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SCUOLA DI DOTTORATO DI RICERCA IN SCIENZE MATEMATICHE
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CICLO XXVI

TOPICS IN INTEREST RATE MODELING

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Abstract

In this thesis, we address some issues in the mathematical modeling of the term structure of interest rates. In Chapter 1, we set the notation, recall some fundamental results and analyze the problems which will be tackled in the thesis, in particular the distinction between instantaneous and discrete rates and the so-called multiple curve framework. In Chapter 2, we propose a multiple-curve model for the instantaneous spot rate and give a fundamental condition to automatically calibrate it to the initial term structure, whereas in Chapter 3 we put forward an HJM multiple-curve model for the instantaneous forward rates and study its freedom from arbitrage opportunities. Finally, in Chapter 4, we introduce the concept of an instantaneous swap rate and build arbitrage-free coterminal and coinitial models around it.

Sunto

In questa tesi affrontiamo alcuni problemi relativi alla modellizzazione matematica della struttura a termine dei tassi di interesse. Nel Capitolo 1, impostiamo la notazione, ricordiamo alcuni risultati fondamentali e analizziamo i problemi che verranno affrontati nella tesi, in particolare la distinzione tra tassi istantanei e tassi discreti e il cosiddetto framework multicurva. Nel Capitolo 2, proponiamo un modello a multicurva per il tasso spot istantaneo e diamo una condizione fondamentale affinchè esso sia automaticamente calibrato alla struttura iniziale, mentre nel Capitolo 3 proponiamo un modello multicurva per i tassi forward istantanei di tipo HJM e studiamo la relativa assenza di opportunità di arbitraggio. Infine, nel Capitolo 4, introduciamo il concetto di tasso swap istantaneo e vi costruiamo attorno dei modelli privi di arbitraggio di tipo coterminal e coinitial.

Contents

Introduction

In this thesis, we address some recent topics about the modeling of the term structure of interest rates. We focus on what has now become known as the multiple curve framework and on the distinction between discrete and instantaneous tenors.

The term structure of interest rates certainly constitutes one of the most important and well investigated subjects of mathematical finance. Inevitably, even the more theoretically oriented analysis do consider the so called LIBOR rate or some idealizations of it. The LIBOR (London Interbank Offered Rate) is an interbank rate at which prime banks lend and borrow unsecured funds in the interbank market for a given currency and a given maturity. Until a few years ago, it was common practice both in the theoretical and in the applied literature, to model the LIBOR rate as a risk-free rate, i.e. a rate which is not subject to the risk of default. As a consequence, it was common practice to deal with a single curve of risk-free discount factors evolving randomly over time, although the classical approaches took different routes with regard to the choice of modeling spot versus forward rates and infinitesimal tenor versus finite (discrete) tenor rates. An instantaneous interest rate is a rate with an infinitesimal tenor, i.e. a rate that applies for an infinitesimal period of time. This is of course a mathematical idealization, but it proves of great utility even in practical applications and it should be noted that the first seminal contributions to the topic of interest rate modeling were indeed oriented towards instantaneous rates, see e.g. Vasicek (1977) and Cox et al. (1985) for the spot instanteneous rate and the classical Heath et al. (1992) on the instantaneous forward rates. Models for discrete tenor rates were in fact formalized years later[1] by Brace et al. (1997), Miltersen et al. (1997) and Jamshidian (1997), who developed what is now referred to as the LIBOR market model. The latter article, in particular, focused not only on (discrete) forward rates, but also on (discrete) forward swap rates, which can be seen as some kind of average of (discrete) forward rates. For a book length treatment of interest rates modeling, see e.g. Musiela and Rutkowski (2005), Hunt and Kennedy (2004) or Brigo and Mercurio (2006).

Mathematical finance is without any doubt a rapidly evolving subject, in which research topics often stem from real world events. For example, as it is well known, the 1987 stock market crash proved that the assumption, based on the classical Black and Scholes

[1] With no doubts, thanks to the develpment of the concept - now pervasive in mathematical finance - of numeraire which was introduced in Geman et al. (1995).

(1973), of stock prices evolving according to a constant volatility geometric Brownian motion was indeed flawed, as stock option prices started to exhibit what is now called the smile or skew effect. The recent financial crisis of 2007, on the other hand, has proved that the assumptions upon which the classical term structure models were build are not sustainable anymore.

In Chapter 1, we attempt to give a detailed overview of why this is the case by first describing the fundamental quantities in interest rate markets and then by giving a series of model free results that should link them. The fact that these results have ceased to hold true in practice is the main motivation for the next two chapters, in which we relax the assumption that a crucial quantity such as the LIBOR rate is risk-free. Since, as it will become clear from our descriptions in Chapter 1, the LIBOR cannot be associated to a single counterparty, we cannot exploit the already known results about the classical defaultable term structure models (see e.g. Bielecki and Rutkowski (2000)) but we will take a more exogenous approach aimed at modeling rather directly the rates themselves, while retaining a no-arbitrage framework.

In fact, the assumption of a risk-free LIBOR has been relaxed already in a discrete (Libor Market Model) forward rate modeling framework by Mercurio (2010b) and by Grbac et al. (2014), which we will review in Chapter 1.

In Chapter 2, we propose a generalization of the classical short rate models. The main issue with such an approach in a classical single curve framework is that we end up with an endogenous model, in which the initial term structure is an output rather than an input of the model. This issue was circumvented in an ad hoc manner for a number of specific models and finally in a comprehensive general manner for every Markovian model by Brigo and Mercurio (2001). The main result of this chapter is to give a corresponding way to achieve the same result in a suitably defined multiple curve framework.

In Chapter 3, we propose the closest possible relative of the celebrated HJM framework, developed in Heath et al. (1992), in a the multiple curve world. The HJM approach overcomes the endogeneity problem by modeling directly the whole forward curve and we overcome the problem in the same way in our generalized approach. We do so by considering some fictitious bond prices which are auxiliary in the definition of the forward LIBOR process. In fact, this kind of bonds have been already considered in the literature by, among others, Crépey et al. (2012). In this chapter we address the important points of their existence, uniqueness and analytical properties, such as differentiability with respect to the maturity. We then take care of the major concern of any HJM-style model, which is the absence of arbitrage: Heath et al. (1992) resolved the issue by imposing a drift condition on each instantaneous forward rate and we derive the analogue of this condition in a multiple curve framework.

In Chapter 4, while retaining a classical single curve approach, we study for the first time in the literature what happens when the tenor of a swap rate tends to zero and by doing so we fill a gap in the existing frameworks for modeling interest rates. This was initially motivated by the desire to better understand the so called OIS's (Overnight Indexed Swaps), which will be described in detail in Chapter 1, in which a floating leg pays (almost) continuously an (almost) infinitesimal tenor rate. The main contribution of this chapter is to develop an infinitesimal version of the Swap Market Model of Jamshidian (1997) by modeling our instantaneous coterminal swap rates. We resolve the problem of absence of arbitrage by a change of numeraire technique, where the key point is to being able to express bond prices in terms of instantaneous swap rates. In fact, we find a drift condition on the swap rates which is the infinitesimal counterpart of Jamshidian (1997). In other words we suitably define and analyze the infinitesimal counterparts of the Swap Market Model and drift condition as the HJM model and drift condition are the infinitesimal counterparts of the LMM. The latter fact is probably overlooked in the literature, but was already known, see e.g. Hull and White (1999).

Chapter 1

Fundamentals of Term-Structure Modeling

1.1 Some Market Interest Rates and Payoffs

In this section we give a fairly detailed overview of some market interest rates and payoffs on them, a knowledge of which is much more important now than in the pre-crisis framework. Since, especially in the interest rate market, contracts might differ by a myriad of features, for each contract we try to describe the market standard (which usually varies geographically). By market standard, we mean some contract specification uniform enough to make it possible to find many transactions using the same specification and thus having something eligible to be called a market price.

1.1.1 LIBOR and EURIBOR

The LIBOR (London Interbank Offered Rate) is an interbank rate at which prime banks lend and borrow unsecured funds in the interbank market for a given currency and a given maturity. As of today, the currencies at which LIBOR is available are Australian dollar (AUD), Canadian dollar (CAD), Swiss franc (CHF), Danish krone (DKK), Euro (EUR), British pound sterling (GBP), Japanese yen (JPY), New Zealand dollar (NZD), Swedish krona (SEK) and U.S. dollar (USD). The maturities are those of the so called money market (i.e. less than 1 year), namely 1 day, 1 and 2 weeks and from 1 up to 12 months. The LIBOR is computed daily by the BBA (British Bankers association) and it is published at 11:30 (GMT time). Specifically, a panel of banks is associated to each currency and components of this panel answer the question: "At what rate could you borrow funds, were you to do so by asking for and then accepting inter-bank offers in a reasonable market size just prior to 11 am?". In the case of USD, the panel is composed as of today of 18 banks, and the LIBOR is computed as the the trimmed average of the submissions with the exclusion of the top and bottom quartile.

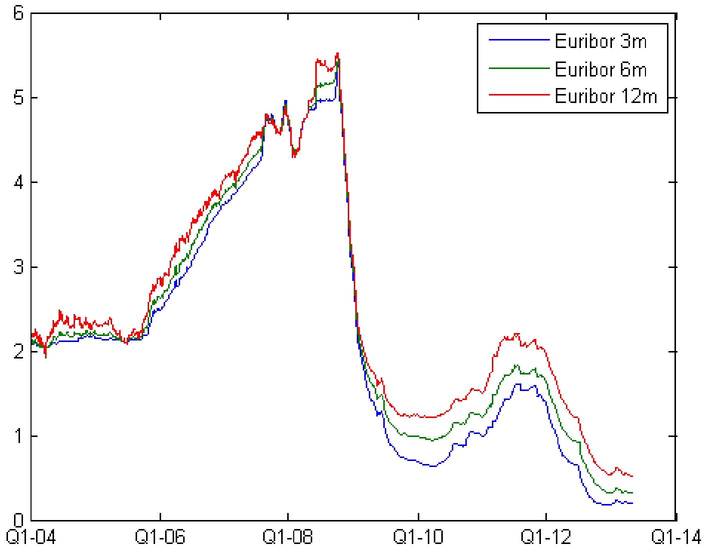

Figure 1.1: EURIBOR 3m, 6m and 12m from January 1, 2004 to April 26, 2013

The EURIBOR is very close in spirit to the LIBOR. The former rate, though, is computed by the EBF (European Banking Federation) and is available only for the Euro with maturities 1, 2 and 3 weeks and from 1 up to 12 months. While the mechanism for the daily creation of EURIBOR is again by submission and it refers to unsecured lending, the panel is bigger (almost 40 institutions) and the wording is slightly different.

A common important point worth noticing is that neither the LIBOR nor the EURIBOR are trade rates: it is perfectly possible that, on a given day, no actual transactions took place at the fixing value.

1.1.2 EONIA Rate and Effective Fed Funds Rate

The EONIA (Euro OverNight Index Average) rate is the effective overnight reference rate for the Euro. It is computed as a transaction-weighted average of all overnight unsecured lending transactions in the interbank market in the European Union. Note that, contrary to the case for the LIBOR and EURIBOR, the computation of the EONIA rate hinges on real market transactions. The analogous rate in the United States is the so-called Federal Funds rate, i.e. the overnight interest rate at which depository institutions trade balances held at the Federal Reserve. Again, this is an uncollateralized rate and its computation is transaction weighted. Basically any currency has its own equivalent for the EONIA rate, e.g. the SONIA for the GBP, the SARON for the CHF and the Mutan Rate for the JPY, but we will not go into these details.

1.1.3 Fixed-vs-floating Interest Rate Swaps

A fixed-vs-floating interest rate swap (IRS) is a contract in which two counterparties exchange a flow of payments based on a predetermined couple of rates, of which one is fixed and the other is floating. The contract must specify the following:

- a floating rate X,
- a fixed rate K,
- a tenor structure[1] for the floating leg, $\mathcal{T}^{fl} = \{T_0^{fl}, T_1^{fl}, \ldots, T_n^{fl}\}$,
- a tenor structure for the fixed leg, $\mathcal{T}^{fix} = \{T_0^{fix}, T_1^{fix}, \ldots, T_m^{fix}\}$,
- a daycount function[2], τ.

We assume in the following that the rates are settled in advance and paid in arrears: this convention implies that the payer of the fixed rate will receive at each time T_i^{fl}

$$\tau(T_{i-1}^{fl}, T_i^{fl}) X_{T_{i-1}^{fl}}$$

and pay at each time T_j^{fix}

$$\tau(T_{j-1}^{fix}, T_j^{fix}) K.$$

Note that we did in no way restrict our attention to the case where the first settlement date coincides with the present date. If that is the case, the swap is called spot-starting, otherwise it is called forward-starting. A very important special case of a swap occurs when the two tenor structures are equal and have just two dates, say T_0 and T_1. Obviously this configuration is non-trivial only in the forward starting case, in which case the swap is referred to as a Forward Rate Agreement (FRA).

Even though every kind of swap could be traded by two hypothetical counterparties, the marked standard is roughly the following. The floating rate in IRSs is normally some LIBOR or EURIBOR rate, the most common case being the 3m USD LIBOR in the USD market and the 6m EURIBOR in the EUR market. The frequency of payments for the floating leg is usually the same as the associated tenor[3], thus quarterly in the USD market and semiannual in the EUR market. The frequency of payments for the fixed leg is usually semiannual in the USD market and annual in the EUR market. Standard swaps are generally spot starting and the most traded maturities are 1, 2, 3, 4, 5, 7, 10, 15, 20, 25 and 30 years. The standard for the FRA is to have them on the 3m reference rate (EURIBOR for EUR and USD LIBOR for USD) with starting date in 1, 3, 6 or 9 months, or on the 6m reference rate with starting date in 1, 2, 6 and 12 months.

[1]By tenor structure we mean an ordered set of increasing dates, not necessarily equally-spaced.
[2]Possibly a different daycount function for each leg.
[3]This has a precise financial motivation, as it will be shown later.

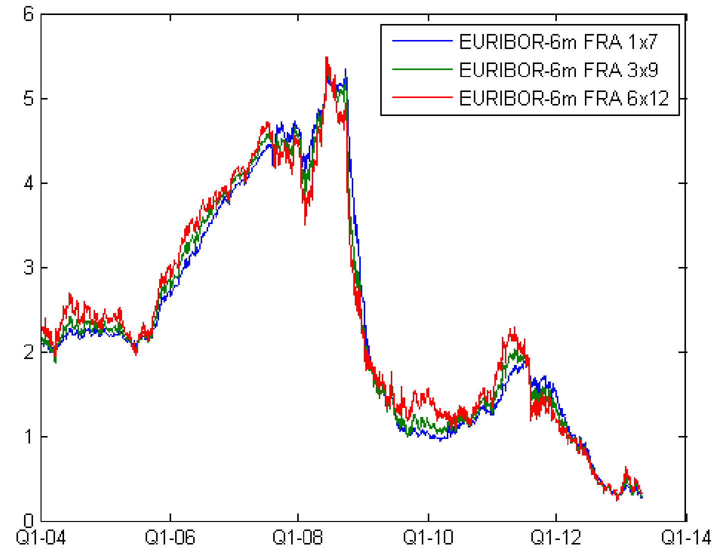

Figure 1.2: EURIBOR-6m FRA 1x7, 3x6 and 6x12 from January 1, 2004 to April 26, 2013

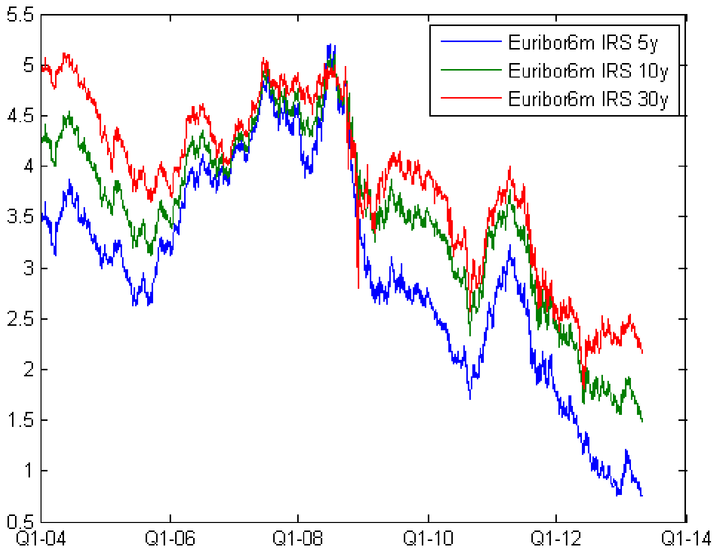

Figure 1.3: EURIBOR-6m IRS 5y, 10y and 30y from January 1, 2004 to April 26, 2013

1.1.4 Basis Swaps

A basis swap (BS) can be defined in different, not necessarily equivalent, ways.

The most natural definition probably consists of two tenor structures, two floating rates and a fixed spread (positive or negative) to be added to the payments of one of the legs. According to this first definition, a BS is basically a floating-vs-floating IRS where one of the leg pays a fixed spread on top of the floating rate (of course, if the spread happens to be negative, then it is actually received).

Alternatively, a BS could be defined as a pair of fixed-vs-floating IRSs with the same tenor structure for the fixed rate but possibly different fixed rates. Manifestly, this specification of the swap does not depend on both fixed rates but only upon their difference: we give it like this to stay closer to market practice, as explained in the sequel[4].

Let us investigate if and under which conditions these two definitions might be reconciled.

If we assume that, in a basis swap according to the first definition, the two tenor structures for the floating legs, call them $\mathcal{T}_a$ and $\mathcal{T}_b$, are such that $\mathcal{T}_a \subset \mathcal{T}_b$, then this swap might be written as a BS according to the second definition. To this end, it is enough to let the floating legs be $\mathcal{T}_a$ and $\mathcal{T}_b$, the common fixed tenor structure the one to which the spread is added and the difference between the fixed rates the spread paid by b. Note that there is no freedom in specifying the tenor structure associated with the fixed payments in the swap: this is forced to be the same as the structure in the leg to which the spread is added. For example if the two legs are equally spaced every 3 and 6 months, there is no way to represent this swap as a portfolio of two fixed-vs-floating IRSs having the fixed tenor structure equally spaced every 12 months.

If we assume that, in a basis swap according to the second definition, the (common) tenor structure for the fixed legs is equal to the tenor structure for one of the floating legs, say $\mathcal{T}_a$, then this swap might be written as a BS according to the first definition. To this end, it is enough to let the floating legs be the same and the spread equal to the difference of the two fixed rates and added to payments of a.

1.1.5 Overnight-Indexed Swaps

Overnight-indexed swaps (OIS) are fixed-vs-floating IRS with the floating rate replaced by a geometric average of some (overnight) rate. The contract must specify the following:

- a floating (overnight) rate X,
- a fixed rate K,
- a tenor structure for the floating leg, $\mathcal{T}^{fl} = \{T_0^{fl}, T_1^{fl}, \ldots, T_n^{fl}\}$, together with a sub tenor structure for each payment date T_i^{fl}, $\mathcal{T}^i = \{t_0^i, t_1^i, \ldots, t_{n_i}^i\}$,

[4] There are even more possible definitions, but we will not go into these details.

- a tenor structure for the fixed leg, $\mathcal{T}^{fix} = \{T_0^{fix}, T_1^{fix}, \dots, T_m^{fix}\}$,
- a daycount function, τ.

The payer of the fixed rate will receive at each time T_i^{fl}

$$\tau(T_{i-1}^{fl}, T_i^{fl})\bar{X}^{\mathcal{T}^i}(T_{i-1}^{fl}, T_i^{fl}),$$

where[5]

$$\bar{X}^{\mathcal{T}}(T,S) = \frac{1}{\tau(T,S)}\left[\prod_{k=0}^{n-1}(1 + (t_{k+1} - t_k)X_{t_k}) - 1\right]$$

if $\mathcal{T} = \{t_0, t_1, \dots, t_n\}$. At each time T_j^{fix}, it will pay

$$\tau(T_{j-1}^{fix}, T_j^{fix})K.$$

As it is the case for fixed-vs-floating IRS, a very important special case occurs when the two tenor structures are equal and have just two dates, say T_0 and T_1. This particular case of OIS will be referred to as OI-FRA. Unlike for a fixed-vs-floating IRS, however, this does make sense even for the spot starting case, since $\bar{X}(t,T)$ is not known at time t.

With regard to OISs the market standard is basically as follows. The variable rate is the Effective Fed Funds rate for the USD market and the EONIA rate for the EUR market. Maturities are of 1, 2 and 3 weeks, from 1 to 12 months and 1, 2, 3, 4, 5, 7, 10, 15, 20, 25 and 30 years. The tenor structures on the two legs are generally the same. When maturity is above 1 year, the frequency is semiannual in the USD market and annual in the EUR market, whereas for maturities below 1 year there is only one payment date. The sub-tenor structure in the floating leg is generally daily spaced, i.e. the t_k's are one day apart one from the other, which is consistent with the fact that the floating rate is an overnight rate.

1.2 Assumptions and Pricing

In this section we aim at pricing the payoffs introduced so far, possibly in a model-free manner. Specifically, under a precise set of assumptions, we will derive model-free results that impose different quantities to be actually equal. In later sections, we will take a close look at market data. The fact that the theoretical results ceased to hold true in the recent years will lead us to develop models in which we relax some of these assumptions. This will be done in Chapters 2 and 3.

We take as given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, supporting all the price processes we are about to introduce and we stick to the assumption that all the markets we consider are frictionless and free of arbitrage opportunities.

[5]This expression comes from a discretization of the exponential of an integral as it will become clear in the subsection about the pricing of OI-FRAs and OISs.

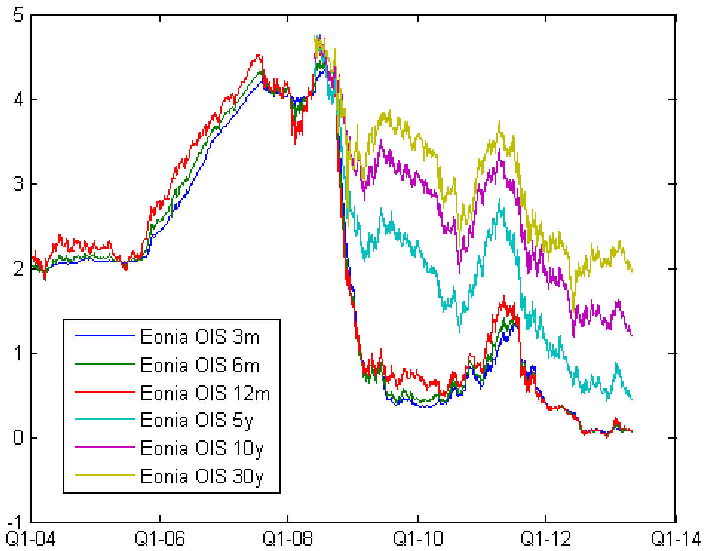

Figure 1.4: EONIA OIS 3m, 6m, 12m, 5y, 10y and 30y from January 1, 2004 to April 26, 2013

We assume the existence at any time t of a risk-free zero-coupon bond $P(\cdot, T)$ for every $T \in [t, T^*]$, where T^* is an arbitrary final date. On one hand, this last assumption about the existence of a continuum of bonds is too strong in order to be able to price most of the stylized contract we defined in the previous section, since it would often be enough to have only two bonds. On the other hand, it will be needed in order to define the instantaneous rates which will play a central role in the following, so that we stick to it unless otherwise stated. We require that $P(T, T) = 1 \quad \forall T \in [0, T^*]$, $P(t, T) \geq 0 \; \forall 0 \leq t \leq T \leq T^*$ and that the mapping $[T, T^*] \ni T \mapsto P(t, T)$ is differentiable $\forall t \in [0, T^*]$.

First of all, at any time t the bond maturing at time $t + \Delta$ can be used to define a simply compounded spot interest rate as follows.

Definition 1.2.1 (Spot rate associated with P)**.** The time-t Δ-tenor (simply compounded) spot rate associated with the curve P is defined as

$$R_t^\Delta := \frac{1}{\Delta}\left(\frac{1}{P(t, t+\Delta)} - 1\right). \tag{1.2.1}$$

In order not to burden notation, we do not explicitly indicate the dependence of R^Δ on P. However this fact is the whole point of the story and should always be kept in mind.

As we said, the positive quantity Δ is called the tenor of the interest rate R^Δ. The following definition ideally lets this tenor tend to zero.

Definition 1.2.2 (Instantaneous spot rate associated with P)**.** The time-t instantaneous spot rate associated with the curve P is defined as

$$r_t := \lim_{\Delta \to 0+} R_t^{\Delta} = -\frac{\partial}{\partial T} \ln P(t,T)|_{T=t}. \tag{1.2.2}$$

In some cases, it will be necessary to assume the possibility of trading in an additional asset, let us call it B for "bank account", whose price process is defined as

$$B_t := e^{\int_0^t r_u du}.$$

This price process might be thought of as a rolling position in the shortest maturing bond, but to make this idea rigorous we should introduce measure-valued portfolios and we refer to Björk et al. (1997) for further details. A discrete-time analogue of the bank-account process was introduced in Jamshidian (1997).

Absence of arbitrage implies that for any numeraire[6] N there exists a probability measure $\mathbb{Q}_N$ equivalent to $\mathbb{P}$ under which the price process of every traded asset A follows a (local) martingale when discounted by N, i.e.

$$\frac{A_t}{N_t} = \mathbb{E}_t^{\mathbb{Q}_N}\left[\frac{A_T}{N_T}\right], \quad \forall t \le T \le T* .$$

When the numeraire in question is the T-bond $P(\cdot, T)$ (respectively, the bank account B), we denote the martingale measure $\mathbb{Q}_T$ (respectively, $\mathbb{Q}_*$).

If we let $\mathcal{T} = \{T_0, T_1, \ldots, T_n\}$, the asset $\sum_{i=i}^n (T_i - T_{i-1}) P(\cdot, T_i)$ can certainly be used as a numeraire, since it is a (finite) linear combination of bonds with constant coefficients. We denote the associated martingale measure by $\mathbb{Q}_{\mathcal{T}}$. Note that $\mathbb{Q}_{\mathcal{T}}$ is indeed a generalization of $\mathbb{Q}_T$, in that we have $\mathbb{Q}_{\{0,T\}} = \mathbb{Q}_T$.

Here and in the following, $\mathbb{E}^T$, $\mathbb{E}^*$ and $\mathbb{E}^{\mathcal{T}}$ will always denote an expectation with respect to $\mathbb{Q}_T$, $\mathbb{Q}_*$ and $\mathbb{Q}_{\mathcal{T}}$, respectively.

1.2.1 Pricing of FRAs

The general payoff of a FRA, let us call it H, can be written as

$$H = \Delta(X_T^{\Delta} - K),$$

paid at $T + \Delta$, where X_T is the time-T value of some interest rate X^{Δ} of tenor Δ.

In some sense, it is "natural" to postpone the payment of R^{Δ} by Δ units of time: if the rate is set at time T, then it is "natural" for it to be paid $T + \Delta$. The reason for this will be clear in a moment.

We are interested in both the time-t price of H, which we will denote by $\Pi_t(H)$, and the strike K that makes the price equal to zero.

[6] A numeraire is a strictly positive price process.

It is well-known by standard results on no-arbitrage, that $\Pi_t(H)$ can be written as

$$\Pi_t(H) = P(t,S)\Delta\mathbb{E}_t^S[X_T - K] = P(t,S)\Delta(F_{X^\Delta}(t,T) - K).$$

In the equation above we already used the following

Definition 1.2.3 (FRA rate on X)**.** The no-arbitrage time-t fair strike in a FRA on the generic rate X^Δ resetting at T and paying at $T+\Delta$ is defined as

$$F_{X^\Delta}(t,T) := \mathbb{E}_t^{T+\Delta}[X_T^\Delta]. \tag{1.2.3}$$

Remark 1.2.4. In the following, it will sometimes be convenient to use the alternative and more general notation with F depending on one more argument:

$$F_{X^\Delta}(t,T,S) := \mathbb{E}_t^S[X_T^\Delta],$$

so that we have

$$F_{X^\Delta}(t,T) = F_{X^\Delta}(t,T,T+\Delta).$$

Note that the FRA rate F_{X^Δ} can be defined for *any* interest rate X^Δ whatsoever. If X_T is $\mathbb{Q}_{T+\Delta}$-integrable, then the process $F_{X^\Delta}(\cdot,T)$ is *a fortiori* a martingale under $\mathbb{Q}_{T+\Delta}$ (by the tower property of conditional expectations) and we obviously have that $F_{X^\Delta}(t,t) = X_t^\Delta$ for every t.

We will now show that, under a precise assumption (on the nature of X, and implicitly on the timing of the payment), $F_{X^\Delta}(t,T)$ and consequently the price $\Pi_t(H)$ can be determined *without any hypothesis on the evolution of the rate itself.*

Assumption 1.2.5. We assume that $X^\Delta = R^\Delta$ for some arbitrary Δ.

This assumption has to be made in order to have some consistence between the curve we use to discount payoffs and the interest rate itself.

Before stating the fundamental proposition of this subsection, let us give a definition which will be useful for the development to come.

Definition 1.2.6 (Forward rate associated to P)**.** The time-t Δ-tenor (simply compounded) forward rate for time T associated to the curve P[7] is defined as

$$R^\Delta(t,T) := \frac{1}{\Delta}\frac{P(t,T) - P(t,T+\Delta)}{P(t,T+\Delta)}. \tag{1.2.4}$$

Note that the process $R^\Delta(\cdot,T)$ must be a $\mathbb{Q}_{T+\Delta}$-martingale by no arbitrage, being the ratio of the price-processes of two traded assets.

We now state the promised representation of $F_{R^\Delta}(\cdot,T)$ under our assumptions

Proposition 1.2.7. *Under Assumption 1.2.5, the fair strike on a FRA on R^Δ setting at time T and paying at time $T+\Delta$ is equal to the forward rate for time T associated with the curve P, namely*

$$F_{R^\Delta}(\cdot,T) = R^\Delta(\cdot,T).$$

[7]Again, its dependence upon P is omitted in the notation but should be kept in mind.

Furthermore, the time-t price $\Pi_t(H)$ is given by

$$\Pi_t(H) = P(t,T) - P(t,T+\Delta) - P(t,T+\Delta)\Delta K.$$

Proof. The crucial point is to note that

$$R_T^\Delta = \frac{1}{\Delta}\frac{P(T,T) - P(T,T+\Delta)}{P(T,T+\Delta)}$$

is the T-value of a ratio of traded assets which, by no arbitrage, has to be a martingale under the measure $\mathbb{Q}_{T+\Delta}$ associated to $P(\cdot, T+\Delta)$ (the asset in the denominator of the ratio). Therefore we have the following closed-form expression for the forward rate

$$F_{R^\Delta}(t,T) = \mathbb{E}_t^{T+\Delta}[R_T^\Delta] = \frac{1}{\Delta}\frac{P(t,T) - P(t,T+\Delta)}{P(t,T+\Delta)},$$

which yields the first part. Now we can substitute this expression in the time-t price to get

$$\begin{aligned}\Pi_t(H) &= P(t,T+\Delta)\Delta(F_{R^\Delta}(t,T) - K)\\ &= P(t,T+\Delta)\Delta\left(\frac{1}{\Delta}\frac{P(t,T) - P(t,T+\Delta)}{P(t,T+\Delta)} - K\right),\end{aligned}$$

so that the second claim is also clear. □

As we did for the *spot* rate R^Δ, we can now let the tenor tend to zero for the *forward* rate $R^\Delta(\cdot, T)$, as we do in the following definition

Definition 1.2.8 (Instantaneous forward rate associated with P)**.** The time-t instantaneous forward rate for the maturity T associated with the curve P is defined as

$$f(t,T) := \lim_{\Delta\to 0+} R^\Delta(t,T) = -\frac{\partial}{\partial T}\ln P(t,T) \tag{1.2.5}$$

Note that the instantaneous forward curve $T \mapsto f(t,T)$ prevailing at time t is uniquely determined by the zero coupon curve $T \mapsto P(t,T)$ prevailing at the same time t and this map is invertible. In fact, we have

$$P(t,T) = e^{-\int_t^T f(t,u)du}$$

and we can recover the zero coupon bond prices from the instantaneous forward rates. This observation should be kept in mind, since it will be important in Chapter 4, where we will propose another parametrization for the term structure.

The next example shows how the assumption of setting the rate in advance and paying in arrears is pivotal in order to have the simple representation for $F_{R^\Delta}(t,T)$.

Example 1.2.9. We do not have a general model-free expression for $\mathbb{E}_t^T[R_T^\Delta] \neq \mathbb{E}_t^{T+\Delta}[R_T^\Delta] = F_{R^\Delta}(t,T)$. This is the fair strike K in a FRA that pays $H = \Delta(R_T^\Delta - K)$ at time T and not at time $T+\Delta$. Note that this is equivalent to the payoff $\Delta(R_T^\Delta - K)(1+\Delta R_T^\Delta)$ to be paid at time $T+\Delta$. The price of the latter payoff cannot be pinned down in a model-free fashion due to the presence of the quadratic term in R_T^Δ. To determine its price and fair strike, it is then necessary to specify (at least) the quadratic variation of the $\mathbb{Q}_{T+\Delta}$-martingale $R^\Delta(\cdot,T)$.

Going back to the case of an arbitrary reference rate, we now give an example of some conditions that allow us to determine a no-arbitrage restriction.

Example 1.2.10. Specifically, let us say that the payoff to be priced is written on X^Δ to be set at T and paid at $T+\Delta$, where the rate X^Δ is generated by some curve P^f different from P (otherwise we are back to the "nice" case), i.e.

$$X_t^\Delta = \frac{1}{\Delta}\left(\frac{1}{P^f(t,t+\Delta)} - 1\right).$$

Of course, the fair strike in a FRA on X^Δ setting at T and paying at $T+\Delta$ is $F_{X^\Delta}(t,T)$, whose definition has in no way changed:

$$F_{X^\Delta}(t,T) = \mathbb{E}_t^{T+\Delta}[X_T^\Delta].$$

Again, it seems impossible to give an explicit expression for $F_{X^\Delta}(t,T)$ unless we have specified the $\mathbb{Q}_{T+\Delta}$ law of the process $X_\cdot^\Delta$ or at least of the variable X_T^Δ. And, again, the problem is that we do not have, *a priori*, any guiding principle in specifying that law, unless we assume that $P^f(\cdot,T)$ is a traded asset $\forall T$. However, a direct assumption of this kind would be pointless because by the law of one price we would end up with $P^f(\cdot,T) = P(\cdot,T) \quad \forall T$. The best we can assume, then, is that the $P^f(\cdot,T)$'s are denominated in a different currency, call it f, which is itself a traded asset, i.e. it has a price process which we naturally call its exchange rate (with the base currency). At this point, we do have a no-arbitrage restriction on

$$X^\Delta(\cdot,T) = \frac{1}{\Delta}\frac{P^f(\cdot,T) - P^f(\cdot,T+\Delta)}{P^f(\cdot,T+\Delta)},$$

namely that it has to be a martingale under $\mathbb{Q}_{T+\Delta}^f$, the forward measure associated to $P^f(\cdot,T)$. In addition the density of the latter measure with respect to $\mathbb{Q}_{T+\Delta}$ is given by[8]

$$\frac{d\mathbb{Q}_{T+\Delta}^f}{d\mathbb{Q}_{T+\Delta}}|\mathcal{F}_t = \frac{S(t,T+\Delta)}{S(0,T+\Delta)},$$

where $S(\cdot,T)$ is the T-forward exchange rate (T-forward price of a unit of foreign currency, namely $S(t,T) = S(t,t)\frac{P^f(t,T)}{P(t,T)}$). Again, no-arbitrage implies that $S(\cdot,T+\Delta)$ must be a $\mathbb{Q}_{T+\Delta}$-martingale.
An extremely simple specification for the processes $X^\Delta(\cdot,T)$ and $S(\cdot,T+\Delta)$ would be

$$X^\Delta(t,T) = X^\Delta(0,T)\mathcal{E}_t\left[\int_0^\cdot \sigma_X(u)dW_u^f\right],$$

[8] If $\mathbb{Q}$ and $\mathbb{P}$ are two probability measures on the same σ-algebra $\mathcal{F}$ with $\mathbb{Q}$ absolutely continuous with respect to $\mathbb{P}$ and $\frac{d\mathbb{Q}}{d\mathbb{P}} = \Lambda$, then, letting $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$, it is straightforward to check that $\mathbb{E}^{\mathbb{P}}[\Lambda|\mathcal{G}] = \frac{d\mathbb{Q}_{|\mathcal{G}}}{d\mathbb{P}_{|\mathcal{G}}}$. We will denote any of the latter quantities by $\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{G}$.

where W^f is a $\mathbb{Q}^f_{T+\Delta}$ Wiener process and that

$$S(t, T+\Delta) = S(0, T+\Delta)\mathcal{E}_t\left[\int_0^{\cdot} \sigma_S(u)dZ_u\right]$$

where Z is a $\mathbb{Q}_{T+\Delta}$ Wiener process such that $[W^f, Z]_t = \rho t$. Here we have that

$$\frac{d\mathbb{Q}_{T+\Delta}}{d\mathbb{Q}^f_{T+\Delta}}|\mathcal{F}_t = \mathcal{E}_t\left[\int_0^{\cdot} -\sigma_S(t)dZ^f_t\right],$$

where $Z^f := Z - \int_0^{\cdot} \sigma_S(u)du$ is a $\mathbb{Q}^f_{T+\Delta}$-Wiener by Girsanov's theorem, so that, by Girsanov theorem again, $W := W^f + \int_0^{\cdot} \rho\sigma_S(u)du$ is a Wiener under $\mathbb{Q}_{T+\Delta}$ and $X^\Delta(\cdot, T)$ satisfies

$$\frac{dX^\Delta(t,T)}{X^\Delta(t,T)} = \sigma_X(t)(dW_t - \rho\sigma_S(t)dt)$$

and we have

$$F_{X^\Delta}(t,T) = \mathbb{E}_t^{T+\Delta}[X_T^\Delta] = \mathbb{E}_t^{T+\Delta}\left[X^\Delta(T,T)\right] = X^\Delta(t,T)e^{-\int_t^T \rho\sigma_X(u)\sigma_S(u)du}.$$

The exponential term in the last formula is often referred to as "convexity adjustment", or "quanto adjustment" when it is related to some FX. See, e.g., Pelsser (2003) for a survey.

1.2.2 Pricing of IRSs

Consider a (possibly forward-starting) fixed-vs-floating IRS on some interest rate X, with fixed rate K and tenor structures $\mathcal{T}^{fl} = \{T_0^{fl}, T_1^{fl}, \ldots, T_n^{fl}\}$ and $\mathcal{T}^{fix} = \{T_0^{fix}, T_1^{fix}, \ldots, T_m^{fix}\}$. Again we are interested in its price and the fixed rate K which makes this price equal to zero. As it was already stressed above, note that the following discussion is a simple generalization of the preceding subsection.

If we make no assumptions on the underlying rate X, the price of the swap is

$$\sum_{i=1}^n \left[P(t, T_i^{fl})(T_i^{fl} - T_{i-1}^{fl})\mathbb{E}_t^{T_i^{fl}}[X_{T_{i-1}^{fl}}]\right] - \sum_{j=1}^m P(t, T_j^{fix})(T_j^{fix} - T_{j-1}^{fix})K$$

and, recalling the definition $F_X(t, T_{i-1}^{fl}, T_i^{fl}) = E_t^{T_i^{fl}}(X_{T_{i-1}^{fl}})$, this might be rewritten as

$$\sum_{i=1}^n \left[P(t, T_i^{fl})(T_i^{fl} - T_{i-1}^{fl})F_X(t, T_{i-1}^{fl}, T_i^{fl})\right] - \sum_{j=1}^m P(t, T_j^{fix})(T_j^{fix} - T_{j-1}^{fix})K. \quad (1.2.6)$$

From the expression above, which is completely model-free, we get the following

Definition 1.2.11 (Swap rate on X)**.** The no-arbitrage time-t fair fixed rate in an IRS on the generic rate X with floating tenor structure $\mathcal{T}^{fl}$ and fixed tenor structure $\mathcal{T}^{fix}$ is defined as

$$S_X(t, \mathcal{T}^{fl}, \mathcal{T}^{fix}) := \frac{\sum_{i=1}^n P(t, T_i^{fl})(T_i^{fl} - T_{i-1}^{fl})F_X(t, T_{i-1}^{fl}, T_i^{fl})}{\sum_{j=1}^m P(t, T_j^{fix})(T_j^{fix} - T_{j-1}^{fix})}. \quad (1.2.7)$$

Note, again, that the swap rate rate S_X can be defined *for any* interest rate X whatsoever. The quantity $S_X(t, \mathcal{T}^{fl}, \mathcal{T}^{fix})$ plays a role that generalizes the role played by $F_X(t,T,S)$ and in fact we have

$$F_X(t,T,S) = S_X(t, \{T,S\}, \{T,S\}).$$

We saw that the process $F_X(\cdot,T,S)$ is necessarily a $\mathbb{Q}_S$-martingale. We have an analogous result for the process $S_X(\cdot, \mathcal{T}^{fl}, \mathcal{T}^{fix})$. In fact, it is easy to see that the numerator in the latter quantity is a linear combination of traded assets with constant coefficients, so that $S_X(\cdot, \mathcal{T}^{fl}, \mathcal{T}^{fix})$ must be a $Q_{\mathcal{T}^{fix}}$ martingale.

Now we show that the counterpart (i.e. generalization) of the hypotheses we made for pricing FRAs will allow to obtain a model-free expression for $S_X(t, \mathcal{T}^{fl}, \mathcal{T}^{fix})$.

The first assumption is simply the same:

Assumption 1.2.12. We assume that the reference floating rate is $X = R^{\Delta}$.

The second assumption generalizes to the following:

Assumption 1.2.13. We assume that $T_i^{fl} = T_{i-1}^{fl} + \Delta \quad \forall i = 1, 2, \cdots, n$. This means that the rate R^{Δ} set at time T_{i-1}^{fl} is paid with a delay of Δ units of time, and this is true for all i's.

Before giving the proposition let us define

Definition 1.2.14 (Swap rate associated to P)**.** The time-t swap rate with floating-leg tenor structure $\mathcal{T}^{fl}$ and fixed-leg tenor structure $\mathcal{T}^{fix}$ associated to the curve P is defined as

$$R(t, \mathcal{T}^{fl}, \mathcal{T}^{fix}) := \frac{P(t, T_0^{fl}) - P(t, T_n^{fl})}{\sum_{j=1}^{m} P(t, T_j^{fix})(T_j^{fix} - T_{j-1}^{fix})}.$$

It is crucial to note that, with this notation, the swap rate $R(t, \mathcal{T}^{fl}, \mathcal{T}^{fix})$ depends on $\mathcal{T}^{fl}$ only through the first and last date. Furthermore, it is indeed a generalization of the forward rate associated to P because

$$F_{R^{\Delta}}(t,T) = R(t, \{T, T+\Delta\}, \{T, T+\Delta\}).$$

By no-arbitrage, $R(\cdot, \mathcal{T}^{fl}, \mathcal{T}^{fix})$, which is a ratio of traded assets, must be a martingale under the measure $\mathbb{Q}_{\mathcal{T}^{fix}}$.

The main proposition on the model free representation of S_X under our assumptions now reads:

Proposition 1.2.15. *Under Assumptions 1.2.12 and 1.2.13, the fair fixed rate on an IRS on R^{Δ} with floating leg tenor structure $\mathcal{T}^{fl} = \{T, T+\Delta, \ldots, S-\Delta, S\}$ and fixed leg tenor structure $\mathcal{T}^{fix}$ is equal to the swap rate associated to P, namely*

$$S_{R^{\Delta}}(\cdot, \{T, T+\Delta, \ldots, S-\Delta, S\}, \mathcal{T}^{fix}) = R(\cdot, \{T, T+\Delta, \ldots, S-\Delta, S\}, \mathcal{T}^{fix}).$$

Furthermore, the time-t price of the swap is given by

$$\left[P(t,T_0^{fl}) - P(t,T_n^{fl})\right] - \sum_{j=1}^{m} P(t,T_j^{fix})(T_j^{fix} - T_{j-1}^{fix})K.$$

Proof. Since we proved in Proposition 1.2.7 that

$$F_{R^\Delta}(t,T_{i-1}^{fl},T_{i-1}^{fl}+\Delta) = \frac{1}{\Delta}\left[\frac{P(t,T_{i-1}^{fl}) - P(t,T_{i-1}^{fl}+\Delta)}{P(t,T_{i-1}^{fl}+\Delta)}\right],$$

we see that the sum appearing in the numerator of equation (1.2.7) is telescoping and we immediately get to the result. This proves also the expression for the price of the swap. □

It is crucial for the following to note that $S_{R^\Delta}(\cdot,\{T,T+\Delta,\dots,S-\Delta,S\},\mathcal{T}^{fix})$ does *not* depend on Δ. In other words, a swap on the risk-free rate always yields the same model-free present value as long as the length between the floating tenor structure dates is constantly equal to the tenor of the rate.

1.2.3 Pricing of Basis Swaps

A basis swap was defined as a pair of fixed-vs-floating IRSs with (possibly) different floating rates, floating tenor structures and fixed rates, but the same fixed tenor structure. There is no new theory needed to price a BS: being able to price each IRS swap separately is enough and we are led to the following

Definition 1.2.16 (Basis swap rate between X and Y)**.** The no-arbitrage time-t basis swap rate on $X/\mathcal{T}_X^{fl}$ and $Y/\mathcal{T}_Y^{fl}$ with fixed tenor structure $\mathcal{T}^{fix}$ is defined as

$$BS_{X/Y}(t,\mathcal{T}_X^{fl},\mathcal{T}_Y^{fl},\mathcal{T}^{fix}) := S_X(t,\mathcal{T}_X^{fl},\mathcal{T}^{fix}) - S_Y(t,\mathcal{T}_Y^{fl},\mathcal{T}^{fix}).$$

It is clear that we will have a model free expression for the BS price as soon as we have model free expressions for the underlying IRSs prices. In particular the main proposition about model-free pricing of IRSs states that $S_{R^\Delta}(t,\{T,T+\Delta,\dots,S-\Delta,S\},\mathcal{T}^{fix})$, the fair strike on a swap on R^Δ with points in the floating tenor structure equally spaced by Δ, does not depend on Δ. Thus, for any two tenors Δ and Λ, we have the following important result:

$$BS_{R^\Delta/R^\Lambda}(\cdot,\{T,T+\Delta,\dots,S-\Delta,S\},\{T,T+\Lambda,\dots,S-\Lambda,S\},\mathcal{T}^{fix}) = 0.$$

1.2.4 Pricing of OI-FRAs

In order to evaluate an OI-FRAs and OISs, we will make a simplifying assumption about the quantity $\bar{X}^{\mathcal{T}}(T,S)$ associated to a generic (overnight) rate X. For ease of reading, we recall that its definition was given by

$$\bar{X}^{\mathcal{T}}(T,S) = \frac{1}{S-T}\left[\prod_{k=0}^{n-1}(1+(t_{k+1}-t_k)X_{t_k})-1\right]$$

for $\mathcal{T} = \{t_0, t_1, \ldots, t_n\}$. In all the sequel, we change the definition of $\bar{X}^{\mathcal{T}}(T,S)$ to read

$$\bar{X}(T,S) = \frac{1}{S-T}\left[e^{\int_T^S X_t dt} - 1\right], \tag{1.2.8}$$

which does not depend on the tenor structure anymore[9].

In this subsection we consider overnight-indexed FRAs, i.e. OISs with a single set date, T, and a single payment date, S, written on the generic (overnight) rate X. Namely, the time S payoff is

$$(S-T)\left[\bar{X}(T,S) - K\right] = (S-T)\left[\frac{1}{S-T}\left(e^{\int_T^S X_t dt} - 1\right) - K\right].$$

As it was the case for a FRA on the generic rate X, also here there is no way to pin down the price in a model-free manner, and we would be led to define the analog of $F_X(\cdot, T, S)$. However, let us limit ourselves to consider the simple case in which some ad-hoc assumptions on the rate and on the timing allow for model-free expressions. The "right" assumption on the rate X turns out to be $X = r$, where we recall the definition of r

$$r_t = -\frac{\partial}{\partial T}\ln p(t,T)|_{T=t}.$$

Note that we have $r_t = \lim_{\Delta \to 0^+} R_t^{\Delta}$, so that we call r the instantaneous rate associated to the curve P. It also turns out that we do not need any assumption on the timing of the payments, so we just assume $S = T + \Delta$ for some Δ. Thus we are led to a payoff at $T + \Delta$ of[10]

$$\Delta\left[\bar{r}(T,T+\Delta) - K\right] = \Delta\left[\frac{1}{\Delta}\left(e^{\int_T^{T+\Delta} r_t dt} - 1\right) - K\right].$$

In this case, it is convenient to use the bank account as a numeraire, to find the time-t

[9] Technically, this is the approximation of a product via a multiplicative integral (just as a sum might be approximated by an integral), but there is limited benefit from pursuing this multiplicative calculus analogy any further.

[10] The $\bar{\cdot}$ notation here is of course the same as in equation (1.2.8)

price as follows:

$$
\begin{aligned}
&B_t \mathbb{E}_t^* \left[\frac{1}{B_{T+\Delta}} \Delta(\bar{r}(T, T+\Delta) - K) \right] \\
&= B_t \mathbb{E}_t^* \left[\frac{1}{B_{T+\Delta}} (\frac{B_{T+\Delta}}{B_T} - (1 + \Delta K)) \right] \\
&= B_t \mathbb{E}_t^* \left[\frac{1}{B_T} - \frac{1 + \Delta K}{B_{T+\Delta}} \right] \\
&= P(t, T) - P(t, T+\Delta) - P(t, T+\Delta)\Delta K).
\end{aligned}
$$

We arrive at the following important result

Proposition 1.2.17. *The time-t fair strike in an OI-FRA on r from T to $T+\Delta$ is equal to $\frac{1}{\Delta}\frac{P(t,T)-P(t,T+\Delta)}{P(t,T+\Delta)}$, i.e. equal to $F_{R^\Delta}(t, T)$.*

Let us compare this result with what we obtained about FRAs on R^Δ: in that case, the payoff (paid at $T + \Delta$) was $\Delta(R_t^\Delta - K)$, now the payoff is $\Delta(\bar{r}(T, T + \Delta) - K)$. We just showed that the fair strike K at time t is the same in both cases and equal to the forward $F_{R^\Delta}(t, T)$. It is convenient to keep in mind this fact and to think of $F_{R^\Delta}(\cdot, T)$ in both ways.

We already noted that, in a FRA on R^Δ, the case in which the reset date coincides with the valuation date t is trivial and the fair strike is R_t^Δ. In an OI-FRA, on the other hand, the situation is not trivial anymore: if the reset date is equal to t, the $t + \Delta$-payoff is not known at time t. However, the results just derived show that its fair strike must be nonetheless R_t^Δ. Again, it is convenient to keep in mind this fact and to think of R_t^Δ in both ways: the time-t, Δ-tenor risk free rate as well as the fair strike on a OI-FRA from t to $t + \Delta$.

1.2.5 Pricing of OIS

Let us consider the case of a proper OIS with two arbitrary tenor-structures, $\mathcal{T}^{fl}$ and $\mathcal{T}^{fix}$. In light of the considerations about OI-FRAs, it is clear that the assumption to be made in order to get a model-free price and fair strike is simply that the floating rate is r (again there are no restrictions on the floating-leg tenor structure). It is then straightforward to see that the OIS price is

$$
\left[P(t, T_0^{fl}) - P(t, T_n^{fl}) \right] - \sum_{j=1}^{m} P(t, T_j^{fix})(T_j^{fix} - T_{j-1}^{fix})K,
$$

so that we can state the following

Proposition 1.2.18. *The time-t fair fixed-rate in an OIS on r with floating leg tenor structure $\mathcal{T}^{fl}$ and fixed leg tenor structure $\mathcal{T}^{fix}$ is equal to $R(t, \mathcal{T}^{fl}, \mathcal{T}^{fix})$*

Again, it is important to keep in mind that $R(\cdot, \mathcal{T}^{fl}, \mathcal{T}^{fix})$ plays a dual role: the fair strike on a IRS on R^Δ with floating-leg tenor points equally spaced by Δ units of time and the fair strike on a OIS on r.

1.2.6 Summary of Definitions

To recapitulate, let us fix a time t and let $T \mapsto P(t,T)$ be a zero-coupon curve. We defined the following quantities out of it:

- $R_t^\Delta := \frac{1}{\Delta}\left(\frac{1}{P(t,t+\Delta)} - 1\right)$ the time-t, Δ-tenor (simply compounded) rate
- $r_t := \lim_{\Delta\to 0+} R_t^\Delta$ the time-t instantaneous spot rate (i.e. the spot rate of infinitesimally small tenor)
- $R^\Delta(t,T) := \frac{1}{\Delta}\left(\frac{P(t,T)}{P(t,T+\Delta)} - 1\right)$ the time-t, Δ-tenor forward rate for time T on the rate R^Δ. This rate has a dual interpretation. The first is the fair strike on a FRA on R^Δ setting at T and paying at $T+\Delta$. The second is the fair strike on an OI-FRA on r from T to $T+\Delta$, paying at $T+\Delta$.
- $f(t,T) := \lim_{\Delta\to 0+} R^\Delta(t,T)$ the instantaneous forward rate (i.e. the forward rate of infinitesimally small tenor)
- $R(t,\mathcal{T}^{fl},\mathcal{T}^{fix}) := \frac{P(t,T_0^{fl}) - P(t,T_n^{fl})}{\sum_{j=1}^m P(t,T_j^{fix})(T_j^{fix} - T_{j-1}^{fix})}$ the time-t swap rate with floating-leg tenor structure $\mathcal{T}^{fl}$ and fixed-leg tenor structure $\mathcal{T}^{fix}$. This rate has a dual interpretation. The first is the fair fixed rate in a IRS on R^Δ with floating tenor structure $T_0^{fl}, T_0^{fl}+\Delta, \ldots, T_n^{fl}-\Delta, T_n^{fl}$ and fixed tenor structure $\mathcal{T}^{fix}$. The second is the fair strike in a OIS on r with arbitrary floating tenor structure and fixed tenor structure $\mathcal{T}^{fix}$.

Note that the swap rates contain the forward rate and the spot rate as special cases, in fact we have

$$R(t,\{T,T+\Delta\},\{T,T+\Delta\}) = R^\Delta(t,T),$$
$$R(t,\{t,t+\Delta\},\{t,t+\Delta\}) = R_t^\Delta$$

and, of course,

$$R^\Delta(t,t) = R_t^\Delta.$$

Naturally the *instantaneous* spot rate is a special case of the *instantaneous* forward rate, in that we have

$$f(t,t) = r_t.$$

1.3 The LIBOR Rate

A spot interest rate of tenor Δ, L^Δ, which we will refer to as LIBOR rate. L^Δ is allowed to be different from R^Δ. If this is the case then, of course, L^Δ cannot be risk-free.

We assume that it is *not* possible to invest at the spot rate L_t^Δ from t to $t+\Delta$, not even subject to some credit risk. On the other hand, we do assume that a family of forward rate agreements (FRA) on L^Δ for every maturity $T \in [0, T^*]$ is traded in the market. A FRA with strike K on L^Δ with maturity T and unit notional has the following payoff to be paid at time $T+\Delta$

$$\Delta(L_T^\Delta - K).$$

The fair strike at time t of a FRA on L^Δ setting at T and paying at $T+\Delta$ is denoted by $L^\Delta(t,T)$ and we recall it is given by

$$F_{L^\Delta}(t,T) := \mathbb{E}_t^{T+\Delta}[L_T^\Delta]$$

or, equivalently,

$$F_{L^\Delta}(t,T) := \frac{\mathbb{E}_t^*[e^{-\int_t^{T+\Delta} r_u du} L_T^\Delta]}{P(t,T+\Delta)}.$$

For ease of notation, in the following we will also use the notation

$$L^\Delta(t,T) := F_{L^\Delta}(t,T).$$

A crucial but simple observation is that $(P(t,T+\Delta)\Delta L^\Delta(t,T))_t$ is the price process of a traded asset. In fact, the latter quantity is exactly the time-t price of the floating leg in a FRA on L^Δ setting at T and paying at $T+\Delta$. It is worth to keep this fact in mind, since it will be used in Chapter 3.

Until a few years ago, it was common practice to assume that the spot LIBOR of tenor Δ could be modeled as a risk-free rate R^Δ and such practice was in fact supported by empirical evidence. Surprisingly enough, at the beginning of the subprime crisis in summer 2007, the basic relations that must hold true if the LIBOR rate were equal to the risk-free rate R^Δ, that we discussed at length in the previous section, suddenly ceased to hold in practice. We will now provide some examples and for a more comprehensive discussion we refer to, e.g., Mercurio (2010a) and Bianchetti (2009).

First, in Figure 1.5, we compare the EURIBOR 3x6 FRA rate versus the standard spot replication with 3m and 6m EURIBOR. The minuscule replication error of a handful of basis points that was present until summer 2007 has now turned into a huge basis of the order of percentage points.

As another example, we show in Figure 1.6 the $3m \times 6m$ basis swap for the EURIBOR. This is simply the difference between the fixed rate to be paid annually to get EURIBOR 6m every 6 months or EURIBOR 3m every 3 months. If EURIBOR were risk-free, this difference should be null as it was indeed the case up to August 2007, but since the explosion of the crisis this financial quantity is definitely an additional risk factor that needs to be modeled for its own sake.

The fact that these two phenomena in Figures 1.5 and 1.6 are actually the two sides

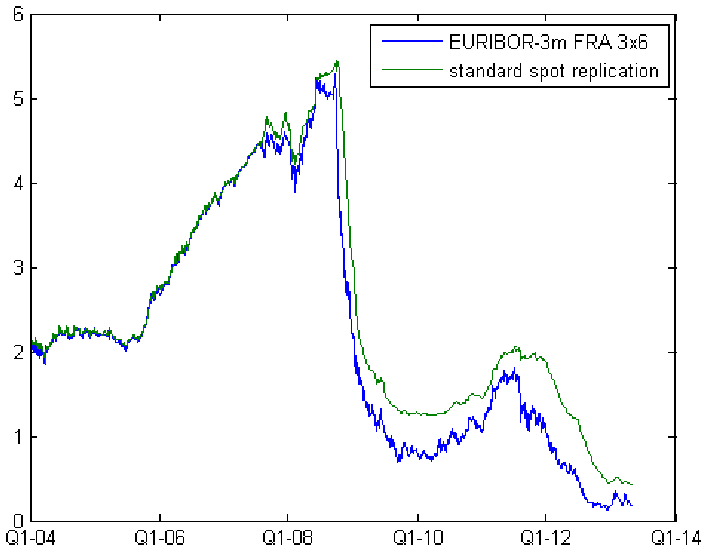

Figure 1.5: EURIBOR 3x6 FRA vs Standard Spot Replication from January 1, 2004 to April 26, 2013.

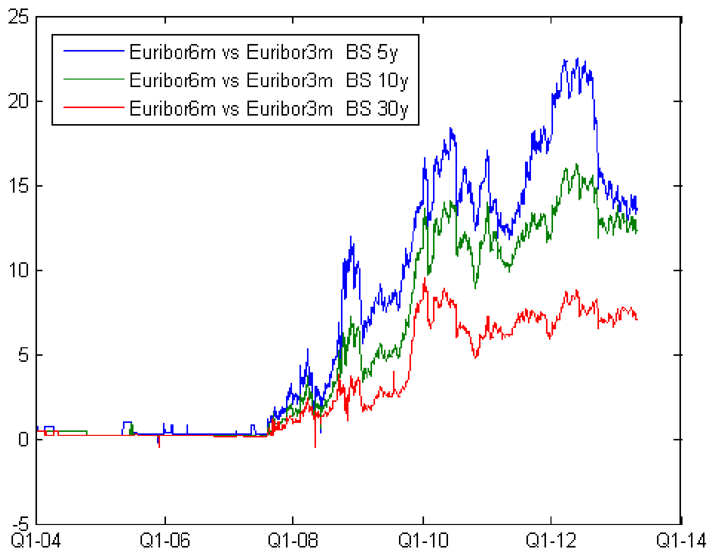

Figure 1.6: EURIBOR 6m vs EURIBOR 3m BS 5y, 10y and 30y from January 1, 2004 to April 26, 2013.

of the same coin was first noted, to the best of our knowledge, by Morini (2009) and we refer to this paper for an explanation.

In this thesis, we do not investigate the economic reasons of these "anomalies", but rather we take an agnostic approach and introduce the spot LIBOR process of some arbitrary but fixed tenor Δ and we refer to it as L^{Δ}. Needless to say, L^{Δ} is allowed to be different from R^{Δ}, but the possibility of having the two to coincide is of course a special case. In other words, we aim at providing a framework where the forward rate implied by two deposits, the corresponding Forward Rate Agreement (FRA) and the forward rate implied by the corresponding OIS quotes should be modeled by a non-negligible spread. Of course, this approach opens the door to a series of non trivial issues since even basic concepts like the construction of zero-coupon curves cannot be longer based on traditional bootstrapping procedures.

Of course the anomalies in the interest market we hinted at have been there for quite a long time now, but very few models to take them into account have been so far published. Since there are no survey papers on the subject available, we find it convenient and useful for the reader to quickly review the existing attempts rather than merely mention them. Before doing so, we review some classical attempts to model the term-structure in a classical single curve framework.

1.4 Single-Curve Term-Structure Modeling

In this section, we present the main existing approaches to the modeling of discrete forward risk-free rates. By *discrete* forward rate we mean a rate that applies to a *strictly* positive accrual period, of which the theoretical forward rates $R^{\Delta}(\cdot, T)$ defined in the previous sections are an example. This must be opposed to the (idealized) concept of *instantaneous* forward rate, which is a forward rate that applies to an *infinitesimal* accrual period. In the non-recent literature, discrete rates were referred to as LIBOR rates and the associated models as LIBOR market models, but we will see that these terms are now inappropriate, if not misleading, since LIBOR rates are to be considered risky. In order to be consistent with the existing literature without being misleading, we will refer to them as "LIBOR" rates and "LIBOR" market models.

In our notation, the rates which are subject to modeling are some $R^{\Delta}(\cdot, T)$ for some Δ's and some T's. We saw that no-arbitrage in the market is equivalent to $R^{\Delta}(\cdot, T)$ being a martingale under the $(T + \Delta)$-forward measure $\mathbb{Q}_{T+\Delta}$. This appears as the only unavoidable property that must be fulfilled by *any* stochastic model. In addition to it, there seem to be two further properties which, though not essential, are of great value:

- the process $R^{\Delta}(\cdot, T)$ must be tractable under as many as possible forward measures
- the process $R^{\Delta}(\cdot, T)$ must be positive.

The first condition is about tractability and is of course made for computational reasons: in fact when evaluating expectations which involve, say, n forward rates (the typical example is the pricing of swaptions), it is clear that one needs to know the law of an n-dimensional process under a single measure: a model in which forward rates are not *all* tractable under *a single* measure is therefore of little interest. The second condition of positivity of rates has attracted different degrees of attention in the literature: some researchers consider it crucial, while others do not even bother about it. These different attitudes probably find their roots in the different environments that major economies have experienced in the last decades, when high interest rate periods in the '90s have left room for record low interest rates in the last years.

The approaches proposed in the literature to model discrete forward rates differ in the following two orthogonal aspects:

- The quantities which are direct object of modeling. Here there are two main alternatives, which are probably more different than it might seem at first glance. The most natural one (and much more developed in the literature) is about modeling directly the forward *rates* $R^{\Delta}(\cdot,T)$. The other possibility is to model the forward *prices* $\frac{P(\cdot,T)}{P(\cdot,T+\Delta)}$ (recall that $\frac{P(\cdot,T)}{P(\cdot,T+\Delta)} = 1+\Delta R^{\Delta}(\cdot,T)$). Broadly speaking, the first approach is more intuitive and more likely to guarantee positivity of rates but also more likely to destroy the analytical tractability of rates under a measure change. The second approach is less intuitive and more likely to produce negative rates but also more likely to preserve the analytical tractability under a measure change.

- The choice of the driving process. In the beginning, research focused on Wiener processes and their (ordinary or stochastic) exponentials. Afterward, Wiener processes left the way to the much more general Levy processes (possibly time-inhomogeneous) and their (ordinary or stochastic) exponentials. Since the "natural environment" for Girsanov-type theorems for changes of measure is that of semimartingales (of which Levy processes are an instance), some attempts have been made to use arbitrary semimartingales as driving processes. This latter framework is, needless to say, purely theoretical but, as we will expound later, in some sense it might even be considered the "right" (i.e. elegant) one, since the class of semimartingales, unlike the class of Levy processes, is closed under the measure changes we will perform: in other words, it might happen that one starts with a process which is Levy under a measure but that is only a semimartingale under a different measure. Another interesting class of processes which has been considered is that of so-called (exponential of) affine processes. Affine processes are, roughly speaking, $\mathbb{R}^d$-valued time-homogenous Markov processes whose semigroup satisfies $\int_{\mathbb{R}^d} P_t(x,dy)e^{<u,y>} = e^{\phi_t(u)+<\psi_t(u),x>}$ for all suitable u's.

In this chapter, we will go over the main approaches which have been proposed in the literature so far. We fix a finite set of dates $T_0, T_1, T_2, \dots, T_n, T_{n+1}$ and, for the sake of simplicity, we assume they are equally spaced of Δ units of time. Recall once more the definition of the Δ-tenor time-t (theoretical) forward rate for maturity T associated with the curve P:

$$R^{\Delta}(t,T) := \frac{1}{\Delta}\left(\frac{P(t,T)}{P(t,T+\Delta)} - 1\right).$$

We assume the existence of a finite family of risk-free bonds

$$P(\cdot,T_0), P(\cdot,T_1), \dots, P(\cdot,T_{n+1}),$$

that define a family of forward rates

$$R^{\Delta}(\cdot,T_0), R^{\Delta}(\cdot,T_1), \dots, R^{\Delta}(\cdot,T_n).$$

Note that here we are using the fact that the dates are equally spaced, otherwise we should write $R^{T_1-T_0}(\cdot,T_0), R^{T_2-T_1}(\cdot,T_1), \dots, R^{T_{n+1}-T_n}(\cdot,T_n)$.

For ease of notation, we will drop the tenor in the rate and write simply $R(t,T)$ for $R^{\Delta}(t,T)$.

We define

$$F(t,T,S) := \frac{P(t,T)}{P(t,S)}$$

and call it the forward price process. Note that it is defined for both $T < S$ and $T > S$.

Recall that $\mathbb{Q}_T$ (the T-forward measure) is defined as the martingale measure associated to the numeraire $P(\cdot,T)$ and that $\frac{d\mathbb{Q}_T}{\mathbb{Q}_S}|\mathcal{F}_t = \frac{F(t,T,S)}{F(0,T,S)}$. Note that this measure is defined only on the σ-algebras $\mathcal{F}_t$ for $t < T$. The goal is to produce, *under a single measure* (generally the "terminal" measure $\mathbb{Q}_{T_{n+1}}$), an n-dimensional model for all the forward rates such that $R(\cdot,T_k)$ is a martingale under $\mathbb{Q}_{T_k}$ for all $k = 0,1,\dots,n$. Note that we emphasize that the model has to be given under a single measure, since it would be trivial, but rather useless, to produce martingales under different measures.

1.4.1 Levy "LIBOR" Models

Here we describe a general approach proposed in Eberlein and Raible (1999) (see also the references therein). We assume we are given a collection of bounded deterministic volatility functions $\sigma(\cdot,T_k)$ for $k = 0,1,\dots n$. Let W^{n+1} be a Wiener process on $\mathbb{R}$ under the measure $\mathbb{Q}_{T_{n+1}}$. Further let J be a Poisson random measure on $[0,\infty)\times\mathbb{R}$ with mean measure

$$A \mapsto \nu^{n+1}(A) := \int_{[0,\infty)}\int_{\mathbb{R}} \mathbb{I}_A(t,x)\lambda_t(dx)dt$$

always under the measure $\mathbb{Q}_{T_{n+1}}$. Here λ_t is a Levy measure for all t's, with the property that

$$\int_{[0,T_{n+1}]}\int_{\mathbb{R}} (x^2 \wedge 1)\lambda_t(dx)dt < \infty$$

Furthermore, we assume that the following integrability condition holds for all real u's:

$$\int_{[0,T_{n+1}]}\int_{|x|>1} e^{ux}\lambda_t(dx)dt < \infty \tag{1.4.1}$$

Define L^{n+1} as follows:

$$L_t^{n+1} = \int_0^t b_s^{n+1} ds + \int_0^t c_s^{\frac{1}{2}} dW_s^{n+1} + \int_{[0,t]\times\mathbb{R}} x(J-\nu^{n+1})(ds,dx),$$

where b^{n+1} is a integrable (deterministic) drift to be specified later and c is a square integrable (deterministic) function. The process L is manifestly a time-inhomogeneous Levy process with triplet $(b_., c_., \lambda_.)$ with respect to the truncation function $x \mapsto |x|$, which is well defined thanks to our assumption (1.4.1), that guarantees integrability (and existence of exponential moments). Note that we restrict ourselves to a one-dimensional driving process only for simplicity, the multidimensional extension being quite straightforward.

We postulate that

$$R(t,T_n) = R(0,T_n)e^{\int_0^t \sigma(s,T_n)dL_s^{n+1}}.$$

The following proposition is exactly what we need in order to make $R(\cdot,T_n)$ a martingale.

Proposition 1.4.1. *Let L be a time-inhomogeneous Levy process on $\mathbb{R}$ with finite exponential moments, so that there exists a function (the "cumulant generating function") $\phi : [0,\infty)\times\mathbb{R}\to\mathbb{R}$ such that*

$$\mathbb{E}(e^{rL_t}) = e^{\int_0^t \phi_s(r)ds} \quad \forall r\in\mathbb{R}.$$

Then for any continuous bounded $r : [0,\infty)\to\mathbb{R}$, the process

$$\left(e^{\int_0^t r(s)dL_s - \int_0^t \phi_s(r(s))ds}\right)_t$$

is a martingale.

The idea of the proof is extremely simple: the result is true almost by definition for constant $r(\cdot)$ and carries over easily to the case in which $r(\cdot)$ is a step function. The technical difficulty is to show the L^1-convergence of (the exponential of) the stochastic integral of the step functions approximating $r(\cdot)$ to (the exponential of) the stochastic integral of $r(\cdot)$. For details we refer to Eberlein and Raible (1999).

To ensure that $R(\cdot,T_n)$ is a martingale under $\mathbb{Q}_{T_{n+1}}$, we impose the sufficient condition

$$\int_0^t \sigma(s,T_n)b_s^{n+1}ds = -\frac{1}{2}\int_0^t \sigma^2(s,T_n)c_s ds \tag{1.4.2}$$

$$-\int_{[0,t]\times\mathbb{R}}\left(e^{\sigma(s,T_n)x} - 1 - \sigma(s,T_n)x\right)\nu^{n+1}(ds,dx). \tag{1.4.3}$$

It is possible to write $R(\cdot,T_n)$ as a stochastic exponential, namely

$$R(t,T_n) = R(0,T_n)\mathcal{E}_t(H(\cdot,T_{n+1}))$$

or in other words $R(\cdot, T_n)$ satisfies

$$dR(t, T_n) = R(t, T_n)dH(t, T_{n+1}),$$

where

$$H(t, T_{n+1}) = \int_0^t \sigma(s, T_n)c_s^{\frac{1}{2}} dW_s^{n+1} + \int_{[0,t]\times\mathbb{R}} (e^{\sigma(s,T_n)x} - 1)(J - \nu^{n+1})(ds, dx)$$

Let us point out that it would be indeed possible to postulate a model in the stochastic exponential form. In this case, though, it is not automatic that the process stays positive. A sufficient condition for strict positivity would be that the Levy measure charges only the interval $(-1, \infty)$.

Since the class of Levy processes is rather large, we feel now compelled to give two very simple special cases which are included in this model, the first one of which will serve also to show that this is indeed a generalization of the classical "LIBOR" Market models.

Example 1.4.2 (Pure Wiener process)**.** Take the function $\sigma(\cdot, T_n)$ to be a constant, i.e. $\sigma(t, T_n) = \sigma \quad \forall t$. Then take $c = 1$ and $\lambda = 0$. This means L^{n+1} is a Wiener process with drift: in particular, the martingale condition on the drift (1.4.2) reads

$$\int_0^t \sigma(s, T_n)b_s^{n+1}ds = -\frac{1}{2}\sigma^2 t$$

so that we have

$$R(t, T_n) = R(0, T_n)e^{\sigma W_t - \frac{1}{2}\sigma^2 t}$$

and

$$H(t, T_{n+1}) = \sigma W_t.$$

This is the classical constant volatility geometric Brownian motion example.

The second simple example is that of a Poisson process.

Example 1.4.3 (Pure Poisson process)**.** Again, take $\sigma(\cdot, T_n)$ a constant, i.e. $\sigma(t, T_n) = \sigma \quad \forall t$. Then take $c = 0$ and $\lambda = \rho\delta_1$, where δ_x is the Dirac measure sitting at x. This means L^{n+1} is a compensated Poisson process with intensity ρ and some drift. We define $N_t = J([0, t] \times \mathbb{R})$, which is finite a.s. since the λ is a finite measure. Now, the martingale condition on the drift (1.4.2) reads

$$\sigma \int_0^t b_s^{n+1}ds = -\rho(e^\sigma - 1 - \sigma)t,$$

so that we have

$$R(t, T_n) = R(0, T_n)e^{\sigma(N_t - \rho t) - \rho(e^\sigma - 1 - \sigma)t} = R(0, T_n)e^{\sigma N_t - \rho(e^\sigma - 1)t}$$

and

$$H(t, T_{n+1}) = (e^\sigma - 1)(N_t - \rho t).$$

In this case, the parameter σ is to be intended as the log-variation of the forward rates at jump times of N. Of course, this is an extremely simplified example, since all the jumps of the driving process L^{n+1} are of the same deterministic size.

It is clear that the likelihood ratio process, call it Λ^{n+1}, between the T_n-forward measure and the T_{n+1}-forward measure is given by

$$\Lambda_t^{n+1} := \frac{d\mathbb{Q}_{T_n}}{d\mathbb{Q}_{T_{n+1}}}|\mathcal{F}_t = 1 + \Delta R(t, T_n)$$

and, since we know the SDE satisfied by $R(\cdot, T_n)$ we can compute the stochastic differential of Λ^{n+1} as follows

$$d\Lambda_t^{n+1} = \Delta dR(t, T_n) = \Lambda_t^{n+1} \frac{\Delta R(t, T_n)}{1 + \Delta R(t, T_n)} dH(t, T_{n+1}),$$

i.e.

$$\Lambda^{n+1} = \mathcal{E}\left(\int_0^{\cdot} \frac{\Delta R(s, T_n)}{1 + \Delta R(s, T_n)} dH(s, T_{n+1})\right).$$

By a Girsanov-type theorem we know that

$$W^n := W^{n+1} - \int_0^{\cdot} \frac{\Delta R(s, T_n)}{1 + \Delta R(s, T_n)} \sigma(s, T_n) c_s^{\frac{1}{2}} ds$$

is a $\mathbb{Q}_{T_n}$-Wiener process and that the $\mathbb{Q}_{T_n}$-compensator of J is

$$\nu^n(dt, dx) := \left(1 + \frac{\Delta R(t, T_n)}{1 + \Delta R(t, T_n)} (e^{\sigma(t, T_n)x} - 1)\right) \nu^{n+1}(dt, dx).$$

Let us now investigate how this change of measure works in our two simplified examples.

Example 1.4.4 (Pure Wiener process)**.** In the pure Wiener example we have that

$$d\Lambda_t^{n+1} = \Lambda_t^{n+1} \frac{\Delta R(t, T_n)}{1 + \Delta R(t, T_n)} \sigma dW_t^{n+1}$$

so that

$$W^n = W^{n+1} - \int_0^{\cdot} \frac{\Delta R(s, T_n)}{1 + \Delta R(s, T_n)} \sigma ds$$

is a Wiener under $\mathbb{Q}_{T_n}$. Note that, even though we took σ to be deterministic and constant, the process W^n has a stochastic time-varying drift.

Example 1.4.5 (Pure Poisson process)**.** In the pure Poisson example we have that

$$d\Lambda_t^{n+1} = \Lambda_t^{n+1} \frac{\Delta R(t, T_n)}{1 + \Delta R(t, T_n)} (e^{\sigma} - 1) d(N_t - \rho t)$$

so that the compensator of N under $\mathbb{Q}_{T_n}$ is

$$\int_0^{\cdot} \left(1 + \frac{\Delta R(s, T_n)}{1 + \Delta R(s, T_n)} (e^{\sigma} - 1)\right) \rho ds.$$

Note that, even though we started from a simple Poisson process and took σ to be deterministic and constant, under $\mathbb{Q}_{T_n}$ the process N is of course still a counting process but has a stochastic compensator, i.e. cannot have independent increments.

Going back to the general case, we have now at our disposal a $\mathbb{Q}_{T_n}$-Wiener and the $\mathbb{Q}_{T_n}$-compensator of J. We define the process L^n in the natural way, namely

$$L_t^n = \int_0^t b_s^n ds + \int_0^t c_s^{\frac{1}{2}} dW_s^n + \int_{[0,t]\times\mathbb{R}} x(J-\nu^n)(ds,dx).$$

Note that this is not, in general, a Levy process under the measure $\mathbb{Q}_{T_n}$, even though it is still a semimartingale. Now we postulate that

$$R(t,T_{n-1}) = R(0,T_{n-1})e^{\int_0^t \sigma(s,T_{n-1})dL_s^n}.$$

To ensure that $R(\cdot,T_{n-1})$ is a martingale under $\mathbb{Q}_{T_n}$, we impose the sufficient condition

$$\begin{aligned}\int_0^t \sigma(s,T_{n-1})b_s^n ds = &-\frac{1}{2}\int_0^t \sigma^2(s,T_{n-1})c_s^{\frac{1}{2}} ds \\ &- \int_{[0,t]\times\mathbb{R}} \left(e^{\sigma(s,T_{n-1})x} - 1 - \sigma(s,T_{n-1})x\right)\nu^n(ds,dx).\end{aligned}$$

It is possible to write $R(\cdot,T_{n-1})$ as a stochastic exponential, namely

$$R(t,T_{n-1}) = R(0,T_{n-1})\mathcal{E}_t(H(\cdot,T_n))$$

or in other words $R(\cdot,T_{n-1})$ satisfies

$$dR(t,T_{n-1}) = R(t,T_{n-1})dH(t,T_n),$$

where

$$H(t,T_n) = \int_0^t \sigma(s,T_{n-1})c_s^{\frac{1}{2}} dW_s^n + \int_{[0,t]\times\mathbb{R}} (J-\nu^n)(ds,dx)(e^{\sigma(s,T_{n-1})x}-1).$$

We are now in position to derive the SDE satisfied by the likelihood ratio process Λ^n. First recall that

$$\Lambda_t^n := \frac{d\mathbb{Q}_{T_{n-1}}}{\mathbb{Q}_{T_n}}|\mathcal{F}_t = 1 + \Delta R(t,T_{n-1}).$$

Now we have that

$$d\Lambda_t^n = \Lambda_t^n \frac{\Delta R(t,T_{n-1})}{1+\Delta R(t,T_{n-1})} dH(t,T_n).$$

By a Girsanov-type theorem we know that

$$W^{n-1} := W^n - \int_0^{\cdot} \frac{\Delta R(s,T_{n-1})}{1+\Delta R(s,T_{n-1})}\sigma(s,T_{n-1})c_s^{\frac{1}{2}} ds$$

is a $\mathbb{Q}_{T_{n-1}}$-Wiener process and that the $\mathbb{Q}_{T_{n-1}}$-compensator of J is

$$\nu^{n-1}(dt,dx) := \left(1 + \frac{\Delta R(t,T_{n-1})}{1+\Delta R(t,T_{n-1})}(e^{\sigma(t,T_{n-1})x}-1)\right)\nu^n(dt,dx).$$

It is now clear how this procedure can be iterated recursively. As soon as we have a $\mathbb{Q}_{T_k}$-Wiener process and the $\mathbb{Q}_{T_k}$-compensator of J, we define the process L^k with a drift b^k to be specified. Then we postulate that $R(\cdot, T_{k-1})$ is the exponential of (an integral transform of) L^k where we pin down the drift in order to obtain a martingale. We find the SDE satisfied by $R(\cdot, T_{k-1})$, which amounts to going from an ordinary exponential to a stochastic exponential, and then the SDE satisfied by Λ^k which is the $\mathbb{Q}_{T_k}$-martingale that defines the measure $\mathbb{Q}_{T_{k-1}}$. By exploiting a Girsanov-type theorem, we are able to find a $\mathbb{Q}_{T_{k-1}}$-Wiener and the $\mathbb{Q}_{T_{k-1}}$-compensator of J and we can continue to the next iteration.

Note that every forward rate $R(\cdot, T_{k-1})$ is *defined* in terms of the process L^k, which is in turn made up by the $\mathbb{Q}_{T_k}$-Wiener W^k and the jump measure J compensated by its $\mathbb{Q}_{T_k}$-compensator ν^k. It is straightforward but crucial to be able to derive an expression for $R(\cdot, T_{k-1})$ that involves only W^n and J: thus to simulate the law of $R(\cdot, T_{k-1})$ under the terminal measure one should simulate the terminal Wiener W^n plus a stochastic drift and the random measure J (which is Poisson under the terminal measure) compensated with a the stochastic compensator ν^k.

Let us now briefly investigate how this model behaves in terms of the two points we made in the previous section, namely the preservation of tractability and the positivity of rates.

With regard to the positivity of rates there is not much to say, in that every forward rate is guaranteed to stay positive.

However, the tractability of forward rates under measures different from their natural one (typically the terminal one) is very limited. First of all the processes L^k's are not even Levy processes under their "own" measures. Second, and more importantly, they are not Levy processes under the terminal measure. Otherwise stated, the tractability we assumed for the $R(\cdot, T_n)$ under its natural measure does not carry over to other rates. Forward price models analyzed in the next section were introduced in the literature with the aim of overcoming this problem.

1.4.2 Levy Forward Price Models

In this subsection, we outline the findings of Eberlein and Özkan (2005) (see also the references therein). Assume we are given the same process L^{n+1} of the previous section and some volatility functions $\sigma(\cdot, T_k)$ for $k = 0, 1, \dots n$. These volatility functions might be (and typically are) different from those of the previous section, even though with a slight abuse of notation we call them in the same way. In the Levy "LIBOR" model, we used the process L^{n+1} to define the forward *rate* process $R(\cdot, T_n)$. In the Levy forward price model, we use it to define the forward *price* process $F(\cdot, T_n, T_{n+1})$ as follows

$$F(t, T_n, T_{n+1}) = F(0, T_n, T_{n+1}) e^{\int_0^t \sigma(s, T_n) dL_s^{n+1}}.$$

Since $F(\cdot, T_n, T_{n+1})$ is the ratio of two traded assets, it must be a martingale under $\mathbb{Q}_{T_{n+1}}$, i.e. the martingale measure associated to the asset in the denominator of the ratio. To ensure this property we make exactly the same assumption we made in the Levy "LIBOR" model, which we recall

$$\int_0^t \sigma(s, T_n) b_s^{n+1} ds = -\frac{1}{2}\int_0^t \sigma^2(s, T_n) c_s^{\frac{1}{2}} ds \\ - \int_{[0,t]\times\mathbb{R}} \nu^{n+1}(ds, dx)\left(e^{\sigma(s,T_n)x} - 1 - \sigma(s, T_n)x\right).$$

As it was possible to express the forward rate as a stochastic exponential, it is now possible to do the same for the forward price, namely

$$F(t, T_n, T_{n+1}) = F(0, T_n, T_{n+1})\mathcal{E}_t(H(\cdot, T_{n+1})),$$

where

$$H(t, T_{n+1}) = \int_0^t \sigma(s, T_n) c_s^{\frac{1}{2}} dW_s^{n+1} + \int_{[0,t]\times\mathbb{R}} (e^{\sigma(s,T_n)x} - 1)(J - \nu^{n+1})(ds, dx).$$

Up to now the two models are basically the same. At this point, in the forward rate model, it was necessary to find the SDE satisfied by $\Lambda^{n+1} = \frac{d\mathbb{Q}_{T_n}}{\mathbb{Q}_{T_{n+1}}}|\mathcal{F}$. In this case, however, we have that

$$\Lambda_t^{n+1} = \frac{F(t, T_n, T_{n+1})}{F(0, T_n, T_{n+1})},$$

so that

$$\Lambda_t^{n+1} = \mathcal{E}_t(H(\cdot, T_{n+1})).$$

Therefore, in the forward price model, the likelihood ratio process between the last two (actually by *any* two) forward measures is the stochastic exponential of H (which is a time-inhomogeneous Levy process) and not, as it was the case in the forward rate model, of some integral transform of H. This is manifestly *the* difference between the two models. We can now use a Girsanov-type theorem to find out a $\mathbb{Q}_{T_n}$-Wiener process W^n defined as

$$W^n := W^{n+1} - \int_0^{\cdot} \sigma(s, T_n) c_s^{\frac{1}{2}} ds$$

and the $\mathbb{Q}_{T_n}$-compensator of J, ν^n, which is defined as

$$\nu^n(dt, dx) := e^{\sigma(t,T_n)x}\nu^{n+1}(dt, dx).$$

Note that $W^{n+1} - W^n$ is deterministic and that the ν^n is a deterministic measure. Neither of these results were true in the forward rate model.

Always in strict analogy with the forward rate process, we now define the process L^n

as

$$L_t^n = \int_0^t b_s^n ds + \int_0^t c_s^{\frac{1}{2}} dW_s^n + \int_{[0,t]\times\mathbb{R}} x(J - \nu^n)(ds, dx).$$

This process is now a time-inhomogeneous Levy process, and this should be compared with the previous section in which this was not the case. The most important point is that L^n is time-inhomogeneous Levy under the terminal measure $\mathbb{Q}_{T_n}$ as well: the tractability of L^{n+1} under the terminal measure is now shared by L^n and by all the L^k's.

Now everything proceeds as in the previous section: we define the process $F(\cdot, T_{n-1}, T_n)$ in the natural way, namely

$$F(t, T_{n-1}, T_n) = F(0, T_{n-1}, T_n) e^{\int_0^t \sigma(s, T_{n-1}) dL_s^n}$$

and all the steps can be carried out iteratively.

The main advantage of this approach with respect to that (more classical and more in line with the previous literature) of the the previous section is now clear and concerns the possibility of having all the forward prices (and thus forward rates) tractable under any forward measure (and in particular under the terminal one).
Let us mention another strong point of this approach: it is not difficult to show that this model can be embedded into a richer one, which we might call Levy HJM model, in which the whole term-structure of bond prices is modeled. In other words, for any choice of the volatility functions in the Levy forward price model, it is possible to find a volatility structure in the Levy HJM model that produces the same (finite-dimensional) forward price process. Since in the Levy HJM model several valuation formulas are available, they simply carry over to the Levy forward rate model.

On the other hand, it is clear that there is no straightforward way to guarantee positivity of rates.

1.4.3 Affine Forward Price Models

In the affine forward price model, introduced by Keller-Ressel et al. (2013), the modeled quantities are the forward prices as in the Levy forward price model that we described in the last section. In particular we postulate the dynamics of the following forward prices

$$\frac{P(\cdot, T_0)}{P(\cdot, T_{n+1})}, \frac{P(\cdot, T_1)}{P(\cdot, T_{n+1})}, \ldots, \frac{P(\cdot, T_n)}{P(\cdot, T_{n+1})}.$$

Note that the forward prices subject to direct modeling are not the same as those in the Levy forward rate model, in that the denominator in all the ratios is the bond with the longest maturity.

We let $(X_t)_{t\in[0,T_{n+1}]}$ be a time-homogeneous affine process on $\mathbb{R}_+^d$ under the measure $\mathbb{Q}_{T_{n+1}}$. We define

$$\mathcal{I}_t := \left\{ u \in \mathbb{R}^d : E^{T_{n+1}}(e^{<u, X_t>}) < \infty \right\}$$

and assume the existence of a neighborhood of $(0,\dots,0)$ contained in $\mathcal{I}_{T_{n+1}}$. We recall that an affine process is a stochastically-continuous Markov process that satisfies

$$\mathbb{E}\left(e^{<u,X_T>}|\mathcal{F}_t\right) = e^{\phi(T-t,u)+<\psi(T-t,u),X_t>}$$

for some $\phi : [0,T_{n+1}] \times \mathcal{I}_T \mapsto \mathbb{R}$ and $\psi : [0,T_{n+1}] \times \mathcal{I}_T \mapsto \mathbb{R}^d$ for all $(t,u,x) \in [0,T_{n+1}] \times \mathcal{I}_T \times \mathbb{R}^d$. The initial value of X does not matter for our purposes and we fix it arbitrarily to $(1,\dots,1)$.
Let us recall the following

Proposition 1.4.6. *The functions ϕ and ψ satisfy the following semi-flow property*

$$\begin{aligned}\phi(t+s,u) &= \phi(t,u) + \phi(s,\psi(t,u)),\\ \psi(t+s,u) &= \psi(s,\psi(t,u)).\end{aligned}$$

Proof. Simply note that by the Markov property we have, for any u,

$$\begin{aligned}&e^{\phi(t+s,u)+<\psi(t+s,u),X_0>} = \mathbb{E}\left[e^{<u,X_{t+s}>}|X_0\right]\\ &= \mathbb{E}\left[\mathbb{E}[e^{<u,X_{t+s}>}|X_s]|X_0\right] = \mathbb{E}\left[e^{\phi(t,u)+<\psi(t,u),X_s>}|X_0\right]\\ &= e^{\phi(t,u)+\phi(s,\psi(t,u))+<\psi(s,\psi(t,u)),X_0>}\end{aligned}$$

□

We define $M_t^u := E^{T_{n+1}}\left(e^{<u,X_{T_{n+1}}>}|\mathcal{F}_t\right)$ for all $u \in \mathcal{I}_{T_{n+1}}$. The M^u's are manifestly martingales under $\mathbb{Q}_{T_{n+1}}$ and positivity of X implies that if $u \in \mathbb{R}^d_+ \cap \mathcal{I}_{T_{n+1}}$ then $M^u \geq 1$ a.s..
We postulate that

$$\frac{P(t,T_k)}{P(t,T_{n+1})} = M_t^{u_k} \quad \forall k \in \{0,1,\dots,n\}.$$

The u_k's are chosen in such a way that we have

$$M_0^{u_k} = \frac{P^*(0,T_k)}{P^*(0,T_{n+1})} \quad \forall k \in \{0,1,\dots,n\}, \tag{1.4.4}$$

where $T \mapsto P^*(0,T)$ is a given initial term structure. We now give a sufficient condition for the existence of such a sequence $(u_k)_{k\in\{0,1,\dots,n\}}$

Proposition 1.4.7. *If the initial forward rates are positive and the following condition holds*

$$\exists u \in \mathbb{R}^d_+ \cap \mathcal{I}_{T_{n+1}} : E^{T_{n+1}}(e^{<u,X_T>}) > \frac{P^*(0,T_0)}{P^*(0,T_{n+1})}$$

then there exists a sequence $(u_k)_{k\in\{0,1,\dots,n\}}$ such that (1.4.4) is satisfied.

Proof. Positivity of forward rates implies that

$$\frac{P^*(0,T_0)}{P^*(0,T_{n+1})} \geq \frac{P^*(0,T_1)}{P^*(0,T_{n+1})} \geq \dots \geq \frac{P^*(0,T_n)}{P^*(0,T_{n+1})}.$$

Now, the condition in the hypothesis implies that we can find some u^* in $\mathcal{I}_{T_{n+1}}$ such that

$$\mathbb{E}\left[e^{<u^*,X_{T_{n+1}}>}\right] > \frac{P^*(0,T_0)}{P^*(0,T_{n+1})}.$$

Now simply note that the map f defined by

$$[0,1] \ni \xi \mapsto M_0^{\xi u} \in \mathbb{R}_+$$

is increasing and continuous and satisfies $f(0) = 1$ and $f(1) > \frac{P^*(0,T_0)}{P^*(0,T_{n+1})}$ and the result follows. □

The no-arbitrage condition that all the ratios must be martingales under the T_{n+1}-forward measure is clearly satisfied by construction.
However, it is clear that the forward prices we are interested in are not those of the type $\frac{P(\cdot,T_k)}{P(\cdot,T_{n+1})} = M^{u_k}$, but rather those of the type $\frac{P(\cdot,T_k)}{P(\cdot,T_{k+1})}$. It is however straightforward to work out the expression of the latter forward price as follows

$$\begin{aligned}
\frac{P(t,T_k)}{P(t,T_{k+1})} &= \frac{M_t^{u_k}}{M_t^{u_{k+1}}} \\
&= \exp\left(\phi(T_{n+1}-t,u_k) - \phi(T_{n+1}-t,u_{k+1}) + <\psi(T_{n+1}-t,u_k) - \psi(T_{n+1}-t,u_{k+1}), X_t>\right) \\
&= \exp\left(A_{T_{n+1}-t}(u_k,u_{k+1}) + <B_{T_{n+1}-t}(u_k,u_{k+1}), X_t>\right)
\end{aligned}$$

where we define

$$\begin{aligned}
A_t(u,v) &:= \phi(t,u) - \phi(t,v) \\
B_t(u,v) &:= \psi(t,u) - \psi(t,v)
\end{aligned}$$

The last thing we need to check is that the forward rates $R(\cdot,T_k)$'s are martingales under their "own" measures, as we now do.

Proposition 1.4.8. *For each $k \in \{0,1,\cdots,n\}$, the process $\frac{P(\cdot,T_k)}{P(\cdot,T_{k+1})}$ is a martingale under $\mathbb{Q}_{T_{k+1}}$. As a consequence $R(\cdot,T_k)$ is a martingale under the same measure.*

Proof. First of all recall that M^{u_k} is a $\mathbb{Q}_{T_{n+1}}$-martingale (if and) only if $\left(M_t^{u_k}\left(\frac{d\mathbb{Q}_{T_{n+1}}}{d\mathbb{Q}_{T_{k+1}}}|\mathcal{F}_t\right)\right)_t$ is a $\mathbb{Q}_{T_{k+1}}$-martingale[11]. Since

$$\frac{d\mathbb{Q}_{T_{n+1}}}{d\mathbb{Q}_{T_{k+1}}}|\mathcal{F}_t = \frac{M_0^{u_{k+1}}}{M_t^{u_{k+1}}},$$

we have that $\left(\frac{M_t^{u_k}}{M_t^{u_{k+1}}}\right)_t$ is a martingale under $\mathbb{Q}_{T_{k+1}}$. The first assertion now follows since, as we already noted,

$$\frac{M_t^{u_k}}{M_t^{u_{k+1}}} = \frac{P(t,T_k)}{P(t,T_{k+1})}.$$

The second assertion follows directly from the definition of the forward rate. □

[11]This is a direct consequence of the so-called abstract Bayes rule.

To conclude, the last proposition investigates the behavior of the process X under measures different the terminal one.

Proposition 1.4.9. *The process X is still a (in general, time-inhomogeneous) affine process under* any *forward measure* $\mathbb{Q}_{T_{n+1}}$

Proof. Let us arbitrarily fix $t < T < T_{n+1}$ and compute the $\mathcal{F}_t$-conditional moment generating function of X_T under the generic measure $\mathbb{Q}_{T_k}$:

$$
\begin{aligned}
&E^{T_k}(e^{<v,X_T>}|\mathcal{F}_t)\\
=&E^{T_{n+1}}\left(\frac{M_T^{u_k}}{M_t^{u_k}}e^{<v,X_T>}|\mathcal{F}_t\right)\\
=&\frac{1}{M_t^{u_k}}E^{T_{n+1}}(e^{\phi(T_{n+1}-T,u_k)+<\psi(T_{n+1}-T,u_k)+v,X_T>}|\mathcal{F}_t)\\
=&\exp\left\{\phi(T_{n+1}-T,u_k)-\phi(T_{n+1}-t,u_k)+\phi(T-t,\psi(T_{n+1}-T,u_k)+v)\right\}\\
&\cdot\exp\left\{<\psi(T-t,\psi(T_{n+1}-T,u_k)+v)-\psi(T_{n+1}-t,u_k),X_t>\right\}\\
=&\exp\left\{\phi(T-t,\psi(T_{n+1}-T,u_k)+v)-\phi(T-t,\psi(T_{n+1}-T,u_k))\right\}\\
&\cdot\exp\left\{<\psi(T-t,\psi(T_{n+1}-T,u_k)+v)-\psi(T-t,\psi(T_{n+1}-T,u_k)),X_t>\right\},
\end{aligned}
$$

where in the last equality we used the "semiflow" property of ϕ and ψ (see proposition 1.4.6). This shows that the conditional moment generating function is the exponential of an affine function of X_t (in general, depending separately on t and T and not only on their difference) and the result follows. □

This model appears to meet both the desirable properties we outlined in the introduction.

Positivity of the process X guarantees that all the M^{u_k} are greater than 1, which is equivalent to the forward rates being positive. Note that if we drop the assumption that X is positive, then the Levy forward price model of the previous section becomes almost a special case of this model, since a Levy process is obviously affine. We say "almost", because, as we already pointed out, in the present section the quantities which are modeled directly are the forward prices with the longest-maturing bond as denominator, which was not the case in the previous section.

With regard to the tractability of the model, we stress again that the dynamics of the process X and consequently of all the forward prices and rates are initially given *directly* under the terminal measure, so that tractability under this measure is obvious. On top of that, we showed that the affine property of X is preserved under a change of measure (losing only the time-homogeneity property), and this is of course extremely important in all the cases in which the terminal measure is not the most natural one to use.

1.5 Multiple-Curve Term-Structure Modeling

In this section, we aim at giving an overview of how the classical approaches for term structure modeling have been adapted so far to cope with the multiple curve framenwork.

Chapters 2 and 3 of this thesis present a contribution to this stream of research.

The classical LIBOR market models (LMMs) (see, e.g., Brace et al. (1997), Jamshidian (1997) and Rutkowski (1999)) have been generalized by Mercurio (2010a) and Mercurio (2010b). In these papers, the approach is to model the riskless forward rates $R(\cdot, T_n)$'s and the forward LIBOR rates, both of discrete tenor. Specifically, having fixed a tenor structure $\{T_0, \dots, T_n\}$, the author postulates a diffusive dynamic for

$$\bar{R} := \{F_{R^{T_1-T_0}}(\cdot, T_0), \dots, F_{R^{T_n-T_{n-1}}}(\cdot, T_{n-1})\}$$

and

$$\bar{L} := \{F_{L^{T_1-T_0}}(\cdot, T_0), \dots, F_{L^{T_n-T_{n-1}}}(\cdot, T_{n-1})\},$$

or alternatively for $\bar{R}$ and the spread $\bar{S} = \bar{L} - \bar{R}$. The simple but crucial observation to restrict the possible dynamics of the processes is that *both* $F_{R^{T_{i+1}-T_i}}(\cdot, T_i)$ and $F_{L^{T_{i+1}-T_i}}(\cdot, T_i)$ must be martingales under the forward measure $\mathbb{Q}_{T_{i+1}}$. A backward induction approach as in Rutkowski (1999) is then used in order to produce consistent dynamics of $\bar{R}$ and $\bar{L}$ under a *single* forward measure.

The *single-curve* approach of Keller-Ressel et al. (2013) is very similar in spirit to the classical LMMs of Brace et al. (1997), but it assigns an explicit dynamic to the forward *prices* instead of the forward *rates*. Specifically, having fixed a tenor structure $\{T_0, \dots, T_n\}$, the authors assume that for $i \in \{1, 2, \dots, n-1\}$

$$M_t^{u_i} = \frac{P(t, T_i)}{P(t, T_n)} = 1 + (T_n - T_i) F_{R^{T_n-T_i}}(\cdot, T_i)$$

is given by

$$M_t^{u_i} = \mathbb{E}_t^{T_n}[e^{u_i \cdot X_{T_n}}],$$

where X is an $\mathbb{R}_+^d$-valued affine Markov process under the measure Q_{T_n} and the u_i are fixed deterministic projection vectors. Then, it is straightforward to check that

$$1 + (T_{i+1} - T_i) F_{R^{T_{i+1}-T_i}}(\cdot, T_i) = \frac{M_t^{u_i}}{M_t^{u_{i+1}}}. \tag{1.5.1}$$

The main advantage of this approach is that the dynamics of forward rates remain tractable under every forward measure $\mathbb{Q}_T$. Following the insight of Mercurio (2010b), this framework has been generalized to the multiple-curve framework in Grbac et al. (2014). Here, the authors model the risk-free forward rates as in (1.5.1) and the forward LIBOR rates as

$$1 + (T_{i+1} - T_i) F_{R^{T_{i+1}-T_i}}(\cdot, T_i) = \frac{M_t^{v_i}}{M_t^{u_{i+1}}} \tag{1.5.2}$$

for some projection vectors $v_1, \dots, v_{n-1}$, a priori different from the u_i. By doing so, the tractability of the model under different forward measure is preserved and, by choosing $v_i \succ u_i$ it is possible to guarantee that $F_{L^{T_{i+1}-T_i}}(\cdot, T_i) > F_{R^{T_{i+1}-T_i}}(\cdot, T_i)$.

A relatively similar approach has been proposed by Cuchiero et al. (2014), the main difference being that the discrete grid is replaced by a continuum of maturities $T \in [0, T^*]$. Specifically, the authors model the continuum of bond prices $(P(\cdot, T))_{T\in[0,T^*]}$ as in a classical HJM framework and, on top of that, the quantity

$$\frac{1+\Delta F_{L^\Delta}(\cdot, T)}{1+\Delta F_{R^\Delta}(\cdot, T)} = \frac{P(\cdot, T+\Delta)(1+\Delta F_{L^\Delta}(\cdot, T))}{P(\cdot, T)}.$$

These quantities, indexed by T, are first assumed to form a family of positive semimartingales and then they are specified to be exponentials of affine processes. This approach is referred to as an HJM approach in the title of Cuchiero et al. (2014), by following the practice (see Carmona and Nadtochiy (2009)) of considering an HJM model any model that evolves a continuum of financial quantities: however this contribution, while outstanding and financially sound, is probably not the closest relative of the HJM approach in the multiple-curve framework.

An approach closer to the original HJM's one is investigated by Crépey et al. (2012), where the authors model the risk-free term structure as in HJM and then introduce some fictitious bonds $(\bar{P}(\cdot, T))_{T\in[0,T^*]}$ such that

$$F_{L^\Delta}(\cdot, T) = \frac{1}{\Delta}\left(\frac{\bar{P}(\cdot, T)}{\bar{P}(\cdot, T+\Delta)} - 1\right).$$

While being closer in spirit to the HJM approach, the essential question of uniqueness of this family of fictitious bonds for a given initial forward LIBOR curve $T \mapsto F_{L^\Delta}(0, T)$ is not discussed. Then, some HJM-style dynamics on the instantaneous forward rates associated with $\bar{P}$ are imposed and the authors first assume that the $\bar{P}$'s are traded in the market, thus getting the classical defaultable HJM drift condition (see, e.g., Bielecki and Rutkowski (2000)). Then they correctly note that the $\bar{P}$'s are not traded instruments, so that the drift condition is relaxed into a more general one, which turns out to be vacous (see equation (29) in the article). We will consider all these issues about non uniqueness of the $\bar{P}$'s and arbitrage-freedom in the HJM approach in Chapter 3.

The short rate approach in the multiple curve setting was first introduced by Kenyon (2010)[12], who models the risk-free discount factors $P(\cdot, T)$ as in the classical short-rate models, i.e.

$$P(t, T) = \mathbb{E}_t\left[e^{-\int_t^T r_u du}\right]$$

and the *forward* LIBORs as

$$F_{L^\Delta}(t, T) := \frac{1}{\Delta}\left(\frac{\bar{P}(t, T)}{\bar{P}(t, T+\Delta)} - 1\right),$$

[12] The model is inspired by the paper of Kijima et al. (2009), not yet written in a modern multiple curve perspective.

where the fictitious bonds $\bar{P}$ are defined as

$$\bar{P}(t,T) := \mathbb{E}_t\left[e^{-\int_t^T (r_u+s_u)du}\right].$$

The dynamics of the processes r and $\bar{r} = r + s$ (s stands for the spread process) are postulated to be two correlated mean reverting Gaussian processes. While the approach is original, it suffers from exactly the same drawback of Crépey et al. (2012) in that the process $F_{L^\Delta}(\cdot, T)$ is not necessarily a $\mathbb{Q}_{T+\Delta}$-martingale, so that the model is not arbitrage-free.

Morino and Runggaldier (2014) take a model similar to that of Kenyon (2010) and Kijima et al. (2009), with one common Vasicek-like factor and 2 idiosyncratic CIR-like processes, all independent. Their model is intrinsically arbitrage free since they use the fictitious bonds to define the *spot* (and not forward) LIBOR as

$$L_t^\Delta := \frac{1}{\Delta}\left(\frac{1}{\bar{P}(t,T+\Delta)} - 1\right).$$

However, their model is intrinsically endogenous in that the initial term structures $T \mapsto P(0,T)$ and $T \mapsto F_{L^\Delta}(0,T)$ are outputs of the model, rather than inputs as they should be. With regard to the first term structure (the riskless one) the problem could be solved by a standard Brigo and Mercurio (2001) approach, while the LIBOR curve requires a non trivial extension of that methodology, which constitutes the main contribution of Chapter 2 of this thesis.

All the contributions reviewed so far take an agnostic approach, in that they do not attempt to explain *why* the old no-arbitrage relations are not satisfied anymore in reality. Morini (2009) has been the first, and so far the only one, to investigate a "structural" approach, based on credit risk, which we do not explain since it is far from what we will develop in this thesis. However, while his results are preliminary and tentative, we deem unfortunate the fact that this possible way-out has not been investigated any further.

Finally, Filipović and Trolle (2013) present an econometric analysis of the issue aimed at assessing the importance of credit risk in it by using data in the CDS market. This study is extremely interesting, though we will not describe it in detail since it is aimed, as said, at giving an econometric explanation of the multiple curve phenomenon and not at pricing interest rate derivatives.

Chapter 2

A Multiple-Curve Instantaneous Spot Rate Model

In this chapter[1], the state variables to which we assign an explicit dynamics are the instantaneous spot rate process $(r_t)_{t\in[0,T^*]}$ and a spread $(s_t^\Delta)_{t\in[0,T^*]}$. The process r will determine the bond price as in the classical theory of short rate modeling (whose most celebrated examples include Vasicek (1977) and Cox et al. (1985)), whereas the process s^Δ will be used to *define* the *spot* LIBOR process $(L_t^\Delta)_t$. The first issue we tackle is, as in any model for the *spot* rate, to determine the forward rates. Since we are modeling the spot LIBOR rate, freedom of arbitrage for the forward rates comes at no cost in this model by the way forward rates are defined as expectation under a forward measure. However, the framework as outlined so far would be an *endogenous* term structure model, in which the initial term structure of bond prices $P(0,\cdot)$ and of forward LIBORs $F_{L^\Delta}(0,\cdot)$ is a model *output*. We turn it into an *exogenous* model in which the initial term structure is an *input* by extending to the multiple-curve framework a deterministic-shift technique proposed (among others) by Dybvig (1997) and Brigo and Mercurio (2001). We then see how affine Markov processes naturally lend themselves to tractable specification of the models and give some concrete examples.

The approach of modeling multiple curves through a "short rate" model was proposed by Morino and Runggaldier (2014). The approach of Kenyon (2010) is apparently similar but is actually more closely related to the approach we develop in the next chapter.

2.1 The Model

Recall that we are holding fixed a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. In this chapter, we take $\mathbb{P} = \mathbb{Q}_*$, so that we are modeling the market directly under the risk-neutral measure. Let $r = (r_t)_{t\in[0,T^*]}$ and $s^\Delta = (s_t^\Delta)_{t\in[0,T^*]}$ be two stochastic processes defined on this space.

[1]This chapter is based on Grasselli and Miglietta (2014)

Let us assume that $e^{-\int_0^t r_u du}$ and $e^{-\int_0^t (r_u + s_u^\Delta) du}$ are well-defined and $\mathbb{Q}^*$-integrable for every $t \in [0, T^*]$.

Since we assume that $\mathbb{Q}_*$ is the martingale measure associated to the numeraire B, the price of the generic T-bond is necessarily given by

$$P(t,T) = B_t \mathbb{E}^*[\frac{1}{B_T}|\mathcal{F}_t] = \mathbb{E}^*[e^{-\int_t^T r_u du}|\mathcal{F}_t].$$

It is obvious that $\frac{P(\cdot,T)}{B}$ is a $\mathbb{Q}_*$-martingale.

The dynamics of the processes $L_\cdot^\Delta$ are postulated by means of the following artifact. We *define* the fictitious bond price process as follows:

$$P^\Delta(t,T) := B_t^\Delta \mathbb{E}^*[\frac{1}{B_T^\Delta}|\mathcal{F}_t] = \mathbb{E}^*[e^{-\int_t^T (r_u + s_u^\Delta) du}|\mathcal{F}_t],$$

where we implicitly defined $B_t^\Delta := e^{\int_0^t (r_u + s_u^\Delta) du}$. It follows trivially that $\frac{P^\Delta(\cdot,T)}{B^\Delta}$ is a $\mathbb{Q}_*$-martingale, but we stress that neither B^Δ nor $P^\Delta(\cdot,T)$ are traded in the market, otherwise no-arbitrage would force s^Δ to be constantly equal to zero.

Now we *define* the spot LIBOR as follows:

$$L_t^\Delta := \frac{1}{\Delta}\left(\frac{1}{P^\Delta(t,t+\Delta)} - 1\right) = \frac{1}{\Delta}\left(\frac{1}{\mathbb{E}^*[e^{-\int_t^{t+\Delta}(r_u + s_u^\Delta)du}|\mathcal{F}_t]} - 1\right).$$

It is clear that we define the spot LIBOR rate L^Δ in this way in order to mimic the definition of the risk-free rate 1.2.1. In the following proposition, we find an (implicit) expression for the forward LIBOR $F_{L^\Delta}(t,T)$.

Proposition 2.1.1. *Under the assumptions above, the forward LIBOR is given by*

$$F_{L^\Delta}(t,T) = \frac{1}{\Delta}\left(\frac{C^\Delta(t,T)}{P(t,T+\Delta)} - 1\right),$$

where

$$C^\Delta(t,T) = \mathbb{E}_t^*\left[e^{-\int_t^T r_u du}\frac{P(T,T+\Delta)}{P^\Delta(T,T+\Delta)}\right] = \mathbb{E}_t^*\left[e^{-\int_t^{T+\Delta} r_u du}\frac{1}{P^\Delta(T,T+\Delta)}\right].$$

Proof. We have that

$$\begin{aligned} F_{L^\Delta}(t,T) &= \frac{\mathbb{E}_t^*\left[e^{-\int_t^{T+\Delta} r_u du}\frac{1}{\Delta}\left(\frac{1}{P^\Delta(T,T+\Delta)} - 1\right)\right]}{P(t,T+\Delta)} \\ &= \frac{\frac{1}{\Delta}\mathbb{E}_t^*\left[e^{-\int_t^T r_u du}P(T,T+\Delta)\left(\frac{1}{P^\Delta(T,T+\Delta)} - 1\right)\right]}{P(t,T+\Delta)} \\ &= \frac{\frac{1}{\Delta}\mathbb{E}_t^*\left[e^{-\int_t^T r_u du}\left(\frac{P(T,T+\Delta)}{P^\Delta(T,T+\Delta)} - P(T,T+\Delta)\right)\right]}{P(t,T+\Delta)} \end{aligned}$$

which yields the result. □

We chose to give the expression for C^Δ in the last proposition under the measure $\mathbb{Q}_*$ since it will be more adapt to our subsequent needs. However, the same quantity can also be represented as an expectation under the $(T+\Delta)$-forward measure, as we show in the next proposition whose proof is straightforward.

Proposition 2.1.2. *An equivalent representation of the factor C^Δ is*

$$C^\Delta(t,T) = P(t,T+\Delta)\mathbb{E}_t^{T+\Delta}\left[\frac{1}{P^\Delta(T,T+\Delta)}\right],$$

so that L^Δ could be equivalently written as

$$F_{L^\Delta}(t,T) = \frac{1}{\Delta}\left(\mathbb{E}_t^{T+\Delta}\left[\frac{1}{P^\Delta(T,T+\Delta)}\right] - 1\right).$$

2.1.1 A simple Representation for the forward LIBOR

It is clear that if the process s is identically equal to zero, then we have that the spot LIBOR rate is equal to the spot risk-free rate, i.e. $L^\Delta = Z^\Delta$ so that a fortiori we have that

$$F_{L^\Delta}(t,T) = F_{R^\Delta}(t,T) = \frac{1}{\Delta}\left(\frac{P(t,T)}{P(t,T+\Delta)} - 1\right)$$

and we are back to the classical single-curve framework.

We now investigate when the simple representation of forward LIBOR

$$F_{L^\Delta}(t,T) = \frac{1}{\Delta}\left(\frac{P^\Delta(t,T)}{P^\Delta(t,T+\Delta)} - 1\right). \tag{2.1.1}$$

holds true.

First we show in the next lemma that (2.1.1) is equivalent to the martingality of a specific process.

Lemma 2.1.3. *The representation* (2.1.1) *is true if and only if the process $\frac{P^\Delta(\cdot,T)}{P^\Delta(\cdot,T+\Delta)}$ is a $\mathbb{Q}_{T+\Delta}$-martingale.*

Proof. Note that

$$F_{L^\Delta}(t,T) = \frac{1}{\Delta}\left(\mathbb{E}_t^{T+\Delta}\left[\frac{P^\Delta(T,T)}{P^\Delta(T,T+\Delta)}\right] - 1\right),$$

so that (2.1.1) is true if and only if

$$\frac{P^\Delta(t,T)}{P^\Delta(t,T+\Delta)} = \mathbb{E}_t^{T+\Delta}\left[\frac{P^\Delta(T,T)}{P^\Delta(T,T+\Delta)}\right],$$

which yields the result. □

Let us now study the case in which the process s is deterministic, which turns out to be a sufficient condition for (2.1.1) as the next proposition shows.

Proposition 2.1.4. *If the process s is deterministic, then the representation* (2.1.1) *holds true*

Proof. If s is deterministic, we have that

$$P^{\Delta}(t,T) = P(t,T)S(t,T),$$

where the deterministic function S is defined as $S(t,T) := e^{-\int_t^T s_u du}$ and enjoys the property

$$S(t,T) = S(t,U)S(U,T) \quad \forall t \leq U \leq T.$$

This can be used to show that $\frac{P^{\Delta}(\cdot,T)}{P^{\Delta}(\cdot,T+\Delta)}$ is a $\mathbb{Q}_{T+\Delta}$-martingale. In fact

$$\frac{P^{\Delta}(t,T)}{P^{\Delta}(t,T+\Delta)} = \frac{P(t,T)S(t,T)}{P(t,T+\Delta)S(t,T+\Delta)} = \frac{P(t,T)}{P(t,T+\Delta)} \frac{1}{S(T,T+\Delta)},$$

which is the product of a $\mathbb{Q}_{T+\Delta}$-martingale times a deterministic term which does not depend on t. Now the result follows by exploiting the last lemma. □

The previous Proposition tells that when the LIBOR rate can be obtained by the risk-free curve through a deterministic shift, all no arbitrage relations valid for the risk-free curve can be translated into the LIBOR curve with the same analogies.

Remark 2.1.5. The converse of Proposition 2.1.4 is indeed false. A counterexample is given by the simple case in which $s^{\Delta} = -r$, with r non-deterministic.

2.2 Calibration in a Markovian Framework

It would be natural, in order to get some explicit formulas for $P(t,T)$ and $F_{L^{\Delta}}(t,T)$, to postulate that (r,s) is a 2-dimensional process driven by some d-dimensional *Markovian* process $(X_t)_t$. In other words, it would be natural, or at least appealing, to take $r_t = \gamma \cdot X_t$ and $s_t = \gamma^{\Delta} \cdot X_t$ for some fixed d-dimensional vectors γ and γ^{Δ}. This approach, while a priori feasible, would suffer from a tremendous drawback, in that the initial term structure of risk-free bonds and forward LIBORs would be an *output* of the model rather than an *input*, as it would be desirable if not mandatory (at least in applications of any use). As a consequence, we circumvent this potential problem by adding a deterministic shift to both r and s^{Δ} in order to match any initial term-structure. By doing so, the model automatically becomes an exogenous term-structure model, rather than an endogenous one. In the single curve case, this path-breaking methodology has been first proposed by Dybvig (1997), Scott (1995) Avellaneda and Newman (1998) and then formalized and extended (among others) by Brigo and Mercurio (2001), so that our approach is an extension of the latter papers in the multiple-curve setting.

Even if one might potentially argue that the initial term structures might be matched by using some parameters in the law of the driving process X, this will never produce a perfect pointwise matching of the whole term structures unless one of the parameter is infinite dimensional: but in such a case we would be back to the approach we are about to describe, which is actually more general and systematic.

In short, the calibration problem of a multiple-curve instantaneous spot rate model is based on the following two ingredients:

- a model consisting of a real time-homogeneous Markov process on $\mathbb{R}^d$, two "projection vectors" and two deterministic shifts;
- a market consisting of an initial term structure of bond prices and forward LIBOR rates.

We now describe the model in more detail.

Let us assume we are given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and that $X = (X_t)_{t\in[0,\infty)}$ is a $\mathbb{R}^d$-valued stochastic processes defined on it enjoying the time-homogeneous Markov property (with respect to $\mathbb{F}$ and $\mathbb{P}$) admitting the transition semigroup $(P_t)_{t\in[0,\infty)}$ acting on $\mathcal{B}(\mathbb{R}^d)_b$ (the space of bounded Borel functions on $\mathbb{R}^d$). This means that X is $\mathbb{F}$-adapted and for any f Borel bounded on $\mathbb{R}^d$ we have

$$\mathbb{E}[f(X_{t+h})|\mathcal{F}_t] = P_h f(X_t) \quad \forall t, h \geq 0.$$

There is no need to assume that $\mathcal{F}_0$ is $\mathbb{P}$-trivial (which would imply that X_0 is $\mathbb{P}$-a.s. a constant), even though this is often the case in practical applications.

We define the mapping $\Pi : \{(t,T) : 0 \leq t \leq T \leq T^*\} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ as follows[2]

$$\Pi(t, T, \gamma, x) := \mathbb{E}[e^{-\int_t^T \gamma\cdot X_u du}|\{X_t = x\}].$$

Thanks to the time-homogeneity of X, it is clear that Π actually depends on its first two "time-arguments" only through their difference. Intuitively, $\Pi(t, T, \gamma, x)$ represents the T-bond price at time t in a model where the instantaneous spot rate is given by $\gamma \cdot X$.

We define the auxiliary mapping

$$\Gamma^\Delta : \{(t,T) : 0 \leq t \leq T \leq T^*\} \times (\mathbb{R}^d \times \mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$$

as follows:

$$\Gamma^\Delta(t, T, \gamma, \gamma^\Delta, x) := \mathbb{E}\left[e^{-\int_t^T \gamma\cdot X_u du} \frac{\Pi(T, T+\Delta, \gamma, X_T)}{\Pi(T, T+\Delta, \gamma+\gamma^\Delta, X_T)}|\{X_t = x\}\right].$$

Again time-homogeneity of X implies that Γ^Δ depends on its first two "time-arguments" only through their difference, but we keep our separate notation for sake of clarity. In-

[2]Here and in the following, $\cdot$ denotes the standard scalar product in $\mathbb{R}^d$.

tuitively, $\Gamma^{\Delta}(t,T,\gamma,\gamma^{\Delta},x)$ can be thought of as the time-t price in a model with instantaneous spot rate $\gamma \cdot X$ and instantaneous spread $\gamma^{\Delta} \cdot X$ of the following portfolio: a $(T+\Delta)$-bond and payment of the spot LIBOR L^{Δ} setting at T and paying at $T+\Delta$.

Finally, we define the mapping $\Lambda^{\Delta} : \{(t,T) : 0 \le t \le T \le T^*\} \times (\mathbb{R}^d \times \mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ as follows:

$$\Lambda^{\Delta}(t,T,\gamma,\gamma^{\Delta},x) := \frac{1}{\Delta}\left(\frac{\Gamma^{\Delta}(t,T,\gamma,\gamma^{\Delta},x)}{\Pi(t,T+\Delta,\gamma,x)} - 1\right).$$

Let us now define the model variables r and s^{Δ} as follows

$$r_t := \theta(t) + \gamma \cdot X_t$$

and

$$s_t^{\Delta} := \theta^{\Delta}(t) + \gamma^{\Delta} \cdot X_t,$$

where γ and γ^{Δ} are two arbitrary constant projection vectors in $\mathbb{R}^d$.

Note that the process (r,s) does not enjoy, in general, the Markov property and when it does, it is a priori time-inhomogeneous.

Recall that $P(t,T)$, $P^{\Delta}(t,T)$ and L_t^{Δ} were defined in terms of r and s^{Δ} in the previous section and, of course, the same definitions do carry over.

The market data is a term structure of risk-free zero coupon bonds $T \mapsto P^{mkt}(0,T)$ and a term structure of forward LIBOR rates $T \mapsto L^{\Delta,mkt}(0,T)$.

The calibration problem can be now stated as follows: for a given Markov process X, projection vectors γ and γ^{Δ} as described above and for any given initial term structure P^{mkt} and $L^{\Delta,mkt}$, find two deterministic shift functions θ and θ^{Δ} such that the equalities

$$P(0,T) = P^{mkt}(0,T) \tag{2.2.1}$$

and

$$F_{L^{\Delta}}(0,T) = L^{\Delta,mkt}(0,T) \tag{2.2.2}$$

hold *for every* $T \in [0,T^*]$. Incidentally, note that equality (2.2.2) on the *forward* LIBOR when $T=0$ actually concerns the *spot* LIBOR L_0^{Δ}, so that there is no need to calibrate the model separately to the current spot LIBOR.

The following simple lemma will be exploited to show that the calibration problem has always a solution.

Lemma 2.2.1. *The $\mathcal{F}_t$-measurable random variables $P(t,T)$, $P^{\Delta}(t,T)$ and $C^{\Delta}(t,T)$ are* *given by the following expressions*

$$P(t,T) = e^{-\int_t^T \theta(u)du}\Pi(t,T,\gamma,X_t); \tag{2.2.3}$$

$$P^{\Delta}(t,T) = e^{-\int_t^T (\theta(u)+\theta^{\Delta}(u))du}\Pi(t,T,\gamma+\gamma^{\Delta},X_t); \tag{2.2.4}$$

$$C^{\Delta}(t,T) = \frac{e^{-\int_t^T \theta(u)du}}{e^{-\int_T^{T+\Delta}\theta^{\Delta}(u)du}}\Gamma^{\Delta}(t,T,\gamma,\gamma^{\Delta},X_t); \tag{2.2.5}$$

$$1+\Delta F_{L^{\Delta}}(t,T) = e^{\int_T^{T+\Delta}(\theta(u)+\theta^{\Delta}(u))du}(1+\Delta\Lambda^{\Delta}(t,T,\gamma,\gamma^{\Delta},X_t)). \tag{2.2.6}$$

Proof. With regard to $P(t,T)$ and $P^{\Delta}(t,T)$ we have that

$$\begin{aligned} P(t,T) &= E[e^{-\int_t^T r_u du}|\mathcal{F}_t] = e^{-\int_t^T \theta(u)du}E[e^{-\int_t^T \gamma\cdot X_u du}|X_t] \\ &= e^{-\int_t^T \theta(u)du}\Pi(t,T,\gamma,X_t) \end{aligned}$$

and analogously for $P^{\Delta}(t,T)$

$$\begin{aligned} P^{\Delta}(t,T) &= E[e^{-\int_t^T (r_u+s_u)du}|\mathcal{F}_t] \\ &= e^{-\int_t^T (\theta(u)+\theta^{\Delta}(u))du}E[e^{-\int_t^T (\gamma+\gamma^{\Delta})\cdot X_u du}|X_t] \\ &= e^{-\int_t^T (\theta(u)+\theta^{\Delta}(u))du}\Pi(t,T,\gamma+\gamma^{\Delta},X_t). \end{aligned}$$

On the other hand, $C^{\Delta}(t,T)$ can now be treated as follows

$$\begin{aligned} C^{\Delta}(t,T) &= \mathbb{E}\left[e^{-\int_t^T r_u du}\frac{P(T,T+\Delta)}{P^{\Delta}(T,T+\Delta)}|\mathcal{F}_t\right] \\ &= e^{-\int_t^T \theta(u)du}\mathbb{E}\left[e^{-\int_t^T \gamma\cdot X_u du}\frac{e^{-\int_T^{T+\Delta}\theta(u)du}\Pi(T,T+\Delta,\gamma,X_T)}{e^{-\int_T^{T+\Delta}(\theta(u)+\theta^{\Delta}(u))du}\Pi^{\Delta}(T,T+\Delta,\gamma+\gamma^{\Delta},X_T)}|X_t\right] \\ &= \frac{e^{-\int_t^T \theta(u)du}}{e^{-\int_T^{T+\Delta}\theta^{\Delta}(u)du}}\Gamma^{\Delta}(t,T,\gamma,\gamma^{\Delta},X_t). \end{aligned}$$

Finally, with regard to $F_{L^{\Delta}}(t,T)$ we have that

$$\begin{aligned} 1+\Delta F_{L^{\Delta}}(t,T) &= \frac{C^{\Delta}(t,T)}{P(t,T+\Delta)} \\ &= \frac{\frac{e^{-\int_t^T \theta(u)du}}{e^{-\int_T^{T+\Delta}\theta^{\Delta}(u)du}}\Gamma^{\Delta}(t,T,\gamma,\gamma^{\Delta},X_t)}{e^{-\int_t^{T+\Delta}\theta(u)du}\Pi(t,T+\Delta,\gamma,X_t)} \\ &= e^{\int_T^{T+\Delta}(\theta(u)+\theta^{\Delta}(u))du}(1+\Delta\Lambda^{\Delta}(t,T,\gamma,\gamma^{\Delta},X_t)). \end{aligned}$$

□

The following theorem gives sufficient conditions on θ and θ^{Δ} to solve the calibration problem.

Theorem 2.2.2. *The calibration problem of a multiple-curve instantaneous spot rate* *model is solved by any two functions θ and θ^{Δ} satisfying*

$$\exp\left(-\int_0^T \theta(u)du\right) = \frac{P^{mkt}(0,T)}{\Pi(0,T,\gamma,X_0)} \quad \forall T \tag{2.2.7}$$

and

$$\exp\left(\int_T^{T+\Delta} \theta^{\Delta}(u)du\right) = \exp\left(-\int_T^{T+\Delta} \theta(u)du\right)\frac{1+\Delta L^{\Delta,mkt}(0,T)}{1+\Delta\Lambda^{\Delta}(0,T,\gamma,\gamma^{\Delta},x)} \quad \forall T. \tag{2.2.8}$$

Proof. In light of the previous lemma, the first equality to be met, (2.2.1), can be rewritten as

$$\exp\left(-\int_0^T \theta(u)du\right)\Pi(0,T,\gamma,X_0) = P^{mkt}(0,T),$$

from which the first condition follows trivially.

Now the second condition (2.2.2) follows easily from the equality

$$1+\Delta F_{L^{\Delta}}(0,T) = 1+\Delta L^{\Delta,mkt}(0,T)$$

by replacing the expression for $F_{L^{\Delta}}(0,T)$ obtained in the previous lemma □

An important point to be noted is that the shift θ applied to the process r is uniquely determined by the available data and is in fact the main result in Brigo and Mercurio (2001), whereas the shift θ^{Δ} applied to the spread is determined only up the first part on $[0,\Delta^*)$. Indeed, by taking logs and differentiating with respect to T condition (2.2.8) we find an expression for $\theta^{\Delta}(T+\Delta)-\theta(T)$ so that θ^{Δ} is uniquely identified once it is defined on $[0,\Delta^*)$. We will find the same kind of indeterminacy in the next chapter, where we will develop a different approach.

2.3 Affine Specification

In this section, we show that all the relevant functions (Π, Γ^{Δ} etc.) can be explicitly computed for an important family of stochastic processes, namely those whose semigroup belongs to the affine class. Affine processes were first studied by Duffie and Kan (1996), Dai and Singleton (2000), Duffie et al. (2000) and then classified by Duffie et al. (2003) in the canonical state space domain $E=\mathbb{R}_+^m\times\mathbb{R}^n$, while they have been recently recovered thanks to the interesting extention to the state space of positive semidefinite matrices (see Bru (1991), Gourieroux and Sufana (2003), Gourieroux and Sufana (2005), Da Fonseca et al. (2007), Da Fonseca et al. (2008), Grasselli and Tebaldi (2008) and Cuchiero et al. (2011)).

We will follow the unified approach as presented in Keller-Ressel and Mayerhofer (2012). Consider a time-homogeneous affine Markov process X taking values in a nonempty convex subset E of $\mathbb{R}^d$ ($d \geq 1$), endowed with the inner product $\langle \cdot, \cdot \rangle$. The Markov process X is affine if it is stochastically continuous and its Laplace transform has exponential-affine dependence on the initial state, that is there exist some deterministic functions $\phi_u : \mathbb{R}_+ \to \mathbb{C}$ and $\psi_u : \mathbb{R}_+ \to \mathbb{C}^d$ such that the semigroup P acts as follows:

$$\int_E e^{\langle u,w \rangle} P_t(x, dw) = e^{\phi_u(t) + \langle \psi_u(t), x \rangle} \tag{2.3.1}$$

for all $t \geq 0$, $x \in E$ and $u \in i\mathbb{R}^d$. It can be shown (see e.g. Cuchiero et al. (2011)) that the process X is a semimartingale with characteristics

$$\begin{aligned} A_t &= \int_0^t a(X_{s-}) ds, \\ B_t &= \int_0^t b(X_{s-}) ds, \\ \nu(\omega, dt, d\xi) &= K(X_{t-}(\omega), d\xi) dt, \end{aligned}$$

with $a(x), b(x), K(x, d\xi)$ affine functions:

$$\begin{aligned} a(x) &= a + x_1 \alpha^1 + \ldots + x_d \alpha^d, \\ b(x) &= b + x_1 \beta^1 + \ldots + x_d \beta^d, \\ K(x, d\xi) &= m(d\xi) + x_1 \mu(d\xi) + \ldots + x_d \mu^d(d\xi), \end{aligned}$$

where $a(x)$ (the diffusion coefficient) is a positive semidefinite $d \times d$ matrix, $b(x)$ is the $\mathbb{R}^d$-vector of the drift, and $K(x, d\xi)$ is a Radon measure on $\mathbb{R}^d$ associated to the affine jump part and it is such that

$$\int_{\mathbb{R}^d} \left(\|\xi\|^2 \wedge 1 \right) K(x, d\xi) < \infty$$

and $K(x, \{0\}) = 0$.

The deterministic functions $\phi_u(t), \psi_u(t)$ solve the *generalized Riccati equations*

$$\frac{\partial}{\partial t} \phi_u(t) = \frac{1}{2} \langle \psi_u(t), a \psi_u(t) \rangle + \langle b, \psi_u(t) \rangle + \int_{\mathbb{R}^d \setminus \{0\}} \left(e^{-\langle \xi, \psi_u(t) \rangle} - 1 - \langle h(\xi), \psi_u(t) \rangle \right) m(d\xi),$$
$$\phi_u(0) = 0,$$

and for all $i = 1, ..., d$:

$$\frac{\partial}{\partial t} \psi_u^i(t) = \frac{1}{2} \langle \psi_u(t), \alpha^i \psi_u(t) \rangle + \langle \beta^i, \psi_u(t) \rangle + \int_{\mathbb{R}^d \setminus \{0\}} \left(e^{-\langle \xi, \psi_u(t) \rangle} - 1 - \langle h(\xi), \psi_u(t) \rangle \right) \mu^i(d\xi),$$
$$\psi_u(0) = u,$$

where $h(\xi) = \mathbf{1}_{\{\|\xi\|\leq 1\}}\xi$ is a truncation function.

In order to compute the functions Π and Γ^{Δ}, it is useful to consider the process $(X, Y^{\gamma}) := (X, \int_0^{\cdot}\langle\gamma, X_u\rangle du)$ which is an affine process with state space $E \times \mathbb{R}$ starting from $(X_0, 0)$.

Lemma 2.3.1. *Let $\tilde{P}^{\gamma}$ be the semigroup of the process (X, Y^{γ}). Then we have for every $u \in i\mathbb{R}^d$ and $v \in i\mathbb{R}$*

$$\int_{E\times\mathbb{R}} e^{\langle u,w\rangle+vz}\tilde{P}_t^{\gamma}((x,y),(dw,dz)) = e^{\Phi_{(u,v)}(t,\gamma)+\langle\Psi_{(u,v)}(t,\gamma),x\rangle+vy},$$

where the functions $\Phi_{(u,v)}(\cdot,\gamma)$ and $\Psi_{(u,v)}(\cdot,\gamma)$ satisfy the following system of generalized Riccati ODEs

$$\begin{aligned}
\frac{\partial}{\partial t}\Phi_{(u,v)}(t,\gamma) &= \frac{1}{2}\langle\Psi_{(u,v)}(t,\gamma), a\Psi_{(u,v)}(t,\gamma)\rangle + \langle b, \Psi_{(u,v)}(t,\gamma)\rangle \\
&\quad + \int_{\mathbb{R}^d\setminus\{0\}}\left(e^{-\langle\xi,\Psi_{(u,v)}(t,\gamma)\rangle} - 1 - \langle h(\xi), \Psi_{(u,v)}(t,\gamma)\rangle\right)m(d\xi), \\
\Phi_{(u,v)}(0,\gamma) &= 0,
\end{aligned}$$

and for $i = 1, ..., d$

$$\begin{aligned}
\frac{\partial}{\partial t}\Psi^i_{(u,v)}(t,\gamma) &= v\gamma^i + \frac{1}{2}\langle\Psi_{(u,v)}(t,\gamma), \alpha^i\Psi_{(u,v)}(t,\gamma)\rangle + \langle\beta^i, \Psi_{(u,v)}(t,\gamma)\rangle \\
&\quad + \int_{\mathbb{R}^d\setminus\{0\}}\left(e^{-\langle\xi,\Psi_{(u,v)}(t,\gamma)\rangle} - 1 - \langle h(\xi), \Psi_{(u,v)}(t,\gamma)\rangle\right)\mu^i(d\xi), \\
\Psi_{(u,v)}(0,\gamma) &= u.
\end{aligned}$$

We are now ready to give an expression of the function Π.

Proposition 2.3.2. *The function Π is given by*

$$\Pi(t,T,\gamma,x) = \exp\left(A(t,T,\gamma) + \langle B(t,T,\gamma), x\rangle\right)$$

where the functions A and B are defined as

$$\begin{aligned}
A(t,T,\gamma) &:= \Phi_{(0,-1)}(T-t,\gamma), \\
B(t,T,\gamma) &:= \Psi_{(0,-1)}(T-t,\gamma).
\end{aligned}$$

Proof. We have that

$$\begin{aligned}
\Pi(t,T,\gamma,x) &= \mathbb{E}[e^{-\int_t^T\langle\gamma,X_u\rangle du}|\{X_t = x\}] \\
&= \mathbb{E}[e^{-\int_0^{(T-t)}\langle\gamma,X_{t+u}\rangle du}|\{X_t = x\}] \\
&= \mathbb{E}[e^{-\int_0^{(T-t)}\langle\gamma,X_u\rangle du}|\{X_0 = x\}] \\
&= \exp\left(\Phi_{(0,-1)}(T-t,\gamma) + \langle\Psi_{(0,-1)}(T-t,\gamma), x\rangle\right)
\end{aligned}$$

where the third equality follows from the Markov property of X. $\square$

By exploiting the exponential structure of Π, we are able to compute explicitly the function Γ^Δ as well.

Proposition 2.3.3. *The function Γ^Δ is given by*

$$\Gamma^\Delta(t,T,\gamma,\gamma^\Delta,x) = \exp\left(A^\Delta(t,T,\gamma,\gamma^\Delta) + \langle B^\Delta(t,T,\gamma,\gamma^\Delta),x\rangle\right)$$

where the functions A and B are defined as

$$\begin{aligned} A^\Delta(t,T,\gamma,\gamma^\Delta) :=& A(T,T+\Delta,\gamma) - A(T,T+\Delta,\gamma+\gamma^\Delta) \\ &+ \Phi_{(B(T,T+\Delta,\gamma)-B(T,T+\Delta,\gamma+\gamma^\Delta),-1)}(T-t,\gamma), \\ B^\Delta(t,T,\gamma,\gamma^\Delta) :=& \Psi_{(B(T,T+\Delta,\gamma)-B(T,T+\Delta,\gamma+\gamma^\Delta),-1)}(T-t,\gamma). \end{aligned}$$

Proof.

$$\begin{aligned} &\Gamma^\Delta(t,T,\gamma,\gamma^\Delta,x) = \\ &= \mathbb{E}\left[e^{-\int_t^T\langle\gamma,X_u\rangle du}\frac{\Pi(T,T+\Delta,\gamma,X_T)}{\Pi(T,T+\Delta,\gamma+\gamma^\Delta,X_T)}|\{X_t=x\}\right] \\ &= \mathbb{E}\left[e^{-\int_0^{T-t}\langle\gamma,X_{t+u}\rangle du}\frac{\Pi(T,T+\Delta,\gamma,X_{t+T-t})}{\Pi(T,T+\Delta,\gamma+\gamma^\Delta,X_{t+T-t})}|\{X_t=x\}\right] \\ &= \mathbb{E}\left[e^{-\int_0^{T-t}\langle\gamma,X_u\rangle du}\frac{\Pi(T,T+\Delta,\gamma,X_{T-t})}{\Pi(T,T+\Delta,\gamma+\gamma^\Delta,X_{T-t})}|\{X_0=x\}\right] \\ &= \mathbb{E}\left[e^{-\int_0^{T-t}\langle\gamma,X_u\rangle du}\frac{e^{A(T,T+\Delta,\gamma)+\langle B(T,T+\Delta,\gamma),X_{T-t}\rangle}}{e^{A(T,T+\Delta,\gamma+\gamma^\Delta)+\langle B(T,T+\Delta,\gamma+\gamma^\Delta),X_{T-t}\rangle}}|\{X_0=x\}\right] \\ &= \exp\left\{(A(T,T+\Delta,\gamma)-A(T,T+\Delta,\gamma+\gamma^\Delta))\right\} \\ &\quad .\exp\left\{\Phi_{(B(T,T+\Delta,\gamma)-B(T,T+\Delta,\gamma+\gamma^\Delta),-1)}(T-t,\gamma)\right\} \\ &\quad .\exp\left\{\langle\Psi_{(B(T,T+\Delta,\gamma)-B(T,T+\Delta,\gamma+\gamma^\Delta),-1)}(T-t,\gamma),x\rangle\right\} \end{aligned}$$

which gives the result. □

2.3.1 Example 1: Ornstein-Uhlenbeck specification

We now explicitly compute the functions A, B, A^Δ and B^Δ for a simple specification of the driving Markov process X that is typically used in the bank industry, namely a purely diffusive Ornstein-Uhlenbeck process in the canonical state space domain $E=\mathbb{R}^2$.

Assume that $X=(X^1,X^2)$ is the unique strong solution of the following SDE

$$\begin{cases} dX_t^1 = -\lambda_1 X_t^1 dt + \sigma_1 dW_t^1, \\ dX_t^2 = -\lambda_2 X_t^2 dt + \sigma_2 dW_t^2, \end{cases}$$

where W^1 and W^2 are two Wiener processes with $d[W^1,W^2]_t = \rho dt$, with $\rho\in[-1,1]$. Here the mean reversion parameters λ_1 and λ_2 and the instantaneous volatilities σ_1 and σ_2 are non-negative reals. We leave the initial condition of the SDE unspecified since we are only interested in the semigroup of X. It is easy to see that we are indeed in the affine case, with the following parameters:

$$b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \beta_1 = \begin{bmatrix} -\lambda_1 \\ 0 \end{bmatrix}, \quad \beta_2 = \begin{bmatrix} 0 \\ -\lambda_2 \end{bmatrix},$$

and

$$a = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}, \quad \alpha_1 = \alpha_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

while there are no jumps, that is $K(x, d\xi) = 0$. Consider the projection vectors given by

$$\gamma = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \gamma^\Delta = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

meaning that X^1 is associated to the short rate and X^2 describes the spread s^Δ.

Our goal is to compute the values of $A(t, T, \gamma)$, $B(t, T, \gamma)$, $A^\Delta(t, T, \gamma, \gamma^\Delta)$ and $B^\Delta(t, T, \gamma, \gamma^\Delta)$ for these values of the parameters.

We first compute $\Phi_{(u,-1)}(t, \gamma)$ and $\Psi_{(u,-1)}(t, \gamma)$ for the generic u as we do in Lemma 2.3.4: this will give us the values of $A(t, T, \gamma)$ and $B(t, T, \gamma)$. Then we compute $\Phi_{(0,-1)}(t, \gamma + \gamma^\Delta)$ and $\Psi_{(0,-1)}(t, \gamma + \gamma^\Delta)$ in Lemma 2.3.5: in this way we will have the values $A(T, T + \Delta, \gamma + \gamma^\Delta)$ and $B(T, T + \Delta, \gamma + \gamma^\Delta)$ which we need to compute the values $A^\Delta(t, T, \gamma, \gamma^\Delta)$ and $B^\Delta(t, T, \gamma, \gamma^\Delta)$.

Lemma 2.3.4. *The functions $\Psi = (\Psi^1, \Psi^2)'$ and Φ solutions of the Riccati ODEs in Lemma 2.3.1 computed at $v = -1$ are given by*

$$\begin{aligned} \Psi^1_{(u,-1)}(t, \gamma) &= u_1 e^{-\lambda_1 t} - \frac{1}{\lambda_1}(1 - e^{-\lambda_1 t}), \\ \Psi^2_{(u,-1)}(t, \gamma) &= u_2 e^{-\lambda_2 t} \end{aligned}$$

and

$$\begin{aligned} \Phi_{(u,-1)}(t, \gamma) = &\frac{1}{2}\sigma_1^2 \left[u_1 \overline{\lambda_1}(t) - \frac{1}{\lambda_1}(t - \overline{\lambda_1}(t)) \right] + \frac{1}{2}\sigma_2^2 \left[u_2 \overline{\lambda_2}(t) \right] \\ &+ \rho\sigma_1\sigma_2 \left[u_1 u_2 \overline{\lambda_1 + \lambda_2}(t) - \frac{u_2}{\lambda_1}(\overline{\lambda_2}(t) - \overline{\lambda_1 + \lambda_2}(t)) \right], \end{aligned}$$

where $\overline{\lambda}(t) := \int_0^t e^{-\lambda s} ds$.

Proof. With regard to the functions Ψ^1, Ψ^2, simply note that they satisfy

$$\begin{cases} \frac{d}{dt}\Psi^1_{(u,-1)}(t, \gamma) = -1 - \lambda_1 \Psi^1_{(u,-1)}(t, \gamma) \\ \Psi^1_{(u,-1)}(0, \gamma) = u_1 \end{cases}$$

and

$$\begin{cases} \frac{d}{dt}\Psi^2_{(u,-1)}(t, \gamma) = -\lambda_2 \Psi^2_{(u,-1)}(t, \gamma) \\ \Psi^2_{(u,-1)}(0, \gamma) = u_2 \end{cases}$$

respectively. On the other hand, for Φ we have

$$\begin{aligned}\Phi_{(u,-1)}(t,\gamma) =& \frac{1}{2}\sigma_1^2 \int_0^t \Psi^1_{(u,-1)}(s,\gamma)ds \\ &+ \frac{1}{2}\sigma_2^2 \int_0^t \Psi^2_{(u,-1)}(s,\gamma)ds \\ &+ \rho\sigma_1\sigma_2 \int_0^t \Psi^1_{(u,-1)}(s,\gamma)\Psi^2_{(u,-1)}(s,\gamma)ds,\end{aligned}$$

which is easily integrated. □

Lemma 2.3.5. *We have that*

$$\begin{aligned}\Psi^1_{(0,-1)}(t,\gamma+\gamma^\Delta) &= -\frac{1}{\lambda_1}(1-e^{-\lambda_1 t}), \\ \Psi^2_{(0,-1)}(t,\gamma+\gamma^\Delta) &= -\frac{1}{\lambda_2}(1-e^{-\lambda_2 t})\end{aligned}$$

and

$$\begin{aligned}\Phi_{(0,-1)}(t,\gamma+\gamma^\Delta) = \frac{1}{2}\sigma_1^2\left[-\frac{1}{\lambda_1}(t-\overline{\lambda_1}(t))\right] + \frac{1}{2}\sigma_2^2\left[-\frac{1}{\lambda_2}(t-\overline{\lambda_2}(t))\right] \\ + \rho\sigma_1\sigma_2\left[\frac{1}{\lambda_1\lambda_2}(t-\overline{\lambda_1}(t)-\overline{\lambda_2}(t)+\overline{\lambda_1+\lambda_2}(t))\right],\end{aligned}$$

where, again, $\overline{\lambda}(t) := \int_0^t e^{-\lambda s}ds$.

Proof. With regard to the functions Ψ^1, Ψ^2, note that they satisfy

$$\begin{cases}\frac{d}{dt}\Psi^1_{(0,-1)}(t,\gamma+\gamma^\Delta) = -1-\lambda_1\Psi^1_{(0,-1)}(t,\gamma+\gamma^\Delta) \\ \Psi^1_{(0,-1)}(0,\gamma+\gamma^\Delta) = 0\end{cases}$$

and

$$\begin{cases}\frac{d}{dt}\Psi^2_{(0,-1)}(t,\gamma+\gamma^\Delta) = -1-\lambda_2\Psi^2_{(0,-1)}(t,\gamma+\gamma^\Delta) \\ \Psi^2_{(0,-1)}(0,\gamma+\gamma^\Delta) = 0\end{cases}$$

respectively. The ODE for Φ has not changed and, again, can be easily integrated. □

We are now in position to give the expressions we are seeking.

Proposition 2.3.6. *The functions $A(\cdot,\cdot,\gamma)$ and $B(\cdot,\cdot,\gamma)$ in Proposition (2.3.2) are given by*

$$\begin{aligned}A(t,T,\gamma) &= -\frac{1}{2}\sigma_1^2\left[\frac{1}{\lambda_1}((T-t)-\overline{\lambda_1}(T-t))\right], \\ B^1(t,T,\gamma) &= -\frac{1}{\lambda_1}(1-e^{-\lambda_1(T-t)}), \\ B^2(t,T,\gamma) &= 0.\end{aligned}$$

Proof. This is straightforward upon using Lemma 2.3.4 with $u = 0$ and the definition given in Proposition 2.3.2. □

Proposition 2.3.7. *The functions $A^{\Delta}(\cdot,\cdot,\gamma,\gamma^{\Delta})$ and $B^{\Delta}(\cdot,\cdot,\gamma,\gamma^{\Delta})$ of Proposition (2.3.3) are given by*

$$\begin{aligned}
A^{\Delta}(t,T,\gamma,\gamma^{\Delta}) =& \frac{1}{2}\sigma_2^2\left[\frac{1}{\lambda_2}(\Delta-\overline{\lambda_2}(\Delta))\right]\\
&-\rho\sigma_1\sigma_2\left[\frac{1}{\lambda_1\lambda_2}(\Delta-\overline{\lambda_1}(\Delta)-\overline{\lambda_2}(\Delta)+\overline{\lambda_1+\lambda_2}(\Delta))\right]\\
&+\frac{1}{2}\sigma_1^2\left[-\frac{1}{\lambda_1}((T-t)-\overline{\lambda_1}(T-t))\right]+\frac{1}{2}\sigma_2^2\left[\overline{\lambda_2}(\Delta)\overline{\lambda_2}(T-t)\right]\\
&+\rho\sigma_1\sigma_2\left[-\frac{\overline{\lambda_2}(\Delta)}{\lambda_1}(\overline{\lambda_2}(T-t)-\overline{\lambda_1+\lambda_2}(T-t))\right],\\
B^{\Delta,1}(t,T,\gamma,\gamma^{\Delta}) =& -\frac{1}{\lambda_1}(1-e^{-\lambda_1(T-t)}),\\
B^{\Delta,2}(t,T,\gamma,\gamma^{\Delta}) =& \frac{1}{\lambda_2}(1-e^{-\lambda_2\Delta})e^{-\lambda_2(T-t)}.
\end{aligned}$$

Proof. Since we have that

$$B(T,T+\Delta,\gamma)-B(T,T+\Delta,\gamma+\gamma^{\Delta})=\begin{bmatrix}0\\ \frac{1}{\lambda_2}(1-e^{-\lambda_2\Delta})\end{bmatrix},$$

the claim on B^{Δ} now follows from its definition which was given in Proposition 2.3.3. With regard to A^{Δ}, we have that

$$\begin{aligned}
A^{\Delta}(T,T+\Delta,\gamma)-A^{\Delta}(T,T+\Delta,\gamma+\gamma^{\Delta}) =& -\frac{1}{2}\sigma_2^2\left[-\frac{1}{\lambda_2}(\Delta-\overline{\lambda_2}(\Delta))\right]\\
&-\rho\sigma_1\sigma_2\left[\frac{1}{\lambda_1\lambda_2}(\Delta-\overline{\lambda_1}(\Delta)-\overline{\lambda_2}(\Delta)+\overline{\lambda_1+\lambda_2}(\Delta))\right],
\end{aligned}$$

so that we can conclude from the definition of A^{Δ} given in Proposition 2.3.3.

□

2.3.2 Example 2: the Wishart specification

Let us now consider a symmetric matrix valued state space domain, that is $E = S_d^+$, the set of positive semidefinite symmetric matrices[3] endowed with the scalar product $(x,y)\to\langle x,y\rangle = tr[xy]$, where tr denotes the trace operator. An important process that is defined on this state space domain is the Wishart process, which has been first introduced by Bru (1991), and then applied to finance by Gourieroux and Sufana (2003) and Gourieroux and Sufana (2005).

The most important property of the Wishart process relies on its ability to describe complex interdependencies among positive stochastic factors. In particular, it allows for the possibility to deal with a stochastic correlation, which represents a crucial tool in

[3] Notice that here the dimension of the vector space E is $\frac{d(d+1)}{2}$, so that for example the diffusion matrix $a(x)$ should be represented as a symmetric $\frac{d(d+1)}{2}\times\frac{d(d+1)}{2}$ matrix and $b(x)$ is a vector in $\mathbb{R}^{\frac{d(d+1)}{2}}$.

order to describe interesting financial phenomena.

In view of the applications, we consider the purely diffusive specification for the Wishart infinitesimal generator, corresponding to the special case $b(x) = x\beta + \beta^\top x$ for a constant matrix $\beta \in M_d^-$ (the set of square matrices whose eigenvalues have negative real part) and a diffusion matrix $\alpha = Q^\top Q$ with $Q \in GL_d$ (the set of invertible $d \times d$ matrices), that is

$$\mathcal{G}f(x) \quad = \quad tr[(b + x\beta + \beta^\top x)Df(x) + 2xDQ^\top QDf(x)],$$

where the constant drift matrix b satisfies $b - (d-1)Q^\top Q \in S_d^+$ (related to the so called Gindikin set) and the differential operator D is defined as follows:

$$D_{ij} = \frac{\partial}{\partial x_{ij}}, \qquad i, j = 1, .., d. \tag{2.3.2}$$

The infinitesimal generator corresponds to the following matrix dynamics for the process X:

$$dX_t = (b + X_t\beta + \beta^\top X_t)dt + \sqrt{X_t}dW_tQ + Q^\top dW_t^\top \sqrt{X_t}, \tag{2.3.3}$$

with $X_0 = x \in S_d^+$ and where W is a matrix Brownian motion, that is a $d \times d$ matrix of independent Brownian motions. Note that the dynamics of X generalizes the CIR process and its stationarity is ensured by the condition on the mean reversion term $\beta \in M_d^-$.

Remark 2.3.8. It is important to notice that the main advantage of the Wishart specification with respect to the canonical state space domain consists in the possibility to allow for a non trivial correlation among the positive factors: for example it is simply checked that (take $d = 2$)

$$d\langle X_{11}, X_{22}\rangle_t = 4X_{12}(Q_{11}Q_{12} + Q_{21}Q_{22})dt,$$

namely the off diagonal elements of X describe the covariation among the (positive) factors along the diagonal.

The (matrix) Riccati ODE satisfied by the functions Ψ_u is the following:

$$\begin{aligned}\frac{\partial}{\partial t}\Psi_u(t) &= v\gamma + \beta^\top\Psi_u(t) + \Psi_u(t)\beta + 2\Psi_u(t)Q^\top Q\Psi_u(t),\\ \Psi_u(0) &= u,\end{aligned}$$

while as usual Φ_u is given by direct integration:

$$\begin{aligned}\frac{\partial}{\partial t}\Phi_u(t) &= tr[b\Psi_u(t)],\\ \Phi_u(0) &= 0.\end{aligned}$$

Using the linearization technique as in Grasselli and Tebaldi (2008), it is possible to express the solution to this Riccati system through an exponentiation that is easy to implement. The procedure is now standard and we just state the result.

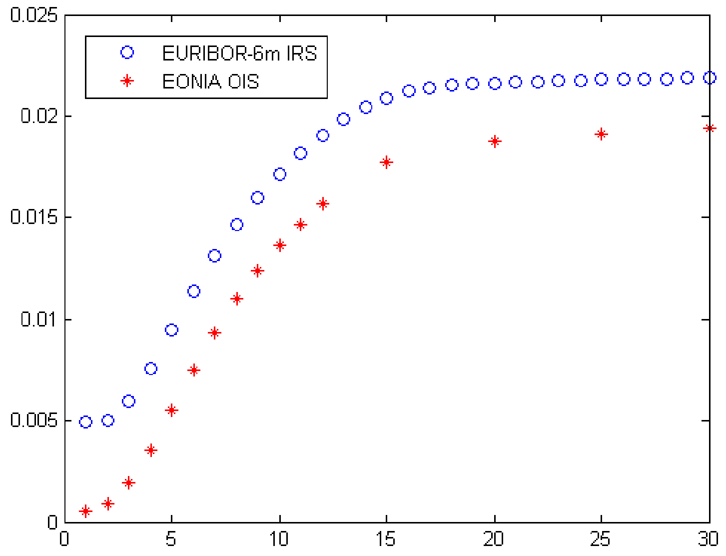

Figure 2.1: EURIBOR 6m IRS vs EONIA OIS.

Proposition 2.3.9. *The deterministic matrix valued functions $\Phi_u(t), \Psi_u(t)$ in Lemma 2.3.1 can be expressed as follows:*

$$\Phi_u(t) = (uA_{12}(t) + A_{22}(t))^{-1}(uA_{11} + A_{21}),$$

where

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \exp t \begin{pmatrix} \beta & -2Q^\top Q \\ v\gamma & -\beta^\top \end{pmatrix}$$

and

$$\Psi_u(t) = -\frac{1}{2} tr[b(\log(uA_{12}(t) + A_{22}(t)) + t\beta^\top (Q^\top Q)^{-1})].$$

Note that following the procedure of the previous subsection, all the relevant functions like Π, Γ^Δ etc. can be efficiently computed thanks to the previous proposition.

2.4 Numerical illustration

In this section we give a numerical example of the functions θ and θ^Δ calibrated to real market data. We take Euro market data as of August, 31^{st}, 2012. Specifically the input data consist of OIS swaps and EURIBOR 6m swaps quotes of maturity up to 30 years. We plot these quotes in figure 2.1.

We obtain the risk-free bond prices P^{mkt} from the OIS quotes on a discrete set of dates

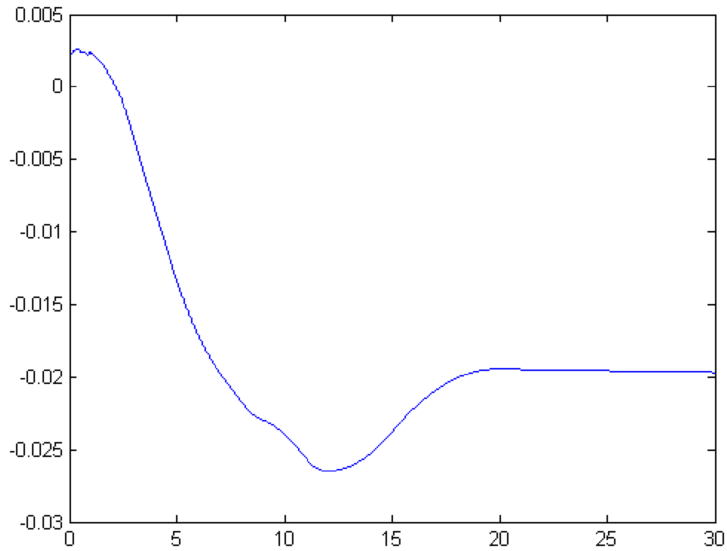

Figure 2.2: θ calibrated to market data 31/08/2012.

and then interpolate their logarithms by a piecewise cubic spline. On the other hand, we use the EURIBOR swap quotes and the risk-free discount factors to compute the forward LIBORs $L^{\Delta,mkt}$.

In this example, we use the Ornstein-Uhlenbeck specification but, as we already noted, numerical values for the functions θ and θ^{Δ} can be computed as soon as a closed form expression for Π and Γ^{Δ} is available. We fix the parameters for the driving process X as follows: $\lambda_1 = \lambda_2 = 0.05$, $\sigma_1 = 0.01$ and $\sigma_2 = 0.0050$. The initial value of X is fixed at

$$X_0 = \begin{bmatrix} 0.0030 \\ 0.0010 \end{bmatrix}.$$

The function θ is determined by λ_1 and σ_1 only and it is plotted in figure 2.2.

On the other hand, the function θ^{Δ} is determined by the full set of parameters it is plotted in figure 2.3 for $\rho = -0.5, 0, 0.5$.

From figure 2.3 it is also possible to appreciate how this function depends continuously on the parameters.

Finally let us stress that, as it is always the case, the law of the driving process X depends on some parameters (in this case λ_1, λ_2, σ_1, σ_2 and ρ) which have to be considered fixed through this stage of the calibration and are free to be calibrated to other instruments (generally caps and swaptions).

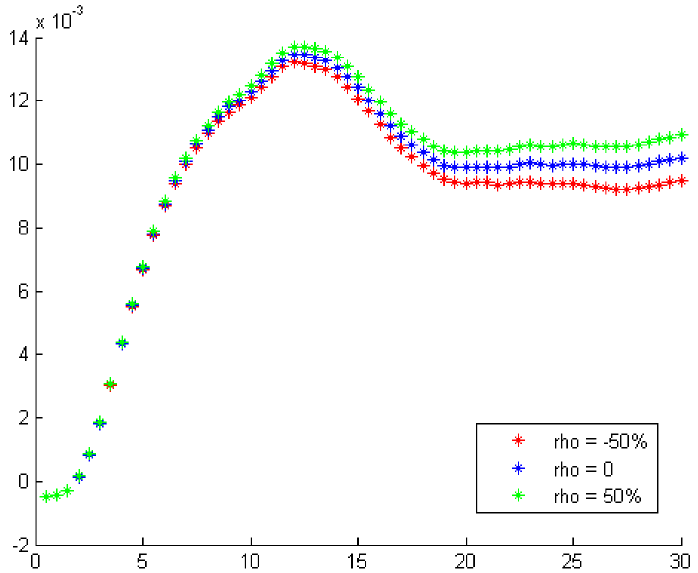

Figure 2.3: θ^{Δ} calibrated to market data 31/08/2012.

Chapter 3

A Multiple-curve Instantaneous Forward Rate Model

The goal of this chapter is to extend the celebrated Heath-Jarrow-Morton model for the dynamics of the term-structure (see Heath et al. (1992)) to a multiple-curve framework.

We start from a continuum of risk-free zero coupon bonds and a continuum of FRA's on a single LIBOR rate of tenor Δ. The fact that there is only one tenor Δ for the LIBOR rate leads to an indeterminacy of the generalization of the instantaneous forward rate curve. This indeterminacy is reflected into an indeterminacy of the drift conditions. In other words, we develop the closest relative of the HJM model in a two-curve framework but show that it suffers from some drawbacks.

Recall from Chapter 1, that we use the notation L_t^{Δ} for the time-t spot LIBOR of tenor Δ and $F_{L^\Delta}(t,T) = L^{\Delta}(t,T)$ for the time-t forward LIBOR of tenor Δ for maturity T.

To begin with, let us state a fact that should already be clear from Chapter 1 and that will be used in the following.

Proposition 3.0.1. *In an arbitrage-free market, for any maturity T, the process*

$$\left(\frac{P(t,T+\Delta)\Delta F_{L^\Delta}(t,T)}{B_t}\right)_t$$

must be a martingale under the risk-neutral measure $\mathbb{Q}_*$.

Proof. The process $(P(t,T+\Delta)\Delta F_{L^\Delta}(t,T))_t$ is the price process of a traded asset, namely the time-t price of the floating leg in a FRA on L^{Δ} setting at T and paying at $T+\Delta$. As a consequence it must be a martingale under $\mathbb{Q}_*$ when divided by the numeraire B. $\square$

In the next section, we analyze the parametrization for the forward LIBOR curve that will be the central theme of this chapter.

3.1 The Fictitious Instantaneous Forward Rates

The following definition deals with forward LIBOR and tries to mimic the definition of forward rates from the zero coupon curve.

Definition 3.1.1 (Fictitious Δ-tenor zero coupon curve)**.** We say $T \mapsto P^{\Delta}(\cdot, T)$ is a fictitious Δ-tenor zero coupon curve consistent with the forward LIBOR curve $T \mapsto F_{L^{\Delta}}(\cdot, T)$ if we have

$$F_{L^{\Delta}}(\cdot, T) = \frac{1}{\Delta}\left(\frac{P^{\Delta}(\cdot, T)}{P^{\Delta}(\cdot, T+\Delta)} - 1\right).$$

This definition is in line with some of the existing literature, see e.g. Bianchetti (2009), Pallavicini and Tarenghi (2010) and Crépey et al. (2012) (see the literature review given in Section 1.5).

We now consider the existence and uniqueness issue arising with the last definition which to the best of our knowledge was overlooked in the literature so far. In the following proposition, since the present time t is held fixed, we temporarily suppress it from the notation (e.g., we write $F_{L^{\Delta}}(T)$ instead of $F_{L^{\Delta}}(t, T)$)

Proposition 3.1.2. *Let us fix an arbitrary forward LIBOR curve $T \mapsto F_{L^{\Delta}}(T)$.*

Then, to any $A : [0, \Delta) \to \mathbb{R}$ we can associate a fictitious zero coupon bond curve $T \mapsto P_A^{\Delta}(T)$ admitted by L^{Δ} defined as[1]

$$P_A^{\Delta}(T) := A(\bar{\epsilon}(T)) \prod_{k=0}^{\bar{n}(T)-1} \frac{1}{1 + \Delta F_{L^{\Delta}}(k\Delta + \bar{\epsilon}(T))},$$

where $\bar{n}(T) := \sup\{n \in \mathbb{N} : n\Delta \leq T\}$ and $\bar{\epsilon}(T) := T - \bar{n}(T)\Delta$.

Conversely, to any P^{Δ} consistent with L^{Δ} we can associate a function A such that the above relation holds.

In other words, there is a bijection between fictitious zero coupon bond curves admitted by L^{Δ} and mappings $A : [0, \Delta) \to \mathbb{R}$.

Proof. First of all note that $\bar{\epsilon}$ is periodic with period Δ, so that $\bar{\epsilon}(T+\Delta) = \bar{\epsilon}(T)$ and that $\bar{n}(T+\Delta) = \bar{n}(T) + 1$. Also, for all T's, we have that $T = \bar{n}(T)\Delta + \bar{\epsilon}(T)$. By the definition of P_A^{Δ} and the above properties we have

$$\begin{aligned}
&\frac{P_A^{\Delta}(T)}{P_A^{\Delta}(T+\Delta)} \\
&= \frac{A(\bar{\epsilon}(T))}{A(\bar{\epsilon}(T+\Delta))} \prod_{k=0}^{\bar{n}(T)-1} \frac{1}{1 + \Delta F_{L^{\Delta}}(k\Delta + \bar{\epsilon}(T))} \prod_{k=0}^{\bar{n}(T+\Delta)-1} 1 + \Delta F_{L^{\Delta}}(k\Delta + \bar{\epsilon}(T+\Delta)) = \\
&= 1 + \Delta F_{L^{\Delta}}(\bar{n}(T)\Delta + \bar{\epsilon}(T)) \\
&= 1 + \Delta F_{L^{\Delta}}(T),
\end{aligned}$$

thus proving the first assertion.

With regard to the second one, for a given P^{Δ} consistent with L^{Δ} it is sufficient to define

$$A(t) := P^{\Delta}(t) \qquad t \in [0, \Delta).$$

[1]here we use the convention that $\prod_{k=0}^{-1} = 1$.

Then we have

$$\begin{aligned}
&A(\bar{\epsilon}(T)) \prod_{k=0}^{\bar{n}(T)-1} \frac{1}{1+\Delta F_{L^\Delta}(k\Delta+\bar{\epsilon}(T))} \\
&= P^\Delta(\bar{\epsilon}(T)) \prod_{k=0}^{\bar{n}(T)-1} \frac{1}{1+\Delta F_{L^\Delta}(k\Delta+\bar{\epsilon}(T))} \\
&= P^\Delta(\bar{\epsilon}(T)) \prod_{k=0}^{\bar{n}(T)-1} \frac{P^\Delta((k+1)\Delta+\bar{\epsilon}(T))}{P^\Delta(k\Delta+\bar{\epsilon}(T))} \\
&= P^\Delta(\bar{\epsilon}(T)) \frac{P^\Delta(\bar{n}(T)\Delta+\bar{\epsilon}(T))}{P^\Delta(\bar{\epsilon}(T))} = P^\Delta(T),
\end{aligned}$$

where the second equality follows from the fact that P^Δ is admitted by L^Δ. This completes the proof. □

It is thus clear that for any forward LIBOR curve, an admitted fictitious curve will always exist but it will be far from being unique. In fact it can be postulated arbitrarily on the first interval $[0, \Delta)$ and then it is fully pinned down by this choice. It also clear that if a fictitious bond curve is given on some $D \subset [0, \Delta)$ then it is fully pinned down on the set $\{n\Delta + D : n = 1, 2, 3, \dots\}$. For example, if we only postulate that $P^\Delta(0) = 1$, then we only have determined the curve on the lattice $\{n\Delta : n = 1, 2, 3, \dots\}$.

This shortcoming will be removed in the sequel by assuming the existence of a continuum of tenors.

For the following it will be important to understand if among all the possible choices of fictitious curves consistent with a given L^Δ there are some with given properties. Here is a first result in this direction.

Proposition 3.1.3. *For any given continuous forward LIBOR curve L^Δ, P_A^Δ is continuous if the function A is continuous and satisfies*

$$A(\Delta-) = A(0)\frac{1}{1+\Delta F_{L^\Delta}(0)}. \tag{3.1.1}$$

Proof. It is clear that, $\bar{n}$ and $\bar{\epsilon}$ being right-continuous on $\mathbb{R}_+$ and continuous on the complement of $\{n\Delta : n = 1, 2, 3, \dots\}$, we only need to check left continuity at the generic point $n\Delta$ for $n \in \mathbb{N}$, i.e. to show that

$$\lim_{\epsilon \to 0+} P_A^\Delta(n\Delta - \epsilon) = P_A^\Delta(n\Delta).$$

By the definition of P_A^Δ, the left limits of $\bar{n}$ and $\bar{\epsilon}$ and our hypothesis, we get to the result by noting that

$$\begin{aligned}
&\lim_{\epsilon\to 0+} P_A^{\Delta}(n\Delta-\epsilon)\\
&= \lim_{\epsilon\to 0+} A(\bar{\epsilon}(n\Delta-\epsilon)) \prod_{k=0}^{\bar{n}(n\Delta-\epsilon)-1} \frac{1}{1+\Delta F_{L^{\Delta}}(k\Delta+\bar{\epsilon}(n\Delta-\epsilon))}\\
&= A(\Delta-) \prod_{k=0}^{(n-1)-1} \frac{1}{1+\Delta F_{L^{\Delta}}(k\Delta+\Delta)}\\
&= A(0)\frac{1}{1+\Delta F_{L^{\Delta}}(0)} \prod_{k=0}^{n-2} \frac{1}{1+\Delta F_{L^{\Delta}}((k+1)\Delta)}\\
&= A(0) \prod_{k=0}^{n-1} \frac{1}{1+\Delta F_{L^{\Delta}}(k\Delta)}\\
&= A(0) \prod_{k=0}^{n-1} \frac{P_A^{\Delta}((k+1)\Delta)}{P_A^{\Delta}(k\Delta)} = P_A^{\Delta}(n\Delta).
\end{aligned}$$

Note that the second equality is justified since $\bar{n}(n\Delta-\epsilon)$ does not depend on ϵ for ϵ sufficiently small. □

In the following proposition, we give sufficient conditions on A in order for P_A^{Δ} to be differentiable.

Proposition 3.1.4. *For any given differentiable forward LIBOR curve L^{Δ}, the mapping P_A^{Δ} is differentiable if the function A is differentiable, satisfies the requirement needed for the continuity* (3.1.1) *and*

$$A'(\Delta-) = \frac{d}{d\epsilon} A(\epsilon)\frac{1}{1+\Delta F_{L^{\Delta}}(\epsilon)}\Big|_{\epsilon=0}. \tag{3.1.2}$$

Proof. Again, thanks to the differentiability of A and L^{Δ}, we only need to care about points that are integer multiples of Δ. First, we exploit the properties of the functions $\bar{n}$ and $\bar{\epsilon}$ used in the definition of P_A^{Δ} to find a more convenient expression for $P_A^{\Delta}(n\Delta+\epsilon)$, $P_A^{\Delta}(n\Delta)$ and $P_A^{\Delta}(n\Delta-\epsilon)$. These expressions will then be used to find the right and left incremental limits.

To compute $P_A^{\Delta}(n\Delta+\epsilon)$ we proceed as follows

$$\begin{aligned}
P_A^{\Delta}(n\Delta+\epsilon) &= A(\bar{\epsilon}(n\Delta+\epsilon)) \prod_{k=0}^{\bar{n}(n\Delta+\epsilon)-1} \frac{1}{1+\Delta F_{L^{\Delta}}(k\Delta+\bar{\epsilon}(n\Delta+\epsilon))}\\
&= A(\epsilon) \prod_{k=0}^{n-1} \frac{1}{1+\Delta F_{L^{\Delta}}(k\Delta+\epsilon)}\\
&= A(\epsilon)\frac{1}{1+\Delta F_{L^{\Delta}}(\epsilon)} \prod_{k=1}^{n-1} \frac{1}{1+\Delta F_{L^{\Delta}}(k\Delta+\epsilon)}.
\end{aligned}$$

With regard to $P_A^\Delta(n\Delta)$ we have

$$\begin{aligned} P_A^\Delta(n\Delta) &= A(0)\prod_{k=0}^{n-1}\frac{1}{1+\Delta F_{L^\Delta}(k\Delta)} \\ &= A(0)\frac{1}{1+\Delta F_{L^\Delta}(0)}\prod_{k=1}^{n-1}\frac{1}{1+\Delta F_{L^\Delta}(k\Delta)} \\ &= A(\Delta-)\prod_{k=1}^{n-1}\frac{1}{1+\Delta F_{L^\Delta}(k\Delta)}. \end{aligned}$$

where we used (3.1.1) to get the last equality.

Finally $P_A^\Delta(n\Delta-\epsilon)$ can be treated as follows

$$\begin{aligned} P_A^\Delta(n\Delta-\epsilon) &= A(\bar{\epsilon}(n\Delta-\epsilon))\prod_{k=0}^{\bar{n}(n\Delta-\epsilon)-1}\frac{1}{1+\Delta F_{L^\Delta}(k\Delta+\bar{\epsilon}(n\Delta-\epsilon))} \\ &= A(\Delta-\epsilon)\prod_{k=0}^{n-2}\frac{1}{1+\Delta F_{L^\Delta}(k\Delta+\Delta-\epsilon)} \\ &= A(\Delta-\epsilon)\prod_{k=1}^{n-1}\frac{1}{1+\Delta F_{L^\Delta}(k\Delta-\epsilon)}. \end{aligned}$$

Now the right-derivative of P_A^Δ at the generic $n\Delta$ can be computed as follows:

$$\begin{aligned} D^+P_A^\Delta(n\Delta) &= \lim_{\epsilon\to 0+}\frac{P_A^\Delta(n\Delta+\epsilon)-P_A^\Delta(n\Delta)}{\epsilon} \\ &= \lim_{\epsilon\to 0+}\frac{A(\epsilon)\frac{1}{1+\Delta F_{L^\Delta}(\epsilon)}\prod_{k=1}^{n-1}\frac{1}{1+\Delta F_{L^\Delta}(k\Delta+\epsilon)}-A(0)\frac{1}{1+\Delta F_{L^\Delta}(0)}\prod_{k=1}^{n-1}\frac{1}{1+\Delta F_{L^\Delta}(k\Delta)}}{\epsilon} \\ &= \frac{d}{d\epsilon}A(\epsilon)\frac{1}{1+\Delta F_{L^\Delta}(\epsilon)}\Big|_{\epsilon=0}f(0)+A(0)\frac{1}{1+\Delta F_{L^\Delta}(0)}f'(0), \end{aligned}$$

where we defined

$$f(\epsilon) := \prod_{k=1}^{n-1}\frac{1}{1+\Delta F_{L^\Delta}(k\Delta+\epsilon)}.$$

On the other hand, the left-derivative of P_A^Δ at the generic $n\Delta$ can be computed as follows:

$$\begin{aligned} D^-P_A^\Delta(n\Delta) &= \lim_{\epsilon\to 0+}\frac{P_A^\Delta(n\Delta)-P_A^\Delta(n\Delta-\epsilon)}{\epsilon} \\ &= \lim_{\epsilon\to 0+}\frac{A(\Delta-)\prod_{k=1}^{n-1}\frac{1}{1+\Delta F_{L^\Delta}(k\Delta)}-A(\Delta-\epsilon)\prod_{k=1}^{n-1}\frac{1}{1+\Delta F_{L^\Delta}(k\Delta-\epsilon)}}{\epsilon} \\ &= A'(\Delta-)f(0)+A(\Delta-)f'(0) \end{aligned}$$

and the proof is complete. □

Once the notion of fictitious zero coupon curve associated to a forward LIBOR curve is defined, it is natural to define a fictitious instantaneous forward rate curve.

Definition 3.1.5 (Fictitious Δ-tenor instantaneous forward rate curve)**.** We say $T \mapsto f^{\Delta}(\cdot, T)$ is a fictitious Δ-tenor instantaneous forward rate curve consistent with the forward LIBOR curve $T \mapsto F_{L^{\Delta}}(\cdot, T)$ if we have

$$f^{\Delta}(t,T) = -\frac{\partial}{\partial T} \ln P^{\Delta}(t,T)$$

for some fictitious Δ-tenor zero coupon curve consistent with the forward LIBOR curve $T \mapsto F_{L^{\Delta}}(\cdot, T)$

We are naturally led to define the Δ-tenor fictitious short rate as $r_t^{\Delta} := f^{\Delta}(t,t)$. Finally we denote the Δ-tenor instantaneous T-forward spread s^{Δ} with

$$s^{\Delta}(\cdot, T) := f^{\Delta}(\cdot, T) - f(\cdot, T).$$

3.2 The Model

Let us assume we are given a probability space $(\Omega, \mathcal{F}, \mathbb{Q}_*)$ and that W is a standard 1-dimensional Wiener process defined on it. We denote by $\mathbb{F} = (\mathcal{F}_t)_t$ the (standard Q_*-augmentation of the) natural filtration of W, so that $\mathbb{F}$ satisfies the usual hypothesis in being complete and right-continuous. We interpret $\mathbb{Q}_*$ as the risk-neutral measure, i.e. the measure associated to the instantaneous bank-account numeraire $B_t = e^{\int_0^t r_u du}$, that was defined in Chapter 1.

In other words we are postulating that the market is free of arbitrage opportunities. Another possibility would be to start from an "objective measure", say $\mathbb{P}$, and then characterize the absence of arbitrage by the existence of a solution to a market price of risk equation, via a Girsanov transformation. In fact, by the martingale representation theorem for Wiener filtrations, any probability on $\mathcal{F}_t$ equivalent to $\mathbb{P}_{|\mathcal{F}_t}$ is given by $\mathcal{E}_t\left(\int_0^{\cdot} \lambda_u du\right) d\mathbb{P}_{|\mathcal{F}_t}$ for some square integrable λ.

For the rest of this chapter, we will work under the following assumptions:

Assumption 3.2.1 (Instantaneous forward rate dynamics)**.** For any $T \in (0, T^*]$, $f(\cdot, T)$ follows an Ito process of the form

$$f(t,T) = f_0(T) + \int_0^t \alpha(u,T)du + \int_0^t \sigma(u,T)dW_u, \qquad t \in [0,T],$$

where:

- $f_0 : [0, T^*] \mapsto \mathbb{R}$ is a fixed, nonrandom, Borel measurable initial forward rate curve
- the drifts $\alpha : \{(t,s) : 0 < t < T < T^*\} \times \Omega \to \mathbb{R}$ are jointly measurable on $\mathcal{B}(\{(t,s) : 0 < t < T < T^*\}) \times \mathcal{F}$ such that $\alpha(\cdot, T)$ is $\mathbb{F}$-adapted and $\mathbb{Q}^*$-a.s.

$$\int_0^T |\alpha(t,T)|dt < \infty;$$

- the volatilities $\sigma : \{(t,s) : 0 < t < T < T^*\} \times \Omega \to \mathbb{R}$ are jointly measurable on $\mathcal{B}(\{(t,s) : 0 < t < T < T^*\}) \times \mathcal{F}$ such that $\sigma(\cdot, T)$ is $\mathbb{F}$-adapted and $\mathbb{Q}^*$-a.s.

$$\int_0^T |\sigma(t,T)|^2 dt < \infty;$$

Assumption 3.2.2 (Instantaneous fictitious Δ-tenor forward rate dynamics)**.** For any $T \in (0, T^*]$, $f^\Delta(\cdot, T)$ follows an Ito process of the form

$$f^\Delta(t,T) = f_0^\Delta(T) + \int_0^t \alpha_\Delta(u,T)du + \int_0^t \sigma_\Delta(u,T)dW_u, \qquad t \in [0,T],$$

where we make the same assumptions on f_0^Δ, α_Δ and σ_Δ that we made about f_0, α and σ.

For ease of notation, in the following we will use the shorthands

$$A(t,T) := \int_t^T \alpha(t,u)du, \quad \Sigma(t,T) := \int_t^T \sigma(t,u)du$$

and

$$A_\Delta(t,T) := \int_t^T \alpha_\Delta(t,u)du, \quad \Sigma_\Delta(t,T) := \int_t^T \sigma_\Delta(t,u)du.$$

3.3 Absence of Arbitrage

We recall the following proposition which is the backbone of the classical HJM drift condition, see Heath et al. (1992). The proof, which is based on the deterministic and stochastic Fubini theorems, is classical, so that we omit it.

Proposition 3.3.1. *For every $T \in (0, T^*]$, let $f(\cdot, T)$ be an Ito process of the form*

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t$$

and define $Y(t,T) := \int_t^T f(t,u)du$ and $P(t,T) := e^{-Y(t,T)}$. Then we have

$$dY(t,T) = (-f(t,t) + A(t,T))dt + \Sigma(t,T)dW_t$$

and

$$\frac{dP(t,T)}{P(t,T)} = (f(t,t) - A(t,T) + \frac{1}{2}\Sigma^2(t,T))dt - \Sigma(t,T)dW_t.$$

The following theorem is the celebrated HJM drift condition, which characterizes absence of arbitrage in the risk-free bond market, see Heath et al. (1992) for details. Note that, for the moment, we are disregarding the LIBOR forward market.

Theorem 3.3.2. *Under assumption 3.2.1, the market consisting of all the zero coupon bonds and the bank account is free of arbitrage opportunities if and only if*

$$A(t,T) = \frac{1}{2}\Sigma^2(t,T) \tag{3.3.1}$$

or, equivalently,

$$\alpha(t,T) = \sigma(t,T)\Sigma(t,T).$$

Proof. It is well known that the absence of arbitrage opportunities within the market in question is equivalent to the processes $\frac{P(\cdot,T)}{B}$ being martingales under $\mathbb{Q}_*$ for all $T \in (0,T^*]$. Let us denote, for the moment, $R = \frac{p(\cdot,T)}{B}$. By the preceding proposition we have

$$\frac{dR_t}{R_t} = (-A(t,T) + \frac{1}{2}\Sigma^2(t,T))dt - \Sigma(t,T)dW_t,$$

so that R is a martingale if and only if (3.3.1) holds, as it was to be shown. □

We now examine the absence of arbitrage of the whole market consisting of the zero coupon bonds $P(\cdot,T)$'s, the bank account B and the FRAs on the LIBOR rate L^Δ. The following theorem is the analogue of the HJM drift condition recalled above.

Theorem 3.3.3. *Under assumptions 3.2.1 and 3.2.2, the market consisting of all the zero coupon bonds, the bank account and the FRAs on the LIBOR rate L^Δ is free of arbitrage opportunities if and only if the HJM drift condition* (3.3.1) *is satisfied and, in addition,*

$$\begin{aligned} &A_\Delta(t,T+\Delta) - A_\Delta(t,T) = \\ &-\frac{1}{2}(\Sigma_\Delta(t,T+\Delta) - \Sigma_\Delta(t,T))^2 + \Sigma(t,T+\Delta)(\Sigma_\Delta(t,T+\Delta) - \Sigma_\Delta(t,T)) \end{aligned} \tag{3.3.2}$$

i.e.,

$$\begin{aligned} &\int_T^{T+\Delta} \alpha_\Delta(t,u)du = \\ &-\frac{1}{2}\left(\int_T^{T+\Delta} \sigma_\Delta(t,u)du\right)^2 + \Sigma(t,T+\Delta)\left(\int_T^{T+\Delta} \sigma_\Delta(t,u)du\right). \end{aligned}$$

Proof. The bond market is taken care of by the HJM drift condition (3.3.1), as it was shown in the previous theorem. As it was already noted in Proposition 3.0.1, absence of arbitrage opportunities within the FRA LIBOR market is equivalent to the processes $\frac{p(\cdot,T+\Delta)\Delta F_{L^\Delta}(\cdot,T)}{B}$ being martingales under $\mathbb{Q}_*$. Note that

$$\frac{P(t,T+\Delta)\Delta F_{L^\Delta}(t,T)}{B_t} = \frac{1}{B_t}P(t,T+\Delta)\frac{P^\Delta(t,T)}{P^\Delta(t,T+\Delta)} - \frac{1}{B_t}P(t,T+\Delta).$$

The second term on the rhs of the last equation is a martingale by the HJM drift condition (3.3.1), so that we need to concentrate only on the first additive term, which for the moment we denote by R, namely

$$R_t := \frac{1}{B_t}P(t,T+\Delta)\frac{P^\Delta(t,T)}{P^\Delta(t,T+\Delta)}.$$

By exploiting proposition 3.3.1 and Ito's lemma, we get the following dynamics for the process R

$$\begin{aligned}\frac{dR_t}{R_t} &= [-A(t,T+\Delta) + A_\Delta(t,T+\Delta) - A_\Delta(t,T) + \frac{1}{2}\Sigma^2(t,T+\Delta) \\ &+ \frac{1}{2}(\Sigma_\Delta(t,T+\Delta) - \Sigma_\Delta(t,T))^2 - \Sigma(t,T+\Delta)(\Sigma_\Delta(t,T+\Delta) - \Sigma_\Delta(t,T))]dt \\ &+ [-\Sigma(t,T+\Delta) + \Sigma_\Delta(t,T+\Delta) - \Sigma_\Delta(t,T)]\, dW_t.\end{aligned}$$

Now the first and fourth terms on the right hand side of the last equation cancel out each other by the HJM drift condition (3.3.1). As a consequence the process R is a martingale if and only if (3.3.2) holds true. □

It is very important to note that, contrary to what happens to the drift process α of the risk-free forward rates, the drift α_Δ is not uniquely specified by the condition (3.3.2). In fact, by differentiating with respect to T this condition we get a constraint on $\alpha_\Delta(t,T+\Delta) - \alpha_\Delta(t,T)$. In exact analogy with the issue of defining a fictitious zero coupon curve from a forward LIBOR curve, also in this case the drift is fully specified only up to the first interval $[0,\Delta)$.

3.4 Alternative Specification

An equivalent way of specifying the model is to postulate directly a stochastic process for the spread s^Δ instead of the fictitious forward rate f^Δ. In other words we replace assumption 3.2.2, with the following

Assumption 3.4.1 (Instantaneous fictitious Δ-tenor forward spread dynamics)**.** For any $T \in (0,T^*]$, $s^\Delta(\cdot,T)$ follows an Ito process of the form

$$ds^\Delta(t,T) = \alpha_S(t,T)dt + \sigma_S(t,T)dW_t, \qquad t \in [0,T],$$

where the (possibly stochastic) coefficients α_S and σ_S are regular enough in order to have s^Δ well defined.

Again for ease of notation, in the following we will use the shorthands

$$A_S(t,T) := \int_t^T \alpha_S(t,u)du, \quad \Sigma_S(t,T) := \int_t^T \sigma_S(t,u)du.$$

We now proceed to derive a no-arbitrage restriction on the drift of $s(\cdot,T)$, which is analogous to the drift condition we found in (3.3.2). In order to do so it will be useful to write down the SDE satisfied by $P_\Delta(\cdot,T)$ in terms of the coefficients A_S and Σ_S, as we do in the following proposition

Proposition 3.4.2. *If the HJM drift condition* (3.3.1) *is satisfied, the fictitious bond* *prices* $P_\Delta(\cdot, T)$ *satisfy*

$$\frac{dP^\Delta(t,T)}{P^\Delta(t,T)} = \left(r_t + s^\Delta(t,t) - A_S(t,T) + \frac{1}{2}\Sigma_S^2(t,T) + \Sigma(t,T)\Sigma_S(t,T)\right)dt$$
$$- (\Sigma(t,T) + \Sigma_S(t,T))dW_t.$$

Proof. First, let us note that $P^\Delta(t,T) = P(t,T)\exp^{-\int_t^T s^\Delta(t,u)du}$. The SDE for the first term has already been derived and, if the HJM drift condition is satisfied, reads

$$\frac{dP(t,T)}{P(t,T)} = r_t dt - \Sigma(t,T)dW_t.$$

The second term can be treated by using the methods of Proposition 3.3.1. The result follows upon an application of Ito's lemma. □

We are now ready to present the main result of this section, i.e. the promised no-arbitrage condition on the drift α_S.

Theorem 3.4.3. *Under assumptions 3.2.1 and 3.4.1, the market consisting of all the zero coupon bonds, the bank account and the FRAs on the LIBOR rate* L^Δ*is free of arbitrage opportunities if and only if the HJM drift condition* (3.3.1) *is satisfied and, in addition,*

$$A_S(t,T+\Delta) - A_S(t,T) = \tag{3.4.1}$$
$$-\frac{1}{2}(\Sigma_S(t,T+\Delta) - \Sigma_S(t,T))^2 + \Sigma(t,T)(\Sigma_S(t,T+\Delta) - \Sigma_S(t,T))$$

i.e.,

$$\int_T^{T+\Delta} \alpha_S(t,u)du =$$
$$-\frac{1}{2}\left(\int_T^{T+\Delta} \sigma_S(t,u)du\right)^2 + \Sigma(t,T)\left(\int_T^{T+\Delta} \sigma_S(t,u)du\right).$$

Proof. Again, the bond market is taken care of by the HJM drift condition (3.3.1). As it was already noted above, absence of arbitrage opportunities within the FRA LIBOR market is equivalent to the processes $\frac{p(\cdot,T+\Delta)\Delta F_{L^\Delta}(\cdot,T)}{B}$ being martingales under $\mathbb{Q}_*$. As in the proof of theorem 3.3.3, we need only to concentrate on the process R that was defined as

$$R_t := \frac{1}{B_t}P(t,T+\Delta)\frac{P^\Delta(t,T)}{P^\Delta(t,T+\Delta)}.$$

By exploiting proposition 3.3.1 and an easy application of Ito's lemma, we get the following dynamics for the process R

$$\frac{dR_t}{R_t} = \mu(t,T)dt - \left[\Sigma(t,T) - (\Sigma_S(t,T+\Delta) - \Sigma_S(t,T))\right]dW_t$$

where, assuming that the HJM drift condition (3.3.1) holds, we have

$$\begin{aligned}\mu(t,T) &= -A_S(t,T) + \frac{1}{2}\Sigma_S^2(t,T) + \Sigma(t,T)\Sigma_S(t,T)\\ &\quad + A_S(t,T+\Delta) - \frac{1}{2}\Sigma_S^2(t,T+\Delta) - \Sigma(t,T+\Delta)\Sigma_S(t,T+\Delta)\\ &\quad + \Sigma_S(t,T+\Delta)\left[-(\Sigma(t,T)+\Sigma_S(t,T)) + (\Sigma(t,T+\Delta)+\Sigma_S(t,T+\Delta))\right].\end{aligned}$$

After some straightforward algebra, we get

$$\begin{aligned}\mu(t,T) &= A_S(t,T+\Delta) - A_S(t,T) + \frac{1}{2}(\Sigma_S(t,T+\Delta) - \Sigma_S(t,T))^2\\ &\quad - \Sigma(t,T)(\Sigma_S(t,T+\Delta) - \Sigma_S(t,T)).\end{aligned}$$

The process R is a martingale if and only if $\mu = 0$, which is equivalent to (3.4.1), as it was to be shown. □

Remark 3.4.4. Note that the condition on the drift of the fictitious *spread* we just derived has the same structure as the condition on the drift of the fictitious *forward rate*. The only minor difference is that the function Σ in the second addend is evaluated in T in the former case and in $T+\Delta$ in the latter.

This alternative formulation of the model lends itself to the analysis of an important special case, namely the case in which the spot LIBOR process L^Δ coincides with the risk-free spot rate Z^Δ, in which case we have that $F_{L^\Delta}(\cdot,T) = F_{R^\Delta}(\cdot,T) \quad \forall T$ and as a consequence $P^\Delta(\cdot,T) = P(\cdot,T) \quad \forall T$ is an admissible fictitious bond process and we have $s^\Delta(0,T) = 0 \quad \forall T$. Now if we assume that there is no volatility in the spread process, i.e. $\sigma_S(t,T) = 0$ we see that the drift

$$\alpha_S(t,T) = 0 \qquad \forall t < T$$

is admitted by our no-arbitrage condition. In this case we have the process $s^\Delta(\cdot,T)$ indistinguishable from zero and we are back to the classical HJM framework. However, there will be other non-zero drifts still compatible with no-arbitrage.

3.5 Further Developments

The above framework should and will be investigated further in the future.

To begin with, as stated in the text, we should understand if there is any room to define *the* canonical fictitious zero coupon bond curve associated with a forward LIBOR curve, by for example restricting the class of functions A. This would allow us to get rid of the indeterminacy in the fictitious forward curve. Also, we should understand if we can link the arbitrariness in defining P^Δ to the one in the drift.

Another interesting point would be to assess under what conditions there is a finite dimensional realization (most likely not including the short rate r^Δ), as this has been a classical subject of research until a few years ago.

Finally, we should attempt to develop a framework for all the $F_{L^\Delta}(\cdot, T)$'s, i.e. for all Δ's in some $[0, \Delta^*]$, in which case we would be in infinite dimension in two directions: the T's and the Δ's.

Chapter 4

Instantaneous Swap Rates

In this chapter, while retaining a classical single curve approach, we explore what happens when the tenor of a swap rate tends to zero. This was initially motivated by the desire to better understand the OIS's (Overnight Indexed Swaps), which were described in detail in Chapter 1, in which a floating leg pays (almost) continuously a rate with an (almost) infinitesimal tenor. Specifically, it is known, though overlooked in the literature, that the HJM drift condition is the infinitesimal limit of the LMM drift condition. The main result of this chapter is to develop an infinitesimal limit of the Swap Market Model drift condition of Jamshidian (1997), again, in a suitably defined framework.

4.1 Bonds, Spot Rates and Forward Rates

We now briefly recall the assumptions and definitions we made in Chapter 1. In a *classical* framework for the term-structure, we model a frictionless market in which trading takes place continuously over the time interval $[0, T^*]$, where T^* is an arbitrary final date. Furthermore, we assume that one risk-free zero-coupon bond $P_\cdot(T)$ is traded in this market for every $T \in [0, T^*]$.

The market being arbitrage-free is equivalent to the existence, for any $T \in [0, T^*]$, of a probability measure $\mathbb{Q}_T$ usually referred to as the T-forward measure, such that price processes discounted by $P_\cdot(T)$ are $\mathbb{Q}_T$-martingales. Expectations with respect to $\mathbb{Q}_T$ will be written as $\mathbb{E}^T$.

As it is classical in term-structure modeling, for a fixed $t \in [0, T^*]$ we use the discount curve[1] $T \mapsto P_t(T)$, which we assume to be smooth enough, to define *spot* rates of discrete and infinitesimal tenor:

[1]In this chapter, in order not to burden the notation, we will always indicate the current time by a subscript as in $P_t(T)$

- R_t^Δ: the time-t (simply compounded) spot rate of tenor Δ

$$R_t^\Delta := \frac{1}{\Delta}\left(\frac{1}{P(t,t+\Delta)} - 1\right). \tag{4.1.1}$$

- r_t: the time-t instantaneous spot rate

$$r_t := \lim_{\Delta\to 0+} R_t^\Delta. \tag{4.1.2}$$

Note that in (4.1.1) we defined a rate with the simple compounding convention, but other conventions might be used instead (e.g. annual or continuous). On the other hand, the defining equation (4.1.2) for r would have in no way changed if we took the limit of a spot rate with a different compounding rule. With the continuous compounding rule, for example, we do have

$$r_t = \lim_{\Delta\to 0}\frac{1}{\Delta}\ln\frac{1}{P_t(t+\Delta)}.$$

As it is also very classical, for a fixed $t \in [0,T^*]$ we use the discount curve $P_t(\cdot)$ to define *forward* rates of discrete and infinitesimal tenor:

- $F_t^\Delta(T)$: time-t (simply compounded) forward rate for time T of tenor Δ. This is the fair strike on a FRA on R^Δ setting at T and paying at $T+\Delta$ and is given by

$$F_t^\Delta(T) := E_t^{T+\Delta}[R_T^\Delta] = \frac{1}{\Delta}\left(\frac{P_t(T)}{P_t(T+\Delta)} - 1\right); \tag{4.1.3}$$

- $f_t(T)$: time-t instantaneous forward rate for time T

$$f_t(T) := \lim_{\Delta\to 0+} F_t^\Delta(T) = \begin{cases} E_t^T[r_T] \\ \frac{-\partial_T P_t(T)}{P_t(T)} = -\partial_T \ln P_t(T). \end{cases} \tag{4.1.4}$$

In equation (4.1.3), note that the first equality is really a definition, whereas the second equality can be deducted from the definition of R^Δ. In equation (4.1.4) the first equality is really a definition which can be manipulated in two ways depending on which expression for the forward rate $F_t^\Delta(T)$ in (4.1.3) one uses.

Here, the same remarks on compounding conventions we made about spot rates still apply.

Thus, for both the spot and the forward cases we defined a rate of *discrete* (positive) tenor and a rate of *instantaneous* (infinitesimal) tenor. Surprisingly enough, another fundamental rate in term-structure modeling, the swap rate, does not have an instantaneous counterpart. We explain the issue and fill the gap in the next section.

4.2 The Instantaneous Swap Rate and the Continuous Annuity

We denote by $S_t^\Delta(\{T,U\}_\Delta)$ the fair strike in a swap on R^Δ, with tenor structure $\{T,U\}_\Delta = (T_i)_{i=0,\dots,\frac{U-T}{\Delta}}$ where $T_i = T + i\Delta$. Note that all the dates T_i are equally spaced. This is the fixed rate K such that a swap for exchanging ΔK for $\Delta R^\Delta_{T_{i-1}}$ for all $i = 1, 2, \dots, \frac{U-T}{\Delta}$ has a null price at time t. Classical no-arbitrage considerations force the swap rate to the following

$$S_t^\Delta(\{T,U\}_\Delta) = \frac{P_t(T) - P_t(U)}{\sum_{i=1}^n (T_i - T_{i-1}) P_t(T_i)}. \tag{4.2.1}$$

Note that the price of the floating leg of the swap (the numerator in the expression above) depends on the tenor structure $\{T,U\}_\Delta$only through its first and last date, T and U. This is due to the fact that the floating rate R^Δ is risk-free and its tenor corresponds to the spacing between payments.

Inspired by the definitions of the instantaneous rates we recalled above, we now introduce the concept of *instantaneous* swap rate $s_t(T,U)$ as follows

Definition 4.2.1 (Instantaneous Swap Rate)**.** The time-t *instantaneous* swap rate for the tenor structure $[T,U]$ is defined as

$$s_t(T,U) := \lim_{\Delta\to 0} S_t^\Delta(\{T,U\}_\Delta) = \frac{P_t(T) - P_t(U)}{\int_T^U du P_t(u)}. \tag{4.2.2}$$

This is the fair swap rate for a swap in which the fixed leg pays continuously whereas the floating leg pays the tenor Δ risk-free rate every Δ units of time. This Δ can be any whatsoever, since in any case the price of the floating leg does not depend on it. It can also be infinitesimal, meaning that the floating leg pays continuously the instantaneous spot rate r. If this is the case the time-t price of the floating leg might be written as

$$\begin{aligned}
\int_T^U du P_t(u) \mathbb{E}_t^u[r_u] &= \int_T^U du P_t(u) f_t(u) \\
&= \int_T^U du P_t(u) \frac{-\partial_T P_t(u)}{P_t(u)} \\
&= P_t(T) - P_t(U).
\end{aligned}$$

which of course yields to the same price. The expression for the instantaneous swap rate

$$s_t(T,U) = \frac{\int_T^U du P_t(u) f_t(u)}{\int_T^U du P_t(u)} \tag{4.2.3}$$

is particularly illuminating since it shows clearly that $s_t(T,U)$ is the average of the function $f_t(\cdot)$ on $[T,U]$ under the positive measure $A \mapsto \int_A du P_t(u)$.

Also note that the floating leg could also be priced using the standard risk-neutral measure $\mathbb{Q}$, i.e. the measure associated to the "bank account" numeraire $B = e^{\int_0^\cdot r_s ds}$, since we could write

$$\begin{aligned}
\int_T^U duP_t(u)\mathbb{E}_t^u[r_u] &= \int_T^U du\mathbb{E}_t[e^{-\int_t^u r_s ds} r_u] \\
&= \int_T^U du\mathbb{E}_t[-\partial_u e^{-\int_t^u r_s ds}] \\
&= \mathbb{E}_t[-\int_T^U du \frac{\partial}{\partial u} e^{-\int_t^u r_s ds}] \\
&= \mathbb{E}_t[e^{-\int_t^T r_s ds} - e^{-\int_t^U r_s ds}] \\
&= P_t(T) - P_t(U),
\end{aligned}$$

which yields once again the same price.

The fixed leg in a standard swap (the term in the denominator of (4.2.1)) is usually referred to as the (discrete) annuity and denoted by $A.(\{T, U\}_\Delta)$. In other words we have

$$A_t(\{T, U\}_\Delta) := \sum_{i=1}^{n} (T_i - T_{i-1}) P_t(T_i). \tag{4.2.4}$$

This quantity can be used as a numeraire, being a linear combination of bonds with constant coefficients and we denote the associated martingale measure by $\mathbb{Q}_{\{T,U\}_\Delta}$. By the expression (4.2.1) for the (discrete) swap rate, we readily see that $S.(\{T, U\}_\Delta)$ must necessarily be a martingale under $\mathbb{Q}_{\{T,U\}_\Delta}$.

Now we push the analogies we made so far a little further and define the continuous analogue of the (discrete) annuity:

Definition 4.2.2 (Continuous annuity process)**.** For fixed $T < U$, the continuous annuity process $a(T, U)$ is defined as

$$a_t(T, U) := \int_T^U duP_t(u), \quad t < T. \tag{4.2.5}$$

We assume that the process $a.(T, U)$ can be used as a numeraire and we denote the associated martingale measure by $Q_{[T,U]}$. Note that this assumption is even more innocuous than the assumption that the "bank account" process B is traded. In fact, the latter process requires the possibility of investing with measure-valued portfolios which have to vary stochastically over time, whereas the continuous annuity process only requires measure-valued portfolios which are constant over time. Thus if one is to accept the assumption that B is traded (which is the cornerstone of all classical short rate models), then he is forced to accept the fact that the continuous annuity $a(T, U)$ is traded. With regard to this point, note that the continuous annuity $a(T, U)$ is to the discrete annuity $A(\{T, U\}_\Delta)$ as the "bank account" process B is to the "spot LIBOR" process of Jamshidian (1997).

Obviously, since the instantaneous swap rate can be written as

$$s_t(T,U) = \frac{P_t(T) - P_t(U)}{A_t(T,U)},$$

no arbitrage forces it to be a $Q_{[T,U]}$-martingale.

Finally, in the next lemma, we now compute the likelihood ratio process of the latter measure with respect to a generic S-forward measure. This will be useful is the sequel.

Lemma 4.2.3. *The $[T,U]$-annuity measure $\mathbb{Q}_{[T,U]}$ has the following likelihood ratio process with respect to the S-forward measure $\mathbb{Q}_S$*

$$\frac{d\mathbb{Q}_{[T,U]}}{d\mathbb{Q}_S}\bigg|\mathcal{F}_t \propto \int_T^U du \frac{P_t(u)}{P_t(S)}.$$

Note that differentiation of $a(\cdot,\cdot)$ with respect to any of its arguments leads to a bond price, i.e.

$$\begin{aligned} P(T) &= -\partial_T a(T,U), \\ P(U) &= \partial_U a(T,U). \end{aligned}$$

4.3 Bond Prices from Instantaneous Swap Rates

In this subsection, we hold fixed the present time t. We will use the notation $\mathcal{T}_t = \{(T,U) \in [t,T^*] : T < U\}$.

We noted already that the map defining the instantaneous forward curve $P \mapsto f(P) := \partial_T \ln P_t(\cdot)$ is a bijection from $\mathcal{C}^1([t,T^*];\mathbb{R}_+)$ into $\mathcal{C}([t,T^*];\mathbb{R})$, with inverse $P(t,T) = e^{-\int_t^T f_t(\cdot)}$.

However, the map defining the instantaneous swap curve $P \mapsto s(P)$ where $s(P)(T,U) := \frac{P_t(T)-P_t(U)}{\int_T^U du P_t(u)}$ is not surjective from $\mathcal{C}^1([t,T^*];\mathbb{R}_+)$ into $\mathcal{C}(\mathcal{T}_t;\mathbb{R})$. In fact the rates $s_t(T,U)$ with $(T,U) \in \mathcal{T}_t$ must satisfy the consistency condition

$$s_t(T,U) = \frac{\int_T^V du P_t(u)}{\int_T^U du P_t(u)} s_t(T,V) + \frac{\int_V^U du P_t(u)}{\int_T^U du P_t(u)} s_t(V,U).$$

In the next proposition we show that, for fixed $t < T$, knowledge of the mapping $U \mapsto s_t(T,U)$ in any right neighborhood of T implies knowledge of $f_t(T)$

Proposition 4.3.1.

$$f_t(T) = \lim_{h \downarrow 0} s_t(T, T+h).$$

Proof. Simply note that

$$s_t(T,T+h) = \frac{\frac{1}{h}(P_t(T) - P_t(T+h))}{\frac{1}{h}\int_T^{T+h} du P_t(u)},$$

so that

$$\lim_{h\downarrow 0} s_t(T, T+h) = \frac{-\partial_T P_t(T)}{P_t(T)}.$$

Alternatively, we could have used the expression for $s_t(T, U)$ given in (4.2.3). $\square$

Corollary 4.3.2.

$$r_t = \lim_{h\downarrow 0} s_t(t, t+h).$$

We indicate prices discounted by the generic T_γ-bond $P_\cdot(T_\gamma)$ with a γ superscript, i.e.

$$P^\gamma(T) := \frac{P(T)}{P(T_\gamma)},$$

$$a^\gamma(T, U) := \frac{a(T, U)}{P(T_\gamma)} = \int_T^U du P^\gamma(u).$$

Note that, in general, we have

$$P^\gamma(T) = -\partial_T a^\gamma(T, U),$$

$$P^\gamma(U) = \partial_U a^\gamma(T, U).$$

It is thus clear that the knowledge of the whole surface $\mathcal{T} \ni (T, U) \mapsto s(T, U)$ is more than enough to recover all the bond prices $[0, T^*] \ni T \mapsto P(T)$. In the next two propositions, we show that a much coarser knowledge is sufficient in order to recover relative bond prices. In particular, in Proposition 4.3.3 we compute the full set of relative bond prices from a set of coterminal swap rates with common maturity T_β and in Proposition 4.3.4 we compute the full set of relative bond prices from a set of coinitial swap rates with common forward start T_α.

Proposition 4.3.3. *For any fixed maturity T_β, given a set of coterminal instantaneous swap rates $\{s(T, T_\beta), T < T_\beta\}$, the $P(T_\beta)$-discounted bond prices and continuous annuities are given by*

$$P^\beta(T) = 1 + s(T, T_\beta) \int_T^{T_\beta} du e^{\int_T^u s(\cdot, T_\beta)}, \tag{4.3.1}$$

$$a^\beta(T, T_\beta) = \int_T^{T_\beta} du e^{\int_T^u s(\cdot, T_\beta)}. \tag{4.3.2}$$

Proof. Since the swap rate can be written as

$$s(T, T_\beta) = \frac{P^\beta(T) - 1}{a^\beta(T, T_\beta)},$$

it is clear that $a^\beta(\cdot, T_\beta)$ satisfies the ODE

$$\begin{cases} \partial_T a^\beta(T, T_\beta) + s(T, T_\beta) a^\beta(T, T_\beta) = -1, & T < T_\beta, \\ a^\beta(T_\beta, T_\beta) = 0, \end{cases}$$

which has solution

$$a^\beta(T,T_\beta) = \int_T^{T_\beta} du e^{\int_T^u s(\cdot,T_\beta)},$$

The expression for $P^\beta(T)$ can be found upon differentiating with respect to T the last equation. □

Proposition 4.3.4. *For any fixed maturity T_β, given a set of coinitial instantaneous swap rates $\{s(T_\alpha,T), T > T_\alpha\}$, the $P(T_\alpha)$-discounted bond prices and continuous annuities are given by*

$$P^\alpha(T) = 1 - s(T_\alpha,T)\int_{T_\alpha}^{T} du e^{-\int_u^T s(T_\alpha,\cdot)}, \tag{4.3.3}$$

$$a^\alpha(T_\alpha,T) = \int_{T_\alpha}^{T} du e^{-\int_u^T s(T_\alpha,\cdot)}. \tag{4.3.4}$$

Proof. Since the swap rate can be written as

$$s(T_\alpha,T) = \frac{1-P^\alpha(T)}{a^\alpha(T_\alpha,T)},$$

it is clear that $a^\alpha(T_\alpha,\cdot)$ satisfies the ODE

$$\begin{cases} \partial_T a^\alpha(T_\alpha,T) + s(T_\alpha,T)a^\alpha(T_\alpha,T) = 1, & T > T_\alpha, \\ a^\alpha(T_\alpha,T_\alpha) = 0, \end{cases}$$

which has solution

$$a^\alpha(T_\alpha,T) = \int_{T_\alpha}^{T} du e^{-\int_u^T s(T_\alpha,\cdot)}.$$

The expression for $P^\alpha(T)$ can be found upon differentiating with respect to T the last equation. □

4.4 Instantaneous Coterminal Swap Rate Model

In his seminal paper, Jamshidian (1997) proposed a model for the coterminal (obviously, discrete) swap rates $(S^\Delta_\cdot(\{T+n\Delta,T_\beta\}_\Delta))_n$ under the measure $\mathbb{Q}_{T_\beta}$.

Throughout this section, we hold fixed a final time T_β, which will represent the final expiry of all the swap rates which constitute the model. We postulate that, for every $T < T_\beta$, the forward swap rate process $s_\cdot(T,T_\beta)$ follows an Ito process of the form

$$ds_t(T,T_\beta) = \mu_t(T,T_\beta)dt + \sigma_t(T,T_\beta)\cdot dW_t^{T_\beta} \quad t\in[0,T] \tag{4.4.1}$$

for some $\mathbb{R}^d$-valued $\mathbb{Q}_{T_\beta}$-Wiener process W^{T_β}. We assume that, for every $T < T_\beta$, the instantaneous drift $(\mu_t(T,T_\beta))_{t\in[0,T]}$ and volatilities $(\sigma_t(T,T_\beta))_{t\in[0,T]}$ processes are adapted to $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ and satisfy the integrability conditions

$$\mathbb{E}\int_0^T dt|\mu_t(T,T_\beta)| < \infty,$$
$$\mathbb{E}\int_0^T dt|\sigma_t(T,T_\beta)|^2 < \infty.$$

For ease of notation we define, for $T < U < T_\beta$, the integrated drifts and volatilities as follows

$$M_t(T,U,T_\beta) := \int_T^U \mu_t(\cdot,T_\beta), \tag{4.4.2}$$
$$\Sigma_t(T,U,T_\beta) := \int_T^U \sigma_t(\cdot,T_\beta). \tag{4.4.3}$$

Note that, as said, the final expiry T_β of all the swaps in the model is held fixed through all this section, but we find it convenient not to suppress it from the notation.

The main concern with a model such as (4.4.1) is indeed to guarantee the absence of arbitrage opportunities, which is equivalent to the fact that the discounted annuity process $a_\cdot^\beta(T,T_\beta)$ is a martingale for every T, since we specified the model directly under the measure $\mathbb{Q}_{T_\beta}$.

In the following technical proposition, which is the key to the proof of the main theorem of this section, we find the dynamics of $a^\beta(T,T_\beta)$ for an arbitrary but fixed T.

Proposition 4.4.1. *For every $T < T_\beta$, the dynamics of $a^\beta(T,T_\beta)$ are given by*

$$da_t^\beta(T,T_\beta) = D_t(T,T_\beta)dt + V_t(T,T_\beta)\cdot dW_t^{T_\beta}, \tag{4.4.4}$$

where

$$D_t(T,T_\beta) := \int_T^{T_\beta} du e^{\int_T^u s_t(\cdot,T_\beta)}\left[M_t(T,u,T_\beta) + \frac{1}{2}\left|\Sigma_t(T,u,T_\beta)\right|^2\right], \tag{4.4.5}$$
$$V_t(T,T_\beta) := \int_T^{T_\beta} du e^{\int_T^u s_t(\cdot,T_\beta)}\Sigma_t(T,u,T_\beta). \tag{4.4.6}$$

Proof. Let us temporally define

$$Y_t(u) := e^{\int_T^u s_t(\cdot,T_\beta)}$$

and use proposition (4.3.2) to write $a_t^\beta(T,T_\beta)$ as

$$a_t^\beta(T,T_\beta) = \int_T^{T_\beta} du Y_t(u).$$

Now, the stochastic differential of the positive process Y can be computed as

$$\frac{dY_t(u)}{Y_t(u)} = \left[M_t(T,u,T_\beta) + \frac{1}{2}\left|\Sigma_t(T,u,T_\beta)\right|^2\right]dt + \Sigma_t(T,u,T_\beta)\cdot dW_t^{T_\beta}.$$

Finally, to compute the instantaneous drift and volatility of $a^\beta(T,T_\beta)$ it is sufficient to integrate those of $Y_\cdot(u)$ with respect to u. $\square$

In light of Proposition 4.4.1 above, we are now in position to characterize the absence of arbitrage in term of the drift processes $\mu(T,T_\beta)$.

Theorem 4.4.2. *The instantaneous coterminal swap rate model specified in* (4.4.1) *is free of arbitrage if and only if for every $T < T_\beta$ the drift $\mu(T,T_\beta)$ is given by*

$$\mu_t(T,T_\beta) = -\frac{1}{a_t^\beta(T,T_\beta)}\sigma_t(T,T_\beta)\cdot V_t(T,T_\beta), \tag{4.4.7}$$

which can be written also as

$$\mu_t(T,T_\beta) = -\frac{1}{a_t^\beta(T,T_\beta)}\sigma_t(T,T_\beta)\cdot \int_T^{T_\beta} du e^{\int_T^u s_t(\cdot,T_\beta)} a_t^\beta(u,T_\beta)\sigma_t(u,T_\beta). \tag{4.4.8}$$

Proof. It is clear that freedom of arbitrage in the whole model is equivalent to the process $a^\beta(T,T_\beta)$ begin a martingale for every T. This in turn is equivalent to the fact that, for every arbitrary but fixed t,

$$D_t(T,T_\beta) = 0 \quad \forall T \in [t,T_\beta].$$

Let us fix such a t. It can be checked that $T \mapsto D_t(T,T_\beta)$ satisfies

$$\begin{cases} \partial_T D_t(T,T_\beta) = \int_T^{T_\beta} du e^{\int_T^u s_t(\cdot,T_\beta)} \left[-\mu_t(T,T_\beta) - \sigma_t(T,T_\beta)\cdot \Sigma_t(T,u,T_\beta)\right] - s_t(T,T_\beta)D_t(T,T_\beta), \\ D_t(,T_\beta,T_\beta) = 0, \end{cases}$$

so that $T \mapsto D_t(T,T_\beta)$ is constantly equal to zero iff the first term on the right-hand-side of the ODE above is zero, i.e. iff

$$\mu_t(T,T_\beta)a_t^\beta(T,T_\beta) = -\sigma_t(T,T_\beta)\cdot \int_T^{T_\beta} du e^{\int_T^u s_t(\cdot,T_\beta)}\Sigma_t(T,u,T_\beta),$$

which is readily seen to be equivalent to (4.4.7). In order to get (4.4.8) it is sufficient to rewrite $V_t(T,T_\beta)$ as follows

$$\begin{aligned} V_t(T,T_\beta) &= \int_T^{T_\beta} du e^{\int_T^u s_t(\cdot,T_\beta)} \int_T^u dv \sigma_t(v,T_\beta) \\ &= \int_T^{T_\beta} dv \sigma_t(v,T_\beta) \int_v^{T_\beta} du e^{\int_T^u s_t(\cdot,T_\beta)} \\ &= \int_T^{T_\beta} dv \sigma_t(v,T_\beta) e^{\int_T^v s_t(\cdot,T_\beta)} \int_v^{T_\beta} du e^{\int_v^u s_t(\cdot,T_\beta)} \end{aligned}$$

where the interchange of order of integration is justified by our assumptions on σ. $\square$

The drift condition just found can be interpreted as the infinitesimal limit of the drift condition in the coterminal Swap Market Model introduced in Jamshidian (1997).

Remark 4.4.3. Both the expressions for the no arbitrage drift process of $s(T,T_\beta)$ in the previous theorem show that $\mu(T,T_\beta)$ is the projection of the instantaneous volatility vector onto the average on $[T, T_\beta]$ of some functional of the volatility process under the measure

$$A \mapsto \int_A due^{\int_T^u s_t(\cdot, T_\beta)},$$

which has mass $a^\beta(T, T_\beta)$ on $[T, T_\beta]$. In the expression (4.4.7), this functional is

$$u \mapsto \Sigma_t(T, u, T_\beta),$$

which is expressed in terms of the integrated volatility whereas in the expression (4.4.8) it is written directly on σ and reads

$$u \mapsto a_t^\beta(u, T_\beta)\sigma_t(u, T_\beta).$$

The crucial point of this model for coterminal instantaneous swap rates is indeed that everything is specified under the *single* measure $\mathbb{Q}_{T_\beta}$, since a model in which every swap rate if specified under a different measure would be of little use. The next proposition shows that, as expected, every $s(T, T_\beta)$ is a $\mathbb{Q}_{[T,T_\beta]}$-martingale by finding a $\mathbb{Q}_{[T,T_\beta]}$ Wiener process.

Proposition 4.4.4. *In the instantaneous coterminal swap rate model specified in* (4.4.1)*, if the no-arbitrage drift condition* (4.4.7) *is satisfied, then, for every $T < T_\beta$, the process*

$$W^{[T,T_\beta]} := W^{T_\beta} - \int_0^\cdot du \frac{1}{a_u^\beta(T, T_\beta)} V_u(T, T_\beta) \tag{4.4.9}$$

is a Wiener under $\mathbb{Q}_{[T,T_\beta]}$. Furthermore $s(T, T_\beta)$ satisfies

$$ds_t(T, T_\beta) = \sigma_t(T, T_\beta) \cdot dW_t^{[T,T_\beta]} \tag{4.4.10}$$

and is then a martingale under $\mathbb{Q}_{[T,T_\beta]}$.

Proof. By Lemma 4.2.3, we have that

$$\left.\frac{d\mathbb{Q}_{[T,T_\beta]}}{d\mathbb{Q}_{T_\beta}}\right| \mathcal{F}_t = a_t^\beta(T, T_\beta),$$

so that it is sufficient to compute the stochastic differential of $a^\beta(T, T_\beta)$ given in Proposition 4.4.1, where the drift vanishes thanks to the no-arbitrage drift condition, and to apply the Girsanov theorem for Wiener processes. □

As it is well known, in the LIBOR Market Model for discrete forward rates, a deterministic volatility function does not imply that the forward rate process is neither lognormal nor normal[2]. The state of affairs for discrete swap rate is analogous to that of discrete forward rates, since in the coterminal model of Jamshidian (1997) the swap rates are neither normal nor lognormal even for a deterministic volatility function. Since this problem

[2]In fact, a large body of literature aimed at finding good approximations to the law of the discrete forward rates (see e.g. Kurbanmuradov et al. (2002), Hunter et al. (2001) and Daniluk and Gatarek (2005)).

does not appear in the context of the HJM model for instantaneous forward rates, where a deterministic volatility process implies that every forward rate is a Gaussian process under the terminal (or the spot) martingale measure, one might be led to conjecture that in our instantaneous version of the swap market model, deterministic volatilities $\sigma(T, T_\beta)$ would lead to deterministic drifts $\mu(T, T_\beta)$ and thus to a family of Gaussian process. Unfortunately, Theorem 4.4.2 makes it clear that this is not the case: a deterministic (or even constant, for that matter) volatility function would still imply a quite complicated non-deterministic drift.

4.5 Instantaneous Coinitial Swap Rate Model

Throughout this section, we hold fixed an initial time T_α, which will represent the forward start of all the swap rates which constitute the model. In other words, we are now considering a coinitial swap model, whose discrete counterpart was introduced by Galluccio and Hunter (2004). The financial reason for introducing a coinitial swap model is that of pricing european derivatives which depend on the realization of two or more swap rates at time T. We now postulate that, for every $T > T_\alpha$, the forward swap rate process $s_\cdot(T_\alpha, T)$ follows an Ito process of the form

$$ds_t(T_\alpha, T) = \mu_t(T_\alpha, T)dt + \sigma_t(T_\alpha, T) \cdot dW_t^{T_\alpha} \quad t \in [0, T_\alpha] \tag{4.5.1}$$

for some $\mathbb{R}^d$-valued $\mathbb{Q}_{T_\alpha}$-Wiener process W^{T_α}. We assume that, for every $T > T_\alpha$, the instantaneous drift $(\mu_t(T_\alpha, T))_{t\in[0,T_\alpha]}$ and volatilities $(\sigma_t(T_\alpha, T))_{t\in[0,T_\alpha]}$ processes are adapted to $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ and satisfy the integrability conditions

$$\mathbb{E}\int_0^{T_\alpha} dt|\mu_t(T_\alpha, T)| < \infty,$$
$$\mathbb{E}\int_0^{T_\alpha} dt|\sigma_t(T_\alpha, T)|^2 < \infty.$$

For ease of notation we define, analogously to the previous section, for $T_\alpha < U < T$, the integrated drifts and volatilities as follows

$$M_t(T_\alpha, U, T) := \int_U^T \mu_t(T_\alpha, \cdot), \tag{4.5.2}$$

$$\Sigma_t(T_\alpha, U, T) := \int_U^T \sigma_t(T_\alpha, \cdot). \tag{4.5.3}$$

It is clear that the absence of arbitrage opportunities is now equivalent to the fact that the discounted annuity process $a_\cdot^\alpha(T_\alpha, T)$ is a martingale for every T, since we specified the model directly under the measure $\mathbb{Q}_{T_\alpha}$.

The following technical proposition, in which we find the dynamics of $a^\alpha(T_\alpha, T)$ for an arbitrary but fixed T, is the counterpart of Proposition 4.4.1. The proof is similar and we omit it.

Proposition 4.5.1. *For every $T > T_\alpha$, the dynamics of $a^\alpha(T_\alpha, T)$ are given by*

$$da_t^\alpha(T_\alpha, T) = D_t(T_\alpha, T)dt - V_t(T_\alpha, T)dW_t^{T_\alpha}, \tag{4.5.4}$$

where

$$D_t(T_\alpha, T) := \int_{T_\alpha}^{T} due^{-\int_u^T s_t(T_\alpha, \cdot)} \left[-M_t(T_\alpha, u, T) + \frac{1}{2} \left| \Sigma_t(T_\alpha, u, T) \right|^2 \right], \tag{4.5.5}$$

$$V_t(T_\alpha, T) := \int_{T_\alpha}^{T} due^{-\int_u^T s_t(T_\alpha, \cdot)} \Sigma_t(T_\alpha, u, T). \tag{4.5.6}$$

In light of the Proposition above, we are now in position to characterize the absence of arbitrage in terms of the drift processes $\mu(T_\alpha, T)$. Again, the proof mimicks the analogue proof for the coterminal model and we omit it.

Theorem 4.5.2. *The instantaneous coinitial swap rate model specified in* (4.5.1) *is free of arbitrage if and only if for every $T > T_\alpha$ the drift $\mu(T_\alpha, T)$ is given by*

$$\mu_t(T_\alpha, T) = \frac{1}{a_t^\alpha(T_\alpha, T)} \sigma_t(T_\alpha, T) \cdot V_t(T_\alpha, T), \tag{4.5.7}$$

which can be written also as

$$\mu_t(T_\alpha, T) = \frac{1}{a_t^\alpha(T_\alpha, T)} \sigma_t(T_\alpha, T) \cdot \int_{T_\alpha}^{T} due^{-\int_u^T s_t(T_\alpha, \cdot)} a_t^\alpha(T_\alpha, u) \sigma_t(T_\alpha, u). \tag{4.5.8}$$

4.6 Conclusions

In this chapter, we filled an important gap in the interest rate literature by introducing the concept of the instantaneous swap rate. We showed how it is possible to recover discounted bond prices from a family of coinitial or coterminal swap rates and we proposed a diffusive model to evolve a continuum of instantaneous swap rate, which can be taken coterminal or coinitial. No arbitrage in this kind of models is guaranteed by a drift condition which makes discounted continuous annuities martingales.

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