UNIVERSITÀ DEGLI STUDI DI PADOVA

Sede Amministrativa: Università degli Studi di Padova

Dipartimento di Matematica Pura ed Applicata

SCUOLA DI DOTTORATO DI RICERCA IN SCIENZE MATEMATICHE
INDIRIZZO MATEMATICA COMPUTAZIONALE
CICLO XX

Algorithms for the computation of the joint spectral radius

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DATA CONSEGNA TESI
31 Gennaio 2008

Alla mia famiglia.

Gli algoritmi sono istruzioni
per eseguire un compito:
persino il computer più potente
ha bisogno di buoni algoritmi.

Gian-Carlo Rota

Introduzione

I sistemi dinamici lineari discreti, della forma

$$x^{(i+1)} = A^{(i)} x^{(i)}, \ i = 0, 1, 2, \ldots \tag{1}$$

dove il vettore iniziale $x^{(0)}$ e le matrici $A^{(i)} \in \mathbb{C}^{n\times n}$, $i = 0, 1, 2, \ldots$, sono assegnate, risultano essere molto importanti in diversi campi della matematica applicata quali, ad esempio, Ingegneria, Fisica, Biologia e Chimica. Molte volte può risultare più conveniente analizzare processi continui mediante le corrispondenti versioni discrete che risultano essere, sostanzialmente, della forma (1). Un aspetto cruciale di un sistema dinamico lineare è costituito dalle sue proprietà di stabilità, ossia, dall'andamento asintotico delle sue soluzioni che, per un fissato punto iniziale $x^{(0)}$, sono esprimibili come

$$x^{(i+1)} = P^{(i)} x^{(0)}, \ \ P^{(i)} = A^{(i)} \cdots A^{(0)}.$$

Risulta quindi chiaro come il prodotto delle matrici della famiglia $\mathcal{F} = \{A^{(i)}\}_{i\geq 0}$ giochi un ruolo essenziale nel comportamento del sistema dinamico lineare. Più precisamente, definendo il *raggio spettrale* della famiglia $\mathcal{F}$ come

$$\rho(\mathcal{F}) = \limsup_{k\to+\infty} \overline{\rho}_k(\mathcal{F}), \quad \overline{\rho}_k(\mathcal{F}) = \sup_{P\in\Sigma_k(\mathcal{F})} \rho(P),$$

dove $\Sigma_k(\mathcal{F}) = \{P \in \mathbb{C}^{n\times n} : P = \prod_{l=1}^{k} A^{(i_l)}\}$, è noto che la stabilità asintotica è garantita dalla condizione $\rho(\mathcal{F}) < 1$. Perciò, per quanto appena detto, si è naturalmente condotti al calcolo del raggio spettrale della famiglia $\mathcal{F}$. Questo compito non è semplice, ma è comunque una importante strategia che permette di analizzare la stabilità del sistema.

Il raggio spettrale di una famiglia di matrici $\mathcal{F}$, introdotto per la prima volta da Rota e Strang [RS60] negli anni 60, ha portato ad una serie di risultati teorici reperibili in letteratura. Una delle questioni teoriche più importanti affrontate negli ultimi anni e relativa a famiglie finite, è la *Congettura di Finitezza*, introdotta da Lagarias e Wang [LW95], e successivamente dimostrata essere falsa nel caso generale, prima da Bousch e Mairesse [BM02] e poi da Blondel, Theys e Vladimirov [BTV03], ma vera per alcune classi di matrici. È possibile interpretare praticamente tale congettura nel seguente modo: tra tutti i possibili sistemi dinamici (2) che possono essere definiti usando le matrici di una famiglia finita $\mathcal{F}$, quelli che generano traiettorie con la massima crescita sono di tipo periodico (ossia, la successione $\{A^{(i)}\}_{i\geq 0}$ è periodica).

In letteratura si possono trovare anche alcuni risultati per il calcolo e l'approssimazione di $\rho(\mathcal{F})$. Tra questi, un primo tipo di algoritmi si basa sulla definizione di raggio spettrale e fornisce un intervallo, arbitrariamente piccolo, contenente $\rho(\mathcal{F})$ (vedi, ad esempio, Gripenberg [Gri96]). Un altro tipo di algoritmi usa invece l'idea di norma estremale $\|\cdot\|_*$. In particolare, è stato dimostrato che, sotto opportune condizioni sulla famiglia, si ha che $\rho(\mathcal{F}) = \|\mathcal{F}\|_*$ per una appropriata norma $\|\cdot\|_*$ la cui palla unitaria è un *politopo complesso bilanciato* (vedi Guglielmi, Wirth and Zennaro [GWZ05]). Per implementare in modo efficiente questo secondo tipo di algoritmi che sfruttano *norme politopiche complesse*, è cruciale avere a disposizione algoritmi efficienti per la costruzione, intesa come rappresentazione geometrica, dei politopi complessi bilanciati e per il calcolo delle relative norme. Per questi motivi, il presente lavoro di tesi è stato indirizzato allo sviluppo di tali strumenti computazionali.

Questo ha richiesto lo studio teorico dei politopi complessi bilanciati in $\mathbb{C}^n$, che ha portato, in primo luogo, ad osservare che la complessità della geometria di tali oggetti cresce con la dimensione n dello spazio. Per questo motivo, nel presente lavoro, ci si è limitati ad esaminare la loro rappresentazione geometrica in $\mathbb{C}^2$. Nel dettaglio, il lavoro di tesi è così suddiviso.

Nel primo capitolo abbiamo richiamato la definizione di raggio spettrale di una famiglia di matrici e riportato i più importanti risultati teorici noti in letteratura.

Nel secondo capitolo abbiamo riportato in modo dettagliato le due tipologie di algoritmi sopra menzionate per il calcolo numerico di $\rho(\mathcal{F})$.

Nel terzo capitolo abbiamo fornito risultati teorici originali inerenti la geometria di politopi complessi bilanciati bidimensionali, al fine di presentare due algoritmi, di cui uno fornisce la loro rappresentazione geometrica in $\mathbb{C}^2$ e l'altro permette di calcolare la relativa norma politopica complessa.

Infine, nel quarto ed ultimo capitolo abbiamo elaborato alcune strategie che, in media, rendono i precedenti algoritmi più efficienti. Tali strategie si basano sull'idea di *cono limite* usata nel metodo Beneath–Beyond per costruire politopi reali. L'incremento delle prestazioni degli algoritmi, ottenuto utilizzando tali strategie, è confermato da alcuni test numerici riportati alla fine del presente capitolo.

In definitiva, in questo lavoro di tesi abbiamo esposto i primi risultati teorici originali che forniscono una completa descrizione della geometria dei politopi complessi bilanciati in $\mathbb{C}^2$. Inoltre, alla luce di questi risultati, abbiamo presentato due algoritmi efficienti, uno per la costruzione di tali politopi e l'altro per il calcolo della relativa norma politopica complessa di un vettore $z \in \mathbb{C}^2$.

Contents

Introduction

The knowledge of discrete linear dynamical systems of the kind

$$x^{(i+1)} = A^{(i)}x^{(i)}, \; i = 0, 1, 2, \dots \tag{2}$$

where $x^{(0)}$ is given and $A^{(i)} \in \mathbb{C}^{n\times n}$, $i = 0, 1, 2, \dots$, are given matrices, is of great importance in many fields of applied mathematics, such as Engineering, Physics, Biology, Chemistry, etc. We note that sometimes, it may be more convenient to perform the analysis of continuous processes on the corresponding discretized ones which are, substantially, of the form (2). A crucial aspect related to a discrete linear dynamical system is its stability properties, that is, the asymptotic behaviour of its solutions. For a given starting point $x^{(0)}$, they are given by

$$x^{(i+1)} = P^{(i)}x^{(0)}\,, \quad P^{(i)} = A^{(i)} \cdots A^{(0)}.$$

Therefore, it is clear that the products of the matrices of the family $\mathcal{F} = \{A^{(i)}\}_{i\geq 0}$ play an essential role in the behaviour of the linear system. Indeed, by defining the *spectral radius* of the family $\mathcal{F}$ as

$$\rho(\mathcal{F}) = \limsup_{k\to+\infty} \overline{\rho}_k(\mathcal{F}), \quad \overline{\rho}_k(\mathcal{F}) = \sup_{P\in\Sigma_k(\mathcal{F})} \rho(P),$$

where $\Sigma_k(\mathcal{F}) = \{P \in \mathbb{C}^{n\times n} : P = \prod_{l=1}^{k} A^{(i_l)}\}$, we have that the asymptotic stability is guaranteed if $\rho(\mathcal{F}) < 1$. So, we are naturally led to the computation of the spectral radius of $\mathcal{F}$. This is not an easy task at all, but it suggests an important way to approach the stability question.

From a historical point of view, the spectral radius of a family of matrices $\mathcal{F}$ was first introduced by Rota and Strang [RS60] in the 60's. Since then, in literature many theoretical results have been given. One of the most important theoretical issues regarding finite families is the *Finiteness Conjecture*, introduced by Lagarias and Wang [LW95] and, later proved to be false in general, first by Bousch and Mairesse [BM02] and then by Blondel, Theys and Vladimirov [BTV03], but true for some classes of matrices. A practical interpretation of this conjecture is the following: among all the possible dynamical systems (2) that can be defined by using the matrices of a finite family $\mathcal{F}$, those which give rise to trajectories with the maximum growth rate are of a periodic type (i.e. the sequence $\{A^{(i)}\}_{i\geq 0}$ is periodic).

In literature we can also find some results on the computation and the approximation of $\rho(\mathcal{F})$. Gripenberg [Gri96] has proposed an algorithm which finds both upper and lower bounds for $\rho(\mathcal{F})$. Another kind of algorithm uses the idea of extremal norm $\|\cdot\|_*$. In particular, it can be proved that, under suitable conditions on the family, $\rho(\mathcal{F}) = \|\mathcal{F}\|_*$ for an appropriate norm $\|\cdot\|_*$ whose unit ball is a *balanced complex polytope* (see Guglielmi, Wirth and Zennaro [GWZ05]). For the efficient implementation of the algorithms that use such *complex polytope norms*, it is crucial to have at our disposal efficient algorithms for the construction (i.e. for the geometric representation) of balanced complex polytopes, and for the computation of the related norms. For these reasons we have decided to address our thesis work to the development of such computational tools.

In order to succeed in our purpose, we first needed to get a deeper theoretical knowledge of the balanced complex polytopes. However, due to the extreme increase in complexity of the geometry of such objects with the dimension n, we have confined ourselves to face the two-dimensional case.

The plan of the present work is as follows.

In the first chapter, we define the spectral radius of a family of matrices and review the most important known theoretical results.

In the second chapter, we recall, in detail, the already mentioned algorithms for the numerical computation of $\rho(\mathcal{F})$.

In the third chapter, we give original theoretical results on the geometry of two-dimensional balanced complex polytopes in order to present our algorithms for their construction in $\mathbb{C}^2$ and for the computation of the related complex polytope norms.

In the fourth chapter, we elaborate some strategies which, in the average case, give rise to much more performing versions of the previous algorithms. These strategies are based on the *limit cone* idea employed in the well known Beneath -Beyond method used in the real case. The speed-up obtained by these improvements is confirmed by the numerical tests reported at the end of the chapter.

To summarise, in this thesis we have presented the first efficient algorithm for the construction of a balanced complex polytope $\mathcal{P}$ in $\mathbb{C}^2$ which completely describes the geometry of $\mathcal{P}$. Furthermore, we have also presented the first efficient algorithm to compute the complex polytope norm of a vector $z \in \mathbb{C}^2$ starting from the knowledge of the boundary of the corresponding unit ball.

Chapter 1

The spectral radius

In this chapter we give some motivations and introduce some basic definitions that are used throughout the rest of the work. We start with the concept of *family of matrices* and the related definitions of *generalised* and *joint spectral radius*. Then we give a characterisation of the spectral radius of a family and we recall some preliminary results from literature. Finally we show two important issues arising in the computation of the radius, i.e., the finiteness conjecture and the normed finiteness conjecture. Both are false in general, but true for some classes of matrices and norms which are introduced at the end of the chapter.

1.1 Motivations

The work of this thesis is motivated by the study of the stability of linear discrete dynamical systems

$$\begin{cases} x^{(i+1)} = A^{(i)} x^{(i)}, \quad i \geq 0 \\ x^{(0)} \text{ given} \end{cases}, \tag{1.1}$$

where $A^{(i)}$, $i = 0, 1, 2, \ldots$, are square $n \times n$ real matrices, and $\mathcal{F} = \{A^{(i)}\}_{i \geq 0}$ denotes its associated family of matrices. We note that the previous system defines a sequence in $\mathbb{R}^n$, where the given vector $x^{(0)}$ acts as the starting point. Easily, we can see that $x^{(i+1)} = P^{(i)} x^{(i)}$, where $P^{(i)} = A^{(i)} \ldots A^{(0)}$. Thus, it is not surprising that the properties of the products of the elements of the family $\mathcal{F}$ will play an important role in the behaviour of the discrete linear dynamical system.

Systems of the type (1.1) are used to model many phenomena in Engineering, Physics, Chemistry and so on. They may also arise as discretizations of continuous models. Furthermore, they may be obtained from linear difference equations with variable coefficients of the kind

$$y_{i+k} = \alpha_{i,k-1} y_{i+k-1} + \alpha_{i,k-2} y_{i+k-2} + \cdots + \alpha_{i,0} y_i, \quad k \in \mathbb{N}, \tag{1.2}$$

which, in turn, may come from a discretization of ordinary differential equations by means of a numerical method. Indeed, equation (1.2) can be written in the form

$$y^{(i+1)} = A^{(i)} y^{(i)}, \quad i = 0, 1, \ldots$$

where

$$y^{(i+1)} = \begin{bmatrix} y_{i+k} \\ y_{i+k-1} \\ \vdots y_{i+1} \end{bmatrix}, \quad y^{(i)} = \begin{bmatrix} y_{i+k-1} \\ y_{i+k-1} \\ \vdots \\ y_i \end{bmatrix}, \quad A^{(i)} = \begin{pmatrix} \alpha_{i,k-1} & \alpha_{i,k-2} & \ldots & \alpha_{i,1} & \alpha_{i,0} \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}. \tag{1.3}$$

We recall that the matrices $A^{(i)},\ i = 0, 1, \ldots$ in (1.3) form the sequence of the *companion matrices* associated with the difference equation (1.2).

One of the most important features of a linear discrete system is its stability properties. Roughly speaking, the stability describes the behaviour of the solutions of the system (1.1) for large values of the index i. More precisely, we have the following definition.

Definition 1.1. *An n-dimensional linear discrete dynamical system is said to be* stable *if, for a given norm $\|\cdot\|$ in $\mathbb{R}^n$, there exists a positive constant C such that $\|x^{(i)}\| \leq C$, $i \geq 0$, for all initial values $x^{(0)}$. Furthermore, if*

$$\lim_{k\to\infty} \|x^{(i)}\| = 0,$$

then the system is said to be asymptotically stable. *Finally, if the system is not stable, then it is said to be* unstable.

As is well known, the stability of system (1.1) is equivalent to the uniform boundedness of the sequence of products $A^{(k)} \ldots A^{(0)}$, $k = 0, 1, \ldots$, whereas the asymptotic stability is equivalent to

$$\lim_{k\to\infty} \prod_{i=0}^{k} A^{(i)} = O. \tag{1.4}$$

As we shall see in the next section, sufficient conditions for stability and asymptotic stability are related to the *spectral radius* of the family $\mathcal{F} = \{A^{(i)}\}_{i\geq 0}$.

Moreover, the spectral radius of a family of matrices is a tool that may be used in other mathematical fields like, for example, wavelets, system approximation and control theory [BM96, BT79, Mae98]. All these applications are of great practical relevance.

Unfortunately, the computation of $\rho(\mathcal{F})$ is, in general, a very difficult task. So, it is of high interest to improve the knowledge of the spectral radius both theoretically and computationally, and this is the main purpose of this work.

1.2 The joint spectral radius

A family of matrices is nothing but a set of matrices. Nevertheless, since it is a basic tool of this work, we prefer to define it precisely.

Definition 1.2 (matrix family). *Let $\mathcal{K}^{n\times n}$ be the vector space of the square $n \times n$ matrices whose elements are in $\mathcal{K}$. Then, any subset of $\mathcal{K}^{n\times n}$ is called a family of matrices and is denoted by $\mathcal{F}$. That is,*

$$\mathcal{F} = \{A^{(i)}\}_{i\in\mathcal{I}}, \tag{1.5}$$

where $\mathcal{I}$ is a possibly infinite set of indexes.

In this work we use the two special cases: $\mathcal{K} = \mathbb{R}$, i.e., the matrices are real, or $\mathcal{K} = \mathbb{C}$, i.e., the matrices are complex.

Clearly, to specify a family of matrices we have to give all its elements. This can be done in an explicit way or by giving a rule to construct each of them. The number of elements of the family $\mathcal{F}$ may be finite or (possibly countable) infinite.

For a family of $n \times n$ complex matrices $\mathcal{F} = \{A^{(i)}\}_{i\in\mathcal{I}}$, where $\mathcal{I}$ is a, possibly infinite, set of indexes, the set of all the possible matrix products of length k whose factors are in $\mathcal{F}$ is denoted by $\Sigma_k(\mathcal{F})$. That is,

$$\Sigma_k(\mathcal{F}) = \left\{ P \in \mathbb{C}^{n\times n} : P = \prod_{l=1}^{k} A^{(i_l)}, \quad i_l \in \mathcal{I} \right\}. \tag{1.6}$$

For $k = 0$, we assume $\Sigma_k(\mathcal{F}) = \{I_n\}$, where I_n is the identity matrix of order n.

Example 1.1. Consider the family $\mathcal{F} = \{A^{(1)}, A^{(2)}, A^{(3)}\}$. We have

$$\Sigma_1(\mathcal{F}) = \mathcal{F},$$

and, recalling that the matrix product is non–commutative,

$$\begin{aligned}\Sigma_2(\mathcal{F}) \quad = \quad & \Big\{(A^{(1)})^2, (A^{(2)})^2, (A^{(3)})^2, A^{(1)} \cdot A^{(2)}, A^{(1)} \cdot A^{(3)}, \\ & A^{(2)} \cdot A^{(3)}, A^{(2)} \cdot A^{(1)}, A^{(3)} \cdot A^{(2)}, A^{(3)} \cdot A^{(1)}\Big\},\end{aligned}$$

and so on. It is interesting to note that the number of elements of $\Sigma_k(\mathcal{F})$ grows exponentially with k. $\diamond$

Remember that for a single square matrix A, as usual, $\rho(A)$ denotes the spectral radius of A, that is

$$\rho(A) = \max\Big\{|\lambda_j|, \lambda_j \in \mathcal{E}_A\Big\},$$

where $|\lambda|$ is the modulus of the (possibly) complex number λ and $\mathcal{E}_A$ is the set of the eigenvalues of A. Since $\rho(A^k) = [\rho(A)]^k$ for all $k \geq 1$, it holds that

$$\rho(A) = \lim_{k\to\infty} \rho(A^k)^{1/k}. \tag{1.7}$$

Nevertheless, given a vector norm $\|\cdot\|$ on $\mathbb{C}^n$, if we consider the induced matrix norm, denoted with the same symbol $\|\cdot\|$ and defined as

$$\|A\| = \sup_{\|x\|=1} \|Ax\|, \tag{1.8}$$

the *Gelfand Formula*

$$\rho(A) = \lim_{k\to\infty} \|A^k\|^{1/k} \tag{1.9}$$

holds.

In order to generalise the properties (1.7) and (1.9) to a family of matrices $\mathcal{F}$, we give the following definitions.

Definition 1.3 (Generalized spectral radius). *Let $\mathcal{F} = \{A^{(i)}\}_{i\in\mathcal{I}}$ be a family of complex $n \times n$ matrices. Setting for each $k \geq 0$*

$$\overline{\rho}_k(\mathcal{F}) = \sup_{P\in\Sigma_k(\mathcal{F})} \rho(P), \tag{1.10}$$

the generalised spectral radius of the family $\mathcal{F}$ is the non-negative real number

$$\overline{\rho}(\mathcal{F}) = \limsup_{k\to\infty} \Big[\,\overline{\rho}_k(\mathcal{F})\,\Big]^{1/k} \tag{1.11}$$

(see Daubechies and Lagarias [DL92]).

Definition 1.4 (Joint spectral radius). *Let $\mathcal{F} = \{A^{(i)}\}_{i\in\mathcal{I}}$ be a family of complex $n \times n$ matrices and let $\|\cdot\|$ be some matrix norm. Setting for each $k \geq 0$*

$$\hat{\rho}_k(\mathcal{F}) = \sup_{P\in\Sigma_k(\mathcal{F})} \|P\|, \tag{1.12}$$

the joint spectral radius of the family $\mathcal{F}$ is the non-negative real number

$$\hat{\rho}(\mathcal{F}) = \limsup_{k\to\infty} [\,\hat{\rho}_k(\mathcal{F})\,]^{1/k} \tag{1.13}$$

(see Rota and Strang [RS60]).

Note that the two definitions are quite similar; in the definition of the spectral radius we use the concept of spectral radius of a matrix whereas for the definition of the joint spectral radius we use some matrix norm. This is the only difference between the two definitions. Regardless of their simplicity, the computation of one or both of the generalised or the joint spectral radius is, in the general case, a very massive task. However, recently it has been shown that the two spectral radii are equal, as stated by the following theorem, which represents the generalisation of the Gelfand Formula (see Berger and Wang [BW92], Elsner [Els95], Shih, Wu and Pang [SWP97], and Shih [Shi99]).

Theorem 1.1. *Let $\mathcal{F} = \{A^{(i)}\}_{i\in\mathcal{I}}$ be a finite family of complex $n \times n$ matrices and let $\|\cdot\|$ be some matrix norm. Then*

$$\overline{\rho}(\mathcal{F}) = \hat{\rho}(\mathcal{F}).$$

As a consequence of this theorem, for a finite family of matrices, we can define the spectral radius of the family $\rho(\mathcal{F})$ as

$$\rho(\mathcal{F}) = \overline{\rho}(\mathcal{F}) = \hat{\rho}(\mathcal{F}). \tag{1.14}$$

Let us give a very simple example.

Example 1.2. Consider the single family $\mathcal{F} = \{A\}$, where A is the real 2×2 diagonal matrix with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$, that is $A = \text{diag}(2, 3)$. Then, for any non-negative integer k, we have

$$\Sigma_k(\mathcal{F}) = \{A^k\} \quad \Rightarrow \quad \overline{\rho}_k(\mathcal{F}) = \rho(A^k) = 3^k,$$

since $A^k = \text{diag}(2^k, 3^k)$. Thus,

$$\overline{\rho}(\mathcal{F}) = \limsup_{k\to\infty} \left[\overline{\rho}_k(\mathcal{F})\right]^{1/k} = 3.$$

In the same way, using, for example, the Euclidean norm, we have

$$\|A^k\|_2 = \sqrt{\rho(A^k(A^k)^T)} = 3^k$$

from which it follows immediately that $\hat{\rho}(\mathcal{F}) = 3$. Note that both the spectral radii are equal to each other, as stated by the previous Theorem 1.1 and, furthermore, to the spectral radius of A. $\diamond$

Now we give a further characterisation of the spectral radius of a bounded family $\mathcal{F} = \{A^{(i)}\}_{i\in\mathcal{I}}$.

Definition 1.5. *Given a norm $\|\cdot\|$ on the vector space $\mathbb{C}^n$ and the corresponding induced $n \times n$-matrix norm the* norm of the family *$\mathcal{F}$ is*

$$\|\mathcal{F}\| = \hat{\rho}_1(\mathcal{F}) = \sup_{i\in\mathcal{I}} \|A^{(i)}\|.$$

The following result can be found, for example, in [RS60] and in [Els95].

Theorem 1.2. *The spectral radius of a bounded family $\mathcal{F}$ of complex $n \times n$-matrices is characterised by the equality*

$$\rho(\mathcal{F}) = \inf_{\|\cdot\|\in\mathcal{N}} \|\mathcal{F}\|, \tag{1.15}$$

where $\mathcal{N}$ denotes the set of all possible induced $n \times n$-matrix norms.

Given a family $\mathcal{F}$, it is important to know whether or not the inf in (1.15) is actually attained by some induced matrix norm. To this purpose, we recall the following definition.

Definition 1.6. *We shall say that a norm $\|\cdot\|_*$ satisfying the condition*

$$\|\mathcal{F}\|_* = \rho(\mathcal{F})$$

is extremal *for the family $\mathcal{F}$.*

It is well known that, for a single family $\mathcal{F} = \{A\}$, the existence of an extremal norm is equivalent to the fact that the matrix A is non-defective, that is, all of the blocks relevant to the eigenvalues of maximum modulus are diagonal in its Jordan canonical form; in other words, all the eigenvalues of maximum modulus are non-defective, that is, their geometric and algebraic multiplicities are equal. Whenever $\rho(A) > 0$, setting $\hat{A} = \rho(A)^{-1}A$, another property equivalent to the definition of non-defective matrix is that the power set $\Sigma(\hat{A}) = \{\hat{A}^k \mid k \geq 1\}$ is bounded.

These results generalise to a bounded family $\mathcal{F} = \{A^{(i)}\}_{i\in\mathcal{I}}$ of complex $n\times n$-matrices as follows. Given a bounded family $\mathcal{F}$ with $\rho(\mathcal{F}) > 0$, let us consider the *normalised* family

$$\hat{\mathcal{F}} = \{\rho(\mathcal{F})^{-1}A^{(i)}\}_{i\in\mathcal{I}},$$

whose spectral radius is $\rho(\hat{\mathcal{F}}) = 1$, and the semigroup of matrices generated by $\hat{\mathcal{F}}$, i.e.,

$$\Sigma(\hat{\mathcal{F}}) = \bigcup_{k\geq 1} \Sigma_k(\hat{\mathcal{F}}).$$

Definition 1.7. *A bounded family $\mathcal{F}$ of complex $n\times n$-matrices is said to be* defective *if the corresponding normalised family $\hat{\mathcal{F}}$ is such that the semigroup $\Sigma(\hat{\mathcal{F}})$ is an unbounded set of matrices. Otherwise, if $\Sigma(\hat{\mathcal{F}})$ is bounded, then the family $\mathcal{F}$ is said to be* non-defective*.*

Remark that the definition of non-defective family does not directly involve the spectral properties of its elements.

The following result can be found, for example, in [BW92].

Proposition 1.1. *A bounded family $\mathcal{F}$ of complex $n \times n$-matrices admits an extremal norm $\|\cdot\|_*$ if and only if it is non-defective.*

Moreover, if $\mathcal{F}$ is non-defective, any given norm $\|\cdot\|$ on $\mathbb{C}^n$ determines the extremal norm

$$\|x\|_* = \sup_{k\geq 0} \sup_{P\in\Sigma_k(\mathcal{F})} \frac{\|Px\|}{\rho(\mathcal{F})^k}. \tag{1.16}$$

Corollary 1.1. *A bounded family $\mathcal{F}$ of complex $n \times n$-matrices is non-defective if and only if there exists an induced norm $\|\cdot\|_*$ such that*

$$\hat{\rho}_k(\mathcal{F}) = \rho(\mathcal{F})^k \qquad \forall\, k \geq 0. \tag{1.17}$$

From Proposition 1.1 it turns out that, for a non-defective family, each vector norm $\|\cdot\|$ canonically determines an extremal norm. However, although (1.16) gives a constructive way of finding an extremal norm, its importance is mainly theoretical since it is often useless from a practical point of view.

Now, we consider non-defective bounded families $\mathcal{F}$ of complex $n \times n$-matrices and recall some properties of extremal norms. The next result is an easy consequence of Definition 1.6.

Lemma 1.1. *Let $\mathcal{F}$ be a non-defective bounded family of complex $n\times n$-matrices and let $\|\cdot\|_*$ be an extremal norm for $\mathcal{F}$. Then, for each $k \geq 1$, it holds that*

$$\|P\|_* \leq [\rho(\mathcal{F})]^k \qquad \forall P \in \Sigma_k(\mathcal{F}).$$

This Lemma and the submultiplicative property of the induced matrix norms yield immediately the following result.

Lemma 1.2. *Let $\mathcal{F}$ be a non-defective bounded family of complex $n\times n$-matrices and let $\|\cdot\|_*$ be an extremal norm for $\mathcal{F}$. If $P \in \Sigma_k(\mathcal{F})$ and $Q \in \Sigma_h(\mathcal{F})$ are such that $\|PQ\|_* = \rho(\mathcal{F})^{k+h}$, then they satisfy the equalities $\|P\|_* = \rho(\mathcal{F})^k$ and $\|Q\|_* = \rho(\mathcal{F})^h$.*

Going back to consider the linear system (1.1), from the foregoing theory it is easy to obtain the following result, which relates the stability of a linear discrete dynamical system to the spectral radius of its associated family $\mathcal{F}$.

Theorem 1.3. *The system (1.1) is asymptotically stable if its associated family $\mathcal{F} = \{A^{(i)}\}_{i\geq 0}$ is such that its spectral radius satisfies the condition*

$$\rho(\mathcal{F}) < 1$$

and is stable if

$$\rho(\mathcal{F}) = 1 \ \text{ and } \mathcal{F} \text{ admits an extremal norm } \|\cdot\|_*.$$

1.3 Finiteness Conjecture and Normed Finiteness Conjecture

For a general family of matrices $\mathcal{F}$, we are not able to compute its spectral radius in an explicit way. Let us, for example, look at the computation of $\overline{\rho}(\mathcal{F})$. The main problem is to find an explicit formula for $\overline{\rho}_k(\mathcal{F})$, $k \in \mathbb{N}$. Indeed, if we can overcome such a problem, then the limit (1.11) may be computed both analytically or via some numerical tool. Unfortunately, the determination of an explicit formula for $\overline{\rho}_k(\mathcal{F})$ is, in general, a very difficult task. It seems to be better if we know the existence of an integer $\hat{k}$ such that $\overline{\rho}(\mathcal{F}) = \overline{\rho}_{\hat{k}}(\mathcal{F})$. In such a case, from a theoretical point of view, we may consider all the possible matrices in $\Sigma_k(\mathcal{F})$, $k \leq \hat{k}$, computing their spectral radius and finding the corresponding maximum value. Clearly, since the cardinality of $\Sigma_k(\mathcal{F})$ grows exponentially with k, this is not a realistic algorithm at all. However, there are other kinds of algorithms which may work well in these conditions. We will see some of them later on in this work. Furthermore, in many important cases the previous situation occurs and so it was firstly conjectured to be true at least for finite families. This was shown later on not to be the case, but, in a first moment, the two following statements, known as the Finiteness Conjectures and the Normed Finiteness Conjecture, were proposed. Note that the name conjecture given to these two statements is only for historical purpose, since both conjectures have been proved, in recent years, to be false.

Conjecture 1.1 (Finiteness Conjecture). *Let $\mathcal{F} = \{A^{(i)}, i \in \mathcal{I}\}$ be a finite family of complex $n\times n$ matrices. Then, there exists a finite, positive, integer k such that*

$$\overline{\rho}(\mathcal{F}) = \left[\overline{\rho}_k(\mathcal{F})\right]^{1/k}.$$

Conjecture 1.2 (Normed Finiteness Conjecture). *Let $\|\cdot\|$ be some matrix norm. Consider the finite family of complex $n \times n$ matrices $\mathcal{F} = \{A^{(i)}, i \in \mathcal{I}\}$. Assume $\|A^{(i)}\| \leq \rho(\mathcal{F})$, $i \in \mathcal{I}$. Then, there is a finite, positive, integer k such that*

$$\bar{\rho}(\mathcal{F}) = \left[\bar{\rho}_k(\mathcal{F})\right]^{1/k}.$$

Note that for a normalised family the non-defectiveness property is equivalent to $\|A^{(i)}\| \leq 1$, $1 \leq i \leq m$, for some matrix norm $\|\cdot\|$.
Both conjectures claim the existence of a finite integer k which may be used to compute the spectral radius of the family $\mathcal{F}$ considering only $\overline{\rho}_k(\mathcal{F})$. This is a great advantage since we deal only with the computation of the spectral radius of the finite number of products belonging to $\Sigma_k(\mathcal{F})$. The two conjectures are equivalent to each other, as stated by the following theorem.

Theorem 1.4. *The finiteness Conjecture is true for any finite family of matrices $\mathcal{F}$ if and only if the normed Finiteness Conjecture is true for any norm $\|\cdot\|$.*

A practical interpretation of these conjectures is given by the following remark.

Remark 1.1. *The Finiteness Conjecture may be practically interpreted as follows: among all the possible dynamical systems (1.1) that can be defined by using the matrices of a finite family $\mathcal{F}$, those which give rise to trajectories with the maximum growth rate are of periodic type (i.e. the sequence $\{A^{(i)}\}_{i\geq 0}$ is periodic).*

As said before, the Finiteness Conjecture was proved to be false first by Bousch and Mairesse [BM02] and, later, by Blondel, Theys and Vladimirov [BTV03]. Indeed, it was disproved first using the following counterexample.

Example 1.3 (Finiteness Conjecture Disproving)**.** Consider the family of matrices

$$\mathcal{F}_\alpha = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \alpha \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

where $\alpha \in \mathbb{R}$. Then there are uncountably many real values of the parameter α for which the Finiteness Conjecture is not satisfied by the corresponding family $\mathcal{F}_\alpha$. ◇

The proof of the previous example is not constructive and, at present, no explicit value of the parameter α for which the Finiteness Conjecture fails is known.

However, we can easily find finite families of matrices for which the previous conjecture is satisfied, as we can see in the following example.

Example 1.4. Consider the following family of two 2×2 matrices $\mathcal{F} = \{A_1, A_2\}$, where

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We can easily check that, for all $k^* \geq 1$, $\bar{P} \in \Sigma_{k^*}(\mathcal{F})$ is given by

$$\bar{P} = A_1^{k_1} A_2^{k_2} = A_2^{k_2} = \begin{pmatrix} 1 & k_2 \\ 0 & 1 \end{pmatrix},$$

where $k_1 \geq 0$ and $k_2 \geq 0$ are, respectively, the numbers of occurrences of A_1 and A_2 in $\bar{P}$, and satisfy $k_1 + k_2 = k^*$. Since both the eigenvalues of P are equal to 1, then $\bar{\rho}_{k^*}(\mathcal{F}) = 1 = \rho(\mathcal{F})$, and so the family $\mathcal{F}$ satisfies the Finiteness Conjecture for each $k^* \geq 1$. ◇

As a consequence of Theorem 1.4, also the Normed Finiteness Conjecture is false. However, in the article of Lagarias and Wang [LW95] this Normed Finiteness Conjecture was proved to be correct for large classes of operator norms. Furthermore, for some of these norms, an upper bound for the k such that $\overline{\rho}(\mathcal{F}) = \left[\overline{\rho}_k(\mathcal{F})\right]^{1/k}$ is also given. However, these bounds are somewhat unrealistically large and complex to obtain; so, we do not report them here. Instead, we give some more details on these norms. Before doing so, we note that a given family of matrices $\mathcal{F}$ may satisfy the finiteness conjecture. In this case we say that this family has the *finiteness property*.

Definition 1.8 (piecewise analytic norm)**.** *A norm* $\|\cdot\|$ *on* $\mathbb{R}^n$ *is a piecewise analytic norm if its unit ball* $B = \{x \in \mathbb{R}^n : \|x\| = 1\}$ *has the boundary* $\partial B \subseteq \mathcal{Z}_f$*, where* $\mathcal{Z}_f = \{\mathbf{z} \in \mathbb{C}^n : f(\mathbf{z}) = 0\}$ *for some holomorphic function* $f : \mathbb{C}^n \to \mathbb{C}$ *such that* $f(\mathbf{0}) \neq 0$.

Definition 1.9 (piecewise algebraic norm)**.** *A norm* $\|\cdot\|$ *on* $\mathbb{R}^n$ *is a piecewise algebraic norm if its unit ball* $B = \{x \in \mathbb{R}^n : \|x\| = 1\}$ *has the boundary* $\partial B \subseteq \mathcal{Z}_p$*, where* $\mathcal{Z}_p = \{\mathbf{z} \in \mathbb{C}^n : p(\mathbf{z}) = 0\}$ *for some polynomial* $p : \mathbb{C}^n \to C$ *such that* $p(\mathbf{0}) \neq 0$.

From the previous two definitions it is clear that a piecewise algebraic norm is also a piecewise analytic norm, since a polynomial function is a holomorphic function, too.

Definition 1.10 (Polyhedron and polytope in $\mathbb{R}^n$)**.** *Consider a point* $b \in \mathbb{R}^n$ *and a real matrix* $A \in \mathbb{R}^{m \times n}$*. A polyhedron* P *in* $\mathbb{R}^n$ *is defined by*

$$P = \{x \in \mathbb{R}^n : Ax \leq b\},$$

where the inequality is understood componentwise.

If the set P *is bounded we have a real polytope (see Ziegler [Zie95] and Grünbaum [Grü67]).*

It is worthwhile to note that a polyhedron may be an unbounded set of $\mathbb{R}^n$. Furthermore, it is a convex set since it is the result of the intersection of m semi-spaces, which are convex subsets of $\mathbb{R}^n$. An example [1] of a classic polytope, an icosahedra, is shown in Figure 1.1.

[1]This figure is taken from 'http://members.aol.com/Polycell/regs.html'.

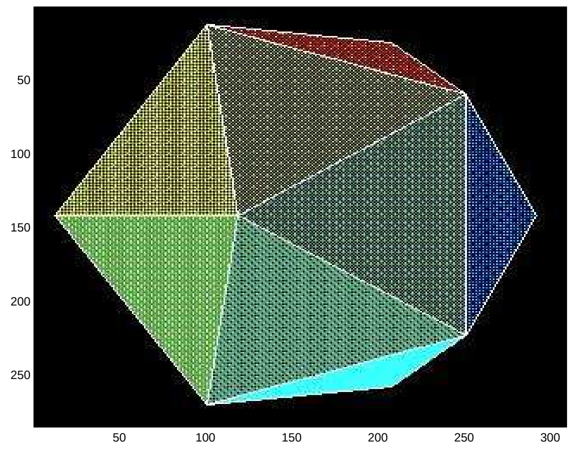

Figure 1.1: An icosahedra, a classic, convex, polytope.

Definition 1.11 (polytope norm)**.** *A norm $\|\cdot\|$ on $\mathbb{R}^n$ is a polytope norm if its unit ball B is a polytope.*

Note that the $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are two classic and simple examples of polytope norms. See Figure 1.2 for a sketch of the corresponding unit balls in $\mathbb{R}^2$. Furthermore, from the same figure, we may see that the Euclidean norm $\|\cdot\|_2$ is not a polytope norm, since its boundary is not the union of a finite set of segments (in the general case of $\mathbb{R}^n$, of pieces of hyperplanes) but is a continuous curve.

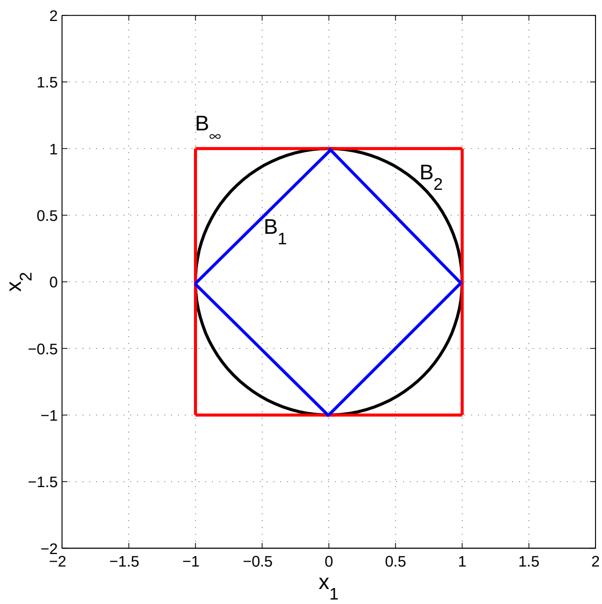

Figure 1.2: The unit balls in $\mathbb{R}^2$ for the the three norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ (blue, black and red lines, respectively).

We have the following theorem.

Theorem 1.5. *Let $\mathcal{F} = \{A^{(i)}\}_{i\in\mathcal{I}}$ be a finite family of real $n \times n$ matrices. Then, the Normed Finiteness Conjecture in true for the matrix norm $\|\cdot\|$ associated with any of the following norms:*

- *piecewise analytic norms;*
- *piecewise algebraic norms;*
- *polytope norms*
- *Euclidean norms.*

In the case of the polytope norms, we can give a simple upper bound for the positive integer $\hat{k}$ for which $\overline{\rho}(\mathcal{F}) = \overline{\rho}_{\hat{k}}(\mathcal{F})$.

Theorem 1.6. *Let $\mathcal{F} = \{A^{(i)}\}_{1\leq i\leq m}$ be a finite family of real $n \times n$ matrices which satisfies $\|A^{(i)}\| \leq \rho(\mathcal{F})$ for the polytope norm $\|\cdot\|$ with a polytope B as the corresponding unit ball. Then*

$$\hat{k} \leq \frac{1}{2}\sum_{i=0}^{n-1} f_i(B), \tag{1.18}$$

where $f_i(B)$ is the number of faces of dimension i in the polytope B.

The previous bound is easy to compute if we know the polytope B and is independent of the cardinality of $\mathcal{F}$.

Example 1.5. Consider a polytope norm $\|\cdot\|$ which has as unit ball the icosahedra of Figure 1.1. Then we have $n = 3$ and the number of faces in eq. (1.18) are

$$\begin{cases} f_0(B) &= 12; \\ f_1(B) &= 30; \\ f_2(B) &= 20. \end{cases}$$

Thus, $\hat{k} \leq (f_0(B) + f_1(B) + f_2(B))/2 = 31$. Note that, if the family has $|\mathcal{F}|$ elements, then we have to consider, in the worse case, all the products of lengths $k = 1, \cdots, 31$. So, we have to compute the spectral radius of

$$\sum_{k=1}^{\hat{k}} |\mathcal{F}|^k = |\mathcal{F}|\frac{|\mathcal{F}|^{\hat{k}} - 1}{|\mathcal{F}| - 1}$$

matrices. For example, if $|\mathcal{F}| = 2$, then the previous number is equal to 4294967294. Thus, to be able to compute, using the simple algorithm, the spectral radius of the family we have to evaluate the spectral radius of 4294967294 matrices, accordingly to the previous bound (1.18), to be sure to find the correct spectral radius of the family. Indeed, this is a massive task. ◇

Useless to say, it is unrealistic to use an algorithm of the previous kind. So, it is better to look for more sophisticated and performance algorithms.

Chapter 2

The computation of the joint spectral radius

This chapter introduces the main ideas underlying the numerical computation of the spectral radius $\rho(\mathcal{F})$ of a finite family of matrices $\mathcal{F}$. We stress that this task is not an easy one, even for a family with some matrices. On the other hand, from a numerical and also practical point of view, it is not so important to get the exact value of $\rho(\mathcal{F})$, but rather to have a sufficiently good approximation of it. This task may be performed by different kinds of algorithms.

In the first section we present a first kind of algorithm which gives a lower and an upper bound for $\rho(\mathcal{F})$, based on the inequalities $\bar{\rho}_k(\mathcal{F}) \leq \rho(\mathcal{F})^k \leq \hat{\rho}_k(\mathcal{F})$ (see [DL92]).

Next, we present a second kind of algorithm which computes $\rho(\mathcal{F})$ using the existence, under suitable hypotheses on the family $\mathcal{F}$, of an extremal complex polytope norm. To be more clear, $\rho(\mathcal{F})$ can be computed using the following chain of equalities

$$\rho(\mathcal{F}) \overset{(1)}{=} \inf_{\|\cdot\| \in \mathcal{N}} \|\mathcal{F}\| \overset{(2)}{=} \inf_{\|\cdot\| \in \mathcal{N}_{pol}} \|\mathcal{F}\| \overset{(3)}{=} \|\mathcal{F}\|_*$$

for some extremal polytope norm $\|\cdot\|_*$. The first equality (see [Els95, RS60]) says that the spectral radius can be found as the infimum over all the induced matrix norms $\mathcal{N}$, whereas, the second one follows from the density of the set of all polytope norms $\mathcal{N}_{pol}$ in the set $\mathcal{N}$ (see [GZ07, GWZ05]). Finally, the last equality, which requires the cited hypotheses on the family $\mathcal{F}$, needs the existence of an extremal norm on the set of the polytope norms. Since we have to deal with complex polytope norms, in the second section we introduce the idea of *balanced complex polytope*, which is the extension of the real symmetric polytope [Zie95] to the complex space, and we give some useful theoretical results and definitions.

In the third section, we extend the concept of *polytope norm* to the complex case.

In the last section, we recall the density results which fulfil the second equality of the previous chain and we give a polytope extremality result, which guarantees the third equality (see [GWZ05]). Finally, we propose an algorithm for the computation of the joint spectral radius through the construction of a polytope norm (see [GZ05]). This algorithm has been successfully applied to the study of the asymptotic stability of linear difference equations with variable coefficients coming from the discretization of differential equations (see, e.g., [GZ01]), as well as for computing the Hölder exponent of wavelets [Mae95, Mae98, Mae05], and seems to have a good potential in view of a large class of applications.

2.1 Lower and upper bounds for the joint spectral radius

The computation of the spectral radius of a family of matrices $\mathcal{F}$ is not an easy task. So, in literature we can find some algorithms that search for an interval which contains $\rho(\mathcal{F})$ instead of computing its exact value. That is, they find an upper and lower bound for $\rho(\mathcal{F})$. More precisely, given a non-negative real number δ, they find two positive real numbers, α and β with $\beta - \alpha \leq \delta$, such that $\alpha \leq \rho(\mathcal{F}) \leq \beta$. In this section, we show one of these algorithms, first presented by Gripenberg [Gri96].

For this purpose we introduce some useful notations. Let $\mathcal{F} = \{A^{(i)}\}_{i\in\mathcal{I}}$, where $\mathcal{I}$ is a finite set of indexes, be, as usual, a finite family of complex $n \times n$ matrices with $n \geq 2$. We set $\mathcal{A} = (A^{(i_1)}, A^{(i_2)}, \cdots, A^{(i_m)})$, where $i_1, \cdots, i_m \in \mathcal{I}$, as an $m-$tuple of matrices of the family $\mathcal{F}$; that is, in a brief notation, $\mathcal{A} \in \mathcal{F}^m$, where $\mathcal{F}^m$ is the cartesian product of $\mathcal{F}$ m times. Furthermore, we denote by $\Pi(\mathcal{A})$ and $p(\mathcal{A})$ the quantities

$$\Pi(\mathcal{A}) = \prod_{k=1}^{m} A^{(i_k)}$$

and

$$p(\mathcal{A}) = \min_{1\leq j\leq m} \left\| \prod_{k=1}^{j} A^{(i_k)} \right\|^{1/j}.$$

Now, we are able to give the algorithm in a compact form.

ALGORITHM 2.1 (Gripenberg algorithm).

% Input: $\mathcal{F}, \delta' > 0$

% Output: α, β

Initialisation
$k \leftarrow 1$
$T \leftarrow \mathcal{F}$
$\alpha \leftarrow \max_{i\in\mathcal{I}} \rho(A^{(i)})$
$\beta \leftarrow \max_{i\in\mathcal{I}} \|A^{(i)}\|$
choose δ *s.t.* $0 \leq \delta < \delta'$

while *(* $\beta - \alpha > \delta'$ *)*

$k \leftarrow k + 1$

$$T \leftarrow \left\{ (\mathcal{A}, A^{(i)}) \in \mathcal{F}^k \middle| \mathcal{A} \in T, A^{(i)} \in \mathcal{F}, p(\mathcal{A}, A^{(i)}) > \alpha + \delta \right\}$$

$$\alpha \leftarrow \max \left\{ \alpha, \sup_{\mathcal{Y}\in T} \rho(\Pi(\mathcal{Y}))^{1/k} \right\}$$

$$\beta \leftarrow \min \left\{ \beta, \max \left\{ \alpha + \delta, \sup_{\mathcal{Y}\in T} p(\mathcal{Y}) \right\} \right\}$$

end

Note that the input parameter δ' is used to determine an interval $[\alpha, \beta]$ of width at the most δ' that includes $\rho(\mathcal{F})$. For this purpose, as stated in the initialisation step, we have to appropriately choose $\delta \in [0, \delta')$.

The algorithm surely stops. Indeed, denoting by α_k and β_k, respectively, the values of α and β at the end of $k-$th iteration, in [Gri96] it is shown that, for each $k \geq 1$,

$$\alpha_k \leq \rho(\mathcal{F}) \leq \beta_k \tag{2.1}$$

and

$$\lim_{k\to\infty}(\beta_k - \alpha_k) \le \delta. \tag{2.2}$$

From (2.2), it follows that, for each $\epsilon > 0$, there exists k^* such that $\beta_{k^*} - \alpha_{k^*} \le \delta + \epsilon$. So, assuming $\epsilon = \delta' - \delta$, we have that the algorithm stops at the k^*-th iteration. Then, using (2.1), we can estimate $\rho(\mathcal{F})$ as $(\alpha_{k^*} + \beta_{k^*})/2$, which differs from the right value for less than $\delta'/2$.

Lastly we remark that the convergence rate of the algorithm depends on the choice of the norm, which, in turn, should depend on $\mathcal{F}$.

An improvement of the Gripemberg algorithm has been proposed by Vermiglio [Ver99].

In order to show the behaviour of the algorithm, we consider the following examples.

Example 2.1. Consider again the finite family of matrices used to disprove the finiteness conjecture

$$\mathcal{F} = \left\{ A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\},$$

and assume $\delta' = 10^{-3}$ and $\delta = 10^{-6}$ in Algorithm 2.1.

Then, for the 2-norm, the algorithm finds optimal lower and an upper bounds for $\rho(\mathcal{F})$ using one iteration, as shown in Table 2.1 and on the left of Figure 2.1.

k	1	2
α_k	1.00000000000000	1.61803398874989
β_k	1.61803398874989	1.61803398874989

Table 2.1: Behaviour of Algorithm 2.1 with $\delta' = 10^{-3}$, $\delta = 10^{-6}$ and $\|\cdot\|_2$ applied to the family $\mathcal{F}$.

This is a very nice result. However, the algorithm performs quite differently using the $\|\cdot\|_1$. In fact, for the same values of δ' and δ, we obtain the result shown on the right of Figure 2.1, where we can see that $k = 256$ iterations are needed to obtain lower and upper bounds for $\rho(\mathcal{F})$ such that $\alpha_k \le \rho(\mathcal{F}) \le \beta_k$ with $\alpha_k = 1.61803398874989$ and $\beta_k = 1.61903105971955$ (which fulfil the stopping condition $\beta_k - \alpha_k \le \delta' = 10^{-3}$).

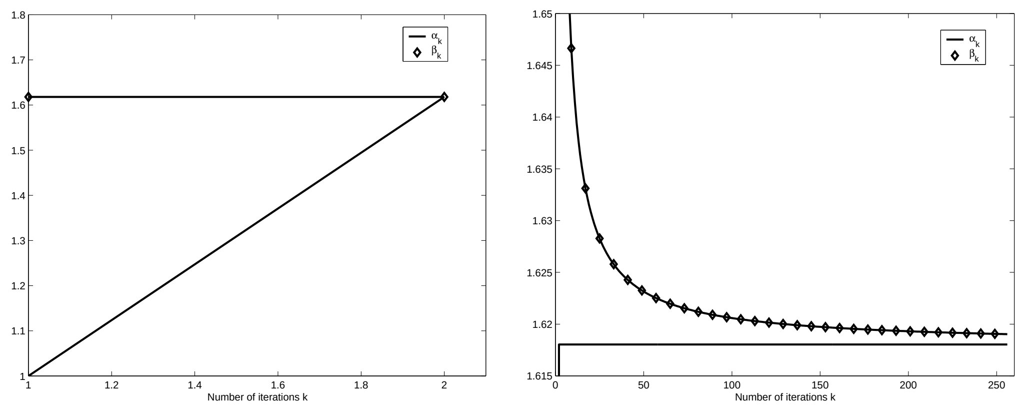

Figure 2.1: Behaviour of Algorithm 2.1 applied to the family $\mathcal{F}$ with $\delta' = 10^{-3}$ and $\delta = 10^{-6}$. The norm used are $\|\cdot\|_2$ on the left and $\|\cdot\|_1$ on the right.

Furthermore, from the same plots, we can see that the lower bound α_k remains almost unchanged during the convergence process, whereas the upper bound β_k decreases slowly. Also, it is interesting to report that the time spent per iteration is very low and constant during the whole convergence process.

From the results just given, we can conclude that the choice of a proper norm seems to be important. We remark also that, it is no wonder that the algorithm performs better using the Euclidean norm, since this norm is extremal for the considered family. $\diamond$

Example 2.2. The performance of Algorithm 2.1 is tested on the family of matrices considered in [Gri96], that is,

$$\mathcal{G} = \left\{ A_1 = \frac{1}{5}\begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix},\ A_2 = \frac{1}{5}\begin{pmatrix} 3 & -3 \\ 0 & -1 \end{pmatrix} \right\},$$

using, as in the previous example, both the 1- and the 2-norm and assuming $\delta' = 10^{-3}$ and $\delta = 10^{-6}$. The quite good results related to the 2-norm are shown in Figure 2.2. At the end of the convergence process, after $k = 22$ iterations the algorithm returns $\alpha_k = 0.65967890895528$ and $\beta_k = 0.66041899785153$, which satisfy the stopping criterion. Moreover, the time spent per iteration is very low and constant during the whole convergence process.

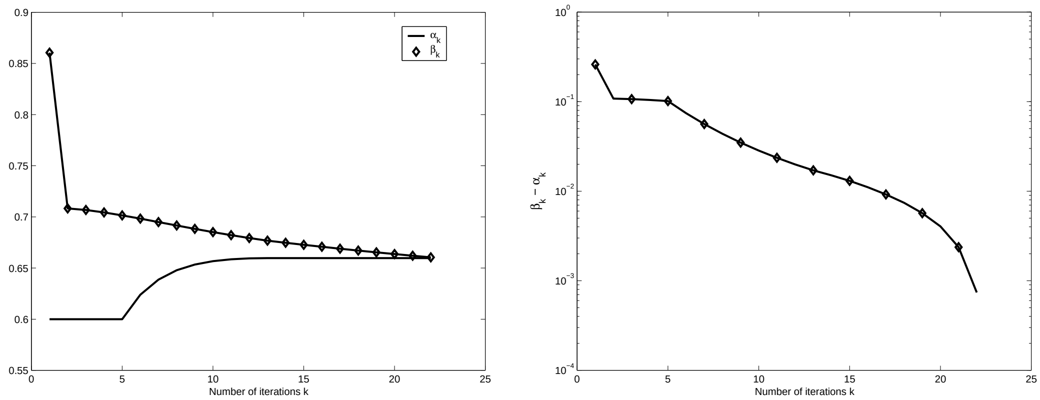

Figure 2.2: Behaviour of Algorithm 2.1 for the family $\mathcal{G}$ with $\delta' = 10^{-3}$, $\delta = 10^{-6}$ and $\|\cdot\|_2$. The lower and the upper bounds α_k and β_k are reported on the left as functions of the number of iterations k. The values of the interval widths $\beta_k - \alpha_k$ are plotted on the right using a semilogarithmic scale.

The algorithm runs differently with the 1-norm, as shown in Figure 2.3. The stopping criterion

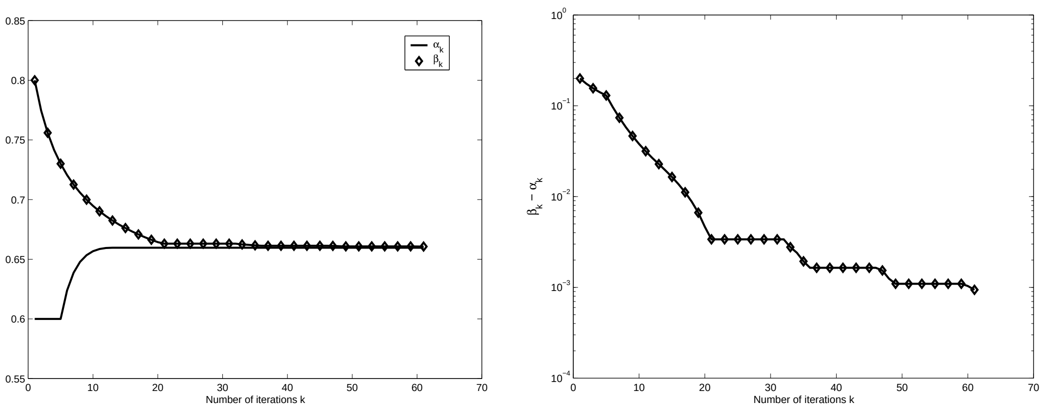

Figure 2.3: Behaviour of Algorithm 2.1 for the family $\mathcal{G}$ with $\delta' = 10^{-3}$, $\delta = 10^{-6}$ and $\|\cdot\|_1$. The values of the lower and upper bounds α_k and β_k are plotted on the left whereas, on the right, we have the values of their difference in a semilogarithmic scale.

$\beta_k - \alpha_k < \delta'$ is fulfilled for $k = 61$ and the corresponding values are $\alpha_k = 0.65967890895528$ and

$\beta_k = 0.66062050028109$. However, the time spent grows exponentially (see Figure 2.4) and so the algorithm becomes impracticable. As a final note, as for the previous example, we may say that the choice of a proper norm is a crucial point for a good convergence behaviour of the algorithm. Moreover, even in this example, the 2-norm is better then the 1-norm. However, we can also find examples in which the 1-norm yields more favorable results. $\diamond$

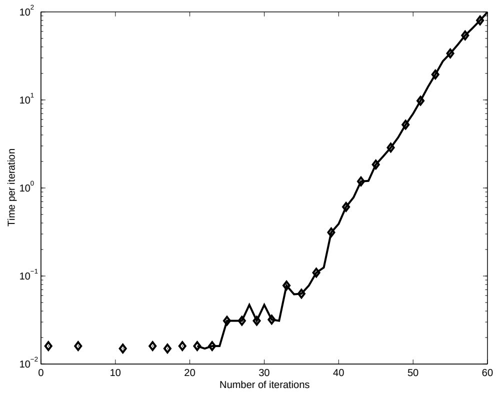

Figure 2.4: Computational time per iteration for a run of Algorithm 2.1 for the family $\mathcal{G}$ with $\delta' = 10^{-3}$, $\delta = 10^{-6}$ and $\|\cdot\|_1$.

2.2 Balanced complex polytopes

In this section we recall from [GZ07] the notion of balanced complex polytope, which is the extension of the real symmetric polytope [Zie95] to the complex space.

Let $\mathcal{X}$ be a set in $\mathbb{C}^n$. The absolutely convex hull of $\mathcal{X}$, denoted by absco($\mathcal{X}$), is the set of all possible absolutely convex linear combinations of vectors of $\mathcal{X}$, that is, $x \in \text{absco}(\mathcal{X})$ if and only if there exist $x^{(1)}, \ldots, x^{(k)} \in \mathcal{X}$ with $k \geq 1$ such that

$$x = \sum_{i=1}^{k} \lambda_i x^{(i)} \qquad \text{with} \qquad \lambda_i \in \mathbb{C} \quad \text{and} \quad \sum_{i=1}^{k} |\lambda_i| \leq 1.$$

In particular, if $\mathcal{X} = \{x^{(i)}\}_{1\leq i\leq k}$ is a finite set of vectors, then

$$\text{absco}(\mathcal{X}) = \{x \in \mathbb{C}^n : x = \sum_{i=1}^{k} \lambda_i x^{(i)}\ ,\ \ \lambda_i \in \mathbb{C}\ ,\ \ \sum_{i=1}^{k} |\lambda_i| \leq 1\}.$$

Definition 2.1. *A bounded set $\mathcal{P} \subset \mathbb{C}^n$ is a* balanced complex polytope (b.c.p) *if there exists a finite set of vectors $\mathcal{X} = \{x^{(i)}\}_{1\leq i\leq k}$ such that* span($\mathcal{X}$) $= \mathbb{C}^n$ *and*

$$\mathcal{P} = \text{absco}(\mathcal{X}). \tag{2.3}$$

Moreover, if absco($\mathcal{X}'$) $\subsetneq$ absco($\mathcal{X}$) *for all $\mathcal{X}' \subsetneq \mathcal{X}$, then $\mathcal{X}$ is called an* essential system of vertexes *for $\mathcal{P}$, whereas any vector ux_i with $u \in \mathbb{C}$, $|u| = 1$, is called a* vertex *of $\mathcal{P}$.*

Remark that, from a geometrical point of view, a b.c.p. $\mathcal{P}$ is not a classical polytope. In fact, if we identify the complex space $\mathbb{C}^n$ with the real space $\mathbb{R}^{2n}$, we can easily see that $\mathcal{P}$ is not bounded by a finite number of hyperplanes. Moreover, if the vertexes of the b.c.p. are not real, even the intersection $\mathcal{P} \bigcap \mathbb{R}^n$ is not a classical polytope. Indeed its boundary is a closed piecewise algebraic hypersurface (see Theorem 3.1 in Section 3.2).

The following property characterises the essential system of vertexes.

Proposition 2.1. *If $\mathcal{P}$ is a b.c.p. and $\mathcal{X} = \{x^{(i)}\}_{1\leq i\leq k}$ is an essential system of vertexes for $\mathcal{P}$, then for each $x^{(i)} \in \mathcal{X}$ the equality*

$$x^{(i)} = \sum_{j=1,\, j\neq i}^{k} \lambda_j x^{(j)}$$

implies

$$\sum_{j=1,\, j\neq i}^{k} |\lambda_j| > 1.$$

Now we consider the concept of *adjoint* set that, in literature, is often referred to as *polarity* or *duality* (see Ziegler or Heuser), to define a b.c.p of *adjoint type*. For this purpose we will use the notation $\langle \cdot\,,\,\cdot \rangle$ to denote the usual Euclidean scalar product in $\mathbb{C}^n$ defined by $\langle x, y\rangle = \sum_{i=1}^{n} x^{(i)}\bar{y}^{(i)}$.

Definition 2.2. *Let $\mathcal{X} \subset \mathbb{C}^n$. The set*

$$\mathrm{adj}(\mathcal{X}) = \{y \in \mathbb{C}^n : |\langle y, x\rangle| \le 1\,,\ \forall\, x \in \mathcal{X}\}$$

is called the adjoint *of $\mathcal{X}$.*

Definition 2.3. *A bounded set $\mathcal{P}^* \subset \mathbb{C}^n$ is a* balanced complex polytope of adjoint type (a.b.c.p) *if there exists a finite set of vectors $\mathcal{X} = \{x^{(i)}\}_{1\le i\le k}$ such that* $\mathrm{span}(\mathcal{X}) = \mathbb{C}^n$ *and*

$$\mathcal{P}^* = \mathrm{adj}(\mathcal{X}) = \{y \in \mathbb{C}^n : |\langle y, x^{(i)}\rangle| \le 1\,,\ i = 1, \ldots, k\} \tag{2.4}$$

Moreover, if $\mathrm{adj}(\mathcal{X}') \supsetneq \mathrm{adj}(\mathcal{X})$ *for all $\mathcal{X}' \subsetneq \mathcal{X}$ then $\mathcal{X}$ is called an* essential system of facets *for $\mathcal{P}^*$, whereas any vector ux_i with $u \in \mathbb{C}$, $|u| = 1$, is called a* facet *of $\mathcal{P}^*$.*

The following property characterises the essential system of facets.

Proposition 2.2. *If $\mathcal{P}^*$ is an a.b.c.p. and $\mathcal{X} = \{x^{(i)}\}_{1\le i\le k}$ is an essential system of facets for $\mathcal{P}^*$, then for each $x^{(i)} \in \mathcal{X}$ there exists $y^{(i)} \in \mathcal{P}^*$ such that*

$$|\langle y^{(i)}, x^{(i)}\rangle| = 1 \quad \textit{and} \quad |\langle y^{(i)}, x^{(j)}\rangle| < 1 \quad \textit{for all}\ \ j \neq i.$$

Now we analyse the mutual relationships between b.c.p.'s and a.b.c.p.'s.

Proposition 2.3. *Let $\mathcal{X} \subset \mathbb{C}^n$. Then*

$$\mathrm{adj}(\mathrm{absco}(\mathcal{X})) = \mathrm{adj}(\mathcal{X}).$$

Corollary 2.1. *Let $\mathcal{P}$ be a b.c.p. and let $\mathcal{X} = \{x^{(i)}\}_{1\le i\le k}$ be a finite set of vectors such that $\mathcal{P} = \mathrm{absco}(\mathcal{X})$. Then $\mathrm{adj}(\mathcal{P})$ is an a.b.c.p. and it holds that*

$$\mathrm{adj}(\mathcal{P}) = \mathrm{adj}(\mathcal{X}).$$

Corollary 2.2. *Let $\mathcal{P}^*$ be an a.b.c.p. and let $\mathcal{X} = \{x^{(i)}\}_{1\le i\le k}$ be a finite set of vectors such that $\mathcal{P}^* = \mathrm{adj}(\mathcal{X})$. Then $\mathrm{adj}(\mathcal{P}^*)$ is a b.c.p. and it holds that*

$$\mathrm{adj}(\mathcal{P}^*) = \mathrm{absco}(\mathcal{X}).$$

Corollaries 2.1 and 2.2 yield the following result, that in literature is often referred to as the *bipolar theorem*.

Theorem 2.1. *Let $\mathcal{P}$ be a b.c.p. and let $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$. Then it holds that*

$$\mathcal{P} = adj(\mathcal{P}^*). \tag{2.5}$$

Conversely, let $\mathcal{P}^*$ be an a.b.c.p. and let $\mathcal{P} = \mathrm{adj}(\mathcal{P}^*)$. Then it holds that

$$\mathcal{P}^* = \mathrm{adj}(\mathcal{P}). \tag{2.6}$$

As we shall see in Chapter 3, the foregoing theorem is very important to create an algorithm for the construction of a b.c.p. by adding one point at a time.

Proposition 2.4. *Let $\mathcal{P}^*$ be a b.c.p. of adjoint type, $\mathcal{X} = \{x^{(i)}\}_{1\le i\le m}$ be an essential system of facets, $\mathcal{P} = \mathrm{adj}(\mathcal{P}^*)$ and $y \in \partial\mathcal{P}^*$. Then the following condition is satisfied:*

(C1) $|\langle y, x\rangle| \le 1 \ \forall x \in \mathcal{P}$ *and* $\exists \hat{x} \in \partial\mathcal{P}$, $\hat{x} = \hat{u}x^{(j)}$ *for some j with* $\hat{u} \in \mathbb{C}$, $|\hat{u}| = 1$, *such that* $\langle y, \hat{x}\rangle = 1$.

Remark that $\hat{x}$ in (C1) is a facet of $\mathcal{P}^*$, i.e., a vertex of $\mathcal{P} = \mathrm{adj}(\mathcal{P}^*)$.

Proposition 2.5. *Let $\mathcal{P}$ be a b.c.p., $\mathcal{X} = \{x^{(i)}\}_{1\le i\le m}$ be an essential system of vertexes, $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$ and $x \in \partial\mathcal{P}$. Then the following conditions are satisfied:*

(C2) $|\langle y, x\rangle| \le 1 \ \forall y \in \mathcal{P}^*$ *and* $\exists \hat{y} \in \partial\mathcal{P}^*$ *such that* $\langle \hat{y}, x\rangle = 1$;

(C3) $\sum_{i=1}^m |\mu_i| \ge 1$ *whenever* $x = \sum_{i=1}^m \mu_i x^{(i)}$ *and* $\exists \lambda_1, \ldots, \lambda_m \in \mathbb{C}$ *such that* $x = \sum_{i=1}^m \lambda_i x^{(i)}$ *with* $\sum_{i=1}^m |\lambda_i| = 1$.

Definition 2.4. *Let $\mathcal{P}$ be a b.c.p., $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$ and $y \in \partial\mathcal{P}^*$. Then the convex set*

$$F_y = \{x \in \mathcal{P} \,\big|\, \langle y, x\rangle = 1\} \tag{2.7}$$

is called a (geometric) face *of $\mathcal{P}$, whereas y is called the* functional *associated with F_y.*

Clearly,

$$F_y \subseteq \partial\mathcal{P} \tag{2.8}$$

and, by property (C1) in Proposition 2.4, the face F_y is never empty.

Definition 2.5. *Let $\mathcal{P}$ be a b.c.p. and $y \in \partial\mathcal{P}^*$, where $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$. Then any vertex of $\mathcal{P}$ belonging to F_y is called a* vertex *of F_y.*

In order to state the next result, we recall that, given a set $\mathcal{X} \subset \mathbb{C}^n$, $\mathrm{co}(\mathcal{X})$ stays for the *convex hull* of $\mathcal{X}$, i.e. the set of all possible convex linear combinations of vectors of $\mathcal{X}$ (with non-negative real coefficients).

Theorem 2.2. *Let $\mathcal{P}$ be a b.c.p. and $y \in \partial\mathcal{P}^*$, where $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$. Then the set $\mathcal{X}_y$ of all the vertexes of F_y is non-empty and*

$$\mathcal{X}_y \subseteq \mathcal{X},$$

where $\mathcal{X} = \{x^{(i)}\}_{1\le i\le m}$ is a suitable essential system of vertexes for $\mathcal{P}$.

Moreover, it holds that

$$F_y = \mathrm{co}(\mathcal{X}_y). \tag{2.9}$$

Definition 2.6. *Let $\mathcal{P}$ be a b.c.p. and y a facet of $\mathcal{P}$. Then the set F_y is called a* (geometric) facet *of $\mathcal{P}$ as well.*

Definition 2.7. *Let $\mathcal{P}$ be a b.c.p. and F_y be a face of $\mathcal{P}$. Then we say that the number*

$$\dim(F_y) = \dim(\mathrm{span}(F_y)) - 1$$

is the dimension *of the face F_y.*

Definition 2.8. *Let $\mathcal{P}$ be a b.c.p. and $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$. Then we say that a vector $y \in \partial\mathcal{P}^*$ is a* facet *of $\mathcal{P}$ if there exist n linearly independent vertexes $x^{(i_1)}, \ldots, x^{(i_n)}$ belonging to an essential system of vertexes $\mathcal{X} = \{x^{(i)}\}_{1\le i\le m}$ of $\mathcal{P}$ such that*

$$\langle y, x^{(i_j)}\rangle = 1, \quad j = 1, \ldots, n,$$

that is,

$$x^{(i_j)} \in F_y, \quad j = 1, \ldots, n.$$

The next theorem states the existence of a (geometric) facet including a given point on the boundary of a b.c.p.

Theorem 2.3. *Let $\mathcal{P}$ be a b.c.p. and let $x \in \partial\mathcal{P}$. Then there exists a (geometric) facet F_y such that $x \in F_y$.*

The foregoing Theorem 2.3 and the inclusion (2.8) allow us to conclude that

$$\partial\mathcal{P} = \bigcup_{y \in F^*(\mathcal{P})} F_y, \tag{2.10}$$

where $F^*(\mathcal{P})$ is the set of all the facets of $\mathcal{P}$. In other words, $\partial\mathcal{P}$ is the union of all the (geometric) facets of $\mathcal{P}$.

Another straightforward consequence of Theorem 2.3 is that

$$\mathcal{P} = \text{adj}(F^*(\mathcal{P})). \tag{2.11}$$

Remark 2.1. *The dimension of a face F_y of a b.c.p. $\mathcal{P}$ can vary between 0 and $n-1$. In particular, a face F_y is a facet if and only if*

$$\dim(F_y) = n - 1,$$

whereas a face F_y is a vertex if and only if

$$\dim(F_y) = 0.$$

The next theorem gives an important upper bound for computational purposes.

Theorem 2.4. *Let $\mathcal{P}$ be a b.c.p. and let F_y be a face of $\mathcal{P}$ of dimension d. Then, for each $x \in F_y$, there exist s vertexes $x^{(i_1)}, \ldots, x^{(i_s)} \in \mathcal{X}_y$ with*

$$s \leq 2d + 1 \tag{2.12}$$

such that

$$x \in \text{co}\left(\left\{x^{(i_1)}, \ldots, x^{(i_s)}\right\}\right).$$

It is well known that, if the b.c.p. $\mathcal{P}$ is real, then (2.12) is replaced by the more stringent inequality

$$s \leq d + 1. \tag{2.13}$$

On the other hand, in the general complex case (2.13) is not true and the upper bound (2.12) can be actually attained (see [GZ07]).

Now, following Miani and Savorgnan [MS06], we consider the intersection of a b.c.p. $\mathcal{P}$ with $\mathbb{R}^n$, i.e. $\mathcal{P} \cap \mathbb{R}^n$, and its projection on $\mathbb{R}^n$, i.e. $Re(\mathcal{P})$.

As is immediately seen, it holds that

$$Re(\mathcal{P}) = \text{absco}\left(\mathcal{P} \cup \overline{\mathcal{P}}\right) \cap \mathbb{R}^n, \tag{2.14}$$

where $\overline{\mathcal{P}}$ is the complex conjugate set of $\mathcal{P}$. Therefore, the computation of $Re(\mathcal{P})$ is a particular case of the computation of $\mathcal{P} \cap \mathbb{R}^n$, namely when $\mathcal{P}$ is self-conjugated.

It is easy to see that, if (and only if) a b.c.p. $\mathcal{P}$ has a real system of vertexes $\mathcal{X} \subset \mathbb{R}^n$, then $\mathcal{P} \cap \mathbb{R}^n$ is a classic symmetric polytope. In the general case, for self-conjugated b.c.p.'s we have the following result, the easy proof of which is given in [MS06].

Proposition 2.6. *If a b.c.p. $\mathcal{P}$ is self-conjugated, i.e. $\mathcal{P} = \overline{\mathcal{P}}$, then $\mathcal{P} \cap \mathbb{R}^n$ is the convex hull of a finite number of vertexes and/or a finite number of two-dimensional ellipses. The vertexes are the real vertexes of $\mathcal{P}$, if any, whereas the ellipses are the intersections with $\mathbb{R}^n$ of the boundaries of the absolutely convex hulls of the pairs of conjugate vertexes of $\mathcal{P}$ (not proportional to real vectors), if any.*

2.3 Complex polytope norms

Now we extend the concept of *polytope norm* to the complex case in a straightforward way.

Lemma 2.1. *Any b.c.p.* $\mathcal{P}$ *is the unit ball of a norm* $\|\cdot\|_{\mathcal{P}}$ *on* $\mathbb{C}^n$.

Proof. Since $\mathrm{span}(\mathcal{X}) = \mathbb{C}^n$, the set $\mathcal{P}$ is *absorbing*. Therefore, since it is absolutely convex and bounded, the Minkowski functional associated with $\mathcal{P}$, defined for all $z \in \mathbb{C}^n$ by

$$\|z\|_{\mathcal{P}} = \inf\{\rho > 0 \mid z \in \rho\mathcal{P}\}, \tag{2.15}$$

is indeed a norm on $\mathbb{C}^n$ (see [Heu82]). ■

Definition 2.9. *We shall call* complex polytope norm *any norm* $\|\cdot\|_{\mathcal{P}}$ *whose unit ball is a b.c.p.* $\mathcal{P}$.

Due to the substantial difference between a b.c.p. $\mathcal{P}$ and a b.c.p. of adjoint type $\mathcal{P}^*$, we need to also introduce the adjoint version of the concept of complex polytope norm.

Lemma 2.2. *Any b.c.p. of adjoint type* $\mathcal{P}^*$ *is the unit ball of a norm* $\|\cdot\|_{\mathcal{P}^*}$ *on* $\mathbb{C}^n$.

Proof. Since also $\mathcal{P}^*$ is absorbing and absolutely convex, as in the proof of Lemma 2.1, we can conclude that the Minkowski functional defined by

$$\|z\|_{\mathcal{P}^*} = \inf\{\rho > 0 \mid z \in \rho\mathcal{P}^*\} \tag{2.16}$$

is a norm on $\mathbb{C}^n$. ■

Definition 2.10. *We shall call* adjoint complex polytope norm *any norm* $\|\cdot\|_{\mathcal{P}^*}$ *whose unit ball is a b.c.p. of adjoint type* $\mathcal{P}^*$.

Definition 2.11. *Let* $\mathcal{P}$ *be a b.c.p.,* $z \in \mathbb{C}^n$ *and*

$$R_z = \{\rho z \mid \rho > 0\}$$

be the ray *of* $\mathbb{C}^n$ *exiting from the origin* 0 *and passing through* z. *Then we say that the (obviously) unique vector* $\hat{z} \in \partial\mathcal{P} \cap R_z$ *is the* radial projection *of* z *onto* $\partial\mathcal{P}$ *and that* z projects *on any face* F_y *of* $\mathcal{P}$ *such that* $\hat{z} \in F_y$.

Formula (2.15) allows us to conclude that

$$z = \|z\|_{\mathcal{P}} \cdot \hat{z} \tag{2.17}$$

where $\hat{z}$ is the radial projection of z onto $\partial\mathcal{P}$.

Remark that the definition of *radial projection* $\hat{z}$ of a vector z onto $\partial\mathcal{B}$ makes sense for any unit ball $\mathcal{B}$ of a norm defined either in $\mathbb{C}^n$ or in $\mathbb{R}^n$ and that also the equality analogous to (2.17) holds true.

Now we illustrate an important link between polytope norms and adjoint polytope norms.

Theorem 2.5. *Let* $\mathcal{P}$ *be a b.c.p. and let* $\|\cdot\|_{\mathcal{P}}$ *be the corresponding complex polytope norm. Then, for any* $z \in \mathbb{C}^n$, *it holds that*

$$\|z\|_{\mathcal{P}} = \min\Big\{\sum_{i=1}^{m} |\lambda_i| \;\Big|\; z = \sum_{i=1}^{m} \lambda_i x^{(i)}\Big\} = \max_{y\in\partial\mathcal{P}^*} |\langle y, z\rangle|, \tag{2.18}$$

where $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$ *and* $\mathcal{X} = \{x^{(i)}\}_{1\le i\le m}$ *is an essential system of vertexes for* $\mathcal{P}$.

Analogously, let $\mathcal{P}^*$ *be a b.c.p. of adjoint type and let* $\|\cdot\|_{\mathcal{P}^*}$ *the corresponding adjoint complex polytope norm. Then, for any* $z \in \mathbb{C}^n$, *it holds that*

$$\|z\|_{\mathcal{P}^*} = \max_{1\le i\le m} |\langle z, x^{(i)}\rangle| = \max_{x\in\partial\mathcal{P}} |\langle z, x\rangle|, \tag{2.19}$$

where $\mathcal{P} = \mathrm{adj}(\mathcal{P}^*)$ *and* $\mathcal{X} = \{x^{(i)}\}_{1\le i\le m}$ *is an essential system of facets for* $\mathcal{P}^*$.

Remark that, in view of Theorem 2.4, formula (2.18) can be rewritten as

$$\|z\|_{\mathcal{P}} = \min\Big\{ \sum_{j=1}^{2n-1} |\lambda_{i_j}| \ \Big| \ z = \sum_{j=1}^{2n-1} \lambda_{i_j} x^{(i_j)} \ \text{ and } \ \{i_1,\dots,i_{2n-1}\} \subset \{1,\dots,m\}\Big\}. \tag{2.20}$$

In view of Definition 2.11, simple geometric arguments allow us to conclude that the minimum in (2.18) is obtained in correspondence to a subset of vertexes $\{e^{i\theta_1}x^{(i_1)},\dots,e^{i\theta_s}x^{(i_s)}\} \subseteq \mathcal{X}_y$, with $s \le 2n-1$, where F_y is any face of $\mathcal{P}$ which z projects on. More precisely, it holds that

$$z = \sum_{j=1}^{s} \alpha_j e^{i\theta_j} x^{(i_j)} \quad \text{with} \quad \alpha_j > 0 \quad \text{for all} \quad j = 1,\dots,s, \tag{2.21}$$

and that

$$\|z\|_{\mathcal{P}} = \sum_{j=1}^{s} \alpha_j. \tag{2.22}$$

Indeed, the equality (2.21) is characterising in the sense that, if it is satisfied and if $\{e^{i\theta_1}x^{(i_1)},\dots,e^{i\theta_s}x^{(i_s)}\} \subseteq \mathcal{X}_y$ for some face F_y of $\mathcal{P}$, then z necessarily projects on F_y and (2.22) holds as well.

Now we consider the so called ∞-*norm* and 1-*norm* which, for all $z \in \mathbb{C}^n$, are defined by $\|z\|_\infty = \max_{1\le i\le n} |z_i|$ and $\|z\|_1 = \sum_{i=1}^n |z_i|$, respectively.

As is well known in the real case, $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are adjoint to each other. In fact, they are associated with the n-dimensional *complex hypercube*

$$\mathcal{H}^* = \text{adj}(\{e^{(1)},\dots,e^{(n)}\}) \tag{2.23}$$

and to its adjoint b.c.p., the n-dimensional *complex crosspolytope*

$$\mathcal{H} = \text{absco}(\{e^{(1)},\dots,e^{(n)}\}), \tag{2.24}$$

respectively, where the $e^{(i)}$'s are the vectors of the canonical basis of $\mathbb{C}^n$. Note that $\mathcal{H}^* \bigcap \mathbb{R}^n$ is the classic n-dimensional hypercube and that $\mathcal{H} \bigcap \mathbb{R}^n$ is the classic n-dimensional crosspolytope.

The two special norms above can be used to conveniently express all the other complex and adjoint complex polytope norms as follows. Given a b.c.p. $\mathcal{P}$ and an essential system of vertexes $\mathcal{X} = \{x^{(i)}\}_{1\le i\le m}$, define the *vertex matrix*

$$V = \left[x^{(1)} \dots x^{(m)}\right]$$

and, for the b.c.p. of adjoint type $\mathcal{P}^* = \text{adj}(\mathcal{P})$, the *facet matrix*, adjoint of V,

$$F = V^*.$$

Then, the first equality in (2.19) yields

$$\|z\|_{\mathcal{P}^*} = \|Fz\|_\infty, \tag{2.25}$$

whereas, with $\lambda = [\lambda_1 \dots \lambda_m]^T$, the first equality in (2.18) leads to

$$\|z\|_{\mathcal{P}} = \min_{V\lambda = z} \|\lambda\|_1. \tag{2.26}$$

Note that, if $m = n$, then (2.26) reduces to

$$\|z\|_{\mathcal{P}} = \|V^{-1}z\|_1. \tag{2.27}$$

Now consider the case $m > n$. In order to compute $\|z\|_{\mathcal{P}}$, assume, without any restriction, that the first n columns of the vertex matrix V are linearly independent and define the matrices

$$V_1 = \left[x^{(1)} \dots x^{(n)}\right] \quad \text{and} \quad V_2 = \left[x^{(n+1)} \dots x^{(m)}\right].$$

Then, if $\lambda \in \mathbb{C}^m$, define also the $(m-n)$-vector

$$\mu = [\lambda_{n+1} \dots \lambda_m]^T,$$

so that any solution of the equation $V\lambda = z$ may be written in the form

$$\lambda = \begin{bmatrix} V_1^{-1}(z - V_2\mu) \\ \mu \end{bmatrix}.$$

In conclusion, we obtain

$$\|z\|_{\mathcal{P}} = \min_{\mu \in \mathbb{C}^{m-n}} \left\| \begin{bmatrix} V_1^{-1}(z - V_2\mu) \\ \mu \end{bmatrix} \right\|_1, \tag{2.28}$$

that is, the computation of $\|z\|_{\mathcal{P}}$ requires the solution of a minimisation problem in $\mathbb{C}^{m-n}$. Therefore, in general, $\|z\|_{\mathcal{P}^*}$ is clearly much easier to compute. However, even if $m \geq 2n$, formula (2.20) reveals that a minimising λ in (2.26) can be found among those which have at the most $2n-1$ non-zero entries.

We conclude this section by considering the real norms the unit balls of which are $\mathcal{P} \cap \mathbb{R}^n$ and $Re(\mathcal{P})$, that we denote by $\|\cdot\|_{\mathcal{P}\cap\mathbb{R}^n}$ and $\|\cdot\|_{Re(\mathcal{P})}$, respectively. They are useful, for example, for the investigation of certain control problems (see [MS06]).

Since the radial projection $\hat{u}$ of a real vector $u \in \mathbb{R}^n$ onto $\partial\mathcal{P}$ is still real, it equals the radial projection of u onto $\partial(\mathcal{P} \cap \mathbb{R}^n)$ and, consequently, formula (2.17) applied to both the norms $\|\cdot\|_{\mathcal{P}}$ and $\|\cdot\|_{\mathcal{P}\cap\mathbb{R}^n}$ yields

$$\|u\|_{\mathcal{P}\cap\mathbb{R}^n} = \|u\|_{\mathcal{P}}. \tag{2.29}$$

Furthermore, the equalities (2.14) and (2.29) lead us to conclude that

$$\|u\|_{Re(\mathcal{P})} = \|u\|_{\text{absco}\left(\mathcal{P}\cup\overline{\mathcal{P}}\right)}. \tag{2.30}$$

2.4 The computation of the joint spectral radius using complex polytope norms

In the first section of the present Chapter we have presented an algorithm which gives an upper and a lower bound for the spectral radius of a complex matrix family $\mathcal{F}$. Here we develop another approach to the problem based on complex polytope norms.

We start by recalling some polytope extremality results and definitions.

A first important result, shown in the next Theorem 2.6, states that the set of the complex polytope norms is *dense* in the set of all norms defined on $\mathbb{C}^n$. As a consequence, the corresponding set of induced matrix complex polytope norms is dense in the set of all induced $n\times n$-matrix norms (see [GZ07]).

Theorem 2.6. *Given a norm $\|\cdot\|$ on the vector space $\mathbb{C}^n$, for any $\epsilon > 0$ there exists a b.c.p. $\mathcal{P}_\epsilon$ whose corresponding complex polytope norm $\|\cdot\|_\epsilon$ satisfies, for all $x \in \mathbb{C}^n$, the inequalities*

$$\|x\| \leq \|x\|_\epsilon \leq (1+\epsilon)\|x\|.$$

Moreover, denoting by $\|\cdot\|$ and $\|\cdot\|_\epsilon$ also the corresponding induced matrix norms, for all $A \in \mathbb{C}^{n\times n}$ it holds that

$$(1+\epsilon)^{-1}\|A\| \leq \|A\|_\epsilon \leq (1+\epsilon)\|A\|.$$

As a consequence we can give the following refinement of Theorem 1.2, which allows us to restrict the search of the infimum from the set $\mathcal{N}$ to the subset $\mathcal{N}_{pol}$.

Theorem 2.7. *The spectral radius of a bounded family $\mathcal{F}$ of complex $n \times n$-matrices is characterised by the equality*

$$\rho(\mathcal{F}) = \inf_{\|\cdot\| \in \mathcal{N}_{pol}} \|\mathcal{F}\|, \tag{2.31}$$

where $\mathcal{N}_{pol}$ denotes the set of all possible induced $n \times n$-matrix complex polytope norms.

Now, we consider under which conditions the construction of an extremal complex polytope norm is feasible for a non-defective family. In order to recall some results in this direction, we start with the following definition.

Definition 2.12. *Given a bounded family $\mathcal{F}$ of complex $n \times n$-matrices, any matrix $P \in \Sigma_k(\mathcal{F})$ satisfying*

$$\rho(\mathcal{F}) = \bar{\rho}_k(\mathcal{F})^{1/k} = \rho(P)^{1/k} \tag{2.32}$$

for some $k \geq 1$ is called a spectrum maximising product *(in short an* s.m.p.*) for $\mathcal{F}$. Moreover an s.m.p. is said to be* minimal *if it is not a power of another s.m.p. of $\mathcal{F}$. Any eigenvector $x \neq 0$ of the s.m.p. P related to the eigenvalue of maximum modulus is said to be a* leading eigenvector *of $\mathcal{F}$.*

An important necessary condition for the existence of an extremal complex polytope norm is given by the next theorem (see [GWZ05]).

Theorem 2.8. *Let $\mathcal{F} = \{A^{(i)}\}_{1 \leq i \leq m}$ be a finite non-defective family of complex $n \times n$-matrices and assume that there exists an extremal complex polytope norm $\|\cdot\|_{\mathcal{P}}$. Then $\mathcal{F}$ has at least an s.m.p. P.*

At the present it is not known whether or not the Theorem 2.8 can be reversed. That is, we have the following *complex polytope extremality (CPE) conjecture*, first presented in [GWZ05].

Conjecture 2.1 (CPE)**.** *Let $\mathcal{F} = \{A^{(i)}\}_{1 \leq i \leq m}$ be a finite non-defective family of complex $n \times n$-matrices and assume that $\mathcal{F}$ has at least an s.m.p. P. Then there exists an extremal complex polytope norm for $\mathcal{F}$.*

This conjecture is still an open problem. However, Guglielmi, Wirth and Zennaro [GWZ05], by adding some hypotheses on the family $\mathcal{F}$, have proved a weaker version of the CPE conjecture called the *Small CPE Theorem*. It uses the concept of *asymptotically simple family* which, in turn, uses some ideas related to the *leading eigenvector* of a family, *trajectory* and *$\mathcal{F}$-cyclicity*.

Before starting to recall these definitions, we scale the family $\mathcal{F} = \{A^{(i)}\}_{i \in \mathcal{I}}$ by the scalar $\rho = \rho(Q_k)^{1/k}$, for some $Q_k \in \Sigma_k(\mathcal{F})$, obtaining the *scaled family*

$$\mathcal{F}^* = \{\rho^{-1} A^{(i)}\}_{i \in \mathcal{I}},$$

which is such that $\rho(\mathcal{F}^*) \geq 1$.

Definition 2.13. *Let us consider a (scaled) family $\mathcal{F}^*$ with $\rho(\mathcal{F}^*) \geq 1$. Then, for any vector $x \in \mathbb{C}^n$, the set*

$$\mathcal{T}[\mathcal{F}^*, x] = \{x\} \cup \{Px | P \in \Sigma(\mathcal{F}^*)\}$$

is called the trajectory *obtained by applying all the products P of matrices of $\mathcal{F}^*$ to the vector x.*

Definition 2.14. *Let $\mathcal{F}$ be a family of complex $n \times n$-matrices and $\hat{\mathcal{F}} = (1/\rho(\mathcal{F}))\mathcal{F}$ the corresponding normalised family. A set $\mathcal{X} \subset \mathbb{C}^n$ is said to be $\mathcal{F}$-*cyclic *if for any pair $(x, y) \in \mathcal{X} \times \mathcal{X}$, there exist two complex numbers α and β with $|\alpha| \cdot |\beta| = 1$ and two (finite) normalised products $\hat{P}, \hat{Q} \in \Sigma(\hat{\mathcal{F}})$ such that $y = \alpha \hat{P} x$ and $x = \beta \hat{Q} y$.*

Definition 2.15. *A non-defective bounded family $\mathcal{F}$ of complex $n \times n$-matrices is said to be* asymptotically simple *if the set $\mathcal{E}$ of its leading eigenvectors is finite (modulo scalar non-zero factors) and $\mathcal{F}$-cyclic.*

Now, we are able to introduce the following theorem which, under some hypotheses on the family $\mathcal{F}$, guarantees the existence of an extremal polytope norm.

Theorem 2.9 (Small CPE Theorem)**.** *Assume that a finite family $\mathcal{F}$ of complex $n \times n$-matrices is non-defective and asymptotically simple. Assume also that for a leading eigenvector x of $\mathcal{F}$ the trajectory $\mathcal{T}[\mathcal{F}^*, x]$ satisfies span$(\mathcal{T}[\mathcal{F}^*, x]) = \mathbb{C}^n$. Then the set*

$$\partial\mathcal{S}[\mathcal{F}^*, x] \bigcap \mathcal{T}[\mathcal{F}^*, x] \tag{2.33}$$

is finite modulo scalar factors of unitary modulus, where $\mathcal{S}[\mathcal{F}^*, x] = \overline{\text{absco}(\mathcal{T}[\mathcal{F}^*, x])}$. As a consequence, there exists a finite number of normalised products $\hat{P}^{(1)}, \cdots, \hat{P}^{(s)} \in \Sigma(\mathcal{F}^*)$ such that

$$\mathcal{S}[\mathcal{F}^*, x] = absco(\{x, \hat{P}^{(1)}x, \cdots, \hat{P}^{(s)}x\}), \tag{2.34}$$

so that $\mathcal{S}[\mathcal{F}^*, x]$ is a b.c.p. which is the unit ball of an extremal norm for $\mathcal{F}$.

Using the previous results, it is possible to develop an algorithm for the construction of the unit ball of an extremal complex polytope norm for a finite non-defective family $\mathcal{F} = \{A^{(i)}\}_{i\in\mathcal{I}}$ (see [GZ05]).

ALGORITHM 2.2. *(construction of the unit ball of an extremal complex polytope norm)*

% *Input:* $\mathcal{F} = \{A^{(i)}\}_{i\in\mathcal{I}}$

% *Output:* $\mathcal{S}[\mathcal{F}^*, x]$

Initialisation

Choose an s.m.p. $Q \in \Sigma_k(\mathcal{F})$ *(for some k).*
$\mathcal{F}^* \leftarrow \{\rho(Q)^{-1/k} A^{(i)}\}_{i\in\mathcal{I}}$
Compute the leading eigenvector v_0 *of* Q
$\mathcal{W}^{(0)} \leftarrow \{v^{(0)}\}, \mathcal{V}^{(0)} \leftarrow \{v^{(0)}\}, \mathcal{X}^{(0)} \leftarrow \{v^{(0)}\}$
$\mathcal{P}^{(0)} \leftarrow \text{absco}\left(\mathcal{X}^{(0)}\right)$.

$s \leftarrow 1$
$stop \leftarrow 0$
while $\sim stop$

$\mathcal{V}^{(s)} \leftarrow \mathcal{F}^*\left(\mathcal{X}^{(s-1)}\right)$.

if $\mathcal{V}^{(s)} \subset \mathcal{P}^{(s-1)}$
$\mathcal{S}[\mathcal{F}^*, x] \leftarrow \mathcal{P}^{(s-1)}$
$stop \leftarrow 1$
else
$\mathcal{P}^{(s)} \leftarrow \text{absco}\left(\mathcal{W}^{(s-1)} \cup \mathcal{V}^{(s)}\right)$
Compute an essential system of vertexes $\mathcal{W}^{(s)}$ *of* $\mathcal{P}^{(s)}$
$\mathcal{X}^{(s)} \leftarrow \mathcal{V}^{(s)} \cap \mathcal{W}^{(s)}$
$s \leftarrow s + 1$
end
end

Remark that, if the algorithm halts in a finite number of steps (i.e., for a finite s), then the set $\mathcal{S}[\mathcal{F}^*, x]$ is a polytope. Moreover, if $\mathcal{X}^{(s)}$ is such that span$(\mathcal{X}^{(s)}) = \mathbb{C}^n$, then $\mathcal{S}[\mathcal{F}^*, x]$ generates an extremal complex polytope norm.

Let us give some comments on how the algorithm works. In the initialization step we choose a candidate s.m.p. Q for $\mathcal{F}$ in order to find a leading eigenvector $v^{(0)}$ for $\mathcal{F}$ which we need to fulfil the assumptions of Theorem 2.9. Then, the algorithm computes a trajectory obtained by applying recursively the scaled family $\mathcal{F}^*$ to $v^{(0)}$, that is, in the s-th step, $s \geq 1$, the trajectory is

$$\mathcal{T}^{(s)}[\mathcal{F}^*, v^{(0)}] = \mathcal{T}^{(s-1)}[\mathcal{F}^*, v^{(0)}] \cup \{P_s v^{(0)} | P_s \in \Sigma_s(\mathcal{F}^*)\},$$

where $\mathcal{T}^{(0)}[\mathcal{F}^*, v^{(0)}] = \{v^{(0)}\}$. Clearly, the whole trajectory $\mathcal{T}[\mathcal{F}^*, v^{(0)}]$ is obtained as a limit for $s \to \infty$. Note also that, since we start from one vector $v^{(0)}$ and the family $\mathcal{F}^*$ is finite, then the set $\mathcal{T}^{(s)}[\mathcal{F}^*, v^{(0)}]$ contains a finite number of points. Such trajectories $\mathcal{T}^{(s)}[\mathcal{F}^*, v^{(0)}]$ are computed by the algorithm until it finds a positive integer s such that $\mathcal{T}^{(s+1)}[\mathcal{F}^*, v^{(0)}] \subset \mathcal{P}^{(s)} = \text{absco}(\mathcal{T}^{(s)}[\mathcal{F}^*, v^{(0)}])$, that is, until it finds a minimal set of points of the trajectory whose absolutely convex hull contains all its remaining points. Clearly, if $\mathcal{T}^{(s+1)}[\mathcal{F}^*, v^{(0)}] \subset \mathcal{P}^{(s)}$, all the subsequent sets $\mathcal{T}^{(s+k)}[\mathcal{F}^*, v^{(0)}]$, $k \geq 1$ are contained in $\mathcal{P}^{(s)}$, that is $P^{(s)}$ is an absolutely convex hull for the trajectory $\mathcal{T}[\mathcal{F}^*, v^{(0)}]$. Moreover, since it is generated by a finite number of points, it is a b.c.p..

Now, we apply Algorithm 2.2 to a two-dimensional model problem, using a Matlab implementation of it.

Example 2.3. Consider again the family used to disprove the finiteness conjecture, that is,

$$\mathcal{F} = \left\{ A^{(1)} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, A^{(2)} = b \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\},$$

where, to be consistent with [GZ05], we choose $b = 0.9$. Computational investigations as well as theoretical statements lead to choose $Q = A^{(1)}A^{(2)} \in \Sigma_2(\mathcal{F})$ as candidate s.m.p. of the family. We easily find that the corresponding eigenpair, used both for the construction of the scaled family and as a starting point in the algorithm, is

$$\begin{cases} \lambda & = & \beta^2 b/4 \\ v^{(0)} & = & [\beta/2 \ 1]^T, \end{cases}$$

where $\beta = 1 + \sqrt{5}$. So the scaled family is $\mathcal{F}^* = \{\lambda^{-1/2}\mathcal{F}\}$ and $v^{(0)}$ is the starting leading eigenvector of the algorithm. The results are summarised step by step in Figures 2.5 and 2.6 and in Tables 2.2 and 2.3.

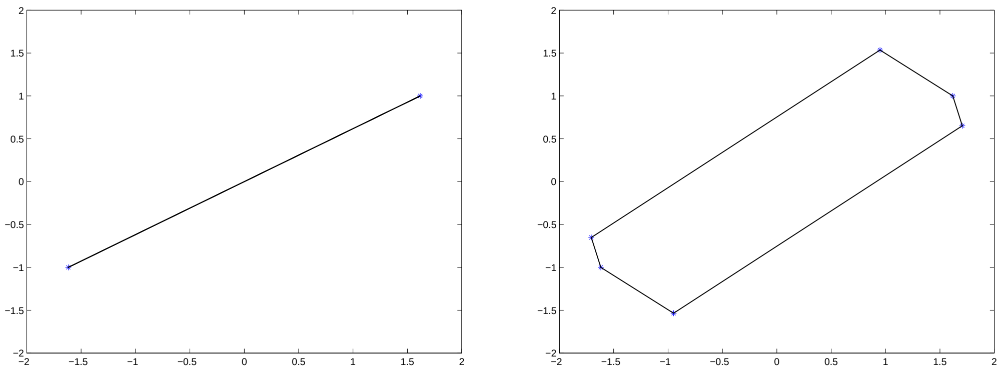

Figure 2.5: At the end of the the first step we obtain $\mathcal{P}^{(0)} = \text{absco}(\{v^{(0)}\})$ (on the left) and at the end of the second step $\mathcal{P}^{(1)} = \text{absco}(\{v^{(0)}, v^{(1)}, v^{(2)}\})$ (on the right).

Since the vectors generated by the algorithm in the fourth step are inside $\mathcal{P}^{(2)}$, then the algorithm stops. Hence, $\mathcal{P}^{(2)}$ is a (real) polytope inducing an extremal norm for $\mathcal{F}$. $\diamond$

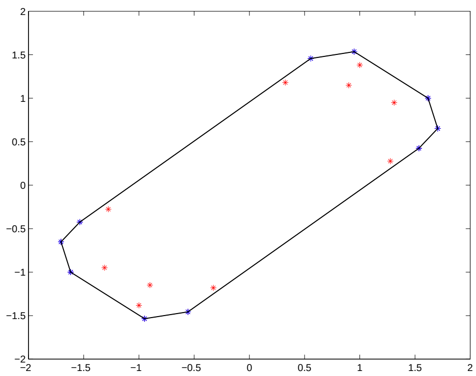

Figure 2.6: At the end of the third step we obtain the b.c.p. $\mathcal{P}^{(2)} = \text{absco}(\{v^{(0)}, v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}\})$. The vertexes of $\mathcal{P}^{(2)}$ are plotted with blue stars, while the remaining set of vectors, computed by the algorithm, is plotted with red stars.

step	essential system of vertexes
1	$v^{(0)}$
2	$v^{(0)}, v^{(1)}, v^{(2)}$
3	$v^{(0)}, v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}$

Table 2.2: Essential systems of vertexes of the b.c.p. generated at each iteration.

	$v^{(0)}$	$v^{(1)}$	$v^{(2)}$	$v^{(3)}$	$v^{(4)}$
x	1.5355	1.7056	1.6180	0.9487	0.5562
y	0.4244	0.6515	1.0000	1.5350	1.4562

Table 2.3: The coordinates x and y (truncated to the fifth digit) of the real points $v^{(k)}$, $k = 0, \cdots, 4$, which form an essential system of vertexes for $\mathcal{P}^{(2)}$.

Chapter 3

The construction of a balanced complex polytope in $\mathbb{C}^2$

In this chapter we present new results, a part of which has been included in [VZ07], concerning the construction of balanced complex polytopes following the ideas of the Beneath-Beyond (B–B) method, which is used to build polytopes in the n-dimensional real space. However, compared with the real case, the complexity of the geometry of a b.c.p. grows much faster with the dimension n (see Theorem 3.1). So we limit ourselves to analyse and construct b.c.p.'s in $\mathbb{C}^2$.

In the first section, we recall the B–B method in $\mathbb{R}^2$, which works in an iterative fashion using the main idea of *limit cone*.

In the second section, we present our original work that extends the iterative approach of the B–B method to the complex case. More precisely, first we describe the boundary of a b.c.p. by analysing some of its geometric features and then we design an algorithm for the actual construction of a b.c.p. However, the proposed algorithm does not extend all the features of the B–B method. In particular, it does not exploit the idea of limit cone. This kind of improvement will be made in the next chapter.

In the third section, we use the geometric features of the b.c.p. in order to give a new algorithm for the computation of the corresponding complex polytope norm.

3.1 The construction of symmetric real polytopes

The purpose of this section is to propose an algorithm for the computation of the symmetric real polytope associated with a set of points in the real plane. As will soon be clear, this algorithm will be inspired by the well known Beneath–Beyond method (see [Ede87]), which is used to construct a real polytope $\mathcal{P} = \text{co}(\mathcal{X})$, which is the convex hull of a set of m real points $\mathcal{X} = \{x^{(1)}, \dots, x^{(m)}\}$, $x^{(k)} \in \mathbb{R}^2$, $k = 1, \cdots, m$. In particular, we consider the special case of the construction of a *symmetric* real polytope $\mathcal{P} = \text{absco}(\mathcal{X})$, which is the absolutely convex hull of $\mathcal{X}$ (i.e. if $x \in \mathcal{P}$, then also $-x \in \mathcal{P}$).

As is well known, the Beneath-Beyond algorithm works in an iterative fashion, starting from a two vertex polytope and then by adding one of the remaining points at a time. Following this idea, we assume, without loss of generality, that the first two points $x^{(1)}, x^{(2)}$ of $\mathcal{X}$ are linearly independent and, starting from $\mathcal{P}^{(2)} = \text{absco}(\{x^{(1)}, x^{(2)}\})$, we construct $\mathcal{P}^{(k)} = \text{absco}(\mathcal{P}^{(k-1)} \cup \{x^{(k)}\})$, for $3 \leq k \leq m$, by adding $x^{(k)}$ to $\mathcal{P}^{(k-1)}$.

Let us give more details on how the algorithm works. We assume that the vertexes of each absolutely convex hull are ordered counterclockwise. This ordering is made to have a representation of the facets of $\mathcal{P}^{(k)}$, $k \geq 2$, and it follows in a natural manner by the construction procedure of the absolutely convex hull, as will soon be clear.

First of all, we construct $\mathcal{P}^{(2)}$, see Figure 3.1, by ordering the set of all its vertexes $\{x^{(1)}, x^{(2)}, -x^{(1)}, -x^{(2)}\}$, observing that, for example, $x^{(2)}$ follows $x^{(1)}$ in the counterclockwise direction if and only if the third component of the outer product $x^{(1)} \times x^{(2)}$ is positive, that is $det([x^{(1)}, x^{(2)}]) = x_1^{(1)} x_2^{(2)} - x_2^{(1)} x_2^{(2)} > 0$. Next, the addition of $x^{(k)}$ to $\mathcal{P}^{(k-1)}$, $3 \leq k \leq m$, is made performing the following steps.

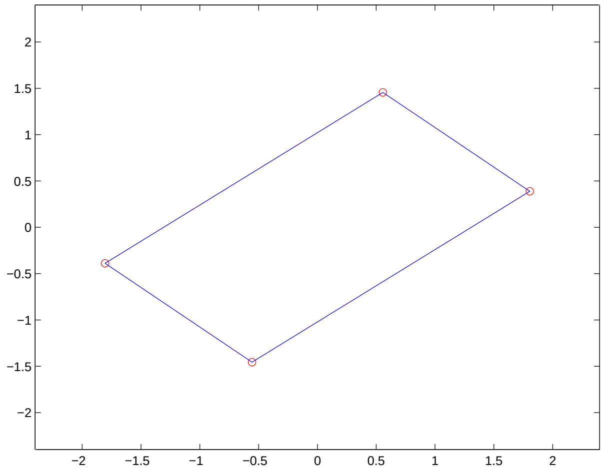

Figure 3.1: $\mathcal{P}^{(2)} = \text{absco}(\{x^{(1)}, x^{(2)}\})$, where $x^{(1)}, x^{(2)} \in \mathbb{R}^2$ are linearly independent.

First of all, we must determine the facet $F_{ij} : x^{(i)} \bullet\!\!-\!\!\bullet\ x^{(j)}$ of $\mathcal{P}^{(k-1)}$ where $x^{(k)}$ *projects*, which is the facet intersected by the line $\lambda x^{(k)}$ for $\lambda > 0$. In other words, if we write $x^{(k)} = \lambda_i x^{(i)} + \lambda_j x^{(j)}$, then $x^{(k)}$ projects on F_{ij} if and only if both λ_i and λ_j are non-negative real numbers.

In the next step, we check if the point $x^{(k)}$ is inside or outside of $\mathcal{P}^{(k-1)}$. Note that this step is not performed by the B-B method which, presorting the points $\mathcal{X}$ with respect to the first coordinate direction, guarantees that the added point $x^{(k)}$ surely lies outside of the current convex hull $\mathcal{P}^{(k-1)}$. Therefore, since we do not perform this presorting, assuming that $x^{(k)}$ projects on F, we have that $x^{(k)}$ is inside $\mathcal{P}^{(k-1)}$ if and only if it is inside the triangle $T_{ij} = \text{co}(O, x^{(i)}, x^{(j)})$, where O is the origin. In other words, if we write $x^{(k)} = \lambda_i x^{(i)} + \lambda_j x^{(j)}$, then $x^{(k)} \in T_{ij}$ if and only if $\lambda_i + \lambda_j \leq 1$.

So, if $x^{(k)}$ is inside $\mathcal{P}^{(k-1)}$, we delete it from $\mathcal{X}$ and we go on to add the next point. Otherwise, in the case that $x^{(k)} \notin \mathcal{P}^{(k-1)}$, the algorithm checks if some of the vertexes of $\mathcal{P}^{(k-1)}$ have to be removed following the addition of the new point. This step uses the *limit cone* which is defined as the cone of apex $x^{(k)}$ tangent to $\mathcal{P}^{(k-1)}$. In Figure 3.2 we show two examples of limit cones and an intuitive idea of how the limit cone is used in the search of the vertexes to be removed. This cone is tangent in two vertexes of $\mathcal{P}^{(k-1)}$, which are called the clockwise (cw) and the counterclockwise (ccw) vertex of the *limit cone* and denoted by $x^{(cw)}$ and $x^{(ccw)}$ respectively; to find these two vertexes we start from the facet F_{ij} where the added point projects and we proceed first with a cw search and then with a ccw search (see Figure 3.2 on the right). Therefore, the facets inside the limit cone are deleted along with their incidence vertexes (except for $x^{(cw)}$ and $x^{(ccw)}$). Since we are constructing the absolutely convex hull, we also have to add $-x^{(k)}$, whose corresponding limit cone is tangent to $\mathcal{P}^{(k-1)}$ in the vertexes $-x^{(cw)}$ and $-x^{(ccw)}$. Also the vertexes inside this last limit cone (except for $-x^{(cw)}$ and $-x^{(ccw)}$) are deleted. The remaining vertexes of $\mathcal{P}^{(k-1)}$ among with $x^{(k)}$ and $-x^{(k)}$ are the vertexes of $\mathcal{P}^{(k)}$. In order to preserve the counterclockwise ordering, we collect the vertexes of $\mathcal{P}^{(k)}$ as they appear in a counterclockwise walk along the boundary of $\mathcal{P}^{(k)}$. More precisely, we link the following vertexes together: $x^{(k)}$, all the vertexes between $x^{(ccw)}$ and $-x^{(cv)}$, then $-x^{(k)}$ and finally all the vertexes between $-x^{(ccw)}$ and $x^{(cv)}$.
Lastly, we must add the ccw and cw facets of the limit cones which are the two segments connecting $x^{(k)}$ with the vertexes $x^{(ccw)}$ and $x^{(cv)}$ along with the corresponding symmetric segments with respect to the origin (see Figure 3.3).

Remark 3.1. *It is important to note that, even if in this section we have explained the main ideas of the*

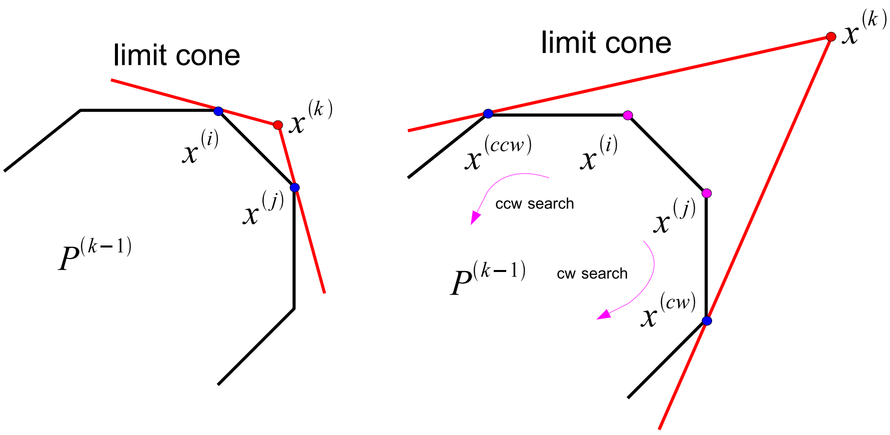

Figure 3.2: The addition of $x^{(k)}$ to $\mathcal{P}^{(k-1)}$. The point projects on the facet $F_{i,j} : x^{(i)} \bullet\!\!-\!\!\bullet\, x^{(j)}$. On the left, none of the existing vertexes have to be removed since the limit cone does not contain any of them (i.e. $x^{(ccw)} = x^{(i)}$ and $x^{(cw)} = x^{(j)}$). On the right, the two magenta vertexes $x^{(i)}$ and $x^{(j)}$ have to be removed since they lie inside the limit cone. These two vertexes no longer belong to the set of vertexes of $\mathcal{P}^{(k)}$.

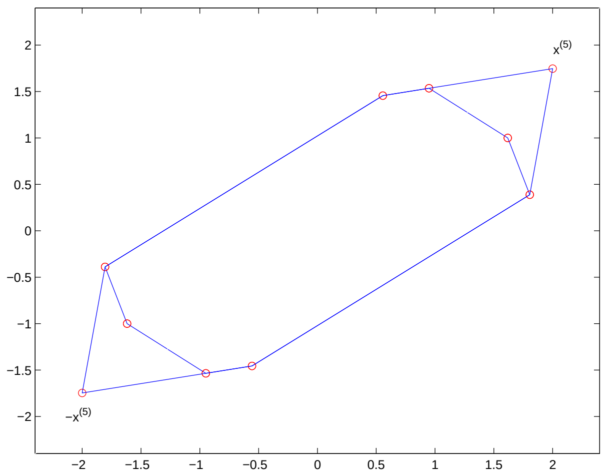

Figure 3.3: Addition of $x^{(5)}$ to $\mathcal{P}^{(4)}$.

Beneath-Beyond method in the real plane, it is feasible to extend this method to higher dimensions. For a detailed treatment of this topic see [Ede87].

3.2 The construction of balanced complex polytopes (b.c.p.)

Given a set $\mathcal{X} = \{x^{(1)}, x^{(2)}, \ldots, x^{(m)}\}$, where $x^{(i)} \in \mathbb{C}^n$, $i = 1, \ldots, m$, our aim is to construct the corresponding absolutely convex hull $\mathcal{P} = \text{absco}(\mathcal{X})$.

As we anticipated in Section 2.2, the boundary of a b.c.p. is a closed piecewise algebraic hypersurface. In particular, we have the following theorem.

Theorem 3.1. *Consider a b.c.p. $\mathcal{P}$ with n vertexes in $\mathbb{C}^n$, that is $\mathcal{P} = \text{absco}(\mathcal{X})$, where $\mathcal{X} = \{x^{(1)}, x^{(2)}, \ldots, x^{(n)}\}$ and $\text{span}(\mathcal{X}) = \mathbb{C}^n$. Then the boundary of $\mathcal{P}$ is a branch of an algebraic hypersurface of order 2^n in $\mathbb{R}^{2n}$ (identified with $\mathbb{C}^n$).*

For the proof of this theorem, we need the following Lemma.

Lemma 3.1. *Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive real variables. Then, the product*

$$P(z; \alpha_1, \alpha_2, \ldots, \alpha_n) = \prod [z - (\pm\sqrt{\alpha_1} \pm \sqrt{\alpha_2}, \ldots, \pm\sqrt{\alpha_n})]$$

of all possible terms of the form $z - (\pm\sqrt{\alpha_1} \pm \sqrt{\alpha_2}, \ldots, \pm\sqrt{\alpha_n})$ is a polynomial in z of degree 2^n whose coefficients are polynomials in the variables $\alpha_1, \alpha_2, \ldots, \alpha_n$ of degree at the most 2^{n-1}.

Proof. The proof is done by using mathematical induction. For $n = 1$, the result holds, since

$$P(z; \alpha_1) = (z - \sqrt{\alpha_1})(z + \sqrt{\alpha_1}) = z^2 - \alpha_1.$$

Now, assume that it holds for $n - 1$ and prove it for n.

Since $P(z; \alpha_1, \alpha_2, \ldots, \alpha_n)$ is the product of 2^n factors, it is homogeneous polynomial of degree 2^n in the variables z, $\sqrt{\alpha_1}$, $\sqrt{\alpha_2}, \ldots, \sqrt{\alpha_n}$. Therefore, we are just left to prove that it does not contain any square root. To this aim, observe that

$$P(z; \alpha_1, \alpha_2, \ldots, \alpha_n) = \prod [z - (\pm\sqrt{\alpha_1} \pm \sqrt{\alpha_2}, \ldots, \pm\sqrt{\alpha_n})] =$$

$$= \prod [z - \sqrt{\alpha_n} - (\pm\sqrt{\alpha_1} \pm \sqrt{\alpha_2}, \ldots, \pm\sqrt{\alpha_{n-1}})] \cdot \prod [z + \sqrt{\alpha_n} - (\pm\sqrt{\alpha_1} \pm \sqrt{\alpha_2}, \ldots, \pm\sqrt{\alpha_{n-1}})] =$$

$$= P(z - \sqrt{\alpha_n}; \alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \cdot P(z + \sqrt{\alpha_n}; \alpha_1, \alpha_2, \ldots, \alpha_{n-1}).$$

By the inductive hypothesis, we have that $P(z - \sqrt{\alpha_n}; \alpha_1, \alpha_2, \ldots, \alpha_{n-1})$ and $P(z + \sqrt{\alpha_n}; \alpha_1, \alpha_2, \ldots, \alpha_{n-1})$ are polynomials in $z - \sqrt{\alpha_n}$ and $z + \sqrt{\alpha_n}$, respectively, whose coefficients are polynomials in the variables $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$. Therefore, it turns out that

$$P(z - \sqrt{\alpha_n}; \alpha_1, \alpha_2, \ldots, \alpha_{n-1}) = q - t\sqrt{\alpha_n},$$

$$P(z + \sqrt{\alpha_n}; \alpha_1, \alpha_2, \ldots, \alpha_{n-1}) = q + t\sqrt{\alpha_n},$$

where $q = q(z; \alpha_1, \alpha_2, \ldots, \alpha_n)$ and $t = t(z; \alpha_1, \alpha_2, \ldots, \alpha_n)$ are polynomials in z and in the variables α_i, $i = 1, \ldots, n$. We can conclude that

$$P(z; \alpha_1, \alpha_2, \ldots, \alpha_n) = q^2 - t^2\alpha_n$$

does not contain any square root, and the proof is complete. ■

Proof of Theorem 3.1 By (2.27), we have that the boundary of $\mathcal{P}$ (in the variable $x \in \mathbb{C}^n$) is

$$\|[x^{(1)} \ldots x^{(n)}]^{-1}x\|_1 = 1.$$

Now, we introduce some notations. We distinguish the real and the imaginary part the vector $x = [x_1, x_2, \cdots, x_n]^T$, that is, we set for each one of its components, $x_k = x_{kR} + \mathrm{i}x_{kI}$, $k = 1, \cdots, n$. Furthermore, we denote

$$[x^{(1)} \dots x^{(n)}]^{-1} = \frac{1}{d} \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1n} \\ \vdots & \ddots & \vdots \\ \gamma_{n1} & \cdots & \gamma_{nn} \end{bmatrix},$$

where $d = \det([x^{(1)} \dots x^{(n)}])$ and $\gamma_{hk} \in \mathbb{C}$, $h, k = 1, \cdots, n$. Thus, we have

$$\|[x^{(1)} \dots x^{(n)}]^{-1} x\|_1 = \frac{1}{|d|} \left\| \begin{bmatrix} \sum_{k=1}^n \gamma_{1k} x_k \\ \vdots \\ \sum_{k=1}^n \gamma_{nk} x_k \end{bmatrix} \right\|_1 = \frac{1}{|d|} \sum_{h=1}^n \left| \sum_{k=1}^n \gamma_{hk} x_k \right| = \frac{1}{|d|} \sum_{h=1}^n f_h(x)$$

where, for $h = 1, \cdots, n$,

$$f_h(x) = \sqrt{\left[\sum_{k=1}^n (Re(\gamma_{hk}) x_{kR} - Im(\gamma_{hk}) x_{kI}) \right]^2 + \left[\sum_{k=1}^n (Re(\gamma_{hk}) x_{kI} + Im(\gamma_{hk}) x_{kR}) \right]^2}$$

is a second degree polynomial in the $2n$ real variables $x_{1R}, \dots, x_{nR}, x_{1I}, \dots, x_{nI}$. So, we can write the boundary of $\mathcal{P}$ as

$$|d| - (f_1(x) + \cdots + f_n(x)) = 0. \tag{3.1}$$

We can eliminate the square roots in the above equation, noting that a polynomial which does not contain any square root may be obtained by multiplying all the possible binomials of the kind

$$|d| - (\pm f_1(x) \pm \cdots \pm f_n(x)),$$

which is (see Lemma 3.1) a polynomial in $|d|$ of degree 2^n whose coefficients are polynomials in the variables $[f_1(x)]^2, [f_2(x)]^2, \dots, [f_n(x)]^2$ of degree at the most 2^{n-1}. Since $[f_1(x)]^2, [f_2(x)]^2, \dots, [f_n(x)]^2$ are polynomials of second degree in the $2n$ real variables $x_{1R}, \dots, x_{nR}, x_{1I}, \dots, x_{nI}$, the proof is complete. ■

Thus, the order of the boundary of an n-vertex b.c.p. in $\mathbb{C}^n$ grows exponentially with the dimension n. This is one of the main reasons for which we limit ourself to the construction of a b.c.p. in $\mathbb{C}^2$. However, we treat the construction of a b.c.p. in $\mathbb{C}^2$ with an arbitrary number of vertexes. More precisely, we analyse some geometric features of the b.c.p.'s, in order to describe their boundary and analyse their facets. Like in the real case, we begin by constructing $\mathcal{P}^{(2)} = \text{absco}(\{x^{(1)}, x^{(2)}\})$ and then we proceed incrementally, adding one of the remaining vertexes at a time. However, in the incremental construction of the b.c.p. $\mathcal{P}$, we distinguish the construction of a b.c.p. with two essential vertexes from the case of a b.c.p. with three essential vertexes and the general case of a b.c.p. with $m \geq 4$ essential vertexes, because these three cases are substantially different from one another. In any case, since we are in $\mathbb{C}^2$, Remark 2.1 reveals that we have only two types of faces: vertexes (of dimension 0) and facets (of dimension 1).

3.2.1 Two-vertex b.c.p.'s

In this section we describe the boundary of a two vertex b.c.p. $\mathcal{P} = \text{absco}(\{x^{(1)}, x^{(2)}\}) \subseteq \mathbb{C}^2$, where $x^{(1)} = [x_1^{(1)}\, x_2^{(1)}]^T$ and $x^{(2)} = [x_1^{(2)}\, x_2^{(2)}]^T$ are linearly independent. By Theorem 2.2, the (geometric) facets of $\mathcal{P}$ are all the possible *segments* of the type

$$F_{y(\theta_1,\theta_2)} = \text{co}\left(\{e^{i\theta_1} x^{(1)}, e^{i\theta_2} x^{(2)}\}\right)$$

denoted also by

$$e^{i\theta_1} x^{(1)} \bullet\!\!-\!\!\bullet\, e^{i\theta_2} x^{(2)},$$

for $\theta_1, \theta_2 \in (-\pi, \pi]$, where $y(\theta_1, \theta_2)$ is the associated functional (see Definition 2.4) determined by

$$\begin{cases} \langle y(\theta_1, \theta_2), e^{i\theta_1} x^{(1)} \rangle = 1, \\ \langle y(\theta_1, \theta_2), e^{i\theta_2} x^{(2)} \rangle = 1, \end{cases}$$

that is,

$$y(\theta_1, \theta_2) = ([x^{(1)}\; x^{(2)}]^*)^{-1} \begin{bmatrix} e^{i\theta_1} \\ e^{i\theta_2} \end{bmatrix}. \tag{3.2}$$

Therefore, in view of formula (2.10), we can conclude that the parametric equation of $\partial\mathcal{P}$ is given by

$$x = \lambda\, e^{i\theta_1} x^{(1)} + (1-\lambda)\, e^{i\theta_2} x^{(2)}, \tag{3.3}$$

where $\theta_1, \theta_2 \in (-\pi, \pi]$ and $\lambda \in [0, 1]$.

On the other hand, the boundary of $\mathcal{P}$ is the set of vectors $x \in \mathbb{C}^2$ such that $\|x\|_{\mathcal{P}} = 1$. Thus, by (2.27), it is given by the following equation in the variable $x = [x_1\; x_2]^T \in \mathbb{C}^2$:

$$\|[x^{(1)} x^{(2)}]^{-1} x\|_1 = 1.$$

Thus, Theorem 3.1 and formula (3.1) imply that $\partial\mathcal{P}$ is a branch of an algebraic hypersurface of order 4 in $\mathbb{R}^4$ (identified with $\mathbb{C}^2$) and its equation is

$$f_1(x) + f_2(x) = d, \tag{3.4}$$

where $f_1(x) = |x_1 x_2^{(1)} - x_2 x_1^{(1)}|$, $f_2(x) = |x_1 x_2^{(2)} - x_2 x_1^{(2)}|$, $d = |x_2^{(2)} x_1^{(1)} - x_2^{(1)} x_1^{(2)}|$.

Remark that, if $x^{(1)}, x^{(2)}, x \in \mathbb{R}^2$, the equation (3.4) represents the boundary of the parallelogram of vertexes $\{x^{(1)}, x^{(2)}, -x^{(1)}, -x^{(2)}\}$. Of course, for $\theta_1 = \theta_2 = 0$, also the parametric equation (3.3) represents the same parallelogram.

As is made in the proof of Theorem 3.1, we can eliminate the square roots in (3.4) by considering the equation

$$(f_1(x) + f_2(x) - d)(f_1(x) - f_2(x) - d)(f_2(x) - f_1(x) - d)(-f_1(x) - f_2(x) - d) = 0. \tag{3.5}$$

Then, by developing the left hand side, we find the fourth order algebraic equation

$$(f_1(x)^2 - f_2(x)^2)^2 - 2d^2(f_1(x)^2 + f_2(x)^2) + d^4 = 0. \tag{3.6}$$

Remark that equation (3.6) may be obtained also by squaring two times equation (3.4), and that equation (3.4) is equivalent to equation (3.6) with the constraint

$$f_1(x)^2 + f_2(x)^2 \le d^2. \tag{3.7}$$

If we identify $\mathbb{C}^2$ with the real space $\mathbb{R}^4$, the equation (3.6) and the constraint (3.7) represent a fourth order and a second order algebraic hypersurface in $\mathbb{R}^4$, respectively. The set of the singular points of the hypersurface (3.6) is given by the vertexes of $\mathcal{P}$.

Clearly, if $x_1, x_2 \in \mathbb{R}$, then the equation (3.6) with the constraint (3.7) represents the intersection of $\partial\mathcal{P}$ with $\mathbb{R}^2$, which is equal to $\partial(\mathcal{P} \cap \mathbb{R}^2)$.

Example 3.1. Let $x^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix}$, $x^{(2)} = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$, $\mathcal{P} = \text{absco}(\{x^{(1)}, x^{(2)}\})$ and $\|\cdot\|_{\mathcal{P}}$ be the corresponding norm.

We want to compute $\partial\mathcal{P} \cap \mathbb{R}^2$. To this purpose, by the previous results we have that

$$\begin{aligned} d^2 &= |x_2^{(2)} x_1^{(1)} - x_2^{(1)} x_1^{(2)}|^2 = 5, \\ f_1(x)^2 &= x_1{}^2 + x_2{}^2, \\ f_2(x)^2 &= 2x_1{}^2 + x_2{}^2 - 2x_1 x_2. \end{aligned}$$

So, by (3.6) and (3.7) we obtain the following fourth order curve in $\mathbb{R}^2$ (plotted in black)

$$(x_1{}^2 - 2x_1x_2)^2 - 10(3x_1{}^2 + 2x_2{}^2 - 2x_1x_2) + 25 = 0$$

with the constraint (plotted in red)

$$3x_1{}^2 + 2x_2{}^2 - 2x_1x_2 \leq 5.$$

Therefore $\partial\mathcal{P} \cap \mathbb{R}^2$ is given by the solid closed curve located inside the dashed closed curve in Figure 3.4. $\diamond$

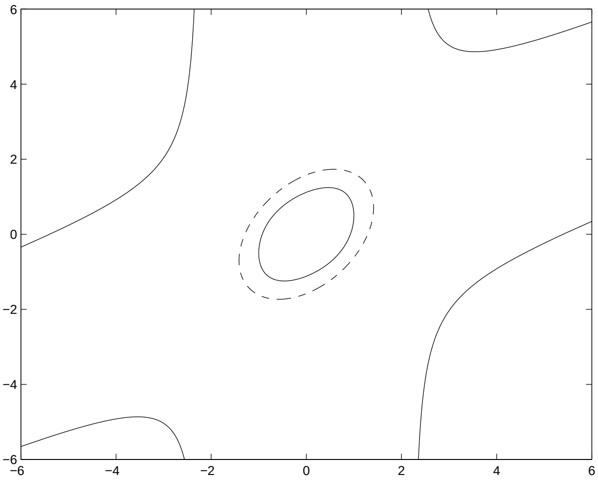

Figure 3.4: Intersection of $\partial\mathcal{P}$ with $\mathbb{R}^2$ in Example 3.1

Now observe that, if $\mathcal{P} = \overline{\mathcal{P}}$ is self-conjugated with $Re(x^{(2)}) = Re(x^{(1)})$ and $Im(x^{(2)}) = -Im(x^{(1)}) \neq 0$, then

$$f_1(x) = f_2(x) \quad \text{for all } x \in \mathbb{R}^2.$$

Consequently, for $x \in \mathbb{R}^2$ the equation (3.4) is equivalent to the second order algebraic equation

$$f_1(x)^2 = \frac{1}{4}d^2, \tag{3.8}$$

which represents an ellipsis, whereas the constraint (3.7) becomes

$$f_1(x)^2 \leq \frac{1}{2}d^2$$

and is automatically satisfied. Remark that, in this case, we have $Re(\mathcal{P}) = \mathcal{P} \cap \mathbb{R}^2$ and that the result is consistent with Proposition 2.6.

Moreover, in this case all the facets are real (modulo scalar factors of unitary modulus). In fact, denoting for simplicity of notation

$$v = x^{(1)} \quad \text{and} \quad \overline{v} = x^{(2)} = \overline{x^{(1)}}, \tag{3.9}$$

the functional $y(\theta_1, \theta_2)$ associated with the general facet

$$F_{y(\theta_1,\theta_2)} = \mathrm{e}^{\mathrm{i}\theta_1}v \bullet\!\!-\!\!\bullet\, \mathrm{e}^{\mathrm{i}\theta_2}\overline{v}$$

is

$$y(\theta_1, \theta_2) = ([v\, \overline{v}]^*)^{-1} \begin{bmatrix} \mathrm{e}^{\mathrm{i}\theta_1} \\ \mathrm{e}^{\mathrm{i}\theta_2} \end{bmatrix} = \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\theta_1}\left(v_1 - \mathrm{e}^{\mathrm{i}(\theta_2-\theta_1)}\overline{v_1}\right)}{2Im\left(\overline{v_1}v_2\right)} \begin{bmatrix} \frac{2\Re\left(\overline{v_2}\left(v_1 - \mathrm{e}^{\mathrm{i}(\theta_2-\theta_1)}\overline{v_1}\right)\right)}{\left|v_1 - \mathrm{e}^{\mathrm{i}(\theta_2-\theta_1)}\overline{v_1}\right|^2} \\ 1 \end{bmatrix},$$

which is proportional to a real vector. Therefore, apart from a scalar factor of unitary modulus, all the facets are given by the set of vectors

$$\mathcal{F} = \{y(\theta)\}_{-\pi<\theta\leq\pi} \subset \mathbb{R}^2,$$

where

$$y(\theta) = \frac{1}{2Im\,(\overline{v_1}v_2)} \begin{bmatrix} \frac{2\Re\left(\overline{v_2}(v_1 - e^{i\theta}\overline{v_1})\right)}{\left|v_1 - e^{i\theta}\overline{v_1}\right|} \\ \left|v_1 - e^{i\theta}\overline{v_1}\right| \end{bmatrix}. \tag{3.10}$$

Observe that, for $\theta = \hat{\theta}$ such that $v_1 - e^{i\hat{\theta}}\overline{v_1} = 0$, formula (3.10) is not well-defined. However, in this case, we have to consider the limit as $\theta \to \hat{\theta}$.

By (2.11), we can conclude that

$$\mathcal{P} = \text{adj}(\mathcal{F})$$

and that

$$\mathcal{P} \cap \mathbb{R}^2 = \text{adj}_{\mathbb{R}}(\mathcal{F}), \tag{3.11}$$

where, for a set $\mathcal{Y} \subset \mathbb{R}^2$, we define

$$\text{adj}_{\mathbb{R}}(\mathcal{Y}) = \left\{x \in \mathbb{R}^2 \;\middle|\; |\langle y, x\rangle| \leq 1 \text{ for all } y \in \mathcal{Y}\right\}$$

as the *real adjoint* of $\mathcal{Y}$.

Remark that the real straight lines of equations

$$\langle \pm y(\theta), x\rangle = 1, \quad y(\theta) \in \mathcal{F},$$

form the family of the straight lines that are tangent to the ellipsis $\partial\mathcal{P} \cap \mathbb{R}^2$.

3.2.2 Three-vertex b.c.p.'s

In this subsection we examine the addition of the third point $x^{(3)}$ to the already available two-vertex b.c.p. $\mathcal{P}^{(2)} = \text{absco}(\{x^{(1)}, x^{(2)}\}) \subseteq \mathbb{C}^2$ when $x^{(3)} \notin \mathcal{P}^{(2)}$ in order to construct $\mathcal{P}^{(3)} = \text{absco}(\{x^{(1)}, x^{(2)}, x^{(3)}\})$. A precise description of this addition operation requires us to analyse several possibilities, each one related to a different reciprocal position between $x^{(3)}$ and $\mathcal{P}^{(2)}$. From a mathematical point of view, these possibilities are easily related to the two complex numbers $\lambda_1, \lambda_2 \in \mathbb{C}$ which are used to express $x^{(3)}$ as a linear combination of $x^{(1)}$ and $x^{(2)}$, i.e.,

$$x^{(3)} = \lambda_1 x^{(1)} + \lambda_2 x^{(2)}.$$

First of all, we observe that the added point $x^{(3)}$ may delete one, but not both, of the previously existing vertexes of $\mathcal{P}^{(2)}$. More precisely, we have

- $x^{(1)} \in \text{absco}(\{x^{(2)}, x^{(3)}\})$ if and only if $|\lambda_1| \geq |\lambda_2| + 1$,
- $x^{(2)} \in \text{absco}(\{x^{(1)}, x^{(3)}\})$ if and only if $|\lambda_2| \geq |\lambda_1| + 1$.

In the former case $x^{(3)}$ deletes $x^{(1)}$, whereas in the latter case it deletes $x^{(2)}$. Thus, in both cases we are left with a b.c.p. with only two vertexes, one of which is $x^{(3)}$ and the other is $x^{(2)}$ in the first case and $x^{(1)}$ in the second case.

On the other hand, the point $x^{(3)}$ may not delete any of the previously existing vertexes, which is equivalent to the condition

$$\big||\lambda_1| - |\lambda_2|\big| < 1. \tag{3.12}$$

So, in this case, we actually construct a three-vertex b.c.p. $\mathcal{P}^{(3)}$ which has, as an essential system of vertexes, the three vertexes $x^{(1)}$, $x^{(2)}$, $x^{(3)}$. Now, in order to compute and describe its boundary, we perform some steps.

The first step is to determine the facets of $\mathcal{P}^{(2)}$ that have to be deleted. We recall that all the facets of $\mathcal{P}^{(2)}$ are of the kind $F_{y_{12}(\theta)} = x^{(1)} \bullet\!\!-\!\!\bullet\, e^{i\theta}x^{(2)}$, where $y_{12}(\theta)$ is the associated functional

$$y_{12}(\theta) = [x^{(1)}\, x^{(2)}]^{-H} \begin{bmatrix} 1 \\ e^{i\theta} \end{bmatrix}.$$

Then, using Theorem 2.1, the facet $F_{y_{12}(\theta)}$ is deleted if and only if it is *seen* from the *circle* generated by $x^{(3)}$, that is, if and only if $|\langle y_{12}(\theta), x^{(3)}\rangle| > 1$. Therefore, we have to find the values of $\theta \in (-\pi, \pi]$ such that

$$\left| ([x^{(1)}\, x^{(2)}]^{-1} x^{(3)})^H \begin{bmatrix} 1 \\ e^{i\theta} \end{bmatrix} \right|^2 > 1$$

or, equivalently,

$$\left| |\lambda_1| e^{-i\arg(\lambda_1)} + |\lambda_2| e^{i(\theta - \arg(\lambda_2))} \right|^2 > 1.$$

This inequality is satisfied if and only if

$$\cos(\theta - \arg(\lambda_2) + \arg(\lambda_1)) > \frac{1 - |\lambda_1|^2 - |\lambda_2|^2}{2|\lambda_1||\lambda_2|}. \tag{3.13}$$

Now, we note that, since $x^{(3)} \notin \mathcal{P}_{12}$, by (2.27) we have

$$\|[x^{(1)}\, x^{(2)}]^{-1} x^{(3)}\|_1 = |\lambda_1| + |\lambda_2| > 1, \tag{3.14}$$

and therefore, using (3.12), we obtain

$$-1 < \frac{1 - |\lambda_1|^2 - |\lambda_2|^2}{2|\lambda_1||\lambda_2|} < 1. \tag{3.15}$$

Thus, the right hand side of (3.13) is a feasible value for the cosine function and we can reverse it obtaining the *deleting interval* $\mathcal{D}_{12}^{(3)}$ generated by $x^{(3)}$ defined as

$$\begin{cases} \mathcal{D}_{12}^{(3)} = \left(-\hat\theta_{12} + \arg(\lambda_2) - \arg(\lambda_1),\, \hat\theta_{12} + \arg(\lambda_2) - \arg(\lambda_1)\right) \quad \text{shift } (-\pi, \pi], \\ \hat\theta_{12} = \arccos\left(\frac{1 - |\lambda_1|^2 - |\lambda_2|^2}{2|\lambda_1||\lambda_2|}\right), \end{cases} \tag{3.16}$$

where we have chosen the principal determination in $[0, \pi]$ for the function arccos. Observe also that, since in (3.15) we have two strict inequalities, $0 < \hat\theta_{12} < \pi$ holds. The notation "shift$(-\pi, \pi]$" used in (3.16) means that the angles are shifted to the reference interval $(-\pi, \pi]$ by $2\pi-$periodicity. This may split the original interval in the union of two intervals. For example, if we consider as the original interval $(\frac{\pi}{2}, \frac{3\pi}{2})$, then we have $(\frac{\pi}{2}, \frac{3\pi}{2})\text{shift}(-\pi, \pi] = (-\pi, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$. Therefore, the facets of $\mathcal{P}^{(2)}$ of the kind $x^{(1)} \bullet\!\!-\!\!\bullet\, e^{i\theta}x^{(2)}$ survive for

$$\theta \in \mathcal{E}_{12}^{(3)} = (-\pi, \pi] \setminus \mathcal{D}_{12}^{(3)},$$

and these are the facets $F_{y_{12}(\theta)}$ of $\mathcal{P}^{(3)}$.

The second step is to find the facets of the kind $F_{y_{13}(\theta)} = x^{(1)} \bullet\!\!-\!\!\bullet\, e^{i\theta}x^{(3)}$ and $F_{y_{23}(\theta)} = x^{(2)} \bullet\!\!-\!\!\bullet\, e^{i\theta}x^{(3)}$ to add. For the first one the associated functional is

$$y_{13}(\theta) = [x^{(1)}\, x^{(3)}]^{-H} \begin{bmatrix} 1 \\ e^{i\theta} \end{bmatrix}. \tag{3.17}$$

Using the equality $\mathcal{P} = (\mathcal{P}^*)^*$ (see Theorem 2.1), we have that the facets to add are those for which

$$\left|\langle y_{13}(\theta), x^{(2)}\rangle\right| \leq 1. \tag{3.18}$$

Therefore, since (3.12) and (3.14) imply

$$-1 < \frac{1+|\lambda_1|^2-|\lambda_2|^2}{2|\lambda_1|} < 1,$$

it turns out that the facets of the kind $F_{y_{13}(\theta)}$ are added for

$$\begin{cases} \theta \in \mathcal{E}_{13}^{(3)} = \left[-\hat{\theta}_{13} - \arg(\lambda_1), \hat{\theta}_{13} - \arg(\lambda_1)\right] \quad \text{shift}(-\pi,\pi] \\ \hat{\theta}_{13} = \arccos\left(\frac{1+|\lambda_1|^2-|\lambda_2|^2}{2|\lambda_1|}\right) \in (0,\pi) \end{cases}. \tag{3.19}$$

In the same way, since (3.12) and (3.14) imply also

$$-1 < \frac{1-|\lambda_1|^2+|\lambda_2|^2}{2|\lambda_2|} < 1,$$

we find that the facets of the kind $F_{y_{23}(\theta)}$ to add are those such that

$$\begin{cases} \theta \in \mathcal{E}_{23}^{(3)} = \left[-\hat{\theta}_{23} - \arg(\lambda_2), \hat{\theta}_{23} - \arg(\lambda_2)\right] \quad \text{shift}(-\pi,\pi] \\ \hat{\theta}_{23} = \arccos\left(\frac{1-|\lambda_1|^2+|\lambda_2|^2}{2|\lambda_2|}\right) \in (0,\pi) \end{cases}. \tag{3.20}$$

Remark 3.2. *The facets $x^{(i)}$ ●—● $e^{i\theta}x^{(j)}$ of the b.c.p. $\mathcal{P}^{(3)}$ exist for θ which varies in the union of, at the most, two intervals. This is due to one or both of the following facts: the first is due to the shift of the original interval into $(-\pi,\pi]$ and the second is due to the set difference used to compute the the facets of $\mathcal{P}^{(3)}$ which remain after the removal from $\mathcal{P}^{(2)}$ of the facets which are* seen *from the added point $x^{(3)}$.*

For further generalisation to come in Section 3.2.3, we introduce some notations and considerations in the following remark.

Remark 3.3. *We denote by $\mathcal{E}_{12}^{(2)} = (-\pi,\pi]$ the* existence interval *of the facets $F_{y_{12}(\theta)}$ of $\mathcal{P}^{(2)}$ (i.e., the facets $F_{y_{12}(\theta)}$ exist for $\theta \in \mathcal{E}_{12}^{(2)}$). Therefore, the* existence interval *of the facets $F_{y_{12}(\theta)}$ of $\mathcal{P}^{(3)}$ is given by*

$$\mathcal{E}_{12}^{(3)} = \mathcal{E}_{12}^{(2)} \backslash \mathcal{D}_{12}^{(3)}.$$

Note that the interval $\mathcal{E}_{12}^{(3)}$ is of the form $[\theta_{12}^-, \theta_{12}^+]$ or $(-\pi, \theta_{12}^+] \cup [\theta_{12}^-, \pi]$, where

$$\begin{aligned} \theta_{12}^- &= \hat{\theta}_{12} + \arg(\lambda_2) - \arg(\lambda_1) \qquad && \text{shift}(-\pi,\pi] \\ \theta_{12}^+ &= -\hat{\theta}_{12} + \arg(\lambda_2) - \arg(\lambda_1) \qquad && \text{shift}(-\pi,\pi]. \end{aligned} \tag{3.21}$$

Analogously, for $i = 1, 2$, we call $\mathcal{E}_{i3}^{(3)}$ the existence intervals *of the facets $F_{y_{i3}(\theta)}$ of $\mathcal{P}^{(3)}$ and we note that these intervals are of the form $[\theta_{i3}^-, \theta_{i3}^+]$ or $(-\pi, \theta_{i3}^+] \cup [\theta_{i3}^-, \pi]$, where*

$$\begin{aligned} \theta_{i3}^- &= -\hat{\theta}_{i3} - \arg(\lambda_i) \qquad && \text{shift}(-\pi,\pi] \\ \theta_{i3}^+ &= \hat{\theta}_{i3} - \arg(\lambda_i) \qquad && \text{shift}(-\pi,\pi]. \end{aligned} \tag{3.22}$$

The angles $\hat{\theta}_{ij}$, $1 \leq i \leq j \leq 3$, and so also the extremal values of the existence intervals $\theta_{ij}^{\pm}$, are related to one another as stated by the following lemma.

Lemma 3.2. *The equality* $\hat\theta_{12} = \hat\theta_{13} + \hat\theta_{23}$*, that is* $\theta^{\pm}_{12} + \theta^{\pm}_{23} = \theta^{\mp}_{13} \mod 2\pi$*, is verified.*

Proof. By (3.16), (3.19) and (3.20) and related discussions, we have that

$$\cos\hat\theta_{12} = \frac{1-|\lambda_1|^2-|\lambda_2|^2}{2|\lambda_1||\lambda_2|},$$
$$\cos\hat\theta_{13} = \frac{1+|\lambda_1|^2-|\lambda_2|^2}{2|\lambda_1|},$$
$$\cos\hat\theta_{23} = \frac{1-|\lambda_1|^2+|\lambda_2|^2}{2|\lambda_2|}.$$

Moreover, since $0 < \hat\theta_{13}, \hat\theta_{23} < \pi$, it holds that $\sin\hat\theta_{13} > 0$ and $\sin\hat\theta_{23} > 0$. For these reasons we have

$$\cos\hat\theta_{13}\cos\hat\theta_{23} = \frac{(1-|\lambda_1|^2-|\lambda_2|^2)(1-|\lambda_1|^2+|\lambda_2|^2)}{4|\lambda_1||\lambda_2|}$$

and

$$\sin\hat\theta_{13}\sin\hat\theta_{23} = \sqrt{\left(1-\frac{\left(1+|\lambda_1|^2-|\lambda_2|^2\right)^2}{4|\lambda_1|^2}\right)\left(1-\frac{\left(1-|\lambda_1|^2+|\lambda_2|^2\right)^2}{4|\lambda_2|^2}\right)}.$$

Therefore, by simple calculations we can conclude that

$$\cos(\hat\theta_{13}+\hat\theta_{23}) = \frac{1-|\lambda_1|^2-|\lambda_2|^2}{2|\lambda_1||\lambda_2|} = \cos\hat\theta_{12},$$

which implies

$$\hat\theta_{12} = \hat\theta_{13} + \hat\theta_{23} \tag{3.23}$$

if $\hat\theta_{13} + \hat\theta_{23} < \pi$ and $\hat\theta_{13} + \hat\theta_{23} = 2\pi - \hat\theta_{12}$ if $\hat\theta_{13} + \hat\theta_{23} > \pi$. If we suppose that the latter case holds, then we can move $x^{(3)}$ in $x^{(3)'}$ in such a way that $\hat\theta'_{13} + \hat\theta'_{23} < \pi$. If we write $x^{(3)}$ and $x^{(3)'}$ as a linear combination with complex coefficients of $x^{(1)}$ and $x^{(2)}$, that is, $x^{(3)} = \lambda_1 x^{(1)} + \lambda_2 x^{(2)}$ and $x^{(3)'} = \mu_1 x^{(1)} + \mu_2 x^{(2)}$, we are allowed to choose $x^{(3)'}$ in such a way that $\arg(\mu_i) = \arg(\lambda_i)$, $i = 1, 2$. Remark that, since $x^{(3)}, x^{(3)'} \notin \mathcal{P}_{12}$, then

$$|\lambda_1| + |\lambda_2| > 1 \quad \text{and} \quad |\mu_1| + |\mu_2| > 1. \tag{3.24}$$

Moreover, since both $x^{(3)}$ and $x^{(3)'}$ do not delete either $x^{(1)}$ or $x^{(2)}$, we have

$$||\lambda_1| - |\lambda_2|| < 1 \quad \text{and} \quad ||\mu_1| - |\mu_2|| < 1. \tag{3.25}$$

The points $w \in \mathbb{C}^2$ which belong to the segment $x^{(3)} \bullet\!\!-\!\!\bullet\, x^{(3)'}$ are given by

$$w = \alpha x^{(3)} + (1-\alpha)x^{(3)'}, \quad \alpha \in [0, 1],$$

that is,

$$w = (\alpha\lambda_1 + (1-\alpha)\mu_1)x^{(1)} + (\alpha\lambda_2 + (1-\alpha)\mu_2)x^{(2)}, \quad \alpha \in [0, 1].$$

Since $x^{(3)'}$ is such that $\arg(\mu_i) = \arg(\lambda_i)$, $i = 1, 2$, by using (3.24) and (3.25), we obtain both the inequalities

$$\left|\alpha\lambda_1 + (1-\alpha)\mu_1\right| + \left|\alpha\lambda_2 + (1-\alpha)\mu_2\right| > 1$$

and

$$\left|\left|(\alpha\lambda_1 + (1-\alpha)\mu_1)\right| - \left|(\alpha\lambda_2 + (1-\alpha)\mu_2)\right|\right| < 1,$$

that is, $\forall \alpha \in [0, 1]$ the b.c.p. $\tilde{\mathcal{P}} = \text{absco}(\{x^{(1)}, x^{(2)}, w\})$ has three essential vertexes. But, since $x^{(3)}$ is such that $\hat\theta_{13} + \hat\theta_{23} > \pi$ and $x^{(3)'}$ is such that $\hat\theta'_{13} + \hat\theta'_{23} < \pi$, there exists a point $x^{(3)''}$ that belongs to the segment $x^{(3)} \bullet\!\!-\!\!\bullet\ x^{(3)'}$ such that $\hat{\theta}''_{13} + \hat{\theta}''_{23} = \pi$, that is, $x^{(3)''}$ deletes $x^{(1)}$ or $x^{(2)}$, which contradicts the fact that all the points of the segment $x^{(3)} \bullet\!\!-\!\!\bullet\ x^{(3)'}$ give rise to a b.c.p. with three essential vertexes. Therefore, (3.23) necessarily holds. ■

Now we observe that, since in the general complex case the upper bound (2.12) can be actually reached (see [GZ07]), there may exist another type of facets, which are different from the already computed facets of the kind $F_{y_{ij}(\theta)}$, $1 \leq i < j \leq 3$. For this reason, we introduce the following definitions.

Definition 3.1. *A facet F_y of a three- vertex b.c.p. $\mathcal{P} = \text{absco}\{x^{(1)}, x^{(2)}, x^{(3)}\} \subset \mathbb{C}^2$ is called* regular *if it contains exactly two vertexes.*

Definition 3.2. *A facet F_y of a three- vertex b.c.p. $\mathcal{P} = \text{absco}\{x^{(1)}, x^{(2)}, x^{(3)}\} \subset \mathbb{C}^2$ is called* special *if it contains three vertexes, that is, if its associated functional y satisfies $\langle y, e^{\mathrm{i}\theta_i} x^{(i)}\rangle = 1$ for suitable θ_i, $i = 1, 2, 3$.*

Observe that the special facet F_y is the *triangle*

$$e^{\mathrm{i}\theta_1}x^{(1)} \blacktriangle e^{\mathrm{i}\theta_2}x^{(2)} \blacktriangle e^{\mathrm{i}\theta_3}x^{(3)} = \text{co}\left(\left\{e^{\mathrm{i}\theta_1}x^{(1)}, e^{\mathrm{i}\theta_2}x^{(2)}, e^{\mathrm{i}\theta_3}x^{(3)}\right\}\right)$$

and that its dimension on $\mathbb{R}$, thought as a part of an affine subspace of $\mathbb{R}^4$, is equal to 2, whereas the dimension on $\mathbb{R}$ of a regular facet is obviously equal to 1.

Definition 3.3. *A segment $e^{\mathrm{i}\theta_i}x^{(i)} \bullet\!\!-\!\!\bullet\ e^{\mathrm{i}\theta_j}x^{(j)}$ containing two vertexes of a three-vertex b.c.p. $\mathcal{P} =$ $\text{absco}(\{x^{(1)}, x^{(2)}, x^{(3)}\})$ is called a* degenerate facet *if it is part of a special facet with three vertexes.*

Now, we are able to give the following result which, for a three-vertex b.c.p. $\mathcal{P}$, guarantees that the upper bound (2.12) is always attained and allows us to compute all the special facets of $\mathcal{P}$.

Theorem 3.2. *Let $\mathcal{P} = \text{absco}(\{x^{(1)}, x^{(2)}, x^{(3)}\}) \subset \mathbb{C}^2$ be a three-vertex b.c.p. Then $\mathcal{P}$ has exactly two special facets with three vertexes (modulo scalar factors of unitary modulus) which are given by*

$$x^{(1)} \blacktriangle e^{\mathrm{i}\theta^+_{12}}x^{(2)} \blacktriangle e^{\mathrm{i}\theta^-_{13}}x^{(3)}$$

and

$$x^{(1)} \blacktriangle e^{\mathrm{i}\theta^-_{12}}x^{(2)} \blacktriangle e^{\mathrm{i}\theta^+_{13}}x^{(3)}$$

where $\theta^{\pm}_{12}$ and $\theta^{\pm}_{13}$ are given by (3.21) and (3.22) and satisfy

$$\theta^+_{12} + \theta^+_{23} = \theta^-_{13} \mod 2\pi \tag{3.26}$$

and

$$\theta^-_{12} + \theta^-_{23} = \theta^+_{13} \mod 2\pi. \tag{3.27}$$

Proof. The extremal values $\theta^{\pm}_{12}$, $\theta^{\pm}_{13}$, $\theta^{\pm}_{23}$, given by (3.21) and (3.22), determine all the degenerate facets of $\mathcal{P}$, that are

$$x^{(1)} \bullet\!\!-\!\!\bullet\ e^{\mathrm{i}\theta^{\pm}_{12}}x^{(2)},$$
$$x^{(1)} \bullet\!\!-\!\!\bullet\ e^{\mathrm{i}\theta^{\pm}_{13}}x^{(3)},$$
$$x^{(2)} \bullet\!\!-\!\!\bullet\ e^{\mathrm{i}\theta^{\pm}_{23}}x^{(3)},$$

and all their multiples by scalar factors of unitary modulus. Such degenerate facets belong to the boundary of the special facets with three vertexes:

$$x^{(1)} \blacktriangle e^{\mathrm{i}\theta^{\pm}_{12}}x^{(2)} \blacktriangle e^{\mathrm{i}\theta^{\pm}_{13}}x^{(3)} \tag{3.28}$$

and all their multiples by scalar factors of unitary modulus. Now we are left to find how to choose the signs. Fixing $x^{(2)}$ in (3.28) yields $\theta^{\pm}_{13} - \theta^{\pm}_{12} = \theta^{\pm}_{23} \mod 2\pi$. By Lemma 3.2 only two of the four possible sign cases of (3.28) are allowed, and this completes the proof. ■

As a consequence, we have to perform a third and last step in the procedure for the construction of a three vertex b.c.p. where we compute the special facets $\mathcal{P}^{(3)}$ which are given by the previous theorem.

Strictly connected to the existence of special facets, it also holds that

$$\mathcal{P}_{12} \cup \mathcal{P}_{13} \cup \mathcal{P}_{23} \subsetneq \mathcal{P}$$

where $\mathcal{P}_{ij} = \text{absco}(\{x^{(i)}, x^{(j)}\})$, for $i, j = 1, 2, 3$ and $i < j$, is a *subpolytope* of $\mathcal{P}$ (see Example 3.2 and also Example 3.1 in [GZ07]).

Now, we examine the intersection of the special facets with $\mathbb{R}^2$. Let $\mathcal{P} = \text{absco}(\{x^{(1)}, x^{(2)}, x^{(3)}\})$ and $y^{(1)}$, $y^{(2)} \in \partial\mathcal{P}^*$ be the functionals associated with two not proportional special facets. Therefore, all the special facets are determined by the vectors $e^{i\theta}y^{(1)}$ and $e^{i\theta}y^{(2)}$ with $\theta \in (-\pi, \pi]$, and so their intersections with $\mathbb{R}^2$ are given by the vectors $x = [x_1, x_2]^T \in \mathcal{P} \cap \mathbb{R}^2$ such that

$$\langle e^{i\theta}y^{(1)}, x\rangle = 1 \quad \text{or} \quad \langle e^{i\theta}y^{(2)}, x\rangle = 1.$$

Then, for each $j = 1, 2$, we have $\langle y^{(j)}, x\rangle = e^{-i\theta}$, that is,

$$\begin{cases} y^{(j)}_{1R}x_1 + y^{(j)}_{2R}x_2 = \cos\theta \\ y^{(j)}_{1I}x_1 + y^{(j)}_{2I}x_2 = -\sin\theta \end{cases}.$$

This system is equivalent to

$$[(y^{(j)}_{1R})^2 + (y^{(j)}_{1I})^2]\, {x_1}^2 + [(y^{(j)}_{2R})^2 + (y^{(j)}_{2I})^2]\, {x_2}^2 + 2(y^{(j)}_{1R}y^{(j)}_{2R} + y^{(j)}_{1I}y^{(j)}_{2I})\, x_1x_2 = 1\,, \tag{3.29}$$

which, for $j = 1, 2$, represents two ellipses, possibly degenerating in pairs of straight lines.

We can conclude that the intersection of the special facets of $\mathcal{P}$ with $\mathbb{R}^2$ is given by two pairs of symmetric arcs of ellipses, where one or even both of them may degenerate into a pair of symmetric straight segments. Consequently, we can also conclude that $\partial\mathcal{P} \cap \mathbb{R}^2$ is the union of arcs of algebraic curves of fourth order and/or arcs of ellipses and/or straight segments. This fact is illustrated in Example 3.2.

In particular, if $\mathcal{P} = \overline{\mathcal{P}}$ is self-conjugated, i.e., for example, $Re(x^{(2)}) = Re(x^{(1)})$, $Im(x^{(2)}) = -Im(x^{(1)}) \neq 0$ and $x^{(3)} \in \mathbb{R}^2$, then the functionals $y^{(1)}, y^{(2)} \in \mathbb{R}^2$ (see (3.10) and (3.9)) and, thus, the ellipses (3.29) reduce to the pairs of straight lines

$$\left(y^{(1)}_1x_1 + y^{(1)}_2x_2 + 1\right)\left(y^{(1)}_1x_1 + y^{(1)}_2x_2 - 1\right) = 0$$

and

$$\left(y^{(2)}_1x_1 + y^{(2)}_2x_2 + 1\right)\left(y^{(2)}_1x_1 + y^{(2)}_2x_2 - 1\right) = 0,$$

respectively. Two of the above four straight lines contain the vertex $x^{(3)}$ and the other two contain the vertex $-x^{(3)}$, and all of them are tangent to the ellipsis $\partial\mathcal{P}_{12} \cap \mathbb{R}^2$ (see (3.9)). Therefore, according to Proposition 2.6, $\partial\mathcal{P} \cap \mathbb{R}^2$ is given by two symmetric arcs of the ellipsis $\partial\mathcal{P}_{12} \cap \mathbb{R}^2$ and by the two pairs of segments coming out of the vertexes $\pm x^{(3)}$ and tangent to the ellipsis $\partial\mathcal{P}_{12} \cap \mathbb{R}^2$.

Example 3.2. We consider the same b.c.p. introduced in Example 3.1 of [GZ07], that is $\mathcal{P} = \text{absco}(\{x^{(1)}, x^{(2)}, x^{(3)}\})$, where

$$x^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix}, \quad x^{(2)} = \begin{bmatrix} 1 \\ 1 - i \end{bmatrix}, \quad x^{(3)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

In order to compute $\partial\mathcal{P} \cap \mathbb{R}^2$, we determine each of the three curves $\partial\mathcal{P}_{ij} \cap \mathbb{R}^2$ for $i, j = 1, 2, 3$ with $i < j$ and, then, the intersection of the special facets with $\mathbb{R}^2$. By (3.6) and (3.7) we obtain that $\partial\mathcal{P}_{13} \cap \mathbb{R}^2$ (represented by the solid closed curve with two cusps in Figure 3.5) is given by the equation

$$4{x_1}^2{x_2}^2 - 8({x_1}^2 + {x_2}^2 - x_1x_2) + 4 = 0$$

with the constraint

$$x_1{}^2 + x_2{}^2 - x_1x_2 \le 1$$

and that $\partial\mathcal{P}_{23} \cap \mathbb{R}^2$ (represented by the dashed closed curve with two cusps in Figure 3.5) is given by the equation

$$x_1{}^4 - 2(3x_1{}^2 + 2x_2{}^2 - 4x_1x_2) + 1 = 0$$

with the constraint

$$3x_1{}^2 + 2x_2{}^2 - 4x_1x_2 \le 1.$$

Finally, $\partial\mathcal{P}_{12} \cap \mathbb{R}^2$, already computed in Example 3.1, is represented by the dash-dotted closed curve in Figure 3.5.

Remark that the two cusps in $\partial\mathcal{P}_{13} \cap \mathbb{R}^2$ and $\partial\mathcal{P}_{23} \cap \mathbb{R}^2$ are the two real vertexes of $\mathcal{P}$.

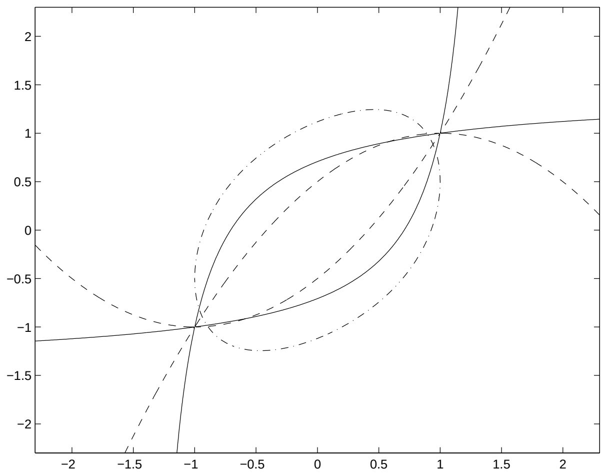

Figure 3.5: Intersection of $\partial\mathcal{P}_{12}, \partial\mathcal{P}_{13}$ and $\partial\mathcal{P}_{23}$ with $\mathbb{R}^2$ in Example 3.2

Two not proportional special facets are determined by the functionals $y^{(1)} = [1, 0]^T$ and $y^{(2)} = [\frac{2-i}{5}, \frac{3+i}{5}]^T$, so that the intersection of all the special facets with $\mathbb{R}^2$ is given by the two ellipses whose equations are

$$x_1{}^2 = 1 \tag{3.30}$$

(degenerating in two straight lines) and

$$x_1{}^2 + 2\,x_2{}^2 + 2\,x_1x_2 = 5 \tag{3.31}$$

(see (3.29)), represented by the dash-dotted and dashed curves in Figure 3.6, respectively.

In conclusion, $\partial\mathcal{P} \cap \mathbb{R}^2$ is the solid closed curve depicted in Figure 3.6, where A , B , A' and B' are the points where the ellipses (3.30) and (3.31) intersect (are tangent to) the curve $\partial\mathcal{P}_{12} \cap \mathbb{R}^2$ and where $\partial\mathcal{P}_{12} \cap \mathbb{R}^2$, $\partial\mathcal{P}_{23} \cap \mathbb{R}^2$ and $\partial\mathcal{P}_{23} \cap \mathbb{R}^2$ are all represented by dotted curves. ◇

Example 3.3. In order to show what happens when the addition of a third point $x^{(3)}$ to the already available b.c.p. $\mathcal{P}^{(2)} = \text{absco}(\{x^{(1)}, x^{(2)}\})$ deletes one of the vertexes of $\mathcal{P}^{(2)}$, we consider as an example

$$x^{(1)} = \begin{bmatrix} 1 \\ \mathrm{i} \end{bmatrix},\ x^{(2)} = \begin{bmatrix} 1 \\ 1-\mathrm{i} \end{bmatrix},\ x^{(3)} = \frac{\sqrt{2}}{2}\begin{bmatrix} (4+4\mathrm{i}) \\ (-1+3\mathrm{i}) \end{bmatrix}.$$

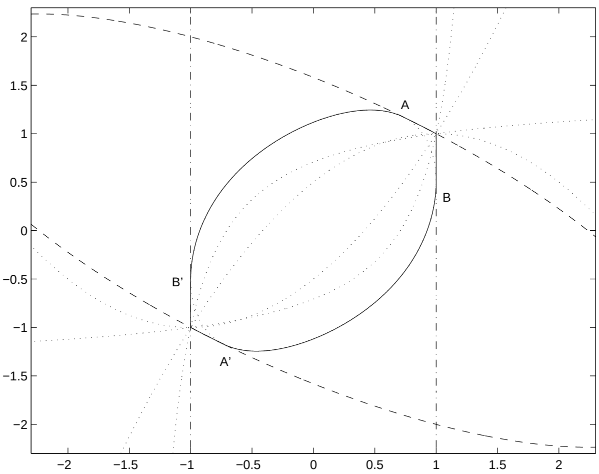

Figure 3.6: $\partial\mathcal{P} \cap \mathbb{R}^2$ in Example 3.2

Since $x^{(3)} \notin \mathcal{P}^{(2)}$ as $\|x^{(3)}\|_{\mathcal{P}^{(2)}} = \|[x^{(1)}, x^{(2)}]^{-1} x^{(3)}\|_1 > 1$, we add $x^{(3)}$ to $\mathcal{P}^{(2)}$ by following the procedure described before. For this purpose we write $x^{(3)}$ in the form $x^{(3)} = \lambda_1\, x^{(1)} + \lambda_2\, x^{(2)}$, with $\lambda_1, \lambda_2 \in \mathbb{C}$. It turns out that $|\lambda_1| > |\lambda_2| + 1$, and so $x^{(1)}$ is deleted.

Now we analyse the intersection of $\partial\mathcal{P}$ with $\mathbb{R}^2$.

Like in Example 3.2 we compute $\partial\mathcal{P}_{12} \cap \mathbb{R}^2$, that is given by the equation

$$({x_1}^2 - 2x_1x_2)^2 - 10(3{x_1}^2 + 2{x_2}^2 - 2x_1x_2) + 25 = 0$$

with the constraint

$$3{x_1}^2 + 2{x_2}^2 - 2x_1x_2 \le 5.$$

Therefore, $\partial\mathcal{P}_{12} \cap \mathbb{R}^2$ is given by the dashed curve (see Figure 3.7).

In the same way we obtain that $\partial\mathcal{P}_{13} \cap \mathbb{R}^2$, represented by the dotted curve (see Figure 3.7), is given by the equation

$$(4{x_1}^2 + 15{x_2}^2 - 8x_1x_2)^2 - 10(6{x_1}^2 + 17{x_2}^2 - 8x_1x_2) + 25 = 0$$

with the constraint

$$6{x_1}^2 + 17{x_2}^2 - 8x_1x_2 \le 5.$$

Finally, $\partial\mathcal{P}_{23} \cap \mathbb{R}^2$, represented by the solid curve (see Figure 3.7), is given by the equation

$$(3{x_1}^2 + 15{x_2}^2 - 6x_1x_2)^2 - 90(7{x_1}^2 + 17{x_2}^2 - 10x_1x_2) + 2025 = 0$$

with the constraint

$$7{x_1}^2 + 17{x_2}^2 - 10x_1x_2 \le 45.$$

Since $\partial\mathcal{P}_{12} \cap \mathbb{R}^2$ and $\partial\mathcal{P}_{13} \cap \mathbb{R}^2$ are both inside $\partial\mathcal{P}_{23} \cap \mathbb{R}^2$, we can conclude that $\partial\mathcal{P} \cap \mathbb{R}^2$ is given by the solid curve which represents $\partial\mathcal{P}_{23} \cap \mathbb{R}^2$. This is consistent to the fact that the addition of $x^{(3)}$ to $\mathcal{P}_{12}$ deletes the vertex $x^{(1)}$ and that, indeed, $\mathcal{P} = \mathcal{P}_{23}$. ◇

3.2.3 The general case

Now, we analyse the construction of a b.c.p. $\mathcal{P} = \mathrm{absco}(\mathcal{X})$, where $\mathcal{X} = \{x^{(k)}\}_{1\le k\le m}$, $m \ge 4$, is a set of vectors in $\mathbb{C}^2$ with at least four essential vertexes. Thus, we begin generalising some definitions introduced in the previous subsection.

Definition 3.4. *A facet F_y of a b.c.p. $\mathcal{P} = \mathrm{absco}(\mathcal{X})$ is called* regular *if it contains exactly two vertexes.*

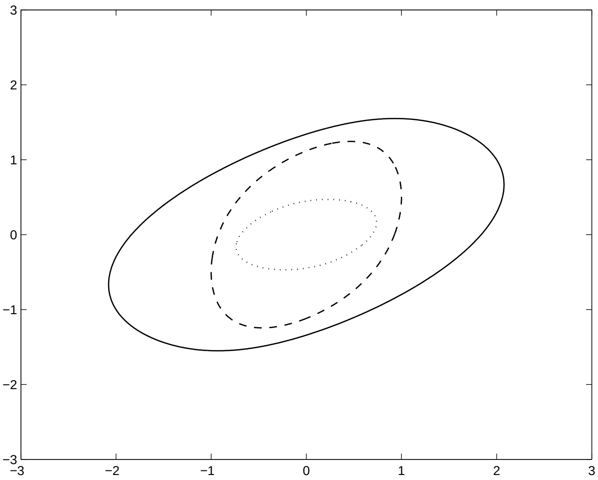

Figure 3.7: Intersection of $\partial\mathcal{P}$ with $\mathbb{R}^2$

Definition 3.5. *Let be $\mathcal{P} = \text{absco}(\mathcal{X}) \subseteq \mathbb{C}^2$, where $\mathcal{X} = \{x^{(k)}\}_{1\le k\le r}$ with $r \ge 4$ is an essential system of vertexes. A facet F_y of $\mathcal{P}$ is called* special *if it contains three or more vertexes, that is, if there exist $x^{(l_1)}, \cdots, x^{(l_s)} \in \mathcal{X}$, with $3 \le s \le m$, such that the functional y associated with F_y satisfies*

$$\langle y, e^{\mathrm{i}\theta_i} x^{(l_i)} \rangle = 1,$$

for suitable θ_i, $i = 1, \cdots, s$.

As a consequence of Theorems 2.3 and 2.4 and Remark 2.1 we have that F_y is the union of all possible *triangles* of the type $e^{\mathrm{i}\theta_{i_1}} x^{(l_{i_1})} \blacktriangle e^{\mathrm{i}\theta_{i_2}} x^{(l_{i_2})} \blacktriangle e^{\mathrm{i}\theta_{i_3}} x^{(l_{i_3})} = \text{co}\left(\left\{e^{\mathrm{i}\theta_{i_1}} x^{(l_{i_1})}, e^{\mathrm{i}\theta_{i_2}} x^{(l_{i_2})}, e^{\mathrm{i}\theta_{i_3}} x^{(l_{i_3})}\right)\right\}$, i.e.

$$F_y = \bigcup_{1\le i_1<i_2<i_3\le s} e^{\mathrm{i}\theta_{i_1}} x^{(l_{i_1})} \blacktriangle e^{\mathrm{i}\theta_{i_2}} x^{(l_{i_2})} \blacktriangle e^{\mathrm{i}\theta_{i_3}} x^{(l_{i_3})}. \tag{3.32}$$

Moreover, we have

$$\mathcal{P} = \bigcup_{1\le i_1<i_2<i_3\le m} \mathcal{P}_{i_1 i_2 i_3}, \tag{3.33}$$

where $\mathcal{P}_{i_1 i_2 i_3} = \text{absco}(\{x^{(i_1)}, x^{(i_2)}, x^{(i_3)}\})$, $1 \le i_1 < i_2 < i_3 \le m$, are all the three-vertex subpolytopes of $\mathcal{P}$. Therefore, the real intersection $\mathcal{P} \cap \mathbb{R}^2$, is given by

$$\mathcal{P} \cap \mathbb{R}^2 = \bigcup_{1\le i_1<i_2<i_3\le m} \mathcal{P}_{i_1 i_2 i_3} \cap \mathbb{R}^2$$

and, consequently,

$$\partial\mathcal{P} \cap \mathbb{R}^2 \subseteq \bigcup_{1\le i_1<i_2<i_3\le m} \partial\mathcal{P}_{i_1 i_2 i_3} \cap \mathbb{R}^2.$$

So, $\partial\mathcal{P} \cap \mathbb{R}^2$ is still the union of arcs of algebraic curves of fourth order and/or arcs of ellipses and/or straight segments.

Definition 3.6. *A segment $\mathrm{e}^{\mathrm{i}\theta_i} x^{(i)} \bullet\!\!-\!\!\bullet\, \mathrm{e}^{\mathrm{i}\theta_j} x^{(j)}$ containing two vertexes of a b.c.p. $\mathcal{P}$ is called a* degenerate facet *if it is part of a special facet.*

Definition 3.7. *The* existence pluri-interval *of the facets of the type $F_{y_{ij}(\theta)} = x^{(i)} \bullet\!\!-\!\!\bullet\, e^{\mathrm{i}\theta} x^{(j)}$ of a b.c.p. $\mathcal{P}$ is the set $\mathcal{E}_{ij} \in (-\pi, \pi]$ such that $\theta \in \mathcal{E}_{ij}$ if and only if the segment $x^{(i)} \bullet\!\!-\!\!\bullet\, e^{\mathrm{i}\theta} x^{(j)}$ is a facet (either regular or degenerate) of $\mathcal{P}$.*

By considering an m-vertex b.c.p. $\mathcal{P}$ as a structure constructed by adding the remaining $m-2$ vertexes one at a time to a two-vertex b.c.p., it is easy to understand that, as the generalisation of the procedure given in Subsection 3.2.2, the existence pluri-interval $\mathcal{E}_{ij}$ may be obtained by subtracting at the most $m-2$ intervals, possibly shifted to our reference interval $(-\pi, \pi]$. Therefore, as will be more clear later, we can state the following result.

Proposition 3.1. *Let $\mathcal{P}$ be a b.c.p. with m essential vertexes. Then, the existence pluri-interval $\mathcal{E}_{ij}$ of the facets $F_{y_{ij}(\theta)}$ is the union of m_{ij} disjoint intervals, $0 \le m_{ij} \le m-1$, that is*

$$\mathcal{E}_{ij} = \bigcup_{l=1}^{m_{ij}} [\theta^-_{ij,l}, \theta^+_{ij,l}] \setminus \{-\pi\} \tag{3.34}$$

for appropriate values $\theta^-_{ij,l}, \theta^+_{ij,l} \in [-\pi, \pi]$ which, apart from the case of equality to $-\pi$ and, possibly, to π, determine all the degenerate facets of the type $F_{y_{ij}(\theta)} = x^{(i)} \bullet\!\!-\!\!\bullet\, e^{\theta} x^{(j)}$.

Note that $m_{ij} = 0$ if and only if $\mathcal{E}_{ij} = \emptyset$, i.e. $\mathcal{P}$ has no facets of the type $F_{y_{ij}(\theta)} = x^{(i)} \bullet\!\!-\!\!\bullet\, e^{\mathrm{i}\theta} x^{(j)}$ at all.

Now, we show how to determine all the special facets of $\mathcal{P}$ with three or more vertexes. Since, according to (3.32), all the special facets with more then three vertexes can be viewed as a union of triangles, the next theorem shows how to compute all of them by using the knowledge of the existence pluri-intervals $\mathcal{E}_{ij}$, $1 \le i < j \le m$.

Lemma 3.3. *Let $\mathcal{P} = \mathrm{absco}(\mathcal{X}) \subset \mathbb{C}^2$, where $\mathcal{X} = \{x^{(i)}\}_{1 \le i \le m}$, $m \ge 3$, is an essential system of vertexes, and let $e^{\mathrm{i}\theta_{\mathrm{i}}} x^{(i)}$, with $i \in \{I, J, K\}$ and $1 \le I < J < K \le m$, be vertexes such that all the segments $e^{\mathrm{i}\theta_{\mathrm{i}}} x^{(i)} \bullet\!\!-\!\!\bullet\, e^{\mathrm{i}\theta_{\mathrm{j}}} x^{(j)}$, for $i < j$ and $j \in \{I, J, K\}$, are included in facets of $\mathcal{P}$. Then the triangle $e^{\mathrm{i}\theta_{\mathrm{I}}} x^{(I)} \blacktriangle e^{\mathrm{i}\theta_{\mathrm{J}}} x^{(J)} \blacktriangle e^{\mathrm{i}\theta_{\mathrm{K}}} x^{(K)}$ is contained in a special facet of $\mathcal{P}$.*

Proof. Since the essential vertexes are pairwise linearly independent, there exist $\alpha_I, \alpha_J \in \mathbb{C}$, $\alpha_I, \alpha_J \neq 0$, such that

$$e^{\mathrm{i}\theta_{\mathrm{K}}} x^{(K)} = \alpha_I e^{\mathrm{i}\theta_{\mathrm{I}}} x^{(I)} + \alpha_J e^{\mathrm{i}\theta_{\mathrm{J}}} x^{(J)}. \tag{3.35}$$

If $y_{IJ} \in \mathcal{P}^*$ is the functional associated with the facet $F_{y_{IJ}}$ including the segment $e^{\mathrm{i}\theta_{\mathrm{I}}} x^{(I)} \bullet\!\!-\!\!\bullet\, e^{\mathrm{i}\theta_{\mathrm{J}}} x^{(J)}$, then

$$\langle y_{IJ}, e^{\mathrm{i}\theta_{\mathrm{K}}} x^{(K)} \rangle = \overline{\alpha_I} + \overline{\alpha_J}$$

and, consequently,

$$|\overline{\alpha_I} + \overline{\alpha_J}| \le 1. \tag{3.36}$$

Moreover, solving (3.35) first for $x^{(I)}$ and then for $x^{(J)}$ and using the fact that also the segments $e^{\mathrm{i}\theta_{\mathrm{J}}} x^{(J)} \bullet\!\!-\!\!\bullet\, e^{\mathrm{i}\theta_{\mathrm{K}}} x^{(K)}$ and $e^{\mathrm{i}\theta_{\mathrm{I}}} x^{(I)} \bullet\!\!-\!\!\bullet\, e^{\mathrm{i}\theta_{\mathrm{K}}} x^{(K)}$ are included in facets of $\mathcal{P}$, we obtain also

$$|1 - \overline{\alpha_J}| \le |\overline{\alpha_I}| \quad \text{and} \quad |1 - \overline{\alpha_I}| \le |\overline{\alpha_J}|.$$

Squaring and summing up these last two inequalities, we find that

$$Re(\overline{\alpha_I} + \overline{\alpha_J}) \ge 1.$$

Therefore, by virtue of (3.36), we can conclude that

$$\langle y_{IJ}, e^{\mathrm{i}\theta_{\mathrm{K}}} x^{(K)} \rangle = \overline{\alpha_I} + \overline{\alpha_J} = 1,$$

that is, $e^{\mathrm{i}\theta_{\mathrm{K}}} x^{(K)} \in F_{y_{IJ}}$, concluding the proof. ■

Theorem 3.3. *For all the triplets (I, J, K) with $1 \le I < J < K \le m$, there exist, at the most, two triangles of the type $x^{(I)} \blacktriangle e^{\mathrm{i}\theta_{\mathrm{IJ}}} x^{(J)} \blacktriangle e^{\mathrm{i}\theta_{\mathrm{IK}}} x^{(K)}$ contained in special facets of $\mathcal{P}$. Such triangles, if any, necessarily have the form*

$$x^{(I)} \blacktriangle e^{\mathrm{i}\theta^+_{\mathrm{IJ}}} x^{(J)} \blacktriangle e^{\mathrm{i}\theta^-_{\mathrm{IK}}} x^{(K)} \quad \textit{only if} \quad \theta^+_{IJ} + \theta^+_{JK} = \theta^-_{IK} \mod 2\pi \tag{3.37}$$

and/or

$$x^{(I)}\blacktriangle e^{i\theta_{IJ}^-}x^{(J)}\blacktriangle e^{i\theta_{IK}^+}x^{(K)} \quad \textit{only if} \quad \theta_{IJ}^- + \theta_{JK}^- = \theta_{IK}^+ \quad \mod 2\pi, \tag{3.38}$$

where $[\theta_{rs}^-, \theta_{rs}^+]$, $r, s \in \{I, J, K\}$, *stands for any of the intervals of the pluri-interval* $\mathcal{E}_{rs}$.

Conversely, if $\theta_{IJ}^+ + \theta_{JK}^+ = \theta_{IK}^- \mod 2\pi$, *then the triangle* $x^{(I)}\blacktriangle e^{i\theta_{IJ}^+}x^{(J)}\blacktriangle e^{i\theta_{IK}^-}x^{(K)}$ *is contained in a special facet of* $\mathcal{P}$ *and, if* $\theta_{IJ}^- + \theta_{JK}^- = \theta_{IK}^+ \mod 2\pi$, *then the triangle* $x^{(I)}\blacktriangle e^{i\theta_{IJ}^-}x^{(J)}\blacktriangle e^{i\theta_{IK}^+}x^{(K)}$ *is contained in a special facet of* $\mathcal{P}$.

Proof. We start by observing that a triangle $x^{(I)}\blacktriangle e^{i\theta_{IJ}}x^{(J)}\blacktriangle e^{i\theta_{IK}}x^{(K)}$ is contained in a special facet of $\mathcal{P}$ only if it is a special facet of the subpolytope $\mathcal{P}^{(IJK)} = \text{absco}(\{x^{(I)}, x^{(J)}, x^{(K)}\})$. Now Theorem 3.2 implies that such triangles are at the most two and that they are obtained respectively for

$$\theta_{IJ} = \theta_{IJ}^+ \quad \wedge \quad \theta_{IK} = \theta_{IK}^-$$

and

$$\theta_{IJ} = \theta_{IJ}^- \quad \wedge \quad \theta_{IK} = \theta_{IK}^+,$$

where $\left[\theta_{IJ}^-, \theta_{IJ}^+\right]$ and $\left[\theta_{IK}^-, \theta_{IK}^+\right]$ are suitable intervals of the pluri-intervals $\mathcal{E}_{IJ}$ and $\mathcal{E}_{IK}$, respectively, (see Proposition 3.1) such that

$$\theta_{IJ}^+ + \theta_{JK}^+ = \theta_{IK}^- \quad \mod 2\pi$$

and

$$\theta_{IJ}^- + \theta_{JK}^- = \theta_{IK}^+ \quad \mod 2\pi.$$

Conversely, let the triangle $x^{(I)}\blacktriangle e^{i\theta_{IJ}^+}x^{(J)}\blacktriangle e^{i\theta_{IK}^-}x^{(K)}$ be such that $\theta_{IJ}^+ + \theta_{JK}^+ = \theta_{IK}^- \mod 2\pi$. Since the segment $x^{(J)} \bullet\!\!-\!\!\bullet\, e^{i\theta_{JK}^+}x^{(K)}$ is a degenerate facet of $\mathcal{P}$ and since $\theta_{JK}^+ = \theta_{IK}^- - \theta_{IJ}^+ \mod 2\pi$, the segment $e^{i\theta_{IJ}^+}x^{(J)} \bullet\!\!-\!\!\bullet\, e^{i\theta_{IK}^-}x^{(K)}$ is a degenerate facet of $\mathcal{P}$, too. Therefore, since also $x^{(I)} \bullet\!\!-\!\!\bullet\, e^{i\theta_{IJ}^+}x^{(J)}$ and $x^{(I)} \bullet\!\!-\!\!\bullet\, e^{i\theta_{IK}^-}x^{(K)}$ are degenerate facets of $\mathcal{P}$, the vertexes $x^{(I)}, e^{i\theta_{IJ}^+}x^{(J)}, e^{i\theta_{IK}^-}x^{(K)}$ determine three segments that satisfy the hypotheses of Lemma 3.3 with respect to the subpolytope $\mathcal{P}_{IJK}$. Therefore, the triangle $x^{(I)}\blacktriangle e^{i\theta_{IJ}^+}x^{(J)}\blacktriangle e^{i\theta_{IK}^-}x^{(K)}$ is a special facet of $\mathcal{P}_{IJK}$ and, consequently, it must be included in a special facet of $\mathcal{P}$.

The case of the triangle $x^{(I)}\blacktriangle e^{i\theta_{IJ}^-}x^{(J)}\blacktriangle e^{i\theta_{IK}^+}x^{(K)}$ may be treated in the same way. ■

An important role in the incoming discussion is played by a new kind of facets, the so called *isolated facets*, which characterise a particular situation that, although being a limiting case, may well occur. These facets are defined as follows.

Definition 3.8. *A segment* $e^{i\theta_i}x^{(i)} \bullet\!\!-\!\!\bullet\, e^{i\theta_j}x^{(j)}$ *containing two vertexes of a b.c.p.* $\mathcal{P}$ *is called an* isolated facet *if it is contained in a facet of* $\mathcal{P}$ *and if all the segments* $e^{i\theta_i}x^{(i)} \bullet\!\!-\!\!\bullet\, e^{i\theta+\theta_j}x^{(j)}$ *are not facets of* $\mathcal{P}$ *for all* $\theta \in (-\theta_0, \theta_0) \setminus \{0\}$ *for some* $\theta_0 > 0$.

Clearly, a two-vertex b.c.p. has no isolated facets. For a three-vertex b.c.p. we can state the following proposition which is an obvious consequence of the results presented in the previous Section 3.2.2.

Proposition 3.2. *Let* $\mathcal{P} = \text{absco}(\{x^{(1)}, x^{(2)}, x^{(3)}\})$ *be a b.c.p. with three essential vertexes. Then* $\mathcal{P}$ *has no isolated facets.*

In general, for b.c.p.'s with four or more essential vertexes, the isolated facets may be related to the special facets as shown in the next proposition.

Proposition 3.3. *Let* $\mathcal{P} = \text{absco}(\mathcal{X})$, *where* $\mathcal{X} = \{x^{(i)}\}_{1\le i\le m}$ *is an essential system of* $m \geq 4$ *vertexes, and let* $e^{i\theta_1}x^{(1)} \bullet\!\!-\!\!\bullet\, e^{i\theta_2}x^{(2)}$ *be an isolated facet. Then there exist at least two vertexes,* $x^{(h)}, x^{(k)} \in \mathcal{X}$, *with* $h, k \in [3, m]$, $h \neq k$, *and the corresponding* $\theta_h, \theta_k \in (-\pi, \pi]$, *such that the triangles* $e^{i\theta_1}x^{(1)}\blacktriangle e^{i\theta_2}x^{(2)}\blacktriangle e^{i\theta_h}x^{(h)}$ *and* $e^{i\theta_1}x^{(1)}\blacktriangle e^{i\theta_2}x^{(2)}\blacktriangle e^{i\theta_k}x^{(k)}$ *are contained in the same special facet* F_y.

Proof. Since the value $\theta = 0$ is extremal for the facets of the kind $e^{i\theta_1}x^{(1)} \bullet\!\!-\!\!\bullet\ e^{i(\theta+\theta_2)}x^{(2)}$, the facet $e^{i\theta_1}x^{(1)} \bullet\!\!-\!\!\bullet\ e^{i\theta_2}x^{(2)}$ is degenerate and is part of a special facet F_y. Thus, by (3.32) it is included in a triangle $e^{i\theta_1}x^{(1)}\blacktriangle e^{i\theta_2}x^{(2)}\blacktriangle e^{i\theta_h}x^{(h)} \subseteq F_y$ with $h \in [3, m]$ and for a suitable θ_h.

Now, we assume by contradiction that $e^{i\theta_1}x^{(1)}\blacktriangle e^{i\theta_2}x^{(2)}\blacktriangle e^{i\theta_h}x^{(h)} = F_y$, that is, $|\langle y, x^{(i)}\rangle| < 1$ for all $i \in [3, m]\backslash\{h\}$. Then, by continuity, we would have that

$$|\langle y(\theta), x^{(i)}\rangle| < 1 \quad \text{for all } i \in [3, m]\backslash\{h\} \text{ and for all } \theta \in (-\theta_0, \theta_0)$$

for some $\theta_0 > 0$, where $y(\theta) \in \mathbb{C}^2$ is such that

$$\langle y(\theta), e^{i\theta_1}x^{(1)}\rangle = \langle y(\theta), e^{i(\theta+\theta_2)}x^{(2)}\rangle = 1.$$

Therefore, since the facet $e^{i\theta_1}x^{(1)} \bullet\!\!-\!\!\bullet\ e^{i\theta_2}x^{(2)}$ is isolated, the fundamental equality $\mathcal{P} = (\mathcal{P}^*)^*$ (see Theorem 2.1) would imply that

$$|\langle y(\theta), x^{(h)}\rangle| > 1 \quad \text{for all } \theta \in (-\theta_0, \theta_0) \setminus \{0\}.$$

But this would mean that the subpolytope $\mathcal{P}_{12h} = \text{absco}(\{x^{(1)}, x^{(2)}, x^{(h)}\})$ has an isolated facet, and this result contradicts Proposition 3.2.

In conclusion, there must exist another vertex $x^{(k)} \in \mathcal{X}$ with $k \geq 3$, $k \neq h$, and a suitable θ_k such that

$$\langle y, e^{i\theta_k}x^{(k)}\rangle = 1,$$

that is, $x^{(k)} \in F_y$ and the triangle $e^{i\theta_1}x^{(1)}\blacktriangle e^{i\theta_2}x^{(2)}\blacktriangle e^{i\theta_k}x^{(k)}$ is contained in the special facet F_y, too. ■

Remark 3.4. *If the segment* $e^{i\theta_i}x^{(i)} \bullet\!\!-\!\!\bullet\ e^{i\theta_j}x^{(j)}$ *is an isolated facet, then it is clearly a degenerate facet and one of the intervals* $[\theta^-_{ij,l}, \theta^+_{ij,l}]$ *that constitute the existence pluri-interval* $\mathcal{E}_{ij}$ *in (3.34) degenerates in a single point, that is,*

$$\theta^-_{ij,l} = \theta^+_{ij,l} = \theta_j - \theta_i.$$

Another key result relates the isolated facets to the vertexes of a b.c.p., and, indeed, it will turn out to be very important in our constructive algorithm of a b.c.p.

Theorem 3.4. *None of the vertexes of a b.c.p.* $\mathcal{P} = \text{absco}(\mathcal{X})$, *where* $\mathcal{X} = \{x^{(i)}\}_{1\leq i\leq m}$ *with* $m \geq 4$, *may belong only to isolated facets.*

For the proof we need the following technical lemmas.

Lemma 3.4. *Let* $S = \left\{s_1, \cdots, s_p\right\}$, *with* $p \geq 4$, *be a generic finite set that satisfies the following two properties:*

(1) *For all pairs* (i, j) *with* $i \neq j$ *the subset* $S_{ij} = S \setminus \left\{s_i, s_j\right\}$ *is divided into two subsets* Σ^+_{ij} *and* Σ^-_{ij}, *one of which can be possibly empty, such that*

$$\Sigma^+_{ij} \cup \Sigma^-_{ij} = S_{ij} \quad \textit{and} \quad \Sigma^+_{ij} \cap \Sigma^-_{ij} = \emptyset\,.$$

(2) *For all triplets* (i, j, k) *with distinct* i, j, k, *it holds that*

$$s_k \in \Sigma^+_{ij} \iff s_j \in \Sigma^-_{ik}.$$

If $s_h \in S$ is such that

$$\Sigma^+_{hj} \neq \emptyset \quad \textit{and} \quad \Sigma^-_{hj} \neq \emptyset \qquad \forall j \neq h, \tag{3.39}$$

then there exists a triplet (α, β, γ) such that

$$s_\beta \in \Sigma^+_{h\alpha}\,, \quad s_\gamma \in \Sigma^+_{h\beta}\,, \quad s_\alpha \in \Sigma^+_{h\gamma}. \tag{3.40}$$

Proof. It is not restrictive to assume that $h = 1$. Trivially, the Lemma holds for $p = 4$, with $(\alpha, \beta, \gamma) = (2, 3, 4)$. Now, by using induction, we suppose that it holds for sets of cardinality $p \geq 4$ and prove it for $p + 1$.

We assume, by contradiction, that the result does not hold for $p + 1$. Let $S = \{s_1, \cdots, s_{p+1}\}$. Then, if we delete the element s_{p+1} from S , we obtain the subset $S_{p+1} = \{s_1, \cdots, s_p\}$. Obviously, the properties (1) and (2), that hold in S, hold in the subset S_{p+1} too. On the countrary, the property (3.39) is not assured any more for S_{p+1}, and the property (3.40), which does not hold in S, does not hold in S_{p+1} either. Indeed, the inductive hypothesis implies that the condition (3.39) cannot hold in S_{p+1}, so that there exists $j \in \{2, \cdots p\}$ such that

$$\Sigma^+_{1j} \cap S_{p+1} = \emptyset \quad \text{or} \quad \Sigma^-_{1j} \cap S_{p+1} = \emptyset.$$

It is not restrictive to assume that $j = p$ and that

$$\Sigma^+_{1p} \cap S_{p+1} = \emptyset.$$

Since the condition (3.39) holds in S, then it must necessarily be

$$\Sigma^+_{1p} = \{s_{p+1}\}\,, \tag{3.41}$$

so that property (1) implies

$$s_j \in \Sigma^-_{1p} \qquad \forall j \in [2, p-1].$$

Consequently, by property (2) we obtain

$$s_p \in \Sigma^+_{1j} \qquad \forall j \in [2, p-1]. \tag{3.42}$$

Analogously, if we eliminate the element s_p from S , we obtain the subset $S_p = \{s_1, \cdots, s_{p-1}, s_{p+1}\}$. Like before, we have that the property (3.39) cannot hold in S_p and, so, there exists $j \in \{2, \cdots p-1, p+1\}$ such that

$$\Sigma^+_{1j} \cap S_p = \emptyset \quad \text{or} \quad \Sigma^-_{1j} \cap S_p = \emptyset.$$

If $j = p + 1$ and $\Sigma^+_{1p+1} \cap S_p = \emptyset$, then, since the property (3.39) holds in S, it must necessarily be $\Sigma^+_{1p+1} = \{s_p\}$ and this contradicts (3.41) because of properties (1) and (2).
If $j = p + 1$ and $\Sigma^-_{1p+1} \cap S_p = \emptyset$, then, since the property (3.39) holds in S, it must necessarily be $\Sigma^-_{1p+1} = \{s_p\}$, from which, by property (2), it follows that $s_{p+1} \in \Sigma^+_{1p}$. Moreover, by property (1) it also follows that $s_{p-1} \in \Sigma^+_{1p+1}$. As a consequence, by (3.42) with $j = p-1$, we have that the property (3.40) holds for the triplet $(\alpha, \beta, \gamma) = (p-1, p, p+1)$, and this is absurd. Therefore, it must be $j \in [2, p-1]$, and it is not restrictive to assume that $j = p-1$. First we suppose that $\Sigma^-_{1p-1} \cap S_p = \emptyset$. Since the property (3.39) holds in S, then we have that $\Sigma^-_{1p-1} = \{s_p\}$, which contradicts (3.42) with $j = p-1$ because of property (1). Lastly, if $\Sigma^+_{1p-1} \cap S_p = \emptyset$, then, since the condition (3.39) holds in S, we have that $\Sigma^+_{1p-1} = \{s_p\}$, so that properties (1) and (2) imply that $s_{p-1} \in \Sigma^+_{1p+1}$. Thus, by (3.41), the property (3.40) holds for the triplet $(\alpha, \beta, \gamma) = (p-1, p, p+1)$, and this is absurd too.

For these reasons, we can conclude that the Lemma is true also for $p + 1$, and hence for all $p \geq 4$. ■

Lemma 3.5. *Assume the hypotheses (1), (2) and (3.39) of Lemma 3.4 and, moreover, that for all pairs* (i, j) *with* $i \neq j$ *it holds that*

$$\Sigma^+_{ij} = \Sigma^-_{ji}. \tag{3.43}$$

Then (3.40) is equivalent to

$$s_h \in \Sigma^+_{\alpha\beta} \cap \Sigma^+_{\beta\gamma} \cap \Sigma^+_{\gamma\alpha}\,. \tag{3.44}$$

Proof. By (3.43) we obtain that $s_\beta \in \Sigma^+_{h\alpha}$ if and only if $s_\beta \in \Sigma^-_{\alpha h}$, that is, by using condition (2) of Lemma 3.4, if and only if $s_h \in \Sigma^+_{\alpha\beta}$. Analogously, we obtain that

$$s_\gamma \in \Sigma^+_{h\beta} \iff s_h \in \Sigma^+_{\beta\gamma}$$

$$s_\alpha \in \Sigma^+_{h\gamma} \iff s_h \in \Sigma^+_{\gamma\alpha}.$$

Therefore (3.40) is equivalent to (3.44). ■

The geometric interpretation of Lemma 3.5 is the following. Let S be a set of $p \geq 4$ points in the two-dimensional real plane such that three of them are never lying on the same straight line. Then, given two points s_i and s_j, it is clear that the straight line passing through them and positively oriented from s_i to s_j divides the plane into two half-planes, namely the left half-plane and the right half-plane. Then we say that a third point s_k belongs to Σ^+_{ij} if it lies in the left half-plane and that it belongs to Σ^-_{ij} if it lies in the right half-plane. It is immediately seen that such a set of points S satisfies the hypotheses (1), (2) and (3.43) of Lemma 3.5. Figure 3.8 illustrates the situation in which the hypothesis (3.39) is satisfied by the point s_1.

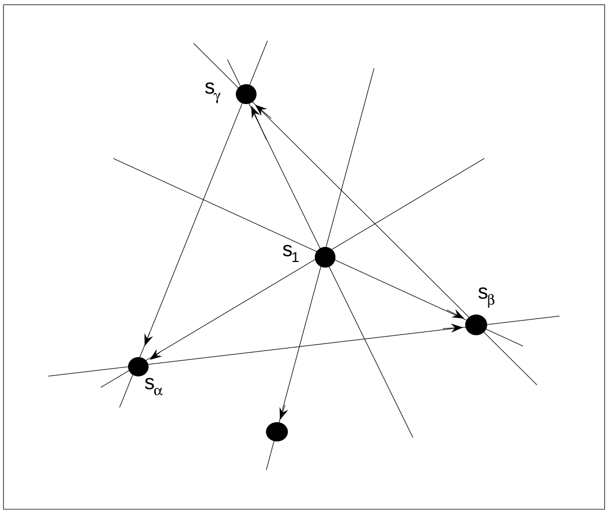

Figure 3.8: Geometric interpretation of Lemma 3.5.

Proof of Theorem 3.4 Assume, by contradiction, that $x^{(1)}$ belongs only to isolated facets. By Proposition 3.3 we know that $x^{(1)}$ belongs to a special facet F_y with $p \geq 4$ vertexes. Let $S = \{x^{(1)}, x^{(2)}, \cdots, x^{(p)}\}$ be the set of such vertexes. In order to use Lemmas 3.4 and 3.5, for all pairs (i, j) with $i \neq j$ we define the sets Σ^+_{ij} and Σ^-_{ij} as follows:

$$\Sigma^+_{ij} = \left\{x^{(k)} \in S \setminus \left\{x^{(i)}, x^{(j)}\right\} \middle| |\langle y(\theta), x^{(k)}\rangle| > 1 \;\; \forall\, \theta \in (0, \theta_0)\right\}$$

$$\Sigma^-_{ij} = \left\{x^{(k)} \in S \setminus \left\{x^{(i)}, x^{(j)}\right\} \middle| |\langle y(\theta), x^{(k)}\rangle| > 1 \;\; \forall\, \theta \in (-\theta_0, 0)\right\}$$

for a suitable $\theta_0 > 0$, where $y(\theta)$ is the functional associated with the facet $x^{(i)} \bullet\!\!-\!\!\bullet\, e^{i\theta}x^{(j)}$, that is,

$$y(\theta) = \left[x^{(i)} \;\; x^{(j)}\right]^{-H} \begin{bmatrix} 1 \\ e^{i\theta} \end{bmatrix}. \tag{3.45}$$

First of all, we examine the property (1). Let y, given by (3.45) for $\theta = 0$, be the functional associated with the special facet F_y, so that

$$\langle y, x^{(k)}\rangle = 1 \quad \forall x^{(k)} \in S. \tag{3.46}$$

For each $x^{(k)} \in S$ we can write

$$x^{(k)} = \lambda_i x^{(i)} + \lambda_j x^{(j)} \quad \text{with} \quad \lambda_i, \lambda_j \in \mathbb{C} \quad \text{and} \quad |\lambda_i| + |\lambda_j| > 1. \tag{3.47}$$

Then (3.46) implies

$$\lambda_i + \lambda_j = 1,$$

so that we can write

$$\lambda_i = \alpha + i\beta \quad \text{and} \quad \lambda_j = 1 - \alpha - i\beta,$$

where $\alpha, \beta \in \mathbb{R}$. We can conclude that

$$|\langle y(\theta), x^{(k)}\rangle| > 1 \;\; \forall\, \theta \in (0, \theta_0) \;\; \text{and} \;\; |\langle y(\theta), x^{(k)}\rangle| < 1 \;\; \forall\, \theta \in (-\theta_0, 0)$$

or vice versa, i.e.

$$|\langle y(\theta), x^{(k)}\rangle| < 1 \;\; \forall\, \theta \in (0, \theta_0) \;\; \text{and} \;\; |\langle y(\theta), x^{(k)}\rangle| > 1 \;\; \forall\, \theta \in (-\theta_0, 0).$$

In fact, if it were not so, the function

$$f(\theta) = |\langle y(\theta), x^{(k)}\rangle|^2 - 1 = |\bar{\lambda}_i + e^{i\theta}\bar{\lambda}_j|^2 - 1 = (1 - \cos\theta)(\alpha^2 + \beta^2 - \alpha) - \beta \sin\theta \tag{3.48}$$

(see (3.45)) would have either a maximum or a minimum at $\theta = 0$, and this would imply that $f'(0) = -\beta = 0$. In this case we would have either $\alpha < 0$ that implies $x^{(j)} = \frac{-\alpha}{1-\alpha}x^{(i)} + \frac{1}{1-\alpha}x^{(k)}$, that is, $x^{(j)} \in \mathrm{co}(x^{(i)}, x^{(k)})$, or $\alpha > 1$ that implies that $x^{(i)} \in \mathrm{co}(x^{(j)}, x^{(k)})$, which are both impossible occurrences. So the hypothesis (1) is proved.

Also the hypothesis (3.43) is verified. In fact, $x^{(k)} \in \Sigma_{ij}^{+}$ if and only if

$$|\langle y(\theta), x^{(k)}\rangle| = |\bar{\lambda}_i + e^{i\theta}\bar{\lambda}_j| > 1 \;\; \forall\, \theta \in (0, \theta_0). \tag{3.49}$$

On the other hand, $x^{(k)} \in \Sigma_{ji}^{-}$ if and only if $|\langle \hat{y}(\theta), x^{(k)}\rangle| > 1 \quad \forall\, \theta \in (-\theta_0, 0)$, where $\hat{y}(\theta) = \left[x^{(j)} \;\; x^{(i)}\right]^{-H} \begin{bmatrix} 1 \\ e^{i\theta} \end{bmatrix}$, that is, if and only if

$$|\bar{\lambda}_j + e^{i\theta}\bar{\lambda}_i| > 1 \;\; \forall\, \theta \in (-\theta_0, 0) \iff$$
$$|e^{-i\theta}\bar{\lambda}_j + \bar{\lambda}_i| > 1 \;\; \forall\, \theta \in (-\theta_0, 0) \iff$$
$$|e^{i\theta}\bar{\lambda}_j + \bar{\lambda}_i| > 1 \;\; \forall\, \theta \in (0, \theta_0),$$

that is the inequality (3.49). Therefore, $\Sigma_{ij}^{+} = \Sigma_{ji}^{-}$.

Then we check the hypothesis (2) as follows. As already seen, we have that $x^{(k)} \in \Sigma_{ij}^{+}$ if and only if the inequality (3.49) is satisfied, that is, if and only if $f'(0) = -\beta > 0$ in (3.48). The equality (3.47) implies that $x^{(j)} = -\frac{\lambda_i}{\lambda_j}x^{(i)} + \frac{1}{\lambda_j}x^{(k)}$. Then $x^{(j)} \in \Sigma_{ik}^{-}$ if and only if $|\langle z(\theta), x^{(j)}\rangle| > 1$ for all $\theta \in (-\theta_0, 0)$, where $z(\theta)$ is the functional associated with the facet $x^{(i)} \bullet\!\!-\!\!\bullet\, e^{i\theta}x^{(k)}$, that is, if and only if

$$\left|-\frac{\bar{\lambda}_i}{\bar{\lambda}_j} + e^{i\theta}\frac{1}{\bar{\lambda}_j}\right| > 1 \quad \forall\, \theta \in (-\theta_0, 0),$$

that is, if and only if

$$g(\theta) = |e^{i\theta} - \bar{\lambda}_i|^2 - |\bar{\lambda}_j|^2 = \alpha^2(1 - \cos\theta) + \beta \sin\theta > 0$$

for all $\theta \in (-\theta_0, 0)$, that is if and only if $\beta < 0$. Therefore, we have shown that

$$x^{(k)} \in \Sigma^+_{ij} \iff x^{(j)} \in \Sigma^-_{ik}.$$

Lastly, since $x^{(1)}$ generates only isolated facets, the condition (3.39) must hold. In fact, for any $j \neq 1$ the segment $x^{(1)} \bullet\!\!-\!\!\bullet\, x^{(j)}$ is a degenerate facet and, it being isolated, there must exist $x^{(k)} \in S$ such that $x^{(k)} \in \Sigma^+_{1j}$ and $x^{(l)} \in S$ such that $x^{(l)} \in \Sigma^-_{1j}$ (with $l \neq k$ because of property (1)). Thus, by Lemmas 3.4 and 3.5, we have that

$$x^{(1)} \in \Sigma^+_{\alpha\beta} \cap \Sigma^+_{\beta\gamma} \cap \Sigma^+_{\gamma\alpha} \tag{3.50}$$

for a suitable triplet (α, β, γ).

Now we show that $x^{(1)} \in \mathrm{co}\left(\left\{\mathrm{x}^{(\alpha)}, \mathrm{x}^{(\beta)}, \mathrm{x}^{(\gamma)}\right\}\right)$. We have

$$\begin{cases} x^{(1)} = \lambda_\alpha x^{(\alpha)} + \lambda_\beta x^{(\beta)} & \text{with} \quad \lambda_\alpha, \lambda_\beta \in \mathbb{C} \\ x^{(1)} = \varphi_\beta x^{(\beta)} + \varphi_\gamma x^{(\gamma)} & \text{with} \quad \varphi_\beta, \varphi_\gamma \in \mathbb{C} \\ x^{(1)} = \xi_\gamma x^{(\gamma)} + \xi_\alpha x^{(\alpha)} & \text{with} \quad \xi_\gamma, \xi_\alpha \in \mathbb{C} \end{cases} \tag{3.51}$$

and then, since $x^{(1)}, x^{(\alpha)}, x^{(\beta)}, x^{(\gamma)} \in S$ are vertexes of the same special facet, we obtain

$$\lambda_\alpha + \lambda_\beta = \varphi_\beta + \varphi_\gamma = \xi_\gamma + \xi_\alpha = 1. \tag{3.52}$$

Remark that (3.50) is equivalent to

$$\begin{cases} |\langle y_{\alpha\beta}(\theta), x^{(1)}\rangle| > 1 \\ |\langle y_{\beta\gamma}(\theta), x^{(1)}\rangle| > 1 \\ |\langle y_{\gamma\alpha}(\theta), x^{(1)}\rangle| > 1 \end{cases}, \quad \forall\theta \in (0, \theta_0) \tag{3.53}$$

for a suitable θ_0, where $y_{\alpha\beta}(\theta), y_{\beta\gamma}(\theta), y_{\gamma\alpha}(\theta)$ are the functionals associated with the facets $x^{(\alpha)} \bullet\!\!-\!\!\bullet\, e^{i\theta} x^{(\beta)}$, $x^{(\beta)} \bullet\!\!-\!\!\bullet\, e^{i\theta} x^{(\gamma)}$, $x^{(\gamma)} \bullet\!\!-\!\!\bullet\, e^{i\theta} x^{(\alpha)}$, respectively. In turn, the system (3.53) is equivalent to

$$\begin{cases} |\bar{\lambda}_\alpha + e^{i\theta}\bar{\lambda}_\beta| > 1 \\ |\bar{\varphi}_\beta + e^{i\theta}\bar{\varphi}_\gamma| > 1 \\ |\bar{\xi}_\gamma + e^{i\theta}\bar{\xi}_\alpha| > 1 \end{cases} \quad \forall\theta \in (0, \theta_0),$$

which implies

$$\begin{cases} Im(\lambda_\alpha) < 0 \\ Im(\varphi_\beta) < 0 \\ Im(\xi_\gamma) < 0 \end{cases}. \tag{3.54}$$

Furthermore, (3.52) implies

$$\begin{cases} Im(\lambda_\alpha) = -Im(\lambda_\beta) \\ Im(\varphi_\beta) = -Im(\varphi_\gamma) \\ Im(\xi_\gamma) = -Im(\xi_\alpha) \end{cases}. \tag{3.55}$$

Moreover, by the last two equations of (3.51) we obtain that

$$\varphi_\beta x^{(\beta)} + (\varphi_\gamma - \xi_\gamma)x^{(\gamma)} - \xi_\alpha x^{(\alpha)} = 0$$

with $\varphi_\gamma - \xi_\gamma \neq 0$, because $x^{(\alpha)}, x^{(\beta)}$ are linearly independent. Thus,

$$x^{(1)} = \frac{\xi_\alpha \varphi_\gamma}{\varphi_\gamma - \xi_\gamma} x^{(\alpha)} + \frac{\varphi_\beta \xi_\gamma}{\varphi_\gamma - \xi_\gamma} x^{(\beta)},$$

from which it follows that

$$\lambda_\alpha = \frac{\xi_\alpha \varphi_\gamma}{\varphi_\gamma - \xi_\gamma} \quad \text{and} \quad \lambda_\beta = \frac{-\varphi_\beta \xi_\gamma}{\varphi_\gamma - \xi_\gamma}. \tag{3.56}$$

Since we can write

$$\lambda_\alpha = \frac{\xi_\alpha \varphi_\gamma (\bar{\varphi}_\gamma - \bar{\xi}_\gamma)}{|\varphi_\gamma - \xi_\gamma|^2},$$

we have that $Im(\lambda_\alpha) < 0$ (see (3.54)) if and only if $Im(\xi_\alpha \varphi_\gamma (\bar{\varphi}_\gamma - \bar{\xi}_\gamma)) < 0$, that is, if and only if

$$\left(|\varphi_\gamma|^2 - Re(\varphi_\gamma)\right) Im(\xi_\alpha) + \left(|\xi_\alpha|^2 - Re(\xi_\alpha)\right) Im(\varphi_\gamma) < 0. \tag{3.57}$$

Therefore, since $Im(\xi_\alpha) > 0$ and $Im(\varphi_\gamma) > 0$, at least one between

$$|\varphi_\gamma|^2 - Re(\varphi_\gamma) < 0 \tag{3.58}$$

and

$$|\xi_\alpha|^2 - Re(\xi_\alpha) < 0 \tag{3.59}$$

holds. Analogously, if we compute ξ_γ and φ_β from (3.56), then we obtain that $Im(\xi_\gamma) < 0$ (see (3.54)) implies that at least one between (3.58) and

$$|\lambda_\beta|^2 - Re(\lambda_\beta) < 0 \tag{3.60}$$

holds, and $Im(\varphi_\beta) < 0$ (see (3.54)) implies that at least one between (3.59) and (3.60) holds.
In conclusion, at the most one of the conditions (3.58), (3.59), (3.60) may not hold. So we can suppose without restriction that (3.58) and (3.59) hold, which in turn imply

$$0 < Re(\varphi_\gamma) < 1 \tag{3.61}$$

and

$$0 < Re(\xi_\alpha) < 1. \tag{3.62}$$

Now, observe that we can write $x^{(1)}$ as

$$x^{(1)} = u\xi_\alpha^{-1}(\xi_\gamma x^{(\gamma)} + \xi_\alpha x^{(\alpha)}) + v\varphi_\beta^{-1}(\varphi_\beta x^{(\beta)} + \varphi_\gamma x^{(\gamma)})$$

with $u, v \in \mathbb{R}$ such that

$$u\xi_\alpha^{-1} + v\varphi_\beta^{-1} = 1, \tag{3.63}$$

that is,

$$x^{(1)} = ux^{(\alpha)} + vx^{(\beta)} + (1 - u - v)x^{(\gamma)} \quad \text{with } u, v \in \mathbb{R}. \tag{3.64}$$

It is easy to show that (3.63), (3.54), (3.55), (3.61) and (3.62) imply $u > 0$ and $v > 0$. Moreover, since (3.63) implies

$$(u + v)\left[Im(\xi_\alpha)Re(\varphi_\beta) - Re(\xi_\alpha)Im(\varphi_\beta)\right] = Im(\xi_\alpha)|\varphi_\beta|^2 - Im(\varphi_\beta)|\xi_\alpha|^2,$$

using (3.58) and (3.59) and the fact that

$$|\varphi_\gamma|^2 - Re(\varphi_\gamma) = |\varphi_\beta|^2 - Re(\varphi_\beta),$$

we obtain $1 - u - v > 0$.

Therefore, we can conclude that $x^{(1)} \in \text{co}\left(\left\{\mathrm{x}^{(\alpha)}, \mathrm{x}^{(\beta)}, \mathrm{x}^{(\gamma)}\right\}\right)$, that is, $x^{(1)}$ is not a vertex of $\mathcal{P}$, which makes it absurd. ■

The constructive procedure of a b.c.p. $\mathcal{P} = \text{absco}(\mathcal{X})$, where $\mathcal{X} = \{x^{(k)}\}_{1 \le k \le r} \subset \mathbb{C}^2$, $r \geq 4$, works in an iterative fashion. From the already available b.c.p. $\mathcal{P}^{(k-1)} = \text{absco}(V^{(k-1)})$, where $k \geq 4$ and, by possibly redefining the indexes, $V^{(k-1)} = \{x^{(1)}, \cdots, x^{(k-1)}\}$ contains the first $k-1$ vectors of $\mathcal{X}$ which determine an essential system of vertexes, we construct $\mathcal{P}^{(k)}$ adding $x^{(k)}$ to $\mathcal{P}^{(k-1)}$.

To perform this addition, we consider all the two-vertex subpolytopes $\mathcal{P}_{ij} = \text{absco}(\{x^{(i)}, x^{(j)}\})$, $1 \le i < j \le k-1$, of $\mathcal{P}^{(k-1)}$, and write

$$x^{(k)} = \lambda_{ij}^{(k)} x^{(i)} + \mu_{ij}^{(k)} x^{(j)}, \tag{3.65}$$

where $\lambda_{ij}^{(k)}, \mu_{ij}^{(k)} \in \mathbb{C}$.

If $\|x^{(k)}\|_{\mathcal{P}_{ij}} = \|[x^{(i)} x^{(j)}]^{-1} x^{(k)}\|_1 \le 1$ for some $j = 2, \cdots, k-1$ and $i = 1, \cdots, j-1$, then necessarily $x^{(k)} \in \mathcal{P}^{(k-1)}$. In this case, the point $x^{(k)}$ is deleted, we go on to add $x^{(k+1)}$ to $\mathcal{P}^{(k-1)}$ and the set $\mathcal{X}$ is redefined as $\mathcal{X} \backslash \{x^{(k)}\}$.

Otherwise, we set

$$V^{(k)} = V^{(k-1)} \cup \{x^{(k)}\}$$

and perform the following steps:

(step 1) deletion from $\mathcal{X}$ and from $V^{(k)}$ of all those vectors which are deleted as vertexes of a two-vertex subpolytope because of the addition of $x^{(k)}$ and the computation of the existence pluri-intervals of those regular facets of $\mathcal{P}^{(k)}$ which were already regular facets of $\mathcal{P}^{(k-1)}$;

(step 2) addition of the new facets of $\mathcal{P}^{(k)}$ whose second vertex is $x^{(k)}$;

(step 3) deletion from $V^{(k)}$ and $\mathcal{X}$ of all the vectors which belong only to isolated facets or to no facets at all.

Now, we illustrate the procedure in more detail beginning with (step 1). First of all, we remark that the facets of $\mathcal{P}^{(k-1)}$ are represented by the already computed existence pluri-intervals $\mathcal{E}_{ij}^{(k-1)}$, $1 \le i < j \le k-1$. By generalising the deleting procedure of Section 3.2.2, we consider all the two-vertex subpolytpes $\mathcal{P}_{ij}$, $1 \le i < j \le k-1$, as follows.

Observe that the vector $x^{(k)}$ may directly delete some of the previous vertexes. As a matter of facts, with reference to (3.65), it may happen that $|\mu_{ij}^{(k)}| \ge |\lambda_{ij}^{(k)}| + 1$, in which case $x^{(i)}$ is deleted, or $|\lambda_{ij}^{(k)}| \ge |\mu_{ij}^{(k)}| + 1$, in which case $x^{(j)}$ is deleted. The deleted vertex, if any, is removed both from $\mathcal{X}$ and $V^{(k)}$ and, of course, also all the existence pluri-intervals that involve it are not considered any more.

Otherwise, we have

$$\left| |\lambda_{ij}^{(k)}| - |\mu_{ij}^{(k)}| \right| < 1$$

and, extending the discussion of the previous subsection, we observe that the point $x^{(k)}$ may see only some regular facets $F_{y_{ij}(\theta)}$ of $\mathcal{P}^{(k-1)}$, $j = 2, \cdots, k-1$ and $i = 1, \cdots, j-1$. These facets have to be deleted from $\mathcal{E}_{ij}^{(k-1)}$ whereas the remaining facets are the new corresponding facets $F_{y_{ij}(\theta)}$ of the b.c.p. $\mathcal{P}^{(k)}$.

The facets of the kind $F_{y_{ij}(\theta)} = x^{(i)} \bullet\!\!-\!\!\bullet\, e^{i\theta} x^{(j)}$ of $\mathcal{P}_{ij}$, whose associated functional is given by

$$y_{ij}(\theta) = [x^{(i)}\, x^{(j)}]^{-H} \begin{bmatrix} 1 \\ e^{i\theta} \end{bmatrix},$$

are "seen" by the "circle" generated by $x^{(k)}$ and so deleted if and only if

$$|\langle y_{ij}(\theta), x^{(k)} \rangle| > 1,$$

that is, if and only if

$$\theta \in \mathcal{D}_{ij}^{(k)} = \left(-\hat{\theta}_{ij}^{(k)} - \arg(\lambda_{ij}^{(k)}) + \arg(\mu_{ij}^{(k)}), \hat{\theta}_{ij}^{(k)} - \arg(\lambda_{ij}^{(k)}) + \arg(\mu_{ij}^{(k)}) \right) \quad \text{shift}(-\pi, \pi], \tag{3.66}$$

where

$$\hat{\theta}_{ij}^{(k)} = \arccos\left(\frac{1 - |\lambda_{ij}^{(k)}|^2 - |\mu_{ij}^{(k)}|^2}{2|\lambda_{ij}^{(k)}||\mu_{ij}^{(k)}|}\right) \in (0, \pi).$$

The set $\mathcal{D}_{ij}^{(k)}$ is the *deleting interval* generated by $x^{(k)}$ and associated with the facets $F_{y_{ij}(\theta)}$. Consequently, the existence pluri-interval $\mathcal{E}_{ij}^{(k)}$ of the facets $F_{y_{ij}(\theta)}$ of $\mathcal{P}^{(k)}$, $1 \leq i < j \leq k-1$, is given by

$$\mathcal{E}_{ij}^{(k)} = \mathcal{E}_{ij}^{(k-1)} \backslash \mathcal{D}_{ij}^{(k)}. \tag{3.67}$$

Remark that $\mathcal{E}_{ij}^{(k)}$ may be empty, in which case all the facets $F_{y_{ij}(\theta)}$ are deleted.

We observe that, after processing all the subpolytopes $\mathcal{P}_{ij}$, $1 \leq i < j \leq k-1$, the vertex set $V^{(k)}$ may be reduced to just two elements. In this case, we have to go back to the very beginning of the procedure and go on to add $x^{(k+1)}$ as the third vertex. Moreover, if $V^{(k)}$ is reduced to three elements, then we complete the iteration from $\mathcal{P}^{(k-1)}$ to $\mathcal{P}^{(k)}$ by using the procedure of Subsection 3.2.2.

In (step 2), the computation of the new facets $F_{y_{ik}(\theta)} = x^{(i)} \bullet\!\!-\!\!\bullet\, e^{\mathrm{i}\theta} x^{(k)}$, $i \leq k-1$, of $\mathcal{P}^{(k)}$ whose second vertex is $x^{(k)}$, is made by using the following system of $k-2$ inequalities in the variable θ:

$$|\langle y_{ik}(\theta), x^{(q)}\rangle| \leq 1\,, \quad q \leq k-1\,,\, q \neq i, \tag{3.68}$$

where $y_{ik}(\theta) = [x^{(i)}\, x^{(k)}]^{-H} \begin{bmatrix} 1 \\ e^{\mathrm{i}\theta} \end{bmatrix}$ is the functional associated with the facets $F_{y_{ik}(\theta)}$. The existence pluri-interval of the facets $F_{y_{ik}(\theta)}$ of $\mathcal{P}^{(k)}$, which is the solution of the previous system, will be denoted by $\mathcal{E}_{ik}^{(k)}$. Remark that (3.68) is a generalisation of (3.18), since we must add all the facets $F_{y_{ik}(\theta)}$ which are not seen from the circles generated by all the remaining $k-2$ vertexes $x^{(q)}$ with $1 \leq q \leq k-1, q \neq i$.

It turns out that the minima of the intervals which constitute $\mathcal{E}_{ik}^{(k)}$ belong to the set of numbers

$$\left\{\theta_{ik}^{(k)-}(q) = -\hat{\theta}_{ik}^{(k)}(q) - \arg(\lambda_{iq}^{(k)}) \quad \text{shift } (-\pi, \pi]\right\}_{1 \leq q \leq k-1,\ q \neq i} \cup \{-\pi\},$$

where

$$\hat{\theta}_{ik}^{(k)}(q) = \arccos\left(\frac{1 + |\lambda_{iq}^{(k)}|^2 - |\mu_{iq}^{(k)}|^2}{2|\lambda_{iq}^{(k)}|}\right) \in (0, \pi),$$

whereas the maxima belong to the set

$$\left\{\theta_{ik}^{(k)+}(q) = \hat{\theta}_{ik}^{(k)}(q) - \arg(\lambda_{iq}^{(k)}) \quad \text{shift } (-\pi, \pi]\right\}_{1 \leq q \leq k-1,\ q \neq i} \cup \{\pi\}.$$

In (step 3) we analyse all the existence pluri-intervals $\mathcal{E}_{ij}^{(k)}$, $1 \leq i < j \leq k-1$, in order to delete all the vectors that do not belong to any facet of $\mathcal{P}^{(k)}$ or which produce only isolated facets from $V^{(k)}$ and from $\mathcal{X}$. These vectors, although not belonging to any of the subpolytopes generated by all the pairs of other elements of $V^{(k)}$, are either inside or on the boundary of the b.c.p. generated by all the other vertexes.

More precisely, we first look for those vectors $x^{(s)} \in V^{(k)}$, if any, such that all the pluri-intervals $\mathcal{E}_{is}^{(k)}$, $1 \leq i < s$, and $\mathcal{E}_{sj}^{(k)}$, $s < j \leq k$, are empty. These vectors do not belong to any facets of $\mathcal{P}^{(k)}$ and hence, by Theorem 2.3, they are not vertexes of $\mathcal{P}^{(k)}$ and must be deleted.

Then we look for those vectors $x^{(s)} \in V^{(k)}$, if any, such that all the pluri-intervals $\mathcal{E}_{is}^{(k)}$, $1 \leq i < s$, and $\mathcal{E}_{sj}^{(k)}$, $s < j \leq k$, are constituted exclusively by intervals that degenerate to single points. Since a single point represents an isolated facet, these vectors only belong to isolated facets and then, by Theorem 3.4, they are not vertexes of $\mathcal{P}^{(k)}$ either and must be deleted as well.

Once we have processed the last vector $x^{(r)} \in \mathcal{X}$, we have found the wanted b.c.p. $\mathcal{P} = \text{absco}(\mathcal{X})$. At this stage the set $\mathcal{X}$ has been possibly reduced in the number of elements and we have

$$\mathcal{X} = V^{(m)} = \{x^{(i_p)}\}_{1\le p\le m}\,,$$

with $\{i_1, \cdots, i_m\} \subseteq \{1, \cdots, r\}$, which represents an essential system of vertexes of $\mathcal{P}$. In order to simplify the notation, possibly redefining the indexes, we rename

$$\mathcal{X} = \{x^{(i)}\}_{1\le i\le m}.$$

At this point, we determine all the (triangles included in) special facets of $\mathcal{P}$ by analysing its existence pluri-intervals $\mathcal{E}_{ij}$, $1 \le i < j \le m$, in the light of Theorem 3.3. More precisely, for all the triplets (i, j, k) with $1 \le i < j < k \le m$, we check the validity of the equalities

$$\theta^+_{ij} + \theta^+_{jk} = \theta^-_{ik} \qquad \text{mod } 2\pi \tag{3.69}$$

and

$$\theta^-_{ij} + \theta^-_{jk} = \theta^+_{ik} \qquad \text{mod } 2\pi, \tag{3.70}$$

where $[\theta^-_{rs}, \theta^+_{rs}]$, $r, s \in \{i, j, k\}$, stands for any of the m_{rs} intervals of the existence pluri-interval $\mathcal{E}_{rs}$. The triangle $x^{(i)} \blacktriangle e^{i\theta^+_{ij}} x^{(j)} \blacktriangle e^{i\theta^-_{ik}} x^{(k)}$ is (included in) a special facet if and only if the equality (3.69) holds, whereas the triangle $x^{(i)} \blacktriangle e^{i\theta^-_{ij}} x^{(j)} \blacktriangle e^{i\theta^+_{ik}} x^{(k)}$ is (included in) a special facet if and only if the equality (3.70) holds. Note that, for a given triplet (i, j, k), three different cases are possible: either none or one or both equalities (3.69) and (3.70) may hold. Then, for each triplet (i, j, k) with $1 \le i < j < k \le m$, we define the *existence pairs* of triangles included in special facets of $\mathcal{P}$ as follows:

$$\mathcal{S}_{ijk} = \begin{cases} \emptyset & \text{if neither (3.69) nor (3.70) holds,} \\ \{(\theta^+_{ij}, \theta^-_{ik})\} & \text{if only (3.69) holds,} \\ \{(\theta^-_{ij}, \theta^+_{ik})\} & \text{if only (3.70) holds,} \\ \{(\theta^+_{ij}, \theta^-_{ik}), (\theta^-_{ij}, \theta^+_{ik})\} & \text{if both (3.69) and (3.70) hold.} \end{cases} \tag{3.71}$$

Example 3.4. We show how to construct the b.c.p. $\mathcal{P} = \text{absco}(\mathcal{V})$, where

$$\mathcal{V} = \{x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}\}, \quad x^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix}, \quad x^{(2)} = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}, \quad x^{(3)} = \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}, \quad x^{(4)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Since

$$x^{(3)} = \frac{1}{3}x^{(1)} + \frac{1}{3}x^{(2)} + \frac{1}{3}x^{(4)}, \tag{3.72}$$

we should find the same b.c.p. $\mathcal{P} = \text{absco}(\{x^{(1)}, x^{(2)}, x^{(4)}\})$ we considered in Example 3.2.

Since $x^{(1)}$ and $x^{(2)}$ are linearly independent, we set

$$\mathcal{X}^{(2)} = \{x^{(1)}, x^{(2)}\}$$

and add $x^{(3)}$ to $\mathcal{P}^{(2)}$ using the procedure described in Section 3.2.2. The vector $x^{(3)}$ is such that

$$\|[x^{(1)}x^{(2)}]^{-1}x^{(3)}\|_1 = 1.008888370\ldots > 1,$$

so that $x^{(3)} \notin \mathcal{P}^{(2)}$ and we define

$$\mathcal{X}^{(3)} = \mathcal{X}^{(2)} \cup \{x^{(3)}\}.$$

Moreover, neither $x^{(1)}$ nor $x^{(2)}$ is deleted by $x^{(3)}$ and, hence, we compute

$$\begin{aligned} \mathcal{E}^{(3)}_{12} &= (-\pi,\ 0] \cup [0.532504098\ldots,\ \pi], \\ \mathcal{E}^{(3)}_{13} &= [0,\ 0.283794109\ldots], \\ \mathcal{E}^{(3)}_{23} &= [-0.248709989\ldots,\ 0]. \end{aligned}$$

Then we go on to add $x^{(4)}$ to $\mathcal{P}^{(3)}$. For this purpose, we consider the two-vertex subpolytopes $\mathcal{P}_{ij} = \text{absco}(\{x^{(i)}, x^{(j)}\})$, $1 \le i < j \le 3$, of $\mathcal{P}^{(3)}$ and compute

$$\begin{aligned}
\|[x^{(1)}x^{(2)}]^{-1}x^{(4)}\|_1 &= 1.079669127\ldots > 1,\\
\|[x^{(1)}x^{(3)}]^{-1}x^{(4)}\|_1 &= 1.454046908\ldots > 1,\\
\|[x^{(2)}x^{(3)}]^{-1}x^{(4)}\|_1 &= 1.264911064\ldots > 1.
\end{aligned}$$

Therefore, we define

$$\mathcal{X}^{(4)} = \mathcal{X}^{(3)} \cup \{x^{(4)}\}$$

and proceed with (step 1). It turns out that none among $x^{(1)}, x^{(2)}, x^{(3)}$ is deleted by $x^{(4)}$ and, hence, by using formulae (3.66) and (3.67), we compute

$$\begin{aligned}
\mathcal{E}_{12}^{(4)} &= (-\pi,\ 0] \cup [1.570796326\ldots,\ \pi],\\
\mathcal{E}_{13}^{(4)} &= [0,\ 0],\\
\mathcal{E}_{23}^{(4)} &= [0,\ 0].
\end{aligned}$$

Then we proceed with (step 2) and compute

$$\begin{aligned}
\mathcal{E}_{14}^{(4)} &= [0,\ 0.927295218\ldots],\\
\mathcal{E}_{24}^{(4)} &= [-0.643501108\ldots,\ 0],\\
\mathcal{E}_{34}^{(4)} &= [0,\ 0].
\end{aligned}$$

The analysis of (step 3) reveals that vector $x^{(3)}$ belongs only to isolated facets, namely the facets $x^{(1)} \bullet\!\!-\!\!\bullet\ x^{(3)}$, $x^{(2)} \bullet\!\!-\!\!\bullet\ x^{(3)}$ and $x^{(3)} \bullet\!\!-\!\!\bullet\ x^{(4)}$, which is consistent with the fact that it is a convex linear combination of the remaining vertexes (see (3.72)). Thus, we remove $x^{(3)}$ from $\mathcal{X}^{(4)}$ to obtain

$$\mathcal{X}^{(4)} = \{x^{(1)}, x^{(2)}, x^{(4)}\},$$

and we also remove the existence pluri-intervals $\mathcal{E}_{13}^{(4)}, \mathcal{E}_{23}^{(4)}, \mathcal{E}_{34}^{(4)}$.

Finally, the analysis of the surviving existence pluri-intervals $\mathcal{E}_{12}^{(4)}, \mathcal{E}_{14}^{(4)}, \mathcal{E}_{24}^{(4)}$ leads to the conclusion that the special facets of $\mathcal{P}$ are represented by the existence pairs

$$\mathcal{S}_{124} = \{(1.570796326\ldots,\ 0.927295218\ldots), (0,\ 0)\}.$$

◇

We conclude this subsection by giving a useful result on the width of existence pluri-intervals of the regular facets of a b.c.p. $\mathcal{P}$ with an arbitrary number of essential vertexes. For this purpose, we need the following definition.

Definition 3.9. *Let $\mathcal{P}$ be a b.c.p. The width of an existence pluri-interval*

$$\mathcal{E}_{ij} = \bigcup_{l=1}^{m_{ij}} [\theta_{ij,l}^{-}, \theta_{ij,l}^{+}] \setminus \{-\pi\}$$

of the facets of the kind $F_{y_{ij}(\theta)}$ is

$$|\mathcal{E}_{ij}| = \sum_{l=1}^{m_{ij}} (\theta_{ij,l}^{+} - \theta_{ij,l}^{-})$$

which is the sum of the widths of each one of its intervals. In particular, if $\mathcal{P}$ has no facets of the type $F_{y_{ij}(\theta)}$ at all, that is, $\mathcal{E}_{ij} = \emptyset$, we set $|\mathcal{E}_{ij}| = 0$.

Theorem 3.5. *Let $\mathcal{P}$ be a b.c.p. with $m \geq 2$ essential vertexes. Then the sum of the widths of the existence pluri-intervals of all its regular facets is 2π, that is,*

$$\sum_{1\leq i<j\leq m} |\mathcal{E}_{ij}| = 2\pi.$$

Proof. Definition 3.9 implies

$$\sum_{1\leq i<j\leq m} |\mathcal{E}_{ij}| = \sum_{1\leq i<j\leq m} \sum_{l=1}^{m_{ij}} (\theta^+_{ij,l} - \theta^-_{ij,l}).$$

Moreover, for all pairs (i, j) such that $1 \leq i < j \leq m$ and for all $l = 1, \ldots, m_{ij}$, each extremal value $\theta^\pm_{ij,l}$ of an existence interval, different from $-\pi$ and, possibly, from π, determines a degenerate facet. Therefore, in view of (3.37) and (3.38), each extremal value $\theta^+_{ij,l}$, different from $-\pi$ and, possibly from π, necessarily satisfies one of the following conditions:

- $\theta^+_{ij,l} - \theta^-_{ih,l'} - \theta^-_{hj,l''} = 0 \mod 2\pi$, for suitable $i < h < j$ and $1 \leq l' \leq m_{ih}$, $1 \leq l'' \leq m_{hj}$;
- $\theta^+_{ij,l} + \theta^+_{jh,l'} - \theta^-_{ih,l''} = 0 \mod 2\pi$, for suitable $h > j$ and $1 \leq l' \leq m_{jh}$, $1 \leq l'' \leq m_{ih}$;
- $\theta^+_{hi,l'} + \theta^+_{ij,l} - \theta^-_{hj,l''} = 0 \mod 2\pi$, for suitable $h < i$ and $1 \leq l' \leq m_{hi}$, $1 \leq l'' \leq m_{hj}$.

Analogously, every value $\theta^-_{ij,l}$ verifies one of the following conditions:

- $-\theta^-_{ij,l} + \theta^+_{ik,l'} + \theta^+_{kj,l''} = 0 \mod 2\pi$, for suitable $i < k < j$ and $1 \leq l' \leq m_{ik}$, $1 \leq l'' \leq m_{kj}$;
- $-\theta^-_{ij,l} - \theta^-_{jk,l'} + \theta^+_{ik,l''} = 0 \mod 2\pi$, for suitable $k > j$ and $1 \leq l' \leq m_{jk}$, $1 \leq l'' \leq m_{ik}$;
- $-\theta^-_{ki,l'} - \theta^-_{ij,l} + \theta^+_{kj,l''} = 0 \mod 2\pi$, for suitable $k < i$ and $1 \leq l' \leq m_{ki}$, $1 \leq l'' \leq m_{kj}$.

Moreover, each value $\theta^\pm_{ij,l}$ may be involved just once in the above formulae. As a consequence, we have that $\sum_{1\leq i<j\leq m} |\mathcal{E}_{ij}| = 0 \mod 2\pi$.

Lastly, consider the function f that associates the quantity $\sum_{1\leq i<j\leq m} |\mathcal{E}_{ij}|$ with each b.c.p. $\mathcal{P}$. If we introduce the usual Euclidean distance among bounded subsets of $\mathbb{C}^n$ in the set of all the b.c.p.'s in $\mathbb{C}^n$, the function f turns out to be continuous. This follows from the continuity of the functions that define the extremal values $\theta^\pm_{ij,l}$. Therefore, since for a 2-vertex b.c.p. $\mathcal{P}^{(2)}$ we always have $\sum_{1\leq i<j\leq 2} |\mathcal{E}_{ij}| = |\mathcal{E}_{12}| = |(-\pi, \pi]| = 2\pi$, the proof is complete. ■

We conclude this section by giving a relationship among the number N_T of all the possible triangles of $\mathcal{P}$, the number N_{EI} of the existence intervals of $\mathcal{P}$ and the number N_{IF} of its isolated facets. Using Proposition 3.1, we have that the number of all the intervals which constitute the existence pluri-intervals of $\mathcal{P}$ is given by

$$N_{EI} = \sum_{1\leq i<j\leq m} m^*_{ij},$$

where

$$m^*_{ij} = \begin{cases} m_{ij} - 1 & \text{if } \mathcal{E}_{ij} \supseteq (-\pi, a] \cup [b, \pi] \text{ for some } -\pi < a < b < \pi \\ m_{ij} & \text{otherwise.} \end{cases}$$

Note that every triangle of a b.c.p. $\mathcal{P}$ has three sides that are degenerate facets, which are the facets obtained in correspondence of the extremal values of the existence intervals of the regular facets of $\mathcal{P}$. Clearly, the number of all the degenerate facets of $\mathcal{P}$ is $2N_{EI}$.

Now we relate the number N_T of all the triangles of $\mathcal{P}$ to N_{IF} and N_{EI}.

Clearly, if $N_{IF} = 0$, i.e. there are no isolated facets, then we have $3N_T = 2N_{EI}$. In fact, in this case two different triangles cannot have a common side.

Otherwise, if $N_{IF} \geq 1$, Proposition 3.3 states that each isolated facet is a diagonal of a special facet F with $p \geq 4$ vertexes. Thus, the isolated facets included in the special facet F are nothing but its diagonals and so we can conclude that their number is

$$N_{IF_p} = \frac{p(p-3)}{2} \geq 2.$$

Therefore, F must have at least 2 diagonals (isolated facets) and so the case $N_{IF} = 1$ is not feasible.

Now we consider a b.c.p. $\mathcal{P}$ with just one p-vertex special facet, $p \geq 4$, and analyse how the numbers N_T, N_{EI} and N_{IF} are related to one another. Remark that each diagonal (isolated facet) is counted double in $2N_{EI}$, since it corresponds to an interval that degenerates to a single point, whereas each side (degenerate but not isolated facet) is counted single. Indeed, since every diagonal and every side of a p-vertex special facet is a side of $p-2$ distinct triangles, then the number $3N_T$ of the degenerate facets that we need to obtain N_T triangles includes, besides the $2N_{EI}$ degenerate facets of $\mathcal{P}$, further $p-4$ sides for every diagonal (altogether $N_{IF_p}(p-4)$ sides) and further $p-3$ sides for every side (altogether $p(p-3)$ sides), that is,

$$3N_T = 2N_{EI} + \frac{p(p-3)}{2}(p-4) + p(p-3)$$

or, equivalently,

$$3N_T = 2N_{EI} + (p-2)N_{IF_p}.$$

Now, if we consider the general case of a b.c.p. which, for $p = 4, \ldots, M$, has α_p special facets with p vertexes, then the number N_{IF} of its isolated facets is

$$N_{IF} = \sum_{p=4}^{M} \alpha_p N_{IF_p} = 2\alpha_4 + 5\alpha_5 + \cdots + \frac{M(M-3)}{2}\alpha_M. \tag{3.73}$$

As a consequence, even the case $N_{IF} = 3$ is not feasible.

We can conclude that, in general, the relationship among N_T, N_{EI} and N_{IF} is

$$3N_T = 2N_{EI} + \sum_{p=4}^{M} \alpha_p N_{IF_p}(p-2). \tag{3.74}$$

Remark 3.5. *As we have seen before, any b.c.p. with more then three vertexes cannot have either $N_{IF} = 1$ or $N_{IF} = 3$ isolated facets. Indeed, these are the only two values for N_{IF} which are, a priori, not admissible, that is, $N_{IF} \in \mathbb{N}\backslash\{1,3\}$. To see this, we note that every even value of N_{IF} may be obtained as $2\alpha_4$ for a proper choice of α_4. On the other hand, every odd value of N_{IF} greater than 3, i.e. $N_{IF} = 2j+1$ with $j \geq 2$, may be obtained considering only the first two terms of (3.73); in fact, we may write*

$$N_{IF} = 2j + 1 = 2\alpha_4 + 5\alpha_5 = 2(\alpha_4 + 2\alpha_5) + \alpha_5$$

which is satisfied, for example, by taking $\alpha_5 = 1$ and $\alpha_4 = j-2$.

Formulae (3.73) and (3.74) may be used to improve the information about the geometry of $\mathcal{P}$, finding how many special facets are present in the b.c.p. and how many vertexes each of them has. We illustrate this in the following example.

Example 3.5. Let us consider a b.c.p. $\mathcal{P}$ with $N_{IF} = 10$ isolated facets, $N_T = 20$ triangles and $N_{EI} = 15$ existence intervals. Now, to use equation (3.74), we have to compute the values of α_p for $p \geq 4$. Now, from equation (3.73) we have

$$10 = 2\alpha_4 + 5\alpha_5 + 9\alpha_6 + \ldots.$$

Therefore, it must necessarily be $\alpha_p = 0$ for $p \geq 6$ and we have two possible cases:

- $\mathcal{P}$ has $\alpha_4 = 5$ quadrilaterals and $\alpha_5 = 0$ pentagons;
- $\mathcal{P}$ has $\alpha_4 = 0$ quadrilaterals and $\alpha_5 = 2$ pentagons.

The feasible case is the one which fulfils equation (3.74), that is, $\alpha_4 = 0$ and $\alpha_5 = 2$. ◇

Note that the numbers N_T, N_{IF} and N_{EI} are easily computable looking at the existence pluri-intervals of the regular facets of the b.c.p. and by using Theorem 3.3.

3.2.4 An algorithm for the construction of a 2-d b.c.p.

Now, we present a detailed algorithm for the construction of the b.c.p. $\mathcal{P} = \text{absco}(\mathcal{X})$, where $\mathcal{X} = \{x^{(1)}, x^{(2)}, \cdots, x^{(r)}\}$, with $r \geq 2$ and $x^{(i)} \in \mathbb{C}^2$ for $i = 1, \cdots, r$, based directly on the theory given in the previous subsections.

The input of the algorithm is the set $\mathcal{X}$ and the outputs are the updated set $\mathcal{X} = \{x^{(l_1)}, x^{(l_2)}, \cdots, x^{(l_m)}\}$, where $\{l_1, l_2, \cdots, l_m\}$ is a subset of $\{1, \cdots, r\}$, the two-index array RF and the three-index array SF.

The two-index array RF of dimension $(m-1) \times m$ contains in the position (i, j) the existence pluri-interval $\mathcal{E}_{ij}$ associated with the facets $x^{(i)} \bullet\!\!-\!\!\bullet\, e^{\mathrm{i}\theta} x^{(j)}$, that is, $RF(i, j) = \mathcal{E}_{ij}$ with $j = 1, \cdots, m$ and $i = 1, \cdots, j-1$.

The three-index array SF of dimension $(m-2) \times (m-1) \times m$ contains in the position (i, j, k) none or one or two pairs $(\theta_{ij}, \theta_{ik})$ associated with (triangles included in) the special facets F_{ijk} : $x^{(i)} \blacktriangle e^{\mathrm{i}\theta_{ij}} x^{(j)} \blacktriangle e^{\mathrm{i}\theta_{ik}} x^{(k)}$, $k = 1, \cdots, m$, $j = 1, \cdots, k-1$ and $i = 1, \cdots, j-1$. Clearly, $SF(i, j, k)$ is empty if and only if no special facets F_{ijk} exist. Otherwise, if $SF(i, j, k) = \{(\theta_{ij}^{(1)}, \theta_{ik}^{(1)})\}$, then only the special facet $F_{ijk} : x^{(i)} \blacktriangle e^{\mathrm{i}\theta_{ij}^{(1)}} x^{(j)} \blacktriangle e^{\mathrm{i}\theta_{ik}^{(1)}} x^{(k)}$ exists. Finally, if $SF(i, j, k) = \{(\theta_{ij}^{(1)}, \theta_{ik}^{(1)}), (\theta_{ij}^{(2)}, \theta_{ik}^{(2)})\}$, then there exist both the special facets $F_{ijk}^{(1)} : x^{(i)} \blacktriangle e^{\mathrm{i}\theta_{ij}^{(1)}} x^{(j)} \blacktriangle e^{\mathrm{i}\theta_{ik}^{(1)}} x^{(k)}$ and $F_{ijk}^{(2)} : x^{(i)} \blacktriangle e^{\mathrm{i}\theta_{ij}^{(2)}} x^{(j)} \blacktriangle e^{\mathrm{i}\theta_{ik}^{(2)}} x^{(k)}$.

Now we give a brief description on how the algorithm works. In the initialisation section the set of vertexes V takes the first two points of $\mathcal{X}$ which are assumed to be linearly independent. These first two vectors, named $x^{(1)}$ and $x^{(2)}$, represent the vertexes of $\mathcal{P}^{(2)}$, and thus $RF(1, 2)$ is set to $(-\pi, \pi]$. As a consequence the cardinality of V, denoted by N_V, is set to 2. If the set $\mathcal{X}$ has more than two vertexes, i.e. its cardinality $N_{\mathcal{X}} > 2$, we begin the **while** loop which performs in an iterative way the addition of the remaining points $\mathcal{X} \backslash V$ to the already available b.c.p. The basic steps performed at each iteration of the **while** loop are summarised as follow:

- *if* $N_V = 2$ we find in $\mathcal{X} \backslash V$ the first vector, named $x^{(3)}$, which is outside of $\mathcal{P}^{(2)}$ and does not delete any of its vertexes and then we construct the regular facets of $\mathcal{P}^{(3)} = \text{absco}(\{x^{(1)}, x^{(2)}, x^{(3)}\})$;
- *if* $N_V > 2$ and $x^{(k)} \in \mathcal{P}^{(k-1)}$, $k \geq 4$, then $x^{(k)}$ is deleted from $\mathcal{X}$;
- in the case that $N_V > 2$ and $x^{(k)} \notin \mathcal{P}^{(k-1)}$, $k \geq 4$, the algorithm builds $\mathcal{P}^{(k)}$ by adding $x^{(k)}$ to $\mathcal{P}^{(k-1)}$ through the following basic steps:
  - find the vertexes of $\mathcal{P}^{(k-1)}$, if any, which are deleted following the addition of $x^{(k)}$ and then delete them from $\mathcal{X}$ and V;
  - update the regular facets of $\mathcal{P}^{(k-1)}$ by deleting those which are seen by the circle generated by $x^{(k)}$;
  - add the new facets of $\mathcal{P}^{(k)}$ whose second vertex is $x^{(k)}$;
  - delete from V and $\mathcal{X}$ all the vectors which only belong to isolated facets or to no facets at all.

At the end of each iteration the set V is an essential system of vertexes of $\mathcal{P}^{(k)}$. Furthermore, observe that, whenever a point $x^{(l)}$ is deleted from $\mathcal{X}$, all the indexes of the subsequent vectors are shifted backward by one unit, and so at the end of the algorithm, the output $\mathcal{X}$ is equal to V.

At the end of the **while** loop, if $N_V > 2$, the algorithm computes the special facets of the final b.c.p. $\mathcal{P} = \text{absco}(\mathcal{X})$. We remark that the algorithm does not compute the special facets of $\mathcal{P}^{(k)}$ at the end of each iteration (from $\mathcal{P}^{(k-1)}$ to $\mathcal{P}^{(k)}$), because the special facets of $\mathcal{P}^{(k)}$ are not used for the constructive procedure of the b.c.p.

Remark 3.6. *Due to the necessity of checking several crucial equalities and inequalities between real numbers, it is necessary to fix a tolerance, suitably greater than the machine precision, under which we consider two floating point numbers to be equal to each other. In other words, the problems arising from the finite precision of the machine arithmetic must be handled with a lot of care just to avoid very unpleasant mistakes.*

ALGORITHM 3.1.

% Input: $\mathcal{X} = \{x^{(1)}, x^{(2)}, \cdots, x^{(r)}\}$, with $r \geq 2$ and $x^{(i)} \in \mathbb{C}^2$ for $i = 1, \cdots, r$

% Output: $\mathcal{X}$, RF, SF

Initializations
    $V \leftarrow \{x^{(1)}, x^{(2)}\}$
    $N_{\mathcal{X}} \leftarrow r$, $N_V \leftarrow 2$
    $RF(1,2) \leftarrow (-\pi, \pi]$
    $k \leftarrow 3$

while $k \leq N_{\mathcal{X}}$
    $stop \leftarrow 0$
    $ins \leftarrow 0$
    if $N_V = 2$
        % find the first vertex $x^{(i)}$ of $\mathcal{X}$, $i \geq 3$, which does not belong to $\mathcal{P}^{(2)}$
        while $(x^{(3)} \in \mathcal{P}^{(2)}$ & $N_{\mathcal{X}} \geq 3)$
            % delete $x^{(3)}$
            $\mathcal{X} \leftarrow \mathcal{X} \backslash \{x^{(3)}\}$, $N_{\mathcal{X}} \leftarrow N_{\mathcal{X}} - 1$
        end
        if $N_{\mathcal{X}} = 2$
            % the algorithm halts ($\mathcal{P} = \mathcal{P}^{(2)}$)
            break
        end
        if $x^{(3)}$ *sees* $x^{(1)}$
            % delete $x^{(1)}$
            $V \leftarrow \{x^{(2)}, x^{(3)}\}$
            $\mathcal{X} \leftarrow \mathcal{X} \backslash \{x^{(1)}\}$, $N_{\mathcal{X}} \leftarrow N_{\mathcal{X}} - 1$
            if $N_{\mathcal{X}} = 2$
                % the algorithm halts ($\mathcal{P} = \mathcal{P}^{(2)}$)
                break
            end
            $stop \leftarrow 1$
        elseif $x^{(3)}$ *sees* $x^{(2)}$
            % delete $x^{(2)}$
            $V \leftarrow \{x^{(1)}, x^{(3)}\}$
            $\mathcal{X} \leftarrow \mathcal{X} \backslash \{x^{(2)}\}$, $N_{\mathcal{X}} \leftarrow N_{\mathcal{X}} - 1$
            if $N_{\mathcal{X}} = 2$
                % the algorithm halts ($\mathcal{P} = \mathcal{P}^{(2)}$)
                break
            end

```
        stop ← 1
    else
        % construct the regular facets of 𝒫^(3)
        V ← {x^(1), x^(2), x^(3)}, N_V ← 3
        RF(1,2) ← ℰ_12^(2) \ 𝒟_12^(3),    RF(i,3) ← ℰ_i3^(3),  i = 1,2
    end
else
    % add x^(k) to 𝒫^(k−1)
    n_rs ← min_{1≤r<s≤k−1} ||x^(k)||_{𝒫_rs}
    if n_rs ≤ 1
        % delete x^(k) ∈ 𝒫_rs ⊆ 𝒫^(k−1)
        𝒳 ← 𝒳\{x^(k)}, N_𝒳 ← N_𝒳 − 1
        ins ← 1
    else
        % x^(k) ∉ 𝒫_rs; add x^(k) to 𝒫^(k−1)
        find the possible vertexes {x^(d_1), ··· , x^(d_s)} of 𝒫^(k−1) seen by x^(k)
        if s ≥ 1
            % delete x^(d_1), ··· , x^(d_s)
            𝒳 ← 𝒳\{x^(d_1), ··· , x^(d_s)}, N_𝒳 ← N_𝒳 − s
            V ← V\{x^(d_1), ··· , x^(d_s)}, N_V ← N_V − s
            k ← max{k − s, 3}
            delete rows and columns of indexes d_1, ··· , d_s from RF
            if k = 3
                V ← {x^(1), x^(2)}, N_V ← 2
            end
        end
        if k > 3
            % find the regular facets of 𝒫^(k)
            compute 𝒟_ij^(k) using (3.66) and ℰ_ik^(k) solving the system (3.68)
            RF(i,j) ← ℰ_ij^(k−1) \ 𝒟_ij^(k),  i = 1,··· j − 1,   j = 1,··· k − 1
            RF(i,k) ← ℰ_ik^(k),   i = 1,··· k − 1
            V ← V ∪ {x^(k)}, N_V ← N_V + 1
        end
    end
end
if (N_V > 2 & stop = 0 & ins = 0)
    by analysing RF, find the subset {x^(i_1), ··· , x^(i_r)} of the vectors {x^(1), ··· , x^(k)} which
    belong to none or only to isolated facets of 𝒫^(k)
    if r ≥ 1
        % delete {x^(i_1), ··· , x^(i_r)}
        𝒳 ← 𝒳\{x^(i_1), ··· , x^(i_r)}, N_𝒳 ← N_𝒳 − r
        V ← V\{x^(i_1), ··· , x^(i_r)}, N_V ← N_V − r
        k ← max{k − r, 2}
        delete the rows and columns of indexes i_1, ··· , i_r from RF
        if k = 2
            V ← {x^(1), x^(2)}, N_V ← 2
        end
    end
    k ← k + 1
end
if N_V = 2
    RF(1,2) ← (−π, π]
```

    end
end
if $N_{\mathcal{X}} > 2$
    compute SF using RF according to Theorem 3.3
else
    $RF(1,2) \leftarrow (-\pi, \pi]$
    $SF \leftarrow \emptyset$
end

Note that the keyword **break** terminates the execution of the **while** loop. That is, the algorithm jumps to the first instruction, if any, which follows the **end** of the **while** loop.

3.3 Computation of the complex polytope norm

Let $\mathcal{X} = \{x^{(i)}\}_{1 \le i \le m}$, with $x^{(i)} \in \mathbb{C}^2$, be an essential system of vertexes of $\mathcal{P} = \text{absco}(\mathcal{X})$ and $z \in \mathbb{C}^2$. We want to compute $\|z\|_{\mathcal{P}}$, once we know the existence pluri-intervals $\mathcal{E}_{ij}$, $1 \le i < j \le m$, of the facets of $\mathcal{P}$ and the existence pairs of values $SF(i,j,k)$, $1 \le i < j < k \le m$, of the triangles included in special facets of $\mathcal{P}$.

Preliminarily, we check whether z is proportional to a vertex or not. To do this we compute

$$d_i = \det\left(\begin{bmatrix} z & x^{(i)} \end{bmatrix}\right) = z_1 x_2^{(i)} - z_2 x_1^{(i)}, \qquad i = 1, \ldots, m. \tag{3.75}$$

Obviously, z is proportional to a vertex $x^{(r)}$ if and only if $d_r = 0$, in which case we have

$$\|z\|_{\mathcal{P}} = \|z\| / \|x^{(r)}\|$$

for any norm defined in $\mathbb{C}^2$.

Now, we assume that z is not proportional to any vertex, that is,

$$d_i \neq 0 \quad \text{for all} \quad i \in \{1, \ldots, m\}. \tag{3.76}$$

Therefore, we must solve the minimisation problem (2.28) in $\mathbb{C}^{m-2}$ or, equivalently, find those facets F_y of $\mathcal{P}$ which z projects on (see Definition 2.11) by using the characterising formula (2.18) and, consequently, the equality (2.20). On the other hand, since we are in $\mathbb{C}^2$, two different facets of $\mathcal{P}$ may not have any common intersection other than a vertex. Thus, the facet F_y which z projects on is unique.

Therefore, first of all, we check whether z projects on a two-vertex facet. For this purpose we write z as a linear combination of all the pairs of vertexes $(x^{(i)}, x^{(j)})$ of $\mathcal{P}$, with $i \neq j$, that is,

$$z = \lambda_{ij}\, x^{(i)} + \mu_{ij}\, x^{(j)} \tag{3.77}$$

with $\lambda_{ij}, \mu_{ij} \in \mathbb{C} \setminus \{0\}$, and then we compute

$$\min_{1 \le i < j \le m} \|z\|_{\mathcal{P}_{ij}} = \min_{1 \le i < j \le m} \left\| \begin{bmatrix} x^{(i)} & x^{(j)} \end{bmatrix}^{-1} z \right\|_1 \tag{3.78}$$

along with the pair $(x^{(r)}, x^{(s)})$ which reaches this minimum.

Then we analyse the pair $(\arg(\lambda_{rs}), \arg(\mu_{rs}))$ and we check whether the condition

$$\arg(\mu_{rs}) - \arg(\lambda_{rs}) \quad \text{shift}(-\pi, \pi] \in \mathcal{E}_{rs} \tag{3.79}$$

is satisfied, where $\mathcal{E}_{rs}$, given by (3.34), represents the existence pluri-interval of the facets $x^{(r)} \bullet\!\!-\!\!\bullet\, e^{i\theta} x^{(s)}$.

If (3.79) is satisfied, then we located the facet which z projects on and, in this case, we have that

$$\|z\|_{\mathcal{P}} = |\lambda_{rs}| + |\mu_{rs}| = \left\| \left[x^{(i)} \; x^{(j)} \right]^{-1} z \right\|_1 . \tag{3.80}$$

On the contrary, if condition (3.79) is not satisfied, then

$$\|z\|_{\mathcal{P}} < |\lambda_{rs}| + |\mu_{rs}|$$

and z necessarily projects on the internal part of a special facet. More precisely, by virtue of (3.32), it projects on the internal part of a triangle $e^{i\theta_r}x^{(r)} \blacktriangle e^{i\theta_s}x^{(s)} \blacktriangle e^{i\theta_t}x^{(t)}$, $r < s < t$. Then the vector z belongs to the three-dimensional linear subspace of $\mathbb{R}^4$ spanned by the vertexes $e^{i\theta_r}x^{(r)}$, $e^{i\theta_s}x^{(s)}$, $e^{i\theta_t}x^{(t)}$ and, more precisely, there exists a unique triplet of real numbers $\beta_r > 0$, $\beta_s > 0$, $\beta_t > 0$ such that

$$z = \beta_r e^{i\theta_r} x^{(r)} + \beta_s e^{i\theta_s} x^{(s)} + \beta_t e^{i\theta_t} x^{(t)} \tag{3.81}$$

and

$$\|z\|_{\mathcal{P}} = \beta_r + \beta_s + \beta_t. \tag{3.82}$$

Moreover, the pair $(\theta_s - \theta_r, \theta_t - \theta_r)$ necessarily represents the existence values of the triangle $e^{i\theta_r}x^{(r)} \blacktriangle e^{i\theta_s}x^{(s)} \blacktriangle e^{i\theta_t}x^{(t)}$ included in a special facet of $\mathcal{P}$, that is,

$$(\theta_s - \theta_r, \theta_t - \theta_r) \in \mathcal{S}_{rst}. \tag{3.83}$$

On the other hand, (3.81) is characterising in the sense that, if a triangle $e^{i\theta_r}x^{(r)} \blacktriangle e^{i\theta_s}x^{(s)} \blacktriangle e^{i\theta_t}x^{(t)}$ included in a special facet satisfies (3.81) with real numbers $\beta_r > 0$, $\beta_s > 0$, $\beta_t > 0$, then the special facet F_y that includes this triangle is the unique facet of $\mathcal{P}$ which z projects on. However, observe that, if the facet F_y is given by the union of more than one triangle, then the triangle which z projects on is not unique.

Therefore, in order to compute $\|z\|_{\mathcal{P}}$, we proceed as follows. We consider the existence values $(\theta^{\pm}_{ij}, \theta^{\mp}_{ik}) \in \mathcal{S}_{ijk}$, $1 \leq i < j < k \leq m$, and we write the vector z in the form

$$z = e^{i\theta_i} \left(\beta_i x^{(i)} + \beta_j e^{i\theta^{\pm}_{ij}} x^{(j)} + \beta_k e^{i\theta^{\mp}_{ik}} x^{(k)} \right), \tag{3.84}$$

looking for $\theta_i \in (-\pi, \pi]$ and real coefficients $\beta_i > 0$, $\beta_j > 0$, $\beta_k > 0$, until such an equality holds. Then the corresponding triplet of indexes, say (r, s, t), is such that

$$\theta_s = \theta_r + \theta^{\pm}_{rs} \quad \text{and} \quad \theta_t = \theta_r + \theta^{\mp}_{rt}$$

in (3.81) and that (3.82) holds.

We are left to show how to analyse equation (3.84). To this aim, we rename the vertexes as

$$v^{(i)} = x^{(i)}, \quad v^{(j)} = e^{i\theta^{\pm}_{ij}} x^{(j)}, \quad v^{(k)} = e^{i\theta^{\mp}_{ik}} x^{(k)},$$

so that its (equivalent) componentwise form is

$$\begin{cases} e^{-i\theta_i} z_1 = \beta_i v^{(i)}_1 + \beta_j v^{(j)}_1 + \beta_k v^{(k)}_1, \\ e^{-i\theta_i} z_2 = \beta_i v^{(i)}_2 + \beta_j v^{(j)}_2 + \beta_k v^{(k)}_2. \end{cases} \tag{3.85}$$

Multiplying the first equation by z_2, the second equation by z_1 and summing them up, we find the homogeneous equation

$$\hat{d}_i \beta_i + \hat{d}_j \beta_j + \hat{d}_k \beta_k = 0 \tag{3.86}$$

in the three real unknowns $\beta_i, \beta_j, \beta_k$, where

$$\hat{d}_i = d_i \neq 0, \quad \hat{d}_j = e^{i\theta^{\pm}_{ij}} d_j \neq 0, \quad \hat{d}_k = e^{i\theta^{\mp}_{ik}} d_k \neq 0$$

(see (3.75) and (3.76)).

Now we rewrite (3.86) as the system of two real homogeneous equations

$$\begin{cases} Re(d_i)\beta_i + Re(d_j)\beta_j + Re(d_k)\beta_k = 0, \\ Im(d_i)\beta_i + Im(d_j)\beta_j + Im(d_k)\beta_k = 0, \end{cases} \tag{3.87}$$

and, for $p, q \in \{i, j, k\}$, $p < q$, we set

$$D_{pq} = \begin{bmatrix} Re(d_p) & Re(d_q) \\ Im(d_p) & Im(d_q) \end{bmatrix} \tag{3.88}$$

and observe that

$$\det(D_{pq}) = 0 \quad \text{if and only if} \quad \hat{d}_p = \rho_{pq}\hat{d}_q \quad \text{for some} \quad \rho_{pq} \in \mathbb{R}. \tag{3.89}$$

On the other hand, in this case we would have

$$\hat{d}_p - \rho_{pq}\hat{d}_q = \det\left(\begin{bmatrix} z & v^{(p)} - \rho_{pq}v^{(q)} \end{bmatrix}\right) = 0,$$

that is,

$$z = \gamma(v^{(p)} - \rho_{pq}v^{(q)}) \quad \text{for some} \quad \gamma \in \mathbb{C}.$$

Since z is not proportional to any vertex, it must be $\rho_{pq} \neq 0$. Moreover, since $e^{\mathrm{i}\cdot\arg(\gamma)}v^{(p)} \bullet\!\!-\!\!\bullet$ $e^{\mathrm{i}\cdot\arg(\gamma)}v^{(q)}$ is a degenerate facet of $\mathcal{P}$, it must necessarily be

$$\rho_{pq} > 0, \tag{3.90}$$

otherwise z would satisfy the equality

$$z = |\gamma|e^{\mathrm{i}\cdot\arg(\gamma)}v^{(p)} - \rho_{pq}|\gamma|e^{\mathrm{i}\arg(\gamma)}v^{(q)},$$

which is of the type (3.77) with positive coefficients. But this would imply that z projects on the degenerate facet $e^{\mathrm{i}\cdot\arg(\gamma)}v^{(p)} \bullet\!\!-\!\!\bullet$ $e^{\mathrm{i}\cdot\arg(\gamma)}v^{(q)}$, which has already been excluded.

Now we observe that, if two of the three matrices (3.88) are singular, then (3.89) implies that all of them are singular and that, by (3.76), the homogeneous equation (3.86) reduces to

$$\rho_{ik}\beta_i + \rho_{jk}\beta_j + \beta_k = 0,$$

which, since (3.90) holds for all $p, q \in \{i, j, k\}$, does not admit a triplet of positive solutions. Therefore, in this case, we stop the analysis with the underlying pair of existence values $(\theta_{ij}^{\pm}, \theta_{ik}^{\mp})$ and go on to consider the next one.

At some point we necessarily find a pair of existence values $(\theta_{ij}^{\pm}, \theta_{ik}^{\mp})$ for which at least two of the matrices D_{pq} in (3.88) are not singular. Without loss of generality, we may assume that D_{ij} is not singular, so that, for $\beta_k \neq 0$, (3.87) yields

$$\begin{bmatrix} \beta_i \\ \beta_j \end{bmatrix} = \beta_k \begin{bmatrix} b_{ik} \\ b_{jk} \end{bmatrix}, \quad \text{where} \quad \begin{bmatrix} b_{ik} \\ b_{jk} \end{bmatrix} = -D_{ij}^{-1} \begin{bmatrix} Re(d_k) \\ Im(d_k) \end{bmatrix}. \tag{3.91}$$

If

$$b_{ik} > 0 \quad \text{and} \quad b_{jk} > 0, \tag{3.92}$$

then we get the desired solution $\theta_i \in (-\pi, \pi]$, $\beta_i > 0$, $\beta_j > 0$, $\beta_k > 0$ to (3.85) by setting

$$\beta_k = \frac{|z_l|}{|b_{ik}v_l^{(i)} + b_{jk}v_l^{(j)} + v_l^{(k)}|} \tag{3.93}$$

and
$$e^{-\mathrm{i}\theta_{\mathrm{i}}} = \frac{z_l}{\beta_k(b_{ik}v_l^{(i)} + b_{jk}v_l^{(j)} + v_l^{(k)})}$$
for any index $l \in \{1, 2\}$ such that $z_l \neq 0$.

Otherwise, if (3.92) is not satisfied, then we go on to consider the next pair of existence values $(\theta_{ij}^{\pm}, \theta_{ik}^{\mp})$.

In any case, it is clear that, sooner or later, we find the unique pair of existence values $(\theta_{ij}^{\pm}, \theta_{ik}^{\mp})$ which allows equation (3.85) to have the desired solution.

After the location of the special facet which the point x projects on, we have that
$$\|x\|_{\mathcal{P}} = \beta_i + \beta_j + \beta_k.$$

Example 3.6. We consider the same b.c.p. introduced in Example 3.2, that is, the b.c.p. $\mathcal{P} = \mathrm{absco}(\{x^{(1)}, x^{(2)}, x^{(3)}\})$, where
$$x^{(1)} = \begin{bmatrix} 1 \\ \mathrm{i} \end{bmatrix}, \quad x^{(2)} = \begin{bmatrix} 1 \\ 1-\mathrm{i} \end{bmatrix}, \quad x^{(3)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
and we want to compute the norm $\|\cdot\|_{\mathcal{P}}$ of two vectors, namely
$$z^{(1)} = \begin{bmatrix} 2+\mathrm{i} \\ 1-\mathrm{i} \end{bmatrix}, \quad z^{(2)} = \begin{bmatrix} 3 \\ 2 \end{bmatrix},$$
which are not proportional to any vertex of $\mathcal{P}$.

To this aim, we recall that the construction carried out in Example 3.4 shows that the existence pluri-intervals and existence pairs of $\mathcal{P}$ are given by
$$\begin{aligned} \mathcal{E}_{12} &= (-\pi,\ 0] \cup [1.570796326\ldots,\ \pi], \\ \mathcal{E}_{13} &= [0,\ 0.927295218\ldots], \\ \mathcal{E}_{23} &= [-0.643501108\ldots,\ 0], \end{aligned}$$
and
$$\mathcal{S}_{123} = \{(1.570796326\ldots,\ 0.927295218\ldots), (0,\ 0)\},$$
respectively.

We begin with $z^{(1)}$. To this aim, for $1 \leq i < j \leq 3$, we write it in the form
$$z^{(1)} = \lambda_{ij}^{(1)} x^{(i)} + \mu_{ij}^{(1)} x^{(j)}$$
and compute the minimum in (3.78). It turns out that
$$\min_{1\leq i<j\leq 3} \|[x^{(i)} x^{(j)}]^{-1} z^{(1)}\|_1 = \|[x^{(1)} x^{(2)}]^{-1} z^{(1)}\|_1 = 2.506878740\ldots$$
and that
$$\arg(\mu_{12}^{(1)}) - \arg(\lambda_{12}^{(1)}) = -0.982793723\ldots \in \mathcal{E}_{12}.$$
Therefore, by (3.80) we can conclude that
$$\|z^{(1)}\|_{\mathcal{P}} = 2.506878740\ldots$$

Then we go on to consider $z^{(2)}$ and, for $1 \leq i < j \leq 3$, we write it in the form
$$z^{(2)} = \lambda_{ij}^{(2)} x^{(i)} + \mu_{ij}^{(2)} x^{(j)}$$

and compute the minimum in (3.78). It turns out that

$$\min_{1\le i<j\le 3} \|[x^{(i)}x^{(j)}]^{-1}z^{(2)}\|_1 = \|[x^{(1)}x^{(2)}]^{-1}z^{(2)}\|_1 = 3.026665112\ldots$$

and

$$\arg(\mu_{12}^{(2)}) - \arg(\lambda_{12}^{(2)}) = 0.266252049\ldots \notin \mathcal{E}_{12},$$

so that $z^{(2)}$ necessarily projects on the internal part of a special facet.

Therefore, we consider the first existence pair of $\mathcal{S}_{123}$, that is (1.570796326 ..., 0.927295218 ...), and, with reference to (3.86), we obtain

$$\hat{d}_1\beta_1 + \hat{d}_2\beta_2 + \hat{d}_3\beta_3 = 0, \tag{3.94}$$

where

$$\hat{d}_1 = -2 + 3\,\mathrm{i}, \quad \hat{d}_2 = 3 + \mathrm{i}, \quad \hat{d}_3 = 0.6 + 0.8\,\mathrm{i}.$$

The relevant matrices D_{12}, D_{13}, D_{23} are all non-singular and, hence, by using (3.91), we compute

$$b_{13} = -0.163636363\ldots, \quad b_{23} = -0.309090909\ldots$$

Since this solution is not positive, we have to go on to consider the second existence pair of $\mathcal{S}_{123}$, that is (0, 0). Indeed, we know that this pair must necessarily be the right one, since there are no more existence pairs. In fact, this time, the coefficients in (3.94) are given by

$$\hat{d}_1 = -2 + 3\,\mathrm{i}, \quad \hat{d}_2 = 1 - 3\,\mathrm{i}, \quad \hat{d}_3 = 1,$$

and, again, the relevant matrices D_{12}, D_{13}, D_{23} are all non-singular. Therefore, by using (3.91), we compute

$$b_{13} = 1, \quad b_{23} = 1,$$

which, as was expected, is a positive solution. Finally, by also using (3.93), we get

$$\beta_1 = 1, \quad \beta_2 = 1, \quad \beta_3 = 1,$$

so that, by (3.82), we can conclude that

$$\|z^{(2)}\|_{\mathcal{P}} = 3.$$

◇

We remark that, by virtue of (2.29) and (2.30), this procedure can also be used to compute the norms $\|\cdot\|_{\mathcal{P}\cap\mathbb{R}^2}$ and $\|\cdot\|_{\Re(\mathcal{P})}$ of real vectors.

3.3.1 An algorithm for the computation of the complex polytope norm

In this section we show an algorithm for the computation of a complex polytope norm. The inputs of the algorithm are the set $\mathcal{X} = \{x^{(1)}, x^{(2)}, \cdots, x^{(m)}\}$, the two-index array *RF*, the three-index array *SF*, which are the outputs of the Algorithm 3.1, and the point $z \in \mathbb{C}^2$. The output of the algorithm is $\|z\|_{\mathcal{P}}$.

Recall that $\mathcal{X}$ is an essential system of vertexes of $\mathcal{P} = \mathrm{absco}(\mathcal{X})$, *RF* is a two-index array which contains the existence pluri-intervals of the regular facets of $\mathcal{P}$ and *SF* is a three-index array which contains pairs of numbers associated with the special facets of $\mathcal{P}$ (see Section 3.2.4).

Also in this case it is necessary to take care of the problems arising from the finite precision of the machine arithmetic.

ALGORITHM 3.2.

% Input: $\mathcal{X} = \{x^{(1)}, x^{(2)}, \cdots, x^{(m)}\}$, *RF*, *SF*, *z*

% Output: $||z||_{\mathcal{P}}$, where $\mathcal{P} = \text{absco}(\mathcal{X})$

while $l \leq m$
  if $det([z\ x^{(l)}]) = 0$
    % z is proportional to the vertex $x^{(l)}$
    $||z||_{\mathcal{P}} \leftarrow ||z||/||x^{(l)}||$
    return
  end
  $l = l + 1$
end
for $j = 1 : m$
  for $i = 1 : j - 1$
    $\begin{bmatrix} \lambda_{ij} \\ \mu_{ij} \end{bmatrix} \leftarrow [x^{(i)}\ x^{(j)}]^{-1} z$
  end
end
compute $n_{rs} \leftarrow \min_{1 \leq i < j \leq m} ||z||_{\mathcal{P}_{ij}} = \min_{1 \leq i < j \leq m}(|\lambda_{ij}| + |\mu_{ij}|)$, *where the indexes r and s reach the minimum*
$spec \leftarrow 1$
if $\arg(\mu_{rs}) - \arg(\lambda_{rs})$ $shift(-\pi, \pi] \in RF(r, s)$
  % z projects on the regular facet $e^{\mathrm{i} \arg(\lambda_{rs})} x^{(r)} \bullet\!\!-\!\!\bullet\ e^{\mathrm{i} \arg(\mu_{rs})} x^{(s)}$
  $spec \leftarrow 0$
  $||z||_{\mathcal{P}} \leftarrow |\lambda_{rs}| + |\mu_{rs}|$
end
if $spec = 1$
  % z projects on a special facet
  locate the special facet on which z projects and determine λ_I, λ_J, λ_K *by using the results of Section 3.3*
  $||z||_{\mathcal{P}} \leftarrow \lambda_I + \lambda_J + \lambda_K$
end

Chapter 4

Improving the algorithms

Although inspired by the Beneath-Beyond method introduced in Section 3.1, the algorithm presented in the previous chapter for the construction of the b.c.p. $\mathcal{P}$ does not actually exploit all of its potentialities in terms of efficiency. In fact, at each iteration, in order to add $x^{(k)}$ to the b.c.p. $\mathcal{P}^{(k-1)}$, it involves all the facets of all the two-vertex subpolytopes $\mathcal{P}_{ij} = \text{absco}(\{x^{(i)}, x^{(j)}\})$ of $\mathcal{P}^{(k-1)}$. This procedure may uselessly be too time consuming. For this reason, in this chapter, we present an improvement, where, in general, only a subset of the regular facets of $\mathcal{P}^{(k-1)}$ is involved. This improvement is based on the use in $\mathbb{C}^2$ of the main idea of the B–B method in $\mathbb{R}^2$, that is, the idea of limit cone.

Also the algorithm presented in Section 3.3.1 for the computation of the polytope norm unnecessarily involves all the two-vertex subpolytopes $\mathcal{P}_{ij}$ of $\mathcal{P}$. Therefore, in this chapter we propose a variant which, on a probabilistic basis, has high chances to give an important speed-up.

4.1 The improved procedure for the construction of a b.c.p.

In this section we recall the main steps performed by the B–B method and by Algorithm 3.1 in the addition of $x^{(k)}$ to $\mathcal{P}^{(k-1)}$, $k \geq 4$. Then we define the *balanced limit cone* in $\mathbb{C}^2$ and use it to improve Algorithm 3.1.

We start recalling the main steps of the B–B method, that are:

(1) determine the facet $x^{(r)} \bullet\!\!-\!\!\bullet\ x^{(c)}$ which $x^{(k)}$ projects on;

(2) if $x^{(k)} \in \mathcal{P}_{rc} = \text{absco}(\{x^{(r)}, x^{(c)}\})$, then delete $x^{(k)}$, or else perform the following steps:

  (2.1) delete the facets of $\mathcal{P}^{(k-1)}$ which are seen by $x^{(k)}$, i.e. the facets which are inside the limit cone of apex $x^{(k)}$;

  (2.2) add the facets whose second vertex is $x^{(k)}$, involving only the survived vertexes lying on the boundary of the limit cone.

The main steps of Algorithm 3.1 are:

(1) determine two vertexes $x^{(r)}, x^{(c)}$ that minimise the quantity $\|x^{(k)}\|_{\mathcal{P}_{ij}}$ on the set of all pairs (i, j) with $i < j$;

(2) if $x^{(k)} \in \mathcal{P}_{rc} = \text{absco}\{x^{(r)}, x^{(c)}\}$, then delete $x^{(k)}$, or else perform the following steps:

  (2.1) delete the regular facets of $\mathcal{P}^{(k-1)}$ which are seen by $x^{(k)}$, checking all the facets of all the two-vertex subpolytopes P_{ij} of $\mathcal{P}^{(k-1)}$;

  (2.2) add the regular facets whose second vertex is $x^{(k)}$, checking all the facets of all the two-vertex subpolytopes P_{ij} of $\mathcal{P}^{(k-1)}$ determined by the vertexes not deleted by $x^{(k)}$.

Since in the latter algorithm each step is performed in an exhaustive fashion, which is a too time consuming operation, our aim is to use the limit cone idea also in $\mathbb{C}^2$, so as to be allowed to check only a minimal subset of the regular facets of $\mathcal{P}^{(k-1)}$.

The *balanced limit cone* in $\mathbb{C}^2$ of apex $x^{(k)}$, which is tangent to $\mathcal{P}^{(k-1)}$, delimits the set D of the regular facets of $\mathcal{P}^{(k-1)}$ which are *seen* (and so deleted) by the *circle* generated by the point $x^{(k)} \notin \mathcal{P}^{(k-1)}$.

Since $\mathcal{P}^{(k-1)}$ is a convex set, D is a connected set and thus, in order to update the regular facets of $\mathcal{P}^{(k-1)}$ due to the addition of $x^{(k)}$, we can start from any seen facet and then find, moving by connection, all the other regular facets which are seen by the circle generated by $x^{(k)}$. Consequently, we do not need to check those pairs (i, j) whose corresponding vertexes $x^{(i)}, x^{(j)}$ produce facets which are not seen by $x^{(k)}$ or which are not associated with any regular facet at all. Moreover, we have to add only the facets $x^{(l)} \bullet\!\!-\!\!\bullet\, e^{i\theta}x^{(k)}$, where $x^{(l)}$ belongs to the set of the non-deleted vertexes of seen facets. Recall that the deleting procedure of the regular facets of $\mathcal{P}^{(k-1)}$ may delete some vertexes of $\mathcal{P}^{(k-1)}$.

To perform the search of a seen facet, if any, we propose a first criterion in order to guess those regular facets of $\mathcal{P}$ that have the greatest chances to be seen by $x^{(k)}$. This criterion is based on the reasonable assumption that, in most cases, a facet which is seen by $x^{(k)}$ includes vertexes that are among the closest to $R_{x^{(k)}}$ in the Euclidean distance (see Definition 2.11).

We proceed as follows. Denoting by $\|\cdot\|_2 = \sqrt{\langle\cdot,\cdot\rangle}$ the usual Euclidean norm in $\mathbb{C}^2$, for all $i = 1, \ldots, k-1$, we compute the Euclidean distance

$$\delta_i = \min_{-\pi<\theta\leq\pi,\, \rho>0} \|e^{i\theta}x^{(i)} - \rho x^{(k)}\|_2$$

of $R_{x^{(k)}}$ from the circle generated by the vertex $x^{(i)}$. Simple calculations lead to

$$\delta_i = \sqrt{\|x^{(i)}\|_2^2 - |\langle x^{(i)}, x^{(k)}\rangle|^2/\|x^{(k)}\|_2^2}\,.$$

Then, we reorder the indexes in non-decreasing order with respect to the distances δ_i and, subsequently, we define a total order relation "$\prec$" on the set of reordered index pairs (i, j) in reversed lexicographical way, i.e.,

$$(i, j) \prec (h, k) \iff j < k \quad \text{or} \quad (j = k \;\&\; i < h).$$

Then we perform the following steps:

(1) following the total order "$\prec$", we find the first pair (r, c), if any, such that some facets of the type $F_{y_{rc}(\theta)} = x^{(r)} \bullet\!\!-\!\!\bullet\, e^{i\theta}x^{(c)}$ are seen by the circle generated by $x^{(k)}$;

(2) if $x^{(k)} \in \mathcal{P}_{rc}$ or no pair (r, c) is found at step (1), then delete $x^{(k)}$, or else perform the following two steps:

(2.1) update the set of the facets $F_{y_{rc}(\theta)}$ and, moving by connection, all the other regular facets of $\mathcal{P}^{(k-1)}$;

(2.2) add the regular facets of the kind $F_{y_{lk}(\theta)} = x^{(l)} \bullet\!\!-\!\!\bullet\, e^{i\theta}x^{(k)}$ with $x^{(l)}$ belonging to the set of the non-deleted vertexes of seen facets.

Note that, if $x^{(k)} \notin \mathcal{P}_{rc}$ and no pair (r, c) can be found at step (1), we delete $x^{(k)}$ because, although being outside of all the two-vertex subpolytopes of $\mathcal{P}^{(k-1)}$, it is inside a three-vertex subpolytope of $\mathcal{P}^{(k-1)}$.

Now, we give a more detailed description on how the improved algorithm works when $x^{(k)} \notin \mathcal{P}_{rc}$ and a pair (r, c) is found at step (1).

In this case, in step (1) we create a list of vertexes, called *vertexlist*, as follows. If $x^{(k)}$ deletes $x^{(r)}$ or $x^{(c)}$, then *vertexlist* contains the non-deleted vertex and all the vertexes of $\mathcal{P}^{(k-1)}$ which have a common facet with the deleted vertex. Or else, *vertexlist* contains the vertexes $x^{(r)}$ and $x^{(c)}$.

Next, we perform step (2.1) in an iterative fashion as follows: we consider, until exhaustion of *vertexlist*, the first vertex $x^{(v)}$ in *vertexlist* and append to *vertexlist* all those vertexes which have a common facet with $x^{(v)}$, are not deleted by $x^{(k)}$ and are associated with deleted regular facets of $\mathcal{P}^{(k-1)}$. If, during this process, the vertex $x^{(v)}$ is deleted by $x^{(k)}$, then we remove it from *vertexlist*, or else we still remove it from *vertexlist* and insert it into another list, called *visitedvertexlist*. At the end of this iterative process *vertexlist* is empty and *visitedvertexlist* contains all the vertexes of *vertexlist* which have not been deleted by $x^{(k)}$.

Finally, in step (2.2), we perform the addition of the regular facets of the kind $F_{y_{lk}(\theta)} = x^{(l)} \bullet\!\!-\!\!\bullet\, e^{i\theta}x^{(k)}$, where $x^{(l)}$ belongs to *visitedvertexlist*.

We conclude this section by presenting the pseudo-code of the improved algorithm for the construction of the b.c.p. $\mathcal{P} = \text{absco}(\mathcal{X})$, where $\mathcal{X} = \{x^{(1)}, x^{(2)}, \cdots, x^{(r)}\}$, with $r \geq 2$.

Like in Algorithm 3.1, the input of the algorithm is the set $\mathcal{X}$ and the outputs are the updated set $\mathcal{X} = \{x^{(l_1)}, x^{(l_2)}, \cdots, x^{(l_m)}\}$, where $\{l_1, l_2, \cdots, l_m\}$ is a subset of $\{1, \cdots, r\}$, the two-index array *RF* and the three-index array *SF*.

ALGORITHM 4.1. *(An improvement of Algorithm 3.1)*

% Input: $\mathcal{X} = \{x^{(1)}, x^{(2)}, \cdots, x^{(r)}\}$, with $r \geq 2$ and $x^{(i)} \in \mathbb{C}^2$, $i = 1, \cdots, r$

% Output: $\mathcal{X}$, *RF*, *SF*

```
Initialisations
   V ← {x^(1), x^(2)}
   N_X ← r, N_V ← 2
   RF(1,2) ← (−π, π]
   k ← 3

while k ≤ N_X
   if N_V = 2
      % find the first vertex x^(i) of X, i ≥ 3, which does not belong to P^(2)
      while (x^(3) ∈ P^(2) & N_X ≥ 3)
         % delete x^(3)
         X ← X\{x^(3)}, N_X ← N_X − 1
      end
      if N_X = 2
         % the algorithm halts (P = P^(2))
         break
      end
      if x^(3) sees x^(1)
         % delete x^(1)
         V ← {x^(2), x^(3)}
         X ← X\{x^(1)}, N_X ← N_X − 1
         if N_X = 2
            % the algorithm halts (P = P^(2))
            break
         end
      elseif x^(3) sees x^(2)
         % delete x^(2)
         V ← {x^(1), x^(3)}
         X ← X\{x^(2)}, N_X ← N_X − 1
         if N_X = 2
            % the algorithm halts (P = P^(2))
            break
```

end
else
% construct the regular facets of $\mathcal{P}^{(3)}$
$V \leftarrow \{x^{(1)}, x^{(2)}, x^{(3)}\}, N_V \leftarrow 3 \ \ , k \leftarrow 4$
$RF(1,2) \leftarrow \mathcal{E}_{12}^{(2)} \backslash \mathcal{D}_{12}^{(3)}, \quad RF(i,3) \leftarrow \mathcal{E}_{i3}^{(3)}, \ i = 1,2$
end
else
% add $x^{(k)}$ to $\mathcal{P}^{(k-1)}$
modified $\leftarrow 0$
newpointinside $\leftarrow 0$
SizeVis2 $\leftarrow 0$
StateM $\leftarrow$ *zero-matrix of dimension*$(k-2) \times (k-1)$
vertexlist $\leftarrow \emptyset$
visitedvertexlist $\leftarrow \emptyset$
% compute the Euclidean distances δ_i of $R_{x^{(k)}}$ from the *circles* generated by the vertexes $x^{(i)}$
% of $\mathcal{P}^{(k-1)}$
$\delta_i = \sqrt{\|x^{(i)}\|_2^2 - |\langle x^{(i)}, x^{(k)} \rangle|^2 / \|x^{(k)}\|_2^2}\ , i = 1, \ldots, k-1$
ord $\leftarrow$ *vector of the indexes reordered in non-decreasing order with respect to the distances* δ_i
BLOCK 1
if $N_{\mathcal{X}} = 2$
% the algorithm halts ($\mathcal{P} = \mathcal{P}^{(2)}$)
break
end
% if $x^{(k)}$ does not delete any regular facets, it is inside $\mathcal{P}^{(k-1)}$
if $modified = 0$
newpointinside $\leftarrow 1$
end
if *newpointinside* $= 0$
% the procedure starts from the first vertex in *vertexlist*
while *(vertexlist* $\neq \emptyset$ & $N_{\mathcal{X}} \geq 3$ & $SizeVis2 = 0$ *)*
$v \leftarrow vertexlist(1)$
outv $\leftarrow 0$
BLOCK 2
if *outv* $= 0$
$vertexlist(1) \leftarrow \emptyset$
$visitedvertexlist \leftarrow [visitedvertexlist, v]$
end
end
if $N_{\mathcal{X}} = 2$
% the algorithm halts ($\mathcal{P} = \mathcal{P}^{(2)}$)
break
end
if $SizeVis2 = 0$
% add the facets of the kind $F_{y_{lk}(\theta)} = x^{(l)} \bullet\!\!-\!\!\bullet\ e^{i\theta} x^{(k)}, \ \ l \in visitedvertexlist$
compute $\mathcal{E}_{lk}^{(k)}$, $l \in$ *visitedvertexlist, solving the system (3.68)*
$RF(l,k) \leftarrow \mathcal{E}_{lk}^{(k)}, \ \ l \in visitedvertexlist$
$V \leftarrow V \cup \{x^{(k)}\}, N_V \leftarrow N_V + 1$
by analysing RF, find the subset $\{x^{(i_1)}, \cdots, x^{(i_r)}\}$ *of the vectors* $\{x^{(1)}, \cdots, x^{(k)}\}$ *which belong to none or only to isolated facets of* $\mathcal{P}^{(k)}$
if $r \geq 1$
% delete $\{x^{(i_1)}, \cdots, x^{(i_r)}\}$
$\mathcal{X} \leftarrow \mathcal{X} \backslash \{x^{(i_1)}, \cdots, x^{(i_r)}\}, N_{\mathcal{X}} \leftarrow N_{\mathcal{X}} - r$

```
                V ← V\{x^(i_1), ··· , x^(i_r)}, N_V ← N_V − r
                if N_X = 2
                    % the algorithm halts (P = P^(2))
                    break
                end
                if N_V = 2
                    SizeVis2 ← 1
                end
                delete the rows and columns of indexes i_1, ··· , i_r from RF
                k ← k − r
            end
            k ← k + 1
        end
        if SizeVis2
            RF(1, 2) ← (−π, π]
            k ← 3
        end
    else
        % delete x^(k) from X
        X ← X \ {x^(k)},   N_X ← N_X − 1
        if N_X = 2
            % the algorithm halts (P = P^(2))
            break
        end
    end
  end
end
if N_X > 2
    compute SF using RF according to Theorem 3.3
else
    RF(1, 2) ← (−π, π]
    SF ← ∅
end
```

BLOCK 1 : Find, if any, a facet which is deleted by $x^{(k)}$ and create the list of vertexes *vertexlist*.

```
ic ← 2
while(modified = 0  &  newpointinside = 0  &  ic ≤ k − 1  &  N_X ≥ 3  &  SizeVis2 = 0)
  ir ← 1
  while (modified = 0  &  newpointinside = 0  &  ir ≤ ic − 1)
    % find the indexes r, c corresponding to ir, ic
    r ← ord(ir),  c ← ord(ic)
    if ||x^(k)||_{P_rc} ≤ 1
      newpointinside ← 1
    else
      if x^(k) deletes x^(d) with d = r or d = c
        modified ← 1
        % update vertexlist by adding the vertexes of P^(k−1) which have a common facet
        % with the deleted vertex x^(d)
        addvertexes ← indexes of the vertexes of P^(k−1) which have a common facet with x^(d)
        vertexlist ← [vertexlist, addvertexes]
        % delete the point x^(d) from X and V
        X ← X \ {x^(d)},  N_X ← N_X − 1
        V ← V \ {x^(d)},  N_V ← N_V − 1
        if N_X = 2
          % the algorithm halts (P = P^(2))
          break
        end
        if N_V = 2
          SizeVis2 ← 1,  k ← 3
          break
        end
        delete the row and the column of index d from RF and StateM
        k ← k − 1
      else
        if RF(r, c) ≠ ∅   % some facets of the type F_{y_rc(θ)} exist
          compute D_rc^(k) using (3.66)
          if RF(r, c) ∩ D_rc^(k) ≠ ∅
            % the pair (r, c) is visited and some facets F_{y_rc(θ)} are deleted
            modified ← 1
            StateM(r, c) ← 2
            RF(r, c) ← E_rc^(k−1) \ D_rc^(k)
            vertexlist ← [r, c]
          else
            % the pair (r, c) is visited but none of the facets F_{y_rc(θ)} is deleted
            StateM(r, c) ← 1
          end
        else
          % the pair (r, c) has been visited but, since the facets F_{y_rc(θ)} do not exist, none
          % of them has been deleted
          StateM(r, c) ← 1
        end
      end
    end
    ir ← ir + 1
  end
  ic ← ic + 1
end
```

BLOCK 2 : Starting from the vertexes $x^{(r)}, x^{(c)}$ of a seen facet (given by BLOCK 1), if any, and moving by connection from one seen facet to another, update the regular facets of $\mathcal{P}^{(k-1)}$ which are seen by $x^{(k)}$.

```
k1 ← 1
while(outv = 0  &  k1 ≤ v − 1  &  N_X ≥ 3 )
  % visit the not yet visited pairs (k1, v)
  if StateM(k1, v) = 0
    if x^(k) deletes x^(d) with d = k1 or d = v
      % update vertexlist by deleting, if any, the deleted vertex x^(d) and by adding
      % the vertexes of P^(k−1) which have a common facet with x^(d)
      vertexlist ← vertexlist \ {x^(d)}
      addvertexes ← indexes of the vertexes of P^(k−1) which have a common facet with x^(d)
      vertexlist ← [vertexlist, addvertexes]
      if x^(k) deletes x^(v)
        outv ← 1
      end
      % delete the point x^(d) from X and V
      X ← X \ {x^(d)},  N_X ← N_X − 1
      V ← V \ {x^(d)},  N_V ← N_V − 1
      if N_X = 2
        % the algorithm halts (P = P^(2))
        break
      end
      if N_V = 2
        SizeVis2 ← 1, k ← 3
        break
      end
      delete the row and the column of index d from RF and StateM
      k ← k − 1
    else
      if RF(k1, v) ≠ ∅  % some facets of the type F_{y_{k1 v}(θ)} exist
        compute D^(k)_{k1 v} using (3.66)
        if RF(k1, v) ∩ D^(k)_{k1 v} ≠ ∅
          % the pair (k1, v) is visited and some facets F_{y_{k1 v}(θ)} are deleted
          StateM(k1, v) ← 2
          RF(k1, v) ← E^(k−1)_{k1 v} \ D^(k)_{k1 v}
          if (k1 ∉ vertexlist  &  k1 ∉ visitedvertexlist)
            % add k1 to vertexlist
            vertexlist ← [vertexlist, k1]
          end
        else
          % the pair (k1, v) is visited but none of the facets F_{y_{k1 v}(θ)} is deleted
          StateM(k1, v) ← 1
        end
      else
        % the pair (k1, v) has been visited but, since the facets F_{y_{k1 v}(θ)} do not exist, none
        % of them has been deleted
        StateM(k1, v) ← 1
      end
    end
  end
  k1 ← k1 + 1
```

end
$k2 \leftarrow v + 1$
while($outv = 0$ & $k2 \leq k - 1$ & $N_{\mathcal{X}} \geq 3$ & $SizeVis2 = 0$)
% visit the not yet visited pairs $(kv, k2)$
if $StateM(v, k2) = 0$
if $x^{(k)}$ *deletes* $x^{(d)}$ *with* $d = k2$ *or* $d = kv$
% update *vertexlist* by deleting, if any, the deleted vertex $x^{(d)}$ and by adding
% the vertexes of $\mathcal{P}^{(k-1)}$ which have a common facet with $x^{(d)}$
$vertexlist \leftarrow vertexlist \setminus \{x^{(d)}\}$
$addvertexes \leftarrow$ *indexes of the vertexes of* $\mathcal{P}^{(k-1)}$ *which have a common facet with* $x^{(d)}$
$vertexlist \leftarrow [vertexlist, addvertexes]$
% update *visitedvertexlist* by deleting $x^{(d)}$, if any
$visitedvertexlist \leftarrow visitedvertexlist \setminus \{x^{(d)}\}$
if $x^{(k)}$ deletes $x^{(v)}$
$outv \leftarrow 1$
end
% delete the point $x^{(d)}$ from $\mathcal{X}$ and V
$\mathcal{X} \leftarrow \mathcal{X} \setminus \{x^{(d)}\},\ N_{\mathcal{X}} \leftarrow N_{\mathcal{X}} - 1$
$V \leftarrow V \setminus \{x^{(d)}\},\ N_V \leftarrow N_V - 1$
if $N_{\mathcal{X}} = 2$
% the algorithm halts ($\mathcal{P} = \mathcal{P}^{(2)}$)
break
end
if $N_V = 2$
$SizeVis2 \leftarrow 1,\quad k \leftarrow 3$
break
end
delete the row and the column of index d from RF and StateM
$k \leftarrow k - 1$
else
if $RF(v, k2) \neq \emptyset$ % some facets of the type $F_{y_{v\,k2}(\theta)}$ exist
compute $\mathcal{D}^{(k)}_{v\,k2}$ *using (3.66)*
if $RF(v, k2) \cap \mathcal{D}^{(k)}_{v\,k2} \neq \emptyset$
% the pair $(v, k2)$ is visited and some facets $F_{y_{v\,k2}(\theta)}$ are deleted
$StateM(v, k2) \leftarrow 2$
$RF(v, k2) \leftarrow \mathcal{E}^{(k-1)}_{v\,k2} \setminus \mathcal{D}^{(k)}_{v\,k2}$
if ($k2 \notin vertexlist$ & $k2 \notin visitedvertexlist$)
% add $k2$ to *vertexlist*
$vertexlist \leftarrow [vertexlist, k2]$
end
else
% the pair $(v, k2)$ is visited but none of the facets $F_{y_{v\,k2}(\theta)}$ is deleted
$StateM(v, k2) \leftarrow 1$
end
else
% the pair $(v, k2)$ has been visited but, since the facets $F_{y_{v\,k2}(\theta)}$ do not exist, none
% of them has been deleted
$StateM(v, k2) \leftarrow 1$
end
end
end
$k2 \leftarrow k2 + 1$
end

4.2 Expected results for Algorithm 4.1 and numerical experiments

In this section we show the numerical experiments we have made using our MATLAB implementations of the algorithms proposed in Section 3.2.4 and in Section 4.1 for the construction of the b.c.p.. The main goal of such experiments is to illustrate the better performance of the improved Algorithm 4.1 with respect to Algorithm 3.1 in dependence of the specific characteristics of the b.c.p. to construct.

Given a b.c.p. $\mathcal{P} = \text{absco}(\mathcal{X})$, where $\mathcal{X} = \{x^{(i)}\}_{1\leq i\leq m}$ is an essential system of vertexes, we denote by N_{EP} the number of the existence pluri-intervals of $\mathcal{P}$ (that is, the total number of the connected vertex pairs), and define

$$AC = \frac{2N_{EP}}{m},$$

which denotes the *average connection* of each vertex with the others, and

$$PAC = \frac{2N_{EP}}{m(m-1)},$$

which represents the *percentage of average connection* of each vertex with the others. Furthermore, we again use the numbers N_{EI} of the existence intervals, N_{IF} of isolated facets and N_T of triangles contained in special facets with three or more vertexes, which we have defined at the end of Section 3.2.3. The above quantities characterise the geometry of the b.c.p.

As for the performance of the two algorithms to compare, the following numbers play a primary role. They are the number of all index pairs (i, j) *checked* during the execution of the algorithms, denoted by C for Algorithm 3.1 and by $\hat{C}$ for Algorithm 4.1, and the number of all the existence pluri-intervals $\mathcal{E}_{ij}^{(k)}$, $k \geq 4$, *updated* during the execution of the algorithms (possibly even without deletion of any of the corresponding facets $F_{y_{ij}(\theta)}$), denoted by U for Algorithm 3.1 and by $\hat{U}$ for Algorithm 4.1. Obviously, for both algorithms, it is always the case that

$$U \leq C \quad \text{and} \quad \hat{U} \leq \hat{C}.$$

Note that the complexity of the work needed to compute all the existence pluri-intervals $\mathcal{E}_{ik}^{(k)}$, $k \geq 4$, related to the facets which are added at each iteration, is proportional to U and $\hat{U}$ in the two algorithms, respectively. Therefore, we do not need to use another counter to measure such computational work.

Of course, we expect Algorithm 4.1 to be characterised by values of $\hat{C}$ and $\hat{U}$ which, in most cases, are less than the corresponding values C and U relevant to Algorithm 3.1. Therefore, as *indicators of saved work* relevant to the former algorithm, we define the *percentage of saved checked index pairs*

$$PSC = \frac{C - \hat{C}}{C}$$

and the *percentage of saved updated existence pluri-intervals*

$$PSU = \frac{U - \hat{U}}{U}.$$

However, the main criterion to judge the performance is the CPU time, denoted by t for Algorithm 3.1 and by $\hat{t}$ for Algorithm 4.1. In order to reduce, as much as possible, the effects of random noises and of any other event which is out of our control (such as, for example, the execution of other processes controlled by the operating system), we shall make the measure of t and $\hat{t}$ by running each experiment ten times and taking the average of the corresponding execution times. Since the final computation of the triangles included in special facets is common to both algorithms, we do not count it in both t and $\hat{t}$, which are expressed in seconds. Therefore, as a measure of the improvement yielded by Algorithm 4.1 with respect to Algorithm 3.1, we consider the *percentage of saved CPU time*

$$PST = \frac{t - \hat{t}}{t}.$$

In any case, the two algorithms have a different structure and some different small parts. Therefore, in the light of the previous discussion, it is reasonable to expect that the relationship between the percentages PST, PSC and PSU is, in first approximation, of the type

$$PST = \alpha + \beta PSC + \gamma PSU \tag{4.1}$$

for suitable coefficients α, β, γ with $\beta, \gamma > 0$.

Another reasonable expectation we have is that in the improved Algorithm 4.1, at each iteration when $x^{(k)}$ is added to $\mathcal{P}^{(k-1)}$, the numbers of checked index pairs and of updated pluri-intervals are, more or less, proportional to the average connection of the b.c.p. $\mathcal{P}^{(k-1)}$. On the contrary, in Algorithm 3.1, which works in an exhaustive fashion, in many cases such numbers are both equal to the total number of pairs of essential vertexes of $\mathcal{P}^{(k-1)}$. Therefore, in general, we expect that PSC and PSU are decreasing functions of PAC and, in particular, often of linear type such as

$$PSC = \alpha_c - \beta_c PAC \quad \text{and} \quad PSU = \alpha_u - \beta_u PAC \tag{4.2}$$

for suitable coefficients $\alpha_c, \beta_c, \alpha_u, \beta_u > 0$. Consequently, in this case, putting (4.1) and (4.2) together, we would also have

$$PST = \alpha_t - \beta_t PAC \tag{4.3}$$

for suitable coefficients $\alpha_t, \beta_t > 0$. In any case, in general, PST is a decreasing function of PAC.

Example 4.1. We consider the set of m vectors $\mathcal{X} = \left\{x^{(i)}\right\}_{1\leq i\leq m}$, where

$$x_1^{(i)} = \cos\left(\frac{(i-1)\pi}{m}\right) \qquad \text{and} \qquad x_2^{(i)} = \sin\left(\frac{(i-1)\pi}{m}\right) e^{\mathrm{i}\left(\frac{(i-1)\pi}{m}\right)}.$$

These points are chosen in such a way that their first components are real and their second components are complex and uniformly distributed on a circle of the complex plane. For this particular choice, the set $\mathcal{X}$ is an essential system of vertexes for $\mathcal{P} = \mathrm{absco}(\mathcal{X})$. In Table 4.1 we report the quantities N_{EP}, N_{EI}, N_T, N_{IF}, AC and PAC, which characterise the b.c.p. $\mathcal{P}$. Then we

m	10	20	30	40	50	60	70
N_{EP}	27	62	97	132	167	202	237
N_{EI}	27	62	97	132	167	202	237
N_T	22	52	82	112	142	172	202
N_{IF}	6	16	26	36	46	56	66
AC	5.4	6.2	6.5	6.6	6.7	6.7	6.8
PAC	60 %	33%	22%	17%	14%	11%	10%

Table 4.1: The characterising quantities associated with $\mathcal{P}$ for some values of m.

begin to analyse and compare the behaviour of the two algorithms.

m	10	20	30	40	50	60	70
t	0.11	1.04	3.92	11.62	24.79	42.23	72.03
$\hat{t}$	0.07	0.41	1.25	2.94	6.56	11.62	17.83
PST	36 %	60 %	68 %	75 %	74%	72 %	75 %

Table 4.2: CPU times t and $\hat{t}$ (in seconds) for Algorithms 3.1 and 4.1, respectively, and percentage PST of the saved CPU time for some values of m.

The values of the CPU times t and $\hat{t}$ and of the percentage of saved CPU time PST are reported in Table 4.2. As shown in Figure 4.1, the CPU times t and $\hat{t}$ obviously increase with m and, more interestingly, the CPU time saved by the improved Algorithm 4.1 seems to be almost 75% for $m \geq 40$.

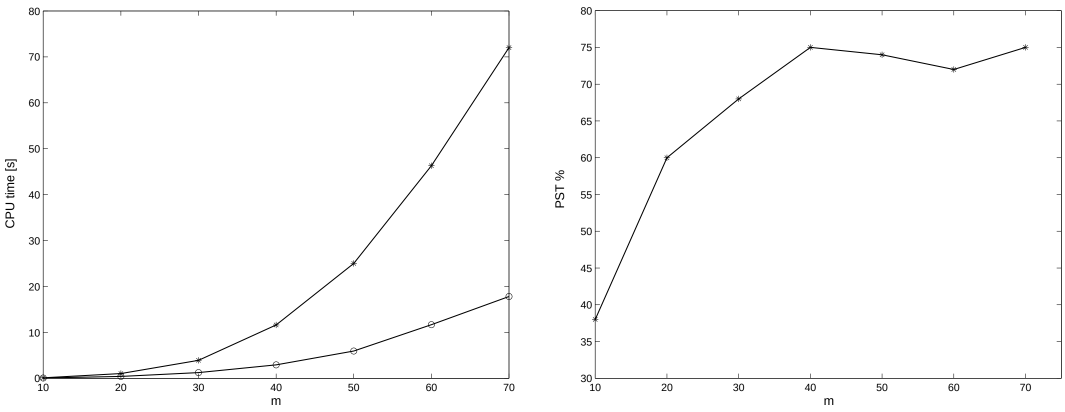

Figure 4.1: CPU time t for Algorithm 3.1, plotted with $*$, and $\hat{t}$ for Algorithm 4.1, plotted with $\circ$, (on the left) and percentage PST of saved CPU time (on the right) versus the number m of essential vertexes of $\mathcal{P}$.

As shown in Table 4.3, we have that

$$\hat{U} < \hat{C} < U = C.$$

Note that $U = C$ since Algorithm 3.1 works in an exhaustive way and all the starting points are essential vertexes of $\mathcal{P}$. In Figure 4.2 we also give the graphic of $C = U$, $\hat{C}$ and $\hat{U}$ versus the number of vertexes m.

According to (4.1), the percentage of saved time PST is almost a linear function of PSC and PSU. Thus, we have computed the corresponding regression plane and the related Pearson coefficient R^2 (see Figure 4.3). Remember that R^2 varies in $[0, 1]$ and that $R^2 = 1$ means perfect linearity. Moreover, the better the linear model, the closer R^2 is to one.

It is also interesting to note that the percentage of saved checked index pairs PSC and the percentage of saved updated existence pluri-intervals PSU are both almost linear functions of the percentage of average connection PAC and, so, we have computed the corresponding regression lines and the related Pearson coefficients, denoted respectively with R^2_C and R^2_U (see Figure 4.4).

As expected, according to (4.3), also the percentage of saved time PST is almost a linear function of PAC and, so, we have computed the corresponding regression line and the related Pearson coefficient R^2_T (see Figure 4.5).

m	10	20	30	40	50	60	70
C	119	1139	4059	9879	19599	34219	54739
$\hat{C}$	106	841	2756	6839	12299	21038	33136
U	119	1139	4059	9879	19599	34219	54739
$\hat{U}$	76	370	891	1635	2609	3811	5238
PSC	11 %	26 %	32 %	31 %	37 %	38 %	39 %
PSU	36 %	68 %	78 %	83 %	87 %	89 %	90 %

Table 4.3: The numbers C, $\hat{C}$ of the index pairs which are checked and U, $\hat{U}$ of the existence pluri-intervals which are updated by Algorithms 3.1 and 4.1, respectively, and the indicators of saved work PSC and PSU for some values of m.

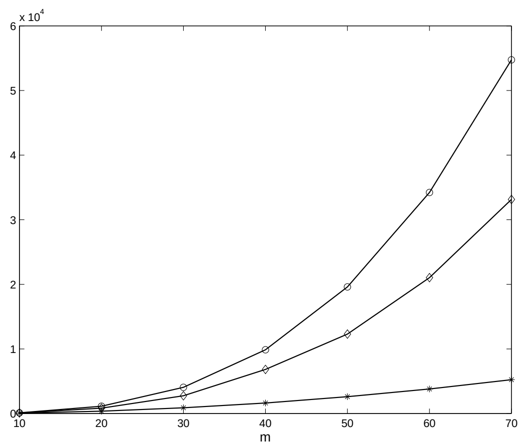

Figure 4.2: The numbers $C = U$ (plotted with $\circ$), $\hat{C}$ (plotted with $\diamond$), $\hat{U}$ (plotted with $*$) as a function of the number of points m.

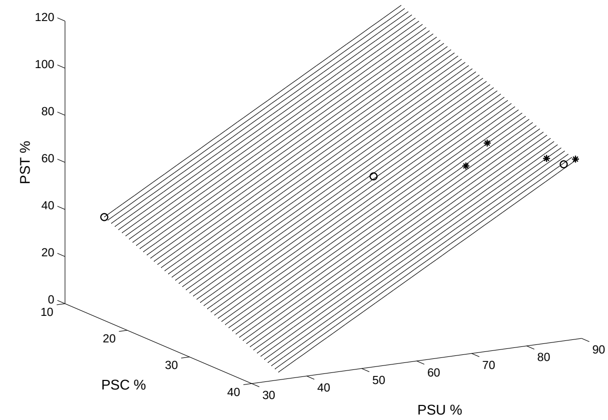

Figure 4.3: The points (PSC, PSU, PST), for some values of m, and the corresponding regression plane; the related Pearson coefficient is $R^2 = 0.99$. The points above the regression plane are plotted with $*$, whereas the ones below with $\circ$.

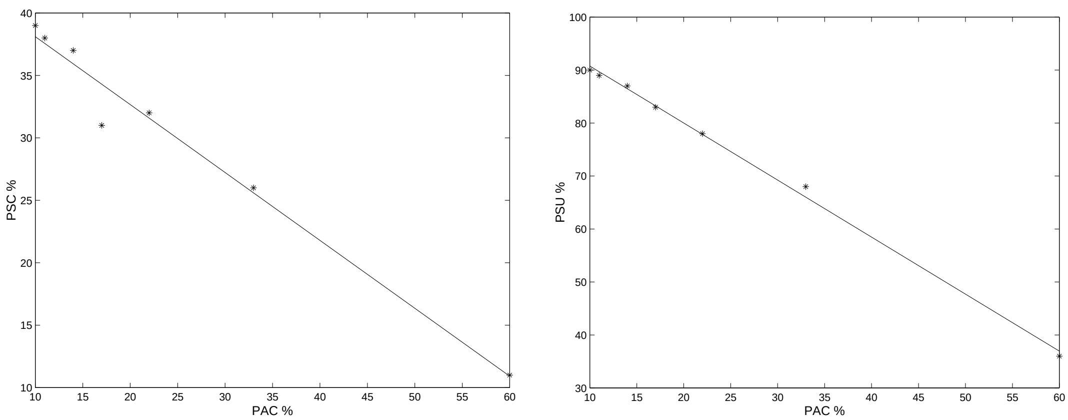

Figure 4.4: The percentages PSC (left) and PSU (right) versus the percentage of average connection PAC and the corresponding regression lines. The related Pearson coefficients are respectively $R_C^2 = 0.98$ and $R_U^2 = 0.997$.

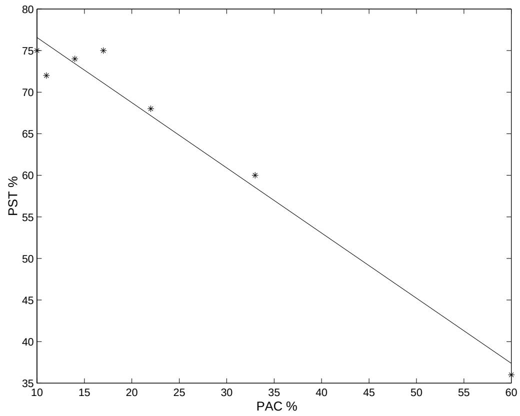

Figure 4.5: The percentage *PST* as a function of the percentage *PAC* and the corresponding regression line. The related Pearson coefficient is $R_T^2 = 0.97$.

◇

Example 4.2. We consider the set $\mathcal{X} = \left\{x^{(i)}\right\}_{1\leq i\leq m}$ of the m vertexes of a regular real polygon, where

$$x_1^{(i)} = \cos\left(\frac{(i-1)\pi}{m}\right) \quad \text{and} \quad x_2^{(i)} = \sin\left(\frac{(i-1)\pi}{m}\right).$$

For this choice, the set $\mathcal{X}$ is an essential system of real vertexes for $\mathcal{P} = \text{absco}(\mathcal{X})$. Note that, for each m the percentage of average connection PAC is 100% (see Table 4.4). In the same table, we report, for some values of m, the quantities N_{EP}, N_{EI}, N_T, N_{IF} and AC.

m	10	20	30	40	50	60	70
N_{EP}	45	190	435	780	1225	1770	2415
N_{EI}	80	360	840	1520	2400	3480	4760
N_T	240	2280	8120	19760	39200	68440	109480
N_{IF}	70	340	810	1480	2350	3420	4690
AC	9	19	29	39	49	59	69
PAC	100%	100%	100%	100%	100%	100%	100%

Table 4.4: The characterising quantities associated with $\mathcal{P}$ for some values of m.

In Table 4.5, we report the average values of the CPU times t for Algorithm 3.1 and $\hat{t}$ for Algorithm 4.1 and the corresponding percentage of saved CPU time PST, which are plotted in Figure 4.6 versus the number m of essential vertexes of $\mathcal{P}$.

As shown in Table 4.6, for each m we have that

$$\hat{U} = \hat{C} = U = C,$$

because, since the m vertexes of $\mathcal{P}$ have maximum connection, all the index pairs are checked and all the corresponding existence pluri-intervals are updated by both the algorithms at each iteration. As a consequence, the percentages PSC and PSU are equal to zero for all m. Therefore, for this example, we cannot determine any dependence among the quantities PAC, PSC, PSU and PST.

◇

m	10	20	30	40	50	60	70
t	0.14	1.14	4.97	10.97	23.10	49.11	85.56
$\hat{t}$	0.13	1.07	4.54	9.78	19.12	40.68	67.15
PST	9 %	6 %	9 %	11 %	17 %	17 %	22 %

Table 4.5: The CPU times t and $\hat{t}$ (in seconds) for Algorithms 3.1 and 4.1, respectively, and the value of percentage PST of the saved CPU time, for some values of m.

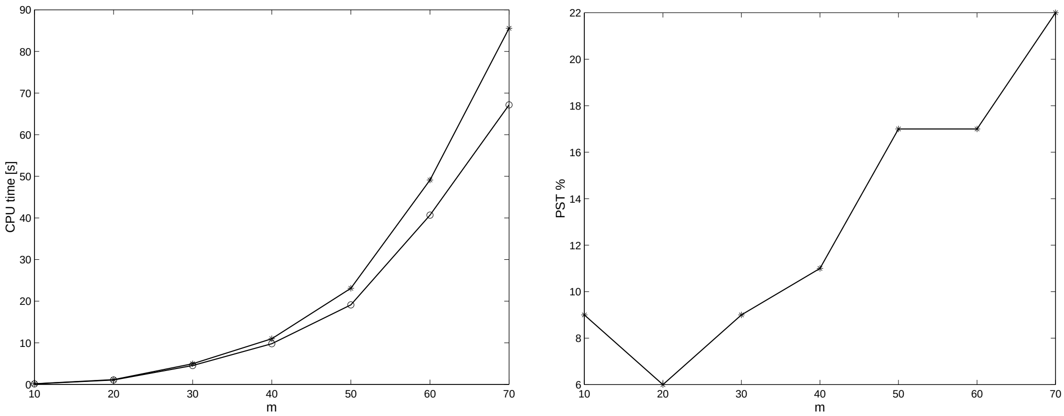

Figure 4.6: The CPU times t (plotted with $*$) and $\hat{t}$ (plotted with $\circ$) on the left and, on the right, the percentage PST of saved CPU time, versus m.

m	10	20	30	40	50	60	70
C	119	1139	4059	9879	19599	34219	54739
$\hat{C}$	119	1139	4059	9879	19599	34219	54739
U	119	1139	4059	9879	19599	34219	54739
$\hat{U}$	119	1139	4059	9879	19599	34219	54739
PSC	0	0	0	0	0	0	0
PSU	0	0	0	0	0	0	0

Table 4.6: The numbers C, $\hat{C}$, U and $\hat{U}$ and the indicators of saved work PSC and PSU for some values of m.

Example 4.3. We consider the set of vectors $\mathcal{X} = \left\{x^{(i)}\right\}_{1\leq i\leq m}$ so that the corresponding b.c.p. $\mathcal{P}$ has the existence pluri-interval $\mathcal{E}_{12}$ with the maximum possible number of intervals, that is $m-2$. We choose this set as

$$x^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad x^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$x_1^{(i)} = \frac{R}{2} \quad \text{and} \quad x_2^{(i)} = \frac{Re^{\mathrm{i}\theta}}{2},$$

with

$$R = 1.01 \ , \ \theta = -\pi + \frac{2\pi(i-2)}{m-1},$$

for $3 \leq i \leq m$. For this choice, $\mathcal{X}$ is an essential system of vertexes for $\mathcal{P} = \text{absco}(\mathcal{X})$. In Table 4.7 we report the quantities N_{EP}, N_{EI}, N_T, N_{IF}, AC and PAC which characterise $\mathcal{P}$ for some values of m.

m	10	20	30	40	50	60	70
N_{EP}	17	54	84	114	144	174	204
N_{EI}	24	54	84	114	144	174	204
N_T	16	36	56	76	96	116	136
N_{IF}	0	0	0	0	0	0	0
AC	3.4	5.4	5.6	5.7	5.8	5.8	5.8
PAC	38%	28%	19%	15%	12%	10%	8%

Table 4.7: The characterising quantities associated with $\mathcal{P}$.

In Table 4.8 we report the CPU times t and $\hat{t}$ related to the Algorithms 3.1 and 4.1, respectively, and the corresponding percentage of saved CPU time PST for some values of m. We see that t and $\hat{t}$ increase with m and that PST is almost 81% for $m \geq 40$ (see Figure 4.7).

m	10	20	30	40	50	60	70
t	0.13	1.12	4.15	10.78	23.53	43.57	75.73
$\hat{t}$	0.04	0.30	0.94	2	4.19	7.48	13.32
PST	67 %	73 %	77 %	81 %	82 %	83 %	82 %

Table 4.8: The CPU times t and $\hat{t}$ (in seconds) and percentage PST of the saved CPU time, for some values of m.

Like in Example 4.1 and as shown in Table 4.9, we have $\hat{U} < \hat{C} < C = U$. A graphic of $C = U$, $\hat{C}$ and $\hat{U}$ as functions of m is reported in Figure 4.8.

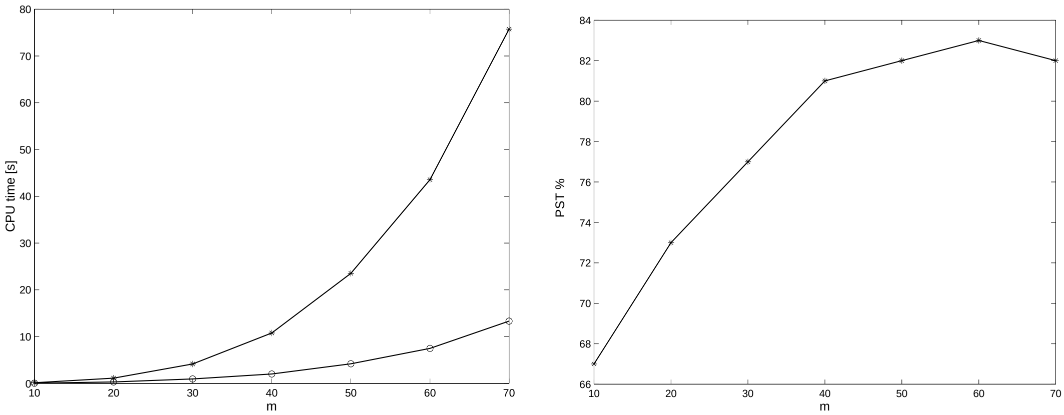

Figure 4.7: The CPU times t and $\hat{t}$ plotted with $*$ and $\circ$, respectively, (on the left) and the percentage PST of saved CPU time (on the right), versus m.

m	10	20	30	40	50	60	70
C	119	1139	4059	9879	19599	34219	54739
$\hat{C}$	119	956	3568	7588	14977	24225	38969
U	119	1139	4059	9879	19599	34219	54739
$\hat{U}$	63	435	1095	1990	3192	4627	6385
PSC	0 %	16 %	12 %	23 %	24 %	29 %	29 %
PSU	47 %	62 %	73 %	80 %	84 %	86 %	88 %

Table 4.9: The numbers C, $\hat{C}$, U, $\hat{U}$ and the indicators of saved work PSC and PSU for some values of m.

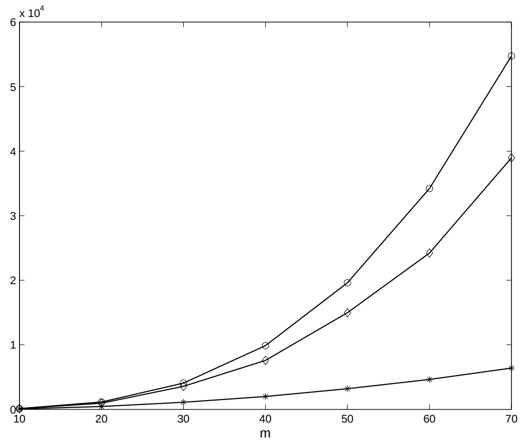

Figure 4.8: The numbers $C = U$ (plotted with $\circ$), $\hat{C}$ (plotted with $\diamond$), $\hat{U}$ (plotted with $*$) versus m.

According to (4.1), the percentage of saved time *PST* is almost a linear function of *PSC* and *PSU*. Thus, we have computed the corresponding regression plane and the related Pearson coefficient R^2 (see Figure 4.9).

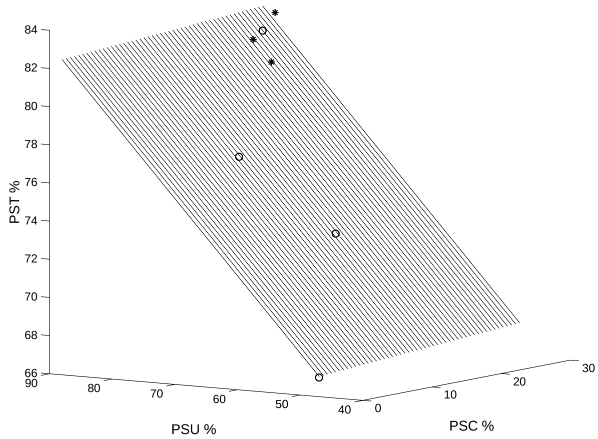

Figure 4.9: The points (PSC, PSU, PST), for some values of m, and the corresponding regression plane with Pearson coefficient $R^2 = 0.99$. The points above the regression plane are plotted with $*$, whereas the ones below with $\circ$.

Moreover, since *PSC*, *PSU* and *PST* are almost linear functions of *PAC*, we have computed the corresponding regression lines and the related Pearson coefficients R_C^2, R_U^2 and R_T^2 (see Figures 4.10 and 4.11).

◇

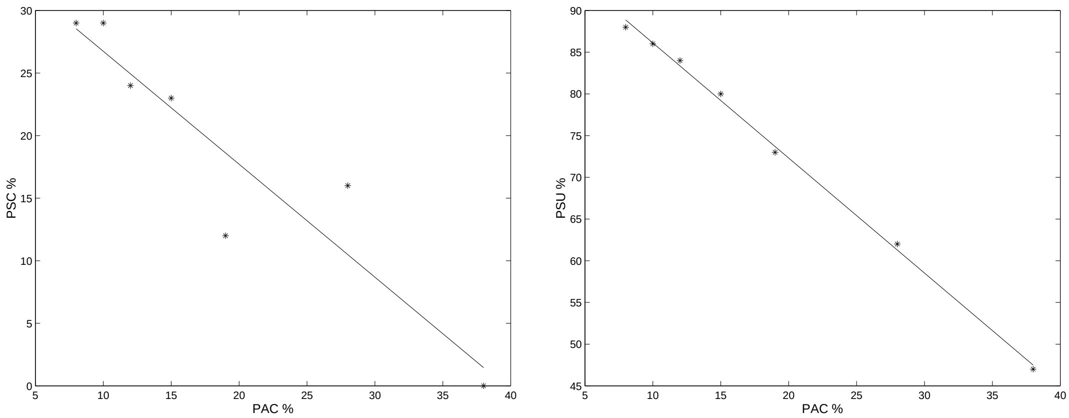

Figure 4.10: *PSC* (left) and *PSU* (right) versus *PAC* and the corresponding regression lines. Respectively, the Pearson coefficients are $R_C^2 = 0.87$ and $R_U^2 = 0.997$.

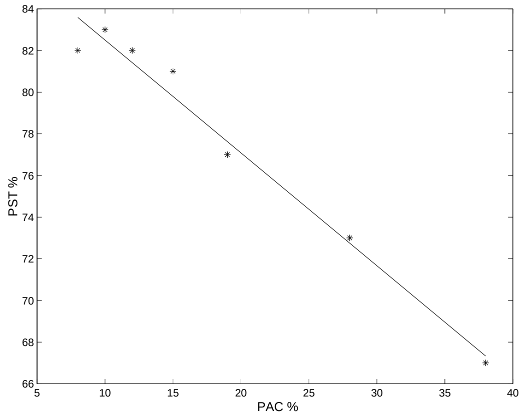

Figure 4.11: *PST* as a function of *PAC* and the corresponding regression line. The related Pearson coefficient is $R_T^2 = 0.98$.

Example 4.4. We consider the set of m points $\mathcal{X} = \left\{x^{(i)}\right\}_{1 \le i \le m}$ randomly generated and distributed on the boundary of a hypersphere centered in the origin. Also for this choice of the points, the set $\mathcal{X}$ is an essential system of vertexes for $\mathcal{P} = \text{absco}(\mathcal{X})$. In Table 4.10 we report, for some values of m, the quantities N_{EP}, N_{EI}, N_T, N_{IF}, AC and PAC.

m	10	20	30	40	50	60	70
N_{EP}	24	54	84	114	144	174	204
N_{EI}	24	54	84	114	144	174	204
N_T	16	36	56	76	96	116	136
N_{IF}	0	0	0	0	0	0	0
AC	4.8	5.4	5.6	5.7	5.8	5.8	5.8
PAC	53%	28%	19%	15%	12%	10%	8%

Table 4.10: The characterising quantities associated with $\mathcal{P}$.

The CPU times t and $\hat{t}$ and the percentage *PST* are shown in Table 4.11 and the corresponding plots as functions of m are presented in Figure 4.12. We can see that the CPU times increase with m and the *PST* is almost 90% for $m \ge 40$.

Also in this case, due to the improvement procedure, according to Table 4.12 we have that $\hat{U} < \hat{C} < C = U$. A graphic of these quantities versus m is shown in Figure 4.13.

According to (4.1), the percentage of saved time *PST* is almost a linear function of *PSC* and *PSU*. Thus, we have computed the corresponding regression plane and Pearson coefficient R^2 (see Figure 4.14).

Since *PSC* and *PSU* are both almost linear functions of *PAC* then, according to (4.3), *PST* is also almost a linear function of *PAC*. Therefore, we have computed the corresponding regression lines and the related Pearson coefficients R_C^2, R_U^2 and R_T^2 (see Figure 4.15 and Figure 4.16, respectively). ◇

m	10	20	30	40	50	60	70
t	0.13	1.15	4.28	11.04	23.57	44.62	76.91
$\hat{t}$	0.07	0.28	0.61	1.18	2.12	3.83	6.41
PST	47 %	76 %	86 %	89 %	91 %	91 %	92 %

Table 4.11: The CPU times t and $\hat{t}$ (in seconds) and the value of percentage *PST* of the saved CPU time, for some values of m.

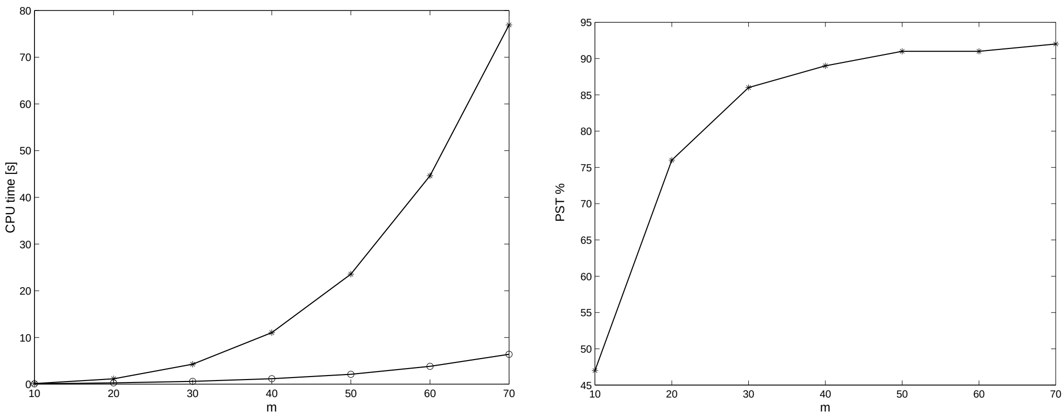

Figure 4.12: The average CPU times t and $\hat{t}$, respectively plotted with $*$ and $\circ$, (left) and the percentage *PST* of saved CPU time (right), versus m.

m	10	20	30	40	50	60	70
C	119	1139	4059	9879	19599	34219	54739
$\hat{C}$	113	686	1825	3315	5973	9427	12493
U	119	1139	4059	9879	19599	34219	54739
$\hat{U}$	79	265	514	689	1000	1379	1551
PSC	5 %	40 %	55 %	66 %	70 %	72 %	77 %
PSU	34 %	77 %	87 %	93 %	95 %	96 %	97 %

Table 4.12: The numbers C, $\hat{C}$, U, $\hat{U}$ and the indicators of saved work *PSC* and *PSU* for some values of m.

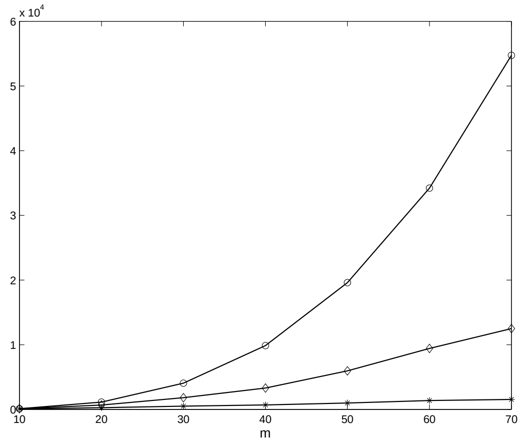

Figure 4.13: The numbers $C = U$ (plotted with $\circ$), $\hat{C}$ (plotted with $\diamond$), $\hat{U}$ (plotted with $*$) versus m.

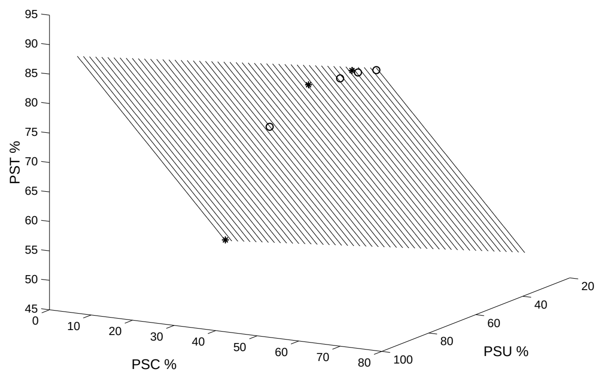

Figure 4.14: The points (PSC, PSU, PST), for some values of m, and the corresponding regression plane. The related Pearson coefficient is $R^2 = 0.997$. The points above the regression plane are plotted with $*$, whereas the ones below with $\circ$.

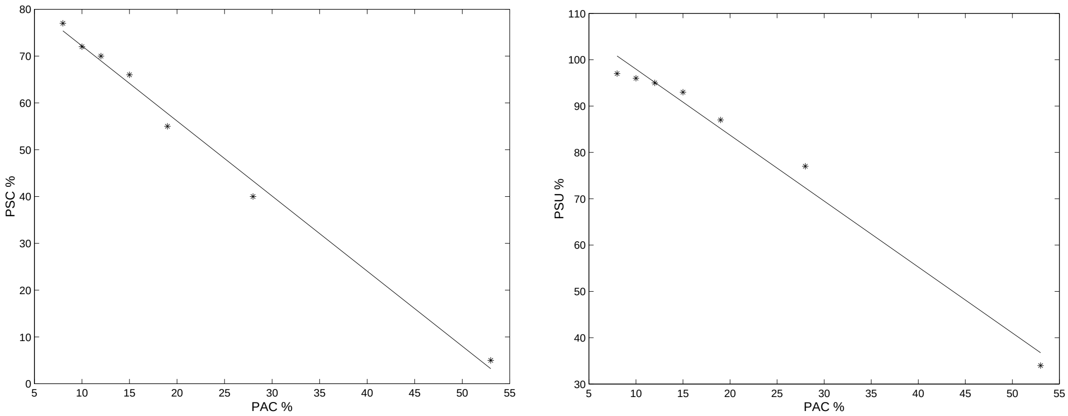

Figure 4.15: *PSC* (left) and *PSU* (right) versus *PAC* and the corresponding regression lines. Respectively, the related Pearson coefficients are $R_C^2 = 0.99$ and $R_U^2 = 0.98$.

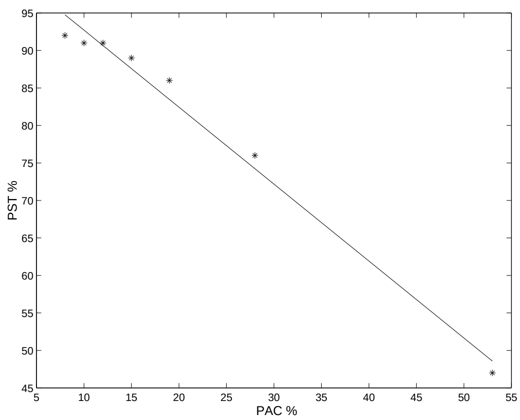

Figure 4.16: The *PST* as a function of *PAC* and the corresponding regression line with Pearson coefficient $R_T^2 = 0.98$.

Example 4.5. We consider a set $\mathcal{X} = \{x^{(i)}\}_{1\leq i\leq 1000}$ of 1000 points randomly generated which gives rise to a b.c.p. $\mathcal{P} = \text{absco}(\mathcal{X})$ with only $m = 129$ essential vertexes. The usual characterising quantities of this b.c.p. $\mathcal{P}$ are summarised in Table 4.13.

N_{EP}	N_{EI}	N_T	N_{IF}	AC	PAC
381	381	254	0	5.9	5%

Table 4.13: The characterising quantities associated with $\mathcal{P}$.

As we can see, the b.c.p. has no isolated facets, only 5 % of all the possible index pairs (i, j) produce regular facets, whose existence pluri-intervals have always only one interval.

The CPU times for both Algorithms are reported on the right of Table 4.14 and given in seconds. Remarkably, the percentage of saved CPU time PST is 90 %.

C	$\hat{C}$	U	$\hat{U}$	PSC	PSU
2809492	421203	1562622	41532	85 %	97 %

t	$\hat{t}$	PST
4080	419	90 %

Table 4.14: The numbers of checked pairs and updated existence pluri-intervals for both Algorithm 3.1 and 4.1 (left) and the corresponding CPU times in seconds (right).

It is also interesting to note that the number of checked pairs, shown on the left of Table 4.14, is less than the number of updated existence pluri-intervals for both algorithms. This is due to the deletion, during the construction procedure, of many vertexes of the original set of points. The percentages PSC of saved checked pairs and PSU of saved updated existence pluri-intervals are, respectively, 85 % and 97 %. Thus, the improved version of the algorithm works extremely well on this example.

◇

4.3 The improved procedure for the computation of the norm

The procedure proposed in Section 3.3 for the computation of $\|z\|_{\mathcal{P}}$ involves in any case the solution of the minimisation problem (3.78), whose computational complexity is $O(m^2)$ with respect to the number m of essential vertexes of $\mathcal{P}$. Moreover, if (3.79) is not satisfied, then we have to start processing the existence pairs $(\theta_{ij}^{\pm}, \theta_{ik}^{\mp})$, the number of which may be even $O(m^3)$. Therefore, the minimum complexity is $O(m^2)$ and, in the most unfortunate cases, even $O(m^3)$. For very large numbers of essential vertexes, this is not very satisfactory.

In order to reduce the average computational complexity, a possible variant of the procedure is not to compute the minimising pairs (r, s) to (3.78) a priori and, instead, to process the pairs (i, j), $1 \leq i < j \leq m$, one at a time until the relationship

$$\arg(\mu_{ij}) - \arg(\lambda_{ij}) \in \mathcal{E}_{ij} \tag{4.4}$$

is satisfied. In fact, since (3.77) is characterising, such a pair must necessarily be a minimising pair (r, s) satisfying (3.80).

However, it is not clear a priori whether this variant is more convenient or not. If we are able (or lucky) to meet a minimising pair (r, s) soon enough and if it satisfies (3.79), then this variant is better. On the contrary, if we meet it quite late or if it does not satisfy (3.79), then it is the original procedure that is less expensive, since it does not involve the check (4.4) for each pair (i, j).

Therefore, we need a criterion in order to guess those facets of $\mathcal{P}$, either regular or special, that have the greatest chances to be the facet which the vector z projects on. This criterion is the same we have already used in Algorithm 4.1 to determine a facet of $\mathcal{P}^{(k-1)}$ which is seen by the new point $x^{(k)}$ at each iteration.

Hence, as in Section 4.1, we compute the Euclidean distance

$$\delta_i = \sqrt{\|x^{(i)}\|_2^2 - |\langle x^{(i)}, z\rangle|^2/\|z\|_2^2}$$

of the ray R_z from the circle generated by the vertex $x^{(i)}$ and, then, we reorder the indexes in non-decreasing order with respect to the distances δ_i and, subsequently, we define a total order relation "≺" on the set of reordered index pairs (i, j) in reversed lexicographical way, i.e.,

$$(i, j) \prec (h, k) \iff j < k \quad \text{or} \quad (j = k \ \& \ i < h).$$

Finally, we add the set of the triplets of indexes (i, j, k), $1 \le i < j < k \le m$, and extend the total order relation "≺" by inserting a triplet (r, s, t) soon after the last of the three pairs (r, s), (r, t), (s, t). If more than one triplet is completed simultaneously, all of them are inserted in lexicographical order.

The resulting procedure consists in processing, one after the other, pairs and triplets of indexes following the total order "≺" defined above until (4.4) is satisfied by a pair (i, j) or (3.92) is satisfied by an existence pair $(\theta^{\pm}_{ij}, \theta^{\mp}_{ik})$ related to a triplet (i, j, k).

ALGORITHM 4.2. *(An improvement of Algorithm 3.2)*

% Input: $\mathcal{X} = \{x^{(1)}, x^{(2)}, \cdots, x^{(m)}\}$, RF, SF, z

% Output: $\|z\|_{\mathcal{P}}$, where $\mathcal{P} = \text{absco}(\mathcal{X})$

$foundface \leftarrow 0$
while $l \le m$
  if $det([z\ x^{(l)}]) = 0$
    % z is proportional to the vertex $x^{(l)}$
    $\|z\|_{\mathcal{P}} \leftarrow \|z\|/\|x^{(l)}\|$
    $foundface \leftarrow 1$
  end
  $l = l + 1$
end
if $foundface = 0$
  % compute the distances δ_i of R_z from the *circle* generated by all the vertexes $x^{(i)}$ of $\mathcal{P}$ with
  % respect to the Euclidean distance
  $\delta_i = \sqrt{\|x^{(i)}\|_2^2 - |\langle x^{(i)}, z\rangle|^2/\|z\|_2^2}\ , \quad i = 1, \ldots, m$
  $ord \leftarrow$ *vector of the indexes reordered in non-decreasing order with respect to* δ_i
  $VisitM \leftarrow$ *zero-matrix of dimension* $m \times m$
  $ic \leftarrow 2$
  while $(foundface = 0\ \&\ ic \le m)$
    $ir \leftarrow 1$
    while $(foundface = 0\ \&\ ir \le ic - 1)$
      % find the indexes r and c of the facet obtained in correspondence of the indexes ir and ic
      $r \leftarrow ord(ir), \quad c \leftarrow ord(ic)$
      if $RF(r, c) \ne \emptyset$ % *some facets* $F_{y_{rc}(\theta)}$ *exist*
        $VisitM(r, c) \leftarrow 1, \quad VisitM(c, r) \leftarrow 1$
        $\begin{bmatrix} \lambda_{rc} \\ \mu_{rc} \end{bmatrix} \leftarrow [x^{(r)}\ x^{(c)}]^{-1} z$
        if $\arg(\mu_{rc}) - \arg(\lambda_{rc})$ *shift*$(-\pi, \pi] \in RF(r, c)$
          % z projects on the facet $e^{\mathrm{i} \arg(\lambda_{rc})} x^{(r)} \bullet\!\!-\!\!\bullet\ e^{\mathrm{i} \arg(\mu_{rc})} x^{(c)}$
          $foundface \leftarrow 1$

```
            ||z||_P ← |λ_rc| + |μ_rc|
        end
        if foundface = 0
            k ← 1
            while (foundface = 0  &  k ≤ m)
                if (VisitM(r,k)=1 & VisitM(c,k)=1)
                    % find if z projects on a triangle determined by the index triplet (r, c, k),
                    % by using the results of Section 3.3
                    if z projects on a triangle determined by the index triplet (r, c, k)
                        determine λ_r, λ_c, λ_k by using the results of Section 3.3
                        ||z||_P ← λ_r + λ_c + λ_k
                        foundface ← 1
                    end
                end
                k ← k + 1
            end
        end
    end
    ir ← ir + 1
  end
  ic ← ic + 1
 end
end
```

4.4 Expected results for Algorithm 4.2 and numerical experiments

We have seen that the improvement of Algorithm 4.1 with respect to Algorithm 3.1 for the construction of a b.c.p. $\mathcal{P}$ is strictly related to the percentage of average connection *PAC*: the smaller *PAC* is, the greater the percentages *PST*, *PSC* and *PSU* are.

The situation for Algorithm 4.2 with respect to Algorithm 3.2 for the computation of the norm is slightly different. In fact, the possible improvement is clearly related to the ability of the adopted criterion to detect, as soon as possible, the facet of $\mathcal{P}$ which the vector z projects on. And this, in turn, seems to be more related to a possible uniform spheric-like distribution of the vertexes of $\mathcal{P}$, rather than to a possible low mutual connection, even if in many cases these two characteristics seem to be quite connected to each other.

In any case, the performance of the two algorithms to compare depends primarily on the number of checked index pairs (i, j) corresponding to regular facets of $\mathcal{P}$, denoted by R for Algorithm 3.2 and by $\hat{R}$ for Algorithm 4.2, and on the number of checked existence pairs $(\theta^{\pm}_{ij}, \theta^{\mp}_{ik})$ corresponding to triangles included in special facets of $\mathcal{P}$, denoted by S for Algorithm 3.2 and by $\hat{S}$ for Algorithm 4.2. Therefore, as in Section 4.2, if we define the *percentage of saved checked index pairs*

$$PSR = \frac{R - \hat{R}}{R}$$

and the *percentage of saved checked existence pairs*

$$PSS = \frac{S - \hat{S}}{S},$$

we expect that the percentage of saved CPU time *PST* follows, in first approximation, a linear rule of the type

$$PST = \alpha + \beta PSR + \gamma PSS \tag{4.5}$$

for suitable coefficients α, β, γ with $\beta, \gamma > 0$. However, in principle, we do not always expect very strong relationships between the triplet (PST, PSR, PSS) and the percentage of average connection PAC. In particular, this applies to PSS because, if a point z projects on a regular facet, then Algorithm 3.2 does not check any special facet, whereas Algorithm 4.2 may also check some special facets.

We have organised our numerical experiments on the same classes of b.c.p.'s that we considered in Section 4.2. This time, we think it is reasonable to perform, for each b.c.p. $\mathcal{P}$, the computation of the norm of a certain number N (say, one hundred) of vectors $\{z^{(i)}\}$ randomly generated. In such a way, by taking the average of the various CPU times t_i, $\hat{t}_i$ and of the quantities R_i, $\hat{R}_i$, S_i, $\hat{S}_i$, related to each vector $z^{(i)}$, which we keep on denoting by t, $\hat{t}$, R, $\hat{R}$, S, $\hat{S}$, and are given by

$$t = \frac{\sum_{i=1}^{N} t_i}{N}, \quad R = \frac{\sum_{i=1}^{N} R_i}{N}, \quad S = \frac{\sum_{i=1}^{N} S_i}{N} \tag{4.6}$$

and

$$\hat{t} = \frac{\sum_{i=1}^{N} \hat{t}_i}{N}, \quad \hat{R} = \frac{\sum_{i=1}^{N} \hat{R}_i}{N}, \quad \hat{S} = \frac{\sum_{i=1}^{N} \hat{S}_i}{N}, \tag{4.7}$$

we should still have a linear relationship of type (4.5). Remark that, by taking the averages, we should get an idea of the average behaviour of the algorithms and we should also considerably reduce the random noises in measuring the CPU times (see the analogous discussion in Section 4.2).

We shall use the same set of one hundred vectors for all examples.

Example 4.6. In this example we compare the performances of the Algorithms 4.2 and 3.2 for the computation of the polytope norm of one hundred given points, when the b.c.p. $\mathcal{P}$ is generated by the set of m essential vertexes defined in Example 4.1. In Table 4.15 we report the percentages PSR, PSS, PST, which are computed using the mean values R, $\hat{R}$, S, $\hat{S}$ defined in (4.6) and (4.7); furthermore, we report, for convenience, also the percentage PAC which has already been shown in Example 4.1.

As we can see in Figures 4.17 and 4.18 associated with Table 4.15 , the mean values R, $\hat{R}$, S, $\hat{S}$, the percentage of saved time PST, and the percentage of saved index pairs PSR increase with the number m of essential vertexes of the b.c.p. $\mathcal{P}$. On the other hand, the percentage of saved existence pairs PSS slowly decreases as m increases.

From figure 4.19, we can see that there is a good linear relation both between PST and PAC and between PSR and PAC, and so we have computed the corresponding regression lines and the related Pearson coefficients, denoted respectively with R_T^2 and R_R^2; on the other hand, there is no clear linear relation between PSS and PAC (see Figure 4.20).

Finally, as expected and as shown in Figure 4.21, since there is a global linear relation between PST, PSR and PSS, we have computed the corresponding regression plane and the Pearson coefficient R^2.

◇

m	10	20	30	40	50	60	70
R	36	171	406	741	1176	1711	2346
$\hat{R}$	8	19	30.08	41.29	52.80	64.05	75.18
S	11.09	26.08	46.44	59.90	64.53	89.90	104.51
$\hat{S}$	3.69	10.86	18.31	26.03	34.04	41.75	49.45
PSR	78 %	89 %	93 %	94 %	96 %	96 %	97 %
PSS	67 %	58 %	61 %	57 %	47 %	54 %	53 %
PST	61 %	80 %	88 %	91 %	92 %	94 %	95 %
PAC	60 %	33%	22%	17%	14%	11%	10%

Table 4.15: The percentages PST, PSR, PSS and PAC for some values of m.

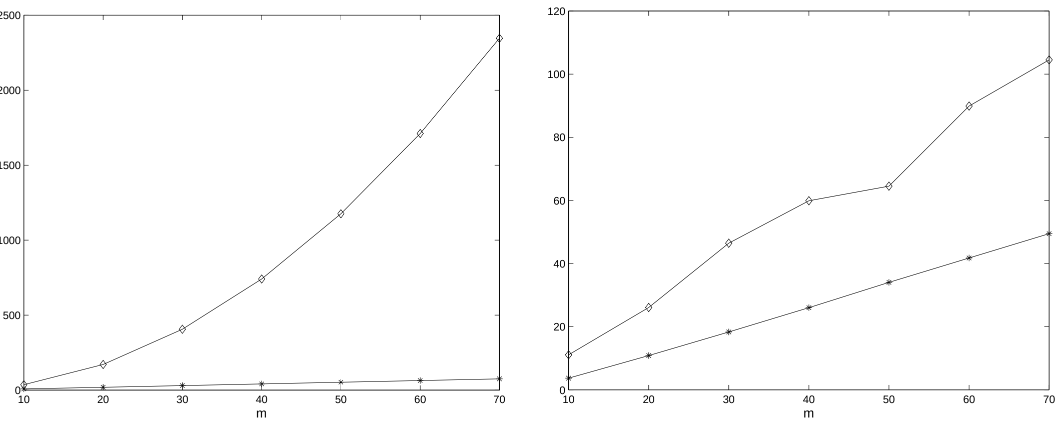

Figure 4.17: The mean values R and $\hat{R}$ (left), respectively plotted with $\diamond$ and $*$, and S, $\hat{S}$ (right), respectively plotted with $\diamond$ and $*$, versus m.

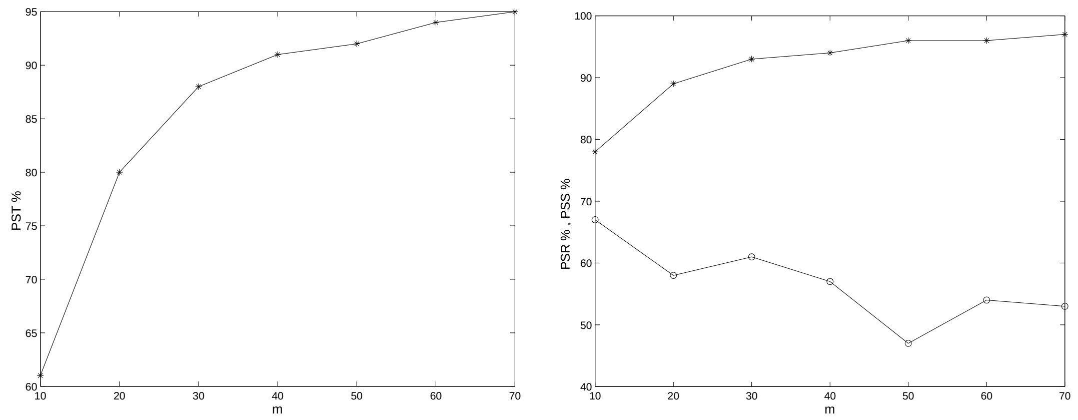

Figure 4.18: *PST* (left) and *PSR*, *PSS* (right) versus m. The percentage *PSR* is plotted with $*$, whereas *PSS* with $\circ$.

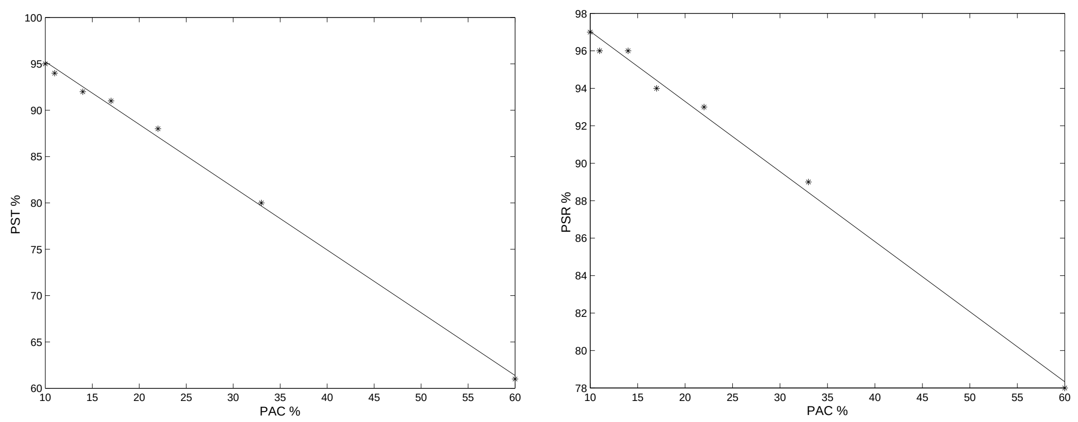

Figure 4.19: *PST* (left) and *PSR* (right) versus *PAC* and the corresponding regression lines. The Pearson coefficients are, respectively, $R_T^2 = 0.997$ and $R_R^2 = 0.99$.

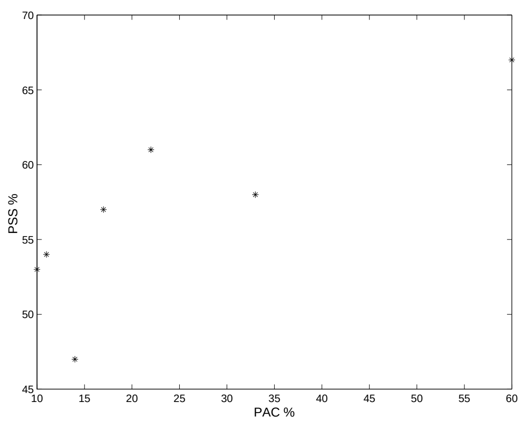

Figure 4.20: *PSS* versus *PAC*.

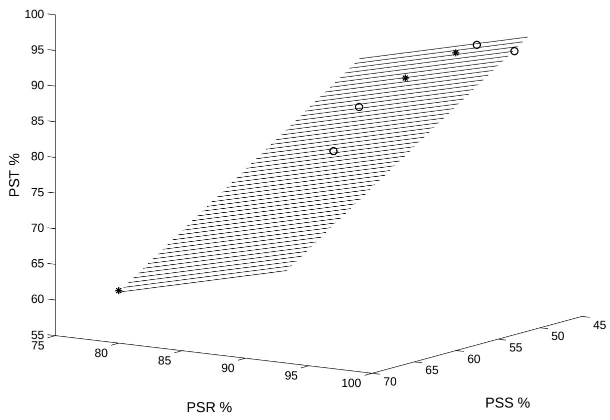

Figure 4.21: The points (PSR, PSS, PST), for some values of m, and the corresponding regression plane with Pearson coefficient $R^2 = 0.997$. The points above the regression plane are plotted with $*$, whereas the ones below with $\circ$.

Example 4.7. In this example we perform the same analysis made in the previous one considering the same b.c.p. of Example 4.2. In Table 4.16 we report the quantities R, $\hat{R}$, S, $\hat{S}$, PST, PSR, PSS and PAC for different values of m. As shown in Figure 4.22, the quantities R, $\hat{R}$, S, $\hat{S}$ increase with m. On the left of Figure 4.23 we can see the little differences among the values of PST obtained from different values of m, while on the right are reported the percentages PSR and PSS which have a slight increase with m. In any case PSR, PSS and PST are almost constant with respect to m. For this reason, we have not reported the dependence of PST with respect to PSR and PSS. Note that, we have not reported either the percentages PST, PSR, PSS versus PAC, since in this case PAC has the constant value 100%.

$\diamond$

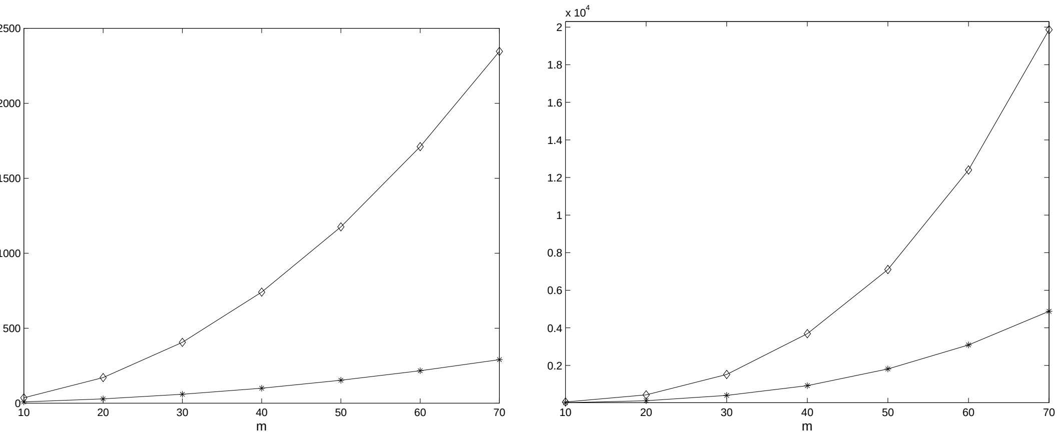

Figure 4.22: The mean values R and $\hat{R}$ (left), respectively plotted with $\diamond$ and $*$, and S, $\hat{S}$ (right), respectively plotted with $\diamond$ and $*$, versus m.

m	10	20	30	40	50	60	70
R	36	171	406	741	1176	1711	2346
$\hat{R}$	9.1	28.71	59.50	99.84	153.32	216.41	290.8
S	52.39	434.43	1518.80	3688.40	7103.30	12402	19864
$\hat{S}$	15.34	121.23	403.27	923.49	1810.30	3089.40	4879
PSR	75 %	83 %	85 %	87 %	87 %	87 %	88 %
PSS	71 %	72 %	73 %	75 %	75 %	75 %	75 %
PST	59 %	67 %	69 %	66 %	57 %	55 %	60 %
PAC	100%	100%	100%	100%	100%	100%	100%

Table 4.16: The percentages PST, PSR, PSS and PAC for some values of m.

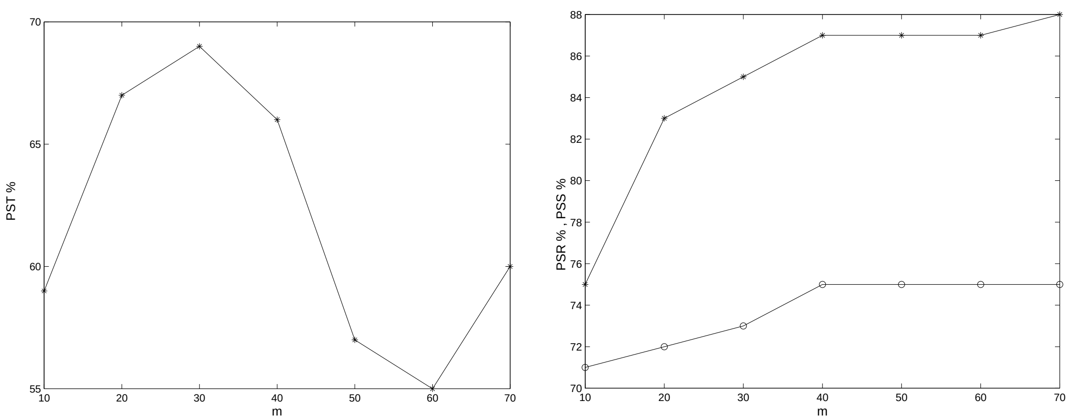

Figure 4.23: PST (left) and PSR, PSS (right) versus m. The percentage PSR is plotted with $*$, whereas PSS with $\circ$.

Example 4.8. We again consider the b.c.p. defined in Example 4.3. In Table 4.17 we can see that Algorithm 4.2 works very well, saving almost 99% of both checked index and existence pairs. As shown in Figure 4.24, R and S increase, $\hat{R}$ has a slight increase and $\hat{S}$ has a slight decrease with m. Moreover, the percentages PST, PSR, PSS increase with m (see Figure 4.25). Furthermore, disregarding the case $m = 10$, we can see a linear relation between PST and PAC and between PST and PAC whereas there is no such relation between PSS and PAC. Since plotting the values of PST as a function of the values of PSR and PSS and computing the corresponding regression plane we have that the Pearson coefficient is $R^2 = 0.999$, then our expectation (4.5) is confirmed. ◇

m	10	20	30	40	50	60	70
R	36	171	406	741	1176	1711	2346
$\hat{R}$	12.08	13.73	16.19	21.57	27.03	32.51	37.81
S	10.80	21.74	34.03	45.30	57.90	68.26	86.66
$\hat{S}$	2.78	3.35	0.85	0.83	0.85	0.89	0.81
PSR	66 %	92 %	96 %	97 %	98 %	98 %	98 %
PSS	74 %	85 %	98 %	98 %	99 %	99 %	99 %
PST	52 %	85 %	92 %	94 %	95 %	96 %	96 %
PAC	38%	28%	19%	15%	12%	10%	8%

Table 4.17: The percentages PST, PSR, PSS and PAC for some values of m.

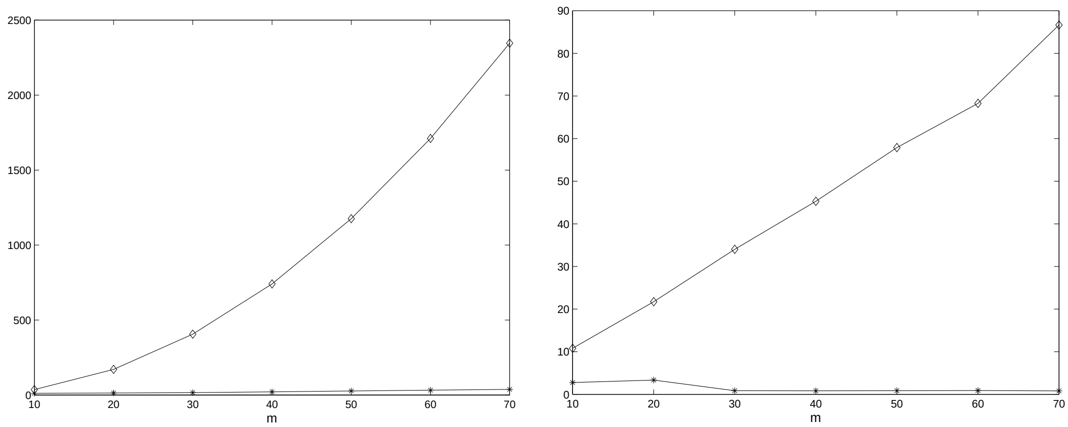

Figure 4.24: The mean values R and $\hat{R}$ (left), respectively plotted with ⋄ and ∗, and S, $\hat{S}$ (right), respectively plotted with ⋄ and ∗, versus m.

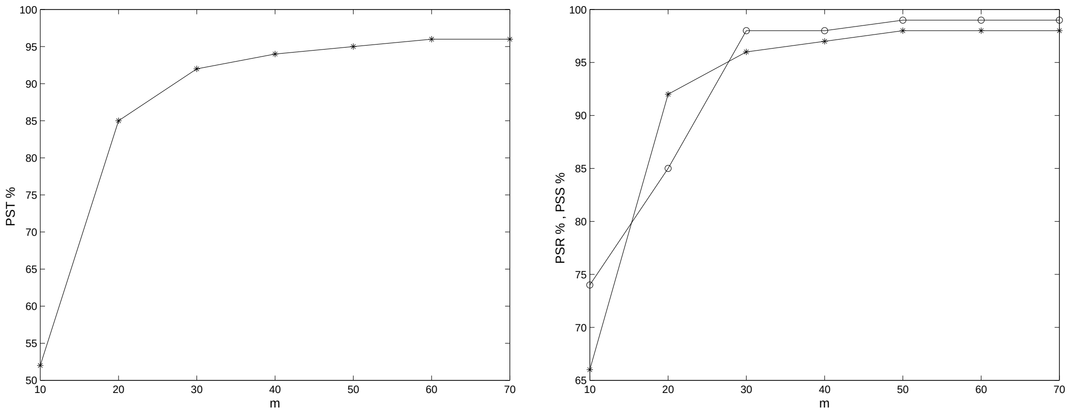

Figure 4.25: *PST* (left) and *PSR*, *PSS* (right) versus m. The percentage *PSR* is plotted with $*$, whereas *PSS* with $\circ$.

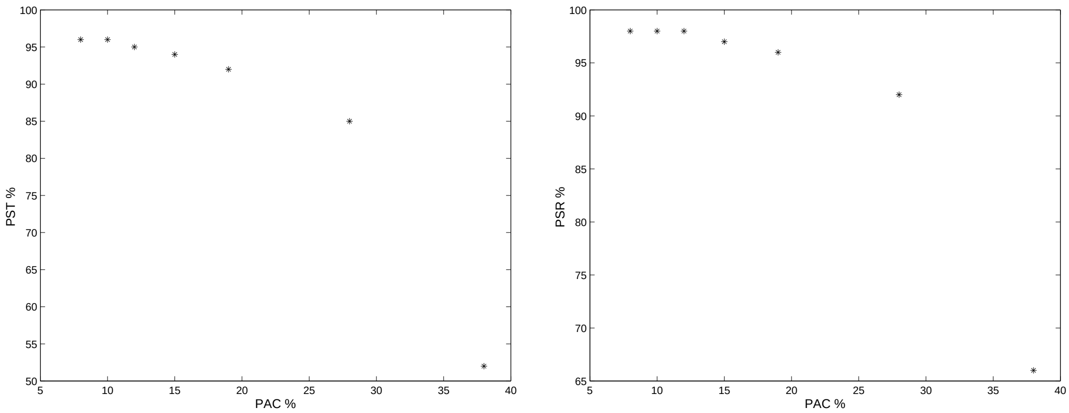

Figure 4.26: *PST* (left) and *PSR* (right) versus *PAC*.

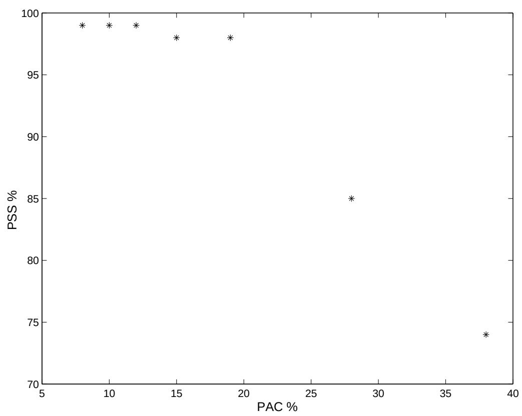

Figure 4.27: *PSS* versus *PAC*.

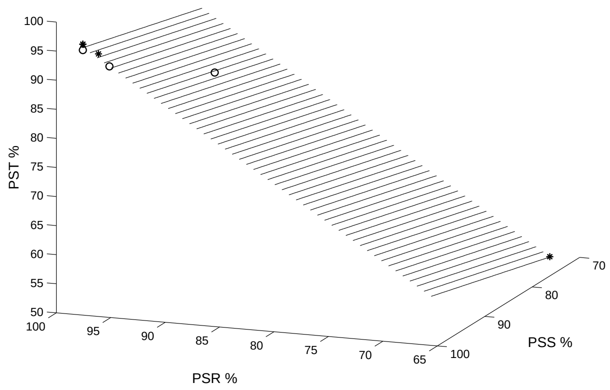

Figure 4.28: The points (PSR, PSS, PST), for some values of m, and the corresponding regression plane with Pearson coefficient $R^2 = 0.999$. The points above the regression plane are plotted with $*$, whereas the ones below with $\circ$.

Example 4.9. In this example we consider the same b.c.p. of Example 4.4, which is associated with an essential system of vertexes which are randomly generated and distributed on a hypersphere. The improvement of Algorithm 4.2 with respect to Algorithm 3.2 is summarised in Table 4.18 and in the next Figure 4.30, where we can see that the percentages PSR, PSS, PST increase with m. Furthermore, it is interesting to note that we save almost 99% of the CPU time; thus Algorithm 4.2 works in an exellent way for this kind of b.c.p. In the next Figures 4.31 and 4.32, we can see that there is a linear relation between PST and PAC, PSR and PAC, PSS and PAC. As expected, we can also see in Figure 4.33 the linear relation which occurs between PST, PSR and PSS.

$\diamond$

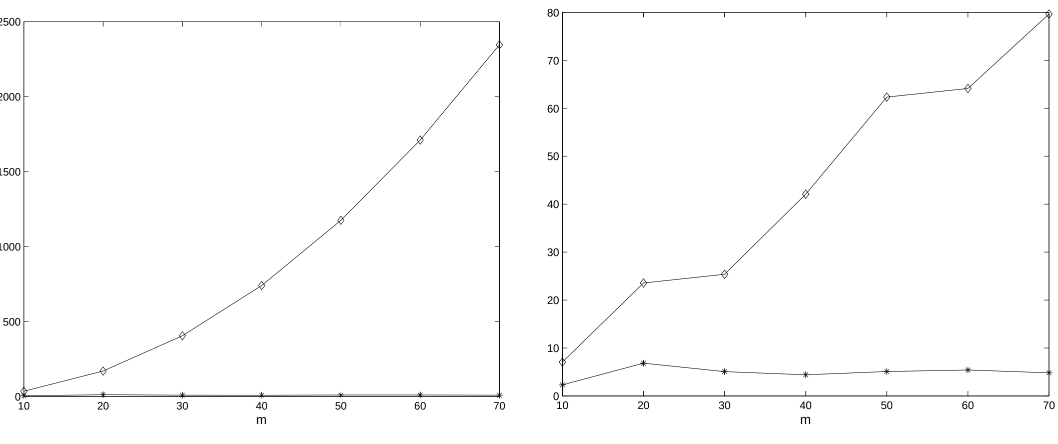

Figure 4.29: The mean values R and $\hat{R}$ (left), respectively plotted with $\diamond$ and $*$, and S, $\hat{S}$ (right), respectively plotted with $\diamond$ and $*$, versus m.

m	10	20	30	40	50	60	70
R	36	171	406	741	1176	1711	2346
$\hat{R}$	5.75	13.08	10.29	9.33	11.03	11.44	10.23
S	7.06	23.56	25.37	42.08	62.33	64.13	79.64
$\hat{S}$	2.32	6.83	5.07	4.41	5.1	5.41	4.83
PSR	84 %	92 %	97 %	99 %	99 %	99 %	100 %
PSS	67 %	71 %	80 %	90 %	92 %	92 %	94 %
PST	76 %	89 %	95 %	98 %	99 %	99 %	99 %
PAC	53%	28%	19%	15%	12%	10%	8%

Table 4.18: The percentages *PST*, *PSR*, *PSS* and *PAC* for some values of m.

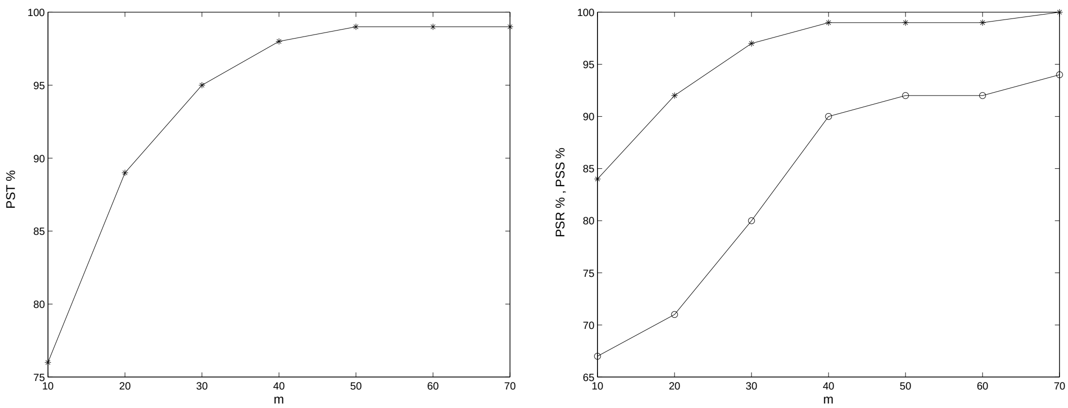

Figure 4.30: *PST* (left) and *PSR*, *PSS* (right) versus m. The percentage *PSR* is plotted with $*$, whereas *PSS* with $\circ$.

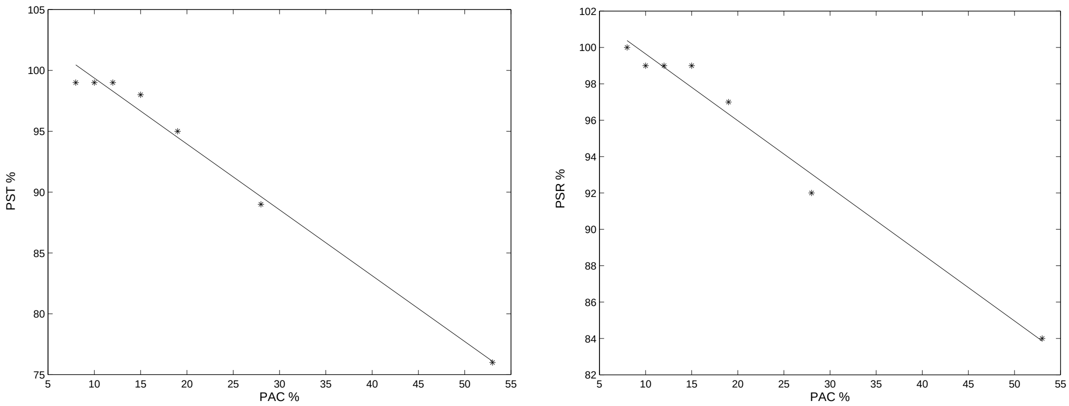

Figure 4.31: *PST* (left) and *PSR* (right) versus *PAC* and the corresponding regression lines. The Pearson coefficients are, respectively, $R_T^2 = 0.99$ and $R_R^2 = 0.98$.

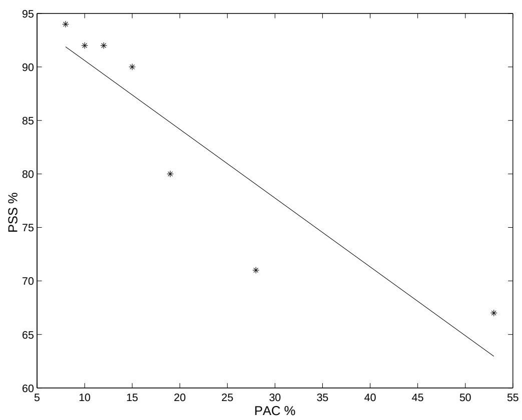

Figure 4.32: *PSS* versus *PAC* and the corresponding regression line with Pearson coefficient $R_S^2 = 0.83$.

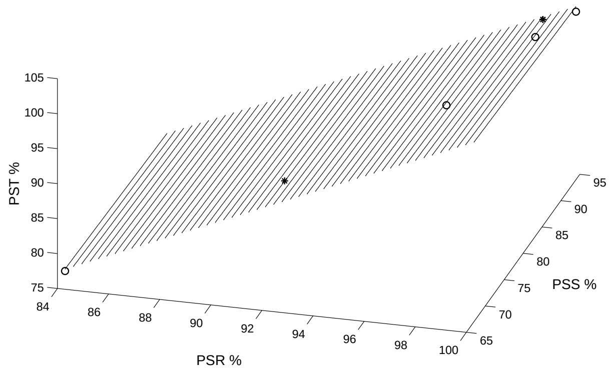

Figure 4.33: The points (PSR, PSS, PST), for some values of m, and the corresponding regression plane with Pearson coefficient $R^2 = 0.99$. The points above the regression plane are plotted with $*$, whereas the ones below with $\circ$.

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