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Goldstone Fields with Spins Higher than 1/2

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Declaration

The work in this thesis is based on research carried out in the Department of Physics and Astronomy at the University of Padua. The results presented have already appeared in the following publication:

- Sukruti Bansal and Dmitri Sorokin, *Can Chern-Simons or Rarita-Schwinger be a Volkov-Akulov Goldstone?* JHEP, 07:106, 2018. doi: 10.1007/JHEP07(2018)106.

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Abstract

We study three-dimensional non-linear models of vector and vector-spinor Goldstone fields associated with the spontaneous breaking of certain higher-spin counterparts of supersymmetry. The Lagrangians in these models are of a Volkov-Akulov type. Goldstone fields in these models transform non-linearly under the spontaneously broken rigid symmetries. We find that the leading term in the action of the vector Goldstone model is the Abelian Chern-Simons action whose gauge symmetry is broken by a quartic term. As a result, the model has a propagating degree of freedom which, in a decoupling limit, is a quartic Galileon scalar field. The vector-spinor goldstino model turns out to be a non-linear generalization of the three-dimensional Rarita-Schwinger action. In contrast to the vector Goldstone case, this non-linear model retains the gauge symmetry of the free Rarita-Schwinger action and eventually reduces to the latter by a non-linear field redefinition. We thus find that the free Rarita-Schwinger action is invariant under a hidden rigid supersymmetry generated by fermionic vector-spinor operators and acting non-linearly on the Rarita-Schwinger goldstino.

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Chapter 1

Introduction

One of the long-term goals of the collective studies and research in high energy physics is to formulate a consistent theory of quantum gravity. Up till now we have consistent theories of quantum mechanics and quantum field theory on the one hand and general relativity on the other. But the unification of the quantum theories with general relativity remains yet to be achieved.

Three of the four fundamental forces of nature, viz. electromagnetism, the weak force and the strong force, along with the elementary particles in the universe, can be described through the theory of the Standard Model. The three forces can be unified at high energy scales via the Grand Unified Theories (GUT). The Standard Model and GUTs being quantum theories aim at the description of physics of the world around us at the microscopic scale. However, the theory describing the force of gravity, i.e. general relativity, is a classical theory which explains the gravitational force only at the macroscopic scale. Theorists aim at finding a theory which can unify the Standard Model (or its extensions) with gravity within a unique quantum theory. String theory is a candidate for such a theory.

One of the important ingredients of string theory is supersymmetry, which is a conjectured symmetry of the universe. It proposes that all the matter and force particles in the universe have superpartner particles.

There are two kinds of particles in nature – fermions, the matter particles – and bosons, the force carriers. Fermions have half-integer spins while bosons have integer spins. Supersymmetry states that every fermion has a bosonic superpartner and vice-versa.

Some of the elementary fermions that we know well by now are electrons, muons, etc. and some of the elementary bosons that we know well are photons, $W^{\pm}$ bosons, Z bosons, gluons and the Higgs boson. Though we know of both fermionic and bosonic particles in nature, we don't yet know of any particles that form a supersymmetric pair. It is believed that the superpartner particles are too heavy to get detected by us using the experiments that have been used up till now.

Since supersymmetry is not explicitly observable in nature, it is believed to be spontaneously broken. There is supposedly a supersymmetry-breaking energy-scale M_s such that at energies $E > M_s$ the theory behaves in a supersymmetric way, while at energies $E < M_s$ supersymmetry is spontaneously broken. If M_s is not too high (e.g. around 1-10 TeV which is the maximum energy scale reachable by LHC) it may be possible for us to observe a spontaneously broken supersymmetry which would then serve as proof for the existence of original supersymmetry.

In quantum field theory the values of the coupling parameters vary or "run" with varying energy. This is known as *renormalization group flow*. Supersymmetry renormalizes the coupling parameters such that they converge to a single value at a very high energy scale, known as the GUT scale. So supersymmetry may be responsible for the grand unification of three of the four fundamental forces.

Attempts at formulating a supersymmetric theory of gravity yielded the theory of supergravity. In supergravity, supersymmetry is a local symmetry generalizing the reparametrization invariance [1]. Supergravity consists of fields with spin 2 or lower. A field with spin higher than 2 is called a *higher-spin field*. So we can say that supergravity does not have any higher spin fields. The classical dynamics of supergravity is well known but when we try to quantize it, we run into a problem because we find that it is non-renormalizable. In recent years, however, it has been found that a maximal N=8 supergravity in four space-time dimensions is finite up to at least five loops (see [2] and references therein).

Theoretical high energy physicists have been working hard trying to get closer to a consistent theory of quantum gravity. As we have already mentioned, string theory is a candidate theory of quantum gravity. It looks very promising. Superstring theory is a quantum theory which is believed to be renormalizable and even finite in the ultraviolet limit. So it can consistently describe quantum gravity. One main feature of this theory which distinguishes it from supergravity and makes it renormalizable is the quantum corrections contributed to it by an infinite tower of massive higher spin fields. For this reason a better understanding of the dynamics of higher spin fields is required.

Quantum field theory has shown that massive fields with spin 1 are renormalizable only if their mass is generated by the spontaneous breaking of the gauge symmetry associated with the corresponding massless gauge fields. One may assume that the same feature applies to higher spin fields. For this reason string theory has been conjectured to be a spontaneously broken phase of an underlying gauge theory of massless higher spin fields. So we need to study the higher spin fields which are massless.

In string theory the mass squared of the higher spin fields is proportional to string tension. So spontaneous breaking of higher-spin symmetry would generate both the mass and the tension of a string.

One approach for studying higher spin string states is the framework of String Field Theory. It is still under construction with regards to supersymmetric and closed strings. Another possible approach is to derive an effective field theory of higher spin fields and study it using conventional field theoretical methods. In the last few decades, the study of massless higher spin fields has revealed a profound and rich geometrical and conformal structure underlying their dynamics.

One of the main problems in higher spin field theory that needs to be addressed is the description of the higher-spin field interactions of different kinds, e.g. three- and four-vertex higher-spin interactions [3–5], interactions with electromagnetic fields [6, 7] and with gravity [8, 9]. When we describe interactions, we need to consider the S-matrix which relates the initial state and the final state of the system undergoing the interaction process. However there are certain restrictions regarding what values the entries of an S-matrix can take. The general no-go theorems like Coleman-Mandula theorem and Haag-Lopuszanski-Sohnius theorem do not allow conserved currents associated with the symmetries of higher spin fields to non-trivially contribute to the unitary S-matrix in D=4 Minkowski space.

One way to circumvent the no-go theorems is to spontaneously break higher spin symmetry, as probably happens in string theory. Another way around is to study the theory in a spacetime different from Minkowski space, a spacetime with non-zero cosmological constant, such as the Anti de Sitter space. This assertion has been successfully used by Fradkin and Vasiliev in [10] to describe the interaction of massless higher spin fields with gravity, up to cubic order.

The construction of higher spin theory was put forward by Dirac in 1936 when he generalized his famous spin-1/2 Dirac equation to the description of free higher spin fields. In 1939 Fierz and Pauli initiated a systematic study of higher-spin particles. They used a field theoretic approach demanding the conditions of Lorentz invariance and positivity of energy for physical consistency. They also proposed a general structure of Lagrangians describing free massive higher-spin fields (involving auxiliary fields of lower spins) and gave an explicit form of the Lagrangian for a massive field of spin 2. Their work was developed by Rarita and Schwinger [11] who proposed a description of the theory of fields of half-integer spins (in particular of spin 3/2) which is simpler than the formalism of Fierz and Pauli based on the use of auxiliary fields. Wigner [12] and Bargmann and Wigner [13] put the description of higher-spin particles on a solid group-theoretical basis by associating them with irreducible unitary representations of the Poincaré group.

Fronsdal [14] and Chang [15] elucidated a procedure for introducing auxiliary fields to construct higher-spin Lagrangians. Further contributions were made by Weinberg [16–18] and others. In 1974 Singh and Hagen [7, 19] managed to write down an explicit form of the Lagrangian for a free massive field of arbitrary spin. The Singh-Hagen Lagrangian for a half-integer spin incorporates symmetric gamma-traceless tensor-spinors and for an integer spin it is written in terms of a set of symmetric traceless tensor fields. Performing a massless limit of these Lagrangians Fronsdal [20], and Fang and Fronsdal [21], respectively, constructed Lagrangians for free massless integer and half-integer higher spin fields. For a review of the historical developments and various aspects of higher-spin field theory look at references [22–47].

Theoretical studies, in particular by Vasiliev (see e.g. [22, 30] for a review), have lead to the conclusion that interacting higher-spin gauge theories should be based on infinite dimensional symmetries and involve an infinite number of fields of increasing spin. The study of the effects of spontaneous symmetry breaking and the appearance of higher-spin Goldstone fields in such theories is a highly non-trivial problem. For this reason we have studied this problem in a simplified set-up. Firstly, we have started studying it in D=3. And secondly, we consider a special class of higher spin algebras, which unlike the conventional higher spin algebras, are finite. We have constructed spin-1 and spin-3/2 Goldstone field models based on these algebras and analysed their properties. The results of this research were published in [48].

1.1 Supersymmetry Breaking

There are two ways in which supersymmetry can be broken – spontaneously and explicitly.

i) **Spontaneous supersymmetry breaking:** In this process it is the vacuum state of the system for which the symmetry is broken. In such vacua one or more scalar fields acquire a vacuum expectation value of the order of the energy scale required for breaking supersymmetry.

ii) **Explicit supersymmetry breaking:** In this process supersymmetry is broken explicitly by adding non-supersymmetric terms in the Lagrangian. In order to preserve the renormalizability of the theory, only a specific kind of terms can be added. On adding such terms supersymmetry is said to be softly broken.

Generally even soft supersymmetry breaking models are assummed to arise as low energy effective descriptions of models where supersymmetry is broken spontaneously. Therefore, we will now discuss spontaneous symmetry breaking. In this thesis, we describe effects of spontaneous supersymmetry breaking using Volkov-Akulov construction [49, 50], which will be explained in further detail ahead.

1.2 Spontaneous Symmetry Breaking

Spontaneous symmetry breaking is a process through which a symmetric system ends up in an asymmetric state. The Lagrangian and the equations of motion of the system continue to obey the symmetries even after the spontaneous breaking of symmetry but a lowest-energy state does not retain the symmetry. In field theory the lowest energy states are the vacua.

Symmetry can be broken spontaneously only for a system with a non-unique vacuum state. This was first suggested by Nambu et. al [51]. If the lowest energy state of a system is degenerate, then its corresponding eigenstates linearly transform among themselves under the symmetry transformations. So there is no unique eigenstate to represent the ground state. Consequently each choice results in breaking the symmetry spontaneously.

When supersymmetry is broken spontaneously, the charges generating the symmetry do not annihilate the vacuum, i.e.,

$$Q\,|0\rangle \neq 0\,. \tag{1.1}$$

Supersymmetry algebra implies that $Q^\dagger Q + QQ^\dagger = H$. Then as a consequence of eq. (1.1), the vacuum expectation value of the Hamiltonian H of the system is positive-definite

$$\langle 0|H|0\rangle > 0. \tag{1.2}$$

The Hamiltonian comprises of kinetic energy terms and potential energy terms. In the vacuum the vacuum expectation values of the kinetic energy terms can be trivially set to zero. So the potential energy of the vacuum state should have a positive definite expectation value

$$\langle 0|V|0\rangle > 0\,. \tag{1.3}$$

The requirement of the vacuum states to be invariant under Poincaré transformations necessitates that they be a function of a scalar field $\phi(x)$ and its vacuum expectation value be constant

$$\langle 0|\phi(x)|0\rangle = c \neq 0\,. \tag{1.4}$$

On the other hand, the vacuum expectation values of the vector fields $A^a(x)$ and spinor fields $\psi^\alpha(x)$ present in the system, should vanish

$$\begin{aligned}\langle 0|A^a(x)|0\rangle &= 0\,,\\ \langle 0|\psi^\alpha(x)|0\rangle &= 0,.\end{aligned} \tag{1.5}$$

Conditions (1.3) and (1.4) can be summed up to say that the condition for the spontaneous breaking of supersymmetry is that the scalar field configuration should be such that the vacuum expectation value of its potential energy is positive definite

$$\langle 0|V(\phi(x))|0\rangle > 0\,. \tag{1.6}$$

Consider the simplest possible case of a scalar field ϕ in terms of which the potential

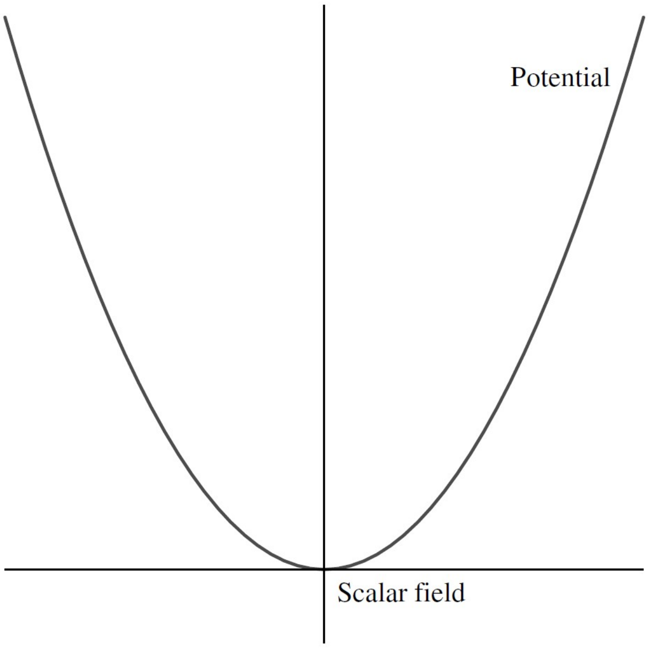

FIGURE 1.1: No symmetry broken

$V(\phi)$ is defined. The expectation values of the potential for four different cases regarding the breaking of supersymmetry and gauge symmetry are shown in Figures 1.1 to 1.4.

In Figure 1.1 since the vacuum expectation value of the potential vanishes for a certain value of the scalar field ϕ, supersymmetry is not broken. Since the potential is minimised for $\langle 0|\phi|0\rangle = \langle\phi\rangle = 0$, the gauge symmetry of the system (if it acts on ϕ) also stays intact.

In Figure 1.2 since the vacuum expectation value of the potential is positive definite, supersymmetry is spontaneously broken. However, since the potential is minimised for $\langle\phi\rangle = 0$, the gauge symmetry stays intact.

In Figure 1.3 the potential vanishes for two configurations of the scalar field ϕ. So its vacuum expectation value is zero and supersymmetry is not broken. Since the potential is minimised at $\langle\phi\rangle \neq 0$, gauge symmetry is broken.

In Figure 1.4 the potential is positive definite. So supersymmetry is broken spontaneously. Gauge symmetry is also broken because the potential is minimised at $\langle\phi\rangle \neq 0$.

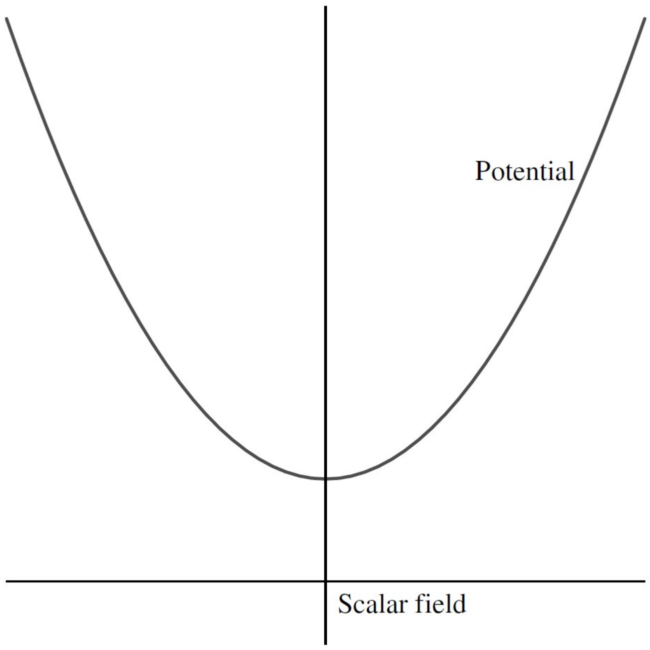

FIGURE 1.2: Supersymmetry broken but gauge symmetry intact

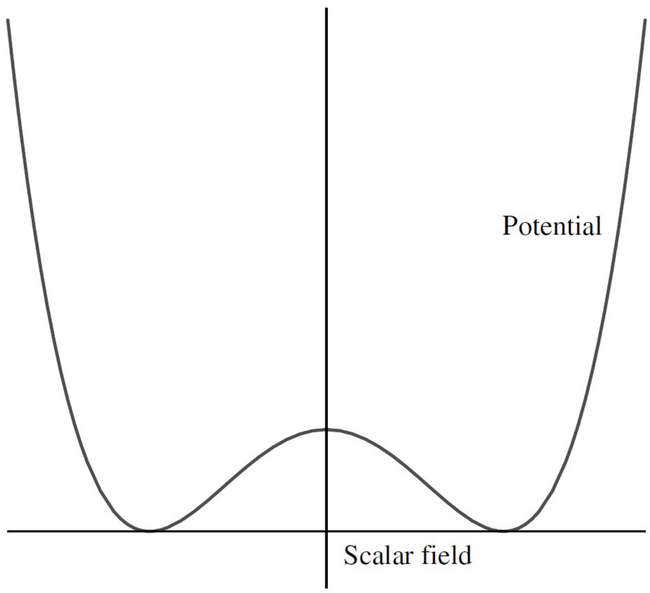

FIGURE 1.3: Supersymmetry intact but gauge symmetry broken

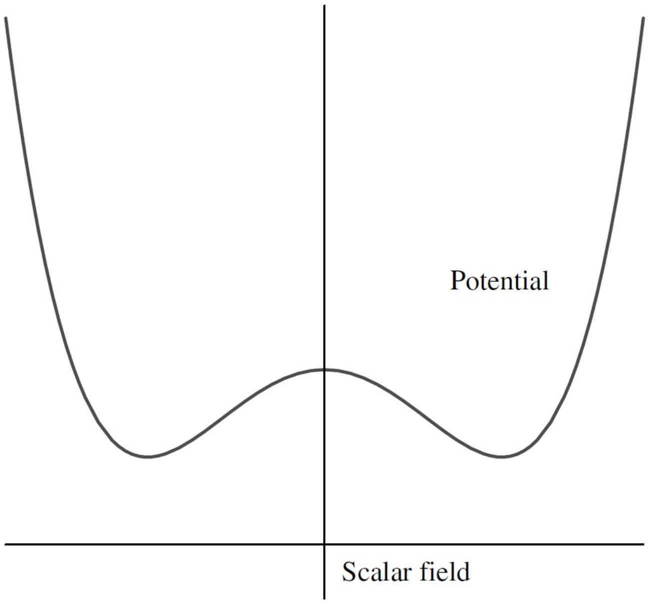

FIGURE 1.4: Both supersymmetry and gauge symmetry broken

Every time a continuous symmetry is broken spontaneously, be it a bosonic symmetry or supersymmetry, a massless particle is produced by virtue of Goldstone's theorem.

Goldstone's Theorem: For every spontaneously broken continuous symmetry, the theory must contain a massless particle known as a Goldstone field.

Proof: Suppose the system has a number of scalar fields ϕ_i transformed under a certain representation of the symmetry group G. $V(\phi)$ is minimised at $\phi = \langle\phi\rangle$, i.e.,

$$\left.\frac{\partial V}{\partial \phi}\right|_{\phi=\langle\phi\rangle} = 0\,. \tag{1.7}$$

Let the generators which break the vacuum symmetry be T^a, i.e. $T^a\langle\phi\rangle \neq 0$. The mass matrix is defined as following:

$$M^{ij} = \left.\frac{\partial^2 V}{\partial\phi_i\,\partial\phi_j}\right|_{\phi=\langle\phi\rangle} \tag{1.8}$$

The symmetry of the Lagrangian implies

$$\begin{aligned} V(\phi) &= V(\phi + i\,\epsilon_a T^a\phi) \\ &= V(\phi) + i\,\epsilon_a T^a_i\phi\,\frac{\partial V}{\partial\phi_i} + \mathcal{O}(|\epsilon|^2) \quad \text{(using Taylor expansion)} \end{aligned}$$

Retaining terms up to the 1^{st} order in ϵ_a, the above equation gives us,

$$T_i^{aj}\phi_j\,\frac{\partial V}{\partial\phi_i}=0 \tag{1.9}$$

Differentiating the above equation with respect to ϕ_k and evaluating it at $\phi=\langle\phi\rangle$ gives

$$\begin{aligned}&\frac{\partial}{\partial\phi_k}\big(T_i^{aj}\phi_j\,\frac{\partial V}{\partial\phi_i}\big)\Big|_{\phi=\langle\phi\rangle}=0\\ =\;&T_i^{ak}\,\frac{\partial V}{\partial\phi_i}\Big|_{\phi=\langle\phi\rangle}+T_i^{aj}\phi_j\,\frac{\partial V}{\partial\phi_k\,\partial\phi_i}\Big|_{\phi=\langle\phi\rangle}=0\end{aligned} \tag{1.10}$$

From eq. (1.7) we know that the first term on the LHS of eq. (1.10) is 0. Therefore, the second term should also be equal to 0.

$$T_i^{aj}\phi_j\,\frac{\partial V}{\partial\phi_k\,\partial\phi_i}\Big|_{\phi=\langle\phi\rangle}=T_i^{aj}\langle\phi_j\rangle\,M^{ki}=0 \tag{1.11}$$

We already know that $T^a\langle\phi\rangle\neq 0$. This implies that M^{ki} has a null eigenvector which has eigenvalue equal to 0. Since the eigenvalue of M^{ki} corresponds to the mass of the particle, the particle's mass is 0. Therefore spontaneously breaking the symmetry produces a massless particle, known as a Goldstone field.

A Goldstone field with integer spin is called a Goldstone boson while a Goldstone field with half-integer spin is called a goldstino. In this thesis we will construct models for both – a Goldstone boson and a goldstino. A distinguished feature of the effective Lagrangians for Goldstone fields (including the Volkov-Akulov Lagrangians considered below) is that they describe low energy dynamics of the Goldstone fields in a universal way independently of the details of the nature of the spontaneous symmetry breaking mechanisms, which, in general, can be yet unknown.

1.3 Poincaré Group Representations and Spin

We consider Minkowski spacetime with the following line element,

$$ds^2=\eta_{mn}\,dx^m\,dx^n\,, \tag{1.12}$$

where x^m are space-time coordinates ($m=0,1,2,...D-1$) and D is the number of spacetime dimensions.

In D dimensions the Minkowski metric η_{mn} is a $D \times D$ matrix as following.

$$\eta_{mn} = \begin{pmatrix} -1 & 0 \cdots & 0 \\ 0 & 1 \cdots & 0 \\ \vdots & \vdots \ddots & \vdots \\ 0 & 0 \cdots & 1 \end{pmatrix}$$

The coordinates of two arbitrary inertial reference frames, say x^m and x'^m, are related to each other by the following linear non-homogeneous transformation:

$$x'^m = \Lambda^m{}_n \, x^n + b^m \, . \tag{1.13}$$

This transformation leaves the Minkowski metric invariant. The invariance under the tranformation by the matrix Λ in particular is expressed by the following relation.

$$\Lambda^T \, \eta \, \Lambda = \eta \tag{1.14}$$

where Λ^T is the matrix transpose of Λ.

Taking the determinant of eq. (1.14), we can see that,

$$\begin{aligned} &\det \Lambda = \pm 1 \\ \Rightarrow\ &(\Lambda^0{}_0)^2 - (\Lambda^1{}_0)^2 - (\Lambda^2{}_0)^2 - (\Lambda^3{}_0)^2 = 1 \end{aligned} \tag{1.15}$$

In order to preserve the direction of time one must demand

$$\Lambda^0{}_0 \geq 1 \, . \tag{1.16}$$

Preservation of parity or spatial orientation requires that

$$\det \Lambda = 1 \, . \tag{1.17}$$

The transformations (1.13) obeying the constraints (1.14, 1.16 and 1.17), are called the "*Poincaré transformations*". In the homogeneous case, when $b^m = 0$, they are called the "*Lorentz transformations*". From here onwards the Poincaré transformations will be symbolically denoted as (Λ, b) and the Lorentz transformations simply by Λ.

A *Lie group* is a differentiable manifold that is also a group which respects the continuum properties of the manifold. On taking the union of all the Poincaré transformations under the following multiplication law,

$$(\Lambda_2, b_2) \times (\Lambda_1, b_1) = (\Lambda_2\,\Lambda_1, b_2 + \Lambda_2\, b_1) \tag{1.18}$$

we get a real Lie group which is called the "*Poincarè group*". Similarly, the union of all the Lorentz transformations forms a real (semisimple) Lie group, called the "*Lorentz group*" denoted as $O(1, D-1)$. The Lorentz transformations physically cause rotations and boosts. The transformation provided by the parameter b^m gives translations.

The generators of Lorentz transformations are denoted as M_{ab} and those of translations are denoted as P_a. They form the Poincaré algebra characterized by the following commutation relations:

$$\begin{aligned}
[M_{ab}\,, M_{cd}] &= i\,(\eta_{bc}\, M_{ad} - \eta_{ac}\, M_{bd} - \eta_{bd}\, M_{ac} + \eta_{ad}\, M_{bc}), \\
[M_{ab}\,, P_c] &= i\,(\eta_{bc}\, P_a - \eta_{ac}\, P_b). \\
[P_a, P_b] &= 0,
\end{aligned} \tag{1.19}$$

These commutation relations define an arbitrary representation of the Poincarè algebra.

Now that we have the group representation and its algebra, let us look at its Casimir operators. A *Casimir operator* is an operator that commutes with every generator of the Lie group. To be concrete we will consider the most physically relevant case of $D = 4$ in which the Poincaré group has two Casimir operators (commuting with M_{ab} and P_a), which are,

$$C_1 = P^a P_a\,, \qquad C_2 = W^a W_a\,, \tag{1.20}$$

where W^a is the Pauli-Lubanski vector

$$W_a = \frac{1}{2}\,\varepsilon_{abcd}\, M^{bc} P^d\,. \tag{1.21}$$

Using eq. (1.19) one can prove the following properties of the Pauli-Lubanski vector:

$$W^a P_a = 0\,, \tag{1.22a}$$
$$[W_a, P_b] = 0\,, \tag{1.22b}$$
$$[M_{ab}, W_c] = i\,\eta_{ac}\,W_b - i\,\eta_{bc}\,W_a\,, \tag{1.22c}$$
$$[W_a, W_b] = i\,\varepsilon_{abcd}\,W^c P^d\,. \tag{1.22d}$$

In quantum field theory, Poincarè invariance means that any Poincarè transformation (Λ, b) induces a unitary transformation

$$U(\Lambda, b) = \exp\,\left[\,i\,(-\,\hat{b}^a P_a + \tfrac{1}{2}\,K^{ab} M_{ab})\right] \tag{1.23}$$

acting in a Hilbert space of particle states. The union of operators $U(\Lambda, b)$ provides us with a unitary representation of the Poincare group. The generators of this unitary group are the same as those of the Poincaré group – P_a and M_{ab}.

1.3.1 Stability Group

Let us consider a Hilbert space of one-particle states with mass m. Then we consider its subspace V_q consisting of particle states with a given four-momentum q_a, such that,

$$P_a\,|q\rangle = q_a\,|q\rangle \tag{1.24}$$

for any state $|q\rangle \in V_q$. The vector q_a lies on the mass-shell surface associated with a value of the Casimir operator C_1 in (1.20)

$$p^a p_a = -\,m^2\,, \qquad p^0 < 0\,, \tag{1.25}$$

in momentum-space. Next we define a set of group elements (Λ, b) in V_q such that the unitary operators $U(\Lambda, b)$ acting on them transform V_q onto itself. We call this group H_q. It forms a subgroup of the Poincaré group. Since it is a group of automorphisms that act as an identity on each V_q, it is the *stability subgroup* for V_q.

If we take a unitary operator $\exp\left(\frac{i}{2}\, K^{ab} M_{ab}\right)$ and make it act on some state $|q\rangle$, then we get,

$$
\begin{aligned}
|q'\rangle &= \exp\left(\tfrac{i}{2}\, K^{ab} M_{ab}\right)|q\rangle \\
&\qquad \text{where} \\
q'^{a} &= \left(\exp \tfrac{i}{2}\, K\right)^{a}{}_{b}\, q^{b}\,.
\end{aligned}
\tag{1.26}
$$

Since H_q maps V_q onto itself, we should have $q' = q$. Therefore,

$$
\begin{aligned}
&\left(\exp \tfrac{i}{2}\, K^{a}\right)_{b} q^{b} = q^{a} \\
\Rightarrow &\left(\exp \tfrac{i}{2}\, K^{a}\right)_{b} q^{b} - \delta^{a}{}_{b}\, q^{b} = 0 \\
\Rightarrow &\, K^{a}{}_{b}\, q^{b} = 0
\end{aligned}
\tag{1.27}
$$

The above equation has the following general solution:

$$
K_{ab} = \varepsilon_{abcd}\, q^{c}\, n^{d}
\tag{1.28}
$$

where n^d is an arbitrary vector. So the elements of the stability group H_q are of the following form:

$$
\exp\,\left[\,i\,(-\,\hat{b}^{a} P_{a} + \tfrac{1}{2}\,\varepsilon^{abcd}\, q_{c}\, n_{d}\, M_{ab})\right],
\tag{1.29}
$$

where b and n are arbitrary vectors. It can be expressed in terms of the Pauli-Lubanski vector (1.21) acting on a state $|q\rangle$ in V_q, i.e., as

$$
\exp(-\,i\,\alpha)\;\exp(-i\, n_{a}\, W^{a})
\tag{1.30}
$$

where $\alpha = b^a\, q_a$. Now if we look at identity (1.22d), we can see that the components of the Pauli-Lubanski vector form a Lie algebra restricted to V_q.

1.3.2 Massive Irreducible Representations

We now proceed to find the massive irreducible representations of the Poincaré group. It is sufficient to construct all the unitary irreducible finite-dimensional representations of the stability subgroup H_q on the 'mass-shell', i.e., when $p^2 = -\,m^2$, $p_0 < 0$. We consider the simple case of a particle at rest in $D = 4$. Its four-momentum vector takes the following value:

$$q_a = (-\,m, 0, 0, 0)\,. \tag{1.31}$$

When the Pauli-Lubanski vector $W_a = \frac{1}{2}\,\varepsilon_{abcd}\,M^{bc}P^d$ is restricted to the subspace V_q, its components should have the following forms:

$$W_0 = 0 \qquad \text{and } W_I = m\,S_I \quad I = 1, 2, 3 \tag{1.32}$$

where the operator S_I is

$$S_I = \frac{1}{2}\,\varepsilon_{IJK}\,M^{JK}\,. \tag{1.33}$$

It satisfies the following algebra:

$$[S_I\,, S_J] = i\,\varepsilon_{IJK}\,S_K \tag{1.34}$$

This algebra is the same as the angular momentum algebra $su(2)$. As is well-known, the finite-dimensional irreducible representations of $su(2)$ obey the following condition:

$$(S_1)^2 + (S_2)^2 + (S_3)^2 = s(s+1)\,I \tag{1.35}$$

where s takes the values $s = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ 2, \ldots$. The dimension of a representation for a given s is $(2s+1)$.

Given the expression for the Pauli-Lubanski vector in terms of the vector S_I along with condition (1.35), the Casimir operator C_2 takes the following form:

$$W^a\,W_a = m^2\,s\,(s+1)\,I \qquad \text{where } s = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ 2, \ldots \tag{1.36}$$

The quantum number s is called the spin of a particle. In the massive case the irreducible representations of the Poincaré group are characterized by mass m and spin s.

1.3.3 Massless Irreducible Representations

In the massless case, where

$$p^a p_a = 0\,, \tag{1.37}$$

let us choose the Lorentz frame where the four-momentum is,

$$q_a = (-E, 0, 0, E) \qquad E \neq 0\,. \tag{1.38}$$

Using relations (1.21, 1.22a and 1.22b) we find that in the subspace V_q the components of the Pauli-Lubanski vector take the following forms:

$$\begin{aligned} W_0 &= -E\, M_{12} \\ W_1 &= E\,(M_{23} + M_{20}) \equiv E\, R_1 \\ W_2 &= E\,(M_{13} + M_{10}) \equiv E\, R_2 \\ W_3 &= E\, M_{12} \end{aligned} \tag{1.39}$$

Because of eq. (1.22d) the operators M_{12}, R_1 and R_2 obey the following algebra:

$$\begin{aligned} [M_{12},\, R_1] &= -\,i\, R_2 \\ [M_{12},\, R_2] &= -\,i\, R_1 \\ [R_1,\, R_2] &= 0 \end{aligned} \tag{1.40}$$

Algebra (1.40) is the Lie algebra of the group E_2 of translations and rotations on $2D$ plane. E_2 has the following Casimir operator:

$$(R_1)^2 + (R_2)^2\,. \tag{1.41}$$

It obeys the following condition:

$$(R_1)^2 + (R_2)^2 = \mu^2\, I \qquad \mu^2 \geq 0\,. \tag{1.42}$$

For the subspace V_q to be finite-dimensional, we should have $\mu^2 = 0$. Then R_1 and R_2 become trivial on V_q and we get,

$$W_1 = W_2 = 0 \tag{1.43}$$

The non-zero components of the Pauli-Lubanski vector we are left with are W_0 and W_3. Now the only Lorentz generator belonging to H_q (whose elements are of the type (1.30)), is M_{12} as can be seen from equations (1.39). Since the action of H_q on V_q is irreducible, it has only one non-trivial state, which is,

$$M_{12}\,|\lambda\rangle = \lambda\,|\lambda\rangle \tag{1.44}$$

where λ takes the following values:

$$\lambda = 0,\ \pm\frac{1}{2},\ \pm 1,\ \pm\frac{3}{2},\ ... \tag{1.45}$$

The quantum number λ is called the helicity of a particle and $|\lambda|$ is called the spin of a massless particle.

Helicity is a Poincaré invariant characteristic of massless particles. Therefore, the massless irreducible representations of the Poincaré group are classified by helicity. Physically, the helicity of a particle is the projection of the spin of the particle along its direction of motion. Therefore, while the spin of a particle is always a positive number, its helicity can be either positive or negative.

1.4 Higher-Spin Hietarinta (Super)algebras

In general, higher-spin algebras involve an infinite number of fields of increasing spin (see e.g. [30] for a review and references)

$$[T_{s_1}\,, T_{s_2}] = T_{s_1+s_2-2} + T_{s_1+s_2-4} + ... + T_{|s_1-s_2|+2} \tag{1.46}$$

Here T_s is the symmetry generator of spin-s.

In 1975 Hietarinta [52] constructed graded Lie algebras with supersymmetry generators of arbitrary spin. Poincaré superalgebra is a special case of this general algebra. This general superalgebra consists of anticommutators of spinor-tensor 'supersymmetry' generators for fermionic fields and commutators of tensor generators for bosonic fields.

$$\begin{aligned}
\text{Fermionic generators}: \{Q_\alpha^{a_1\ldots a_n}, Q_\beta^{b_1\ldots b_m}\} &= f_{\alpha\beta}^{a_1\ldots a_n, b_1\ldots b_m, c} P_c\,,\\
\text{Bosonic generators}: \quad [S^{a_1\ldots a_p}, S^{b_1\ldots b_q}] &= f^{a_1\ldots a_n, b_1\ldots b_m, c} P_c\,,\\
[Q,P]=0\,, \qquad [S,P]=0\,, \qquad & [Q,S]=0\,,
\end{aligned} \tag{1.47}$$

Here $a_1, a_2, ..., b_1, b_2, ... = 0, 1, ..., D-1$ are vector indices and α, β are spinor indices. A spinor index represents spin-$\frac{1}{2}$. Every vector index added to a generator represents spin-1 added to it, be it a fermionic generator or a bosonic generator. So a generator of spin-s has $[s]$[1] vector indices and a spinor index if s-[s]=$\frac{1}{2}$. $Q_\alpha^{a_1...a_n}$ is a fermionic tensor-spinor generator of spin-$(n+\frac{1}{2})$ and $S^{a_1...a_p}$ is a bosonic tensor generator of spin-p. P_c is the translation generator. The generators transform under certain representations of the Lorentz group $SO(1, D-1)$. The structure constants $f_{\alpha\beta}^{a_1...a_n,b_1...b_m,c}$ and $f^{a_1...a_n,b_1...b_m,c}$ are constructed with the use of the Minkowski metric, Levi-Civita tensor and gamma-matrices. They are invariant under the symmetries of the group $SO(1, D-1)$.

The algebras (1.47) are finite-dimensional higher-spin algebras. Their finiteness distinguishes them from the more familiar infinite-dimensional higher-spin algebras (1.46) in which the commutators of higher-spin generators close on generators carrying yet higher spins.

We will consider the spontaneous breaking of symmetries associated with Hietarinta algebras. These symmetries are realized non-linearly on the corresponding models of Goldstone fields. To construct higher-spin Goldstone Lagrangians we will use the method put forward by Volkov and Akulov for the description of spin-1/2 goldstini associated with the spontaneous breaking of conventional supersymmetry [49, 50].

1.5 Brief Historical Review

We will be studying spin-1 and spin-$\frac{3}{2}$ models in D=3. The non-linear realisation of spin-$\frac{3}{2}$ superalgebra has been considered in $D = 4$ independently by Baaklini in [53] and by Pilot and Rajpoot in [54, 55]. It was exploited further in [56] and the references therein. However, the properties of these non-linear generalizations of the Rarita-Schwinger action have never been explored.

Issues related to the consistent gravitational coupling in $D = 4$ of a massless spin-$\frac{5}{2}$ field, which might be regarded as a gauge field of the local spin-$\frac{3}{2}$ supersymmetry, were studied in [9, 57–59]. Aragone and Deser [60] have successfully constructed a consistent model of "hypergravity" in $D = 3$. Their model is invariant under the local symmetry transformations associated with a spin-$(n+\frac{1}{2})$ superalgebra where $n = 0, 1, \ldots$. It describes the interaction between a non-propagating graviton and a

[1] $[s]$ is the greatest integer which gives the largest integer less than or equal to s.

spin-$(n + \frac{3}{2})$ gauge field[2]. More recently this model was extended to an AdS_3 background including an additional spin-4 field by Zinoviev [61] who also constructed its higher-spin generalizations. Different aspects of higher-spin superalgebras of this kind in $D \geq 3$ and associated models have been considered in [62–65]. We intend to study the effect of spontaneous symmetry breaking in such models.

As is well-known, the construction of interacting higher-spin theories in space-time dimensions higher than three is a highly non-trivial problem, even when working with the superalgebras provided by Hietarinta in [52]. In [66] Shima et. al showed (for the spin-$\frac{3}{2}$ case in $D = 4$) that these algebras do not have non-trivial linear unitary representations. But there is still the question of whether the higher-spin Goldstone field constructions based on the non-linear realizations of these algebras produce physically consistent interacting models. A priori, such a possibility is not excluded since non-linearly realized symmetry may act only on positive-norm states while the negative-norm states of the corresponding linear multiplets get cut off.

1.6 Hietarinta Spin-3/2 Algebras in $D = 3$

We will be constructing Volkov-Akulov Lagrangians in D=3 for Goldstone fields with three different spins 1/2, 1 and 3/2. Spin-1/2 Volkov-Akulov Lagrangian will be considered as an instructive example before moving on to the new cases of Goldstone models with spin-1 and spin-3/2.

In Chapter 4 we will consider the case of a spin-3/2 goldstino field model associated with the spin-3/2 superalgebra [52–54] whose most general form in $D = 3$ is as following

$$
\begin{aligned}
[M_{ab}\,, M_{cd}] &= i\,(\eta_{bc}\, M_{ad} - \eta_{ac}\, M_{bd} - \eta_{bd}\, M_{ac} + \eta_{ad}\, M_{bc})\,,\\
[M^{ab}\,, Q^c_\alpha] &= i\,(\eta^{bc}\, Q^a_\alpha - \eta^{ac}\, Q^b_\alpha) - \frac{i}{2}\,(\Gamma^{ab})_\alpha{}^\beta\, Q^c_\beta\,,\\
[M_{ab}\,, P_c] &= i\,(\eta_{bc}\, P_a - \eta_{ac}\, P_b)\,,\\
[Q^a_\alpha, P_b] &= 0\,,\\
[P_a, P_b] &= 0\,,
\end{aligned}
\tag{1.48}
$$

[2]Strictly speaking massless particles in $D = 3$ do not have a spin. However, as is often adopted in higher-spin literature for any space-time dimension, we loosely call symmetric tensor fields $A_{a_1\ldots a_s}$ of rank s as fields with integer spin-s and symmetric-tensor spinor fields $\Psi^\alpha_{a_1\ldots a_s}$ as fields with half-integer spin-$(s+\frac{1}{2})$.

$$\{Q^a_\alpha, Q^b_\beta\} = 2\,\mathtt{a}\,C_{\alpha\beta}\,\varepsilon^{abc}\,P_c + \mathtt{b}\,\Gamma^{(a}_{\alpha\beta}P^{b)} + \mathtt{c}\,\eta^{ab}\,\Gamma^c_{\alpha\beta}\,P_c\,. \tag{1.49}$$

where $\mathtt{a}$, $\mathtt{b}$ and $\mathtt{c}$ are arbitrary real parameters.

Here M_{ab} is a Lorentz generator, P_a is a translation generator and Q^a_α $(\alpha = 1, 2)$ are Hietarinta symmetry generators which are Majorana (real) vector-spinors. The matrices Γ^a are as following:

$$\Gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Γ matrices obey Clifford algebra. To see the properties of these matrices, refer to Appendix A.4.

The charge conjugation matrix $C_{\alpha\beta}$ is defined as following:

$$C_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In eq. (1.49) one of the parameters $\mathtt{a}$, $\mathtt{b}$ and $\mathtt{c}$ can always be set to a given number by re-scaling the fermionic generators Q^a_α or the momentum P_a. The generator Q^a_α belongs to a reducible representation of the Lorentz group which splits into the following irreducible parts:

$$Q^a_\alpha = \hat{Q}^a_\alpha + \frac{1}{3}\,(\Gamma^a Q)_\alpha\,, \tag{1.50}$$

where Q_α is a Majorana-spinor generator and $\hat{Q}^a_\alpha$ is gamma-traceless $(\Gamma_a \hat{Q}^a = 0)$.

Depending on the choice of the parameters $\mathtt{a}$, $\mathtt{b}$ and $\mathtt{c}$, the superalgebra (1.49) can be reduced to simpler superalgebras. Three specific cases are the following ones.

1. **Case 1:** $\mathtt{a} = -\frac{5}{12}$, $\mathtt{b} = \frac{1}{3}$ and $\mathtt{c} = -\frac{2}{3}$

   In this case the only non-trivial anti-commutator in eq. (1.49) is between the gamma-traceless $\hat{Q}^a_\alpha$, i.e., $\{\hat{Q}^a_\alpha, \hat{Q}^b_\beta\}$. The spin-1/2 generators Q_α anti-commute both with themselves and with $\hat{Q}^a_\alpha$, i.e.,

$$\begin{aligned} &\{Q_\alpha, Q_\beta\} = 0 \\ &\{(\Gamma^a Q)_\alpha, \hat{Q}^b_\beta\} = 0 \end{aligned}$$

This superalgebra was exploited in [63].

2. **Case 2:** $\mathtt{b} = 4\,\mathtt{a}$ and $\mathtt{c} = -\,2\,\mathtt{a}$

   In this case only the spin-$\frac{1}{2}$ generators Q_α have a non-trivial commutator which is

   $$\{Q_\alpha\,, Q_\beta\} = -\,2\,(\Gamma^a C^{-1})_{\alpha\beta}\, P_a\,.$$

   The gamma-traceless generators $\hat{Q}^a_\alpha$ anti-commute both with themselves and with Q_α and hence decouple

   $$\{\hat{Q}^a_\alpha, \hat{Q}^b_\beta\} = 0$$
   $$\{\hat{Q}^a_\alpha, (\Gamma^b Q)_\beta\} = 0\,.$$

   Therefore, in this case, the superalgebra (1.49) reduces to the conventional $N = 1$ superalgebra.

3. **Case 3:** $\mathtt{b} = \mathtt{c} = 0$ and $\mathtt{a} = 1$

   In this case the algebra (1.49) reduces to

   $$\{Q^a_\alpha, Q^b_\beta\} = 2\, C_{\alpha\beta}\, \varepsilon^{abc} P_c\,. \tag{1.51}$$

   This is the algebra that we will use to construct our vector-spin goldstino model in Chapter 4.

We choose to work with algebra (1.51) because (as we will see) the quadratic part of the non-linear Lagrangian associated with this algebra coincides with the Rarita-Schwinger (or Chern-Simons-like) Lagrangian for a massless vector-spinor field χ^a_α. The gamma-traceless case can be associated with the gauge-fixed Rarita-Schwinger action in which $\Gamma_a \chi^a = 0$, while for other (inequivalent) choices of parameters (except those corresponding to the conventional supersymmetry), the spin-3/2 superalgebra does not seem to produce physically consistent models even in the free (quadratic) approximation because of the absence of gauge symmetry and the presence of negative energy states.

We will show that, in contrast to the spin-1 case, higher-order contributions to the spin-3/2 goldstino action do not break the gauge symmetry of its quadratic Rarita-Schwinger part but only require a non-linear modification of the gauge variation of the spin-3/2 field. Moreover, the non-linear action reduces to the free Rarita-Schwinger action by an invertible non-linear field redefinition, which means that the Rarita-Schwinger action itself is non-manifestly invariant under the non-linearly realized spin-3/2 supersymmetry (1.51).

Chapter 2

Fermion Goldstino Model

In this chapter, we will construct and analyze the Goldstone model for the familiar spin-1/2 fermion in order to demonstrate the procedure. For the construction of the Goldstone model, we will use the Volkov-Akulov formalism. For the analysis of the model, we will try a couple of ways including the Hamiltonian analysis using Dirac formalism.

In 1972, two Soviet physicists Volkov and Akulov developed a special kind of Lagrangian formalism for describing Goldostone models associated with the spontaneous breaking of supersymmetry [49, 50].

In a standard transformation of linear supersymmetry a fermion transforms into a boson and vice versa. But in the non-linear realisation of supersymmetry a fermion gets shifted by a parameter and in addition, transforms by a term which is non-linear in the field itself. Such a transformation is characteristic of the Goldstone field.

The Volkov-Akulov Lagrangian can be generalised to higher-spin Goldstone fields associated with Hietarinta algebras. There is a well-defined algorithm for constructing the Lagrangian starting from the algebra.

Below we will demonstrate the construction of the Volkov-Akulov Lagrangian for the simplest case of a spin-1/2 field, the Goldstone fermion we are most familiar with.

2.1 Volkov-Akulov Model of Spin-1/2 Goldstino

The conventional $N=1$ superalgebra for a spin-1/2 Majorana fermion χ^α in $D=3$ is as following:

$$
\begin{aligned}
[M_{ab}\,,M_{cd}] &= \mathrm{i}(\eta_{bc}\,M_{ad}-\eta_{ac}\,M_{bd}-\eta_{bd}\,M_{ac}+\eta_{ad}\,M_{bc}),\\
[M_{ab}\,,P_c] &= \mathrm{i}(\eta_{bc}\,P_a-\eta_{ac}\,P_b),\\
[M_{ab}\,,Q_\alpha] &= -\frac{\mathrm{i}}{2}\,(\Gamma_{ab})_\alpha{}^\beta\,Q_\beta,\\
\{Q_\alpha\,,Q_\beta\} &= -\,2\,(\Gamma^a C^{-1})_{\alpha\beta}\,P_a,\\
[Q_\alpha\,,P_a] &= 0,\\
[P_a,P_b] &= 0,
\end{aligned}
\tag{2.1}
$$

where M_{ab} is a Lorentz generator, Q_α ($\alpha=1,2$) is a Majorana spinor generator of supersymmetry transformations and P_a is a translation generator.

The supersymmetry transformations of x^a and $\chi^\alpha(x)$ generated by the algebra (2.1) are:

$$
x'^a = x^a - \mathrm{i}\,f^{-2}\,\epsilon^\alpha\,\Gamma^a_{\alpha\beta}\,\chi^\beta(x)\,,\qquad \chi'^\alpha(x') = \chi^\alpha(x)+\epsilon^\alpha\,, \tag{2.2}
$$

where ϵ^α is a constant spinor parameter, f is a supersymmetry breaking parameter of mass-dimension $M^{\frac{3}{2}}$ and χ^α has the canonical dimension of M in $D=3$. The infinitesimal transformation of the form of the goldstino field $\chi^\alpha(x)$ is[1]

$$
\delta\chi^\alpha(x) = \epsilon^\alpha + \mathrm{i}\,f^{-2}\left(\epsilon\,\Gamma^a\,\chi(x)\right)\partial_a\chi^\alpha(x)\,. \tag{2.3}
$$

This shows that the goldstino transforms non-linearly under supersymmetry.

The commutator of two variations (2.3) closes on the translations *off the mass shell,* i.e. without the use of the equations of motion

$$
[\delta_2\,,\delta_1]\,\chi^\alpha = 2\,\mathrm{i}\,f^{-2}\,(\epsilon_1\,\Gamma^a\,\epsilon_2)\,\partial_a\chi^\alpha\,. \tag{2.4}
$$

We take a supersymmetry group element in the form $g=e^{i\,x^m P_m}\,e^{if^{-1}\chi^\alpha Q_\alpha}$. Then

[1] As a shorthand notation, in what follows, we define the contraction of the spinors with a single gamma-matrix as $\chi\,\Gamma^a\,\psi \equiv \chi^\alpha\,\Gamma^a_{\alpha\beta}\,\psi^\beta = -\,\chi^\alpha\,\Gamma^{a\,\beta}_{\alpha}\,\psi_\beta$. For other rules regarding the notation of the spinor indices see Appendix A.

the Cartan one-form $g^{-1}dg$ is invariant under the rigid supersymmetry transformations $g' = h(\epsilon)g$. It can be written as follows:

$$\begin{aligned}
g^{-1}dg &= e^{-if^{-1}\chi^\alpha Q_\alpha}\, e^{-ix^m P_m}\, d\,(e^{ix^n P_n}\, e^{if^{-1}\chi^\beta Q_\beta}) \\
&= i\, P_n\, dx^n + e^{-if^{-1}\chi^\alpha Q_\alpha}\, d\, e^{if^{-1}\chi^\beta Q_\beta} \\
&= i\, P_n\, dx^n + i f^{-1} d\chi^\alpha Q_\alpha - f^{-2}\chi^\alpha d\chi^\beta\, (\Gamma^a C^{-1})_{\alpha\beta}\, P_a \\
&\equiv i E^m P_m + i\, E^\alpha Q_\alpha
\end{aligned} \tag{2.5}$$

$$g^{-1}dg \equiv i\, E^m P_m + i\, E^\alpha Q_\alpha \tag{2.6}$$

E^m is the one-form[2] that is used as a building block to construct the Volkov-Akulov Lagrangian. The Lagrangian constructed using E^m gives a non-linear generalisation of Dirac Lagrangian. If we were also to use E^α, we would get a Lagrangian with higher-derivative kinetic terms, not having a well-known analogue.

E^m has the following form:

$$E^m = dx^m + i f^{-2}\chi^\alpha\, d\chi^\beta\, (\Gamma^m C^{-1})_{\alpha\beta} \tag{2.7}$$

$$\begin{aligned}
&= dx^a\, (\delta^m_a + i f^{-2}\, \chi^\alpha \partial_a \chi^\beta\, (\Gamma^m C^{-1})_{\alpha\beta}) \\
&= dx^a\, E^m_a
\end{aligned}$$

$$\Rightarrow E^m_a = \delta^m_a + i f^{-2}\, \chi^\alpha \partial_a \chi^\beta\, (\Gamma^m C^{-1})_{\alpha\beta} \tag{2.8}$$

Having obtained the invariant one-form E^m, we can proceed to construct the Volkov-Akulov action with it. The form of the Volkov-Akulov action in any dimension is

$$S = -f^2 \int d^D x\, \det E^a_b\, .$$

In $D = 3$ the action reduces to

$$S = -f^2 \int d^3 x\, \det E^a_b = \frac{f^2}{3!} \int \varepsilon_{abc}\, E^a \wedge E^b \wedge E^c\, , \tag{2.9}$$

or explicitly

$$S_{1/2} = \int d^3 x \left(-f^2 - \mathrm{i}\, \chi\, \Gamma^a\, \partial_a \chi + \frac{f^{-2}}{2}\, \varepsilon^{abc}\, (\chi\chi)\, \partial_a \chi\, \Gamma_b\, \partial_c \chi \right), \tag{2.10}$$

[2]For a recent review of the different aspects and realizations of the Volkov-Akulov model and its coupling to supergravity, see [67, 68] and the references therein.

where $\chi\,\chi \equiv \chi^{\alpha}\,C_{\alpha\beta}\,\chi^{\beta} \equiv \chi^{\alpha}\,\chi_{\alpha}$.

For the purpose of our analyses of Volkov-Akulov action and the ensuing results, the constant term f^2 in the action above will not bring any qualitative difference to our results. Therefore from here onwards we will omit it in the Volkov-Akulov actions. It becomes important when the goldstino couples to gravity since it gives a positive contribution to the cosmological constant. In that scenario it cannot be omitted.

Using variational calculus we can write down the equation of motion for the goldstino χ^{α} from the action (2.10). It is

$$i\,\Gamma^{a}_{\alpha\beta}\,\partial_a\chi^{\beta} = f^{-2}\,\chi_{\alpha}\,\varepsilon^{abc}\,\partial_a\chi\,\Gamma_b\,\partial_c\chi + f^{-2}\,\chi^{\gamma}\,\Gamma^{a}_{\gamma\alpha}\,\partial_a\chi\,\Gamma^{b}\,\partial_b\chi. \tag{2.11}$$

We wish to find out how many physical degrees of freedom the system has and whether it has gauge symmetries.

In the case of complicated dynamical systems, the most reliable method of analysing their dynamical properties is the Dirac Hamiltonian formalism [69, 70].

2.2 Dirac Hamiltonian Formalism

If we have a Lagrangian $L(q_i, \dot{q}_i)$, function of the variables q_i - position coordinate and $\dot{q}_i = \partial_t q_i$ - time derivative of the position coordinate or the velocity, then the conjugate momentum p_i is defined as

$$p^i = \frac{\partial L}{\partial \dot{q}_i}\,. \tag{2.12}$$

Now let us construct a quantity that can be expressed exclusively in terms of the position and momenta coordinates – q_i and p^i. Consider the quantity $p^i\,\dot{q}_i - L$. Let us vary this quantity with respect to the variables q_i and $\dot{q}_i$, the coordinates and the velocities. This will bring a variation in the momentum variables p_i also. It is as following:

$$\begin{aligned} \delta(p^i\,\dot{q}_i - L) &= \delta p^i\,\dot{q}_i + p^i\,\delta\dot{q}_i - \left(\frac{\partial L}{\partial q_i}\right)\delta q_i - \left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta\dot{q}_i && (2.13)\\ &= \delta p^i\,\dot{q}_i - \left(\frac{\partial L}{\partial q_i}\right)\delta q_i && (2.14) \end{aligned}$$

by eq. (2.12). This shows us that the variation of the quantity $p^i\,\dot{q}_i - L$ depends only on the variation of the position coordinates q_i and the momenta coordinates p^i but not on the velocities $\dot{q}_i$. So $p^i\,\dot{q}_i - L$ is a quantity that is expressible exclusively in terms of the position and momentum variables. It is called the canonical Hamiltonian H.

$$H = p^i\,\dot{q}_i - L \tag{2.15}$$

It is possible that the position and momenta variables are not independent of each other. They may satisfy the following kind of relations:

$$\phi_m(q_i, p^j) = 0 \tag{2.16}$$

Such relations are the constraints in the theory. One way of classifying them is on the basis of how they are derived. Doing so they can be classified as primary and secondary constraints.

i) **Primary constraint**
A primary constraint is derived directly from the conjugate momentum without using an equation of motion. For e.g., if the conjugate momentum p^i is equal to a function $f^i(q)$, then the primary constraint corresponding to p^i is simply $C^i = p^i - f^i(q) = 0$.

ii) **Secondary constraint**
A secondary constraint is derived from the equation of motion involving the primary constraint. Let us see below how it is derived.

The total Hamiltonian H_T is defined as:

$$H_T = H + u^m \phi_m \tag{2.17}$$

where H is the canonical Hamiltonian, ϕ_m is a primary constraint and u^m is the Lagrange multiplier. The equations of motion are written as following:

$$\begin{aligned}\dot{F} &= \{F\,, H_T\} \\ &= \{F\,, H\} + u^m\,\{F\,, \phi_m\}\,,\end{aligned} \tag{2.18}$$

where F is an arbitrary function of the canonical variables and the Poisson bracket[3] of the functions F and G is defined as

$$\{F\,,G\} = \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p^i} - \frac{\partial G}{\partial q_i}\frac{\partial F}{\partial p^i}\,. \tag{2.19}$$

Since a constraint ϕ_m is supposed to be preserved in time, it should obey the following equation of motion:

$$\begin{aligned}\dot{\phi}_m &= \{\phi_m\,, H_T\} \\ &= \{\phi_m\,, H\} + u^{m'}\{\phi_m\,, \phi_{m'}\} = 0\end{aligned} \tag{2.20}$$

This equation can be solved to provide an expression or a restriction for the Lagrangian multipliers $u^{m'}$. However, if it so happens that the equation reduces to a relation independent of $u^{m'}$, involving only p^i and q_i, then that relation is a secondary constraint.

Once we obtain the secondary constraints, we can repeat the procedure of solving their equations of motion to see if there are tertiary constraints also present. This procedure can be continued to higher order constraints till we reach the point where the time derivative of the highest order constraint only imposes a restriction on the Lagrange multipliers but does not yield yet another constraint.

Another way of classifying the constraints is as first-class and second-class constraints.

i) **First-class constraint**
A constraint F is said to be first-class if its Poisson bracket with every constraint vanishes weakly,

$$\{F\,,\phi_i\} \approx 0, \quad i = 1, 2, \ldots \tag{2.21}$$

where ϕ_i is a constraint of the system. A quantity is said to be weakly vanishing if it is restricted to be zero on the constraint surface but does not identically vanish throughout the phase space. More generally, any function F of the canonical variables q and p, that commutes with all the constraints is called a first-class function.

[3] In this section, for denoting the Poisson brackets we use braces $\{..\}$ without distinguishing whether the dynamical variables are bosonic or fermionic. However, for concrete models discussed below we will use $[..]$ for the Poisson brackets of the bosonic variables and $\{..\}$ for the fermionic ones.

ii) **Second-class constraint**
A constraint is second-class if there is at least one constraint with which its Poisson bracket does not vanish weakly. In other words, a constraint that is not first-class, is a second-class constraint.

This classification is necessary for counting the number of the degrees of freedom in the system. Let us see ahead how that is made possible.

2.2.1 First-class constraints generate gauge transformations

Because of the time conservation of constraints ϕ_i, the Lagrange multipliers u^m have the following restrictions on them:

$$\{\phi_i\,, H\} + u^m\{\phi_i\,, \phi_m\} \approx 0. \tag{2.22}$$

These are a system of non-homogeneous linear equations with the unknowns u^m. Their general solution is of the form

$$u^m = U^m + V^m \tag{2.23}$$

where U^m is a particular solution that is fixed by the consistency conditions derived from the requirement that the constraints be preserved in time. V^m is the general solution of the associated homogeneous system

$$V^m\{\phi_i\,, \phi_m\} \approx 0. \tag{2.24}$$

The most general solution of this equation is the linear combination of A linearly independent solutions $V_a{}^m$, $a = 1, 2, ...A$. Therefore,

$$u^m \approx U^m + v^a\, V_a{}^m \tag{2.25}$$

where the coefficients v^a are totally arbitrary. Thus u^m is the sum of a fixed term (U^m) and arbitrary terms. Therefore

$$H_T = H + U^m\,\phi_m + v^a\,V_a{}^m\,\phi_m. \tag{2.26}$$

Eq. (2.22) can be rewritten as

$$\{\phi_i\,, H\} + U^m\{\phi_i\,, \phi_m\} + v^a\,V_a{}^m\,\{\phi_i\,, \phi_m\} \approx 0. \tag{2.27}$$

Because of relation (2.24) we have,

$$\{\phi_i\,,H\} + U^m\{\phi_i\,,\phi_m\} \approx 0$$
$$\Rightarrow \{\phi_i\,,H'\} \approx 0 \quad \text{where } H' = H + U^m\,\phi_m. \tag{2.28}$$

This shows us that H' is a first-class function.

Eq. (2.24) can be rewritten as

$$v^a\,V_a{}^m\,\{\phi_i\,,\phi_m\} \approx 0 \tag{2.29}$$
$$\Rightarrow v^a\,\{\phi_i\,,\phi_a\} \approx 0 \quad \text{where } V_a{}^m\,\phi_m = \phi_a. \tag{2.30}$$

As visible from eq. (2.30), ϕ_a are first-class constraints. In fact, since $v^a\,V_a{}^m$ is the general solution to eq. (2.29), ϕ_a is the complete set of first-class primary constraints.

The total Hamiltonian H_T in (2.26) can be written as

$$H_T = H' + v^a\,\phi_a \tag{2.31}$$

The equations of motion can be written as

$$\dot{F} = \{F\,,H_T\} = \{F\,,H' + v^a\,\phi_a\} \tag{2.32}$$

where $F(q,p)$ is an arbitrary function of the canonical variables.

Let us now see which functions of the dynamical variables (p,q) are physical observables. The classical physical observables are such functions of the canonical varibles for which the initial values of the latter completely define the behaviour of these observables in time through the Hamiltonian equations of motion (2.32). In the systems with gauge symmetries a given observable (or a physical state), which is gauge invariant, can be represented by different sets of canonical variables, playing the role of gauge potentials.

Consider this in more detail. Since the total Hamiltonian H_T contains arbitrary functions v^a, the time-dependence of $F(q,p)$ is uniquely determined by (2.32) only if the equations of motion do not depend on v^a. For this to happen, the function $F(q,p)$ should Poisson-commute (at least weakly) with the first-class constraints

$$\{F\,,\phi_a\} \approx 0.$$

Then the equations of motion reduce to

$$\dot{F} = \{F\,, H' + v^a\,\phi_a\} \approx \{F\,, H'\}. \tag{2.33}$$

On the other hand, if $F(q,p)$ does not Poisson-commute with ϕ_a, its equation of motion contains the arbitrary functions v^a and there is an ambiguity in the time dependence of $F(q,p)$. But this ambiguity should be physically irrelevant, i.e. it should correspond to different choices of the canonical variables related by a gauge transformation and defining the same physical state. To see this, let us suppose we have an initial (fixed) value of a canonical variable F at time t_1. At later time $t_2 > t_1$, if any ambiguity arises in the value of the canonical variables, it should be physically irrelevant, i.e. correspond to a gauge transformation. Indeed, consider in particular $t_2 = t_1 + \delta t$. Since v^a is an arbitrary function of time, let us take two different choices v^a and $\tilde{v}^a$. As follows from (2.32), the difference in the values of the variable F at time t_2 corresponding to the different choices of v^a at t_1 is:

$$\delta F = (v^a(t_1) - \tilde{v}^a(t_1))\,\delta t\,\{F(t_1)\,, \phi_a(t_1)\}. \tag{2.34}$$

This does not change the time dependence of the physical state governed by (2.33), and, therefore, the variations (2.34) produced by first-class constraints are gauge transformations.

A first-class constraint not only knocks down a degree of freedom by virtue of being a constraint, but it also knocks down another degree of freedom by gauging it away. So a first-class constraint knocks down two degrees of freedom in the system. A second-class constraint on the other hand, cancels only one degree of freedom as usual.

Obtaining and classifying all the constraints in the system enables us to count the number of physical degrees of freedom in the system, which conclusively determines the number of observables in the system.

2.3 Hamiltonian Analysis of the Spin-1/2 Goldstino Model

Now let us perform the Hamiltonian analysis of the spin-1/2 goldstino model using Dirac formalism. We perform this analysis mainly for the following two reasons:

1. to find out how many degrees of freedom the system has

2. to check whether the energy of the system is bounded by looking at the positive-definiteness property of the Hamiltonian.

2.3.1 Hamiltonian Analysis of Free Spin-1/2 Theory

Let us first consider the Hamiltonian of the free spin-1/2 theory where $f = 0$. This case is simpler than that of the full Volkov-Akulov spin-1/2 Lagrangian which includes interaction terms in it. We will consider the latter case in the next section.

We split the $D = 3$ space-time indices into time and space indices $a = (0, i)$, and define $\varepsilon^{0ij} \equiv \varepsilon^{ij}$. Then the free Lagrangian in eq. (2.10) gets written in the following form:

$$\mathcal{L}_{\frac{1}{2}free} = \mathrm{i}\,\partial_0\chi\,\Gamma^0\chi - \mathrm{i}\,\chi\,\Gamma^i\,\partial_i\chi \tag{2.35}$$

The conjugate momentum is,

$$p_\rho(t, \mathbf{x}) = \frac{\delta\mathcal{L}(t, \mathbf{x})}{\delta(\partial_0\chi^\rho)} = i\chi^\alpha(t, \mathbf{x})\,\Gamma^0_{\alpha\rho}\,, \tag{2.36}$$

where $\mathbf{x}$ and $\mathbf{y}$ stand for the spatial coordinates x^i and y^i.

The canonical Hamiltonian density is

$$\begin{aligned}\mathcal{H}_c(t, \mathbf{x}) &= \partial_0\chi^\rho(t, \mathbf{x})\,p_\rho(t, \mathbf{x}) - \mathcal{L}(t, \mathbf{x}) \\ &= -\,i\,\chi^\alpha(t, \mathbf{x})\,\Gamma^i_{\alpha\beta}\,\partial_i\chi^\beta(t, \mathbf{x})\,.\end{aligned} \tag{2.37}$$

$\chi^\alpha(t, \mathbf{x})$ and $p_\beta(t, \mathbf{x})$ have the following equal-time (anti-commuting) Poisson bracket relation:

$$\{\chi^\alpha(t, \mathbf{x})\,, p_\beta(t, \mathbf{y})\} = \delta^\alpha_\beta\,\delta^{(2)}(\mathbf{x} - \mathbf{y})\,. \tag{2.38}$$

Knowing the expression for the conjugate momentum from eq. (2.36), we can write down the primary constraint, which is,

$$C_\rho(t, \mathbf{x}) = p_\rho(t, \mathbf{x}) + i\,\chi^\alpha(t, \mathbf{x})\,\Gamma^0_{\alpha\rho} = 0\,. \tag{2.39}$$

The equal-time Poisson bracket relation between the primary constraints is:

$$\{C_\alpha(t,\mathbf{x})\,,C_\beta(t,\mathbf{x})\} = 2\,i\,\Gamma^0_{\alpha\beta}\,\delta^{(2)}(\mathbf{x}-\mathbf{y})\,. \tag{2.40}$$

Since the Poisson bracket above is non-zero, the constraints $C_\alpha(t,\mathbf{x})$ are of the second-class.

Now we wish to find if this system has secondary constraints. For that we need the total Hamiltonian as defined in eq. (2.17). First we write the total Hamiltonian density which is the following:

$$\mathcal{H}_T(t,\mathbf{x}) = \mathcal{H}_C(t,\mathbf{x}) + u^\alpha(t,\mathbf{x})\,C_\alpha(t,\mathbf{x})\,. \tag{2.41}$$

On integrating the equation above with respect to $\mathbf{x}$, we get the total Hamiltonian, which is,

$$H_T(t) = H_c(t) + \int d^2\mathbf{x}\,u^\alpha(t,\mathbf{x})\,C_\alpha(t,\mathbf{x})\,. \tag{2.42}$$

Next we impose the time-conservation of the primary constraint $C_\alpha(t,\mathbf{x})$ for which we equate its Poisson bracket with the total Hamiltonian to be zero as shown in the following.

$$\begin{aligned}
&\int d^2\mathbf{y}\,[C_\rho(t,\mathbf{x})\,,\mathcal{H}_T(t,\mathbf{y})]\\
&= \int d^2\mathbf{y}\,[C_\rho(t,\mathbf{x})\,,\mathcal{H}_c(t,\mathbf{y})] + \int d^2\mathbf{y}\,u^\alpha(t,\mathbf{x})\,\{C_\rho(t,\mathbf{y})\,,C_\alpha(t,\mathbf{y})\} = 0\\
&= 2\,i\,\partial_i\chi^\alpha(t,\mathbf{x})\,\Gamma^i_{\alpha\rho} + u^\alpha(t,\mathbf{x})\,2\,i\,\Gamma^0_{\alpha\rho} = 0\,.
\end{aligned} \tag{2.43}$$

The last equation above can be solved for $u^\alpha(t,\mathbf{x})$. It gives,

$$u^\alpha(t,\mathbf{x}) = \varepsilon^{ij}\,\partial_i\chi^\beta(t,\mathbf{x})\,\Gamma^\alpha_{j\beta}\,. \tag{2.44}$$

Since $[C_\rho(t,\mathbf{x})\,,H_T(t,\mathbf{y})]$ vanishes for $u^\alpha(t,\mathbf{x}) \;=\; -\,\varepsilon^{ij}\,\partial_i\chi^\beta(t,\mathbf{x})\,\Gamma^\alpha_{j\beta}$, it does not give a secondary constraint. Therefore, the linear spin-1/2 theory has only two constraints, the primary constraints, which are second-class.

Now that we know all the constraints present in this system, we count the number of canonical degrees of freedom (DoF).

$$\begin{aligned}
\text{DoF to begin with} &: \ 4 \text{ because of } \chi_1,\ \chi_2,\ p^1,\ \text{and } p^2\,. \\
\text{DoF cancelled by } 1^{st}\text{-class constraints} &: \ 0 \text{ because there is no } 1^{st}\text{-class constraint.} \\
\text{DoF cancelled by } 2^{nd}\text{-class constraints} &: \ 2 \text{ because there are two } 2^{nd}\text{-class} \\
&\quad\ \text{constraints – } C_1 \text{ and } C_2\,.
\end{aligned} \tag{2.45}$$

So the total DoF $= 4 - 2 = 2$. These two canonical degrees of freedom correspond to position and momentum, making up one physical degree of freedom. Therefore the free fermionic theory has one physical degree of freedom which belongs to the on-shell Majorana fermion.

Now let us move to the case of the full-fermionic Lagrangian which involves self-interactions.

2.3.2 Hamiltonian Analysis of Full Volkov-Akulov Spin-1/2 Theory

Following is the full Lagrangian obtained after splitting the spacetime indices into space and time indices separately.

$$\mathcal{L}_{1/2} = \mathrm{i}\,\partial_0\chi\,\Gamma^0\chi - \mathrm{i}\,\chi\,\Gamma^i\,\partial_i\chi\, - \frac{f^{-2}}{2}\,\varepsilon^{ij}\,\chi\chi\,\big(\partial_i\chi\,\Gamma_0\,\partial_j\chi - 2\,\partial_0\chi\,\Gamma_i\,\partial_j\chi\big)\,. \tag{2.46}$$

The conjugate momentum is

$$p_\alpha = \frac{\delta L}{\delta\partial_0\chi^\alpha} = \mathrm{i}\,\Gamma^0_{\alpha\beta}\,\chi^\beta + f^{-2}\,\Gamma_{i\alpha\beta}\,\partial_j\chi^\beta(\chi\chi) \tag{2.47}$$

and the canonical Hamiltonian density is

$$\begin{aligned}
\mathcal{H}_{1/2} &= \partial_0\chi^\alpha\,p_\alpha - \mathcal{L}_{1/2} \\
&= \mathrm{i}\,\chi\,\Gamma^i\,\partial_i\chi\, + \frac{f^{-2}}{2}\,\varepsilon^{ij}\,\chi\chi\,\partial_i\chi\Gamma^0\,\partial_j\chi\,.
\end{aligned} \tag{2.48}$$

The primary constraint is:

$$F_\alpha = p_\alpha - i\,\Gamma^0_{\alpha\beta}\,\chi^\beta - f^{-2}\,\varepsilon^{ij}\,\Gamma_{i\alpha\beta}\,\partial_j\chi^\beta(\chi\chi) = 0\,, \tag{2.49}$$

The anti-commutator of the primary constraint with itself is:

$$
\begin{aligned}
&\{F_\alpha(t,\mathbf{x})\,,F_\beta(t,\mathbf{y})\}\\
=&-\,2\,i\,\Gamma^0_{\alpha\beta}\,\delta^{(2)}(\mathbf{x}-\mathbf{y})\\
&-2\,f^{-2}\,\varepsilon^{ij}\big(\,\chi_\alpha(t,\mathbf{x})\,\Gamma_{i\eta\beta}+\chi_\beta(t,\mathbf{x})\,\Gamma_{i\eta\alpha}\,\big)\,\partial_j\chi^\eta(t,\mathbf{x})\,\delta^{(2)}(\mathbf{x}-\mathbf{y})\\
&+2\,f^{-2}\,\varepsilon^{ij}\,\Gamma_{i\alpha\beta}\chi^\eta(t,\mathbf{x})\,C_{\eta\lambda}\,\partial_j\chi^\lambda(t,\mathbf{x})\,\delta^{(2)}(\mathbf{x}-\mathbf{y})\\
&-\,f^{-2}\,\varepsilon^{ij}\big(\chi^\eta(t,\mathbf{x})\,C_{\eta\lambda}\,\chi^\lambda(t,\mathbf{x})\big)\Gamma_{i\alpha\beta}(\partial_{x_j}+\partial_{y_j})\,\delta^{(2)}(\mathbf{x}-\mathbf{y})
\end{aligned}
\tag{2.50}
$$

It is weakly non-vanishing. Therefore, it is of the second-class as was also the case for the primary constraint in the free fermionic theory discussed in the last section.

Next, we attempt to look for secondary constraints. Following is the total Hamiltonian:

$$
H_T(t)=H_c(t)+\int d^2\mathbf{x}\,u^\alpha(t,\mathbf{x})\,F_\alpha(t,\mathbf{x})\,.
\tag{2.51}
$$

We impose the time conservation of the primary constraint $F_\alpha(t,\mathbf{x})$ by taking its Poisson bracket with the total Hamiltonian and equating it with zero.

$$
\begin{aligned}
&\int d^2\mathbf{y}\,[F_\rho(t,\mathbf{x})\,,\mathcal{H}_T(t,\mathbf{y})]=0\\
=&\int d^2\mathbf{y}\,[F_\rho(t,\mathbf{x})\,,\mathcal{H}_c(t,\mathbf{y})]-\int d^2\mathbf{y}\,u^\alpha(t,\mathbf{y})\,\{F_\rho(t,\mathbf{y})\,,F_\alpha(t,\mathbf{y})\}=0\\
=&\,2\,i\,\partial_i\chi^\alpha(t,\mathbf{y})\,\Gamma^i_{\alpha\rho}+f^{-2}\,\varepsilon^{ij}\,\chi_\rho(t,\mathbf{x})\,\partial_i\chi^\beta(t,\mathbf{x})\,\partial_j\chi^\nu(t,\mathbf{x})\,\Gamma_{0\beta\nu}\\
&-2f^{-2}\,\varepsilon^{ij}\partial_i\chi^\eta(t,\mathbf{x})\,\chi_\eta(t,\mathbf{x})\,\partial_j\chi^\nu(t,\mathbf{x})\,\Gamma_{0\rho\nu}+u^\alpha(t,\mathbf{x})\,\Big(2\,i\,\Gamma^0_{\rho\alpha}\\
&+2\,\varepsilon^{ij}f^{-2}\,\big(\,\chi_\rho(t,\mathbf{x})\,\Gamma_{i\beta\alpha}+\chi_\alpha(t,\mathbf{x})\,\Gamma_{i\beta\rho}\,\big)\,\partial_j\chi^\beta(t,\mathbf{x})\\
&-2\,\varepsilon^{ij}\,f^{-2}\,\Gamma_{i\rho\alpha}\,\chi^\eta(t,\mathbf{x})\,\partial_j\chi_\eta(t,\mathbf{x})\Big)=0
\end{aligned}
\tag{2.52}
$$

The equation above can be solved for $u^\alpha(t,\mathbf{x})$. It gives,

$$
\begin{aligned}
u^\alpha(t,\mathbf{x})=&\Big(2\,i\,\Gamma^0_{\rho\alpha}+2\,f^{-2}\,\varepsilon^{ij}\big(\,\chi_\rho(t,\mathbf{x})\,\Gamma_{i\beta\alpha}+\chi_\alpha(t,\mathbf{x})\,\Gamma_{i\beta\rho}\,\big)\,\partial_j\chi^\beta(t,\mathbf{x})\\
&-2\,f^{-2}\,\varepsilon^{ij}\,\Gamma_{i\rho\alpha}\,\chi^\eta(t,\mathbf{x})\,\partial_j\chi_\eta(t,\mathbf{x})\Big)^{-1}\Big(2\,i\,\partial_i\chi^\alpha(t,\mathbf{y})\,\Gamma^i_{\alpha\rho}\\
&+f^{-2}\,\varepsilon^{ij}\,\chi_\rho(t,\mathbf{x})\,\partial_i\chi^\beta(t,\mathbf{x})\,\partial_j\chi^\nu(t,\mathbf{x})\,\Gamma_{0\beta\nu}\\
&-2\,f^{-2}\,\varepsilon^{ij}\partial_i\chi^\eta(t,\mathbf{x})\,\chi_\eta(t,\mathbf{x})\,\partial_j\chi^\nu(t,\mathbf{x})\,\Gamma_{0\rho\nu}\Big)
\end{aligned}
\tag{2.53}
$$

The time conservation of the constraint $F_\alpha(t, \mathbf{x})$ gives us the value of the Lagrange multiplier for the total Hamiltonian. Therefore we don't get a secondary constraint.

Since F_α are the only constraints we have in the system, we can now start counting the degrees of freedom. We begin with the fermions χ^α and their corresponding momenta. We have $\chi_1,\ \chi_2,\ p^1$ and p^2 . Therefore, we have four phase-space degrees of freedom to begin with. The two second-class constraints F_1 and F_2 cancel two degrees of freedom. So we are left with two canonical degrees of freedom. One of the canonical degrees of freedom is that of the position and the other one of the momentum. So two canonical degrees of freedom correspond to one physical degree of freedom in the configuration space.

The counting of the degrees of freedom using Dirac Hamiltonian formalism verifies the presence of only one physical degree of freedom in the non-linear Volkov-Akulov goldstino model.

2.3.3 On-Shell Hamiltonian Value of Full Fermion Goldstino Model

Depending on whether the energy of the system is positive, zero or negative, we get to know whether or not the system is physical and if it has dynamical degrees of freedom. For this purpose, we evaluate the Hamiltonian density on the mass-shell. As shown in eq. (2.48) the canonical Hamiltonian density is

$$\mathcal{H}_{1/2} = i\,\chi\,\Gamma^i\,\partial_i\chi + \frac{f^{-2}}{2}\,\varepsilon^{ij}\,\chi\chi\,\partial_i\chi\Gamma^0\,\partial_j\chi\,. \tag{2.54}$$

The quartic order term in the above expression can be expressed differently using Γ- matrix identities (A.4). Modulo a total derivative, eq. (2.54) becomes

$$\mathcal{H}_{1/2} = i\,\chi\,\Gamma^i\,\partial_i\chi + \frac{f^{-2}}{2}\,\chi\chi\,\partial_i\chi\,\Gamma^i\,\Gamma^j\,\partial_j\chi - \frac{f^{-2}}{4}\,\partial_i(\chi\chi)\,\partial^i(\chi\chi) \tag{2.55}$$

Now note that the equation of motion (2.11) implies that

$$\Gamma^i\,\partial_i\chi = -\,\Gamma^0\,\partial_0\chi + \mathcal{O}\left(\chi\,\partial\chi\,\partial\chi\right). \tag{2.56}$$

Substituting this expression into eq. (2.55) we get the on-shell value of the Hamiltonian density:

$$\mathcal{H}_{1/2}|_{\text{on-shell}} = \mathrm{i}\,\chi\,\Gamma^i\,\partial_i\chi\; + \frac{f^{-2}}{4}\,\partial_i(\mathrm{i}\,\chi\chi)\,\partial^i(\mathrm{i}\,\chi\chi) + \frac{f^{-2}}{4}\,\partial_0(\mathrm{i}\,\chi\chi)\,\partial_0(\mathrm{i}\,\chi\chi)\,. \tag{2.57}$$

In this expression the leading order quadratic term is the standard free Hamiltonian of a massless Majorana fermion. The quartic terms are positive semi-definite since $i\,\chi^{\alpha}\chi_{\alpha}$ is a real nilpotent scalar. The positive semi-definiteness of the Hamiltonian density shows that the energy of the system is bounded from below and is either zero or positive. This verifies that the system is stable.

So we have seen that the Volkov-Akulov goldstino model is physically consistent and does not have any unphysical ghost degrees of freedom.

Now let us move to the Volkov-Akulov Goldstone model of the field with the next higher spin — spin-1 boson in $D = 3$.

Chapter 3

Vector Goldstone Model

Now we construct the Volkov-Akulov goldstone model with the spin-1 goldstone $A_a(x)$ where $a = 0, 1, 2$. The corresponding Hietarinta spin-1 algebra is generated by Poincarè generators and a bosonic vector operator S_a satisfying the following commutation relations:

$$\begin{aligned}
[M_{ab}\,, M_{cd}] &= \mathrm{i}\,(\eta_{bc}\, M_{ad} - \eta_{ac}\, M_{bd} - \eta_{bd}\, M_{ac} + \eta_{ad}\, M_{bc}), \\
[M_{ab}\,, P_c] &= \mathrm{i}\,(\eta_{bc}\, P_a - \eta_{ac}\, P_b), \\
[M^{ab}\,, S^c] &= \mathrm{i}\,(\eta^{bc}\, S^a - \eta^{ac}\, S^b)\,, \\
[S^a, S^b] &= 2\,\mathrm{i}\,\varepsilon^{abc} P_c\,, \\
[S^a, P_b] &= 0\,, \\
[P_a, P_b] &= 0\,.
\end{aligned} \tag{3.1}$$

The algebra above can also be regarded as an Inonu-Wigner contraction of the $so(2,2) \oplus so(1,2)$-algebra.

The symmetry transformations of x^a and $A_a(x)$ generated by the algebra above are:

$$x'^a = x^a - f^{-2}\,\varepsilon^{abc}\, s_b\, A_c(x)\,, \tag{3.2}$$

$$A'_a(x') = A_a(x) + s_a\,, \tag{3.3}$$

where s_a is a constant vector parameter.

The infinitesimal transformation of the form of the goldstone field $A_a(x)$,

$$\delta A_a(x) = s_a + f^{-2}\,\varepsilon^{dbc}\big(s_b\, A_c(x)\big)\,\partial_d A_a(x)\,, \tag{3.4}$$

shows that it transforms non-linearly under the symmetry. The commutator of two variations (3.4) closes on the translation of A_a,

$$[\delta_2\,,\delta_1]\,A_a(x) = 2\,f^{-2}\,\varepsilon^{dbc}\,(s^1_b\,s^2_c)\,\partial_d A_a(x)\,. \tag{3.5}$$

In this case the invariant one-form is

$$\begin{aligned} E^a &= dx^a + f^{-2}\,\varepsilon^{abc}A_b(x)\,dA_c(x) \\ &= dx^m(\delta^a_m + f^{-2}\,\varepsilon^{abc}A_b(x)\,\partial_m A_c(x)) \\ &\equiv dx^m E^a_m\,. \end{aligned} \tag{3.6}$$

3.1 Action and Equation of Motion

Using the Volkov-Akulov formalism as explained in sec. 2.1, we obtain the action. Subtracting the constant term and modulo a total derivative the action is

$$\begin{aligned} S_1 &= -\,f^2\int d^3x\,(\det E^a_d - 1) \\ &= \int d^3x\,\Big(\varepsilon^{abc}A_a\,\partial_b A_c - \frac{f^{-2}}{2}\,\varepsilon^{abc}\varepsilon^{def}\,A_a\,A_d\,\partial_e A_b\,\partial_f A_c \\ &\qquad + \frac{f^{-4}}{6}\,\varepsilon^{abc}(\varepsilon^{def}\varepsilon^{klm}A_d\,A_e\,A_l\,\partial_a A_k\,\partial_b A_f\,\partial_c A_m \\ &\qquad\qquad - \varepsilon^{kef}\varepsilon^{dlm}A_d\,A_e\,A_l\,\partial_a A_k\,\partial_b A_f\,\partial_c A_m)\Big) \end{aligned} \tag{3.7}$$

In the expression above the terms of the sixth order in A_a vanish due to the anti-symmetry of the Levi-Civita tensors and the Abelian nature of the vector field A_a. Therefore, the action becomes

$$S_1 = \int d^3x\,\Big(\varepsilon^{abc}A_a\,\partial_b A_c - \frac{f^{-2}}{2}\,\varepsilon^{abc}\varepsilon^{def}\,A_a\,A_d\,\partial_e A_b\,\partial_f A_c\Big) \tag{3.8}$$

We can see that the leading order quadratic term in the action above is the Abelian Chern-Simons action. We know that it is gauge invariant under the following gauge transformation:

$$A'_a(x) = A_a(x) + \partial_a\lambda(x). \tag{3.9}$$

Therefore, as is well known, the leading order Chern-Simons action describes a gauge theory with no local degrees of freedom. Its equation of motion is

$$\varepsilon^{abc}\,\partial_b A_c = 0 \tag{3.10}$$

On taking the divergence of the equation of motion, we get the following identity, which is the Noether identity:

$$\partial_a(\varepsilon^{abc}\,\partial_b A_c) \equiv 0 \tag{3.11}$$

This identity exists because of the anti-symmetry of the Levi-Civita tensor in the indices a, b and c and the symmetry of the partial derivatives ∂_a and ∂_b. A Noether identity is a differential relation that shows that the equations of motions in the system are not independent of each other. This tells us that there is a gauge symmetry present in the system. So the existence of Noether identity (3.11) verifies that the leading order Chern-Simons action is gauge invariant.

Equation of motion (3.10) can also be read as equating the field strength $F_{ab} = \partial_a A_b - \partial_b A_a$ to zero. With that perspective eq. (3.11) is a Bianchi identity.

We should now figure out whether the non-linear action (3.8) is invariant under a non-linear generalization of the local symmetry. If it were so, then also in the non-linear theory there would be no physical degree of freedom of $A_a(x)$ as in the free case.

If a Noether identity exists for a system, then it has gauge symmetry. So let us try to see if we can get a Noether identity for the non-linear action.

3.1.1 Noether Identity Test

We try to find the non-linear generalization of the Noether identity (3.11). Let us first write the equation of motion for action (3.8), derived by extremizing the action. It is

$$\varepsilon^{abc}\,\partial_b A_c - \varepsilon^{abc}\varepsilon^{def}\,A_d\,\partial_e A_b\,\partial_f A_c = 0 \tag{3.12}$$

Note that in the linear case the identity in question is the divergence of the left hand side of the equation (3.10) identically equal to zero. Trying to follow a similar pattern here for the non-linear generalization, we re-express eq. (3.12) in the following form:

$$\varepsilon^{abc}\,\mathcal{D}_b A_c = 0 \tag{3.13}$$

where $\mathcal{D}_b$ is the covariant derivative of the form

$$\mathcal{D}_b = (E^{-1})^d_b\,\partial_d + \frac{1}{2E}\,\partial_d(E\,(E^{-1})^d_b), \tag{3.14}$$

$(E^{-1})^d_b$ is the matrix inverse of E^a_b defined in (3.6) and $E := \det E^a_b$. It is natural to assume that the non-linear generalization of the Noether identity of the free theory is the vanishing of the divergence of the left hand side of (3.13) with respect to $\mathcal{D}_a$. We check if the operator $\mathcal{D}_a$ can replace the partial derivative ∂_a in the sought after non-linear generalization of the Noether identity (3.11). We find that it doesn't do so.

$$\varepsilon^{abc}\mathcal{D}_a\mathcal{D}_b A_c \not\equiv 0 \tag{3.15}$$

We were not able to find a more general form of $\mathcal{D}_b$ which would produce the Noether identity. If the Noether identity does not exist, then the non-linear Lagrangian does not have a local gauge symmetry and the field $A_a(x)$ is a propagating physical field. To confirm this assumption and to understand the properties of the possible propagating degrees of freedom we will now carry out a perturbative analysis of the equation of motion.

3.1.2 Perturbative Analysis

The solution of the non-linear equation of motion (3.12) can be studied order-by-order in f^{-2}. At the zeroth order in f^{-2}, the solution is

$$A^{(0)}_a = \partial_a\varphi \tag{3.16}$$

where φ is a scalar field. Up to the order f^{-2} the solution is

$$A_a = \partial_a\varphi + f^{-2}A^{(1)}_a + \mathcal{O}(f^{-4})\,. \tag{3.17}$$

Plugging this into eq. (3.12) we get the following expression for the field-strength of $A^{(1)}_a$ in terms of φ :

$$\varepsilon^{abc}\,\partial_b A^{(1)}_c - \varepsilon^{abc}\varepsilon^{def}\,\partial_d\varphi\,\partial_e\partial_b\varphi\,\partial_f\partial_c\varphi\; = 0. \tag{3.18}$$

Upon taking the divergence of the equation above, we get,

$$\varepsilon^{abc}\,\partial_a\partial_b A_c^{(1)} - \varepsilon^{abc}\varepsilon^{def}\,\partial_a\partial_d\varphi\,\partial_e\partial_b\varphi\,\partial_f\partial_c\varphi \;= 0. \tag{3.19}$$

The first term in the equation above, i.e. $\varepsilon^{abc}\,\partial_a\partial_b A_c^{(1)}$, is identically zero because the Levi-Civita tensor is anti-symmetric in a, b and c while the partial derivatives ∂_a and ∂_b commute with each other. The φ term in the equation above can be re-written as following on multiplying with a constant factor of 1/6:

$$\begin{aligned}
& -\frac{1}{6}\,\varepsilon^{abc}\varepsilon^{def}\,\partial_a\partial_d\varphi\,\partial_e\partial_b\varphi\,\partial_f\partial_c\varphi = 0 \\
&= \det\big(\partial_a\partial^b\varphi\big) = 0 && (3.20)\\
&= (\Box\varphi)^3 - 3\,\Box\varphi\,\partial_a\partial^b\varphi\,\partial_b\partial^a\varphi + 2\,\partial_a\partial^b\varphi\,\partial_b\partial^c\varphi\,\partial_c\partial^a\varphi = 0\,. && (3.21)
\end{aligned}$$

This equation can be regarded as the higher-order equation of motion of φ. It has only second-order time derivatives of φ, though. This tells us that φ is a propagating scalar field albeit not the usual kind of scalar field since it does not have a usual kinetic term $\partial_a\varphi\,\partial^a\varphi$.

At the end of section (3.1.1) we mentioned that the absence of gauge symmetry should result in the presence of a propagating field. Now that we have obtained a propagating scalar field φ in the system by performing perturbative analysis, we see that it is this field which is associated with the broken gauge symmetry.

3.1.3 Stückelberg Trick for Vector Goldstone Model

Now we apply the Stückelberg trick to study the properties of the system. Stückelberg trick brings a local symmetry in the Lagrangian by introducing an auxiliary field, also known as the Stückelberg field. Let us introduce an auxiliary field $\hat{\varphi}$ into the system and replace the vector field A_a with $\hat{A}_a$ as following:

$$A_a \to \hat{A}_a = A_a - f^{\frac{1}{2}}\,\partial_a\hat{\varphi} \tag{3.22}$$

$\hat{A}_a$ is invariant under the following transformations:

$$\begin{aligned}
\delta A_a &= \partial_a\lambda\,,\\
\delta\hat{\varphi} &= f^{-\frac{1}{2}}\lambda\,.
\end{aligned} \tag{3.23}$$

Hence, by introducing the auxiliary field $\hat{\varphi}$ we made the Lagrangian gauge invariant. In order to keep the Lagrangian finite in the field $\hat{\varphi}$ under the limit $f \to \infty$, which we take ahead, the exponential power of f in eq. (3.22) should be $1/2$.

In the Lagrangian (3.8) we substitute the vector field A_a with $\hat{A}_a$ to get the Stückelberg Lagrangian. On taking the limit $f \to \infty$ in the Lagrangian, we find that the vector field A_a decouples from the Stückelberg field $\hat{\varphi}$.

$$\mathcal{L}(\hat{A}_a)|_{f\to\infty} = \varepsilon^{abc}\, A_a\, \partial_b A_c - \frac{1}{2}\, \varepsilon^{abc}\varepsilon^{def}\, \partial_a\hat{\varphi}\, \partial_d\hat{\varphi}\, \partial_e\partial_b\hat{\varphi}\, \partial_f\partial_c\hat{\varphi}\,. \tag{3.24}$$

Therefore the limit $f \to \infty$ is the decoupling limit for the Stückelberg Lagrangian.

Now that we have the Lagrangian, we can find the dimensionality of each field in it and see if the field $\hat{\varphi}$ needs to be rescaled to make it canonical. We have the following dimensions for the fundamental scales of mass, length and time:

$$[M] = 1 \quad [L] = [T] = -1 \tag{3.25}$$

This implies that the dimension of a derivative ∂_a is 1.

The action $S = \int d^3x\, \mathcal{L}$ is a dimensionless quantity which implies that the Lagrangian density $\mathcal{L}$ (also referred to as Lagrangian in this text) must have canonical dimension 3. Using this we can calculate the dimensions of the fields appearing in the Lagrangian. For the fields A_a and $\hat{\varphi}$ appearing in the Lagrangian (3.24), on counting the dimensions we get,

$$\begin{aligned} [A_a] &= \big(3 - 1\,(\text{from } \partial_b)\big)/2 = 1 \\ [\hat{\varphi}] &= \big(3 - 6\,(\text{from } \partial_a)\big)/4 = -\,3/4 \end{aligned} \tag{3.26}$$

The kinetic Lagrangian for a canonical massless scalar field ϕ is $(\partial\phi)^2$. So the dimension of a canonical scalar field is:

$$[\phi] = \big(3 - 2\,(\text{from } \partial)\big)/2 = 1/2 \tag{3.27}$$

But in (3.26) we got the dimension of $\hat{\varphi}$ to be $-3/4$. That means $\hat{\varphi}$ is not canonical yet. It should be rescaled with a mass parameter in order for its dimension to become $1/2$.

Let $\hat{\varphi}$ be rescaled by M^y such that now $\hat{\varphi}$ has the canonical dimension $1/2$. Then, the $\hat{\varphi}$ term in Lagrangian (3.24) becomes

$$-\frac{1}{2}\,\varepsilon^{abc}\varepsilon^{def}\,M^{4y}\,\partial_a\hat{\varphi}\,\partial_d\hat{\varphi}\,\partial_e\partial_b\hat{\varphi}\,\partial_f\partial_c\hat{\varphi}\,. \tag{3.28}$$

On counting the dimensions of M^{4y} in the above Lagrangian term, we get,

$$\begin{aligned}&[M^{4y}] = 3 - 6\,(\text{from }\,\partial_a) - 2\,(\text{from }\,\hat{\varphi}) = -\,5\\ \Rightarrow\ & y = -\,5/4\,.\end{aligned} \tag{3.29}$$

So the scalar part of Lagrangian (3.24) becomes

$$-\frac{1}{2}\,\varepsilon^{abc}\varepsilon^{def}\,M^{5}\,\partial_a\hat{\varphi}\,\partial_d\hat{\varphi}\,\partial_e\partial_b\hat{\varphi}\,\partial_f\partial_c\hat{\varphi}\,, \tag{3.30}$$

where the scalar field $\hat{\varphi}$ has the canonical dimension $1/2$. We rewrite the scalar field Lagrangian by integrating expression (3.30) by parts, to get the following:

$$\begin{aligned}\mathcal{L}(\hat{\varphi}) &= \frac{M^{-5}}{2}\,\hat{\varphi}\,\varepsilon^{abc}\varepsilon^{def}\,\partial_a\partial_d\hat{\varphi}\,\partial_e\partial_b\hat{\varphi}\,\partial_f\partial_c\hat{\varphi}\\ &= -\,3\,M^{-5}\hat{\varphi}\,\det\big(\partial_a\partial^b\hat{\varphi}\big)\\ &= -\,\frac{M^{-5}}{2}\hat{\varphi}\,\big((\Box\hat{\varphi})^3 - 3\,\Box\hat{\varphi}\,\partial_a\partial^b\hat{\varphi}\,\partial_b\partial^a\hat{\varphi} + 2\,\partial_a\partial^b\hat{\varphi}\,\partial_b\partial^c\hat{\varphi}\,\partial_c\partial^a\hat{\varphi}\big)\,.\end{aligned} \tag{3.31}$$

We can notice that the Lagrangian above has expressions that appeared earlier in the perturbative analysis written in (3.20)! So the propagating scalar field φ that we obtained in the last section is a Stückelberg field. Equation (3.21) is the equation of motion for the Stückelberg Lagrangian in the decoupling limit (3.31) written above.

Lagrangian (3.31) happens to be the Lagrangian of a galileon.

3.2 Galileon

A galileon is a scalar field, generally called $\pi(x)$, that arises in theories of modified gravity. It modifies general relativity on perturbed Minkowski spacetime. It couples to the metric $g_{\mu\nu}$ resulting in the Einstein-Hilbert action $\sqrt{-g}R$ getting replaced by the action $\sqrt{-g}(1-2\pi)R$ plus self-interaction terms that are derivatives of π. It is not coupled to matter directly but indirectly because of its coupling to $g_{\mu\nu}$. This is necessary to maintain the universality of gravitational interactions. It enables a model independent analysis of a large class of modified gravity models [71].

General relativity is an effective theory that is assumed to hold at low energy scales below an ultraviolet cut-off limit and at large distances, such as solar system distances. Also, the accuracy of the predictions of general relativity increases with time, so it is valid for late time scales such as the present but perhaps not at early time scales near the Big Bang. Because of its validity at low energy scales, general relativity is said to be a theory valid in the infrared regime.

The supernovae experiments [72, 73] have shown that the universe is expanding at an accelerated rate. If ones tries to justify the accelerated expansion through general relativity, one needs to provide a certain value to the cosmological constant Λ. Giving that particular value to Λ physically means giving a correction to the vacuum energy of the universe. This correction in the vacuum energy can be accounted for by the presence of dark energy.

The value of Λ obtained in general relativity [74] on positing the existence of dark energy to explain the accelerated expansion, is smaller than the value provided by quantum field theory by the order of magnitude 10^{120}. This enormous annatural discrepancy could mean that instead of a dark energy (cosmological constant) component being responsible for the accelerated expansion, gravity is modified at cosmological distances so as to produce an accelerating universe.

A number of different theories of modified gravity [75, 76] have been proposed till date, for e.g., Dvali-Gabadadze-Porrati (DGP) model [77], massive gravity [78], etc. All the theories of modified gravity have a scalar field universally coupled to general relativity. The additional degree of freedom introduced by the scalar field extends the validity of the theory to sub-cosmological distances, hence beyond the infrared regime. For a review on galileons, see [79–82] and the references therein.

3.2.1 Modifying Gravity

Let us see how galileons arise in the theories of modified gravity [83]. A galileon is supposed to modify the linearized gravitational potential produced by an energy momentum source or the Hubble flow at the order $O(1)$, i.e., at cosmological scales. But since general relativity gives us the correct predictions at large-scale distances, we need to ensure that modified gravity does not deviate from general relativity at solar system distances beyond the order $O(10^{-3})$. With linear dynamics of the scalar field, it is not possible to satisfy both the above conditions simultaneously. However, the introduction of non-linearities enables the extrapolation between different length scales.

In the infrared regime when gravity is locally modified by a universally coupled scalar, we need to ensure that the solar system tests are recovered. They can be recovered by the Vainshtein effect [84] which is the decoupling of the scalar from matter in gravitationally bound systems.

Let us denote the universally coupled scalar field by π_c . We need to make the following assumptions for π_c:

1. The dynamics of π_c should be non-linear. Up to the quadratic order, it should be expressed as following:

$$\pi_c = C + B_\mu x^\mu + A_{\mu\nu} x^\mu x^\nu + O(x^3 H^3) \,. \tag{3.32}$$

This quadratic approximation is invariant under the combined action of a spacetime translation, i.e. $\pi_c(x) \to \pi_c(x+\delta)$ and a shift, i.e. $\pi_c(x) \to \pi_c(x) + b_\mu x^\mu + c$. When π_c as expressed in eq. (3.32) is transformed under these transformations, we get,

$$\pi_c(x) = \pi_c(x+\delta) - 2A_{\mu\nu}\,\delta^\mu x^\nu - B_\mu \delta^\mu \,, \tag{3.33}$$

This tells us that $b_\mu = -2A_{\mu\nu}\delta^\nu$ and $c = -B_\mu \delta^\mu$. Generally the symmetry of a solution follows from an invariance of the equations of motion. That invariance constrains the form of the Lagrangian. It can be deduced that the Lagrangian is invariant under the following shift:

$$\pi_c(x) \to \pi_c(x) + b_\mu x^\mu + c \,. \tag{3.34}$$

The symmetry condition (3.34) is obtained as a result of demanding the solution to have a quadratic approximation (3.32). A solution up to the quadratic order can be obtained even from a linear equation of motion via Taylor expansion. For e.g., consider an ordinary two derivative Lagrangian:

$$-\frac{1}{2}(\partial\rho)^2 - V(\rho) \,,$$

where the equation of motion linearized in $\varphi = \rho(x) - \rho(0)$, is

$$\Box\varphi = V'(\rho(0)) = \mathrm{const} \,. \tag{3.35}$$

This is a linear equation of motion which admits the quadratic solution (3.32). It is invariant under the symmetry (3.34). In fact, generally the symmetry

$\rho(x) \to \rho(x) + b_\mu x^\mu + c$ of the approximated solution follows trivially from the shift invariance of the equations of motion.

But as mentioned before, we want π_c dynamics even at the local quadratic order in the coordinates to be described by a non-linear equation of motion. So we need to ensure that the equations of motion of π_c have the shift symmetry (3.34) while being non-linear. This leads us to our second non-trivial assumption.

2. The equations of motion of π_c should be invariant under the shift symmetry (3.34) while each π_c in the equations of motion is acted upon by at least two derivatives.

3. The third and final assumption we make is that the equations of motion of π_c should be only of the second order (in time derivatives). This is equivalent to demanding that the equations of motion with $\mathcal{L}_{\pi_c}$ being the Lagrangian, be of the following form:
$$\frac{\delta \mathcal{L}_{\pi_c}}{\delta \pi_c} = F(\partial_\mu \partial_\nu \pi_c) \tag{3.36}$$
where F is a non-linear Lorentz invariant function of the tensor $\partial_\mu \partial_\nu \pi$. The need for this assumption arises by comparing the ghost-free DGP model with the ghost-containing Fierz-Pauli massive gravity. The DGP model contains equations of motion in π_c only of the second order while in the Fierz-Pauli case the analogue of π_c dynamics leads to fourth order equations of motion giving a ghost at the onset of non-linearity [85].

The three conditions listed above correspond to the three main properties of the DGP model that guarantee its viabilty at solar system distances.

In the shift transformation (3.34) the vectorial parameter b_μ corresponds to the shift of the gradient of π by a constant vector, i.e.,

$$\partial_\mu \pi_c \to \partial_\mu \pi_c + b_\mu\,.$$

This is the space-time generalization of the Galilean symmetry

$$\dot{x} \to \dot{x} + v \Rightarrow x \to x + vt$$

of non-relativistic mechanics (0+1 field theory), where x is the position coordinate, v is the velocity and t is the time coordinate. By analogy the transformation $\pi_c \to$

$\pi_c + b\,x$ is called the Galilean transformation. This is why π_c is called galileon. From now on we can write it as just π.

So now we have seen how modifying gravity gives a galileon.

The shift corresponding to c in (3.34) suggests that π_c may be a Goldstone boson. In massive gravity models it is associated with the breaking of diffeomorphism invariance, while in our model of the vector Goldstone field it is associated with the spontaneous breaking of $U(1)$ gauge symmetry.

3.2.2 Galileon Lagrangian

As mentioned before, the galileon Lagrangian must be such that its variation with respect to π is of the following form:

$$\Rightarrow \frac{\delta \mathcal{L}_\pi}{\delta \pi} = F(\partial_\mu \partial_\nu \pi) \tag{3.37}$$

with F being a non-linear function that is Lorentz invariant. When we try to construct functions of the kind F, we find that there is only a small number of such functions. In three dimensional spacetime only four such functions exist. We get one extra such function every time we go one dimension higher. In D dimensions, there exist $D+1$ Lagrangian terms of the kind F, which we can refer to as Galileo-invariant Lagrangian terms. The most general Lagrangian is the linear combination of such terms multiplied by arbitrary functions of π.

In $D = 3$ there are four Galileo-invariant (modulo total derivatives) Lagrangian terms:

$$\mathcal{L}_1 \equiv \pi\,, \tag{3.38}$$
$$\mathcal{L}_2 \equiv \pi\,\Box\pi\,, \tag{3.39}$$
$$\mathcal{L}_3 \equiv \pi\left[(\Box\pi)^2 - (\partial_\mu\partial_\nu\pi)(\partial^\mu\partial^\nu\pi)\right], \tag{3.40}$$
$$\mathcal{L}_4 \equiv \pi\left[(\Box\pi)^3 - 3(\Box\pi)(\partial_\mu\partial_\nu\pi)(\partial^\mu\partial^\nu\pi) + 2(\partial_\mu\partial^\nu\pi\,\partial_\nu\partial^\rho\pi\,\partial_\rho\partial^\mu\pi)\right]. \tag{3.41}$$

Higher order Galileo-invariants are total derivatives, so they are trivial.

The complete Lagrangian for π is a linear combination of the above invariants:

$$\mathcal{L}_\pi = \sum_{n=1}^{4} a_n\,\mathcal{L}_n\,, \tag{3.42}$$

where the a_n's are generic coefficients.

$\mathcal{L}_4$ in eq. (3.41) is the same as the Lagrangian (3.31) which we obtained on performing the Stückelberg trick on the vector Goldstone in section 3.1.3. So our Goldstone is like a galileon.

It is more convenient to look at the equations of motion derived from the extremisation of the Lagrangian terms above. The equations of motion do not have any ambiguities on account of the presence of total derivative terms obtained from integration by parts. Also, all the terms in the equations are composed of second derivatives of π.

Extremising the Lagrangian terms, $\mathcal{E}_n \equiv \frac{\delta \mathcal{L}_n}{\delta \pi}$, we get the following contributions to the left-hand-side of the equations of motion:

$$\mathcal{E}_1 = 1 \tag{3.43}$$

$$\mathcal{E}_2 = \Box\pi \tag{3.44}$$

$$\mathcal{E}_3 = (\Box\pi)^2 - (\partial_\mu\partial_\nu\pi)^2 \tag{3.45}$$

$$\mathcal{E}_4 = (\Box\pi)^3 - 3\,\Box\pi(\partial_\mu\partial_\nu\pi)^2 + 2(\partial_\mu\partial_\nu\pi)^3 \tag{3.46}$$

where $(\partial_\mu\partial_\nu\pi)^n$ denotes the cyclic contraction
$(\partial_\mu\partial_\nu\pi)^n \equiv \partial_\mu\partial_\nu\pi\,\partial^\nu\partial^\rho\pi\,\partial_\rho\partial_\sigma\pi\cdots\partial^\tau\partial^\mu\pi$ (with n πs).

The complete equation of motion is the following linear combination:

$$\mathcal{E} \equiv \frac{\delta\mathcal{L}_\pi}{\delta\pi} = \sum_{n=1}^{4} c_n\,\mathcal{E}_n = 0\,. \tag{3.47}$$

We can see from $\mathcal{E}_4$ in eq. (3.46) that it is the same as eq. (3.21) that was obtained in the perturbative analysis of the vector Goldstone model, which is also the equation of motion of the Lagrangian obtained in the Stückelberg trick. So the form of $\mathcal{L}_4$ and $\mathcal{E}_4$ tell us that the Stückeblerg field in our vector Goldstone model is like a galileon.

Below we rewrite the Stückelberg Lagrangian and its equation of motion shown earlier:

$$\mathcal{L}(\hat\varphi) = -\frac{M^{-5}}{2}\hat\varphi\left((\Box\hat\varphi)^3 - 3\,\Box\hat\varphi\,\partial_a\partial^b\hat\varphi\,\partial_b\partial^a\hat\varphi + 2\,\partial_a\partial^b\hat\varphi\,\partial_b\partial^c\hat\varphi\,\partial_c\partial^a\hat\varphi\right)\,, \tag{3.48}$$

$$(\Box\hat\varphi)^3 - 3\,\Box\hat\varphi\,\partial_a\partial^b\hat\varphi\,\partial_b\partial^a\hat\varphi + 2\,\partial_a\partial^b\hat\varphi\,\partial_b\partial^c\hat\varphi\,\partial_c\partial^a\hat\varphi = 0\,. \tag{3.49}$$

We did not get the terms corresponding to $\mathcal{L}_1,\ \mathcal{L}_2$, and $\mathcal{L}_3$ in the above.

The equation of motion of Stückelberg Lagrangian, i.e., (3.49), has two kinds of simple non-trivial solutions. One set of such solutions is that of the fields that are static:

$$\partial_t \hat{\varphi}(t, x^i) = 0, \tag{3.50}$$

The other set of solutions consists of plane-wave-like solutions:

$$\hat{\varphi} = e^{ip_a x^a} \phi(p) + e^{-ip_a x^a} \phi^*(p), \tag{3.51}$$

where p_a is an arbitrary time-like, space-like or light-like momentum. It is a priori not subject to the mass-shell condition $p_a p^a - m^2 = 0$ since the Lagrangian (3.48) does not contain the quadratic kinetic term $\mathcal{L}_2 = -\frac{1}{2}\left(\partial_a \hat{\varphi}\, \partial^a \hat{\varphi} + m^2\, \hat{\varphi}^2\right)$. Hence, there is no corresponding term in the equation of motion. So, this higher-order model contains tachyons, unless they are excluded by imposing appropriate mass-shell conditions on $\hat{\varphi}$.

We wish to confirm that $\hat{\varphi}$ is the only propagating field in this vector Goldstone model and study its dynamical properties. For this purpose we now move to the Hamiltonian analysis based on Dirac formalism.

3.3 Hamiltonian Analysis of the Abelian Chern-Simon Theory

First we carry out the analysis of the Hamiltonian for the simpler case of the free Chern-Simons vector field. Doing so will give us a better idea of the scheme of the analysis. Also, it will enable us to distinctly understand the factors that make the non-linear model different from the linear one.

In the Hamiltonian formalism, time and position coordinates are considered separately. Therefore, as in the spin-1/2 case analysis, we split the space-time indices into time and space indices – $a = (0, i)$. We have $\varepsilon^{0ij} \equiv \varepsilon^{ij}$.

The leading order Abelian Chern-Simons Lagrangian has the following form:

$$\begin{aligned}\mathcal{L}_{CS} &= \varepsilon^{abc} A_a(t, \mathbf{x})\, \partial_b A_c(t, \mathbf{x}) \\ &= \varepsilon^{jk} A_0(t, \mathbf{x})\, \partial_j A_k(t, \mathbf{x}) - \varepsilon^{ik} A_i(t, \mathbf{x})\, \partial_0 A_k(t, \mathbf{x}) + \varepsilon^{ij} A_i(t, \mathbf{x})\, \partial_j A_0(t, \mathbf{x}),\end{aligned} \tag{3.52}$$

where x stands for the spacial coordinates $x^i = (x^1, x^2)$, so that $x^a = (t, \mathbf{x})$.

The conjugate momenta are:

$$\begin{aligned} p^i(t,\mathbf{x}) &= \frac{\delta L}{\delta(\partial_0 A_i(t,\mathbf{x}))} = \varepsilon^{ij} A_j(t,\mathbf{x})\,, \\ p^0(t,\mathbf{x}) &= \frac{\delta L}{\delta(\partial_0 A_0(t,\mathbf{x}))} = 0\,. \end{aligned} \tag{3.53}$$

The equal-time Poisson bracket relation between A_a and p^b is:

$$[A_a(t,\mathbf{x}), p^b(t,\mathbf{y})] = \delta_a^b\,\delta^{(2)}(\mathbf{x}-\mathbf{y}), \qquad (a=0,i)\,. \tag{3.54}$$

The canonical Hamiltonian density is

$$\begin{aligned} \mathcal{H}_c(t,\mathbf{x}) &= p^i(t,\mathbf{x})\,\partial_0 A_i(t,\mathbf{x}) - \mathcal{L}_{CS}(t,\mathbf{x}) \\ &= \varepsilon^{ij}\left(A_0(t,\mathbf{x})\,\partial_j A_i(t,\mathbf{x}) + A_j(t,\mathbf{x})\,\partial_i A_0(t,\mathbf{x})\right) \end{aligned} \tag{3.55}$$

The above expression can be integrated by parts with respect to x to yield the canonical Hamiltonian. It comes out to be,

$$H_c(t) = 2\int d^2\mathbf{x}\,\varepsilon^{ij} A_0(t,\mathbf{x})\,\partial_j A_i(t,\mathbf{x})\,. \tag{3.56}$$

The primary constraints are:

$$C^i(t,\mathbf{x}) = p^i(t,\mathbf{x}) - \varepsilon^{ij} A_j(t,\mathbf{x}) = 0 \tag{3.57}$$

$$C^0(t,\mathbf{x}) = p^0(t,\mathbf{x}) = 0\,. \tag{3.58}$$

The Poisson bracket relations between these primary constraints are,

$$\begin{aligned} &[C^i(t,\mathbf{x}), C^j(t,\mathbf{y})] = -\,2\,\varepsilon^{ij}\,\delta^{(2)}(\mathbf{x}-\mathbf{y})\,, \\ &[C^0(t,\mathbf{x}), C^0(t,\mathbf{y})] = 0\,, \\ &[C^i(t,\mathbf{x}), C^0(t,\mathbf{y})] = 0\,. \end{aligned} \tag{3.59}$$

We see that the constraints C^i are of the second-class while C^0 is of the first-class. So C^i are not associated with a gauge symmetry of the system while C^0 is.

To find the secondary constraint following the Dirac procedure, we consider the total Hamiltonian which includes the primary constraints with their corresponding

Lagrange multipliers.

$$\mathcal{H}_T(t,\mathbf{x}) = \mathcal{H}_c(t,\mathbf{x}) + u_k(t,\mathbf{x})\, C^k(t,\mathbf{x}) + u_0(t,\mathbf{x})\, C^0(t,\mathbf{x})\,. \tag{3.60}$$

We impose the time-conservation condition on the primary constraint C^i. For that we take its Poisson bracket with the total Hamiltonian and equate it to 0.

$$\begin{aligned}
&\int d^2\mathbf{y}\, [C^i(t,\mathbf{x}), \mathcal{H}_T(t,\mathbf{y})] \\
&= \int d^2\mathbf{y}\, [C^i(t,\mathbf{x}), \mathcal{H}_c(t,\mathbf{y})] + \int d^2\mathbf{y}\, u_k(t,\mathbf{y})[C^i(t,\mathbf{x}), C^k(t,\mathbf{y})] \\
&\quad + \int d^2\mathbf{x}\, u_0(t,\mathbf{y})[C^i(t,\mathbf{x}), C^0(t,\mathbf{y})] = 0\,.
\end{aligned} \tag{3.61}$$

Solving the above equation provides us with the value of the Lagrange multiplier $u_k(t,\mathbf{x})$, which comes out to be,

$$u_k(t,\mathbf{x}) = \partial_k A_0(t,\mathbf{x})\,. \tag{3.62}$$

So the time conservation equation (3.61) does not provide us with a secondary constraint.

Now let us see if we can get a secondary constraint from the time conservation equation for C^0.

$$\begin{aligned}
&\int d^2\mathbf{y}\, [C^0(t,\mathbf{x}), \mathcal{H}_T(t,\mathbf{y})] \\
&= \int d^2\mathbf{y}\, [C^0(t,\mathbf{x}), \mathcal{H}_c(t,\mathbf{y})] + \int d^2\mathbf{y}\, u_k(t,\mathbf{y})[C^0(t,\mathbf{x}), C^k(t,\mathbf{y})] \\
&\quad + \int d^2\mathbf{y}\, u_0(t,\mathbf{y})[C^0(t,\mathbf{x}), C^0(t,\mathbf{y})] = 0 \\
&\Rightarrow \varepsilon^{ij}\partial_i A_j(t,\mathbf{x}) = 0
\end{aligned}$$

So we get a secondary constraint:

$$\varepsilon^{ij}\partial_i A_j(t,\mathbf{x}) = 0. \tag{3.63}$$

Let us compute the Poisson brackets of the above secondary constraint with the other constraints to check whether it belongs to the first-class or the second-class.

The Poisson bracket of the secondary constraint with $C^0(t,\mathbf{x})$ turns out to be 0.

$$[C^0(t,\mathbf{x})\,,\varepsilon^{ij}\partial_i A_j(t,\mathbf{y})] = 0 \tag{3.64}$$

The Poisson bracket of the secondary constraint with $C^i(t,\mathbf{x})$ is as following.

$$[\varepsilon^{ij}\partial_i A_j(t,\mathbf{x})\,,C^k(t,\mathbf{y})] = \varepsilon^{ik}\partial_i\,\delta^{(2)}(\mathbf{x}-\mathbf{y}) \tag{3.65}$$

The above Poisson bracket has a non-zero value, but we will see that the secondary constraint can be modified such that it becomes a first-class constraint.

The expression for the secondary constraint can be written as the derivative of the second term of the primary constraint $C^i(t,\mathbf{x})$ (eq. (3.57)). This fact enables us to guess that the Poisson bracket in eq. (3.65) can also be expressed as following:

$$\begin{aligned}[\varepsilon^{ij}\partial_i A_j(t,\mathbf{x})\,,C^k(t,\mathbf{y})] &= \varepsilon^{ik}\partial_i\,\delta^{(2)}(\mathbf{x}-\mathbf{y})\\ &= -\frac{1}{2}\,[\partial_i C^i(t,\mathbf{x})\,,C^k(t,\mathbf{y})]\end{aligned} \tag{3.66}$$

This tells us that,

$$[\varepsilon^{ij}\partial_i A_j(t,\mathbf{x}) + \frac{1}{2}\,\partial_i C^i(t,\mathbf{x})\,,C^k(t,\mathbf{y})] = 0 \tag{3.67}$$

Let us denote the expression $\varepsilon^{ij}\partial_i A_j(t,\mathbf{x}) + \frac{1}{2}\,\partial_i C^i(t,\mathbf{x})$ by $D(t,\mathbf{x})$. The Poisson brackets of $D(t,\mathbf{x})$ with $C^0(t,\mathbf{y})$ and with $D(t,\mathbf{y})$ are,

$$\begin{aligned}&[D(t,\mathbf{x})\,,C^0(t,\mathbf{y})] = 0\,,\\ &[D(t,\mathbf{x})\,,D(t,\mathbf{y})] = 0\,,\end{aligned} \tag{3.68}$$

respectively. So $D(t,\mathbf{x})$ qualifies to be a first-class constraint.

Now let us try to check if there are any tertiary constraints present in this system. For this purpose, we write down the time conservation equation for the secondary constraint $D(t,\mathbf{x})$, which is,

$$\begin{aligned}&\int d^2\mathbf{y}\,[D(t,\mathbf{x}),\mathcal{H}_T(t,\mathbf{y})]\\ &= \int d^2\mathbf{y}\,[D(t,\mathbf{x}),\mathcal{H}_c(t,\mathbf{y})] + \int d^2\mathbf{y}\,u_k(t,\mathbf{y})[D(t,\mathbf{x}),C^k(t,\mathbf{y})]\\ &\quad + \int d^2\mathbf{y}\,u_0(t,\mathbf{y})[D(t,\mathbf{x}),C^0(t,\mathbf{y})] = 0\end{aligned}$$

Since $[D(t,\mathbf{x}), \mathcal{H}_c(t,\mathbf{y})] = 0$ and $D(t,\mathbf{x})$ commutes with all the constraints, the above equation does not yield any new information. Therefore, this system has constraints only up till the secondary level and no further.

Now that we have all the constraints in this system, we can count the number of canonical degrees of freedom. To begin with, we have 6 canonical degrees from the vector field components A_1 , A_2 and A_3 and their corresponding momenta p^1, p^2 and p^3 . We have two firs-class constraints – C^0 and D. Each of them cancels two canonical degrees of freedom, leaving us with $6-2-2=2$ canonical degrees of freedom. We have two second-class constraints – C^1 and C^2, each of which knocks down one degree of freedom. So we are left with $2-1-1=0$ canonical degrees of freedom. This implies that there is no dynamical field in this theory.

It verifies that the abelian Chern-Simons action does not have a physical degree of freedom due to the gauge symmetry (3.9).

3.3.1 Hamiltonian Value on the Constraint Surface of the Chern-Simons Model

Let us try to see what value the Hamiltonian takes on the constraint surface. We rewrite the Hamiltonian from eq. (3.56) below.

$$H_c(t) = 2\int d^2\mathbf{x}\, \varepsilon^{ij} A_0(t,\mathbf{x})\, \partial_j A_i(t,\mathbf{x})\,. \tag{3.69}$$

Substituting the secondary constraint from eq. (3.63), i.e., $\varepsilon^{ij}\partial_i A_j(t,\mathbf{x}) = 0$, into the Hamiltonian above, we find that it vanishes.

$$H(t)_{\text{free 1}}|_{cs} = 0\,. \tag{3.70}$$

This verifies the absence of propagating modes in the system.

Now let us move to the Hamiltonian analysis of the full vector-Goldstone model that includes higher-order self-interaction terms.

3.4 Hamiltonian Analysis of the Full Vector Goldstone Model

Action (3.7) takes the following form:

$$\begin{aligned}\mathcal{S}_1 = \int d^3x\, \varepsilon^{ij}\, (2A_0\, \partial_i A_j + A_j\, \partial_0 A_i) \\ + f^{-2} \int d^3x\, \varepsilon^{ij}\varepsilon^{kl} \Big(A_j A_k\, (\partial_i A_l\, \partial_0 A_0 - \partial_l A_0\, \partial_0 A_i) \\ + A_0 A_k\, \partial_0 A_i\, \partial_l A_j - A_0^2\, \partial_l A_j\, \partial_k A_i \Big).\end{aligned} \tag{3.71}$$

We can notice that this action is first-order in time derivatives, as was also the case for the fermionic action discussed in Section 2.3. This implies that the conjugate momenta obtained from this action would be devoid of dependence on time derivatives of the vector field, i.e., they will be expressed in terms of only the spatial derivatives of the vector field components A_0 and A_i . The conjugate momenta are:

$$\begin{aligned} p^i &= \frac{\delta \mathcal{L}}{\delta(\partial_0 A_i)} = \varepsilon^{ij} A_j - \varepsilon^{ij}\varepsilon^{kl} (A_j\, A_k\, \partial_l A_0 - A_0\, A_k\, \partial_l A_j) \\ p^0 &= \frac{\delta \mathcal{L}}{\delta(\partial_0 A_0)} = \varepsilon^{ij}\varepsilon^{kl}\, A_j\, A_k\, \partial_i A_l \end{aligned} \tag{3.72}$$

The canonical Hamiltonian density is

$$\mathcal{H}_c = p^0\, \partial_0 A_0 + p^i\, \partial_0 A_i - \mathcal{L}_1 \tag{3.73}$$

$$= 2\, \varepsilon^{ij} A_0\, \partial_i A_j - \varepsilon^{ij}\varepsilon^{kl} A_0^2\, \partial_k A_i\, \partial_l A_j \tag{3.74}$$

The primary constraints are:

$$C^i = p^i - \varepsilon^{ij} A_j + f^{-2} \varepsilon^{ij}\varepsilon^{kl} (A_j\, A_k\, \partial_l A_0 - A_0\, A_k\, \partial_l A_j) = 0\,, \tag{3.75}$$

$$C^0 = p^0 - f^{-2}\, \varepsilon^{ij}\varepsilon^{kl}\, A_j\, A_k\, \partial_i A_l = 0\,. \tag{3.76}$$

The Poisson bracket relations between the primary constraints are:

$$[C^i(t,\mathbf{x})\,,C^j(t,\mathbf{y})] = 2\,\varepsilon^{ij}\,\delta^{(2)}(\mathbf{x}-\mathbf{y}) + 4\,\varepsilon^{ij}\varepsilon^{kl}A_l(t,\mathbf{x})\,\partial_k A_0(t,\mathbf{x})\,\delta^{(2)}(\mathbf{x}-\mathbf{y}) \\ - \varepsilon^{ij}\varepsilon^{kl}A_0(t,\mathbf{x})\,A_l(t,\mathbf{x})\,(\partial_{x_k}+\partial_{y_k})\,\delta^{(2)}(\mathbf{x}-\mathbf{y}) \qquad (3.77)$$

$$[C^i(t,\mathbf{x})\,,C^0(t,\mathbf{y})] = 4\,\varepsilon^{ji}\varepsilon^{kl}A_l(t,\mathbf{x})\,\partial_k A_j(t,\mathbf{x})\,\delta^{(2)}(\mathbf{x}-\mathbf{y}) \\ - \varepsilon^{kl}\varepsilon^{ji}A_l(t,\mathbf{x})\,A_j(t,\mathbf{x})\,(\partial_{x_k}+\partial_{y_k})\,\delta^{(2)}(\mathbf{x}-\mathbf{y}) \qquad (3.78)$$

$$[C^0(t,\mathbf{x})\,,C^0(t,\mathbf{y})] = 0\,. \qquad (3.79)$$

Since the constraints C^i have non-zero Poisson brackets above, they are of the second-class, as in the linear case. On comparing these constraints with the ones obtained in the case of leading order Chern-Simons action in equations (3.59), we can see that the first-class constraint C^0 of the leading order case does not commute with C^i, but we will see below that it can be modified by terms involving C^i such that the modified constraint commutes with C^i.

Now let us see which secondary constraints this system has. To find the secondary constraints following the Dirac procedure, we consider the Hamiltonian density which includes the primary constraints with their corresponding Lagrange multipliers.

$$\mathcal{H}_T(t,\mathbf{x}) = \mathcal{H}_c(t,\mathbf{x}) + u_i(t,\mathbf{x})\,C^i(t,\mathbf{x}) + u_0(t,\mathbf{x})\,C^0(t,\mathbf{x}) \qquad (3.80)$$

Here $\mathcal{H}_T(t,\mathbf{x})$ is the total Hamiltonian density, $\mathcal{H}_c(t,\mathbf{x})$ is the canonical Hamiltonian density, $u_i(t,\mathbf{x})$ and $u_0(t,\mathbf{x})$ are the Lagrange multipliers corresponding to the primary constraints $C^i(t,\mathbf{x})$ and $C^0(t,\mathbf{x})$ respectively.

On integrating the above, we get the expression for the total Hamiltonian:

$$H_T(t) = \int d^2\mathbf{x}\,\mathcal{H}_T(t,\mathbf{x}) = H_c(t) + \int d^2\mathbf{x}\,u_i(t,\mathbf{x})\,C^i(t,\mathbf{x}) + \int d^2\mathbf{x}\,u_0(t,\mathbf{x})\,C^0(t,\mathbf{x}) \qquad (3.81)$$

As explained before, in order to get a secondary constraint, we need to impose the time conservation of the primary constraint (see eq. (2.20)). So we take the time derivative of the primary constraint $C^i(t,\mathbf{x})$ by taking its Poisson bracket with $H_T(t)$

and equate it to zero as following:

$$
\begin{aligned}
&[C^i(t,\mathbf{x})\,,H_T(t,\mathbf{y})] \\
&= \int d^2\mathbf{y}\,[C^i(t,\mathbf{x})\,,\mathcal{H}_T(t,\mathbf{y})] \\
&= \int d^2\mathbf{y}\,[C^i(t,\mathbf{x})\,,\mathcal{H}_c(t,\mathbf{y})] + \int d^2\mathbf{y}\,u_j(t,\mathbf{y})\,[C^i(t,\mathbf{x})\,,C^j(t,\mathbf{y})] \\
&\quad + \int d^2\mathbf{y}\,u_0(t,\mathbf{y})\,[C^i(t,\mathbf{x})\,,C^0(t,\mathbf{y})] = 0
\end{aligned} \tag{3.82}
$$

Let $\int d^2\mathbf{y}\,[C^i(t,\mathbf{x}),\,C^j(t,\mathbf{y})]$ be denoted by $C^{ij}(t,\mathbf{x})$. Let $C_{jk}(t,\mathbf{x})$ be the inverse such that $C^{ij}(t,\mathbf{x})\,C_{jk}(t,\mathbf{x}) = \delta^i_k$. Then eq. (3.82) becomes:

$$
[C^i(t,\mathbf{x})\,,H_c(t)] + u_j(t,\mathbf{x})\,C^{ij}(t,\mathbf{x}) + u_0(t,\mathbf{x})\int d^2\mathbf{y}\,[C^i(t,\mathbf{x})\,,C^0(t,\mathbf{y})] = 0 \tag{3.83}
$$

We can solve the above equation to get an expression for $u_j(t,\mathbf{x})$.

$$
\begin{aligned}
u_j(t,\mathbf{x})\,C^{ij}(t,\mathbf{x}) &= -\,[C^i(t,\mathbf{x})\,,H_c(t)] - u_0(t,\mathbf{x})\int d^2\mathbf{y}\,[C^i(t,\mathbf{x})\,,C^0(t,\mathbf{y})] \\
\Rightarrow u_j(t,\mathbf{x}) &= -\,[C^i(t,\mathbf{x})\,,H_c(t)]\,C_{ji}(t,\mathbf{x}) \\
&\quad - u_0(t,\mathbf{x})\int d^2\mathbf{y}\,[C^i(t,\mathbf{x})\,,C^0(t,\mathbf{y})]\,C_{ji}(t,\mathbf{x})
\end{aligned} \tag{3.84}
$$

On substituting the values of $\mathcal{H}_c(t,\mathbf{x})$, $C^i(t,\mathbf{x})$ and $C^0(t,\mathbf{x})$ into the above equation, we get the following:

$$
\begin{aligned}
u_j(t,\mathbf{x}) = \frac{1}{\varepsilon^{ij}\big(1 + 2\,\varepsilon^{k'l'}A_{l'}(t,\mathbf{x})\,\partial_{k'}A_0(t,\mathbf{x})\big)}\Big(&\varepsilon^{ik}\partial_k A_0(t,\mathbf{x}) \\
&- 2\,\varepsilon^{ik}\varepsilon^{lm}A_0(t,\mathbf{x})\,\partial_m A_0(t,\mathbf{x})\,\partial_l A_k(t,\mathbf{x}) \\
&- 2\,u_0(t,\mathbf{x})\,\varepsilon^{kl}\varepsilon^{mi}A_l(t,\mathbf{x})\,\partial_k A_m(t,\mathbf{x})\Big)
\end{aligned} \tag{3.85}
$$

We have obtained a restriction on the Lagrange multiplier $u_j(t,\mathbf{x})$. This means that eq. (3.82) is not going to provide us with a secondary constraint.

Now when we substitute the expression for $u_j(t,\mathbf{x})$ from eq. (3.85) into the total Hamiltonian (3.81), one can check that the term proportional to $u_0(t,\mathbf{x})$, which will be of the form $C^0(t,\mathbf{x})+F(t,\mathbf{x})$, will commute with $C^i(t,\mathbf{y})$. Let us denote the term

$C^0(t,\mathbf{x}) + F(t,\mathbf{x})$ by $\hat{C}^0(t,\mathbf{x})$. We have,

$$\hat{C}^0(t,\mathbf{x}) = C^0(t,\mathbf{x}) + F(t,\mathbf{x}) \quad \text{where}$$
$$F(t,\mathbf{x}) = \frac{2\,\varepsilon^{i'j'} A_{j'}(t,\mathbf{x})\,\partial_{i'} A_k(t,\mathbf{x})}{\left(1 + 2\,\varepsilon^{ij} A_j(t,\mathbf{x})\,\partial_i A_0(t,\mathbf{x})\,\right)}\, C^k(t,\mathbf{x}) \tag{3.86}$$

On taking the commutator of $\hat{C}^0(t,\mathbf{x})$ with $C^l(t,\mathbf{y})$ we get,

$$\begin{aligned}
&[\hat{C}^0(t,\mathbf{x})\,, C^l(t,\mathbf{y})] \\
&= \frac{1}{\left(1 + 2\,\varepsilon^{i'j'} A_{j'}(t,\mathbf{x})\,\partial_{i'} A_0(t,\mathbf{x})\,\right)} \Big(4\,\varepsilon^{ij}\varepsilon^{kl} A_j(t,\mathbf{x})\,\partial_i A_k(t,\mathbf{x}) \\
&\quad - 4\,\varepsilon^{ij}\varepsilon^{kl} A_j(t,\mathbf{x})\,\partial_i A_k(t,\mathbf{x}) \\
&\quad + 8\,\varepsilon^{ij}\varepsilon^{kl}\varepsilon^{mn} A_n(t,\mathbf{x})\,\partial_m A_k(t,\mathbf{x})\, A_j(t,\mathbf{x})\,\partial_i A_0(t,\mathbf{x}) \\
&\quad - 8\,\varepsilon^{ij}\varepsilon^{kl}\varepsilon^{mn} A_n(t,\mathbf{x})\,\partial_m A_0(t,\mathbf{x})\, A_j(t,\mathbf{x})\,\partial_i A_k(t,\mathbf{x}) \\
&\quad + \left(1 + 2\,\varepsilon^{mn} A_n(t,\mathbf{x})\,\partial_m A_0(t,\mathbf{x})\,\right) \varepsilon^{ij}\varepsilon^{kl} A_j(t,\mathbf{x})\, A_k(t,\mathbf{x})\,(\partial_{x^i} + \partial_{y^i})\Big)\, \delta^{(2)}(\mathbf{x}-\mathbf{y})
\end{aligned} \tag{3.87}$$

This gives us the following result:

$$[\hat{C}^0(t,\mathbf{x})\,, C^l(t,\mathbf{y})] = \varepsilon^{ij}\varepsilon^{kl} A_j(t,\mathbf{x})\, A_k(t,\mathbf{x})\,(\partial_{x^i} + \partial_{y^i})\,\delta^{(2)}(\mathbf{x}-\mathbf{y}) \tag{3.88}$$

The right hand side of the above equation vanishes on integrating it with respect to $\mathbf{x}$ while using the properties of Dirac-Delta function. Please refer to Appendix B for further details. The terms in the Poisson brakets which are proportional to $(\partial_{x_i} + \partial_{y_i})\,\delta^{(2)}(\mathbf{x}-\mathbf{y})$ are effectively zero and can be omitted.

So, effectively eq. (3.88) reduces to

$$[\hat{C}^0(t,\mathbf{x})\,, C^l(t,\mathbf{y})] = 0\,. \tag{3.89}$$

So we have managed to modify the original constraint $C^0(t,\mathbf{x})$ to the constraint $\hat{C}^0(t,\mathbf{x})$ such that it effectively commutes with the other primary constraint $C^l(t,\mathbf{x})$.

Let us now check if we get a secondary constraint by imposing the time conservation of the primary constraint $C^0(t,\mathbf{x})$. Taking the Poisson bracket of $C^0(t,\mathbf{x})$ with $H_T(t)$

we get,

$$\begin{aligned}
&\left[C^0(t,\mathbf{x})\,,H_T(t,\mathbf{y})\right] \\
&= \int d^2\mathbf{y}\,\left[C^0(t,\mathbf{x})\,,\mathcal{H}_T(t,\mathbf{y})\right] \\
&= \int d^2\mathbf{y}\,\left[C^0(t,\mathbf{x})\,,\mathcal{H}_c(t,\mathbf{y})\right] + \int d^2\mathbf{y}\,u_j(t,\mathbf{y})\,\left[C^0(t,\mathbf{x})\,,C^j(t,\mathbf{y})\right] \\
&\quad + \int d^2\mathbf{y}\,u_0(t,\mathbf{y})\,\left[C^0(t,\mathbf{x})\,,C^0(t,\mathbf{y})\right] = 0
\end{aligned} \tag{3.90}$$

Since $[C^0(t,\mathbf{x})\,,C^0(t,\mathbf{y})] = 0$, we have,

$$\int d^2\mathbf{y}\,\left[C^0(t,\mathbf{x})\,,\mathcal{H}_c(t,\mathbf{y})\right] + u_j(t,\mathbf{x})\int d^2\mathbf{y}\,\left[C^0(t,\mathbf{x})\,,C^j(t,\mathbf{y})\right] = 0 \tag{3.91}$$

Substituting the expression for $u_j(t,\mathbf{x})$ from eq. (3.84) into the equation above, we get,

$$\begin{aligned}
&\left[C^0(t,\mathbf{x})\,,H_c(t)\right] - \left[C^i(t,\mathbf{x})\,,H_c(t)\right]C_{ji}(t,\mathbf{x})\int d^2\mathbf{y}\,\left[C^0(t,\mathbf{x})\,,C^j(t,\mathbf{y})\right] \\
&- u_0(t,\mathbf{x})\left(\int d^2\mathbf{z}\,\left[C^i(t,\mathbf{x})\,,C^0(t,\mathbf{z})\right]\right)C_{ji}(t,\mathbf{x})\left(\int d^2\mathbf{y}\,\left[C^0(t,\mathbf{x})\,,C^j(t,\mathbf{y})\right]\right) = 0
\end{aligned} \tag{3.92}$$

Let us denote $\int d^2\mathbf{y}\,[C^i(t,\mathbf{x})\,,C^0(t,\mathbf{y})]$ by $V^i(t,\mathbf{x})$. Then,

$$\begin{aligned}
\Rightarrow &\left[C^0(t,\mathbf{x})\,,H_c(t)\right] - \left[C^i(t,\mathbf{x})\,,H_c(t)\right]C_{ji}(t,\mathbf{x})\int d^2\mathbf{y}\,\left[C^0(t,\mathbf{x})\,,C^j(t,\mathbf{y})\right] \\
&+ u_0(t,\mathbf{x})\,V^i(t,\mathbf{x})\,C_{ji}(t,\mathbf{x})\,V^j(t,\mathbf{x}) = 0
\end{aligned} \tag{3.93}$$

$V^i(t,\mathbf{x})$ and $V^j(t,\mathbf{x})$ are vectors which can commute with each other. So they are symmetric in i and j. $C_{ji}(t,\mathbf{x})$ on the other hand is antisymmetric in i and j. Therefore, the last term in the equation above is equal to zero.

$$u_0(t,\mathbf{x})\,V^i(t,\mathbf{x})\,C_{ji}(t,\mathbf{x})\,V^j(t,\mathbf{x}) = 0 \tag{3.94}$$

This give us,

$$\left[C^0(t,\mathbf{x})\,,H_c(t)\right] - \left[C^i(t,\mathbf{x})\,,H_c(t)\right]C_{ji}(t,\mathbf{x})\int d^2\mathbf{y}\,\left[C^0(t,\mathbf{x})\,,C^j(t,\mathbf{y})\right] = 0 \tag{3.95}$$

In the equation above we can see that there is no Lagrange multiplier, $u_j(t,\mathbf{x})$ or $u_0(t,\mathbf{x})$, present. This means that this equation is a secondary constraint.

Now let us try to simplify equation (3.95) and write down its expression in terms of $A_0(t,\mathbf{x})$ and $A_i(t,\mathbf{x})$. The equation can be rewritten as following:

$$[C^0(t,\mathbf{x})\,,H_c(t)] - \int d^2\mathbf{y}\,[C^0(t,\mathbf{x})\,,C^j(t,\mathbf{y})]\,C_{ji}(t,\mathbf{x})\,[C^i(t,\mathbf{x})\,,H_c(t)] = 0$$
$$= [C^0(t,\mathbf{x})\,,H_c(t)]_D = 0 \tag{3.96}$$

where $[..]_D$ is a Dirac bracket[1].

From eqn (3.77) we know that

$$\begin{aligned} C^{ij}(t,\mathbf{x}) &= 2\,\varepsilon^{ij} + 4\,\varepsilon^{ij}\varepsilon^{kl}A_l(t,\mathbf{x})\,\partial_k A_0(t,\mathbf{x}) \\ &= 2\,\varepsilon^{ij}\left(1 + 2\,\varepsilon^{kl}A_l(t,\mathbf{x})\,\partial_k A_0(t,\mathbf{x})\right), \end{aligned} \tag{3.97}$$

which gives

$$C_{ij}(t,\mathbf{x}) = -\frac{\varepsilon_{ij}}{2\left(1 + 2\,\varepsilon^{kl}A_l(t,\mathbf{x})\,\partial_k A_0(t,\mathbf{x})\right)}\,. \tag{3.98}$$

Substituting the expression for $C_{ij}(t,\mathbf{x})$ shown above into eq. (3.96), we get,

$$\begin{aligned} &[C^0(t,\mathbf{x})\,,H_c(t)]_D = 0 \\ &= -2\,\varepsilon^{ij}\partial_i A_j(t,\mathbf{x}) - 2\,\varepsilon^{ij}\varepsilon^{kl}A_0(t,\mathbf{x})\,\partial_k A_j(t,\mathbf{x})\,\partial_l A_i(t,\mathbf{x}) \\ &\quad + 4\,\frac{\varepsilon^{ij}\varepsilon^{kl}A_j(t,\mathbf{x})\,\partial_i A_l(t,\mathbf{x})\left(\partial_k A_0(t,\mathbf{x}) - 2\,\varepsilon^{mn}\,A_0(t,\mathbf{x})\,\partial_n A_0(t,\mathbf{x})\,\partial_m A_k(t,\mathbf{x})\right)}{1 + 2\,\varepsilon^{i'j'}A_{j'}(t,\mathbf{x})\,\partial_{i'}A_0(t,\mathbf{x})} = 0 \end{aligned} \tag{3.99}$$

$$\begin{aligned} &= -\frac{1}{1 + 2\,\varepsilon^{i'j'}A_{j'}\,\partial_{i'}A_0}\Big(\varepsilon^{ij}\partial_i A_j + 2\,\varepsilon^{ij}\varepsilon^{kl}\partial_k A_0\,(A_l\,\partial_i A_j - A_j\,\partial_i A_l) \\ &\quad + \varepsilon^{ij}\varepsilon^{kl}A_0\,\partial_k A_j\,\partial_l A_i + 2\,\varepsilon^{ij}\varepsilon^{kl}\varepsilon^{mn}A_0\,\partial_l A_i\,\partial_m A_0\,(A_n\,\partial_k A_j - A_k \partial_n\,A_j)\Big) = 0 \end{aligned} \tag{3.100}$$

Therefore, the secondary constraint is,

$$\begin{aligned} B(t,\mathbf{x}) = \varepsilon^{ij}\partial_i A_j(t,\mathbf{x}) &- 2\,\varepsilon^{ij}\varepsilon^{kl}\,A_j(t,\mathbf{x})\,\partial_k A_0(t,\mathbf{x})\,\partial_l A_i(t,\mathbf{x}) \\ &+ \varepsilon^{ij}\varepsilon^{kl}A_0(t,\mathbf{x})\,\partial_k A_j(t,\mathbf{x})\,\partial_l A_i(t,\mathbf{x}) = 0 \end{aligned} \tag{3.101}$$

[1] Dirac bracket is defined as follows:

$$[F\,,G]_D = [F\,,G] - [F\,,J^\alpha]\,C_{\alpha\beta}\,[J^\beta,G]\,,$$

where $C_{\alpha\beta}$ is the inverse of the matrix $C^{\alpha\beta} = [J^\alpha, J^\beta]$.

So the system has one secondary constraint similar to the case of the leading order Cherns-Simons case.

The Poisson bracket of $B(t\,,\mathbf{x})$ with $C^i(t\,,\mathbf{y})$ is

$$\begin{aligned}&[B(t\,,\mathbf{x}),C^i(t\,,\mathbf{y})]\\&\quad=-\,\varepsilon^{ij}\,\partial_{x^j}\delta(\mathbf{x}-\mathbf{y})+6\,f^{-2}\,\varepsilon^{ij}\varepsilon^{kl}\,\partial_kA_0(t,\mathbf{x})\,\partial_lA_j(t,\mathbf{x})\,\delta(\mathbf{x}-\mathbf{y})\\&\qquad+2\,f^{-2}\,\varepsilon^{ij}\varepsilon^{kl}\,\partial_{x^l}\big(A_0(t,\mathbf{x})\,\partial_kA_j(t,\mathbf{x})-A_j(t,\mathbf{x})\,\partial_kA_0(t,\mathbf{x})\big)\,\delta(\mathbf{x}-\mathbf{y})\,. \quad (3.102)\end{aligned}$$

We can make this Poisson bracket vanish by modifying the constraint B (3.101) as follows

$$\begin{aligned}\hat{B}(t\,,\mathbf{x})&=B(t\,,\mathbf{x})-6\,f^{-2}\,\varepsilon^{kl}\,\partial_kA_0(t\,,\mathbf{x})\,\partial_lA_j(t\,,\mathbf{x})\,\hat{C}^j(t\,,\mathbf{x})\\&\quad+\partial_j\hat{C}^j(t\,,\mathbf{x})-2f^{-2}\varepsilon^{kl}\partial_l\big((A_0(t\,,\mathbf{x})\,\partial_kA_j(t\,,\mathbf{x})\\&\quad-A_j(t\,,\mathbf{x})\,\partial_kA_0(t\,,\mathbf{x}))\,\hat{C}^j(t\,,\mathbf{x})\big), \quad (3.103)\end{aligned}$$

where

$$\hat{C}^j(t\,,\mathbf{x})=\frac{C^j(t\,,\mathbf{x})}{2\,(1-2\,f^{-2}\varepsilon^{kl}A_k(t\,,\mathbf{x})\,\partial_lA_0(t\,,\mathbf{x}))}\,,\quad \text{such that,}$$

$$[\hat{C}^j(t,\mathbf{x}),C^i(t,\mathbf{y})]=\varepsilon^{ij}\,\delta(\mathbf{x}-\mathbf{y})\,. \quad (3.104)$$

Thus,

$$[\hat{B}(t\,,\mathbf{x}),C^i(t\,,\mathbf{y})]=0\,. \quad (3.105)$$

However, $B(t\,,\mathbf{x})$ has a non-vanishing Poisson bracket with $C^0(t\,,\mathbf{y})$:

$$\begin{aligned}[B(t,\mathbf{x}),C^0(t,\mathbf{y})]&=-f^{-2}\varepsilon^{ij}\varepsilon^{kl}\,\partial_kA_i(t\,,\mathbf{x})\,\partial_lA_j(t\,,\mathbf{x})\,\delta(\mathbf{x}-\mathbf{y})\\&\quad-2\,f^{-2}\varepsilon^{ij}\varepsilon^{kl}A_j(t\,,\mathbf{x})\,\partial_lA_i(t\,,\mathbf{x})\,\partial_{x^k}\delta(\mathbf{x}-\mathbf{y})\,. \quad (3.106)\end{aligned}$$

If we take the linear combination of the constraints $B(t\,,\mathbf{x})$ and $C^0(t\,,\mathbf{x})$, namely $B_1(t\,,\mathbf{x})=\frac{1}{2}\,(B(t\,,\mathbf{x})+C^0(t\,,\mathbf{x}))$ and $B_2(t\,,\mathbf{x})=\frac{1}{2}\,(B(t\,,\mathbf{x})-C^0(t\,,\mathbf{x}))$, the Poisson bracket simplifies to

$$[B_1(t,\mathbf{x}),B_2(t,\mathbf{y})]=f^{-2}\varepsilon^{ij}\varepsilon^{kl}\,\partial_kA_i(t\,,\mathbf{x})\,\partial_lA_j(t\,,\mathbf{x})\,\delta(\mathbf{x}-\mathbf{y}). \quad (3.107)$$

As we had seen in the case of the free Chern-Simons theory, there were two first-class constraints. However, we do not have any first-class constraint in this system, since the right hand side of eq. (3.107) is in general not zero. We should be able to recover the constraints of the leading order case by setting $f^{-2} = 0$ here.

We can see that on setting $f^{-2} = 0$, the Poisson brackets (3.106) and (3.107) vanish. So then the constraints $C^0(t\,,\mathbf{x})$ and $B(t\,,\mathbf{x})$ (or equivalently $B_1(t\,,\mathbf{x})$ and $B_2(t\,,\mathbf{x})$) be come of the the first-class and generate the local symmetry of the Chern-Simons action.

In the non-linear case in which $f^{-2} \neq 0$, the Poisson brackets (3.106) and (3.107) are non-zero for a generic field $A_a(t\,,\mathbf{x})$. Therefore the constraints $C^0(t\,,\mathbf{x})$ and $B(t\,,\mathbf{x})$ belong to the second-class.

One can also check that the non-linear model does not have tertiary constraints, i.e. that the Poisson brackets of the primary and the secondary constraints with the total Hamiltonian (3.80) vanish provided the Lagrange multipliers $u_n(t\,,\mathbf{x})$ and $u_0(t\,,\mathbf{x})$ are appropriate functions of $A_a(t\,,\mathbf{x})$ and its derivatives.

Let us count the number of canonical degrees of freedom in the non-linear case. There are 6 canonical degrees of freedom to begin with, coming from the three vector field components $A_0(t\,,\mathbf{x})\,,A_1(t\,,\mathbf{x})$ and $A_2(t\,,\mathbf{x})$ and their corresponding momenta. There are four second-class constraints $-$ $C^0(t\,,\mathbf{x})\,,C^1(t\,,\mathbf{x})\,,C^2(t\,,\mathbf{x})$ and $B(t\,,\mathbf{x})$. Each of them cancels one canonical degree of freedom. So we are left with $6-4=2$ canonical degrees of freedom. These 2 canonical degrees of freedom contained in $A_a(t\,,\mathbf{x})$ and $p^a(t\,,\mathbf{x})$ correspond to a single degree of freedom in the Lagrangian formulation. This is the scalar mode discussed in sections 3.1.2 and 3.1.3.

3.4.1 Hamiltonian Value on the Constraint Surface of the Full Vector Goldstone Model

We should evaluate the value of the canonical Hamiltonian on the constraint surface for getting an insight about the energy of the system. Let us once again look at the expressions for the canonical Hamiltonian density (3.73) and the constraint $B(t\,,\mathbf{x})$ (3.101).

$$\begin{aligned}\mathcal{H}_c(t,\mathbf{x}) &= 2\,\varepsilon^{ij}A_0(t,\mathbf{x})\,\partial_i A_j(t,\mathbf{x}) + \varepsilon^{ij}\varepsilon^{kl}A_0^2\,(t,\mathbf{x})\,\partial_k A_j(t,\mathbf{x})\,\partial_l A_i(t,\mathbf{x})\\ B(t,\mathbf{x}) &= \varepsilon^{ij}\partial_i A_j(t,\mathbf{x}) - 2\,\varepsilon^{ij}\varepsilon^{kl}\,A_j(t,\mathbf{x})\,\partial_k A_0(t,\mathbf{x})\,\partial_l A_i(t,\mathbf{x})\\ &\quad + \varepsilon^{ij}\varepsilon^{kl}A_0(t,\mathbf{x})\,\partial_k A_j(t,\mathbf{x})\,\partial_l A_i(t,\mathbf{x}) = 0\,. \end{aligned} \tag{3.108}$$

On rewriting the expression for $\int d^2\mathbf{x}\,\mathcal{H}_c(t,\mathbf{x})$ using integration by parts and adding and subtracting terms, it can be written in the following form, modulo total derivatives.

$$\begin{aligned}\mathcal{H}_c(t,\mathbf{x})_1|_{cs} &= 2A_0(t,\mathbf{x})\,B(t,\mathbf{x}) + 3A_0^2(t,\mathbf{x})\,\varepsilon^{ij}\varepsilon^{kl}\,\partial_k A_i(t,\mathbf{x})\,\partial_l A_j(t,\mathbf{x})\\ &= 6A_0^2(t,\mathbf{x})\ \det\left(\partial_i A_j(t,\mathbf{x})\right).\end{aligned} \tag{3.109}$$

Note that this Hamiltonian density is non-zero for the perturbative solution (3.17)-(3.20), and it is not bounded from below for generic classical values of the field $A_a(t,\mathbf{x})$, since $\det \partial_i A_j(t,\mathbf{x})$ is not positive definite. This implies that the system may be classically unstable.

Let us look at the form of the Hamiltonian in the decoupling limit as described in Section 3.1.3. The Lagrangian of the scalar field $\hat{\varphi}$ in the decoupling limit is,

$$\begin{aligned}\mathcal{L}(\hat{\varphi}) &= \frac{M^{-5}}{2}\,\hat{\varphi}\,\varepsilon^{abc}\varepsilon^{def}\,\partial_a\partial_d\hat{\varphi}\,\partial_e\partial_b\hat{\varphi}\,\partial_f\partial_c\hat{\varphi}\\ &= -\,3\,M^{-5}\,\hat{\varphi}\ \det\left(\partial_a\partial^b\hat{\varphi}\right)\end{aligned} \tag{3.110}$$

The conjugate momentum for this Lagrangian is,

$$p_{\hat{\varphi}} = -\,6\,M^{-5}\left(\det \partial_i\partial_j\hat{\varphi}\right)\partial_0\hat{\varphi} \tag{3.111}$$

The canonical Hamiltonian density we get, is,

$$\begin{aligned}\mathcal{H}_{\hat{\varphi}} &= p_{\hat{\varphi}}\,\partial_0\hat{\varphi} - \mathcal{L}(\hat{\varphi})\\ &= -\,6\,M^{-5}\left(\partial_0\hat{\varphi}\right)^2\left(\det \partial_i\partial_j\hat{\varphi}\right)\end{aligned} \tag{3.112}$$

It can also be expressed in the following form:

$$\mathcal{H}_{\hat{\varphi}} = -\,\frac{M^5 p_{\hat{\varphi}}^2}{6\det \partial_i\partial_j\hat{\varphi}}. \tag{3.113}$$

Equation (3.113) is the three-dimensional counterpart of the quartic galileon term in the Hamiltonian of the generic $D=4$ galileon theory derived in [86, 87].

Let us take a simple static solution which sets the Hamiltonian and hence the energy to be zero.

$$\hat{\varphi}_0 = \frac{1}{2}\, x^i x^i$$
$$p_{\hat{\varphi}} = 0 \qquad (3.114)$$

Now we perturbe this solution by $\delta\phi$.

$$\hat{\varphi} = \hat{\varphi}_0 + \delta\phi\,. \qquad (3.115)$$

Then, to the second order in $\delta\phi$ we have

$$\mathcal{H}_{\delta\phi} = -\,\frac{p^2_{\delta\phi}}{6} = -\,6\,\delta\dot{\phi}^2, \qquad (3.116)$$

which is negative. However, if we take another zero-energy static solution, i.e. of the form $\hat{\varphi}_0 = e^{a_i x^i} b + \text{c.c.}$ (where a_i and b are complex constants) and consider fluctuations around it, then the Hamiltonian density turns out to be positive.

Also, if we had started with the initial Lagrangian (3.8) with the opposite sign (which is a priori admissible since the Chern-Simons term may have any sign), then the Hamiltonian density in (3.112) and (3.113) due to fluctuations around $\hat{\varphi}_0 = \frac{1}{2}\, x^i x^i$, $p_{\hat{\varphi}} = 0$ would be positive and that with the fluctuations around $\hat{\varphi}_0 = e^{a_i x^i} b + \text{c.c.}$ would be negative.

So the fluctuations around the zero-energy static solutions can be either negative or positive. This, in general, leads to instabilities. These instabilities are not of the (higher-derivative) Ostrogradski type, since the higher-order galileon Lagrangians are quadratic in time derivatives.

To summarize, the vector Goldstone model describing the spontaneous breaking of the rigid symmetry generated by the Hietarinta algebra does not maintain the local gauge symmetry of the quadratic Chern-Simons action. Due to the presence of the non-linear terms in the action there is a propagating scalar degree of freedom. The scalar field is like a galileon field which appears in modified theories of gravity. The Hamiltonian of the system is not bounded from below. This, in general, makes this model classically unstable, even though the Lagrangian is linear in the time derivative of $A_a(x)$.

Chapter 4

Vector-Spinor Goldstino Model

Now we consider the vector-spinor or the spin-3/2 goldstino model. We use the following algebra to construct this model, which was shown earlier in equations (1.48) and (1.51),

$$
\begin{aligned}
[M_{ab}\,, M_{cd}] &= i\,(\eta_{bc}\,M_{ad} - \eta_{ac}\,M_{bd} - \eta_{bd}\,M_{ac} + \eta_{ad}\,M_{bc}),\\
[M_{ab}\,, P_c] &= i\,(\eta_{bc}\,P_a - \eta_{ac}\,P_b),\\
[M^{ab}\,, Q^c_\alpha] &= i\,(\eta^{bc}\,Q^a_\alpha - \eta^{ac}\,Q^b_\alpha) - \frac{i}{2}\,(\Gamma^{ab})_\alpha{}^\beta\,Q^c_\beta\,,\\
\{Q^a_\alpha, Q^b_\beta\} &= 2\,C_{\alpha\beta}\,\varepsilon^{abc}P_c\,,\\
[Q^a_\alpha, P_b] &= 0\,,\\
[P_a, P_b] &= 0\,.
\end{aligned}
\tag{4.1}
$$

where Q^a_α ($\alpha = 1, 2$) is the Majorana vector-spinor generator of the spin-3/2 symmetry transformations.

The spin-3/2 symmetry transformations of the spacetime coordinate x^a and the vector-spinor field $\chi^\alpha_a(x)$ generated by the algebra above are:

$$
\begin{aligned}
x'^a &= x^a - i\,f^{-2}\,\varepsilon^{abc}\,\zeta^\alpha_b\,\chi_{\alpha c}\,,\\
\chi'^\alpha_a(x') &= \chi^\alpha_a(x) + \zeta^\alpha_a\,,
\end{aligned}
\tag{4.2}
$$

where ζ^α_a is a constant parameter.

The infinitesimal transformation of the form of the goldstino field $\chi^\alpha_a(x)$,

$$
\delta\chi^\alpha_a(x) = \zeta^\alpha_a + i\,f^{-2}\varepsilon^{dbc}\left(\zeta_b\,\chi_c(x)\right)\partial_d\chi^\alpha_a(x) \tag{4.3}
$$

shows that it transforms non-linearly under the symmetry. Hence, the symmetry is spontaneously broken.

Note that, as for all the other cases, the commutator of two variations (4.3) closes on the translations off the mass shell, i.e. without the use of the equations of motion:

$$[\delta_2, \delta_1]\, \chi_a^\alpha = \xi^d\, \partial_d \chi_a^\alpha\,, \qquad \xi^d = 2\, i\, f^{-2}\, \varepsilon^{dbc}\, \zeta_b^1\, \zeta_c^2\,. \tag{4.4}$$

The one-form that is invariant under the symmetry transformations (4.2) is,

$$\begin{aligned} E^a &= dx^a + i\, f^{-2}\, \varepsilon^{abc}\, \chi_b\, d\chi_c \\ &= dx^d\, (\delta_d^a + i\, f^{-2}\, \varepsilon^{abc}\, \chi_b\, \partial_d \chi_c) \\ &= dx^d\, E_d^a\,. \end{aligned} \tag{4.5}$$

This one-form is used to construct the spin-3/2 goldstino Lagrangian via the Volkov-Akulov Lagrangian formalism.

4.1 Action and Equation of Motion

We construct the action using the Volkov-Akulov formalism as explained in Section 2.1. On subtracting the constant term f^{-2} from it, we get,

$$\begin{aligned} S_{3/2} &= -f^2 \int d^3x\, (\det E_d^a - 1) \\ &= \int d^3x\, \Big(\mathrm{i}\, \varepsilon^{abc}\, \chi_a\, \partial_b \chi_c + \frac{f^{-2}}{2}\, \varepsilon^{abc} \varepsilon^{dfg}\, \big((\chi_a\, \partial_b \chi_c)\, (\chi_d\, \partial_f \chi_g) - (\chi_b\, \partial_d \chi_c)\, (\chi_f\, \partial_a \chi_g) \big) \\ &\quad + \frac{\mathrm{i}\, f^{-4}}{6}\, \varepsilon^{a'b'c'}\, (\varepsilon^{abc} \varepsilon^{def} - \varepsilon^{abf} \varepsilon^{dec})\, (\chi_c\, \partial_{a'} \chi_f)\, (\chi_a\, \partial_{b'} \chi_b)\, (\chi_d\, \partial_{c'} \chi_e) \Big)\,. \end{aligned} \tag{4.6}$$

The leading order term in the action above, i.e. $i\, \varepsilon^{abc}\, \chi_a\, \partial_b \chi_c$, is the action for a $D = 3$ Rarita-Schwinger spin-3/2 free massless field. We know that the free Rarita-Schwinger action is invariant under the following gauge transformation:

$$\chi_a^{\alpha\,\prime} = \chi_a^\alpha + \partial_a \epsilon^\alpha \tag{4.7}$$

The free Rarita-Schwinger action does not have local degrees of freedom due to the gauge invariance. Its equation of motion is

$$\varepsilon^{abc}\,\partial_b \chi_c^\alpha = 0\,. \tag{4.8}$$

On taking the divergence of the equation of motion, we get the following identity:

$$\partial_a(\varepsilon^{abc}\,\partial_b \chi_c^\alpha) \equiv 0\,. \tag{4.9}$$

It is a Noether identity that implies the presence of gauge symmetry in the system.

The equation of motion for the full non-linear action (4.6) has a form similar to eq. (3.13), i.e.,

$$\varepsilon^{abc}\,\mathcal{D}_b \chi_c = 0\,. \tag{4.10}$$

Unlike the vector Goldstone case where we found that such a general equation does not satisfy a Noether identity, in the case of the vector-spinor goldstino model here we find that the solution does exist. We will not present it explicitly, but will simply find that the spin-3/2 model retains the local gauge symmetry.

Let us figure out if in contrast to the spin-1 case, the spin-3/2 goldstino action can be invariant under a non-linear generalization of the Rarita-Schwinger gauge symmetry. We need to know if there are any extra degrees of freedom present in the system that break the gauge symmetry of the free Rarita-Schwinger action. On performing the Hamiltonian analysis order by order up till the order f^{-2}, we find that the on-shell Hamiltonian vanishes on the constraint surface. Please refer to Appendix C to see how this result is obtained.

Since the Hamiltonian analysis beyond the order f^{-2} becomes very complicated and involved, it has not been shown. However, the vanishing of the Hamiltonian on the constraint surface up to the order f^{-2} indicates that the system may have gauge symmetry. This observation prompts us to use the Stückelberg trick and verify this possibility.

4.1.1 Stückelberg Trick for Vector-Spinor Goldstino Model

With the aim of bringing a local symmetry into the Lagrangian we introduce an auxiliary field ψ^α. We define a new vector-spinor field $\hat{\chi}_a^\alpha$ in terms of the original field

χ_a^α and the Stückelberg spinor field ψ^α, as following,

$$\hat{\chi}_a^\alpha = \chi_a^\alpha + f^{\frac{2}{3}}\,\partial_a \psi^\alpha\,. \tag{4.11}$$

The factor $f^{2/3}$ is chosen to perform a certain non-singular limit $f \to \infty$ in the action. By construction $\hat{\chi}_a^\alpha(x)$ is invariant under the gauge transformations:

$$\begin{aligned}\delta\chi_a^\alpha(x) &= \partial_a \epsilon^\alpha(x)\\ \delta\psi^\alpha(x) &= -f^{-\frac{2}{3}}\,\epsilon^\alpha(x)\end{aligned} \tag{4.12}$$

Since $\epsilon^\alpha(x)$ is arbitrary, it can be given a value such that $\psi^\alpha(x)$ vanishes.

Now we take the limit of the coupling parameter f in which $f \to \infty$. In this limit the action (4.6) takes the following form:

$$S_{f\to\infty} = \int d^3x \left(i\,\varepsilon^{abc}\,\chi_a\,\partial_b\chi_c + 2\,\varepsilon^{abc}\varepsilon^{dfg}(\chi_a\,\partial_d\partial_c\psi)\,(\partial_f\psi\,\partial_b\partial_g\psi) - \frac{1}{3}\,\mathrm{Tr}(M^3)\right), \tag{4.13}$$

where ${M^a}_d = i\,\varepsilon^{abc}\,\partial_b\psi\,\partial_d\partial_c\psi\,.$ As we can see from the second term on the right hand side in the equation above, there is a term in the action having χ_a^α and ψ^α terms coupled to each other. So unlike the case of the vector Goldstone model where the Stückelberg action gets decoupled under the limit $f \to \infty$, in the vector-spinor model the Stückelberg action does not attain decoupling in the limit $f \to \infty$.

There is a quartic term present in $S_{f\to\infty}$ in terms of only ψ^α which has not been shown in eq. (4.13) because that term can be re-expressed as a total derivative, which on getting integrated, does not contribute to the expression. That term is,

$$\int d^3x\,\varepsilon^{abc}\varepsilon^{dfg}\,(\partial_b\psi\,\partial_d\partial_c\psi)\,(\partial_f\psi\,\partial_a\partial_g\psi)\,. \tag{4.14}$$

It can be re-written as following after integrating by parts:

$$\begin{aligned}&\varepsilon^{abc}\varepsilon^{dfg}\,(\partial_b\psi\,\partial_d\partial_c\psi)\,(\partial_f\psi\,\partial_a\partial_g\psi)\\ &= \partial_b\left(\varepsilon^{abc}\varepsilon^{dfg}(\psi\,\partial_d\partial_c\psi)\,(\partial_f\psi\,\partial_a\partial_g\psi)\right) + \varepsilon^{abc}\varepsilon^{dfg}(\psi\,\partial_c\partial_d\psi)\,(\partial_b\partial_f\psi\,\partial_a\partial_g\psi)\end{aligned} \tag{4.15}$$

The second term on the right hand side of the equation above, vanishes because of the anti-commutativity of ψ and the total symmetry of this expression in the exchange of the pairs of the indices (cd), (bf) and (ag).

$$\varepsilon^{abc}\varepsilon^{dfg}(\partial_c\partial_d\psi^\alpha)\,(\partial_b\partial_f\psi\,\partial_a\partial_g\psi) \equiv 0 \tag{4.16}$$

So term (4.14) can be re-expressed as a total derivative which can be discarded.

Due to identity (4.16), we can now see that the quartic term of even the original action (4.6), i.e.,

$$\int d^3x\,\frac{f^{-2}}{2}\,\varepsilon^{abc}\varepsilon^{dfg}\left((\chi_a\,\partial_b\chi_c)\,(\chi_d\,\partial_f\chi_g) - (\chi_b\,\partial_d\chi_c)\,(\chi_f\,\partial_a\chi_g)\right) \tag{4.17}$$

vanishes modulo a total derivative on the solution of the free Rarita-Schwinger field equation, which is $\chi^\alpha_a(x) = \partial_a\epsilon^\alpha(x)$.

This provides us a hint that the action $S_{f\to\infty}$ (4.13) has a gauge symmetry. On taking the following gauge transformation,

$$\begin{aligned}
&\chi^\alpha_a \to \chi^\alpha_a + \partial_a\lambda^\alpha\,,\\
&\psi^\alpha \to \psi^\alpha\,,
\end{aligned} \tag{4.18}$$

action (4.13) transforms into the following:

$$\begin{aligned}
S_{f\to\infty} \to \int d^3x\,\Big(&i\,\varepsilon^{abc}\,(\chi_a\,\partial_b\chi_c + \partial_a\lambda\,\partial_b\chi_c) + 2\,\varepsilon^{abc}\varepsilon^{dfg}(\chi_a\,\partial_d\partial_c\psi\\
&+ \partial_a\lambda\,\partial_d\partial_c\psi)\,(\partial_f\psi\,\partial_b\partial_g\psi) - \frac{1}{3}\,\mathrm{Tr}(M^3)\Big)\\
= \int d^3x\,\Big(&i\,\varepsilon^{abc}\,(\chi_a\,\partial_b\chi_c + \partial_a(\lambda\,\partial_b\chi_c) - \cancelto{0}{\lambda\,\partial_a\partial_b\,\chi_c)} + 2\,\varepsilon^{abc}\varepsilon^{dfg}(\chi_a\,\partial_d\partial_c\psi\\
&+ \partial_a(\lambda\,\partial_d\partial_c\psi) - \cancelto{0}{\lambda\,\partial_d\partial_a\partial_c\psi)}\,(\partial_f\psi\,\partial_b\partial_g\psi) - \frac{1}{3}\,\mathrm{Tr}(M^3)\Big)\\
= S_{f\to\infty}&
\end{aligned} \tag{4.19}$$

Hence, we see the gauge invariance of $S_{f\to\infty}$.

Now we look at the equations of motion for the action (4.13). The equation of motion obtained by extremising the action with respect to χ^α_a is,

$$\varepsilon^{abc}\,\partial_b\chi^\alpha_c = i\,\varepsilon^{abc}\,\varepsilon^{dfg}\,\partial_d\partial_c\psi^\alpha\,(\partial_f\psi\,\partial_b\partial_g\psi)\,. \tag{4.20}$$

Using the following identity,

$$\begin{aligned}\varepsilon^{abc}\,\varepsilon^{dfg}\,\partial_d\partial_c\psi^\alpha\,(\partial_f\psi\,\partial_b\partial_g\psi) &\equiv -\,\frac{1}{2}\,\varepsilon^{abc}\varepsilon^{dfg}\,(\partial_d\partial_c\psi\,\partial_b\partial_g\psi)\,\partial_f\psi^\alpha \\ &\equiv -\,\frac{1}{3}\,\varepsilon^{abc}\,\varepsilon^{dfg}\,\partial_b\,(\partial_d\psi^\alpha\,(\partial_f\psi\,\partial_c\partial_g\psi))\;,\end{aligned} \tag{4.21}$$

the expression on the right hand side of eq. (4.20) can be written as a total derivative. Therefore, we find that the general solution of eq. (4.20) is

$$\chi^\alpha_c = \partial_c\epsilon^\alpha - \frac{\mathrm{i}}{3}\,\varepsilon^{dfg}\,\partial_d\psi^\alpha\,(\partial_f\psi\,\partial_c\partial_g\psi)\,. \tag{4.22}$$

This implies that, modulo the pure gauge degree of freedom ϵ^α, the field χ_a is completely determined in terms of the derivatives of ψ.

The equation of motion obtained by extremising action (4.13) with respect to ψ is identically satisfied by the solution (4.22). Therefore, ψ is completely arbitrary in the limit $f\to\infty$.

The form of the solution (4.22) enables us to guess that action (4.13) can be recast into the free Rarita-Schwinger form as following:

$$S_{f\to\infty} = \mathrm{i}\int d^3x\,\varepsilon^{abc}\Big(\chi^\alpha_a+\frac{\mathrm{i}}{3}\varepsilon^{dfg}\,\partial_d\psi^\alpha\,(\partial_f\psi\,\partial_a\partial_g\psi)\Big)\partial_b\Big(\chi_{c\alpha}+\frac{\mathrm{i}}{3}\,\varepsilon^{pqr}\,\partial_p\psi_\alpha\,(\partial_q\psi\,\partial_c\partial_r\psi)\Big). \tag{4.23}$$

This conjecture turns out to hold true.

We know that the free Rarita-Schwinger action is gauge invariant. Therefore, result (4.23) implies that $S_{f\to\infty}$ is gauge invariant. It is invariant under the following gauge transformation:

$$\begin{aligned}\delta\psi^\alpha &= \epsilon^\alpha(x),\\ \delta\chi^\alpha_a &= \partial_a\lambda^\alpha(x) - \frac{\mathrm{i}}{3}\,\varepsilon^{dfg}\,\Big(\partial_d\epsilon^\alpha\,(\partial_f\psi\,\partial_a\partial_g\psi) + \partial_d\psi^\alpha\,(\partial_f\epsilon\,\partial_a\partial_g\psi) + \partial_d\psi^\alpha\,(\partial_f\psi\,\partial_a\partial_g\epsilon)\Big)\\ &\equiv \partial_a\Big(\lambda^\alpha(x) - \frac{\mathrm{i}}{3}\varepsilon^{dfg}\,\partial_d\psi^\alpha\,(\partial_f\psi\,\partial_g\epsilon)\Big) - \mathrm{i}\,\varepsilon^{dfg}\,(\partial_d\epsilon\,\partial_a\partial_f\psi)\,\partial_g\psi^\alpha\,,\end{aligned} \tag{4.24}$$

where $\lambda^\alpha(x)$ and $\epsilon^\alpha(x)$ are independent parameters. Hence, ψ is a pure gauge.

So we have found that the Stückelberg trick for the vector-spinor goldstino model does not have a decoupling limit, and the action obtained in the limit $f\to\infty$ can be recast into the Rarita-Schwinger form. The Stückelberg case is a simpler, special case of the original, general action. Analysing the whole general action head-on is not easy. But if we find certain symmetries to hold for the special case of the

Stückelberg action, it provides us with a hint that that kind of symmetries might hold true for the general action as well. So having found the Rarita-Schwinger form and gauge symmetry of the action in the Stückelberg limit, we can now try to check if similar results hold for the general spin-3/2 action as well.

4.2 General Spin-3/2 Action Redefined as Free Rarita-Schwinger Action

The analysis in the previous section prompts us the form of the perturbative solution of the full non-linear equation of motion (4.10), i.e., $\varepsilon^{abc}\,\mathcal{D}_b\chi_c = 0$. Up to the order f^{-2} it is obtained from eq. (4.22) by re-scaling $\psi \to f^{-\frac{2}{3}}\psi$ and taking $\epsilon = \psi$:

$$\chi_a^\alpha = \partial_a\psi^\alpha - \frac{\mathrm{i}f^{-2}}{3}\,\varepsilon^{dfg}\partial_d\psi^\alpha\,(\partial_f\psi\partial_a\partial_g\psi) + \mathcal{O}(f^{-4})\,. \tag{4.25}$$

The way in which we guessed the free Rarita-Schwinger form of the Stückelberg limit by looking at the solution of the equation of motion, we can do the same here again. We find that the full action (4.6) can be recast into the free Rarita-Schwinger form.

$$S_{3/2} = \mathrm{i}\int d^3x\,\varepsilon^{abc}\Big(\chi_a^\alpha + \frac{\mathrm{i}f^{-2}}{3}\varepsilon^{dfg}\chi_d^\alpha\,(\chi_f\partial_a\chi_g)\Big)\partial_b\Big(\chi_{c\alpha} + \frac{\mathrm{i}f^{-2}}{3}\varepsilon^{pqr}\chi_{p\alpha}\,(\chi_q\partial_c\chi_r)\Big). \tag{4.26}$$

Action (4.26) is equal to the original action (4.6) modulo a total derivative due to the following identities:

$$\begin{aligned}
\varepsilon^{abc}\,\varepsilon^{dfg}(\chi_c\,\chi_d)(\partial_b\chi_f\,\partial_a\chi_g) &= -\,2\,\varepsilon^{abc}\,\varepsilon^{dfg}\,(\chi_b\,\partial_c\chi_d)(\chi_f\,\partial_a\chi_g),\\
\varepsilon^{abc}\,\varepsilon^{dfg}\,\varepsilon^{pqr}\,(\chi_f\,\partial_a\chi_g)(\chi_d\,\chi_p)(\partial_b\chi_q\,\partial_c\chi_r) &= 2\,\varepsilon^{abc}\,\varepsilon^{dfg}\,\varepsilon^{pqr}\,(\chi_f\,\partial_a\chi_g)(\chi_d\,\partial_b\chi_p)(\chi_q\,\partial_c\chi_r)\,.
\end{aligned} \tag{4.27}$$

Action (4.26) can be re-expressed as following:

$$S_{RS} = i\int d^3x\,\varepsilon^{abc}\,\hat{\chi}_a\,\partial_b\hat{\chi}_c \tag{4.28}$$

where the field $\hat{\chi}$ is,

$$\hat{\chi}_a^\alpha = \chi_a^\alpha + \frac{i\,f^{-2}}{3}\,\varepsilon^{dfg}\,\chi_d^\alpha\,(\chi_f\,\partial_a\chi_g). \tag{4.29}$$

So we can see that upon the field redefinition (4.29), the non-linear action can be shown more concisely to be of the Rarita-Schwinger form.

4.2.1 Gauge Symmetry of the Vector-Spinor Action

The existence of the free Rarita-Schwinger form of the non-linear action implies the existence of gauge symmetry of the original non-linear action. Gauge symmetry has further implications with regards to the nature of the goldstino field – whether it is a pure gauge field or a dynamical field. It affects the number of degrees of freedom present in the system. We will see ahead what the gauge transformations for action (4.26) are like.

Equation (4.29) is invertible. An explicit expression for χ_a as a polynomial in $\hat{\chi}_a$ and $\partial_b \hat{\chi}_a$ can be found using an iteration procedure. The expression stops at most at the sixth order in $\hat{\chi}$, because of the Grassmann nilpotency of $\hat{\chi}$. Up to the order f^{-4}, we get,

$$\begin{aligned}\chi^\alpha_a &= \hat{\chi}^\alpha_a - \frac{\mathrm{i}f^{-2}}{3}\,\varepsilon^{dfg}\,\hat{\chi}^\alpha_d\,(\hat{\chi}_f\,\partial_a\hat{\chi}_g) \\ &\quad - \frac{f^{-4}}{3}\,\varepsilon^{dfg}\,\varepsilon^{pqr}\Big(\hat{\chi}^\alpha_g\,(\hat{\chi}_q\,\partial_d\hat{\chi}_r)(\hat{\chi}_p\,\partial_a\hat{\chi}_f) + \frac{1}{3}\,\partial_a\Big(\hat{\chi}^\alpha_d(\hat{\chi}_f\,\hat{\chi}_p)(\hat{\chi}_q\,\partial_g\hat{\chi}_r)\Big)\Big) + \mathcal{O}(f^{-6})\,.\end{aligned} \tag{4.30}$$

Now let us look at the gauge variations for the non-linear action. As we already know, the free Rarita-Schwinger action is invariant under the following gauge variation:

$$\delta\hat{\chi}^\alpha_a = \partial_a\epsilon^\alpha \tag{4.31}$$

Action (4.26) is invariant under the above gauge transformation.

Knowing the relation between $\hat{\chi}^\alpha_a$ and χ^α_a from eq. (4.29), we can express the above gauge variation in terms of χ^α_a as well. It is,

$$\begin{aligned}\delta\hat{\chi}^\alpha_a &= \partial_a\epsilon^\alpha \\ &= \delta\chi^\alpha_a + \frac{\mathrm{i}f^{-2}}{3}\,\varepsilon^{dfg}\,\partial_a\Big(\chi^\alpha_d\,(\chi_f\,\delta\chi_g)\Big) + \mathrm{i}\,f^{-2}\,\varepsilon^{dfg}\,(\delta\chi_d\,\partial_a\chi_f)\,\chi^\alpha_g\,,\end{aligned} \tag{4.32}$$

We can also express the gauge variation exclusively in terms of χ^α_a for the action (4.6). Using the same iteration procedure as used to get eq. (4.30), we get the gauge variation of χ_a as following:

$$\delta\chi_a^\alpha = \partial_a\Big(\epsilon^\alpha - \frac{\mathrm{i}f^{-2}}{3}\,\varepsilon^{dfg}\,\chi_d^\alpha\,(\chi_f\,\partial_g\epsilon)\Big) - \mathrm{i}f^{-2}\,\varepsilon^{dfg}\,(\partial_d\epsilon\,\partial_a\chi_f)\,\chi_g^\alpha + \mathcal{O}(f^{-4})\,. \quad (4.33)$$

The commutator of two transformations (4.33) is exactly zero to all orders.

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\,\chi_a^\alpha \equiv 0\,. \quad (4.34)$$

This is an important result as it tells us that the gauge invariance of the non-linear action exists up till all orders in the vector-spinor field χ_a^α and that it is an abelian symmetry as in the Rarita-Schwinger case.

We can see that the vector-spinor goldstino model is different from the vector Goldstone model in that the former is gauge invariant while the latter is not.

4.2.2 Invariance under Spin-3/2 Symmetry Transformation

Another respect in which the redefinition of the non-linear action as the free Rarita-Schwinger action is insightful is with regards to rigid spin-3/2 supersymmetry transformation. The invariance of the non-linear action under the rigid spin-3/2 supersymmetry variation (4.3) implies the invariance of action (4.28) also under the same transformation with χ_a^α being the function of $\hat{\chi}_a^\alpha$ as in (4.30).

The rigid spin-3/2 symmetry transformation (4.3) can be written in terms of $\hat{\chi}_a^\alpha$ using relation (4.29), as following:

$$\delta\hat{\chi}_a^\alpha = \zeta_a^\alpha + \mathrm{i}\,f^{-2}\,\varepsilon^{dbc}\,(\zeta_b\,\hat{\chi}_c)\,\partial_d\hat{\chi}_a^\alpha + \frac{\mathrm{i}f^{-2}}{3}\,\varepsilon^{dbc}\,\Big((\hat{\chi}_b\,\partial_a\hat{\chi}_c)\,\zeta_d^\alpha + (\zeta_b\,\partial_a\hat{\chi}_c)\hat{\chi}_d^\alpha\Big) + \mathcal{O}(f^{-4}) \quad (4.35)$$

The commutator of two variations (4.35) closes on the translations off the mass shell, i.e. without the use of the equations of motion:

$$[\delta_2, \delta_1]\,\hat{\chi}_a^\alpha \equiv \xi^d\,\partial_d\,\hat{\chi}_a^\alpha\,, \qquad \text{where } \xi^d = 2\,\mathrm{i}\,f^{-2}\,\varepsilon^{dbc}\,\zeta_b^1\,\zeta_c^2\,. \quad (4.36)$$

We have thus found that the free Rarita-Schwinger action (4.28) is non-manifestly invariant under the rigid spin-3/2 supersymmetry with the Rarita-Schwinger field being its goldstino transforming non-linearly under the symmetry as in (4.35).

Chapter 5

Conclusions and Outlook

We started in Chapter 1 with a general introduction followed by a literature review on higher-spin theories. We then reviewed the process of supersymmetry breaking, in particular spontaneous supersymmetry breaking, as it is assumed to occur in nature, accounting for the absence of a direct proof for the existence of supersymmetry at the energy scales within the reach of present experiments. Then we looked at the representations of the Poincaré group. It showed us how the spin is defined for massive and massless particles and what all transformations the particles undergo.

It was then followed by a brief discussion on the literature based on Hietarinta algebras. Chapter 2 served the instructive material for demonstrating the construction of Volkov-Akulov Goldstone models and Dirac Hamiltonian formalism using a spin-1/2 fermion as an example. Then we constructed and analysed the Goldstone models for spin-1 and spin-3/2 fields.

We have found that the simplest Goldstone models constructed by the spontaneous breaking of the symmetries introduced by Hietarinta [52], are certain non-linear generalisations of the Chern-Simons and Rarita-Schwinger Lagrangians.

In the case of the vector Goldstone model, the spontaneous breaking of the rigid symmetry leads to the breaking of the gauge symmetry of the Abelian Chern-Simons action. The resulting Goldstone boson propagates a scalar mode which turns out to be a galileon field that appears in the theories of modified gravity.

In view of this result it would be interesting to couple the Chern-Simons Goldstone boson to a $3D$ gravity model which is invariant under the local symmetry associated with the algebra (3.1). As mentioned earlier in Chapter 3, the bosonic algebra with the generators S^a and P_a in (3.1) is a contraction of $so(2,2) = sl(2,\mathbf{R}) \oplus sl(2,\mathbf{R})$

on which the Chern-Simons description of the conventional $3D$ gravity is based [88, 89]. But the full algebra also includes the Lorentz generators. Therefore, our $3D$ gravity model would contain two spin-2 gauge fields, the conventional gravity dreibein, written as,

$$e^a(x) = dx^m\, e^a_m(x)\,,$$

associated with the translation generator P_a, and another dreibein, written as,

$$f^a(x) = dx^m\, f^a_m(x)\,,$$

associated with the generator S_a. The model will also contain the spin connection,

$$\omega^a(x) = dx^m\, \omega^a_m(x)\,,$$

associated with the Lorentz generators $M_a = \frac{1}{2}\,\varepsilon_{abc}\,M^{bc}$.

An action for these (a priori) independent fields, which is invariant under the local symmetries generated by (3.1), has the following form:

$$S = \int (e^a \wedge R_a + \frac{1}{2} f^a \wedge D f_a), \tag{5.1}$$

where

$$R^a = d\omega^a + \frac{1}{2}\,\varepsilon^{abc}\,\omega_b \wedge \omega_c \tag{5.2}$$

is the curvature and

$$D f_a = d\, f_a + \varepsilon_{abc}\,\omega^b \wedge f^c \tag{5.3}$$

is the covariant derivative associated with the local Lorentz transformations.

The local symmetry variations of the fields are,

$$\begin{aligned}
\delta e^a &= D\xi^a(x) + \epsilon^{abc} e_b \lambda_c(x) + \varepsilon^{abc} f_b s_c(x), \\
\delta f^a &= D s^a(x) + \epsilon^{abc} f_b \lambda_c(x), \\
\delta \omega^a &= D\lambda^a(x),
\end{aligned} \tag{5.4}$$

where $\xi^a(x)$, $s^a(x)$ and $\lambda^a(x)$ are the parameters associated with the generators P_a, S_a and M_a, respectively. All the gauge fields in this model are non-dynamical as can be easily seen by analysing the equations of motion. Recently, we have learned that the most general action (including action (5.1)) for so-called Maxwell-Chern-Simons gravity invariant under the transformations (5.4) was constructed several years ago in [90] (see also [91]).

The action (5.1) can be straightforwardly generalised to describe similar couplings between higher-spin fields and gravity. To this end one should just promote the one-form field $f^a(x)$ and the gauge parameter $s^a(x)$ to (generically mixed-symmetry) tensors $f^{ab_1\ldots b_n}$ and $s^{ab_1\ldots b_n}$, and appropriately adjust the contraction of the indices and the Lorentz transformations of $f^{ab_1\ldots b_n}$ in equations (5.1)-(5.4).

Once we construct the model describing the coupling of the Goldstone boson $A_a(x)$ with the gravity action (5.1), generating a Higgs effect, we can try to find out what kind of $3D$ massive gravity or bi-gravity we get. We can try to check if it has a relation with one of the three-dimensional gravity models considered in [92–95].

In contrast to the vector Goldstone model, in the spin-3/2 goldstino model gauge symmetry remains preserved on spontaneously breaking the rigid spin-3/2 supersymmetry. We have found that upon a non-linear field redefinition the non-linear action reduces to the free Rarita-Schwinger action, which itself turns out to be non-manifestly invariant under the rigid spin-3/2 supersymmetry (1.51). The symmetry is non-linearly realized on the variations of the Rarita-Schwinger goldstino (4.35).

If we couple the spin-3/2 goldstino to other fields, then the non-linear field redefinition may no longer remove the non-linear terms, and the two forms of the spin-3/2 goldstino models may not be equivalent anymore. To check this out, we can couple the Rarita-Schwinger goldstino to other matter and gauge fields such as (super)gravity and Hypergravity with spin-2 and spin-5/2 gauge fields and study the properties of these models.

Another interesting problem is to consider a four-dimensional Rarita-Schwinger goldstino model associated with the following algebra:

$$\{Q^a_\alpha, Q^b_\beta\} = 2\,\varepsilon^{abcd}\,(\Gamma_5\,\Gamma_c)_{\alpha\beta}\,P_d \quad (\alpha,\beta=1,...,4), \quad (a,b,...=0,1,2,3), \tag{5.5}$$

to figure out if in this case also, as in $D=3$, the non-linear action for spin-3/2 goldstino possesses a local gauge symmetry. If it does possess gauge symmetry, then one should check if the Lagrangian can be redefined as the quadratic Rarita-Schwinger Lagrangian through a non-linear field redefinition. Then one can check whether the non-linearly realised symmetry (5.5) can fit into the formulation of $N=1, D=4$ supergravity as a non-linear realization of two complex finite-dimensional supergroups considered in [96–98].

Further still, it would be insightful to generalise the constructions used in this work for studying the Goldstone models with yet higher-spins and for studying the models in AdS spacetime as well.

Appendix A

Tensor and Matrix Identities

All of the following conventions and identities are in $D = 3$ Minkowski spacetime with the signature $(-\,,\,+\,,\,+)$.

a, b, c, d, e, f are spacetime indices. $(a, b, c, d, e, f) \in \{0, 1, 2\}$.
i, j, k, l are space indices. $(i, j, k, l) \in \{1, 2\}$.
$\alpha, \beta, \rho, \sigma$ are spinor indices. $(\alpha, \beta, \rho, \sigma) \in \{1, 2\}$.

A.1 Levi-Civita Tensor Identities

$$\varepsilon^{012} = -\,\varepsilon_{012} = 1$$

$$\varepsilon^{abc}\varepsilon_{abc} = -\,3\,! \tag{A.1}$$

$$\varepsilon^{abc}\varepsilon_{ade} = -\,2\,!\,\delta^{b}_{[d}\,\delta^{c}_{e]} = -\,(\delta^{b}_{d}\,\delta^{c}_{e} - \delta^{b}_{e}\,\delta^{c}_{d}) \tag{A.2}$$

$$\varepsilon^{abc}\varepsilon_{abd} = -\,2\,!\,\delta^{c}_{d} \tag{A.3}$$

$$\begin{aligned}\varepsilon^{abc}\varepsilon_{def} &= -\,3\,!\,\delta^{a}_{[d}\,\delta^{b}_{e}\,\delta^{c}_{f]}\\ &= -\,(\delta^{a}_{d}\,\delta^{b}_{e}\,\delta^{c}_{f} - \delta^{a}_{e}\,\delta^{b}_{d}\,\delta^{c}_{f} + \delta^{a}_{e}\,\delta^{b}_{f}\,\delta^{c}_{d} - \delta^{a}_{f}\,\delta^{b}_{e}\,\delta^{c}_{d} + \delta^{a}_{f}\,\delta^{b}_{d}\,\delta^{c}_{e} - \delta^{a}_{d}\,\delta^{b}_{f}\,\delta^{c}_{e})\end{aligned} \tag{A.4}$$

$$\varepsilon^{12} = \varepsilon_{12} = 1$$

$$\varepsilon^{ij}\varepsilon_{ij} = 2\,! \tag{A.5}$$

$$\varepsilon^{ij}\varepsilon_{ik} = \delta^j_k \tag{A.6}$$

$$\varepsilon^{ij}\varepsilon_{kl} = 2\,!\,\delta^i_{[k}\,\delta^j_{l]} = \delta^i_k\,\delta^j_l - \delta^i_l\,\delta^j_k \tag{A.7}$$

A.2 Charge Conjugation Matrix Identities

$$C^{-1}_{\alpha\beta} = C_{\beta\alpha} = -\,C_{\alpha\beta} \tag{A.8}$$

$$C^{\alpha\beta}\,C_{\alpha\gamma} = \delta^\beta_\gamma \tag{A.9}$$

$$C^{\alpha\beta}\,C_{\alpha\beta} = 2 \tag{A.10}$$

$$\chi_\alpha = C_{\alpha\beta}\,\chi^\beta \tag{A.11}$$

$$\chi^\alpha = -\,C^{\alpha\beta}\,\chi_\beta \tag{A.12}$$

A.3 Index Contraction Notation

$$(\chi\,\chi) = \chi^\alpha\,\chi_\alpha = \chi^\alpha\,C_{\alpha\beta}\,\chi^\beta \tag{A.13}$$

$$\chi^\alpha\,\chi_\alpha = -\,\chi_\alpha\,\chi^\alpha \tag{A.14}$$

$$(\chi_a\,\chi_b) = \chi^\alpha_a\,\chi_{b\,\alpha} = -\,\chi_{b\,\alpha}\,\chi^\alpha_a = \chi^\alpha_b\,\chi_{a\,\alpha} = (\chi_b\,\chi_a) \tag{A.15}$$

$$\chi\,\Gamma^a\,\psi \equiv \chi^\alpha\,\Gamma^a_{\alpha\beta}\,\psi^\beta = -\,\chi^\alpha\,\Gamma^{a\,\beta}_{\alpha}\,\psi_\beta \tag{A.16}$$

A.4 Γ- Matrix Identities

$$\Gamma^a_{\alpha\beta} = \Gamma^a_{\beta\alpha}$$
$$\Gamma^a_{\alpha\beta} \equiv (\Gamma^a C^{-1})_{\alpha\beta} = -\,\Gamma^{a\gamma}_{\alpha} C_{\gamma\beta} \tag{A.17}$$

$$\Gamma^T_a = -\,C_{\alpha\beta}\,\Gamma^{\beta\rho}_a\,C^{-1}_{\rho\sigma} \tag{A.18}$$

$$\{\Gamma^a\,,\Gamma^b\} = 2\,\eta^{ab}$$

or more explicitly

$$\{\Gamma^a\,,\Gamma^b\}_\alpha{}^\beta = \Gamma^a{}_\alpha{}^\rho\,\Gamma^b{}_\rho{}^\beta + \Gamma^b{}_\alpha{}^\rho\,\Gamma^a{}_\rho{}^\beta = 2\,\eta^{ab}\,\delta^\beta_\alpha \tag{A.19}$$

$$\Gamma^{ab} = \Gamma^{[a}\,\Gamma^{b]} \; = \frac{1}{2}\,[\Gamma^a\,,\Gamma^b]$$
$$\Gamma^{abc\ldots n} = \Gamma^{[a}\,\Gamma^b\,\Gamma^c \ldots \Gamma^{n]} \tag{A.20}$$

$$\Gamma^a\,\Gamma^b = \varepsilon^{abc}\,\Gamma_c + \eta^{ab} \tag{A.21}$$
$$\varepsilon_{abc}\,\Gamma^a\,\Gamma^b = -\,2\,\Gamma_c \tag{A.22}$$
$$\Gamma^a\,\Gamma^b\,\Gamma^c = \varepsilon^{abc} I + \eta^{ab}\,\Gamma^c + \eta^{bc}\,\Gamma^a - \eta^{ac}\,\Gamma^b \tag{A.23}$$

A.5 Matrix Determinant

The determinant of a 3×3 matrix $E^a_m = \delta^a_m + M^a_m$ is

$$\begin{aligned} & \det E^a_m \\ = \; & \det(1 + M) \qquad\qquad (\text{A.24}) \\ = \; & 1 + \mathrm{Tr}\,M + \frac{1}{2}\left[(\mathrm{Tr}\,M)^2 - \mathrm{Tr}(M^2)\right] + \frac{1}{6}\left[(\mathrm{Tr}\,M)^3 - 3\,\mathrm{Tr}\,M\;\;\mathrm{Tr}(M^2) + 2\,\mathrm{Tr}(M^3)\right]\;, \end{aligned}$$

where $\mathrm{Tr}\,(M) = M_\alpha{}^\alpha = -\,M^\alpha{}_\alpha\,.$

Appendix B

Integral Identities

Let us consider a function $f^i(x)$. We wish to compute the following integral:

$$\int d^2\mathbf{x}\, f^i(x)\, (\partial_{x^i} + \partial_{y^i})\, \delta^{(2)}(\mathbf{x} - \mathbf{y})\,. \tag{B.1}$$

It can be written as following:

$$\begin{aligned}
&\int d^2\mathbf{x}\, f^i(x)\, (\partial_{x^i} + \partial_{y^i})\, \delta^{(2)}(\mathbf{x} - \mathbf{y}) \\
&= \int d^2\mathbf{x}\, f^i(x)\, \partial_{x^i}\, \delta^{(2)}(\mathbf{x} - \mathbf{y}) + \int d^2\mathbf{x}\, f^i(x)\, \partial_{y^i}\, \delta^{(2)}(\mathbf{x} - \mathbf{y})\,.
\end{aligned} \tag{B.2}$$

The first integral in the above expression can be evaluated as following:

$$\begin{aligned}
&\int d^2\mathbf{x}\, f^i(x)\, \partial_{x^i}\, \delta^{(2)}(\mathbf{x} - \mathbf{y}) \\
&= -\int d^2\mathbf{x}\, \partial_{x^i} f^i(x)\, \delta^{(2)}(\mathbf{x} - \mathbf{y}) \qquad \text{(by integration by parts)} \\
&= -\, \partial_{y^i} f^i(y)
\end{aligned} \tag{B.3}$$

The second integral in eq. (B.2) can be evaluated as following:

$$\begin{aligned}
&\int d^2\mathbf{x}\, f^i(x)\, \partial_{y^i}\, \delta^{(2)}(\mathbf{x} - \mathbf{y}) \\
&= \partial_{y^i} \int d^2\mathbf{x}\, f^i(x)\, \delta^{(2)}(\mathbf{x} - \mathbf{y}) \quad \text{(taking } \partial_{y^i} \text{ outside the integral which is w.r.t. } \mathbf{x}\text{)} \\
&= \partial_{y^i} f^i(y)
\end{aligned} \tag{B.4}$$

Adding up expressions (B.3) and (B.4), we get 0. Therefore, we see that the integral in eq. (B.2) is 0.

Similarly, if instead of $f^i(x)$ we have $f^i(y)$, then,

$$\int d^2\mathbf{y}\, f^i(y)\, (\partial_{x^i} + \partial_{y^i})\, \delta^{(2)}(\mathbf{x} - \mathbf{y}) = 0\,. \tag{B.5}$$

Appendix C

Hamiltonian Analysis of Vector-Spinor Goldstino Model

We first analyse the Hamiltonian of the free vector-spinor model where $f^{-2} = 0$.

C.1 Hamiltonian Analysis of Free Rarita-Schwinber Model

First we separate in the action (4.6) the spacetime indices $(a = 0, 1, 2)$ into space and time indices $(0,\ i = 1, 2)$.

The leading order Lagrangian, which is the free Rarita-Schwinger Lagrangian, is,

$$\begin{aligned}\mathcal{L}_{free}(t, \mathbf{x}) &= -\, i\, \varepsilon^{abc}\, \chi_a^\alpha(t, \mathbf{x})\, \partial_c \chi_{\alpha b}(t, \mathbf{x}) \\ &= -\, i\, \varepsilon^{ij} \left(\chi_i^\alpha(t, \mathbf{x})\, \partial_0 \chi_{\alpha j}(t, \mathbf{x}) + \chi_0^\alpha(t, \mathbf{x})\, \partial_j \chi_{\alpha i}(t, \mathbf{x}) - \chi_i^\alpha(t, \mathbf{x})\, \partial_j \chi_{\alpha 0}(t, \mathbf{x}) \right)\end{aligned} \tag{C.1}$$

The conjugate momenta are:

$$\begin{aligned} p_\alpha^i(t, \mathbf{x}) &= \frac{\delta \mathcal{L}(t, \mathbf{x})}{\delta(\partial_0 \chi_i^\alpha(t, \mathbf{x}))} = -\, i\, \varepsilon^{ij} \chi_{\alpha j}(t, \mathbf{x})\,, \\ p_\alpha^0(t, \mathbf{x}) &= \frac{\delta \mathcal{L}(t, \mathbf{x})}{\delta(\partial_0 \chi_0^\alpha(t, \mathbf{x}))} = 0\,. \end{aligned} \tag{C.2}$$

$\chi_i^\alpha(t,\mathbf{x})$ and $p_\beta^j(t,\mathbf{y})$ have the following equal-time Poisson bracket relation:

$$\{\chi_i^\alpha(t,\mathbf{x}), p_\beta^j(t,\mathbf{y})\} = \delta_\beta^\alpha\, \delta_i^j\, \delta^{(2)}(\mathbf{x}-\mathbf{y})\,. \tag{C.3}$$

The canonical Hamiltonian $\mathcal{H}_c(t,\mathbf{x})$ density is,

$$\begin{aligned} \mathcal{H}_c(t,\mathbf{x}) &= p_\alpha^i\, \partial_0 \chi_i^\alpha(t,\mathbf{x}) + p_\alpha^0\, \partial_0 \chi_0^\alpha(t,\mathbf{x}) - \mathcal{L}(t,\mathbf{x}) \\ &= i\, \varepsilon^{ij} \big(\chi_0^\alpha(t,\mathbf{x})\, \partial_j \chi_{\alpha i}(t,\mathbf{x}) - \chi_i^\alpha(t,\mathbf{x})\, \partial_j \chi_{\alpha 0}(t,\mathbf{x})\big)\,. \end{aligned} \tag{C.4}$$

Following are the primary constraints we derive from the conjugate momenta (C.2):

$$\begin{aligned} C_\alpha^i(t,\mathbf{x}) &= p_\alpha^i(t,\mathbf{x}) + i\, \varepsilon^{ij} \chi_{\beta j}(t,\mathbf{x}) = 0 \\ C_\alpha^0(t,\mathbf{x}) &= p_\alpha^0(t,\mathbf{x}) = 0\,. \end{aligned} \tag{C.5}$$

The Poisson bracket relations between these two primary constraints are:

$$\begin{aligned} &\{C_\alpha^i(t,\mathbf{x}), C_\beta^j(t,\mathbf{y})\} = 2\, i\, C_{\alpha\beta}\, \varepsilon^{ij}\, \delta^{(2)}(\mathbf{x}-\mathbf{y})\,, \\ &\{C_\alpha^0(t,\mathbf{x}), C_\beta^0(t,\mathbf{y})\} = 0\,, \\ &\{C_\alpha^i(t,\mathbf{x}), C_\beta^0(t,\mathbf{y})\} = 0\,. \end{aligned} \tag{C.6}$$

We can see from these Poisson bracket relations that the constraints $C_\alpha^i(t,\mathbf{x})$ belong to the second-class whereas the constraints $C_\alpha^0(t,\mathbf{x})$ belong to the first-class.

Now we proceed to look for secondary constraints. For this purpose we write down the expression for the total Hamiltonian density $\mathcal{H}_T(t,\mathbf{x})$, which is as following.

$$\mathcal{H}_T(t,\mathbf{x}) = \mathcal{H}_c(t,\mathbf{x}) + u_k^\alpha(t,\mathbf{x})\, C_\alpha^k(t,\mathbf{x}) + u_0^\alpha(t,\mathbf{x})\, C_\alpha^0(t,\mathbf{x}) \tag{C.7}$$

Here $u_k^\alpha(t,\mathbf{x})$ is the Lagrange multiplier associated with the constraint $C_\alpha^k(t,\mathbf{x})$ and $u_0^\alpha(t,\mathbf{x})$ the one associated with the constraint $C_\alpha^0(t,\mathbf{x})$.

We write down the equation for the time-conservation of the constraint $C_\alpha^k(t,\mathbf{x})$ by taking its Poisson bracket with the total Hamiltonian. It gives us the following:

$$\begin{aligned}
&\int d^2\mathbf{y}\,\{C_\alpha^i(t,\mathbf{x}),\mathcal{H}_T(t,\mathbf{y})\}\\
&=\int d^2\mathbf{y}\,\{C_\alpha^i(t,\mathbf{x}),\mathcal{H}_c(t,\mathbf{y})\}+\int d^2\mathbf{y}\,u_k^\gamma(t,\mathbf{y})\,\{C_\alpha^i(t,\mathbf{x}),C_\gamma^k(t,\mathbf{y})\}\\
&\qquad+\int d^2\mathbf{y}\,u_0^\gamma(t,\mathbf{y})\{C_\alpha^i(t,\mathbf{x}),C_\gamma^0(t,\mathbf{y})\}=0\\
&=\int d^2\mathbf{y}\,\big(-\,2\,i\,C_{\alpha\gamma}\,\varepsilon^{ik}\,\partial_k\chi_0^\gamma(t,\mathbf{y})-u_k^\gamma(t,\mathbf{y})\,2\,i\,C_{\alpha\gamma}\,\varepsilon^{ik}\big)\,\delta^{(2)}(\mathbf{x}-\mathbf{y})=0
\end{aligned}\tag{C.8}$$

Solving the above equation gives,

$$u_k^\gamma(t,\mathbf{x})=-\,\partial_k\chi_0^\gamma(t,\mathbf{x})\,.\tag{C.9}$$

We get the expression for the Lagrange multiplier $u_k^\gamma(t,\mathbf{x})$. Therefore, we do not get a secondary constraint from the time conservation equation for the constraint $C_\alpha^k(t,\mathbf{x})$.

Now let us try to see what we get by solving the time-conservation equation for the constraint $C_\beta^0(t,\mathbf{y})$.

$$\begin{aligned}
&\int d^2\mathbf{y}\,\{C_\alpha^0(t,\mathbf{x}),\mathcal{H}_T(t,\mathbf{y})\}\\
&=\int d^2\mathbf{y}\,\{C_\alpha^0(t,\mathbf{x}),\mathcal{H}_c(t,\mathbf{y})\}-\int d^2\mathbf{y}\,u_k^\gamma(t,\mathbf{y})\{C_\alpha^0(t,\mathbf{x}),C_\gamma^k(t,\mathbf{y})\}\\
&\qquad-\int d^2\mathbf{y}\,u_0^\gamma(t,\mathbf{y})\{C_\alpha^0(t,\mathbf{x}),C_\gamma^0(t,\mathbf{y})\}=0\\
&=\int d^2\mathbf{y}\,\delta^{(2)}(\mathbf{x}-\mathbf{y})\,2\,i\,C_{\alpha\gamma}\,\varepsilon^{ik}\,\partial_i\chi_k^\gamma(t,\mathbf{y})=0
\end{aligned}$$

This gives us the following secondary constraint:

$$\varepsilon^{ik}\,\partial_i\chi_{k\alpha}(t,\mathbf{x})=0\,.\tag{C.10}$$

Let us denote this secondary constraint by $D_\alpha(t,\mathbf{x})$. The equal-time Poisson brackets of $D_\alpha(t,\mathbf{x})$ with the other constraints are as following:

$$\begin{aligned}
&\{D_\alpha(t,\mathbf{x})\,,D_\beta(t,\mathbf{y})\}=0\,,\\
&\{D_\alpha(t,\mathbf{x})\,,C_\beta^k(t,\mathbf{y})\}=\varepsilon^{ik}\,C_{\alpha\beta}\,\partial_{x^i}\delta^{(2)}(\mathbf{x}-\mathbf{y})\,,\\
&\{D_\alpha(t,\mathbf{x})\,,C_\beta^0(t,\mathbf{y})\}=0\,.
\end{aligned}\tag{C.11}$$

The Poisson bracket of $D_\alpha(t,\mathbf{x})$ with $C'^k_\beta(t,\mathbf{y})$ does not vanish, but, as in the case of the Chern-Simons theory discussed in the main text, the constraint D_α can be modified by adding a term proportional to the divergence of C^k such that the Poisson bracket of the modified constraint with C^k vanishes. Therefore the secondary constraint $D_\alpha(t,\mathbf{x})$ belongs to the first-class.

Now we need to check if there are any tertiary constraints present. For checking this, we need to solve the time conservation equation for the secondary constraint $D_\alpha(t,\mathbf{x})$.

The time conservation equation for $D_\alpha(t,\mathbf{x})$ is as following:

$$\begin{aligned}
&\int d^2\mathbf{y}\,\{D_\alpha(t,\mathbf{x}),\mathcal{H}_T(t,\mathbf{y})\}\\
&=\int d^2\mathbf{y}\,\{D_\alpha(t,\mathbf{x}),\mathcal{H}_c(t,\mathbf{y})\}+\int d^2\mathbf{y}\,u^\gamma_k(t,\mathbf{y})\,\{D_\alpha(t,\mathbf{x}),C^k_\gamma(t,\mathbf{y})\}\\
&\quad+\int d^2\mathbf{y}\,u^\gamma_0(t,\mathbf{y})\{D_\alpha(t,\mathbf{x}),C^0_\gamma(t,\mathbf{y})\}=0
\end{aligned}\tag{C.12}$$

The left hand side of the above equation comes out to be zero. Therefore, the above equation does not give us any new information. It means that the free Rarita-Schwinger action has constraints only up till the secondary order and no higher.

Now that we know all the constraints present in the system for the case when $f^{-2}=0$, we can count the number of degrees of freedom. We have the six-component vector-spinor field χ^a_α along with the corresponding momenta. So we begin with 12 canonical phase-space degrees of freedom. There are four first-class constraints present in the system — C^0_α and D_α. Each of them generates gauge transformations and hence cancels two degrees of freedom. So we are left with $12-2\times 4=4$ canonical degrees of freedom. Then there are four second-class constraints — C^i_α each of which cancels one degree of freedom. So that leaves us with $4-4=0$ degrees of freedom. This confirms that the gauge invariant free Rarita-Schwinger action does not have any physical degree of freedom.

C.1.1 Hamiltonian Value on the Constraint Surface of the Free Rarita-Schwinger Model

Now we try to see what value the canonical Hamiltonian takes on the constraint surface. Doing so provides us an insight regarding the energy and physical consistency of the system.

The canonical Hamiltonian density, as shown in eq. (C.4), is,

$$\mathcal{H}_c(t,\mathbf{x}) = i\,\varepsilon^{ij}\big(\chi_0^\alpha(t,\mathbf{x})\,\partial_j\chi_{\alpha i}(t,\mathbf{x}) - \chi_i^\alpha(t,\mathbf{x})\,\partial_j\chi_{\alpha 0}(t,\mathbf{x})\big)\,. \tag{C.13}$$

Using integration by parts, it can be written as following:

$$\mathcal{H}_c(t,\mathbf{x}) = 2\,i\,\varepsilon^{ij}\,\chi_0^\alpha(t,\mathbf{x})\,\partial_j\chi_{\alpha i}(t,\mathbf{x}) - i\,\varepsilon^{ij}\,\partial_j\big(\chi_i^\alpha(t,\mathbf{x})\,\chi_{\alpha 0}(t,\mathbf{x})\big) \tag{C.14}$$

On integrating the above with respect to $\mathbf{x}$ we get the canonical Hamiltonian, which is,

$$H_c(t) = 2\,i\,\varepsilon^{ij}\int \mathrm{d}^2\mathbf{x}\,\chi_0^\alpha(t,\mathbf{x})\,\partial_j\chi_{\alpha i}(t,\mathbf{x})\,. \tag{C.15}$$

On substituting the secondary constraint (C.9), i.e. $\varepsilon^{ij}\,\partial_i\chi_{j\alpha}(t,\mathbf{x}) = 0$, into the Hamiltonian above, we find that it vanishes.

$$H(t)_{\text{free}\,3/2}|_{cs} = 0\,. \tag{C.16}$$

This verifies the fact that the free Rarita-Schwinger Lagrangian does not have dynamical degrees of freedom.

Now we perform the Hamiltonian analysis for the case with next higher order in f^{-2}, that is the action up to the terms of the order f^{-2}.

C.2 Hamitonian Analysis of Vector-Spinor Goldstino Model up to the Order f^{-2}

Following is the Volkov-Akulov Lagrangian for the vector-spinor field χ_a^α up to the terms of the order f^{-2}.

$$\begin{aligned}\mathcal{L}_{f^{-2}} = i\,\varepsilon^{ij}\,\big(\,\chi_0^\alpha\,\partial_i\chi_{j\alpha} + \chi_i^\alpha\,\partial_j\chi_{0\alpha} - \chi_i^\alpha\,\partial_0\chi_{j\alpha}\,\big) + \frac{\varepsilon^{ij}\varepsilon^{kl}}{f^2}\Big(&(\chi_i\partial_0\chi_j)(\chi_0\partial_k\chi_l)\\ - (\chi_i\partial_k\chi_j)(\chi_0\partial_0\chi_l) - (\chi_i\partial_0\chi_j)(\chi_l\partial_k\chi_0) + &(\chi_i\partial_k\chi_j)(\chi_l\partial_0\chi_0)\\ - (\chi_0\partial_l\chi_j)(\chi_0\partial_k\chi_i) - (\chi_j\partial_l\chi_0)(\chi_i\partial_k\chi_0)\Big)&\end{aligned} \tag{C.17}$$

The conjugate momenta are:

$$\begin{aligned} p^i_\alpha(t,\mathbf{x}) &= \frac{\delta\mathcal{L}(t,\mathbf{x})}{\delta\,(\,\partial_0\chi^\alpha_i(t,\mathbf{x})\,)} \\ &= i\,\varepsilon^{ij}\,\chi_{j\alpha}(t,\mathbf{x}) - \frac{1}{f^2}\,\varepsilon^{ij}\varepsilon^{kl}\left(\chi_{j\alpha}(\chi_0\partial_k\chi_l) - \chi_{j\alpha}(\chi_l\partial_k\chi_0) - \chi_{0\alpha}(\chi_k\partial_j\chi_l)\right) \end{aligned} \tag{C.18}$$

$$\begin{aligned} p^0_\alpha(t,\mathbf{x}) &= \frac{\delta\mathcal{L}(t,\mathbf{x})}{\delta\,(\,\partial_0\chi^\alpha_0(t,\mathbf{x})\,)} \\ &= \frac{1}{f^2}\,\varepsilon^{ij}\varepsilon^{kl}\,\chi_{l\alpha}(\chi_i\partial_k\chi_j) \end{aligned} \tag{C.19}$$

The canonical Hamiltonian is

$$\begin{aligned} &\mathcal{H}_c(t,\mathbf{x}) \\ &= \partial_0\chi^\alpha_i(t,\mathbf{x})\,p^i_\alpha(t,\mathbf{x}) + \partial_0\chi^\alpha_0(t,\mathbf{x})\,p^0_\alpha(t,\mathbf{x}) - \mathcal{L}(t,\mathbf{x}) \qquad \text{(C.20)} \\ &= i\,\varepsilon^{ij}\left(\,\chi^\alpha_0\,\partial_j\chi_{i\alpha} - \chi^\alpha_i\,\partial_j\chi_{0\alpha}\,\right) + \frac{1}{f^2}\,\varepsilon^{ij}\varepsilon^{kl}\left((\chi_0\partial_l\chi_j)(\chi_0\partial_k\chi_i) + (\chi_j\partial_l\chi_0)(\chi_i\partial_k\chi_0)\right) \end{aligned}$$

The primary constraints derived from the conjugate momenta (C.18), are:

$$\begin{aligned} F^i_\alpha(t,\mathbf{x}) = p^i_\alpha(t,\mathbf{x}) - i\,\varepsilon^{ij}\,\chi_{j\alpha}(t,\mathbf{x}) + \frac{1}{f^2}\,\varepsilon^{ij}\varepsilon^{kl}\Big(&\chi_{j\alpha}(t,\mathbf{x})\big(\chi_0(t,\mathbf{x})\,\partial_k\chi_l(t,\mathbf{x})\big) \\ &- \chi_{j\alpha}(t,\mathbf{x})\big(\chi_l(t,\mathbf{x})\,\partial_k\chi_0(t,\mathbf{x})\big) - \chi_{0\alpha}(t,\mathbf{x})\big(\chi_k(t,\mathbf{x})\,\partial_j\chi_l(t,\mathbf{x})\big)\Big) \end{aligned}$$

$$F^0_\alpha(t,\mathbf{x}) = p^0_\alpha(t,\mathbf{x}) - \frac{1}{f^2}\,\varepsilon^{ij}\varepsilon^{kl}\,\chi_{l\alpha}(t,\mathbf{x})\big(\chi_i(t,\mathbf{x})\,\partial_k\chi_j(t,\mathbf{x})\big) \tag{C.21}$$

The Poisson bracket relations between these primary constraints are,

$$
\begin{aligned}
&\{F^i_\alpha(t,\mathbf{x})\,,F^{i'}_\beta(t,\mathbf{y})\}\\
=&-\;2\,i\,\varepsilon^{ii'}\,C_{\alpha\beta}\,\delta^{(2)}(\mathbf{x}-\mathbf{y})+\frac{2}{f^2}\,\varepsilon^{ii'}\varepsilon^{kl}\,C_{\alpha\beta}\big((\chi_0(t,\mathbf{x})\,\partial_k\chi_l(t,\mathbf{x}))\\
&-(\partial_k\chi_0(t,\mathbf{x})\chi_l(t,\mathbf{x}))\big)\,\delta^{(2)}(\mathbf{x}-\mathbf{y})+\frac{2}{f^2}\,\varepsilon^{ik}\varepsilon^{i'j}C_{\rho\alpha}C_{\sigma\beta}\Big(\partial_k\big(\chi^\rho_0(t,\mathbf{x})\chi^\sigma_j(t,\mathbf{x})\big)\\
&+\partial_j\big(\chi^\sigma_0(t,\mathbf{x})\chi^\rho_k(t,\mathbf{x})\big)\Big)\,\delta^{(2)}(\mathbf{x}-\mathbf{y})-\frac{1}{f^2}\,\varepsilon^{ik}\varepsilon^{i'j}C_{\rho\alpha}C_{\beta\sigma}\big(\chi^\rho_0(t,\mathbf{x})\,\chi^\sigma_j(t,\mathbf{x})(\partial_{x^k}+\partial_{y^k})\\
&+\chi^\sigma_0(t,\mathbf{x})\,\chi^\rho_k(t,\mathbf{x})(\partial_{x^j}+\partial_{y^j})\big)\,\delta^{(2)}(\mathbf{x}-\mathbf{y})
\end{aligned}
\tag{C.22}
$$

$$
\{F^0_\alpha(t,\mathbf{x})\,,F^0_\beta(t,\mathbf{y})\}=0 \tag{C.23}
$$

$$
\begin{aligned}
&\{F^i_\alpha(t,\mathbf{x})\,,F^0_\beta(t,\mathbf{y})\}\\
=&\,\frac{2}{f^2}\,\varepsilon^{ij}\varepsilon^{kl}\,C_{\alpha\beta}\big(\chi_l(t,\mathbf{x})\,\partial_j\chi_k(t,\mathbf{x})\big)\,\delta^{(2)}(\mathbf{x}-\mathbf{y})\\
&+\frac{2}{f^2}\,\varepsilon^{ij}\varepsilon^{kl}\,C_{\rho\alpha}\,C_{\sigma\beta}\,\partial_k\big(\chi^\sigma_l(t,\mathbf{x})\chi^\rho_j(t,\mathbf{x})\big)\,\delta^{(2)}(\mathbf{x}-\mathbf{y})\\
&+\frac{2}{f^2}\,\varepsilon^{ij}\varepsilon^{kl}C_{\rho\alpha}\,C_{\sigma\beta}\,\chi^\rho_j(t,\mathbf{x})\chi^\sigma_l(t,\mathbf{x})\,(\partial_{x^k}+\partial_{y^k})\,\delta^{(2)}(\mathbf{x}-\mathbf{y})
\end{aligned}
\tag{C.24}
$$

The terms containing $(\partial_x+\partial_y)\,\delta^{(2)}(\mathbf{x}-\mathbf{y})$ in the expressions above, vanish upon integration as explained in Appendix B . We can see that the Poisson brackets of the primary constraints are non-vanishing.

To find the secondary constraints following the Dirac procedure, we consider the total Hamiltonian density $\mathcal{H}_T(t,\mathbf{x})$ which includes the primary constraints with the corresponding Lagrange multipliers.

$$
\mathcal{H}_T(t,\mathbf{x})=\mathcal{H}_c(t,\mathbf{x})+u^\alpha_i(t,\mathbf{x})\,F^i_\alpha(t,\mathbf{x})+u^\alpha_0(t,\mathbf{x})\,F^0_\alpha(t,\mathbf{x}) \tag{C.25}
$$

Integrating the above equation with respect to x, we get,

$$
H_T(t)=H_c(t)+\int d^2\mathbf{x}\,u^\alpha_i(t,\mathbf{x})\,F^i_\alpha(t,\mathbf{x})+\int d^2\mathbf{x}\,u^\alpha_0(t,\mathbf{x})\,F^0_\alpha(t,\mathbf{x})\,. \tag{C.26}
$$

Please note that the Poisson bracket of the constraint $F^i_\alpha(t,\mathbf{x})$ with the canonical Hamiltonian $\mathcal{H}_c(t,\mathbf{y})$ is,

$$
\begin{aligned}
&[F^i_\alpha(t,\mathbf{x})\,,\mathcal{H}_c(t,\mathbf{y})] \\
&= i\,\varepsilon^{ij}\,C_{\alpha\beta}\,\big(\,\partial_j\,(\chi^\beta_0(t,\mathbf{x})\,\delta^{(2)}(\mathbf{x}-\mathbf{y})) - 2\,\partial_j\chi^\beta_0(t,\mathbf{x})\,\delta^{(2)}(\mathbf{x}-\mathbf{y})\big) \\
&\quad + \frac{2}{f^2}\,\varepsilon^{ij}\varepsilon^{kl}\,C_{\alpha\beta}\bigg(\,\chi^\beta_0(t,\mathbf{x})\,\big(\chi_0(t,\mathbf{x})\,\partial_l\chi_j(t,\mathbf{x})\big)\,\partial_{y^k}\,\delta^{(2)}(\mathbf{x}-\mathbf{y}) \\
&\quad + \partial_k\chi^\beta_0(t,\mathbf{x})\,\big((\chi_j(t,\mathbf{x})\,\partial_l\chi_0(t,\mathbf{x})) - (\partial_l\chi_j(t,\mathbf{x})\,\chi_0(t,\mathbf{x})\big) \\
&\quad - \chi^\beta_0(t,\mathbf{x})\,\big(\partial_k\chi_0(t,\mathbf{x})\,\partial_l\chi_j(t,\mathbf{x})\big)\,\delta^{(2)}(\mathbf{x}-\mathbf{y})\bigg)\,,
\end{aligned}
\tag{C.27}
$$

and that between the constraint $F^0_\alpha(t,\mathbf{x})$ and $\mathcal{H}_c(t,\mathbf{y})$ is the following:

$$
\begin{aligned}
&[F^0_\alpha(t,\mathbf{x})\,,\mathcal{H}_c(t,\mathbf{y})] \\
&= i\,\varepsilon^{ij}\,C_{\alpha\beta}\big(2\,\partial_j\chi^\beta_i(t,\mathbf{x})\,\delta^{(2)}(\mathbf{x}-\mathbf{y}) - \chi^\beta_i(t,\mathbf{x})\,\partial_{y^j}\,\delta^{(2)}(\mathbf{x}-\mathbf{y})\big) \\
&\quad + \frac{2}{f^2}\,\varepsilon^{ij}\varepsilon^{kl}\,C_{\alpha\beta}\bigg(\chi^\beta_j(t,\mathbf{x})\,(\chi_i(t,\mathbf{x})\,\partial_k\chi_0(t,\mathbf{x}))\,\partial_{y^l}\,\delta^{(2)}(\mathbf{x}-\mathbf{y}) \\
&\quad + \partial_l\chi^\beta_j(t,\mathbf{x})\,\big((\chi_0(t,\mathbf{x})\,\partial_k\chi_i(t,\mathbf{x})) - (\chi_i(t,\mathbf{x})\,\partial_k\chi_0(t,\mathbf{x}))\big)\,\delta^{(2)}(\mathbf{x}-\mathbf{y}) \\
&\quad - \chi^\beta_j(t,\mathbf{x})\,\big(\partial_l\chi_i(t,\mathbf{x})\,\partial_k\chi_0(t,\mathbf{x})\big)\,\delta^{(2)}(\mathbf{x}-\mathbf{y})\bigg)\,.
\end{aligned}
\tag{C.28}
$$

Now we write down the equation for the time conservation of the constraint $F^i_\alpha(t,\mathbf{x})$. It is as following:

$$
\begin{aligned}
&\int d^2\mathbf{y}\,[F^i_\alpha(t,\mathbf{x})\,,\mathcal{H}_T(t,\mathbf{y})] \\
&= \int d^2\mathbf{y}\,[F^i_\alpha(t,\mathbf{x})\,,\mathcal{H}_c(t,\mathbf{y})] - \int d^2\mathbf{y}\,u^\beta_{i'}(t,\mathbf{y})\,\{F^i_\alpha(t,\mathbf{x})\,,F^{i'}_\beta(t,\mathbf{y})\} \\
&\quad - \int d^2\mathbf{y}\,u^\beta_0(t,\mathbf{y})\,\{F^i_\alpha(t,\mathbf{x})\,,F^0_\beta(t,\mathbf{y})\} = 0\,.
\end{aligned}
\tag{C.29}
$$

Let $\int d^2\mathbf{y}\,\{F^i_\alpha(t,\mathbf{x})\,,F^{i'}_\beta(t,\mathbf{y})\}$ be denoted by $M^{ii'}_{\alpha\beta}(t,\mathbf{x})$. Then, the above equation can be re-written as following:

$$
\begin{aligned}
&[F^i_\alpha(t,\mathbf{x})\,,H_c(t)] - u^\beta_{i'}(t,\mathbf{x})\,M^{ii'}_{\alpha\beta}(t,\mathbf{x}) - u^\rho_0(t,\mathbf{x})\int d^2\mathbf{y}\,\{F^i_\alpha(t,\mathbf{x})\,,F^0_\rho(t,\mathbf{y})\} = 0 \\
&\Rightarrow u^\beta_{i'}(t,\mathbf{x})\,M^{ii'}_{\alpha\beta}(t,\mathbf{x}) = [F^i_\alpha(t,\mathbf{x})\,,\mathcal{H}_c(t)] - u^\rho_0(t,\mathbf{x})\int d^2\mathbf{y}\,\{F^i_\alpha(t,\mathbf{x})\,,F^0_\rho(t,\mathbf{y})\}
\end{aligned}
$$

This gives us,

$$
\begin{aligned}
&u^{\beta}_{i'}(t,\mathbf{x}) \\
&= [F^i_\alpha(t,\mathbf{x})\,, H_c(t)]\, M^{\beta\alpha}_{i'i}(t,\mathbf{x}) - u^{\rho}_0(t,\mathbf{x}) \int d^2\mathbf{y}\, \{F^i_\alpha(t,\mathbf{x})\,, F^0_\rho(t,\mathbf{y})\}\; M^{\beta\alpha}_{i'i}(t,\mathbf{x})\,,
\end{aligned} \tag{C.30}
$$

where

$$
\begin{aligned}
M^{\beta\alpha}_{i'i}(t,\mathbf{x}) &= \left(M^{ii'}_{\alpha\beta}(t,\mathbf{x})\right)^{-1} = \{F^i_\alpha(t,\mathbf{x})\,, F^{i'}_\beta(t,\mathbf{y})\}^{-1} \\
&= \frac{i}{2}\,\varepsilon_{i'i}\, C^{\beta\alpha} + \frac{1}{2\,f^2}\,\varepsilon_{i'i}\,\varepsilon^{kl}\, C^{\beta\alpha}\left(\chi_0(t,\mathbf{x})\,\partial_k\chi_l(t,\mathbf{x}) - \partial_k\chi_0(t,\mathbf{x})\,\chi_l(t,\mathbf{x})\right) \\
&\quad + \frac{1}{2\,f^2}\left(\partial_i(\chi^\alpha_0(t,\mathbf{x})\,\chi^\beta_{i'}(t,\mathbf{x})) + \partial_{i'}(\chi^\beta_0(t,\mathbf{x})\,\chi^\alpha_i(t,\mathbf{x}))\right)
\end{aligned} \tag{C.31}
$$

So from eq. (C.30) we see that we get a restriction on the expression for the Lagrange multiplier $u^{\beta}_{i'}(t,\mathbf{x})$ by solving the time conservation equation for the primary constraint $F^i_\alpha(t,\mathbf{x})$. This means that the equation does not yield a secondary constraint.

Now let us see what we get by solving the time conservation equation for the constraint $F^0_\alpha(t,\mathbf{x})$.

$$
\begin{aligned}
&\int d^2\mathbf{y}\,[F^0_\alpha(t,\mathbf{x})\,, \mathcal{H}_T(t,\mathbf{y})] \\
&= \int d^2\mathbf{y}\,[F^0_\alpha(t,\mathbf{x})\,, \mathcal{H}_c(t,\mathbf{y})] - \int d^2\mathbf{y}\, u^\beta_i(t,\mathbf{y})\,\{F^0_\alpha(t,\mathbf{x})\,, F^i_\beta(t,\mathbf{y})\} \\
&\qquad - \int d^2\mathbf{y}\, u^\beta_0(t,\mathbf{y})\,\{F^0_\alpha(t,\mathbf{x})\,, F^0_\beta(t,\mathbf{y})\} = 0
\end{aligned} \tag{C.32}
$$

Since $\{F^0_\alpha(t,\mathbf{x})\,, F^0_\beta(t,\mathbf{y})\} = 0$ (as shown in eq. (C.23)), the above equation becomes,

$$
\int d^2\mathbf{y}\,[F^0_\alpha(t,\mathbf{x})\,, \mathcal{H}_c(t,\mathbf{y})] - \int d^2\mathbf{y}\, u^\beta_i(t,\mathbf{y})\,\{F^0_\alpha(t,\mathbf{x})\,, F^i_\beta(t,\mathbf{y})\} = 0 \tag{C.33}
$$

Substituting the expression for the Lagrange multiplier $u^\beta_i(t,\mathbf{y})$ from eq. (C.30), into the equation above, we get,

$$
\begin{aligned}
&[F^0_\rho(t,\mathbf{x})\,, H_c(t)] - [F^i_\alpha(t,\mathbf{x})\,, H_c(t)]\, M^{\beta\alpha}_{i'i}(t,\mathbf{x}) \int d^2\mathbf{y}\,\{F^0_\rho(t,\mathbf{x})\,, F^{i'}_\beta(t,\mathbf{y})\} \\
&\quad + u^\sigma_0(t,\mathbf{x}) \int d^2\mathbf{y}\,\{F^i_\alpha(t,\mathbf{x})\,, F^0_\sigma(t,\mathbf{y})\}\, M^{\beta\alpha}_{i'i}(t,\mathbf{x}) \int d^2\mathbf{y}\,\{F^0_\rho(t,\mathbf{x})\,, F^{i'}_\beta(t,\mathbf{y})\} = 0
\end{aligned} \tag{C.34}
$$

Let $\int d^2\mathbf{y}\,\{F^i_\alpha(t,\mathbf{x})\,,F^0_\sigma(t,\mathbf{y})\}$ be denoted by $V^i_{\alpha\sigma}(t,\mathbf{x})$. Then,

$$[F^0_\rho(t,\mathbf{x})\,,H_c(t)] - [F^i_\alpha(t,\mathbf{x})\,,H_c(t)]\,M^{\beta\alpha}_{i'i}(t,\mathbf{x})\int d^2\mathbf{y}\,\{F^0_\rho(t,\mathbf{x})\,,F^{i'}_\beta(t,\mathbf{y})\}$$
$$+\,u^\sigma_0(t,\mathbf{x})\,V^i_{\alpha\sigma}(t,\mathbf{x})\,M^{\beta\alpha}_{i'i}(t,\mathbf{x})\,V^{i'}_{\beta\rho}(t,\mathbf{x}) = 0 \tag{C.35}$$

Since $V^i_{\alpha\sigma}(t,\mathbf{x})$ is of the order f^{-2}, the last term in the equation above, i.e., $u^\sigma_0(t,\mathbf{x})\,V^i_{\alpha\sigma}(t,\mathbf{x})\,M^{\beta\alpha}_{i'i}(t,\mathbf{x})\,V^{i'}_{\beta\rho}(t,\mathbf{x})$, is of the order f^{-4}. We are retaining terms only up to the order f^{-2}, due to which $u^\sigma_0(t,\mathbf{x})\,V^i_{\alpha\sigma}(t,\mathbf{x})\,M^{\beta\alpha}_{i'i}(t,\mathbf{x})\,V^{i'}_{\beta\rho}(t,\mathbf{x}) \approx 0$

Therefore, we get,

$$[F^0_\rho(t,\mathbf{x})\,,H_c(t)] - [F^i_\alpha(t,\mathbf{x})\,,H_c(t)]\,M^{\beta\alpha}_{i'i}(t,\mathbf{x})\int d^2\mathbf{y}\,\{F^0_\rho(t,\mathbf{x})\,,F^{i'}_\beta(t,\mathbf{y})\} = 0$$

The above equation is in fact a secondary constraint. Let us denote it by $G_\alpha(t,\mathbf{x})$. It can be expressed as follows:

$$\begin{aligned}
&G_\alpha(t,\mathbf{x})\\
&= 2\,i\,C_{\alpha\beta}\,\varepsilon^{ij}\partial_j\chi^\beta_i(t,\mathbf{x}) + \frac{2}{f^2}\,\varepsilon^{ij}\varepsilon^{kl}\,C_{\alpha\beta}\,\Big(\partial_l\chi^\beta_j(t,\mathbf{x})\,(\chi_0(t,\mathbf{x})\,\partial_k\chi_i(t,\mathbf{x}))\\
&\quad + \partial_k\chi^\beta_j(t,\mathbf{x})\,(\chi_l(t,\mathbf{x})\partial_i\chi_0(t,\mathbf{x})) + \chi^\beta_j(t,\mathbf{x})\,(\partial_k\chi_l(t,\mathbf{x})\,\partial_i\chi_0(t,\mathbf{x}))\\
&\quad + \partial_i\chi^\beta_0(t,\mathbf{x})\,(\chi_l(t,\mathbf{x})\,\partial_j\chi_k(t,\mathbf{x}))\Big) = 0
\end{aligned} \tag{C.36}$$

In section 3.4, we had modified the constraint $C^0(t,\mathbf{x})$ to give $\hat{C}^0(t,\mathbf{x})$ such that $\hat{C}^0(t,\mathbf{x})$ commuted with more constraints than $C^0(t,\mathbf{x})$ commuted with. We can do a similar modification here. The constraint F^0_ρ can be modified to give $\hat{F}^0_\rho$ as following.

$$\begin{aligned}
\hat{F}^0_\rho &= F^0_\rho - \int d^2\mathbf{y}\,\{F^i_\alpha(t,\mathbf{x})\,,F^0_\rho(t,\mathbf{y})\}\,M^{\beta\alpha}_{i'i}\,F^{i'}_\beta\\
\hat{F}^0_\rho &= p^0_\rho + \frac{i}{f^2}\,\varepsilon^{kl}\,C_{\sigma\rho}\,p^i_\beta\,\partial_k(\chi^\sigma_l\,\chi^\beta_i)\\
&\quad - \frac{i}{f^2}\,\varepsilon^{kl}\,p^i_\rho\,(\chi_l\,\partial_i\chi_k) - \frac{1}{f^2}\,\varepsilon^{ij}\varepsilon^{kl}\,C_{\rho\beta}\,C_{\sigma\rho}\,\chi^\rho_j\,\partial_k(\chi^\sigma_l\,\chi^\beta_i)\,.
\end{aligned} \tag{C.37}$$

$\hat{F}^0_\rho(t,\mathbf{x})$ anti-commutes with $F^i_\alpha(t,\mathbf{y})$.

$$\{\hat{F}^0_\rho(t,\mathbf{x})\,,F^i_\alpha(t,\mathbf{y})\} = 0 \tag{C.38}$$

However $\hat{F}^0_\rho(t,\mathbf{x})$ does not anti-commute with the remaining constraints, and it is technically quite involved to guess whether a further modification exists which would make it first class. Instead of trying to find it let us look at the on-shell value of the Hamiltonian at order f^{-2}.

C.2.1 Hamiltonian Value on the Constraint Surface of the Full Vector-Spinor Goldstino Model up to the Order f^{-2}

From the constraint (C.36) we get,

$$\begin{aligned} &i\,\varepsilon^{ij}\partial_j\chi^\alpha_i(t,\mathbf{x}) \\ &= -\ f^{-2}\,\varepsilon^{ij}\varepsilon^{kl}\Big(\partial_l\chi^\alpha_j(t,\mathbf{x})\,(\chi_0(t,\mathbf{x})\,\partial_k\chi_i(t,\mathbf{x})) + \partial_k\chi^\alpha_j(t,\mathbf{x})\,(\chi_l(t,\mathbf{x})\partial_i\chi_0(t,\mathbf{x})) \\ &\quad + \chi^\alpha_j(t,\mathbf{x})\,(\partial_k\chi_l(t,\mathbf{x})\,\partial_i\chi_0(t,\mathbf{x})) + \partial_i\chi^\alpha_0(t,\mathbf{x})\,(\chi_l(t,\mathbf{x})\,\partial_j\chi_k(t,\mathbf{x})\Big) \end{aligned} \tag{C.39}$$

Plugging the above into the expression for the canonical Hamiltonian density $\mathcal{H}_c(t,\mathbf{x})$ in eq. (C.20), we get,

$$\begin{aligned} \mathcal{H}_c(t,\mathbf{x}) = \frac{1}{2\,f^2}\Big(&\big(\chi_0(t,\mathbf{x})\,\chi_0(t,\mathbf{x})\big)\big(\partial_k\chi_j(t,\mathbf{x})\,\partial_l\chi_i(t,\mathbf{x})\big) \\ &-\big(\chi_0(t,\mathbf{x})\,\chi_0(t,\mathbf{x})\big)\,\partial_i\partial_k\big(\chi_j(t,\mathbf{x})\,\chi_l(t,\mathbf{x})\big)\Big) \end{aligned} \tag{C.40}$$

It turns out that this Hamiltonian density vanishes for the perturbative solution $\chi^\alpha_a = \partial_a\psi^\alpha + O(f^{-2})$ of the spin- 3/2 field equations of motion.

$$H(t)_{3/2}\big|_{on-shell} = 0\,. \tag{C.41}$$

Therefore, the on-shell field configurations have zero energy at least at the order f^{-2}. This observation prompts us that the spin-3/2 goldstino model may do not have dynamical degrees of freedom because of a hidden local symmetry, as in the case of the Rarita-Schwinger action. The results given in the main text of the Thesis prove this observation.

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