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SEPARATE ANALYTICITY
AND HOLOMORPHIC SECTORS

Direttore della Scuola: Ch.mo Prof. Paolo Dai Pra

Coordinatore d'indirizzo: Ch.mo Prof. Franco Cardin

Supervisore: Ch.mo Prof. Giuseppe Zampieri

Dottorando: Dott. Raffaele Marigo

Introduzione

Nel primo capitolo di questa tesi studieremo il fenomeno della separata analiticità: nel caso complesso è ben noto (Hartogs, 1906) che una funzione di n variabili complesse è olomorfa se e solo se è separatamente olomorfa in ogni variabile (vedi [11, 12, 16]). Dopo aver osservato che ÃÍ sufficiente supporre $n = 2$ (possiamo in seguito iterare le conclusioni, aggiungendo una alla volta le variabili), dimostriamo il passaggio fondamentale del teorema di Hartogs: se f è una funzione definita in $\Delta \times \Delta \subset \mathbb{C}^2$, olomorfa per $|z_2| < \epsilon$ e separatamente olomorfa in z_2 quando z_1 è fissato, allora f è olomorfa nel complesso delle due variabili. La convergenza normale della serie di Taylor di f è data dal lemma di Hartogs per funzioni subarmoniche. Tale risultato è stato generalizzato in più direzioni; nel lavoro presente si considera il caso in cui f è separatamente olomorfa lungo le rette complesse, uscenti da una curva reale γ, che fogliano un'ipersuperficie reale $M \subset \mathbb{C}^2$ e olomorfa in un intorno di γ. Allora f è olomorfa in un intorno di M. Questa generalizzazione del lemma di Hartogs offre una nuova interpretazione geometrica di un teorema di Siciak ([17]) sulla separata analiticità reale: se una funzione in $\mathbb{R}^2$ è separatamente analitica reale in una variabile, e si estende ad una funzione olomorfa in una striscia uniforme nella seconda, allora è analitica reale nel complesso delle due variabili (Baracco-Zampieri, [5]).

Nella seconda parte trattiamo l'estensione di funzioni olomorfe definite in un intorno di un wedge V con edge non generico in una varietà generica M. Viene definito l'angolo complesso $\alpha\pi$ di V in un punto $p \in \partial V$ come il massimo angolo di intersezione del cono tangente a V in quel punto con una retta complessa. Nel caso in cui V sia senza bordo ($\alpha = 2$), o se l'edge di V e generico ($\alpha = 1$), le teorie classiche di Boggess-Polking ([8]) e Tumanov ([19]) assicurano l'estensione delle funzioni olomorfe in un intorno di V ad un wedge V' su V. In [21] e [22] Zaitsev e Zampieri hanno generalizzato il problema al caso $\frac{1}{2} < \alpha < 1$: le funzioni olomorfe nell'intorno del wedge, in questa situazione, si estendono ad un cosiddetto α-wedge su V (tale insieme può essere visto come un wedge la cui componente normale ha un andamento $\frac{1}{\alpha}$). Per ottenere questo risultato viene introdotta una nuova teoria di dischi analitici con una singolarità α-Lipschitz in un punto di bordo: proprietà fondamentale di tali dischi α-lipschitziani è che la loro componente normale viene resa regolare dalla composizione con la funzione h di cui M è il grafo. Grazie a questo fatto è possibile controllare la direzione di tali α-dischi nel momento in cui vengono attaccati alla varietà. Nel nostro lavoro viene presentata la naturale generalizzazione della teoria al caso $\alpha \leq \frac{1}{2}$: per rendere regolare la composizione della componente normale dei dischi con h, chiederemo che $h = O^k$ (cioè M piatta e rigida all'ordine k) per $k > \frac{1}{\alpha}$.

Introduction

In the first chapter of this thesis we study separate analyticity, starting from the complex setting: it is a well known fact, proved by Hartogs in 1906 (see [11, 12, 16]), that a function of n-complex variables is holomorphic if and only if it is separately holomorphic in each variable. First we remark that, by use of iteration, it is not restrictive to assume $n = 2$. Once we are in dimension 2, we observe that the main step in the proof of Hartogs' theorem consists in showing that if a function f defined in $\Delta \times \Delta \subset \mathbb{C}^2$ is holomorphic for $|z_2| < \epsilon$ and separately holomorphic in z_2 when z_1 is kept fixed, then it is jointly holomorphic; the normal convergence of the Taylor series of f is obtained through the celebrated Hartogs' lemma on subharmonic functions. This result has been generalized in various directions and following different approaches; in our work we consider the case where f is separately holomorphic along the complex lines issued from a real curve γ, which foliate a real hypersurface $M \subset \mathbb{C}^2$, and holomorphic in a neighborhood of γ. Then it is holomorphic in a neighborhood of M. This generalization of Hartogs' lemma also offers a geometric interpretation of a theorem by Siciak ([17]) about separate real analyticity: it is proved that a function in $\mathbb{R}^2$ which is separately real analytic in one variable and CR extendible in the other (that is separately holomorphically extendible to a uniform strip), is real analytic (see Baracco and Zampieri, [5]).

In the second part we deal with the extension of holomorphic functions defined in a neighborhood of a wedge V with non generic edge on a generic manifold M. We define the complex angle $\alpha\pi$ of V at a point $p \in \partial V$ as the maximal angle of the intersection of the tangent cone to V at p with a complex line. If V has no boundary ($\alpha = 2$), or if the edge of V is generic ($\alpha = 1$), the classical theories of Boggess-Polking ([8]) and Tumanov ([19]) yield the extension

of holomorphic functions defined in a neighborhood of V to a wedge V' over V. In [21] and [22], Zaitsev and Zampieri generalized the problem to the case $1/2 < \alpha < 1$: in this situation, holomorphic functions defined in a neighborhood of the wedge extend to a so-called α-wedge over V (this can be viewed as a wedge in the space where the normal directions have a weight $1/\alpha$). To obtain this result, a new theory of analytic discs with an α-Lipschitz singularity at a boundary point was introduced: the main property of this new class of α-Lipschitz discs is that the conposition of their normal component with the function h which graphs M is smooth. Hence it is possible to control the direction of these α-discs when they are attached to the manifold. In this work we present the natural generalization of this theory to the case $\alpha \leq 1/2$: to keep the composition of the normal component of the discs with h regular, we will ask that $h = O^k$ (i.e. M is flat and rigid to the order k) for $k > 1/\alpha$.

Contents

Chapter 1

Separate Analyticity

1.1 Separate complex analyticity

We study complex valued functions defined in open sets of $\mathbb{C}^n$. Identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, writing complex coordinates $z = x + iy$, with $z = (z_1, \ldots, z_n)$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ e $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, and where $i = \sqrt{-1}$. We also write $x = \operatorname{Re} z$ e $y = \operatorname{Im} z$. Defining the conjugate of z as $\bar{z} = x - iy$, we have a linear change of real coordinates given by

$$(x, y) \mapsto (z, \bar{z}) = (x + iy, x - iy),$$

with inverse

$$(z, \bar{z}) \mapsto \left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i} \right).$$

This transformation can be read on the derivatives as

$$\begin{cases} \partial_x = \partial_z + \partial_{\bar{z}} \\ \partial_y = i(\partial_z - \partial_{\bar{z}}), \end{cases}$$

with inverse

$$\begin{cases} \partial_z = \frac{1}{2}\left(\partial_x - i\partial_y\right) \\ \partial_{\bar{z}} = \frac{1}{2}\left(\partial_x + i\partial_y\right), \end{cases}$$

and in the dual base of differentials as

$$\begin{cases} dx = \frac{1}{2}(dz + d\bar{z}) \\ dy = \frac{1}{2i}(dz - d\bar{z}), \end{cases}$$

$$\begin{cases} dz = dx + idy \\ d\bar{z} = dx - idy \end{cases}$$

If $f : \mathbb{C}^n \to \mathbb{C}$ is a C^1 function, we can write it as

$$df = \partial f + \bar{\partial} f = \sum_{j=1}^{n} \left(\partial_{z_j} f dz_j + \partial_{\bar{z}_j} f d\bar{z}_j \right).$$

Definition 1.1. *Let Ω be an open subset of $\mathbb{C}^n$. A function $f \in C^1(\Omega)$ is said to be* holomorphic *when $\bar{\partial} f = 0$, that is*

$$\partial_{\bar{z}_j} f = 0 \tag{1.1}$$

for any $j = 1, \dots, n$.

We are interested in studying problems of separate analyticity: f is said to be *separately holomorphic* in the variable z_j if it is holomorphic in z_j when the other $n-1$ variables are kept fixed. As we will see in the next section, if f is separately analytic in each of the n complex variables, then it is automatically jointly holomorphic; that is, the C^1 regularity required in the definition above is a direct consequence of the separate holomorphy. All the results in this chapter are stated in $\mathbb{C}^2$: they can easily be generalized by iteration.

We begin by noticing that, if we assume continuity, or even less as boundedness, the conclusion is immediate

Proposition 1.1. *Let f be a continuous function in a domain $\Omega \subset \mathbb{C}^2$, separately holomorphic in both variables when the other is kept fixed. Then $f \in C^\infty(\Omega)$, and in particular f is holomorphic.*

Proof. Let D_1 and D_2 two discs such that their product $D = D_1 \times D_2$ is contained in Ω, where f is separately holomorphic. Then, by Cauchy's formula, we can write

$$\begin{aligned} f(z) &= \int_{\partial D_1} \left(\int_{\partial D_2} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta_2 \right) d\zeta_1 \\ &= \int\int_{\partial D_1 \times \partial D_2} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta_1 \wedge d\zeta_2, \end{aligned}$$

where the second equality is given by Fubini's theorem (that we can apply because f is continuous). Differientiating under the integral sign, we obtain our conclusion. $\square$

Proposition 1.2. *Let f be separately holomorphic and bounded on compact subsets of Ω. Then f is holomorphic.*

Proof. By Proposition 1.1 we just need to prove that f is continuous. Let $z^0 \in \Omega$, and $z_1^\nu \to z_1^0$, and let z_2 move near z_2^0 so that (z_1^ν, z_2) stays at a distance bigger of r from the boundary of Ω. Let c be a uniform bound for $|f|$ in the r-neighborhood of these points. Set $F_\nu(z_2) := f(z_1^\nu, z_2)$, then for Cauchy's inequalities we have

$$|\partial_{z_2} F_\nu(z_2)| \leq \frac{c}{r};$$

hence $\{F_\nu\}$ is equicontinuous and, if we also take $z_2^\nu \to z_2^0$, $f(z_1^\nu, z_2^\nu) \to f(z_1^0, z_2^0)$. $\square$

1.2 Hartogs theorem

We now state Hartogs' celebrated result of [11], where the equivalence of joint and separate complex analyticity is proved (see also [12], [16]).

Theorem 1.1. *Let Ω be an open set in $\mathbb{C}^2$ and $f : \Omega \mapsto \mathbb{C}$ a function such that*

$$\begin{cases} z_1 \mapsto f(z_1, z_2^0) \text{ is holomorphic, for all } z_2^0 \\ z_2 \mapsto f(z_1^0, z_2) \text{ is holomorphic, for all } z_1^0. \end{cases}$$

Then f is holomorphic.

Notice that the statement is local, hence we can prove it for f defined in polydiscs. The first step of the proof consists in gaining a small region of joint analyticity.

Proposition 1.3. *Let $f : \Delta \times \Delta \to \mathbb{C}$ be a function separately holomorphic in each of the two variables. Then there exists $\delta > 0$ such that $f \in hol(\Delta \times \Delta_\delta)$.*

Proof. Define

$$E_l = \{z_2 \in \Delta : |f(z_1, z_2)| \leq l, \forall z_1 \in \Delta\};$$

E_l is closed because f is continuous in the variable z_2 for a fixed z_1. Moreover

$$\cup_l E_l = \Delta.$$

But then, for Baire category theorem, there exists l_0 such that E_l has a non-empty interior whenever $l \geq l_0$. We can find a dense open set $B \subset \Delta$ such that, applying Proposition 1.2, f is holomorphic in $\Delta \times B$. Assuming that $0 \in B$ (up to an arbitrarily small shrinking of the analyticity domain), we take a disc $\Delta_\delta \subset B$, centered at 0: then $f \in hol(\Delta \times \Delta_\delta)$. $\square$

We are now in the following situation: we have a function $f : \Delta \times \Delta \to \mathbb{C}$, jointly holomorphic in a strip $\Delta \times \Delta_\delta$, and separately holomorphic in the variable z_2 for any fixed value of $z_1 \in \Delta$ (we will not need the separate analyticity in z_1 anymore).

The Taylor expansion of f with respect to z_2 and center in $z_2 = 0$ is:

$$f(z_1, z_2) = \sum_{\nu=0}^{+\infty} \frac{\partial_{z_2}^{\nu} f(z_1, 0)}{\nu!} z_2^{\nu}. \tag{1.2}$$

We would like it to converge normally in $\Delta \times \Delta$, making f holomorphic there. The coefficients $\frac{\partial_{z_2}^{\nu} f(z_1,0)}{\nu!}$ are holomorphic in z_1, hence the functions

$$\varphi_\nu(z_1) = \left| \frac{\partial_{z_2}^{\nu} f(z_1, 0)}{\nu!} \right|^{\frac{1}{\nu}} \tag{1.3}$$

are subharmonic.

By the separate holomorphy in z_2 and Cauchy-Hadamard criterion on the convergence radius of power series, we have:

$$\limsup_{\nu \to \infty} \varphi_\nu(z_1) \leq 1, \text{ for any fixed } z_1 \in \Delta,$$

while the joint analyticity in $\Delta \times \Delta_\delta$, along with Cauchy inequalities, gives us the following uniform estimate:

$$\limsup_{\nu \to \infty} \sup_{z_1 \in \Delta} \varphi_\nu(z_1) \leq \delta^{-1}.$$

These conditions allow us to gain the uniformity in z_1 of the normal convergence in z_2 of the series (1.2), making use of the following fundamental result on subharmonic functions:

Lemma 1.1. *Let $\{\varphi_\nu\}$ be a sequence of subharmonic functions defined on Δ, and suppose there exist constants $m < M$ such that*

$$\limsup_{\nu \to \infty} \sup_{z \in \Delta} \varphi_\nu(z) \leq M$$

and

$$\limsup_{\nu \to \infty} \varphi_\nu(z) \leq m, \quad \forall z \in \Delta.$$

Then, for fixed $r < 1$, we have

$$\limsup_{\nu \to \infty} \sup_{|z| \leq r} \varphi_\nu(z) \leq m.$$

Proof. Fix $r < \rho < 1$, and let $\alpha > 0$. First we prove that, for any $\alpha' > 0$, there exist a measurable subset E of $\partial\Delta_\rho$ with $\lambda(E) < \alpha'$ (where λ is the Lebesgue measure on $\partial\Delta_\rho$) and $\nu_{\alpha,\alpha'} \in \mathbb{N}$, such that $\varphi_\nu(\zeta) < m + \alpha$ for all $\zeta \in \partial\Delta_\rho \setminus E$, when $\nu \geq \nu_{\alpha,\alpha'}$. Define

$$E_\nu = \bigcup_{\mu \geq \nu} \{\zeta \in \partial\Delta_\rho : \varphi_\mu(\zeta) \geq m + \alpha\};$$

then $E_{\nu+1} \subset E_\nu$ and, since $\limsup_{\nu\to\infty} \varphi_\nu(\zeta) \leq m$ for $\zeta \in \partial\Delta_\rho$, $\bigcap_{\nu=1}^\infty E_\nu = \emptyset$. Hence we can find $\nu_{\alpha,\alpha'} \in \mathbb{N}$ with $\lambda(E_{\nu_{\alpha,\alpha'}}) < \alpha'$. Take $E = E_{\nu_{\alpha,\alpha'}}$, $P_z(\zeta)$ the Poisson kernel of Δ_ρ, and $C = \sup P_z(\zeta)$ for $|z| \leq r$ and $\zeta \in \partial\Delta_\rho$.
If $|z| \leq r$, then

$$\varphi_\nu(z) \leq \int_E P_z(\zeta)\varphi_\nu(\zeta)d\lambda(\zeta) \; + \int_{\partial\Delta_\rho\setminus E} P_z(\zeta)\varphi_\nu(\zeta)d\lambda(\zeta)$$

by subharmonicity of φ_ν.
For a big enough ν, the first integral is:

$$\begin{aligned} \int_E P_z(\zeta)\varphi_\nu(\zeta)d\lambda(\zeta) &\leq \int_E C(M+\alpha)d\lambda(\zeta) \\ &\leq C(M+\alpha)\lambda(E). \end{aligned}$$

As for the second integral, assuming $m \geq 0$ (up to translation), we notice that for $\nu \geq \nu_{\alpha,\alpha'}$:

$$\begin{aligned} \int_{\partial\Delta_\rho\setminus E} P_z(\zeta)\varphi_\nu(\zeta)d\lambda(\zeta) &\leq \int_{\partial\Delta_\rho\setminus E} P_z(\zeta) \sup_{\partial\Delta_\rho\setminus E} \varphi_\nu d\lambda(\zeta) \\ &\leq \int_{\partial\Delta_\rho\setminus E} P_z(\zeta)(m+\alpha)d\lambda(\zeta) \\ &\leq (m+\alpha)\int_{\partial\Delta_\rho\setminus E} P_z(\zeta)d\lambda(\zeta) \\ &\leq m+\alpha, \end{aligned}$$

where the first inequality is trivial, the second follows from the definition of E, the third one from the positivity of $P_z(\zeta)$ and the fourth from the fact that

$$\int_{\partial\Delta_\rho} P_z(\zeta)d\lambda(\zeta) = 1.$$

Choose α' satisfying $C(M+\alpha)\alpha' \leq \alpha$: recalling that
$\lambda(E) < \alpha'$, for $\nu > \nu_{\alpha,\alpha'} = \nu_\alpha$ we have

$$\varphi_\nu(z) \leq m + 2\alpha,$$

uniformly for $|z| \leq r$. □

Theorem 1.2. *Let* $f : \Delta \times \Delta \to \mathbb{C}$ *satisfying*

$$\begin{cases} f \in hol(\Delta \times \Delta_\delta) \\ f \in hol(\{z_1^0\} \times \Delta), \text{ for all } z_1^0 \in \Delta. \end{cases}$$

Then $f \in hol(\Delta \times \Delta)$.

Proof. We saw that under these hypotheses the estimates of Lemma 1.1 for the subharmonic functions $\{\varphi_\nu\}$ defined in (1.3) hold with $m = 1$ e $M = \delta^{-1}$; then we have

$$\limsup_{\nu \to \infty} \sup_{|z_1| \leq r} \varphi_\nu(z_1) \leq 1$$

for all $r < 1$. This implies we have a ν_r such that, for $\nu \geq \nu_r$ we have

$$\sup_{|z_1| \leq r} \varphi_\nu(z_1) \leq \frac{1}{r},$$

hence

$$\sup_{|z_1| \leq r} \left| \frac{\partial_{z_2}^\nu f(z_1, 0)}{\nu!} \right| r^\nu \leq 1, \text{ for } \nu \geq \nu_r,$$

which proves the normal convergence of (1.2) in $\Delta_r \times \Delta_r$ for any $r < 1$. Since all terms of the series are holomorphic and r is arbitrary, f is holomorphic in $\Delta \times \Delta$. $\square$

As a direct consequence, Theorem 1.1 is proved.

1.3 Separate real analyticity

We have seen that if f is separately holomorphic in z_1 and z_2, then it is holomorphic. It is a well known fact that this doesn't hold for the real analytic case: for example, the function

$$f(x_1, x_2) = x_1 x_2 \exp\left(-\frac{1}{x_1^2 + x_2^2}\right)$$

is a C^∞-function, real analytic in each variable, but not jointly real analytic.

In $\mathbb{C}^2 = \mathbb{R}^2 + i\mathbb{R}^2$ with coordinates $z = (z_1, z_2)$, $z = x + iy$, we consider a function f on a domain $\Omega \subset \mathbb{R}^2$. We adopt the following terminology:

- f is *separately real analytic* in x_j if its restriction to the section of Ω with each line parallel to the x_j-axis is real analytic. This means that, when the other real coordinate is fixed, f extends to $|y_j| < \epsilon_x$.

- f is *separately CR extendible* to y_j if it is separately real analytic in x_j, with a holomorphic extension to $|y_j| < \epsilon$ for a uniform ϵ.

The issue in the previous example was that, while f extends in the directions y_1 and y_2, the extension is not uniform when x approaches 0.

The aim of this section is to prove the following result, stated in [5]:

Theorem 1.3. *Let f be a continuous function defined in a domain $\Omega \subset \mathbb{R}^2$, which is separately real analytic in x_1 and CR-extendible to y_2. Then f is real analytic.*

This result improves Siciak's theorem of [17], where f was required to be CR-extendible both to y_1 and y_2.

The statement being local, we can suppose that Ω is the unit square $I \times I$, where $I = (-1, 1) \subset \mathbb{R}$; we denote by I_δ the interval $(-\delta, \delta)$, and by U_δ the strip $I + iI_\delta \subset \mathbb{R}^2$. We begin by proving the following result:

Proposition 1.4. *Let f be a function as in Theorem 1.3. Then there is a positive δ such that f extends holomorphically to $U_\delta \times \Delta_\delta$*

Proof. Up to rescaling, we can suppose f has holomorphic extension to the sets $\cup_{x_1 \in I}\{x_1\} \times \Delta$ and $\cup_{x_2 \in I}\Delta_{\epsilon_{x_2}} \times \{x_2\}$; we can even assume that f extends to discs of radius slightly bigger than 1 and ϵ_{x_2}. We first prove that there are an open interval I_δ and an open strip $U_\delta = I + iI_\delta$ such that f is continuous in $I_\delta \times \Delta$ and in $U_\delta \times I_\delta$, hence it is a continuous CR function therein. We start from the proof of the continuity on $I_\delta \times \Delta$. Let

$$K_l = \{x_1 \in I : \sup_{z_2 \in \Delta} |f(x_1, z_2)| \leq l\}.$$

We note that $K_l \subset K_{l+1}$ and that $\cup_l K_l = I$ since $\sup_{z_2 \in \Delta} |f(x_1, z_2)| < +\infty$ for each x_1. We claim that K_l is closed and f is continuous on $K_l \times \Delta$. In fact, let $x_1^\nu \to x_1^o$ with $x_1^\nu \in K_l$; we want to show that then $x_1^o \in K_l$. We use the notation $F_\nu(z_2) := f(x_1^\nu, z_2) - f(x_1^o, z_2)$. The sequence $\{F_\nu\}_\nu$ is equicontinuous on the compact subsets of Δ, as a consequence of Cauchy's inequalities and the hypothesis of boundedness of f on $K_l \times \Delta$. We claim that $F_\nu \to 0$ uniformly on compact sets. Otherwise, by the equicontinuity, there is a subsequence $\{F_{\nu_k}\}_k$ which converges to a limit $F \neq 0$. But this limit is holomorphic on Δ and equal to 0 on I, which is a contradiction that proves f continuity on $K_l \times \Delta$. By Baire's Theorem, since $\cup_l K_l = I$, the sets K_l must contain an open interval for large l; also, such an interval can be found in a neighborhood of any point and we may assume it contains 0, by means of a small translation. Thus there is a positive δ such that f

extends as a continuous function on $I_\delta \times \Delta$, holomorphic in z_2: hence it is a continuous CR function therein.

We now prove that f is a continuous CR function on $U_\delta \times I_\delta$. For this purpose, we define

$$J_l = \{x_2 : f(\cdot, x_2) \text{ extends to } |y_1| < \frac{1}{l} \text{ and } |f(\cdot, x_2)| < l\}.$$

If $x_2^\nu \to x_2^o$ with $x_2^\nu \in J_l$, then by boundedness there is a subsequence which converges to a holomorphic function on $U_{\frac{1}{l}}$; this must be $f(\cdot, x_2^o)$. As before we have $|f(\cdot, x_2)| \leq l$ and $f|_{U_{\frac{1}{l}} \times I_l}$ is continuous. By Baire's theorem we still conclude that for large l, the set J_l contains an open interval that we can suppose to be centered at 0. This concludes the proof of the claim.

Now we can use Ajrapetyan-Henkin's edge of the wedge theorem of [1] (following the presentation of [20]) to prove that f has a holomorphic extension to $\Delta_\delta \times \Delta_\delta$, for a possibly smaller δ. We show first how to extend f for $0 \leq \mathrm{Im}\, z_1 < \delta$, $0 \leq \mathrm{Im}\, z_2 < \delta$. In fact, choose smooth functions $y_j(e^{i\vartheta}) \geq 0$ with $\mathrm{supp}(y_1) \subset [0, \pi]$, $\mathrm{supp}(y_2) \subset [\pi, 2\pi]$ and with unit mean value, take (λ_j) with $0 \leq \lambda_j < \delta$, $j = 1, 2$, write $y_\lambda = (\lambda_1 y_1, \lambda_2 y_2)$ and consider the discs $A_{x_o,\lambda}(\tau)$ which are the holomorphic extensions of $(x_o - T_0 y_\lambda) + i y_\lambda$ from $\tau = e^{i\vartheta} \in \partial\Delta$ to $\tau \in \Delta$. (Here T_0 is the Hilbert transform normalized by $T_0(\cdot)(0) = 0$.) Note that the boundaries of these discs, corresponding to the values $\tau = e^{i\vartheta}$ of the parameter, are contained in the union of $\Delta^+ \times I_\delta$ and $I_\delta \times \Delta^+$ (where Δ^+ is the half disc defined by $\mathrm{Im}\,(\tau) > 0$). Also, the set of their centers $\{A_{x_o,\lambda}(0)\} = \{x_o + i\lambda\}$ is the set described by $0 \leq y_1 < \delta$, $0 \leq y_2 < \delta$. On the other hand f is uniformly approximated over the set of the boundaries by a sequence of polynomials according to the Baouendi-Treves approximation theorem (see [3]). This sequence is also convergent in the inside of these discs, in particular in the set of their centers, by the maximum principle. The limit of the sequence provides the desired extension of f to the first quadrant $0 \leq y_1 < \delta$, $0 \leq y_2 < \delta$; in the same way we prove extension to the other quadrants.

We have seen that f is continuous and CR on $U_\delta \times I_\delta$ and extends holomorphically to $\Delta_\delta \times \Delta_\delta$. We notice that $U_\delta \times I_\delta$ is foliated by the complex leaves $U_\delta \times \{x_2\}$, for $x_2 \in I_\delta$, which meet the set of holomorphic extension $\Delta_\delta \times \Delta_\delta$. But then the propagation of the holomorphic extendibility of CR functions along complex leaves yields extension of f to an open domain $U_\delta \times \Delta_\delta$ of $\mathbb{C}^2$ for a small δ. This can be referred to Hanges-Treves theorem of [10]; however, in the case of a plane, there is a simpler proof which uses convergence radii of Taylor expansions, that we will present below (Lemma 1.2). The proposition is proved. □

Lemma 1.2. *Let $M = \{(z_1, z_2) \in \Delta \times \Delta : \operatorname{Im} z_1 = 0, |z_2| \geq \epsilon\}$. If f is a holomorphic function in $(\Delta \times \Delta) \setminus M$, then it extends holomorphically to $\Delta \times \Delta$.*

Proof. Define $(\Delta \times \Delta)^+ = (\Delta \times \Delta) \cap \{\operatorname{Im} z_1 > 0\}$ and $(\Delta \times \Delta)^- = (\Delta \times \Delta) \cap \{\operatorname{Im} z_1 < 0\}$, and let $f^\pm = f|_{(\Delta\times\Delta)^\pm}$. Take $\lambda \ll 1$ and write the Taylor expansion of f^- with respect to z_1 centered at $x_1 - i\lambda$. The functions

$$\psi_\nu(z_2) = \left| \frac{\partial^\nu_{z_1} f^-(x_1 - i\lambda, z_2)}{\nu!} \right|^{\frac{1}{\nu}}$$

are subharmonic. We have:

$$\limsup_{\nu\to\infty} \psi_\nu(z_2) \leq \lambda^{-1} \quad \forall z_2 \in \Delta$$

and

$$\limsup_{\nu\to\infty} \psi_\nu(z_2) \leq (1-\lambda)^{-1} \quad \text{if } |z_2| < \epsilon.$$

Now take a disc D centered at $z_2^0 \in \Delta$ (with $|z_2^0| > \epsilon$) and of radius ρ, such that $D \cap \Delta_\epsilon \neq \emptyset$. Then, by Fatou's lemma and for solid submean property of subharmonic functions, naming χ the function that is $(1-\lambda)^{-1}$ in $D \cap \Delta_\epsilon$ and λ^{-1} in $D \setminus \Delta_\epsilon$, we have:

$$\begin{aligned}\limsup_{\nu\to\infty} \psi_\nu(z_2^0) &= \limsup_{\nu\to\infty} \frac{1}{\pi\rho^2} \int_D \psi_\nu(z_2) dz_2 \\ &\leq \frac{1}{\pi\rho^2} \int_D \chi(z_2) dz_2 \;\leq\; \lambda'^{-1},\end{aligned}$$

for some $\lambda' > \lambda$.
Iterating and applying Theorem 1.2, we prove that f^- extends to a holomorphic function in a neighborhood of M in $\Delta \times \Delta$; but f is holomorphic in $(\Delta \times \Delta) \setminus M$, hence the extension must coincide with f^+ in $(\Delta \times \Delta)^+$. □

We are now in a situation similar to the one of Theorem 1.2, with the difference that the separate extension takes place only for $x_1 \in I$ instead of $z_1 \in \Delta$; $f : U_\delta \times \Delta_\delta \to \mathbb{C}$ is a holomorphic function, whose restrictions $z_2 \mapsto f(x_1^0, z_2)$ extend holomorphically to $|z_2| < 1$, for any fixed value of $x_1^0 \in I \subset U_\delta$. As in the previous section, we can write the Taylor expansion of f in z_2 centered at $z_2 = 0$ and consider, for $\nu \in \mathbb{N}$, the subharmonic functions

$$\varphi_\nu(z_1) = \left| \frac{\partial^\nu_{z_2} f(z_1, 0)}{\nu!} \right|^{\frac{1}{\nu}}. \tag{1.4}$$

For the joint analyticity in $U_\delta \times \Delta_\delta$, we have

$$\limsup_{\nu\to\infty} \sup_{z_1\in U_\delta} \varphi_\nu(z_1) \le \delta^{-1}, \tag{1.5}$$

while, for any fixed $x_1 \in I$, separate analyticity in z_2 gives us

$$\limsup_{\nu\to\infty} \varphi_\nu(x_1) \le 1. \tag{1.6}$$

We now state the following modified version of Lemma 1.1:

Lemma 1.3. *Let $\Delta^+ = \{z = x + iy \in \mathbb{C} : |z| < 1, y > 0\}$, $I = (-1, 1) \subset \partial\Delta^+$. Suppose that $\{\varphi_\nu\}$ is a sequence of functions, subharmonic in Δ^+ and upper semicontinuous on $\bar{\Delta}^+$, such that*

$$\limsup_{\nu\to\infty} \varphi_\nu(x) \le 1, \quad \forall x \in I \tag{1.7}$$

and

$$\limsup_{\nu\to\infty} \sup_{z\in\partial\Delta^+} \varphi_\nu(z) \le \delta^{-1}. \tag{1.8}$$

Then there exists a uniform constant k such that

$$\limsup_{\nu\to\infty} \varphi_\nu(z) \le 1 + k\delta^{-1}y,$$

uniformly on every compact subset of Δ^+.

Proof. We adapt the proof of Lemma 1.1 to our case. Let $\alpha > 0$ and K be a compact subset of Δ^+. For every $\alpha' > 0$ we can find a measurable subset $E \subset I$ with $\lambda(E) < \alpha'$ (where λ is the Lebesgue measure on $\partial\Delta^+$) and $\nu_{\alpha,\alpha'}$ such that $\sup_{z\in I\setminus E} \varphi_\nu(z) \le 1 + \alpha$ if $\nu > \nu_{\alpha,\alpha'}$, since the measurable sets

$$E_\nu = \bigcup_{\mu\ge\nu} \{\zeta \in I : \varphi_\mu(\zeta) \ge 1 + \alpha\}$$

form a decreasing sequence with $\bigcap_\nu E_\nu = \emptyset$ (for (1.7)).
Let $P_z(\zeta)$ be the Poisson kernel of Δ^+, and set $C_K = \sup P_z(\zeta)$ for $z \in K$ and $\zeta \in \partial\Delta^+$. If $z \in K$, for the submean property we have:

$$\begin{aligned}\varphi_\nu(z) \;\le\; & \int_{I\setminus E} P_z(\zeta)\varphi_\nu(\zeta)d\lambda(\zeta) + \int_E P_z(\zeta)\varphi_\nu(\zeta)d\lambda(\zeta) + \\ & + \int_{\partial\Delta^+\setminus I} P_z(\zeta)\varphi_\nu(\zeta)d\lambda(\zeta).\end{aligned}$$

The first integral, by definition of E, is

$$\begin{aligned}\int_{I\setminus E} P_z(\zeta)\varphi_\nu(\zeta)d\lambda(\zeta) &\leq (1+\alpha)\int_{I\setminus E} P_z(\zeta)d\lambda(\zeta)\\ &\leq 1+\alpha\end{aligned}$$

for $\nu > \nu_{\alpha,\alpha'}$. If ν is big enough, we have

$$\begin{aligned}\int_{E} P_z(\zeta)\varphi_\nu(\zeta)d\lambda(\zeta) &\leq (\delta^{-1}+\alpha)\int_{E} P_z(\zeta)d\lambda(\zeta)\\ &\leq (\delta^{-1}+\alpha)C_K\lambda(E)\end{aligned}$$

and

$$\begin{aligned}\int_{\partial\Delta^+\setminus I} P_z(\zeta)\varphi_\nu(\zeta)d\lambda(\zeta) &\leq (\delta^{-1}+\alpha)\int_{\partial\Delta^+\setminus I} P_z(\zeta)d\lambda(\zeta)\\ &= \frac{2(\delta^{-1}+\alpha)}{\pi}\arg\frac{1+z}{1-z},\end{aligned}$$

both by estimate (1.8). Notice that

$$\int_{\partial\Delta^+\setminus I} P_z(\zeta)d\lambda(\zeta) = \frac{2}{\pi}\arg\frac{1+z}{1-z}, \tag{1.9}$$

since the second member is harmonic and is equal to the characteristic function of $\partial\Delta^+\setminus I$ on the boundary of Δ^+ (see [9]).
Choose α' such that $(\delta^{-1}+\alpha)C_K\alpha' \leq \alpha$; it follows that, for $\nu > \nu_{\alpha,\alpha'} = \nu_{\alpha,K}$,

$$\begin{aligned}\varphi_\nu(z) &\leq 1+\alpha+(\delta^{-1}+\alpha)C_K\alpha' + \frac{2(\delta^{-1}+\alpha)}{\pi}\arg\frac{1+z}{1-z}\\ &\leq 1+3\alpha+\frac{2\delta^{-1}}{\pi}\arg\frac{1+z}{1-z}\\ &\leq 1+3\alpha+k\delta^{-1}y,\end{aligned}$$

for a constant $k>0$, uniformly in K. □

We are now ready to prove the following

Theorem 1.4. *Let f be a holomorphic function in $U_\delta\times\Delta_\delta$, whose restrictions given by $z_2\mapsto f(x_1^0,z_2)$ extend holomorphically to Δ, for any fixed $x_1^0\in I$. Then f extends to a holomorphic function defined in a neighborhood of $I\times\Delta$.*

Proof. Given a small $\alpha>0$, choose $\eta<k^{-1}\delta\alpha$. Reasoning as in Theorem 1.2, and applying Lemma 1.3, we get normal convergence of the Taylor series of f for $|z_2|<1-\alpha$, uniformly in z_1 when $0<|y_1|<\eta_\alpha$. Hence f turns out to be holomorphic in $(I+i\dot{I}_{\eta_\alpha})\times\Delta_{1-\alpha}$, where $\cdot$ means that 0 is removed. But in fact f is holomorphic also at $y_1=0$ because this is true when $z_2\in\Delta_\delta$ (see Lemma 1.2). □

1.4 Further results

In this section we present some results of [4] and [14]. In Lemma 1.3, the interval I does not play a special role: any curve $\mathcal{I} \subset \Delta \times \Delta$ serves the purpose.

Let γ be a C^1 curve and M a real C^1 hypersurface in $\mathbb{C}^2$ foliated by parallel complex lines Γ_z issued transversally from each $z \in \gamma$. We show the following

Theorem 1.5. *Let $f : M \to \mathbb{C}$ be a function, which is separately holomorphic along each line Γ_z and extends holomorphically to a neighborhood U of γ. Then f extends as a holomorphic function to a neighborhood of M.*

Proof. Let (z_1, z_2) be the coordinates in $\mathbb{C}^2$; we can suppose $\gamma \subset \mathbb{C} \times \{0\}$, $\Gamma_{z_1} = \{z_1\} \times \Delta$ and $M = \gamma \times \Delta$. The statement is local in z_1, therefore we can suppose that U is a neighborhood of $\{|z_2| < \epsilon\}$; f is holomorphic in a neighborhood of γ, say $\mathcal{U}_\epsilon \times \Delta_\epsilon$ (where $\mathcal{U}_\epsilon$ is the ϵ-neighborhood of γ in $\mathbb{C} \times \{0\}$ and Δ_ϵ is the disc of radius ϵ). We can write the Taylor expansion of f in $\mathcal{U}_\epsilon \times \Delta_\epsilon$ and define the subharmonic functions φ_ν as in (1.4), obtaining the pointwise estimate (1.6) for $z \in \gamma$ and the uniform estimate (1.5).

We need a variant of Lemma 1.3. Let $\mathcal{I}$ be a curve in Δ^+ contained in the strip $\{0 < y < \eta\}$ with end points in $\partial\Delta^+ \setminus I$, and denote by $\tilde{\Delta}^+$ the region bounded by $\mathcal{I}$ and $\partial\Delta^+ \setminus I$. In this discussion $\mathcal{I}$ needs not to be a C^1 curve; it must have just the regularity which is required for the Dirichlet problem in $\tilde{\Delta}^+$ to be solved.

Let $\{\varphi_\nu\}$ be a uniformly bounded sequence of subharmonic functions in $\tilde{\Delta}^+$ such that

$$\begin{cases} \limsup\limits_{\nu \to +\infty} \varphi_\nu(x) \le 1, \quad \forall x \in \mathcal{I} \\ \limsup\limits_{\nu \to +\infty} \sup\limits_{z \in \partial\tilde{\Delta}^+} \varphi_\nu(z) \le \epsilon^{-1}; \end{cases}$$

we claim that

$$\limsup_{\nu \to \infty} \varphi_\nu(z) \le 1 + 2k\epsilon^{-1}\eta, \tag{1.10}$$

uniformly for $y < 2\eta$. In order to prove (1.10), denote with $\tilde{P}_z(\zeta)$ the Poisson kernel of $\tilde{\Delta}^+$; for $z \in \partial\tilde{\Delta}^+$, we have

$$\int_{\partial\tilde{\Delta}^+ \setminus \mathcal{I}} \tilde{P}_z(\zeta) d\lambda(\zeta) \le \int_{\partial\tilde{\Delta}^+ \setminus \mathcal{I}} P_z(\zeta) d\lambda(\zeta), \tag{1.11}$$

since, for $z \in \mathcal{I}$, the first integral vanishes and the second is positive, while for $z \in \partial\tilde{\Delta}^+ \setminus \mathcal{I}$ they are both equal to 1. Since these integrals are harmonic functions of the variable z, the inequality (1.11) holds for all $z \in \tilde{\Delta}^+$. It follows

$$\limsup_{\nu\to\infty} \varphi_\nu(z) \leq 1 + k\epsilon^{-1}y,$$

uniformly on the compact subsets of $\tilde{\Delta}^+$; this last estimate yields (1.10) for $y < 2\eta$.

We now turn our attention to the curve γ. For $z_0 \in \gamma$ and $\alpha > 0$, the C^1 regularity of γ assures us of the existence of a positive δ_α such that

$$\operatorname{dist}(z, T_{z_0}\gamma) < \frac{\alpha\delta_\alpha\epsilon}{2k}$$

for $z \in \gamma$, $|z - z_0| < \delta_\alpha$. After rescaling by a factor ${\delta_\alpha}^{-1}$, we interpret $T_{z_0}\gamma$ as I and γ as the curve $\mathcal{I}$.

By the argument above, applied to δ_α-half discs with center z_0, we get

$$\limsup_{\nu\to\infty} \varphi_\nu(z) \leq 1 + 2k\epsilon^{-1}{\delta_\alpha}^{-1}\eta$$

uniformly for $y < 2\eta$ (notice that a factor ${\delta_\alpha}^{-1}$ enters into play because of the rescaling). At this point we just need to choose

$$\eta = \frac{\alpha\delta_\alpha\epsilon}{2k}$$

and to use a finite covering of γ by δ_α-half discs. Taking the inf of the η's needed for this procedure, and reasoning simmetrically for negative half discs, we end up with

$$\limsup_{\nu\to\infty} \varphi_\nu(z) \leq 1 + \alpha$$

uniformly in the compact subsets of $\mathcal{U}_\eta \setminus \gamma$, where $\mathcal{U}_\eta$ is the η-strip around γ. Therefore the Taylor series of f converges uniformly for $z_1 \in \mathcal{U}_\eta \setminus \gamma$, normally in $z_2 \in \Delta_{1-\alpha}$:

its sum is then a holomorphic function in $(\mathcal{U}_\eta \setminus \gamma) \times \Delta_{1-\alpha}$. But f extends across M for $|z_2| < \epsilon$, and so it extends as a holomorphic function to $\mathcal{U}_\eta \times \Delta_{1-\alpha}$, thus proving our statement. □

Remark 1. In the proof of Theorem 1.5 we needed to assume a C^1-regularity for the curve γ. We can obtain an alternative proof by using Carathéodory's theorem: since the biholomorphic equivalence (Riemann map) between Δ^+ and $\tilde{\Delta}^+$ is continuous up to the boundary, it interchanges the distance to I (that is $\operatorname{Im}\tau$) with the distance to $\mathcal{I}$. Hence the estimate of Lemma 1.3 can be rewritten in a neighborhood of γ, and the proof of our theorem can be concluded as before.

We now pass to prove an extension of Hartogs theorem for real analytic foliations.

Remark 2. If we apply Theorem 1.4 to a family of lines $\{y_1 = \text{const}\}$, whose corresponding stripes U_ϵ form a covering of Δ, we still have the conclusion of Theorem 1.1. As we are going to see, this new proof is invariant under real analytic transformations.

Let $\{\Gamma_\lambda\}$ be a foliation of holomorphic curves depending in a C^ω fashion from a parameter $\lambda \in \Lambda$ (where Λ is a connected open subset of $\mathbb{R}^2$), and define $\Omega = \cup_\lambda \Gamma_\lambda$. With this we mean there exists a real analytic diffeomorphism

$$\Phi : \Lambda \times \Delta \to \Omega$$

which is holomorphic for $\lambda = \textit{const}$. Let $\Omega' \subset \Omega$ be an open set with $\Omega' \cap \Gamma_\lambda \neq \emptyset$ for all $\lambda \in \Lambda$.

Since the results we will prove are local in λ, we can choose holomorphic coordinates such that Ω' is a neighborhood of $z_2 = 0$ and the leaves are transversal to such plane, and choos the parameter $\lambda = z_1 \in \Delta_\epsilon$. We will also write $\Phi_{z_1}(\tau)$ instead of $\Phi(z_1, \tau)$ and normalize the parametrization with the condition $\Phi_{z_1}(0) = (z_1, 0)$.

Theorem 1.6. *Let f be a holomorphic function in Ω' which extends along Γ_{x_1} for all $x_1 \in I_\epsilon$. Then f extends to a holomorphic function in M.*

Proof. $\Phi|_{I_\epsilon \times \Delta}$ is a real analytic function, holomorphic in τ: it can be locally represented as a power series in x_1 e τ. Changing x_1 with z_1 in the series, we obtain a function $\tilde{\Phi}$, holomorphic in a neighborhood of $I_\epsilon \times \Delta$, say $V_{\epsilon,\delta} \times \Delta$ (where $V_{\epsilon,\delta} = I_\epsilon + iI_\delta$), such that $\tilde{\Phi}|_{I_\epsilon \times \Delta} = \Phi$. We will write $\tilde{\Gamma}_{z_1} = \tilde{\Phi}_{z_1}(\Delta)$. Notice that

$$\tilde{\Gamma}_{x_1} = \Gamma_{x_1}, \tag{1.12}$$

for $x_1 \in I_\epsilon$. Up to taking a slightly smaller δ, we can suppose

$$\cup_{z_1 \in V_{\epsilon,\delta}} \tilde{\Phi}_{z_1}(\Delta_\delta) \subset \Omega'.$$

Consider the function $f \circ \tilde{\Phi}$ defined on $V_{\epsilon,\delta} \times \Delta_\delta$: it is holomorphic, and for (1.12), its restriction $\tau \mapsto f \circ \tilde{\Phi}(x_1, \tau)$ extends to a holomorphic function in Δ, for any fixed $x_1 \in I_\epsilon \subset V_{\epsilon,\delta}$. Under these hypotheses we can apply Theorem 1.4: for any $\alpha > 0$ there is a positive δ_α such that $f \circ \tilde{\Phi}$ extends to a holomorphic function in $V_{\epsilon,\delta_\alpha} \times \Delta_{1-\alpha}$. $\square$

As an immediate corollary, we obtain:

Theorem 1.7. *Let $f : \Omega \to \mathbb{C}$ be a holomorphic function in Ω', which is separately holomorphic along each leaf Γ_{z_1}. Then f is holomorphic in Ω.*

Proof. We can apply Theorem 1.6 to any line $\{\operatorname{Im} z_1 = c\}$: for any $\alpha > 0$, we find a holomorphic extension of $f \circ \tilde{\Phi}$ for z_1 satisfying $c - \delta_\alpha < \operatorname{Im} z_1 < c + \delta_\alpha$ and $|\tau| < 1 - \alpha$. Now any compact subset of Ω has a finite covering of open sets where f is holomorphic, thus proving our statement. $\square$

Chapter 2

Holomorphic sectors

2.1 CR manifolds and CR functions

Let M be a smooth real submanifold in $\mathbb{C}^n$. The complex tangent space at a point $p \in M$ is defined as the maximal complex subspace in T_pM, that is

$$T_p^c M = T_pM \cap JT_pM,$$

where $J : \mathbb{C}^n \to \mathbb{C}^n$ is the operator of multiplication by i. The manifold M is called a *CR manifold* if dim T_p^cM is independent on $p \in M$. Moreover, M is called *generic* if T_pM spans $T_p\mathbb{C}^n = \mathbb{C}^n$ over $\mathbb{C}$ for all $p \in M$, that is

$$T_pM + JT_pM = \mathbb{C}^n.$$

We observe that all real hypersurfaces are generic, and that a generic manifold is always a CR manifold.

Definition 2.1. *Let f be a C^1 function defined on a CR manifold M. f is called a* CR function *if df is $\mathbb{C}$-linear on T^cM.*

In other words, if $M \subset \mathbb{C}^n$ is a CR manifold defined by $r = 0$ for $r = (r_j)_{j=1,\dots,m}$, a C^1 function $f : M \to \mathbb{C}$ is CR if and only if for any extension of f to $\mathbb{C}^n$, which we still denote by f, we have $\bar\partial f \wedge \bar\partial r_1 \wedge \dots \wedge \bar\partial r_m = 0$. For a function f which is only continuous, we say that f is a CR function if the above condition holds in the distributional sense.

The celebrated Baouendi-Treves theorem of [3] states that a CR function can be locally approximated by holomorphic polynomials.

Theorem 2.1. *Let $M \subset \mathbb{C}^n$ be a generic manifold. For any point $p \in M$ there exists a neighborhood U of p in M such that for every continuous CR function f defined on M there is a sequence of holomorphic polynomials f_λ that converges uniformly to f on the compact subsets of U.*

Proof. First we prove the statement for a maximally real submanifold M_0 throuh p, that is $T^c M_0 = 0$. We set coordinates in $\mathbb{C}^n$ such that $p = 0$ and $T_p M_0 = \mathbb{R}^n \subset \mathbb{C}^n$. Up to shrinking M_0, we can suppose there exists $0 < c < 1$ such that

$$|\mathrm{Im}\,(z - w)| \leq c|\mathrm{Re}\,(z - w)|, \tag{2.1}$$

for all $z, w \in M_0$. We define the entire functions

$$f_\lambda(z) = \left(\frac{\lambda}{\pi}\right)^{\frac{n}{2}} \int_{M_0} f(w) e^{-\lambda(z-w)^2} dw_1 \wedge \ldots \wedge dw_n,$$

where $(z - w)^2 = \sum(z_j - w_j)^2$ and $\lambda > 0$. (2.1) tells us that, as $\lambda \to \infty$, $f_\lambda(z) \to f(z)$ for $z \in M_0$. We now pass to prove that $f_\lambda \to f$ in a neighborhood of $p \in M$. We view M_0 as a manifold with boundary, and consider a small perturbation M_1 of M_0 with the same boundary. Let

$$\tilde{f}_\lambda(z) = \left(\frac{\lambda}{\pi}\right)^{\frac{n}{2}} \int_{M_1} f(w) e^{-\lambda(z-w)^2} dw_1 \wedge \ldots \wedge dw_n;$$

then $\tilde{f}_\lambda(z) \to f(z)$ for $z \in M_1$.

Let $\tilde{M} \subset M$ be the manifold bounded by M_0 and M_1, that is $\partial\tilde{M} = M_0 - M_1$; since $e^{-\lambda(z-w)^2}$ is holomorphic, the integrand is a closed form on M. Then, by Stokes theorem, it is immediate to conclude that $f_\lambda(z) = \tilde{f}_\lambda(z)$ for all $z \in \mathbb{C}^n$. We have proved that the sequence f_λ converges to f on every small perturbation of M_0, and then in a neighborhood of p in M. Using the Taylor expansions of f_λ, we obtain the desired approximation of f by polynomials. □

We now discuss the normal form for a generic manifold. Suppose M is generic: then we can choose holomorphic coordinates $(z' = x + iy, z'') \in \mathbb{C}^m \times \mathbb{C}^{n-m}$ around a point p such that $p = 0$, $T_p M$ has the equation $y = 0$, and $T_p^c M$ has the equation $z' = 0$. Then M is defined by a local equation

$$y = h(x, z''), \tag{2.2}$$

where h is a smooth function with $h(0) = 0$ and $dh(0) = 0$. Moreover (see [7] and [23]) we have:

Proposition 2.1. *Let $M \subset \mathbb{C}^n$ be a generic submanifold of codimension m and class C^k and let p be a point in M. Then there is a holomorphic change of coordinates such that, in the new coordinates, M is graphed over x, z'' at $p = 0$ by*

$$y_j = h_j(x, z''), \quad j = 1, ..., m,$$

with

$$\partial^{|I|+|J|}_{x^I z''^J} h(0) = \partial^{|I|+|J|}_{x^I \bar{z}''^J} h(0) = 0 \quad \mathrm{if|I| + |J| \leq k.} \tag{2.3}$$

Proof. As we stated before, we can suppose M is graphed by a smooth function h with $h(0) = 0$ and $dh(0) = 0$. For the Taylor expansion of $h = (h_j)$ up to order k, we have

$$h = \sum_{|I|+|J|+|K|\geq 2}^{k} a_{I\,J\,K} x^I z''^J \bar{z}''^K + o^k,$$

where $a = a_{I\,J\,K}$ is an l-vector $a = (a_j)$. We "complexify" from $x \in \mathbb{R}^m$ to $z' \in \mathbb{C}^m$ and from $(z'', \bar{z}'') \in \mathbb{C}^{n-m} \times_{\mathbb{C}^{n-m}} \bar{\mathbb{C}}^{n-m}$ to $(z'', \bar{w}'') \in \mathbb{C}^{n-m} \times \bar{\mathbb{C}}^{n-m}$. Consider the polynomial map

$$h^k : (z', z'', \bar{w}'') \mapsto \sum_{|I|+|J|+|K|\geq 2}^{k} a_{IJK} z'^I z''^J \bar{w}''^K :$$

by the implicit function theorem, there is a unique map $\Phi = \Phi(z', z'')$ such that

$$z' = \Phi(z' + ih^k(z', z'', 0), z''). \tag{2.4}$$

Define a holomorphic change of coordinates in a neighborhood of 0 by

$$\begin{cases} \tilde{z}' = z' - ih^k(\Phi(z', z''), z'', 0), \\ \tilde{z}'' = z''. \end{cases} \tag{2.5}$$

If $z \in M$, that is, $y = h(x, z'', \bar{z}'')$, then

$$\begin{aligned} \tilde{y} &= y - \operatorname{Re} h^k(\Phi(z', z''), z'', 0) \\ &= \operatorname{Re} \big(h(x, z'', \bar{z}'') - h^k \left(\Phi \left(x + ih(x, z'', \bar{z}''), z'' \right), z'', 0 \right) \big) . \end{aligned} \tag{2.6}$$

To obtain an equation for the image $\tilde{M}$ of M under the coordinates defined by (2.5), we must replace (x, z'') by $(x(\tilde{z}', \tilde{z}''), \tilde{z}'')$, which yields

$$\begin{aligned} \tilde{y} = \operatorname{Re} &\Big(h \left(x(\tilde{z}', \tilde{z}''), \tilde{z}'', \bar{\tilde{z}}'' \right) \\ &-h^k \left(\Phi \left(x(\tilde{z}', \tilde{z}'') + ih(x(\tilde{z}', \tilde{z}''), \tilde{z}'', \bar{\tilde{z}}''), \tilde{z}'' \right), \tilde{z}'', 0 \right) \Big) . \end{aligned} \tag{2.7}$$

We write (2.7) as $\tilde{y} = \operatorname{Re} \tilde{h}$. By the implicit function theorem, we can remove $\tilde{y}'$ from $\tilde{h}$; hence $\operatorname{Re} \tilde{h}$ can be seen a graphing function for $\tilde{M}$. We just have to see that $\tilde{h}$ satisfies (2.3). Consider the function

$$(\tilde{x}, \tilde{z}'') \mapsto x(\tilde{x}, \tilde{z}'', \overline{\tilde{z}''}). \tag{2.8}$$

Take x^k, of (2.8) and complexify the variables from $(\tilde{x}, \tilde{z}'', \overline{\tilde{z}}'')$ to $(\tilde{z}', \tilde{z}'', \overline{\tilde{w}}'')$: we get for $x^k = x^k(\tilde{z}', \tilde{z}'', \overline{\tilde{w}}'')$

$$\begin{aligned}\tilde{h}(\tilde{z}', \tilde{z}'', \overline{\tilde{w}}'') = h^k\left(x^k, \tilde{z}'', \overline{\tilde{w}}''\right)\\ - h^k\left(\Phi\left(x^k + ih^k\left(x^k, \tilde{z}'', \overline{\tilde{w}}''\right), \tilde{z}''\right), \tilde{z}'', 0\right) + o^k. \quad (2.9)\end{aligned}$$

We want to prove

$$\tilde{h}(\tilde{z}', \tilde{z}'', 0) = o^k$$

and

$$\tilde{h}(\tilde{z}', 0, \overline{\tilde{w}}'') = o^k.$$

If we prove the first, the second follows conjugating. Using (2.4), we have, for x and z'' replaced by $x^k = x^k(\tilde{z}', \tilde{z}'', 0)$ and $\tilde{z}''$, respectively,

$$x^k = \Phi\left(x^k + ih^k(x^k, \tilde{z}'', 0), \tilde{z}''\right) + o^k. \quad (2.10)$$

But then, using (2.10) into (2.9) and evaluating h^k and Φ for $\overline{\tilde{w}}'' = 0$, we have

$$\begin{aligned}\tilde{h} &= h^k(x^k, \tilde{z}'', 0) - h^k\left(\Phi\left(x^k + ih^k(x^k, \tilde{z}'', 0), \tilde{z}''\right), \tilde{z}'', 0\right) + o^k\\ &= h^k(x^k, \tilde{z}'', 0) - h^k\left(x^k + o^k, \tilde{z}'', 0\right) + o^k \quad (2.11)\\ &= o^k,\end{aligned}$$

which completes our proof. □

2.2 Analytic discs

We review the classical results about extension of CR functions by the analytic disc technique. Our presentation closely follows those of [20] and [23] (see also [2] and [7]).

Let Δ be the standard unit disc in $\mathbb{C}$, and $\partial\Delta$ the unit circle.

Definition 2.2. *An* analytic disc *in $\mathbb{C}^n$ is a holomorphic map*

$$A : \Delta \to \mathbb{C}^n$$

continuous up to the boundary. If $M \subset \mathbb{C}^n$ is a submanifold, we say that A is attached to M if its boundary $A(\partial\Delta)$ is contained in M.

Let f be a CR function defined on a generic manifold $M \subset \mathbb{C}^n$; then by Theorem 2.1 f is a uniform limit of a sequence f_λ of polynomials in a neighborhood U of a point p. If A is a small disc such that $A(\partial\Delta) \subset U$, then f_λ converges to f on $A(\partial\Delta)$; but then, since $f_\lambda \circ A$ converge on $\partial\Delta$, they must converge in the interior Δ by the maximun principle. Hence f_λ converge on the image $A(\overline{\Delta})$. Suppose now that Ω is an open set filled up by sufficiently small analytic discs attached to M: then f_λ converge uniformly on Ω to a holomorphic function that extends f. Analogously, if Ω is a CR manifold filled by discs, the limit of f_λ is a CR-extension of f.

We now introduce the Hilbert transform. If $u : \partial\Delta \to \mathbb{R}$ is a smooth function, it has a unique harmonic extension to $\bar{\Delta}$: this extension u has a harmonic conjugate v on $\bar{\Delta}$, that is, a function v such that $u + iv$ is holomorphic in Δ. v is uniquely determined up to an additive constant. The Hilbert transform (normalized at 1) is the map $T_1 : u|_{\partial\Delta} \mapsto v|_{\partial\Delta}$ normalized by the condition $v(1) = 0$. Note that $u = -T_1 v + u(1)$. T_1 is not a continuous functional over the spaces C^k of functions with integer regularity; it finds its natural settings in the Hölder spaces $C^{k,\alpha}(\partial\Delta)$, for $k \geq 0$ and $0 < \alpha < 1$, of functions f endowed with continuous derivatives up to order k which satisfy

$$\|f\|_{k,\alpha} = ||f||_k + \sup_{\vartheta_1,\vartheta_2} \frac{|\partial_\tau^k f(e^{i\vartheta_1}) - \partial_\tau^k f(e^{i\vartheta_2})|}{|\vartheta_1 - \vartheta_2|^\alpha} < +\infty.$$

With this norm, the space $C^{k,\alpha}$ is a Banach space; moreover, if f and g are two maps in $C^{k,\alpha}$, it is immediate to show that

$$\|fg\|_\alpha \leq \|f\|_\alpha \|g\|_\alpha.$$

Hence $C^{k,\alpha}$ is a Banach algebra.

We prove the following classical result, due to Privalov:

Theorem 2.2. *The functional $T_1 : C^{k,\alpha}(\partial\Delta) \to C^{k,\alpha}(\partial\Delta)$ is continuous.*

Proof. We write the harmonic extension of u through Poisson integral as

$$\begin{aligned} u(re^{i\varphi}) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)u(e^{i\vartheta})}{(1+r^2-2r\cos(\vartheta-\varphi))} d\vartheta \\ &= \frac{1}{2\pi i}\int_{\{|\tau|=1\}} \frac{1-|z|^2}{|\tau - z|^2} u(\tau)\frac{d\tau}{\tau} \\ &= \mathrm{Re}\left[\frac{1}{2\pi i}\int_{\{|\tau|=1\}} \left(\frac{\tau+z}{\tau-z}\right) u(\tau)\frac{d\tau}{\tau}\right] \end{aligned}$$

for $r \leq 1$. Set

$$F = \frac{1}{2\pi i}\int_{\{|\tau|=1\}} \left(\frac{\tau+z}{\tau-z}\right) u(\tau)\frac{d\tau}{\tau};$$

F is holomorphic in z, and since u is real, we have $\operatorname{Im} F(0) = 0$. But then

$$T_0 u|_{z=e^{i\vartheta}} = \operatorname{Im}\left[\frac{1}{2\pi i}\int_{\{|\tau|=1\}}\left(\frac{\tau+z}{\tau-z}\right)u(\tau)\frac{d\tau}{\tau}\right]|_{z=e^{i\vartheta}},$$

where T_0 is the Hilbert transform normalized at 0.

Since $\frac{\tau+z}{\tau}$ is smooth for $|\tau| = 1$, $|z| < 1$, it is sufficient to prove the continuity of the Cauchy integral K defined by

$$Ku(z) = \frac{1}{2\pi i}\int_{\{|\tau|=1\}}\frac{u(\tau)}{\tau - z}d\tau, \quad |z| < 1.$$

It is immediate to estimate $\|Ku\|_k$ by $\|u\|_{k,\alpha}$. So, we have to estimate $\tilde{u} := \partial_\vartheta^k u$. For $z_1 = e^{i\vartheta_1}$ and $z_2 = e^{i\vartheta_2}$ in $\partial\Delta$, set $\epsilon = |z_1 - z_2|$ and $B_{2\epsilon}(z_1) = \{\tau : |\tau - z_1| < 2\epsilon\}$. We have

$$\begin{aligned}
K\tilde{u}(z_1) - K\tilde{u}(z_2) &= \frac{1}{2\pi i}\int_{\partial\Delta}\left(\left(\frac{\tilde{u}(\tau)-\tilde{u}(z_1)}{\tau - z_1}\right) - \left(\frac{\tilde{u}(\tau)-\tilde{u}(z_2)}{\tau - z_2}\right)\right)d\tau \\
&+ (\tilde{u}(z_1) - \tilde{u}(z_2)) \\
&= \frac{1}{2\pi i}\int_{\partial\Delta\cap B_{2\epsilon}(z_1)}\cdot \ + \frac{1}{2\pi i}\int_{\partial\Delta\setminus B_{2\epsilon}(z_1)}\cdot \ + (\tilde{u}(z_1) - \tilde{u}(z_2)).
\end{aligned}$$

We use

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\partial\Delta\setminus B_{2\epsilon}(z_1)}\cdot \ &+ \ (\tilde{u}(z_1) - \tilde{u}(z_2)) \\
&= \frac{1}{2\pi i}\int_{\partial\Delta\setminus B_{2\epsilon}(z_1)}(\tilde{u}(\tau) - \tilde{u}(z_1))\left[\frac{1}{\tau - z_1} - \frac{1}{\tau - z_2}\right]d\tau \\
&+ \frac{1}{2\pi i}\int_{\partial\Delta\setminus B_{2\epsilon}(z_1)}\frac{(-\tilde{u}(z_1) + \tilde{u}(z_2))}{\tau - z_2}d\tau + (\tilde{u}(z_1) - \tilde{u}(z_2));
\end{aligned}$$

since $\frac{1}{2\pi i}\int_{\partial\Delta}\frac{1}{\tau - z_2}d\tau = 1$, the sum of the second and third terms in the right-hand side is $\int_{B_{2\epsilon}(z_1)\cap\partial\Delta}\frac{\tilde{u}(z_1)-\tilde{u}(z_2)}{\tau - z_2}d\tau$. Its absolute value is then estimated by $\|\tilde{u}\|_\alpha |z_1 - z_2|^\alpha$. The absolute value of the first term on the right-hand side can be estimated by

$$\|\tilde{u}\|_\alpha |z_1 - z_2|\int_{\partial\Delta\setminus B_{2\epsilon}(z_1)} 2|\vartheta - \vartheta_1|^{-2+\alpha}d\vartheta \lesssim ||\tilde{u}||_\alpha |z_1 - z_2|^\alpha.$$

As for the remaining integral over $\partial\Delta \cap B_{2\epsilon}(z_1)$, we have

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\partial\Delta\cap B_{2\epsilon}(z_1)}\cdot \ &\lesssim \ \|\tilde{u}\|_\alpha \int_{\partial\Delta\cap B_{2\epsilon}(z_1)}(|\vartheta - \vartheta_1|^{\alpha-1} + |\vartheta - \vartheta_2|^{\alpha-1})d\vartheta \\
&\lesssim \ \|\tilde{u}\|_\alpha |z_1 - z_2|^\alpha.
\end{aligned}$$

The proof is complete. □

We are now ready to construct analytic discs attached to a generic manifold M, as described by Bishop in [6].

Proposition 2.2. *Let M be defined by (2.2) with $h \in C^{k+l+2}$. Then for any small $w \in C^{k,\alpha}(\partial\Delta, \mathbb{C}^{n-m})$ with $w(1) = 0$, and for any z_0 close to 0, there is a unique $u \in C^{k,\alpha}(\partial\Delta, \mathbb{R}^m)$ which is a solution of*

$$u = -T_1 h(u, z_0'' + w(\tau)) + x_0'. \tag{2.12}$$

Moreover, if w depends on some parameter $\eta \in \mathbb{R}^d$ so that $\mathbb{R}^d \to C^{k,\alpha}$, $\eta \mapsto w_\eta$ is C^l, then also $(\eta, x_o', z_o'') \mapsto u_{\eta, x_o', z_o''}$, $\mathbb{R}^d \times \mathbb{C}^{n-m} \to C^{k,\alpha}(\partial\Delta)$ is C^l.

Proof. Consider the mapping

$$F: \ C^{k,\alpha}(\partial\Delta, \mathbb{R}^m) \times C^{k,\alpha}(\partial\Delta, \mathbb{C}^{n-m}) \times \mathbb{R}^m \times \mathbb{C}^{n-m} \to C^{k,\alpha}(\partial\Delta, \mathbb{R}^m),$$
$$(u, w, x_0', z_0'') \mapsto u + T_1 h(u, z_0'' + w) - x_0'.$$

F is a C^1-functional between function spaces. For the Jacobian $\partial_u F$ with respect to u, we have

$$\partial_u F : \dot{u} \mapsto \dot{u} - T_1 \partial_x h \dot{u}.$$

Evaluation at $(0,0,0)$ implies that $\partial_u F$ is invertible since $\partial_x h(0) = 0$. Hence the implicit function theorem in Banach spaces yields the solvability of (2.12), along with the required dependence of the solution on all the parameters. $\square$

Let $N_p M = T_p \mathbb{C}^n / T_p M$ be the normal space to M at $p \in M$, and $\Gamma \subset \mathbb{N}_p M$ an open cone. A *wedge* W with edge M and direction cone Γ is a set of the form

$$W = ((M \cap U) + \Gamma) \cap U,$$

where U is a neighborhood of p in $\mathbb{C}^n$. We state the following version of the edge-of-the-wedge theorem, due to Ajrapetyan and Henkin ([1]).

Theorem 2.3. *Let $M \subset \mathbb{C}^n$ be a generic submanifold, and $p \in M$. Let M_j ($1 \le j \le m$, $m = \mathrm{codimM}$) be manifolds with boundary M (hence $\mathrm{dimM_j} = \mathrm{dimM} + 1$). Suppose there are $\xi_1, \ldots, \xi_m$, $\xi_j \in T_p M_j / T_p M$ pointing inside M_j, such that $\mathrm{span}\{\xi_1, \ldots, \xi_\mathrm{m}\} = \mathrm{N_p M}$. Then all CR functions on $M \cup M_1 \cup \ldots \cup M_m$ extend holomorphically to a wedge W' with direction cone Γ', where Γ' is any finer cone than $\Gamma = \mathrm{conv}\{\xi_1, \ldots, \xi_\mathrm{m}\}$.*

Proof. We take $p = 0$, and describe our situation by taking M defined by (2.2) and adding the extra directions $\xi_1, \ldots, \xi_m$. We suppose M and M_j are defined by the equation

$$y = h(x, w, t),$$

where $t \in \mathbb{R}^m$ and h is a smooth function defined in a neighborhood of 0 in $\mathbb{R}^m \times \mathbb{C}^{n-m} \times \mathbb{R}^m$: M has the equation $y = h(x, w, 0)$, while M_j is described by $t_j > 0$ and $t_i = 0$ for $i \neq j$. We can assume $h(0) = 0$, $\partial_x h(0) = 0$, $\partial_w h(0) = 0$, and $\partial_t h(0) = \text{id}$; then the cone Γ, defined as the convex span of the ξ_j's, is turned into

$$\Gamma = \{t \in \mathbb{R}^m : t_j \geq 0, 1 \leq j \leq m\}.$$

We observe that (x, w, t) is a set of local coordinates of $\mathbb{C}^n$ in a neighborhood of 0. Let $A(\tau) = (u(\tau) + iv(\tau), w(\tau)$ be a disc attached to $M \cup M_1 \cup \ldots \cup M_m$, and $t(\tau) = (t_1(\tau), \ldots, t_m(\tau))$ be the t-component of A in the coordinates described above. Then all $t_j(\tau) \geq 0$ for $|\tau| = 1$, but for any $\tau \in \partial\Delta$ only one of the $t_j(\tau)$ can be different from 0. We decompose $\partial\Delta$ into a union $\bigcup_{j=1}^{m} \gamma_j$ of arcs of length $\frac{2\pi}{m}$, take $\varphi_j(\tau) \geq 0$ with $\operatorname{supp} \varphi_j \subset \gamma_j$ and $\frac{1}{2\pi} \int \varphi_j d\vartheta = 1$ and define

$$t_\lambda(\tau) = (\lambda_1 \varphi_1(\tau), ..., \lambda_m \varphi_m(\tau)),$$

for $(\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+$.

Take parameters λ, w_0 and x_0. Then we have a family of discs A_{λ, x_0, w_0} given by the solution of the Bishop equation

$$u = -T_1 h(u, w_0, t) + x_0.$$

Consider the evaluation mapping of the centers of the discs

$$\mathcal{E} : \ (x_o, \lambda, w_o) \mapsto A_{\lambda, x_o, w_0}(0) = (x_o + iv(0), w_o);$$

we prove that for the Jacobian of $\mathcal{E}$ at 0 we have

$$J\mathcal{E} = \begin{pmatrix} \text{id} & i\partial_t h & * \\ 0 & 0 & \text{id} \end{pmatrix}, \tag{2.13}$$

where the asterisk denotes unimportant elements. Thus, when the parameters (λ, x_0, w_o) describe $\mathbb{R}^m_+ \times \mathbb{R}^m \times \mathbb{C}^{n-m}$, the union of the centers of the discs covers a wedge W with direction cone Γ' for every Γ' finer than $\Gamma = \mathbb{R}^m_+$, and we can conclude by the Baouendi-Treves approximation theorem (Theorem 2.1).

From $v(0) = \frac{1}{2\pi} \int_0^{2\pi} v(e^{i\vartheta}) d\vartheta$ for $v = h$, we get

$$\partial_{x_0} v(0) = (2\pi)^{-1} \int_0^{2\pi} \partial_x h \partial_{x_o} u d\vartheta.$$

We also have $\partial_\lambda v(0) = \frac{1}{2\pi} \int_0^{2\pi} (\partial_x h \partial_\lambda u + \partial_t h \partial_\lambda t_\lambda) d\vartheta$. At $\lambda = 0$, $x_o = 0$, $w_o = 0$, we have $\partial_x h = 0$, $\partial_\lambda t_\lambda = \text{id}$. Hence, if $z(0) = u(0) + iv(0)$, $\partial_{x_o} z(0) = \text{id}$, $\partial_\lambda z(0) = i\partial_t h$, which proves (2.13). □

We now treat the Lewy extension theorem of [13], and its extension to manifolds of higher codimension due to Boggess-Polking (see [8]).

Let $M \subset \mathbb{C}^n$ be a smooth real hypersurface given by the equation $r = 0$.

Definition 2.3. *The* Levi form *of $M = \partial\Omega$ at a point p is the hermitian form*

$$L_M(p)(X, \overline{Y}) = \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(p) X_j \overline{Y_k},$$

for $X, Y \in T_p^c M$.

The Levi form is invariant under holomorphic change of coordinates, and its rank and signature are well defined (that is, independent of the defining function r), up to the choice of an orientation: as a convention, we will suppose that the open set Ω is given by $\{r < 0\}$.

When the Levi form of M has at least one negative eigenvalue, holomorphic functions defined in Ω extend across the boundary:

Theorem 2.4. *Let $M = \partial\Omega$ a real hypersurface of class C^5. Suppose that*

$$\partial_{w_0} \partial_{\bar{w}_0} h(p) < 0$$

for a complex tangential vector w_0. Then there is a full neighborhood U of p in $\mathbb{C}^n$ with an extension map

$$hol(U \cap \Omega) \longrightarrow hol(U).$$

Proof. We will construct a family of discs $\{A\}$ attached to M, with $A(1) = z$ describing a neighborhood of p, and prove they are transversal to M at 1 with a uniform bound for the angle they form with TM; then the rays $A([0, 1])$ will fill up the desired neighborhood of p, forcing the extension of the holomorphic functions defined on Ω.

Let M be defined bas in Proposition 2.1 in coordinates $(x + iy, z'')$, and define the z''-component of a disc $A_{z,\eta}$ (for $z = (x + iy, z'')$ close to p and η small) as $w_\eta(\tau) = \eta w_0(1 - \tau)$. By Theorem 2.2, we can find a disc $A_{z,\eta}(\cdot) = (u_\eta(\cdot) + iv_\eta(\cdot), z'' + w_\eta(\cdot))$ attached to M and such that $A_{z,\eta}(1) = z$. Fix $z = p = 0$; it is easy to see (by the normal form of the hypersurface) that the Taylor development of $\partial_t v_\eta$ (for $\tau = te^{i\vartheta} \in \Delta$) with respect to η reduces to

$$\partial_t v_\eta = \partial_t \partial_\eta^2 v_\eta|_{\eta=0} \frac{\eta^2}{2} + o^2.$$

Recalling that $v_\eta = h$ on $\partial\Delta$, and applying the vanishing of derivatives of Proposition 2.1, we can prove that

$$\partial_\eta^2 v_\eta = 2\partial_{w_0} \partial_{\bar{w}_0} h |1 - \tau|^2 \text{ on } \partial\Delta.$$

Since $|1-\tau|^2|_{\partial\Delta} = 2\mathrm{Re}\,(1-\tau)|_{\partial\Delta}$, we have

$$\partial_t \partial_\eta^2 v_\eta|_{\eta=0} = -4\partial_{w_0}\partial_{\bar{w}_0} h > 0,$$

that is, the ray of the disc A_η is transversal to $\partial\Omega$ and points outside Ω. The final step of the proof consists in moving z near 0 for a fixed small η_0, obtaining the desired family of discs. □

Let now $M \subset \mathbb{C}^n$ be a generic, higher-codimensional, submanifold, locally given by an equation $r = 0$, where $r = (r_1, \ldots, r_m)$ is a smooth $\mathbb{R}^m$-valued function in a neighborhood of $0 \in \mathbb{C}^n$ such that $\partial r_1 \wedge \cdots \wedge \partial r_m \neq 0$. Define the Levi form of M as

$$L_M(p)(X, \overline{Y}) = \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(p) X_j \overline{Y_k} \in \mathbb{C}^m,$$

for $X, Y \in T_p^c M$; we identify $N_p M$ with $\mathbb{R}^m$ by the differential $dr(p)$. The *Levi cone* of M at p is the cone

$$\Gamma = \mathrm{conv}\{\mathrm{L_M(p)(X, \overline{X}) : X \in T_p^c M}\};$$

Γ is independent on the choice of r.

Theorem 2.5. *Let M be a generic submanifold of $\mathbb{C}^n$ of class C^5, and let Γ be the Levi cone of M. Then all continuous CR functions on M extend to a wedge W with edge M and direction cone Γ.*

2.3 Nonsmooth analytic discs

Following [21] and [22], we introduce spaces of functions on $\partial\Delta$ and on $\overline{\Delta}$ that are C^1 everywhere outside the point $1 \in \partial\Delta$ and have a prescribed singularity at $\tau = 1$.

In the sequel, for $0 < \alpha < 1$, we take the principle branch of $(1-\tau)^\alpha$ on $\overline{\Delta}$ which is real positive on the segment $[-1, 1]$. For each α, we denote by $d = d(\alpha)$ the unique positive integer such that $d\alpha < 1 \leq (d+1)\alpha$. Then d is the maximal power such that $(1-\tau)^{d\alpha} \notin C^{1,\beta}$ for any $0 < \beta < 1$. Fix any β satisfying

$$\begin{cases} 0 < \beta \leq (d+1)\alpha - 1 & \text{if } (d+1)\alpha > 1, \\ 0 < \beta \leq (d+2)\alpha - 1 & \text{if } (d+1)\alpha = 1. \end{cases} \tag{2.14}$$

Then we have $\beta < \alpha$ and $(1-\tau)^{j\alpha} \in C^{1,\beta}$ if and only if $j = 0$ or $j > d$.

Denote by $\mathbb{C}_d[(1-\tau)^\alpha]$ and $\mathbb{C}_d[(1-\tau)^\alpha, (1-\bar{\tau})^\alpha]$ the spaces of complex polynomials of degree at most d in the corresponding variables. By a slight abuse we use the same notation for the spaces of restrictions of the polynomials to $\partial\Delta$ and to $\overline{\Delta}$ respectively. In order to exclude constant functions from $C^{1,\beta}$, we consider the subspace of all functions $f \in C^{1,\beta}$ with $f(1) = 0$ denoted by $C_1^{1,\beta}$.

We define

$$\begin{aligned} \mathcal{P}^\alpha(\partial\Delta) &:= \mathbb{C}_d[(1-\tau)^\alpha] + C_1^{1,\beta}(\partial\Delta) \subset C(\partial\Delta), \\ \mathcal{P}^\alpha(\overline{\Delta}) &:= \mathbb{C}_d[(1-\tau)^\alpha, (1-\bar{\tau})^\alpha] + C_1^{1,\beta}(\overline{\Delta}) \subset C(\overline{\Delta}). \end{aligned} \tag{2.15}$$

Our definition is given an important motivation by the following property

Lemma 2.1. *Both sums in (2.15) are direct, i.e. any function $f \in \mathcal{P}^\alpha(\partial\Delta)$ (resp. $f \in \mathcal{P}^\alpha(\overline{\Delta})$) is uniquely decomposed as a sum $f = p + g$ with $g \in C_1^{1,\beta}(\partial\Delta)$ (resp. $g \in C_1^{1,\beta}(\overline{\Delta})$) and p a polynomial in the corresponding space.*

Proof. We prove that the decomposition $f = p + g$ is uniquely determined by the asymptotics of f at $1 \in \partial\Delta$. For $\tau = e^{i\vartheta} \in \partial\Delta$ and any j, we have

$$(1-\tau)^{j\alpha} = (1 - \cos\vartheta - i\sin\vartheta)^{j\alpha} = (-i\vartheta)^{j\alpha}(1 + \vartheta r_j(\vartheta))$$

with $r_j(\vartheta)$ real analytic in $[-\pi, \pi]$ (we used the power series expansions of $\sin\vartheta$ and $\cos\vartheta$ at $\vartheta = 0$). Hence, if we take two decompositions $f = p_1 + g_1 = p_2 + g_2$, we must have $p_1((-i\vartheta)^\alpha) - p_2((-i\vartheta)^\alpha) \in C_1^{1,\beta}$ which is only possible for $p_1 = p_2$, and therefore $g_1 = g_2$. The uniqueness of the decomposition in $\mathcal{P}^\alpha(\overline{\Delta})$ also follows from the asymptotics of the powers $(1-\tau)^{j\alpha}$ at $\tau = 1$ for $\tau \in \overline{\Delta}$. □

Let $f \in \mathcal{P}^\alpha$, and $f = p + g$ its decomposition as in Lemma 2.1, with $p = \sum_{j=1}^d c_j(1-\tau)^{j\alpha}$. Thanks to the uniqueness of such decomposition, we can define the norm

$$\|f\|_{(\alpha)} = \sum_{j=1}^d |c_j| + \|g\|_{C^{1,\beta}}.$$

This norm makes $\mathcal{P}^\alpha(\partial\Delta)$ (resp. $\mathcal{P}^\alpha(\overline{\Delta})$) a Banach space. Moreover

Lemma 2.2. *There exists a constant $C > 0$ such that the spaces $\mathcal{P}^\alpha(\partial\Delta)$ and $\mathcal{P}^\alpha(\overline{\Delta})$ with the norm $C\|\cdot\|_{(\alpha)}$ become Banach algebras.*

Proof. We prove the statement for $\mathcal{P}^\alpha(\partial\Delta)$; the case of $\mathcal{P}^\alpha(\overline{\Delta})$ is analogous. The only nontrivial statement is the behaviour with respect to the multiplication. If f and g are either polynomials or functions in $C_1^{1,\beta}$, it is easy to check that

$$C\|fg\|_{(\alpha)} \le C^2\|f\|_{(\alpha)}\|g\|_{(\alpha)}. \tag{2.16}$$

It remains to consider the case when $f(\tau) = (1-\tau)^{j\alpha}$ and $g \in C_1^{1,\beta}$. After removing the linear terms, we may suppose $g(1) = g'(1) = 0$ and hence $|g(\tau)| \leq \|g\|_{1,\beta}|1-\tau|$, $|g'(\tau)| \leq \|g\|_{1,\beta}|1-\tau|^\beta$. We have

$$(fg)'(\tau) = j\alpha(1-\tau)^{j\alpha-1}g(\tau) + (1-\tau)^{j\alpha}g'(\tau). \tag{2.17}$$

Since $j\alpha \geq \alpha > \beta$, the second term is a product of functions in C^β, hence it is in C^β with its norm estimated by $\|f\|_{(\alpha)}\|g\|_{(\alpha)}$. To show that the first term is also in C^β, we estimate its derivative

$$|(j\alpha(1-\tau)^{j\alpha-1}g(\tau))'| \lesssim |(1-\tau)^{j\alpha-2}g(\tau)|+|(1-\tau)^{j\alpha-1}g'(\tau)| \lesssim \|g\|_{1,\beta}|1-\tau|^{\beta-1}$$

which implies, by integration, that also the first term on the right-hand side of (2.17) is in C^β with its norm estimated by $\|g\|_{1,\beta}$. We have $fg \in C_1^{1,\beta}(\partial\Delta)$ with $\|fg\|_{(\alpha)} \lesssim \|f\|_{(\alpha)}\|g\|_{(\alpha)}$. Then there exists a suitable constant C such that we easily obtain the estimate (2.16). □

From now on we rescale the $\mathcal{P}^\alpha$-norm according to Lemma 2.2 to obtain the inequality $\|fg\|_{(\alpha)} \leq \|f\|_{(\alpha)}\|g\|_{(\alpha)}$ for all $f, g \in \mathcal{P}^\alpha(\partial\Delta)$ (resp. $f, g \in \mathcal{P}^\alpha(\overline{\Delta})$) without any constant C.

It is an immediate consequence of the construction of the spaces $\mathcal{P}^\alpha$ and of Privalov's theorem (2.2) that the Hilbert transform

$$T_1 : \mathcal{P}^\alpha(\partial\Delta) \to \mathcal{P}^\alpha(\partial\Delta)$$

is a continuous linear operator.

Lemma 2.3. *Let $d \geq 1$ and $0 < \beta < \alpha$ be chosen as before, and let K be either $\partial\Delta$ or $\overline{\Delta}$. If $f \in \mathcal{P}^\alpha(K, \mathbb{R}^n)$ with $f(1) = 0$ and $h \in C^{d+2}(\mathbb{R}^n, \mathbb{R}^m)$ with vanishing derivatives $h^{(j)}(0)$, $1 \leq j \leq d$, the composition $h \circ f$ is always in $C^{1,\beta}(K, \mathbb{R}^m)$.*

Proof. Without loss of generality, we can suppose $h(0) = 0$. We want to show that the estimate

$$\|h \circ f\|_{1,\beta} \lesssim \left(1 + \|f\|_{(\alpha)}^{d+1}\right)\|h\|_{d+2} \tag{2.18}$$

holds. Write $f = p + g$ as in Lemma 2.1; we have

$$(h(f))'(\tau) = h'(f(\tau))[p'(\tau)] + h'(f(\tau))[g'(\tau)]. \tag{2.19}$$

Since $\beta < \alpha$ we always have $f \in C^\alpha \subset C^\beta$, with $\|f\|_\beta \lesssim \|f\|_{(\alpha)}$; therefore $h' \circ f \in C^\beta$ with

$$\|h' \circ f\|_\beta \lesssim (1 + \|f\|_{(\alpha)})\|h\|_2.$$

Since g' is in C^β (which is a Banach algebra), the second term on the right-hand side of (2.19) is C^β and its norm is estimated by the right-hand side of (2.18). We need to show the same estimates holds for the first term.

Since $(1-\tau)^\alpha \in C^\beta$, the multiplication with $(1-\tau)^\alpha$ or with $(1-\bar\tau)^\alpha$ preserves the class C^β. Hence it suffices to prove the estimate

$$\|h'(f(\tau))[p'(\tau)]\|_\beta \lesssim \big(1+\|f\|_{(\alpha)}^{d+1}\big)\|h\|_{d+2} \tag{2.20}$$

for $p(\tau)=(1-\tau)^\alpha$ (the case $p(\tau)=(1-\bar\tau)^\alpha$ is analogous). We write the derivative

$$\begin{aligned}(h'(f(\tau))[(1-\tau)^{\alpha-1}])'(\tau) = h''(f(\tau))[f'(\tau)][(1-\tau)^{\alpha-1}]\\ +(\alpha-1)h'(f(\tau))[(1-\tau)^{\alpha-2}];\end{aligned} \tag{2.21}$$

by the vanishing hypothesis on the derivatives, we have

$$|h(x)|\lesssim \|h\|_{d+2}|x|^{d+2},\quad |h'(x)|\lesssim \|h\|_{d+2}|x|^{d+1},\quad |h''(x)|\lesssim \|h\|_{d+2}|x|^{d}.$$

Using these estimates, along with

$$|f(\tau)|\lesssim \|f\|_{(\alpha)}|1-\tau|^\alpha \quad |f'(\tau)|\lesssim \|f\|_{(\alpha)}|1-\tau|^{\alpha-1},$$

we obtain

$$|h''(f(\tau))[f'(\tau)][(1-\tau)^{\alpha-1}]| \lesssim \|f\|_{(\alpha)}^{d+1}\|h\|_{d+2}|1-\tau|^{(d+2)\alpha-2} \tag{2.22}$$

and

$$|h'(f(\tau))[(1-\tau)^{\alpha-2}]| \lesssim \|f\|_{(\alpha)}^{d+1}\|h\|_{d+2}|1-\tau|^{(d+2)\alpha-2}. \tag{2.23}$$

By an integration of (2.22) and (2.23), we get the estimate (2.20), that concludes our proof. □

The C^{d+2}-smoothness in Lemma 2.3 is not necessary if we don't have $(d+1)\alpha=1$; in any other case it is sufficient to replace $d+2$ with $d+1$ to obtain the same result. We write $d'=d'(\alpha)=d$ for $(d+1)\alpha>1$ and $d'=d+1$ otherwise; hence the conclusion of Lemma 2.3 holds for $d+2$ replaced replaced by $d'+1$.

In [21] the following more general result is proved on the differentiability of the composition operator acting on $\mathcal{P}^\alpha$.

Proposition 2.3. *For $l\geq 1$, the composition $(h,f)\mapsto h\circ f$ defines a C^l map $c\colon C^{d'+l+1}(\mathbb{R}^n,\mathbb{R}^m)\times\mathcal{P}^\alpha(K,\mathbb{R}^n)\to\mathcal{P}^\alpha(K,\mathbb{R}^m)$ whose first derivative is given by*

$$c'(h,f)[\dot h,\dot f](\vartheta)=\dot h(f(\vartheta))+h'(f(\vartheta))[\dot f(\vartheta)].$$

Let $M \subset \mathbb{C}^n$ be a generic submanifold of class $C^{d'+l+1}$ ($l \geq 1$) through 0 that is locally represented as a graph

$$M = \{(x+iy, w) \in \mathbb{C}^{n-m} \times \mathbb{C}^m : y = h(x,w)\} \tag{2.24}$$

with $h(0) = 0$, $h'(0) = 0$. We know from the classical theory (Proposition 2.2) that it is possible to attach C^α-discs $A(\cdot) = (z(\cdot), w(\cdot))$ to M for a prescribed component $w(\cdot) \in C^\alpha$. We want to show that, when the w-component is in $\mathcal{P}^\alpha$, the whole disc is in $\mathcal{P}^\alpha$; moreover, we will prove that the discs smoothly depend on their parameters.

Proposition 2.4. *Let $h = h_0$ be of class $C^{d'+2}$ and $M = M_0$ be given by (2.24). For every sufficiently small $x_0 \in \mathbb{R}^{n-m}$, $w(\cdot) \in \mathcal{P}^\alpha$, where $w(\cdot)$ is holomorphically extendible to Δ, and for every h sufficiently close to h_0 in $C^{d'+2}$, there exists a unique sufficiently small disc $A(\cdot) = (z(\cdot), w(\cdot))$ in $\mathcal{P}^\alpha$ attached to M such that $\operatorname{Re} z(1) = x$. For $h \in C^{d'+l+1}$ ($l \geq 1$), the disc $A \in \mathcal{P}^\alpha$ depends in a C^l fashion on the parameters $x_0 \in \mathbb{R}^{n-m}$, $w \in \mathcal{P}^\alpha$ and $h \in C^{d'+l+1}$.*

Proof. The required disc $A(\cdot) = (x(\cdot)+iy(\cdot), w(\cdot))$ can be obtained by solving the Bishop equation

$$x(\cdot) + T_1 h\big(x(\vartheta), w(\vartheta)\big)(\cdot) - x_0 = 0. \tag{2.25}$$

Call F the left-hand side of 2.25: it follows from Lemma 2.3, Proposition 2.3, and the continuity of T_1 on the space $\mathcal{P}^\alpha$, that F is a differentiable mapping. Moreover, we have

$$\partial_x F[\dot{x}] = \dot{x} - T_1 h(x,w)(\cdot) - x_0;$$

in particular, evaluation at $(0,0,0)$ implies that $\partial_x F$ is invertible, since $\partial_x h(0) = 0$. Thus, for the implicit function theorem, we have a unique solution to 2.25. □

Consider the cotangent bundle $T^*\mathbb{C}^n$: we identify it with the space of all $(1,0)$ covectors. Then the conormal bundle $T^*_M\mathbb{C}^n$ of M in $\mathbb{C}^n$ is the set of all covectors in $T^*\mathbb{C}^n|_M$ which are purely imaginary when restricted to TM; $T^*_M\mathbb{C}^n$ is a real (not necessarily CR) submanifold of $T^*\mathbb{C}^n$. If $\pi : T^*\mathbb{C}^n \to \mathbb{C}^n$ is the natural projection: each fiber $(T^*_M\mathbb{C}^n)_p$ is naturally identified with a maximal totally real linear subspace of $T^*_p\mathbb{C}^{n-m}$. Denote by $\mathsf{GL}(\mathbb{C}^{n-m})$ the general linear group on $\mathbb{C}^n$ (that is, the group of all linear automorphisms of $\mathbb{C}^{n-m}$); given $p \in M$, the set $\mathcal{G}^*(p)$ of all $(q, G^*) \in M \times \mathsf{GL}(\mathbb{C}^{n-m})$ with $G^*((T^*_M\mathbb{C}^n)_p) = (T^*_M\mathbb{C}^n)_q$ is a generic submanifold in $\mathbb{C}^n \times \mathsf{GL}(\mathbb{C}^{n-m})$ with maximal totally real fibers over M. Hence we can identify $T^*_M\mathbb{C}^n$ with a generic submanifold of $T^*\mathbb{C}^{n-m}$. Applying Proposition 2.4 to the generic submanifold $\mathcal{G}^*(p)$ for $p = A(1)$, we get:

Lemma 2.4. *Let A be a small $\mathcal{P}^\alpha$ disc in attached to a generic submanifold $M \subset \mathbb{C}^n$ of class $C^{d'+3}$. Then there is a unique disc $G^* : \overline{\Delta} \to \mathsf{GL}(\mathbb{C}^{n-m})$ of class $\mathcal{P}^\alpha$ such that $G^*(1) = \mathsf{id}$ and, for $\vartheta \in \partial\Delta$,*

$$G^*(\vartheta)((T^*_M\mathbb{C}^n)_{A(1)}) = (T^*_M\mathbb{C}^n)_{A(\vartheta)}.$$

We say that a generic manifold M, graphed by a function h in a neighborhood of 0 as in (2.24), is flat and rigid up to the order k if

$$h(x,w) = O(|x|^k + |w|^k);$$

we remark that by (2.2) all generic manifolds are flat and rigid up to the order 2. This observation led in [21] to prove that, if $1/2 < \alpha < 1$, the normal component of discs in $\mathcal{P}^\alpha$ is smoothed by composition with the function h graphing M. More generally, making use of Lemma 2.3:

Lemma 2.5. *Let A be a sufficiently small $\mathcal{P}^\alpha$-disc attached to a generic submanifold $M \subset \mathbb{C}^n$, flat and rigid up to the order k and of class C^{k+1}, and set $p = A(1) \in M$. Then there is a unique representation*

$$A(\tau) = p + (1-\tau)^\alpha A_1 + \ldots + (1-\tau)^{d\alpha} A_d + B(\tau),$$

with $A_1, \ldots, A_d \in T^c_pM$, $B(\cdot) \in C^{1,\beta}$ and $B(1) = 0$.

Proof. We can suppose $p = 0$ and M is given by (2.24) with $h^{(j)}(0) = 0$ for $1 \le j \le k-1$. Since A is attached to M, we have $y(\tau) = h(x(\tau), w(\tau))$ for $\tau \in \partial\Delta$. Then, by Lemma 2.3, $y(\cdot) \in C^{1,\beta}(\partial\Delta)$; hence even its Hilbert transform $x(\cdot)$ is in $C^{1,\beta}(\partial\Delta)$. Therefore the holomorphic extension $z(\cdot) = x(\cdot) + iy(\cdot) \in C^{1,\beta}(\overline{\Delta})$. Then the existence and uniqueness of the representation above are given by the definition of the space $\mathcal{P}^\alpha$ and Lemma 2.1. □

We denote by $[v] \in T_M\mathbb{C}^n$ the equivalence class defined by a tangent vector v. Since under the conditions of Lemma 2.5, the normal component of A is $C^{1,\beta}$, it makes sense to write $[\partial_r A(1)] \in (T_M\mathbb{C}^n)_{A(1)}$, even though $\partial_r A(1)$ (the radial derivative in Δ) may not exist; it makes sense now to discuss the directions of $\mathcal{P}^\alpha$-discs at their singular points.

Let G^* be the "connection" on $T^*_M\mathbb{C}^n$ over $\partial\Delta$ defined by Lemma 2.4: for each $\vartheta_1, \vartheta_2 \in \partial\Delta$, we have a linear isomorphism between $(T^*_M\mathbb{C}^n)_{A(\vartheta_1)}$ and $(T^*_M\mathbb{C}^n)_{A(\vartheta_2)}$. By duality, we can define an isomorphism between $(T_M\mathbb{C}^n)_{A(\vartheta_1)}$ and $(T_M\mathbb{C}^n)_{A(\vartheta_2)}$; we call G the corresponding $\mathsf{GL}(\mathbb{C}^{n-m})$-valued analytic disc that gives the dual connection on $T_M\mathbb{C}^n$ over A. We now show that G describes the direction of the deformation of a $\mathcal{P}^\alpha$-disc attached to M.

Proposition 2.5. *Let M be a generic manifold, rigid and flat up to the order k, of class C^l $(l \geq k+1)$, and let A be a small analytic disc of class $\mathcal{P}^\alpha$ attached to M with $p = A(1)$. Let M' be a C^l submanifold with boundary M at a point $q \in A(\partial\Delta)$, with $C_q M' = T_q M \oplus \mathbb{R}_+ v$ for a $v \in T_q\mathbb{C}^n$. For any $\epsilon > 0$ there is a C^l family of submanifolds $M_\eta \subset M \cup M'$, for $0 \leq \eta \leq \eta_0$, such that $M_0 = M$ and the analytic disc $A_\eta(\tau) = (z_\eta(\tau), w(\tau))$ attached to M, with the same w-component $w(\tau)$ as A and $A_\eta(1) = p$, satisfies*

$$[\partial_r A_\eta(1)] = [\partial_r A(1)] + \eta\big(G(\tau_0)^{-1}[v] + [v_0]\big) + o(\eta) \tag{2.26}$$

when $\eta \to 0$, for some $v_0 \in T_p\mathbb{C}^n$ with $|v_0| < \epsilon$.

Proof. We consider real coordinates $x \in \mathbb{C}^n$ with $q = 0$, in which M is given by $x_1 = \cdots = x_m = 0$ and M' by $x_1 = \cdots = x_{m-1} = 0$, $x_m \geq 0$. Let $\varphi \geq 0$ be a function with compact support in a sufficiently small neighborhood of q in M, and define M_η, $0 \leq \eta \leq \eta_0$, as the deformation of M that coincides with M outside the support of φ, and is given by $x_s = \eta\varphi(x_{s+1}, \ldots, x_{2n})$ near q.

Take the analytic disc A_η attached to M_η with $A_\eta(1) = p$ and with the same w-component as A. By Proposition 2.4, the derivative $\dot{A}$ of A_η with respect to η for $\eta = 0$ exists and belongs to $\mathcal{P}^\alpha$. Since the "w-component" of A_η is fixed, we have $\dot{A} = (\dot{z}, 0)$. Let G^* as in Lemma 2.4; for any $\xi \in (T^*_M\mathbb{C}^n)_p$, the function $\psi(\tau) := (G^*(\tau)\xi)\,[\dot{A}(\tau)]$ is holomorphic in Δ, and, since the real part of $G^*(\tau)\xi$ is 0 on $\partial\Delta$, the real part of ψ vanishes on $\partial\Delta$ away from the support of φ, where $M_\eta = M$.

Take ξ such that $\mathrm{Re}\,(G^*(\tau_0)\xi)[v] \geq \delta|\xi|$ for a fixed small $\delta > 0$. If the deformation of M takes place only in a sufficiently small neighborhood of q, the direction of $[\dot{A}(\tau)]$ differs only slightly from $[v]$ for $\tau \sim \tau_0$ (while $[\dot{A}(\tau)] = 0$ for τ far from τ_0). Then, since also $\mathrm{Re}\,\psi(1) = 0$, $\mathrm{Re}\,\psi(\tau) \geq 0$ for all $\tau \in \partial\Delta$. But then, by Hopf lemma, the radial derivative $\mathrm{Re}\,\xi[\partial_r\dot{A}(1)]$ is positive. Since ξ is arbitrarily chosen satisfying $\mathrm{Re}\,(G^*(\tau_0)\xi)[v] \geq \delta|\xi|$, we have

$$\big|[\partial_r\dot{A}(1)] - \lambda G(\tau_0)^{-1}[v]\big| < \varepsilon$$

for some $\lambda > 0$ and sufficiently small $\delta > 0$. By a linear change of the parameter η we can achieve $\lambda = 1$. It remains to remark that the radial derivative $[\partial_r A_\eta(1)]$ is continuous in η with $\frac{\partial}{\partial\eta}[\partial_r A_\eta(1)]|_{\eta=0} = [\partial_r\dot{A}(1)]$. □

It is now immediate to state the following:

Proposition 2.6. *Let $M \subset \mathbb{C}^n$ be a generic, k-flat and rigid, C^l-smooth submanifold $(l \geq k+1)$ through $p = 0$ and let A be a small analytic disc of class $\mathcal{P}^\alpha$ attached to M. If $M'_1, \ldots, M'_s$ are C^l-smooth submanifolds with boundary M at a point $q \in A(\partial\Delta)$ in s linearly independent directions*

$[v_1], \dots, [v_s] \in (T_M\mathbb{C}^n)_q$, we can find s submanifolds $M_j \subset M \cup M'_j$ of class C^l, $\dim \mathrm{M_j} = \dim \mathrm{M}$ *for all j, and arbitrarily close to M in the C^l norm such that, for the discs $A_1, \dots, A_s$ of class $\mathcal{P}^\alpha$ attached to $M_1, \dots, M_s$ respectively, with $A_j(1) = A(1)$ and with the same w-component as A, we have that $[\partial_r A_1(1)], \dots, [\partial_r A_s(1)]$ are linearly independent.*

2.4 Baouendi-Treves approximation for sectors

Definition 2.4. *Let $V \subset \mathbb{C}^n$ be a generic submanifold. V is a* Baouendi-Treves submanifold *if, for every $j = 0, 1, \dots, \infty, \omega$, every CR-function on V of class C^j can be uniformly approximated by holomorphic polynomials on the compact subsets of V in the C^s topology.*

In [19] it was observed that the original proof of the Baouendi-Treves approximation theorem (Theorem 2.1) can be adapted to the situation of a submanifold with generic edge; however, in the following section we will consider a case where the edge is not generic. Here we show, following [21], that neighborhoods of certain sectors in V are submanifolds of Baouendi-Treves.

Theorem 2.6. *Let $M \subset \mathbb{C}^n$ be a generic submanifold and $V \subset M$ an open subset with Lipschitz boundary at $p = 0 \in \partial V$. Suppose we can find $v \in \mathbb{C}^n$ and $0 < \alpha < 1$ such that the sector $S_v(\alpha) := \{z^\alpha v : \operatorname{Re} z \geq 0\}$ is contained in C_pV. Then there exists $\varepsilon_0 > 0$ such that the open subset*

$$\{z \in M : \operatorname{dist}(z, S_v(\alpha)) < \varepsilon_0 \operatorname{dist}(z, p) < \varepsilon_0^2\} \tag{2.27}$$

is a Baouendi-Treves submanifold in $\mathbb{C}^n$.

We want to adapt the proof of Theorem 2.1 to our situation. Denote by B_0^n (resp. B^n) the open (resp. closed) unit ball in $\mathbb{R}^n$.

Lemma 2.6. *Let $M \subset \mathbb{C}^n$ be a generic submanifold of CR-dimension m. Set $d := n - m$. Suppose that for any compact $K \subset M$, there exist a holomorphic nondegenerate quadratic form $\omega(z)$ in $\mathbb{C}^n$ and a smooth map $\varphi\colon B^n \times B^d \to M$ such that the following hold:*

(i) the image $\varphi(B_0^n \times B_0^d)$ contains K;

(ii) for each $y \in B^d$, the restriction $\varphi(\cdot, y)$ is an embedding of B^n into M as a maximally totally real submanifold N_y (with boundary) such that the restriction $\operatorname{Re}\omega|_{N_y}$ is positive definite.

(iii) $\varphi(x, \cdot) = \mathrm{const}$ *for every* $x \in \partial B^n$*, in particular, the boundaries of* N_y*'s are the same for all* $y \in B^d$*.*

Then M *is a submanifold of Baouendi-Treves.*

Proof. Take a CR-function f on M and $y \in B^d$ and define the sequence of entire functions

$$f_{\lambda,y}(z) := \left(\frac{\lambda}{\pi}\right)^{n/2} \int_{N_y} f(\zeta) e^{-\lambda\omega(\zeta - z)} d\zeta_1 \wedge \cdots \wedge d\zeta_n, \quad l = 1, 2, \ldots.$$

The positivity condition for $\mathrm{Re}\,\omega|_{N_y}$ implies that $f_{\lambda,y}$ converges to f as $\lambda \to \infty$, uniformly on compacta in the interior of N_y. It is easy to see by using (i), that the convergence is uniform on K. Moreover, it follows from the fact that f is CR, from (iii) and from the Stokes theorem that the functions $f_{\lambda,y}$ are independent on $y \in B^d$. Thus, $f_{\lambda,y}$ is a sequence of entire functions that uniformly converges to f on K, and we can approximate entire functions by taking their Taylor polynomials. □

In order to apply Lemma 2.6 to our situation, we need to construct φ satisfying the requirements. For $\epsilon > 0$,we define the real convex cone

$$A_\varepsilon := \{x_1^2 > \frac{\varepsilon}{2}(x_2^2 + \cdots + x_n^2)\} \subset \mathbb{R}^n \subset \mathbb{C}^n$$

and the ball

$$B_\varepsilon := \left\{ iy : y_1^2 + \cdots + y_n^2 < \frac{1}{1+\varepsilon} \right\} \subset i\mathbb{R}^n \subset \mathbb{C}^n.$$

Moreover, we set $\widetilde{A}_\varepsilon := (-1 + A_\varepsilon) \cap (1 - A_\varepsilon) \subset \mathbb{R}^n$, (where $1 = (1, 0, \ldots, 0) \in \mathbb{C}^n$). If $iy \in B_\varepsilon$, let $C_\varepsilon(y) \subset \mathbb{C}^n$ be the union of all real line segments connecting iy with boundary points of $\widetilde{A}_\varepsilon$ and let C_ε be the union of the subsets $C_\varepsilon(y)$ for $iy \in B_\varepsilon$.

Lemma 2.7. *For any* $\varepsilon > 0$ *and any* $iy \in B_\varepsilon$*, the standard form* $\mathrm{Re}\,\omega = \mathrm{Re} \sum_j z_j^2 = \sum_j (x_j^2 - y_j^2)$ *is positive on tangent vectors to* $C_\varepsilon(y)$*. Moreover, for any* $1 \leq d \leq n$ *and* $\delta > 0$*, the exist a smooth map* $\varphi\colon B^n \times B^d \to \mathbb{R}^n \oplus i\mathbb{R}^d$ *satisfying conditions (ii) and (iii) in Lemma 2.6 with* ω *as above and such that*

$$C_\varepsilon \cap (\mathbb{R}^n \oplus i\mathbb{R}^d) \subset \varphi(B_0^n \times B_0^d) \subset (1 + \delta) C_\varepsilon \cap (\mathbb{R}^n \oplus i\mathbb{R}^d).$$

Proof. Any tangent vector v to $C_\varepsilon(y_0)$ at a point $z = x + iy$ is a sum $v_1 + v_2$, where v_1 is tangent to the segment connecting z with a boundary point $a \in \partial\widetilde{A}_\varepsilon$ and v_2 is tangent to $\partial\widetilde{A}_\varepsilon$ at a. If $v_1 = 0$, the claim is clear, since

$\mathrm{Re}\,\omega$ is positive on $\mathbb{R}^n \subset \mathbb{C}^n$. Otherwise, by rescaling v, we may assume $v_1 = a - iy$. By the convexity of $\widetilde{A}_\varepsilon$, $a + v_2 \notin \widetilde{A}_\varepsilon$. Then, by the construction of A_ε and B_ε, we have $|y| < |a + v_2|$. Since $v = a + v_2 - iy$, this shows $\mathrm{Re}\,\omega(v) > 0$ as required.

For the second statement, remark that $C_\varepsilon(y)$ can be written as $C_\varepsilon(y) = \{x + i\xi(x)y : x \in \widetilde{A}_\varepsilon\}$ for a suitable continuous function $\xi(x)$. We replace ξ by a smooth function $\widetilde{\xi}$ that approximates ξ in the C^1 norm such that the submanifold $\widetilde{C}_\varepsilon(y) := \{x + i\widetilde{\xi}(x)y : x \in \widetilde{A}_\varepsilon\}$ still satisfies the above positivity condition. It remains to choose $\varphi(x, y) := \sqrt{1 + \delta}(x + i\widetilde{\xi}(x)y)$. □

We are now ready to prove Theorem 2.6. Denote by d the codimension of M in $\mathbb{C}^n$. Without loss of generality, $p = 0$. The proof will depend on the case whether α is larger or smaller than $1/2$. Suppose first $\alpha > 1/2$. Then $\varepsilon > 0$ can be chosen such that, for $y := (1 + \varepsilon)^{-1/2}(1, 0, \ldots, 0)$, the intersection $I := C_\varepsilon(y) \cap (\mathbb{C} \times \{0\})$ has the angle α at the point iy. Then, for any $\lambda > 0$, there exists a complex affine automorphism $F_\lambda \colon \mathbb{C}^n \to \mathbb{C}^n$ sending $i\lambda y$ to 0, $\lambda C_\varepsilon(y) \cap (\mathbb{C} \times \{0\})$ to $S_v(\alpha)$ and $\lambda C_\varepsilon \cap (\mathbb{R}^n \oplus i\mathbb{R}^d)$ into the interior of C_pV. It follows from the definition of C_pV that for $\lambda > 0$ sufficiently small, the map F_λ can be approximated on the closure $\overline{\lambda C_\varepsilon}$ in C^1 norm by a diffeomorphism $\widetilde{F}_\lambda \colon \mathbb{C}^n \to \mathbb{C}^n$ sending $\lambda C_\varepsilon \cap (\mathbb{R}^n \oplus i\mathbb{R}^d)$ into the closure $\overline{V} \subset M$ such that $\widetilde{F}_\lambda(i\lambda y) = 0$ and $d\widetilde{F}_\lambda(i\lambda y) = dF_\lambda(i\lambda y)$. We can now use $\widetilde{F}$ to transfer the standard form ω and the family φ constructed in Lemma 2.7 to the image $U := \widetilde{F}_\lambda(\lambda C_\varepsilon \cap (\mathbb{R}^n \oplus i\mathbb{R}^d)) \subset \overline{V}$ in order to have data satisfying the assumptions of Lemma 2.6. Then Lemma 2.6 yields the required conclusion. The proof for $\alpha \le 1/2$ is analogous to that in the first case $\alpha > 1/2$ with the only exception that the above map F_λ must be chosen to sends the point $(\lambda, 0, \ldots, 0) \in \lambda C_\varepsilon$ instead of $i\lambda y$ into $p = 0$.

2.5 Extension of CR functions into weighted wedges

The celebrated theorem of Boggess-Polking of [8] extending classical results of Hans Lewy (see [13]) (see Theorems 2.4 and 2.5) states that CR-functions defined on a generic submanifold $M \subset \mathbb{C}^n$ extend holomorphically to a wedge in the direction of the convex cone spanned by the values of the Levi form of M. Here one starts with a submanifold M and ends with a wedge. A natural question is to obtain generalizations of this result within the category of wedges.

In [19], Tumanov shows that holomorphic functions defined in a fixed neighborhood of a wedge V with generic edge E in a submanifold $M \subset \mathbb{C}^n$, extend holomorphically to a fixed wedge in $\mathbb{C}^n$ with edge E. This conclusion does not hold if the edge E is not generic, or if it is not smooth. We want to give conditions on V that yield holomorphic extension to regions more general than usual wedges.

We say that an open subset V in a smooth manifold M has Lipschitz boundary at a point $p \in \partial V$ if, in suitable coordinates near p, ∂V is represented by the graph of a Lipschitz function. One can see that V has Lipschitz boundary at p if and only if there is an open cone $\Gamma \subset T_pM$ such that, for any strictly finer subcone $\Gamma' \subset \Gamma$, one has (in local coordinates) $x + y \in V$ for all $x \in V$ and $y \in \Gamma'$ sufficiently close to p and 0 respectively. It is clear that, if two cones $\Gamma_1, \Gamma_2 \subset T_pM$ satisfy the above property, so does their sum $\Gamma_1 + \Gamma_2$. Furthermore, among all such cones there is a unique maximal one, namely the sum of all of them that is automatically convex. We call it the *tangent cone* to V at p and denote by C_pV. We define the *complex angle* of V at p to be the maximal angle of the intersection of C_pV with a complex line in T_pM. If all intersections are empty, we say that the complex angle is 0. It is clear that the complex angle is a local biholomorphic invariant of V at p.

In this section the edge of V plays a secondary role. It can be seen as a subset of the Lipschitz boundary of V:

Definition 2.5. *Let M be a submanifold of $\mathbb{R}^m$ and $p \in M$. A wedge with edge M at p is an open subset in $\mathbb{R}^m$ with Lipschitz boundary at $p \in \partial V$ such that ∂V contains a neighborhood of p in M.*

A basic notion in our exposition is that of α-wedge, as defined in [21]: they can be viewed as wedges with the normal directions to M that have a weight $0 < \alpha < 1$.

Definition 2.6. *Let $M \subset \mathbb{R}^m$ be a submanifold, $V \subset M$ an open subset and $p \in \partial V$. Fix $0 < \alpha < 1$. An* α-wedge *in $\mathbb{R}^m$ over V at p is an open subset $V' \subset \mathbb{R}^m$ for which there exist a neighborhood Ω of p in $\mathbb{R}^m$, a wedge W with edge M at p and a constant $C > 0$ such that*

$$V' \cap \Omega \supset \{x \in W : \operatorname{dist}(x, V) < C \operatorname{dist}(x, \partial V)^{1/\alpha}\}. \tag{2.28}$$

The main result of this chapter is the following:

Theorem 2.7. *Let $M \subset \mathbb{C}^n$ be a generic k-flat and rigid submanifold of class C^{k+1} and $V \subset M$ an open subset with Lipschitz boundary at $p \in \partial V$, with complex angle $\pi\alpha$ for some $1/k < \alpha < 1$. Then for every neighborhood V' of V in $\mathbb{C}^n$ there exists an α-wedge V'' in $\mathbb{C}^n$ over V at p such that all holomorphic functions in V' extend holomorphically to V''.*

This is a generalization of the result contained in [21], where the case $k = 2$ was considered. We recall that a generic manifold M is always 2-rigid and flat.

We will make use of the following abstract lemma for families of real curve: the aim of this result is to prove that, in the context we are considering, a family of radii of nonsmooth analytic discs (as defined in the previous section) attached to V fills an α-wedge over V.

Proposition 2.7. *Let $V \subset \mathbb{R}^m \times 0 \subset \mathbb{R}^m \times \mathbb{R}^{n-m}$ be an open set with Lipschitz boundary at* 0*, and take a map of the form*

$$\begin{array}{rcl}\varphi : [0,1] \times \overline{V} & \to & \mathbb{R}^n \\ (t,p) & \mapsto & p + t^{\alpha} a_1(p) + \ldots + t^{d\alpha} a_d(p) + b(t,p)\end{array}$$

with $a(\cdot)$, $b(\cdot, p)$ of class $C^{1,\gamma}$ and $d = d(\alpha)$ as defined before. Suppose that

- $a_j(p) \in C_0 V \times \{0\}$ *for all* $j = 1, \ldots, d$*;*
- $b(0,p) = 0$ *for all* $p \in \overline{V}$*;*
- $\partial_t b(0,0) \notin \mathbb{R}^m \times \{0\}$*;*
- *the map* $p \mapsto b(\cdot,p)$, $\overline{V} \to C^{1,\gamma}$ *is of class* $C^{1,\gamma}$.

Then there exist $\epsilon > 0$, a neighborhood U of 0 *in $\mathbb{R}^m \times \{0\}$ and a submanifold M' of class C^{1,γ^2} with boundary M at* 0 *and additional direction $\partial_t b(0,0)$ such that φ is a homeomorphism between $(0,\epsilon) \times (V \cap U)$ and an α-wedge over V at* 0 *in M'.*

Proof. We can suppose (up to a linear change of coordinates) $\partial_t b(0,0) \in 0 \times \mathbb{R}^{n-m}_+$ and define the map

$$\widetilde{\varphi}(\tau,p) := \big(\varphi_1(\tau^{1/\alpha},p), \ldots, \varphi_m(\tau^{1/\alpha},p), (\varphi_{m+1}(\tau^{1/\alpha},p))^{\alpha}, \ldots, (\varphi_n(\tau^{1/\alpha},p))^{\alpha}\big)$$

for small $\tau \geq 0$ and $p \in \overline{V}$ close to 0. We prove that $\widetilde{\varphi}$ is C^{1,γ^2} in a neighborhood of $(0,0)$ in $[0,1] \times \overline{V}$. The first m components clearly satisfy our claim; we just have to check $(\varphi_{m+j}(\tau^{1/\alpha},p))^{\alpha}$, $j = 1, \ldots, n-m$. If we write $\varphi_{m+j}(t,p) = t\widehat{\varphi}_{m+j}(t,p)$ for a $\widehat{\varphi} \in C^{\gamma}$ with $\widehat{\varphi}(0,0) \neq 0$, we have

$$\begin{array}{rcl}\partial_\tau(\varphi_{m+j}(\tau^{1/\alpha},p))^{\alpha}) & = & (\varphi_{m+j}(\tau^{1/\alpha},p))^{\alpha-1}\,(\partial_t\varphi_{m+j})(\tau^{1/\alpha},p)\,\tau^{(1/\alpha)-1} \\ & = & (\widehat{\varphi}_{m+j}(\tau^{1/\alpha},p))^{\alpha-1}\,(\partial_t\varphi_{m+j})(\tau^{1/\alpha},p).\end{array}$$

Recalling that the composition $h \circ f$ of two maps in C^{γ} is in C^{γ^2}, and that it depends smoothly on h, we can conclude that the map $p \mapsto \partial_\tau \widetilde{\varphi}(\cdot,p)$ between

$\overline{V}$ and C^{γ^2} is of class $C^{1,\gamma}$. In particular, the map $p \mapsto \widetilde{\varphi}(\cdot, p)$ between $\overline{V}$ and C^{γ^2} is also of class $C^{1,\gamma}$. But then both derivatives $\partial_\tau \widetilde{\varphi}$ and $\partial_p \widetilde{\varphi}$ are in C^{γ^2} with respect to $(\tau, p) \in [0,1] \times \overline{V}$ and the regularity of $\widetilde{\varphi}$ follows. Now, since

$$d\widetilde{\varphi}(0,0)[\mathbb{R} \oplus \{0\}] = (a(0), \partial_t b_{m+1}(0,0), \ldots, \partial_t b_n(0,0))$$

and $d\widetilde{\varphi}(0,0)[\mathbb{R} \oplus \mathbb{R}^m] = \mathbb{R}^m \oplus \mathbb{R}\partial_t b(0,0)$, we can use the rank theorem to find a submanifold M' of class C^{1,γ^2} with boundary M at 0 such that $C_0 M' = \mathbb{R}^m \oplus \mathbb{R}_+ \partial_t b(0,0)$ and, for some $C > 0$, the set

$$\widetilde{V}' := \{x \in M' : \operatorname{dist}(x, V) < C \operatorname{dist}(x, \partial V)\}$$

is contained in $\widetilde{\varphi}((0,1] \times \overline{V})$. Then the set

$$V' := \{x \in \mathbb{R}^m \times \mathbb{R}^{n-m}_+ : (x_1, \ldots, x_m, x^\alpha_{m+1}, \ldots, x^\alpha_n) \in \widetilde{V}'\}$$

contains an α-wedge over V at 0. □

As in [19], we prove our extendibility result through a propagation principle for wedge-extendibility of CR functions:

Theorem 2.8. *Let $M \subset \mathbb{C}^n$ be a generic k-flat and rigid submanifold of class C^l ($l \geq k+1$) through $p = 0$, $V \subset M$ an open subset with Lipschitz boundary at $p \in \partial V$ with complex angle $\pi\alpha$ for $\frac{1}{k} < \alpha < 1$, and v a vector in T_pM such that $\{\zeta^\alpha v : \operatorname{Re}\zeta \geq 0, \zeta \neq 0\} \subset C_pV$. Then for any sufficiently small analytic disc A attached to M of the form*

$$A(\zeta) = (1-\zeta)^\alpha A_1 + \ldots + (1-\zeta)^{d\alpha} A_d + B(\zeta), \tag{2.29}$$

where d is the unique positive integer such that $d\alpha < 1 \leq (d+1)\alpha$, $A_j \in \mathbb{R}^+\tilde{v}$ and $B \in C^{1,\beta}(\overline{\Delta})$ with $B(1) = p$, and for any $q \in A(\partial\Delta) \cap V$, the following hold:

(i) For any wedge $V' \subset \mathbb{C}^n$ with edge V at q, there exists an α-wedge $V'' \subset \mathbb{C}^n$ at p over V such that any continuous CR-function on V that has a holomorphic extension to V', has also a holomorphic extension to V''.

(ii) For any wedge V' with edge $E \subset V$ at q in a submanifold M' with boundary V of class C^l, there exists an α-wedge V'' over V at 0 in a submanifold $M'' \subset \mathbb{C}^n$ with boundary M of class C^1 such that any continuous CR-function on V that has a CR-extension to V', has also a CR-extension to V''. Moreover, given several wedges $V'_1, \ldots, V'_s$ as above in s linearly independent directions in $T_q\mathbb{C}^n/T_qM$, the corresponding submanifolds $M''_1, \ldots, M''_s$ can be chosen in s linearly independent directions in $T_p\mathbb{C}^n/T_pM$.

Proof. Take A, p, q and V' as in the statement of part (ii): then $A \in \mathcal{P}^\alpha$ for a suitable $0 < \beta \leq (d'+1)\alpha - 1$. By Proposition 2.4, we can find a C^l-family of $\mathcal{P}^\alpha$-discs attached to M, $x \mapsto A_x$, defined for $x \in M$ in a neighborhood of $p = 0$, with $A_0 = A$ and $A_x(1) = x$. From the assumptions on the complex angle, we can suppose that $A_x(\partial\Delta) \subset V$ if $x \in V$ is sufficiently close to p. Applying Proposition 2.6, we can assume that A is transversal to M at $\tau = 1$, up to an arbitrarily small deformation of M in V'. Now we can apply Proposition 2.7, finding $\epsilon > 0$ and a neighborhood U of 0 in M such that the map $(\tau, x) \mapsto A_x(\tau)$ defines a homeomorphism between $(1-\epsilon, 1) \times (V \cap U)$ and an α-wedge V'' over V at 0 in a submanifold M'' with boundary M at 0 of class $C^{1,\delta}$ for a suitably chosen $0 < \delta < 1$. Finally, if ϵ_0 is given by Theorem 2.6 with v as in the statement, and if A is small enough, $A_x(\partial\Delta)$ is contained in the set (2.27) for any $x \in V$ sufficiently close to p. If we take any such $x = x_0$, any CR-function f on V can be uniformly approximated by a sequence of polynomials in a neighborhood of $A_{x_0}(\partial\Delta) \subset V$. By the maximum principle, the sequence of polynomials converges uniformly on $A_x(\partial\Delta)$ for $x \in V$ sufficiently close to x_0 to a CR extension of f in a neighborhood of $A_{x_0}((1-\epsilon, 1))$ in V''. Moreover, any such sequence of polynomials yields the same limit function. We have obtained a covering of $V \cap U$ by open subsets V_j, such that f extends to a CR function on the interior of each subset

$$V_j'' := \{A_x(\tau) : x \in V_j,\, \tau \in (1-\epsilon, 1)\} \subset V''.$$

We can now choose the covering $\{V_j\}$ so small that, whenever $V_j \cap V_k \neq \emptyset$, there exists a sequence of polynomials as above that converges uniformly on the union $V_j'' \cup V_k''$. Then, by the uniqueness property of the limit, the CR-extensions of f to V_j'' and V_k'' must coincide on the intersection, yielding a well-defined CR-extension of f to V''.

We now pass to prove part (i): we take a wedge $V' \subset \mathbb{C}^n$ with edge V at q, and we observe that we can choose submanifolds $V_1', \dots, V_m' \subset \mathbb{C}^n$ of class C^l with boundary V at q in m linearly independent directions in $T_q\mathbb{C}^n/T_qM$, where m is the codimension of M in $\mathbb{C}^n$. Part (ii) proves that we have extension of any CR-function f on $V \cup (\cup_j V_j')$ to α-wedges $V_1'', \dots, V_m''$ over V at p in submanifolds $M_1'', \dots, M_m''$ respectively, each with boundary M, whose directions in $T_p\mathbb{C}^n/T_pM$ are also linearly independent. Then near each point $p_0 \in V$ close enough to 0 we can apply the edge of the wedge theorem of Ajrapetyan-Henkin (Theorem 2.3), extending f to a wedge W_{p_0} with edge V at p_0 whose direction cone is an arbitrarily smaller cone than the convex linear span of the directions of $M_1'', \dots, M_m''$ at p_0. In order to obtain an α-wedge over V as required we need to estimate the size of W_{p_0} as p_0 approaches the boundary ∂V. To do this, we use the deformation version of the edge-of-the-wedge theorem stated in [15, Proposition 3.3], and then we apply linear rescaling (i.e. linear maps $z \mapsto \lambda z$), to show that the size of the wedge W in the edge-of-th-wedge theorem is proportional to the size of the given submanifolds. Since each V_j'' is an α-wedge over V at 0, its size near p_0 in all directions can be estimated from below by $\mathrm{dist}(p_0, \partial V)^{1/\alpha}$ up to a constant. Hence also the size of W_{p_0} has a proportional estimate from below. It follows from the definitionof α-wedge that the wedges W_{p_0} cover an α-wedge V'' over V at 0. Furthermore, by choosing W_{p_0} in a suitable way and using the uniqueness of a holomorphic extension of functions into wedges, we get holomorphic extension of f to V''. □

We recall that a CR-curve in M is a piecewise-smooth curve $\gamma : [0,1] \to M$ with $\gamma'(t) \in T^c_{\gamma(t)}M$ for all $t \in [0,1]$. By approximating CR curves and using Theorem 2.8, we get the following more general result:

Theorem 2.9. *Let $M \subset \mathbb{C}^n$ be a generic submanifold of class C^{k+1}, flat and rigid up to the order k, and $V \subset M$ an open subset with Lipschitz boundary at $p \in \partial V$. Let $\gamma : [0,1] \to \{p\} \cup V$ be a CR-curve with $\gamma(0) = p$ such that $\gamma'(0) \in C_pV$ and suppose the angle of the sector $C_pV \cap \mathbb{C}\gamma'(0)$ is $\pi\alpha$ for some $1/k < \alpha < 1$. Then, for any wedge $V' \in \mathbb{C}^n$ with edge V at $q = \gamma(1)$, there exists an α-wedge $V'' \in \mathbb{C}^n$ over V at p such that continuous CR-functions on V that extend holomorphically V', also extend holomorphically to V''.*

Proof. Suppose $p = 0$. We approximate γ by a chain of arbitrarily small discs $\{A_j\}$ attached to M, for $1 \le j \le s$, such that $A_j(\partial\Delta) \cap A_{j+1}(\partial\Delta) \neq \emptyset$ for $1 \le j \le s-1$, A_1 is of the kind (2.29), and all other discs are of class $C^{1,\beta}$. We start from $p_1 = p = 0$ and we construct $A_1 \in \mathcal{P}^\alpha$ with $A_1(1) = 0$ and whose projection on $T^c_{p_1}M$ is $\dot\gamma(1)(1-\tau)^\alpha$. The distance of $A_1(-1)$ to some point $p_2 = \gamma(t_2)$ in γ is $o(\mathrm{diam}(\mathrm{A}_1))$. Next, we take A_2 with $A_2(1) = A_1(-1)$ and whose projection in $T^c_{p_2}M$ is $p_2 + \dot\gamma(t_2)(1-\tau)$. In this way we find a chain ending at $A_s(-1)$ with $|A_s(-1) - q| < \epsilon$. If V' is a wedge with edge in $\mathbb{C}^n$ with edge V at q, then V' is a submanifold with boundary M at some point in the boundary of the disc A_s. But then we can apply Theorem 2.8 and the classical propagation of wedge extendibility by Tumanov ([18]) to reach our conclusion. □

It is now immediate to prove our main result:

Proof of Theorem 2.7. By the definition of the complex angle, there must exist a CR curve γ satisfying the assumptions of Theorem 2.9, and the conclusion follows immediately from that theorem. □

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