L.N. GUMILYOV
EURASIAN
NATIONAL
UNIVERSITY

UNIVERSITÀ
DEGLI STUDI
DI PADOVA

Sede Amministrativa: L. N. Gumilyov Eurasian National University (Kazakhstan)

Sede di co-tutela: Università degli Studi di Padova (Italia)

SCUOLA DI DOTTORATO DI RICERCA IN: Scienze Matematiche

INDIRIZZO: Matematica

CICLO: XXIII

# REGULARITY AND APPROXIMATION PROPERTIES OF THE SOLUTIONS OF SECOND ORDER DEGENERATE AND NONLINEAR ELLIPTIC SYSTEMS

**Direttore della Scuola:** Ch.mo Prof. Paolo Dai Pra

**Coordinatore d'indirizzo:** Ch.mo Prof. Franco Cardin

**Supervisore nella sede amministrativa:**

Ch.mo Prof. Mukhtarbay Otelbayev

**Supervisore nella sede di co-tutela:**

Ch.mo Prof. Massimo Lanza de Cristoforis

**Dottoranda:** Lyailya Zhapsarbayeva

# Abstract

This Dissertation consists of seven chapters.

Chapter 1 is an introduction, where, in particular, the importance of studying the second order degenerate systems is discussed and motivated. In this chapter we present some well-known auxiliary facts and necessary notation.

In Chapter 2, we obtain the conditions of the unique solvability of the semiperiodical Dirichlet problems in the rectangle for second order degenerate systems with the right hand side in $L_2$.

In Chapter 3, we establish a coercive estimate for the solutions of the semiperiodical Dirichlet problem for a second order degenerate system in the rectangle.

In Chapter 4, we prove the existence, uniqueness and regularity in the Sobolev space $W_2^2(G, \mathbb{R}^2)$ of the solutions of second order singular degenerate systems with variable principal coefficients.

Chapter 5 is devoted to the questions of coercive estimates for the solutions of second order singular degenerate systems.

Chapter 6 is devoted to the questions of compactness and approximation properties of the solutions of second order singular degenerate systems. We also obtain double-sided estimates for the distribution function of the approximation numbers of the corresponding operator. We extend the main results of K.Ospanov on approximation properties of the solutions of an elliptic operator [42] for Bitsadze-type systems with variable lower order coefficients to the case of degenerate systems.

The unique solvability of the semiperiodical nonlinear problems for second order singular elliptic systems is proved in Chapter 7.

# Riassunto

Questa Tesi consiste di sette capitoli.

Il Capitolo 1 è una introduzione, dove, in particolare, viene discussa e motivata l'importanza dello studio di sistemi degeneri del secondo ordine. In questo capitolo presentiamo alcuni risultati ausiliari noti e notazioni necessarie.

Nel Capitolo 2, otteniamo condizioni di risolubilità con unicità dei problemi di Dirichlet semiperiodici nel rettangolo per sistemi del secondo ordine degeneri con il dato in $L_2$.

Nel Capitolo 3, stabiliamo una stima coerciva per le soluzioni del problema di Dirichlet semiperiodico per sistemi del secondo ordine degeneri nel rettangolo.

Nel Capitolo 4, dimostriamo l'esistenza, l'unicità e la regolarità nello spazio di Sobolev $W_2^2(G, \mathbb{R}^2)$ delle soluzioni di sistemi degeneri del secondo ordine singolari con i coefficienti principali variabili.

Il Capitolo 5 è dedicato a questioni di stime coercive per le soluzioni di sistemi degeneri del secondo ordine singolari.

Il Capitolo 6 è dedicato a questioni di compattezza e a proprietà di approssimazione delle soluzioni di sistemi degeneri del secondo ordine singolari. Otteniamo anche stime sia dal basso che dall'alto per la funzione di distribuzione dei numeri di approssimazione dell'operatore corrispondente. Estendiamo i risultati principali di K. Ospanov sulle proprietà di approssimazione delle soluzioni di un operatore ellittico [42] per sistemi di tipo Bitsadze con coefficienti di ordine inferiore variabili al caso di sistemi ellittici degeneri.

La risolubilità con unicità dei problemi nonlineari semiperiodici per sistemi ellittici del secondo ordine singolari è dimostrata nel Capitolo 7.

# Acknowledgements

First of all I want to express my deepest and sincere gratitude to my supervisors Professor Massimo Lanza de Cristoforis (Dipartamento di Matematica Pura ed Applicata, Università Degli Studi di Padova) and Professor Mukhtarbay Otelbayev (L.N. Gumilyov Eurasian National University, Kazakhstan) for their patience, for valuable knowledge in the field, and for their constant support during my studies. It is a great honor for me to be their student.

I also wish to thank my co-supervisor Professor Kordan Ospanov (Kazakhstan) and Professor Ryskul Oinarov (Kazakhstan) who gave me suggestions, advice and support. I would like to thank Università Degli Studi di Padova and L.N. Gumilyov Eurasian National University, for financial support and for signing the international PhD program, which made my PhD studies possible.

Finally, I thank my family for their constant love and support.

# Contents

**1 Introduction** **3**

1.1 Notation . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 9

**2 The unique solvability of the semiperiodical Dirichlet problem for second order degenerate systems** **17**

**3 On the regularity of the solution of the semiperiodical Dirichlet problem for system (2.1)** **31**

**4 The solvability of the semiperiodical problem for second order degenerate system on the strip** **47**

**5 A coercive estimate for the solutions of a singular degenerate system** **67**

**6 Compactness of the resolvent and properties of the Kolmogorov diameters of the set $M = \{w \in D(L) : \|Lw\|_{2,G} + \|w\|_{2,G} \leq 1\}$** **87**

**7 The solvability of the semiperiodical nonlinear problem for second order elliptic systems** **93**

**Bibliography** **100**

# Chapter 1

# Introduction

This work is devoted to study the questions of regularity and approximation properties of the solutions of second order degenerate singular systems. The analysis of singular differential equations had began with the work of H.Weil (1910). The Schrödinger equations and the Dirac system are the basic mathematical models of quantum mechanics. The problem of the selfadjointness of the differential operators is important in the theory of partial differential equations and it leads to the problem of existence and uniqueness of square summable generalized solutions. The effective conditions for unique solvability of the equations of quantum mechanics were obtained in the works of P. Hartman (1948), B.M. Levitan (1953), R.S. Ismagilov (1962) and others.

For the treatment of elliptic equations on bounded domains and with regular coefficients, we refer to the papers of J. L. Lions and E. Magenes [27], L. Bers, S. Bochner and F. John [4], A. V. Bitsadze [5], [7], F.E. Browder [10], L. Garding [17], V.P. Glushko and Yu.B. Savchenko [18], O. A. Ladyzhenskaya [24], O.A. Ladyzhenskaya and N.N. Uraltseva [25], S.G. Mikhlin [33], O.A. Oleĭnik and E.V. Radkevič [37] and others. M.I. Vishik [64], and O. A. Ladyzhenskaya [24], and L. Nirenberg [35], and K.O. Friedrichs [16] used the Hilbert space method to study boundary value problems for second order elliptic equations.

We consider the linear equation

$$\sum_{i,j=1}^{n} A_{i,j}(x_1,\dots,x_n)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i}^{n} B_i(x_1,\dots,x_n)\frac{\partial u}{\partial x_i} + C(x_1,\dots,x_n)u$$

$$= F(x_1, \dots, x_n), \quad (1.1)$$

with the unknown function $u(x_1, \dots, x_n)$. The functions $A_{i,j}$, $B_i$, $C$, $F$ are real and defined in some region $G$ of the Euclidean space $\mathbb{R}^n$. We assume $A_{i,j} = A_{j,i}$.

The equation (1.1) is said to be *elliptic* in a region $G \subset \mathbb{R}^n$ if

$$\left| \sum_{i,j=1}^{n} A_{i,j} t_i t_j \right| \geq \mu(x) \sum_{i=1}^{n} t_i^2,$$

where $\mu(x)$ is a positive function of the point $x \in G$ and for arbitrary real numbers $t_1, \dots, t_n$ such that $\sum\limits_{i=1}^{n} t_i^2 \neq 0$. If, in addition, there exists a constant $\mu_0$, such that $\mu(x) \geq \mu_0$ for all $x \in G$, the elliptic equation (1.1) is said to be *uniformly elliptic.*

Obviously, $\inf \mu(x) > 0$ for an uniformly elliptic equation and we may set $\mu_0 = \inf\limits_{x \in G} \mu(x)$.

If $\inf\limits_{x \in G} \mu(x) = 0$, the elliptic equation is said to be *degenerate.*

The equation (1.1) is said to be *strongly elliptic* at a given point $x \in G$, if for arbitrary real numbers $t_1, \dots, t_n$ such that $\sum\limits_{i=1}^{n} t_i^2 \neq 0$,

$$\sum_{i,j=1}^{n} A_{i,j} t_i t_j \geq \mu_0 \sum_{i=1}^{n} t_i^2.$$

Every strongly elliptic equation is elliptic.

The strongly elliptic systems and degenerate systems are well studied in the works of M.I. Vishik [64], M.I. Vishik and V.V. Grushin [65], N.E. Tovmasyan [59] -[62], L. Nirenberg [35], Ya.B. Lopatinskiy [29], F.E. Browder [10], A.P.Soldatov [58]. Among the last works devoted to study the strongly elliptic systems, we mention the works of N.E. Tovmasyan [61], [62], A.P. Soldatov [57] and K.N. Ospanov [39]. Among elliptic systems only some classes, which were considered in the works of M.I. Vishik [64] and of Ya.B. Lopatinskiy [29] have a theory which is anolog to the theory of the second order single equations. The well-known Bitsadze system [7]

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - 2\frac{\partial^2 v}{\partial x \partial y} = 0, \\ 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = 0 \end{cases} \quad (1.2)$$

is an example of elliptic system which does not satisfy the conditions of [64], [29]. The first boundary value problem with homogeneous boundary conditions for an elliptic system (1.2) is not well posed (i.e. it may have an infinite number of linearly independent solutions in an arbitrarily small circle [5]). Under the theoretical point of view, it is important to know what is the nature of the well posed problems for system of Bitsadze-type in an unbounded region. In particular, for the problem considered in this work, which is analog to the Dirichlet problem. The functional methods are used successfully to study linear boundary problems for strongly elliptic systems. Such methods do not apply in the case of degenerate systems, and in particular, in the case of unbounded domains.

One of the basic methods of research of the boundary value problems and of the Cauchy problem for partial differential equations is the integral equation method. Beside the existence and uniqueness theorems, the integral equation method gives a manner of finding the approximate solutions of the problems considered. However this method does not carry over for singular differential equations because the corresponding integrals do not converge in the domain, or the corresponding operators are not compact. At the same time, in the singular case one can sometimes establish properties of the resolvent, which allow to obtain good properties of approximation of the solutions for the problem. Such approaches are based on the general theory of the operators [20], and are connected with spectral and approximation properties of the resolvent.

One of the effective methods to study a singular differential equation with unbounded coefficients is the method of coercive estimates of the solutions, which is the analog of the famous ‘second basic inequality’ for boundary value problems [24]. There are two approaches to establish the coercive estimates of the solutions. Namely, the variational and the Tichmarsh methods. In order to apply the variational method, it is necessary to prove the well posedness of certain classes of boundary value problems for the equations we consider. The resolvent of the singular problems is constructed via resolvent of the corresponding boundary value problems. The method of coercive estimates of the solutions for different classes of singular elliptic equations was found and developed in the works of W.N. Everitt and M. Giertz [13]-[15], M. Otelbaev [48], K.Kh. Boimatov [9]. Later, it was extended to elliptic systems of equations with higher order derivatives (M. Otelbaev [44]-[47], R. Oinarov [36], K.N. Ospanov [38]-[43] and others). In such a case coercive estimates of the solutions of singular differential operators enable

a) to obtain the differential properties of the solutions;

b) to establish weighted estimates of the norm of the solutions and their derivatives;

c) to obtain the estimates of the approximation numbers of the solutions;

d) to find the effective conditions of solvability of the quasilinear generalization of the given systems.

The questions of the spectrum and of the approximation properties have the importance in the spectral theory of differential operators. A.M. Molchanov [31], Don B. Hinton [21], M. Otelbayev [48]-[51], J.V. Baxley [3], D.E. Edmunds and W.D. Evans [12], O.D. Apyshev and M. Otelbayev [2] have studied these problems for second and higher order singular elliptic operators. M. Otelbayev [44], [45] and K.N. Ospanov have studied this problem for first order multidimensional systems, and for generalized Cauchy-Riemann type systems [40], and for Beltrami-type systems.

The interest is growing up to the problem connected with studding nonlinear system given in unbounded domains. The questions of existence and uniqueness of the solutions of the boundary value problem of hydrodynamics are studied O. A. Ladyzhenskaya, V.A. Solonnikov, K.I.Piletskas and V. Kalantarov. The conditions of the coercive solvability of the singular Sturm-Liouville equation with nonlinear potential have been obtained in the work of M. Otelbaev and M.B.Muratbekov (1981). The analogous results for multidimensional equations of Schrödinger-type and higher order equations have been established in the works of M. Otelbaev, and M.B.Muratbekov, and E.Z.Grinshpun, and R. Oinarov.

The so far developed methods do not apply in the case of degenerate systems especially if defined on unbounded domains and if with variable coefficients. We consider the system

$$\begin{cases} k(y)\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - 2\frac{\partial^2 v}{\partial x \partial y} + \varphi(y)\frac{\partial u}{\partial x} + a(y)u + b(y)v = f(x,y), \\ 2\frac{\partial^2 u}{\partial x \partial y} + k(y)\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} + \psi(y)\frac{\partial v}{\partial x} + c(y)u + d(y)v = g(x,y) \end{cases} \tag{1.3}$$

which is a nonstrongly degenerate system.

System of equations of the form given by (1.3) may be classified at a point as follows. Let us consider a particular point $(x_0, y_0)$ in the domain $G \subset \mathbb{R}$ (see Notation, p.8) and construct the quadratic form

$$F = A^0_{11} dx^2 + 2A^0_{12} dxdy + A^0_{22} dy^2,$$

where

$$A^0_{11} = \begin{pmatrix} k(y_0) & 0 \\ 0 & k(y_0) \end{pmatrix}, \quad A^0_{12} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad A^0_{22} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This quadratic form can be rewritten in the following form

$$Q(\lambda) = A^0_{11}\lambda^2 + 2A^0_{12}\lambda + A^0_{22}, \tag{1.4}$$

where $\lambda = \frac{dx}{dy}$,

or

$$Q(\lambda) = \begin{pmatrix} k(y_0)\lambda^2 - 1 & -2\lambda \\ 2\lambda & k(y_0)\lambda^2 - 1 \end{pmatrix}.$$

If $A_{12} = A_{21}$, we can use the Sylvester theorem in order to determinant type of the system (1.3) in the domain $G$ without reducing the quadratic form (1.4) to its canonical form. System of equations (1.3) is of the elliptic type at the point $(x_0, y_0)$ if at this point the quadratic form (1.4) is non-singular and positive definite [6, p.17], [8, p.14], [19, p.6], [63]. That means that determinant of of the quadratic form (1.4) is positive. Namely $detQ(\lambda) > 0$ and $A_{11} > 0$ at the point $(x_0, y_0)$. So, if $k(y) > 0$ and $detQ(\lambda) = (k(y)\lambda^2 - 1)^2 + 4\lambda^2 > 0$ the system (1.3) is elliptic at the point $(x_0, y_0)$. In this case the characteristic equation $(k(y)\lambda^2 - 1)^2 + 4\lambda^2 = 0$ has four solutions. That means that elliptic system has four characteristic curves.

If $k(y) = 0$ and the quadratic form (1.4) is singular at the point $(x_0, y_0) \in G$ is called the system (1.3) of parabolic degeneracy. Thus, the determinant of the quadratic form (1.4) is equal to zero:

$$4\lambda^2 + 1 = 0.$$

This characteristic equation has two solutions. That means that corresponding parabolic system has two characteristic curves.

We note that the corresponding traditional questions such as the existence, uniqueness, regularity and approximation properties of the solutions , the conditions of solvability of the nonlinear generalization of the given systems have not been studied completely. The importance of studying such problems have been explained in the monographs [24], [5], [52].

In the applications, there is an interest in the question of extending the coercive estimate method to more general systems of partial differential equations, and in particular, to second order singular degenerate systems. Such systems are important in problems of hydrodynamics, quantum mechanics, membrane theory of shells and geometry.

## 1.1 Notation

Below we introduce some notation and terminology with the corresponding definitions.

We denote by $G$ a domain in Euclidean space $\mathbb{R}^n$ and by $X = (x_1, \ldots, x_n)$ a point in $\mathbb{R}^n$.

We denote by $G$ a subset of $\mathbb{R}^2$.

We denote by $\bar{G}$ the closure of the set $G$.

We denote by $\mathbb{C}$ the set of complex numbers.

We denote by $C(G)$ the class of continuous real valued functions in $G$.

We denote by $C^{(k)}(G)$ the class of real valued functions which are continuous in $G$ together with their derivatives up to order $k$.

We denote by $C^{\infty}(G)$ the class of infinitely differentiable functions in $G$.

We denote by $P_k$ the set of polynomials of degree up to $k$.

We denote by $C_0^{\infty}(G)$ the class of infinitely differentiable functions, which may differ from zero only a compact subset of the domain $G$.

We denote by $\nu$ a Borel measure in $G$.

A set $M$ in the metric space $\mathbb{R}$ is said to be *precompact* if every sequence of elements in $M$ contains a subsequence which converges to some $x \in \mathbb{R}$. If from every sequence of elements in $M$ it is possible to select a subsequence which converges to some $x$ belonging to $M$, then the set $M$ is said to be *compact*.

We denote by $L_p(G, \mathbb{R}^2)$ the space of measurable functions defined on $G$ such that the following norm is finite

$$\|f\|_{L_p(G,\mathbb{R}^2)} = \left[ \int_G |f(X)|^p dX \right]^{1/p}, \ 1 \leq p < \infty.$$

For brevity, we set $L_p = L_p(\mathbb{R}^2)$ and $\|f\|_p = \|f\|_{L_p(\mathbb{R}^2)}$.

We denote by $L_p^{loc}(G, \mathbb{R}^2)$ the class of locally $p$-summable functions in $G$ with respect to Lebesgue measure $dX = dxdy$ in $\mathbb{R}^2$, i.e., the class of measurable functions $f$ in $G$ which are $p$-summable $\int_K |f(X)|^p dX < \infty, \ (1 \leq p < \infty)$ for every compact set $K \subset G$.

We denote by $C^l(\bar{G})$ the subset of $C^l(G)$ of those functions $f$ such that $D^{\alpha} f$ admits a continuous extension to $\bar{G}$ for all $\alpha \in \mathbb{N}^n$, with $|\alpha| \leq l$. We introduce the following norm

$$\|f\|_{W_p^l(G)} = \left[ \int\limits_G \sum_{k=0}^{l} \sum_{(k)} \left| \frac{\partial^k f}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \right|^p dx \right]^{1/p},$$

where $p \geq 1$ and where $\sum\limits_{(k)}$ is the sum on all of possible derivatives of order $k$. We define the Sobolev space $W_p^l(G)$ to be the closure of $C^l(\bar{G})$ under norm $\|\cdot\|_{W_p^l(G)}$.

Let $f(x)$ be a complex-valued function defined in an open set $\Omega$ of $\mathbb{R}^n$. By the *support* of $f$, denoted by *supp*$f$, we mean the smallest closed set containing the set $\{x \in \Omega; f(x) \neq 0\}$. It may be equivalently defined as the smallest closed set of $\Omega$ outside which $f$ vanishes identically.

We denote by $D(A)$ and by $R(A)$ the domain of definition and range of the operator $A$, respectively.

We denote by $E$ the identity operator or matrix. For example, $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in the space $\mathbb{R}^2$.

We denote by $KerA = \{x \in D(A) : A(x) = 0\}$ the kernel of the operator $A$.

The following inequality

$$|ab| \leq \frac{\epsilon}{2}|a|^2 + \frac{1}{2\epsilon}|b|^2$$

is said to be '*the Cauchy inequality with weight* $\epsilon$'and holds for all $\epsilon > 0$ and for arbitrary $a, b$.

Let $x(t)$ and $y(t)$ be functions measurable on the set $X$. The following inequality holds:

$$\int\limits_X |x(t)y(t)|dt \leq \left[ \int\limits_X |x(t)|^p dt \right]^{\frac{1}{p}} \left[ \int\limits_X |y(t)|^q dt \right]^{\frac{1}{q}}.$$

It is said to be the Hölder's inequality, where $p$ and $q$ be real positive numbers such that $\frac{1}{p} + \frac{1}{q} = 1$.

A subset $M$ of a metric space $X$ is said to be *dense in the set* $X_0 \subset X$ if there exists, for every $x \in X_0$ and $\epsilon > 0$, a point $z \in M$ such that $\rho(x, z) < \epsilon$. $M$ is dense in $X_0$ if and only if the closure $\bar{M}$ of the set $M$ contains $X_0$, i.e. $\bar{M} \supset X_0$.

A set $X$ is called a *linear space* over a field $K$ if the following conditions are satisfies

I. A sum is defined: for every $x, y \in X$ there is an element of $X$, denoted by $x + y$, such that

1)$(x + y) + z = x + (y + z) \ \ (x, y, z \in X)$;

2)$x + y = y + x \ \ (x, y \in X)$;

3) an element 0 exists in $X$ such that $0 + x = x$ for any $x \in X$;

II. A scalar multiplication is defined: for every $x, y \in X$ and each $\alpha \in K$ there is an element of $X$, denoted by $\alpha x$, such that

4) $\alpha(x + y) = \alpha x + \alpha y \ \ (x, y \in X, \ \alpha \in K)$;

5) $(\alpha + \beta)x = \alpha x + \beta x \ \ (x \in X, \ \alpha, \beta \in K)$;

6) $(\alpha\beta)x = \alpha x(\beta x) \ \ (x \in X, \ \alpha, \beta \in K)$;

7) $1 \cdot x = x$ (1 is the unit element of the field $K$).

A linear space will be said *real* or *complex* according as the field $K$ is the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$.

A linear space $X$ is called a *Banach* space if it is complete, i.e., if every Cauchy sequence $\{x_n\}$ of $X$ converges strongly to a point $x$ of $X$:

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

Let $A$ be an operator acting from $\Omega \subset X$ to $Y$, where $X$, $Y$ are metric spaces. The operator $A$ is said to be *continuous* at the point $x_0 \in \Omega$ if $A(x_n) \to A(x_0)$ as $x_n \to x_0$ $(x_n \in \Omega)$. If the operator $A$ is continuous at every point of a set $E \subset \Omega$, we simply say that $A$ is continuous on $E$.

If $X, Y$ are normed spaces and $\Omega$ is a linear set contained in $X$. An operator $A$ acting from $\Omega \subset X$ to $Y$ is said to be *homogeneous* if

$$A(\lambda x) = \lambda A(x) \ \ (\forall\ \lambda \in \mathbb{R}, \ x \in \Omega).$$

The operator $A$ is described as *additive* if

$$A(x_1 + x_2) = A(x_1) + A(x_2) \ \ (\forall\ x_1, x_2 \in \Omega).$$

The operator $A$ is called *linear* if it is additive and homogeneous on $\Omega$. An operator $A$ whose range is a set of numbers is called a *functional*.

Let $A_1$ and $A_2$ be linear operators with domain of definition $D(A_1)$ and $D(A_2)$ both contained in a linear space $X$, and with ranges $R(A_1)$ and $R(A_2)$ both contained in a linear space $Y$, respectively. If $D(A_1) \subseteq D(A_2)$ and $A_1x = A_2x$ for all $x \in D(A_1)$, then $A_2$ is called an *extension* of $A_1$, and $A_1$ a *restriction* of $A_2$.

The set of all bounded linear functionals acting from a normed space $X$ into a (generally speaking complex) Banach space $Y$ is called the *conjugate* space of $X$ and denoted by $X^*$.

Let $A$ be a linear operator acting from $X$ to $Y$, where $X$,$Y$ are Banach spaces. Let the operator $A$ have a domain $D(A)$ which is dense in $X$. Let $f$ be a bounded linear functional on $Y$. We consider the functional $f(Ax)$ defined for all $x \in D(A)$. Since $f$ is bounded on $D(A)$ and $D(A)$ is dense in $X$ by assumption, the functional $f$ can be extended uniquely to a bounded linear functional $g$ on $X$ by the Hahn-Banach theorem. In this case we can say that the *adjoint operator* $A^*$ is defined on the functional $f$, and denote by $g = A^*f$. The following formula defines the functional $A^*f$

$$(A^*f)(x) = f(Ax)$$

for all $x \in D(A)$.

An operator $A$ that coincides with its adjoint is said to be *self-adjoint*. A self-adjoint operator is characterized by the equation

$$(Ax, y) = (x, Ay),$$

for $x, y \in D(A)$.

Let $A$ be a linear operator acting from $X$ to $Y$, where $X$, $Y$ are Banach spaces. The linear operator $A$ is said to be *completely continuous* if it is defined on the whole of the space $X$ and maps every bounded subset of $X$ into precompact subsets of $Y$.

Consider the equations

$$Ax = y, \tag{1.5}$$

where $x \in D(A)$, $y \in R(A)$ and

$$A^*g = f, \tag{1.6}$$

where $g \in D(A^*) \subset Y^*,\ f \in R(A^*) \subset X^*$.

If a linear operator $A$ gives a one-to-one map of $D(A)$ onto $R(A)$, the inverse map $A^{-1}$ gives a linear operator on $R(A)$ onto $D(A)$:

$$A^{-1}Ax = x \text{ for } x \in D(A)$$

and

$$AA^{-1}y = y \text{ for } y \in R(A).$$

$A^{-1}$ is said to be the *inverse* operator of $A$.

The operator $A_l^{-1}$ is called the *left-hand inverse* of the operator $A$ if

$$A_l^{-1}A = E.$$

Similarly, the operator $A_r^{-1}$ is called the *right-hand inverse* of the operator $A$ if

$$AA_r^{-1} = E.$$

If the left-hand inverse operator $A_l^{-1}$ exists, the solution of the equation (1.5) is unique, if it exists. Similarly, the existence of the right-hand inverse operator can be shown to involve the (generally not unique) solvability of the equation (1.5) for any $y \in R(A)$.

Equation (1.5) is *uniquely solvable on* $R(A)$ provided that the homogeneous equation $Ax = 0$ has only the null solution, i.e., if $KerA^* = 0$.

Equation (1.5) is said to be *well posed on* $R(A)$ if there exists $k(\lambda) > 0$ such that the inequality $\|x\|_X \leq k\,\|Ax\|_Y$ holds for all $x \in D(A)$. Well posedness implies unique solvability.

If equation (1.5) is well posed, then the operator $A$ has a bounded inverse on $R(A)$.

Equation (1.5) is said to be *densely solvable* if $R(A)$ is dense in $Y$: $\overline{R(A)} = Y$.

**Theorem 1.1.** *[23] Equation (1.5) is densely solvable if and only if equation (1.6) is uniquely solvable* ($KerA^* = 0$).

A linear operator $A$ is called *closed* if whenever $\{x_n\}$ is a sequence in $D(A)$ such that $x_n \to x$ and $Ax_n \to y$, we have $x \in D(A)$ and $Ax = y$. A nonclosed operator is said to be *closable* (or to admit a closure) if it can be extended to a *closed operator*. A linear operator $A$ is closable if and only if given a sequence $x_n \to 0$ with $x_n \in D(A)$ and $Ax_n \to y$ we always have $y = 0$. The least closed extension of the operator $A$ is said to be the *closure* of $A$. The closure of the operator $A$ is denoted by $\bar{A}$. If an operator $A$ admits a closure then $x_n \to x$, $x_n \in D(A)$ and $Ax_n \to y$ imply that $x \in D(\bar{A})$ and $\bar{A}x = y$. Also, if both $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} Ax_n$ exist, we can write $\bar{A} \lim_{n\to\infty} x_n = \lim_{n\to\infty} Ax_n$.

It is not difficult to show the truth of the following assertion: if the operator $A$ is closed, then each operator $A - \lambda E$ is closed, and if the inverse operator $A^{-1}$ exists then it is closed.

Each bounded linear operator, defined on the whole space is closed.

Next we assume that $H$ is a Hilbert space.

Let $A$ be a linear operator whose domain $D_A$ and range $R(A)$ both lie in the linear space $X$. We consider the operator equation

$$Ax - \lambda x = y, \tag{1.7}$$

where $\lambda$ is a complex number.

We denote by $\Delta_A(\lambda)$ the range of the operator $A - \lambda E$. The operator $A - \lambda E = A_\lambda$ defines a(not necessarily one-to-one) correspondence between $D_A$ and $\Delta_A(\lambda)$. If this correspondence is one-to-one, then the operator $A - \lambda E$ has an inverse operator $(A - \lambda E)^{-1}$ with domain $\Delta_A(\lambda)$ and range $D_A$.

If $\lambda$ is such that the range $\Delta_A(\lambda)$ is dense in $X$ and $A_\lambda$ has a continuous inverse $(A - \lambda E)^{-1}$, we say that $\lambda$ is in the *resolvent set* $\rho(A)$ of $A$. We denote the inverse $(A - \lambda E)^{-1}$ by $R_\lambda(A)$ and call it the *resolvent* of $A$. All complex numbers $\lambda$ not in $\rho(A)$ form a set $\sigma(A)$ called the *spectrum* of $A$. The resolvent set is open.

Let $X$ be a complex Banach space and $A$ a closed linear operator with domain $D_A$ and range $R(A)$ both in $X$. Then the resolvent $(A - \lambda E)^{-1}$ is an everywhere defined continuous operator for any $\lambda \in \rho(A)$.

If a (real or complex) number $\lambda$ is in the resolvent set $\rho(A)$ of the operator $A$, then there exists a constant $k = k(\lambda) > 0$ such that

$$\|(A - \lambda E)f\| \geq k \|f\|,$$

for all $f \in D(A)$.

If $A$ is a symmetric operator and $z = x + iy$ ($y \neq 0$), then

$$\|(A - zE)f\|^2 = \|(A - xE)f\|^2 + y^2\|f\|^2 \geq y^2\|f\|^2$$

for all $f \in D(A)$. Hence, the upper and lower $z$-half-planes are connected subsets of the resolvent set of an arbitrary symmetric operator.

**Theorem 1.2.** *[1, p. 92] If $\Gamma$ is a connected subset of the resolvent set of a linear operator $A$, then the dimension of the subspace $H \ominus \Delta_A(\lambda)$ is the same for each $\lambda \in \Gamma$.*

[We note that in [1, p. 92] a resolvent set is called 'field of regularity'.]

Equation (1.7) above can be rewritten in the form

$$(A - \lambda E)x = y.$$

For example, if $H = \mathbb{R}^2$, then $\lambda E = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ for $x \in D(A) \subset \mathbb{R}^2$.

**Lemma 1.3.** *[30] Let $M$ be a subspace of a Hilbert space $H$. Then $M$ is dense in $H$ if and only if the null element of $H$ is the only element of $H$ which is orthogonal to $M$.*

**Proposition 1.4.** *[23] The kernel of the adjoint operator is the orthogonal complement of the range of the initial operator.*

**Theorem 1.5.** *Let $X$ be a Banach space. Let $E$ be the identity operator in $X$, and $A$ be a bounded linear operator, of $X$ to itself such that $\|A\| \leq q < 1$. Then the operator $(E - A)^{-1}$ exists, is bounded and*

$$\left\|(E - A)^{-1}\right\| \leq \frac{1}{1 - q}.$$

**Theorem 1.6.** *[34] Let $1 \leq q < \infty$. Let $K \subset L_p(\mathbb{R}^n)$. Then $K$ is precompact in $L_p(\mathbb{R}^n)$ if and only if all the following three conditions are satisfied*

$$1) \sup_{f \in K} \|f\|_{L_p(\mathbb{R}^n)} < \infty$$

*(boundness of $K$);*

$$2) \sup_{f \in K} \sup_{|h| \leq \delta} \|f(\cdot + h) - f(\cdot)\|_{L_p(\mathbb{R}^n)} \to 0, \ \delta \to 0$$

*(uniform continuity by displacement of the translations);*

$$3)\ \lim_{N\to\infty} \sup_{f\in K} \|f(x)\|_{L_p(\mathbb{R}^n/B(0,N))} = 0$$

*(uniform decay at infinity).*

**Theorem 1.7.** *[22, p.645] A continuous operator $A$ mapping a closed convex set $\Omega$ in a Banach space $X$ into a compact set $\Delta \subset \Omega$ has a fixed point.*

# Chapter 2

# The unique solvability of the semiperiodical Dirichlet problem for second order degenerate systems

We consider the following semi-periodical problem

$$\begin{cases} k(y)\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - 2\frac{\partial^2 v}{\partial x \partial y} + \varphi(y)\frac{\partial u}{\partial x} + a(y)u + b(y)v = f(x,y), \\ 2\frac{\partial^2 u}{\partial x \partial y} + k(y)\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} + \psi(y)\frac{\partial v}{\partial x} + c(y)u + d(y)v = g(x,y), \end{cases} \tag{2.1}$$

$$w(-\pi, y) = w(\pi, y), w_x(-\pi, y) = w_x(\pi, y), w(x, \alpha) = w(x, \beta) = 0, \tag{2.2}$$

in the rectangle $G_0 = \{(x,y) \in \mathbb{R}^2 : \; -\pi < x < \pi, \; \alpha < y < \beta\}$, where $w(x,y) \equiv u(x,y), v(x,y)$. Here $k(y)$ is a continuous and bounded real valued function such that $\inf\limits_{y \in [\alpha,\beta]} k(y) \geq 0$, $f, g \in L_2(G_0)$. Let the functions $\varphi, \psi, a, b, c, d$ be continuous from $[\alpha, \beta]$ to $\mathbb{R}$.

Now we introduce the following notation

$$B_{xy} = \begin{pmatrix} k(y)\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} & -2\frac{\partial^2}{\partial x \partial y} \\ 2\frac{\partial^2}{\partial x \partial y} & k(y)\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \end{pmatrix},$$

$$P(y) = \begin{pmatrix} \varphi(y) & 0 \\ 0 & \psi(y) \end{pmatrix},$$

$$Q(y) = \begin{pmatrix} a(y) & b(y) \\ c(y) & d(y) \end{pmatrix},$$

$$w=(u,v)\,,\;F=(f,g)\,,\;X=(x,y).$$

System (2.1) can be written in the following form

$$L_0w=B_{xy}w+P(y)w_x+Q(y)w=F(X). \tag{2.3}$$

**Assumption 1.** We assume that the real valued functions $\varphi,\psi,a,b,c,d$ of $[\alpha,\beta]$ to $\mathbb{R}$ satisfy the following conditions

$$\inf_{y\in[\alpha,\beta]}\{-\varphi(y),a(y),d(y)\}=\delta>0; \tag{2.4}$$

$$\frac{1}{2}\left(|b(y)|+|c(y)|\right)^{2r}\leq\frac{a(y)}{3}, \tag{2.5}$$
$$\frac{1}{2}\left(|b(y)|+|c(y)|\right)^{2q}\leq\frac{d(y)}{3},$$
$$\vartheta\psi(y)>d(y),$$

where $r,q$ and $\vartheta$ are constants such that $r>0,\ q>0,\ r+q=1,\ 0<\vartheta<3.$

We denote by $C^2_{\pi,0}(G_0,\mathbb{R}^2)$ the set of twice continuously differentiable functions in $\tilde{G}_0=\{(x,y)\in\mathbb{R}^2:\ -\pi\leq x\leq\pi,\ \alpha<y<\beta\}$ with values in $\mathbb{R}^2$ which are periodic in the variable $x$ with period $2\pi$ and which have compact support in $(\alpha,\beta)$ in the variable $y$ for each fixed value of $x$ in $[-\pi,\pi]$. We denote by $L$ the closure of the operator $L_0$ defined in the domain $D(L_0)=C^2_{\pi,0}(G_0,\mathbb{R}^2)$ in the space $L_2(G_0,\mathbb{R}^2)$.

**Definition 2.1.** *A function $w=(u,v)\in L_2(G_0,\mathbb{R}^2)$ is said to be a solution of the problem (2.1),(2.2), if there exists a sequence $\{w_n\}_{n=1}^{\infty}$ in $C^2_{\pi,0}(G_0,\mathbb{R}^2)$ such that $\|w_n-w\|_{L_2(G_0,\mathbb{R}^2)}\to 0$ and $\|Lw_n-F\|_{L_2(G_0,\mathbb{R}^2)}\to 0$ as $n\to\infty$.*

**Lemma 2.2.** *Let Assumption 1 hold. Then there exists a constant $C_0$ such that*

$$\|w\|^2_{W^1_2(G_0,\mathbb{R}^2)}=\|w_x\|^2_{2,G_0}+\|w_y\|^2_{2,G_0}+\|w\|^2_{2,G_0}\leq C_0\,\|Lw\|^2_{2,G_0}\,, \tag{2.6}$$

*for all functions $w=(u,v)\in D(L)$.*

**Proof.** Let $w=(u,v)\in C^2_{\pi,0}(G_0,\mathbb{R}^2)$. Integrating by parts and using the boundary conditions for the function $w$, we have

$$((L_0+\lambda E)w,w)=$$

$$\int\limits_{G_0} (k(y)u_{xx} - u_{yy} - 2v_{xy} + \varphi(y)u_x +$$
$$a(y)u + \lambda u + b(y)v)\, \bar{u} dx dy +$$
$$\int\limits_{G_0} (2u_{xy} + k(y)v_{xx} - v_{yy} + \psi(y)v_x +$$
$$c(y)u + d(y)v + \lambda v)\, \bar{v} dx dy =$$
$$\int\limits_{\alpha}^{\beta} k(y) \left( \int\limits_{-\pi}^{\pi} u_{xx}\bar{u} dx \right) dy - \int\limits_{-\pi}^{\pi} \left( \int\limits_{\alpha}^{\beta} u_{yy}\bar{u} dy \right) dx -$$
$$2\int\limits_{G_0} v_{xy}\bar{u} dx dy + \int\limits_{\alpha}^{\beta} \varphi(y) \left( \int\limits_{-\pi}^{\pi} u_x\bar{u} dx \right) dy +$$
$$\int\limits_{G_0} a(y)|u|^2 dx dy + \int\limits_{G_0} b(y)|u||v| dx dy + 2\int\limits_{G_0} u_{xy}\bar{v} dx dy +$$
$$\int\limits_{\alpha}^{\beta} k(y) \left( \int\limits_{-\pi}^{\pi} v_{xx}\bar{v} dx \right) dy - \int\limits_{-\pi}^{\pi} \left( \int\limits_{\alpha}^{\beta} v_{yy}\bar{v} dy \right) dx +$$
$$\int\limits_{\alpha}^{\beta} \psi(y) \left( \int\limits_{-\pi}^{\pi} v_x\bar{v} dx \right) dy + \int\limits_{G_0} c(y)|u||v| dx dy +$$
$$\int\limits_{G_0} d(y)|v|^2 dx dy + \int\limits_{G_0} \lambda(|u|^2 + |v|^2) dx dy =$$
$$\int\limits_{\alpha}^{\beta} k(y) \left( u_x\bar{u}|_{-\pi}^{\pi} - \int\limits_{-\pi}^{\pi} |u_x|^2 dx \right) dy -$$
$$\int\limits_{-\pi}^{\pi} \left( u_y\bar{u}|_{\alpha}^{\beta} - \int\limits_{\alpha}^{\beta} |u_y|^2 dy \right) dx -$$
$$2\int\limits_{-\pi}^{\pi} \left( v_x\bar{u}|_{\alpha}^{\beta} - \int\limits_{\alpha}^{\beta} v_x u_y dy \right) dx +$$
$$\int\limits_{\alpha}^{\beta} \varphi(y) \left( \int\limits_{-\pi}^{\pi} u_x\bar{u} dx \right) dy +$$
$$\int\limits_{G_0} a(y)|u|^2 dx dy + \int\limits_{G_0} b(y)|u||v| dx dy +$$
$$2\int\limits_{-\pi}^{\pi} \left( u_x\bar{v}|_{\alpha}^{\beta} - \int\limits_{\alpha}^{\beta} u_x v_y dy \right) dx -$$

$$\int_{\alpha}^{\beta} k(y)\left(v_x\bar{v}|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}|v_x|^2dx\right)dy -$$
$$\int_{-\pi}^{\pi}\left(v_y\bar{v}|_{\alpha}^{\beta} - \int_{\alpha}^{\beta}|v_y|^2dy\right)dx + \int_{\alpha}^{\beta}\psi(y)\left(\int_{-\pi}^{\pi}v_x\bar{v}dx\right)dy +$$
$$\int_{G_0} c(y)|u||v|dxdy + \int_{G_0} d(y)|v|^2dxdy +$$
$$\int_{G_0}\lambda(|u|^2+|v|^2)dxdy = -\int_{G_0} k(y)|u_x|^2dxdy + \int_{G_0}|u_y|^2dxdy +$$
$$2\int_{G_0} u_yv_xdxdy + \int_{\alpha}^{\beta}\varphi(y)\left(\frac{u^2}{2}\Big|_{-\pi}^{\pi}\right)dy + \int_{G_0} a(y)u^2dxdy +$$
$$\int_{G_0}\lambda u^2dxdy + \int_{G_0} b(y)uvdxdy - 2\int_{G_0} u_yv_xdxdy -$$
$$\int_{G_0} k(y)|v_x|^2dxdy + \int_{G_0}|v_y|^2dxdy + \int_{\alpha}^{\beta}\psi(y)\left(\frac{v^2}{2}\Big|_{-\pi}^{\pi}\right)dy +$$
$$\int_{G_0} c(y)uvdxdy + \int_{G_0} d(y)v^2dxdy + \int_{G_0}\lambda v^2dxdy =$$
$$-\int_{G_0} k(y)|w_x|^2dxdy + \int_{G_0}|w_y|^2dxdy +$$
$$\int_{G_0} a(y)|u|^2dxdy + \int_{G_0} d(y)v^2dxdy +$$
$$\int_{G_0}\lambda(u^2+v^2)dxdy + \int_{G_0}(b(y)+c(y))uvdxdy.$$

$$((L_0+\lambda E)w,w) = -\int_{G_0} k(y)|w_x|^2dxdy + \tag{2.7}$$
$$\int_{G_0}|w_y|^2dxdy + \int_{G_0}(a(y)u^2+d(y)v^2)dxdy +$$
$$\int_{G_0}\lambda|w|^2dxdy + \int_{G_0}(b(y)+c(y))uvdxdy.$$

By applying the Hölder and Cauchy-Bunyakovsky inequalities to the last term of (2.7), we obtain

$$\left|\int_{G_0}(b(y)+c(y))uvdxdy\right| \tag{2.8}$$

$$\leq \left(\int\limits_{G_0} (|b(y)|+|c(y)|)^{2r}u^2dxdy\right)^{\frac{1}{2}} \cdot \left(\int\limits_{G_0} (|b(y)|+|c(y)|)^{2q}v^2dxdy\right)^{\frac{1}{2}}$$
$$\leq \frac{1}{2}\int\limits_{G_0} (|b(y)|+|c(y)|)^{2r}u^2dxdy + \frac{1}{2}\int\limits_{G_0} (|b(y)|+|c(y)|)^{2q}v^2dxdy.$$

By (2.8), we can transform equality (2.7) in the following way

$$((L_0+\lambda E)w,w) \geq -\max_{y\in[\alpha,\beta]}|k(y)|\int\limits_{G_0}|w_x|^2dxdy+ \tag{2.9}$$
$$\int\limits_{G_0}|w_y|^2dxdy+\int\limits_{G_0}\lambda|w|^2dxdy+\int\limits_{G_0}(a(y)u^2+d(y)v^2)dxdy-$$
$$\frac{1}{2}\int\limits_{G_0}(|b(y)|+|c(y)|)^{2r}u^2dxdy-\frac{1}{2}\int\limits_{G_0}(|b(y)|+|c(y)|)^{2q}v^2dxdy.$$

We now apply to the left hand side of the last inequality 'the Cauchy inequality with weight $\epsilon$' for some $\epsilon=\gamma_0>0$. Then by condition (2.5), we obtain

$$\frac{1}{2\gamma_0}\left\|(L_0+\lambda E)w\right\|^2_{2,G_0}+\frac{\gamma_0}{2}\left\|w\right\|^2_{2,G_0} \tag{2.10}$$
$$\geq -\max_{y\in[\alpha,\beta]}|k(y)|\int\limits_{G_0}|w_x|^2dxdy+\int\limits_{G_0}|w_y|^2dxdy+$$
$$\int\limits_{G_0}\lambda|w|^2dxdy+\int\limits_{G_0}\frac{2a(y)}{3}u^2dxdy+\int\limits_{G_0}\frac{2d(y)}{3}v^2dxdy.$$

Below, we consider the functional $((L_0+\lambda E)w,\tilde{w}_x)$, where $\tilde{w}=(-u,v)$,

$$((L_0+\lambda E)w,\tilde{w}_x)=$$
$$\int\limits_{G_0}(k(y)u_{xx}-u_{yy}-2v_{xy}+\varphi(y)u_x+$$
$$a(y)u+\lambda u+b(y)v)\,(-\bar{u}_x)dxdy+$$
$$\int\limits_{G_0}(2u_{xy}+k(y)v_{xx}-v_{yy}+\psi(y)v_x+$$
$$c(y)u+d(y)v+\lambda v)\,\bar{v}_xdxdy=$$
$$-\int\limits_{\alpha}^{\beta}k(y)\left(\int\limits_{-\pi}^{\pi}u_{xx}\bar{u}_xdx\right)dy+\int\limits_{-\pi}^{\pi}\left(\int\limits_{\alpha}^{\beta}u_{yy}\bar{u}_xdy\right)dx+$$
$$2\int\limits_{G_0}v_{xy}\bar{u}_xdxdy-\int\limits_{\alpha}^{\beta}\varphi(y)\left(\int\limits_{-\pi}^{\pi}|u_x|^2dx\right)dy-$$
$$\int\limits_{G_0}a(y)u\bar{u}_xdxdy-\int\limits_{G_0}b(y)v\bar{u}_xdxdy+$$

$$2\int\limits_{G_0} u_{xy}\bar{v}_x dxdy + \int\limits_{\alpha}^{\beta} k(y)\left(\int\limits_{-\pi}^{\pi} v_{xx}\bar{v}_x dx\right) dy -$$

$$\int\limits_{-\pi}^{\pi}\left(\int\limits_{\alpha}^{\beta} v_{yy}\bar{v}_x dy\right) dx + \int\limits_{\alpha}^{\beta}\psi(y)\left(\int\limits_{-\pi}^{\pi} v_x\bar{v}_x dx\right) dy +$$

$$\int\limits_{G_0} c(y)u\bar{v}_x dxdy + \int\limits_{G_0} d(y)v\bar{v}_x dxdy +$$

$$\int\limits_{G_0} (-\lambda u\bar{u}_x + \lambda v\bar{v}_x) dxdy =$$

$$-\int\limits_{\alpha}^{\beta} k(y)\left(\left.\frac{u_x^2}{2}\right|_{-\pi}^{\pi}\right) dy + \int\limits_{G_0} u_{yy}u_x dxdy +$$

$$2\int\limits_{G_0} v_{xy}u_x dxdy - \int\limits_{G_0}\varphi(y)u_x^2 dxdy - \int\limits_{G_0} a(y)uu_x dxdy -$$

$$\int\limits_{G_0}\lambda uu_x dxdy - \int\limits_{G_0} b(y)vu_x dxdy + 2\int\limits_{G_0} u_{xy}v_x dxdy +$$

$$\int\limits_{\alpha}^{\beta} k(y)\left(\left.\frac{v_x^2}{2}\right|_{-\pi}^{\pi}\right) dy - \int\limits_{G_0} v_{yy}v_x dxdy + \int\limits_{G_0}\psi(y)v_x^2 dxdy +$$

$$\int\limits_{G_0} c(y)uv_x dxdy + \int\limits_{G_0} d(y)vv_x dxdy + \int\limits_{G_0}\lambda vv_x dxdy =$$

$$-\int\limits_{G_0} u_y\left(u_y\right)_x dxdy - 2\int\limits_{G_0} v_x u_{xy} dxdy - \int\limits_{G_0}\varphi(y)u_x^2 dxdy -$$

$$\int\limits_{G_0} b(y)vu_x dxdy + 2\int\limits_{G_0} v_x u_{xy} dxdy + \int\limits_{G_0} v_y\left(v_x\right)_y dxdy +$$

$$\int\limits_{G_0}\psi(y)v_x^2 dxdy + \int\limits_{G_0} c(y)uv_x dxdy.$$

$$((L_0+\lambda E)w,\tilde{w}_x) = -\int\limits_{G_0}\varphi(y)u_x^2 dxdy +$$

$$\int\limits_{G_0}\psi(y)v_x^2 dxdy + \int\limits_{G_0}(b(y)+c(y))uv_x dxdy.$$

Hence, inequality (2.8) implies that

$$((L_0+\lambda E)w,\tilde{w}_x) \geq -\int\limits_{G_0}\varphi(y)u_x^2 dxdy + \int\limits_{G_0}\psi(y)v_x^2 dxdy -$$

$$\frac{1}{2}\int\limits_{G_0}(|b(y)|+|c(y)|)^{2r}u^2 dxdy - \frac{1}{2}\int\limits_{G_0}(|b(y)|+|c(y)|)^{2q}v_x^2 dxdy.$$

By condition (2.5) and by applying 'the Cauchy inequality with weight $\epsilon > 0$' to the left hand side of the last equation, we obtain

$$\frac{1}{2\epsilon}\left\|(L_0+\lambda E)\,w\right\|^2_{2,G_0}+\frac{\epsilon}{2}\left\|w_x\right\|^2_{2,G_0}\geq \tag{2.11}$$
$$-\int\limits_{G_0}\varphi(y)u_x^2dxdy+\int\limits_{G_0}\psi(y)v_x^2dxdy-$$
$$\int\limits_{G_0}\frac{a(y)}{3}u^2dxdy-\vartheta\int\limits_{G_0}\frac{\psi(y)}{3}v_x^2dxdy.$$

By combining inequalities (2.10), (2.11) and by condition (2.4), we obtain

$$\left(\frac{1}{2\gamma_0}+\frac{1}{2\epsilon}\right)\|(L_0+\lambda E)w\|^2_{2,G_0}\geq$$
$$\left(\delta\left(1-\frac{\vartheta}{3}\right)-\frac{\epsilon}{2}-\max_{y\in[\alpha,\beta]}|k(y)|\right)\|w_x\|^2_{2,G_0}+$$
$$\|w_y\|^2_{2,G_0}+\left[\frac{\delta}{3}+\lambda-\frac{\gamma_0}{2}\right]\|w\|^2_{2,G_0}.$$

Thus

$$\tilde{C}\,\|(L_0+\lambda E)w\|^2_{2,G_0}\geq\mu\|w_x\|^2_{2,G_0}+\|w_y\|^2_{2,G_0}+\gamma\|w\|^2_{2,G_0},$$

where $\tilde{C}=\frac{1}{2\gamma_0}+\frac{1}{2\epsilon}$, $\mu=\delta\left(1-\frac{\vartheta}{3}\right)-\frac{\epsilon}{2}-\max\limits_{y\in[\alpha,\beta]}|k(y)|$, $\gamma=\frac{\delta}{3}+\lambda-\frac{\gamma_0}{2}$. Hence, inequality (2.6) follows and the proof is complete. □

**Remark 2.3.** *Lemma 2.2 holds, if condition (2.5) is replaced by the following inequalities*

$$\frac{1}{2}(|b(y)|+|c(y)|)^{2r}\leq\frac{d(y)}{3},$$
$$\frac{1}{2}(|b(y)|+|c(y)|)^{2q}\leq\frac{a(y)}{3},$$
$$-\vartheta\varphi(y)>a(y),$$

*where $r,q$ and $\vartheta$ are constants such that $r>0,\ q>0,\ r+q=1,\ 0<\vartheta<3$.*

**Remark 2.4.** *If $b(y)=-c(y)$, then one can prove Lemma 2.2 with condition (2.5) replaced by the following*

$$\inf_{y\in[\alpha,\beta]}\psi(y)=\delta>0.$$

We now write the functions $f$ and $g$ in the right hand side of (2.1) in the following form

$$f = \sum_{n=-\infty}^{\infty} f_n(y)e^{inx},\ g = \sum_{n=-\infty}^{\infty} g_n(y)e^{inx}. \tag{2.12}$$

We will search for a solution $w = (u, v)$ of the problem (2.1), (2.2) as a limit in the norm $L_2(G_0, \mathbb{R}^2)$ of the sequence $\{(\tilde{u}_N, \tilde{v}_N)\}_{N=-\infty}^{\infty}$, where

$$\tilde{u}_N = \sum_{n=-N}^{N} u_n(y)e^{inx},\ \tilde{v}_N = \sum_{n=-N}^{N} v_n(y)e^{inx}. \tag{2.13}$$

By replacing $u, v, f, g$ by the corresponding expression of (2.12) and (2.13), we obtain that

$$\begin{cases} -\sum\limits_{n=-N}^{N} u_n^{''}e^{inx} - 2in\sum\limits_{n=-N}^{N} v_n^{'}e^{inx} - n^2k(y)\sum\limits_{n=-N}^{N} u_n(y)e^{inx}+ \\ \qquad in\varphi(y)\sum\limits_{n=-N}^{N} u_n(y)e^{inx} + a(y)\sum\limits_{n=-N}^{N} u_n(y)e^{inx}+ \\ \qquad b(y)\sum\limits_{n=-N}^{N} v_n(y)e^{inx} = \sum\limits_{n=-N}^{N} f_n(y)e^{inx}, \\ -\sum\limits_{n=-N}^{N} v_n^{''}(y)e^{inx} + 2in\sum\limits_{n=-N}^{N} u_n^{'}(y)e^{inx} + c(y)\sum\limits_{n=-N}^{N} u_n(y)e^{inx}- \\ \qquad n^2k(y)\sum\limits_{n=-N}^{N} u_n(y)e^{inx} + in\psi(y)\sum\limits_{n=-N}^{N} v_n(y)e^{inx}+ \\ \qquad d(y)\sum_{n=-N}^{N} v_n(y)e^{inx} = \sum\limits_{n=-N}^{N} g_n(y)e^{inx}, \end{cases}$$

$$w_n(\alpha) = 0,\ w_n(\beta) = 0,$$

and by equating the coefficients of $e^{inx}$, we obtain the following problem for $w_n = (u_n(y), v_n(y))$ $(n = 0, \pm1, \pm2, \ldots)$

$$\begin{cases} -u_n^{''} - 2inv_n^{'} + (-n^2k(y) + in\varphi(y) + a(y))u_n + b(y)v_n = f_n(y), \\ -v_n^{''} + 2inu_n^{'} + c(y)u_n + (-n^2k(y) + in\psi(y) + d(y))v_n = g_n(y), \end{cases} \tag{2.14}$$

$$w_n(\alpha) = 0,\ w_n(\beta) = 0, \tag{2.15}$$

where $f_n, g_n \in L_2(\alpha, \beta)$.

Next we set

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$Q(y) = \begin{pmatrix} -n^2k(y) + in\varphi(y) + a(y) & b(y) \\ c(y) & -n^2k(y) + in\psi(y) + d(y) \end{pmatrix}.$$

Then we consider the differentiation operator $l_n + \lambda E$ defined by

$$(l_n + \lambda E)w = -w^{''} + 2inTw^{'} + Q_n(y)w + \lambda w,$$

for all functions $w(y)$ in the space $C_0^2((\alpha,\beta),\mathbb{C}^2)$ of twice continuously differentiable of $[\alpha,\beta]$ to $\mathbb{C}^2$ which satisfy the boundary conditions (2.15). We denote also by $l_n + \lambda E$ the closure of $l_n + \lambda E$ in the norm of $L_2 \equiv L_2((\alpha,\beta),\mathbb{C}^2)$.

**Lemma 2.5.** *Let $\lambda \geq 0$. Let Assumption 1 hold. Then there exists a constant $C_0$ such that*

$$\|(l_n + \lambda E)w\|_2^2 \geq C_0 \left[ \int_\alpha^\beta \left|w_n^{'}\right|^2 dy + \int_\alpha^\beta \left(\frac{\delta}{3} + \lambda + n^2\right) |w_n|^2 \, dy \right], \qquad (2.16)$$

*for all $w_n = (u_n(y), v_n(y)) \in D(l_n + \lambda E)$, where we have denoted by $\|\cdot\|_2$ the norm of $L_2 \equiv L_2((\alpha,\beta),\mathbb{C}^2)$.*

**Proof.** Let $w = (u,v) \in C_0^2((\alpha,\beta),\mathbb{C}^2)$. By the conditions (2.4), (2.5) and (2.15), we obtain

$$\begin{aligned}
&\mathrm{Re}\,((l_n + \lambda E)w_n, w_n) = \\
&\qquad \mathrm{Re}\left[\int_\alpha^\beta \left\{-u_n^{''} - 2inv_n^{'} + \right.\right. \\
&\qquad (-n^2k(y) + in\varphi(y) + a(y) + \lambda)u_n + b(y)v_n\big\}\, \bar{u}_n dy + \\
&\qquad \int_\alpha^\beta \left\{-v_n^{''} + 2inu_n^{'} + c(y)u_n + \right. \\
&\qquad \left.\left.(-n^2k(y) + in\psi(y) + d(y) + \lambda)v_n\right\}\bar{v}_n dy\right].
\end{aligned}$$

Further

$$\begin{aligned}
&\mathrm{Re}\,((l_n + \lambda E)w_n, w_n) \geq \\
&\qquad \int_\alpha^\beta \left|u_n^{'}\right|^2 dy + \int_\alpha^\beta \left(\lambda - n^2 \max_{y\in[\alpha,\beta]} |k(y)|\right) |u_n|^2 \, dy + \\
&\qquad \int_\alpha^\beta \left|v_n^{'}\right|^2 dy + \int_\alpha^\beta \left(\lambda - n^2 \max_{y\in[\alpha,\beta]} |k(y)|\right) |v_n|^2 \, dy + \\
&\qquad \int_\alpha^\beta \left(a(y)u_n^2 + (b(y)+c(y))u_nv_n + d(y)v_n^2\right) dy.
\end{aligned}$$

Hence,

$$\operatorname{Re}\left((l_n+\lambda E)w_n,w_n\right) \geq \int\limits_{\alpha}^{\beta}\left|w'\right|^2 dy + \int\limits_{\alpha}^{\beta}\left(\frac{2\delta}{3}+\lambda-n^2\max_{y\in[\alpha,\beta]}|k(y)|\right)|w|^2\,dy, \tag{2.17}$$

by conditions (2.4) and (2.5).

Furthermore, the following holds

$$\begin{aligned}
&\operatorname{Im}\left[(-f_n,nu_n)+(g_n,nv_n)\right]= \\
&\operatorname{Im}\Bigg[\int\limits_{\alpha}^{\beta}\Big\{u_n''+2inv_n'- \\
&(-n^2k(y)+in\varphi(y)+a(y))u_n-b(y)v_n\Big\}\,n\bar{u}_n dy+ \\
&\int\limits_{\alpha}^{\beta}\Big\{-v_n''+2inu_n'+c(y)u_n+ \\
&(-n^2k(y)+in\psi(y)+d(y))v_n\Big\}\,n\bar{v}_n dy\Bigg] \geq \\
&2n^2\int\limits_{\alpha}^{\beta}v_n'u_n dy-n^2\int\limits_{\alpha}^{\beta}\varphi(y)|u|_n^2 dy+ \\
&2n^2\int\limits_{\alpha}^{\beta}u_n'v_n dy+n^2\int\limits_{\alpha}^{\beta}\psi(y)|v|_n^2 dy= \\
&2n^2\left(u_nv_n|_{\alpha}^{\beta}-\int\limits_{\alpha}^{\beta}u_n'v_n dy\right)+2n^2\int\limits_{\alpha}^{\beta}u_n'v_n dy+ \\
&\delta n^2\int\limits_{\alpha}^{\beta}(u_n^2+v_n^2)dy=\delta n^2\|w\|_{2,G_0}^2,
\end{aligned}$$

or

$$\operatorname{Im}\left[(-f_n,nu_n)+(g,nv_n)\right]\geq \delta n^2\|w\|_{2,G_0}^2 \tag{2.18}$$

By multiplying both hand sides of (2.18) by $\rho>0$ and by invoking (2.17), we obtain

$$\begin{aligned}
&\operatorname{Re}\left((l_n+\lambda E)w_n,w_n\right)+\rho Im\left[(-f_n,nu_n)+(g_n,nv_n)\right]\geq \\
&\int\limits_{\alpha}^{\beta}\left|w_n'\right|^2 dy+\int\limits_{\alpha}^{\beta}\left(\rho\delta n^2+\frac{2\delta}{3}+\lambda-n^2\max_{y\in[\alpha,\beta]}|k(y)|\right)|w_n|^2\,dy.
\end{aligned}$$

Hence, the 'Cauchy's inequality with weight $\epsilon$' implies that

$$\frac{3}{4\delta}\int\limits_{\alpha}^{\beta}|(l_n+\lambda E)w_n|^2\,dy+\frac{\delta}{3}\int\limits_{\alpha}^{\beta}|w_n|^2\,dy+$$

$$\frac{\rho}{2\epsilon}\int_{\alpha}^{\beta}\left[|f_n(y)|^2+|g_n(y)|^2\right]dy+\frac{\rho\epsilon}{2}n^2\int_{\alpha}^{\beta}|w_n|^2\,dy\geq$$

$$\int_{\alpha}^{\beta}\left|w_n^{'}\right|^2dy+\int_{\alpha}^{\beta}\left(\rho\delta n^2+\frac{2\delta}{3}+\lambda-n^2\max_{y\in[\alpha,\beta]}|k(y)|\right)|w_n|^2\,dy.$$

Finally,

$$\left(\frac{3}{4\delta}+\frac{\rho}{2\epsilon}\right)\|(l_n+\lambda E)w_n\|_2^2\geq\int_{\alpha}^{\beta}\left|w_n^{'}\right|^2dy$$

$$+\int_{\alpha}^{\beta}\left(\rho\delta n^2+\frac{\delta}{3}+\lambda-n^2\max_{y\in[\alpha,\beta]}|k(y)|-\frac{\rho\epsilon}{2}n^2\right)|w_n|^2\,dy.$$

We now choose $\rho$ and $\epsilon$ so that $\rho\delta-\max\limits_{y\in[\alpha,\beta]}|k(y)|-\frac{\rho\epsilon}{2}\geq 1$ and assume that $C_0=\left(\frac{3}{4\delta}+\frac{\rho}{2\epsilon}\right)^{-1}$. Then the last inequality implies (2.16) and the proof of the lemma is complete. □

**Lemma 2.6.** *Let $\lambda\geq 0$. Let Assumption 1 hold. Then the operator $l_n+\lambda E$ has an inverse defined on the whole of $L_2\left((\alpha,\beta),\mathbb{C}^2\right)$. Namely, the operator $(l_n+\lambda E)^{-1}$.*

**Proof.** The existence of the inverse operator $(l_n+\lambda E)^{-1}$ is ensured by inequality (2.16).

We assume by contradiction that the range $R\left(l_n+\lambda E\right)$ of the operator $l_n+\lambda E$ is not dense in $L_2\left((\alpha,\beta),\mathbb{C}^2\right)$. Then in accordance with Lemma 1.3, there exists a nonzero element $U=(p,s)$ in $L_2\left((\alpha,\beta),\mathbb{C}^2\right)$ such that $((l_n+\lambda E)w,U)=0$ for all $w\in D\left(l_n+\lambda E\right)$. Then the density of $D\left(l_n+\lambda E\right)$ in $L_2\left((\alpha,\beta),\mathbb{C}^2\right)$, and Proposition 1.4 of Chapter 1 imply that
$U\in D\left((l_n+\lambda E)^*\right)$ and $(l_n+\lambda E)^*U=0$, where $(l_n+\lambda E)^*$ is the adjoint operator to $l_n+\lambda E$, i.e.

$$\begin{cases}-p^{''}+2ins^{'}+(-n^2k(y)-in\varphi(y)+a(y)+\lambda)p+c(y)s=0,\\ -s^{''}-2inp^{'}+b(y)p+(-n^2k(y)-in\psi(y)+d(y)+\lambda)s=0,\ y\in(\alpha,\beta).\end{cases}$$

Hence, the following inclusions hold

$$-p^{''}+2ins^{'},-s^{''}-2inp^{'}\in L_2(\alpha,\beta),\tag{2.19}$$

and thus

$$-p^{'}(y) + 2ins(y) + p^{'}(y_0) - 2ins(y_0) \in C(\alpha, \beta)$$

and

$$-s^{'}(y) - 2inp(y) + s^{'}(y_0) + 2inp(y_0) \in C(\alpha, \beta).$$

It follows, that $p^{'} \in L_2(\alpha, \beta)$ and $s^{'} \in L_2(\alpha, \beta)$, respectively. Hence, by (2.19) we obtain $p^{''}, s^{''} \in L_2(\alpha, \beta)$ . Thus, the function $U = (p, s)$ belongs to the Sobolev space $W_2^2((\alpha, \beta), \mathbb{C}^2)$. We now show that $U = (p, s)$ satisfies the boundary conditions (2.15). Clearly

$$0 = (w, (l_n + \lambda E)^* U) = u^{'}(\beta)\bar{p}(\beta) - u^{'}(\alpha)\bar{p}(\alpha) +$$
$$v^{'}(\beta)\bar{s}(\beta) - v^{'}(\alpha)\bar{s}(\alpha) + ((l_n + \lambda E)w, U),$$

for all $w = (u, v)$ from $D(l_n + \lambda E)$. Hence by definition of the adjoint operator, we obtain the following equality

$$u^{'}(\beta)\bar{p}(\beta) - u^{'}(\alpha)\bar{p}(\alpha) + v^{'}(\beta)\bar{s}(\beta) - v^{'}(\alpha)\bar{s}(\alpha) = 0. \tag{2.20}$$

The last relation holds if and only if $p(\alpha) = p(\beta) = 0$, $s(\alpha) = s(\beta) = 0$. In order to show such an equality, we now make a different choice of the 'test' function. We take the following functions

$$w_1(y) = ((y - \alpha)^2(y - \beta), \sin^k(y - \alpha)(y - \beta)),$$
$$w_2(y) = ((y - \alpha)(y - \beta)^2, \sin^k(y - \alpha)(y - \beta)),$$
$$w_3(y) = ((y - \alpha)^2(y - \beta)^2, (y - \beta)\sin^k(y - \alpha)),$$
$$w_4(y) = ((y - \alpha)^2(y - \beta)^2, (y - \alpha)\sin^k(y - \beta))$$

(where $k \geq 2, k \in \mathbb{N}$), each of which belongs to $D(l_n + \lambda E)$ and we substitute them into equality (2.20). Thus, the function $U = (p, s)$ belongs to the Sobolev space $W_2^2((\alpha, \beta), \mathbb{C}^2)$ and satisfies conditions (2.15). Hence, by arguing as in the proof of Lemma 2.5, we obtain the inequality

$$\|(l_n + \lambda E)^* U\|_{L_2((\alpha,\beta),\mathbb{C}^2)} \geq C_2 \|U\|_{L_2((\alpha,\beta),\mathbb{C}^2)},$$

for all $U = (p, s) \in D((l_n + \lambda E)^*)$. Consequently $U = 0$, a contradiction. Thus the proof of the lemma is complete. □

We now have the main statement of this Chapter.

**Theorem 2.7.** *Let the coefficients of the system (2.1) satisfy Assumption 1. Then the problem (2.1), (2.2) has an unique solution $w = (u, v)$ in the Sobolev space $W_2^1(G_0, \mathbb{R}^2)$ for every right hand side $F = (f, g) \in L_2(G_0, \mathbb{R}^2)$.*

**Proof.** Let $(u_n, v_n)(n \in \mathbb{Z})$ be a solution of problem (2.14), (2.15). Then the function $w_N = \left(\sum_{k=-N}^{N} u_k(y)e^{ikx},\ \sum_{k=-N}^{N} v_k(y)e^{ikx}\right)$ is the solution of problem (2.1), (2.2), where $F(x, y)$ is replaced on $F_N = \left(\sum_{k=-N}^{N} f_k(y)e^{ikx}, \sum_{k=-N}^{N} g_k(y)e^{ikx}\right)$. Since the sequence $\{F_N\}$ converges to the right hand side $F(x, y)$ of system (2.1), it is a Cauchy sequence. Then by inequality (2.6), $\{w_N\}_{N=-\infty}^{\infty}$ is a Cauchy sequence also in $W_2^1(G_0, \mathbb{R}^2)$. Since $W_2^1(G_0, \mathbb{R}^2)$ is complete, then the sequence $\{w_N\}_{N=-\infty}^{\infty}$ has a limit $w = (u, v)$ in $W_2^1(G_0, \mathbb{R}^2)$. By definition $w = (u, v)$ is a solution of problem (2.1), (2.2). The uniqueness of the solution follows by inequality (2.6). Thus the proof is complete. □

# Chapter 3

# On the regularity of the solution of the semiperiodical Dirichlet problem for system (2.1)

Let $\lambda, \hat{\lambda}$ be constants such that $\hat{\lambda} \geq \lambda \geq 0$. Let

$$\breve{E} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We consider the operator $l_n + in\hat{\lambda}\breve{E} + \lambda E$ defined by

$$(l_n + in\hat{\lambda}\breve{E} + \lambda E)w = -w^{''} + 2inTw^{'} + Q_n(y)w + in\hat{\lambda}\tilde{w} + \lambda w,$$

for all functions $w = (u, v)$ in the space $C_0^2\left((\alpha, \beta), \mathbb{C}^2\right)$ of twice continuously differentiable of $[\alpha, \beta]$ to $\mathbb{C}$ which satisfy the boundary conditions (2.15), where $\tilde{w} = (-u, v)$, $T$ and $Q_n(y)$ are the matrices associated to system (2.14). We denote by $l_n + in\hat{\lambda}\breve{E} + \lambda E$ the closure of the operator $l_n + in\hat{\lambda}\breve{E} + \lambda E$ in the norm of $L_2 \equiv L_2\left((\alpha, \beta), \mathbb{C}^2\right)$ .

**Lemma 3.1.** *Let $\hat{\lambda} \geq \lambda \geq 0$. Let Assumption 1 hold. Let*

$$\inf_{y,\eta\in[\alpha,\beta],|y-\eta|\leq\mu} \frac{\varphi^2(y)}{a(\eta)} \geq c_1 > 0, \quad \inf_{y,\eta\in[\alpha,\beta],|y-\eta|\leq\mu} \frac{\psi^2(y)}{d(\eta)} \geq c_2 > 0, \tag{3.1}$$

$$\sup_{y,\eta\in[\alpha,\beta],|y-\eta|\leq\mu} \left\{ \frac{\varphi(y)}{\varphi(\eta)}, \frac{\psi(y)}{\psi(\eta)}, \frac{a(y)}{a(\eta)}, \frac{d(y)}{d(\eta)}, \frac{a(y)}{d(\eta)} \right\} \leq c_3 < \infty. \tag{3.2}$$

*Then*

$$\left\| |n| \left(P(\cdot) + \hat{\lambda}\breve{E}\right) \left(l_n + in\hat{\lambda}\breve{E} + \lambda E\right)^{-1} \right\|_{L_2 \to L_2} + \left\| (Q(\cdot) + \lambda E) \left(l_n + in\hat{\lambda}\breve{E} + \lambda E\right)^{-1} \right\|_{L_2 \to L_2} < \infty. \tag{3.3}$$

*Here $P(y)$ and $Q(y)$ are matrices of (2.1) and $\mu = \frac{\beta-\alpha}{2}$.*

**Proof.** By arguing as in the proof of Lemma 2.6, one can show that if Assumption 1 holds, then the operator $l_n + in\hat{\lambda}\breve{E} + \lambda E$ has a bounded inverse. Also, by the argument of the proof of Lemma 2.5, we obtain

$$\left\| w' \right\|_{L_2}^2 + \left( \frac{2}{3} \inf_{y\in[\alpha,\beta]} [a(y), d(y)] + \lambda - n^2 \max_{y\in[\alpha,\beta]} |k(y)| \right) \|w\|_{L_2}^2 - \frac{\frac{2}{3} \inf_{y\in[\alpha,\beta]} [a(y), d(y)] + \lambda}{4C_0} \|w\|_{L_2}^2 \le \frac{C_0}{\frac{2}{3} \inf_{y\in[\alpha,\beta]} [a(y), d(y)] + \lambda} \left\| \left(l_n + in\hat{\lambda}\breve{E} + \lambda E\right) w \right\|_{L_2}^2, \tag{3.4}$$

$$|n| \frac{\inf_{y\in[\alpha,\beta]} [-\varphi(y), \psi(y)] + \hat{\lambda}}{\sqrt{\frac{2}{3} \inf_{y\in[\alpha,\beta]} [a(y), d(y)] + \lambda}} \|w\|_{L_2} \le \frac{1}{\sqrt{\frac{2}{3} \inf_{y\in[\alpha,\beta]} [a(y), d(y)] + \lambda}} \left\| \left(l_n + in\hat{\lambda}\breve{E} + \lambda E\right) w \right\|_{L_2}, \tag{3.5}$$

for all $w \in D(l_n + in\hat{\lambda}\breve{E} + \lambda E)$ . Hence, we have

$$\left\| w' \right\|_{L_2}^2 + \left( \inf_{y\in[\alpha,\beta]} [a(y), d(y)] + \lambda \right) \left(1 - \frac{1}{4C_0}\right) \|w\|_{L_2}^2 + n^2 \left( \frac{C_1 \left( \inf_{y\in[\alpha,\beta]} [-\varphi(y), \psi(y)] + \hat{\lambda} \right)^2}{\inf_{y\in[\alpha,\beta]} [a(y), d(y)] + \lambda} - \max_{y\in[\alpha,\beta]} |k(y)| \right) \|w\|_{L_2}^2 \le \frac{C_0 + C_1}{\inf_{y\in[\alpha,\beta]} [a(y), d(y)] + \lambda} \left\| \left(l_n + in\hat{\lambda}\breve{E} + \lambda E\right) w \right\|_{L_2}^2,$$

which we can rewrite as

$$\left\| \left(l_n + in\hat{\lambda}\breve{E} + \lambda E\right)^{-1} \right\|_{L_2 \to L_2} \le \frac{C_3}{\inf_{y\in[\alpha,\beta]} [a(y), d(y)] + \lambda}.$$

Hence, condition (3.2) implies that

$$\left\|(Q(\cdot)+\lambda E)\left(l_n+in\hat{\lambda}\breve{E}+\lambda E\right)^{-1}\right\|_{L_2\to L_2}\le$$
$$C_1\max_{y\in[\alpha,\beta]}\{a(y)+\lambda,d(y)+\lambda\}\left\|\left(l_n+in\hat{\lambda}\breve{E}+\lambda E\right)^{-1}\right\|_{L_2\to L_2}<\infty.$$

Next we note that inequality (3.5) implies that

$$\left\|\left(l_n+in\hat{\lambda}\breve{E}+\lambda E\right)^{-1}\right\|_{L_2\to L_2}\le\frac{1}{|n|\left(\inf\limits_{y\in[\alpha,\beta]}[-\varphi(y),\psi(y)]+\hat{\lambda}\right)}.$$

Hence, by condition (3.2), we obtain

$$\left\||n|\left(P(\cdot)+\hat{\lambda}\breve{E}\right)\left(l_n+in\hat{\lambda}\breve{E}+\lambda E\right)^{-1}\right\|_{L_2\to L_2}\le$$
$$|n|\sup_{y\in[\alpha,\beta]}\left(|\varphi(y)+\hat{\lambda}|,|\psi(y)+\hat{\lambda}|\right)\left\|\left(l_n+in\hat{\lambda}\breve{E}+\lambda E\right)^{-1}\right\|_{L_2\to L_2}\le$$
$$|n|\sup_{y\in[\alpha,\beta]}\left(|\varphi(y)|+\hat{\lambda},\psi(y)+\hat{\lambda}\right)\frac{1}{|n|\inf\limits_{y\in[\alpha,\beta]}[|\varphi(y)|+\hat{\lambda},\psi(y)+\hat{\lambda}]}<\infty,$$

and the proof is complete. □

We now consider the operator $L_{\lambda,\hat{\lambda}}$ defined by

$$L_{\lambda,\hat{\lambda}}w=B_{xy}w+\left(P(y)+\hat{\lambda}\breve{E}\right)w_x+(Q(y)+\lambda E)\,w,$$

for all functions $w=(u,v)$ in the space $C^2_{\pi,0}(G_0,\mathbb{R}^2)$. We denote by $L_{\lambda,\hat{\lambda}}$ the closure of $L_{\lambda,\hat{\lambda}}$ in the norm of $L_2(G_0,\mathbb{R}^2)$.

**Definition 3.2.** *The operator $L_{\lambda,\hat{\lambda}}$ is said to be separable, if the following inequality holds*

$$\|w_{xx}\|_{2,G_0}+\|w_{yy}\|_{2,G_0}+\|w_{xy}\|_{2,G_0}+\left\|\left(P(y)+\hat{\lambda}\breve{E}\right)w_x\right\|_{2,G_0}+$$
$$\|(Q(y)+\lambda E)\,w\|_{2,G_0}\le C\left(\left\|L_{\lambda,\hat{\lambda}}w\right\|_{2,G_0}+\|w\|_{2,G_0}\right),$$

*for all $w\in D(L_{\lambda,\hat{\lambda}})$.*

We now prove the following intermediate statement.

**Lemma 3.3.** *Let the folowing conditions hold.*

*a) The coefficients $\varphi,\psi,a,b,c,d$ of system (2.1) satisfy Assumption 1.*

*b) The function $k(y)$ of $[\alpha,\beta]$ to $[0;+\infty)$ is twice continuously differentiable and satisfies one and only one of the following three conditions*

*i)* $\sqrt{2} < k(y) < 2, \quad \min\limits_{y\in[\alpha,\beta]} \{\varphi^2(y), \psi^2(y)\} + k''(y) > 2\left[k'(y)\right]^2;$

*ii)* $k(y) < 2,\ \frac{\sqrt{2}k'(y)}{k(y)} \le 1, \quad \min\limits_{y\in[\alpha,\beta]} \{\varphi^2(y), \psi^2(y)\} + k''(y) > 2;$

*iii)* $k(y) < 2,\ k^2(y) > 2k'(y), \quad \min\limits_{y\in[\alpha,\beta]} \{\varphi^2(y), \psi^2(y)\} + k''(y) > 2k'(y).$

*c) There exist non-negative constants $\lambda$ and $\hat{\lambda}$ such that the following inequality holds*

$$\|B_{xy}w\|_{2,G_0} + \left\|\left(P(y) + \hat{\lambda}\breve{E}\right) w_x\right\|_{2,G_0} + \|(Q(y) + \lambda E)\, w\|_{2,G_0} \le C\left(\left\|(L + \hat{\lambda}\breve{E}\frac{\partial}{\partial x} + \lambda E)w\right\|_{2,G_0} + \|w\|_{2,G_0}\right), \tag{3.6}$$

*for all $w = (u,v) \in D(L_{\lambda,\hat{\lambda}})$.*

*Then the operator $L_{\lambda,\hat{\lambda}}$ is separable.*

**Proof.** Let $w = (u,v) \in C^2_{\pi,0}(G_0,\mathbb{R}^2)$. By simple computations, we obtain

$$\|B_{xy}w\|^2_{2,G_0} = \int\limits_{G_0} k^2(y)[u^2_{xx} + v^2_{xx}]dxdy + \int\limits_{G_0} [u^2_{yy} + v^2_{yy}]dxdy + 4\int\limits_{G_0} v^2_{xy}dxdy + 4\int\limits_{G_0} u^2_{xy}dxdy - 2\int\limits_{G_0} k(y)u_{xx}u_{yy}dxdy - 4\int\limits_{G_0} k(y)u_{xx}v_{xy}dxdy + 4\int\limits_{G_0} u_{yy}v_{xy}dxdy + 4\int\limits_{G_0} k(y)u_{xy}v_{xx}dxdy - 4\int\limits_{G_0} u_{xy}v_{yy}dxdy - 2\int\limits_{G_0} k(y)v_{xx}v_{yy}dxdy. \tag{3.7}$$

We now introduce some notation. Let

$$I_1 = -2\int\limits_{G_0} k(y)u_{xx}u_{yy}dxdy,$$
$$I_2 = -4\int\limits_{G_0} k(y)u_{xx}v_{xy}dxdy,$$
$$I_3 = 4\int\limits_{G_0} u_{yy}v_{xy}dxdy,$$
$$I_4 = 4\int\limits_{G_0} k(y)u_{xy}v_{xx}dxdy,$$
$$I_5 = -4\int\limits_{G_0} u_{xy}v_{yy}dxdy,$$
$$I_6 = -2\int\limits_{G_0} k(y)v_{xx}v_{yy}dxdy.$$

Integrating by parts, we obtain $I_3 = 4\int\limits_{G_0} u_{xy}v_{yy}dxdy$, and thus $I_3 + I_5 = 0$.

By similar computations, we obtain

$$I_2 = -4\int\limits_{G_0} k(y)u_{xy}v_{xx}dxdy - 4\int\limits_{G_0} k^{'}(y)u_x v_{xx}dxdy,$$

then $I_2 + I_4 = -4\int\limits_{G_0} k^{'}(y)u_x v_{xx}dxdy$.

$$\begin{aligned} I_1 = &-2\int\limits_{\alpha}^{\beta}\left(k(y)u_x u_{yy}|_{-\pi}^{\pi}\right)dy + 2\int\limits_{G_0} k(y)u_x u_{yyx}dxdy \\ &-\int\limits_{G_0} k^{'}(y)\left(u_x^2\right)_y dxdy - 2\int\limits_{G_0} k(y)u_{xy}^2 dxdy = \\ &-\int\limits_{-\pi}^{\pi}\left(k^{'}(y)u_x^2\Big|_{\alpha}^{\beta}\right)dx + \int\limits_{G_0} k^{''}(y)u_x^2 dxdy - 2\int\limits_{G_0} k(y)u_{xy}^2 dxdy = \\ &\int\limits_{G_0} k^{''}(y)u_x^2 dxdy - 2\int\limits_{G_0} k(y)u_{xy}^2 dxdy. \end{aligned}$$

$$\begin{aligned} I_6 = &-2\int\limits_{\alpha}^{\beta}\left(k(y)v_{yy}v_x|_{-\pi}^{\pi}\right)dy + 2\int\limits_{G_0} k(y)v_x v_{yyx}dxdy = \\ &-\int\limits_{G_0} k^{'}(y)\left(v_x^2\right)_y dxdy - 2\int\limits_{G_0} k(y)v_{xy}^2 dxdy = \\ &-\int\limits_{-\pi}^{\pi}\left(k^{'}(y)v_x^2\Big|_{\alpha}^{\beta}\right)dx + \int\limits_{G_0} k^{''}(y)v_x^2 dxdy - 2\int\limits_{G_0} k(y)v_{xy}^2 dxdy. \end{aligned}$$

Hence, $I_1 + I_6 = \int\limits_{G_0} k^{''}(y)|w_x|^2 dxdy - 2\int\limits_{G_0} k(y)|w_{xy}|^2 dxdy$.

Then by (3.7), we obtain

$$\begin{aligned} \|B_{xy}w\|_{2,G_0}^2 = &\int\limits_{G_0} k^2(y)|w_{xx}|^2 dxdy + \int\limits_{G_0} |w_{yy}|^2 dxdy + \\ &4\int\limits_{G_0} |w_{xy}|^2 dxdy + \int\limits_{G_0} k^{''}(y)|w_x|^2 dxdy - \\ &2\int\limits_{G_0} k(y)|w_{xy}|^2 dxdy - 4\int\limits_{G_0} k^{'}(y)u_x v_{xx}dxdy, \end{aligned}$$

and thus

$$\|B_{xy}w\|_{2,G_0}^2 + \left\|\left(P(y) + \hat{\lambda}\breve{E}\right)w_x\right\|_{2,G_0}^2 \geq$$

$$\int_{G_0} k^2(y)|w_{xx}|^2 dxdy + \int_{G_0} |w_{yy}|^2 dxdy +$$
$$4\int_{G_0} |w_{xy}|^2 dxdy + \int_{G_0} k^{''}(y)|w_x|^2 dxdy - 2\int_{G_0} k(y)|w_{xy}|^2 dxdy -$$
$$4\int_{G_0} k^{'}(y)u_x v_{xx} dxdy + \int_{G_0} \varphi^2(y)u_x^2 dxdy + \int_{G_0} \psi^2(y)v_x^2 dxdy.$$

Hence,

$$\|B_{xy}w\|_{2,G_0}^2 + \left\|\left(P(y)+\hat{\lambda}\breve{E}\right)w_x\right\|_{2,G_0}^2 \geq \tag{3.8}$$
$$\int_{G_0} k^2(y)|w_{xx}|^2 dxdy + \int_{G_0} |w_{yy}|^2 dxdy +$$
$$4\int_{G_0} |w_{xy}|^2 dxdy + \int_{G_0} k^{''}(y)|w_x|^2 dxdy - 2\int_{G_0} k(y)|w_{xy}|^2 dxdy -$$
$$4\int_{G_0} k^{'}(y)u_x v_{xx} dxdy + \min_{y\in[\alpha,\beta]}\left\{\varphi^2(y),\psi^2(y)\right\}\int_{G_0} |w_x|^2 dxdy.$$

We now consider the following three cases, which we label as 1), 2), 3).

1) Let $k(y)$ satisfy conditions i) of the statement. In accordance with 'the Cauchy inequality', we have

$$\left|\int_{G_0} k^{'}(y)u_x v_{xx} dxdy\right| \leq \frac{1}{2}\int_{G_0}\left[k^{'}(y)\right]^2 u_x^2 dxdy + \frac{1}{2}\int_{G_0} v_{xx}^2 dxdy.$$

Therefore inequality (3.8) implies that

$$\|B_{xy}w\|_{2,G_0}^2 + \left\|\left(P(y)+\hat{\lambda}\breve{E}\right)w_{xx}\right\|_{2,G_0}^2 \geq$$
$$\int_{G_0} k^2(y)|w_{xx}|^2 dxdy + \int_{G_0} |w_{yy}|^2 dxdy + 4\int_{G_0} |w_{xy}|^2 dxdy +$$
$$\int_{G_0} k^{''}(y)|w_x|^2 dxdy - 2\int_{G_0} k(y)|w_{xy}|^2 dxdy +$$
$$\min_{y\in[\alpha,\beta]}\left\{\varphi^2(y),\psi^2(y)\right\}\int_{G_0} |w_x|^2 dxdy -$$
$$2\int_{G_0}\left[k^{'}(y)\right]^2 u_x^2 dxdy - 2\int_{G_0} v_{xx}^2 dxdy$$

and

$$\|B_{xy}w\|_{2,G_0}^2 + \left\|\left(P(y)+\hat{\lambda}\breve{E}\right)w_x\right\|_{2,G_0}^2 \geq \tag{3.9}$$

$$\int_{G_0} [k^2(y)-2]|w_{xx}|^2 dxdy +$$
$$\int_{G_0} |w_{yy}|^2 dxdy + 2\int_{G_0} [2-k(y)]|w_{xy}|^2 dxdy +$$
$$\int_{G_0} \left( \min_{y\in[\alpha,\beta]} \{\varphi^2(y),\psi^2(y)\} + k''(y) - 2\left[k'(y)\right]^2 \right) |w_x|^2 dxdy.$$

Hence, condition i) of the statement implies the validity of the following inequality

$$\|B_{xy}w\|^2_{2,G_0} + \left\| \left(P(y)+\hat{\lambda}\breve{E}\right) w_x \right\|^2_{2,G_0} \geq \tag{3.10}$$
$$C_1 \|w_{xx}\|^2_{2,G_0} + \|w_{yy}\|^2_{2,G_0} + C_2 \|w_{xy}\|^2_{2,G_0} +$$
$$C_3 \left(C_4 + \hat{\lambda}\right) \|w_x\|^2_{2,G_0}.$$

2) Let condition ii) of the statement hold. We estimate the last term in the right hand side of (3.8) by applying ‘the Cauchy inequality’ in the following form

$$\left| \int_{G_0} k'(y) u_x v_{xx} dxdy \right| \leq \frac{1}{2}\int_{G_0} u_x^2 dxdy + \frac{1}{2}\int_{G_0} \left[k'(y)\right]^2 v_{xx}^2 dxdy.$$

Then we have

$$\|B_{xy}w\|^2_{2,G_0} + \left\| \left(P(y)+\hat{\lambda}\breve{E}\right) w_x \right\|^2_{2,G_0} \geq$$
$$\int_{G_0} k^2(y)|w_{xx}|^2 dxdy + \int_{G_0} |w_{yy}|^2 dxdy +$$
$$4\int_{G_0} |w_{xy}|^2 dxdy + \int_{G_0} k''(y)|w_x|^2 dxdy -$$
$$2\int_{G_0} k(y)|w_{xy}|^2 dxdy + \min_{y\in[\alpha,\beta]} \left\{\varphi^2(y),\psi^2(y)\right\} \int_{G_0} |w_x|^2 dxdy -$$
$$2\int_{G_0} u_x^2 dxdy - 2\int_{G_0} \left[k'(y)\right]^2 v_{xx}^2 dxdy$$

and

$$\|B_{xy}w\|^2_{2,G_0} + \left\| \left(P(y)+\hat{\lambda}\breve{E}\right) w_x \right\|^2_{2,G_0} \geq \tag{3.11}$$
$$\int_{G_0} \left[k^2(y) - 2(k'(y))^2\right] |w_{xx}|^2 dxdy +$$
$$\int_{G_0} |w_{yy}|^2 dxdy + 2\int_{G_0} \left[2-k(y)\right] |w_{xy}|^2 dxdy +$$

$$\int_{G_0} \left( \min_{y\in[\alpha,\beta]} \left\{\varphi^2(y), \psi^2(y)\right\} + k^{''}(y) - 2 \right) |w_x|^2 dxdy.$$

Hence, by condition ii) of this lemma, we obtain inequality (3.10).

3) Let condition iii) of the statement hold. By using 'the Cauchy inequality'

$$\left| \int_{G_0} k^{'}(y) u_x v_{xx} dxdy \right| \leq \frac{1}{2} \int_{G_0} k^{'}(y) u_x^2 dxdy + \frac{1}{2} \int_{G_0} k^{'}(y) v_{xx}^2 dxdy,$$

we transform inequality (3.8) in the following form

$$\begin{aligned} &\|B_{xy} w\|^2_{2,G_0} + \left\| \left( P(y) + \hat{\lambda} \breve{E} \right) w_x \right\|^2_{2,G_0} \geq \\ &\quad \int_{G_0} k^2(y) |w_{xx}|^2 dxdy + \int_{G_0} |w_{yy}|^2 dxdy + \\ &\quad 4 \int_{G_0} |w_{xy}|^2 dxdy + \int_{G_0} k^{''}(y) |w_x|^2 dxdy - \\ &\quad 2 \int_{G_0} k(y) |w_{xy}|^2 dxdy + \min_{y\in[\alpha,\beta]} \left\{\varphi^2(y), \psi^2(y)\right\} \int_{G_0} |w_x|^2 dxdy - \\ &\quad 2 \int_{G_0} k^{'}(y) u_x^2 dxdy - 2 \int_{G_0} k^{'}(y) v_{xx}^2 dxdy \end{aligned}$$

and thus

$$\begin{aligned} &\|B_{xy} w\|^2_{2,G_0} + \left\| \left( P(y) + \hat{\lambda} \breve{E} \right) w_x \right\|^2_{2,G_0} \geq \\ &\quad \int_{G_0} \left[ k^2(y) - 2k^{'}(y) \right] |w_{xx}|^2 dxdy + \int_{G_0} |w_{yy}|^2 dxdy + \\ &\quad 2 \int_{G_0} \left[ 2 - k(y) \right] |w_{xy}|^2 dxdy + \\ &\quad \int_{G_0} \left( \min_{y\in[\alpha,\beta]} \left\{\varphi^2(y), \psi^2(y)\right\} + k^{''}(y) - 2k^{'}(y) \right) |w_x|^2 dxdy. \end{aligned} \tag{3.12}$$

Hence, by condition iii) of the statement, we obtain the inequality (3.10).

Consequently, from inequalities (3.6) and (3.10) we obtain

$$\begin{aligned} &\|w_{xx}\|_{2,G_0} + \|w_{yy}\|_{2,G_0} + \|w_{xy}\|_{2,G_0} + \\ &\quad \left\| \left( P(y) + \hat{\lambda} \breve{E} \right) w_x \right\|_{2,G_0} + \| (Q(y) + \lambda E) w \|_{2,G_0} \leq \\ &\quad \|B_{xy} w\|_{2,G_0} + \left\| \left( P(y) + \hat{\lambda} \breve{E} \right) w_x \right\|_{2,G_0} + \| (Q(y) + \lambda E) w \|_{2,G_0} \leq \\ &\quad C_2(\lambda, \hat{\lambda}) \left( \left\| L_{\lambda,\hat{\lambda}} w \right\|_{2,G_0} + \|w\|_{2,G_0} \right) \end{aligned}$$

and the proof of the lemma is complete. □

We now introduce the main result of this chapter.

**Theorem 3.4.** *Let $\hat{\lambda} \geq \lambda \geq 0$. Let the coefficients of system (2.1) satisfy Assumption 1, conditions (3.1), (3.2). Let $k(y)$ be twice continuously differentiable on $[\alpha, \beta]$ and satisfy one and only one of conditions i), ii), iii). Then the operator $L_{\lambda,\hat{\lambda}}$ is separable.*

**Proof.** By Lemma 3.3 it is enough to show the correctness of (3.6). The operator $L$ is bounded and invertible by the assumptions of the theorem. The operator $L + \hat{\lambda}\breve{E}\frac{\partial}{\partial x}$ satisfies all the conditions of Theorem 3.4. Hence, $L + \hat{\lambda}\breve{E}\frac{\partial}{\partial x}$ is bounded and invertible. Furthermore, the following inequality holds

$$\left\|(L + \hat{\lambda}\breve{E}\frac{\partial}{\partial x} + \lambda E)w\right\|_{2,G_0} \geq C\,\|w\|_{2,G_0},$$

for all $w \in D(L + \hat{\lambda}\breve{E}\frac{\partial}{\partial x} + \lambda E)$. Then by the well-known Theorem 1.2, the operator $L+\hat{\lambda}\breve{E}\frac{\partial}{\partial x}+\lambda E$ is bounded and invertible in $L_2(G_0, \mathbb{R}^2)$. Furthermore, we have

$$(L + \hat{\lambda}\breve{E}\frac{\partial}{\partial x} + \lambda E)^{-1}F = \sum_{n=-\infty}^{\infty} (l_n + in\hat{\lambda}\breve{E} + \lambda E)^{-1}F_n e^{inx},$$

by construction. Here $F = \sum_{n=-\infty}^{\infty} F_n e^{inx}$, $F = (f, g)$, $F_n = (f_n, g_n)$.

Hence, by the orthonormality of the system $\{e^{inx}\}_{n=-\infty}^{\infty}$ in $L_2[-\pi, \pi]$, we obtain

$$\left\|\rho(y)D_x^\tau\left(L + \hat{\lambda}\breve{E}\frac{\partial}{\partial x} + \lambda E\right)^{-1}\right\|_{L_2(G_0,\mathbb{R}^2)\to L_2(G_0,\mathbb{R}^2)} =$$
$$\sup_n \left\||n|^\tau\rho(y)\left(l_n + in\hat{\lambda}\breve{E} + \lambda E\right)^{-1}\right\|_{L_2((\alpha,\beta),\mathbb{C}^2)\to L_2((\alpha,\beta),\mathbb{C}^2)}.$$

Here $D_x^\tau = \frac{\partial^\tau}{\partial x^\tau}$, $\tau = 0, 1$ and $\rho(y)$ is a $2 \times 2$-matrix with continuous elements. Since (3.3) holds, we have

$$\left\|\left(P(\cdot) + \hat{\lambda}\breve{E}\right)\frac{\partial}{\partial x}\left(L + \hat{\lambda}\breve{E}\frac{\partial}{\partial x} + \lambda E\right)^{-1}\right\|_{L_2(G_0,\mathbb{R}^2)\to L_2(G_0,\mathbb{R}^2)} +$$
$$\left\|(Q(\cdot) + \lambda E)\frac{\partial}{\partial x}\left(L + \hat{\lambda}\breve{E}\frac{\partial}{\partial x} + \lambda E\right)^{-1}\right\|_{L_2(G_0,\mathbb{R}^2)\to L_2(G_0,\mathbb{R}^2)} =$$

$$\sup_n \left\| |n| \left(P(\cdot)+\lambda E\right)\left(l_n + in\hat{\lambda}\breve{E}+\lambda E\right)^{-1}\right\|_{L_2((\alpha,\beta),\mathbb{C}^2)\to L_2((\alpha,\beta),\mathbb{C}^2)} +$$
$$\sup_n \left\| \left(Q(\cdot)+\lambda E\right)\left(l_n + in\hat{\lambda}\breve{E}+\lambda E\right)^{-1}\right\|_{L_2((\alpha,\beta),\mathbb{C}^2)\to L_2((\alpha,\beta),\mathbb{C}^2)} < \infty.$$

Then we obtain

$$\left\| B_{xy}\left(L+\hat{\lambda}\breve{E}\frac{\partial}{\partial x}+\lambda E\right)^{-1}\right\|_{L_2(G_0,\mathbb{R}^2)\to L_2(G_0,\mathbb{R}^2)} < \infty$$

from system (2.1). Hence, inequality (3.6) follows and the proof is complete.

□

**Corollary 3.5.** *Let the coefficients $\varphi, \psi, a, b, c, d, k$ of system (2.1) satisfy the conditions of Theorem 3.4. Then the following inequality holds*

$$\begin{aligned} &\|u_{xx}\|^2_{2,G_0} + \|u_{yy}\|^2_{2,G_0} + \|u_{xy}\|^2_{2,G_0} + \|v_{xx}\|^2_{2,G_0} + \\ &\quad \|v_{yy}\|^2_{2,G_0} + \|v_{xy}\|^2_{2,G_0} + \|\varphi(y)u_x\|^2_{2,G_0} + \|u_y\|^2_{2,G_0} + \\ &\quad \|\psi(y)v_x\|^2_{2,G_0} + \|v_y\|^2_{2,G_0} + \|a(y)u\|^2_{2,G_0} + \|b(y)v\|^2_{2,G_0} + \\ &\quad \|c(y)u\|^2_{2,G_0} + \|d(y)v\|^2_{2,G_0} \le C\,\|F\|^2_{2,G_0}, \end{aligned} \tag{3.13}$$

*for the solution $w=(u,v)$ of problem (2.1), (2.2).*

**Remark 3.6.** *The definition of separability ensures the validity of the following inequality*

$$\begin{aligned} &\|u_{xx}\|^2_{2,G_0} + \|u_{yy}\|^2_{2,G_0} + \|u_{xy}\|^2_{2,G_0} + \|v_{xx}\|^2_{2,G_0} + \\ &\quad \|v_{yy}\|^2_{2,G_0} + \|v_{xy}\|^2_{2,G_0} + \|\varphi(y)u_x\|^2_{2,G_0} + \|u_y\|^2_{2,G_0} + \\ &\quad \|\psi(y)v_x\|^2_{2,G_0} + \|v_y\|^2_{2,G_0} + \|a(y)u\|^2_{2,G_0} + \|b(y)v\|^2_{2,G_0} + \\ &\quad \|c(y)u\|^2_{2,G_0} + \|d(y)v\|^2_{2,G_0} \le C\left(\|Lw\|^2_{2,G_0} + \|w\|^2_{2,G_0}\right). \end{aligned} \tag{3.14}$$

*If ineqality (2.6) is holds, then (3.14) is equivalent to (3.13).*

By the well-known norm of the Sobolev space $W_2^2(G_0,\mathbb{R}^2)$, one can rewrite (3.13) in the following compact form

$$\begin{aligned} &\|w\|^2_{W_2^2(G_0,\mathbb{R}^2)} + \|\varphi(y)u_x\|^2_{2,G_0} + \|\psi(y)v_x\|^2_{2,G_0} + \\ &\quad \|(|a|+|c|)\,u\|^2_{2,G_0} + \|(|b|+|d|)\,v\|^2_{2,G_0} \le C_1\,\|F\|^2_{2,G_0}. \end{aligned}$$

**Example 3.7.** *We consider the following problem*

$$\begin{cases} y^3\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}-2\frac{\partial^2 v}{\partial x\partial y}-e^{y^2+21}\frac{\partial u}{\partial x}+ \\ \qquad (arctgy+2^y)u+yv=f(x,y), \\ 2\frac{\partial^2 u}{\partial x\partial y}+y^3\frac{\partial^2 v}{\partial x^2}-\frac{\partial^2 v}{\partial y^2}+\chi(y)2^{y^2+15}\frac{\partial v}{\partial x}+ \\ \qquad \frac{y-3}{5\sqrt{4+y^2}}u+(y+siny)v=g(x,y), \end{cases}$$

$$w(-\pi,y)=w(\pi,y), w_x(-\pi,y)=w_x(\pi,y), w(x,\sqrt[6]{2})=w(x,\sqrt[3]{1.9})=0$$

in the rectangle $G_1=\left\{(x,y)\in\mathbb{R}^2:\ -\pi<x<\pi,\ \sqrt[6]{2}<y<\sqrt[3]{1.9}\right\}$. Here $\chi(y)$ is an arbitrary function such that $2\le\chi(y)\le 3$ for all $y\in[\sqrt[6]{2},\sqrt[3]{1.9}]$ and $f,g\in L_2(G_1)$.

By Theorem 2.7 this problem has an unique solution $w=(u,v)\in L_2$ for any data $f,g\in L_2(G_1)$. Moreover, the solutions of the above system satisfy the coercive inequality with the norm of space $L_2(G_1)$ in the form (3.13).

We now show that the following functions

$$a(y)=arctgy+2^y,$$
$$b(y)=y,$$
$$c(y)=\frac{y-3}{5\sqrt{4+y^2}},$$
$$d(y)=y+siny,$$
$$\varphi(y)=-e^{y^2+21},$$
$$\psi(y)=\chi(y)2^{y^2+15}$$

satisfy Assumption 1.

Indeed

1)there exists a constant $\delta>0$ such that

$$\inf_{y\in[\sqrt[6]{2},\sqrt[3]{1.9}]}\{-\varphi(y),a(y),d(y)\}=$$
$$\inf_{y\in[\sqrt[6]{2},\sqrt[3]{1.9}]}\left\{e^{y^2+21},arctgy+2^y,y+siny\right\}=\delta>0;$$

2) there exist constants $r,q,(r>0,\ q>0,\ r+q=1)$ and $\vartheta(0<\vartheta<3)$ such that

$$\frac{1}{2}\left(|y|+|\frac{y-3}{5\sqrt{4+y^2}}|\right)^{2r}\le\frac{arctgy+2^y}{3},$$
$$\frac{1}{2}\left(|y|+|\frac{y-3}{5\sqrt{4+y^2}}|\right)^{2q}\le\frac{y+siny}{3},$$
$$\vartheta\chi(y)2^{y^2+15}>y+siny$$

for all $y \in [\sqrt[6]{2}, \sqrt[3]{1.9}]$.

Further, we can see that the functions $a(y), d(y), \varphi(y), \psi(y)$ of $\mathbb{R}$ satisfy the conditions (3.1) and (3.2) of Lemma 3.1. There exist constants $c_0, c_1$ and $C$ such that

$$\inf_{y,\eta\in[\sqrt[6]{2},\sqrt[3]{2}],|y-\eta|\leq\mu} \frac{e^{2(y^2+21)}}{(arctg\eta + 2^\eta)} \geq c_0 > 0,$$

$$\inf_{y,\eta\in[\sqrt[6]{2},\sqrt[3]{2}],|y-\eta|\leq\mu} \frac{\chi^2(y)2^{2(y^2+15)}}{(\eta + sin\eta)} \geq c_1 > 0$$

and

$$\sup_{y,\eta\in[\sqrt[6]{2},\sqrt[3]{2}],|y-\eta|\leq\mu} \left\{ \frac{e^{y^2+21}}{e^{\eta^2+21}}, \frac{\chi(y)2^{y^2+15}}{\chi(\eta)2^{\eta^2+15}}, \frac{arctgy + 2^y}{arctg\eta + 2^\eta}, \frac{y + siny}{\eta + sin\eta}, \frac{arctgy + 2^y}{\eta + sin\eta} \right\} \leq C < \infty,$$

where $\mu = \frac{\sqrt[3]{1.9} - \sqrt[6]{2}}{2}$.

The function $k(y) = y^3$ on $[\sqrt[6]{2}, \sqrt[3]{1.9}]$ is twice continuously differentiable and satisfies the condition (i) of Lemma 3.3.

First of all we show that

$$\min_{y\in[\sqrt[6]{2},\sqrt[3]{1.9}]} \{\varphi^2(y), \psi^2(y)\} > 2[k^{'}(y)]^2 - k^{''}(y).$$

Indeed,

$2[k^{'}(y)]^2 = 18y^4$ and $k^{''}(y) = 6y$.

Then

$$\max_{y\in[\sqrt[6]{2},\sqrt[3]{1.9}]} \left\{2[k^{'}(y)]^2 - k^{''}(y)\right\} = \max_{y\in[\sqrt[6]{2},\sqrt[3]{1.9}]} \left\{18y^4 - 6y\right\} < 84.$$

Thus

$$\min_{y\in[\sqrt[6]{2},\sqrt[3]{1.9}]} \left\{e^{2(y^2+21)}, \chi^2(y)2^{2(y^2+15)}\right\} > 84$$

holds for any $y \in [\sqrt[6]{2}, \sqrt[3]{1.9}]$.

**Example 3.8.** *We consider the following problem*

$$\begin{cases} -y^3lny\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - 2\frac{\partial^2 v}{\partial x\partial y} + (2^{\frac{1}{3tgy}} - 23)\frac{\partial u}{\partial x} + e^{|y|+1}u + \frac{\sqrt{y}}{5}v = f(x,y), \\ 2\frac{\partial^2 u}{\partial x\partial y} - y^3lny\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} + (2^{\frac{1}{arcsiny}} + 15)\frac{\partial v}{\partial x} + \frac{\sqrt{y+1}}{5}u + \chi(y)2^{|y|}v = g(x,y), \end{cases}$$

$$w(-\pi,y)=w(\pi,y), w_x(-\pi,y)=w_x(\pi,y), w(x,e^{-\frac{1}{3}})=w(x,1)=0$$

in the rectangle $G_2=\{(x,y)\in\mathbb{R}^2:\ -\pi<x<\pi,\ e^{-\frac{1}{3}}\le y\le 1\}$. Here $\inf\limits_{y\in[e^{-\frac{1}{3}},1]} k(y)=\inf\limits_{y\in[e^{-\frac{1}{3}},1]}(-y^3 lny)=0$ and $\chi(y)$ is an arbitrary function such that $1\le\chi(y)\le 2$ for all $y\in[e^{-\frac{1}{3}},1]$, and $f,g\in L_2(G_2)$.

By Theorem 2.7 this problem has an unique solution $w=(u,v)\in L_2$ for any data $f,g\in L_2(G_2)$. Moreover, the solutions of the above system satisfy the coercive inequality with the norm of space $L_2(G_2)$ in the form (3.13).

We now show that the following functions

$$\begin{aligned} a(y)&=e^{|y|+1},\\ b(y)&=\frac{\sqrt{y}}{5},\\ c(y)&=\frac{\sqrt{y+1}}{5},\\ d(y)&=\chi(y)2^{|y|},\\ \varphi(y)&=2^{\frac{1}{3tgy}}-23,\\ \psi(y)&=2^{\frac{1}{arcsiny}}+15 \end{aligned}$$

satisfy Assumption 1.

Indeed

1)there exists a constant $\delta>0$ such that

$$\inf_{y\in[e^{-\frac{1}{3}},1]}\{-\varphi(y),a(y),d(y)\}=$$
$$\inf_{y\in[e^{-\frac{1}{3}},1]}\left\{-2^{\frac{1}{3tgy}}+23,e^{|y|+1},\chi(y)2^{|y|}\right\}=\delta>0;$$

2) there exist constants $r,q,(r>0,\ q>0,\ r+q=1)$ and $\vartheta(0<\vartheta<3)$ such that

$$\begin{aligned} \tfrac{1}{2}\left(|\tfrac{\sqrt{y}}{5}|+|\tfrac{\sqrt{y+1}}{5}|\right)^{2r}&\le\tfrac{e^{|y|+1}}{3},\\ \tfrac{1}{2}\left(|\tfrac{\sqrt{y}}{5}|+|\tfrac{\sqrt{y+1}}{5}|\right)^{2q}&\le\tfrac{\chi(y)2^{|y|}}{3},\\ \vartheta(2^{\frac{1}{arcsiny}}+15)&>\chi(y)2^{|y|} \end{aligned}$$

for all $y\in[e^{-\frac{1}{3}},1]$.

Also we can see that the functions $a(y),d(y),\varphi(y),\psi(y)$ of $[e^{-\frac{1}{3}},1]$ satisfy the conditions (3.1) and (3.2) of Lemma 3.1. There exist constants $c_0,c_1$ and

$C$ such that

$$\inf_{y,\eta\in[e^{-\frac{1}{3}},1],|y-\eta|\le\frac{1-e^{-\frac{1}{3}}}{2}}\frac{(2^{\frac{1}{3tgy}}-23)^2}{e^{|\eta|+1}}\ge c_0>0,$$

$$\inf_{y,\eta\in[e^{-\frac{1}{3}},1],|y-\eta|\le\frac{1-e^{-\frac{1}{3}}}{2}}\frac{(2^{\frac{1}{arcsiny}}+15)^2}{\chi(\eta)2^{|\eta|}}\ge c_1>0$$

and

$$\sup_{y,\eta\in[e^{-\frac{1}{3}},1],|y-\eta|\le\frac{1-e^{-\frac{1}{3}}}{2}}\left\{\frac{2^{\frac{1}{3tgy}}-23}{2^{\frac{1}{3tg\eta}}-23},\frac{2^{\frac{1}{arcsiny}}+15}{2^{\frac{1}{arcsin\eta}}+15},\right.$$
$$\left.\frac{e^{|y|+1}}{e^{|\eta|+1}},\frac{\chi(y)2^{|y|}}{\chi(\eta)2^{|\eta|}},\frac{e^{|y|+1}}{\chi(\eta)2^{|\eta|}}\right\}\le C<\infty.$$

The function $k(y)=-y^3lny$ is twice continuously differentiable and satisfies the condition (ii) of Lemma 3.3.

Really

1)$k(y)=-y^3lny<1$;

2) $\frac{\sqrt{2}k'(y)}{k(y)}=-\frac{\sqrt{2}(3lny+1)}{ylny}<0$ for all $y\in[e^{-\frac{1}{3}},1)$;

3) And $k''(y)=-6ylny-5y$.

Then we show that

$$\min_{y\in[e^{-\frac{1}{3}},1]}\left\{\varphi^2(y),\psi^2(y)\right\}>2-k''(y),$$

namely

$$\min_{y\in[e^{-\frac{1}{3}},1]}\left\{\left(2^{\frac{1}{3tgy}}-23\right)^2,\left(2^{\frac{1}{arcsiny}}+15\right)^2\right\}>2+y6lny+5y.$$

We can see that

$$\max_{y\in[e^{-\frac{1}{3}},1]}\{2+y6lny+5y\}\le\max_{y\in[e^{-\frac{1}{3}},1]}\{2+5y\}\le 7,$$

since $y>0$ and $lny<0$ on $[e^{-\frac{1}{3}},1]$.

Then

$$\min_{y\in[e^{-\frac{1}{3}},1]}\left\{\left(2^{\frac{1}{3tgy}}-23\right)^2,\left(2^{\frac{1}{arcsiny}}+15\right)^2\right\}>7$$

holds for any $y\in[e^{-\frac{1}{3}},1]$.

**Example 3.9.** *We consider the following problem*

$$\begin{cases}-y\frac{\partial^2u}{\partial x^2}-\frac{\partial^2u}{\partial y^2}-2\frac{\partial^2v}{\partial x\partial y}+(2y^3-17)\frac{\partial u}{\partial x}+\\ \qquad 2^{cosy}u+\frac{1}{10}sinyv=f(x,y),\\ 2\frac{\partial^2u}{\partial x\partial y}-y\frac{\partial^2v}{\partial x^2}-\frac{\partial^2v}{\partial y^2}+\chi(y)(y^3+10)\frac{\partial v}{\partial x}+\\ \qquad \frac{1}{10}cos\frac{y}{5}u+(\frac{1}{2}sin2y+cosy)v=g(x,y),\end{cases}$$

$$w(-\pi, y) = w(\pi, y), w_x(-\pi, y) = w_x(\pi, y), w(x, -1) = w(x, 0) = 0$$

on the rectangle $G_3 = \{(x, y) \in \mathbb{R}^2 : \; -\pi < x < \pi, \; -1 \leq y \leq 0\}$. Here $\inf\limits_{y\in[-1,0]} k(y) = \inf\limits_{y\in[-1,0]} (-y) = 0$ and $\chi(y)$ is an arbitrary function such that $1 \leq \chi(y) \leq 2$ for all $y \in [-1, 0]$, and $f, g \in L_2(G_3)$.

By Theorem 2.7 this problem has an unique solution $w = (u, v) \in L_2$ for any data $f, g \in L_2(G_3)$. Moreover, the solutions of the above system satisfy the coercive inequality with the norm of space $L_2(G_3)$ in the form (3.13).

We now show that the following functions

$$\begin{aligned} a(y) &= 2^{cosy}, \\ b(y) &= \frac{1}{10} siny, \\ c(y) &= \frac{1}{10} cos\frac{y}{5}, \\ d(y) &= \frac{1}{2} sin2y + cosy, \\ \varphi(y) &= 2y^3 - 17, \\ \psi(y) &= \chi(y)(y^3 + 10) \end{aligned}$$

satisfy Assumption 1.

Indeed

1)there exists a constant $\delta > 0$ such that

$$\inf_{y\in[-1,0]} \{-\varphi(y), a(y), d(y)\} = \inf_{y\in[-1,0]} \left\{-2y^3 + 17, 2^{cosy}, \frac{1}{2} sin2y + cosy\right\} = \delta > 0;$$

2) there exist constants $r, q, (r > 0, \; q > 0, \; r + q = 1)$ and $\vartheta(0 < \vartheta < 3)$ such that

$$\begin{gathered} \tfrac{1}{2}\left(|\tfrac{1}{10} siny| + |\tfrac{1}{10} cos\tfrac{y}{5}|\right)^{2r} \leq \tfrac{2^{cosy}}{3}, \\ \tfrac{1}{2}\left(|\tfrac{1}{10} siny| + |\tfrac{1}{10} cos\tfrac{y}{5}|\right)^{2q} \leq \tfrac{\frac{1}{2} sin2y + cosy}{3}, \\ \vartheta\chi(y)(y^3 + 10) > \tfrac{1}{2} sin2y + cosy \end{gathered}$$

for all $y \in [-1, 0]$.

Also we can see that the functions $a(y), d(y), \varphi(y), \psi(y)$ of $[-1, 0]$ satisfy the conditions (3.1) and (3.2) of Lemma 3.1. There exist constants $c_0, c_1$ and

$C$ such that

$$\inf_{y,\eta\in[-1,0],|y-\eta|\le\frac{1}{2}} \frac{(2y^3-17)^2}{2^{cos\eta}} \ge c_0 > 0,$$

$$\inf_{y,\eta\in[-1,0],|y-\eta|\le\frac{1}{2}} \frac{\chi^2(y)(y^3+10)^2}{\frac{1}{2}sin2\eta + cos\eta} \ge c_1 > 0$$

and

$$\sup_{y,\eta\in[-1,0],|y-\eta|\le\frac{1}{2}} \left\{ \frac{2y^3-17}{2\eta^3-17}, \frac{\chi(y)(y^3+10)}{\chi(\eta)(\eta^3+10)}, \frac{2^{cosy}}{2^{cos\eta}}, \frac{\frac{1}{2}sin2y + cosy}{\frac{1}{2}sin2\eta + cos\eta}, \frac{2^{cosy}}{\frac{1}{2}sin2\eta + cos\eta} \right\} \le C < \infty.$$

The function $k(y) = -y$ is twice continuously differentiable and satisfies the condition (iii) of Lemma 3.3.

Really

1)$k(y) = -y < 2$;

2) $k'(y) = -1 < 0$, then $k^2(y) > 2k'(y)$ holds, i.e. $y^2 > -2$ for all $y \in [-1, 0]$;

3) And $k''(y) = 0$.

Then we verify that

$$\min_{y\in[-1,0]} \left\{\varphi^2(y), \psi^2(y)\right\} > 2k'(y) - k''(y).$$

We can see that

$$\max_{y\in[-1,0]} \left\{2k'(y) - k''(y)\right\} \le \max_{y\in[-1,0]} \left\{-2 - y^2\right\} < 0.$$

Then

$$\min_{y\in[-1,0]} \left\{ \left(2y^3 - 17\right)^2, \left(\chi(y)(y^3+10)\right)^2 \right\} > 0$$

holds for any $y \in [-1, 0]$.

# Chapter 4

# The solvability of the semiperiodical problem for second order degenerate system on the strip

We consider the following problem

$$\begin{cases} k(y)\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - 2\frac{\partial^2 v}{\partial x \partial y} + \varphi(y)\frac{\partial u}{\partial x} + a(y)u + b(y)v = f(x,y), \\ 2\frac{\partial^2 u}{\partial x \partial y} + k(y)\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} + \psi(y)\frac{\partial v}{\partial x} + c(y)u + d(y)v = g(x,y), \end{cases} \tag{4.1}$$

$$w(-\pi, y) = w(\pi, y), w_x(-\pi, y) = w_x(\pi, y) \tag{4.2}$$

on the strip $G = \{(x,y) \in \mathbb{R}^2 : \ -\pi < x < \pi, \ -\infty < y < +\infty\}$. Here $k(y)$ is a continuous and bounded real valued function such that $\inf\limits_{y \in \mathbb{R}} k(y) \geq 0$, $f, g \in L_2(G)$. Let functions $\varphi, \psi, a, b, c, d$ be continuous on $\mathbb{R}$.

The system (4.1) can be written in the following form

$$L_0 w = B_{xy} w + P(y) w_x + Q(y) w = F(X), \tag{4.3}$$

here

$$B_{xy} = \begin{pmatrix} k(y)\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} & -2\frac{\partial^2}{\partial x \partial y} \\ 2\frac{\partial^2}{\partial x \partial y} & k(y)\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \end{pmatrix},$$

$$P(y) = \begin{pmatrix} \varphi(y) & 0 \\ 0 & \psi(y) \end{pmatrix},$$

$$Q(y) = \begin{pmatrix} a(y) & b(y) \\ c(y) & d(y) \end{pmatrix},$$

$$U = (u, v)\,,\ F = (f, g)\,,\ X = (x, y).$$

**Assumption 2.** We assume that the real valued functions $\varphi, \psi, a, b, c, d$ on $\mathbb{R}$ satisfy the following conditions

$$\inf_{y\in\mathbb{R}} \{-\varphi(y), a(y), d(y)\} = \delta > 0; \tag{4.4}$$

$$\frac{1}{2}\left(|b(y)| + |c(y)|\right)^{2\alpha} \le \frac{a(y)}{3}, \tag{4.5}$$

$$\frac{1}{2}\left(|b(y)| + |c(y)|\right)^{2\beta} \le \frac{d(y)}{3},$$

$$\vartheta\psi(y) > d(y),$$

where $\alpha, \beta$ and $\vartheta$ are constants such that $\alpha > 0,\ \beta > 0,\ \alpha + \beta = 1,\ \vartheta < 3$.

We denote by $C^2_{\pi,0}(G, \mathbb{R}^2)$ the set of twice continuously differentiable real-valued vector-functions $w = (u, v)$ from
$\tilde{G} = \{(x, y) \in \mathbb{R}^2 :\ -\pi \le x \le \pi,\ \infty < y < \infty\}$ to $\mathbb{R}^2$ satisfy (4.2) which are periodic in the variable $x$ and which have compact support in the variable $y$, for each fixed value of $x$ in $[-\pi, \pi]$. We denote by $L$ the closure under the norm of $L_2(G, \mathbb{R}^2)$ of the differential operator $L_0$ with domain $D(L_0) = C^2_{\pi,0}(G, \mathbb{R}^2)$.

**Definition 4.1.** *A function $w = (u, v) \in L_2(G, \mathbb{R}^2)$ is said to be a solution of the problem (4.1), (4.2), if there exists a sequence $\{w_n\}_{n=1}^{\infty}$ in $C^2_{\pi,0}(G, \mathbb{R}^2)$ such that $\|w_n - w\|_{L_2(G,\mathbb{R}^2)} \to 0$ and $\|Lw_n - F\|_{L_2(G,\mathbb{R}^2)} \to 0$ as $n \to \infty$.*

**Lemma 4.2.** *Let Assumption 2 hold. Then there exists a constant $C_0$ such that the following inequality holds*

$$\|w\|^2_{W_2^1(G,R^2)} = \|w_x\|^2_{2,G} + \|w_y\|^2_{2,G} + \|w\|^2_{2,G} \le C_0 \,\|Lw\|^2_{2,G}\,, \tag{4.6}$$

*for all functions $w = (u, v) \in D(L)$.*

**Proof.** Let $w = (u, v) \in C^2_{\pi,0}(G, \mathbb{R}^2)$ and $\tilde{w} = (-u, v)$. Integrating by parts and exploiting the boundary conditions for the function $w$, we obtain

$$((L_0 + \lambda E)w, w) = \int_G (k(y)u_{xx} - u_{yy} - 2v_{xy} + \varphi(y)u_x +$$
$$a(y)u + \lambda u + b(y)v)\,\bar{u}dxdy +$$

$$\int\limits_G (2u_{xy} + k(y)v_{xx} - v_{yy} + \psi(y)v_x +$$
$$c(y)u + d(y)v + \lambda v)\,\bar{v}dxdy =$$
$$\int\limits_{-\infty}^{+\infty} k(y)\left(\int\limits_{-\pi}^{\pi} u_{xx}\bar{u}dx\right)dy - \int\limits_{-\pi}^{\pi}\left(\int\limits_{-\infty}^{+\infty} u_{yy}\bar{u}dy\right)dx -$$
$$2\int\limits_G v_{xy}\bar{u}dxdy + \int\limits_{-\infty}^{+\infty}\varphi(y)\left(\int\limits_{-\pi}^{\pi} u_x\bar{u}dx\right)dy +$$
$$\int\limits_G a(y)|u|^2dxdy + \int\limits_G b(y)|u||v|dxdy +$$
$$2\int\limits_G u_{xy}\bar{v}dxdy + \int\limits_{-\infty}^{+\infty} k(y)\left(\int\limits_{-\pi}^{\pi} v_{xx}\bar{v}dx\right)dy -$$
$$\int\limits_{-\pi}^{\pi}\left(\int\limits_{-\infty}^{+\infty} v_{yy}\bar{v}dy\right)dx + \int\limits_{-\infty}^{+\infty}\psi(y)\left(\int\limits_{-\pi}^{\pi} v_x\bar{v}dx\right)dy +$$
$$\int\limits_G c(y)|u||v|dxdy + \int\limits_G d(y)|v|^2dxdy +$$
$$\int\limits_G \lambda(|u|^2 + |v|^2)dxdy =$$
$$\int\limits_{-\infty}^{+\infty} k(y)\left(u_x\bar{u}|_{-\pi}^{\pi} - \int\limits_{-\pi}^{\pi}|u_x|^2dx\right)dy -$$
$$\int\limits_{-\pi}^{\pi}\left(u_y\bar{u}|_{-\infty}^{+\infty} - \int\limits_{-\infty}^{+\infty}|u_y|^2dy\right)dx -$$
$$2\int\limits_{-\pi}^{\pi}\left(v_x\bar{u}|_{-\infty}^{+\infty} - \int\limits_{-\infty}^{+\infty} v_xu_ydy\right)dx -$$
$$\int\limits_{-\infty}^{+\infty}\varphi(y)\left(\int\limits_{-\pi}^{\pi} u_x\bar{u}dx\right)dy + \int\limits_G a(y)|u|^2dxdy +$$
$$\int\limits_G b(y)|u||v|dxdy + 2\int\limits_{-\pi}^{\pi}\left(u_x\bar{v}|_{-\infty}^{+\infty} - \int\limits_{-\infty}^{+\infty} u_xv_ydy\right)dx -$$
$$\int\limits_{-\infty}^{+\infty} k(y)\left(v_x\bar{v}|_{-\pi}^{\pi} - \int\limits_{-\pi}^{\pi}|v_x|^2dx\right)dy -$$
$$\int\limits_{-\pi}^{\pi}\left(v_y\bar{v}|_{-\infty}^{+\infty} - \int\limits_{-\infty}^{+\infty}|v_y|^2dy\right)dx +$$

$$\int_{-\infty}^{+\infty} \psi(y) \left( \int_{-\pi}^{\pi} v_x \bar{v} dx \right) dy + \int_G c(y)|u||v| dxdy +$$
$$\int_G d(y)|v|^2 dxdy + \int_G \lambda(|u|^2 + |v|^2) dxdy =$$
$$- \int_G k(y)|u_x|^2 dxdy + \int_G |u_y|^2 dxdy +$$
$$2 \int_G u_y v_x dxdy + \int_{-\infty}^{+\infty} \varphi(y) \left( \frac{u^2}{2} \Big|_{-\pi}^{\pi} \right) dy + \int_G a(y) u^2 dxdy +$$
$$\int_G \lambda u^2 dxdy + \int_G b(y) uv dxdy - 2 \int_G u_y v_x dxdy -$$
$$\int_G k(y)|v_x|^2 dxdy + \int_G |v_y|^2 dxdy + \int_{-\infty}^{+\infty} \psi(y) \left( \frac{v^2}{2} \Big|_{-\pi}^{\pi} \right) dy +$$
$$\int_G c(y) uv dxdy + \int_G d(y) v^2 dxdy + \int_G \lambda v^2 dxdy$$

and

$$((L_0 + \lambda E)w, \tilde{w}_x) = - \int_G u_y (u_y)_x dxdy - 2 \int_G u_{xy} v_x dxdy -$$
$$\int_G \varphi(y) u_x^2 dxdy + \int_G b(y) u v_x dxdy +$$
$$2 \int_G v_x u_{xy} dxdy + \int_G v_x \left(v_x\right)_y dxdy +$$
$$\int_G \psi(y) v_x^2 dxdy + \int_G c(y) u v_x dxdy.$$

Since the functions $u, v, u_x, v_x$ have compact support in the variable $y$, the integrals in the previous equalities are actually integrals are bounded thus converge. Then we obtain

$$((L_0 + \lambda E)w, w) = - \int_G k(y)|w_x|^2 dxdy + \tag{4.7}$$
$$\int_G |w_y|^2 dxdy + \int_G \lambda |w|^2 dxdy +$$
$$\int_G (a(y)u^2 + d(y)v^2) dxdy + \int_G (b(y) + c(y)) uv dxdy,$$

$$((L_0 + \lambda E)w, \tilde{w}_x) = - \int_G \varphi(y) u_x^2 dxdy + \tag{4.8}$$

$$\int\limits_G \psi(y)v_x^2 dxdy + \int\limits_G (b(y)+c(y))uv_x dxdy.$$

By applying the Hölder and Cauchy-Bunyakovski inequalities to the last term of (4.7), we have

$$\left|\int\limits_G (b(y)+c(y))uvdxdy\right| \leq \tag{4.9}$$

$$\int\limits_G |(b(y)+c(y))uv|\, dxdy =$$

$$\int\limits_G |b(y)+c(y)|^{\alpha}|u||b(y)+c(y)|^{\beta}|v|dxdy \leq$$

$$\left(\int\limits_G (|b(y)|+|c(y)|)^{2\alpha}u^2 dxdy\right)^{\frac{1}{2}} \cdot \left(\int\limits_G (|b(y)|+|c(y)|)^{2\beta}v^2 dxdy\right)^{\frac{1}{2}} \leq$$

$$\frac{1}{2}\int\limits_G (|b(y)|+|c(y)|)^{2\alpha}u^2 dxdy + \frac{1}{2}\int\limits_G (|b(y)|+|c(y)|)^{2\beta}v^2 dxdy.$$

Then by arguing on (4.8) so as to obtain (4.9) from (4.7), we have the following inequality

$$\left|\int\limits_G (b(y)+c(y))uv_x dxdy\right| \leq \tag{4.10}$$

$$\frac{1}{2}\int\limits_G (|b(y)|+|c(y)|)^{2\alpha}u^2 dxdy + \frac{1}{2}\int\limits_G (|b(y)|+|c(y)|)^{2\beta}v_x^2 dxdy.$$

By conditions (4.4) and (4.5), and by applying inequalities (4.9) and (4.10), and by applying 'the Cauchy inequality with weight $\epsilon$' for some $\gamma_0 > 0$ and $\epsilon > 0$ to the left hand side of equalities (4.7) and (4.8), respectively, we have

$$\frac{1}{2\gamma_0}\left\|(L_0+\lambda E)w\right\|_{2,G}^2 \geq \tag{4.11}$$

$$-\frac{\gamma_0}{2}\left\|w\right\|_{2,G}^2 - \sup_{y\in\mathbb{R}}|k(y)|\int\limits_G |w_x|^2 dxdy +$$

$$\int\limits_G |w_y|^2 dxdy + \int\limits_G \lambda|w|^2 dxdy +$$

$$\int\limits_G \frac{2a(y)}{3}u^2 dxdy + \int\limits_G \frac{2d(y)}{3}v^2 dxdy$$

and

$$\frac{1}{2\epsilon}\left\|(L_0+\lambda E)\,w\right\|_{2,G}^2 + \frac{\epsilon}{2}\left\|w_x\right\|_{2,G}^2 \geq \tag{4.12}$$

$$\delta\int\limits_G u_x^2dxdy+\int\limits_G \psi(y)v_x^2dxdy-$$
$$\int\limits_G \frac{a(y)}{3}u^2dxdy-\int\limits_G \frac{\vartheta\psi(y)}{3}v_x^2dxdy.$$

By combining inequalities (4.11) and (4.12), we obtain

$$\left(\frac{1}{2\gamma_0}+\frac{1}{2\epsilon}\right)\|(L_0+\lambda E)\,w\|_{2,G}^2\geq$$
$$\left[\frac{\delta}{3}+\lambda-\frac{\gamma_0}{2}\right]\|w\|_{2,G}^2+\|w_y\|_{2,G}^2+$$
$$\left(\delta\left(1-\frac{\vartheta}{3}\right)-\frac{\epsilon}{2}-\sup_{y\in\mathbb{R}}|k(y)|\right)\|w_x\|_{2,G}^2.$$

Hence, for $C_1=\frac{1}{2\gamma_0}+\frac{1}{2\epsilon}$, $C_2=\frac{\delta}{3}+\lambda-\frac{\gamma_0}{2}$, $C_3=\delta\left(1-\frac{\vartheta}{3}\right)-\frac{\epsilon}{2}-\sup\limits_{y\in\mathbb{R}}|k(y)|$ inequality (4.6) follows. Thus the proof of the lemma is complete. □

**Remark 4.3.** *Lemma 4.2 holds, if the condition (4.5) is replaced by the following inequalities*

$$\frac{1}{2}(|b(y)|+|c(y)|)^{2\alpha}\leq\frac{d(y)}{3},$$
$$\frac{1}{2}(|b(y)|+|c(y)|)^{2\beta}\leq\frac{a(y)}{3},$$
$$-\vartheta\varphi(y)>a(y),$$

*where $\alpha,\beta$ and $\vartheta$ are constants such that $\alpha>0,\ \beta>0,\ \alpha+\beta=1,\ \vartheta<3$.*

**Remark 4.4.** *If $b(y)=-c(y)$, then one can prove Lemma 4.2 with condition (4.5) replaced by the following*

$$\inf_{y\in\mathbb{R}}\psi(y)=\delta>0.$$

We now write the functions $f$ and $g$ in the right hand side of (4.1) in the following form

$$f=\sum_{n=-\infty}^{\infty}f_n(y)e^{inx},\ g=\sum_{n=-\infty}^{\infty}g_n(y)e^{inx}. \tag{4.13}$$

We will search for a solution $w=(u,v)$ of the problem (4.1), (4.2) as a limit in the norm of $L_2(G,\mathbb{R}^2)$ the sequence $\{(\tilde{u}_N,\tilde{v}_N)\}_{N=-\infty}^{\infty}$, where

$$\tilde{u}_N=\sum_{n=-N}^{N}u_n(y)e^{inx},\ \tilde{v}_N=\sum_{n=-N}^{N}v_n(y)e^{inx}, \tag{4.14}$$

$$\frac{\partial^2}{\partial x^2}\left(\sum_{n=-N}^{N} u_k(y)e^{ikx}\right) = -\sum_{n=-N}^{N} k^2 u_k(y)e^{ikx},$$

$$\frac{\partial^2}{\partial x^2}\left(\sum_{n=-N}^{N} v_k(y)e^{ikx}\right) = -\sum_{n=-N}^{N} k^2 v_k(y)e^{ikx},$$

$$\frac{\partial}{\partial x}\left(\sum_{n=-N}^{N} u_k(y)e^{ikx}\right) = i\sum_{n=-N}^{N} u_k(y)e^{ikx},$$

$$\frac{\partial}{\partial x}\left(\sum_{n=-N}^{N} v_k(y)e^{ikx}\right) = i\sum_{n=-N}^{N} v_k(y)e^{ikx},$$

$$\frac{\partial}{\partial y}\left(\sum_{n=-N}^{N} u_k(y)e^{ikx}\right) = \sum_{n=-N}^{N} u_k^{'}(y)e^{ikx},$$

$$\frac{\partial^2}{\partial y^2}\left(\sum_{n=-N}^{N} u_k(y)e^{ikx}\right) = \sum_{n=-N}^{N} u_k^{''}(y)e^{ikx},$$

$$\frac{\partial^2 \tilde{u}_N}{\partial y \partial x} = i\sum_{n=-N}^{N} k u_k^{'}(y)e^{ikx},$$

$$\frac{\partial^2 \tilde{v}_N}{\partial y \partial x} = i\sum_{n=-N}^{N} k v_k^{'}(y)e^{ikx}.$$

By replacing $u, v, f, g$ by the corresponding expression of (4.13) and (4.14), we obtain that

$$\begin{cases} -\sum\limits_{n=-N}^{N} u_n^{''}e^{inx} - 2in\sum\limits_{n=-N}^{N} v_n^{'}e^{inx} - n^2k(y)\sum\limits_{n=-N}^{N} u_n(y)e^{inx} + \\ \qquad in\varphi(y)\sum\limits_{n=-N}^{N} u_n(y)e^{inx} + a(y)\sum\limits_{n=-N}^{N} u_n(y)e^{inx} + \\ \qquad b(y)\sum\limits_{n=-N}^{N} v_n(y)e^{inx} = \sum\limits_{n=-N}^{N} f_n(y)e^{inx}, \\ -\sum\limits_{n=-N}^{N} v_n^{''}(y)e^{inx} + 2in\sum\limits_{n=-N}^{N} u_n^{'}(y)e^{inx} + c(y)\sum\limits_{n=-N}^{N} u_n(y)e^{inx} - \\ \qquad n^2k(y)\sum\limits_{n=-N}^{N} u_n(y)e^{inx} + in\psi(y)\sum\limits_{n=-N}^{N} v_n(y)e^{inx} + \\ \qquad d(y)\sum\limits_{n=-N}^{N} v_n(y)e^{inx} = \sum\limits_{n=-N}^{N} g_n(y)e^{inx}, \end{cases}$$

and by equating the coefficients of $e^{inx}$, we obtain the following problem for $w_n = (u_n(y), v_n(y))$ $(n = 0, \pm 1, \pm 2, \ldots)$

$$\begin{cases} -u_n^{''} - 2inv_n^{'} + (-n^2k(y) + in\varphi(y) + a(y))u_n + b(y)v_n = f_n(y), \\ -v_n^{''} + 2inu_n^{'} + c(y)u_n + (-n^2k(y) + in\psi(y) + d(y))v_n = g_n(y). \end{cases} \tag{4.15}$$

Next we consider the operator $l_n + \lambda E$ defined by

$$(l_n + \lambda E)w = -w^{''} + 2inTw^{'} + Q_n(y)w + \lambda w,$$

for all functions $w(y)$ in the space $C_0^2(\mathbb{R}, \mathbb{C}^2)$ of twice continuously differentiable functions $w(y)$ of $\mathbb{R}$ to $\mathbb{C}^2$ with compact support in $\mathbb{R}$. Here

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$Q_n(y) = \begin{pmatrix} -n^2k(y) + in\varphi(y) + a(y) & b(y) \\ c(y) & -n^2k(y) + in\psi(y) + d(y) \end{pmatrix}.$$

We denote also by $l_n+\lambda E$ the closure of $l_n+\lambda E$ in the norm of $L_2 \equiv L_2(\mathbb{R}, \mathbb{C}^2)$.

**Lemma 4.5.** *Let $\lambda \geq 0$. Let Assumption 2 hold. Then there exists a constant $C_0$ such that*

$$\|(l_n + \lambda E)w\|_2^2 \geq C_0 \left[ \int_{\mathbb{R}} \left| w^{'} \right|^2 dy + \int_{\mathbb{R}} \left( \frac{\delta}{3} + \lambda + n^2 \right) |w|^2 dy \right], \qquad (4.16)$$

*for all $w_n = (u_n(y), v_n(y)) \in D(l_n + \lambda E)$, where we denote by $\|\cdot\|_2$ the norm of $L_2 \equiv L_2(\mathbb{R}, \mathbb{C}^2)$.*

**Proof.** Let $w = (u, v) \in C_0^2(\mathbb{R}, \mathbb{C}^2)$. By Assumption 2, we obtain

$$\begin{aligned}
&\operatorname{Im}\left[(-f_n, nu_n) + (g_n, nv_n)\right] = \\
&\quad \operatorname{Im}\left[ \int_{\mathbb{R}} \left\{ u_n^{''} + 2inv_n^{'} - \right.\right. \\
&\quad \left.\left. (-n^2k(y) + in\varphi(y) + a(y))u_n - b(y)v_n \right\} n\bar{u}_n dy \right] + \\
&\quad \operatorname{Im}\left[ \int_{\mathbb{R}} \left\{ -v_n^{''} + 2inu_n^{'} + c(y)u_n + \right.\right. \\
&\quad \left.\left. (-n^2k(y) + in\psi(y) + d(y))v_n \right\} n\bar{v}_n dy \right] = \\
&\quad 2n^2 \int_{\mathbb{R}} v_n^{'} u_n dy - n^2 \int_{\mathbb{R}} \varphi(y) u_n^2 dy + \\
&\quad 2n^2 \int_{\mathbb{R}} u_n^{'} v_n dy + n^2 \int_{\mathbb{R}} \psi(y) v_n^2 dy \geq \\
&\quad 2n^2 \left( u_n v_n|_{-\infty}^{+\infty} - \int_{\mathbb{R}} u_n^{'} v_n dy \right) + 2n^2 \int_{\mathbb{R}} u_n^{'} v_n dy + \\
&\quad \delta n^2 \int_{\mathbb{R}} (u_n^2 + v_n^2) dy = \delta n^2 \|w\|_{2,G}^2,
\end{aligned}$$

or

$$\operatorname{Im}\left[(-f_n, nu_n) + (g_n, nv_n)\right] \geq \delta n^2 \|w\|^2_{2,G}. \tag{4.17}$$

And accordingly

$$\begin{aligned}
&\operatorname{Re}\left((l_n + \lambda E)w, w\right) \geq \\
&\quad \operatorname{Re}\left[\int\limits_{\mathbb{R}} \left\{-u_n^{''} - 2inv_n^{'} + \right.\right. \\
&\quad (-n^2k(y) + in\varphi(y) + a(y) + \lambda)u_n + b(y)v_n\big\} \bar{u}_n dy + \\
&\quad \int\limits_{\mathbb{R}} \left\{-v_n^{''} + 2inu_n^{'} + c(y)u_n + \right. \\
&\quad \left.\left.(-n^2k(y) + in\psi(y) + d(y) + \lambda)v_n\right\} \bar{v}_n dy\right] = \\
&\quad \int\limits_{\mathbb{R}} \left|u_n^{'}\right|^2 dy + \int\limits_{\mathbb{R}} \left(-n^2 \sup_{y\in\mathbb{R}} |k(y)| + \lambda\right) |u_n|^2 dy + \\
&\quad \int\limits_{\mathbb{R}} \left|v_n^{'}\right|^2 dy + \int\limits_{\mathbb{R}} \left(-n^2 \sup_{y\in\mathbb{R}} |k(y)| + \lambda\right) |v_n|^2 dy + \\
&\quad \int\limits_{\mathbb{R}} \left(a(y)u_n^2 + (b(y) + c(y))u_nv_n + d(y)v_n^2\right) dy \geq \\
&\quad \int\limits_{\mathbb{R}} \left|w^{'}\right|^2 dy + \int\limits_{\mathbb{R}} \left(-n^2 \sup_{y\in\mathbb{R}} |k(y)| + \frac{2\delta}{3} + \lambda\right) |w|^2 dy
\end{aligned}$$

or

$$\begin{aligned}
&\operatorname{Re}\left((l_n + \lambda E)w, w\right) \geq \\
&\quad \int\limits_{\mathbb{R}} \left|w^{'}\right|^2 dy + \int\limits_{\mathbb{R}} \left(-n^2 \sup_{y\in\mathbb{R}} |k(y)| + \frac{2\delta}{3} + \lambda\right) |w|^2 dy.
\end{aligned} \tag{4.18}$$

By multiplying both hand sides of (4.17) by $\rho > 0$ and by invoking inequality (4.18), we obtain that

$$\begin{aligned}
&\operatorname{Re}\left((l_n + \lambda E)w, w\right) + \rho \operatorname{Im}\left[(-f_n, nu_n) + (g_n, nv_n)\right] \geq \\
&\quad \int\limits_{\mathbb{R}} \left|w_n^{'}\right|^2 dy + \int\limits_{\mathbb{R}} \left(\rho\delta n^2 + \frac{2\delta}{3} + \lambda - n^2 \sup_{y\in\mathbb{R}} |k(y)|\right) \left|w_n^{'}\right|^2 dy.
\end{aligned}$$

Hence, 'the Cauchy inequality with weight $\epsilon$' implies that

$$\begin{aligned}
&\left(\frac{3}{4\delta} + \frac{\rho}{2\epsilon}\right) \|(l_n + \lambda E)w\|_2^2 \geq \int\limits_{\mathbb{R}} \left|w^{'}\right|^2 dy + \\
&\quad \int\limits_{\mathbb{R}} \left(\rho\delta n^2 + \frac{\delta}{3} + \lambda - n^2 \sup_{y\in\mathbb{R}} |k(y)| - \frac{\rho\epsilon}{2} n^2\right) |w|^2 dy.
\end{aligned}$$

We now choose $\rho$ and $\epsilon$ so that $\rho\delta - \sup\limits_{y\in\mathbb{R}} |k(y)| - \frac{\rho\epsilon}{2} \geq 1$. Hence, the last inequality implies (4.16) for all $w = (u(y), v(y)) \in D(l_n + \lambda E)$ and the proof of lemma is complete. $\square$

Let $\Delta_j = (j-1, j+1)$, $j \in \mathbb{Z}$. We consider the operator $l_{n,j}^{(0)} + \lambda E$ defined by

$$(l_{n,j}^{(0)} + \lambda E)w = -w^{''} + 2inTw^{'} + Q_n(y)w + \lambda w,$$

for all functions $w(y)$ in the space $C^2\left(\bar{\Delta}_j, \mathbb{C}^2\right)$ of twice continuously differentiable functions of $\bar{\Delta}_j = [j-1, j+1]$, $j \in \mathbb{Z}$ to $\mathbb{C}^2$ which satisfy the following conditions

$$w(j-1) = 0, \; w(j+1) = 0. \tag{4.19}$$

We denote by $l_{n,j} + \lambda E$ the closure of $l_{n,j}^{(0)} + \lambda E$ in the norm of $L_2\left(\Delta_j, \mathbb{C}^2\right)$ . By arguing as in the proof of Lemma 4.5, we obtain

$$\|(l_{n,j} + \lambda E)w\|^2_{L_2(\Delta_j, \mathbb{C}^2)} \geq \tag{4.20}$$
$$C_1 \left[ \int\limits_{\Delta_j} \left|w^{'}\right|^2 dy + \int\limits_{\Delta_j} \left(\frac{\delta}{3} + \lambda + n^2\right) |w|^2 \, dy \right],$$

for all functions $w \in D(l_{n,j} + \lambda E)$, where $C_1$ is a constant which does not depend on $j, w, \lambda$.

**Lemma 4.6.** *Let $\lambda \geq 0$. Let Assumption 2 hold. Then the operator $l_{n,j} + \lambda E$ has an inverse, defined on the whole of $L_2\left(\Delta_j, \mathbb{C}^2\right)$. Namely, the operator $(l_{n,j} + \lambda E)^{-1}$.*

**Proof.** The existence of the inverse operator $(l_{n,j} + \lambda E)^{-1}$ is ensured by the inequality (4.20).

We assume by contradiction, that the range $D\left((l_{n,j} + \lambda E)^{-1}\right) = R(l_{n,j} + \lambda E)$ is not dense in $L_2\left(\Delta_j, \mathbb{C}^2\right)$. Then under Lemma 1.3 there exists a nonzero element $U = (p, s)$ in $L_2\left(\Delta_j, \mathbb{C}^2\right)$ such that $((l_{n,j} + \lambda E)w, U) = 0$ for all $w \in D(l_{n,j} + \lambda E)$. Then the density of $D(l_{n,j} + \lambda E)$ in $L_2\left(\Delta_j, \mathbb{C}^2\right)$, and Proposition 1.4 of Chapter 1 implies that $U \in D\left((l_{n,j} + \lambda E)^*\right)$ and $(l_{n,j} + \lambda E)^* U = 0$, where $(l_{n,j} + \lambda E)^*$ is the adjoint operator to $l_{n,j}+\lambda E$, i.e.

$$\begin{cases} -p^{''}+2ins^{'}+(-n^2k(y)-in\varphi(y)+a(y)+\lambda)p+c(y)s=0, \\ -s^{''}-2inp^{'}+b(y)p+(-n^2k(y)+in\psi(y)+d(y)+\lambda)s=0,\ y\in\Delta_j. \end{cases}$$

The function $U=(p,s)$ belongs to the Sobolev space $W_2^2(\Delta_j,\mathbb{C}^2)$ and satisfies the boundary conditions (4.19), as one can show by arguing as in the proof of Lemma 2.6. In order to show that $U=(p,s)$ satisfies the boundary conditions (4.19), we now make a different choice of the 'test' function. We take the following functions

$$w_1(y)=((y-(j-1))^2(y-(j+1)),\sin^k(y-(j-1))(y-(j+1))),$$

$$j=0,\pm1,\pm2,\ldots(k\geq 2,k\in\mathbb{N}),$$

$$w_2(y)=((y-(j-1))(y-(j+1))^2,\sin^k(y-(j-1))(y-(j+1))),$$

$$w_3(y)=((y-(j-1))^2(y-(j+1))^2,(y-(j+1))\sin^k(y-(j-1))),$$

$$w_4(y)=((y-(j-1))^2(y-(j+1))^2,(y-(j-1))\sin^k(y-(j+1))),$$

each of which belongs to $D(l_{n,j}+\lambda E)$ and we substitute them into the equality which correspond to equality (2.20) in our argument.

Since $U=(p,s)\in W_2^2(\Delta_j,\mathbb{C}^2)$, we can argue as in the proof of Lemma 4.5 and obtain the inequality $\|(l_{n,j}+\lambda E)^*U\|_{L_2(\Delta_j,\mathbb{C}^2)}\geq C_3\|U\|_{L_2(\Delta_j,\mathbb{C}^2)}$ for all $U=(p,s)\in D((l_{n,j}+\lambda E)^*)$. Consequently $U=0$, a contradiction. Thus the proof of the lemma is complete. □

**Lemma 4.7.** *Let Assumption 2 hold. Then there exists a number $\lambda_0>0$ such that*

$$\left\|(l_{n,j}+\lambda E)^{-1}\right\|_{L_2(\Delta_j,\mathbb{C}^2)\to L_2(\Delta_j,\mathbb{C}^2)}\leq\frac{\tilde{C}_0}{\sqrt{\lambda}},\tag{4.21}$$

$$\left\|\frac{d}{dy}(l_{n,j}+\lambda E)^{-1}\right\|_{L_2(\Delta_j,\mathbb{C}^2)\to L_2(\Delta_j,\mathbb{C}^2)}\leq\frac{\tilde{C}_1}{\lambda^{1/4}},\tag{4.22}$$

*for all $\lambda\geq\lambda_0$.*

**Proof.** We note that inequality (4.20) implies that

$$\|(l_{n,j}+\lambda E)w\|^2_{L_2(\Delta_j,\mathbb{C}^2)}\geq C_1\lambda\|w\|^2_{L_2(\Delta_j,\mathbb{C}^2)}.$$

Hence,

$$C_1'\sqrt{\lambda}\,\|w\|_{L_2(\Delta_j,\mathbb{C}^2)} \le \|(l_{n,j}+\lambda E)w\|_{L_2(\Delta_j,\mathbb{C}^2)}\,,\ \ w\in D(l_{n,j}+\lambda E).$$

If we set $(l_{n,j}+\lambda E)w = v$, then $w = (l_{n,j}+\lambda E)^{-1}v$ , for all $v \in R(l_{n,j}+\lambda E) = D((l_{n,j}+\lambda E)^{-1})$.

Hence,

$$\frac{\|(l_{n,j}+\lambda E)^{-1}v\|_{L_2(\Delta_j,\mathbb{C}^2)}}{\|v\|_{L_2(\Delta_j,\mathbb{C}^2)}} \le \frac{\tilde{C}_0}{\sqrt{\lambda}},$$

and thus

$$\sup_{v\neq 0, v\in D((l_{n,j}+\lambda E)^{-1})} \frac{\|(l_{n,j}+\lambda E)^{-1}v\|_{L_2(\Delta_j,\mathbb{C}^2)}}{\|v\|_{L_2(\Delta_j,\mathbb{C}^2)}} \le \frac{\tilde{C}_0}{\sqrt{\lambda}},$$

and accordingly (4.21).

Furthermore, (4.18) implies that

$$|((l_{n,j}+\lambda E)w,w)| \ge \left(\lambda + \frac{2\delta}{3} - n^2\sup_{y\in\Delta_j}|k(y)|\right)\|w\|_{L_2(\Delta_j,\mathbb{C}^2)}\,.$$

We now multiply both hand sides of the last inequality by $\frac{1}{\sqrt{\lambda+\frac{2\delta}{3}}} > 0$. Then by applying 'the Cauchy inequality with weight $\epsilon$' for $\epsilon = \gamma > 0$, we obtain that

$$\frac{\gamma}{2\sqrt{\lambda+\frac{2\delta}{3}}}\|(l_{n,j}+\lambda E)w\|^2_{L_2(\Delta_j,\mathbb{C}^2)} + \frac{1}{2\gamma\sqrt{\lambda+\frac{2\delta}{3}}}\|w\|^2_{L_2(\Delta_j,\mathbb{C}^2)} \ge \left(\sqrt{\lambda+\frac{2\delta}{3}} - \frac{n^2\sup\limits_{y\in\Delta_j}|k(y)|}{\sqrt{\lambda+\frac{2\delta}{3}}}\right)\|w\|^2_{L_2(\Delta_j,\mathbb{C}^2)}\,.$$

Hence,

$$\frac{\gamma}{2\sqrt{\lambda+\frac{2\delta}{3}}}\|(l_{n,j}+\lambda E)w\|^2_{L_2(\Delta_j,\mathbb{C}^2)} \ge \left(\sqrt{\lambda+\frac{2\delta}{3}} - \frac{1}{2\gamma\sqrt{\lambda+\frac{2\delta}{3}}} - \frac{n^2\sup\limits_{y\in\Delta_j}|k(y)|}{\sqrt{\lambda+\frac{2\delta}{3}}}\right)\|w\|^2_{L_2(\Delta_j,\mathbb{C}^2)}\,. \tag{4.23}$$

Inequality (4.17) extends to all the functions $w$ in $D(l_{n,j}+\lambda E)$. We now multiply both hand sides of inequality (4.23) by $\frac{1}{\sqrt{\lambda+\frac{2\delta}{3}}} > 0$ and apply 'the Cauchy inequality with weight $\epsilon$' for $\epsilon = \tilde{\mu} > 0$, and we obtain that

$$\frac{\tilde{\mu}}{2\sqrt{\lambda+\frac{2\delta}{3}}}\int\limits_{\Delta_j}\left[|f_n|^2+|g_n|^2\right]dy + \frac{n^2}{2\tilde{\mu}\sqrt{\lambda+\frac{2\delta}{3}}}\|w\|^2_{L_2(\Delta_j,\mathbb{C}^2)} \ge \frac{n^2\delta}{\sqrt{\lambda+\frac{2\delta}{3}}}\|w\|^2_{L_2(\Delta_j,\mathbb{C}^2)}\,.$$

Hence,

$$\frac{\tilde{\mu}}{2\sqrt{\lambda+\frac{2\delta}{3}}}\|(l_{n,j}+\lambda E)w\|^2_{L_2(\Delta_j,\mathbb{C}^2)} \geq \tag{4.24}$$

$$n^2\left(-\frac{1}{2\tilde{\mu}\sqrt{\lambda+\frac{2\delta}{3}}}+\frac{\delta}{\sqrt{\lambda+\frac{2\delta}{3}}}\right)\|w\|^2_{L_2(\Delta_j,\mathbb{C}^2)}.$$

Next we note that inequality (4.18) implies that

$$|((l_{n,j}+\lambda E)w,w)| \geq$$

$$C_1\left[\int_{\Delta_j}|w'|^2dy+\int_{\Delta_j}\left(\frac{2\delta}{3}+\lambda-n^2\sup_{y\in\Delta_j}|k(y)|\right)|w|^2dy\right],$$

for all $w \in D(l_{n,j}+\lambda E)$.

By applying 'the Cauchy inequality with weight $\epsilon$' for some $\epsilon = \frac{\rho}{\sqrt{\lambda+\frac{2\delta}{3}}} > 0$, we obtain

$$\frac{\rho}{2\sqrt{\lambda+\frac{2\delta}{3}}}\|(l_{n,j}+\lambda E)w\|^2_{L_2(\Delta_j,\mathbb{C}^2)}+\frac{\sqrt{\lambda+\frac{2\delta}{3}}}{2\rho}\|w\|^2_{L_2(\Delta_j,\mathbb{C}^2)} \geq \tag{4.25}$$

$$C_1\|w'\|^2_{L_2(\Delta_j,\mathbb{C}^2)}+C_1\left(\frac{2\delta}{3}+\lambda-n^2\sup_{y\in\Delta_j}|k(y)|\right)\|w\|^2_{L_2(\Delta_j,\mathbb{C}^2)}.$$

Now by combining (4.23), (4.24) and (4.25), we have

$$\frac{\tilde{\mu}+\gamma+\rho}{2\sqrt{\lambda+\frac{2\delta}{3}}}\|(l_{n,j}+\lambda E)w\|^2_{L_2(\Delta_j,\mathbb{C}^2)} \geq$$

$$C_1\|w'\|^2_{L_2(\Delta_j,\mathbb{C}^2)}+\left(C_1(\frac{2\delta}{3}+\lambda)+\sqrt{\lambda+\frac{2\delta}{3}}-\right.$$

$$\left.\frac{\sqrt{\lambda+\frac{2\delta}{3}}}{2\rho}-\frac{1}{2\gamma\sqrt{\lambda+\frac{2\delta}{3}}}\right)\|w\|^2_{L_2(\Delta_j,\mathbb{C}^2)}+$$

$$n^2\left(\frac{\delta}{\sqrt{\lambda+\frac{2\delta}{3}}}-C_1\sup_{y\in\Delta_j}|k(y)|-\right.$$

$$\left.\frac{1}{2\tilde{\mu}\sqrt{\lambda+\frac{2\delta}{3}}}-\frac{n^2\sup_{y\in\Delta_j}|k(y)|}{\sqrt{\lambda+\frac{2\delta}{3}}}\right)\|w\|^2_{L_2(\Delta_j,\mathbb{C}^2)}.$$

Hence, there exists a number $\lambda_0 > 0$ such that

$$\frac{C_2'}{\sqrt{\lambda+\frac{2\delta}{3}}}\|(l_{n,j}+\lambda E)w\|^2_{L_2(\Delta_j,\mathbb{C}^2)} \geq C_1\|w'\|^2_{L_2(\Delta_j,\mathbb{C}^2)},$$

for all $\lambda \geq \lambda_0$, where $C_2' = \frac{\tilde{\mu}+\gamma+\rho}{2}$ and thus

$$\frac{\|w'\|_{L_2(\Delta_j,\mathbb{C}^2)}}{\|(l_{n,j}+\lambda E)w\|_{L_2(\Delta_j,\mathbb{C}^2)}} \leq \frac{\tilde{C}_2}{\sqrt[4]{\lambda + \frac{2\delta}{3}}},$$

for all $w \in D(l_{n,j}+\lambda E)$. If we set $(l_{n,j}+\lambda E)w = v$, then $w' = \frac{d}{dy}(l_{n,j}+\lambda E)^{-1}v$, for all $v \in \mathbb{R}(l_{n,j}+\lambda E) = D((l_{n,j}+\lambda E)^{-1})$ and the last inequality implies that

$$\frac{\|\frac{d}{dy}(l_{n,j}+\lambda E)^{-1}v\|_{L_2(\Delta_j,\mathbb{C}^2)}}{\|v\|_{L_2(\Delta_j,\mathbb{C}^2)}} \leq \frac{\tilde{C}_2}{\sqrt[4]{\lambda + \frac{2\delta}{3}}},$$

that is equivalent to (4.22). The proof of the lemma is complete. □

Let $\theta_1, \theta_2, \ldots$ be non-negative functions in $C^2_{\pi,0}(G,\mathbb{R}^2)$ such that

$$0 \leq \theta_j(y) < 1,\ supp\,\theta_j \in \Delta_j, j \in \mathbb{Z},\ \sum_{j=-\infty}^{\infty} \theta_j^2(y) = 1.$$

We assume that the conditions of Lemma 4.6 hold. We now introduce the operators $K$, $M_{s,n}(\lambda)(s = 1,2,3)$ defined by

$$KF = \sum_{j=-\infty}^{\infty} \theta_j(y)(l_{n,j}+\lambda E)^{-1}\theta_j F,\ F \in L_2(G,\mathbb{R}^2),$$

$$M_{1,n}(\lambda)F = -\sum_{j=-\infty}^{\infty} \theta_j''(y)(l_{n,j}+\lambda E)^{-1}\theta_j F,$$

$$M_{2,n}(\lambda)F = -2\sum_{j=-\infty}^{\infty} \theta_j' \frac{d}{dy}(l_{n,j}+\lambda E)^{-1}\theta_j F,$$

$$M_{3,n}(\lambda)F = 2in\sum_{j=-\infty}^{\infty} \theta_j' T(l_{n,j}+\lambda E)^{-1}\theta_j F.$$

By virtue of properties of the functions $\theta_1, \theta_2, \ldots$ at each point $y \in \mathbb{R}$ the right hand side of these expressions consists of finite numbers terms (no more then three). Since $l_{n,j}$ is the restriction of the operator $l_n$ to $\Delta_j$, then
$KF \in D(l_n+\lambda E)$ and

$$(l_n+\lambda E)KF = \sum_{j=-\infty}^{\infty} (l_{n,j}+\lambda E)[\theta_j(y)(l_{n,j}+\lambda E)^{-1}\theta_j F] =$$
$$\sum_{j=-\infty}^{\infty} \Big\{ -[\theta_j(y)(l_{n,j}+\lambda E)^{-1}\theta_j F]'' +$$

$$2inT\left(\theta_j(l_{n,j}+\lambda E)^{-1}\theta_j F\right)^{'}+$$
$$Q_n(y)[\theta_j(y)(l_{n,j}+\lambda E)^{-1}\theta_j F]+$$
$$\lambda E[\theta_j(y)(l_{n,j}+\lambda E)^{-1}\theta_j F]\Big\}=$$
$$\sum_{j=-\infty}^{\infty}\Big\{-\theta_j^{''}(y)(l_{n,j}+\lambda E)^{-1}\theta_j F-$$
$$2\theta_j^{'}\frac{d}{dy}(l_{n,j}+\lambda E)^{-1}\theta_j F-$$
$$\theta_j\frac{d^2}{dy^2}(l_{n,j}+\lambda E)^{-1}\theta_j F+$$
$$2inT\theta_j^{'}(y)(l_{n,j}+\lambda E)^{-1}\theta_j F+$$
$$2inT\theta_j(y)\frac{d}{dy}(l_{n,j}+\lambda E)^{-1}\theta_j F+$$
$$\theta_j(y)Q_n(y)(l_{n,j}+\lambda E)^{-1}\theta_j F+$$
$$\theta_j[\lambda E(l_{n,j}+\lambda E)^{-1}\theta_j F]\Big\}=$$
$$\sum_{j=-\infty}^{\infty}\Big\{-\theta_j\frac{d^2}{dy^2}(l_{n,j}+\lambda E)^{-1}\theta_j F+$$
$$2inT\theta_j(y)\frac{d}{dy}(l_{n,j}+\lambda E)^{-1}\theta_j F+$$
$$\theta_j(y)Q_n(y)(l_{n,j}+\lambda E)^{-1}\theta_j F+$$
$$\theta_j[\lambda E(l_{n,j}+\lambda E)^{-1}\theta_j F]\Big\}-$$
$$\sum_{j=-\infty}^{\infty}\theta_j^{''}(y)(l_{n,j}+\lambda E)^{-1}\theta_j F-$$
$$2\sum_{j=-\infty}^{\infty}\theta_j^{'}\frac{d}{dy}(l_{n,j}+\lambda E)^{-1}\theta_j F+$$
$$2in\sum_{j=-\infty}^{\infty}T\theta_j^{'}(y)(l_{n,j}+\lambda E)^{-1}\theta_j F=$$
$$\sum_{j=-\infty}^{\infty}\theta_j(y)(l_{n,j}+\lambda E)(l_{n,j}+\lambda E)^{-1}\theta_j F-$$
$$\sum_{j=-\infty}^{\infty}\theta_j^{''}(y)(l_{n,j}+\lambda E)^{-1}\theta_j F-$$
$$2\sum_{j=-\infty}^{\infty}\theta_j^{'}\frac{d}{dy}(l_{n,j}+\lambda E)^{-1}\theta_j F+$$
$$2in\sum_{j=-\infty}^{\infty}T\theta_j^{'}(y)(l_{n,j}+\lambda E)^{-1}\theta_j F=$$
$$F+M_{1,n}(\lambda)F+M_{2,n}(\lambda)F+M_{3,n}(\lambda)F.$$

By our assumptions on the functions $\theta_k(k=1,2,\ldots)$ and by Lemmas 4.5, 4.6, we can estimate the norms of operators $M_{s,n}(\lambda)(s=1,2,3)$ as follows

$$\|M_{1,n}(\lambda)F\|^2_{L_2} =$$
$$\int\limits_{\mathbb{R}} \left|\sum_{k=-\infty}^{\infty} \theta''_k(y)(l_{n,k}+\lambda E)^{-1}\theta_k F\right|^2 dy =$$
$$\sum_{k=-\infty}^{\infty}\int\limits_{\Delta_k} \Big|\ldots + \theta''_{-3}(y)(l_{n,-3}+\lambda E)^{-1}\theta_{-3}F(y)+$$
$$\theta''_{-2}(y)(l_{n,-2}+\lambda E)^{-1}\theta_{-2}F(y)+$$
$$\theta''_{-1}(y)(l_{n,-1}+\lambda E)^{-1}\theta_{-1}F(y)+$$
$$\theta''_{0}(y)(l_{n,0}+\lambda E)^{-1}\theta_{0}F(y)+$$
$$\theta''_{1}(y)(l_{n,1}+\lambda E)^{-1}\theta_{1}F(y)+$$
$$\theta''_{2}(y)(l_{n,2}+\lambda E)^{-1}\theta_{2}F(y)+\ldots\Big|^2 dy.$$

Hence, by inequality $(A+B+C)^2 \le 3(A^2+B^2+C^2)$, which holds for any numbers $A,B,C$, we obtain the following inequality

$$\|M_{1,n}(\lambda)F\|^2_{L_2} =$$
$$\sum_k \int\limits_{k-1}^{k+1} \Big|\theta''_{k-1}(y)(l_{n,k-1}+\lambda E)^{-1}\theta_{k-1}F(y)+$$
$$\theta''_{k}(y)(l_{n,k}+\lambda E)^{-1}\theta_{k}F(y)+$$
$$\theta''_{k+1}(y)(l_{n,k+1}+\lambda E)^{-1}\theta_{k+1}F(y)\Big|^2 dy \le$$
$$3\sum_k \int\limits_{k-1}^{k+1} \Big|[\theta''_{k-1}(y)(l_{n,k-1}+\lambda E)^{-1}\theta_{k-1}F(y)]^2+$$
$$[\theta''_{k}(y)(l_{n,k}+\lambda E)^{-1}\theta_{k}F(y)]^2+$$
$$[\theta''_{k+1}(y)(l_{n,k+1}+\lambda E)^{-1}\theta_{k+1}F(y)]^2\Big| dy \le$$
$$3\sum_{k+1=-\infty}^{\infty}\int\limits_{k-1}^{k+1}[\theta''_k(y)(l_{n,k}+\lambda E)^{-1}\theta_k F(y)]^2 dy +$$
$$3\sum_{k=-\infty}^{\infty}\int\limits_{k-1}^{k+1}[\theta''_k(y)(l_{n,k}+\lambda E)^{-1}\theta_k F(y)]^2 dy +$$
$$3\sum_{k-1=-\infty}^{\infty}\int\limits_{k-1}^{k+1}[\theta''_k(y)(l_{n,k}+\lambda E)^{-1}\theta_k F(y)]^2 dy \le$$

$$9\sum_k \int\limits_{k-1}^{k+1} [\theta_k''(y)(l_{n,k}+\lambda E)^{-1}\theta_k F(y)]^2 dy.$$

We now set $C_7 = 9\max\limits_k \max\limits_{y\in\delta_k}\left(|\theta_k''(y)|^2\right)$, and thus we obtain

$$\|M_{1,n}(\lambda)F\|_{L_2}^2 \le$$
$$C_7\sum_{k=-\infty}^{\infty}\int\limits_{\Delta_k} |(l_{n,k}+\lambda E)^{-1}\theta_k F(y)|^2 dy \le$$
$$C_7\sum_{k=-\infty}^{\infty} \|(l_{n,k}+\lambda E)^{-1}\theta_k F(y)\|_{L_2(\Delta_k,\mathbb{C}^2)}^2.$$

Hence, by inequality (4.21), we have

$$\|M_{1,n}(\lambda)F\|_{L_2}^2 \le \frac{C_7'}{\lambda}\sum_{k=-\infty}^{\infty}\|\theta_k F(y)\|_{L_2(\Delta_k,\mathbb{C}^2)}^2 \le$$
$$\frac{C_7'}{\lambda}\sum_{k=-\infty}^{\infty}\|\theta_k F(y)\|_{L_2(\mathbb{R},\mathbb{C}^2)}^2.$$

Thus

$$\|M_{1,n}(\lambda)F\|_{L_2}^2 \le \frac{\tilde{C}_7'}{\lambda}\|F(y)\|_{L_2}^2, \quad \tilde{C}_7' = 6C_7'.$$

Moreover

$$\|M_{3,n}(\lambda)F\|_{L_2}^2 =$$
$$\int\limits_{\mathbb{R}} \left|2in\sum_{k=-\infty}^{\infty}\theta_k' T(l_{n,k}+\lambda E)^{-1}\theta_k F(y)\right|^2 dy =$$
$$4n^2\sum_{k=-\infty}^{\infty}\int\limits_{\Delta_k}\left|\sum_{k=-\infty}^{\infty}\theta_k' T(l_{n,k}+\lambda E)^{-1}\theta_k F(y)\right|^2 dy =$$
$$4n^2\sum_{k=-\infty}^{\infty}\int\limits_{\Delta_k}\Big|\theta_{k-1}' T(l_{n,k-1}+\lambda E)^{-1}\theta_{k-1}F(y)+$$
$$\theta_k' T(l_{n,k}+\lambda E)^{-1}\theta_k F(y)+$$
$$\theta_{k+1}' T(l_{n,k+1}+\lambda E)^{-1}\theta_{k+1}F(y)\Big|^2 dy \le$$
$$12n^2\sum_{k=-\infty}^{\infty}\int\limits_{\Delta_k}\left[\theta_{k-1}' T(l_{n,k-1}+\lambda E)^{-1}\theta_{k-1}F(y)\right]^2 dy +$$
$$12n^2\sum_{k=-\infty}^{\infty}\int\limits_{\Delta_k}\left[\theta_k' T(l_{n,k}+\lambda E)^{-1}\theta_k F(y)\right]^2 dy +$$
$$12n^2\sum_{k=-\infty}^{\infty}\int\limits_{\Delta_k}\left[\theta_{k+1}' T(l_{n,k+1}+\lambda E)^{-1}\theta_{k+1}F(y)\right]^2 dy \le$$

$$12n^2 \sum_{k+1=-\infty}^{\infty} \int_{k-1}^{k+1} \left[\theta_k' T(l_{n,k}+\lambda E)^{-1}\theta_k F(y)\right]^2 dy +$$

$$12n^2 \sum_{k=-\infty}^{\infty} \int_{k-1}^{k+1} \left[\theta_k' T(l_{n,k}+\lambda E)^{-1}\theta_k F(y)\right]^2 dy +$$

$$12n^2 \sum_{k-1=-\infty}^{\infty} \int_{k-1}^{k+1} \left[\theta_k' T(l_{n,k}+\lambda E)^{-1}\theta_k F(y)\right]^2 dy \leq$$

$$36n^2 \sum_{k=-\infty}^{\infty} \int_{k-1}^{k+1} \left[\theta_k' T(l_{n,k}+\lambda E)^{-1}\theta_k F(y)\right]^2 dy \leq$$

$$36n^2 \max_k \max_{y\in\delta_k} |\theta_k'(y)|^2 \sum_{k=-\infty}^{\infty} \int_{\Delta_k} \left|T(l_{n,k}+\lambda E)^{-1}\theta_k F(y)\right|^2 dy \leq$$

$$C_8 \sum_{k=-\infty}^{\infty} \|T(l_{n,k}+\lambda E)^{-1}\theta_k F(y)\|^2_{L_2(\Delta_k,\mathbb{C}^2)}.$$

Consequently

$$\|M_{3,n}(\lambda)F\|^2_{L_2} \leq$$
$$C_8 \sum_{k=-\infty}^{\infty} \|T\|^2_{L_2(\Delta_k,\mathbb{C}^2)} \|(l_{n,k}+\lambda E)^{-1}\|^2_{L_2(\Delta_k,\mathbb{C}^2)} \|\theta_k F(y)\|^2_{L_2(\Delta_k,\mathbb{C}^2)}.$$

Then by inequality (4.21), we obtain

$$\|M_{3,n}(\lambda)F\|^2_{L_2} \leq$$
$$\frac{\tilde{C}_8}{\lambda} \sum_{k=-\infty}^{\infty} \|\theta_k F(y)\|^2_{L_2(\Delta_k,\mathbb{C}^2)} \leq$$
$$\frac{\tilde{C}_8}{\lambda} \sum_{k=-\infty}^{\infty} \|\theta_k F(y)\|^2_{L_2(\mathbb{R},\mathbb{C}^2)}.$$

Hence,

$$\|M_{3,n}(\lambda)F\|^2_{L_2} \leq \frac{\tilde{C}_8'}{\lambda} \|F(y)\|^2_{L_2}, \quad \tilde{C}_8' = 6C_8'.$$

We now consider $M_{2,n}(\lambda)F$.

$$\|M_{2,n}(\lambda)F\|^2_{L_2} =$$
$$\int_{\mathbb{R}} \left|2\sum_{k=-\infty}^{\infty} \theta_k' \frac{d}{dy}(l_{n,k}+\lambda E)^{-1}\theta_k F(y)\right|^2 dy \leq$$
$$4\sum_{k=-\infty}^{\infty} \int_{\Delta_k} \left|\sum_{k=-\infty}^{\infty} \theta_k' \frac{d}{dy}(l_{n,k}+\lambda E)^{-1}\theta_k F(y)\right|^2 dy \leq$$

$$4\sum_{k=-\infty}^{\infty}\int_{k-1}^{k+1}\Big|\theta_{k-1}^{'}\frac{d}{dy}(l_{n,k-1}+\lambda E)^{-1}\theta_{k-1}F(y)+$$

$$\theta_{k}^{'}\frac{d}{dy}(l_{n,k}+\lambda E)^{-1}\theta_{k}F(y)+$$

$$\theta_{k+1}^{'}\frac{d}{dy}(l_{n,k+1}+\lambda E)^{-1}\theta_{k+1}F(y)\Big|^2\,dy\le$$

$$12\sum_{k=-\infty}^{\infty}\int_{k-1}^{k+1}\Bigg|\left(\theta_{k-1}^{'}\frac{d}{dy}(l_{n,k-1}+\lambda E)^{-1}\theta_{k-1}F(y)\right)^2+$$

$$\left(\theta_{k}^{'}\frac{d}{dy}(l_{n,k}+\lambda E)^{-1}\theta_{k}F(y)\right)^2+$$

$$\left(\theta_{k+1}^{'}\frac{d}{dy}(l_{n,k+1}+\lambda E)^{-1}\theta_{k+1}F(y)\right)^2\Bigg|\,dy\le$$

$$12\sum_{k+1=-\infty}^{\infty}\int_{k-1}^{k+1}\left(\theta_{k}^{'}\frac{d}{dy}(l_{n,k}+\lambda E)^{-1}\theta_{k}F(y)\right)^2+$$

$$12\sum_{k=-\infty}^{\infty}\int_{k-1}^{k+1}\left(\theta_{k}^{'}\frac{d}{dy}(l_{n,k}+\lambda E)^{-1}\theta_{k}F(y)\right)^2dy+$$

$$12\sum_{k-1=-\infty}^{\infty}\int_{k-1}^{k+1}\left(\theta_{k}^{'}\frac{d}{dy}(l_{n,k}+\lambda E)^{-1}\theta_{k}F(y)\right)^2dy\le$$

$$36\sum_{k=-\infty}^{\infty}\int_{k-1}^{k+1}\left(\theta_{k}^{'}\frac{d}{dy}(l_{n,k}+\lambda E)^{-1}\theta_{k}F(y)\right)^2dy\le$$

$$36\max_{k}\max_{y\in\delta_k}|\theta_k^{'}|^2\sum_{k=-\infty}^{\infty}\int_{k-1}^{k+1}\left|\frac{d}{dy}(l_{n,k}+\lambda E)^{-1}\theta_{k}F(y)\right|^2dy\le$$

$$C_9\sum_{k=-\infty}^{\infty}\|\frac{d}{dy}(l_{n,k}+\lambda E)^{-1}\theta_{k}F(y)\|^2_{L_2(\Delta_k,\mathbb{C}^2)}.$$

Hence, by inequality (4.22), we obtain

$$\|M_{2,n}(\lambda)F\|^2_{L_2}\le$$

$$\frac{C_9^{'}}{\sqrt{\lambda}}\sum_{k=-\infty}^{\infty}\|\theta_kF(y)\|^2_{L_2(\Delta_k,\mathbb{C}^2)}\le$$

$$\frac{C_9^{'}}{\sqrt{\lambda}}\sum_{k=-\infty}^{\infty}\|\theta_kF(y)\|^2_{L_2(\mathbb{R},\mathbb{C}^2)}\le\frac{\tilde{C}_9^{'}}{\sqrt{\lambda}}\|F(y)\|^2_{L_2},$$

where $\tilde{C_9^{'}}=6C_9^{'}$. Consequently

$$\|M_{1,n}(\lambda)+M_{2,n}(\lambda)+M_{3,n}(\lambda)\|_{L_2\to L_2}=\|S_n(\lambda)\|_{L_2\to L_2}\le\frac{C_{10}}{\sqrt{\lambda}},$$

for $\lambda > 1$. Therefore, there exists a number $\lambda_0 > 1$ such that

$$\|S_n(\lambda)\|_{L_2\to L_2} \leq \frac{1}{2},$$

$$\frac{1}{2} \leq \|E + S_n(\lambda)\|_{L_2\to L_2} \leq \frac{3}{2}$$

and

$$\|(E + S_n(\lambda))^{-1}\|_{L_2\to L_2} \leq 2,$$

for all $\lambda \geq \lambda_0$. Hence, it follows that $E + S_n(\lambda) : L_2(\mathbb{R}, \mathbb{C}^2) \to L_2(\mathbb{R}, \mathbb{C}^2)$ is an one-to-one map. We set $(E + S_n(\lambda))F = h$. Clearly,

$$(l_n + \lambda E)K(E + S_n(\lambda))^{-1}h = h,$$

for all $h \in L_2(\mathbb{R}, \mathbb{C}^2)$, $\lambda \geq \lambda_0$. Thus, the operator $l_n + \lambda E$ has an inverse for all $\lambda \geq \lambda_0$, and the inverse operator $K(E + S_n(\lambda))^{-1}$ is defined on the whole of $L_2(\mathbb{R}, \mathbb{C}^2)$. Hence, and inequality (4.20), and the well-known Theorem 1.2 [1, p. 92], implies that

**Lemma 4.8.** *Let Assumption 2 hold. Then the operator $l_n$ has an inverse, defined on the whole of $L_2(\mathbb{R}, \mathbb{C}^2)$. Namely, the operator $l_n^{-1}$.*

We now have the main statement of this chapter.

**Theorem 4.9.** *Let the coefficients of system (4.1) satisfy Assumption 2. Then the problem (4.1),(4.2) has an unique solution $w = (u, v)$ in the Sobolev space $W_2^1(G, \mathbb{R}^2)$ for every right hand side $F = (f, g) \in L_2(G, \mathbb{R}^2)$.*

**Proof.** Let $(u_n, v_n)$ $(n \in \mathbb{Z})$ be a solution of system (4.15). Then the function $w_N = \left(\sum\limits_{k=-N}^{N} u_k(y)e^{ikx}, \sum\limits_{k=-N}^{N} v_k(y)e^{ikx}\right)$ is the solution of problem (4.1), (4.2), where $F(x, y)$ is replaced on $F_N = \left(\sum\limits_{k=-N}^{N} f_k(y)e^{ikx}, \sum\limits_{k=-N}^{N} g_k(y)e^{ikx}\right)$. Since the sequence $\{F_N\}$ converges to the right hand side of system (4.1), it is a Cauchy sequence. Then by inequality (4.6), $\{w_N\}_{N=-\infty}^{\infty}$ is a Cauchy sequence also in $W_2^1(G, \mathbb{R}^2)$. Since $W_2^1(G, \mathbb{R}^2)$ is complete, then the sequence $\{w_N\}_{N=-\infty}^{\infty}$ has a limit $w = (u, v) \in W_2^1(G, \mathbb{R}^2)$. By definition $w = (u, v)$ is a solution of problem (4.1), (4.2). The uniqueness of the solution follows by inequality (4.6). Hence, the proof is complete. $\square$

# Chapter 5

# A coercive estimate for the solutions of a singular degenerate system

Let $\Delta_j = (j-1, j+1)$, $j \in \mathbb{Z}$. Let $\lambda, \hat{\lambda}$ be constants such that $\hat{\lambda} \geq \lambda \geq 0$. Let

$$\breve{E} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For each $j \in \mathbb{Z}$ we consider the operator $l_{n,j}^{(0)} + in\hat{\lambda}\breve{E} + \lambda E$ defined by

$$(l_{n,j}^{(0)} + in\hat{\lambda}\breve{E} + \lambda E)w = -w^{''} + 2inTw^{'} + Q_n(y)w + in\hat{\lambda}\tilde{w} + \lambda w,$$

for all functions $w = (u, v)$ in the space $C^2\left(\bar{\Delta}_j, \mathbb{C}^2\right)$ of twice continuously differentiable functions of $\bar{\Delta}_j = [j-1, j+1]$, $j \in \mathbb{Z}$ to $\mathbb{C}^2$ which satisfy the boundary conditions (4.19), where $\tilde{w} = (-u, v)$, $T$ and $Q_n(y)$ are the matrices associated to system (4.15).

We denote by $l_{n,j} + in\hat{\lambda}\breve{E} + \lambda E$ the closure of the operator $l_{n,j}^{(0)} + in\hat{\lambda}\breve{E} + \lambda E$ in the norm of $L_2 \equiv L_2(\Delta_j, \mathbb{C}^2)$ . We denote by $\varphi_j(y)$, $\psi_j(y)$, $a_j(y)$, $b_j(y)$, $c_j(y)$, $d_j(y)$ the extensions to $\mathbb{R}$ of the restrictions of functions $\varphi(y)$, $\psi(y)$, $a(y)$, $b(y)$, $c(y)$, $d(y)$ to $\Delta_j$ with period 2.

**Lemma 5.1.** *Let $\hat{\lambda} \geq \lambda \geq 0$. Let Assumption 2 hold. Let*

$$\inf_{y,\eta \in \mathbb{R}, |y-\eta| \leq 2} \frac{\varphi^2(y)}{a(\eta)} \geq c_0 > 0, \tag{5.1}$$

$$\inf_{y,\eta\in\mathbb{R},|y-\eta|\le 2}\frac{\psi^2(y)}{d(\eta)}\ge c_1>0.$$

*Then there exist constants $C_1, C_2$ such that*

$$\left\|\left(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E\right)^{-1}\right\|_{L_2(\Delta_j,\mathbb{C}^2)\to L_2(\Delta_j,\mathbb{C}^2)}\le\frac{C_1}{|n|\lambda}, \tag{5.2}$$

$$\left\|\frac{d}{dy}\left(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E\right)^{-1}\right\|_{L_2(\Delta_j,\mathbb{C}^2)\to L_2(\Delta_j,\mathbb{C}^2)}\le\frac{C_2}{\sqrt{\lambda}}. \tag{5.3}$$

**Proof.** Let Assumption 2 hold. By arguing as in the proof of Lemma 4.6 one can prove that the operator $l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E$ is bounded and invertible. Let $w=(u,v)\in C^2\left(\bar{\Delta}_j,\mathbb{C}^2\right)$. We denote by $\gamma_j=\left(\inf_{y\in\Delta_j}(-\varphi_j(y)),\ \inf_{y\in\Delta_j}\psi_j(y),\ \inf_{y\in\Delta_j}a_j(y),\ \inf_{y\in\Delta_j}d_j(y)\right)$. Then

$$\begin{aligned}&\mathrm{Re}\left((l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)w,w\right)\ge \\ &\int_{\Delta_j}\left|u_n'\right|^2dy+\int_{\Delta_j}\left(\lambda-n^2\sup_{y\in\Delta_j}|k(y)|\right)|u_n|^2\,dy+ \\ &\int_{\Delta_j}\left|v_n'\right|^2dy+\int_{\Delta_j}\left(\lambda-n^2\sup_{y\in\Delta_j}|k(y)|\right)|v_n|^2dy+ \\ &\int_{\Delta_j}\left((a_j(y))u_n^2+(b_j(y)+c_j(y))u_nv_n+d_j(y))v_n^2\right)dy\ge \\ &\int_{\Delta_j}\left|w'\right|^2dy+\int_{\Delta_j}\left(\frac{2\gamma_j}{3}+\lambda-n^2\sup_{y\in\Delta_j}|k(y)|\right)|w|^2\,dy,\end{aligned} \tag{5.4}$$

and accordingly

$$\begin{aligned}&\left|\left((l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)w,w\right)\right|\ge \\ &\left|\left((l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)w,\tilde{w}\right)\right|\ge|n|(\gamma_j+\hat{\lambda})\|w\|^2_{L_2(\Delta_j,\mathbb{C}^2)}.\end{aligned} \tag{5.5}$$

Hence, we obtain the following inequalities

$$\begin{aligned}&\|w'\|^2_{L_2(\Delta_j,\mathbb{C}^2)}+ \\ &C_1\left(\frac{2\gamma_j}{3}+\lambda-n^2\sup_{y\in\Delta_j}|k(y)|\right)\|w\|^2_{L_2(\Delta_j,\mathbb{C}^2)}- \\ &\frac{\frac{2\gamma_j}{3}+\lambda}{4C_0}\|w\|^2_{L_2(\Delta_j,\mathbb{C}^2)}\le\frac{C_0}{\frac{2\gamma_j}{3}+\lambda}\left\|(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)w\right\|^2_{L_2(\Delta_j,\mathbb{C}^2)},\end{aligned} \tag{5.6}$$

$$\begin{aligned}&|n|\frac{\gamma_j+\hat{\lambda}}{\sqrt{\frac{2\gamma_j}{3}+\lambda}}\|w\|_{L_2(\Delta_j,\mathbb{C}^2)}\le \\ &\frac{1}{\sqrt{\frac{2\gamma_j}{3}+\lambda}}\left\|(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)w\right\|_{L_2(\Delta_j,\mathbb{C}^2)}.\end{aligned} \tag{5.7}$$

Hence, inequality (5.2) follows. By taking the square of both hand sides of inequality (5.7) and by inequality (5.6), we obtain

$$\|w^{'}\|^2_{L_2(\Delta_j,\mathbb{C}^2)} + \left(\frac{2\gamma_j}{3}+\lambda\right)\left(1-\frac{1}{4C_0}\right)\|w\|^2_{L_2(\Delta_j,\mathbb{C}^2)} + \tag{5.8}$$
$$n^2\left(\frac{C_1(\gamma_j+\hat{\lambda})^2}{\frac{2\gamma_j}{3}+\lambda} - \sup_{y\in\Delta_j}|k(y)|\right)\|w\|^2_{L_2(\Delta_j,\mathbb{C}^2)} \le$$
$$\frac{C_0+C_1}{\frac{2\gamma_j}{3}+\lambda}\left\|(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)w\right\|^2_{L_2(\Delta_j,\mathbb{C}^2)}.$$

We now choose $C_0$ so that $C_0 > \frac{1}{4}$. Hence, by condition (5.1) inequality (5.3) holds for all $\hat{\lambda} \ge \lambda \ge 0$ and the proof is complete. □

We now consider the operator $l_n + in\hat{\lambda}\breve{E} + \lambda E$ defined by

$$(l_n + in\hat{\lambda}\breve{E} + \lambda E)w = -w^{''} + 2inTw^{'} + Q_n(y)w + in\hat{\lambda}\tilde{w} + \lambda w,$$

for all functions $w = (u, v)$ in the space $C^2(\mathbb{R}, \mathbb{C}^2)$ of twice continuously differentiable of $\mathbb{R}$ to $\mathbb{C}^2$ which have compact support.

We denote by $l_n + in\hat{\lambda}\breve{E} + \lambda E$ the closure of the operator $l_n + in\hat{\lambda}\breve{E} + \lambda E$ in the norm of $L_2 \equiv L_2(\mathbb{R}, \mathbb{C}^2)$ .

**Lemma 5.2.** *Let $\hat{\lambda} \ge \lambda \ge 0$. Let Assumption 2 and condition (5.1) hold. Then the following equality holds*

$$(l_n + in\hat{\lambda}\breve{E} + \lambda E)P_n(\hat{\lambda},\lambda)F = \tag{5.9}$$
$$F + P_{1,n}(\hat{\lambda},\lambda)F + P_{2,n}(\hat{\lambda},\lambda)F + P_{3,n}(\hat{\lambda},\lambda)F,$$

*for all $F \in L_2(\mathbb{R}, \mathbb{C}^2)$, where*

$$P_n(\hat{\lambda},\lambda)F = \sum_j \theta_j(y)(l_{n,j} + in\hat{\lambda}\breve{E} + \lambda E)^{-1}\theta_j F,$$
$$P_{1,n}(\hat{\lambda},\lambda)F = -\sum_j \theta_j^{''}(y)(l_{n,j} + in\hat{\lambda}\breve{E} + \lambda E)^{-1}\theta_j F,$$
$$P_{2,n}(\hat{\lambda},\lambda)F = \sum_j 2inT\theta_j^{'}(y)(l_{n,j} + in\hat{\lambda}\breve{E} + \lambda E)^{-1}\theta_j F,$$
$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$P_{3,n}(\hat{\lambda},\lambda)F = -2\sum_j \frac{d}{dy}\theta_j^{'}(y)(l_{n,j} + in\hat{\lambda}\breve{E} + \lambda E)^{-1}\theta_j F.$$

**Proof.** By simple computations, we obtain that

$$
\begin{aligned}
&(l_n + in\hat{\lambda}\breve{E} + \lambda E)P_n(\hat{\lambda}, \lambda)F = \\
&\quad \sum_{j=-\infty}^{\infty} (l_n + in\hat{\lambda}\breve{E} + \lambda E)[\theta_j(y)(l_n + in\hat{\lambda}\breve{E} + \lambda E)^{-1}\theta_j F] = \\
&\quad \sum_{j=-\infty}^{\infty} \Big\{ -[\theta_j(y)(l_n + in\hat{\lambda}\breve{E} + \lambda E)^{-1}\theta_j F]'' + \\
&\quad 2inT\left(\theta_j \frac{d}{dy}(l_n + in\hat{\lambda}\breve{E} + \lambda E)^{-1}\theta_j F\right)' \Big\} + \\
&\quad \sum_{j=-\infty}^{\infty} \Big\{ Q_n(y)[\theta_j(y)(l_n + in\hat{\lambda}\breve{E} + \lambda E)^{-1}\theta_j F] + \\
&\quad in\hat{\lambda}\breve{E}[\theta_j(y)(l_n + in\hat{\lambda}\breve{E} + \lambda E)^{-1}\theta_j F] + \\
&\quad \lambda E[\theta_j(y)(l_n + in\hat{\lambda}\breve{E} + \lambda E)^{-1}\theta_j F] \Big\} = \\
&\quad \sum_{j=-\infty}^{\infty} \theta_j (l_n + in\hat{\lambda}\breve{E} + \lambda E)(l_n + in\hat{\lambda}\breve{E} + \lambda E)^{-1}\theta_j F - \\
&\quad \sum_j \theta_j''(y)(l_{n,j} + in\hat{\lambda}\breve{E} + \lambda E)^{-1}\theta_j F + \\
&\quad \sum_j 2inT\theta_j'(y)(l_{n,j} + in\hat{\lambda}\breve{E} + \lambda E)^{-1}\theta_j F - \\
&\quad 2\sum_j \frac{d}{dy}\theta_j'(y)(l_{n,j} + in\hat{\lambda}\breve{E} + \lambda E)^{-1}\theta_j F = \\
&\quad F + P_{1,n}(\hat{\lambda}, \lambda)F + P_{2,n}(\hat{\lambda}, \lambda)F + P_{3,n}(\hat{\lambda}, \lambda)F.
\end{aligned}
$$

and thus the proof of the lemma is complete. □

**Lemma 5.3.** *Let Assumption 2 and condition (5.1) hold. Then there exists a number $\lambda_0 > 0$ such that*

$$
(l_n + in\hat{\lambda}\breve{E} + \lambda E)^{-1} = P_n(\hat{\lambda}, \lambda)\left[E + \sum_{k=1}^{3} P_{k,n}(\hat{\lambda}, \lambda)\right]^{-1}, \tag{5.10}
$$

*for all $\lambda \in [\lambda, \hat{\lambda}]$.*

**Proof.** By definition of $P_{1,n}(\hat{\lambda}, \lambda)$ in (5.9) and by the properties of the functions $\theta_j$ $(j \in \mathbb{Z})$ , we obtain

$$
\begin{aligned}
&\left\| P_{1,n}(\hat{\lambda}, \lambda)F \right\|_{L_2}^2 \le \\
&\quad \int_{\mathbb{R}} \sum_j \left| \theta_j''(y)(l_{n,j} + in\hat{\lambda}\breve{E} + \lambda E)^{-1}\theta_j F \right|^2 dy \le
\end{aligned}
$$

$$\sup_{j\in\mathbb{Z}} \left\|\theta_j''(y)(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)^{-1}\right\|^2_{L_2\to L_2} \sum_j \int\limits_{\Delta_j} |\theta_j F|^2\,dy \le$$

$$2\max_{j\in\mathbb{Z}} |\theta_j''(y)| \sup_{j\in\mathbb{Z}} \left\|(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)^{-1}\right\|^2_{L_2\to L_2} \|F\|^2_{L_2}.$$

Hence, by applying inequality (5.2), we have

$$\left\|P_{1,n}(\hat{\lambda},\lambda)\right\|_{L_2\to L_2} \le \frac{\tilde{C}_1}{\lambda|n|}. \tag{5.11}$$

Now by definitions $P_{2,n}(\hat{\lambda},\lambda)$ and $P_{3,n}(\hat{\lambda},\lambda)$, and by the properies of the matrix $T$ and of functions $\theta_j$ $(j\in\mathbb{Z})$ , we obtain

$$\left\|P_{2,n}(\hat{\lambda},\lambda)F\right\|^2_{L_2} \le$$

$$\int\limits_{\mathbb{R}} \sum_j \left|2inT\theta_j'(y)(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)^{-1}\theta_j F\right|^2 dy \le$$

$$4n^2 \sup_{j\in\mathbb{Z}} \left\|T\theta_j'(y)(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)^{-1}\right\|^2_{L_2\to L_2} \sum_j \int\limits_{\Delta_j} |\theta_j F|^2\,dy \le$$

$$8n^2 \max_{j\in\mathbb{Z}} |\theta_j'(y)| \sup_{j\in\mathbb{Z}} \left\|(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)^{-1}\right\|^2_{L_2\to L_2} \|F\|^2_{L_2}$$

and

$$\left\|P_{3,n}(\hat{\lambda},\lambda)F\right\|^2_{L_2} \le$$

$$\int\limits_{\mathbb{R}} \sum_j \left|2\frac{d}{dy}\theta_j'(y)(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)^{-1}\theta_j F\right|^2 dy \le$$

$$4\sup_{j\in\mathbb{Z}} \left\|\frac{d}{dy}\theta_j'(y)(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)^{-1}\right\|^2_{L_2\to L_2} \sum_j \int\limits_{\Delta_j} |\theta_j F|^2\,dy \le$$

$$8\max_{j\in\mathbb{Z}} |\theta_j'(y)| \sup_{j\in\mathbb{Z}} \left\|\frac{d}{dy}(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)^{-1}\right\|^2_{L_2\to L_2} \|F\|^2_{L_2}.$$

Hence, by inequalities (5.2), (5.3), we have

$$\left\|P_{2,n}(\hat{\lambda},\lambda)\right\|_{L_2\to L_2} \le \frac{\tilde{C}_2}{\lambda} \tag{5.12}$$

and

$$\left\|P_{3,n}(\hat{\lambda},\lambda)\right\|_{L_2\to L_2} \le \frac{\tilde{C}_3}{\sqrt{\lambda}}. \tag{5.13}$$

If Assumption 2 and condition (5.1) hold, then $\lim\limits_{\hat{\lambda},\lambda\to\infty} \left\|P_{k,n}(\hat{\lambda},\lambda)\right\|_{L_2\to L_2} = 0$ $(k=1,2,3)$. By inequalities (5.11)-(5.13), it follows that there exists a number $\lambda_0>0$ such that

$$\left\|P_{1,n}(\hat{\lambda},\lambda)\right\|_{L_2\to L_2} + \left\|P_{2,n}(\hat{\lambda},\lambda)\right\|_{L_2\to L_2} + \left\|P_{3,n}(\hat{\lambda},\lambda)\right\|_{L_2\to L_2} \le \frac{1}{2},$$

for all $\lambda \in [\lambda, \hat{\lambda}]$.

Then the operator $S_{\hat{\lambda},\lambda} = E + P_{1,n}(\hat{\lambda},\lambda) + P_{2,n}(\hat{\lambda},\lambda) + P_{3,n}(\hat{\lambda},\lambda)$ is bounded and invertible for $\hat{\lambda} \geq \lambda \geq \lambda_0$ by Theorem 1.5. Moreover, $\left\| S_{\hat{\lambda},\lambda} \right\|_{L_2 \to L_2} \leq 2$, $\left\| S^{-1}_{\hat{\lambda},\lambda} \right\|_{L_2 \to L_2} \leq 2$. This means that the operator $S_{\hat{\lambda},\lambda}$ is a bijection of the whole of $L_2(\mathbb{R}, \mathbb{C}^2)$ onto itself. We now set $G_{\hat{\lambda},\lambda}(T) = (S_{\hat{\lambda},\lambda} F)(T)$ $(\hat{\lambda} \geq \lambda \geq \lambda_0)$. By the equality of Lemma 5.3, we obtain that

$$(l_n + in\hat{\lambda}\breve{E} + \lambda E)(P_n(\hat{\lambda},\lambda) S^{-1}_{\hat{\lambda},\lambda} G_{\hat{\lambda},\lambda})(T) = G_{\hat{\lambda},\lambda}(T),$$

for all $G_{\hat{\lambda},\lambda}(T) \in L_2(\mathbb{R}, \mathbb{C}^2)$.

Thus, the operator $P_n(\hat{\lambda},\lambda) \left[ E + \sum_{k=1}^{3} P_{k,n}(\hat{\lambda},\lambda) \right]^{-1}$ coincides with operator $(l_n + in\hat{\lambda}\breve{E} + \lambda E)^{-1}$ for $\hat{\lambda} \geq \lambda \geq \lambda_0$. □

**Lemma 5.4.** *Let Assumption 2 and condition (5.1) hold. Then there exists a number* $\lambda_0 > 0$ *such that*

$$\left\| (l_n + in\hat{\lambda}\breve{E} + \lambda E)^{-1} \right\|_{L_2 \to L_2} \leq \sqrt{8} \sup_{j \in \mathbb{Z}} \left\| (l_{n,j} + in\hat{\lambda}\breve{E} + \lambda E)^{-1} \right\|_{L_2(\Delta_j, \mathbb{C}^2) \to L_2(\Delta_j, \mathbb{C}^2)}, \tag{5.14}$$

*for all* $\lambda \in [\lambda, \hat{\lambda}]$.

**Proof.** By virtue of the properties of the functions $\theta_1, \theta_2, \ldots$ at each point $y \in \mathbb{R}$ the sum $P_n(\hat{\lambda},\lambda)\, h(y)$ consists of a finite numbers of terms for all $h \in L_2$ (no more then three). Therefore, (5.10) implies that

$$\begin{aligned}
&\left\| (l_n + in\hat{\lambda}\breve{E} + \lambda E)^{-1} h \right\|^2_{L_2} \leq \\
&\quad \left\| \left[ E + \sum_{k=1}^{3} P_{k,n}(\hat{\lambda},\lambda) \right]^{-1} \right\|^2_{L_2 \to L_2} \times \\
&\quad \int_{\mathbb{R}} \sum_j \left| \theta_j(y)(l_{n,j} + in\hat{\lambda}\breve{E} + \lambda E)^{-1} \theta_j h \right|^2 dy \leq \\
&\quad 4 \int_{\mathbb{R}} \sum_j \left| \theta_j(y)(l_{n,j} + in\hat{\lambda}\breve{E} + \lambda E)^{-1} \theta_j h \right|^2 dy \leq \\
&\quad 4 \sum_j \left\| \theta_j(y)(l_{n,j} + in\hat{\lambda}\breve{E} + \lambda E)^{-1} \right\|^2_{L_2 \to L_2} \|\theta_j h\|^2_{L_2(\Delta_j, \mathbb{C}^2)} \leq \\
&\quad 4 \sup_{j \in \mathbb{Z}} \left\| \theta_j(y)(l_{n,j} + in\hat{\lambda}\breve{E} + \lambda E)^{-1} \right\|^2_{L_2(\Delta_j, \mathbb{C}^2) \to L_2(\Delta_j, \mathbb{C}^2)} \times
\end{aligned}$$

$$\sum_j \int_{\Delta_j} |\theta_j(y)h(y)|^2\,dy \le$$
$$8\sup_{j\in\mathbb{Z}} \left\|\theta_j(y)(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)^{-1}\right\|^2_{L_2(\Delta_j,\mathbb{C}^2)\to L_2(\Delta_j,\mathbb{C}^2)} \|h\|^2_{L_2},$$

(see definition (5.10)). Thus the proof of the lemma is complete. □

**Lemma 5.5.** *Let Assumption 2 and conditions (5.1) hold. Let*

$$\sup_{y,\eta\in\mathbb{R},|y-\eta|\le 2}\left\{\frac{\varphi(y)}{\varphi(\eta)},\frac{\psi(y)}{\psi(\eta)},\frac{a(y)}{a(\eta)},\frac{d(y)}{d(\eta)},\frac{a(y)}{d(\eta)}\right\}\le C<\infty. \tag{5.15}$$

*Then*

$$\left\||n|\left(P(\cdot)+\hat{\lambda}\breve{E}\right)\left(l_n+in\hat{\lambda}\breve{E}+\lambda E\right)^{-1}\right\|_{L_2\to L_2}+ \tag{5.16}$$
$$\left\|(Q(\cdot)+\lambda E)\left(l_n+in\hat{\lambda}\breve{E}+\lambda E\right)^{-1}\right\|_{L_2\to L_2}<\infty.$$

*Here $P(y)$ and $Q(y)$ are matrices of (4.1).*

**Proof.** Without loss of generality, we can assume that $\hat{\lambda}\ge\lambda\ge\lambda_0$, where $\lambda_0$ is a constant of Lemma 5.4. By Lemma 5.4 and by the properties of the functions $\theta_j$ $(j\in\mathbb{Z})$ , we obtain

$$\left\|(Q(\cdot)+\lambda E)\left(l_n+in\hat{\lambda}\breve{E}+\lambda E\right)^{-1}h\right\|_{L_2}\le$$
$$\max\left\{\sup_{y\in\Delta_j}(a_j(y)+\lambda),\sup_{y\in\Delta_j}(d_j(y)+\lambda)\right\}\times$$
$$\left\|\left[E+\sum_{k=1}^{3}P_{k,n}(\hat{\lambda},\lambda)\right]^{-1}\right\|_{L_2\to L_2}\times$$
$$\left(\int_{\mathbb{R}}\sum_j\left|\theta_j(y)(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)^{-1}\theta_j h\right|^2dy\right)^{\frac{1}{2}}\le$$
$$C_1\max\left\{\sup_{y\in\Delta_j}(a_j(y)+\lambda),\sup_{y\in\Delta_j}(d_j(y)+\lambda)\right\}\times$$
$$\sup_{j\in\mathbb{Z}}\left\|\theta_j(y)(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)^{-1}\right\|_{L_2(\Delta_j,\mathbb{C}^2)\to L_2(\Delta_j,\mathbb{C}^2)}\times$$
$$\left(\sum_j\int_{\Delta_j}|\theta_j(y)h(y)|^2\,dy\right)^{\frac{1}{2}}\le$$

$$C_2 \max\left\{\sup_{y\in\Delta_j}(a_j(y)+\lambda), \sup_{y\in\Delta_j}(d_j(y)+\lambda)\right\}\times$$
$$\sup_{j\in\mathbb{Z}}\left\|(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)^{-1}\right\|_{L_2(\Delta_j,\mathbb{C}^2)\to L_2(\Delta_j,\mathbb{C}^2)}\|h\|_{L_2},$$

or

$$\left\|(Q(\cdot)+\lambda E)\left(l_n+in\hat{\lambda}\breve{E}+\lambda E\right)^{-1}h\right\|_{L_2}\le \tag{5.17}$$
$$C_2 \max\left\{\sup_{y\in\Delta_j}(a_j(y)+\lambda), \sup_{y\in\Delta_j}(d_j(y)+\lambda)\right\}\times$$
$$\sup_{j\in\mathbb{Z}}\left\|(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)^{-1}\right\|_{L_2(\Delta_j,\mathbb{C}^2)\to L_2(\Delta_j,\mathbb{C}^2)}\|h\|_{L_2},$$

Accordingly

$$\left\||n|\left(P(\cdot)+\hat{\lambda}\breve{E}\right)\left(l_n+in\hat{\lambda}\breve{E}+\lambda E\right)^{-1}h\right\|_{L_2}\le$$
$$|n|\max\left\{\sup_{y\in\Delta_j}(|\varphi_j(y)|+\lambda), \sup_{y\in\Delta_j}(\psi_j(y)+\lambda)\right\}\times$$
$$\left\|\left[E+\sum_{k=1}^{3}P_{k,n}(\hat{\lambda},\lambda)\right]^{-1}\right\|_{L_2\to L_2}\times$$
$$\left(\int_{\mathbb{R}}\sum_j\left|\theta_j(y)(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)^{-1}\theta_j h\right|^2 dy\right)^{\frac{1}{2}}\le$$
$$C_3|n|\max\left\{\sup_{y\in\Delta_j}(|\varphi_j(y)|+\lambda), \sup_{y\in\Delta_j}(\psi_j(y)+\lambda)\right\}\times$$
$$\sup_{j\in\mathbb{Z}}\left\|(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)^{-1}\right\|_{L_2(\Delta_j,\mathbb{C}^2)\to L_2(\Delta_j,\mathbb{C}^2)}\|h\|_{L_2},$$

or

$$\left\||n|\left(P(\cdot)+\hat{\lambda}\breve{E}\right)\left(l_n+in\hat{\lambda}\breve{E}+\lambda E\right)^{-1}h\right\|_{L_2}\le \tag{5.18}$$
$$C_3|n|\max\left\{\sup_{y\in\Delta_j}(|\varphi_j(y)|+\lambda), \sup_{y\in\Delta_j}(\psi_j(y)+\lambda)\right\}\times$$
$$\sup_{j\in\mathbb{Z}}\left\|(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)^{-1}\right\|_{L_2(\Delta_j,\mathbb{C}^2)\to L_2(\Delta_j,\mathbb{C}^2)}\|h\|_{L_2}.$$

Inequality (5.8) implies that

$$\tilde{C}_0\left(\frac{2\gamma_j}{3}+\lambda\right)\|w\|^2_{L_2(\Delta_j,\mathbb{C}^2)}\le\frac{\tilde{C}_1}{\frac{2\gamma_j}{3}+\lambda}\left\|(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)w\right\|^2_{L_2(\Delta_j,\mathbb{C}^2)},$$

for all $w\in D(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)$. Hence, if we set $(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)w=v$, then $w=(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)^{-1}v$, for all $v\in R(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)=D((l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)^{-1})$.

So we obtain that

$$\left\|(l_{n,j}+in\hat{\lambda}\breve{E}+\lambda E)^{-1}\right\|_{L_2(\Delta_j,\mathbb{C}^2)\to L_2(\Delta_j,\mathbb{C}^2)}\leq\frac{C_3}{\frac{2\gamma_j}{3}+\lambda}.$$

Hence, conditions (5.15) and inequality (5.17) imply that

$$\left\|(Q(\cdot)+\lambda E)\left(l_n+in\hat{\lambda}\breve{E}+\lambda E\right)^{-1}\right\|_{L_2\to L_2}\leq$$
$$C_2\max\left\{\sup_{y\in\Delta_j}(a_j(y)+\lambda),\sup_{y\in\Delta_j}(d_j(y)+\lambda)\right\}\frac{C_3}{\frac{2\gamma_j}{3}+\lambda}<\infty.$$

Inequality (5.7) implies that

$$\left\|\left(l_n+in\hat{\lambda}\breve{E}+\lambda E\right)^{-1}\right\|_{L_2(\Delta_j,\mathbb{C}^2)\to L_2(\Delta_j,\mathbb{C}^2)}\leq\frac{1}{|n|(\gamma_j+\lambda)}.$$

Hence, conditions (5.15) and inequality (5.18) imply that

$$\left\||n|\left(P(\cdot)+\hat{\lambda}\breve{E}\right)\left(l_n+in\hat{\lambda}\breve{E}+\lambda E\right)^{-1}\right\|_{L_2\to L_2}\leq$$
$$C_4|n|\max\left\{\sup_{y\in\Delta_j}(|\varphi_j(y)|+\lambda),\sup_{y\in\Delta_j}(\psi_j(y)+\lambda)\right\}\times$$
$$\sup_{y,\eta\in\mathbb{R},|y-\eta|\leq 2}\left(\frac{|\varphi_j(y)|+\lambda}{\min\left(\inf_{y\in\Delta_j}|\varphi_j(\eta)|+\lambda,\inf_{y\in\Delta_j}|\varphi_j(\eta)|+\lambda\right)},\right.$$
$$\left.\frac{\psi_j(y)+\lambda}{\min\left(\inf_{y\in\Delta_j}|\varphi_j(\eta)|+\lambda,\inf_{y\in\Delta_j}|\varphi_j(\eta)|+\lambda\right)}\right)\times$$
$$\frac{1}{|n|(\gamma_j+\lambda)}<\infty,$$

and the proof of the lemma is complete. □

Let $G=\{(x,y)\in\mathbb{R}^2:\ -\pi<x<\pi,\ -\infty<y<+\infty\}$. We now consider the operator $L_{\lambda,\hat{\lambda}}$ defined by

$$L_{\lambda,\hat{\lambda}}w=B_{xy}w+\left(P(y)+\hat{\lambda}\breve{E}\right)w_x+(Q(y)+\lambda E)\,w,$$

for all functions $w=(u,v)$ in the space $C^2_{\pi,0}(G,\mathbb{R}^2)$. We denote by $L_{\lambda,\hat{\lambda}}$ the closure of $L_{\lambda,\hat{\lambda}}$ in the norm of $L_2(G,\mathbb{R}^2)$.

**Definition 5.6.** *The operator $L_{\lambda,\hat{\lambda}}$ is said to be separable, if the following inequality holds*

$$\|w_{xx}\|_{2,G} + \|w_{yy}\|_{2,G} + \|w_{xy}\|_{2,G} + \left\|\left(P(y) + \hat{\lambda}\breve{E}\right) w_x\right\|_{2,G} +$$
$$\|(Q(y) + \lambda E)\, w\|_{2,G} \le C\left(\left\|L_{\lambda,\hat{\lambda}} w\right\|_{2,G} + \|w\|_{2,G}\right),$$

*for all $w \in D(L_{\lambda,\hat{\lambda}})$.*

We now prove the following intermediate statement.

**Lemma 5.7.** *Let the following conditions hold.*

*a) The coefficients $\varphi$, $\psi$, $a, b$, $c, d$ of the system (4.1) satisfy Assumption 2.*

*b) The function $k(y)$ of $\mathbb{R}$ is twice continuously differentiable and satisfies one and only one of the following three conditions*

$$i)\quad \sqrt{2} < k(y) < 2, \qquad \min_{y\in\mathbb{R}}\left\{\varphi^2(y), \psi^2(y)\right\} + k^{''}(y) > 2\left[k^{'}(y)\right]^2;$$
$$ii)\quad k(y) < 2,\ \frac{\sqrt{2}k^{'}(y)}{k(y)} \le 1, \qquad \min_{y\in\mathbb{R}}\left\{\varphi^2(y), \psi^2(y)\right\} + k^{''}(y) > 2;$$
$$iii)\quad k(y) < 2,\ k^2(y) > 2k^{'}(y), \quad \min_{y\in\mathbb{R}}\left\{\varphi^2(y), \psi^2(y)\right\} + k^{''}(y) > 2k^{'}(y).$$

*c) There exist non-negative constants $\lambda$ and $\hat{\lambda}$ such that the following inequality holds*

$$\|B_{xy} w\|_{2,G} + \left\|\left(P(y) + \hat{\lambda}\breve{E}\right) w_x\right\|_{2,G} + \tag{5.19}$$
$$\|(Q(y) + \lambda E)\, w\|_{2,G} \le C\left(\left\|(L + \hat{\lambda}\breve{E}\frac{\partial}{\partial x} + \lambda E)w\right\|_{2,G} + \|w\|_{2,G}\right),$$

*for all $w = (u, v) \in D(L_{\lambda,\hat{\lambda}})$.*

*Then the operator $L_{\lambda,\hat{\lambda}}$ is separable.*

**Proof.** Let $w = (u, v) \in C^2_{\pi,0}(G, \mathbb{R}^2)$. By simple computations, we obtain that

$$\|B_{xy} w\|^2_{2,G} = \int_G [k(y)u_{xx} - u_{yy} - 2v_{xy}]^2 dxdy +$$
$$\int_G [2u_{xy} - k(y)v_{xx} - v_{yy}]^2 dxdy =$$
$$\int_G [k^2(y)u^2_{xx} + u^2_{yy} - 2k(y)u_{xx}u_{yy} +$$

$$4v_{xy}^2 - 4(k(y)u_{xx} - u_{yy})v_{xy}]dxdy +$$
$$\int_G [4u_{xy}^2 + 4(k(y)v_{xx} - v_{yy})u_{xy} +$$
$$k^2(y)v_{xx}^2 - 2k(y)v_{xx}v_{yy} + v_{yy}^2]dxdy =$$
$$\int_G k^2(y)[u_{xx}^2 + v_{xx}^2]dxdy + \int_G [u_{yy}^2 + v_{yy}^2]dxdy +$$
$$4\int_G v_{xy}^2 dxdy + 4\int_G u_{xy}^2 dxdy - 2\int_G k(y)u_{xx}u_{yy}dxdy -$$
$$4\int_G k(y)u_{xx}v_{xy}dxdy + 4\int_G u_{yy}v_{xy}dxdy +$$
$$4\int_G k(y)u_{xy}v_{xx}dxdy - 4\int_G u_{xy}v_{yy}dxdy -$$
$$2\int_G k(y)v_{xx}v_{yy}dxdy.$$

Hence, by arguing as the derivation of (3.3) from (3.2), and by exploiting the fact that the function $w$ has compact support in $\mathbb{R}$ in the variable $y$, we obtain

$$\|B_{xy}w\|_{2,G}^2 + \left\|\left(P(y) + \hat{\lambda}\breve{E}\right) w_x\right\|_{2,G}^2 \geq \tag{5.20}$$
$$\int_G k^2(y)|w_{xx}|^2 dxdy + \int_G |w_{yy}|^2 dxdy +$$
$$4\int_G |w_{xy}|^2 dxdy + \int_G k''(y)|w_x|^2 dxdy - 2\int_G k(y)|w_{xy}|^2 dxdy -$$
$$4\int_G k'(y)u_x v_{xx}dxdy + \inf_{y\in\mathbb{R}}\left\{\varphi^2(y), \psi^2(y)\right\}\int_G |w_x|^2 dxdy.$$

The last term of (5.20) satisfies the following inequalities

$$\left|\int_G k'(y)u_x v_{xx}dxdy\right| \leq \frac{1}{2}\int_G \left[k'(y)\right]^2 u_x^2 dxdy + \frac{1}{2}\int_G v_{xx}^2 dxdy,$$

$$\left|\int_G k'(y)u_x v_{xx}dxdy\right| \leq \frac{1}{2}\int_G u_x^2 dxdy + \frac{1}{2}\int_G \left[k'(y)\right]^2 v_{xx}^2 dxdy,$$

$$\left|\int_G k'(y)u_x v_{xx}dxdy\right| \leq \frac{1}{2}\int_G k'(y)u_x^2 dxdy + \frac{1}{2}\int_G k'(y)v_{xx}^2 dxdy.$$

Hence, we obtain

$$\|B_{xy}w\|_{2,G}^2 + \left\|\left(P(y) + \hat{\lambda}\breve{E}\right) w_x\right\|_{2,G}^2 \geq \tag{5.21}$$

$$\int_G [k^2(y)-2]|w_{xx}|^2dxdy+$$
$$\int_G |w_{yy}|^2dxdy+2\int_{G_0}[2-k(y)]|w_{xy}|^2dxdy+$$
$$\int_G \left(\inf_{y\in\mathbb{R}}\left\{\varphi^2(y),\psi^2(y)\right\}+k^{''}(y)-2\left[k^{'}(y)\right]^2\right)|w_x|^2dxdy,$$

$$\|B_{xy}w\|^2_{2,G}+\left\|\left(P(y)+\hat{\lambda}\breve{E}\right)w_x\right\|^2_{2,G}\geq \tag{5.22}$$
$$\int_G \left[k^2(y)-2(k^{'}(y))^2\right]|w_{xx}|^2dxdy+$$
$$\int_G |w_{yy}|^2dxdy+2\int_G [2-k(y)]\,|w_{xy}|^2dxdy+$$
$$\int_G \left(\inf_{y\in\mathbb{R}}\left\{\varphi^2(y),\psi^2(y)\right\}+k^{''}(y)-2\right)|w_x|^2dxdy.$$

$$\|B_{xy}w\|^2_{2,G}+\left\|\left(P(y)+\hat{\lambda}\breve{E}\right)w_x\right\|^2_{2,G}\geq \tag{5.23}$$
$$\int_G \left[k^2(y)-2k^{'}(y)\right]|w_{xx}|^2dxdy+$$
$$\int_G |w_{yy}|^2dxdy+2\int_G [2-k(y)]\,|w_{xy}|^2dxdy+$$
$$\int_G \left(\inf_{y\in\mathbb{R}}\left\{\varphi^2(y),\psi^2(y)\right\}+k^{''}(y)-2k^{'}(y)\right)|w_x|^2dxdy.$$

By conditions i), ii) and iii) and the inequalities (5.21), and (5.22) and (5.23) by arguing as to prove Lemma 3.3, we obtain that

$$\|B_{xy}w\|^2_{2,G}+\left\|\left(P(y)+\hat{\lambda}\breve{E}\right)w_x\right\|^2_{2,G}\geq \tag{5.24}$$
$$C_1\|w_{xx}\|^2_{2,G}+C_2\|w_{yy}\|^2_{2,G}+C_3\|w_{xy}\|^2_{2,G}+(C_4+\hat{\lambda})\|w_x\|^2_{2,G}.$$

and inequality $\left\|L_{\lambda,\hat{\lambda}}w\right\|_{2,G}\geq C_0(\lambda,\hat{\lambda})\left(\left\|\hat{\lambda}w_x\right\|_{2,G}+\|\lambda w\|_{2,G}\right)$ holds. Hence, inequalities (5.19) and (5.24) imply that

$$\|w_{xx}\|_{2,G}+\|w_{yy}\|_{2,G}+\|w_{xy}\|_{2,G}+$$
$$\left\|\left(P(y)+\hat{\lambda}\breve{E}\right)w_x\right\|_{2,G}+\|(Q(y)+\lambda E)\,w\|_{2,G}\leq$$
$$\|B_{xy}w\|_{2,G}+\left\|\left(P(y)+\hat{\lambda}\breve{E}\right)w_x\right\|_{2,G}+\|(Q(y)+\lambda E)\,w\|_{2,G}\leq$$
$$C_2(\lambda,\hat{\lambda})\left(\left\|L_{\lambda,\hat{\lambda}}w\right\|_{2,G}+\|w\|_{2,G}\right),$$

and the proof of the lemma is complete. □

We now introduce the main result of this Chapter.

**Theorem 5.8.** *Let $\hat{\lambda} \geq \lambda \geq 0$. Let the coefficients of system (4.1) satisfy Assumption 2, conditions (5.1), (5.15). Let $k(y)$ be twice continuously differentiable on $\mathbb{R}$ and satisfy one and only one of conditions i), ii), iii). Then the operator $L_{\lambda,\hat{\lambda}}$ is separable.*

**Proof.** By Lemma 5.7 it is enough to show that (5.19) holds. By the assumptions of this theorem the operator $L$ is bounded and invertible. The operator $L+\hat{\lambda}\breve{E}\frac{\partial}{\partial x}$ satisfies all the conditions of Theorem 5.8. Hence, $L+\hat{\lambda}\breve{E}\frac{\partial}{\partial x}$ is bounded and invertible. Furthermore, the following inequality holds

$$\left\|(L+\hat{\lambda}\breve{E}\frac{\partial}{\partial x}+\lambda E)w\right\|_{2,G} \geq C\left\|w\right\|_{2,G},$$

for all $w \in D(L+\hat{\lambda}\breve{E}\frac{\partial}{\partial x}+\lambda E)$. Then by the well-known Theorem 1.2, the operator $L+\hat{\lambda}\breve{E}\frac{\partial}{\partial x}+\lambda E$ is bounded and invertible in $L_2(G,\mathbb{R}^2)$ for $\hat{\lambda} \geq \lambda \geq 0$. Furthermore, we have

$$(L+\hat{\lambda}\breve{E}\frac{\partial}{\partial x}+\lambda E)^{-1}F = \sum_{n=-\infty}^{+\infty}(l_n+in\hat{\lambda}\breve{E}+\lambda E)^{-1}F_n e^{inx},$$

by construction. Here $F = \sum\limits_{n=-\infty}^{\infty} F_n e^{inx}$, $F=(f,g)$, $F_n=(f_n,g_n)$.

Hence, by the orthonormality of the system $\{e^{inx}\}_{n=-\infty}^{+\infty}$ in $L_2(-\pi,\pi)$, we obtain

$$\left\|\rho(y)D_x^\tau\left(L+\hat{\lambda}\breve{E}\frac{\partial}{\partial x}+\lambda E\right)^{-1}\right\|_{L_2(G,\mathbb{R}^2)\to L_2(G,\mathbb{R}^2)} = \sup_n\left\||n|^\tau\rho(y)\left(l_n+in\hat{\lambda}\breve{E}+\lambda E\right)^{-1}\right\|_{L_2((\alpha,\beta),\mathbb{C}^2)\to L_2((\alpha,\beta),\mathbb{C}^2)}.$$

Here $D_x^\tau = \frac{\partial^\tau}{\partial x^\tau}$, and $\tau = 0,1$, and $\rho(y)$ is a $2\times 2$-matrix with continuous elements. Since (5.16) holds, we have

$$\left\|\left(P(\cdot)+\hat{\lambda}\breve{E}\right)\frac{\partial}{\partial x}\left(L+\hat{\lambda}\breve{E}\frac{\partial}{\partial x}+\lambda E\right)^{-1}\right\|_{L_2(G,\mathbb{R}^2)\to L_2(G,\mathbb{R}^2)} + \left\|\left(Q(\cdot)+\lambda E\right)\frac{\partial}{\partial x}\left(L+\hat{\lambda}\breve{E}\frac{\partial}{\partial x}+\lambda E\right)^{-1}\right\|_{L_2(G,\mathbb{R}^2)\to L_2(G,\mathbb{R}^2)} =$$

$$\sup_n \sup_j \left\| |n| \left(P(\cdot) + \lambda E\right) \left(l_{n,j} + in\hat{\lambda}\breve{E} + \lambda E\right)^{-1} \right\|_{L_2(\Delta_j,\mathbb{C}^2)\to L_2(\Delta_j,\mathbb{C}^2)} +$$
$$\sup_n \sup_j \left\| \left(Q(\cdot) + \lambda E\right) \left(l_{n,j} + in\hat{\lambda}\breve{E} + \lambda E\right)^{-1} \right\|_{L_2(\Delta_j,\mathbb{C}^2)\to L_2(\Delta_j,\mathbb{C}^2)} < \infty.$$

Then we obtain

$$\left\| B_{xy} \left( L + \hat{\lambda}\breve{E}\frac{\partial}{\partial x} + \lambda E \right)^{-1} \right\|_{L_2(G,\mathbb{R}^2)\to L_2(G,\mathbb{R}^2)} < \infty,$$

by system (4.3). Hence, inequality (5.19) follows and the proof is complete. □

**Corollary 5.9.** *Let the coefficients $\varphi, \psi, a, b, c, d, k$ of system (4.1) satisfy the conditions of Theorem 5.8. Then the following inequality holds*

$$\begin{aligned} &\|u_{xx}\|^2_{2,G} + \|u_{yy}\|^2_{2,G} + \|u_{xy}\|^2_{2,G} + \\ &\quad \|v_{xx}\|^2_{2,G} + \|v_{yy}\|^2_{2,G} + \|v_{xy}\|^2_{2,G} + \|\varphi(y)u_x\|^2_{2,G} + \\ &\quad \|u_y\|^2_{2,G} + \|\psi(y)v_x\|^2_{2,G} + \|v_y\|^2_{2,G} + \|a(y)u\|^2_{2,G} + \\ &\quad \|b(y)v\|^2_{2,G} + \|c(y)u\|^2_{2,G} + \|d(y)v\|^2_{2,G} \leq C\,\|F\|^2_{2,G}\,, \end{aligned} \tag{5.25}$$

*for the solution $w = (u, v)$ of problem (4.1), (4.2).*

**Remark 5.10.** *The definition of separability ensures the validity of the following inequality*

$$\begin{aligned} &\|u_{xx}\|^2_{2,G} + \|u_{yy}\|^2_{2,G} + \|u_{xy}\|^2_{2,G} + \\ &\quad \|v_{xx}\|^2_{2,G} + \|v_{yy}\|^2_{2,G} + \|v_{xy}\|^2_{2,G} + \|\varphi(y)u_x\|^2_{2,G} + \\ &\quad \|u_y\|^2_{2,G} + \|\psi(y)v_x\|^2_{2,G} + \|v_y\|^2_{2,G} + \|a(y)u\|^2_{2,G} + \\ &\quad \|b(y)v\|^2_{2,G} + \|c(y)u\|^2_{2,G} + \|d(y)v\|^2_{2,G} \leq \\ &\quad C\left(\|Lw\|^2_{2,G} + \|w\|^2_{2,G}\right). \end{aligned} \tag{5.26}$$

*If ineqality (4.6) is holds, then (5.26) is equivalent to (5.25).*

By the well-known norm of the Sobolev space $W_2^2(G,\mathbb{R}^2)$, one can rewrite (5.25) in the following compact form

$$\begin{aligned} &\|w\|^2_{W_2^2(G,\mathbb{R}^2)} + \|\varphi(y)u_x\|^2_{2,G} + \|\psi(y)v_x\|^2_{2,G} + \\ &\quad \|(|a| + |c|)\,u\|^2_{2,G} + \|(|b| + |d|)\,v\|^2_{2,G} \leq C_1\,\|F\|^2_{2,G}\,. \end{aligned}$$

**Example 5.11.** *The following system satisfies the conditions of Theorem 5.8.*

$$\begin{cases} \sqrt{2-e^{-y^2}}\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - 2\frac{\partial^2 v}{\partial x \partial y} - \sqrt{2+\frac{e^y(y^4+3)}{2y^2+1}}\frac{\partial u}{\partial x} + (y^2+1)u + \frac{1}{3}v = f(x,y), \\ 2\frac{\partial^2 u}{\partial x \partial y} + \sqrt{2-e^{-y^2}}\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} + \sqrt{2+\frac{e^y(y^4+3)}{2y^2+1}}\frac{\partial v}{\partial x} + \frac{1}{3}u + (y^2+1)v = g(x,y), \end{cases}$$

*where* $f, g \in L_2(G)$.

**Example 5.12.** *We consider the following problem*

$$\begin{cases} k(y)\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - 2\frac{\partial^2 v}{\partial x \partial y} - [(1+8y^2)e^{7y^2-2y+1} - 2]\frac{\partial u}{\partial x} + \\ \qquad (11y^4 - y + 8)u + \frac{1}{4}sinyv = f(x,y), \\ 2\frac{\partial^2 u}{\partial x \partial y} + k(y)\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} + [\chi(y)e^{5y^4+1} + 7]\frac{\partial v}{\partial x} + \\ \qquad \frac{1}{3}cosyu + (5y^4 - 2y + 1)v = g(x,y), \end{cases}$$

$$w(-\pi, y) = w(\pi, y), w_x(-\pi, y) = w_x(\pi, y)$$

on the strip $G = \{(x, y) \in \mathbb{R}^2 : \ -\pi < x < \pi, \ -\infty < y < +\infty\}$. Here

$$k(y) = \begin{cases} \frac{1}{2}e^{-y^2} \text{ for } y \geq 0 \\ 1 - \frac{1}{2}e^{-y^2} \text{ for } y < 0 \end{cases}$$

such that $\inf_{y \in \mathbb{R}} k(y) = 0$, and $\chi(y)$ is an arbitrary function such that $1 \leq \chi(y) \leq 2$ for all $y \in \mathbb{R}$ and $f, g \in L_2(G)$.

By Theorem 4.9 this problem has an unique solution $w = (u, v) \in L_2$ for any data $f, g \in L_2(G)$. Moreover, the solutions of the above system satisfy the coercive inequality with the norm of space $L_2(G)$ in the form (5.25).

We now show that the following functions

$$\begin{aligned} a(y) &= 11y^4 - y + 8, \\ b(y) &= \frac{1}{4}siny, \\ c(y) &= \frac{1}{3}cosy, \\ d(y) &= 5y^4 - 2y + 1, \\ \varphi(y) &= -(1+8y^2)e^{7y^2-2y+1} + 2, \\ \psi(y) &= \chi(y)e^{5y^4+1} + 7 \end{aligned}$$

satisfy Assumption 2.

Indeed

1)there exists a constant $\delta > 0$ such that

$$\inf_{y\in\mathbb{R}} \{-\varphi(y), a(y), d(y)\} =$$

$$\inf_{y\in\mathbb{R}} \left\{(1+8y^2)e^{7y^2-2y+1} - 2, 11y^4 - y + 8, 5y^4 - 2y + 1\right\} = \delta > 0;$$

2) there exist constants $r, q, (r > 0,\ q > 0,\ r + q = 1)$ and $\vartheta(0 < \vartheta < 3)$ such that

$$\begin{gathered} \tfrac{1}{2}\left(|\tfrac{1}{4}siny| + |\tfrac{1}{3}cosy|\right)^{2r} \leq \tfrac{11y^4-y+8}{3}, \\ \tfrac{1}{2}\left(|\tfrac{1}{4}siny| + |\tfrac{1}{3}cosy|\right)^{2q} \leq \tfrac{5y^4-2y+1}{3}, \\ \vartheta[\chi(y)e^{5y^4+1} + 7] > 5y^4 - 2y + 1 \end{gathered}$$

for all $y \in \mathbb{R}$.

Further, we can see that the functions $a(y), d(y), \varphi(y), \psi(y)$ of $\mathbb{R}$ satisfy the condition (5.1) of Lemma 5.1 and the condition (5.15) of Lemma 5.5. There exist constants $c_0, c_1$ and $C$ such that

$$\inf_{y,\eta\in\mathbb{R},|y-\eta|\leq 2} \frac{[(1+8y^2)e^{7y^2-2y+1} - 2]^2}{11\eta^4 - \eta + 8} \geq c_0 > 0,$$

$$\inf_{y,\eta\in\mathbb{R},|y-\eta|\leq 2} \frac{[\chi^2(y)e^{5y^4+1} + 7]^2}{5\eta^4 - 2\eta + 1} \geq c_1 > 0$$

and

$$\sup_{y,\eta\in\mathbb{R},|y-\eta|\leq 2} \left\{ \frac{(1+8y^2)e^{7y^2-2y+1} - 2}{(1+8\eta^2)e^{7\eta^2-2\eta+1} - 2}, \frac{\chi(y)e^{5y^4+1} + 7}{\chi(\eta)e^{5\eta^4+1} + 7}, \frac{11y^4 - y + 8}{11\eta^4 - \eta + 8}, \right.$$

$$\left. \frac{5y^4 - 2y + 1}{5\eta^4 - 2\eta + 1}, \frac{11y^4 - y + 8}{5\eta^4 - 2\eta + 1} \right\} \leq C < \infty.$$

The function $k(y)$ is twice continuously differentiable and satisfies the condition (ii) of Lemma 5.7.

Really

1)$k(y) < 2$;

2) If $y \geq 0$ then $\frac{\sqrt{2}k'(y)}{k(y)} = -2\sqrt{2}y \leq 0$,

if $y < 0$ then $\frac{\sqrt{2}k'(y)}{k(y)} = \frac{2\sqrt{2}y}{2e^{y^2}-1} < 0$.

Thus $\frac{\sqrt{2}k'(y)}{k(y)} < 1$ for all $y \in \mathbb{R}$.

And we show that

$$\min_{y\in\mathbb{R}} \left\{\varphi^2(y), \psi^2(y)\right\} > 2 - k''(y).$$

Indeed,

if $y \geq 0$ then $k''(y) = -2\sqrt{2} \leq 0$,

if $y < 0$ then $k''(y) = \frac{1-2y}{e^{y^2}} < 1$.

So $k''(y) < 1$ for all $y \in \mathbb{R}$.

Then

$$\min_{y\in\mathbb{R}} \left\{\varphi^2(y), \psi^2(y)\right\} > 3,$$

namely

$$\min_{y\in\mathbb{R}} \left\{[(1+8y^2)e^{7y^2-2y+1} - 2]^2, [\chi(y)e^{5y^4+1} + 7]^2\right\} > 3,$$

holds for any $y \in \mathbb{R}$.

**Example 5.13.** *We consider the following problem*

$$\begin{cases} \left(\frac{\pi}{2} - arctgy\right) \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - 2\frac{\partial^2 v}{\partial x \partial y} - \left|\sqrt{(1+2y^4)e^{6y^2} + 11}\right| \frac{\partial u}{\partial x} + \\ \qquad e^{|y|+1}u + \frac{\chi(y)}{9} sinycos3yv = f(x,y), \\ 2\frac{\partial^2 u}{\partial x \partial y} + \left(\frac{\pi}{2} - arctgy\right) \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} + \left|\sqrt{(3+y^4)e^{6y^2+2y-3} + 7}\right| \frac{\partial v}{\partial x} + \\ \qquad \frac{1}{9} cosysin3yu + e^{8|y|+0.5}v = g(x,y), \end{cases}$$

$$w(-\pi, y) = w(\pi, y), w_x(-\pi, y) = w_x(\pi, y)$$

on the strip $G = \{(x, y) \in \mathbb{R}^2 : \ -\pi < x < \pi, \ -\infty < y < +\infty\}$. Here $\inf\limits_{y\in\mathbb{R}} k(y) = \inf\limits_{y\in\mathbb{R}} \left(\frac{\pi}{2} - arctgy\right) = 0$ and $\chi(y)$ is an arbitrary function such that $1 \leq \chi(y) \leq 2$ for all $y \in \mathbb{R}$, and $f, g \in L_2(G)$.

By Theorem 4.9 this problem has an unique solution $w = (u, v) \in L_2$ for any data $f, g \in L_2(G)$. Moreover, the solutions of the above system satisfy the coercive inequality with the norm of space $L_2(G)$ in the form (5.25).

We now show that the following functions

$$\begin{aligned} a(y) &= e^{|y|+1}, \\ b(y) &= \frac{\chi(y)}{9} sinycos3y, \\ c(y) &= \frac{1}{9} cosysin3y, \\ d(y) &= e^{8|y|+0.5}, \\ \varphi(y) &= -\left|\sqrt{(1+2y^4)e^{6y^2} + 11}\right|, \\ \psi(y) &= \left|\sqrt{(3+y^4)e^{6y^2+2y-3} + 7}\right| \end{aligned}$$

satisfy Assumption 2.

Indeed

1)there exists a constant $\delta > 0$ such that

$$\inf_{y\in\mathbb{R}} \{-\varphi(y), a(y), d(y)\} =$$

$$\inf_{y\in\mathbb{R}} \left\{ \left| \sqrt{(1+2y^4)e^{6y^2}+11} \right|, e^{|y|+1}, e^{8|y|+0.5} \right\} = \delta > 0;$$

2) there exist constants $r, q, (r > 0,\ q > 0,\ r + q = 1)$ and $\vartheta(0 < \vartheta < 3)$ such that

$$\frac{1}{2}\left( |\frac{\chi(y)}{9} sinycos3y| + |\frac{1}{9} cosysin3y| \right)^{2r} \leq \frac{e^{|y|+1}}{3},$$

$$\frac{1}{2}\left( |\frac{\chi(y)}{9} sinycos3y| + |\frac{1}{9} cosysin3y| \right)^{2q} \leq \frac{e^{8|y|+0.5}}{3},$$

$$\vartheta \left| \sqrt{(3+y^4)e^{6y^2+2y-3}+7} \right| > e^{8|y|+0.5}$$

for all $y \in \mathbb{R}$.

Also we can see that the functions $a(y), d(y), \varphi(y), \psi(y)$ of $\mathbb{R}$ satisfy the condition (5.1) of Lemma 5.1 and the condition (5.15) of Lemma 5.5. There exist constants $c_0, c_1$ and $C$ such that

$$\inf_{y,\eta\in\mathbb{R}, |y-\eta|\leq 2} \frac{(1+2y^4)e^{6y^2}+11}{e^{|\eta|+1}} \geq c_0 > 0,$$

$$\inf_{y,\eta\in\mathbb{R}, |y-\eta|\leq 2} \frac{(3+y^4)e^{6y^2+2y-3}+7}{e^{8|\eta|+0.5}} \geq c_1 > 0$$

and

$$\sup_{y,\eta\in\mathbb{R}, |y-\eta|\leq 2} \left\{ \frac{\left|\sqrt{(1+2y^4)e^{6y^2}+11}\right|}{\left|\sqrt{(1+2\eta^4)e^{6\eta^2}+11}\right|}, \frac{\left|\sqrt{(3+y^4)e^{6y^2+2y-3}+7}\right|}{\left|\sqrt{(3+y^4)e^{6\eta^2+2\eta-3}+7}\right|}, \frac{e^{|y|+1}}{e^{|\eta|+1}}, \right.$$

$$\left. \frac{e^{8|y|+0.5}}{e^{8|\eta|+0.5}}, \frac{e^{|y|+1}}{e^{8|\eta|+0.5}} \right\} \leq C < \infty.$$

The function $k(y) = \frac{\pi}{2} - arctgy$ is twice continuously differentiable and satisfies the condition (ii) of Lemma 5.7.

Really

1)$k(y) < 2$;

2) $\frac{\sqrt{2}k'(y)}{k(y)} = -\frac{2}{(1+y^2)(\frac{\pi}{2}-arctgy)} < 1$ for all $y \in \mathbb{R}$.

3) If $y < 0$ then $k''(y) = \frac{2y}{(1+y^2)^2} < 0$,

if $y \geq 0$ then $k''(y) = \frac{2y}{(1+y^2)^2} \leq 1$.

So $k''(y) = \frac{2y}{(1+y^2)^2} \leq 1$ for all $y \in \mathbb{R}$

Then we see that

$$\min_{y\in\mathbb{R}} \left\{ \varphi^2(y), \psi^2(y) \right\} > 3,$$

namely

$$\min_{y\in\mathbb{R}} \left\{(1+2y^4)e^{6y^2}+11, (3+y^4)e^{6y^2+2y-3}+7\right\} > 3,$$

holds for any $y \in \mathbb{R}$.

# Chapter 6

# Compactness of the resolvent and properties of the Kolmogorov diameters of the set $M = \left\{w \in D(L) : \|Lw\|_{2,G} + \|w\|_{2,G} \leq 1\right\}$

**Theorem 6.1.** *Let the conditions of Theorem 5.1 hold. Let*

$$\lim_{|y|\to+\infty} a(y) = +\infty \quad \textit{and} \quad \lim_{|y|\to+\infty} d(y) = +\infty. \tag{6.1}$$

*Then the inverse $L^{-1}$ of the operator $L$ is completely continuously in the space $L_2(G, \mathbb{R}^2)$.*

**Proof.** We denote by $W^2_{2,P,Q}(G, \mathbb{R}^2)$ the weighted space of functions $w = (u, v)$ with the norm

$$\|w\|_{2,P,Q} = \Big[\|u_{xx}\|^2_{2,G} + \|u_{yy}\|^2_{2,G} + \|u_{xy}\|^2_{2,G} + \|v_{xx}\|^2_{2,G} + \|v_{yy}\|^2_{2,G} + \|v_{xy}\|^2_{2,G} + \|\varphi(y)u_x\|^2_{2,G} + \|\psi(y)v_x\|^2_{2,G} + \|a(y)u\|^2_{2,G} + \|b(y)v\|^2_{2,G} + \|c(y)u\|^2_{2,G} + \|d(y)v\|^2_{2,G}\Big]^{\frac{1}{2}}.$$

By Corollary 5.9 inequality (5.25) holds for all $w$ in $D(L)$, which is contained in $W^2_{2,P,Q}(G,\mathbb{R}^2)$. Since the elements of the matrices $P(y)$ and $Q(y)$ are continuous and condition (6.1) holds then Theorem 1.6 implies that the space $W^2_{2,P,Q}(G,\mathbb{R}^2)$ is compactly imbedded into $L_2(G,\mathbb{R}^2)$. We now prove the compactness of the unit ball $M$ of $W^2_{2,P,Q}(G,\mathbb{R}^2)$ in $L_2(G,\mathbb{R}^2)$ by exploiting of the Fréchet-Kolmogorov theorem, see Theorem 1.6 of Chapter 1. We first verify conditions 2) and 3) of such statement. Let $w=(u,v)\in W^2_{2,P,Q}(G,\mathbb{R}^2)$. Let $h_1,h_2\neq 0$ and $N>0$. Clearly

$$\int_G |w(t+h_1,\tau+h_2)-w(t,\tau)|^2\,dtd\tau \le$$

$$\int_G \left|\int_t^{t+h_1} w'_\xi(\xi,\tau+h_2)d\xi + \int_\tau^{\tau+h_2} w'_\eta(t,\eta)d\eta\right|^2 dtd\tau \le$$

$$2\int_G \left|\int_t^{t+h_1} w'_\xi(\xi,\tau+h_2)d\xi\right|^2 dtd\tau + 2\int_G \left|\int_\tau^{\tau+2} w'_\eta(t,\eta)d\eta\right|^2 dtd\tau \le$$

$$2\int_G\int_t^{t+h_1} d\xi \int_t^{t+h_1} \left|w'_\xi(\xi,\tau+2\right|^2 d\xi dtd\tau +$$

$$2\int_G\int_\tau^{\tau+h_2} 1^2 d\eta \int_\tau^{\tau+h_2} \left|w'_\eta(t,\eta)\right|^2 d\eta dtd\tau \le$$

$$2|h_1|\int_t^{t+h_1} dt \int_G \left|w'_\xi(\xi,\tau+2h_2)\right|^2 d\xi d\tau +$$

$$2|h_2|\int_\tau^{\tau+h_2} d\tau \int_G \left|w'_\eta(t,\eta)\right|^2 d\eta dt =$$

$$2|h_1|^2\left\|w_x\right\|^2_{2,G} + 2|h_2|^2\left\|w_y\right\|^2_{2,G} \to 0 \quad \text{as} \quad h_1\to 0, h_2\to 0.$$

Now

$$\int_{\sqrt{\xi^2+\eta^2}\ge N} |u|^2 d\xi d\eta \le$$

$$\left(\inf_{|t|\ge N}[a(t)+|c(t)|]\right)^{-1}\int_G [a(t)+|c(t)|]|u|^2 d\xi d\eta \le$$

$$\left(\inf_{|t|\ge N} a(t)\right)^{-1}\left\|(a(t)+c(t))^{\frac{1}{2}}u\right\|^2_{2,G} \to 0 \quad \text{as} \quad N\to+\infty$$

and

$$\int\limits_{\sqrt{\xi^2+\eta^2}\geq N} |v|^2 d\xi d\eta \leq$$

$$\left(\inf_{|t|\geq N}[b(t)+|d(t)|]\right)^{-1}\int\limits_G [b(t)+|d(t)|]|v|^2 d\xi d\eta \leq$$

$$\left(\inf_{|t|\geq N} d(t)\right)^{-1}\left\|(b(t)+d(t))^{\frac{1}{2}}u\right\|^2_{2,G} \to 0 \text{ as } N\to+\infty.$$

This means, that operator $L^{-1}$ maps the whole of $L_2(G,\mathbb{R}^2)$ into the weighted space $W^2_{2,P,Q}$, i.e. the operator $L^{-1}$ is completely continuously. Thus the proof is complete. □

Theorems 5.8 and 6.1 allow us to consider the problem of estimating the diameters of the set $M = \{w \in D(L) : \|Lw\|_{2,G} + \|w\|_{2,G} \leq 1\}$, which is a part of the domain of definition of the operator $L$. By definition, the $k$-diameter of Kolmogorov of a set $M$ in $L_2$ is a number, which is equal to

$$d_k(M) = \inf_{G_k}\sup_{w\in M}\inf_{\omega\in P_k}\|w-\omega\|_{L_2},\ k=1,2,\ldots,$$

where $G_k$ is the set of the all subsets of $L_2$ with dimension no more than $k$. The estimate of the diameters can be used to understand the rate of convergence of the approximate solutions of equation $L\omega = F$ to the exact solution.

We denote by $N(\lambda)$ the number of diameters $d_k(M)$ greater than $\lambda > 0$.

**Theorem 6.2.** *Let the functions $\varphi, \psi, a, b, c, d, k$ satisfy Assumption 2 and conditions (5.1) and (5.15) and (6.1) and one and only one of conditions i), ii), iii). Then there exist constants $c_0, c$ such that*

$$c_0\lambda^{-1} meas\left(X : \|Q(X)\| \leq c_1^{-1}\lambda^{-\frac{1}{2}}\right) \leq$$
$$N(\lambda) \leq c\lambda^{-1} meas\left(X : \|Q(X)\| \leq c_2\lambda^{-\frac{1}{2}}\right),$$

*for the function $N(\lambda)$, where meas is the Lebesgue measure on $\mathbb{R}^2$.*

We introduce the following sets

$$\tilde{M}_s = \{w \in L_2(G,\mathbb{R}^2) : \|w\|_{W^2_{2,P,Q}} \leq s\},$$

$$\dot{M}_p = \{w \in L_2(G,\mathbb{R}^2) : \|w\|_{W^2_{2,P,Q}} \leq p\}.$$

In order to prove Theorem 6.2, we need some lemmas.

**Lemma 6.3.** *Let the assumptions of Theorem 6.2 hold. There exists a constant $C_1 > 1$ such that*

$$\dot{M}_{C_1^{-1}} \subseteq M \subseteq \tilde{M}_{C_1}.$$

**Proof.** Let $w = (u, v) \in \dot{M}_{C_1^{-1}}$. Since the conditions of Theorem 6.2 hold, the operator $Lw = B_{xy}w + P(y)w_x + Q(y)w$ is separable in the space $L_2(G, \mathbb{R}^2)$. Hence, we obtain that

$$\begin{aligned}
&\|Lw\|_{2,G} + \|w\|_{2,P,Q} \leq \\
&\qquad \|Lw\|_{2,G} + \|B_{xy}w\|_{2,G} + \|P(y)w_x\|_{2,G} \leq \\
&\qquad C_2 \Big[ \|u_{xx}\|_{2,G}^2 + \|u_{yy}\|_{2,G}^2 + \|u_{xy}\|_{2,G}^2 + \\
&\qquad \|v_{xx}\|_{2,G}^2 + \|v_{yy}\|_{2,G}^2 + \|v_{xy}\|_{2,G}^2 + \|\varphi(y)u_x\|_{2,G}^2 + \\
&\qquad \|u_y\|_{2,G}^2 + \|\psi(y)v_x\|_{2,G}^2 + \|v_y\|_{2,G}^2 + \|a(y)u\|_{2,G}^2 \\
&\qquad + \|b(y)v\|_{2,G}^2 + \|c(y)u\|_{2,G}^2 + \|d(y)v\|_{2,G}^2 \Big]^{\frac{1}{2}} \leq C_2 C_0^{-1},
\end{aligned}$$

for all $w = (u, v) \in \dot{M}_{C_1^{-1}}$. We set $C_2 = C_0$, then $\dot{M}_{C_0^{-1}} \subseteq M$.

Let $w = (u, v) \in M$. By inequality (5.25), we have

$$\begin{aligned}
&C \geq C \left( \|Lw\|_{2,G} + \|w\|_{2,P,Q} \right) \geq \\
&\qquad C_3 \left( \|B_{xy}w\|_{2,G} + \|P(y)w_x\|_{2,G} + \|Q(y)w\|_{2,G} \right)^{\frac{1}{2}} \geq \\
&\qquad C_4 \Big[ \|u_{xx}\|_{2,G}^2 + \|u_{yy}\|_{2,G}^2 + \|u_{xy}\|_{2,G}^2 + \\
&\qquad \|v_{xx}\|_{2,G}^2 + \|v_{yy}\|_{2,G}^2 + \|v_{xy}\|_{2,G}^2 + \\
&\qquad \|\varphi(y)u_x\|_{2,G}^2 + \|\psi(y)v_x\|_{2,G}^2 + \|a(y)u\|_{2,G}^2 + \\
&\qquad \|b(y)v\|_{2,G}^2 + \|c(y)u\|_{2,G}^2 + \|d(y)v\|_{2,G}^2 \Big]^{\frac{1}{2}},
\end{aligned}$$

for all $w = (u, v) \in D(L)$. Thus $M \subseteq \tilde{M}_c$. We now choose a constant $C_1$ so that $C_1 \geq C$ and $C_1^{-1} \leq C_0^{-1}$, and the proof is complete. □

**Lemma 6.4.** *Let the assumptions of Theorem 6.2 hold. The diameters $d_k(M)$ satisfy the following properties:*

*a)* $d_0 \geq d_1 \geq d_2 \geq \ldots$;

*b)* $d_k \tilde{M} \leq d_k(M)$, *if* $\tilde{M} \subseteq M$;

*c)* $d_k(nM) = nd_k(M)$, $n > 0$, *here* $nM = \{z : z = n\theta,\ \theta \in M\}$.

The proof of Lemma 6.4 is an immediate consequence of the definition of the diameters.

**Lemma 6.5.** *Let* $\dot{d}_k$, $\tilde{d}_k$ *be* $k$*-diameters of the sets* $\dot{M}_{C_1^{-1}}, \tilde{M}_{C_1}$*, respectively. Then the following inequalities hold*

$$C_{-1}\dot{d}_k \leq d_k(M) \leq C\tilde{d}_k, \quad k = 1, 2, .... \tag{6.2}$$

The inequalities (6.2) follow by Lemma 6.3 and by condition b) of Lemma 6.4.

We now introduce the functions $N(\lambda) = \sum\limits_{d_k > \lambda} 1$, $\tilde{N}(\lambda) = \sum\limits_{\tilde{d}_k > \lambda} 1$, $\dot{N}(\lambda) = \sum\limits_{\dot{d}_k > \lambda} 1$, which equal the number of diameters $d_k(M)$, $\tilde{d}_k(M)$, $\dot{d}_k(M)$ greater than $\lambda > 0$, respectively. From estimates (6.2), we easily deduce the validity of the following

**Lemma 6.6.** *Let* $N(\lambda)$ *be a number of diameters* $d_k(M)$ *greater than* $\lambda > 0$*. Let* $N(\lambda) = \sum\limits_{d_k > \lambda} 1$*. Then there exists a constant* $C > 1$ *such that*

$$\dot{N}(C\lambda) \leq N(\lambda) \leq \tilde{N}(C^{-1}\lambda).$$

We now recall the following well known result of M.Otelbayev [51].

**Theorem 6.7.** *Let* $N(\lambda)$ *be the number of diameters* $d_k(M)$ *greater than* $\lambda > 0$ *of imbedding* $L_p^l(\mathbb{R}^n, q) \to L_p(\mathbb{R}^n)$*, for* $1 < p < +\infty$*,* $pl > n$*,* $l > 0$*. Here* $L_p^l(\mathbb{R}^n, q)$ *the completion of* $C_0^\infty(\mathbb{R}^n)$ *in the following norm*

$$\|u\|_{L_p^l(\mathbb{R}^n,q)} = \left( \|(-\Delta)^{\frac{l}{2}} u\|_p^p + \int\limits_{\mathbb{R}^n} q(t)|u(t)|^p dt \right)^{\frac{1}{p}}.$$

*Then the following estimates hold*

$$C^{-1}N(C\lambda) \leq \lambda^{-\frac{n}{l}} mes\left(x \in \mathbb{R}^n : q^*(x) \leq \lambda^{-\frac{1}{l}}\right) \leq CN(C^{-1}\lambda).$$

*Here*

$$q^*(x) = \inf_{Q_d(x) \subseteq \mathbb{R}^n} \left( d^{-1} : d^{-pl+n} \geq \int\limits_{\bar{Q}_d(x)} q(t)dt \right),$$

$Q_d(x)$ *is a cube in* $\mathbb{R}^n$ *with center at* $x$ *and sides equal to* $d$*.*

**Proof of Theorem 6.2.** By Theorem 6.7 and by Lemma 6.6 the number $N(\lambda)$ of diameters $d_k(M)$ of the unit ball greater than $\lambda > 0$ satisfies the following inequalities

$$C^{-1}\dot{N}(C\lambda) \leq \lambda^{-1} meas \left(x \in \mathbb{R}^2 : \|Q(X)\| \leq \lambda^{-\frac{1}{2}}\right) \leq C\tilde{N}(C^{-1}\lambda).$$

Hence, by inequalities (6.2), the proof of the theorem is complete. □

By taking the inverse functions, and by Theorem 6.2 we can prove an asymptotic formula for the eigenvalues. If we denote by
$F(\lambda) = \lambda^{-1} meas \left(x \in \mathbb{R}^2 : \|Q(X)\| \leq \lambda^{-\frac{1}{2}}\right)$,
then by Theorem 6.2 implies that the functions $N(\lambda_n)$ satisfy the following estimates

$$F(c_2\lambda_n) \leq N(\lambda_n) \leq F(c_0\lambda_n).$$

**Corollary 6.8.** *Let the conditions of Theorem 6.2 hold. Then*

$$c_2^{-1}F^{-1}(n) \leq \lambda_n \leq c_0^{-1}F^{-1}(n),$$

*where $F^{-1}$ is the inverse function of the strongly momotone non-negative function $F$.*

# Chapter 7

# The solvability of the semiperiodical nonlinear problem for second order elliptic systems

We consider the folowing problem

$$\begin{cases} k(y)\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - 2\frac{\partial^2 v}{\partial x \partial y} + \varphi(y)\frac{\partial u}{\partial x} + a(y,u,v)u + b(y,u,v)v = f(x,y), \\ 2\frac{\partial^2 u}{\partial x \partial y} + k(y)\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} + \psi(y)\frac{\partial v}{\partial x} + c(y,u,v)u + d(y,u,v)v = g(x,y), \end{cases} \tag{7.1}$$

$$w(-\pi, y) = w(\pi, y), w_x(-\pi, y) = w_x(\pi, y), \tag{7.2}$$

in the strip $G = \{(x, y) \in \mathbb{R}^2 : \ -\pi < x < \pi, \ -\infty < y < +\infty\}$. Here $k(y)$ is a continuous and bounded real value function such that $\inf\limits_{y \in \mathbb{R}} k(y) > 0$, $f, g \in L_2(G)$. Let the functions $\varphi, \psi, a, b, c, d$ be continuous on $\mathbb{R}$.

The system (7.1) can be written in the following form

$$L_0 w = B_{xy} w + P(y) w_x + Q(y, w) w = F(X), \tag{7.3}$$

here

$$B_{xy} = \begin{pmatrix} k(y)\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} & -2\frac{\partial^2}{\partial x \partial y} \\ 2\frac{\partial^2}{\partial x \partial y} & k(y)\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \end{pmatrix},$$

$$P(y) = \begin{pmatrix} \varphi(y) & 0 \\ 0 & \psi(y) \end{pmatrix},$$

$$Q(y,w)=\left(\begin{array}{cc} a(y,u,v) & b(y,u,v) \\ c(y,u,v) & d(y,u,v) \end{array}\right),$$
$$U=(u,v)\,,\ F=(f,g)\,,\ X=(x,y).$$

**Assumption 3.** We assume that the real valued functions $\varphi, \psi, a, b, c, d$ on $\mathbb{R}$ satisfy the following conditions

$$\inf_{y\in\mathbb{R}}\{-\varphi(y), a(y,\xi,\zeta), d(y,\xi,\zeta)\}=\delta>0,\ \ \xi,\zeta\in\mathbb{R}; \tag{7.4}$$

$$\frac{1}{2}\left(|b(y,\xi,\zeta)|+|c(y,\xi,\zeta)|\right)^{2\alpha}\leq\frac{a(y,\xi,\zeta)}{3}, \tag{7.5}$$
$$\frac{1}{2}\left(|b(y,\xi,\zeta)|+|c(y,\xi,\zeta)|\right)^{2\beta}\leq\frac{d(y,\xi,\zeta)}{3},$$
$$\vartheta\psi(y)>d(y,\xi,\zeta),$$

where $\alpha, \beta$ and $\vartheta$ are constants such that $\alpha>0,\ \beta>0,\ \alpha+\beta=1,\ \vartheta<3.$

We denote by $L$ the operator with domain $D(L)=\{w(X)\in L_2 : Lw\in L_2\}$ defined by the formula

$$Lw=B_{xy}w+P(y)w_x+Q(y,w)w=F(X). \tag{7.6}$$

Let $W_2^2(G,\mathbb{R}^2)$ be the space of functions belonging to $L_2(G,\mathbb{R}^2)$ together their generalized derivatives up to second order. The norm of the space $W_2^2(G,\mathbb{R}^2)$ is defined as follows

$$\|w\|_{W_2^2(G,\mathbb{R}^2)}=\left[\left(\int\limits_G \|w_{xx}\|^2+\|w_{yy}\|^2+\right.\right.$$
$$\left.\|w_{xy}\|^2+\|w_x\|^2+\|w_y\|w^2+\|w\|^2\right)dxdy\Big]^{\frac{1}{2}}.$$

We denote by $W_{2,loc}^2(G,\mathbb{R}^2)$ the class of vector valued functions $\Phi=(\varphi(x,y),\psi(x,y))$, such that $\Phi\cdot\theta(x,y)\in W_2^2(G,\mathbb{R}^2)$ for all functions $\theta(x,y)\in C_0^\infty(G,\mathbb{R}^2)$.

We denote by $W_{2,P,Q}^2(G,\mathbb{R}^2)$ the space of functions $w=(u,v)$ with the norm

$$\|w\|_{2,P,Q}=\Big[\|u_{xx}\|_{2,G}^2+\|u_{yy}\|_{2,G}^2+\|u_{xy}\|_{2,G}^2+$$
$$\|v_{xx}\|_{2,G}^2+\|v_{yy}\|_{2,G}^2+\|v_{xy}\|_{2,G}^2+$$
$$\|\varphi(y)u_x\|_{2,G}^2+\|\psi(y)v_x\|_{2,G}^2+\|a(y)u\|_{2,G}^2+$$
$$\|b(y)v\|_{2,G}^2+\|c(y)u\|_{2,G}^2+\|d(y)v\|_{2,G}^2\Big]^{\frac{1}{2}}.$$

**Definition 7.1.** *A function $w = (u, v) \in L_2(G, \mathbb{R}^2)$ is said to be a solution of the problem (7.1), (7.2), if there exists a sequence $\{w_n\}_{n=1}^{\infty}$ of functions in the class $W_2^1(G, \mathbb{R}^2) \cap W_{2,loc}^2(G, \mathbb{R}^2)$ such that $\|\theta(w_n - w)\|_{L_2(G,\mathbb{R}^2)} \to 0$ and $\|\theta(Lw_n - F)\|_{L_2(G,\mathbb{R}^2)} \to 0$ as $n \to \infty$ for all $\theta \in C_0^{\infty}(G, \mathbb{R}^2)$.*

**Theorem 7.2.** *Let the following conditions hold.*

*a) The coefficients $\varphi, \psi, a, b, c, d$ of system (7.1) satisfy Assumption 3.*

*b) Let there exist constants $K_1(\sigma), K_2(\sigma), K_3(\sigma)$ such that ($U = (p, s)$, $U_1 = (p_1, s_1)$, $U_2 = (p_2, s_2)$)*

$$\inf_{y,\eta\in\mathbb{R},|y-\eta|\le 2,U\in\mathbb{R}^2} \frac{\varphi^2(y)}{a(\eta, p, s)} \ge K_1(\sigma) > 0, \tag{7.7}$$

$$\inf_{y,\eta\in\mathbb{R},|y-\eta|\le 2,U\in\mathbb{R}^2} \frac{\psi^2(y)}{d(\eta, p, s)} \ge K_2(\sigma) > 0;$$

$$\sup_{y,\eta,U_1,U_2\in\mathbb{R},|y-\eta|\le 2} \left\{ \frac{\varphi(y)}{\varphi(\eta)}, \frac{\psi(y)}{\psi(\eta)}, \frac{a(y,p_1,s_1)}{a(\eta,p_2,s_2)}, \frac{d(y,p_1,s_1)}{d(\eta,p_2,s_2)}, \frac{a(y,p_1,s_1)}{d(\eta,p_2,s_2)} \right\} \le K_3(\sigma) < \infty, \tag{7.8}$$

*for all $\sigma > 0$;*

*c)*

$$\lim_{|y|\to+\infty} a(y, p, s) = +\infty \ \text{and} \ \lim_{|y|\to+\infty} d(y, p, s) = +\infty, \tag{7.9}$$

*for all $U = (p, s) \in \mathbb{R}^2$.*

*d) The function $k(y)$ of $\mathbb{R}$ is twice continuously differentiable and satisfies one and only one of the following three conditions*

$$(i) \quad \sqrt{2} < k(y) < 2, \qquad \min_{y\in\mathbb{R}} \{\varphi^2(y), \psi^2(y)\} + k''(y) > 2\left[k'(y)\right]^2;$$

$$(ii) \quad k(y) < 2, \ \frac{\sqrt{2}k'(y)}{k(y)} \le 1, \qquad \min_{y\in\mathbb{R}} \{\varphi^2(y), \psi^2(y)\} + k''(y) > 2;$$

$$(iii) \quad k(y) < 2, \ k^2(y) > 2k'(y), \quad \min_{y\in\mathbb{R}} \{\varphi^2(y), \psi^2(y)\} + k''(y) > 2k'(y).$$

*Then problem (7.1), (7.2) has a solution $w = (u, v)$ in the Sobolev space $W_{2,P,Q}^2(G, \mathbb{R}^2)$ for every right hand side $F(X) \in L_2(G, \mathbb{R}^2)$ of system (7.1). Moreover, there exists a constant $\tilde{C} > 0$ independent of $F$ such that*

$$\begin{aligned} &\|u_{xx}\|_{2,G}^2 + \|u_{yy}\|_{2,G}^2 + \|u_{xy}\|_{2,G}^2 + \|v_{xx}\|_{2,G}^2 + \\ &\quad \|v_{yy}\|_{2,G}^2 + \|v_{xy}\|_{2,G}^2 + \|\varphi(y)u_x\|_{2,G}^2 + \|u_y\|_{2,G}^2 + \\ &\quad \|\psi(y)v_x\|_{2,G}^2 + \|v_y\|_{2,G}^2 + \|a(y,u,v)u\|_{2,G}^2 + \|b(y,u,v)v\|_{2,G}^2 + \\ &\quad \|c(y,u,v)u\|_{2,G}^2 + \|d(y,u,v)v\|_{2,G}^2 \le \tilde{C} \|F\|_{2,G}^2 . \end{aligned} \tag{7.10}$$

In order to prove Theorem 7.2, we need the following lemmas.

We consider the following 'system with weight '

$$L_\epsilon w_\epsilon = B_{xy} w_\epsilon + P(y)(w_\epsilon)_x + Q(y, w_\epsilon) + \epsilon(1 + |X|^2) w_\epsilon = F(X), \qquad (7.11)$$

where $F(X) \in L_2(G, \mathbb{R}^2)$.

We now choose the following ball

$$S_{H(\sigma)} = \{\omega(u, v) : \|\omega(u, v)\|_{C(\mathbb{R}^2, \mathbb{R}^2)} < H(\sigma)\}$$

in the space $C(\mathbb{R}^2, \mathbb{R}^2)$. Here $H(\sigma) = 2C\|F\| + 1 \equiv T$ (C is a constant in Corollory 5.9). In this ball we define the operator $\Phi_\epsilon(\omega)$ ($\epsilon > 0$), by following formula

$$\Phi_\epsilon(\omega) = (L_\epsilon(\omega))^{-1} F(X),$$

where $L_\epsilon(\omega)$ is the operator generated as system (7.3) and with the following coefficients

$$\tilde{a}_\epsilon(y) = a(y, \omega(y)) + \epsilon(1 + |y|^2),$$
$$\tilde{b}(y) = b(y, \omega(y)),$$
$$\tilde{c}(y) = c(y, \omega(y)),$$
$$\tilde{d}_\epsilon(y) = d(y, \omega(y)) + \epsilon(1 + |y|^2),$$

for all $\omega(y) \in S_{H(\sigma)}$ and for fixed $F(X) \in L_2(G, \mathbb{R}^2)$.

**Lemma 7.3.** *Let condition (7.9) of Theorem 7.2 hold. Then there exist constants $C_0, C_1 > 0$ such that*

$$\sup_{y,\eta \in \mathbb{R}, |y-\eta| \le 2} \left\{ \frac{\tilde{a}_\epsilon(y)}{\tilde{a}_\epsilon(\eta)} \right\} \le C_0 < \infty,$$

$$\sup_{y,\eta \in \mathbb{R}, |y-\eta| \le 2} \left\{ \frac{\tilde{d}_\epsilon(y)}{\tilde{d}_\epsilon(\eta)} \right\} \le C_1 < \infty.$$

**Proof.** If $|y - \eta| \le 2$ we have

$$|\omega(y) - \omega(\eta)| \le 2|\omega(y)| \le 2T,$$

for all $\omega \in S_{H(\sigma)}$.

Then by condition b) of Theorem 7.2 we obtain

$$\sup_{y,\eta \in \mathbb{R}, |y-\eta| \le 2} \left\{ \frac{\tilde{a}_\epsilon(y)}{\tilde{a}_\epsilon(\eta)} \right\} \le$$

$$\sup_{y,\eta\in\mathbb{R},|y-\eta|\le 2}\ \sup_{|U_1-U_2|\le\sigma}\left\{\frac{a(y,p_1,s_1)+\epsilon(1+|y|^2)}{a(\eta,p_2,s_2)+\epsilon(1+|\eta|^2)}\right\}\le$$
$$\sup_{y,\eta\in\mathbb{R},|y-\eta|\le 2}\ \sup_{|U_1-U_2|\le\sigma}\left\{\frac{a(y,p_1,s_1)}{a(\eta,p_2,s_2)+\epsilon(1+|\eta|^2)}\right\}+$$
$$\sup_{y,\eta\in\mathbb{R},|y-\eta|\le 2}\ \sup_{|U_1-U_2|\le\sigma}\left\{\frac{\epsilon(1+|y|^2)}{a(\eta,p_2,s_2)+\epsilon(1+|\eta|^2)}\right\}\le$$
$$\sup_{y,\eta\in\mathbb{R},|y-\eta|\le 2}\ \sup_{|U_1-U_2|\le\sigma}\left\{\frac{a(y,p_1,s_1)}{a(\eta,p_2,s_2)}\right\}+$$
$$\sup_{y,\eta\in\mathbb{R},|y-\eta|\le 2}\ \sup_{|U_1-U_2|\le\sigma}\left\{\frac{\epsilon(1+|y|^2)}{\epsilon(1+|\eta|^2)}\right\}\le K(2T)+4<\infty,$$

where $U_1=(p_1,s_1)$, $U_2=(p_2,s_2)\in G$. The second inequality of this lemma can be proved analogously. □

**Lemma 7.4.** *Let the assumptions of Theorem 7.2 hold. Let $F(X)\in L_2(G,\mathbb{R})$ be fixed. Then the operator $\Phi_\epsilon(\omega)$ maps the ball $S_{H(\sigma)}$ into itself and is completely continuous for all $\epsilon>0$.*

**Proof.** By the assumptions of Theorem 7.2, which hold for the functions $\tilde a_\epsilon(y)$, $\tilde b(y)$, $\tilde c(y)$, $\tilde d_\epsilon(y)$ and by Lemma 7.3 the operator $L_\epsilon(\omega)$ is bounded and invertible in the space $L_2(G,\mathbb{R})$ for every function $\omega\in S_{H(\sigma)}$. So, the existence of the operator $\Phi_\epsilon(\omega)$ has been proved.

By Lemma 7.3 and by Theorem 6.1, we have that

$$\|\Phi_\epsilon(\omega)\|_{C(\mathbb{R}^2,\mathbb{R}^2)}\le c\|F(X)\|_{W^2_{2,P,Q}}\le H(\sigma)-1.$$

Hence it follows that the operator $\Phi_\epsilon(\omega)$ maps the ball $S_{H(\sigma)}$ into itself.

Corollary 5.9 also implies that the operator $\Phi_\epsilon(\omega)$ maps the ball $S_{H(\sigma)}$ into some of part of the Sobolev space $W^2_{2,P,Q}$ with the norm

$$\|w\|_{2,P,Q}=$$
$$\Big[\|u_{xx}\|^2_{2,G}+\|u_{yy}\|^2_{2,G}+\|u_{xy}\|^2_{2,G}+$$
$$\|v_{xx}\|^2_{2,G}+\|v_{yy}\|^2_{2,G}+\|v_{xy}\|^2_{2,G}+\|\varphi(y)u_x\|^2_{2,G}+$$
$$\|u_y\|^2_{2,G}+\|\psi(y)v_x\|^2_{2,G}+\|v_y\|^2_{2,G}+\|\tilde a_\epsilon(y)u\|^2_{2,G}+$$
$$\left\|\tilde b(y)v\right\|^2_{2,G}+\|\tilde c(y)u\|^2_{2,G}+\left\|\tilde d_\epsilon(y)v\right\|^2_{2,G}\Big]^{\frac12}.$$

Since $\lim_{|y|\to+\infty}\tilde a_\epsilon(y)=+\infty$, $\lim_{|y|\to+\infty}\tilde d_\epsilon(y)=+\infty$, Theorem 6.1 implies that the operator $(L_\epsilon(\omega))^{-1}$ is compact. Hence it follows that the operator $\Phi_\epsilon(\omega)$ is compact in $C(\mathbb{R}^2, \mathbb{R}^2)$. Finally, the operator $\Phi_\epsilon(\omega)$ is continuous. Indeed, the functions $\tilde{a}_\epsilon(y)$, $\tilde{b}(y)$, $\tilde{c}(y)$ and $\tilde{d}_\epsilon(y)$ depend continuously on $\omega$.

**Proof of Theorem 7.1** By Lemma 7.4 the operator $\Phi_\epsilon(\omega)$ is completely continuous and maps the ball $S_{H(\sigma)}$ into itself. Thus, all the conditions of the well-known Schauder Fixed Point Theorem hold for the operator $\Phi_\epsilon(\omega)$ (cf. e.eg., Theorem 1.7 of Chapter 1). Hence, $\Phi_\epsilon(\omega)$ has fixed point $w_{\epsilon,0} = w_{\epsilon,0}(u, v)$ in the ball $S_{H(\sigma)}$, i.e,

$$\Phi_\epsilon(w_{\epsilon,0}) = (L_\epsilon(w_{\epsilon,0}))^{-1} F = w_{\epsilon,0}.$$

Now we set

$$\begin{aligned}
\tilde{a}_{\epsilon,0}(y) &= a(y, w_{\epsilon,0}) + \epsilon(1 + |y|^2),\\
\tilde{b}_{\epsilon,0}(y) &= b(y, w_{\epsilon,0}),\\
\tilde{c}_{\epsilon,0}(y) &= c(y, w_{\epsilon,0}),\\
\tilde{d}_{\epsilon,0}(y) &= d(y, w_{\epsilon,0}) + \epsilon(1 + |y|^2).
\end{aligned}$$

We now show that $w_{\epsilon,0} = w_{\epsilon,0}(u_{\epsilon,0}, v_{\epsilon,0})$ is a solution of (7.11) and that it satisfies the following inequality

$$\begin{aligned}
&\|(u_{\epsilon,0})_{xx}\|^2_{2,G} + \|(u_{\epsilon,0})_{yy}\|^2_{2,G} + \|(u_{\epsilon,0})_{xy}\|^2_{2,G} +\\
&\quad \|(v_{\epsilon,0})_{xx}\|^2_{2,G} + \|(v_{\epsilon,0})_{yy}\|^2_{2,G} + \|(v_{\epsilon,0})_{xy}\|^2_{2,G} +\\
&\quad \|\varphi(y)(u_{\epsilon,0})_x\|^2_{2,G} + \|(u_{\epsilon,0})_y\|^2_{2,G} + \|\psi(y)(v_{\epsilon,0})_x\|^2_{2,G} +\\
&\quad \|(v_{\epsilon,0})_y\|^2_{2,G} + \|\tilde{a}_{\epsilon,0}(y)u_{\epsilon,0}\|^2_{2,G} + \left\|\tilde{b}_{\epsilon,0}(y)v_{\epsilon,0}\right\|^2_{2,G} +\\
&\quad \|\tilde{c}_{\epsilon,0}(y)u_{\epsilon,0}\|^2_{2,G} + \left\|\tilde{d}_{\epsilon,0}(y)v_{\epsilon,0}\right\|^2_{2,G} \le \tilde{C}\,\|F\|^2_{2,G}.
\end{aligned}$$

We show that if $\epsilon \to 0$, then $w_{\epsilon,0}$ converges to the solution of system (7.1).

Let $\epsilon_k \to 0$ as $k \to \infty$. For every $\epsilon_k > 0$ we have $w_{\epsilon_k} \in W^2_{2P,Q_\epsilon}(G, \mathbb{R}^2)$. The space $W^2_{2,P,Q_\epsilon}(G, \mathbb{R}^2)$ is compactly imbeddes in $L_2(\Omega, \mathbb{R}^2)$, where $\Omega$ is a bounded subset of $G$. Hence, there exists a subsequence of $\{w_{\epsilon_k}\}$ which converges in $L_2(\Omega, \mathbb{R}^2)$. We denote it by $\{w_{\epsilon_k}\}$. Let $w$ be a limit of the sequence $\{w_{\epsilon_k}\}$. Then $\|\theta(w_{\epsilon_k} - w)\|_{L_2(G,\mathbb{R}^2)} \to 0$ as $k \to \infty$. By choosing the operator $\Phi_\epsilon(w)$, it follows that

$$\Phi_\epsilon(w_{\epsilon_k}) = (L_\epsilon(w_{\epsilon_k}))^{-1} F = w_{\epsilon_k}$$

or

$$L_\epsilon w_{\epsilon_k} = B_{xy} w_{\epsilon_k} + P(y)(w_\epsilon)_x + Q(y, w_{\epsilon_k}) + \epsilon(1 + |X|^2) w_{\epsilon_k} = F(X).$$

This means that $\|\theta(L_\epsilon w_{\epsilon_k} - F)\|_{L_2(G,\mathbb{R}^2)} \to 0$ as $k \to \infty$ for all $\theta \in C_0^\infty(G, \mathbb{R}^2)$. Thus, by definition $w$ is the solution of the system (7.1).

Since $w$ belongs to the set $\{w_{\epsilon_k}\}$, then $w$ satisfies the inequality (7.10). The proof of the theorem is complete. □

# Bibliography

AkGl47 [1] N.I. Akhiezer and I.M. Glazman, Theory of linear operators in Hilbert space, New York: Dover, 1993, XI, 147, 218 p.

ApOt43 [2] O.D. Apyshev and M. Otelbayev, On the spectrum of one class of differential operators and some embedding theorems, Izv. Akad. Nauk SSSR, Ser. Mat., 43(1979), P.739-764 (in Russian).

Baxl28 [3] J.V. Baxley, Some partial differential operators with discrete spectra, Spectr. Theory Differ.Oper. Proc. Conf. Birmingham, Ala., March 26-28, 1981, P.53-59.

BBJh54 [4] L. Bers, S. Bochner and F. John, Contributions to the theory of partial differential equations, Princeton: Princeton university press, 1954, 257 p.

Bits21 [5] A.V. Bitsadze, Boundary value problems for second order elliptic equations, Amsterdam: North-Holland, 1968, 211 p.

Bits3 [6] A.V. Bitsadze, Equations of mathematical phisics, Mir Publisher, Moskow, 1980, 319 p.

Bits48 [7] A.V. Bitsadze, On the uniqueness of the solutions of the Dirichlet problem for elliptic partial differential equations, Ukr. Mat. Zhur., III, 1948, no.6, P.211-212 (in Russian).

Bits82 [8] A.V. Bitsadze, Partial differential equations, Moskow, Nauka, 1982, 336 p. (in Russian).

Boim23 [9] K.Kh. Boimatov, Theory of separability, Dokl. Akad. Nauk SSSR, 1975, v.223, no.3, P. 521-524 (in Russian).

Brdr15 [10] F.E. Browder, Strongly elliptic system of differential equations, Ann. Math. Studies, no.33, Princeton (1954), P.15-51.

Brdr35 [11] F.E. Browder, On the regularity properties of solutions of elliptic differential equation, Comm. Pure Appl.Math., 9(1956), P. 351-361.

EdEv79 [12] D.E. Edmunds and W.D. Evans, Spectral theory and embeddings of Sobolev spaces, Quart. J.Math., 1979, v.30, no.120, P. 431-453.

EvGt75 [13] W.N. Everitt and M. Giertz, On some properties of the domains of powers of certain differential operators, Proc. London Math. Soc., v. 24(3), 1972, P. 756-768.

EvGt14 [14] W.N. Everitt and M. Giertz, On some properties of the powers of a formally selfadjoint differential expression, Proc. London Math. Soc., 1972, v.24(3), P. 149-170.

EvGt30 [15] W.N. Everitt and M. Giertz, Some properties of the domains of certain differ-ential operators, Proc. London Math. Soc., 1971, v.23(3), P. 301-324.

Frdh29 [16] K.O. Friedrichs, On the differentiability of solutions of linear elliptic differen-tial equations, Comm. Pure Appl.Math., 6(1953), P. 299-326.

Grdg55 [17] L. Garding, The Dirichlet problem for linear elliptic partial differential equations, Math.Scan.,1(1953), P.55-72.

GlSv30 [18] V.P. Glushko and Yu.B. Savchenko, Higher-order degenerate elliptic equations: spaces, operators, boundary-value problems, J. Soviet Math., 1987, v.39, no. 6, P. 3088-3148.

GdZl74 [19] S.K. Godunov and E.V. Zolotoreva, Sbornik zadach po uravneniam matematicheskoi phiziki, Nauka: Sibirskoe otdelenie, 1974, XV, 76 p.

GbKr37 [20] I.C. Gohberg and M. G. Krein, Introduction to the theory of linear nonselfadjoint operators, Providence R. I.: American mathematical society, 1969, XV, 378 p.

Hntn42 [21] Don B. Hinton, Molchanov's discrete spectra criterion for a weighted operator, Canad.Math.Bull., 1979, v.22, no.4, P. 425-431.

KaAk64 [22] L.N. Kantorovich and G.P. Akilov , 1964, Functional analysis in normed spaces, N.Y: Pergamon Press Ltd., 773 p.

Krei10 [23] S.G. Krein, Linear equations in Banach spaces, Boston: Birkhauser, 1982, XII, 102 p.

Ldzh40 [24] O.A. Ladyzhenskaya, Boundary value problems of mathematical physics, Moskow, Nauka, 1973, 408 p. (in Russian).

LdUr49 [25] O.A. Ladyzhenskaya and N.N. Uraltseva, Linear and quasilinear elliptic equations, New York: Academic press, 1968, XVIII, 495 p.

Leve19 [26] S. Levendorskii, Degenerate elliptic equations, Kluwer Academic Publisher, Dortrecht-Boston-London, 1993.

LiMa24 [27] J.L. Lions, E. Magenes, Non-homogeneous boundary value problems and applications, Berlin: Springer, v.

LiOt08 [28] P.I. Lisorkin and M. Otelbayev, Imbedding and compactness theorems for Sobolev-type space with weights, Mat. Sb. (N.S.), 108 (1979), no.3, P. 358-377 (in Russian).

Lopa12 [29] Ya.B. Lopatinskiy, A method of reducing the boundary-value problems for a system of elliptic differential equations to regular integral equations, Ukr. mat.zhurn., 1953, no.2, P. 123-151 (in Russian).

LySo51 [30] L.A. Lyusternik and V.I. Sobolev, The elements of functional analysis, Moskow-Leningrad, Gosudarstv. Izdat. Tehn.-Teor. Lit.,1951 (in Russian).

Molc11 [31] A.M. Molchanov, On conditions of discretness of the spectrum of second order selfadjoint differential equations, Dokl. Akad. Nauk SSSR, 1973, v.211, n.4, P. 787-790 (in Russian).

MuOt47 [32] M.B. Muratbekov and M. Otelbayev, Separability and estimates of the singu-lar numbers of the mixed-type operator, Izv. Vuz. Ser. Mat., 1989, no.3, P. 44-47 (in Russian).

Mikl67 [33] S.G. Mikhlin, Linear equations of mathematical physics, Holt, Rinehart and Winston, ins., 1967, 318 p.

MnOt88 [34] K. Mynbayev and Otelbayev, Weighted Functional spaces and differential operator spectrum, Moskow, Nauka, 1988, 288 p.(in Russian).

Nrb85 [35] L. Nirenberg, Remarks on strongly elliptic partial differential equations, Comm. Pure Appl. Math.,1955, no.8, P. 648-674.

Onr85 [36] R. Oinarov, The separability of the Schrödinger operator in the space of summable functions, Dokl. Akad. Nauk SSSR, 1985, v.285, no. 5, P. 1062-1064 (in Russian).

OlRd73 [37] O.A. Oleǐnik and E.V. Radkevič, Second order equations with nonnegative characteristic form, Providence, R.I.: American mathematical society and Plenum press, New York, 1973, 259 p.

Osp82 [38] K.N. Ospanov, On the properties of solutions of the Schrödinger-type stationary equations, Izv. Akad. Nauk KSSR, Ser. Phys.-math., 1982, no.3, P. 60-62 (in Russian).

Osp04 [39] K.N. Ospanov, On one well posed problem for singular Bitsadze-type system, Mat. Zhur. RK, 2004, v.4, no.3 (13), P. 68-73 (in Russian).

Osp96 [40] K.N. Ospanov, Coercivity and solvability of the generalized Cauchy-Riemann system in the space $L_p(E)$, Ukr. Mat. Zhur., 1996, no.11 (15), P. 1564-1569 (in Russian).

Ospv4 [41] K.N. Ospanov, On conditions of coercivity and solvability of second order singular systems of mixed type, Mat. Mezh. Ross.-Kaz. Symp., Nalchik-Elbrus, 2004, P. 138-140 (in Russian).

Osp05 [42] K.N. Ospanov, On approximation properties of on elliptic operator, Tez. Mezhd. Konf. ‘Prob. Sovr. Mat. I mech.‘, Almaty, 2005, P. 109-110 (in Russian).

OpZh06 [43] K.N. Ospanov and L.K. Zhapsarbayeva, On the problem of coercivity and solvability of second order singular systems of mixed type, Tez. Dokl. IV mezhd. Konf., Aktobe, 2006, P. 92-93 (in Russian).

Otl77 [44] M. Otelbayev, On the separability of the elliptic operators, Dokl. Akad. Nauk SSSR, 1977, v. 243, 1977, v.243, no. 3, P. 540-543 (in Russian).

Otl83 [45] M. Otelbayev, Coercivity estimates and separability theorems of elliptic equations, Tr. Mat. Inst. AN SSSR, 1983, v. 161, P. 195-217 (in Russian).

Otl7 [46] M. Otelbayev, On the regularity of the solutions of differential equations, Izv. Akad. Nauk KazSSR, ser. Phys.-mat., 1977, no.5, P. 45-48 (in Russian).

Otl79 [47] M. Otelbayev, Nonlocal estimates of the derivatives of solutions of the elliptic equations in unbounded domains, Prepr. Inst. Mat. Sib. Otd. Akad. Nauk SSSR, 1979, 18 p. (in Russian).

Otl73 [48] M. Otelbayev, On the Titchmarsch method of estimating the resolvent, Dokl. Akad. Nauk SSSR, 1973, v. 211, no.4, P.787-790 (in Russian).

Otl90 [49] M. Otelbayev. Estimates of the spectrum of the Sturm-Liouville operator, Almaty, Gylym, 1990, 192 p. (in Russian).

Otl13 [50] M. Otelbayev, Criteria of discreteness of the spectrum of one singular operator and some embedding theorems, Differential Equations, 13 (1977), no.1, P.111-120 (in Russian).

Otl150 [51] M. Otelbayev, Emdedding theorems of the spaces with weight and their applications to study of the spectrum of the Schrödinger operator, Tr. Mat. Ins. Akad. Nauk SSSR, 1979, v.150, P.265-305 (in Russian).

Ptr67 [52] I.G. Petrovskii, Partial differential equations, Iliffe Books Ltd., London, 1967, 410 p.

Sch59 [53] M. Schechter, General boundary value problems for elliptic partial differential equations, Comm. Pure Appl.Math., 12(1959), P.457-486.

Smv78 [54] M.M. Smirnov, Equations of mixed type, Providence, R.I.: American mathe-matical society, 1978, III, 232 p.

Smv66 [55] M.M. Smirnov, Second-order partial differential equations, Groningen: Noordhoff, 1966, 192 p.

Sld72 [56] A.P. Soldatov, On the uniqueness of the solution of Bitsadze problem, Diff. Equat., 1972, v.8, P.143-146 (in Russian).

Sld04 [57] A.P. Soldatov, The Dirichlet and Neumann problems for second order elliptic systems in the halfplane, Tez. Mezhd. Konf. 'Dif. Urav. I smezh. Vopr.' Moskow, 2004, P.218-219 (in Russian).

Sld64 [58] A.P. Soldatov, The Dirichlet problem for second order elliptic differential equations, Dokl. Akad. Nauk SSSR, 1964, v.159, 995 p.(in Russian).

Tvm65 [59] N.E. Tovmasyan, Some boundary problem for second order elliptic type system of equations in the plane, Dokl. Akad. Nauk SSSR, 1965, v. 160, P.1275-1278 (in Russian).

Tvm66 [60] N.E. Tovmasyan, The general boundary value problem for second order elliptic systems of equations with constant coefficients, Diff. Equat., 1966, v. 2, P. 163-171 (in Russian).

Tvm04 [61] N.E. Tovmasyan, The Poincaré problem with shift for irregular second order elliptic equations, Mat. Mezh. Ross.-Kaz. Symp., Nalchik-Elbrus, 2004, P.170-172 (in Russian).

TvBb04 [62] N.E. Tovmasyan, A.O. Babayan, The Dirichlet problem for second order elliptic systems in the class of the functions with polynomial growth, Mat. Mezh. Ross.-Kaz. Symp., Nalchik-Elbrus, 2004, P.172-174 (in Russian).

Vk62 [63] I.N. Vekua, Generalized analytic functions, Pergamon press, 1962, XXIX, 668 p.

Vsh51 [64] M.I. Vishik, On strongly elliptic systems of differential equations, Mat. Sb., 1951, v. 29, P.615-676 (in Russian).

Vsh69 [65] M.I. Vishik and V.V. Grushin, Boundary value problems for elliptic equations that degenerate on the boundary of the region, Mat. Sb. SSSR, 1969, v.9, no.4, P.423-454 (in Russian).

Ysd78 [66] K. Yoshida, Functional analysis, Berlin: Springer-Verlag, 1978, XII, 500 p.

Zhp05 [67] L.K. Zhapsarbayeva, Second order degenerate systems with right hand side in $L_2$, Vestnik ENU, 2005, no.4(44), P.36-41 (in Russian).