *For each $r\in R$, choose $g_r,h_r\in G$ such that $\mathfrak{t}(r)\in g_r(\mathcal{E})$ and $\mathfrak{s}(r)\in h_r(\mathcal{E})$. Then*

$$\eta\frac{(\Bbbk Q)*G}{\langle R\otimes 1\rangle}\eta=\frac{\eta((\Bbbk Q)*G)\eta}{\langle g_r^{-1}(r)\otimes h_rg_r^{-1}\mid r\in R\rangle}.$$

Proof. We have

$$g_r^{-1}(r)\otimes h_rg_r^{-1}=(1\otimes g_r^{-1})(r\otimes 1)(1\otimes h_r)$$

so that $R\otimes 1$ generates the same ideal in $(\Bbbk Q)*G$ as the set $\{g_r^{-1}(r)\otimes h_rg_r^{-1}\mid r\in R\}$. Now

$$\eta(g_r^{-1}(r)\otimes h_rg_r^{-1})\eta=\hat{\varepsilon}g_r^{-1}(r)(h_rg_r^{-1})(\hat{\varepsilon})\otimes h_rg_r^{-1}=g_r^{-1}(r)\otimes h_rg_r^{-1},$$

so the claim follows from Proposition 3.4.1. □

Lemma 3.4.4. *In the assumptions (A1)-(A7), we have*

$$\eta(\Lambda*G)\eta\cong\frac{\eta((\Bbbk Q)*G)\eta}{\langle\partial_{g^{-t(\alpha)}\alpha}W\otimes g^{-t(\alpha)}\mid\alpha\text{ of type }(1),(2),(3),(4)\rangle}.$$

Proof. Since G acts on W, the ideal of $\Bbbk Q$ generated by $\{\partial_\alpha W\}\otimes 1$ is also generated by

$$\{\partial_\alpha W\mid\alpha\text{ of type }(1),(2),(3),(4)\}\otimes 1,$$

since $h(\partial_\alpha W)=\partial_{h(\alpha)}W$ for any $h\in G$. Notice that α is of type (1), (2), (3), (4) precisely if $\mathfrak{s}(\partial_\alpha W)=\mathfrak{t}(\alpha)\in\mathcal{E}$, and then $\mathfrak{t}(\partial_\alpha W)=\mathfrak{s}(\alpha)\in g^{t(\alpha)}(\mathcal{E})$. Then we can apply Lemma 3.4.3 with $g_r=g^{t(\alpha)}$ and $h_r=1$, and we get the claim. □

3.4.2 Derivatives of W_G as elements of $\eta(\Lambda * G)\eta$

In this section we shall express elements of the form $\partial_{g^{-t(\alpha)}\alpha} W \otimes g^{-t(\alpha)}$ for α of type (1), (2), (3), (4) in terms of the derivatives of the potential W_G. Precisely, identifying $\eta((\Bbbk Q) * G)\eta$ with $\Bbbk Q_G$ via the isomorphism J of §3.3.2, each $\partial_{g^{-t(\alpha)}\alpha} W \otimes g^{-t(\alpha)}$ corresponds to $\sum_{i,j\in(Q_G)_0} x_{ij}$, where x_{ij} is a linear combination of paths in Q_G from vertex i to vertex j (i.e., a relation of Q_G). In Lemma 3.4.7 we describe the elements x_{ij} in terms of the derivatives of W_G, in a way that depends on the type of α. This will be the last ingredient we need in order to prove Theorem 3.3.7. We advise the reader to compare Lemma 3.4.7 with the computations carried out in [46, §4.5].

In the proof of Lemma 3.4.7, we will use the following identities.

Lemma 3.4.5. *If $\alpha \in Q_1$ and β is an arrow of type (4), then*

$$(\alpha \otimes e_\mu)(\beta \otimes e_\nu) = \begin{cases} \alpha\beta \otimes e_\nu, & \text{if } \nu = \mu + b(\beta); \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We compute

$$\begin{aligned}(\alpha \otimes e_\mu)(\beta \otimes e_\nu) &= \frac{1}{n}\sum_{i=0}^{n-1} \zeta^{i\mu}(\alpha \otimes g^i)(\beta \otimes e_\nu) = \\ &= \frac{1}{n}\sum_{i=0}^{n-1} \zeta^{i\mu}\alpha g^i(\beta) \otimes g^i e_\nu = \\ &= \frac{1}{n}\sum_{i=0}^{n-1} \zeta^{i(\mu+b(\beta))}\alpha\beta \otimes g^i e_\nu = \\ &= \alpha\beta \otimes \frac{1}{n}\sum_{i=0}^{n-1} \zeta^{i(\mu+b(\beta))} g^i e_\nu = \\ &= \alpha\beta \otimes e_{\mu+b(\beta)} e_\nu\end{aligned}$$

and this proves the claim. □

Lemma 3.4.6. *If c is a cycle of type (iii), then $a(g * c) = \zeta^{q(c)} a(c)$.*

Proof. From assumption (A5), it follows that $g(a(c)c) = a(g * c) g * c$. Then we get the claim since $g(c) = \zeta^{q(c)} g * c$. □

Now we use the identification $\Bbbk Q_G \cong \eta((\Bbbk Q) * G)\eta$ to see cyclic derivatives of W_G as elements of $\eta((\Bbbk Q) * G)\eta$. To avoid clogging the notation, we will at times write $h\alpha$ and hc instead of $h(\alpha)$ and $h(c)$ for $h \in G$.

Lemma 3.4.7. *1. Let α be an arrow of Q of type (1). Let $\beta = g^{-t(\alpha)}(\alpha)$. Then*

$$\partial_\beta W \otimes g^{-t(\alpha)} = \partial_{\tilde{\alpha}} W_G.$$

2. *Let α be an arrow of Q of type (2). Then*

$$\partial_\alpha W \otimes 1 = \sum_{\mu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} W_G.$$

In particular,

$$\eta^{\mathfrak{s}(\alpha)}(\partial_\alpha W \otimes 1)\eta_\mu^{\mathfrak{t}(\alpha)} = \partial_{\tilde{\alpha}^\mu} W_G$$

for every $\mu = 0, \dots, n-1$.

3. *Let α be an arrow of Q of type (3). Then*

$$\partial_\alpha W \otimes 1 = \sum_{\mu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} W_G.$$

In particular,

$$\eta_\mu^{\mathfrak{s}(\alpha)}(\partial_\alpha W \otimes 1)\eta^{\mathfrak{t}(\alpha)} = \partial_{\tilde{\alpha}^\mu} W_G$$

for every $\mu = 0, \dots, n-1$.

4. *Let α be an arrow of type (4). Then*

$$\partial_\alpha W \otimes 1 = n\sum_{\mu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} W_G.$$

In particular,

$$\eta_\mu^{\mathfrak{s}(\alpha)}(\partial_\alpha W \otimes 1)\eta_{\mu-b(\alpha)}^{\mathfrak{t}(\alpha)} = n\partial_{\tilde{\alpha}^\mu} W_G$$

for every $\mu = 0, \dots, n-1$.

Proof. First notice that the second part of statements (2), (3), (4) follows directly by multiplying $\sum \partial_{\tilde{\alpha}^\mu} W_G$, which is a linear combination of paths in Q_G, with idempotents corresponding to vertices of Q_G.

It will be convenient to use the following notation: for integers $t_1, \dots, t_l$, write

$$t_{i,j} = \begin{cases} t_i + t_{i+1} + \cdots + t_j, \text{ if } j \geq i; \\ t_i + t_{i+1} + \cdots + t_l + t_1 + t_2 + \cdots + t_j, \text{ if } j < i. \end{cases}$$

1. We have that

$$\partial_\beta W \otimes g^{-t(\alpha)} = \sum_{c \text{ of type (i)}} a(c)\partial_\beta c \otimes g^{-t(\alpha)} + \sum_{c \text{ of type (ii)}} a(c)\partial_\beta c \otimes g^{-t(\alpha)}$$

and

$$\partial_{\tilde{\alpha}} W_G = \frac{|Gc|}{n} \sum_{c \in \mathcal{C}(\mathrm{i})} a(c) \partial_{\tilde{\alpha}} \tilde{c} + \sum_{c \in \mathcal{C}(\mathrm{ii})} a(c) \sum_{\mu=0}^{n-1} \zeta^{-p(c)\mu} \partial_{\tilde{\alpha}} \tilde{c}^{\mu}.$$

The statement will be proved using the following two claims:

Claim (a1). If $c \in \mathcal{C}(\mathrm{i})$, then

$$\sum_{r=0}^{n-1} \partial_{\beta} g^r c \otimes g^{-t(\alpha)} = \partial_{\tilde{\alpha}} \tilde{c}.$$

Claim (b1). If $c \in \mathcal{C}(\mathrm{ii})$, then

$$\sum_{r=0}^{n-1} \partial_{\beta} g^r c \otimes g^{-t(\alpha)} = \sum_{\mu=0}^{n-1} \zeta^{-p(c)\mu} \partial_{\tilde{\alpha}^{\mu}} \tilde{c}^{\mu}.$$

Assuming these claims hold, let us prove the statement. Recall that by assumption (A6), $gc = g * c$ if c is of type (i) or (ii). We have

$$\begin{aligned}
\sum_{c \text{ of type (i)}} a(c) \partial_{\beta} c \otimes g^{-t(\alpha)} &= \sum_{c \in \mathcal{C}(\mathrm{i})} \sum_{r=0}^{|Gc|-1} a(g^r * c) \partial_{\beta} (g^r * c) \otimes g^{-t(\alpha)} = \\
&= \sum_{c \in \mathcal{C}(\mathrm{i})} \frac{|Gc|}{n} \frac{n}{|Gc|} \sum_{r=0}^{|Gc|-1} a(c) \partial_{\beta} g^r c \otimes g^{-t(\alpha)} = \\
&= \sum_{c \in \mathcal{C}(\mathrm{i})} \frac{|Gc|}{n} \sum_{r=0}^{n-1} a(c) \partial_{\beta} g^r c \otimes g^{-t(\alpha)} = \\
&= \frac{|Gc|}{n} \sum_{c \in \mathcal{C}(\mathrm{i})} a(c) \partial_{\tilde{\alpha}} \tilde{c}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{c \text{ of type (ii)}} a(c) \partial_{\beta} c \otimes g^{-t(\alpha)} &= \sum_{c \in \mathcal{C}(\mathrm{ii})} \sum_{r=0}^{|Gc|-1} a(g^r * c) \partial_{\beta} (g^r * c) \otimes g^{-t(\alpha)} = \\
&= \sum_{c \in \mathcal{C}(\mathrm{ii})} \sum_{r=0}^{n-1} a(c) \partial_{\beta} g^r c \otimes g^{-t(\alpha)} = \\
&= \sum_{c \in \mathcal{C}(\mathrm{ii})} a(c) \sum_{\mu=0}^{n-1} \zeta^{-p(c)\mu} \partial_{\tilde{\alpha}^{\mu}} \tilde{c}^{\mu}
\end{aligned}$$

which together imply that

$$\partial_\beta W \otimes g^{-t(\alpha)} = \partial_{\tilde{\alpha}} W_G.$$

It remains to prove the claims (a1) and (b1).

Proof of (a1). Since $c \in \mathcal{C}(\mathrm{i})$ we can write

$$c: \qquad \varepsilon_0 = \varepsilon_l \xrightarrow{g^{t_{1,l-1}}(\alpha_l)} g^{t_{1,l-1}}(\varepsilon_{l-1}) \longrightarrow \cdots \longrightarrow g^{t_1}(\varepsilon_1) \xrightarrow{\alpha_1} \varepsilon_0.$$

Let $M = \{m \in \{1,\dots,l\} \mid \alpha = \alpha_m\}$. Then

$$\begin{aligned}\partial_{\tilde{\alpha}}\tilde{c} = \partial_{\tilde{\alpha}}\tilde{\alpha}_1\cdots\tilde{\alpha}_l &= \sum_{m\in M}\tilde{\alpha}_{m+1}\cdots\tilde{\alpha}_{m-1} = \\ &= \sum_{m\in M}(\alpha_{m+1}\otimes g^{t_{m+1}})\cdots(\alpha_{m-1}\otimes g^{t_{m-1}}) = \\ &= \sum_{m\in M}\alpha_{m+1}g^{t_{m+1}}(\alpha_{m+2})\cdots g^{t_{m+1,m-2}}(\alpha_{m-1})\otimes g^{-t_m}.\end{aligned}$$

Note that $t_m = t(\alpha)$ for all $m \in M$, so we are left to prove that

$$\sum_{m\in M}\alpha_{m+1}g^{t_{m+1}}(\alpha_{m+2})\cdots g^{t_{m+1,m-2}}(\alpha_{m-1}) = \sum_{r=0}^{n-1}\partial_\beta g^r c.$$

For each $r = 0,\dots,n-1$ and $m \in M$, the path $g^r c$ contains the arrow $g^{r+t_{1,m-1}}\alpha_m = g^{r+t_{1,m}}\beta$. Hence, if we define $M_r = \{m \in M \mid r = -t_{1,m}\}$, we have that

$$\partial_\beta g^r c = \sum_{m\in M_r}\alpha_{m+1}g^{t_{m+1}}(\alpha_{m+2})\cdots g^{t_{m+1,m-2}}(\alpha_{m-1}).$$

So the equality we wanted to show becomes

$$\begin{aligned}&\sum_{m\in M}\alpha_{m+1}g^{t_{m+1}}(\alpha_{m+2})\cdots g^{t_{m+1,m-2}}(\alpha_{m-1}) = \\ &= \sum_{r=0}^{n-1}\sum_{m\in M_r}\alpha_{m+1}g^{t_{m+1}}(\alpha_{m+2})\cdots g^{t_{m+1,m-2}}(\alpha_{m-1}),\end{aligned}$$

but this holds because $M = \bigsqcup_{r=0}^{n-1} M_r$.

Proof of (b1). Since $c \in \mathcal{C}(\mathrm{ii})$ we can write

$$c: \varepsilon_0 = \varepsilon_l \xrightarrow{g^{t_{1,l-1}}(\alpha_l)} g^{t_{1,l-1}}(\varepsilon_{l-1}) \longrightarrow \cdots \longrightarrow g^{t_2}(\varepsilon_2) \xrightarrow{\alpha_2} \varepsilon_1^{\alpha_1} \longrightarrow \varepsilon_0.$$

Recall that by definition $p(c) = t_2$. Let $M = \{m \in \{1, \ldots, l\} \mid \alpha = \alpha_m\}$. Then

$$\begin{aligned}\partial_{\tilde{\alpha}}\tilde{c}^{\mu} &= \partial_{\tilde{\alpha}}\tilde{\alpha}_1^{\mu}\widetilde{g^{-p(c)}(\alpha_2)}^{\mu}\tilde{\alpha}_3\cdots\tilde{\alpha}_l = \\ &= \sum_{m\in M}\tilde{\alpha}_{m+1}\cdots\tilde{\alpha}_1^{\mu}\widetilde{g^{-p(c)}(\alpha_2)}^{\mu}\tilde{\alpha}_3\cdots\tilde{\alpha}_{m-1} = \\ &= \sum_{m\in M}(\alpha_{m+1}\otimes g^{t_{m+1}})\cdots(\alpha_1\otimes e_{\mu})(g^{-t_2}(\alpha_2)\otimes 1)\cdots(\alpha_{m-1}\otimes g^{t_{m-1}}).\end{aligned}$$

Now, recalling that $\sum_{\mu=0}^{n-1} e_{\mu} = 1$ and $\zeta^{-t_2\mu}e_{\mu} = g^{t_2}e_{\mu}$, we get

$$\begin{aligned}\sum_{\mu=0}^{n-1}\zeta^{-t_2\mu}\partial_{\tilde{\alpha}}\tilde{c}^{\mu} &= \sum_{m\in M}\sum_{\mu=0}^{n-1}(\alpha_{m+1}\otimes g^{t_{m+1}})\cdots(\alpha_1\otimes\zeta^{-t_2\mu}e_{\mu})(g^{-t_2}(\alpha_2)\otimes 1)\cdots \\ &\quad\cdots(\alpha_{m-1}\otimes g^{t_{m-1}}) = \\ &= \sum_{m\in M}(\alpha_{m+1}\otimes g^{t_{m+1}})\cdots(\alpha_1\otimes g^{t_2})(g^{-t_2}(\alpha_2)\otimes 1)\cdots(\alpha_{m-1}\otimes g^{t_{m-1}}) = \\ &= \sum_{m\in M}(\alpha_{m+1}\otimes g^{t_{m+1}})\cdots(\alpha_1\otimes g^{t_1})(\alpha_2\otimes g^{t_2})\cdots(\alpha_{m-1}\otimes g^{t_{m-1}}) = \\ &= \sum_{m\in M}\alpha_{m+1}g^{t_{m+1}}(\alpha_{m+2})\cdots g^{t_{m+1,m-2}}(\alpha_{m-1})\otimes g^{-t_m}.\end{aligned}$$

The rest of the proof of part (b1) is analogous to that of part (a1).

2. We have that

$$\partial_{\alpha}W\otimes 1 = \sum_{c\text{ of type (ii)}} a(c)\partial_{\alpha}c\otimes 1 + \sum_{c\text{ of type (iii)}} a(c)\partial_{\alpha}c\otimes 1$$

and

$$\sum_{\mu=0}^{n-1}\partial_{\tilde{\alpha}^{\mu}}W_G = \sum_{\mu=0}^{n-1}\sum_{c\in\mathcal{C}(\text{ii})}a(c)\sum_{\nu=0}^{n-1}\zeta^{-p(c)\nu}\partial_{\tilde{\alpha}^{\mu}}\tilde{c}^{\nu} + \sum_{\mu=0}^{n-1}\sum_{c\in\mathcal{C}(\text{iii})}a(c)\sum_{\nu=0}^{n-1}\partial_{\tilde{\alpha}^{\mu}}\tilde{c}^{\nu}.$$

The statement will be proved using the following two claims:

Claim (a2). If $c \in \mathcal{C}(\text{ii})$, then $\partial_{\tilde{\alpha}^{\mu}}\tilde{c}^{\nu} = 0$ for $\mu \neq \nu$ and

$$\sum_{r=0}^{n-1}\partial_{\alpha}g^{r}c\otimes 1 = \sum_{\mu=0}^{n-1}\zeta^{-p(c)\mu}\partial_{\tilde{\alpha}^{\mu}}\tilde{c}^{\mu}.$$

Claim (b2). If $c \in \mathcal{C}(\text{iii})$, then $\partial_{\tilde{\alpha}^{\mu}}\tilde{c}^{\nu} = 0$ for $\mu \neq \nu$ and

$$\partial_{\alpha}c\otimes 1 = \sum_{\mu=0}^{n-1}\partial_{\tilde{\alpha}^{\mu}}\tilde{c}^{\mu}.$$

Assuming these claims hold, let us prove the statement. First notice that if $c \in \mathcal{C}(\text{iii})$ and $\alpha \in h * c$, then $h = 1$. We have

$$\begin{aligned}\sum_{c \text{ of type (ii)}} a(c)\partial_\alpha c \otimes 1 &= \sum_{c\in\mathcal{C}(\text{ii})} \sum_{r=0}^{|Gc|-1} a(g^r * c)\partial_\alpha (g^r * c) \otimes 1 = \\ &= \sum_{c\in\mathcal{C}(\text{ii})} \sum_{r=0}^{n-1} a(c)\partial_\alpha g^r c \otimes 1 = \\ &= \sum_{c\in\mathcal{C}(\text{ii})} a(c) \sum_{\mu=0}^{n-1} \zeta^{-p(c)\mu} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\mu = \\ &= \sum_{\mu=0}^{n-1} \sum_{c\in\mathcal{C}(\text{ii})} a(c) \sum_{\nu=0}^{n-1} \zeta^{-p(c)\nu} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu\end{aligned}$$

and

$$\begin{aligned}\sum_{c \text{ of type (iii)}} a(c)\partial_\alpha c \otimes 1 &= \sum_{c\in\mathcal{C}(\text{iii})} \sum_{r=0}^{|Gc|-1} a(g^r * c)\partial_\alpha (g^r * c) \otimes 1 = \\ &= \sum_{c\in\mathcal{C}(\text{iii})} a(c)\partial_\alpha c \otimes 1 = \\ &= \sum_{c\in\mathcal{C}(\text{iii})} a(c) \sum_{\mu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\mu = \\ &= \sum_{\mu=0}^{n-1} \sum_{c\in\mathcal{C}(\text{iii})} a(c) \sum_{\nu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu\end{aligned}$$

which together imply that

$$\partial_\alpha W \otimes 1 = \sum_{\mu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} W_G.$$

It remains to prove the claims (a2) and (b2).

Proof of (a2). Since $c \in \mathcal{C}(\text{ii})$, we can write it as

$$c : \varepsilon_0 = \varepsilon_l \xrightarrow{g^{t_{1,l-1}}(\alpha_l)} g^{t_{1,l-1}}(\varepsilon_{l-1}) \longrightarrow \cdots \longrightarrow g^{t_2}(\varepsilon_2) \xrightarrow{\alpha_2} \varepsilon_1 \xrightarrow{\alpha_1} \varepsilon_0.$$

If $\alpha \notin g^r c$ for all r then the statement is trivially true. Otherwise, suppose $\alpha \in g^r c$ for some r. Then, since α is of type (2), we necessarily have that $r = -t_2$ and $\alpha = g^{-t_2}(\alpha_2)$ is the only copy of α in $g^{-t_2}c$. Hence

$$\tilde{c}^\nu = \tilde{\alpha}_1^\nu \widetilde{g^{-t_2}(\alpha_2)}^{\nu} \tilde{\alpha}_3 \cdots \tilde{\alpha}_l = \tilde{\alpha}_1^\nu \tilde{\alpha}^\nu \tilde{\alpha}_3 \cdots \tilde{\alpha}_l$$

and $\partial_{\tilde{\alpha}^\mu}\tilde{c}^\nu = 0$ for $\mu \neq \nu$. We have

$$\begin{aligned}\partial_{\tilde{\alpha}^\mu}\tilde{c}^\mu &= \partial_{\tilde{\alpha}^\mu}\tilde{\alpha}^\mu\tilde{\alpha}_3\cdots\tilde{\alpha}_l\tilde{\alpha}_1^\mu = \\ &= \tilde{\alpha}_3\cdots\tilde{\alpha}_l\tilde{\alpha}_1^\mu = \\ &= (\alpha_3\otimes g^{t_3})\cdots(\alpha_l\otimes g^{t_l})(\alpha_1\otimes e_\mu)\end{aligned}$$

so that (recall that $t_{2,l} = 0 \pmod n$)

$$\begin{aligned}\sum_{\mu=0}^{n-1}\zeta^{-t_2\mu}\partial_{\tilde{\alpha}^\mu}\tilde{c}^\mu &= \sum_{\mu=0}^{n-1}(\alpha_3\otimes g^{t_3})\cdots(\alpha_l\otimes g^{t_l})(\alpha_1\otimes g^{t_2}e_\mu) = \\ &= \sum_{\mu=0}^{n-1}(\alpha_3 g^{t_3}(\alpha_4)\cdots g^{t_{3,l-1}}(\alpha_l)\otimes g^{-t_2})(\alpha_1\otimes e_\mu)(1\otimes g^{t_2}) = \\ &= \alpha_3 g^{t_3}(\alpha_4)\cdots g^{t_{3,l-1}}(\alpha_l)g^{t_{3,l}}(\alpha_1)\otimes 1 = \\ &= \partial_\alpha g^{-t_2}c\otimes 1 = \\ &= \sum_{r=0}^{n-1}\partial_\alpha g^r c\otimes 1,\end{aligned}$$

which proves the claim.

Proof of (b2). We have, since $c \in \mathcal{C}(\text{iii})$,

$$c: \qquad \varepsilon_0 = \varepsilon_l \xrightarrow{\alpha_l} \varepsilon_{l-1} \longrightarrow \cdots \longrightarrow \varepsilon_1 \xrightarrow{\alpha_1} \varepsilon_0,$$

with $\alpha = \alpha_1$, and observe that this is the only instance of α in c. Setting $b_i = b(\alpha_i) + \cdots + b(\alpha_l)$ for $i \geq 3$, we have $\tilde{c}^\nu = \tilde{\alpha}^\nu\tilde{\alpha}_2^{\nu-b_3}\tilde{\alpha}_3^{\nu-b_4}\cdots\tilde{\alpha}_l^\nu$, so $\partial_{\tilde{\alpha}^\mu}\tilde{c}^\nu = 0$ for $\mu \neq \nu$. We can compute

$$\begin{aligned}\partial_{\tilde{\alpha}^\mu}\tilde{c}^\mu &= \partial_{\tilde{\alpha}^\mu}\tilde{\alpha}^\mu\tilde{\alpha}_2^{\mu-b_3}\tilde{\alpha}_3^{\mu-b_4}\cdots\tilde{\alpha}_l^\mu = \\ &= \tilde{\alpha}_2^{\mu-b_3}\tilde{\alpha}_3^{\mu-b_4}\cdots\tilde{\alpha}_l^\mu = \\ &= (\alpha_2\otimes e_{\mu-b_3})(\alpha_3\otimes e_{\mu-b_4})\cdots(\alpha_l\otimes e_\mu) = \\ &= \alpha_2\alpha_3\cdots\alpha_l\otimes e_\mu = \\ &= \partial_\alpha c\otimes e_\mu\end{aligned}$$

so that

$$\sum_{\mu=0}^{n-1}\partial_{\tilde{\alpha}^\mu}\tilde{c}^\mu = \partial_\alpha c\otimes 1$$

as claimed.

3. We have that

$$\partial_\alpha W \otimes 1 = \sum_{c \text{ of type (ii)}} a(c)\partial_\alpha c \otimes 1 + \sum_{c \text{ of type (iii)}} a(c)\partial_\alpha c \otimes 1$$

and

$$\sum_{\mu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} W_G = \sum_{\mu=0}^{n-1} \sum_{c\in\mathcal{C}(\text{ii})} a(c) \sum_{\nu=0}^{n-1} \zeta^{-p(c)\nu} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu + \sum_{\mu=0}^{n-1} \sum_{c\in\mathcal{C}(\text{iii})} a(c) \sum_{\nu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu.$$

The statement will be proved using the following two claims:

Claim (a3). If $c \in \mathcal{C}(\text{ii})$, then $\partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu = 0$ for $\mu \neq \nu$ and

$$\partial_\alpha c \otimes 1 = \sum_{\mu=0}^{n-1} \zeta^{-p(c)\mu} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\mu.$$

Claim (b3). If $c \in \mathcal{C}(\text{iii})$, then $\partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu = 0$ for $\mu \neq \nu - q(c)$ and

$$\partial_\alpha c \otimes 1 = \sum_{\mu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} \tilde{c}^{\mu+q(c)}.$$

Assuming these claims hold, let us prove the statement. First notice that if $c \in \mathcal{C}(\text{ii}) \cup \mathcal{C}(\text{iii})$ and $\alpha \in h * c$, then $h = 1$. We have

$$\begin{aligned}
\sum_{c \text{ of type (ii)}} a(c)\partial_\alpha c \otimes 1 &= \sum_{c\in\mathcal{C}(\text{ii})} \sum_{r=0}^{|Gc|-1} a(g^r * c)\partial_\alpha (g^r * c) \otimes 1 = \\
&= \sum_{c\in\mathcal{C}(\text{ii})} a(c)\partial_\alpha c \otimes 1 = \\
&= \sum_{c\in\mathcal{C}(\text{ii})} a(c) \sum_{\mu=0}^{n-1} \zeta^{-p(c)\mu} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\mu = \\
&= \sum_{\mu=0}^{n-1} \sum_{c\in\mathcal{C}(\text{ii})} a(c) \sum_{\nu=0}^{n-1} \zeta^{-p(c)\nu} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu
\end{aligned}$$

and

$$\sum_{c \text{ of type (iii)}} a(c)\partial_\alpha c \otimes 1 = \sum_{c\in\mathcal{C}(\text{iii})} \sum_{r=0}^{|Gc|-1} a(g^r * c)\partial_\alpha(g^r * c) \otimes 1 =$$
$$= \sum_{c\in\mathcal{C}(\text{iii})} a(c)\partial_\alpha c \otimes 1 =$$
$$= \sum_{c\in\mathcal{C}(\text{iii})} a(c) \sum_{\mu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} \tilde{c}^{\mu+q(c)} =$$
$$= \sum_{\mu=0}^{n-1} \sum_{c\in\mathcal{C}(\text{iii})} a(c) \sum_{\nu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu$$

which together imply that

$$\partial_\alpha W \otimes 1 = \sum_{\mu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} W_G.$$

It remains to prove the claims (a3) and (b3).

Proof of (a3). We have, for $c \in \mathcal{C}(\text{ii})$,

$$c : \varepsilon_0 = \varepsilon_l \xrightarrow{g^{t_{1,l-1}}(\alpha_l)} g^{t_{1,l-1}}(\varepsilon_{l-1}) \longrightarrow \cdots \longrightarrow g^{t_2}(\varepsilon_2) \xrightarrow{\alpha_2} \varepsilon_1 \xrightarrow{\alpha_1} \varepsilon_0,$$

where $\alpha = \alpha_1$, and this is the only copy of α_1 in c. Hence $\tilde{c}^\nu = \tilde{\alpha}^\nu \widetilde{g^{-t_2}(\alpha_2)}^{\nu} \tilde{\alpha}_3 \cdots \tilde{\alpha}_l$ and $\partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu = 0$ for $\mu \neq \nu$. Then

$$\partial_{\tilde{\alpha}^\mu} \tilde{c}^\mu = \partial_{\tilde{\alpha}^\mu} \tilde{\alpha}^\mu \widetilde{g^{-t_2}(\alpha_2)} \tilde{\alpha}_3 \cdots \tilde{\alpha}_l =$$
$$= \widetilde{g^{-t_2}(\alpha_2)} \tilde{\alpha}_3 \cdots \tilde{\alpha}_l =$$
$$= (1 \otimes e_\mu)(g^{-t_2}(\alpha_2) \otimes 1)(\alpha_3 \otimes g^{t_3}) \cdots (\alpha_l \otimes g^{t_l})$$

and so

$$\sum_{\mu=0}^{n-1} \zeta^{-t_2\mu} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\mu = \sum_{\mu=0}^{n-1} (1 \otimes e_\mu)(1 \otimes g^{t_2})(g^{-t_2}(\alpha_2) \otimes 1)(\alpha_3 \otimes g^{t_3}) \cdots (\alpha_l \otimes g^{t_l}) =$$
$$= (\alpha_2 \otimes g^{t_2})(\alpha_3 \otimes g^{t_3}) \cdots (\alpha_l \otimes g^{t_l}) =$$
$$= \partial_\alpha c \otimes 1$$

as claimed.

Proof of (b3). We have

$$c : \qquad \varepsilon_0 = \varepsilon_l \xrightarrow{\alpha_l} \varepsilon_{l-1} \longrightarrow \cdots \longrightarrow \varepsilon_1 \xrightarrow{\alpha_1} \varepsilon_0,$$

with $\alpha = \alpha_2$, and again observe that this is the only instance of α in c. Write $b_i = b(\alpha_i) + \cdots + b(\alpha_l)$ for $i \geq 3$, and recall that $b_3 = q(c)$. Then $\tilde{c}^\nu = \tilde{\alpha}_1^\nu \tilde{\alpha}^{\nu-b_3} \tilde{\alpha}_3^{\nu-b_4} \cdots \tilde{\alpha}_l^\nu$, so $\partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu = 0$ for $\mu \neq \nu - q(c)$. Hence

$$\begin{aligned}
\partial_{\tilde{\alpha}^\mu} \tilde{c}^{\mu+q(c)} &= \partial_{\tilde{\alpha}^\mu} \tilde{\alpha}_1^{\mu+b_3} \tilde{\alpha}^\mu \tilde{\alpha}_3^{\mu+b_3-b_4} \cdots \tilde{\alpha}_l^{\mu+b_3} = \\
&= \tilde{\alpha}_1^{\mu+b_3} \tilde{\alpha}^\mu \tilde{\alpha}_3^{\mu+b_3-b_4} \cdots \tilde{\alpha}_l^{\mu+b_3} \tilde{\alpha}_1^{\mu+b_3} = \\
&= (\alpha_3 \otimes e_{\mu+b_3-b_4}) \cdots (\alpha_l \otimes e_{\mu+b_3})(1 \otimes e_{\mu+b_3})(\alpha_1 \otimes 1) = \\
&= (\alpha_3 \cdots \alpha_l \otimes e_{\mu+b_3})(1 \otimes e_{\mu+b_3})(\alpha_1 \otimes 1) = \\
&= (\alpha_3 \cdots \alpha_l \otimes e_{\mu+b_3})(\alpha_1 \otimes 1)
\end{aligned}$$

so

$$\sum_{\mu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} \tilde{c}^{\mu+q(c)} = \alpha_3 \cdots \alpha_l \alpha_1 \otimes 1 = \partial_\alpha c \otimes 1$$

which concludes the proof.

4. We have that

$$\partial_\alpha W \otimes 1 = \sum_{c \text{ of type (iii)}} a(c) \partial_\alpha c \otimes 1 + \sum_{c \text{ of type (iv)}} a(c) \partial_\alpha c \otimes 1$$

and

$$n \sum_{\mu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} W_G = n \sum_{\mu=0}^{n-1} \sum_{c \in \mathcal{C}(\text{iii})} a(c) \sum_{\nu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu + n \sum_{\mu=0}^{n-1} \sum_{c \in \mathcal{C}(\text{iv})} a(c) \sum_{\nu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu.$$

The statement will be proved using the following two claims:

Claim (a4). If $c \in \mathcal{C}(\text{iii})$, then

$$\sum_{r=0}^{n-1} \partial_\alpha g^r c \otimes 1 = \sum_{\mu=0}^{n-1} \sum_{\nu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu.$$

Claim (b4). If $c \in \mathcal{C}(\text{iv})$, then

$$\partial_\alpha c \otimes 1 = \sum_{\mu=0}^{n-1} \sum_{\nu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu.$$

Assuming these claims hold, let us prove the statement. We have that

$$\sum_{c \text{ of type (iii)}} a(c)\partial_\alpha c \otimes 1 = \sum_{c\in\mathcal{C}(\text{iii})}\sum_{r=0}^{n-1} a(g^r * c)\partial_\alpha(g^r * c) \otimes 1 =$$
$$= \sum_{c\in\mathcal{C}(\text{iii})}\sum_{r=0}^{n-1} \zeta^{rq(c)} a(c)\partial_\alpha(g^r * c) \otimes 1 =$$
$$= \sum_{c\in\mathcal{C}(\text{iii})}\sum_{r=0}^{n-1} a(c)\partial_\alpha g^r c \otimes 1 =$$
$$= n\sum_{\mu=0}^{n-1}\sum_{c\in\mathcal{C}(\text{iii})} a(c)\sum_{\nu=0}^{n-1}\partial_{\tilde{\alpha}^\mu}\tilde{c}^\nu$$

and

$$\sum_{c \text{ of type (iv)}} a(c)\partial_\alpha c \otimes 1 = \sum_{c\in\mathcal{C}(\text{iv})}\sum_{r=0}^{n-1} a(g^r * c)\partial_\alpha(g^r * c) \otimes 1 =$$
$$= \sum_{c\in\mathcal{C}(\text{iv})}\sum_{r=0}^{n-1} a(c)\partial_\alpha c \otimes 1 =$$
$$= n\sum_{c\in\mathcal{C}(\text{iv})} a(c)\partial_\alpha c \otimes 1 =$$
$$= n\sum_{\mu=0}^{n-1}\sum_{c\in\mathcal{C}(\text{iv})} a(c)\sum_{\nu=0}^{n-1}\partial_{\tilde{\alpha}^\mu}\tilde{c}^\nu$$

which together imply that

$$\partial_\alpha W \otimes 1 = n\sum_{\mu=0}^{n-1}\partial_{\tilde{\alpha}^\mu} W_G.$$

It remains to prove the claims (a4) and (b4).

Proof of (a4). Let us write, for $c \in \mathcal{C}(\text{iii})$,

$$c: \qquad \varepsilon_0 = \varepsilon_l \xrightarrow{\alpha_l} \varepsilon_{l-1} \longrightarrow \cdots \longrightarrow \varepsilon_1 \xrightarrow{\alpha_1} \varepsilon_0,$$

where $\varepsilon_1 \in \mathcal{E}'$ and $\varepsilon_i \in \mathcal{E}''$ for $i \neq 1$.

Let $M = \{m \in \{1,\dots,l\} \mid \alpha = \alpha_m\}$ and put $b_i = b(\alpha_i) + \cdots + b(\alpha_l)$ for all $i \geq 3$. We have

$$\tilde{c}^\nu: \qquad \eta_\nu^{\varepsilon_l} \xrightarrow{\tilde{\alpha}_l^\nu} \eta_{\nu-b_l}^{\varepsilon_{l-1}} \xrightarrow{\tilde{\alpha}_{l-1}^{\nu-b_l}} \cdots \xrightarrow{\tilde{\alpha}_2^{\nu-b_3}} \eta^{\varepsilon_1} \xrightarrow{\tilde{\alpha}_1^\nu} \eta_\nu^{\varepsilon_0},$$

so we may note that, if $m \in M$, the m-th arrow of $\tilde{c}^{\nu}$ is $\tilde{\alpha}_m^{\nu-b_{m+1}}$, and it coincides with $\tilde{\alpha}^{\mu}$ if and only if $\nu = \mu + b_{m+1}$. Hence

$$\begin{aligned}
\sum_{\mu=0}^{n-1}\sum_{\nu=0}^{n-1}\partial_{\tilde{\alpha}^{\mu}}\tilde{c}^{\nu} &= \sum_{\mu=0}^{n-1}\sum_{\nu=0}^{n-1}\partial_{\tilde{\alpha}^{\mu}}\tilde{\alpha}_1^{\nu}\tilde{\alpha}_2^{\nu-b_3}\tilde{\alpha}_3^{\nu-b_4}\cdots\tilde{\alpha}_{l-1}^{\nu-b_l}\tilde{\alpha}_l^{\nu} = \\
&= \sum_{\mu=0}^{n-1}\sum_{m\in M}\tilde{\alpha}_{m+1}^{\mu+b_{m+1}-b_{m+2}}\cdots\tilde{\alpha}_l^{\mu+b_{m+1}}\tilde{\alpha}_1^{\mu+b_{m+1}}\tilde{\alpha}_2^{\mu+b_{m+1}-b_3}\cdots\tilde{\alpha}_{m-1}^{\mu+b_{m+1}-b_m} = \\
&= \sum_{\mu=0}^{n-1}\sum_{m\in M}(\alpha_{m+1}\otimes e_{\mu+b_{m+1}-b_{m+2}})\cdots(\alpha_l\otimes e_{\mu+b_{m+1}}) \\
&\quad (1\otimes e_{\mu+b_{m+1}})(\alpha_1\otimes 1)(\alpha_2\otimes e_{\mu+b_{m+1}-b_3})\cdots(\alpha_{m-1}\otimes e_{\mu+b_{m+1}-b_m}) = \\
&= \sum_{\mu=0}^{n-1}\sum_{m\in M}(\alpha_{m+1}\cdots\alpha_l\otimes e_{\mu+b_{m+1}})(\alpha_1\alpha_2\cdots\alpha_{m-1}\otimes e_{\mu+b_{m+1}-b_m}) = \\
&= \sum_{\mu=0}^{n-1}\sum_{m\in M}\frac{1}{n}\sum_{i=0}^{n-1}\zeta^{i(\mu+b_{m+1})}(\alpha_{m+1}\cdots\alpha_l\otimes g^i)(\alpha_1\alpha_2\cdots\alpha_{m-1}\otimes e_{\mu+b_{m+1}-b_m}) = \\
&= \sum_{\mu=0}^{n-1}\sum_{m\in M}\frac{1}{n}\sum_{i=0}^{n-1}\zeta^{i(\mu+b_{m+1})}\alpha_{m+1}\cdots\alpha_l g^i(\alpha_1\alpha_2\cdots\alpha_{m-1})\otimes g^i e_{\mu+b_{m+1}-b_m} = \\
&= \sum_{\mu=0}^{n-1}\sum_{m\in M}\frac{1}{n}\sum_{i=0}^{n-1}\zeta^{ib_m}\alpha_{m+1}\cdots\alpha_l g^i(\alpha_1\alpha_2\cdots\alpha_{m-1})\otimes e_{\mu+b_{m+1}-b_m} = \\
&= \sum_{\mu=0}^{n-1}\sum_{m\in M}\frac{1}{n}\sum_{i=0}^{n-1}\zeta^{ib_3}\alpha_{m+1}\cdots\alpha_l g^i(\alpha_1\alpha_2)\cdots\alpha_{m-1}\otimes e_{\mu+b_{m+1}-b_m} = \\
&= \sum_{\mu=0}^{n-1}\sum_{m\in M}\frac{1}{n}\sum_{i=0}^{n-1}\partial_{\alpha}g^i c\otimes e_{\mu+b_{m+1}-b_m} = \\
&= \sum_{m\in M}\frac{1}{n}\sum_{i=0}^{n-1}\partial_{\alpha}g^i c\otimes 1 = \\
&= \frac{1}{n}\sum_{i=0}^{n-1}\partial_{\alpha}g^i c\otimes 1
\end{aligned}$$

which is what we wanted to prove.

Proof of (b4). Let

$$c \quad : \quad \varepsilon_0 = \varepsilon_l \xrightarrow{\alpha_l} \varepsilon_{l-1} \longrightarrow \cdots \longrightarrow \varepsilon_1 \xrightarrow{\alpha_1} \varepsilon_0,$$

where $\varepsilon_i \in \mathcal{E}''$ for all $i = 1, \ldots, l$. Let $M = \{m \in \{1, \ldots, l\} \,|\, \alpha = \alpha_m\}$, and put as usual $b_i = b(\alpha_i) + \cdots + b(\alpha_l)$ for all i. We have

$$\tilde{c}^\nu : \qquad \eta_\nu^{\varepsilon_l} \xrightarrow{\tilde{\alpha}_l^\nu} \eta_{\nu-b_l}^{\varepsilon_{l-1}} \xrightarrow{\tilde{\alpha}_{l-1}^{\nu-b_l}} \cdots \xrightarrow{\tilde{\alpha}_2^{\nu-b_3}} \eta_{\nu-b_2}^{\varepsilon_1} \xrightarrow{\tilde{\alpha}_1^{\nu-b_2}} \eta_\nu^{\varepsilon_0},$$

hence

$$\begin{aligned}
\sum_{\mu=0}^{n-1}\sum_{\nu=0}^{n-1} \partial_{\tilde{\alpha}^\mu}\tilde{c}^\nu &= \sum_{\mu=0}^{n-1}\sum_{\nu=0}^{n-1} \partial_{\tilde{\alpha}^\mu}\tilde{\alpha}_1^{\nu-b_2}\tilde{\alpha}_2^{\nu-b_3}\cdots\tilde{\alpha}_{l-1}^{\nu-b_l}\tilde{\alpha}_l^\nu = \\
&= \sum_{\mu=0}^{n-1}\sum_{m\in M} \tilde{\alpha}_{m+1}^{\mu+b_{m+1}-b_{m+2}}\cdots\tilde{\alpha}_{m-1}^{\mu+b_{m+1}-b_m} = \\
&= \sum_{\mu=0}^{n-1}\sum_{m\in M} (\alpha_{m+1}\otimes e_{\mu+b_{m+1}-b_{m+2}})\cdots(\alpha_{m-1}\otimes e_{\mu+b_{m+1}-b_m}) = \\
&= \sum_{\mu=0}^{n-1}\sum_{m\in M} \alpha_{m+1}\cdots\alpha_{m-1}\otimes e_{\mu+b_{m+1}-b_m} = \\
&= \sum_{m\in M} \alpha_{m+1}\cdots\alpha_{m-1}\otimes 1 = \\
&= \partial_\alpha c\otimes 1,
\end{aligned}$$

and the claim is proved. □

3.4.3 Isomorphism of algebras

We are now ready to prove our main result.

Proof of Theorem 3.3.7. We will first prove that

$$\frac{\Bbbk Q_G}{\langle \partial_\gamma W_G \mid \gamma \in (Q_G)_1\rangle} \cong \eta(\Lambda * G)\eta.$$

By Lemma 3.4.4, the right-hand side is isomorphic to

$$\frac{\eta((\Bbbk Q) * G)\eta}{\langle \partial_{g^{-t(\alpha)}\alpha} W \otimes g^{-t(\alpha)} \mid \alpha \text{ of type } (1),(2),(3),(4)\rangle},$$

and by [57, §2.2,§2.3] we have that $\Bbbk Q_G \cong \eta((\Bbbk Q) * G)\eta$ via the isomorphism J of §3.3.2. For every arrow α of Q of type (1),(2),(3),(4), we can write (recall that for types (2),(3),(4) we set $t(\alpha) = 0$)

$$J^{-1}\left(\partial_{g^{-t(\alpha)}\alpha} W \otimes g^{-t(\alpha)}\right) = \sum_{i,j\in(Q_G)_0} x_{ij}$$

such that x_{ij} are linear combinations of paths from i to j in $\Bbbk Q_G$. By Lemma 3.4.7, every nonzero x_{ij} is associated in $\Bbbk Q_G$ to a unique element of the form $\partial_\gamma W_G$ for some $\gamma \in (Q_G)_1$, and moreover every nonzero $\partial_\gamma W_G$ appears in this way for some α. This means that

$$J\left(\langle \partial_\gamma W_G \mid \gamma \in (Q_G)_1 \rangle\right) = \langle \partial_{g^{-t(\alpha)}\alpha} W \otimes g^{-t(\alpha)} \mid \alpha \text{ of type } (1), (2), (3), (4) \rangle$$

so the claim is proved. Now notice that by Lemma 3.4.2, the ideal $\langle \partial_\gamma W_G \mid \gamma \in (Q_G)_1 \rangle \subseteq \Bbbk Q_G$ is admissible, so by Proposition 3.1.1 we conclude that

$$\mathcal{P}(Q_G, W_G) \cong \frac{\Bbbk Q_G}{\langle \partial_\gamma W_G \mid \gamma \in (Q_G)_1 \rangle}$$

and we are done. □

3.5 Dual group action

It was proved in [57] that we can always recover the algebra Λ from $\Lambda * G$ by applying another skew group algebra construction. In this section we will show that in our case this construction satisfies again the assumptions (A1)-(A7), and the potential we obtain corresponds to the potential we started with.

Let Λ be a finite dimensional algebra and G be a finite abelian group acting on Λ by automorphisms. We denote by $\hat{G}$ the dual group of G. Its elements are the group homomorphism $\chi\colon G \to \Bbbk^*$.

Theorem 3.5.1 ([57, Corollary 5.2]). *Define an action of $\hat{G}$ on $\Lambda * G$ by $\chi(\lambda \otimes g) = \chi(g)\lambda \otimes g$, $\lambda \in \Lambda$, $g \in G$. Then the skew group algebra $(\Lambda * G) * \hat{G}$ is Morita equivalent to Λ.*

We want to apply Theorem 3.5.1 to our setting, so we retain the notation of Section 3.3 (in particular we are assuming that $\Lambda = \mathcal{P}(Q, W)$). Since G is finite and cyclic, there is an isomorphism $G \cong \hat{G}$. We can write $\hat{G} = \{\chi_0, \dots, \chi_{n-1}\}$, where we define χ_μ to be the homomorphism which sends g to ζ^μ. Put $\chi = \chi_1$ and note that it is a generator of $\hat{G}$.

Recall that, by Theorem 3.3.7, we have an isomorphism $\mathcal{P}(Q_G, W_G) \cong \eta(\Lambda * G)\eta$, where $\eta \in \Lambda * G$ is an idempotent such that $\eta(\Lambda * G)\eta$ is Morita equivalent to $\Lambda * G$ and (Q_G, W_G) is the QP described in §3.3.2.

We will now show that the process of getting back Λ from $\Lambda * G$ is achieved via a construction which satisfies the assumptions (A1)-(A7).

Proposition 3.5.2. *The dual group $\hat{G}$ acts on $\mathcal{P}(Q_G, W_G)$ by automorphisms and the skew group algebra $\mathcal{P}(Q_G, W_G) * \hat{G}$ is Morita equivalent to Λ. Moreover this action satisfies the assumptions (A1)-(A7).*

Proof. Since $\eta = \hat{\varepsilon} \otimes 1$ for an idempotent $\hat{\varepsilon} \in \Lambda$, we have that $\hat{G}$ acts trivially on η and so the action of $\hat{G}$ on $\Lambda * G$ restricts to an action on $\eta(\Lambda * G)\eta \cong \mathcal{P}(Q_G, W_G)$. Hence, by [57, Lemma 2.2], we have that $(\eta(\Lambda * G)\eta) * \hat{G}$ is Morita equivalent to $(\Lambda * G) * \hat{G}$, and the latter is Morita equivalent to Λ by Theorem 3.5.1. So the first assertion is proved and we are left to check that the action of $\hat{G}$ on (Q_G, W_G) satisfies the assumptions (A1)-(A7).

Assumption (A1) holds because $\hat{G}$ has the same order of G.

If $\varepsilon \in \mathcal{E}'$, then $\chi(\eta^\varepsilon) = \chi(\varepsilon \otimes 1) = \eta^\varepsilon$. If $\varepsilon' \in \mathcal{E}''$ and $0 \leq \mu \leq n-1$, then

$$\chi(\eta^\varepsilon_\mu) = \chi(\varepsilon \otimes e_\mu) = \frac{1}{n}\sum_{i=0}^{n-1} \zeta^{i\mu}\chi(\varepsilon \otimes g^i) = \frac{1}{n}\sum_{i=0}^{n-1} \zeta^{i(\mu+1)}\varepsilon \otimes g^i = \varepsilon \otimes e_{\mu+1} = \eta^\varepsilon_{\mu+1}.$$

Hence $\hat{G}$ permutes the vertices of Q_G. In particular assumption (A4) holds.

Now we consider the action on the arrows of Q_G. Four cases have to be analysed.

(1) Let α be an arrow of type (1) in Q. Then we have an arrow $\tilde{\alpha} = \alpha \otimes g^{t(\alpha)}$ in Q_G and $\hat{G}$ acts on it as

$$\chi(\tilde{\alpha}) = \chi(\alpha \otimes g^{t(\alpha)}) = \chi(g^{t(\alpha)})\alpha \otimes g^{t(\alpha)} = \zeta^{t(\alpha)}\alpha \otimes g^{t(\alpha)} = \zeta^{t(\alpha)}\tilde{\alpha}.$$

(2) Let α be an arrow of type (2) in Q and $0 \leq \mu \leq n-1$. Then $\hat{G}$ acts on $\tilde{\alpha}^\mu = (1 \otimes e_\mu)(\alpha \otimes 1)$ as

$$\chi(\tilde{\alpha}^\mu) = \chi((1 \otimes e_\mu)(\alpha \otimes 1)) = (1 \otimes e_{\mu+1})(\alpha \otimes 1) = \tilde{\alpha}^{\mu+1}.$$

(3),(4) Let α be an arrow of type either (3) or (4) in Q and $0 \leq \mu \leq n-1$. Then $\hat{G}$ acts on $\tilde{\alpha}^\mu = \alpha \otimes e_\mu$ as

$$\chi(\tilde{\alpha}^\mu) = \chi(\alpha \otimes e_\mu) = \alpha \otimes e_{\mu+1} = \tilde{\alpha}^{\mu+1}.$$

This proves assumptions (A2) and (A3).

From these calculations we can deduce how $\hat{G}$ acts on the cycles of W_G. Again we distinguish four cases.

(i) Let c be a cycle of type (i) and write $\tilde{c} = \tilde{\alpha}_1 \cdots \tilde{\alpha}_l$. Then, observing that $t(\alpha_1) + \cdots + t(\alpha_l) = 0 \pmod{n}$, we get $\chi(\tilde{c}) = \zeta^{t(\alpha_1)+\cdots+t(\alpha_l)}\tilde{c} = \tilde{c}$.

(ii) Let c be a cycle of type (ii) and $0 \leq \mu \leq n-1$. Write $\tilde{c}^\mu = \tilde{\alpha}_1^\mu \widetilde{g^{-p(c)}(\alpha_2)}^\mu \tilde{\alpha}_3 \cdots \tilde{\alpha}_l$. Then we get $\chi(\tilde{c}^\mu) = \zeta^{t(\alpha_3)+\cdots+t(\alpha_l)}\tilde{c}^{\mu+1} = \zeta^{-t(\alpha_2)}\tilde{c}^{\mu+1} = \zeta^{-p(c)}\tilde{c}^{\mu+1}$, since $t(\alpha_1) + \cdots + t(\alpha_l) = 0 \pmod{n}$ and $t(\alpha_1) = 0$.

(iii) Let c be a cycle of type (iii) and $0 \leq \mu \leq n-1$. Write $\tilde{c}^\mu = \tilde{\alpha}_1^\mu \tilde{\alpha}_2^\mu \tilde{\alpha}_3^\mu \cdots \tilde{\alpha}_l^\mu$. Then we get $\chi(\tilde{c}^\mu) = \tilde{c}^{\mu+1}$.

(iv) Let c be a cycle of type (iv) and $0 \leq \mu \leq n-1$. Write $\tilde{c}^\mu = \tilde{\alpha}_1^{\mu-b_2}\tilde{\alpha}_2^{\mu-b_3} \cdots \tilde{\alpha}_{l-1}^{\mu-b_l}\tilde{\alpha}_l^\mu$. Then we get $\chi(\tilde{c}^\mu) = \tilde{c}^{\mu+1}$.

So assumption (A7) is proved.

Finally we get that

$$\begin{aligned}\chi(W_G) =& \sum_{c\in\mathcal{C}(\mathrm{i})} a(c)\chi(\tilde{c}) + \sum_{c\in\mathcal{C}(\mathrm{ii})} a(c)\sum_{\mu=0}^{n-1}\zeta^{-p(c)\mu}\chi(\tilde{c}^{\mu})+\\ &+\sum_{c\in\mathcal{C}(\mathrm{iii})} a(c)\sum_{\mu=0}^{n-1}\chi(\tilde{c}^{\mu}) + \sum_{c\in\mathcal{C}(\mathrm{iv})} a(c)\sum_{\mu=0}^{n-1}\chi(\tilde{c}^{\mu}) =\\ =&\sum_{c\in\mathcal{C}(\mathrm{i})} a(c)\tilde{c} + \sum_{c\in\mathcal{C}(\mathrm{ii})} a(c)\sum_{\mu=0}^{n-1}\zeta^{-p(c)(\mu+1)}\tilde{c}^{\mu+1}+\\ &+\sum_{c\in\mathcal{C}(\mathrm{iii})} a(c)\sum_{\mu=0}^{n-1}\tilde{c}^{\mu+1} + \sum_{c\in\mathcal{C}(\mathrm{iv})} a(c)\sum_{\mu=0}^{n-1}\tilde{c}^{\mu+1} =\\ =&W_G,\end{aligned}$$

so the potential W_G is fixed by $\hat{G}$ and thus assumption (A5) holds. □

To sum up, we have an action of $\hat{G}$ on the Jacobian algebra $\mathcal{P}(Q_G, W_G)$ which satisfies the assumptions (A1)-(A7). Using the procedure described in Section 3.3 we can construct from it a new QP $((Q_G)_{\hat{G}}, (W_G)_{\hat{G}})$ whose Jacobian algebra is Morita equivalent to Λ. Now we want to construct an explicit isomorphism $\mathcal{P}((Q_G)_{\hat{G}}, (W_G)_{\hat{G}}) \cong \Lambda$.

Firstly, let us give an explicit description of $((Q_G)_{\hat{G}}, (W_G)_{\hat{G}})$.

Let $\mathcal{E}_G = \mathcal{E}'_G \sqcup \mathcal{E}''_G$, where $\mathcal{E}'_G = \{\eta_0^{\varepsilon} \,|\, \varepsilon \in \mathcal{E}''\}$ and $\mathcal{E}''_G = \{\eta^{\varepsilon} \,|\, \varepsilon \in \mathcal{E}'\}$. Then $\mathcal{E}_G$ is a set of representatives for the orbits of the action of $\hat{G}$ on Q_G. The elements of $\mathcal{E}'_G$ and $\mathcal{E}''_G$ have orbits of cardinality n and 1 respectively.

The arrows of Q_G can be divided into four families, according to whether their starting and ending points are fixed or not by the action of $\hat{G}$.

(1) Arrows between two non-fixed vertices. These are all the arrows of the form $\tilde{\alpha}^{\mu}\colon \eta_{\mu}^{\varepsilon} \to \eta_{\mu-b(\alpha)}^{\varepsilon'}$, where $\alpha : \varepsilon \to \varepsilon'$ is an arrow of type (4) in Q and $0 \le \mu \le n-1$. Among them, the arrows which are of type (1) with respect to the action of $\hat{G}$ on Q_G are the ones which end in $\mathcal{E}'_G$, i.e., the ones of the form $\tilde{\alpha}^{b(\alpha)}\colon \eta_{b(\alpha)}^{\varepsilon} \to \eta_0^{\varepsilon'}$. Since $\eta_{b(\alpha)}^{\varepsilon} = \chi^{b(\alpha)}(\eta_0^{\varepsilon})$, we have that $t(\tilde{\alpha}^{b(\alpha)}) = b(\alpha)$.

(2) Arrows from a non-fixed vertex to a fixed one. These are all the arrows of the form $\tilde{\alpha}^{\mu}\colon \eta_{\mu}^{\varepsilon} \to \eta^{\varepsilon'}$, where $\alpha : \varepsilon \to \varepsilon'$ is an arrow of type (3) in Q and $0 \le \mu \le n-1$. Among them, the arrows which are of type (2) with respect to the action of $\hat{G}$ on Q_G are the ones which start in $\mathcal{E}'_G$, i.e., the ones of the form $\tilde{\alpha}^{0}\colon \eta_0^{\varepsilon} \to \eta^{\varepsilon'}$.

(3) Arrows from a fixed vertex to a non-fixed one. These are all the arrows of the form $\tilde{\alpha}^{\mu}\colon \eta^{\varepsilon} \to \eta_{\mu}^{\varepsilon'}$, where $\alpha : \varepsilon \to \varepsilon'$ is an arrow of type (2) in Q and $0 \le \mu \le n-1$. Among them, the arrows which are of type (3) with respect to the action of $\hat{G}$ on Q_G are the ones which end in $\mathcal{E}'_G$, i.e., the ones of the form $\tilde{\alpha}^{0}\colon \eta^{\varepsilon} \to \eta_0^{\varepsilon'}$.

(4) Arrows between two fixed vertices. These are all the arrows of the form $\tilde{\alpha}\colon \eta^{\varepsilon} \to \eta^{\varepsilon'}$, where $\alpha : \varepsilon \to \varepsilon'$ is an arrow of type (1) in Q. All of them are of type (4) with respect to the action of $\hat{G}$ on Q_G. Since $\chi(\tilde{\alpha}) = \zeta^{t(\alpha)}\tilde{\alpha}$, we have that $b(\tilde{\alpha}) = t(\alpha)$.

We deduce that the quiver $(Q_G)_{\hat{G}}$ is made as follows. Its vertices are $\eta_0^{\varepsilon} \otimes 1$ for $\varepsilon \in \mathcal{E}''$ and $\eta^{\varepsilon} \otimes e_{\nu}$ for $\varepsilon \in \mathcal{E}'$, $0 \leq \nu \leq n-1$, while its arrows are the following:

(1) $\tilde{\beta}\colon \eta_0^{\varepsilon} \otimes 1 \to \eta_0^{\varepsilon'} \otimes 1$, where $\beta = \tilde{\alpha}^{b(\alpha)}$ and $\alpha : \varepsilon \to \varepsilon'$ is an arrow of type (4) in Q,

(2) $\tilde{\beta}^{\nu}\colon \eta_0^{\varepsilon} \otimes 1 \to \eta^{\varepsilon'} \otimes e_{\nu}$, where $\beta = \tilde{\alpha}^0$, $0 \leq \nu \leq n-1$ and $\alpha : \varepsilon \to \varepsilon'$ is an arrow of type (3) in Q,

(3) $\tilde{\beta}^{\nu}\colon \eta^{\varepsilon} \otimes e_{\nu} \to \eta_0^{\varepsilon'} \otimes 1$, where $\beta = \tilde{\alpha}^0$, $0 \leq \nu \leq n-1$ and $\alpha : \varepsilon \to \varepsilon'$ is an arrow of type (2) in Q,

(4) $\tilde{\beta}^{\nu}\colon \eta^{\varepsilon} \otimes e_{\nu} \to \eta^{\varepsilon'} \otimes e_{\nu - t(\alpha)}$, where $\beta = \tilde{\alpha}$, $0 \leq \nu \leq n-1$ and $\alpha : \varepsilon \to \varepsilon'$ is an arrow of type (1) in Q.

Proposition 3.5.3. *Let $\phi\colon (Q_G)_{\hat{G}} \to Q$ be the morphism of quivers defined as follows.*

- $\phi(\eta_0^{\varepsilon} \otimes 1) = \varepsilon$ *for* $\varepsilon \in \mathcal{E}''$.
- $\phi(\eta^{\varepsilon} \otimes e_{\mu}) = g^{\mu}(\varepsilon)$ *for* $\varepsilon \in \mathcal{E}'$, $0 \leq \mu \leq n-1$.
- $\phi(\tilde{\beta}) = \alpha$, *where* $\beta = \tilde{\alpha}^{b(\alpha)}$ *and* α *is an arrow of type (4) in* Q.
- $\phi(\tilde{\beta}^{\nu}) = g^{\nu}(\alpha)$, *where* $\beta = \tilde{\alpha}^0$, $0 \leq \nu \leq n-1$ *and* α *is an arrow of type (3) in* Q.
- $\phi(\tilde{\beta}^{\nu}) = g^{\nu}(\alpha)$, *where* $\beta = \tilde{\alpha}^0$, $0 \leq \nu \leq n-1$ *and* α *is an arrow of type (2) in* Q.
- $\phi(\tilde{\beta}^{\nu}) = g^{\nu - t(\alpha)}(\alpha)$, *where* $\beta = \tilde{\alpha}$, $0 \leq \nu \leq n-1$ *and* α *is an arrow of type (1) in* Q.

Then ϕ is an isomorphism and, if we extend it to an isomorphism between the corresponding path algebras, we have $\phi((W_G)_{\hat{G}}) = W$.

Proof. We first note that ϕ is a well defined morphism of quivers. Moreover, by what we observed earlier in this section, ϕ is a bijection on both the sets of vertices and arrows, thus it is an isomorphism.

Given the set $\mathcal{E}_G$ defined above, we can choose a set $\mathcal{C}_G = \{\hat{d} \mid d \text{ cycle in } W_G\}$ of representatives for the $*$ action of $\hat{G}$ on cycles as in §3.3.3. We have that $\mathcal{C}_G = \mathcal{C}_G(\mathrm{i}) \sqcup \mathcal{C}_G(\mathrm{ii}) \sqcup \mathcal{C}_G(\mathrm{iii}) \sqcup \mathcal{C}_G(\mathrm{iv})$. We now describe each of these four subsets and show where their elements are sent by ϕ. We use the notation $t_{i,j}$ of the proof of Lemma 3.4.7.

(i) Cycles of type (i) in Q_G are the ones of the form $d = \tilde{c}^{\mu}$, where $c \in \mathcal{C}(\mathrm{iv})$. If we write $c = \alpha_1 \cdots \alpha_l$ for some arrows α_i of type (4) in Q, then $\tilde{c}^{\mu} = \tilde{\alpha}_1^{\mu - b_2}\tilde{\alpha}_2^{\mu - b_3}\tilde{\alpha}_3^{\mu - b_4} \cdots \tilde{\alpha}_{l-1}^{\mu - b_l}\tilde{\alpha}_l^{\mu}$, where $b_i = b(\alpha_i) + \cdots + b(\alpha_l)$. Hence we can choose $\hat{d} = \tilde{\alpha}_1^{-b_2}\tilde{\alpha}_2^{-b_3}\tilde{\alpha}_3^{-b_4} \cdots \tilde{\alpha}_{l-1}^{-b_l}\tilde{\alpha}_l^{0} =$

$\tilde{c}^0$, and $\mathcal{C}_G(\mathrm{i})$ is the subset of all the cycles of this kind. Moreover we have that $\tilde{d} = \tilde{\beta}_1 \cdots \tilde{\beta}_l$, where $\beta_i = \tilde{\alpha}_i^{b(\alpha_i)}$. It follows that

$$\phi(\tilde{d}) = \phi(\tilde{\beta}_1 \cdots \tilde{\beta}_l) = \alpha_1 \cdots \alpha_l = c.$$

Let us now look at the coefficient $a(d)$ of d as a summand of W_G. The cycle c of W gives rise to a number $x = |\hat{G}\tilde{c}^\mu|$ of distinct cycles in W_G (this does not depend on the choice of μ). Then $a(d) = a(c)\frac{n}{x}$.

(ii) Cycles of type (ii) in Q_G are the ones of the form $d = \tilde{c}^\mu$, where $c \in \mathcal{C}(\mathrm{iii})$. If we write $c = \alpha_1\alpha_2 \cdots \alpha_l$ for α_1 of type (2), α_2 of type (3), and $\alpha_3, \ldots, \alpha_l$ of type (4) in Q, then $\tilde{c}^\mu = \tilde{\alpha}_1^\mu \tilde{\alpha}_2^{\mu-b_3} \tilde{\alpha}_3^{\mu-b_4} \cdots \tilde{\alpha}_{l-1}^{\mu-b_l} \tilde{\alpha}_l^\mu$, where we write $b_i = b(\alpha_i) + \cdots + b(\alpha_l)$. Hence we obtain that $\hat{d} = \tilde{\alpha}_1^0 \tilde{\alpha}_2^{-b_3} \tilde{\alpha}_3^{-b_4} \cdots \tilde{\alpha}_{l-1}^{-b_l} \tilde{\alpha}_l^0 = \tilde{c}^0$, and $\mathcal{C}_G(\mathrm{ii})$ is the subset of all the cycles of this kind. Moreover we have that $\tilde{d}^\nu = \tilde{\beta}_1^\nu \tilde{\beta}_2^\nu \tilde{\beta}_3 \cdots \tilde{\beta}_l$, where $\beta_1 = \tilde{\alpha}_1^0$, $\beta_2 = \tilde{\alpha}_2^0$ and $\beta_i = \tilde{\alpha}_i^{b(\alpha_i)}$ for $i \geq 3$. It follows that (recall that by definition $q(c) = b_3$)

$$\phi(\tilde{d}) = \phi(\tilde{\beta}_1^\nu \tilde{\beta}_2^\nu \tilde{\beta}_3 \cdots \tilde{\beta}_l) = g^\nu(\alpha_1) g^\nu(\alpha_2) \alpha_3 \cdots \alpha_l = \zeta^{-b_3} g^\nu(c) = \zeta^{-q(c)} g^\nu(c).$$

Note that $\beta_2 = \chi^{b_3} \tilde{\alpha}_2^{-b_3}$. This implies that $p(d) = -q(c)$ and so $\phi(\tilde{d}) = \zeta^{p(d)} g^\nu(c)$.

(iii) Cycles of type (iii) in Q_G are the ones of the form $d = \tilde{c}^\mu$, where $c \in \mathcal{C}(\mathrm{ii})$. If we write $c = \alpha_1\alpha_2 g^{t_2}(\alpha_3) \cdots g^{t_{2,l-1}}(\alpha_l)$ for α_1 of type (3), α_2 of type (2), and $\alpha_3, \ldots, \alpha_l$ of type (1) in Q, then $\tilde{c}^\mu = \tilde{\alpha}_1^\mu \widetilde{g^{-t_2}(\alpha_2)}^\mu \tilde{\alpha}_3 \cdots \tilde{\alpha}_l$. Hence $\hat{d} = \tilde{\alpha}_1^0 \widetilde{g^{-t_2}(\alpha_2)}^0 \tilde{\alpha}_3 \cdots \tilde{\alpha}_l = \tilde{c}^0$, and $\mathcal{C}_G(\mathrm{iii})$ is the subset of all the cycles of this kind. Now define $\beta_1 = \tilde{\alpha}_1^0$, $\beta_2 = \widetilde{g^{-t_2}(\alpha_2)}^0$ and $\beta_i = \tilde{\alpha}_i$ for $i \geq 3$. Recall that, for $i \geq 3$, $\chi(\beta_i) = \zeta^{t(\alpha_i)} \beta_i$, so $b(\beta_i) = t(\alpha_i)$. If we put $b'_i = b(\beta_i) + \cdots + b(\beta_l)$ for $i \geq 3$, we have that $\tilde{d}^\nu = \tilde{\beta}_1^\nu \tilde{\beta}_2^{\nu-b'_3} \tilde{\beta}_3^{\nu-b'_4} \cdots \tilde{\beta}_{l-1}^{\nu-b'_l} \tilde{\beta}_l^\nu$. Then

$$\begin{aligned}
\phi(\tilde{d}^\nu) &= \phi(\tilde{\beta}_1^\nu \tilde{\beta}_2^{\nu-b'_3} \tilde{\beta}_3^{\nu-b'_4} \cdots \tilde{\beta}_{l-1}^{\nu-b'_l} \tilde{\beta}_l^\nu) = \\
&= g^\nu(\alpha_1) g^{\nu-b'_3}(g^{-t_2}(\alpha_2)) g^{\nu-b'_4-t(\alpha_3)}(\alpha_3) \cdots g^{\nu-t(\alpha_l)}(\alpha_l) = \\
&= g^\nu(\alpha_1 g^{-t_{2,l}}(\alpha_2) g^{-t_{3,l}}(\alpha_3) \cdots g^{-t_l}(\alpha_l)) = \\
&= g^\nu(\alpha_1 \alpha_2 g^{t_2}(\alpha_3) \cdots g^{t_{2,l-1}}(\alpha_l)) = \\
&= g^\nu(c).
\end{aligned}$$

(iv) Cycles of type (iv) in Q_G are the ones of the form $d = \tilde{c}$, where $c \in \mathcal{C}(\mathrm{i})$. If we write $c = \alpha_1 g^{t_1}(\alpha_2) g^{t_{1,2}}(\alpha_3) \cdots g^{t_{1,l-1}}(\alpha_l)$ for α_i of type (1) in Q, then $\tilde{c} = \tilde{\alpha}_1 \cdots \tilde{\alpha}_l$. Hence $\hat{d} = d$, and $\mathcal{C}_G(\mathrm{iv})$ is the subset of all the cycles of this kind. If we put $\beta_i = \tilde{\alpha}_i$ for all i, then $\tilde{d}^\nu = \tilde{\beta}_1^{\nu-b'_2} \tilde{\beta}_2^{\nu-b'_3} \tilde{\beta}_3^{\nu-b'_4} \cdots \tilde{\beta}_{l-1}^{\nu-b'_l} \tilde{\beta}_l^\nu$. It follows that

$$\begin{aligned}
\phi(\tilde{d}^\nu) &= \phi(\tilde{\beta}_1^{\nu-b'_2} \tilde{\beta}_2^{\nu-b'_3} \tilde{\beta}_3^{\nu-b'_4} \cdots \tilde{\beta}_{l-1}^{\nu-b'_l} \tilde{\beta}_l^\nu) = \\
&= g^{\nu-b'_2-t(\alpha_1)}(\alpha_1) g^{\nu-b'_3-t(\alpha_2)}(\alpha_2) \cdots g^{\nu-t(\alpha_l)}(\alpha_l) = \\
&= g^\nu(\alpha_1 g^{t_1}(\alpha_2) g^{t_{1,2}}(\alpha_3) \cdots g^{t_{1,l-1}}(\alpha_l)) = \\
&= g^\nu(c).
\end{aligned}$$

Now we can write $(W_G)_{\hat{G}}$ as follows:

$$\begin{aligned}(W_G)_{\hat{G}} = & \sum_{d\in\mathcal{C}_G(\mathrm{i})} a(d)\frac{|\hat{G}d|}{n}\tilde{d} + \sum_{d\in\mathcal{C}_G(\mathrm{ii})} a(d)\sum_{\nu=0}^{n-1}\zeta^{-p(d)\nu}\tilde{d}^{\nu} + \\ & + \sum_{d\in\mathcal{C}_G(\mathrm{iii})} a(d)\sum_{\nu=0}^{n-1}\tilde{d}^{\nu} + \sum_{d\in\mathcal{C}_G(\mathrm{iv})} a(d)\sum_{\nu=0}^{n-1}\tilde{d}^{\nu} = \\ & = \sum_{c\in\mathcal{C}(\mathrm{iv}),d=\tilde{c}^0} a(c)\tilde{d} + \sum_{c\in\mathcal{C}(\mathrm{iii}),d=\tilde{c}^0} a(c)\sum_{\nu=0}^{n-1}\zeta^{q(c)\nu}\tilde{d}^{\nu} + \\ & + \sum_{c\in\mathcal{C}(\mathrm{ii}),d=\tilde{c}^0} a(c)\sum_{\nu=0}^{n-1}\tilde{d}^{\nu} + \sum_{c\in\mathcal{C}(\mathrm{i}),d=\tilde{c}} a(c)\frac{|Gc|}{n}\sum_{\nu=0}^{n-1}\tilde{d}^{\nu}.\end{aligned}$$

Applying ϕ we get

$$\begin{aligned}\phi((W_G)_{\hat{G}}) = & \sum_{c\in\mathcal{C}(\mathrm{iv}),d=\tilde{c}^0} a(c)\phi(\tilde{d}) + \sum_{c\in\mathcal{C}(\mathrm{iii}),d=\tilde{c}^0} a(c)\sum_{\nu=0}^{n-1}\zeta^{q(c)\nu}\phi(\tilde{d}^{\nu}) + \\ & + \sum_{c\in\mathcal{C}(\mathrm{ii}),d=\tilde{c}^0} a(c)\sum_{\nu=0}^{n-1}\phi(\tilde{d}^{\nu}) + \sum_{c\in\mathcal{C}(\mathrm{i}),d=\tilde{c}} a(c)\frac{|Gc|}{n}\sum_{\nu=0}^{n-1}\phi(\tilde{d}^{\nu}) = \\ & = \sum_{c\in\mathcal{C}(\mathrm{iv}),d=\tilde{c}^0} a(c)c + \sum_{c\in\mathcal{C}(\mathrm{iii}),d=\tilde{c}^0} a(c)\sum_{\nu=0}^{n-1}\zeta^{q(c)\nu}\zeta^{-q(c)\nu}g^{\nu}(c) + \\ & + \sum_{c\in\mathcal{C}(\mathrm{ii}),d=\tilde{c}^0} a(c)\sum_{\nu=0}^{n-1}g^{\nu}(c) + \sum_{c\in\mathcal{C}(\mathrm{i}),d=\tilde{c}} a(c)\frac{|Gc|}{n}\sum_{\nu=0}^{n-1}g^{\nu}(c) = \\ & = \sum_{c\in\mathcal{C}(\mathrm{iv})} a(c)c + \sum_{\nu=0}^{n-1}g^{\nu}\left(\sum_{c\in\mathcal{C}(\mathrm{iii})} a(c)c + \sum_{c\in\mathcal{C}(\mathrm{ii})} a(c)c + \sum_{c\in\mathcal{C}(\mathrm{i})} a(c)\frac{|Gc|}{n}c\right) = \\ & = W.\end{aligned}$$

□

Corollary 3.5.4. *Let θ be the idempotent $\sum_{s\in\mathcal{E}_G} s\otimes 1$ in $(\eta(\Lambda * G)\eta) * \hat{G}$. Then the isomorphism of quivers $\phi : (Q_G)_{\hat{G}} \to Q$ induces an isomorphism of algebras*

$$\theta\left((\eta\,(\Lambda * G)\,\eta) * \hat{G}\right)\theta \cong \Lambda,$$

where $\Lambda = \mathcal{P}(Q,W)$.

Proof. Applying Theorem 3.3.7 to $\eta(\Lambda * G)\eta$ with the action of $\hat{G}$, we get

$$\theta\left((\eta\,(\Lambda * G)\,\eta) * \hat{G}\right)\theta \cong \mathcal{P}((Q_G)_{\hat{G}}, (W_G)_{\hat{G}}),$$

and the latter is isomorphic to $\mathcal{P}(Q,W)$ by Proposition 3.5.3. □

3.6 Planar rotation-invariant QPs

Our main result Theorem 3.3.7 is about skew group algebras of Jacobian algebras of QPs, but it only applies under some assumptions on the group action. There is however a class of QPs which satisfy these assumptions, as well as a way of generating many examples in this class. To define this class, we follow [29] and associate a CW-complex to a QP called its canvas. First we need to fix some notation.

We denote by D^d the d-disk and by $S^{d-1} = \partial D^d$ the $(d-1)$-sphere in $\mathbf{R}^d$. We suppose that $D^1 = [0,1]$ and $S^0 = \{0,1\}$. A CW-complex is a topological space realized as a union $\bigcup_{d\in\mathbb{Z}_{\geq 0}} X^d$, where X^0 is a discrete space and each X^d is obtained from X^{d-1} in the following way. For each d there are a set $\{D^d_a\}_{a\in I_d}$ of copies of the d-disk and continuous maps $\phi_a\colon S^{d-1}_a = \partial D^d_a \to X^{d-1}$, such that we have a pushout diagram

$$\begin{array}{ccc} \bigsqcup_{a\in I_d} S^{d-1}_a & \xrightarrow{(\phi_a)} & X^{d-1} \\ \downarrow & & \downarrow \\ \bigsqcup_{a\in I_d} D^d_a & \xrightarrow{(\varepsilon_a)} & X^d \end{array}$$

in the category of topological spaces with continuous maps (the left vertical map is given by the inclusions of S^{d-1}_a as boundaries of D^d_a). For $d \geq 1$ the image of the interior of D^d_a under ε_a is called a d-cell. The elements of X_0 are called 0-cells. We say that X has dimension m if $X = X^m$.

Definition 3.6.1 ([29, Definition 8.1])**.** Let (Q,W) be a QP and let Q_2 be a set of representatives modulo com_Q of the cycles which appear in W. The *canvas* of (Q,W) is the 2-dimensional CW-complex $X_{(Q,W)}$ defined in the following way. Its cells are indexed by the sets $X_0 = Q_0$, $I_1 = Q_1$, $I_2 = Q_2$. For each $\alpha \in Q_1$ we have an attaching map $\phi_\alpha\colon S^0_\alpha \to X_0$ defined by $\phi_\alpha(0) = \mathfrak{s}(\alpha)$ and $\phi_\alpha(1) = \mathfrak{t}(\alpha)$. If $c = \alpha_0 \cdots \alpha_{l-1} \in Q_2$, we define the attaching map $\phi_c\colon S^1_c \to X_1$ by

$$\phi_c\left(\cos\left(\frac{2\pi}{l}(i+t)\right), \sin\left(\frac{2\pi}{l}(i+t)\right)\right) = \varepsilon_{\alpha_i}(t)$$

for $i = 0,\dots,l-1$ and $t \in [0,1)$.

Remark 3.6.2. In other (imprecise) words, the 1-skeleton of $X_{(Q,W)}$ is the underlying graph of Q, and we attach 2-cells along the cycles appearing in W.

Definition 3.6.3 ([29, Definition 9.1])**.** A QP (Q,W) is *planar* if it is simply connected and there exists an embedding of $X_{(Q,W)}$ into $\mathbf{R}^2$. We call it *strongly planar* if it is planar and $X_{(Q,W)}$ is homeomorphic to a disk.

If (Q, W) is a planar QP, then by [29, Proposition 9.3] the embedding of the quiver Q in $\mathbf{R}^2$ determines the Jacobian algebra, so we can assume that the coefficients in W are +1 for the clockwise faces, and -1 for the anticlockwise faces.

Definition 3.6.4. Let (Q, W) be a planar QP and G be a cyclic group acting on Q. We say that G *acts on* (Q, W) *by rotations* if:

- there is an embedding of $X_{(Q,W)}$ in $\mathbf{R}^2$ such that the action of a generator of G is induced by a rotation of the plane;
- the action of G is faithful;
- assumption (A7) is satisfied.

Notice that in this case the image $\mathrm{im}(G) \subseteq \mathrm{Aut}(Q)$ is necessarily finite. For simplicity, we will identify G with $\mathrm{im}(G)$.

We remark some facts which follow immediately from the definition, and directly imply that this class of quivers falls within the scope of Theorem 3.3.7.

Lemma 3.6.5. *Let G act on a planar QP (Q, W) by rotations. Then the action of G satisfies the assumptions (A2)-(A7).*

Proof. A rotation permutes the vertices and maps arrows to arrows, so assumptions (A2) and (A6) are satisfied. There is at most one fixed vertex, and there can be no loops at that vertex because we are assuming (A7), so assumption (A3) holds. Since we are assuming that G acts faithfully, we have that every vertex which is not fixed has order the order of a rotation generating G, hence assumption (A4) is satisfied. Assumption (A5) holds because G maps faces of $X_{(Q,W)}$ to faces. Finally, assumption (A7) holds by definition. □

There is a way of producing strongly planar QPs with a group acting by rotations by means of so-called Postnikov diagrams (see [56], [9], [54]). A Postnikov diagram is a collection of oriented curves in a disk subject to some axioms depending on two integer parameters $a, n \geq 1$, and it naturally gives rise to a planar QP. For this result we need to assume that $\Bbbk = \mathbb{C}$.

Theorem 3.6.6 ([54, Corollary 7.3])**.** *An (a, n)-Postnikov diagram is invariant under rotation by $\frac{2\pi a}{n}$ if and only if the corresponding QP is self-injective. In this case, a Nakayama automorphism is given by this rotation.*

In particular, there is a finite cyclic group acting by rotations on a planar QP, so we can apply our construction. We include an unpublished result to justify the claim that Postnikov diagrams give rise to many examples. Namely, rotation-invariant Postnikov diagrams exist and in fact abound.

Theorem 3.6.7. *[55] There exists an (a, n)-Postnikov diagram which is invariant under rotation by $\frac{2\pi a}{n}$ if and only if a is congruent to -1, 0 or 1 modulo $n / \mathrm{GCD}(n, a)$. In particular there are infinitely many self-injective planar QPs with Nakayama automorphism of order d, for any choice of d.*

Remark 3.6.8. There exist self-injective planar QPs with Nakayama automorphism acting by rotation which do not come from Postnikov diagrams. For instance, the quiver of the 3-preprojective algebra of type A_n (see Example 3.8.1) with n odd.

We conclude this section by observing that Theorem 3.3.7 can be naturally applied to any self-injective QP where the Nakayama automorphism satisfies our assumptions. In this case we get:

Proposition 3.6.9. *Let (Q, W) be a self-injective QP with Nakayama automorphism φ of finite order. Call $G = \langle\varphi\rangle \subseteq \mathrm{Aut}(\mathcal{P}(Q, W))$, and assume that the assumptions (A1)-(A7) are satisfied. Then $\mathcal{P}(Q_G, W_G)$ is symmetric.*

Proof. By Theorem 3.3.7, $\mathcal{P}(Q_G, W_G)$ is a self-injective algebra which is Morita equivalent to $\Lambda * G$. The latter is symmetric by Corollary 3.2.4 using Lemma 3.2.5. □

Combining this with our previous discussion, we remark that by Theorem 3.6.6 there is a symmetric Jacobian algebra associated to every rotation-invariant Postnikov diagram.

Corollary 3.6.10. *If (Q, W) is a self-injective QP coming from a Postnikov diagram with Nakayama automorphism φ, then $\mathcal{P}(Q_{\langle\varphi\rangle}, W_{\langle\varphi\rangle})$ is symmetric.*

These results are illustrated in Example 3.8.3.

3.7 Cuts and 2-representation finite algebras

Let (Q, W) be a QP assume that a finite cyclic group G acts on $\mathcal{P}(Q, W)$ satisfying the assumptions (A1)-(A7). We want to understand when a cut in (Q_G, W_G) can be induced from one in (Q, W). We call a cut in (Q, W) invariant under the $*$ action of G a *G-invariant cut.*

Proposition 3.7.1. *Let C be a G-invariant cut in (Q, W). Then the subset $C_G = C_1 \cup C_2 \cup C_3 \cup C_4$ of $(Q_G)_1$ defined by*

$$C_1 = \{\tilde{\alpha} \mid \alpha \in C \text{ of type } (1)\}, \quad C_x = \{\tilde{\alpha}^\mu \mid \alpha \in C \text{ of type } (x),\ 0 \le \mu \le n-1\},\ x = 2, 3, 4,$$

is a cut in (Q_G, W_G).

Proof. In order to show that C_G is a cut in (Q_G, W_G), we shall prove that every cycle in W_G has degree 1 with respect to d_{C_G}. Thus we have four different cases to consider.

(i) Let $c \in \mathcal{C}(\mathrm{i})$, so $c = \alpha_1 g^{t_1}(\alpha_2) \cdots g^{t_1+\cdots+t_{l-1}}(\alpha_l)$ for some arrows $\alpha_i \in Q_1$ of type (1). Then W_G contains the cycle $\tilde{c} = \tilde{\alpha}_1 \cdots \tilde{\alpha}_l$ and, since C is G-invariant, we have

$$d_{C_G}(\tilde{c}) = \sum_{i=1}^{l} d_{C_G}(\tilde{\alpha}_i) = \sum_{i=1}^{l} d_C(\alpha_i) = \sum_{i=1}^{l} d_C(g^{t_1+\cdots+t_{i-1}}(\alpha_i)) = d_C(c) = 1.$$

(ii) Let $c \in \mathcal{C}(\text{ii})$, so $c = \alpha_1\alpha_2 g^{t_2}(\alpha_3)\cdots g^{t_2+\cdots+t_{l-1}}(\alpha_l)$ for α_1 of type (3), α_2 of type (2) and $\alpha_3,\ldots,\alpha_l$ of type (1). For each $\mu = 0,\ldots,n-1$ we have a cycle $\tilde{c}^\mu = \tilde{\alpha}_1^\mu \widetilde{g^{-t_2}(\alpha_2)}^\mu \tilde{\alpha}_3\cdots\tilde{\alpha}_l$ in W_G and

$$d_{C_G}(\tilde{c}^\mu) = d_{C_G}(\tilde{\alpha}_1^\mu) + d_{C_G}(\widetilde{g^{-t_2}(\alpha_2)}^\mu) + \sum_{i=3}^{l} d_{C_G}(\tilde{\alpha}_i) = \sum_{i=1}^{l} d_C(\alpha_i) = \\ = \sum_{i=1}^{l} d_C(g^{t_1+\cdots+t_{i-1}}(\alpha_i)) = d_C(c) = 1.$$

(iii) Let $c \in \mathcal{C}(\text{iii})$, so $c = \alpha_1\alpha_2\cdots\alpha_l$ for α_1 of type (2), α_2 of type (3) and $\alpha_3,\ldots,\alpha_l$ of type (4). For each $\mu = 0,\ldots,n-1$ we have a cycle $\tilde{c}^\mu = \tilde{\alpha}_1^\mu\tilde{\alpha}_2^{\mu-b_3}\cdots\tilde{\alpha}_{l-1}^{\mu-b_l}\tilde{\alpha}_l^\mu$ in W_G, where $b_i = b(\alpha_i)+\cdots+b(\alpha_l)$. Hence

$$d_{C_G}(\tilde{c}^\mu) = d_{C_G}(\tilde{\alpha}_1^\mu) + \sum_{i=2}^{l} d_{C_G}(\tilde{\alpha}_i^{\mu-b_{i+1}}) = \sum_{i=1}^{l} d_C(\alpha_i) = d_C(c) = 1.$$

(iv) Let $c \in \mathcal{C}(\text{iv})$, so $c = \alpha_1\alpha_2\cdots\alpha_l$ for α_i of type (4). For each $\mu = 0,\ldots,n-1$ we have a cycle $\tilde{c}^\mu = \tilde{\alpha}_1^{\mu-b_2}\tilde{\alpha}_2^{\mu-b_3}\cdots\tilde{\alpha}_{l-1}^{\mu-b_l}\tilde{\alpha}_l^\mu$ in W_G, where $b_i = b(\alpha_i)+\cdots+b(\alpha_l)$. Hence

$$d_{C_G}(\tilde{c}^\mu) = \sum_{i=1}^{l} d_{C_G}(\tilde{\alpha}_i^{\mu-b_{i+1}}) = \sum_{i=1}^{l} d_C(\alpha_i) = d_C(c) = 1. \qquad \square$$

Observe that from [46, Corollary 1.6(1)], 2-representation finiteness is preserved by taking skew group algebras. Thus it follows from Theorem 3.1.3 that the property of being a truncated Jacobian algebra is also preserved. In our setting, the corresponding cut on (Q_G, W_G) is precisely C_G:

Proposition 3.7.2. *Let C be a G-invariant cut in (Q,W) and let C_G be the cut constructed in Proposition 3.7.1. Then the action of G on $\mathcal{P}(Q,W)$ restricts to an action on $\mathcal{P}(Q,W)_C$, and the skew group algebra $(\mathcal{P}(Q,W)_C) * G$ is Morita equivalent to $\mathcal{P}(Q_G,W_G)_{C_G}$.*

Proof. Call $\Lambda = \mathcal{P}(Q,W)$ and let Λ_0 be its degree 0 part with respect to the grading d_C, so $\Lambda_0 = \mathcal{P}(Q,W)_C$. The fact that C is G-invariant implies that G preserves the grading, so the first assertion holds.

Now note that we can define a grading on $\Lambda * G$ by assigning degree $d_C(x)$ to $x \otimes h$ for all $h \in G$ and all homogeneous elements $x \in \Lambda$. Moreover this induces a grading on $\eta(\Lambda * G)\eta$ and we have that $(\eta(\Lambda * G)\eta)_0 = \eta(\Lambda_0 * G)\eta$. Hence, in order to prove the claim, it is enough to show that the grading on $\eta(\Lambda * G)\eta$ coincides with the grading d_{C_G} on $\mathcal{P}(Q_G,W_G)$ under the isomorphism $\eta(\Lambda * G)\eta \cong \mathcal{P}(Q_G,W_G)$. But this follows immediately from the definition of C_G, since both algebras are generated in degree 0 and 1 and the elements of degree 1 in $\eta(\Lambda * G)\eta$ are exactly the ones given by C_G. $\square$

Let (Q, W) be a self-injective QP with a group G acting as per the assumptions (A1)-(A7). Then (Q_G, W_G) is self-injective, so its truncated Jacobian algebras are 2-representation finite. In the spirit of [29, §7], we will give sufficient conditions on (Q, W) for the truncated Jacobian algebras of (Q_G, W_G) to be derived equivalent to each other.

In the following discussion we do not need to assume self-injectivity.

Definition 3.7.3. We say that (Q, W) *has enough cuts* if every arrow of Q is contained in a cut. We say that (Q, W) *has enough G-invariant cuts* if every arrow of Q is contained in a G-invariant cut (cf. [29, Definition 7.4]).

Lemma 3.7.4. *If (Q, W) has enough G-invariant cuts, then (Q_G, W_G) has enough cuts.*

Proof. Let $\beta \in (Q_G)_1$, so $\beta = \tilde{\alpha}$ or $\beta = \tilde{\alpha}^\mu$ for some $\alpha \in Q_1$. Let C be a G-invariant cut in (Q, W) containing α, then the cut C_G in (Q_G, W_G) constructed in Proposition 3.7.1 contains β. □

To use the results of [29], we need to study the topology of the canvas of (Q_G, W_G). We will do this in the case of G acting by rotations on a strongly planar QP.

Proposition 3.7.5. *Let (Q, W) be a strongly planar QP with a group G acting by rotations, and assume that there is a vertex of Q fixed by G. Then $X_{(Q_G, W_G)}$ is simply connected.*

Proof. Let us decompose $X_{(Q,W)} = \mathcal{U} \cup \mathcal{V}$, where $\mathcal{V}$ is the subcomplex consisting of all the faces adjacent to the central vertex, and $\mathcal{U}$ is the subcomplex consisting of the other faces. Since (Q, W) is strongly planar, $X_{(Q,W)}$ is homeomorphic to a disk. Note that if G is trivial, then the statement is immediate. Otherwise, this implies that the central vertex Ω has a neighbourhood in $X_{(Q,W)}$ which is itself homeomorphic to a disk. So $\mathcal{V}$ is homeomorphic to a disk as well. Thus $\mathcal{V}$ looks as in Figure 3.1, where α_i, β_i are arrows, γ_i, δ_i are paths, and all cycles $\alpha_i\gamma_i\beta_i$, $\alpha_{i+1}\delta_i\beta_i$, and $\alpha_1\delta_l\beta_l$ bound faces. The action of a generator g of G is given by adding a to indices. By picking g suitably, we can assume that $an = l$, where $n = |G|$. We choose as representatives of vertices a set $\mathcal{E}$ which contains $\{\Omega, P_1, \dots, P_a, Q_1, \dots, Q_a\}$. Observe that G acts freely on $\mathcal{U}$, and it also acts freely on $\mathcal{U} \cap \mathcal{V}$. Then $X_{(Q_G,W_G)} = \tilde{\mathcal{U}} \cup \tilde{\mathcal{V}}$, where $\tilde{\mathcal{V}}$ is as in Figure 3.2 and $\tilde{\mathcal{U}} \cong \mathcal{U}/G$ is the quotient space of $\mathcal{U}$ by G, by our construction of Q_G. In the picture we denote by $\tilde{\delta}_i$ the product of $\tilde{d}$, where d is an arrow of δ_i, and similarly for $\tilde{\gamma}_i$. We have that $\tilde{\mathcal{U}}$ is attached to $\tilde{\mathcal{V}}$ along $(\mathcal{U} \cap \mathcal{V})/G = \tilde{\mathcal{U}} \cap \tilde{\mathcal{V}}$. Now observe that since $X_{(Q,W)}$ is simply connected, it must retract to $\mathcal{V}$. In particular there is a deformation retraction F between $\mathcal{U}$ and $\mathcal{U} \cap \mathcal{V}$. We choose F such that it commutes with the action of G on $\mathcal{U}$. Then there is an induced deformation retraction $\tilde{F}$ between $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{U}} \cap \tilde{\mathcal{V}}$. In particular $X_{(Q_G,W_G)}$ retracts to $\tilde{\mathcal{V}}$, so they have the same homotopy type.

We need to describe the faces of $\tilde{\mathcal{V}}$. Let us look at the set of cycles in W involving only vertices in $\mathcal{V}$. These are $\gamma_1\beta_1\alpha_1, \dots, \gamma_a\beta_a\alpha_a, \delta_1\beta_1\alpha_2, \dots, \delta_a\beta_a\alpha_{a+1}$ and their orbits. These cycles are all of type (ii), so we have

$$\widetilde{\gamma_i\beta_i\alpha_i}^{\mu} = \tilde{\gamma}_i\tilde{\beta}_i^{\mu}\tilde{\alpha}_i^{\mu}$$
$$\widetilde{\delta_i\beta_i\alpha_{i+1}}^{\mu} = \tilde{\delta}_i\tilde{\beta}_i^{\mu}\tilde{\alpha}_{i+1}^{\mu},$$

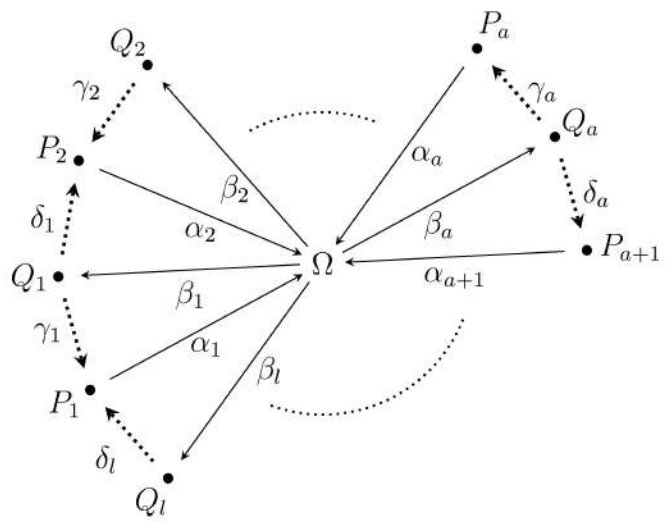


Figure 3.1. The subcomplex $\mathcal{V}$ of $X_{(Q,W)}$.

for $i = 1, \dots, a$, with the notation $\tilde{\alpha}^{\mu}_{a+1} = \tilde{\alpha}^{\mu}_1$. Now fix $\mu \in \{0, \dots, n-1\}$. Then Ω^{μ} is contained in every $\tilde{\gamma}_i\tilde{\beta}^{\mu}_i\tilde{\alpha}^{\mu}_i$, in every $\tilde{\delta}_i\tilde{\beta}^{\mu}_i\tilde{\alpha}^{\mu}_{i+1}$, and no other cycle in W_G. The subcomplex consisting of the faces corresponding to these $2a$ cycles is a disk with center Ω^{μ}. Thus $\tilde{\mathcal{V}}$ consists of n disks glued along their boundary $\tilde{\delta}_a \cdots \tilde{\gamma}_2^{-1}\tilde{\delta}_1\tilde{\gamma}_1^{-1}$, and therefore has the homotopy type of a bouquet of spheres. In particular it is simply connected, which concludes the proof. □

In Example 3.8.3 we proceed as in the proof of Proposition 3.7.5 to determine the canvas of (Q_G, W_G).

Remark 3.7.6. If G acts on a planar QP (Q, W) by rotations and (Q, W) has a G-invariant cut, then Q must have a central vertex. Indeed, Q has either a central vertex or a central cycle, but on a central cycle one cannot choose exactly one arrow in a way which is invariant under rotations.

In the self-injective case we have the following result.

Theorem 3.7.7. *Let (Q, W) be a strongly planar self-injective QP, with a group G acting by rotations and enough G-invariant cuts. Then all the truncated Jacobian algebras of (Q_G, W_G) are derived equivalent to each other.*

Proof. By Lemma 3.7.4, (Q_G, W_G) has enough cuts. By Proposition 3.7.5, $X_{(Q_G,W_G)}$ is simply connected. Then we conclude by [29, Theorem 8.7]. □

In particular, note that this result applies to QPs coming from Postnikov diagrams, provided they have enough G-invariant cuts. It should be noted that we know of no examples of a self-injective QP with a cut that does not have enough cuts, nor of a self-injective QP with a G-invariant cut that does not have enough G-invariant cuts.

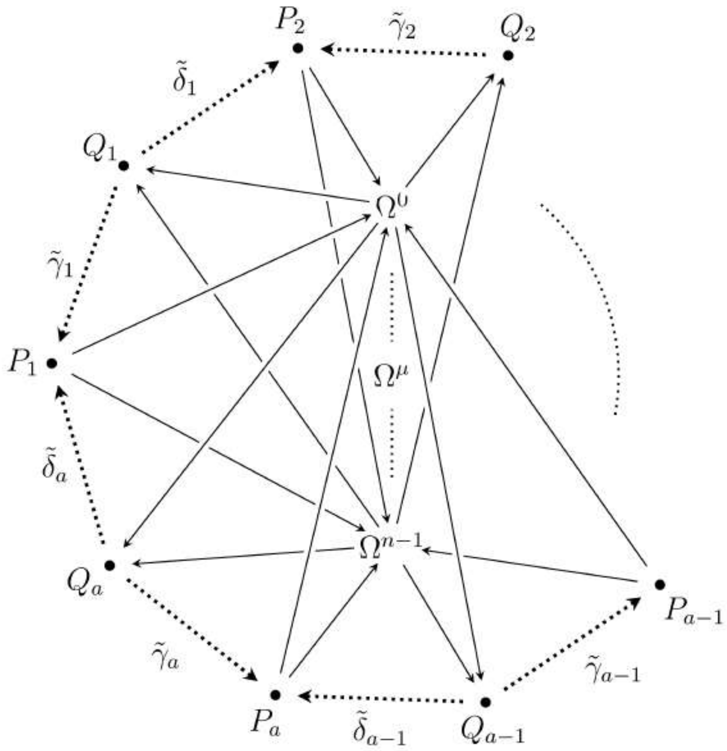


Figure 3.2. The subcomplex $\tilde{\mathcal{V}}$ of $X_{(Q_G,W_G)}$.

3.8 Examples

In this section we will illustrate our construction with some examples. For simplicity we will assume that $\Bbbk = \mathbb{C}$, so the assumption (A1) will be always satisfied.

3.8.1 Examples from planar rotation-invariant QPs

As we have seen in Section 3.6, many examples where our construction may be applied are given by quivers embedded in the plane with a group acting by rotations. Let us illustrate some of them.

Example 3.8.1 (2-representation finite algebras of type A)**.** A family of examples of self-injective planar QPs is given by 3-preprojective algebras of 2-representation finite algebras of type A, which were introduced in [40] and are defined as follows.

Let $s \geq 1$ and $Q = Q^{(s)}$ be the quiver defined by

$$Q_0 = \{(x_1, x_2, x_3) \in \mathbb{Z}^3_{\geq 0} \mid x_1 + x_2 + x_3 = s - 1\},$$

$$Q_1 = \{\alpha_i \colon x \to x + f_i \mid 1 \leq i \leq 3,\ x, x + f_i \in Q_0\},$$

where $f_1 = (-1, 1, 0)$, $f_2 = (0, -1, 1)$, $f_3 = (1, 0, -1)$. The potential W is given by the sum of all cycles of the form $\alpha_1\alpha_2\alpha_3$ minus the ones of the form $\alpha_1\alpha_3\alpha_2$.

The Nakayama automorphism of $\Lambda = \mathcal{P}(Q,W)$ is induced by the unique automorphism of Q given on vertices by $(x_1, x_2, x_3) \mapsto (x_3, x_1, x_2)$. Then the group G generated by it acts on Q by an anticlockwise rotation by $2\pi/3$. We may note that this action has a (unique) fixed vertex if and only if $s \equiv 1 \pmod 3$. In that case the vertex $(\frac{s-1}{3}, \frac{s-1}{3}, \frac{s-1}{3})$ is fixed.

Proposition 3.8.2. *If* $s \equiv 1 \pmod 3$*, then* $Q^{(s)}$ *has enough* G*-invariant cuts.*

Proof. Call $x_0 = (\frac{s-1}{3}, \frac{s-1}{3}, \frac{s-1}{3})$ the unique fixed vertex. Let $L = \{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid x_1 + x_2 + x_3 = 0\}$ and note that it is a free abelian group of rank 2 with basis $\{f_1, f_2\}$. We may embed Q_0 in L via the map $x \mapsto x - x_0$. Note that the action of G on Q_0 can be naturally extended to an action on L, which is again given by $(x_1, x_2, x_3) \mapsto (x_3, x_1, x_2)$.

Let $\omega \colon L \to \mathbb{Z}/3\mathbb{Z}$ be the group homomorphism defined by $\omega(f_i) = 1$ for $i = 1, 2, 3$. For each $j \in \mathbb{Z}/3\mathbb{Z}$ we define the following subset of Q_1:

$$C_j = \{\alpha_i \colon x \to x + f_i \mid \omega(x - x_0) = j\}.$$

Then C_j is a cut (cf. [31, Example 5.8]). It is symmetric because ω is invariant on G-orbits. Moreover every arrow is contained in a cut of this type, so the statement follows. $\square$

As an example, we illustrate the cut C_0 of $Q^{(7)}$ in Figure 3.3.

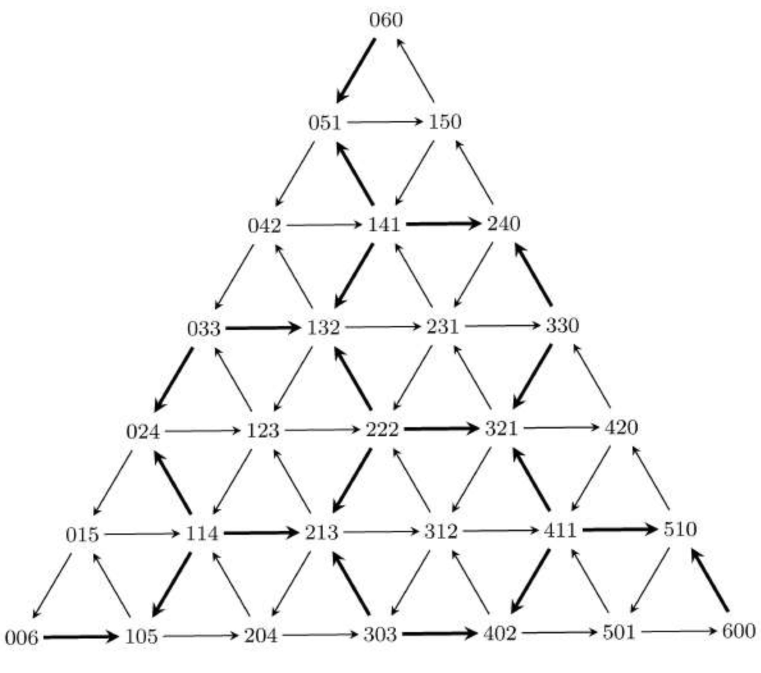


Figure 3.3. The quiver $Q^{(7)}$. The cut C_0 is given by the thick arrows.

Now we will describe our skew group algebra construction for the quiver $Q = Q^{(4)}$ (which is depicted in Figure 3.4).

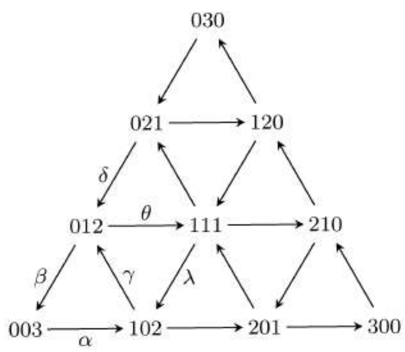


Figure 3.4. The quiver $Q^{(4)}$.

We can choose, for example, $\mathcal{E} = \{(0,0,3),(0,1,2),(1,0,2),(1,1,1)\}$ as a set of representatives of vertices. For simplicity we shall denote the elements of this set by $\{1,2,3,4\}$ respectively. Then Q_G (depicted in Figure 3.5) has vertices $\eta^1,\eta^2,\eta^3,\eta^4_0,\eta^4_1,\eta^4_2$, which will be denoted respectively by $1,2,3,4^0,4^1,4^2$. We will also rename the arrows of type (1), (2), (3) in Q. These are

$$\alpha\colon 1\to 3,\quad \beta\colon 2\to 1,\quad \gamma\colon 3\to 2,\quad \delta\colon g^2(3)\to 2 \quad \text{of type (1)},$$
$$\theta\colon 2\to 4 \quad \text{of type (2)},$$
$$\lambda\colon 4\to 3 \quad \text{of type (3)}.$$

We take $\mathcal{C} = \{c_1, c_2, c_3\}$, where $c_1 = \alpha\beta\gamma$ is of type (i) and $c_2 = \lambda\theta\gamma$, $c_3 = \lambda g(\theta)g(\delta)$ are of type (ii). Note that $p(c_2) = 0$ and $p(c_3) = 1$. Then we get

$$W_G = -\tilde{\alpha}\tilde{\beta}\tilde{\gamma} + \sum_{\mu=0}^{2} \tilde{\lambda}^{\mu}\tilde{\theta}^{\mu}\tilde{\gamma} - \sum_{\mu=0}^{2} \zeta^{-\mu}\tilde{\lambda}^{\mu}\tilde{\theta}^{\mu}\tilde{\delta}.$$

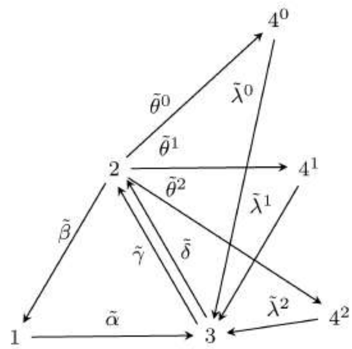


Figure 3.5. The quiver $Q_G^{(4)}$.

By the results in Section 3.5, the dual group $\hat{G} = \langle\chi\rangle$ acts on Q_G as follows. The vertices $1,2,3$ are fixed, while $\chi(4^{\mu}) = 4^{\mu+1}$, $\mu = 0,1,2$. The arrows $\tilde{\alpha},\tilde{\beta},\tilde{\gamma}$ are fixed,

$\chi(\tilde{\theta}^\mu) = \tilde{\theta}^{\mu+1}$ and $\chi(\tilde{\lambda}^\mu) = \tilde{\lambda}^{\mu+1}$, $\mu = 0, 1, 2$. Since $t(\delta) = 2$, we have $\chi(\tilde{\delta}) = \zeta^2\tilde{\delta}$. Note that, in the process of getting back the initial quiver using the isomorphism ϕ of Proposition 3.5.3, the vertices $4^0, 4^1, 4^2$ give rise to the vertex $(1, 1, 1)$ of Q, the vertex 2 gives rise to $(0, 1, 2), (1, 2, 0), (2, 0, 1)$, 3 to $(0, 2, 1), (2, 1, 0), (1, 0, 2)$ and 1 to $(0, 0, 3), (0, 3, 0), (3, 0, 0)$.

Example 3.8.3 (Self-injective QPs from Postnikov diagrams)**.** In this example we illustrate Corollary 3.6.10 and (the proof of) Proposition 3.7.5. Let Q be the quiver of Figure 3.6, with the potential W given by the sum of the clockwise faces minus the sum of the anticlockwise faces. Thus (Q, W) is a strongly planar quiver with potential. It is constructed from a rotation-invariant $(4, 16)$-Postnikov diagram, see [54, Figure 19]. By Theorem 3.6.6, its Jacobian algebra Λ is therefore self-injective, with Nakayama automorphism φ induced by a rotation by $\frac{\pi}{2}$. Let us consider the group $G = \langle\varphi^2\rangle$. Then the skew group algebra $\Lambda * G$ is Morita equivalent to the Jacobian algebra $\mathcal{P}(Q_G, W_G)$, where Q_G is depicted in Figure 3.7. The canvas $X_{(Q_G,W_G)}$ is given by an octahedron in the middle attached to an annulus made of all the remainining faces. Note that this describes the potential W_G completely up to signs. This algebra is self-injective with Nakayama automorphism given by $\varphi \otimes 1$, but it is not symmetric since its Nakayama permutation has order 2.

If we instead take the skew group algebra construction with respect to $\langle\varphi\rangle$, we get the quiver of Figure 3.8. Its canvas is an annulus consisting of the outer cycles, attached to four disks sharing their boundary circle. These disks are subdivided into two triangles each. Again note that describing the canvas determines the potential up to fourth roots of unity. This algebra is symmetric by Corollary 3.6.10.

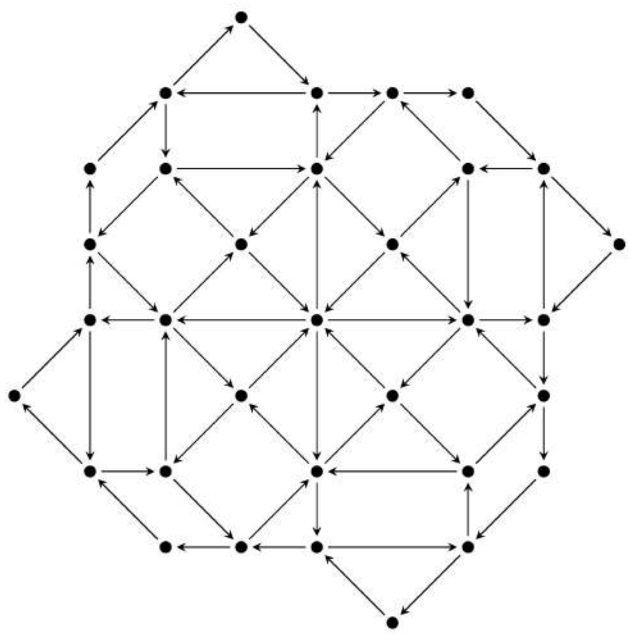


Figure 3.6. A self-injective QP with Nakayama automorphism φ of order 4.

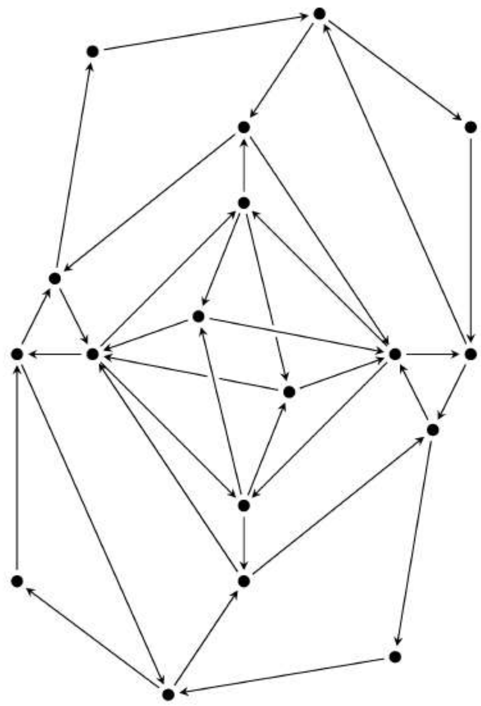


Figure 3.7. The quiver of the skew group algebra $\Lambda * \langle \varphi^2 \rangle$.

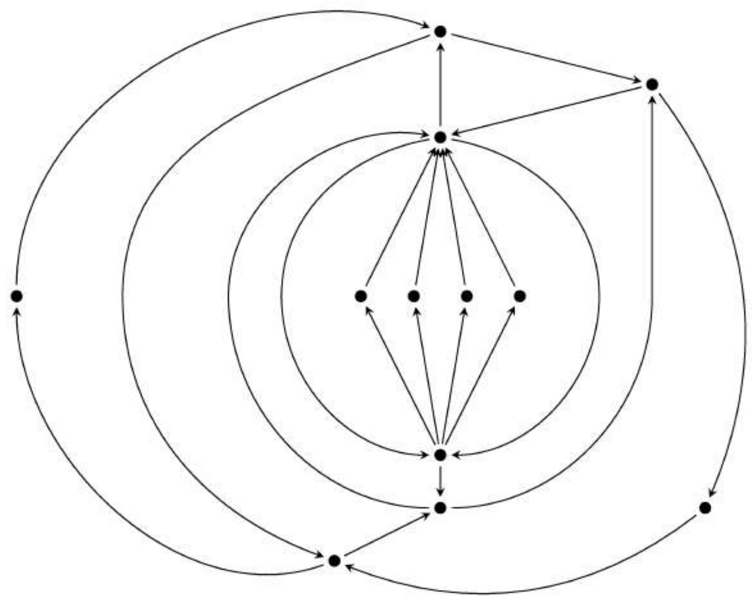


Figure 3.8. The quiver of the skew group algebra $\Lambda * \langle \varphi \rangle$.

3.8.2 Examples from tensor products of quivers

The following family of self-injective QPs was introduced in [29, §5.2]. Let us recall their definition.

Given two quivers Q^1, Q^2 without oriented cycles we can define a new quiver $Q = Q^1 \tilde{\otimes} Q^2$ with $Q_0 = Q_0^1 \times Q_0^2$ and $Q_1 = (Q_0^1 \times Q_1^2) \sqcup (Q_1^1 \times Q_1^2) \sqcup (Q_1^1 \times Q_0^2)$. The starting and ending points of the arrows of Q are given by

$$\mathfrak{s}(\alpha, y) = (\mathfrak{s}(\alpha), y), \quad \mathfrak{s}(x, \beta) = (x, \mathfrak{s}(\beta)), \quad \mathfrak{s}(\alpha, \beta) = (\mathfrak{t}(\alpha), \mathfrak{t}(\beta)),$$

$$\mathfrak{t}(\alpha, y) = (\mathfrak{t}(\alpha), y), \quad \mathfrak{t}(x, \beta) = (x, \mathfrak{t}(\beta)), \quad \mathfrak{t}(\alpha, \beta) = (\mathfrak{s}(\alpha), \mathfrak{s}(\beta)),$$

for $x \in Q_0^1$, $y \in Q_0^2$, $\alpha \in Q_1^1$, $\beta \in Q_1^2$. We define a potential on Q by

$$W = W^{\tilde{\otimes}}_{Q^1,Q^2} = \sum_{\alpha \in Q_1^1, \beta \in Q_1^2} (\alpha, \mathfrak{t}(\beta))(\mathfrak{s}(\alpha), \beta)(\alpha, \beta) - (\mathfrak{t}(\alpha), \beta)(\alpha, \mathfrak{s}(\beta))(\alpha, \beta).$$

Now we consider group actions on $\Bbbk Q$. Let $G_1 = \langle g_1 \rangle$ and $G_2 = \langle g_2 \rangle$ be finite cyclic groups and suppose that the following condition holds:

($*$) either one of G_1 or G_2 is trivial, or $G_1 \cong G_2$.

We denote by n the maximum of the orders of G_1 and G_2. Let G be the subgroup of $G_1 \times G_2$ generated by (g_1, g_2), and note that it is cyclic of order n.

Lemma 3.8.4. *Let Q^1, Q^2, G_1, G_2 as above. Suppose we have actions of G_i on $\Bbbk Q^i$, $i = 1, 2$, which satisfy the assumptions (A1), (A2), (A4), (A6), and:*

(A3') every arrow in Q^i between two fixed vertices is fixed by G_i.

Then the induced action of G on $\Bbbk Q$ satisfies the assumptions (A1)-(A7).

Proof. Assumption (A1) holds by the assumptions on the orders of G_1 and G_2. The assumptions (A2), (A3) and (A4) follow immediately by hypothesis. Now note that G permutes the cycles of the potential $W^{\tilde{\otimes}}_{Q^1,Q^2}$, and every cycle is sent to a cycle with the same coefficient. Hence $GW = W$. Finally, assumption (A7) is satisfied because all cycles have length 3. □

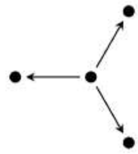
If Q^1 and Q^2 are Dynkin quivers with the same Coxeter number and which are stable under their canonical involutions (see [29, §5.2] for definitions), then $(Q, W) = (Q^1 \tilde{\otimes} Q^2, W^{\tilde{\otimes}}_{Q^1,Q^2})$ is a self-injective QP by [29, Proposition 5.1]. Let g_1 and g_2 be the unique automorphisms of, respectively, Q^1 and Q^2 given by extending to arrows their canonical involutions.

Proposition 3.8.5. *Let Q^1 and Q^2 be Dynkin quivers which are stable under their canonical involutions and have the same Coxeter number. Let G be the cyclic group generated by (g_1, g_2) and consider the induced action of G on $Q = Q^1 \tilde{\otimes} Q^2$. Then (Q_G, W_G) is a self-injective QP with enough cuts.*

Proof. Note that g_1 and g_2 have order either 1 or 2, so the condition $(*)$ for $G_1 = \langle g_1 \rangle$ and $G_2 = \langle g_2 \rangle$ is satisfied. The assumptions (A1), (A2), (A3'), (A4), and (A6) for G_1 and G_2 are immediately checked, so by Lemma 3.8.4 we can apply the construction of Section 3.3 to (Q, W) and G. By [29, Proposition 5.1] (Q, W) is self-injective, hence so is (Q_G, W_G).

From the definition of $W^{\tilde{\otimes}}_{Q^1,Q^2}$ it follows that the subsets (Q^1_0, Q^2_1), (Q^1_1, Q^2_1) and (Q^1_1, Q^2_0) of Q_1 are all G-invariant cuts. Since every arrow of Q is contained in one of them, we have that (Q, W) has enough G-invariant cuts. Hence (Q_G, W_G) has enough cuts by Lemma 3.7.4. □

Example 3.8.6. Consider the following Dynkin quivers:

Q^1: Q^2: 

Here Q^1 is of type A_5 and Q^2 of type D_4, so they have the same Coxeter number. The canonical involution of Q^1 is the reflection with respect to the central vertex, while the one of Q^2 is the identity. Hence the two quivers are stable and, by Proposition 3.8.5, (Q_G, W_G) is a self-injective QP with enough cuts. The quivers Q and Q_G are illustrated respectively in Figures 3.9 and 3.10.

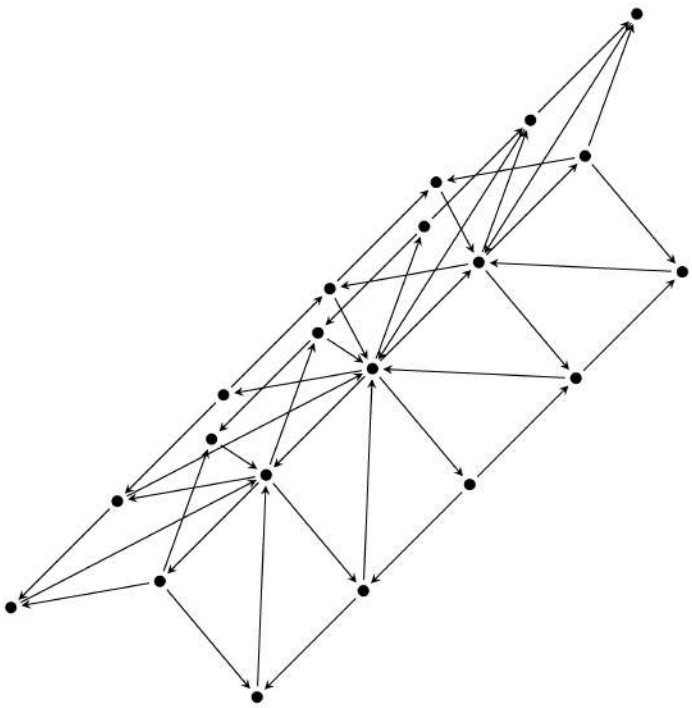


Figure 3.9. The quiver $Q^1 \tilde{\otimes} Q^2$.

All examples we have illustrated so far are related to self-injective QPs. In the next one we will consider a case where the QP we start with is not self-injective.

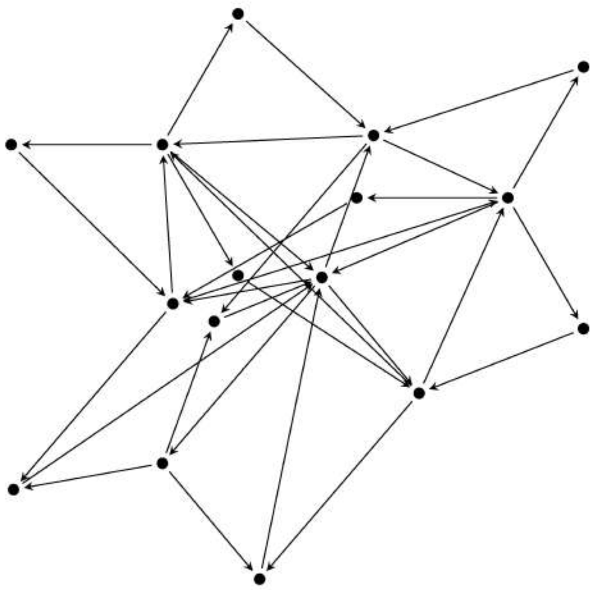


Figure 3.10. The quiver $(Q^1\tilde{\otimes}Q^2)_G$.

Example 3.8.7. Consider the Dynkin quivers

$$Q^1\colon \quad 2 \xleftarrow{\beta} 1 \xleftarrow{\alpha} 0 \xrightarrow{\alpha'} 1' \xrightarrow{\beta'} 2' \qquad\qquad Q^2\colon \quad 0 \xleftarrow{\gamma_1} 1 \xleftarrow{\gamma_2} 2$$

and let $Q = Q^1\tilde{\otimes}Q^2$ (see Figure 3.11).

Let g be the unique automorphism of Q^1 given on vertices by $g(0) = 0$, $g(i) = i'$ and $g(i') = i$, $i = 1, 2$. Then we can consider the action of the cyclic group $G = \langle (g, \mathrm{id})\rangle$ of order 2 on Q. If we apply the construction of Section 3.3 choosing as a set of representatives of the vertices $\mathcal{E} = \{(i, j) \mid i, j = 0, 1, 2\}$, then we obtain the quiver Q_G of Figure 3.12. We can take

$$\mathcal{C} = \{(\alpha, i{-}1)(0, \gamma_i)(\alpha, \gamma_i), (1, \gamma_i)(\alpha, i)(\alpha, \gamma_i), (\beta, i{-}1)(0, \gamma_i)(\beta, \gamma_i), (1, \gamma_i)(\beta, i)(\beta, \gamma_i) \mid i = 1, 2\}$$

and obtain the potential

$$\begin{aligned} W_G = &\sum_{i=1}^{2} \widetilde{(\beta, i-1)}\widetilde{(0, \gamma_i)}\widetilde{(\beta, \gamma_i)} - \widetilde{(1, \gamma_i)}\widetilde{(\beta, i)}\widetilde{(\beta, \gamma_i)} + \\ &+ \sum_{i=1}^{2}\sum_{\mu=0}^{1} \widetilde{(\beta, i-1)}\widetilde{(0, \gamma_i)}^{\mu}\widetilde{(\beta, \gamma_i)} - \widetilde{(1, \gamma_i)}^{\mu}\widetilde{(\beta, i)}\widetilde{(\beta, \gamma_i)}. \end{aligned}$$

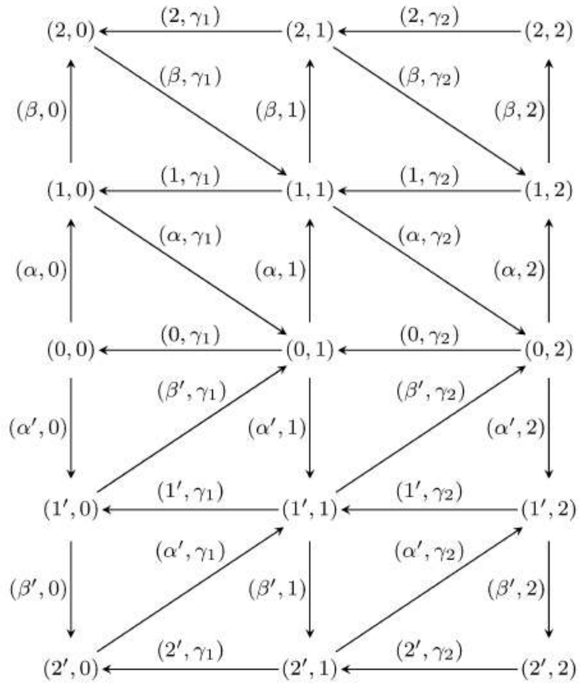


Figure 3.11. The quiver $Q^1\tilde{\otimes}Q^2$.

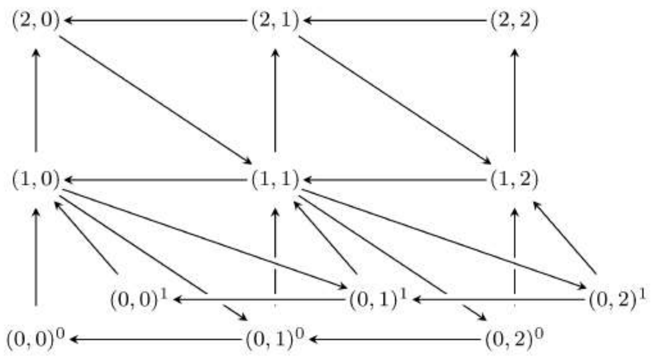


Figure 3.12. The quiver $(Q^1\tilde{\otimes}Q^2)_G$.

Remark 3.8.8. We may choose another basis for $\operatorname{rad}\mathcal{P}(Q,W)/\operatorname{rad}^2\mathcal{P}(Q,W)$ by replacing (α',i) with $-(\alpha',i)$ and (β',i) with $-(\beta',i)$, $i=0,1,2$. In this way we get that $\mathcal{P}(Q,W)\cong\mathcal{P}(Q,W')$, where W' is the potential defined as the sum of all the clockwise 3-cycles minus the sum of all the anticlockwise ones. We have an action of G on $\mathcal{P}(Q,W')$ such that $\mathcal{P}(Q,W)*G\cong\mathcal{P}(Q,W')*G$, but note that in this case the assumption (A6) is no longer satisfied.

Now let us consider the G-invariant cut $C=Q^0\times Q^1$ in Q. We may note that the truncated Jacobian algebra $\mathcal{P}(Q,W')_C$ is isomorphic to the Auslander algebra of Q^1. Moreover, by Proposition 3.7.2 and what we observed above, we have that $(\mathcal{P}(Q,W)_C)*G\cong$

$(\mathcal{P}(Q, W')_C) * G$ is Morita equivalent to $\mathcal{P}(Q_G, W_G)_{C_G}$. Notice that $\mathcal{P}(Q_G, W_G)_{C_G}$ is isomorphic to the Auslander algebra of a Dynkin quiver of type D_4. This is no surprise, since we know by [57, Theorem 1.3(c)(iv)] that skew group algebras of Auslander algebras are again Auslander algebras.

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