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CYCLICALLY PRESENTED MODULES, AUTOMORPHISM-INVARIANT MODULES AND POOR MODULES

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ABSTRACT

In this thesis, we study three kinds of modules: cyclically presented modules in relation to factorization of elements in a noncommutative integral domain, automorphism-invariant modules and poor modules. First, we investigate projective covers of cyclically presented modules, characterizing the rings over which every cyclically presented module has a projective cover. Such rings R are Von Neumann regular modulo their Jacobson radical $J(R)$ and their idempotents can be lifted modulo $J(R)$. Then we study the modules M_R whose endomorphism ring $E := \mathrm{End}(M_R)$ is Von Neumann regular modulo $J(E)$ and their idempotents lift modulo $J(E)$. Next, the endomorphism rings of automorphism-invariant modules and their injective envelopes are investigated. We consider some cases where automorphism-invariant modules are quasi-injective and a connection between automorphism-invariant modules and boolean rings. Finally, we give some necessary conditions for rings over which every non-zero cyclic module is poor.

SOMMARIO

In questa tesi studiamo tre tipi di moduli: i moduli ciclicamente presentati in relazione alla fattorizzazione degli elementi di un dominio di integrità non commutativo, i moduli automorphism-invariant e i moduli poveri. Innanzitutto studiamo i ricoprimenti proiettivi dei moduli ciclicamente presentati, caratterizzando gli anelli sui quali ogni modulo ciclicamente presentato ha un ricoprimento proiettivo. Tali anelli R sono regolari alla Von Neumann modulo il loro radicale di Jacobson $J(R)$ e i loro idempotenti si sollevano modulo $J(R)$. Poi studiamo i moduli M_R il cui anello degli endomorfismi $E := \mathrm{End}(M_R)$ è regolare alla Von Neumann modulo $J(R)$ e i loro idempotenti si sollevano modulo $J(R)$. Studiamo quindi gli anelli degli endomorfismi dei moduli automorphism-invariant e i loro inviluppi iniettivi, consideriamo alcuni casi in cui i moduli automorphism-invariant sono quasi-iniettivi ed una relazione tra i moduli automorphism-invariant e gli anelli booleani. Infine diamo alcune condizioni necessarie per gli anelli sui quali ogni modulo ciclico è povero.

To my parents

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LIST OF SYMBOLS

R	an associative ring with $1 \neq 0$.
$J(R)$	the Jacobson radical of the ring R.
$Rad(M_R)$	the radical of the module M_R.
$N(R)$	the prime radical of the ring R.
$U(R)$	the group of invertible elements of the ring R.
$Z(R)$	the right singular ideal of the ring R.
$Z(M_R)$	the singular submodule of the module M_R.
$Z_2(M_R)$	the second singular submodule of the module M_R.
$Soc(M_R)$	the socle of the module M_R.
$E(M)$	the injective envelope of the module M_R.
$M^{(I)}$	the direct sum $\oplus_{i \in I} M_i$ with $M_i \cong M$ for all $i \in I$.
$Hom(A, B)$	the set of all morphisms from A to B.
$Tr_A(B)$	$\sum_{f \in Hom(B,A)} Imf$.
$A \leq_e B$	A is an essential submodule of B.
$A << B$	A is a superfluous submodule of B.
$\mathrm{End}(M_R)$	the endomorphism ring of the module M_R.
$\Delta(M, M)$	the set of all module morphisms $f\colon M \to M$ whose kernel $\mathrm{Ker}(f)$ is an essential submodu
$\mathcal{A}^{\perp}$	the class of modules orthogonal to all members of $\mathcal{A}$.
$M_n(R)$	the ring of $n \times n$ matrices over the ring R.
$In^{-1}(M)$	the injectivity domain of the module M.
Mod-R	the category of all right R-modules.
$\lvert A \rvert$	the cardinality of the set A.
$c(M)$	the composition length of the module M.

INTRODUCTION

In recent years, some new powerful techniques have been introduced in Module Theory which can be conveniently subdivided as follows:

- the study of modules over arbitrary rings,
- the study of modules over special rings,
- the study of rings R by way of the category Mod-R, or subcategories of it.

The aim of this thesis is to study three kinds of modules: cyclically presented modules, automorphism-invariant modules and poor modules. The organization of the thesis is given as follows.

In the first chapter of the thesis we review the background knowledge needed for studying our targets that would appear in the rest three chapters.

In the first section of chapter two, we recall some properties of cyclically presented modules over a local ring R. The material for this section is from the paper [AAF08]. The rest of this chapter contains the material from my joint paper with Alberto Facchini and Daniel Smertnig [FDT14]. We study some natural connections between cyclically presented R-modules, their submodules, their projective covers and factorizations of elements in the ring R. That is, we find some results on projective covers of cyclically presented modules and apply them to the study of factorizations of elements in a ring. In this way, we are naturally led to the class of 2-firs. Recall that a ring R is a 2-fir if every right ideal of R generated by at most 2 elements is free of unique rank. This condition is right/left symmetric, and a ring R is a 2-fir if and only if it is a domain and the sum of any two principal right ideals with non-zero intersection is again a principal right ideal [Coh85, Theorem 1.5.1]. P. M. Cohn investigated factorization of elements in 2-firs, applying the Artin-Schreier Theorem and the Jordan-Hölder-Theorem to the corresponding cyclically presented modules [Coh85]. One of the main ideas developed in this chapter is to characterize the submodules of a cyclically presented module M_R that, under a suitable cyclic presentation $\pi_M \colon R_R \to M_R$, lift to principal right ideals of R that are generated by a left cancellative element (Lemmas 2.2.2, 2.3.1 and 2.4.3). The key role is played by a class of cyclically presented submodules of a cyclically presented module M_R, which we call *π_M-exact submodules* of M_R. We show (Theorem 2.3.8) that, for every cyclically presented right R-module M_R and every cyclic presentation $\pi_M \colon R_R \to M_R$ with non-zero kernel, the set of all cyclically presented π_M-exact submodules is closed under finite sums if and only if R is a 2-fir. As we have said above, when sums and intersections of exact submodules are again exact submodules, we can use the Artin-Schreier and the Jordan-Hölder Theorems to study factorizations of elements. We also study the rings over which every cyclically presented module has a projective cover. We characterize these rings as the rings R that are Von Neumann regular modulo their Jacobson radical $J(R)$ and in which idempotents can be lifted modulo $J(R)$ (Theorem 2.4.1). Finally, in the last Section, we consider the modules M_R whose endomorphism rings E are Von Neumann regular modulo the Jacobson radical $J(E)$ and in which idempotents can be lifted modulo $J(E)$. In particular, this applies to the case in which the module M_R in question is quasi-projective (Lemma 2.5.3 and Proposition 2.5.5).

The third chapter is devoted to automorphism-invariant modules. We review some basic facts of automorphism-invariant modules in section 3.1. The main result of section 3.2 is that every automorphism-invariant module is the direct sum of a quasi-injective module and a square-free module [ESS13, Theorem 3]. In section 3.3, automorphism-invariant modules are proved to satisfy Condition (C_2) and (C_3) as well as satisfy Condition (C_1) if and only if they are quasi-injective. In the next section, we will see that automorphism-invariant modules have the exchange property [AS13], so that indecomposable automorphism-invariant modules have a local endomorphism ring. Moreover, idempotents can be lifted modulo every right ideal both in $\mathrm{End}(M)$ and in $\mathrm{End}(E(M))$ [Nic77]. The main theorem of section 3.2 leads us to study, for an automorphism-invariant square-free module M, the relation between M being quasi-injective and the existence of factors isomorphic to $\mathbb{F}_2$ in $\mathrm{End}(M)$ and in $\mathrm{End}(E(M))$ in section 3.5. Notice that if M is an automorphism-invariant right R-module and $\mathrm{End}(M)$ has no factor isomorphic to $\mathbb{F}_2$, then M is quasi-injective [AS14, Theorem 3]. In the last section, we study the connection between automorphism-invariant modules and boolean rings. The existence of such a connection was suggested to us by the results in Section 5 of [Vam05], where Vámos considers modules whose endomorphism ring (or endomorphism ring modulo the Jacobson radical) is a boolean ring. He studies modules in which the identity endomorphism is the sum of two automorphisms. This condition is related to the existence of factors of the endomorphism ring isomorphic to the field $\mathbb{F}_2$ with two elements [KS07]. The part of results in this chapter is taken from my joint paper with Adel Alahmadi and Alberto Facchini [AFT15]

Chapter four is about poor modules introduced for the first time by Alahmadi, Alkan and López Permouth in the paper [AAL10]. In the first section of chapter 4, we mention some basic properties of poor modules and investigate rings having projective semisimple poor modules. In Section 4.2, we characterize rings having semisimple poor modules and give some examples for such rings. In section 4.3, we consider rings whose modules are either injective or poor and give necessary conditions for such rings. Moreover, we give some sufficient conditions for those rings, too. The material for the first three sections is all taken from the paper [AAL10] and the paper [ELS11]. The last section is devoted to characterizing rings over which every non-zero cyclic module is poor. The results in this section are based on the unpublished paper [ELT], which is presently being prepared for submission.

Chapter 1

Preliminaries

1.1 Basic concepts

All rings we consider are associative rings R with $1_R \neq 0_R$ and modules are unital right R-modules unless we state differently.

Proposition 1.1.1. *The following conditions are equivalent for a ring R:*

1. *For every element $x \in R$, there is an element $y \in R$ such that $xyx = x$.*
2. *Every principal right ideal is generated by an idempotent of R.*
3. *Every finitely generated right ideal is generated by an idempotent of R.*

"Right" can be replaced by "left" everywhere in the conditions of this proposition because of the left-right symetric condition (1). A ring R satisfying one of these equivalent is said to be *Von Neumann regular* .

Definition 1.1.2. A ring R is *abelian* if all its idempotents are central.

Definition 1.1.3. An element $e \in R$ is an *idempotent* if $e^2 = e$. Moreover, if $ex = xe$ for every $x \in R$, then e is called a *central idempotent* of R.

Superfluous submodules

Definition 1.1.4. Let M_R be a right R-module. A submodule N_R of M_R is *superfluous* in M_R if, for every submodule L_R of M_R, $N + L = M$ implies that $L = M$. To denote that N_R is superfluous in M_R, we will write $N << M$.

Example 1.1.5. Let I be a non-zero submodule of $\mathbb{Z}_\mathbb{Z}$ and n be a non-zero element in I. Let p be a prime that does not divide n. Then $I + p\mathbb{Z} = \mathbb{Z}$ and $p\mathbb{Z}$ is proper. Hence I is not superfluous in $\mathbb{Z}$, so that the only superfluous submodule of $\mathbb{Z}_\mathbb{Z}$ is 0.

Definition 1.1.6. Let M_R, N_R be two right R-module. An epimorphism $f : M_R \to N_R$ is said to be *superfluous* if $\operatorname{Ker} f$ is superfluous in M_R.

Proposition 1.1.7. *A surjective morphism $g : M \to N$ is superfluous if and only if for every morphism h such that gh is epic, then h is surjective.*

Proof. Assume that g is superfluous and h is a morphism such that gh is surjective. Then $g(Imh) = N = g(M)$, so that $Imh + \ker g = M$. Hence $Imh = M$ because $\ker g$ is superfluous in M.

Conversely, let K be a submodule of M such that $K + \ker g = M$. Hence $g(K) = g(M) = N$. Now let $h : K \to M$ be the canonical injection. Then $Imgh = g(Imh) = g(K) = N$, that is, gh is surjective. It follows that h is surjective, that is, $K = M$. ■

Local rings and the exchange property

Definition 1.1.8. A ring R is *local* if it has a unique maximal right ideal. Equivalently, if $R/J(R)$ is a division ring.

Definition 1.1.9. Given a cardinal λ, an R module M is said to have the *λ-exchange property* if for any R-module G and any two direct sum decompositions

$$G = M' \oplus N = \oplus_{i \in I} A_i,$$

where $M' \cong M$ and $|I| \leq \lambda$, there are submodule B_i of $A_i, i \in I$ such that $G = M' \oplus (\oplus_{i \in I} B_i)$.

A module has the *exchange property* if it has the λ-exchange property for every cardinal λ. A module has the *finite exchange property* in case it has the exchange property for every finite cardinal λ. A ring R is an *exchange* ring if R_R has the exchange property.

Lemma 1.1.10. *Let λ be a cardinal and $M = M_1 \oplus M_2$. Then M has the λ-exchange property if and only if M_1 and M_2 have the λ-exchange property.*

Proof. Assume that M has the λ-exchange property. Let G, M_1', N and $A_i (i \in I)$ be modules such that $G = M_1' \oplus N = \oplus_{i \in I} A_i$ where $M_1' \cong M_1$ and $|I| \leq \lambda$. Set $G' = G \oplus M_2$. Then $G' = M' \oplus N = M_2 \oplus (\oplus_{i \in I} A_i)$ where $M' = M_1' \oplus M_2 \cong M_1 \oplus M_2 = M$. Fix an element $k \in I$ and set $I' = I \backslash k$. Then $G' = M' \oplus N = (M_2 \oplus A_k) \oplus (\oplus_{i \in I'} A_i)$. Because M has the λ-exchange property, there exist a submodule B of $M_2 \oplus A_k$ and submodules B_i of A_i for all $i \in I'$. such that $G' = M' \oplus B \oplus (\oplus_{i \in I} B_i)$. As $M_2 \leq M_2 \oplus B \leq M_2 \oplus A_k$, we have that $M_2 \oplus B = M_2 \oplus B_k$ where $B_k = (M_2 \oplus B) \cap A_k$. It follows that $M' \oplus B = (M_1' \oplus M_2) \oplus B = M_1' \oplus M_2 \oplus B_k$. Thus $G' = M_1' \oplus M_2 \oplus B_k \oplus (\oplus_{i \in I'} B_i) = M_1' \oplus M_2 \oplus (\oplus_{i \in I} B_i)$. Since $M_1' \oplus (\oplus_{i \in I} B_i) \leq G$, we obtain that $G \cap (M_2 + (M_1' \oplus (\oplus_{i \in I} B_i))) = G \cap M_2 + (M_1' \oplus (\oplus_{i \in I} B_i)) = M_1' \oplus (\oplus_{i \in I} B_i)$, and hence $G = M_1' \oplus (\oplus_{i \in I} B_i)$.

Conversely, assume that M_1 and M_2 have the λ-exchange property. Let G, M', N and $A_i (i \in I)$ be modules such that $G = M' \oplus N = \oplus_{i \in I} A_i$ where $M' \cong M$ and $|I| \leq \lambda$. Hence there are two submodules M_1', M_2' of M' such that $M_j' \cong M_j (j = 1, 2)$ and $M' = M_1' \oplus M_2'$. We have that $G = M_1' \oplus M_2' \oplus N = \oplus_{i\in I} A_i$. As M_1 has the λ-exchange property, there exist submodules $A_i' \leq A_i (i \in I)$ such that $M_1' \oplus M_2' \oplus N = G = M_1' \oplus (\oplus_{i\in I} A_i')$. Hence

$$G/M_1' = [(M_2' \oplus M_1)/M_1'] \oplus [(N \oplus M_1')/M_1'] = \oplus_{i\in I}(A_i' \oplus M_1')/M_1'.$$

Because M_2 has the λ-exchange property, there are submodules B_i of $A_i'(i \in I)$ such that $G/M_1' = [M_2' \oplus M_1'/M_1'] \oplus [\oplus_{i\in I}(B_i \oplus M_1')/M_1']$. This implies that $G = M_2' + M_1' + (\oplus_{i\in I} B_i)$. In order to prove that this sum is direct it suffices to show that if $m_2' + m_1' + \sum_{i\in I} b_i = 0$ for some $m_2' \in M_2', m_1' \in M_1$ and $b_i \in B_i$ almost all zero. We have that $(m_2' + M_1') + (\sum_{i\in I}(b_i + M_1')) = 0$ in G/M_1', so that $m_2' \in M_1'$ and $b_i \in M_1'$ for every $i \in I$. Hence $m_2' \in M_1' \cap M_2' = 0$ and $b_i \in B_i \cap M_1' \subseteq A_i \cap M_1' = 0$, and therefore $m_1' = 0$. So $G = M_1' \oplus M_2' \oplus (\oplus_{i\in I} B_i)$. ■

Recall that a module M is called an *indecomposable* module if 0 and M are its only direct summands.

Lemma 1.1.11. *Let M be a module with the 2-exchange property. Then M has the finite exchange property.*

Theorem 1.1.12. *The following conditions are equivalent for an indecomposable module M_R.*

1. *The endomorphism ring of M_R is local.*
2. *M_R has the finite exchange property.*
3. *M_R has the exchange property.*

Let M be a module. Suppose that $\{M_i | i \in I\}$ and $\{N_j | j \in J\}$ are two families of submodules of M such that $M = \oplus_{i\in I} M_i = \oplus_{j\in J} N_j$. These two decompositions are said to be *isomorphic* if there is a bijection $\phi : I \to J$ such that $M_i \cong N_{\phi(i)}$ for every $i \in I$, and the second decomposition is a *refinement* of the first if there is a surjective map $\varphi : J \to I$ such that $N_j \subseteq M_{\varphi(j)}$ for every $j \in J$.

Proposition 1.1.13. *Let λ be a cardinal and M be a module with the λ-exchange property. If $M = \oplus_{i\in I} M_i = \oplus_{j\in J} N_j$ are two direct sum decompositions of M with I finite and $|J| \leq \lambda$, then these two direct sum decompositions of M have isomorphic refinements.*

Lemma 1.1.14. *Let M be a direct sum of modules with local endomorphism ring. Then every indecomposable direct summand of M has local endomorphism ring.*

Proof. Assume that $M = A \oplus B = \oplus_{i\in I} M_i$, where A is indecomposable and all the modules M_i have local endomorphism rings. Let F be a finite subset of I such that $A \cap \oplus i \in F M_i \neq 0$ and set $C = \oplus_{i\in F} M_i$. Because C has the exchang property, there exist direct sum decompositions $A = A' \oplus A''$ and $B = B' \oplus B''$ such that $M = C \oplus A' \oplus B'$. Since $A \cap C \neq 0$ and $A' \cap C = 0$, A' is a proper submodule of A. As A is indecomposable, $A' = 0$. Hence $M = C \oplus B'$ and $C \cong A \oplus B''$. Therefore A is isomorphic to a direct summand of C. This gives that A has the exchange property by 1.1.10. Hence A has local endomorphism ring by 1.1.12 ■

Theorem 1.1.15 (Krull-Schmidt-Azumaya)**.** *Let M be a direct sum of modules with local endomorphism rings. Then any two direct-sum decompositions of M into indecomposable direct summand are isomorphic.*

PROOF. Assume that $M = \oplus_{i\in I} M_i = \oplus_{j\in J} N_j$, where M_i and N_j are indecomposable. By 1.1.14 all the modules M_i and N_j have local endomorphism rings. For $I' \subseteq I$ and $J' \subseteq J$, set $M(I') = \oplus_{i\in I'} M_i$ and $N(J') = \oplus_{j\in J'} N_j$. Then $M(I')$ and $N(J')$ have the exchange property whenever I' and J' are finite. Since N_j is indecomposable for every $j \in J$, for every finite subset $I' \subseteq I$ there is a subset $J' \subseteq J$ such that $M = M(I') \oplus N(J\backslash J')$ and hence $M(I') \cong N(J')$. By 1.1.13, the two decompositions $M(I') = \oplus_{iinI'} M_i$ and $N(J') = \oplus_{j\in J'} N_j$ have isomorphic refinements. Because of the indecomposability of the M_i and N_j, we obtain that there is a bijection $\varphi : I' \to J'$ such that $M_i \cong N_{\varphi(i)}$ for every $i \in I'$. For every module A set

$$I(A) = \{i \in I | M_i \cong A\} \text{ and } J(A) = \{j \in J | N_j \cong A\} .$$

It follows that $I(A)$ finite implies that $|I(A)| \leq |J(A)|$ and if $I(A) \neq \emptyset$, then $J(A) \neq \emptyset$. In order to prove the theorem it is sufficient to show that $|I(A)| = |J(A)|$ for every module A.

Assume first that $I(A)$ is finite. In this case we argue by induction on $|I(A)$. If $|I(A)| = 0$, then $|J(A)| = 0$. If $|I(A)| \geq 1$, fix an index $i_0 \in I(A)$. Then there is an index $j_0 \in J$ such that $M = M(\{i_0\}) \oplus N(J\backslash \{j_0\})$. Hence $N(J\backslash \{j_0\}) \cong M(I\backslash \{i_0\})$.

By the inductive hypothesis we get that $|I(A)\backslash \{i_0\}| = |J(A)\backslash \{j_0\}|$ and hence $|I(A)| = |J(A)|$. By symmetry, we can conclude that $J(A)$ finite implies that $|I(A)| = |J(A)|$.

Thus we may assume that both $I(A)$ and $J(A)$ are infinite sets. By symmetry, it suffices to show that $|J(A)| \leq |I(A)|$ for an arbitrary module A.

For each $i \in I(A)$, set $J_i = \{j \in J | M = M_i \oplus N(J\backslash \{j\})\}$. Then $J_i \subseteq J(A)$. If x is a non-zero element of M_i, then there is a finite subset J'' of J such that $x \in N(J'')$. Therefore $M_i \cap N(K) \neq 0$ for every $K \subseteq J$ that contains J''. Thus $J_i \subseteq J''$, so that J_i is finite.

We claim that $\bigcup_{i\in I(A)} J_i = J(A)$. In order to prove the claim, fix $j \in J(A)$. Then there is a finite subset I' of I such that $N_j \cap M(I') \neq 0$. Thus there exists a finite subset J' of J such that $M = M(I') \oplus N(J\backslash J')$. Since $j \in J'$ and $N(J'\backslash \{j\})$ has the exchange property, we obtain that for every $i \in I'$ there is a direct summand M_i' of M_i such that $M = N(J'\backslash \{j\}) \oplus (\oplus_{i\in I'} M_i') \oplus N(J\backslash J')$. Then $N_j \cong \oplus_{i\in I'} M_i'$, so that there is an index $k \in I'$ with $M_k' = M_k$ and $M_i' = 0$ for every $i \in I', i \neq k$. Note that $M_k \cong N_j \cong A$, so that $k \in I(A)$. Thus

$$M = N(J\backslash \{j\}) \oplus M_k \oplus N(J\backslash J') = M_k \oplus N(J\backslash \{j\}),$$

that is, $j \in J_k$. Hence $j \in \bigcup_{i\in I(A)} J_i$. This proves the claim.

It follows that

$$|J(A)| = |\bigcup_{i\in I(A)} J_i| \leq |I(A)|.$$

■

Projective modules Let M_R be a right R-module, X a subset of M_R, and $\mathcal{F}$ the family of all the submodules of M_R that contain X. The family $\mathcal{F}$ is always non-empty because it contains M_R. The intersection of all the submodules in $\mathcal{F}$ is the smallest submodule of M_R that contains X. It is called the submodule of M_R generated by X and is denoted by XR. If X is empty, then XR is the zero submodule of M_R. Otherwise, $XR = \{x_1r_1 + \cdots + x_nr_n | n \geq 1, x_i \in X, r_i \in R \text{ for } i = 1, \ldots, n\}$.

We say that a subset X of a right R-module is a set of generators of M_R if $XR = M_R$.

Definition 1.1.16. Let X be a set of generators of a right R-module M_R. The set X is called a *free set of generators* if, for every $n \geq 1, x_1, \ldots, x_n$ distinct elements of X and $r_1, \ldots, r_n$ in R, one has that $x_1r_1 + \cdots + x_nr_n = 0$ implies that $r_1 = \cdots = r_n = 0$.

Definition 1.1.17. A right R-module M_R is *free* if it has a free set of generators.

Proposition 1.1.18 (Universal Property of free modules)**.** *Let M_R be a free right R-module, X a free set of generators for M_R and $\varepsilon : X \to M_R$ the embedding of X into M_R. Then for every right R-module M'_R and every mapping $f : X \to M'_R$, there is a unique right R-module morphism $\overline{f} : M_R \to M'_R$ such that $f = \overline{f} \circ \varepsilon$.*

Definition 1.1.19. Let P_R, M_R be two right R-module. We say that P_R is *projective relative* to M_R (or P_R is M-projective) if, for each epimorphism $f : M_R \to K_R$ and each morphism $g : P_R \to K_R$, there exists a morphism $h : P_R \to M_R$ with $f \circ h = g$.

Proposition 1.1.20. *Let M_R be a right R-module and $(M_\alpha)_{\alpha \in A}$ be a family of right R module. Then $\oplus_{\alpha \in A} M_\alpha$ is M-projective if and only if M_α is M-projective.*

Proposition 1.1.21. *Let U_R be a right R-module.*

1. *Assume that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence and U is M-projective. Then U is projective relative to both M' and M''.*

2. *If U is projective relative to $M_1, \ldots, M_n$, then U is projective relative to $\oplus_{i=1}^n M_i$*

Definition 1.1.22. A right R-module P_R is *projective* if it is projective relative to every right R-module, that is, for every epimorphism $f : M_R \to K_R$ and every morphism $g : P_R \to K_R$, there exists a morphism $h : P_R \to M_R$ with $f \circ h = g$.

Lemma 1.1.23.
1. *Every free module is projective.*

2. *Every direct summand of a projective module is projective.*

3. *Every direct sum of projective modules is projective.*

Proof. (1) Let F_R be a free module. Then F_R has a free set of generators X. Let $f : M_R \to N_R$ be an epimorphism. For every $x \in X$, let m_x be an element of M_R such that $f(m_x) = g(x)$. Let $h : X \to M_R,\ x \mapsto m_x$. By the universal property of free R-modules, there exists a unique morphism $\overline{h} : F_R \to M_R$ that extends h. Because, for every $x \in X, f(\overline{h}(x)) = fh(x) = f(m_x) = g(x)$, we have $f \circ h = g$. This proves that F_R is projective.

(2) Let P_R be a projective module and assume $P = A \oplus B$. We claim that A is projective. Let $f : M_R \to N_R$ be an epimorphism and $g : A_R \to N_R$ be a morphism. Let $\varepsilon : A \to P$ and $\pi : P \to A$ be the embedding and the canonical projection, so that $\pi \circ \varepsilon = 1_A$. Since P is projective, there exists $h : P \to M$ such that $f \circ h = g \circ \pi$. It follows that $f \circ h \circ \varepsilon = g \circ \pi \circ \varepsilon = g \circ 1_A = g$. Hence A is projective.

(3) It follows from the fact that $Hom(\oplus_{i \in I} M_i, N) \cong \oplus_{i \in I} Hom(M_i, N)$. ■

Lemma 1.1.24. *Let $f : M \to N$ and $g : N \to M$ be morphisms such that $gf = 1_M$. Then $N = f(M) \oplus \operatorname{Ker} g$.*

Proposition 1.1.25. *The following conditions are equivalent for a right R-module P_R :*

1. *P_R is projective.*
2. *Every short exact sequence $0 \to M_R \to N_R \to P_R \to 0$ splits.*
3. *P_R is isomorphic to a direct summand of a free module.*

Proof. (1) $\Rightarrow$ (2) : Let P_R be a projective module and

$$0 \longrightarrow M_R \longrightarrow N_R \xrightarrow{g} P_R \longrightarrow 0$$

be a short exact sequence. Since g is surjective and P_R is projective, there exists a morphism $h : P_R \to N_R$ such that $gh = 1_P$. This means that g is right invertible, which implies that the short exact sequence above splits.

(2) $\Rightarrow$ (3) : Assume that (1) holds. Since every R-module is a homomorphic image of a free module, there exist a free R-module F_R and an epimorphism $g : F_R \to P_R$. Now consider the following short exact sequence:

$$0 \longrightarrow \operatorname{Ker}(g) \longrightarrow F_R \xrightarrow{g} P_R \longrightarrow 0$$

This exact sequence splits by (2), and hence $F_R \cong P_R \oplus \operatorname{Ker} g$.

(3) $\Rightarrow$ (1) : It follows from the fact that every free module is projective, and every direct summand of a projective module is projective. ■

Proposition 1.1.26. *If P is a projective R-module, then $Rad(P) = J(R)P$.*

Proof. By 1.1.25, one may assume that P is a direct summand of a free module $R^{(A)} = P \oplus P'$. Then $Rad(P) \oplus Rad(P') = Rad(R^{(A)}) = (Rad(R_R))^{(A)} = (J(R))^{(A)} = R^{(A)}J(R) = (P \oplus P')J(R) = PJ(R) \oplus P'J(R)$. So, since $PJ(R) \leq Rad(P)$, we must have $Rad(P) = PJ(R)$. ■

Proposition 1.1.27. *Let P be a projective R-module with endomorphism ring $S = End(P)$ and $a \in S$. Then $a \in J(S)$ if and only if $Im(a) << P$.*

PROOF. Let $a \in J(S)$ and assume that $K \leq P$ with $Im(a) + K = P$. Then we readily see that if $\eta_K : P \to P/K$ is the natural epimorphism, $a\eta_K : P \to P/K$ is epic. So, there is an element $s \in S$ such that $a\eta_K s = \eta_K$. Hence $(1 - sa)\eta_K = 0$. But, since $a \in J(S)$, $1 - sa$ is invertible. Therefore $\eta_K = 0$, that is, $K = P$.

Conversely, assume that $Im(a) << P$. Then it suffices to show that $aS << S_S$. Let $I \leq S_S$ such that $aS + I = S$. Hence $1_P = sa + b$ for some $s \in S$ and $b \in I$. Then $P = P1_P \leq Psa + Pb \leq Im(a) + Pb$, so that $Pb = P$. But then b is an epimorphism $b : P \to P$. Thus, since P is projective, this epimorphism splits and there is some $c \in S$ with $1_P = cb \in I$. Therefore $I = S$ and $Sa << S$. ■

Proposition 1.1.28. *If P is a non-zero projective R-module, then $Rad(P)$ is a proper submodule of P.*

PROOF. By 1.1.25, we may assume that there is a free R-module F with $F = P \oplus P'$. If $Rad(P) = P$, then P has no maximal submodule. By 1.1.26 we have $P = J(R)P \subseteq J(R)F$. Let $x \in P$ and e be an idempotent endomorphism of F such that $Fe = P$. Let $(x_\alpha)_{\alpha \in A}$ be a free basis for F. Then, for some finite subset $H \subseteq A$, and some $r_\alpha \in R \quad (\alpha \in R)$,

$$x = \sum_{\alpha \in H} r_\alpha x_\alpha.$$

Also, for each $\alpha \in H$, there are finite sets $H_\alpha \subseteq A$ and $a_{\alpha\beta} \in J \quad (\beta \in H_\alpha)$ such that $x_\alpha e = \sum_{\beta \in H_\alpha} a_{\alpha\beta} x_\beta$. Now, inserting $0's$ where necessary, we may assume that all of these sums are taken over a common finite subset $K \subseteq A$ to get

$$\begin{aligned} 0 &= = x - xe = (\sum_{\alpha \in K} r_\alpha x_\alpha) - (\sum_{\alpha \in K} r_\alpha x_\alpha e) \\ &= (\sum_{\alpha \in K} r_\alpha (\sum_{\beta \in K} \delta_{\alpha\beta} x_\beta)) - (\sum_{\alpha \in K} r_\alpha (\sum_{\beta \in K} a_{\alpha\beta} x_\beta)) \\ &= \sum_{\beta \in K} (\sum_{\alpha \in K} r_\alpha (\delta_{\alpha\beta} - a_{\alpha\beta})) x_\beta. \end{aligned}$$

Since the x_β are independent this equation yields the matrix equation

$$(r_\alpha)(I_n - (a_{\alpha\beta})) = (0) \in M_{1\times n}(R)$$

where $n = card(K)$ and I_n is the identity matrix in $M_n(R)$. But $(a_{\alpha\beta}) \in J(M_n(R))$ and hence $I_n - (a_{\alpha\beta})$ is invertible. Thus $(r_\alpha) = (0) \in M_{1\times n}(R)$, so that $x = \sum_{\alpha \in K} r_\alpha x_\alpha = 0$. ■

Definition 1.1.29. A *projective cover* of a right R-module M_R is a pair (P_R, f) where P_R is a projective right R-module and $f : P \to M$ is a superfluous epimorphism.

Lemma 1.1.30. *Assume M has a projective cover $p : P \to M$. If Q is projective and $q : Q \to M$ is an epimorphism, then Q has a decomposition $Q = P' \oplus P''$ such that*

1. $P' \cong P$;

2. $P'' \leq Kerq$;

3. $(q|'_P) : P' \to M$ *is a projective cover for* M.

Moreover, if $f : M_1 \to M_2$ *is an isomorphism and if* $p_1 : P_1 \to M_1$ *and* $p_2 : P_2 \to M_2$ *are projective covers, then there is an isomorphism* $\bar{f} : P_1 \to P_2$ *such that* $p_2\bar{f} = fp_1$.

Proposition 1.1.31. *The following conditions are equivalent for two idempotents* e, f *of* R*:*

1. $eR \cong fR$.

2. $eR/eJ(R) \cong fR/fJ(R)$.

Proposition 1.1.32. *Let* P *be a projective* R*-module. Then the following conditions are equivalent:*

1. P *is the projective cover of a simple* R*-module.*

2. $PJ(R)$ *is a superfluous maximal submodule of* P.

3. $End(P)$ *is a local ring.*

Moreover, if these condition hold, then $P \cong Re$ *for some idempotent* $e \in R$.

Proof. (1) $\Rightarrow$ (2). Clearly P is the projective cover of a simple module if and only if P contain a superfluous maximal submodule. But PJ is contained in every maximal submodule of P; and PJ contains every superfluous submodule of P by 1.1.26.

(2) $\Rightarrow$ (3). Assume that $End(P)$ is local. Then $P \neq 0$. By 1.1.28 there is a maximal submodule $K < P$. We claim that the natural epimorphism $P \to P/K \to 0$ is a projective cover, that is, $K << P$. Suppose that $K + L = P$ for some $L \leq P$. Then $P/K \cong (L + K)/K \cong L/(L \cap K)$. So there is a nonzero morphism $f : P \to L/(L \cap K)$. Thus, since P is projective there is an endomorphism $s : P \to L \leq P$ such that $f = \pi s$ where $\pi : L \to L/(L \cap K)$ is a canonical projection. Since $0 \neq f = s\pi$, Ims is not contained K; from which it follows that Ims is not superfluous in P. Therefore $s \notin J(End(P))$ by 1.1.27, s is an invertible endomorphism of P, $L = P$; and we have shown that $K << P$.

Moreover, every simple module is a factor of R, so by 1.1.30, a projective cover P of a simple module must be isomorphic to a direct summand of R_R, that is, $P \cong eR$ for some idempotent e of R. ■

Corollary 1.1.33. *Let* e *be an idempotent of* R. *Set* $J = J(R)$. *Then the following conditions are equivalent:*

1. eR/Je *is simple.*

2. eJ *is the unique maximal submodule of* eR.

3. *eRe is a local ring.*

Semisimple rings and modules

Definition 1.1.34. A right R-module M_R is *simple* if it is non-zero and has exactly two submodules M_R and 0.

Definition 1.1.35. A right R-module M_R is said to be *semisimple* if every submodule of M_R is a direct summand.

Note that the zero right module is semisimple but not simple. Moreover, every simple right module is semisimple.

Lemma 1.1.36. *Every submodule and every quotient module of a semisimple right module is semisimple.*

PROOF. Let M_R be a semisimple right module and N_R be a submodule of M_R. If N' is a submodule of N, then we have $M = N' \oplus N''$ for some $N'' \leq M$. Thus

$$N = N \cap M = N \cap (N' \oplus N'') = N' \oplus (N \cap N'').$$

■

Lemma 1.1.37. *Let M_R be a right semisimple right R-module. Then M_R contains a simple module.*

PROOF. Since M is non-zero, then there exists a non-zero element m of M. By our previous Lemma, it suffices to prove the statement for the case $M = mR$. There exists a maximal submodule K of M. Now we have $M = K \oplus K'$ for some submodule K' of M because M is semisimple. Since $K' \cong M/K$ and K is a maximal submodule of M, it follows that K' is a simple submodule of M. ■

Proposition 1.1.38. *Let M_R be a right R-module. Then the following conditions are equivalent:*

1. *M is semisimple;*

2. *M is the sum of a family of simple submodules;*

3. *M is a direct sum of a family of simple submodules.*

Definition 1.1.39. A ring R is called *semisimple* (or *semisimple artinian*) if R_R is a right semisimple module.

Theorem 1.1.40. *Let R be a ring. The following conditions are equivalent:*

1. *The ring R is semisimple artinian.*

2. *Every right R-module is semisimple.*

3. *The ring R is right artinian and $J(R) = 0$.*

4. *There exists a finite number of division rings $D_1, \ldots, D_t$ an positive integers $n_1, \ldots, n_t$ such that $R \cong \prod_{i=1}^{t} M_{n_i}(D_i)$.*

Since condition (4) is left-right symmetric, it follows that "right" can be replaced by "left" everywhere in the conditions of Theorem 1.1.40.

Semiperfect rings

Let R be a ring and I be an ideal of R. Let $g + I$ be an idempotent of R/I. We say that *idempotents can be lifted modulo I* if, for every idempotent $g + I$ of R/I, there exists an idempotent $e \in R$ such that $g + I = e + I$.

Definition 1.1.41. A ring R is said to be *semiperfect* if $R/J(R)$ is semisimple and idempotents can be lifted modulo $J(R)$.

Lemma 1.1.42. *Let $M_R = M_1 \oplus \cdots \oplus M_n$ where M_i has a projective cover. Then $p : P \to M$ is a projective cover if and only if P has a decomposition $P = P_1 \oplus \cdots \oplus P_n$ such that for each $i = 1, \ldots, n$*

$$(p|P_i) : P_i \to M_i$$

is a projective cover.

Lemma 1.1.43. *A cyclic module M has a projective cover if and only if $M \cong Re/Ie$ for some idempotent $e \in R$ and some right ideal $I \subseteq J(R)$. For e and I satisfying this condition the natural map $Re \to Re/Ie \to 0$ is a projective cover.*

Proof. The natural map $Re \to Re/Ie$ has kernel Ie. So if $I \subseteq J(R)$, then $Ie \subset J(R)e << Re$. Conversely, suppose M has a projective cover $p : P \to M$. If M is cyclic, then there is an epimorphism $f : R \to M$. So by 1.1.30, we may assume that $R = P \oplus P'$ with $p = (f|P)$. Thus for some idempotent $e \in R, P = Re$ and $Ie = Kerp << Re$. Whence $Ie \subseteq J(R)e \subseteq J(R)$ and $M \cong Re/Ie$. ■

Proposition 1.1.44. *Let R be a ring and let I be an ideal of R with $I \subseteq J(R)$. Then the following are equivalent:*

1. *Idempotents can be lifted modulo I.*

2. *Every direct summand of the R-module R/I has a projective cover.*

3. *Every (complete) finite orthogonal set of idempotents in R/I can be lifted to a (complete) orthogonal set of idempotents in R.*

Proof. (1) $\Rightarrow$ (2). *A direct summand of R/I_R is also one of $R/I_{R/I}$ and so is generated by an idempotent of R/I. Assuming (a), we can lift any such idempotent, so it suffices to prove that if $e \in R$ is idempotent, then $(Re + I)/I$ has a projective cover in M_R. But $(Re + I)/I \cong Re/(I \cap Re) = Re/Ie$ and so 1.1.43 applies.*

$(2) \Rightarrow (3)$. *Let $g_1, ..., g_n$ be a complete orthogonal set of idempotents module I. (This will suffice since any finite orthogonal set can be expanded to a complete orthogonal set.) Since $I \leq J(R) << R$, the natural map $\eta_I : R \to R/I$ is a projective cover. By hypothesis each term in $R/I = R/I(g_1 + I) \oplus ... \oplus (R/I)(g_n + I)$ has a projective cover, so by 1.1.42 there is a complete orthogonal set of idempotents $e_1, ..., e_n \in R$ such that $(R/I)(e_i + I) = \eta_I(Re_i) = (R/I)(g_i + I) \quad (i = 1, ...n)$. But then we have $e_i + I = g_i + I \quad (i = 1, ...n)$.*

$(3) \Rightarrow (1)$*: This is clear.* ■

Lemma 1.1.45. *Let $f : M \to N$ be a superfluous epimorphism and $p : P \to M$ be a morphism. Then $p : P \to M$ is a projective cover if and only if $fp : P \to N$ is a projective cover.*

Theorem 1.1.46. *The following conditions are equivalent for a ring R:*

1. *R is semiperfect.*
2. *R has a complete set $e_1, \ldots, e_n$ of orthogonal idempotents such that each e_iRe_i is a local ring.*
3. *Every simple right R-module has a projective cover.*
4. *Every finitely generated right R-module has a projective cover.*

Proof. Set $J = J(R)$.

$(1) \Rightarrow (2)$. If R is semiperfect, then we can, by 1.1.44, lift the idempotents for a semisimple decomposition of R/J to obtain a complete orthogonal set $e_1, ..., e_n$ of idempotents in R with each $Re_i/Je_i \cong (R/J)(e_i + J)$ simple. Then by 1.1.33 each e_iRe_i is local.

$(2) \Rightarrow (3)$. Given (2), each Re_i/Je_i is simple by 1.1.33, and has a projective cover by 1.1.43. But each simple R-module is isomorphic to a factor of $R/J \cong Re_1/Je_1 \oplus ... \oplus Re_n/Je_n$ and so is isomorphic to one of the Re_i/Je_i. (See 1.1.36).

$(3) \Rightarrow (4)$. Assume (3) and let $\mathbb{P}$ be a complete set of projective covers of simple R-modules. Then $\mathbb{P}$ generates every R-module. Let M be finitely generated. Then there is a sequence $P_1, ...P_n$ in $\mathbb{P}$ and an epimorphism

$$P = P_1 \oplus ... \oplus P_n \to^f M \to 0$$

Since $f(JP) = JM$, we infer that there is an epimorphism

$$P_1/JP_1 \oplus ... \oplus P_n/JP_n \cong P/JP \to^f M/JM \to 0.$$

But each P_i/JP_i is simple by 1.1.32, so M/JM is a finite direct sum of simple modules 1.1.36. Therefore, by 1.1.42, M/JM has a projective cover. But $JM << M$ by Nakayama's Lemma, so $M \to M/JM$ is a superfluous epimorphism. Now apply 1.1.45.

$(4) \Rightarrow (1)$. Assume (4). Since this implies in particular that every direct summand of R/J has a projective cover, idempotents can be lifted modulo J by 1.1.44. To see that R/J is semisimple, let $J \leq K \leq R_R$. Then, since the cyclic R-module R/K has a projective cover, we have by 1.1.43 $R/K \cong Re/Ie$ for some left ideal $le \subseteq Je$. But then $J.Re/Ie \cong JR/K = 0$ so that

$Je = JRe \subseteq Ie$. Thus $Ie = Je$ and $R/K \cong Re/Je \cong (R/J)(e + J)$ is projective over R/J. Hence K/J is a direct summand of R/J. Thus R/J is semisimple. ■

Essential submodules

Definition 1.1.47. Let M_R be a right R-module. A submodule N_R of M_R is *essential* in M_R if, for every submodule L_R of M_R, $L \cap N = 0$ implies that $L = 0$.

Proposition 1.1.48. *Let M be a module with submodules K, N, H such that $K \leq N \leq M$ and $H \leq M$. Then*

(a) *If $N \leq_e M$, then $H \cap N \leq_e H$.*

(b) *$K \leq_e N$ and $N \leq_e M$ if and only if $K \leq_e M$.*

(c) *$H \leq_e M$ and $K \leq_e M$ if and only if $H \cap K \leq_e M$.*

(d) *If $f : M \to M'$ and $N' \leq_e M'$, then $f^{-1}(N') \leq_e M$.*

Lemma 1.1.49. *Let $K \leq M$. Then K is essential in M if and only if for every $0 \neq x \in M$ there exists an element $r \in R$ such that $0 \neq xr \in K$.*

Proof. If $K \leq_e M$ and $0 \neq x \in M$, then xR is a non-zero submodule of M, and hence $K \cap xR \neq 0$.

Conversely, let L be a non-zero submodule of M. Then there exists $0 \neq x \in L$, and hence there is an element $r \in R$ such that $0 \neq xr \in K$. It follows that $0 \neq xr \in K \cap L$, that is, $K \cap L \neq 0$. ■

Definition 1.1.50. A monomorphism $f : K \to M$ is called *essential monomorphism* if $Imf \leq_e M$.

Proposition 1.1.51. *Let $f : L \to M$ be an injective morphism. Then f is essential if and only if, for every morphism h such that hf is injective, then h is injective.*

Proof. Assume that f is essential and h is a morphism such that hf is injective. Then $f^{-1}(\ker h) = \ker fh = 0$, so that $\ker h \cap Imf = 0$. Hence $\ker h = 0$, that is, h is injective.

Conversely, let $K \leq M$ such that $K \cap Imf = 0$ and $h : M \to M/K$ be the canonical projection. Then hf is injective since $\ker hf = f^{-1}(\ker h) = f^{-1}(K) = 0$. Hence h is injective, that is, $K = \ker h = 0$. ■

Proposition 1.1.52. *Let $M = M_1 \oplus M_2$, $K_1 \leq M_1 \leq M$ and $K_2 \leq M_2 \leq M$. Then $K_1 \oplus K_2 \leq_e M_1 \oplus M_2$ if and only if $K_1 \leq_e M_1$ and $K_2 \leq_e M_2$.*

Proposition 1.1.53. *Let $(L_\alpha)_{\alpha \in A}$ be a set of independent submodules of M and $(M_\alpha)_{\alpha \in A}$ be a set of submodules of M such that $L_\alpha \leq_e M_\alpha$ for every $\alpha \in A$. Then $(M_\alpha)_{\alpha \in A}$ is independent, and $\oplus_{\alpha \in A} L_\alpha \leq_e M_\alpha$.*

PROOF. We claim that the proposition holds for every finite subset F of A. By induction, it suffices to prove the claim in the case that F has two elements. Assume that L_1 and L_2 are independent submodules of M with $L_1 \leq_e M_1$ and $L_2 \leq_e M_2$. Then $(L_1 \cap M_2) \cap L_2 = L_1 \cap L_2 = 0$ implies that $L_1 \cap M_2 = 0$ because $L_2 \leq_e M_2$. Moreover, we have $(M_1 \cap M_2) \cap L_1 \leq L_1 \cap M_2 = 0$, so that $M_1 \cap M_2 = 0$ since $L_1 \leq_e M_1$. Therefore (M_1, M_2) is an independent set of submodules of M and $L_1 \oplus L_2 \leq_e M_1 \oplus M_2$ by 1.1.52. Now we will show that $(M_\alpha)_{\alpha \in A}$ is independent and $\oplus_{\alpha \in A} L_\alpha \leq_e M_\alpha$. Let $\alpha \in A$ and $x \in M_\alpha \cap \sum_{\beta \neq \alpha} M_\beta$. Then there exists a finite subset F of $A \setminus \{\alpha\}$ such that $x \in M_\alpha \cap \sum_{\beta \in F} M_\beta = 0$. Hence $M_\alpha \cap \sum_{\beta \neq \alpha} M_\beta = 0$. Let $0 \neq y \in \oplus_{\alpha \in A} M_\alpha$. Then there exists a finite subset G of A such that $0 \neq y \in \oplus_{\alpha \in G} M_\alpha$, so that there is an element $r \in R$ such that $0 \neq yr \in \oplus_{\alpha \in F} L_\alpha \leq \oplus_{\alpha \in A} L_\alpha$. It follows from 1.1.49 that $\oplus_{\alpha \in A} L_\alpha \leq_e \oplus_{\alpha \in A} M_\alpha$.
■

Injective modules

Definition 1.1.54. Let E_R, M_R be two right R-module. We say that E_R is *injective relative* to M_R (or E_R is M-injective) if, for each monomorphism $f : K_R \to M_R$ and each morphism $g : K_R \to E_R$, there exists a morphism $h : M_R \to E_R$ with $h \circ f = g$.

Definition 1.1.55. Let M be a right R-module. The *injectivity domain* of M is the class $In^{-1}(M) = \{N \in \text{Mod-}R \mid M \text{ is } N\text{-injective}\}$.

Proposition 1.1.56. *Let E_R be a right R-module. Then*

1. *Assume that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence and E is M-injective. Then E is injective relative to both M' and M''.*
2. *If E is injective relative to each of the R-modules M_α ($\alpha \in A$), then E is $\bigoplus_A M_\alpha$-injective.*

Proposition 1.1.57. *A module E is M-injective if and only if E is aR-injective for every $a \in M$.*

PROOF. If E is M-injective, then, by 1.1.56, E is aR-injective for every $a \in M$.

Conversely, if E is aR-injective for every $a \in M$, then E is $\oplus_{a \in M} aR$-injective by 1.1.56. Since there is an epimorphism $f : \oplus_{a \in M} aR \to M$, E is M-injective (see 1.1.56). ■

Definition 1.1.58. A right R-module E is *injective* if E is injective relative to every right R-module, that is, for every monomorphism $f : K \to M$ and every morphism $g : K \to E$, there exists a morphism $h : M \to E$ such that $g = h \circ f$.

Proposition 1.1.59. *The following conditions are equivalent for a right R-module E_R :*

1. *For every exact sequence $0 \to M'_R \to M''_R \to 0$ of right R-modules, the sequence of abelian groups $0 \to Hom(M''_R, E_R) \to Hom(M_R, E_R) \to Hom(M'_R, E_R) \to 0$ is exact.*
2. *For every monomorphism $M'_R \to M_R$ of right R-modules,*

$$Hom(M_R, E_R) \to Hom(M'_R, E_R)$$

is an epimorphism of abelian groups.

3. *For every submodule M'_R of a right R-module M_R, every morphism $M'_R \to E_R$ extends to a morphism $M_R \to E_R$.*

4. *For every monomorphism $f : M'_R \to M_R$ and every morphism $g : M'_R \to E_R$, there exists a morphism $h : M_R \to E_R$ such that $h \circ f = g$.*

PROOF. It follows immediately from the definition of injective modules. ■

In the next Proposition, we give a further criterion to determine injective modules, that is, a further characterization of injective modules.

Proposition 1.1.60 (Baer's criterion)**.** *The following about a right R-module E are equivalent:*

1. *E is injective.*

2. *E is injective relative to R.*

3. *For every right ideal $I \leq R_R$ and every morphism $h : I \to E$ there exists an $x \in E$ such that $h(a) = xa$ $(a \in I)$.*

PROOF. (1) $\Leftrightarrow$ (2) : It follows from 1.1.56.

(2) $\Rightarrow$ (3) : If E is injective and $I \leq R_R$ with $h : I \to E$, then there exists an $\overline{h} : R \to E$ such that $\overline{h}|_I = h$. Let $x = \overline{h}(1)$. Then $h(a) = \overline{h}(a) = \overline{h}(1)a = xa$ for all $a \in I$.

(3) $\Rightarrow$ (2) : Let $I \leq R_R$ and $h : I \to E$. Then there exists an element $x \in E$ such that $h(a) = xa$ for all $a \in I$, then left multiplication by x extends h. Hence E is injective. ■

Definition 1.1.61. An abelian group G is *divisible* if $nG = G$ for every non zero integer n. Hence G is divisible if and only if, for every $g \in G$ and every $n > 0$, there exists $h \in G$ such that $g = nh$. A $\mathbb{Z}$-module G is injective if and only if it is a divisible abelian group.

Definition 1.1.62. An *injective envelope* of a right R-module M_R is a pair (E_R, i) where E_R is an injective right R-module and $i : M_R \to E_R$ is an essential monomorphism.

Lemma 1.1.63. *Let M_R be a right R-module and assume that $i : M \to E$ is an injective envelope of M_R. If Q_R is injective and $q : M \to Q$ is a monomorphism, then Q has a decomposition $Q = E' \oplus E''$ such that*

1. *$E' \cong E$;*

2. *$Imq \leq E'$;*

3. *$q : M \to E'$ is an injective envelope of M_R.*

Furthermore, if $f : M_1 \to M_2$ is an isomorphism and $i_1 : M_1 \to E_1$ and $i_2 : M_2 \to E_2$ are injective envelopes, then there exists an isomorphism $\overline{f} : E_1 \to E_2$ such that $\overline{f} i_1 = i_2 f$.

Theorem 1.1.64. *Every right R-module has a unique injective envelope up to isomorphism.*

Proposition 1.1.65. *Let E be an injective module with endomorphism ring $S = End(E)$ and $a \in S$. Then the Jacobson radical of S is the set of all endomorphisms whose kernels are essential in E.*

Let N be a submodule of M. We say that L is a *complement* of N in M if $L \leq M$ is maximal with respect to the property that $N \cap L = 0$.

Proposition 1.1.66. *Let M be a module. Then every submodule N of M has a complement L and $N \oplus L \leq_e M$. Furthermore, $(N \oplus L)/L \leq_e M/L$.*

PROOF. Set $\mathcal{F} = \{K \leq M | N \cap L = 0\}$. Then $0 \in \mathcal{F}$, that is, $\mathcal{F}$ is non-empty. By Zorn Lemma, there is a maximal element L of $\mathcal{F}$ and L is a complement of N in M. Let $K \leq M$ such that $(N \oplus L) \cap K = 0$. Hence $N \cap (L \oplus K) = 0$, so that $L \oplus K = L$ because of the maximality of L. It follows that $K = 0$. This proves that $N \oplus L \leq_e M$.

Assume that $K/L \cap (N \oplus L)/L = 0$. Then $K \cap (N + L) = L$, which implies that $K \cap N + L = L$. Hence $K \cap N \subseteq L$, so that $K \cap N \subseteq L \cap N = 0$. By the maximality of L, we get that $K = L$, that is, $K/L = 0$. ■

Definition 1.1.67. Let M_R be a right R-module. A submodule N_R of M_R is said to be *closed* in M_R if N_R has no proper essential extension within M_R.

Proposition 1.1.68. *Let $C \leq M$. Then the following conditions are equivalent:*

1. *C is closed in M;*
2. *C is a complement of a module D in M;*
3. *$C = X \cap M$ for some direct summand X of an injective envelope $E(M)$ of M.*

PROOF. (1) $\Rightarrow$ (2): Let D be a complement of C in M. Then, by 1.1.66, $C \oplus D \leq_e M$. Let $C' \geq C$ be a complement of D in M. Then $C \oplus D \leq_e C' \oplus D \leq_e M$, which implies that $C \leq_e C'$ (see 1.1.52). But C is closed in M. Therefore, $C = C'$ is a complement of D in M.

(2) $\Rightarrow$ (3): Assume that C is a complement of D in M. Hence $C \oplus D \leq_e M$ by 1.1.66 and $E(M) = E(C) \oplus E(D)$ where $E(M), E(C), E(D)$ denote injective envelopes of M, C, D respectively. Because $E(C) \cap E(D) = 0$, we get that $(E(C) \cap M) \cap D = 0$. Moreover, $C \leq E(C) \cap M$ and C is a complement of D in M. Therefore $C = E(C) \cap M$.

(3) $\Rightarrow$ (1): Assume that $C = X \cap M$ where X is a direct summand of an injective envelope of M. Hence X is injective and $C \leq_e X$, that is, X is an injective envelope of C. Let C' be an essential extension of C' in M. Then X is also an injective envelope of C', so that $C' \leq X \cap M = C$. This proves that C is closed in M. ■

Proposition 1.1.69. *Let E be an injective module. Then $End(E(M))/J(End(E(M))$ is a von Neumann regular ring and idempotents can be lifted module $J(End(E(M))$.*

PROOF. This follows trivially from 1.4.10 ■

1.2 Quasi-injective modules

Definition 1.2.1. A right R-module M_R is called *quasi-injective* if M is M-injective.

Definition 1.2.2. A module M satisfies *Condition* (C_1) if every submodule of M is essential in a direct summand of M.

A module M satisfies *Condition* (C_2) if every submodule of M isomorphic to a direct summand of M is also a direct summand of M.

A module M satisfies *Condition* (C_3) if, for any two direct summands N_1, N_2 of M with $N_1 \cap N_2 = 0$, the direct sum $N_1 \oplus N_2$ is a direct summand of M.

Lemma 1.2.3. *Let $M = K \oplus K'$ and L be a submodule of M. Let $\pi_K : M \to K$ be a canonical projection. Then $M = L \oplus K'$ if and only if $(\pi_K|_L) : L \to K$ is an isomorphism. If these equivalent conditions hold, the canonical projection $\pi_L : M \to L$ with respect to the decompostion $M = L \oplus K'$ is $(\pi_K|_L)^{-1} \circ \pi_K$.*

Proof. The morphism $\pi_K|_L$ is injective if and only if $K' \cap L = 0$, and is surjective if and only if for every $k \in K$, there exists $l \in L$ such that $l = k + k'$ for some $k' \in K$ if and only if $K \subseteq L + K'$ if and only if $K + L = K' + L$. This proves the first part of this proposition.

For the second part, let $m \in M$. We have that $m = k' + \pi_L(m)$ for some $k' \in K'$, so that $\pi_K(m) = \pi_K(k') + \pi_K|_L(\pi_L(m)) = \pi_K|_L(\pi_L(m))$. It follows that $\pi_K = (\pi_K|_L)\pi_L$, and hence $(\pi_K|_L)^{-1}\pi_K = \pi_L$. ■

Proposition 1.2.4. *Let M_R be a right R-module. If M_R has Condition (C_2), then it satisfies Condition (C_3).*

Proof. Let M_1, M_2 be direct summands of M such that $M_1 \cap M_2 = 0$. Write $M = M_1 \oplus M_1'$ and let $\pi : M_1 \oplus M_1' \to M_1'$ be the canonical projection. By 1.2.3, we have $M_1 \oplus M_2 = M_1 \oplus \pi(M_2)$. Since $\pi|_{M_2}$ is a monomorphism, $\pi(M_2)$ is a direct summand of M by Condition (C_2). Because $\pi(M_2) \leq M_1'$, there exists $K \leq M_1 \leq M$ such that $pi(M_2) \oplus K = M_1'$. Hence $M = M_1 \oplus M_1' = M_1 \oplus \pi(M_2) \oplus K = M_1 \oplus M_2 \oplus K$, so that $M_1 \oplus M_2$ is a direct summand of M. This proves that M satisfies Condition (C_3). ■

Theorem 1.2.5. *Let M be a module. Then M is quasi-injective if and only if it is invariant under every endomorphism of $E(M)$.*

Proof. Assume M is quasi-injective and let f be an element of $\mathrm{End}(E(M))$. Then $L = \{m \in M | f(m) \in M\}$ is a submodule of M. Since $f \in Hom(L, M)$, there exists $g \in End(M)$ such that $f|_L = g|_L$. Since g extend to an element of $End(E(M))$, without loss of generality we may assume that $g \in End(E(M))$. Suppose that $(g - f)M \neq 0$. Then $M \cap (g - f)M \neq 0$, so that $(g - f)m = m' \neq 0$ for some $m, m' \in M$. Now $f(m) = g(m) - m' \in M$ implies that $m \in M$, and $f|_L = g|_L$ leads to $m' = 0$, a contradiction. Thus we have $(g - f)(M) = 0$, and hence $f(M) = g(M) \subseteq M$.

Conversely, let $L \leq M$ and $f \in Hom(L, M)$. Then f extend to an endomorphism g of $E(M)$ since $E(M)$ is injective. Hence $g(M) \subseteq M$, so that $g|_M \in End(M)$ extends f. This show that M is quasi-injective. ■

Proposition 1.2.6. *Let M be a quasi-injective module. Assume that $E(M) = \oplus_{i \in I} X_i$ is a direct sum decomposition of $E(M)$. Then $M = \oplus_{i \in I}(M \cap X_i)$.*

PROOF. It suffices to show that $M \subseteq \oplus_{i \in I}(M \cap X_i)$. Let $m \in M$ and π_i be the canonical projection from $E(M)$ to X_i. Then $m = x_{i_1} + \cdots + x_{i_n}$ where $x_{i_j} \in M_{i_j}$. Hence by 1.2.5, $x_{i_j} \in \pi_{i_j}(m) \in M \cap X_{i_j}$, so that $m \in \oplus_{i \in I}(M \cap X_i$. This completes the proof. ■

Corollary 1.2.7. *Let M be a quasi-injective module. Then M satisfies Condition (C_1).*

PROOF. Let N be a submodule of M and T be a complement of N in M. Then $N \oplus T \leq_e M$ by 1.1.66, so that $E(M) = E(N) \oplus E(T)$. By 1.2.6, we get that $M = (M \cap E(N)) \oplus (M \cap E(T))$. Since $N \leq_e E(N)$, we obtain that $N \leq_e M \cap E(N)$ by 1.1.48. This proves that N is essential in a direct summand of M, that is, M satisfies Condition (C_1). ■

Proposition 1.2.8. *Every quasi-injective module satisfies Condition (C_2) and (C_3).*

PROOF. Let M be a quasi-injective module. By 1.2.4, it suffices to show that M satisfies Condition (C_2). Let M_1 be a direct summand of M and M_2 be a submodule of M isomorphic to M_1. We have M_1 is M-injective, so that M_2 is also M-injective. Hence M_2 is a direct summand of M. This proves that M satisfies Condition (C_2). ■

Theorem 1.2.9. *Every quasi-injective has the exchange property.*

PROOF. Let $A = M \oplus N = \oplus_{i \in I} A_i$. Set $X_i = A_i \cap N$ and $X = \oplus_{i \in I} X_i$. By Zorn's Lemma, we can find $B \leq A$ maximal with respect to the following properties:

1. $B = \oplus_{i \in I} B_i$ with $X_i \leq B_i \leq A_i$,
2. $M \cap B = 0$.

Now we claim that $A = M \oplus B$. For every submodule Y of A, we denote $\overline{Y}$ the image of Y under the natural morphism $A \to A/B$. In order to prove the claim, it suffices to show that $\overline{M} \leq_e \overline{A}$ and $\overline{M}$ is a direct summand of $\overline{A}$. Let D be an arbitrary submodule of A_j such that B_j is a proper submodule of D. Then B is a proper submodule of $D + B = D \oplus (\oplus_{j \neq i} B_i)$. By maximality of B, we deduce that $M \cap (D + B) \neq 0$. Since $M \cap B = 0$, $M \cap (D + B)$ is not a submodule of B. Hence $(\overline{M} \cap \overline{A_i}) \cap \overline{D} = \overline{M} \cap \overline{D} \neq 0$. Thus $\overline{M} \cap \overline{A_j} \leq_e \overline{A_j}$ for all $j \in I$. Therefore $\oplus_{j \in I}(\overline{M} \cap A_j) \leq_e \oplus_{j \in J} \overline{A_j} = \overline{A}$, which implies that $\overline{M} \leq_e \overline{A}$. Let π be the canonical projection from $M \oplus N$ to M. The restriction of π to A_i has kernel X_i, and therefore A_i/X_i is isomorphic to a submodule of M. Since M is quasi-injective, M is A_i/X_i-injective (see 1.1.56). Because $A/X \cong \oplus_{i \in I} A_i/X_i$, we get that M is A/X-injective, hence M is A/B injective by 1.1.56. As $\overline{M} = (M + B)/B \cong M$, $\overline{M}$ is $\overline{A}$-injective, so that $\overline{M}$ is a direct summand of $\overline{A}$. ■

1.3 Quasi-continuous modules

Definition 1.3.1. A right R-module M_R is said to be *quasi-continuous* if it satisfies Condition (C_1) and Condition (C_3).

Theorem 1.3.2. *The following conditions are equivalent for a module M:*

1. *M is quasi-continuous.*
2. *$M = X \oplus Y$ where X and Y are two submodules of M which are complements of each other.*
3. *M is invariant under every idempotent of $End(E(M))$.*
4. *If $E(M) = \bigoplus_{i\in I} E_i$, then $M = \bigoplus_{i\in I} M \cap E_i$.*

Proof. (1) $\Rightarrow$ (2) : Since X and Y are complements of each other, X and Y are closed. Hence X and Y are direct summands of M because M satisfies Condition (C_1). Note that $X \oplus Y$ is a direct summand of M thanks to the fact that M satisfies Conditon (C_3). Furthermore, $X \oplus Y \leq_e M$. Therefore $M = X \oplus Y$.

(2) $\Rightarrow$ (3) : Set $A_1 = M \cap f(E(M))$ and $A_2 = M \cap (1-f)(E(M))$. Let B_1 be a complement of A_2 that contains A_1 and B_2 be a complement of B_1 that contains A_2. Hence $M = B_1 \oplus B_2$. Let $\pi : B_1 \oplus B_2 \to B_1$ be the canonical projection. We will show that $M \cap (f-\pi)(M) = 0$. Let $x, y \in M$ be such that $(f-\pi)(x) = y$. Then $f(x) = y + \pi(x) \in M$, and therefore $f(x) \in A_1$. Hence $(1-f)(x) \in M$, and hence $(1-f)(x) \in A_2$. Thus $\pi(x) = f(x)$, so that $y = 0$. It follows that $M \cap (f-\pi)(M) = 0$. Since $M \leq_e E(M)$, $(f-\pi)(M) = 0$, which implies that $f(M) = \pi(M) \leq M$.

(3) $\Rightarrow$ (4) : It suffices to show that $M \leq \oplus_{i\in I} M \cap E_i$. Let m be an arbitrary elment of M. Then $m \in \oplus_{i\in F} E_i$ for a finite subset $F \subseteq I$. Write $E(M) = (\oplus_{i\in F} E_i) \oplus E'$ where $E' = \oplus_{i\in I\setminus F} E_i$. Then there exists orthogonal idempotents $f_i \in End(E(M))(i \in F)$ such that $E_i = f_i(E(M))$. Since $f_i(M) \leq M$ by assumption, we get that

$$m = (\sum_{i\in F} f_i)(m) = \sum_{i\in F} f_i(m) \in \oplus_{i\in F} M \cap E_i.$$

Hence $M \leq \oplus_{i\in I} M \cap E_i$.

(4) $\Rightarrow$ (1) : Let $A \leq M$. Write $E(M) = E(A) \oplus E'$. Then $M = (M \cap E(A)) \oplus M \cap E'$ with $A \leq_e M \cap E(A)$. Hence M satisfies Condition (C_1). Let M_1, M_2 be direct summands of M. with $M_1 \cap M_2 = 0$. Write $E(M) = E(M_1) \oplus E(M_2) \oplus E''$. Then $M = (M \cap E_1) \oplus (M \cap E_2) \oplus (M \cap E'')$. Since $M_i (i = 1, 2)$ are direct summands of M and $M_i \leq_e M \cap E(M_i)$, $M_i = M \cap E_i$ $(i = 1, 2)$. Hence M satisfies Condition (C_3). ■

Proposition 1.3.3. *An indecomposable module M satisfies Condition (C_1) if and only if M is uniform. Any uniform module is quasi-continous.*

PROOF. If M is indecomposable and satisfies Condition (C_1), then every submodule of M is essential in a direct summand of M and every direct summand of M is either 0 or M. This gives every non-zero submodule of M is essential in M, that is, M is uniform.

Conversely, if M is uniform, then every non-zero submodule of M is essential in M. Hence M satisfies Condition (C_1). Assume that M is not indecomposable, that is, there is two non-zero submodules N and K such that $M = N \oplus K$. But $N \leq_e M$. This leads to $N = M$, so that $K = 0$, a contradiction. Therefore M is indecomposable.

The last statement is obvious. ■

1.4 Continuous modules

Throughout this section, denote Δ the set of all endomorphisms with essential kernels.

Definition 1.4.1. A right R-module M_R is called *continuous* if it satisfies Conditions (C_1) and (C_2).

Proposition 1.4.2. *Let M be a module. Then the following conditions are equivalent:*

1. *M has the exchange property.*
2. *If $M \oplus N = \oplus_{i \in I} A_i$ with $A_i \cong M$ for all $i \in I$, then there are submodules $C_i \leq A_i$ such that $M \oplus N = M \oplus (\oplus_{i \in I} C_i)$.*
3. *For every summable family $(f_i)_{i \in I}$ in S with $\sum f_i = 1$, there are orthogonal idempotents $e_i \in Sf_i$ such that $\sum e_i = 1$.*

PROOF. (1) $\Rightarrow$ (2) follows from the definition of the exchange property.

(2) $\Rightarrow$ (3) : Let $(f_i)_{i \in I}$ be a summable faimly of elements of S such that $\sum f_i = 1$. Set $A = \oplus_{i \in I} A_i$ with $A_i = M$ for all $i \in I$. Define $f : M \to A$ via $f(m) = (f_i(m))_{i \in I}$ and $g : A \to M$ via $g((m_i)_{i \in I} = \sum_{i \in I} m_i$. Then $gf = 1_M$, and hence $A = f(M) \oplus \operatorname{Ker} g$. By hypothesis, $A_i = B_i \oplus C_i$ such that $A = fM \oplus (\oplus_{i \in I} C_i)$.

Let $p : A \to \oplus_{i \in I} B_i$ be the canonical projection with respect to the decomposition $A = (\oplus_{i \in I} B_i) \oplus (\oplus_{i \in I} C_i)$. Then the restriction of p to $f(M)$ is an isomorphism and $pfgp^{-1}$ is the identity on $\oplus_{i \in I} B_i$. Let $\pi_j : \oplus_{i \in I} B_i \to B_j$ be the canonical projection and set $e_i = gp^{-1}\pi_i pf$. Then $e_i e_j = gp^{-1}\pi_i pfgp^{-1}\pi_j pf = gp^{-1}\pi_i \pi_j pf$. Hence $e_i e_j = 0$ for $j \neq i$ and $e_i^2 = e_i$.

Let p_i be the canonical projection from $B_i \oplus C_i \to B_i$. For any $m \in M$,

$$\pi_i pf(m) = \pi_i p(f_j(m))_{j \in I} = \pi_i (p_j f_j(m))_{j \in J} = p_i f_i(m).$$

Thus $\pi_i pf = p_i f_i$, and therefore $e_i = gp^{-1} p_i f_i \in Sf_i$. In particular, the family $(e_i)_{i \in I}$ is summable. By construction, we have $\sum e_i = 1$.

(3) $\Rightarrow$ (1) : Let $X = M \oplus Y = \oplus_{i \in I} X_i$. Let $\mu_j : \oplus_{i \in I} \to X_j$ and $q : M \oplus Y \to M$ be the canonical projections, and set $h_i = q\mu_j|_M$. Then $h_i \in S = End(M)$, the family $(h_i)_{i \in I}$ is

summable, and $\sum h_i = 1$. By hypothesis, we can find orthogonal idempotents $\gamma_i = s_i h_i \in Sh_i$ with $\sum \gamma_i = 1$. Define $\varphi_i : X \to M$ by $\varphi_i = \gamma_i s_i q \mu_i$. We claim that $X = M \oplus (\oplus_{i \in I}(X_i \cap \operatorname{Ker} \varphi_i))$.

Once this is established, (1) follows.

Note that $(\varphi_i)_{i \in I}$ is summable. Let $\varphi = \sum \varphi_i$. Then $\varphi_i|_M = \gamma_i$; indeed $\varphi_i(m) = \gamma_i s_i q \mu_i(m) = \gamma_i s_i h_i(m) = \gamma_i \gamma_i(m) = \gamma_i(m)$ for every $m \in M$. Hence $\varphi|_M = (\sum \varphi_i)|_M = \sum \gamma_i = 1_M$. Therefore $X = M \oplus \operatorname{Ker} \varphi$. Now we have $\varphi_i \varphi_j = \varphi_i(\gamma_j s_j q \mu_j) = \gamma_i \gamma_j s_j q \mu_j = 0$.

Using this, one can check that $\operatorname{Ker} \varphi = \oplus_{i \in I} X_i \cap \operatorname{Ker} \varphi_i$. ∎

We say that two R-modules are *orthogonal* if they have no non-zero isomophic submodules.

Lemma 1.4.3. *Let N and $\oplus_{i \in I} X_i$ be submodules of a module M. If $N \cap (\oplus_{i \in I} X_i) \neq 0$, then there exists $j \in I$ such that X_j and N are not orthogonal.*

Proof. Since $N \cap (\oplus_{i \in I} X_i) \neq 0$, we get that $N \cap (\oplus_{i \in F} X_i) \neq 0$ for a finite subset $F \subseteq I$. Let K be a maximal subset of F such that $N \cap (\oplus_{i \in K} X_i) = 0$. Fix $j \in F \backslash K$ and let π be the canonical projection from $X_j \oplus (\oplus_{i \in K})$ to X_j. Hence $N' = N \cap (X_j \oplus (\oplus_{i \in K} X_i)) \neq 0$ and $N \geq N' \cong \pi(N') \leq X_j$. This proves that X_j and N are not orthogonal. ∎

For any class $\mathcal{A}$ of modules, $\mathcal{A}^{\perp}$ denotes the class of modules orthogonal to all members of $\mathcal{A}$. A pair of classes $\mathcal{A}$ and $\mathcal{B}$ is said to be *orthogonal* if $\mathcal{A}^{\perp} = \mathcal{B}$ and $\mathcal{B}^{\perp} = \mathcal{A}$.

Lemma 1.4.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be an orthogonal pair classes of modules. Let M be a module. If M satisfies Condition (C_1), then $M = A \oplus B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.*

Proof. By Zorn's Lemma, M has a submodule A maximal with the property $A \in \mathcal{A}$. Since $\mathcal{A}$ is closed under essential extensions, A is closed submodule of M. Hence A is a direct summand of M because M satisfies Condition (C_1). Applying the same argument to B, we get that $B = C \oplus D$ where C is maximal with the property $C \in \mathcal{B}$. Assume that $D \neq 0$. Since $D \notin \mathcal{B}$, D contains a non-zero submodule $Z \in \mathcal{A}$, a contradiction to the maximality of A. Therefore $D = 0$, and hence $M = A \oplus B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. ∎

Definition 1.4.5. Let M be an R-module. Then M is said to be *square-free* if it does not contain a direct sum of two non-zero isomorphic submodules. The module M is called a *square-full* module if every non-zero submodule N o M contains a non-zero submodule K such that K^2 embeds in N.

Theorem 1.4.6. *A quasi-continuous module M decomposes as a direct sum $M = M_1 \oplus M_2$ where M_1 is square-free, M_2 is square-full and M_1 is orthogonal to M_2. Moreover, M_2 is quasi-injective.*

Lemma 1.4.7. *If M is a quasi-continous, then idempotents modulo Δ can be lifted.*

Proof. Let $a + \Delta$ be an idempotent of $End(M)/\Delta$. Then $a^2 - a \in \Delta$. Set $K = \operatorname{Ker}(a^2 - a)$. Since $aK \cap (1-a)K = 0$ and M is quasi-continuous, then there exist two submodules M_1, M_2 of M such that $M = M_1 \oplus M_2$, $aK \leq M_1$ and $(1-a)K \leq M_2$. Let e be the canonical projection from $M_1 \oplus M_2$ to M_1. Then $(e-a)K \leq (e-a)aK + (e-a)(1-a)K = 0$. Furthermore, $K \leq_e M$. Hence $e - a \in \Delta$, which completes the proof. ∎

Corollary 1.4.8. *Let M be a quasi-continuous module. Then any family of orthogonal idempotents of $\overline{S} = End(M)/\Delta$ can be lifted to a family of orthogonal idempotents of $S = End(M)$.*

Lemma 1.4.9. *1. If $(g_j)_{j\in J}$ and $(f_i)_{i\in I}$ are both summable, then so is $(g_j f_i)_{J\times I}$.*

2. If $(g_i)_{i\in I}$ is summable, and $(f_i)_{i\in I}$ is finitely valued, that is, $\{f_i(m)|i\in I\}$ is finite for each $m \in M$, then $(g_i f_i)_{i\in I}$ is summable.

3. If $(g_i)_{i\in I}$ and $(f_i)_{i\in I}$ are both summable and $g_i \equiv f_i$ (modulo Δ) for all $i \in I$, then $\sum g_i \equiv \sum f_i$.

Proof. For $m \in M$, set $F(m) = \{i|f_i(m) \neq 0\}$ and $G(m) = \{j|g_j(m) \neq 0\}$.

(1) If $g_j f_i(m) \neq 0$, then $f_i(m) \neq 0$, and therefore $i \in F(m)$, as well as $j \in G(f_i(m))$. Since $g(f_i(m)) \subseteq \cup_{k\in F(m)} G(f_k(m))$, which is finite. Hence $g_j f_i$ is summable.

(2) Let $\{f_i(m)|i \in I\} = \{u_1, \ldots, u_t\}, u_i \in M$. If $g_i f_i(m) \neq 0$, then $i \in G(f_i(m)) \subseteq \bigcup_{k=1}^{t} G(u_k)$, which is finite. Hence $g_j f_i$ is summable.

(3) Without loss of generality, we may assume that $g_i = 0$, that is, $f_i \in \Delta$. Let $0 \neq m \in M$. Then $\cap_{i\in F(m)} \operatorname{Ker} f_i \leq_e M$, and hence the intersection contains $0 \neq mr$ for some $r \in R$. Since $f_i(m) = 0$ for all $i \notin F(m)$, we get that $mr \in \cap_{i\in I} \operatorname{Ker} f_i$. This gives that $\cap_{i\in I} \operatorname{Ker} f_i \leq_e M$. Because $\sum f_i(\cap_{i\in I} \operatorname{Ker} f_i) = 0$, it follows that $\sum f_i \in \Delta$. ∎

Proposition 1.4.10. *If M is continuous, then S/Δ is a Von Neumann regular ring and Δ equals the Jacobson radical J of S.*

Proof. Let $\alpha \in S$ and let L be a complement of $K = Ker\alpha$. By Condition (C_1), $L \leq_\oplus M$. Since $\alpha|_L$ is a monomorphism, $\alpha L \leq_\oplus M$ by Condition (C_2). Hence there exists $\beta \in S$ such that $\beta\alpha = 1_L$. Then $(\alpha - \alpha\beta\alpha)(K \oplus L) = (\alpha - \alpha\beta\alpha)L = 0$, and so $K \oplus L \leq Ker(\alpha - \alpha\beta\alpha)$. Since $K \oplus L \leq_e M$, $\alpha - \alpha\beta\alpha \in \Delta$. Therefore S/Δ is a Von Neumann regular ring. This also proves that $J \leq \Delta$.

Let $a \in \Delta$. Since $\ker a \cap \ker(1-a) = 0$ and $\ker a \leq_e M$, $\ker(1-a) = 0$. Hence $(1-a)M \leq_\oplus M$ by Condition (C_2). However $(1-a)M \leq_e M$ since $\ker a \leq (1-a)M$ Thus $(1-a)M = M$, and therefore $1-a$ is a unit in S. It then follows that $a \in J$, and hence $\Delta \leq J$. ∎ A ring R is said to be *reduced* if R has no non-zero nilpotent elements.

Lemma 1.4.11. *Let M be a square free module. Then* $\operatorname{End}(M)/\Delta$ *is reduced. In particular, all idempotents of $End(M)/\Delta$ are central.*

Proof. It suffices to show that if $\alpha \in S$ such that $\alpha^2 \in \Delta$, then $\alpha \in \Delta$. Let L be a complement of $\ker\alpha$ in M. Then $\ker\alpha \oplus L \leq_e M$. Since $\ker\alpha \cap L = 0$, we obtain that $\ker\alpha^2 \cap L \cong \alpha(\ker\alpha^2 \cap L) \leq Ker\alpha$. Because M is square free, $\ker\alpha^2 \cap L = 0$, and hence $L = 0$ because $\ker\alpha^2 \leq_e M$. Thus $\ker\alpha \leq_e M$, that is, $\alpha \in \Delta$.

Let e be an idempotent of $End(M)$ and $a \in End(M)$. We have $(ea(1-e))^2 = ea(1-e)ea(1-e) = 0$ because $e^2 = e$. Since $End(M)$ is reduced, we get that $ea(1-e) = 0$, that is, $ea = eae$. By a similar argument, we also deduce that $ae = eae$. It follows that $ea = ae$, and hence e is central. This completes the proof. ∎

Theorem 1.4.12. *Every continuous module has the exchange property.*

PROOF. By 1.4.6, 1.1.10 and 1.2.9, it suffices to prove the exchange property for a square-free continuous module M. By 1.4.11, all idempotents of $\overline{S} = S/\Delta$ are central. Furthermore, $J(R) = \Delta$ and $\overline{S}$ is von Neumann regular by 1.4.10.

We establish the result by verifying (3) of 1.4.2. Let I be a set of ordinals, and $f_i \in S$ $(i \in I)$ be a summable family with $\sum f_i = 1$. Since $\overline{S}$ is von Neumann regular, there exists $\alpha_i \in S$ such that $f_i \equiv f_i\alpha_i f_i$ (modulo Δ). Let $h_i = \alpha_i f_i$. Then $(h_i)_i \in I$ is a summable family and the $\overline{h}_i$ are central idempotents in $\overline{S}$.

We define inductively $\gamma_k = (1 - \sum_{i<k} \gamma_i)h_k \in Sf_k$. By induction, we see that the γ_k are well defined, summable, and are orthogonal idempotents modulo Δ. By 1.4.8, the γ_k lift to orthogonal idempotents $\gamma_k \in S$. Now

$$h_k = \gamma_k + (\sum_{i<k} \gamma_i)h_k \equiv \gamma_k + h_k \sum_{i<k} \gamma_i.$$

By 1.4.9, we have

$$\begin{aligned} 1 &= \sum_k f_k \equiv \sum_k f_k h_k \\ &\equiv \sum_k (\gamma_k + h_k \sum_{i<k} \gamma_i) \\ &\equiv \sum_k (f_k\gamma_k + f_k \sum_{i<k} \gamma_i) \\ &= \sum_k \sum_{i\le k} f_k\gamma_i \\ &= \sum_i \sum_{k\ge i} f_k\gamma_i. \end{aligned}$$

Let $\varphi_i = \sum_{k\ge i} f_k$. Then $1 \equiv \sum_i \varphi_i\gamma_i \equiv \sum_i g_i\varphi_i\gamma_i$. Thus $\sum g_i\varphi_i\gamma_i = 1 + x$ for some $x \in \Delta$, so that $\sum_i (1+x)^{-1} g_i\varphi_i\gamma_i = 1 = \sum_i g_i\varphi_i\gamma_i(1+x)^{-1}$. Hence $M = \oplus_i g_i M$, and $M = (1+x)^{-1}M = \oplus_i (1+x)^{-1} g_i M$.

Let $(e_i)_{i\in I}$ be the canonical projections of M with respect to the decomposition $M = \oplus(1+x)^{-1}g_iM$. Since $\sum_i(1+x)^{-1}g_i\varphi_i\gamma_i = 1$, we get that $e_i = (1+x)^{-1}g_i\varphi_i\gamma_i \in Sf_i$ for all $i \in I$. By (3) of 1.4.2, M has the exchange property. ■

1.5 Morita equivalent rings

Definition 1.5.1. A *category* $\mathcal{C}$ consists of:

1. A class object $Ob(\mathcal{C})$, whose elements will be called the *objects* of $\mathcal{C}$;

2. For each pair (A, B) of objects of $\mathcal{C}$, a set $Hom(A, B)$, whose elements will be called *morphisms* of A to B;

3. For each triple (A, B, C) of objects of $\mathcal{C}$, a mapping

$$\circ : Hom_{\mathcal{C}}(B, C) \times Hom_{\mathcal{C}}(A, B) \to Hom_{\mathcal{C}}(A, C),$$

   called *composition* .

Definition 1.5.2. Let $\mathcal{C}, \mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ assigns to every object $C \in Ob(\mathcal{C})$ an object $F(C) \in Ob(\mathcal{D})$, and to every morphism $f : C \to C'$ in $\mathcal{C}$ a morphism $F(f) : F(C) \to F(C')$ in $\mathcal{D}$, and the following axioms are satisfied:

1. If $f : C_1 \to C_2$ and $g : C_2 \to C_3$ are morphisms in $\mathcal{C}$, then

$$F(g \circ f) = F(g) \circ F(f);$$

2. $F(1_C) = 1_{F(C)}$ for every $C \in Ob(\mathcal{C})$.

Definition 1.5.3. Let $\mathcal{C}$ and $\mathcal{D}$ be arbitrary categories. Then a functor $F : \mathcal{C} \to \mathcal{D}$ is a *category equivalence* if there is a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $GF \cong 1_{\mathcal{C}}$ and $FG \cong 1_{\mathcal{D}}$.

We say that two categories are *equivalent* if there exists a category equivalence from one to the other. We write $\mathcal{C} \approx \mathcal{D}$ in case $\mathcal{C}$ and $\mathcal{D}$ is equivalent.

Definition 1.5.4. Let R and S be two rings. We say that R is *Morita equivalent* to S if Mod-$R \approx$ Mod-S.

Proposition 1.5.5. *Let R, S be two rings and F : Mod-$R \to$ Mod-S be a category equivalence. Then a sequence*

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

is (split) exact in Mod-R if and only if the following sequence

$$0 \longrightarrow F(M_1) \xrightarrow{F(f)} F(M_2) \xrightarrow{F(g)} F(M_3) \longrightarrow 0$$

is (split) exact in Mod-S.

Proposition 1.5.6. *Let R, S be two rings and F : Mod-$R \to$ Mod-S be a category equivalence. Then*

1. *A pair $(M, (\pi_\alpha)_{\alpha \in A})$ is a direct product of $(M_\alpha)_{\alpha \in A}$ if and only if $(F(M), (F(\pi_\alpha))_{\alpha \in A})$ is a direct product of $(F(M_\alpha))_{\alpha \in A}$.*

2.

3. *A pair $(M, (j_\alpha)_{\alpha \in A})$ is a direct sum of $(M_\alpha)_{\alpha \in A}$ if and only if $(F(M), (F(j_\alpha))_{\alpha \in A})$ is a direct sum of $(F(M_\alpha))_{\alpha \in A}$.*

4. *An R-module M is N-projective (N-injective) if and only if $F(M)$ is $F(N)$-projective ($F(N)$-injective).*

5. *An R-module M is projective (injective) if and only if $F(M)$ is projective (injective).*

6. *A monomorphism (epimorphism) $f : M \to M'$ is essential (superfluous) if and only if $F(f) : F(M) \to F(M')$ is essential (superfluous).*

7. *$f : M \to M'$ is a projective cover (injective envelope) if and only if $F(f) : F(M) \to F(M')$ is a projective cover (injective envelope).*

8. *An R-module M is simple (semisimple, finitely generated, artinian, noetherian, indecomposable) if and only if $F(M)$ is simple (semisimple, finitely generated, artinian, noetherian, indecomposable).*

9. *Two modules M and $F(M)$ have the same composition length.*

1.6 Singular modules and right SI-rings

Definition 1.6.1. Let M be a right R-module. An element $m \in M$ is said to be a *singular element* of M if the right annihilator $ann(m)$ is essential in R_R. Denote the set of all singular element of M by $Z(M)$.

Proposition 1.6.2. *Let M be a right R-module. Then*

1. *$Z(M)$ is a submodule, called the singular submodule of M.*

2. *$Z(M).soc(R_R) = 0$, where $soc(R_R)$ is the socle of R_R.*

3. *If $f \in Hom_R(M, N)$, then $f(Z(M)) \leq Z(N)$.*

4. *If $M \leq N$, then $Z(M) = Z(N) \cap M$.*

Definition 1.6.3. 1. A module M is *singular* if $Z(M) = M$.

2. A module M is *nonsingular* if $Z(M) = 0$.

3. A ring R is *right nonsingular* if $Z(R_R) = 0$.

4. A ring R is *right SI* if every singular right R-module is injective.

Proposition 1.6.4. *[Goo72, Proposition 3.3] If R is a right SI ring, then:*

1. *$Rad(R_R) \leq Soc(R_R)$.*

2. *$Rad(R_R)^2 = 0$.*

3. *$I^2 = I$ for all essential right ideals of R.*

4. R is right hereditary.

Theorem 1.6.5. *[Goo72, Theorem 3.6] Let R be a right SI ring. Then* $\frac{R}{Soc(R_R)}$ *is a right noetherian ring*

Theorem 1.6.6. *[Goo72, Theorem 3.11] Let R be a ring. Then R is right SI if and only if R is isomorphic to* $K \times R_1 \times \cdots \times R_n$ *such that* $K/soc(K_K)$ *is semisimple artinian and each* R_i *is Morita equivalent to a right SI-domain.*

1.7 Semiartinian modules and right semiartinian rings

Let M be a right R-module. The *Loewy series (or socle series)* of M is the ascending chain of submodules

$$0 = S_0(M) \subseteq S_1(M) \subseteq S_2(M) \subseteq \cdots \subseteq S_\alpha(M) \subseteq \ldots$$

where, for each ordinal $\alpha \geq 0$, $S_{\alpha+1}(M)/S_\alpha(M) = soc(M/S_\alpha(M))$, and, if α is a limit ordinal, then $S_\alpha(M) = \cup_{0 \leq \beta < \alpha} S_\beta(M)$. Note that the Loewy series is always stationary, that is, for every module M there exists an ordinal α such that $S_\alpha(M) = S_\beta(M)$ for every $\beta \geq \alpha$ (for instance, let α be any ordinal whose cardinality is greater than the cardinality of M).

Definition 1.7.1. A module M is *semiartinian* if every factor of M has essential socle.

Theorem 1.7.2. *Let M be a right R-module. The following conditions are equivalent:*

1. *M is semiartinian.*

2. *Every factor of M has non-zero socle.*

3. $S_\lambda(M) = M$ *for some ordinal* $\lambda \geq 0$.

Proposition 1.7.3. *Let M be a noetherian right R-module. If M is semiartinian, then M is artinian.*

Definition 1.7.4. A ring R is said to be *right semiartinian* if R_R is semiartinian.

Theorem 1.7.5. *A ring R is right semiartinian if and only if every right R-module is semiartinian.*

PROOF. $(\Leftarrow)$: Obvious.
$(\Rightarrow)$: By [DHSW94, 3.12] ■

Chapter 2

Cyclically presented modules, projective covers and factorizations

2.1 Preliminaries

Definition 2.1.1. An R-module M_R is said to be *cyclically presented* if $M_R \cong R/aR$ for some $a \in R$.

For the rest of this section, we will review some results in [AAF08]

Remark 2.1.2. The endomorphism ring of a non-zero cyclically presented module R/aR is canonically isomorphic to E/aR where $E = \{r \in R | ra \in aR\}$ is the *idealizer* of aR and the right ideal aR turns out to be an ideal in the subring E of R.

Theorem 2.1.3. *Let a be a non-zero non-invertible element of a local ring R and E be the idealizer of aR. Let $I = \{r \in R | ra \in aJ(R)\}$ and $K = J(R) \cap E$. Then I and K are completely prime ideals of E containing aR, the union $(I/aR) \cup (K/aR)$ is the set of all non-invertible elements of the endomorphism ring E/aR of R/aR, and every proper right ideal of E/aR and every proper left ideal of E/aR is contained either in I/aR or in K/aR. Moreover, exactly one of the following two conditions hold:*

1. *Either the ideals I and K are comparable, so that E/aR is a local ring with maximal ideal $(I/aR) \cup (K/aR)$, or*

2. *I and K are not comparable, $J(E/aR) = (I \cap K)/aR$, and $\frac{(E/aR)}{J(E/aR)}$ is canonically isomorphic to the direct product of the two division rings E/I and E/K.*

Proof. Set $K = J(R) \cap E$. Then K is an ideal of E because K is the intersection of the maximal ideal $J(R)$ of R with the subring E of R. We conclude that K is a proper, completely prime ideal of E containing aR thanks to the fact that E/K is a subring of the division ring $R/J(R)$.

Consider the morphism $\varphi : E \to End(aR/aJ(R))$ that sends an element $r \in E$ to the endomorphism $\varphi(r)$ of the right R-module $aR/aJ(R)$ defined by $\varphi(r)(x + aJ(R)) = rx + aJ(R)$ for every $x \in aR$. Set $I = \operatorname{Ker}\varphi$. Then E/I is isomorphic to a subring of the division ring $End(aR/aJ(R))$, so that I is a proper, completely prime ideal of E containing aR. Hence I/aR and K/aR are proper ideals of E/aR. In particular, all the elements of $(I/aR) \cup (K/aR)$ are non-invertible elements of E/aR. Conversely, let $r \notin I \cup K$ be an element of E, so tha there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & aR & \longrightarrow & R & \longrightarrow & R/aR & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & aR & \longrightarrow & R & \longrightarrow & R/aR & \longrightarrow & 0 \end{array}$$

in which the vertical arrows are the morphisms induced by left multiplication by r.

Since $r \notin K = J(R) \cap E$ and $r \in E$, it follows that $r \notin J(R)$, so that r is invertible in R. This gives that the vertical arrow in the middle is an isomorphism, and hence the vertical arrow on the right is an epimorphism. As $r \notin I$, the vertical arrow on the left is an epimorphism. By the Snake Lemma, the vertical arrow on the right is injective, and hence it is an automorphism of R/aR. It follows that $r + aR$ is invertible in the endomorphism ring E/aR of R/aR. Therefore $(I/aR) \cup (K/aR)$ is exactly the set of all non-invertible elements of E/aR.

Thus every proper right or left ideal L/aR of E/aR is contained in $(I/aR) \cup (K/aR)$. If there exist $x \in L \backslash I$ and $y \in L \backslash K$, then $x + y \in L$, $x \in K$ and $y \in I$. Thus $x + y \notin I$ and $x + y \notin K$, so that $x + y \notin I \cup K$, which contradicts the fact that $x + y \in L$. Therefore L is contained either in I or in K. In particular, the unique maximal right ideals of E/aR are at most I/aR and K/aR. Similarly, the unique maximal left ideals of E/aR are at most I/aR and K/aR.

If I and K are comparable, then $(I/aR) \cup (K/aR)$ is the unique maximal right (and left) ideal of E/aR. If I and K are not comparable, then E/aR has exactly two maximal right ideals I/aR and K/aR, so that $J(E/aR) = (I \cap K)/aR$, and there is a canonical injective morphism $\pi : (E/aR)/J((E/aR)) \to E/I \times E/K$. Since I and K are incomparable maximal ideal of E, we get that $I + K = E$, and hence π is surjective thanks to the Chinese Remainder Theorem. ■

Lemma 2.1.4. *Let R be a local ring and r, s be two elements of R. Then $R/rR \cong R/sR$ if and only if there are invertible elements u, v of R such that $urv = s$.*

Proof. Assume that there are invertible elements $u, v \in R$ such that $urv = s$. Define a morphism $f : R \to R/urR$ via $f(x) = ux + urR$. It is clear that f is onto and $\operatorname{Ker} f = rR$. Hence $R/rR \cong R/urR$. Moreover, $urR = sv^{-1}R = sR$. Therefore $R/rR \cong R/sR$.

Conversely, let $f : R/rR \to R/sR$ be an isomorphism. Then there is an element $u \in R$ such that left multiplication by u is a morphism $R_R \to R_R$ that induces the isomorphism f. Since f is onto, we have $R = uR + sR$. If s is invertible, so is r. Set $u = r^{-1}, v = s$. Then u, v are invertible and $urv = s$. If s is not invertible, then u is invertible. Thus we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & rR & \longrightarrow & R & \longrightarrow & R/rR & \longrightarrow & 0 \\
& & {\scriptstyle u|_{rR}}\downarrow & & {\scriptstyle u}\downarrow & & {\scriptstyle f}\downarrow & & \\
0 & \longrightarrow & sR & \longrightarrow & R & \longrightarrow & R/sR & \longrightarrow & 0
\end{array}$$

Applying the Snake Lemma, we have an exact sequence

$$\operatorname{Ker} f \longrightarrow \operatorname{Coker}(u|_{rR}) \longrightarrow \operatorname{Coker} u.$$

Because f is an isomorphism and u is invertible, $\operatorname{Ker} f = \operatorname{Coker} u = 0$, so that $\operatorname{Coker}(u|rR) = 0$. Hence $urR = sR$. By [Kap49, Lemma 2.1], there is an invertible element $v \in R$ with $urv = s$. ∎

The following corollary is immediate.

Corollary 2.1.5. *Let R be a local ring and r, s be two elements of R. Then $R/rR \cong R/sR$ if and only if $R/Rr \cong R/Rs$.*

Definition 2.1.6. Two $m \times n$ matrices A and B over a ring R are said to be *equivalent* if there exist an $m \times m$ invertible matrix P and an $n \times n$ invertible matrix Q such that $A = PBQ$.

Definition 2.1.7.

1. Let A, B be two modules. We say that A and B have *the same epigeny class* and write $[A]_e = [B]_e$ if there are an epimorphism from A to B and an epimorphism from B to A.

2. Let R be a local ring. Two cyclically presented modules R/aR and R/bR have *the same lower part* and write $[R/aR]_l = [R/bR]_l$ if there are $r, s \in R$ such that $raR = bR$ and $sbR = aR$.

3. For cyclically presented left modules over a local ring, we say that R/Ra and R/Rb have the same lower part, and write $[R/Ra]_l = [R/Rb]_l$ if there are $r, s \in R$ such that $Rar = Rb$ and $Rbs = Ra$.

Remark 2.1.8. The unique cyclically presented module, up to isomorphism, with the same epigeny class as 0 is 0, and R_R is the unique cyclically presented module, up to isomorphism, with the same epigeny class as R_R. Similarly for the lower part. Note that, if a, b are elements of a local ring R, then $[R/aR]_e = [R/bR]_e$ if and only if there are $u, v \in U(R)$ with $ua \in bR$ and $vb \in aR$, if and only if there are $u, v \in U(R)$ and $r, s \in R$ with $ua = br$ and $vb = as$. Moreover, for $a, b \in R$, $[R/aR]_l = [R/bR]_l$ if and only if there are $u, v \in U(R)$ and $r, s \in R$ with $au = rb$ and $bv = sa$.

Lemma 2.1.9. *Let a, b be elements of a local ring R. Then $R/aR \cong R/bR$ if and only if $[R/aR]_l = [R/bR]_l$ and $[R/aR]_e = [R/bR]_e$.*

PROOF. Assume that $[R/aR]_l = [R/bR]_l$ and $[R/aR]_e = [R/bR]_e$. Then there exist two invertible elements $u, v \in R$ and two elements $r, s \in R$ with $ua = br, sa = bv$. If either r or s is invertible, then $R/aR \cong R/bR$ by 2.1.4. Hence we may suppose that both r and s are in $J(R)$, in which case $u + s$ and $r + u$ are invertible. Because $(u + s)a = b(r + v)$, we obtain that $R/aR \cong R/bR$ by 2.1.4.

The converse follows from 2.1.4. ∎

Corollary 2.1.10. *Let a, b be elements of a local ring R. Then:*

1. $[R/aR]_l = [R/bR]_l$ *if and only if* $[R/Ra]_e = [R/Rb]_e$.
2. $[R/aR]_e = [R/bR]_e$ *if and only if* $[R/Ra]_l = [R/Rb]_l$.

PROOF. (1) $[R/aR]_l = [R/bR]_l$ if and only if there are $r, s \in R$ and $u, v \in U(R)$ such that $ra = bu$ and $sb = av$. if and only $[R/Ra]_e = [R/Rb]_e$.

(2) is exactly (1) applied to the opposite ring R^{op} of R. ∎

Proposition 2.1.11. *Let R be a local ring. If R is either a commutative ring, or a chain ring, or it it has the acc on principal right ideals, or it has the dcc on principal right ideals, or $J(R)$ is nil, then $[R/aR]_e = [R/bR]_e$ implies that $R/aR \cong R/bR$ for every $a, b \in R$.*

PROOF. Case 1 : R is a chain ring. The proof is given in the proof of [Fac10, Theorem 9.19].

Case 2 : R is a commutative ring. Since $[R/aR]_e = [R/bR]_e$, it follows that R/aR and R/bR have the same annihilator, so that $aR = bR$.

Case 3 : R has the acc on principal right ideals or it has the dcc on principal right ideals or its Jacobson radical $J(R)$ is nil. Let $a, b \in R$ be two elements such that $[R/aR]_e = [R/bR]_e$. Assume that $R/aR \ncong R/bR$. By 2.1.8, both a and b are non-zero and in $J(R)$.

$[R/aR]_e = [R/bR]_e$ and $R/aR \ncong R/bR$ imply that R/aR has an endomorphism which is epi but not mono. Hence there is an element $u \in U(R)$ with $uaR \subseteq aR$ and $aR \subset u^{-1}aR$, so that $uaR \subset aR$. Multiplying by the unit u^n for some arbitrary integer n, we obtain that $u^{n+1}aR \subset u^naR$. Thus $\{u^naR | n = 1, 2, \dots\}$ is a strictly descending chain and $\{u^naR | n = -1, -2, \dots\}$ is a strictly ascending chain. It follows that R does not have the dcc and the acc on principal right ideals, which implies that $J(R)$ is nil. Now $uaR \subset aR$ implies that $ua = ar$ for some $r \in J(R)$. Hence $u^na = ar^n$ for every positive integer n. Since $u^na \neq 0$, we get that $r^n \neq 0$, which contradicts the fact that $J(R)$ is nil. This proves that $R/aR \cong R/bR$. ∎

From 2.1.10, we immediately obtain the following result:

Corollary 2.1.12. *Let R be a local ring. If R is either a commutative ring, or a chain ring, or it it has the acc on principal right ideals, or it has the dcc on principal right ideals, or $J(R)$ is nil, then $[R/aR]_l = [R/bR]_l$ implies that $R/aR \cong R/bR$ for every $a, b \in R$.*

Proposition 2.1.13. *Let $a, c_1, \dots, c_n$ $(n \geq 2)$ be non-invertible elements of a local ring R. If R/aR is a direct summand of $R/c_1R \oplus \cdots \oplus R/c_nR$ and $R/aR \ncong R/c_iR$ for every $i = 1, 2, \dots, n$, then there are two distinct indices $i, j = 1, \dots, n$ such that $[R/aR]_l = [R/c_iR]_l$ and $[R/aR]_e = [R/c_jR]_e$.*

PROOF. Since R/aR is a direct summand of $R/c_1R \oplus \cdots \oplus R/c_nR$ and $R/aR \not\cong R/c_iR$ for every $i = 1, 2, \ldots, n$, it follows that the endomorphism ring of R/aR is not local. Let $\varepsilon : R/aR \to R/c_1R \oplus \cdots \oplus R/c_nR$ and $\pi : R/c_1 \oplus \cdots \oplus R/c_nR \to R/aR$ be morphisms with the composite mapping $\pi\varepsilon = 1_{R/aR}$. Then there are elements $r_1, \ldots, r_n, s_1, \ldots, s_n \in R$ with $r_ia \in c_iR$ and $s_ic_i \in aR$ whose residue classes are the entries of the matrices representing ε and π, that is, such that

$$(s_1 \ldots s_n) \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} - 1 \in aR$$

Note that $s_ir_ia \in s_ic_iR \subseteq aR$, so that $s_ir_i \in E$, the idealizer of aR. Hence $\sum_{i=1}^n s_ir_i - 1 \in aR \subseteq I$, so that $\sum_{i=1}^n s_ir_i \notin I$, and therefore there exists an index i with $s_ir_i \notin I$. Similarly, $\sum_{i=1}^n s_ir_i - 1 \in K$, so that there is an index j with $s_jr_j \notin K$. Assume that $i = j$. Then $s_ir_i \notin I \cup K$, so that s_ir_i represents an invertible element of the endomorphism ring E/aR of R/aR. Thus there are morphisms $R/aR \to R/c_iR$ and $R/c_iR \to R/aR$ whose composition is an automorphism of R/aR. It follows that R/aR is isomorphic to a direct summand of $\oplus_{i=1}^n R/c_iR$. Because both R/aR and R/c_iR have dual Goldie dimension one, R/aR and R/c_iR are isomorphic, a contradiction. This proves that $i \neq j$.

Since $s_ir_i \in E \backslash I$, $s_ir_ia \in aR \backslash aJ(R)$, so that $s_ir_iaR = aR$. Furthermore, $r_iaR \subseteq c_iR$. Assume that $r_iaR \subset c_iR$. Then $r_iaR \subseteq c_iJ(R)$. Hence $aR = s_ir_iaR \subseteq s_ic_iJ(R) \subseteq aJ(R)$, a contradiction. Therefore $r_iaR = c_iR$. Similarly, $s_ic_iR = aR$. This gives $[R/aR]_l = [R/c_iR]_l$.

Similarly, $s_jr_j \in E \backslash K$ implies that $s_jr_j \notin J(R)$, so that $s_j, r_j \notin J(R)$. Hence $[R/aR]_e = [R/c_jR]_e$. ∎

We say that a ring R is *semilocal* if $R/J(R)$ is semisimple.

Lemma 2.1.14. *Let R be a local ring and a, b, c be non-invertible elements of R. Assume that $[R/aR]_l = [R/bR]_l$ and $[R/aR]_e = [R/cR]_e$. Then:*

1. *There exists a module D such that $R/aR \oplus D \cong R/bR \oplus R/cR$.*

2. *The module D in* (1) *is unique up to isomorphism and is cyclically presented.*

3. $[D]_l = [R/cR]_l$ *and* $[D]_e = [R/bR]_e$.

PROOF. Since $[R/aR]_l = [R/bR]_l$, there exists $r, s \in R$ such that $raR = bR$ and $sbR = aR$. Because $[R/aR]_e = [R/cR]_e$, there are $r', s' \in U(R)$ with $r'a \in cR$ and $s'c \in aR$. If one of the elements a, b, c is zero, then so are the others. Hence the statement is trivial. So we may suppose that a, b, c are all non-zero.

(1) If r is invertible, then $raR = bR$ implies that $ra = bv$ for some invertible element v by [Kap49, Lemma 2.1]. Hence $R/aR \cong R/bR$ by 2.1.4. It suffices to take $D = R/cR$ in this case. Now we may suppose $r \in J(R)$, and so both rs and sr are in $J(R)$. Then sr belongs to the ideal K of the idealizer E of aR, but not to the ideal I.

If $r'aR = cR$, then there exists a unit $u \in R$ with $r'au = c$, so that $R/aR \cong R/cR$ by 2.1.4. It is sufficient to take $D = R/bR$ in this case. Thus we may suppose $r'aR \subseteq cJ(R)$. Therefore

$s'r'aR \subseteq aJ(R)$. It follows that $s'r'$ belongs to the ideal I of the idealizer E of aR, but not to the ideal K.

Now matrix muliplication

$$R \xrightarrow{\begin{pmatrix} r \\ r' \end{pmatrix}} R \oplus R \xrightarrow{\begin{pmatrix} s & s' \end{pmatrix}} R$$

induces morphisms

$$R/aR \longrightarrow R/bR \oplus R/cR \longrightarrow R/aR,$$

whose composite mapping is the endomorphism of R/aR given by left multiplication by $sr+s'r' \notin I \cup K$. Hence this composite mapping is an automorphism of R/aR, so that R/aR is isomorphic to a direct summand of $R/bR \oplus R/cR$.

(2) If $R/aR \oplus D \cong R/bR \oplus RcR$ and $R/aR \oplus D' \cong R/bR \oplus R/cR$, then $R/aR \oplus D \cong R/aR \oplus D'$. Hence $D \cong D'$ because the endomorphism ring of R/aR is semilocal, and therefore R/aR cancels from direct sums [Fac10, Corollary 4.6]. This proves that D is unique up to isomorphism.

Assume $R/aR \oplus D \cong R/bR \oplus R/cR$. Then D is finitely generated. Moreover, $R/aR, R/bR, R/cR$ are right vector spaces of dimension one over the division ring $R/J(R)$. Hence $D/DJ(R)$ is also a one dimensional right vector space over $R/J(R)$. By Nakayama's Lemma, D is cyclic. Therefore $D \cong R/T$. Now there are exact sequences

$$0 \longrightarrow bR \oplus cR \longrightarrow R \oplus R \longrightarrow R/bR \oplus R/cR \longrightarrow 0$$

$$0 \longrightarrow aR \oplus T \longrightarrow R \oplus R \longrightarrow R/aR \oplus D \longrightarrow 0$$

Since $R/aR \oplus D \cong R/bR \oplus R/cR$, Schanuel's Lemma implies that $bR \oplus cR \cong aR \oplus T$. It follows that T is finitely generated. Moreover, aR, bR and cR are one-dimiensional over $R/J(R)$, so that $T/TJ(R)$ also is one-dimensional. By Nakayama's Lemma, T is cyclic. This proves that $D \cong R/T$ is cyclically presented.

(3) If $D \cong R/cR$, then $R/aR \cong R/bR$ and hence $[D]_e = [R/cR]_e = [R/aR]_e = [R/bR]_e$ and $[D]_l = [R/cR]_l$. Similarly, the statement holds in the case $D \cong R/bR$. Therefore we may assume that D is not isomorphic to R/bR and R/cR.

By 2.1.13, $[R/bR]_e = [R/aR]_e$ or $[R/bR]_e = [D]_e$. If $[R/bR]_e = [R/aR]_e$, then $R/aR \cong R/bR$, so that $R/cR \cong D$, a contradiction. Hence $[R/bR]_e = [D]_e$. Similarly, $[R/cR]_l = [D]_l$. ∎

Theorem 2.1.15. *Let $a_1, \dots, a_n, b_1, \dots, b_t$ be non-invertible elements of a local ring R. Then $R/a_1R \oplus \dots \oplus R/a_nR \cong R/b_1R \oplus \dots \oplus R/b_tR$ if and only if $n = t$ and there are two permutations σ, τ of $\{1, 2, \dots, n\}$ such that $[R/a_iR]_l = [R/b_{\sigma(i)}R]_l$ and $[R/a_iR]_e = [R/b_{\tau(i)}R]_e$ for every $i = 1, 2, \dots, n$.*

Proof. $(\Rightarrow)$: If a is not invertible, R/aR is couniform, that is, has dual Goldie dimension one. If $R/a_1R \oplus \dots \oplus R/a_nR \cong R/b_1R \oplus \dots \oplus R/b_tR$, then $n = t$.

For the existence of the permutation σ and τ, we argue by induction on n. The case $n = 1$ is trivial. Assume that R/a_iR is isomorphic to one of the R/b_jR's. We can cancel the isomorphic modules R/a_iR and R/b_jR because they have semilocal endomorphism rings. Now we can clearly proceed by induction. Hence we can assume that for every $i, j = 1, 2, \ldots, n$, R/a_iR is not isomorphic to R/b_jR. In particular, the endomorphism rings of R/a_iR and R/b_iR are not local.

Since R/a_1R is isomorphic to a direct summand of $R/b_1R \oplus \cdots \oplus R/b_nR$, by 2.1.13,there exist two distinct indeces $i, j = 1, 2, \ldots, n$ such that $[R/a_1R]_l = [R/b_iR]_l$ and $[R/a_1R]_e = [R/b_jR]_e$. Applying 2.1.14 to the three cyclically presented modules $R/a_1R, R/b_iR, R/b_jR$, we can find a cyclically presented module R/dR, unique up to isomorphism, such that $R/a_1R \oplus R/dR \cong R/b_iR \oplus R/b_jR$, $[R/dR]_l = [R/b_jR]_l$ and $[R/dR]_e = [R/b_iR]_e$. Hence $R/a_1R \oplus \cdots \oplus R/a_nR \cong R/b_1R \oplus \cdots \oplus R/b_nR \cong R/a_1R \oplus R/dR \oplus (\oplus_{k \in \{1,2,\ldots,n\} \setminus \{i,j\}} R/b_kR)$. It follows that $R/a_2 \oplus \cdots \oplus R/a_nR \cong R/dR \oplus (\oplus_{k \in \{1,2,\ldots,n\} \setminus \{i,j\}} R/b_kR)$. Now we deal with direct sums of $n-1$ cyclically presented modules, and again we can conclude by induction.

$(\Leftarrow)$: We argue by induction on $n = t$. The case $n = t = 1$ is obvious. Assume that $a_1, \ldots, a_n, b_1, \ldots, b_n$ are non-invertible elements of R and there are two permutations σ, τ of $\{1, 2, \ldots, n\}$ with $[R/a_iR]_l = [R/b_{\sigma(i)}R]_l$ and $[R/a_iR]_e = [R/b_{\tau(i)}R]_e$ for every i. If $\sigma(1) = \tau(1)$, then $R/a_1R \cong R/b_{\sigma(1)}R$. Thus σ and τ, induce two bijections from $\{2, 3, \ldots, n\}$ to $\{1, 2, \ldots, n\} \setminus \{\sigma(1)\}$, with the same properties as σ and τ. Now, by induction, $R/a_2R \oplus \cdots \oplus R/a_nR \cong \oplus_{j \in \{1,2,\ldots,n\} \setminus \{\sigma(1)\}} R/b_jR$.

Hence we can suppose $\sigma(1) \neq tau(1)$. Applying 2.1.14, we obtain that there exists a cyclically presented module R/a_0R, unique up to isomorphism, such that $R/a_1R \oplus R/a_0R \cong R/b_{\sigma(1)}R \oplus R/b_{\tau(1)}R$, $[R/a_0R]_l = [R/b_{\tau(1)}R]_l$ and $[R/a_0R]_e = [R/b_{\sigma(1)}R]_e$. That is, the modules R/a_1R, R/a_0R and the modules $R/b_{\sigma(1)}, R/b_{\tau(1)}R$ have the same lower parts and the same epigeny classes, counting multiplicities. The modules $R/a_0R, R/a_1R, \ldots, R/a_nR$ and the modules R/a_0R, $R/b_1R, \ldots, R/b_nR$ have the same lower parts and the same epigeny classes as well, so that the modules $R/a_2R, R/a_3R, \ldots, R/a_nR$ and the modules $R/a_0R, R/b_1R, \ldots, \widehat{R/b_{\sigma(1)}}R, \ldots, \widehat{R/b_{\tau(1)}}R$, $\ldots, R/b_nR$ have the same lower parts and the same epigeny classes. By the inductive hypothesis, $R/a_2R \oplus R/a_3R \oplus \cdots \oplus R/a_nR \cong R/a_0R \oplus (\oplus_{j \in \{1,2,\ldots,n\} \setminus \{\sigma(1),\tau(1)\}} R/b_jR)$. Thus $R/a_0R \oplus R/b_1R \oplus \cdots \oplus R/b_nR \cong R/a_2R \oplus R/a_3R \oplus \cdots \oplus R/b_{\sigma(1)}R \oplus R/b_{\tau(1)}R \cong R/a_0R \oplus R/a_1R \oplus R/a_2R \oplus \cdots \oplus R/a_nR$. It follows that $R/b_1R \oplus \cdots \oplus R/b_nR \cong R/a_1R \oplus R/a_2R \oplus \cdots \oplus R/a_nR$. ∎

Proposition 2.1.16. *Let R be a local ring and $a_1, \ldots, a_n, b_1, \ldots, b_t$ be non-zero non-inertible elements of R. Then $[R/a_1R \oplus \cdots \oplus R/a_nR]_l = [R/b_1R \oplus \cdots \oplus R/b_tR]_l$ if and only if $n = t$ and there is a permutation σ of $\{1, 2, \ldots, n\}$ such that $[R/a_iR]_l = [R/b_{\sigma(i)}R]_l$ for every $i = 1, 2, \ldots, n$.*

Proposition 2.1.17. *Let R be a local ring and M_R, N_R be finite direct sums of cyclically presented R-modules. Then $M_R \cong N_R$ if and only if $[M_R]_l = [N_R]_l$ and $[M_R]_e = [N_R]_e$.*

Proof. Assume that $[M_R]_l = [N_R]_l$ and $[M_R]_e = [N_R]_e$. By hypothesis, we can write $M_R = R/a_1R \oplus \cdots \oplus R/a_mR$ and $N_R = R/b_1R \oplus \cdots \oplus R/b_nR$ with $a_1, \ldots, a_p, b_1, \ldots, b_t \in J(R) \setminus \{0\}$ and $a_{p+1} = \cdots = a_m = b_{t+1} = \cdots = b_n = 0$.

Applying [DF02, Theorem 2] to M_R and N_R, we find that the epigeny classes $[R/a_iR]_e, i = 1, \ldots, m$ coincide with the epigeny classes $[R/b_jR]_e, j = 1, \ldots, n$, counting mulitiplicity. Note that $[R/aR]_e \neq [R_R]_e$ for any $a \in J(R)\backslash\{0\}$. Thus $p = t, m = n$ and there exists a permutation τ of $\{1, \dots, p\}$ such that $[R/a_iR]_e = [R/b_{\tau(i)}R]_e$ for every $i \in \{1, \dots, p\}$.

Now we have $[R/a_1R \oplus \cdots \oplus R/a_pR]_l = [M_R]_l = [N_R]_l = [R/b_1R \oplus \cdots \oplus R/b_tR]_l$. Applying 2.1.16 to $R/a_1R \oplus \cdots \oplus R/a_pR$ and $R/b_1R \oplus \cdots \oplus R/b_tR$, we obtain that there is a permutation σ of $\{1, \dots, p\}$ such that $[R/a_iR]_l = [R/b_{\sigma(i)}R]_l$ for every $i \in \{1, \dots, p\}$. Hence, by 2.1.15, we have $R/a_1R \oplus \cdots \oplus R/a_pR \cong R/b_1R \oplus \cdots \oplus R/b_tR$. It follows that $M_R \cong N_R$. ■

The material for the rest of this chapter is based on my joint paper with Alberto Facchini and Daniel Smertnig [FDT14].

2.2 Factorization of elements

Let R be a ring. An element $a \in R$ is *left cancellative* if, for all $b, c \in R$, $ab = ac$ implies $b = c$. Equivalently, $a \in R$ is left cancellative if it is non-zero and is not a left zero-divisor. A (non-necessarily commutative) ring R is a *domain* if every non-zero element is left cancellative (equivalently, if every non-zero element is right cancellative). If $a \in R$, the right R-module homomorphism $\lambda_a \colon R_R \to aR, x \mapsto ax$, is an isomorphism if and only if a is left cancellative. More precisely, $aR \cong R_R$ if and only if there exists a left cancellative element $a' \in R$ with $a'R = aR$. If $a, a' \in R$ are two left cancellative elements, then $aR = a'R$ if and only if $a = a'\varepsilon$ for some $\varepsilon \in U(R)$.

Let $a, x_1, \dots, x_n \in R\backslash U(R)$ be $n+1$ left cancellative elements and assume that $a = x_1 \cdot \ldots \cdot x_n$. If $\varepsilon_1, \dots, \varepsilon_{n-1} \in U(R)$, then obviously also $a = (x_1\varepsilon_1) \cdot (\varepsilon_1^{-1}x_2\varepsilon_2) \cdot \ldots \cdot (\varepsilon_{n-1}^{-1}x_n)$. This gives an equivalence relation on finite ordered sequences of left cancellative elements whose product is a. More precisely, if $F_a := \{\, (x_1, \dots, x_n) \mid n \geq 1,\ x_i \in R \setminus U(R)$ is left cancellative for every $i = 1, 2, \dots, n$ and $a = x_1 \cdot \ldots \cdot x_n \,\}$, then the equivalence relation $\sim$ on F_a is defined by $(x_1, \dots, x_n) \sim (x'_1, \dots, x'_m)$ if $n = m$ and there exist $\varepsilon_1, \dots, \varepsilon_{n-1} \in U(R)$ such that $x'_1 = x_1\varepsilon_1$, $x'_i = \varepsilon_{i-1}^{-1}x_i\varepsilon_i$ for all $i = 2, \dots, n-1$ and $x'_n = \varepsilon_{n-1}^{-1}x_n$. We call an equivalence class of F_a modulo $\sim$ a *factorization of a up to insertion of units*. Notice that the factors need not be irreducible. When this causes no confusion, we will simply call a representative of such an equivalence class a factorization.

A factorization $a = x_1 \cdot \ldots \cdot x_n$ gives rise to an ascending chain of principal right ideals, generated by left cancellative elements and containing aR:

$$aR \subsetneq x_1 \cdot \ldots \cdot x_{n-1}R \subsetneq \ldots \subsetneq x_1R \subsetneq R,$$

hence to an ascending chain of cyclically presented submodules

$$0 = aR/aR \subsetneq x_1 \cdot \ldots \cdot x_{n-1}R/aR \subsetneq \ldots \subsetneq x_1R/aR \subsetneq R/aR$$

of the cyclically presented R-module R/aR. Notice that $x_1 \cdot \ldots \cdot x_{i-1}R/aR \cong R/x_i \cdot \ldots \cdot x_nR$ is cyclically presented because the elements x_i are left cancellative.

The next lemma shows that, conversely, every chain of principal right ideals generated by left cancellative elements in $aR \subset R$, determines a factorization of a into left cancellative elements, which is unique up to insertion of units.

Lemma 2.2.1. *Let $a \in R$ be a left cancellative element, $aR = y_nR \subsetneq y_{n-1}R \subsetneq \dots \subsetneq y_1R \subsetneq y_0R = R$ be an ascending chain of principal right ideals of R, where $y_1, \dots, y_{n-1} \in R$ are left cancellative elements, $y_0 = 1$ and $y_n = a$. For every $i = 1, \dots, n$, let $x_i \in R$ be such that $y_{i-1}x_i = y_i$. Then $x_1, \dots, x_n$ are left cancellative elements and $a = x_1 \cdot \ldots \cdot x_n$.*

Moreover, if $y'_1, \dots, y'_{n-1} \in R$ are also left cancellative elements with $y'_iR = y_iR$, $y'_0 = 1$ and $y'_n = a$, and we similarly define x'_i by $y'_{i-1}x'_i = y'_i$ for every $i = 1, 2, \dots, n$, then there exist $\varepsilon_1, \dots, \varepsilon_{n-1} \in U(R)$ such that $x'_1 = x_1\varepsilon_1$, $x'_i = \varepsilon_{i-1}^{-1}x_i\varepsilon_i$ for all $i = 2, \dots, n-1$ and $x'_n = \varepsilon_{n-1}^{-1}x_n$.

Proof. Assume that x_i is not left cancellative for some $i = 1, 2, \dots, n$. Then there exists $b \neq 0$ such that $x_ib = 0$. Therefore $y_ib = y_{i-1}x_ib = 0$. This is a contradiction because y_i is left cancellative. Notice that $a = y_{n-1}x_n = y_{n-2}x_{n-1}x_n = \dots = y_0x_1 \dots x_n = x_1 \dots x_n$.

Now if $y'_iR = y_iR$ for every $i = 1, \dots, n-1$, then there exists $\varepsilon_1, \dots, \varepsilon_{n-1} \in U(R)$ such that $y'_i = y_i\varepsilon_i$. Therefore $y'_{i-1}x'_i = y_{i-1}x_i\varepsilon_i = y'_{i-1}\varepsilon_{i-1}^{-1}x_i\varepsilon_i$. But y'_{i-1} is left cancellative, so that $x'_i = \varepsilon_{i-1}^{-1}x_i\varepsilon_i$ for every $i = 2, \dots, n-1$.

Moreover, $y_1 = y_0x_1 = x_1$ and, similarly, $y'_1 = x'_1$, so that $y'_1 = y_1\varepsilon_1$ implies $x'_1 = x_1\varepsilon_1$. Finally, $y_{n-1}x_n = y_n = a = y'_n = y'_{n-1}x'_n = y_{n-1}\varepsilon_{n-1}x'_n$. Thus $x_n = \varepsilon_{n-1}x'_n$ and $x'_n = \varepsilon_{n-1}^{-1}x_n$. □

We will characterize, in Lemmas 2.3.1 and 2.4.3, the submodules of cyclically presented modules M_R that, under a suitable cyclic presentation $\pi\colon R_R \to M_R$, that is, a suitable epimorphism $\pi\colon R_R \to M_R$, lift to principal right ideals of R generated by a left cancellative element. The following lemma will prove to be helpful to this end.

Lemma 2.2.2. *Let A_R, B_R, M_R, N_R be modules over a ring R, $\pi_M\colon A_R \to M_R$ and $\pi_N\colon B_R \to N_R$ be epimorphisms, $\lambda\colon B_R \to A_R$ be a homomorphism and $\varepsilon\colon N_R \to M_R$ be a monomorphism such that $\pi_M\lambda = \varepsilon\pi_N$, so that there is a commutative diagram*

$$\begin{array}{ccc} B_R & \xrightarrow{\lambda} & A_R \\ {\scriptstyle \pi_N}\downarrow & & \downarrow{\scriptstyle \pi_M} \\ N_R & \overset{\varepsilon}{\hookrightarrow} & M_R. \end{array}$$

Then the following three conditions are equivalent:

(a) $\pi_M^{-1}(\varepsilon(N_R)) = \lambda(B_R)$.

(b) $\lambda(\ker(\pi_N)) = \ker(\pi_M)$.

(c) π_M *induces an isomorphism* $\operatorname{coker}(\lambda) \to \operatorname{coker}(\varepsilon)$.

If, moreover, A'_R, B'_R are right R-modules such that there exist isomorphisms $\varphi_A\colon A'_R \to A_R$ and $\varphi_B\colon B'_R \to B_R$, and one defines $\pi'_N := \pi_N\varphi_B$, $\pi'_M := \pi_M\varphi_A$ and $\lambda' := \varphi_A^{-1}\lambda\varphi_B$, then the three conditions (a), (b) *and* (c) *are equivalent also to the the three conditions*

(d) $(\pi'_M)^{-1}(\varepsilon(N_R)) = \lambda'(B'_R)$.

(e) $\lambda'(\ker(\pi'_N)) = \ker(\pi'_M)$.

(f) π'_M *induces an isomorphism* $\operatorname{coker}(\lambda') \to \operatorname{coker}(\varepsilon)$.

Proof. (a) $\Leftrightarrow$ (b): We have $\pi_M\lambda(B_R) = \varepsilon\pi_N(B_R) = \varepsilon(N_R)$. It follows that $\pi_M^{-1}(\varepsilon(N_R)) = \lambda(B_R) + \ker\pi_M$. Thus (a) is equivalent to $\ker\pi_M \subseteq \lambda(B_R)$. The inclusion $\lambda(\ker(\pi_N)) \subseteq \ker(\pi_M)$ always holds by the commmutativity of the diagram, so that b is equivalent to $\ker(\pi_M) \subseteq \lambda(\ker(\pi_N))$. Thus (b) $\Rightarrow$ (a) is trivial. Conversely, if (a) holds, and $a \in \ker(\pi_M)$, then $a = \lambda(b)$ for some $b \in B_R$, so that $0 = \pi_M(a) = \pi_M\lambda(b) = \varepsilon\pi_N(b)$. But ε is mono, so $\pi_N(b) = 0$, and $a = \lambda(b) \in \lambda(\ker(\pi_N))$.

(b) $\Leftrightarrow$ (c) Apply the Snake Lemma to the diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker(\pi_N) \arrow[r] \arrow[d, "\lambda|_{\ker}"] & B_R \arrow[r, "\pi_N"] \arrow[d, "\lambda"] & N_R \arrow[r] \arrow[d, "\varepsilon"'] & 0 \\
0 \arrow[r] & \ker(\pi_M) \arrow[r] & A_R \arrow[r, "\pi_M"'] & M_R \arrow[r] & 0,
\end{tikzcd}$$

obtaining a short exact sequence

$$0 = \ker(\varepsilon) \longrightarrow \operatorname{coker}(\lambda|_{\ker}) \longrightarrow \operatorname{coker}(\lambda) \longrightarrow \operatorname{coker}(\varepsilon) \longrightarrow 0.$$

Therefore $\lambda(\ker(\pi_N)) = \ker(\pi_M)$ if and only if $\lambda|_{\ker}$ is surjective, if and only if $\operatorname{coker}(\lambda|_{\ker}) = 0$, if and only if the epimorphism $\operatorname{coker}(\lambda) \to \operatorname{coker}(\varepsilon)$ is injective, if and only if it is an isomorphism.

Now assume that there exist isomorphisms $\varphi_A\colon A'_R \to A_R$ and $\varphi_B\colon B'_R \to B_R$ and set $\pi'_N := \pi_N\varphi_B$, $\pi'_M := \pi_M\varphi_A$ and $\lambda' := \varphi_A^{-1}\lambda\varphi_B$. To conclude the proof, it suffices to show that $\lambda(\ker(\pi_N)) = \ker(\pi_M)$ if and only if $\lambda'(\ker(\pi'_N)) = \ker(\pi'_M)$. This is true, since $\ker(\pi'_M) = \varphi_A^{-1}(\ker(\pi_M))$ and

$$\lambda'(\ker(\pi'_N)) = \lambda'(\varphi_B^{-1}(\ker(\pi_N))) = \varphi_A^{-1}\lambda\varphi_B(\varphi_B^{-1}(\ker(\pi_N))) = \varphi_A^{-1}(\lambda(\ker(\pi_N))). \qquad \square$$

2.3 π-exactness

Let M_R be a cyclically presented right R-module and $\pi_M\colon R_R \to M_R$ a cyclic presentation. We introduce the notion of π_M-exactness to characterize those submodules of M_R that lift, via π_M, to principal right ideals of R, generated by a left cancellative element of R. We give sufficient conditions on R for this notion to be independent from the chosen presentation π_M.

Definition and Lemma 2.3.1 (π-exactness). Let $N_R \leq M_R$ be cyclic right R-modules. Let $F_R \cong R_R$, fix an epimorphism $\pi_M\colon F_R \to M_R$ and let $\varepsilon\colon N_R \hookrightarrow M_R$ denote the embedding. The following conditions are equivalent:

(a) $\pi_M^{-1}(N_R) \cong R_R$.

(b) There exists a monomorphism $\lambda\colon R_R \to F_R$ and an epimorphism $\pi_N\colon R_R \to N_R$ such that $\lambda(\ker(\pi_N)) = \ker(\pi_M)$ and the following diagram commutes:

$$\begin{tikzcd} R_R \arrow[r, "\lambda"] \arrow[d, "\pi_N"'] & F_R \arrow[d, "\pi_M"] \\ N_R \arrow[r, "\varepsilon"'] & M_R. \end{tikzcd} \tag{2.1}$$

(c) There exists a monomorphism $\lambda\colon R_R \to F_R$ and an epimorphism $\pi_N\colon R_R \to N_R$ such that diagram (2.1) commutes and induces an isomorphism $\operatorname{coker}(\lambda) \to \operatorname{coker}(\varepsilon)$.

If these equivalent conditions are satisfied, we call N_R a textitπ_M-exact!submodules of M_R.

Proof. (a) $\Rightarrow$ (b). By (a), there exists an isomorphism $\lambda_0\colon R_R \to \pi_M^{-1}(N_R)$. Let λ be the composite mapping $R_R \xrightarrow{\lambda_0} \pi_M^{-1}(N_R) \hookrightarrow F_R$ and $\varepsilon^{-1}\colon \varepsilon(N_R) \to N_R$ be the inverse of the corestriction of ε to $\varepsilon(N_R)$. Noticing that $\pi_M\lambda(R_R) = \varepsilon(N_R)$, one gets an onto mapping $\pi_N := \varepsilon^{-1}\pi_M\lambda\colon R_R \to N_R$. Then diagram (2.1) clearly commutes and $\lambda(R_R) = \pi_M^{-1}(N_R)$. The statement now follows from Lemma 2.2.2.

(b) $\Leftrightarrow$ (c) and (b) $\Rightarrow$ (a). By Lemma 2.2.2. $\square$

Corollary 2.3.2. *Let $F_R \cong R_R$ and let $\pi_M\colon F_R \to M_R$ be an epimorphism. If $\varphi\colon F'_R \to F_R$ is an isomorphism and $N_R \leq M_R$, then N_R is a π_M-exact submodule of M_R if and only if it is a $\pi_M\varphi$-exact submodule of M_R.*

Proof. Let N_R be a π_M-exact submodule of M_R and let $\lambda\colon R_R \to F_R$ be a monomorphism satisfying condition (b) of Definition and Lemma 2.3.1. Apply Lemma 2.2.2 to $B_R = B'_R = R_R$, $A_R = F_R$, $A'_R = F'_R$, $\varphi_B = 1_R$ and $\varphi_A = \varphi$. Setting $\lambda' := \varphi^{-1}\lambda$, it follows that $\lambda'(\ker(\pi_N)) = \ker(\pi_M\varphi)$ and hence N_R is a $\pi_M\varphi$-exact submodule of M_R. The converse follows applying what we have just shown to φ^{-1}. $\square$

Corollary 2.3.3. *Let $N_R \leq M_R$ be cyclic R-modules, $\pi_M\colon R_R \to M_R$ be an epimorphism and $N_R \leq M_R$ be a π_M-exact submodule. Then M_R/N_R is cyclically presented with presentation induced by π_M.*

Proof. Let $\lambda\colon R_R \to R_R$ be as in condition (c) of Definition and Lemma 2.3.1. Then $M_R/N_R \cong R_R/\lambda(R_R)$, from which the conclusion follows immediately. $\square$

Corollary 2.3.4. *Let $N_R \leq M_R \leq P_R$ be cyclic R-modules and let $\pi_P \colon F_R \to P_R$ be an epimorphism, where $F_R \cong R_R$. If $M_R \leq P_R$ is π_P-exact and $N_R \leq M_R$ is $\pi_P|_{\pi_P^{-1}(M_R)}$-exact, then $N_R \leq P_R$ is π_P-exact.*

Proof. Set $F'_R := \pi_P^{-1}(M_R)$. By condition (a) of Definition and Lemma 2.3.1, $F'_R \cong R_R$. Therefore the notion of $\pi_P|_{F'_R}$-exactness of N_R in M_R is indeed defined. Since $\pi_P^{-1}(N_R) = (\pi_P|_{F'_R})^{-1}(N_R) \cong R_R$, the claim follows. □

Let $c \in R$ be left cancellative and denote by $\mathsf{L}(cR, R)$ the set of all right ideals aR with $a \in R$ left cancellative and $cR \subset aR \subset R$. It is partially ordered by set inclusion. Let $\pi \colon R \to R/cR$ be an epimorphism. Denote by $\mathsf{L}_\pi(R/cR)$ the set of all π-exact submodules of R/cR. This set is also partially ordered by set inclusion.

Lemma 2.3.5. *Let $c \in R$ be left cancellative and let $\pi \colon R_R \to R/cR$ be the canonical epimorphism. Then π induces an isomorphism of partially ordered sets $\mathsf{L}(cR, R) \cong \mathsf{L}_\pi(R/cR)$.*

Proof. It suffices to show that $N_R \subset R/cR$ is π-exact if and only if there exists a left cancellative $a \in R$ with $\pi^{-1}(N_R) = aR$. But this is equivalent to $\pi^{-1}(N_R) \cong R_R$. The statement now follows from condition Definition and Lemma (a) of 2.3.1. □

The following example shows that, in general, the condition of π-exactness indeed depends on the particular choice of the epimorphism $\pi \colon R_R \to M_R$. We refer the reader to any of [MR03], [Rei75] or [Vig80] for the necessary background on quaternion algebras.

Example 2.3.6. Let A be a quaternion algebra over $\mathbb{Q}$ and R be a maximal $\mathbb{Z}$-order in A such that there exists an unramified prime ideal $\mathfrak{P} \subset R$ and maximal right ideals I, J of R with $I, J \supset \mathfrak{P}$, I principal and J non-principal. Then $\mathfrak{p} = \mathfrak{P} \cap \mathbb{Z}$ is principal, say $\mathfrak{p} = p\mathbb{Z}$ with $p \in \mathbb{P}$, $\mathfrak{P} = pR$, $R/\mathfrak{P} \cong M_2(\mathbb{F}_p)$ and $\mathfrak{P} = \mathrm{Ann}(R/\mathfrak{P})$. (E.g., take $A = \left(\frac{-1,-11}{\mathbb{Q}}\right)$, $R = {}_{\mathbb{Z}}\langle 1, i, \frac{1}{2}(i+j), \frac{1}{2}(1+k)\rangle$, $p = 3$, $I = {}_{\mathbb{Z}}\langle \frac{1}{2}(1+5k), \frac{1}{2}(i+5j), 3j, 3k\rangle$ and $J = {}_{\mathbb{Z}}\langle \frac{1}{2}(1+2j+3k), \frac{1}{2}(i+3j+4k), 3j, 3k\rangle$).

The module $R/\mathfrak{P}$ has a composition series (as an $R/\mathfrak{P}$- and hence as an R-module)

$$0 \subsetneq I/\mathfrak{P} \subsetneq R/\mathfrak{P},$$

and there exists an isomorphism $R/\mathfrak{P} \to R/\mathfrak{P}$ mapping $J/\mathfrak{P}$ to $I/\mathfrak{P}$, as is easily seen from $R/\mathfrak{P} \cong M_2(\mathbb{F}_p)$. Therefore there exist epimorphisms $\pi_M \colon R \to R/\mathfrak{P}$ and $\pi'_M \colon R \to R/\mathfrak{P}$ with $\pi_M^{-1}(I/\mathfrak{P}) = I$ and $\pi'^{-1}_M(I/\mathfrak{P}) = J$. This implies that $I/\mathfrak{P}$ is a π_M-exact submodule of $R/\mathfrak{P}$ that is not π'_M-exact.

However, under an additional assumption on R_R, which holds, for instance, whenever R is a semilocal ring, the notion is independent of the choice of π.

Lemma 2.3.7. *Suppose that $R_R \oplus K_R \cong R_R \oplus R_R$ implies $K_R \cong R_R$ for all right ideals K_R of R.*

1. *If $M_R \cong R/aR$ with $a \in R$ left cancellative and $\pi_M \colon R_R \to M_R$ is an epimorphism, then there exists a left cancellative $a' \in R$ such that* $\ker(\pi_M) = a'R$.

2. *If M_R is a cyclic R-module, $\pi_M\colon R_R \to M_R$ and $\pi'_M\colon R_R \to M_R$ are epimorphisms and $N_R \le M_R$, then N_R is a π_M-exact submodule of M_R if and only if it is a π'_M-exact submodule of M_R.*

Proof. (1) Let $\pi_{aR}\colon R_R \to R/aR, 1 \mapsto 1 + aR$ be the canonical epimorphism. Since a is left cancellative, $aR \cong R_R$. Consider the exact sequences

$$0 \to aR \hookrightarrow R_R \xrightarrow{\pi_{aR}} R/aR \to 0$$

and

$$0 \to \ker(\pi_M) \hookrightarrow R_R \xrightarrow{\pi_M} R/aR \to 0.$$

By Schanuel's Lemma, $R_R \oplus aR \cong R_R \oplus \ker(\pi_M)$, and hence by assumption $aR \cong \ker(\pi_M)$. Thus there exists a left cancellative $a' \in R$ with $\ker(\pi_M) = a'R$.

(2) Let $\pi_{M/N}\colon M_R \to M_R/N_R$ be the canonical quotient module epimorphism. There are exact sequences

$$0 \to \pi_M^{-1}(N_R) \to R_R \xrightarrow{\pi_{M/N}\pi_M} M_R/N_R \to 0$$

and

$$0 \to \pi'^{-1}_M(N_R) \to R_R \xrightarrow{\pi_{M/N}\pi'_M} M_R/N_R \to 0,$$

and by Schanuel's Lemma therefore $R_R \oplus \pi_M^{-1}(N_R) \cong R_R \oplus \pi'^{-1}_M(N_R)$. If N_R is a π_M-exact submodule of M_R, then $\pi_M^{-1}(N_R) \cong R_R$ and hence $\pi'^{-1}_M(N_R) \cong R_R$ by our assumption on R, showing that N_R is a π'_M-exact submodule. The converse follows by symmetry. □

Suppose that R has invariant basis number (for all $m, n \in \mathbb{N}_0$, $R_R^m \cong R_R^n$ implies $m = n$). Then the condition of the previous lemma is satisfied if every stably free R-module of rank 1 is free [MR01, §11.1.1]. This is true if R is commutative [MR01, §11.1.16]. The condition is also true if R is semilocal [Fac10, Corollary 4.6] or R is a 2-fir (by [Coh85, Theorem 1.1(e)]).

Let M_R be a right R-module with an epimorphism $\pi_M\colon R_R \to M_R$ with $\ker(\pi_M) = aR$ and $a \in R$ left cancellative. We say that a finite series

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M_R$$

of submodules is *π_M-exact*, if every M_i is an $\pi_M|_{\pi_M^{-1}(M_{i+1})}$-exact submodule of M_{i+1}. By Lemma 2.3.5 the π_M-exact series of submodules of R are in bijection with series of principal right ideals in $\mathsf{L}(aR, R)$. By Lemma 2.2.1 they are therefore in bijection with factorizations of a into left cancellative elements, up to insertion of units.

Recall that a ring R is a 2-*fir* if and only if it is a domain and the sum of any two principal right ideals with non-zero intersection is again a principal right ideal [Coh85, Theorem 1.5.1]. In the next theorem, we will consider, for a cyclically presented right R-module M_R and a cyclic presentation $\pi_M\colon R_R \to M_R$ with non-zero kernel, the set of all submodules of cyclically presented π_M-exact submodules. We say it is *closed under finite sums* if for every two cyclically presented π_M-exact submodules M_1 and M_2 of M_R, the sum $M_1 + M_2$ also is cyclically presented and a π_M-exact submodule of M_R.

Theorem 2.3.8. *Let R be a domain. The following conditions are equivalent.*

1. *For every cyclically presented right R-module M_R and every cyclic presentation $\pi_M\colon R_R \to M_R$ with non-zero kernel, the set of all cyclically presented π_M-exact submodules is closed under finite sums.*

2. *R is a 2-fir.*

Proof. (1) $\Rightarrow$ (2): Let $a, b, c \in R \setminus \{0\}$ be such that $cR \subset aR \cap bR$. We have to show that $aR + bR$ is right principal. Let $M_R = R/cR$, $\pi_M\colon R_R \to R/cR$ be the canonical epimorphism, $M_1 = aR/cR$ and $M_2 = bR/cR$. By Lemma 2.3.5, $M_1 = \pi_M(aR)$ and $M_2 = \pi_M(bR)$ are π_M-exact submodules of M_R. By assumption $M_1 + M_2$ is a π_M-exact submodule of M_R. Again by Lemma 2.3.5, $aR + bR = \pi_M^{-1}(M_1 + M_2)$ is a principal right ideal of R, generated by a left cancellative element.

(2) $\Rightarrow$ (1): We may assume $M_1, M_2 \neq 0$, as the statement is trivial otherwise. Let $\pi_M\colon R_R \to M_R$ be an epimorphism with non-zero kernel. Since M_1 and M_2 are π_M-exact submodules of M_R, there exist $a, b \in R \setminus \{0\}$ such that $\pi^{-1}(M_1) = aR$ and $\pi^{-1}(M_2) = bR$. Because $\ker(\pi) \neq 0$, we have $aR \cap bR \neq 0$. Since R is a 2-fir, there exists $c \in R \setminus \{0\}$ such that $aR + bR = \pi_M^{-1}(M_1 + M_2) = cR$. Therefore $M_1 + M_2$ is cyclically presented and a π_M-exact submodule of M_R. $\square$

Notice that if we assume that sums and intersections of exact submodules are again exact submodules, one may use the Artin-Schreier and Jordan-Hölder-Theorems to study factorizations of elements. As we have just seen, such an assumption leads to the 2-firs investigated by Cohn in [Coh85].

2.4 Projective covers of cyclically presented modules

Let R be a ring and R/xR a cyclically presented right R-module, $x \in R$. The module R/xR does not have a projective cover in general, but if it has one, it has one of the form $\pi|_{eR}\colon eR \to R/xR$, where $e \in R$ is an idempotent that depends on x and $\pi|_{eR}$ is the restriction to eR of the canonical projection $\pi\colon R_R \to R/xR$ (see 1.1.30). More precisely, given any projective cover $p\colon P_R \to R/xR$, there is an isomorphism $f\colon eR \to P_R$ such that $pf = \pi|_{eR}$. The kernel of the projective cover $\pi|_{eR}\colon eR \to R/xR$ is $eR \cap xR$ and is contained in $eJ(R)$ because the kernel of $\pi|_{eR}$ is a superfluous submodule of eR and $eJ(R)$ is the largest superfluous submodule of eR. Considering the exact sequences $0 \to xR \to R_R \to R/xR \to 0$ and $0 \to eR \cap xR \to eR \to R/xR \to 0$, one sees that $R_R \oplus (eR \cap xR) \cong eR \oplus xR$ (Schanuel's Lemma), so that $eR \cap xR$ can be generated with at most two elements.

Recall that every right R-module has a projective cover if and only if the ring R is perfect, and that every finitely generated right R-module has a projective cover if and only every simple right R-module has a projective cover, if and only if the ring R is semiperfect. Denoting by $J(R)$ the Jacobson radical of R, R is semiperfect if and only if $R/J(R)$ is semisimple and idempotents can be lifted modulo $J(R)$ (see 1.1.46). The next result gives a similar characterization for the rings R over which every cyclically presented right module has a projective cover.

Theorem 2.4.1. *The following conditions are equivalent for a ring R with Jacobson radical $J(R)$:*

(1) *Every cyclically presented right R-module has a projective cover.*

(2) *The ring $R/J(R)$ is Von Neumann regular and idempotents can be lifted modulo $J(R)$.*

Proof. Set $J := J(R)$.

(1) $\Rightarrow$ (2) Assume that every cyclically presented right R-module has a projective cover. In order to show that R/J is Von Neumann regular, it suffices to prove that every principal right ideal of R/J is a direct summand of the right R/J-module R/J by **??** and **??**. Let x be an element of R. We will show that $(xR + J)/J$ is a direct summand of R/J as a right R/J-module. By (1), the cyclically presented right R-module R/xR has a projective cover. By 1.1.43, the projective cover is of the form $\pi|_{eR}\colon eR \to R/xR$ for some idempotent e of R, where $\pi\colon R_R \to R/xR$ is the canonical projection.

Applying the right exact functor $-\otimes_R R/J$ to the short exact sequence $0 \to eR\cap xR \to eR \to R/xR \to 0$, we get an exact sequence $(eR \cap xR) \otimes_R R/J \to eR \otimes_R R/J \to R/xR \otimes_R R/J \to 0$, which can be rewritten as $(eR \cap xR)/(eR \cap xR)J \to eR/eJ \to R/(xR + J) \to 0$. It follows that there is a short exact sequence $0 \to ((eR\cap xR)+eJ)/eJ \to eR/eJ \to R/(xR+J) \to 0$. Now the kernel $eR\cap xR$ of the projective cover $\pi|_{eR}$ is superfluous in eR and eJ is the largest superfluous submodule of eR, hence $((eR \cap xR) + eJ)/eJ = 0$ and $eR/eJ \cong R/(xR + J)$.

Now $(e+J)(R/J) = (eR+J)/J \cong eR/(eR\cap J) = eR/eJ$, so that $eR/eJ \cong R/(xR+J)$ is a projective right R/J-module. Thus the short exact sequence $0 \to (x+J)(R/J) = (xR+J)/J \to R/J \to R/(xR + J) \to 0$ splits, and the principal right ideal of R/J generated by $x + J$ is a direct summand of the right R/J-module R/J.

We must now prove that idempotents of R/J lift modulo J. By 1.1.44, this is equivalent to showing that every direct summand of the R-module R/J has a projective cover. Let M_R be a direct summand of $(R/J)_R$. Then it is also a direct summand of $(R/J)_{R/J}$ and hence is generated by an idempotent of R/J. Let $g \in R$ be such that $g + J \in R/J$ is idempotent and $M_{R/J} = (g + J)(R/J)$. Then $R/J = (g + J)(R/J) \oplus (1 - g + J)(R/J)$ as R/J-modules, and hence also as R-modules. The canonical projection $\pi_g\colon R/J \to M_R$ has kernel $\ker(\pi_g) = (1 - g + J)(R/J)$. Let $\pi\colon R_R \to R/J, r \mapsto r + J$ be the canonical epimorphism. Set $f := \pi_g\pi$. Then $\ker(f) = (1 - g)R + J$ and so f factors through an epimorphism $\overline{f}\colon R/(1 - g)R \to M_R$ with $\ker(\overline{f}) = (J + (1 - g)R)/(1 - g)R$. In particular, $\ker(\overline{f})$ is the image of the superfluous submodule J of R_R via the canonical projection $R_R \to R/(1 - g)R$. It follows that $\ker(\overline{f})$ is superfluous in $R/(1 - g)R$, i.e., $\overline{f}$ is a superfluous epimorphism.

By hypothesis, there is a projective cover $p\colon P_R \to R/(1-g)R$. Since the composite mapping of two superfluous epimorphisms is a superfluous epimorphism (this follows easily from 1.1.7), $\overline{f}p\colon P_R \to M_R$ is a superfluous epimorphism and hence a projective cover of M.

(2) $\Rightarrow$ (1) Assume that (2) holds. Let R/xR be a cyclically presented right R-module, where $x \in R$. The principal right ideal $(x+J)(R/J)$ of the Von Neumann regular ring R/J is generated by an idempotent and idempotents can be lifted modulo J. Hence there exists an idempotent

element $e \in R$ such that $(x+J)(R/J) = (e+J)(R/J)$. Let $\pi|_{(1-e)R}$ be the restriction to $(1-e)R$ of the canonical epimorphism $\pi\colon R_R \to R/xR$. We claim that $\pi|_{(1-e)R}\colon (1-e)R \to R/xR$ is onto. To prove the claim, notice that $xR + J = eR + J$, so that $(1-e)R + xR + J = R$. As J is superfluous in R_R, it follows that $(1-e)R + xR = R$ and so $\pi|_{(1-e)R}$ is onto. This proves our claim. Finally, $\ker(\pi|_{(1-e)R}) = (1-e)R \cap xR \subseteq ((1-e)R+J) \cap (xR+J) = ((1-e)R+J) \cap (eR+J) \subseteq J$, so that $\ker(\pi|_{(1-e)R}) \subseteq J \cap (1-e)R = (1-e)J$ is superfluous in $(1-e)R$. Thus $\pi|_{(1-e)R}$ is the required projective cover of the cyclically presented R-module R/xR. □

Corollary 2.4.2. *If R is a domain and every cyclically presented right R-module has a projective cover, then R is local.*

Proof. By the previous Theorem, $R/J(R)$ is Von Neumann regular. Since idempotents can be lifted modulo $J(R)$, and R has only two idempotents $0, 1$ thanks to the fact that R is a domain, the only idempotents of $R/J(R)$ are $0 + J(R)$ and $1 + J(R)$. Let $0 \neq x \in R/J(R)$. Because $R/J(R)$ is von Neumann and $R/J(R)$ has only two idempotents $0 + J(R), 1 + J(R)$, we deduce that $xR/J(R) = R/J(R)$, which implies that x is right invertible. Hence every nonzero element of $R/J(R)$ is right invertible. Let $0 \neq y \in R/J(R)$. Then there is an element $z \in R/J(R)$ such that $yz = 1 + J(R)$. As $z \neq 0$, there is an element $t \in R/J(R)$ such that $zt = 1 + J(R)$. Now we have that $y = y(zt) = (yz)t = t$, and hence $zy = zt = 1 + J(R)$. Thus y is invertible. Therefore $R/J(R)$ is a division ring and so R is local. ■

Notice that, conversely, if R is a local ring and M_R is any non-zero cyclic module, then every epimorphism $\pi\colon R_R \to M_R$ is a projective cover.

Lemma 2.4.3. *Let R be an arbitrary ring, let $N_R \leq M_R$ be cyclic right R-modules with a projective cover and let $\varepsilon\colon N_R \to M_R$ be the embedding. Then the following two conditions are equivalent:*

1. *There exist a projective cover $\pi_N\colon P_R \to N_R$ of N_R, a projective cover $\pi_M\colon Q_R \to M_R$ of M_R and a commutative diagram of right R-module morphisms*

$$\begin{tikzcd} P_R \arrow[r, "\lambda"] \arrow[d, "\pi_N"'] & Q_R \arrow[d, "\pi_M"] \\ N_R \arrow[r, "\varepsilon"'] & M_R, \end{tikzcd} \tag{2.2}$$

   such that the following equivalent conditions hold:

   (a) $\lambda(P_R) = \pi_M^{-1}(\varepsilon(N_R))$;

   (b) $\lambda(\ker(\pi_N)) = \ker(\pi_M)$;

   (c) π_M *induces an isomorphism* $\operatorname{coker}(\lambda) \to \operatorname{coker}(\varepsilon)$.

2. *For every pair of projective covers $\pi_N\colon P_R \to N_R$ of N_R and $\pi_M\colon Q_R \to M_R$ of M_R and every commutative diagram* (2.2) *of right R-module morphisms, the following equivalent conditions hold:*

(a') $\lambda(P_R) = \pi_M^{-1}(\varepsilon(N_R))$*;*

(b') $\lambda(\ker(\pi_N)) = \ker(\pi_M)$*;*

(c') π_M *induces an isomorphism* $\operatorname{coker}(\lambda) \to \operatorname{coker}(\varepsilon)$*.*

Proof. The equivalences (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) and (a') $\Leftrightarrow$ (b') $\Leftrightarrow$ (c') have been proved in Lemma 2.2.2.

(b) $\Rightarrow$ (b'): Assume that $\pi_N\colon P_R \to N_R$, $\pi_M\colon Q_R \to M_R$ and $\lambda\colon P_R \to Q_R$ satisfy condition (b), that is, make diagram (2.2) commute and $\lambda(\ker(\pi_N)) = \ker(\pi_M)$. Let $\pi'_N\colon P'_R \to N_R$ and $\pi'_M\colon Q'_R \to M_R$ be projective covers and $\lambda'\colon P'_R \to Q'_R$ be a morphism that make the diagram corresponding to diagram (2.2) commute, that is, such that $\pi'_M\lambda' = \varepsilon\pi'_N$. Projective covers are unique up to isomorphism and, by Lemma 2.2.2, we may therefore assume $P'_R = P_R$, $Q'_R = Q_R$ and $\pi'_M = \pi_M$, $\pi'_N = \pi_N$.

Then $\pi_M(\lambda - \lambda') = \pi_M\lambda - \varepsilon\pi_N = \varepsilon\pi_N - \varepsilon\pi_N = 0$, so that $(\lambda - \lambda')(P_R) \subseteq \ker\pi_M$. Let $\iota\colon \ker\pi_M \to Q_R$ denote the inclusion. Then there exists a morphism $\psi\colon P_R \to \ker\pi_M$ such that $\lambda - \lambda' = \iota\psi$. As images via module morphisms of superfluous submodules are superfluous submodules and $\ker\pi_N$ is a superfluous submodule of P_R, it follows that $\psi(\ker\pi_N)$ is a superfluous submodule of $\ker\pi_M$. Now $\ker\pi_M = \lambda(\ker\pi_N) = (\lambda' + \iota\psi)(\ker\pi_N) \subseteq \lambda'(\ker\pi_N) + \iota\psi(\ker\pi_N) = \lambda'(\ker\pi_N) + \psi(\ker\pi_N) \subseteq \ker\pi_M$. Thus $\ker\pi_M = \lambda'(\ker\pi_N) + \psi(\ker\pi_N)$. But $\psi(\ker\pi_N)$ is superfluous in $\ker\pi_M$, hence $\ker\pi_M = \lambda'(\ker\pi_N)$, which proves (b').

(b') $\Rightarrow$ (b): Let $\pi_N\colon P_R \to N_R$ and $\pi_M\colon Q_R \to M_R$ be projective covers of N_R, respectively M_R. Since P_R is projective and $\pi_M\colon Q_R \to M$ is an epimorphism, there exists a $\lambda\colon P_R \to Q_R$ such that $\pi_M\lambda = \varepsilon\pi_N$. By (b'), then $\lambda(\ker(\pi_N)) = \ker(\pi_M)$. □

Definition 2.4.4. If $N_R \leq M_R$ are cyclic right R-modules and the equivalent conditions of Theorem 2.4.3 are satisfied, we say that N_R is an *exact submodule* of M_R.

Corollary 2.4.5. *If* $L_R \leq M_R \leq N_R$ *are cyclic right R-modules,* M_R *is exact in* N_R *and* L_R *is exact in* M_R*, then* L_R *is exact in* N_R*.*

Proof. Since L_R is exact in M_R and M_R is exact in N_R, there exist projective covers $\pi_L\colon P_R \to L_R$, $\pi_M\colon Q_R \to M_R$, $\pi'_M\colon Q'_R \to M_R$ and $\pi_N\colon U_R \to N_R$ and homomorphisms $\lambda\colon P_R \to Q_R$ and $\mu\colon Q'_R \to U_R$ such that $\pi_M\lambda = \pi_L$, $\pi_N\mu = \pi'_M$, $\lambda(\ker(\pi_L)) = \ker(\pi_M)$ and $\mu(\ker(\pi'_M)) = \ker(\pi_N)$.

Since the projective cover of M_R is unique up to isomorphism, we may assume by Lemma 2.2.2 that $Q_R = Q'_R$ and $\pi'_M = \pi_M$ (replacing λ accordingly). Then $\pi_N\mu\lambda = \pi_M\lambda = \pi_L$ and $\ker(\pi_N) = \mu(\ker(\pi_M)) = \mu(\lambda(\ker(\pi_L)) = (\mu\lambda)(\ker(\pi_L))$. Therefore N_R is an exact submodule of M_R. ■

Corollary 2.4.6. *If a cyclic module* N_R *is an exact submodule of a cyclic module* M_R *and* M_R *has a projective cover isomorphic to* R_R*, then* M_R/N_R *is cyclically presented.*

Proof. Since N_R is an exact submodule of M_R, there exists a commutative diagram

$$\begin{tikzcd} P_R \arrow[r, "\lambda"] \arrow[d, "\pi_N"'] & Q_R \arrow[d, "\pi_M"] \\ N_R \arrow[r, "\varepsilon"'] & M_R \end{tikzcd}$$

where $\pi_N \colon P_R \to N_R$ and $\pi_M \colon Q_R \to M_R$ are projective covers of N_R and M_R and $\operatorname{coker}(\lambda) \cong \operatorname{coker}(\varepsilon)$. By assumption, there exists an idempotent $e \in R$ such that $P_R \cong eR$ and $Q_R \cong R_R$. By Lemma 2.2.2, we may therefore assume $P_R = eR$ and $Q_R = R_R$ (replacing π_M, π_N and λ accordingly). Therefore $M_R/N_R = \operatorname{coker}(\varepsilon) \cong \operatorname{coker}(\lambda) = R/eR$. Hence M_R/N_R is cyclically presented. ∎

The following example shows that if R is not a domain, then even if a non-unit $x \in R$ is not a zero-divisor, the projective cover of R/xR need not be isomorphic to R_R.

Example 2.4.7. Let D be a discrete valuation ring and $\pi \in D$ a prime element. The unique maximal ideal of D is πD. Let $R = M_2(D)$, $x = \begin{bmatrix} 1 & 0 \\ 0 & \pi \end{bmatrix}$ and $e = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

We have

$$xR = \begin{bmatrix} D & D \\ \pi D & \pi D \end{bmatrix} \quad \text{and} \quad eR = \begin{bmatrix} 0 & 0 \\ D & D \end{bmatrix}.$$

Let $p \colon R_R \to R/xR$ be the canonical projection. We will show that $p|_{eR} : eR \to R/xR$ is a projective cover of R/xR. We have $\ker p|_{eR} = xR \cap eR = \begin{bmatrix} 0 & 0 \\ \pi D & \pi D \end{bmatrix}$. Since $J(R) = M_2(J(D)) = \begin{bmatrix} \pi D & \pi D \\ \pi D & \pi D \end{bmatrix}$, it follows that $\ker p|_{eR} = eJ(R)$. Since e is an idempotent of R, eR is projective and $eJ(R) = J(eR)$. In particular, $\ker p|_{eR}$ is superfluous in eR. Therefore eR is a projective cover of R/xR.

We now show that $eR \not\cong R$. Assume eR is isomorphic to R. Then there exists an isomorphism $f \colon R_R \to eR$. Hence $f(1) = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} \neq 0$.

Let $b = \begin{bmatrix} -d & 0 \\ c & 0 \end{bmatrix}$. Then $b \neq 0$, because $f(1) \neq 0$. But $f(1)b = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} \begin{bmatrix} -d & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ implies $f(b) = 0$. It follows that $b = 0$, which contradicts $b \neq 0$. Thus eR is not isomorphic to R.

The next example shows that the condition for the projective cover of M_R to be isomorphic to R_R is necessary in Corollary 2.4.6.

Example 2.4.8. Let $R = T_2(\mathbb{Z}/2\mathbb{Z})$ be the ring of all upper triangular 2×2 matrices with coefficients in $\mathbb{Z}/2\mathbb{Z}$. Since $J(R)$ consists of all strictly upper triangular matrices, $R/J(R) \cong (\mathbb{Z}/2\mathbb{Z})^2$ is semisimple and obviously idempotents lift modulo $J(R)$. Therefore every finitely generated R-module has a projective cover. Set

$$M_R := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} R = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\},$$

$$N_R := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} R = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\},$$

$$M_R/N_R = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + N_R, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + N_R \right\}.$$

Consider

$$\begin{aligned} \phi \colon N_R &\longrightarrow \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} R \\ \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix} &\longmapsto \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \end{aligned}$$

It is obvious that ϕ is an isomorphism. Since $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is an idempotent of R, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} R$ is a projective R-module. Hence N_R is a projective R-module. On the other hand, M_R is also a projective R-module, because $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is an idempotent of R. Hence $1_N \colon N_R \to N_R$ and $1_M \colon M_R \to M_R$ are projective covers. This implies that the diagram

$$\begin{tikzcd} N_R \arrow[r, "\varepsilon"] \arrow[d, "1_N"'] & M_R \arrow[d, "1_M"] \\ N_R \arrow[r, "\varepsilon"'] & M_R, \end{tikzcd}$$

where $\varepsilon(\ker 1_N) = \ker 1_M$, commutes. Therefore N_R is an exact submodule of M_R.

Assume M_R/N_R is a cyclically presented module. Then M_R/N_R is isomorphic to R/xR, where $x \in R$. Since $|M_R/N_R| = 2$, $|xR| = 4$. We have

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} R = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\},$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} R = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix} \right\} = N_R,$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} R = M_R,$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} R = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & b+c \\ 0 & 0 \end{bmatrix} \right\} = M_R,$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} R = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \right\},$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} R = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 0 & c \\ 0 & c \end{bmatrix} \right\},$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} R = R_R,$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} R = R_R.$$

Thus $xR = M_R$. Hence

$$R/xR = R/M_R = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + M_R, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + M_R \right\},$$

$$\begin{aligned} \operatorname{ann}(M_R/N_R) &= \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in R \;\middle|\; \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in N_R \right\} \\ &= \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in R \;\middle|\; \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \in N_R \right\} \\ &= \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}, \end{aligned}$$

$$\operatorname{ann}(R/xR) = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in R \;\middle|\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in xR = M_R \right\} = M_R.$$

Hence $\operatorname{ann}(M_R/N_R) \neq \operatorname{ann}(R/xR)$. On the other hand, we have $\operatorname{ann}(M_R/N_R) = \operatorname{ann}(R/xR)$ since M_R/N_R is isomorphic to R/xR. This is a contradiction. Therefore M_R/N_R is not a cyclically presented module.

Proposition 2.4.9. *Let R be a local domain. Let $N_R, M_R \neq 0$ be cyclically presented right R-modules and let $\pi_M \colon R_R \to M_R$ be an epimorphism. Then $N_R \subset M_R$ is exact if and only if it is π_M-exact in the sense of Definition and Lemma 2.3.1.*

Proof. Suppose first $N_R \subset M_R$ exact. Let $\pi_N \colon R_R \to N_R$ be any epimorphism. Then π_M and π_N are necessarily projective covers, because $\ker(\pi_M)$ and $\ker(\pi_N)$ are superfluous. Let $\varepsilon \colon N_R \to M_R$ denote the inclusion. By projectivity of R_R, there exists a $\lambda \colon R_R \to R_R$ such that $\pi_M \lambda = \varepsilon \pi_N$. By condition (a) in Lemma 2.4.3, $\lambda(R_R) = \pi_M^{-1}(N_R)$. Since $\pi_M^{-1}(N_R) \neq 0$, it follows that $\pi_M^{-1}(N_R) \cong R_R$ and hence condition (a) in Definition and Lemma 2.3.1 is satisfied.

Suppose now that $N_R \subset M_R$ is π_M-exact. Let $\pi_N \colon R_R \to N_R$ be an epimorphism and $\lambda \colon R_R \to R_R$ a monomorphism satisfying condition (b) of Definition and Lemma 2.3.1. Then π_N is a projective cover of N_R, and condition (b) of Lemma 2.4.3 is satisfied, implying that $N_R \subset M_R$ is exact. □

The previous proposition, together with the results from the previous section, shows that in the special case of R a local domain and $x \in R$ a non-unit, series of exact submodules of R/xR may be used to study factorizations of $x \in R$ up to insertion of units.

2.5 Cokernels of endomorphisms

Let M_R be a right module over a ring R and let $E := \mathrm{End}(M_R)$ be its endomorphism ring. Let s be a fixed element of E. In this section, we investigate the relation between projective covers $eE \to E/sE$ for an idempotent e, induced by the canonical epimorphism $E_E \to E/sE$, and properties of the module $e(M_R)$. This is of particular interest if we assume that $E/J(E)$ is Von Neumann regular and idempotents can be lifted modulo $J(E)$, as in this case for every non-zero $s \in E$ the module E/sE has a projective cover. For instance, every continuous module M_R has this property (see 1.4.10 and 1.4.8), in particular every quasi-injective module has this property, and every module of Goldie dimension one and dual Goldie dimension one has this property (see 1.3.3).

Let $s\colon M_R \to M_R$ be an endomorphism of M_R. We can consider the direct summands M_1 of M_R such that there exists a direct sum decomposition $M_R = M_1 \oplus M_2$ of M_R for some complement M_2 of M_1 with the property that $\pi_2 s\colon M_R \to M_2$ is a split epimorphism. Here $\pi_2\colon M_R \to M_2$ is the canonical projection with kernel M_1. Let $\mathcal{F}$ be the set of all such direct summands, that is,

$$\begin{aligned}\mathcal{F} := \{\, M_1 \mid M_1 \leq M_R,\ &\text{ there exists } M_2 \leq M_R \text{ such that } M_R = M_1 \oplus M_2 \\ &\text{and } \pi_2 s\colon M_R \to M_2 \ \text{ a split epimorphism}\,\}.\end{aligned}$$

The set $\mathcal{F}$ can be partially ordered by set inclusion.

It is well known that there is a one-to-one correspondence between the set of all pairs (M_1, M_2) of R-submodules of M_R such that $M_R = M_1 \oplus M_2$ and the set of all idempotents $e \in E$. If $e \in E$ is an idempotent, the corresponding pair is the pair $(M_1 := e(M_R), M_2 := (1-e)(M_R))$. If $s \in \mathrm{End}(M_R)$, we always denote by $\varphi\colon E_E \to E/sE$ the canonical epimorphism $\varphi(f) = f + sE$.

Lemma 2.5.1. *Let $M_R = M_1 \oplus M_2$, let $\pi_2\colon M_R \to M_2$ be the projection with kernel M_1, and let $e \in \mathrm{End}(M_R)$ be the endomorphism corresponding to the pair (M_1, M_2). If $s\colon M_R \to M_R$ is an endomorphism, then $\pi_2 s$ is a split epimorphism if and only if $\varphi|_{eE}\colon eE \to E/sE$ is surjective.*

Proof. We have to show that $\pi_2 s\colon M_R \to M_2$ is a split epimorphism if and only if $eE + sE = E$. In order to prove the claim, assume that $\pi_2 s\colon M_R \to M_2$ is a split epimorphism, so that there is an R-module morphism $f\colon M_2 \to M_R$ with $\pi_2 s f = 1_{M_2}$. Let $\varepsilon_2\colon M_2 \to M_R$ be the embedding. Then the right ideal $eE + sE$ of E contains the endomorphism

$$\begin{aligned}e(1_M - sf\pi_2) + s(f\pi_2) &= e + (1_M - e)sf\pi_2 = e + \varepsilon_2\pi_2 sf\pi_2 \\ &= e + \varepsilon_2 1_{M_2}\pi_2 = e + (1_M - e) = 1_M,\end{aligned}$$

so that $eE + sE = E$. Conversely, let $e \in E$ be an idempotent with $eE + sE = E$, so that there exist $g, h \in E$ with $1 = eg + sh$. Then $(1-e) = (1-e)sh$, so that $(1-e) = (1-e)sh(1-e)$, that is, $\varepsilon_2\pi_2 = \varepsilon_2\pi_2 sh\varepsilon_2\pi_2$. Since ε_2 is injective and π_2 is surjective, they can be canceled, so that $1_{M_2} = \pi_2 sh\varepsilon_2$. Hence $\pi_2 s$ is a split epimorphism, which proves our claim. $\square$

Proposition 2.5.2. *Let M_R be a right module, and let $E := \mathrm{End}(M_R)$ be its endomorphism ring. Let $s \in E$ and suppose that E/sE has a projective cover. Then*

$$\mathcal{F} := \{\, M_1 \mid M_1 \leq M_R, \quad \textit{there exists } M_2 \leq M_R \textit{ such that } M_R = M_1 \oplus M_2 \\ \textit{and } \pi_2 s \colon M_R \to M_2 \quad \textit{a split epimorphism}\}$$

has minimal elements, and all minimal elements of $\mathcal{F}$ are isomorphic R-submodules of M_R.

Proof. From the previous lemma, it follows that there is a one-to-one correspondence between the set $\mathcal{F}'$ of all pairs (M_1, M_2) of R-submodules of M_R such that $M_R = M_1 \oplus M_2$ and $\pi_2 s \colon M_R \to M_2$ is a split epimorphism and the set of all idempotents $e \in E$ for which the canonical mapping $eE \to E_E/sE$, $x \in eE \mapsto x + sE$, is surjective. In order to prove that $\mathcal{F}$ has minimal elements, it suffices to show that if the canonical mapping $eE \to E_E/sE$ is a projective cover, then $e(M_R)$ is a minimal element of $\mathcal{F}$. Let $e \in E$ be such that $eE \to E_E/sE$ is a projective cover, and let $M_1' \in \mathcal{F}$ be such that $M_1' \subseteq e(M_R)$. Let $e' \in E$ be an idempotent such that $M_1' = e'(M_R)$ and $\pi_2' s \colon M_R \to (1 - e')(M_R)$ is a split epimorphism. Then $M_1' = e'(M_R) \subseteq e(M_R)$, so that $ee' = e'$. Thus $e'E = ee'E \subseteq eE$. If $\varphi|_{eE} \colon eE \to E/sE$ is the projective cover, $\varphi|_{e'E} \colon e'E \to E/sE$ denot! es the canonical epimorphism and $\varepsilon \colon e'E \to eE$ is the embedding, it follows that $\varphi|_{eE}\varepsilon = \varphi|_{e'E}$. Now $\varphi|_{eE}$ is a superfluous epimorphism and $\varphi|_{eE}\varepsilon = \varphi|_{e'E}$ is onto, so that ε is onto, that is, $e'E = eE$. Thus $e = e'f$ for some $f \in E$, so that $e(M_R) \subseteq e'(M_R) = M_1'$ and $M_1' = e(M_R)$. It follows that $e(M_R)$ is a minimal element of $\mathcal{F}$.

Now let M_1'' be any other minimal element of $\mathcal{F}$, and let e'' be an idempotent element of E with $\pi_2'' s \colon M_R \to (1-e'')(M_R)$ a split epimorphism. Then the canonical projection $e''E \to E/sE$ is an epimorphism. As the canonical projection $\varphi|_{eE} \colon eE \to E/sE$ is the projective cover, there is a direct sum decomposition $e''E = P_E' \oplus P_E''$ with the canonical projection $P_E' \to E/sE$ a projective cover. Thus $P_E' = p'E$ for some idempotent p' of E with $p'E + sE = E$, so that $p'(M_R) \in \mathcal{F}$. Now $e''E \supseteq P_E' = p'E$ implies that $p' = e''g$ for some $g \in E$, so that $p'(M_R) \subseteq e''(M_R) = M_1''$. By the minimality of M_1'' in $\mathcal{F}$, it follows that $p'(M_R) = e''(M_R)$, so that $M_1'' = e''(M_R) = p'(M_R) \cong p'E \otimes_E M_R = P' \otimes_E M_R \cong eE \otimes_E M_R \cong e(M_R)$. Thus every minimal element of $\mathcal{F}$ is isomorphic to $e(M_R)$. □

Let M_R be quasi-projective, $E := \mathrm{End}_R(M_R)$ and suppose $s \in E$. In the following, we relate projective covers of the R-module $M_R/s(M_R)$ and the cyclically presented E-module E/sE.

Lemma 2.5.3. *Let M_R be a quasi-projective right R-module, E the endomorphism ring of M_R and let $s \in E$. Let π be the canonical epimorphism of M_R onto $M_R/s(M_R)$ and φ the canonical epimorphism of E_E onto E/sE.*

1. *For every $g \in E$, $\pi|_{g(M_R)}$ is surjective if and only if $\varphi|_{gE}$ is surjective.*
2. *For every $g \in E$, gE is a direct summand of E_E if and only if $g(M_R)$ is a direct summand of M_R.*
3. *Let e, e' be idempotents in E. Then $e(M_R) \cong e'(M_R)$ if and only if $eE \cong e'E$.*

4. Let $e \in E$ be idempotent. Then $\ker(\pi|_{e(M_R)})$ is superfluous if and only if $\ker(\varphi|_{eE})$ is superfluous.

Proof. (1) ($\Leftarrow$) Since $\varphi|_{gE}$ is surjective, there exists h in E such that $gh+sE = 1_M+sE$. Hence there exists h' in E such that $gh = 1_M + sh'$. For all $m \in M_R$ we have $\pi(m) = \pi(1_M(m)) = \pi(g(h(m))$, whence $\pi|_{g(M_R)}$ is surjective.

($\Rightarrow$) Since M_R is quasi-projective and $\pi g\colon M_R \to M_R$ is an epimorphism, there exists $h\colon M_R \to M_R$ such that $\pi gh = \pi$. Therefore $(gh-1_M)(M_R) \subset s(M_R)$. Since $s\colon M_R \to s(M_R)$ is an epimorphism, quasi-projectivity of M_R implies that there exists $h' \in E$ such that $gh - 1_M = sh'$. This implies that $\varphi(gh) = 1_M + sE$. Therefore $\varphi|_{gE}$ is surjective.

(2) ($\Rightarrow$) If gE is a direct summand of E, there exists an idempotent e in E such that $gE = eE$. Hence there exist h, h' in E such that $g = eh$ and $e = gh'$. This implies that $g(M_R) = e(M_R)$. On the other hand, $e(M_R)$ is a direct summand of M_R since e is an idempotent of E. Therefore $g(M_R)$ is a direct summand of M_R.

($\Leftarrow$) If $g(M_R)$ is a direct summand of E, there exists an idempotent e in E such that $g(M_R) = e(M_R)$. Hence $eg = g$. Therefore $gE \subset eE$. Since $g\colon M_R \to e(M_R)$ is an epimorphism and M_R is quasi-projective, there exists $h\colon M_R \to M_R$ such that $e = gh$. This implies that $eE \subset gE$. Hence $eE = gE$.

(3) ($\Leftarrow$) Since $eE \cong e'E$, there exists an isomorphism $\Gamma\colon eE \to e'E$. Consider the two following homomorphisms $f\colon e(M_R) \to e'(M_R)$ defined via $f(m) = e'x(m)$ where $e'x = \Gamma(e)$ and $g\colon e'(M_R) \to e(M_R)$ defined via $g(m) = ey(m)$ where $ey = \Gamma^{-1}(e')$. It suffices to show that $fg = 1_{e'(M_R)}$ and $gf = 1_{e(M_R)}$. For $m \in e'(M_R)$, $fg(m) = f(ey(m)) = e'xey(m) = e'xy(m) = \Gamma(e)y(m) = \Gamma(ey)(m) = \Gamma(\Gamma^{-1}(e'))(m) = e'(m) = m$, it follows that $fg = 1_{e'(M_R)}$. By an argument analogous to the previous one, we get $gf = 1_{e(M_R)}$.

($\Rightarrow$) Since $e(M_R) \cong e'(M_R)$, there exists an isomorphism $h\colon e(M_R) \to e'(M_R)$. Consider the two following homomorphisms $\theta\colon eE \to e'E$ defined via $\theta(ex) = e'hex$, and $\theta'\colon e'E \to eE$ defined via $\theta'(e'x) = eh^{-1}e'x$. It suffices to show that $\theta\theta' = 1_{e'E}$ and $\theta'\theta = 1_{eE}$. Since $\theta\theta'(e'x)(m) = \theta(eh^{-1}e'x)(m) = e'heh^{-1}e'x(m) = e'he(h^{-1}(e'x(m))) = e'h(h^{-1}(e'x(m))) = e'e'(x(m)) = e'(x(m))$, it follows that $\theta\theta'(e'x) = e'x$. Hence $\theta\theta' = 1_{e'E}$. By an argument analogous to the previous one, we get $\theta'\theta = 1_{eE}$.

(4) ($\Rightarrow$) Let K_E be a submodule of eE such that $K_E + \ker(\varphi|_{eE}) = eE$. It suffices to show that $K_E = eE$. There exists $h \in \ker(\varphi|_{eE}) = eE \cap sE$ and $k \in K_E$ such that $e = k + h$. Hence $e(M_R) = k(M_R) + h(M_R)$. This implies that $e(M_R) = k(M_R) + \big(e(M_R) \cap s(M_R)\big)$. Since $e(M_R) \cap s(M_R)$ is superfluous in $e(M_R)$, then $e(M_R) = k(M_R)$. Since $k\colon M_R \to e(M_R)$ is an epimorphism and M_R is quasi-projective, there exists h' in E such that $e = kh'$. This implies that $e \in K_E$. Therefore $K_E = eE$.

($\Leftarrow$) Let N_R be a submodule of M_R such that $N_R + \ker(\pi|_{e(M_R)}) = M_R$. Hence $\pi|_{N_R}$ is surjective. It suffices to show that $N_R = M_R$. Since M_R is quasi-projective and N_R is a submodule of M_R, it follows that M_R is also N_R-projective. Therefore the induced homomorphism $(\pi|_{N_R})_*\colon \mathrm{Hom}(M_R, N_R) \to \mathrm{Hom}(M_R, M_R/s(M_R))$ is surjective and hence there exists $g\colon M_R \to N_R$ such that $\pi g = \pi e$. Again by quasi-projectivity of M_R, there exists $h\colon M_R \to M_R$ such that $g - e = sh$. Since $g(M_R) \subset N_R \subset e(M_R)$, for every $x \in M_R$ there exists $y \in M_R$ such that $g(x) = e(y)$. We have $eg(x) = e(e(y)) = e(y) = g(x)$. Thus $eg = g$. Since $g-e = eg-e = sh$, $eg - e \in eE$ and $sh \in sE$, it follows that $g - e \in eE \cap sE$. From $e = g - (g - e)$, we have $eE = gE + (g - e)E$. Hence $eE = gE + (eE \cap sE)$. Since $eE \cap sE = \ker \varphi|_{eE}$ is superfluous, $eE = gE$. Therefore $e(M_R) = g(M_R) \subset N_R$. Thus $N_R = e(M_R)$. □

Corollary 2.5.4. *Let M_R be a projective right R-module and E the endomorphism ring of M_R. Let $s \in E$, let π be the canonical epimorphism from M_R to $M_R/s(M_R)$ and φ the canonical epimorphism from E to E/sE. Then $\pi|_{e(M_R)}$ is a projective cover of $M_R/s(M_R)$ if and only if $\varphi|_{eE}$ is a projective cover of E/sE.*

Proof. Since M_R is projective, so is $e(M_R)$. Hence $\pi|_{e(M_R)}$ is a projective cover if and only if $\ker(\pi|_{e(M_R)})$ is superfluous. Therefore the corollary follows from the previous lemma. □

Proposition 2.5.5. *Let M_R be a quasi-projective right R-module, let $s \in E = \mathrm{End}(M_R)$ and let $\pi\colon M_R \to M_R/s(M_R)$ be the canonical epimorphism. Suppose that E/sE has a projective cover.*

Consider $\mathcal{E} := \{ N_R \leq M_R \mid \pi|_{N_R}$ is surjective$\}$ and $\mathcal{E}_\oplus := \{ N_R \in \mathcal{E} \mid N_R$ is a direct summand of $M_R\}$, both partially ordered by set inclusion. Then $\mathcal{E}_\oplus$ has minimal elements, any two minimal elements of $\mathcal{E}_\oplus$ are isomorphic as right R-modules and any minimal element of $\mathcal{E}_\oplus$ is minimal in $\mathcal{E}$.

Proof. Let $N_R \leq M_R$ be a direct summand of M_R, let $e \in E$ be an idempotent with $e(M_R) = N_R$ and let $\pi_2\colon M_R \to \ker(e)$ be the canonical projection corresponding to the direct sum decomposition $M_R = N_R \oplus \ker(e)$. Lemma 2.5.3(1) implies that $\pi|_{N_R}\colon N_R \to M_R/s(M_R)$ is surjective if and only if $\varphi|_{eE}\colon eE \to E/sE$ is surjective. By Lemma 2.5.1 this is the case if and only if $\pi_2 s$ is a split epimorphism. This shows that $\mathcal{E}_\oplus = \mathcal{F}$, where the latter is defined as in Proposition 2.5.2. The claims about $\mathcal{E}_\oplus$ therefore follow from the proposition.

It remains to show that the minimal elements of $\mathcal{E}_\oplus$ are minimal in $\mathcal{E}$. Let $N_R \in \mathcal{E}_\oplus$ be minimal, and let $e\colon M_R \to N_R$ be an idempotent with $e(M_R) = N_R$. From the proof of Proposition 2.5.2, we see that $eE \to E/sE$ is a projective cover. Therefore Lemma 2.5.3(4) implies that $\ker(\pi|_{N_R})$ is superfluous. Therefore, if $L_R \leq N_R$ and $\pi|_{L_R}$ is surjective, we have $L_R + \ker(\pi|_{N_R}) = N_R$ and hence $L_R = N_R$, showing that N_R is minimal in $\mathcal{E}$. □

Chapter 3

Automorphism invariant modules

3.1 Basic properties

Definition 3.1.1. A module M is called *automorphism-invariant* if it is invariant under automorphisms of its injective envelope, that is, if $\varphi(M) \subseteq M$ for every $\varphi \in \mathrm{Aut}(E(M))$ (equivalently, if $\varphi(M) = M$ for every $\varphi \in \mathrm{Aut}(E(M))$).

Quasi-injective modules are clearly automorphism-invariant. The following example show that there exists an automorphism-invariant module M_R that it is not quasi-injective.

Example 3.1.2. Let $R = \{(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathbb{Z}_2$: all except finitely many x_n are equal to some $a \in \mathbb{Z}_2\}$. Then R is a ring, and $E(R_R) = \prod_{n\in\mathbb{N}} \mathbb{Z}_2$. Because $End(E(R_R))$ has only one automorphism, namely the identity, R is automorphism-invariant but it is not quasi-injective.

Theorem 3.1.3. *[LZ13, Theorem 2] Let M be an R-module. Then the following conditions are equivalent:*

1. *M is an automorphism-invariant module.*
2. *Every isomorphism between two essential submodules of M extends to an endomorphism of M.*
3. *Every isomorphism between two essential submodules of M extends to an automorphism of M.*

Proof. (1) $\Rightarrow$ (3): Let X, Y be essential submodules of M and $\alpha : X \to Y$ be an isomorphism. Then there is an endomorphism β of $E(M)$ such that $\beta|_X = \alpha$. Because $E(X) = E(Y) = E(M)$ and $\beta|_X$ is an isomorphism, β must be an automorphism of $E(M)$. Since M is automorphism invariant, $\beta(M) \subseteq M$ and $\beta^{-1}(M) \subseteq M$, so $\beta|_M$ is an automorphism of M which extends α.

(3) $\Rightarrow$ (2) : It is clear.

(2) $\Rightarrow$ (1) : Let σ be an automorphism of $E(M)$. Set $Y = \sigma(M) \cap M, X = \sigma^{-1}(Y)$ and $\alpha = \sigma|_X$. Then α is an isomorphism between X and Y. Moreover, by 1.1.48, we deduce that

X and Y are essential submodules of M. By (2), α extends to an endomorphism β of M . Let $y \in Y \cap (\sigma - \beta)(M)$ and write $y = (\sigma - \beta)(x)$ with $x \in M$. Then $\sigma(x) = y + \beta(x) \in Y$ implies that $x \in X$, and hence $y = (\sigma - \beta)(x) = \sigma(x) - \beta(x) = \alpha(x) - \beta(x) = 0$. It follows that $Y \cap (\sigma - \beta)(M) = 0$. Since Y is essential in $E(M)$, we get that $(\sigma - \beta)(M) = 0$. Therefore $\sigma(M) = \beta(M) \subseteq M$. ∎

Proposition 3.1.4. *[LZ13, Lemma 4] Let M be an automorphism invariant module. Then every direct summand of M is automorphism invariant.*

Proof. Let N be a direct summand of M. Then there is a submodule N' of M such that $M = N \oplus N'$. Hence $E(M) = E(N) \oplus E(N')$ where $E(M), E(N)$ and $E(N')$ are injective envelopes of M, N and N' respectively. Let f be an automorphism of $E(N)$. Then $f \oplus 1_{E(N')} : E(M) \to E(M)$ is an isomorphism of $E(M)$. Since M is automorphism invariant, $(f \oplus 1_{E(N')})(N \oplus N') \subseteq N \oplus N'$. It implies that $f(N) \subseteq N$. Hence N is an automorphism invariant module. ∎

Theorem 3.1.5. *[LZ13, Theorem 5] If the direct sum $M = M_1 \oplus M_2$ is automorphism-invariant, then M_1 and M_2 are relatively injective.*

Proof. Let $A \leq M_2$ and $f : A \to M_1$. We wish to show that f extends to a morphism $\overline{f} : M_2 \to M_1$. Let B be a complement of A in M_2. Then $A \oplus B \leq_e M_2$ by 1.1.66, and f extends to a morphism $g : A \oplus B \to M_1$ where $g(B) = 0$. Set $C = A \oplus B$ and define $\alpha : M_1 \oplus C \to M_1 \oplus M_2$ by $\alpha(x, c) = (x + g(c), c)$ for $x \in M_1$ and $c \in C$. Then $\ker \alpha = 0$, that is, α is injective. Furthermore, $\alpha(M_1 \oplus C) = M_1 \oplus C$ is essential in $M_1 \oplus M_2$ (see 1.1.52). Hence α is an automorphism of $M_1 \oplus C \leq_e M_1 \oplus M_2$. As $M_1 \oplus M_2$ is automorphism-invariant, α extends to an endomorphism β of $M_1 \oplus M_2$ by 3.1.3. Set $\overline{f} = \pi\beta i : M_2 \to M_1$ where $i : M_2 \to M_1 \oplus M_2$ is the canonical injection and $\pi : M_1 \oplus M_2 \to M_1$ is the canonical projection. Thus $\overline{f}$ extends f, so that M_1 is M_2-injective. By a similar argument, we also obtain that M_2 is M_1-injective. This complete the proof. ∎

Corollary 3.1.6. *[LZ13, Corollary 6] Let M be a module. Then M is quasi-injective if and only if $M \oplus M$ is automorphism-invariant.*

Let $E(M)$ be the injective envelope of a module M. It is easily seen that

$$\sum_{\varphi \in \mathrm{Aut}(E(M))} \varphi(M)$$

is the smallest automorphism-invariant submodule of $E(M)$ containing M. We call it the *automorphism-invariant envelope* of M, and denote it by $AI(M)$. Clearly, a module is automorphism-invariant if and only if $M = AI(M)$.

Lemma 3.1.7. *[AFT15, Lemma 2.9] Let M, N be arbitrary R-modules. Then every monomorphism $M \to N$ extends to a monomorphism $AI(M) \to AI(N)$.*

PROOF. A monomorphism $\varphi\colon M \to N$ extends to a monomorphism $\varphi'\colon E(M) \to E(N)$, which is necessarily a split monomorphism. Thus there is a direct-sum decomposition $E(N) = \varphi'(E(M)) \oplus C$ and, with respect to this direct-sum decomposition, $\varphi'\colon E(M) \to \varphi'(E(M)) \oplus C$ can be written in matrix form as $\varphi' = \binom{\alpha}{0}$, where $\alpha\colon E(M) \to \varphi'(E(M))$ is an isomorphism. It suffices to show that $\varphi'(AI(M)) \subseteq AI(N)$. Let f be an automorphism of $E(M)$. Then $\begin{pmatrix} \alpha f \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ is an automorphism of $\varphi'(E(M)) \oplus C = E(N)$. Thus $\varphi'(f(M)) = \alpha f(M) = (\alpha f \alpha^{-1})(\alpha(M)) \subseteq \begin{pmatrix} \alpha f \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix}(\alpha(M)) \subseteq \begin{pmatrix} \alpha f \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix}(N) \subseteq AI(N)$. Therefore $\varphi'(AI(M)) \subseteq AI(N)$. ■

3.2 Decomposition of automorphism-invariant modules

Lemma 3.2.1. *[SS14, Lemma 7] Let M be an automorphism-invariant module and $E(M)$ its injective envelope. Assume that $E(M)$ decomposes as a direct sum $E(M) = E_1 \oplus E_2 \oplus E_3$ where $E1 \cong E_2$. Then $M = (M \cap E_1)) \oplus (M \cap E_2) \oplus (M \cap E_3)$.*

PROOF. Let $\sigma : E_1 \to E_2$ be an isomorphism and let $\pi_1 : E(M) \to E_1, \pi_2 : E(M) \to E_2$, and $\pi_3 : E(M) \to E_3$ be the canonical projections. Then $M \cap E_1 \subseteq \pi_1(M), M \cap E_2 \subseteq \pi_2(M)$ and $M \cap E_3 \subseteq \pi_3(M)$.

Let $\eta = \sigma^{-1}$. Consider the map $\lambda_1 : E(M) \to E(M)$ given by $\lambda_1(x_1, x_2, x_3) = (x_1, \sigma(x_1) + x_2, x_3)$. Then λ_1 is an automorphism of $E(M)$. Since M is automorphism invariant, M is invariant under λ_1 and $1_{E(M)}$. Hence M is invariant under $\lambda_1 - 1_{E(M)}$, that is, $(\lambda_1 - 1_{E(M)})(M) \subseteq M$. Consider the map $\lambda_2 : E(M) \to E(M)$ given by $\lambda_2(x_1, x_2, x_3) = (x_1 + (x_2), x_2, x_3)$. Then λ_2 is also an automorphism of $E(M)$. Therefore, as explained above, M is also invariant under $\lambda_2 - 1_{E(M)}$, that is, $(\lambda_2 - 1_{E(M)})(M) \subseteq M$.

Let $x = (x_1, x_2, x_3) \in M$. Then $(\lambda_1 - 1_{E(M)})(x) = (0, \sigma(x_1), 0) \in M$. Also we have $(\lambda_2 - 1_{E(M)})(x) = (\eta(x_2), 0, 0) \in M$. This implies that $(\lambda_1 - 1_{E(M)})(\eta(x_2), 0, 0) = (0, \sigma\eta(x_2), 0) = (0, x_2, 0) \in M$. Hence $\pi_2(M) \subseteq M$. By a similar argument we get that $(\lambda_2 - 1_{E(M)})(0, \sigma(x_1), 0) = (\eta\sigma(x_1), 0, 0) = (x_1, 0, 0) \in M$. Therefore $\pi_1(M) \subseteq M$, so that $(0, 0, x_3) \in M$, that is, $\pi_3(M) \subseteq M$. It follows that $\pi_1(M) \oplus \pi_2(M) \oplus \pi_3(M) \subseteq M$ and hence, $M = \pi_1(M) \oplus \pi_2(M) \oplus \pi_3(M)$. Thus $M = (M \cap E_1) \oplus (M \cap E_2) \oplus (M \cap E_3)$. ■

Recall that a module is square-free if it does not contain a direct sum of two non-zero isomorphic submodules.

Theorem 3.2.2. [ESS13, Theorem 3] *Every automorphism-invariant module M decomposes as a direct sum $M = X \oplus Y$, where X is quasi-injective, Y is a square-free module orthogonal to X, and X and Y are relatively injective modules.*

PROOF. Let $\Gamma = \{(A, B, f) : A, B \leq M, A \cap B = 0, \text{ and } f : A \to B \text{ is an isomorphism}\}$. Define a partial order on as follows: $(A, B, f) \leq (A', B', f')$ if $A \subseteq A', B \subseteq B'$, and f' extends f. By Zorn Lemma, Γ has a maximal element, say (A, B, f). Let C' be a complement of $A \oplus B$ in M. Then $A \oplus B \oplus C' \leq_e M$ and C' is closed in M by 1.1.68. Hence $E(M) = E(A) \oplus E(B) \oplus E(C')$ where $E(M), E(A), E(B)$ and $E(C')$ are injective envelopes of M, A, B and C' respectively. Since

$C' \leq_e E(C') \cap M \leq M$ and C' is closed in M, we get that $C' = E(C') \cap M$. Now we claim that C' is square-free. Assume the contrary. Then there is nonzero submodules X and Y of C' with $X \cap Y = 0$, and an isomorphism: $\varphi : X \to Y$. Hence $(A \oplus X, B \oplus Y, f \oplus \varphi)$ is a maximal element of Γ, which contradicts the maximality of (A, B, f). This proves the claim. Consider $g : A \oplus B \oplus C' \to A \oplus B \oplus C'$ as follows: For each $a \in A, b \in B, c \in C$, $g(a+b+c) = f^{-1}(b)+f(a)+c$. Since $A \oplus B \oplus C' \leq_e M$ and M is automorphism invariant, g extends to an automorphism g' of M (see 3.1.3). Let A' be a closed submodule of M essentially containing A. If A were properly contained in A', $g'|_{A'}$ would contradict the maximality mentioned above. Thus, A must be a closed submodule of M, so that B is closed in M too. Since $A \leq_e E(A) \cap M \leq M$ and $B \leq_e E(B) \cap M \leq M$, we obtain that $A = E(A) \cap A$ and $B = E(B) \cap B$. Hence by 3.2.1, $M = (E(A) \cap M) \oplus (E(B) \cap M) \oplus (E(C') \cap M) = A \oplus B \oplus C'$. It follows that $A \oplus B$ is automorphism-invariant by 3.1.4. Therefore A and B are relatively injective (see 3.1.5). Since $A \cong B$, $A \oplus B$ is then quasi-injective, and hence B is quasi-injective (see 1.1.56). Furthermore, $A \oplus B$ and C' are relatively injective modules by 3.1.5. Hence C' is B-injective. Next, in a similar way to the above argument, we can find a maximal monomorphism $t : B' \to B$ from a submodule $B' \subseteq C'$ into B. Since B is C'-injective, t can be monomorphically extended to every submodule of C' essentially containing B' (see 1.1.56 and 1.1.51). Because of the maximality of t, we deduce that B' is closed in C'. Now we claim that $t(B')$ is a direct summand of B. Because of the fact that B is quasi-injective and 1.2.7, in order to prove the claim it suffices to show that $t(B')$ is closed in B. Let $D \leq B$ such that $t(B') \leq_e D$. Since C' is B-injective, the monomorphism $t^{-1} : t(B') \to C'$ extends monomorphically to $\overline{t^{-1}} : D \to C'$ by 1.1.56 and 1.1.51. Note that $B' = t^{-1}(t(B')) \leq_e \overline{t^{-1}}(D) \leq C'$ because $t(B') \leq_e D$ and t^{-1} is injective. It follows that $B' = \overline{t^{-1}}(D)$, and therefore $t(B') = D$. This proves that $t(B')$ is closed in B. Since B is C'-injective and $t(B')$ is a direct sumand of B, $t(B')$ is C'-injective, so that B' is a C'-injective submodule of C'. Hence $C' = B' \oplus C$ for some C. Now, we will show that C and B are orthogonal. Assume that C and B have nonzero isomorphic submodules C_1 and B_1. Then C_1 and B' are orthogonal thanks to square-freeness of C',and hence so are B_1 and $t(B')$. It follows that $B_1 \cap t(B') = 0$. This contradicts the maximality of the monomorphism t because we can define a monomorphism $\alpha \oplus t : C_1 \oplus B' \to B$ where α is an isomorphism from C_1 to B_1. Therefore C and B are orthogonal. Now we claim that C and $A \oplus B \oplus B'$ are orthogonal. Assume that there are two submodules X, Y such that $X \leq A \oplus B \oplus B'$, $Y \leq C$ and $X \cong Y$ by an isomorphism $\gamma : X \to Y$. If $X \cap B = 0$, we could define an isomorphism $f \oplus \gamma^{-1} : A \oplus Y \to B \oplus X$, which contradict the maximality of (A, B, f). Therefore $X \cap B \neq 0$. But then $X \cap B, \gamma(X \cap B)$ are two isomorphic submodules of B, C respectively, which contradict the fact that B and C are orthogonal. This proves the claim. Now we will show that $A \oplus B \oplus B'$ is quasi-injective. On the one hand, because $A \oplus B$ is quasi-injective and $A \oplus B$ is $C' = B' \oplus C$-injective, $A \oplus B$ is $A \oplus B \oplus B'$-injective. On the other hand, B' is $A \oplus B \oplus B'$-injective since B' is $C' = B' \oplus C$-injective and C' is $A \oplus B$-injective. Therefore, $A \oplus B \oplus B'$ is quasi-injective. The proof is completed by taking $X = A \oplus B \oplus B'$ and $Y = C$. ■

3.3 Conditions (C_i) $(i = 1, 2, 3)$

Definition 3.3.1. A module M is called *pseudo-injective* if, for any submodule A of M, every monomorphism $f : A \to M$ can be extended to an element of $End(M)$.

Lemma 3.3.2. *[Nic77, Lemma 14] Let M be a module such that $M = M_1 \oplus M_2$. Then M_1 is M_2-injective if and only if for any submodule N of M with $N \cap M_1 = 0$, there is some submodule M' of M such that $N \leq M'$ and $M = M_1 \oplus M'$*

PROOF. Assume that M_1 is M_2-injective. Let $\pi_i : M \to M_i$ $(i = 1, 2)$ be canonical projections and N be a submodule of M with $N \cap M_1 = 0$. Because $\pi_2|_N$ is injective and M_1 is M_2-injective, there is a morphism $f : M_2 \to M_1$ such that $\pi_1|_N = \pi_2|_N \circ f$. Set $M' = \{f(m) + m | m \in M_2\}$. Then $N \subseteq M'$ and $M' = eM$ where $e = \begin{pmatrix} 0 & f \\ 0 & 1_{M_2} \end{pmatrix} \in End(M)$. Since $e^2 = e$ and $M_1 = (1 - e)M$, we get that $M = M_1 \oplus M'$. Conversely, let $L \leq M_2$ and $g : L \to M_1$. Now we will show that g extend to a morphism $\overline{g} : M_2 \to M_1$. Set $N = \{-g(x) + x | x \in L\}$. Then $N \leq M$ and $N \cap M_1 = 0$. Now by hypothesis, there is a submodule M' of M such that $N \leq M'$ and $M = M_1 \oplus M'$. Set $\overline{g} = \pi : M \to M_2$ where π is the canonical projection with kernel M'. Hence $\overline{g}$ extend g. This completes the proof. ■

Theorem 3.3.3. *[ESS13, Theorem 16] Let M be a module. Then M is automorphism-invariant if and only if it is pseudo-injective.*

PROOF. The fact that every pseudo-injective is automorphism-invariant follows from 3.1.3. Conversely, assume that M is automorphism-invariant. Then by 3.2.2, M decomposes as a direct sum $M = A \oplus B$ where A is quasi-injective and B is square-free. Hence $E(M) = E(A) \oplus E(B)$ where $E(M), E(A)$ and $E(B)$ are injective envelopes of M, A and B respectively. Let C be a submodule of M and $f : C \to M$ be a monomorphism. Set $D = f(C \cap B) \cap (C \cap B)$. We claim that $D \leq_e f(C \cap B)$ and $D \leq_e C \cap B$. Assume that there is a nonzero submodule $X \leq f(C \cap B)$ such that $X \cap (C \cap B) = 0$. If $X \cap B = 0$, then the restriction of the canonical projection $\pi : A \oplus B \to A$ to X would be injective. Furthermore, X is isomorphic to a submodule of $C \cap B$ by a monomorphism $f^{-1}|_X : X \to C \cap B$. Because A and B are orthogonal, we get that $X = 0$, a contradiction. Therefore $X \cap B$ is a nonzero submodule of B. Now we can embed $(X \cap B) \oplus (X \cap B)$ into $(X \cap B) \oplus (C \cap B) \leq B$, which contradicts the square-freeness of B. This proves that $D \leq_e f(C \cap B)$. Similarly, we also show that $D \leq_e C \cap B$.

Let K be a complement of D in B. Then by 1.1.66, $K \oplus D \leq_e B$, so that $K \oplus D \oplus A \leq_e B \oplus A = M$. Since $K \oplus D \leq_e K \oplus f(C \cap B)$ by 1.1.53 and $(K \oplus D) \cap A \subseteq B \cap A = 0$, we obtain that $(K \oplus f(C \cap B)) \cap A = 0$. By 1.1.52, we get that $(K \oplus f(C \cap B)) \oplus A \leq_e M$.

By 3.1.5, A is B-injective. Now by 3.3.2, there is a submodule B' of M such that $f(C \cap B) \oplus K \subseteq B'$ and $M = A \oplus B'$. Hence $E(M) = E(A) \oplus E(B')$ where $E(B')$ is an injective envelope of B' and B' is closed in M. Since $(K \oplus f(C \cap B)) \oplus A \leq_e M = A \oplus B'$, $K \oplus f(C \cap B) \leq_e B'$ (see 1.1.52). The isomorphism $f|_{C \cap B} \oplus 1_K : (C \cap B) \oplus K \to f(C \cap B) \oplus K$ extends to an isomorphism $\overline{f} : E(B) \to E(B')$, so that $1_{E(A)} \oplus \overline{f} : E(M) \to E(M)$ is an isomorphism. Since M

is automorphism invariant, $(1_{E(A)} \oplus \overline{f})(M) \subseteq M$. It follows that $\overline{f}(B) \subseteq M \cap E(B')$. Moreover, $M \cap E(B') = B'$ because $B' \leq_e M \cap E(B') \leq M$ and B' is closed in M. Therefore, $\overline{f}(B) \subseteq B'$. As $1_{E(A)} \oplus \overline{f}$ is an isomorphism from $E(B)$ to $E(B')$ and B is closed in M, $\overline{f}(B) = (1_{E(A)} \oplus \overline{f})(B)$ is essential in $E(B')$ and closed in M. Hence $\overline{f}(B) \leq_e B'$, which implies that $\overline{f}(B) = B'$. Set $f' = \overline{f}|_B$. Then f' is an isomorphism from B to B' and extends f.

From $f'|_{C \cap B} = \overline{f}|_{C \cap B} = f|_{C \cap B}$, we can define a morphism $g : C + B \to f(C) + B'$ as follows: For $c \in C, b \in B$, $g(c + b) = f(c) + f'(b)$. Then g extends f. Let $\pi : A \oplus B \to A$ be the canonical projection. Hence $B + C = B \oplus \pi(C)$ and $\pi(C) = (B + C) \cap A$. Because A is quasi-injective and B is A-injective by 3.1.5, we obtain that M is A-injective (see **??**). Thus $g|_{\pi(C)} : \pi(C) \to M$ extends to some $g' : A \to M$. Consider the morphism $\overline{f} : M \to M$ defined by $\overline{f}(a + x) = g'(a) + g(x)$ for $a \in A, x \in B + C$. This morphism is well-defined because $g'|_{\pi(C)} = g'|_{(B+C) \cap A} = g|_{\pi(C)} = g|_{(B+C) \cap A}$. Moreover, $\overline{f}$ extends f. This completes the proof. ■

Theorem 3.3.4. *[Din05, Theorem 2.6] Every Pseudo-injective module satisfies Condition (C_2).*

Proof. Let M be a Pseduo-injective module and A be a direct summands of M. Let $B \leq M$ with $B \cong A$. Since A is a direct summand of M, M decomposes as a direct sum $M = A \oplus A'$. Denote an isomorphism from B to A by f. Define $\alpha : M \to B$ as follows: For $a \in A, a' \in A'$, $\alpha(a + a') = f^{-1}(a)$. In order to prove that B is a direct summand of M it suffices to show that the canonical injection $i : B \to M$ is split. As M is Pseudo-injective, there is a morphism $g : M \to M$ such that $f = g \circ i$. Hence $\alpha \circ g \circ i = \alpha \circ f = 1_B$, that is, i is split. ■

Theorem 3.3.5. *Every automorphism invariant satisfies Condition (C_2).*

Proof. It follows from 3.3.4 and 3.3.3. ■

Theorem 3.3.6. *[LZ13, Theorem 12] If M is an automorphism invariant module, then it satisfies Condition (C_3).*

Proof. Assume that A and B are two direct summands of M such that $A \cap B = 0$. We wish to show that $A \oplus B$ is a direct summand of M. Write $M = A \oplus A'$, and let $\pi : M \to A'$ be the canonical projection. Let C be a complement of $A \oplus B$ in M. Then by 1.1.66, $A \oplus B \oplus C \leq_e M$. Set $D = B \oplus C$. Note that $\pi|_D : D \to \pi D$ is an isomorphism. By 1.2.3, $A \oplus D = A \oplus \pi D$. Thus $1_A \oplus \pi|_D : A \oplus D \to A \oplus \pi D$ is an isomorphism. Because M is automorphism invariant and $A \oplus D$ is essential in M , $1_A \oplus \pi|_D$ extends to an automorphism σ of M by Theorem 3.1.3. Since B is a direct summand of M and σ is an automorphism, σB is a direct summand of M, so that $\pi B = \sigma B$ is a direct summand of A'. Therefore $A \oplus B = A \oplus \pi B$ is a direct summand of M. ■

Corollary 3.3.7. *[LZ13, Corollary 13] An automorphism-invariant module M is quasi-injective if and only if it is automorphism invariant and satifies Condition (C_1).*

Proof. If M is quasi-injective, it is automorphism invariant and satisfies Condition (C_1) by 1.2.5 and 1.2.7.

Conversely, if M is automorphism invariant and satifies Condition (C_1), then M is quasi-continuous by Theorem 3.3.6. Hence M is invariant under idempotent endomorphisms of $E(M)$ by 1.3.2. Because M is already invariant under automorphisms of $E(M)$, M is invariant under all endomorphisms of $E(M)$ by [3, Theorem 3.9]. Therefore M is quasi-injective. ■

3.4 The exchange property and the endomorphism ring

From now, let $\Delta(M, M)$ denote the set of all module morphisms $f\colon M \to M$ whose kernel $\mathrm{Ker}(f)$ is an essential submodule of M

Proposition 3.4.1. *[War72, Theorem 2] Let M be a module. Then M has the finite exchange property if and only if $End(M)$ is an exchange ring.*

Theorem 3.4.2. *[Nie10, Theorem 9] Let M be a square-free module with the finite exchange property. Then M has the exchange property.*

Proposition 3.4.3. *[AS13, Proposition 1] Let M be an automorphism-invariant module. Then the Jacobson radical of $End(M)$ is $\Delta(M, M)$, $End(M)/J(End(M))$ is a von Neumann regular ring and idempotents can be lifted modulo $J(End(M))$.*

Proof. Let $r \in End(M)$. Then there is a morphism $s \in \mathrm{End}(E(M))$ such that $s|_M = r$. Set $K = Ker(r)$ and let L be a complement of K in M. Then by 1.1.66, $K \oplus L \subseteq_e M$, so that $E(M) = E(K) \oplus E(L)$. Let $g \in \mathrm{End}(E(M))$ defined by $g|_{E(K)} = 0$ and $g|_{E(L)} = s|_{E(L)}$. Then $(g-s)|_{K \oplus E(L)} = 0$ and hence, $g - s \in J(S)$ by 1.1.65. Therefore $1-(g-s)$ is an automorphism of E. Because M is automorphism-invariant, $(1-(g-s))(M) \subseteq M$. It follows that $(g-s)(M) \subseteq M$. Now since s is an extension of $r \in R$, we get that $s(M) \subseteq M$, so that $g(M) \subseteq M$.

As $L \cap Ker(g) = 0$, $g|_{E(L)}$ is a monomorphism. Let $E' = Im(g) = Im(g|_{E(L)})$. Then $E' \cong E(L)$ is injective. Moreover, as $g|_{E(L)} : E(L) \to E'$ is an isomorphism, there is a morphism $h : E' \to E(L)$ such that $h \circ g \circ u = u \circ 1_{E(L)}$ and hence, $u \circ h \circ g = u \circ \pi$, where $u : E(L) \to E(M)$ and $\pi : E(M) \to E(L)$ are the inclusion and projection associated to the decomposition $E(M) = E(K) \oplus E(L)$. Since L is essential in $E(L)$, $g(L)$ is essential in E' and hence, $N = M \cap g(L)$ is also essential in E', thanks to the fact that M is essential in $E(M)$. It follows that the monomorphism $h|_N : N \to L \subseteq M$ extends to an endomorphism $t : E(M) \to E(M)$. Because M is automorphism-invariant, $t(M) \subseteq M$. Set $t' = t|_M \in End(M)$. Since N is essential in $E = Im(g), g^{-1}(N)$ is essential in $E(M)$ and hence, $N' = (K \oplus L) \cap g^{-1}(N)$ is also essential in $E(M)$ (see 1.1.48). Consider the morphism $\varphi\colon \mathrm{End}(M) \to \mathrm{End}(E(M))/J(\mathrm{End}(E(M)))$ is defined as follows. If $f \in \mathrm{End}(M)$, let $\tilde{f}$ be an endomorphism of $E(M)$ that extends f. Then $\varphi(f) = \tilde{f} + J(\mathrm{End}(E(M)))$ [FH06, §4, p. 412]. It is easily seen that φ is a well-defined ring morphism. Since $J(End(E(M))$ consists of all endomorphisms having essential kernels, we get that $ker\varphi = \Delta(M, M)$, and hence φ factors through an injective morphism $\psi : End(M)/\Delta(M, M) \to End(E(M))/J(End(E(M))$. Let $x \in N'$. Then $g(x) \in N$ and $x = k+l$ where $k \in K$ and $l \in L$. Thus $g(l) = g(k)+g(l) = g(x) \in N \subseteq M$. Therefore $t' \circ g(x) = t \circ g(l) = t' \circ g(l) = h \circ g(l) = l$. As $t \circ g|_{N'} = u \circ \pi|_{N'}$ and N' is essential in $E(M)$, it follows that $t \circ g + J(End(E(M)) = u \circ \pi + J(End(E(M)))$. Thus $s \circ t \circ s + J(End(E(M)) = g \circ t \circ g + J(End(E(M)) = g \circ u \circ \pi + J(End(E(M)) = g + J(End(E(M)) = s + J(End(E(M)))$, so that $\psi((r \circ t' \circ r) + \Delta(M, M)) = (s \circ t \circ s) + J(End(E(M)) = s + J(End(E(M)) = \psi(r + \Delta(M, M))$. Since ψ is injective, we get that $(r \circ t' \circ r) + \Delta(M, M) = r + \Delta(M, M)$. This proves that $End(M)/\Delta(M, M)$ is von Neumann regular.

Since $End(M)/\Delta(M, M)$ is von Neumann regular, $J(End(M)/\Delta(M, M)) = 0$, so that $J(End(M)) \subseteq \Delta(M, M)$. Let $a \in \Delta(M, M)$. Because $Ker(a) \cap Ker(1 - a) = 0$ and $Ker(a) \subseteq_e M$, $Ker(1-a) = 0$. Thus $(1-a)$ is an isomorphism from M to $(1-a)(M)$. As M is automorphism-invariant, M satisfies Condition (C_2) (see 3.3.5), that is, submodules isomorphic to a direct summand of M are direct summands. Hence $(1 - a)(M)$ is a direct summand of M. But $(1 - a)(M) \subseteq_e M$ because $Ker(a) \subseteq (1 - a)(M)$. Thus $(1 - a)(M) = M$ and hence, $1 - a$ is a unit in $End(M)$. It follows that $a \in J(End(M))$ and hence, $\Delta(M, M) \subseteq J(End(E(M)))$. This gives $J(End(M)) = \Delta(M, M)$, so that $End(M)/J(End(M))$ is a von Neumann regular ring.

Now, we will show that idempotents can be lifted modulo $J(End(M))$. Let $e' + J(End(M))$ be an idempotent in $End(M)/J(End(M)$ and $f' + J(End(E(M)) = \psi(e' + J(End(M)))$. Then $f' + J(End(E(M))$ is an idempotent in $End(E(M))/J(End(E(M))$. Because idempotents can be lifted modulo $J(End(E(M))$, there is an idempotent f in $End(E(M))$ such that $f' = f + j$ with $j \in J(End(E(M))$. Now, $1 - j$ is a unit in $End(E(M)$, and hence M is invariant under $1 - j$. Therefore $j(M) \subseteq M$, so that $f(M) \subseteq f'(M) + j(M) \subseteq M$. It follows that $e = f|_M$ belongs to $End(M)$ and it is an idempotent since so is f. By construction, $\psi(e + J(End(M))) = f + J(End(E(M))) = f' + J(End(E(M))) = \psi(e' + J(End(E(M))))$. And, as ψ is an injective morphism, we obtain that $e + J(End(M)) = e' + J(End(M))$. This complete the proof. ■

Theorem 3.4.4. *[AS13, Theorem 3] Every automorphism-invariant module satifies the exchange property.*

PROOF. Let M be an automorphism-invariant module. Set $R = End(M)$. By 3.4.3, $R/J(R)$ is a von Neumann regular ring and idempotents can be liftted modulo $J(R)$. By [Nic77, Proposition 1.6], R is an exchange ring. Hence M has the finite exchange property by 3.4.1. Now M decomposes as a direct sum $M = P \oplus Q$ where Q is quasi-injective and P is square-free (see 3.2.2). Applying 1.1.10, we dedude that P has the finite exchange property, so that P has the exchange property thanks to 3.4.2 and the fact that P is square-free. Applying 1.2.9 to Q, we get that Q has the full exchange property. Now by 1.1.10, M has the exchange property. ■

Theorem 3.4.5. *[AFT15, Theorem 2.1] Let M be an automorphism-invariant module and $E(M)$ be its injective envelope. Then*

(a) *There is a canonical local morphism*

$$\varphi\colon \operatorname{End}(M) \to \operatorname{End}(E(M))/J(\operatorname{End}(E(M)))$$

with kernel $J(\operatorname{End}(M))$, so that φ induces an embedding $\overline{\varphi}$, as a rationally closed subring, of the von Neumann regular ring $\operatorname{End}(M)/J(\operatorname{End}(M))$ into the von Neumann regular right *self-injective ring*

$$\mathrm{End}(E(M))/J(\mathrm{End}(E(M))).$$

(b) *For every invertible element* v *of the ring* $\mathrm{End}(E(M))/J(\mathrm{End}(E(M)))$*, there exists an invertible element* u *of* $\mathrm{End}(M)/J(\mathrm{End}(M))$ *such that* $\overline{\varphi}(u) = v$.

(c) *For every idempotent element* f *of the ring* $\mathrm{End}(E(M))/J(\mathrm{End}(E(M)))$ *there exists an idempotent element* e *of* $\mathrm{End}(M)/J(\mathrm{End}(M))$ *such that* $\overline{\varphi}(e) = f$ *if and only if the module* M *is quasi-injective.*

(d) *If* M *is quasi-injective, then* $\overline{\varphi}$ *is an isomorphism.*

Proof. (a) For any module M, the morphism

$$\varphi\colon \mathrm{End}(M) \to \mathrm{End}(E(M))/J(\mathrm{End}(E(M)))$$

is defined as follows. If $f \in \mathrm{End}(M)$, let $\tilde{f}$ be an endomorphism of $E(M)$ that extends f. Then $\varphi(f) = \tilde{f} + J(\mathrm{End}(E(M)))$ [FH06, §4, p. 412]. It is easily seen that φ is a well-defined ring morphism. Moreover, φ is a local morphism, because if $f \in \mathrm{End}(M)$ and $\varphi(f)$ is invertible in the ring $\mathrm{End}(E(M))/J(\mathrm{End}(E(M)))$, then $\tilde{f}$ is an automorphism of $E(M)$. Since M is automorphism-invariant, it follows that $\tilde{f}(M) = M$; that is, $f(M) = M$. This proves that f is onto. Moreover, $\tilde{f}$ is an automorphism of $E(M)$ implies that its restriction f is an injective endomorphism of M. Thus f is an automorphism, and the ring morphism φ is a local morphism. It follows that the injective morphism $\overline{\varphi}\colon \mathrm{End}(M)/\ker(\varphi) \to \mathrm{End}(E(M))/J(\mathrm{End}(E(M)))$ induced by φ is a local morphism as well. Moreover, $\ker(\varphi) = \Delta(M, M) = J(\mathrm{End}(M))$ by 3.4.3.

(b) If v is an invertible element of $\mathrm{End}(E(M))/J(\mathrm{End}(E(M)))$, then $v = v' + J(\mathrm{End}(E(M)))$ for some element $v' \in \mathrm{End}(E(M))$, necessarily invertible. Therefore v' is an automorphism of $E(M)$. Since M is automorphism-invariant, the restriction u' of v' to M is an automorphism of M. Thus $u := u' + J(\mathrm{End}(M))$ is an invertible element of $\mathrm{End}(M)/J(\mathrm{End}(M))$ and $\overline{\varphi}(u) = v$.

(d) If M is quasi-injective, for every $f \in \mathrm{End}(E(M))$, the restriction f' of f to M is an endomorphism of M. Thus $\overline{\varphi}(f' + J(\mathrm{End}(M))) = f + J(\mathrm{End}(E(M)))$. Hence $\overline{\varphi}$ is onto, and (a) allows us the conclusion.

(c) Assume that for every idempotent element $f \in \mathrm{End}(E(M))/J(\mathrm{End}(E(M)))$ there exists an idempotent element e of $\mathrm{End}(M)/J(\mathrm{End}(M))$ with $\overline{\varphi}(e) = f$. In order to show that M is quasi-injective, we will prove that it satisfies Condition (C_1). Let N be a submodule of M. We must show that N is essential in a direct summand of M. Now $E(M)$ has a direct-sum decomposition $E(M) = E(N) \oplus E$. Thus there is an idempotent $\varepsilon \in \mathrm{End}(E(M))$ with $E(N) = \varepsilon E(M)$ and $E = (1 - \varepsilon)E(M)$. By hypothesis, there exists an idempotent $e \in \mathrm{End}(M)/J(\mathrm{End}(M))$ with $\overline{\varphi}(e) = \varepsilon + J(\mathrm{End}(E(M)))$. As idempotents lift modulo $J(\mathrm{End}(M))$, there is an idempotent $\varepsilon' \in \mathrm{End}(M)$ such that $e = \varepsilon' + J(\mathrm{End}(M))$. The idempotent $\varepsilon' \in \mathrm{End}(M)$ corresponds to a direct-sum decomposition $M = \varepsilon' M \oplus (1 - \varepsilon')M$. This direct-sum decomposition of M induces a direct-sum decomposition $E(M) = E(\varepsilon' M) \oplus E((1 - \varepsilon')M)$. Thus there is an idempotent $\varepsilon'' \in \mathrm{End}(E(M))$ with $E(\varepsilon' M) = \varepsilon'' E(M)$ and $E((1 - \varepsilon')M) = (1 - \varepsilon'')E(M)$. We claim that endomorphism ε'' of $E(M)$ extends the endomorphism ε' of M. To prove this claim, it suffices to show that $\varepsilon''(x) = x$ for every $x \in \varepsilon' M$ and $\varepsilon''(y) = 0$ for every $y \in (1-\varepsilon')M$. Now $\varepsilon' M \subseteq E(\varepsilon' M) = \varepsilon'' E(M)$, so that $\varepsilon''(x) = x$ for every $x \in \varepsilon' M$. Similarly $(1-\varepsilon')M \subseteq E((1-\varepsilon')M) = (1-\varepsilon'')E(M)$, so that for every $y \in (1-\varepsilon')M$ one has that $y \in (1-\varepsilon'')E(M)$. Hence $\varepsilon''(y) = 0$. This proves the claim. Thus $\overline{\varphi}(\varepsilon' + J(\mathrm{End}(M)) = \varepsilon'' + J(\mathrm{End}(E(M_R)))$. But $\overline{\varphi}(e) = \varepsilon + J(\mathrm{End}(E(M)))$ and $e = \varepsilon' + J(\mathrm{End}(M))$, so that $\overline{\varphi}(\varepsilon' + J(\mathrm{End}(M))) = \varepsilon + J(\mathrm{End}(E(M)))$. It follows that $\varepsilon'' + J(\mathrm{End}(E(M))) = \varepsilon + J(\mathrm{End}(E(M)))$; that is, $\varepsilon'' - \varepsilon \in J(\mathrm{End}(E(M)))$, so $1 - \varepsilon'' + \varepsilon$ is an automorphism of $E(M)$. As M is automorphism-invariant, we have that $(1-\varepsilon''+\varepsilon)(M) = M$. Thus $\varepsilon(M) \subseteq (1-\varepsilon''+\varepsilon)(M) + 1(M) + \varepsilon''(M) = M + M + \varepsilon'(M) = M$. It follows that ε restricts to an idempotent endomorphism of M. In particular, $\varepsilon(M)$ is a direct summand of M. Moreover, $N \subseteq E(N) \cap M = \varepsilon E(M) \cap M = \varepsilon(M)$, so that N is a submodule of $\varepsilon(M)$. It remains to show that N is essential in $\varepsilon(M)$. This follows immediately from the fact that $\varepsilon(M) \subseteq \varepsilon E(M) = E(N)$ and N is essential in $E(N)$. This proves that M satisfies Condition (C_1), and hence is quasi-injective by 3.3.7.

The converse follows immediately from (d), noting that the inverse image of an idempotent via an injective morphism is necessarily idempotent. ■

Proposition 3.4.6. *[AFT15, Proposition 2.2]Let M be an automorphism-invariant module. Then*

(a) *If M is indecomposable, then* $\mathrm{End}(M)$ *is a local ring.*

(b) *If M has finite Goldie dimension, then every injective endomorphism of M is an automorphism of M and the endomorphism ring* $\mathrm{End}(M)$ *is a semiperfect ring.*

PROOF. (a) Automorphism-invariant modules have the exchange property by 3.4.4, and indecomposable modules with the exchange property have a local endomorphism ring by 1.1.12.

(b) Let M be an automorphism-invariant module of finite Goldie dimension and let $\varphi\colon M \to M$ be an injective endomorphism of M. Then φ extends to an endomorphism $\varphi_0\colon E(M) \to E(M)$, which is necessarily injective. As M has finite Goldie dimension, φ_0 is an automorphism of $E(M)$. But M is automorphism-invariant, so $\varphi_0(M) = M$. Thus $\varphi(M) = M$, that is, the endomorphism φ is also surjective.

Finally, every module of finite Goldie dimension is a direct sum of indecomposable modules. Thus if $M = M_1 \oplus \cdots \oplus M_n$ is automorphism-invariant and the M_i are indecomposable, then the modules M_i are automorphism-invariant. Hence they have a local endomorphism ring by (a). Since $M = M_1 \oplus \cdots \oplus M_n$, there is a complete set $e_1, ...e_n$ of orthogonal idempotents in $End(M)$ such that $M_i = Me_i$. Moreover, $e_i End(M) e_i \cong End(Me_i) = End(M_i)$ is local for every $i = 1, ..., n$. This proves that $\mathrm{End}(M)$ is semiperfect (see 1.1.46). ■

Corollary 3.4.7. *[AFT15, Corollary 2.3] If M, N are two automorphism-invariant R-modules of finite Goldie dimensions isomorphic to submodules of each other, then M is isomorphic to N.*

PROOF. By the hypothesis, there exists two monomorphisms $f : M \to N$ and $g : N \to M$. So $fg \in \mathrm{End}(N)$ and fg is injective. Hence fg is an automorphism by Proposition 3.4.6(b). Thus f is onto. Since f is a monomorphism, f is an isomorphism. ■

3.5 A connection with quasi-injective modules

Lemma 3.5.1. *[AFT15, Lemma 2.6] Let M be an automorphism-invariant module. If $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$, where each M_i is a quasi-injective module, then M is quasi-injective.*

PROOF. It clearly suffices to prove the case $n = 2$. Assume that $M = M_1 \oplus M_2$ is automorphism-invariant, where M_1 and M_2 are quasi-injective. By 3.1.5, M_1 is M_2-injective and M_2 is M_1-injective. Since M_1 and M_2 are quasi-injective, M is quasi-injective (see 1.1.56). ■

Proposition 3.5.2. *[AFT15, Proposition 2.7] Let $M_1, M_2, \ldots, M_n$ be uniform modules. If $M := M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is automorphism-invariant, then M is quasi-injective.*

PROOF. By the previous lemma, it suffices to show that each M_i is quasi-injective. On the one hand, each M_i is uniform, and each M_i satisfies (C_1). On the other hand, by 3.1.4, each M_i is automorphism-invariant. By 3.3.7, every M_i is quasi-injective. Now apply Lemma 3.5.1. ■

Proposition 3.5.3. *[AFT15, Proposition 2.8] The following conditions are equivalent for a ring R.*

1. *Every automorphism-invariant R-module of finite Goldie dimension is quasi-injective.*
2. *Every automorphism-invariant indecomposable R-module of finite Goldie dimension is uniform.*
3. *Every automorphism-invariant indecomposable R-module of finite Goldie dimension is quasi-injective.*

PROOF. (1) $\Rightarrow$ (2) An automorphism-invariant indecomposable module M of finite Goldie dimension is quasi-injective by (1). Hence it satisfies Condition (C_1) (see 1.2.7). Therefore M is uniform.

(2) $\Rightarrow$ (3) Let M be an automorphism-invariant indecomposable module of finite Goldie dimension. Then M is uniform by (2), and hence it satisfies Condition (C_1). By 3.3.7, M is quasi-injective.

(3) $\Rightarrow$ (1) Let M be an automorphism-invariant module of finite Goldie dimension. So $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$, where each M_i is an automorphism-invariant indecomposable module of finite Goldie dimension. By (3), every M_i is quasi-injective. From Lemma 3.5.1, it follows that M is quasi-injective. ■

Proposition 3.5.4. *[AFT15, Proposition 2.5] If R is a ring of odd characteristic, then every automorphism-invariant R-module is quasi-injective.*

PROOF. Suppose that R is a ring of odd characteristic n with a module M that is automorphism-invariant but not quasi-injective. By Theorem 3.5.8, the endomorphism ring $\mathrm{End}(M)$ has a factor

$\mathrm{End}(M)/\mathcal{M}$ isomorphic to $\mathbb{F}_2$. Then $nR = 0$, so that $nM = 0$. Hence $n\,\mathrm{End}(M) = 0$, so that $n(\mathrm{End}(M)/\mathcal{M}) = 0$. Thus $n\mathbb{F}_2 = 0$, which is a contradiction because n is odd. ∎

By Proposition 3.5.4, every ring R of odd characteristic satisfies the equivalent conditions of Proposition 3.5.3.

Lemma 3.5.5. [MM90, Lemma 3.3] *Let M be a module. Assume that M decomposes as a direct sum $M = M_1 \oplus M_2$ where M_1 and M_2 are orthogonal. Then* $\mathrm{End}(M)/\Delta(M,M) \cong \mathrm{End}(M_1)/\Delta(M_1,M_1) \times \mathrm{End}(M_2)/\Delta(M_2,M_2)$. *The converse holds if M_1 and M_2 are relatively injective.*

Proof. Let $s \in S = End(M)$. We can write $s = \begin{pmatrix} s_1 & \psi \\ \varphi & s_2 \end{pmatrix}$ where $s_1 \in End(M_1), s_2 \in End(M_2), \varphi \in Hom(M_1, M_2)$ and $\psi \in Hom(M_2, M_1)$. Furthermore, we can consider s_1, s_2, φ and ψ as elements of S by defining them to be zero on the other summand. Then $\varphi, \psi \in \Delta(M,M)$ because M_1 and M_2 are orthogonal. We have $\ker s \cap M_1 = \ker s_1 \cap \ker \varphi$ and $\ker s \cap M_2 = \ker s_2 \cap \ker \psi$.

Now we will show that $s \in \Delta(M,M)$ if and only if $s_1 \in \Delta(M_1,M_1)$ and $s_2 \in \Delta(M_2,M_2)$. Assume that $s \in \Delta(M,M)$. Then $\ker s \leq_e M$, so that $\ker s_1 \cap \ker \varphi = \ker s \cap M_1 \leq_e M_1$ by 1.1.48. It follows that $\ker s_1 \leq_e M_1$, that is, $s_1 \in \Delta(M_1,M_1)$. By a similar argument, we also have $s_2 \in \Delta(M_2,M_2)$. Conversely, assume that $s_1 \in \Delta(M_1,M_1)$ and $s_2 \in \Delta(M_2,M_2)$. Since $\ker\varphi \leq_e M_1$, $\ker\varphi \cap \ker s_1 \leq_e M_1$ by 1.1.48, and therefore $\ker s \cap M_1 \leq_e M_1$.Similarly $\mathrm{Ker}\, s \cap M_2 \leq_e M_2$. Thus $\ker s \leq_e M$, that is, $s \in \Delta(M,M)$. It follows that $End(M)/\Delta(M,M) \cong$
$\begin{pmatrix} End(M_1)/\Delta(M_1,M_1) & 0 \\ 0 & End(M_2)/\Delta(M_2,M_2) \end{pmatrix}$ ∎

Lemma 3.5.6. *[AS13, Lemma 1] Let M be an R-module such that $End(M)$ has no factor isomorphic to $\mathbb{F}_2$. Then $End(E(M))$ has no factor isomorphic to $\mathbb{F}_2$.*

Proof. Let M be any R-module such that $End(M)$ has no factor isomorphic to $\mathbb{F}_2$ and set $S = End(E(M))$. Assume that $End(E(M))$ has a factor isomorphic to $\mathbb{F}_2$, that is, there is a ring morphism $\psi : S \to \mathbb{F}_2$. It follows that there is a ring morphism $\psi' : S/J(S) \to \mathbb{F}_2$. Set $f = \psi'\varphi$ where $\varphi : End(M) \to S/J(S)$ as in 3.4.5 (a). Then f is a ring morphism from $End(M)$ to $\mathbb{F}_2$, a contradiction. ∎

Lemma 3.5.7. *[AS14, Lemma 2] Let M be a continuous module over any ring S. Then each element of the endomorphism $R = End(M_S)$ is the sum of two units if and only if R has no factor isomorphic to $\mathbb{F}_2$.*

Proof. Assume that R has no factor isomorphic to $\mathbb{F}_2$. Set $\Delta = \Delta(M,M)$. By 3.5.5, $R/\Delta \cong R_1 \oplus R_2$ where R_1 is von Neumann regular, right self-injective, and R_2 is an exchange ring with no non-zero nilpotent element. ∎

Theorem 3.5.8. [AS14, Theorem 3] *Let M be a right module such that* $\mathrm{End}(M)$ *has no factor isomorphic to $\mathbb{F}_2$. Then M is quasi-injective if and only if M is automorphism-invariant.*

Proof. Let M be an automorphism invariant module such that $End(M)$ has no factor isomorphic to $\mathbb{F}_2$. Then by Lemma 3.5.6, $End(E(M))$ has no factor isomorphic to $\mathbb{F}_2$. Moreoer, by Lemma 3.5.7, each element of $End(E(M))$ is a sum of two units. This means that for every endomorphism $\lambda \in End(E(M))$, we have $\lambda = u_1 + u_2$ where u_1, u_2 are automorphisms in $End(E(M))$. Since M is automorphism-invariant, it is invariant under both u_1 and u_2, and we obtain that M is invariant under λ. This shows that M is quasi-injective by 1.2.5. The converse follows from 1.2.5. ∎

3.6 Boolean rings

Lemma 3.6.1. *[AFT15, Lemma 3.1]Let T be a ring and I the two-sided ideal of T generated by the subset $\{ t - t^2 \mid t \in T \}$ of T. Then*

(a) *The ideal I is the smallest ideal of T with T/I a boolean ring or the zero ring.*

(b) *The ideal I is the intersection of all maximal two-sided ideals $\mathcal{M}$ of T with $T/\mathcal{M} \cong \mathbb{F}_2$.*

(c) *The ideal I contains the Jacobson radical $J(T)$ of T.*

(d) *The kernel of every ring morphism $T \to \mathbb{F}_2$ contains I.*

(e) *I is a proper ideal of T if and only if there exists a ring morphism $T \to \mathbb{F}_2$, if and only if T has a maximal two-sided ideal $\mathcal{M}$ with $T/\mathcal{M} \cong \mathbb{F}_2$.*

Proof. (a) is trivial.

(b) Let us check that

$$I = \bigcap_{T/\mathcal{M} \cong \mathbb{F}_2} \mathcal{M}.$$

($\subseteq$) Since I is generated by the elements $t - t^2$, it suffices to show that $t - t^2 \in \mathcal{M}$ for every $t \in T$ and every maximal two-sided ideal $\mathcal{M}$ with $T/\mathcal{M} \cong \mathbb{F}_2$. Now $\mathbb{F}_2$ is boolean, so that $T/\mathcal{M}$ is boolean, hence $t + \mathcal{M} = t^2 + \mathcal{M}$. It follows that $t - t^2 \in \mathcal{M}$.

($\supseteq$) By (a), the ring T/I is boolean. Boolean rings are isomorphic to subrings of $\mathbb{F}_2^X$ for some set X. Let $\varepsilon\colon T/I \to \mathbb{F}_2^X$ be an embedding and $\pi_x\colon \mathbb{F}_2^X \to \mathbb{F}_2$ $(x \in X)$, $p\colon T \to T/I$ be the canonical projections. Then the morphisms $\varphi_x := \pi_x \varepsilon p\colon T \to \mathbb{F}_2$ have kernels $\ker \varphi_x$, which are maximal two-sided ideals of T, $T/\ker\varphi_x \cong \mathbb{F}_2$ and $\bigcap_{x\in X} \ker\varphi_x = \bigcap_{x \in X} (\varepsilon p)^{-1}(\ker \pi_x) = (\varepsilon p)^{-1}(\bigcap_{x\in X} \ker\varphi_x) = p^{-1}(\ker\varepsilon) = \ker p = I$. Thus $\bigcap_{T/\mathcal{M}\cong\mathbb{F}_2} \mathcal{M} \subseteq \bigcap_{x\in X} \ker\varphi_x = I$.

(c) By (b), I is the intersection of all maximal two-sided ideals $\mathcal{M}$ of T with $T/\mathcal{M} \cong \mathbb{F}_2$, and all maximal two-sided ideals $\mathcal{M}$ of T with $T/\mathcal{M} \cong \mathbb{F}_2$ are maximal right ideals of T. Hence I is an intersection of maximal right ideals of T, so that $I \supseteq J(T)$.

(d) The kernel of every ring morphism $T \to \mathbb{F}_2$ is a maximal two-sided ideal of T with $T/\mathcal{M} \cong \mathbb{F}_2$. Thus (d) follows from (b).

(e) is now trivial. ∎

Lemma 3.6.2. *[AFT15, Corollary 3.3] Let $M = M_1 \oplus M_2$ be an automorphism-invariant R-module where M_1 and M_2 are orthogonal. Then* $\mathrm{End}(M)$ *has no factor isomorphic to* $\mathbb{F}_2$ *if and only if each* $\mathrm{End}(M_i)$ $(i = 1, 2)$ *has no factor isormophic to* $\mathbb{F}_2$*.*

Proof. Let I be the two-sided ideal of $\mathrm{End}(M)$ generated by the set $\{\, x - x^2 \mid x \in \mathrm{End}(M)\,\}$. By Lemma 3.5.5, $\mathrm{End}(M)/\Delta(M, M) \cong \mathrm{End}(M_1)/\Delta(M_1, M_1) \times \mathrm{End}(M_2)/\Delta(M_2, M_2)$. As $\Delta(M, M) = J(\mathrm{End}(M))$ for any automorphism-invariant R-module M (see 3.4.3), it follows that $\mathrm{End}(M)/J(\mathrm{End}(M)) \cong \mathrm{End}(M_1)/J(\mathrm{End}(M_1)) \times \mathrm{End}(M_2)/J(\mathrm{End}(M_2))$ in a canonical way. Thus there is a homomorphism $\mathrm{End}(M) \to \mathbb{F}_2$ if and only if there is a homomorphism $\mathrm{End}(M)/J(\mathrm{End}(M)) \to \mathbb{F}_2$, if and only if there is a homomorphism $\mathrm{End}(M_i)/J(\mathrm{End}(M_i)) \to \mathbb{F}_2$ for an i equal to 1 or 2. The conclusion follows immediately. ∎

Lemma 3.6.3. *[AFT15, Lemma 3.5] If M_1, M_2 are two right modules over a ring R and M_1, M_2 have isomorphic injective envelopes, which are non-zero modules, then M_1 and M_2 have non-zero isomorphic submodules.*

Proof. Let $f\colon E(M_1) \to E(M_2)$ be an isomorphism. Then M_1 and $f^{-1}(M_2)$ are essential submodules of $E(M_1)$. Hence $M_1 \cap f^{-1}(M_2)$ is an essential submodule of $E(M_1)$ by 1.1.48. It follows that $M_1 \cap f^{-1}(M_2)$ is a non-zero submodule of M_1. Via the isomorphism f, we find that $f(M_1 \cap f^{-1}(M_2))$ is an essential submodule of $E(M_2)$ isomorphic to $M_1 \cap f^{-1}(M_2)$. But $f(M_1 \cap f^{-1}(M_2)) = f(M_1) \cap M_2$ is a submodule of M_2. ∎

Corollary 3.6.4. *[AFT15, Corollary 3.6] A module M is square-free if and only if its injective envelope $E(M)$ is square-free.*

Proof. If M is not square-free, then it contains a submodule isomorphic to $N \oplus N$ for some non-zero module N. Hence the same holds for $E(M)$, that is, $E(M)$ is not square-free.

Conversely, assume that $E(M)$ is not square-free. Then $E(M)$ contains a submodule isomorphic to $N \oplus N$ for some non-zero module N. It follows that $E(M) = E_1 \oplus E_2 \oplus E_3$ with $E_1 \cong E_2 \neq 0$. Then by 1.1.48, $M \cap E_i$ is a non-zero essential submodule of E_i for $i = 1, 2$. In particular, $M \cap E_1$ and $M \cap E_2$ have isomorphic injective envelopes, which are non-zero modules. By Lemma 3.6.3, $M \cap E_1$ and $M \cap E_2$ have non-zero isomorphic submodules. Thus M is not square-free. ∎

Corollary 3.6.5. *[AFT15, Corollary 3.7]If M is an automorphism-invariant square-free module, then every injective endomorphism of M is an automorphism of M.*

Proof. Let M be an automorphism-invariant square-free module and let $\varphi\colon M \to M$ be an injective endomorphism of M. Then φ extends to an endomorphism $\varphi_0\colon E(M) \to E(M)$, which is necessarily injective (see 1.1.51). Then $E(M) = \varphi_0(E(M)) \oplus C$, so that $E(M) = \varphi_0^2(E(M)) \oplus \varphi_0(C) \oplus C$ with $\varphi_0(C) \cong C$. By Corollary 3.6.4, $E(M)$ is square-free, so $C = 0$. This proves that φ_0 is an automorphism of $E(M)$. But M is automorphism-invariant, so $\varphi_0(M) = M$. Thus $\varphi(M) = M$, that is, the endomorphism φ of M is also surjective. ∎

Arguing as in Corollary 3.4.7, we find that:

Corollary 3.6.6. *[AFT15, Corollary 3.8] If M, N are two automorphism-invariant square-free R-modules isomorphic to submodules of each other, then M is isomorphic to N.*

Corollary 3.6.7. *[AFT15, Corollary 3.9] Let M be an automorphism-invariant R-module. Assume that $M = M_1 \oplus M_2$, where M_1 and M_2 are orthogonal. Let E_i be an injective envelope of M_i. Then E_1 is orthogonal to E_2.*

Proof. Assume that there exists $0 \neq N_1 \leq E_1$ and $0 \neq N_2 \leq E_2$ such that $N_1 \cong N_2$. Let E_i' be an injective envelope of N_i. Then $E_1 = E_1' \oplus E_1''$ and $E_2 = E_2' \oplus E_2''$ where $E_1' \cong E_2'$. Set $E := E_1 \oplus E_2 = E_1' \oplus E_2' \oplus (E_1'' \oplus E_2'')$. Then E is an injective envelope of M. Since M is automorphism-invariant and $E_1' \cong E_2'$, we get that $M = (M \cap E_1') \oplus (M \cap E_2') \oplus (M \cap (E_1'' \oplus E_2''))$ from 3.2.1. We will show that $M \cap E_1' \leq M_1$. Let $x \in M \cap E_1'$, then $x = x_1 + x_2$ where $x_i \in M_i$ and $x \in E_1' \subseteq E_1$. Hence $x_2 = x - x_1 \in M_2 \cap E_1 \subseteq E_2 \cap E_1 = 0$. Therefore $x = x_1 \in M_1$. By a similar argument, we get that $M \cap E_2' \leq M_2$. As $M \cap E_i'$ is essential in E_i' and E_i' is injective, E_i' is an injective envelope of $M \cap E_i'$. Moreover, $E_1' \cong E_2'$. Hence, by Lemma 3.6.3, there exist non-zero submodules $P_1 \leq M \cap E_1' \leq M_1$ and $P_2 \leq M \cap E_2' \leq M_2$ such that $P_1 \cong P_2$. Therefore M_1 is not orthogonal to M_2. This is a contradiction. ■

Lemma 3.6.8. *Let e, f be two idempotents of R. Then*

1. *If e, f are central idempotents of R, then $eR \cong fR$ if and only if $e = f$.*
2. *$eR \cong fR$ and $(1-e)R \cong (1-f)R$ if and only if there is a unit $u \in R$ such that $e = u^{-1}fu$.*

Proof. (1) It suffices to show that if $eR \cong fR$, then $e = f$. There exists an isomorphism $\varphi : eR \to fR$. Set $a = \varphi(e)$ and $b = varphi^{-1}(f)$. Then $ab = \varphi(e)varphi^{-1}(f) = \varphi(e\varphi^{-1}(f)) = \varphi(\varphi^{-1}(f)) = f$ and $ba = \varphi^{-1}(f)\varphi(e) = \varphi^{-1}(f\varphi(e)) = \varphi^{-1}(\varphi(e)) = e$. Now we have $e = e^2 = baba = bfa = fba = abba = abe = aeb = abab = f^2 = f$.

(2) If $eR \cong fR$ and $(1-e)R \cong (1-f)R$, then there are two isomorphism $h_1 : fR \to eR$, $h_2 : (1-f)R \to (1-e)R$. Note that $eR \oplus (1-e)R = fR \oplus (1-f)R = R$. Then $h_1 \oplus h_2 : R \to R$ is an isomorphism given by left multiplication by some unit $u \in R$. From $uf \in eR$ and $u(1-f) \in (1-e)R$, we get that $ufu^{-1} \in eRu^{-1} = eR$ and $u(1-f)u^{-1} \in (1-e)Ru^{-1} = (1-e)R$, which implies that $ufu^{-1}R \leq fR$ and $u(1-f)u^{-1} \leq (1-f)R$. Since $ufu^{-1},\ u(1-f)u^{-1}$ are two idempotents of R, and $ufu^{-1} + u(1-f)u^{-1} = 1$, we obtain that $ufu^{-1}R \oplus u(1-f)u^{-1}R = R = eR \oplus (1-e)R$. It follows that $ufu^{-1}R = eR$ and $u(1-f)u^{-1}R = (1-e)R$. Hence $(ufu^{-1})e = e = e^2$, so that $(ufu^{-1} - e)e = 0$. Moreover, $(ufu^{-1} - e)(1-e) = 0$. It follows that $ufu^{-1} - e = 0$, that is, $f = u^{-1}eu$.

Conversely, if $f = u^{-1}eu$ for some unit $u \in R$, then $ufR = euR = eR$. Hence left multiplication by u defines an isomorphism from fR to eR. Similarly, we also have $1 - f = u^{-1}(1-f)u$, which implies that $(1-f)R \cong (1-e)R$. ■

Proposition 3.6.9. *[AFT15, Proposition 3.10] Let M be an automorphism-invariant module and $E(M)$ be its injective envelope. The following conditions are equivalent:*

(a) M *is square-free.*

(b) $E(M)$ *is square-free.*

(c) *The von Neumann regular ring* $\mathrm{End}(M)/J(\mathrm{End}(M))$ *is abelian.*

(d) *The von Neumann regular right self-injective ring* $\mathrm{End}(E(M))/J(\mathrm{End}(E(M)))$ *is abelian.*

PROOF. (a) $\Leftrightarrow$ (b) has been proved in Corollary 3.6.4.

(b) $\Rightarrow$ (d) follows from the fact that $\Delta(E,E) = J(\mathrm{End}(E))$ for any injective module E and 1.4.11.

(d) $\Rightarrow$ (c) follows from the fact that every subring of an abelian ring is an abelian ring and Theorem 3.4.5.

(c) $\Rightarrow$ (a) Assume that (c) holds. Set $S := \mathrm{End}(M)$. Suppose that M contains a direct sum $X \oplus Y$ of two isomorphic submodules. Taking the injective envelopes in $E(M)$, one finds that $E(M) = E(X) \oplus E(Y) \oplus C$. If $\varphi\colon X \to Y$ is an isomorphism, φ extends to an isomorphism $\psi\colon E(X) \to E(Y)$ by 1.1.63. Thus there is an isomorphism

$$\omega := \begin{pmatrix} 0 & \psi^{-1} & 0 \\ \psi & 0 & 0 \\ 0 & 0 & 1_C \end{pmatrix} : E(M) = E(X) \oplus E(Y) \oplus C \to E(M) = E(X) \oplus E(Y) \oplus C.$$

The automorphism ω of $E(M)$ restricts to an automorphism ω' of M because M is automorphism-invariant. From 3.2.1, we know that $M = (M \cap E(X)) \oplus (M \cap E(Y)) \oplus (M \cap C)$. Thus $M = e_1M \oplus e_2M \oplus e_3M$ for orthogonal idempotents $e_i \in S$, where $e_1M = M \cap E(X)$ and $e_2M = M \cap E(Y)$. Now $\omega'(M \cap E(X)) = \omega(M \cap E(X)) = \omega(M) \cap \omega(E(X)) = M \cap E(Y)$. Thus $M \cap E(X)) \cong M \cap E(Y)$, that is, $e_1M \cong e_2M$. Applying the functor $\mathrm{Hom}(M, -)\colon$ Mod-$R \to$ Mod-S, one finds that $S_S = e_1S \oplus e_2S \oplus e_3S$ and $e_1S_S \cong e_2S_S$ [Fac10, Theorem 4.7]. If $\overline{e_i}$ is the image of e_i in $S/J(S)$, then $S/J(S) = \overline{e_1}S/J(S) \oplus \overline{e_2}S/J(S) \oplus \overline{e_3}S/J(S)$ and $\overline{e_1}S/J(S) \cong \overline{e_2}S/J(S)$ (see 1.1.31). But $S/J(S)$ is abelian, so that $\overline{e_1}S/J(S) \cong \overline{e_2}S/J(S)$ implies $\overline{e_1} = \overline{e_2}$ by 3.6.8. Thus $e_1 - e_2$ is an idempotent in $J(S)$, from which $e_1 = e_2$. Thus $M \cap E(X) = M \cap E(Y)$, and $X = Y = 0$. ∎

The next Corollary generalizes [Bie14]. Recall that a ring is *duo* if all its right ideals and all its left ideals are two-sided ideals. A ring is *quasi-duo* if all its maximal right ideals and all its maximal left ideals are two-sided ideals.

Corollary 3.6.10. *[AFT15, Corollary 3.11] The endomorphism ring of an automorphism-invariant square-free module is quasi-duo.*

PROOF. Let M be an automorphism-invariant square-free module. By Proposition 3.6.9, $\mathrm{End}(M)/J(\mathrm{End}(M))$ is an abelian von Neumann regular ring. Because every one-sided principal ideal of abelian von Neumann regular ring is generated by a central idempotent, all of one-sided ideal of $End(M)/J(End(M))$ are two-sided. Thus $\mathrm{End}(M)/J(\mathrm{End}(M))$ is a duo ring. The conclusion now follows from the fact that a ring S is quasi-duo if and only if $S/J(S)$ is quasi-duo.
∎

Theorem 3.6.11. *[AFT15, Theorem 3.12] Let M be an automorphism-invariant module and let $E(M)$ be its injective envelope.*

(a) *If M is quasi-injective and* $\mathrm{End}(M)$ *has a factor isomorphic to $\mathbb{F}_2$, then* $\mathrm{End}(E(M))$ *has a factor isomorphic to $\mathbb{F}_2$.*

(b) *If M has finite Goldie dimension and* $\mathrm{End}(M)$ *has a factor isomorphic to $\mathbb{F}_2$, then the following conditions hold.*

  (i) $\mathrm{End}(E(M))$ *has a factor isomorphic to $\mathbb{F}_2$.*

  (ii) *$E(M)$ has a direct-sum decomposition $E(M) = E \oplus C$ with E orthogonal to C, E an indecomposable R-module and* $\mathrm{End}(E)/J(\mathrm{End}(E)) \cong \mathbb{F}_2$.

  (iii) $\mathrm{Aut}(E) = 1 + J(\mathrm{End}(E))$*, so that every automorphism of the R-module E is the identity on an essential R-submodule of E.*

  (iv) *E is the injective envelope of its non-zero R-submodule* $\mathrm{ann}_E(2)$.

Proof. (a) If M is quasi-injective, the mapping

$$\overline{\varphi}\colon \mathrm{End}(M)/J(\mathrm{End}(M)) \to \mathrm{End}(E(M))/J(\mathrm{End}(E(M)))$$

is an isomorphism by Theorem 3.4.5(d). Thus $\mathrm{End}(E(M))$ has a factor isomorphic to $\mathbb{F}_2$.

(i) We first consider the case of M indecomposable. If M is automorphism-invariant indecomposable, then $\mathrm{End}(M)$ is local by Proposition 3.4.6. If $\mathrm{End}(M)$ also has a factor isomorphic to $\mathbb{F}_2$, then

$$\mathrm{End}(M)/J(\mathrm{End}(M)) \cong \mathbb{F}_2.$$

Since M is automorphism-invariant, we get that $M = N \oplus P$, where N is quasi-injective and P is square-free (Theorem 3.2.2). But M is indecomposable, so that either $M = N$ or $M = P$. If $M = N$ is quasi-injective, $\mathrm{End}(E(M))$ has a factor isomorphic to $\mathbb{F}_2$ by (a). In the other case, $M = P$ is square-free, so that $\mathrm{End}(E(M))/J(\mathrm{End}(E(M))$ is abelian. As M has finite Goldie dimension, $E(M)$ has finite Goldie dimension. Hence $\mathrm{End}(E(M))$ is semilocal. Therefore $\mathrm{End}(E(M))/J(\mathrm{End}(E(M)) \cong D_1 \times D_2 \times \cdots \times D_n$, where each D_i is a division ring. Consider the mapping $\varphi\colon \mathrm{End}(M) \to \mathrm{End}(E(M))/J(\mathrm{End}(E(M))$ of Theorem 3.4.5. From $\ker\varphi = J(\mathrm{End}(M))$, it follows that $\mathrm{im}\,\varphi \cong \mathrm{End}(M)/J(\mathrm{End}(M)) \cong \mathbb{F}_2$. Moreover, the group of units of $\mathrm{End}(E(M))/J(\mathrm{End}(E(M))$ is contained in $\mathrm{im}(\varphi)$, because M is automorphism-invariant (see 3.4.5(b)). Hence the group of units of

$$\mathrm{End}(E(M)/J(\mathrm{End}(E(M))$$

has one element. Since it is isomorphic to $D_1 \setminus \{0\} \times \cdots \times D_n \setminus \{0\}$, it follows that $D_i \cong \mathbb{F}_2$ for every $i = 1, \ldots, n$. So $\mathrm{End}(E(M))$ has a factor isomorphic to $\mathbb{F}_2$. This concludes the proof of (i) for M indecomposable.

Now let M be an arbitrary automorphism-invariant module of finite Goldie dimension and assume that $\mathrm{End}(M)$ has a factor isomorphic to $\mathbb{F}_2$. The proof will be by induction on the

Goldie dimension n of M. If $n = 1$, then M is indecomposable, and we are done. Suppose $n > 1$. Since M is automorphism-invariant, we have that $M = N \oplus P$, where N is quasi-injective, P is square-free and N, P are orthogonal. If $P = 0$, then M is quasi-injective, and we conclude by (a). If $N = 0$, then M is square-free. If M is indecomposable, we are done, as we have seen in the previous paragraph. Otherwise $M = M_1 \oplus M_2$ for suitable non-zero submodules M_1, M_2. The modules M_1, M_2 are orthogonal because M is square-free. By Corollary 3.6.2, either $\mathrm{End}(M_1)$ or $\mathrm{End}(M_2)$ has a factor isomorphic to $\mathbb{F}_2$. Without loss of generality, we can assume that $\mathrm{End}(M_1)$ has a factor isomorphic to $\mathbb{F}_2$. Let E_i be an injective envelope of M_i, so that $E(M) = E_1 \oplus E_2$. By the inductive hypothesis, we get that $\mathrm{End}(E_1)$ has a factor isomorphic to $\mathbb{F}_2$. Moreover, E_1, E_2 are orthogonal by Corollary 3.6.7. Thus $\mathrm{End}(E)$ has a factor isomorphic to $\mathbb{F}_2$ by Corollary 3.6.2, and we are done.

It remains to consider the case $M = N \oplus P$ with both N and P non-zero. Then $E(M) = E(N) \oplus E(P)$. Then $E(N)$ and $E(P)$ are orthogonal (Corollary 3.6.7), and either $\mathrm{End}(N)$ or $\mathrm{End}(P)$ has a factor isomorphic to $\mathbb{F}_2$ (Corollary 3.6.2). By the inductive hypothesis, $\mathrm{End}(E(N))$ or $\mathrm{End}(E(P))$ has a factor isomorphic to $\mathbb{F}_2$. The conclusion follows by Corollary 3.6.2.

(ii) Since M is of finite Goldie dimension, $E(M)$ decomposes as $E(M) = E_1 \oplus ... \oplus E_n$, where the E_i are indecomposable injective R-modules. Now $\mathrm{End}(M)$ is semiperfect (Proposition 3.4.6(b)), hence semilocal. By the hypothesis, there exists a ring morphism $\mathrm{End}(M) \to \mathbb{F}_2$, so that there exists a ring morphism $\mathrm{End}(M)/J(\mathrm{End}(M)) \to \mathbb{F}_2$. The semisimple artinian ring $\mathrm{End}(M)/J(\mathrm{End}(M))$ is a finite direct product of rings of matrices $M_{n_j}(D_j)$ over division rings D_j. The kernel of the ring morphism $\mathrm{End}(M)/J(\mathrm{End}(M)) \to \mathbb{F}_2$ is a maximal ideal of this finite direct product of rings of matrices $M_{n_j}(D_j)$. It follows that there exists an index j with $n_j = 1$ and $D_j \cong F_2$. Thus, in the direct-sum decomposition $E(M) = E_1 \oplus ... \oplus E_n$, there exists an index i with $E_i \not\cong E_k$ for every $k = 1, \dots n$ different from i and $\mathrm{End}(E_i)/J(\mathrm{End}(E_i)) \cong \mathbb{F}_2$. Set $E := E_i$ and $C := E_1 \oplus \cdots \oplus E_{i-1} \oplus E_{i+1} \oplus \cdots \oplus E_n$. In order to conclude the proof of (ii), it suffices to show that E is orthogonal to C. Assume the contrary. Then there exist isomorphic non-zero submodules A of E and B of C. Thus $E(B)$ is an indecomposable direct summand of C isomorphic to $E(A) \cong E$. By the Krull-Schmidt-Azumaya Theorem, the module $E(B)$ must be isomorphic to one of the modules $E_1, \dots, E_{i-1}, E_{i+1}, \dots, E_n$. This is a contradiction.

(iii) If $\varphi \in \mathrm{Aut}(E)$, we have that $\varphi + J(\mathrm{End}(E))$ is an invertible element in the ring $\mathrm{End}(E)/J(\mathrm{End}(E))$. But $\mathrm{End}(E)/J(\mathrm{End}(E)) \cong \mathbb{F}_2$, so that $\varphi + J(\mathrm{End}(E)) = 1 + J(\mathrm{End}(E))$. Thus $\varphi \in 1 + J(\mathrm{End}(E))$. This proves that $\mathrm{Aut}(E) = 1 + J(\mathrm{End}(E))$. In particular, every automorphism of the R-module E is the identity on an essential R-submodule of E.

(iv) From $\mathrm{End}(E)/J(\mathrm{End}(E)) \cong \mathbb{F}_2$, it follows that $1_E + 1_E \in J(\mathrm{End}(E))$; that is 2 annihilates an essential submodule of E. Therefore $\mathrm{ann}_E(2)$ is a non-zero R-submodule of E. But E is uniform. ∎

Theorem 3.6.11(b) does not hold when M is not automorphism-invariant. To see this, take $R = \mathbb{Z}$ and $M = \mathbb{Z}_\mathbb{Z}$. Then $\mathbb{Q}_\mathbb{Z}$ is an injective envelope of $\mathbb{Z}_\mathbb{Z}$. The endomorphism ring of $\mathbb{Z}_Z$ is isomorphic to $\mathbb{Z}$. So it has a factor isomorphic to $\mathbb{F}_2$. But the endomorphism ring of $\mathbb{Q}_\mathbb{Z}$ has no factor isomorphic to $\mathbb{F}_2$.

Remark 3.6.12. Let M be any right R-module, let $E(M)$ be its injective envelope and $S :=$

$\mathrm{End}(E(M))$ be the endomorphism ring of $E(M)$, so that $E(M)$ turns out to be a S-R-bimodule. Let I be the two-sided ideal of S generated by the set $\{ s - s^2 \mid s \in S \}$. Then the annihilator $\mathrm{ann}_{E(M)} I := \{ e \in E(M) \mid Ie = 0 \}$ is an S-R-subbimodule of ${}_S E(M)_R$, as is easily seen. Thus there is an R-module direct-sum decomposition $E(M)_R = E_1 \oplus E_2$, where E_1 is an injective envelope $E(\mathrm{ann}_{E(M)} I)$ of $\mathrm{ann}_{E(M)} I$ in $E(M)_R$ and E_2 is a complement of E_1 in $E(M)_R$, so that no non-zero element of E_2 is annihilated by I, i.e., $e_2 \in E_2$ and $Ie_2 = 0$ imply $e_2 = 0$. Assume there are two non-zero R-submodules A_1, A_2 such that $A_1 \leq E_1$, $A_2 \leq E_2$ and $A_1 \cong A_2$. Then their injective envelopes $E(A_1), E(A_2)$ are isomorphic and each $E(A_i)$ is a direct summand of E_i. So $E(M)$ decomposes as a direct sum $E(M) = e_1E(M) \oplus e_2E(M) \oplus e_3E(M)$ for orthogonal idempotents $e_i \in \mathrm{End}(E(M))$ where $e_iE(M) = E(A_i)(i = 1, 2)$. Since $E(A_1) \cong E(A_2)$, $e_1E(M) \cong e_2E(M)$. Applying the functor $\mathrm{Hom}(E(M), -)\colon \text{Mod-}R \to \text{Mod-}S$, one finds that $S_S = e_1S \oplus e_2S \oplus e_3S$ and $e_1S_S \cong e_2S_S$ [Fac10, Theorem 4.7], where $S = \mathrm{End}(E(M))$. So there exists a unit element $u \in S$ such that $e_1 = u^{-1}e_2u$ (see 3.6.8). As $e_2 \,\mathrm{ann}_{E(M)} I = 0$ and $e_1 = u^{-1}e_2u$, it follows that $e_1 \,\mathrm{ann}_{E(M)} I = 0$. But this contradicts $e_1 \,\mathrm{ann}_{E(M)} I \neq 0$, because $e_1 \,\mathrm{ann}_{E(M)} I = E(A_1) \cap \mathrm{ann}_{E(M)} I \neq 0$. Therefore two R-modules E_1 and E_2 are orthogonal. By Lemma 3.5.5, $S/\Delta(E(M), E(M)) \cong S_1/\Delta(E_1, E_1) \times S_2/\Delta(E_2, E_2)$, where S_i denotes the endomorphism ring of the R-module E_i. As $\Delta(E, E) = J(\mathrm{End}(E))$ for any injective R-module E by 1.1.65, it follows that $S/J(S) \cong S_1/J(S_1) \times S_2/J(S_2)$ in a canonical way. If I_i denotes the two-sided ideal of S_i generated by all $x - x^2$ with $x \in S_i$, then $I/J(S) \cong I_1/J(S_1) \times I_2/J(S_2)$.

Now consider the ring morphism $\rho\colon S \to \mathrm{End}(\mathrm{ann}_{E(M)} I)$ that associates to any $f \in S$ its restriction $f|_{\mathrm{ann}_{E(M)} I}$ to $\mathrm{ann}_{E(M)} I$. The ring morphism ρ is well defined because $\mathrm{ann}_{E(M)} I$ is a left S-submodule of $E(M)$. The morphism ρ is clearly an onto mapping, and its kernel is $\ker \rho := \{ f \in S \mid f(\mathrm{ann}_{E(M)} I) = 0 \}$. In particular $I \subseteq \ker \rho$. Since S/I is a boolean ring, the ring $\mathrm{End}(\mathrm{ann}_{E(M)} I)$ is also boolean. Moreover, $J(S) \subseteq I \subseteq \ker \rho$, so that ρ induces a ring morphism $\overline{\rho}\colon S/J(S) \to \mathrm{End}(\mathrm{ann}_{E(M)} I)$. As $S/J(S) \cong S_1/J(S_1) \times S_2/J(S_2)$ and the elements of $S_2/J(S_2)$ are clearly mapped to 0 by $\overline{\rho}$, we get that $0 \times S_2/J(S_2) \subseteq \ker(\rho)$. Thus there is a surjective ring morphism $S_1/I_1 \to \mathrm{End}(\mathrm{ann}_{E(M)} I)$.

From Remark 3.6.12, we get in particular that:

Proposition 3.6.13. *[AFT15, Proposition 3.14]Let M be an R-module, $S := \mathrm{End}(E(M))$ be the endomorphism ring of $E(M)$ and I be the two-sided ideal of S generated by the set $\{ s - s^2 \mid s \in S \}$.*

(a) *If $\mathrm{ann}_{E(M)} I \neq 0$, then $\mathrm{End}(M)$ has a factor isomorphic to $\mathbb{F}_2$.*

(b) *If M is automorphism-invariant and $\mathrm{ann}_{E(M)} I$ is an essential submodule of the R-module $E(M)$, then the ring $\mathrm{End}(M)/J(\mathrm{End}(M))$ is a boolean ring.*

Proof. Compose the ring morphism $\varphi\colon \mathrm{End}(M) \to S/J(S)$ of Theorem 3.4.5 with the morphism $\overline{\rho}\colon S/J(S) \to \mathrm{End}(\mathrm{ann}_{E(M)} I)$ in Remark 3.6.12, obtaining a morphism $\overline{\rho}\varphi\colon \mathrm{End}(M) \to \mathrm{End}(\mathrm{ann}_{E(M)} I)$, where $\mathrm{End}(\mathrm{ann}_{E(M)} I)$ is a boolean ring. If $\mathrm{ann}_{E(M)} I \neq 0$, then $\mathrm{End}(\mathrm{ann}_{E(M)} I)$ is a non-zero boolean ring, so that there is a morphism $\mathrm{End}(\mathrm{ann}_{E(M)} I) \to \mathbb{F}_2$. Thus there is a morphism $\mathrm{End}(M) \to \mathbb{F}_2$, necessarily surjective. Hence $\mathrm{End}(M)$ has a factor isomorphic to $\mathbb{F}_2$. This concludes the proof of (a).

If M is automorphism-invariant and $\operatorname{ann}_{E(M)} I$ is essential in $E(M)$, then in Remark 3.6.12 we have that $E(M) = E_1$, $E_2 = 0$ and $\ker \rho \subseteq \Delta(E(M), E(M)) = J(S)$. As $I \subseteq \ker \rho$ and $J(S) \subseteq I$, it follows that $I = \ker \rho = J(S)$. Thus $S/J(S) \cong \operatorname{End}(\operatorname{ann}_{E(M)} I)$ is a boolean ring. By Theorem(a) 3.4.5, the ring $\operatorname{End}(M)/J(\operatorname{End}(M))$ is isomorphic to a subring of the ring $\operatorname{End}(E(M))/J(\operatorname{End}(E(M))) = S/J(S)$. Thus $\operatorname{End}(M)/J(\operatorname{End}(M))$ is boolean. ∎

Proposition 3.6.14. *[AFT15, Proposition 3.15] Let M be an automorphism-invariant square-free module of finite Goldie dimension. Then M decomposes as a direct sum $M = N \oplus P$, where N is a module orthogonal to P, $\operatorname{End}(N)$ has no factor isomorphic to $\mathbb{F}_2$, and $\operatorname{End}(P)/J(\operatorname{End}(P))$ is isomorphic to a boolean ring $\mathbb{F}_2^n$ for some n.*

Proof. The automorphism-invariant module M of finite Goldie dimension, decomposes as a direct sum $M = M_1 \oplus \cdots \oplus M_t$ of indecomposable modules, necessarily automorphism-invariants by 3.1.4. Let $e_1, \dots, e_t \in \operatorname{End}(M)$ be the orthogonal idempotents corresponding to this direct-sum decomposition of M. Then $\overline{e_1}, \dots, \overline{e_t} \in \operatorname{End}(M)/\Delta(M, M)$ are orthogonal idempotents of $\operatorname{End}(M)/\Delta(M, M)$, which is an abelian ring by Proposition 3.6.9. Thus the idempotents $\overline{e_1}, \dots, \overline{e_t}$ of $\operatorname{End}(M)/\Delta(M, M) = \operatorname{End}(M)/J(\operatorname{End}(M))$ are central, so that

$$\begin{aligned}&\operatorname{End}(M)/J(\operatorname{End}(M)) \cong \\ &\quad \cong \overline{e_1} \operatorname{End}(M)/J(\operatorname{End}(M))\overline{e_1} \times \cdots \times \overline{e_t} \operatorname{End}(M)/J(\operatorname{End}(M))\overline{e_t} \cong \\ &\quad \cong \operatorname{End}(M_1)/J(\operatorname{End}(M_1)) \times \cdots \times \operatorname{End}(M_t)/J(\operatorname{End}(M_t)),\end{aligned}$$

is isomorphic to the direct product of the residue division rings $\operatorname{End}(M_i)/J(\operatorname{End}(M_i))$. Let N be the direct sum of the M_i with the residue division rings $\operatorname{End}(M_i)/J(\operatorname{End}(M_i))$ not isomorphic to $\mathbb{F}_2$ and P be the direct sum of the M_i with the residue division rings $\operatorname{End}(M_i)/J(\operatorname{End}(M_i))$ isomorphic to $\mathbb{F}_2$. Then $M = N \oplus P$, $\operatorname{End}(N)$ has no factor isomorphic to $\mathbb{F}_2$, because $\operatorname{End}(N)/J(\operatorname{End}(N))$ is a direct product of finitely many division rings not isomorphic to $\mathbb{F}_2$, and $\operatorname{End}(P)/J(\operatorname{End}(P))$ isomorphic to a direct product of finitely many copies of $\mathbb{F}_2$.

Finally, N and P are relatively injective by 3.1.5. As

$$\operatorname{End}(M)/\Delta(M, M) \cong \operatorname{End}(N)/\Delta(N, N) \times \operatorname{End}(P)/\Delta(P, P),$$

we conclude that N and P are orthogonal (see 3.5.5). ∎

Chapter 4

Poor modules

4.1 Basic properties

Definition 4.1.1. A module M is *poor* in case, for every module N, if M is N-injective, then N is semisimple. Equivalently a module M is poor if for every non-semisimple module N there exists a submodule N' of N and a morphism $f : N' \to M$ can not be extended to N.

Proposition 4.1.2. *[AAL10, Remark 2.3] The following conditions are equivalent for any ring R :*

1. *R is semisimple artinian.*
2. *Every module is poor.*
3. *There exists an injective poor module E.*

Proof. $(1) \Rightarrow (2)$: It follows from the fact that every right R-module is semisimple.

$(2) \Rightarrow (3)$: Obvious.

$(3) \Rightarrow (1)$: Assume that E is an injective poor module. Then E is R_R-injective, so that R_R is semisimple. This proves that R is semisimple artinian. ■

Proposition 4.1.3. *[AAL10, Proposition 3.1] The intersection of all injectivity domains*

$$\bigcap_{M \in Mod\text{-}R} In^{-1}(M)$$

is the class of all semisimple modules.

Proof. Let N be an element of $\bigcap_{M\in \text{Mod-}R} In^{-1}(M)$ and K be an arbitrary submodule of N. Then $N \in In^{-1}(K)$, so that the embedding map from $K \to N$ has a left inverse. Hence K is a direct summand of N, which implies that N is semisimple.

Conversely, let N be a semisimple module and K be an arbitrary submodule of N. Then $N = K \oplus K'$ for some $K' \leq N$. Let M be an arbitrary module. Then, for every $f \in Hom(K, M)$, the morphism $f \oplus 0 : N \to M$ extends f. Therefore, $N \in In^{-1}(M)$ for every module M. ■

Proposition 4.1.4. *[AAL10, Remark 2.4] Let M be a poor module. Then $M \oplus N$ is poor for every module N.*

PROOF. Assume that $M \oplus N$ is K-injective. Then M is K-injective, so that K is semisimple. This proves $M \oplus N$ is poor. ■

Proposition 4.1.5. *[ELS11, Proposition 1] Every ring has a poor module.*

PROOF. Let $\{A_\alpha | \alpha \in I\}$ be a complete of representatives of isomorphism classes of non-semisimple cyclic R-modules. Since A_α is non-semisimple for each $\alpha \in I$, there exists a proper essential submodule K_α of A_α. Now set $T = \oplus_{\alpha \in I} K_\alpha$. Now we claim that T is poor. Assume the contrary. Then there exists a non-semisimple cyclic module B such that T is B-injective. Hence $B \cong A_\alpha$ for some $\alpha \in I$, so that B has a proper essential submodule, say N, isomorphic to K_α. Because T is B-injective, so is N. This implies that N is a direct summand of B, which contradicts the fact that N is a proper essential submodule of B. ■

Corollary 4.1.6. *[ELS11, Corollary 1] Let R be a ring. Then the following conditions are equivalent.*

1. *R is semisimple artinian.*
2. *All poor right R-modules are semisimple.*
3. *Non-zero direct summands of poor right R-modules are poor.*
4. *Non-zero factors of poor right R-modules are poor.*

PROOF. $(1) \Rightarrow (2)$, $(1) \Rightarrow (3)$ and $(1) \Rightarrow (4)$: follow from 4.1.2.
$(2) \Rightarrow (1)$, $(3) \Rightarrow (1)$ and $(4) \Rightarrow (1)$: follow from 4.1.4. ■

Theorem 4.1.7. *[AAL10, Theorem 4.3] Let M be a projective semisimple poor module. Then any semisimple module B orthogonal to M is injective.*

PROOF. In order to prove this theorem, it suffices to show that M is $E(B)$-injective. We claim, for every $X \leq E(B)$, that $Hom(X, M) = 0$. Let X be a submodule of $E(B)$ and f be a morphism from X to M. Since X is projective, we have $X = Y \oplus \operatorname{Ker} f$ where $Y \cong f(X)$. Assume that $f(X \cap B) \neq 0$. Then $f(X \cap B)$ is a projective submodule of M. Hence $X \cap B \cong f(X \cap B) \oplus \operatorname{Ker} f \cap (X \cap B))$, which contradicts the hypothesis that B is orthogonal to M. Therefore $f(X \cap B) = 0$, so that $X \cap B \leq \operatorname{Ker} f$. Since $X \cap B \leq_e X$, $f(X) = 0$. This gives that M is $E(B)$-injective. ■

Corollary 4.1.8. *[AAL10, Corollary 4.5] Let R be a ring which is not semisimple artinian. If there is a simple projective poor module M, then*

1. *Every direct sum of simple injective modules is injective.*

2. *Every simple module is either injective or poor.*

PROOF. Let V be a simple projective poor module and (V_i) $(i \in I)$ be a family of simple injective modules. If $V_i \cong V$ for some $i \in I$, then V would be an injective poor module, which would implies that R is a semisimple artinian ring, a contradiction. Therefore, for each $i \in I$, V_i is not isomorphic to V, so that $\oplus_{i\in I} V_i$ is orthogonal to V. Applying 4.1.7, we get that $\oplus_{i\in I} V_i$ is injective. This proves (1).

For (2), let U be an arbitrary simple module. Then U is either isomorphic to V or orthogonal to V. For the former case, we deduce that U is poor. For the latter case, we conclude that U is injectve by 4.1.7 ∎

Corollary 4.1.9. *[AAL10, Corollary 4.8] If there is a projective semisimple poor module M, then*

1. *$Soc(R_R)$ is projective.*

2. *The socle of any projective R-module is projective.*

PROOF. (1) It suffices to prove the corollary in the case that R is not semisimple artinian. If $Soc(R_R) = 0$, we are done. Otherwise, let S be a minimal right ideal of R. By 4.1.8,S is either projective or injective. If S is injective, then S is a direct summand of R_R, which implies that S is projective. Therefore, all minimal right ideals of R are projective, so that $Soc(R_R)$ is projective.

(2) follows from the first one and the fact that every projective module is a direct summand of some free module. ∎

4.2 Existence of semisimple poor modules

Definition 4.2.1. Let M be a module. If socles split in all factors of M, we will say that M *crumbles* .

Lemma 4.2.2. *[ELS11, Remark 1] Let B be a cyclic module that crumbles. Then every factor of B has finite Golide dimension.*

PROOF. Let N be a factor of B. Assume that N has infinite Goldie dimension. Then N contain an infinite direct sum $\oplus_{i\in I} A_i$ of non-zero cyclic submodules A_i. For each $i \in I$, there exists a maximal submodule T_i of A_i. Set $T = \oplus_{i\in I} T_i$. Because $\frac{\oplus_{i\in I} A_i}{\oplus_{i\in I} T_i} \cong \oplus_{i\in I}(\frac{A_i}{T_i})$, the factor $\frac{N}{T}$ has an infinite socle, which is not a direct summand of $\frac{N}{T}$, a contradiction. This completes the proof. ∎

Theorem 4.2.3. *[ELS11, Theorem 1] Let R be any ring. The following conditions are equivalent:*

1. *R has a semisimple poor module.*

2. *Every cyclic right R-module that crumbles is semisimple.*

3. *Every right R-module that crumbles is semisimple.*

4. *Every noetherian but not artinian cyclic right R-module has a factor whose radical has non-zero socle.*

5. *Every noetherian but not artinian cyclic right R-module has a factor with non-zero radical.*

PROOF. (1) $\Rightarrow$ (2) : Let S be a semisimple poor module. Assume (2) does not hold. Then there exists a non-semisimple cyclic module B that crumbles. Since S is poor, S is not B-injective. Hence there is a morphism $f : B \to E(S)$ such that $f(B)$ is not contained in S, which implies that $f(B)$ is non-semisimple. Because $Soc(E(S)) = S \leq_e E(S),\ f(B)$ has essential socle. Thus $f(B) = Soc(f(B))$ thanks to the fact that B crumbles, a contradiction.

(2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) : Obvious.

(3) $\Rightarrow$ (2) : follows from the fact that cyclic submodules of crumbling modules are also crumbling.

(2) $\Rightarrow$ (4) : Let N be a noetherian but not artinian cyclic module. Then N is non-semisimple, so that N has a factor B whose socle does not split by assumption. We claim that B contains a simple submodule V which is not a direct summand of B. Assume the contrary. Then every simple submodule of B is a direct summand of B. Hence $Soc(B)$ is a direct summand of B because $Soc(B)$ is finitely generated. This is a contradiction. Now since V is not a direct summand of B, every maximal submodule of B contains B. It follows that $V \subseteq Rad(B)$, and hence $Soc(Rad(B)) \neq 0$.

(5) $\Rightarrow$ (2) : Assume that B is non-semisimple cyclic module that crumbles. By 4.2.2 and [Er09, Proposition 1], B is noetherian. Since B is non-semisimple, B is not artinian. Now suppose there is a factor C of B has non-zero radical. Then $Rad(C)$ contains a non-zero cyclic D and a maximal submodule E of D. Since C is cyclic, $Rad(C)$ is superfluous in C, so that D is superfluous in C. Hence $\frac{D}{E}$ is superfluous in $\frac{C}{E}$. Since B crumbles, $\frac{D}{E}$ is a direct summand of $\frac{C}{E}$, which contradicts the fact that $\frac{D}{E}$ is superfluous in $\frac{C}{E}$.

(2) $\Rightarrow$ (1) : Let Γ be a complete set of representatives of isomorphism classes of simple modules. Set $S = \oplus_{B\in\Gamma} B^{(R)}$. In order prove (1) it suffices to show that S is poor. Assume that S is not poor. Then there exists a non-semisimple cyclic module A such that S is A-injective. By assumption, there is a semisimple subfactor of A, say $\frac{L}{C}$, which does not split in $\frac{A}{C}$. Let $\frac{K}{C}$ be a complement of $\frac{L}{C}$ in $\frac{A}{C}$. Then $\frac{\frac{L}{C}\oplus\frac{K}{C}}{\frac{K}{C}}$ is a proper essential submodule of $\frac{\frac{A}{C}}{\frac{K}{C}}$. Then $\frac{A}{K}$ has a proper essential socle isomorphic to $\frac{L}{C}$. Since S is A-injective, it is $\frac{A}{K}$-injective. Note that $Soc(\frac{A}{K})$ can be embedded in S because of the choice of Γ and S. Therefore the embedding $Soc(\frac{A}{K}) \to S$ extends to some monomorphism $f : \frac{A}{K} \to S$, so that $\frac{A}{K}$ is semisimple, a contradiction. ∎

We say that a module M is said to be *locally noetherian* if every finitely generated submodule of M is noetherian. A module N is a *V-module* if every simple module is N-injective.

Corollary 4.2.4. *[ELS11, Corollary 2] Let R be a ring. The following conditions are equivalent:*

1. R has a semisimple poor module.

2. Every locally noetherian V-module is semisimple.

PROOF. (1) ⇒ (2) : Let M be a locally noetherian V-module and N be an arbitrary factor of M. Then $Soc(N)$ is M-injective by [DHSW94, 2.5], so that $Soc(N)$ is N-injective, and slits in N. Hence M crumbles. By 4.2.3, M is semisimple.

(2) ⇒ (1) : Let M be a cyclic module that crumbles. Then M is noetherian by the proof of (5) ⇒ (4) in 4.2.3. In order to prove (1), it is enough to show that M is semsimple thanks to 4.2.3. Let S be an arbitrary simple module. Let A be a submodule of M, and $f : A \to S$ be any non-zero morphism. Since M crumbles, we have $\frac{M}{\operatorname{Ker} f} = \frac{A}{\operatorname{Ker} f} \oplus B$ for some submodule B of $\frac{M}{\operatorname{Ker} f}$. Hence the composition of the natural maps $M \to \frac{M}{\operatorname{Ker} f}$, $\frac{M}{\operatorname{Ker} f} \to \frac{A}{\operatorname{Ker} f}$, and $\frac{A}{\operatorname{Ker} f} \to S$ extends f. This means that S is M-injective, so that M is V-module. By (2), we have that M is semisimple. ∎

Corollary 4.2.5. *[ELS11, Corollary 3] Let R be a ring such that every noetherian right module is artinian (in particular a right semiartinian ring), then R has a semisimple poor module.*

PROOF. It follows immediately from 4.2.3. ∎

4.3 Rings whose modules are either injective or poor.

Definition 4.3.1. A ring R is said to have *no middle class* if every right R-module is either injective or poor.

Lemma 4.3.2. *[ELS11, Lemma 1] Let R be a ring. If R has no middle class, so is every factor ring of R.*

PROOF. Let I be an ideal of R and $M_{R/I}$ be a non-poor R/I-module. Then there is a non-semisimple R/I-module $N_{R/I}$ such that $M_{R/I}$ is $N_{R/I}$-module, so that M_R is N_R-injective and N_R is non-semisimple R-module. Since R has no middle class, M_R is injective as an R-module, which implies that $M_{R/I}$ is injective as a R/I module. ∎

The *second singular submodule* of a module M is defined to be the singular submodule $Z(M/Z(M))$ of $M/Z(M)$. Denote it by $Z_2(M)$.

Lemma 4.3.3. *[ELS11, Lemma 2] Let R be a non-right SI ring with no middle class. Then:*

1. Every nonsingular module is injective (hence semisimple).

2. The second singular submodule splits in any module.

3. There is a ring direct sum $R = S \oplus T$ such that S is semisimple artinian ring and T_T has essential socle with $Z(T_T) = Soc(T_T)$.

4. R has essential socle.

PROOF. (1) Since R is not a right SI ring, there is a non-injective singular module M. Assume that $E(M)$ is semisimple. Then $M = E(M)$ is injective, a contradiction. Hence $E(M)$ is not semisimple. Because R has no middle class, in order to prove that every nonsingular module is injective it suffices to show that every nonsingular module is $E(M)$-injective. Let A be an arbitrary nonsingular module and B be any submodule of $E(M)$. Assume that there is a non-zero morphism $f : B \to A$. Since A is nonsingular and M is singular, $f(B \cap M) \leq Z(A) = 0$, so that $B \cap M \leq \operatorname{Ker} f$. It follows that $\operatorname{Ker} f \leq_e B$. Hence $B/\operatorname{Ker} f$ is singular. But $B/\operatorname{Ker} f \cong Imf \leq A$ implies that $B/\operatorname{Ker} f$ is nonsingular. Therefore $B/\operatorname{Ker} f = 0$, so that $\operatorname{Ker} f = B$, that is, $f = 0$, a contradiction. So, $Hom(B, A) = 0$, which implies that A is $E(M)$-injective. Note that every submodule of any nonsingular module is also nonsingular and hence injective. Thus every submodule of any nonsingular module is a direct summand, so that all nonsingular module are semisimples.

(2) Let N be an arbitrary module. Then $Z(N) \leq_e Z_2(N)$ and $Z_2(N)$ is closed in N. Hence $Z_2(N)$ is a complement of some submodule C of N, so that $\frac{C \oplus Z_2(N)}{Z_2(N)} \leq_e \frac{N}{Z_2(N)}$. Since $\frac{C \oplus Z_2(N)}{Z_2(N)} \cong C$ and C is nonsingular, $\frac{C \oplus Z_2(N)}{Z_2(N)}$ is injective. It follows that $\frac{C \oplus Z_2(N)}{Z_2(N)} = \frac{N}{Z_2(N)}$, so that $C \oplus Z_2(N) = N$.

(3) Applying (2) to R_R, we get that $R = A \oplus Z_2(R_R)$ for some semisimple right ideal A. Now we claim that A is an ideal. Let r be an arbitrary element of $Z_2(R)$. On the one hand, rA is isomorphic to a factor of A, which implies that rA is isomorphic to a direct summand of A, and hence rA is nonsingular. On the other hand, since $Z_2(R)$ is an ideal of R, $rA \subseteq Z_2(R)$. Therefore $Z(rA) = rA \cap Z(R) = 0$ implies that $rA = 0$. It follows that $Z_2(R)A = 0$, and hence A is an ideal of R_R. This proves the claim. Set $S = A$ and $T = Z_2(R)$. Then we have a ring decomposition $R = S \oplus T$ where S is a semisimple artinian ring. Now we have $Z(R_R) = Z(S_R) \oplus Z(T_R) = Z(T_T)$, so $Z(T_T) \leq_e T_T$. It remains to show that $Soc(T_T) = Z(T_T)$. Note that $Soc(T_T) \leq Z(T_T)$ because $Z(T_T) \leq_e T_T$. Now assume that $Z(T_T)$ is not semisimple. Then $Z(T_T) \neq 0$. Since $Z(E(T_T))$ is a fully invariant submodule of $Z(E(T_T))$, it is quasi-injective, so that $Z(E(T_T))$ is $Z(T_T)$-injective as an T-module. It follows that $Z(T_T)$ is not a poor T-module. Because R has no middle class, so does T by 4.3.2. Hence $Z(T_T)$ is injective, so that $Z(T_T) = T$, a contradiction. Therefore $Z(T_T)$ is semisimple, that is, $Z(T_T) \leq Soc(T_T)$. It follows that $Z(T_T) = Soc(T_T)$.

(4) $Soc(R_R) = Soc(S_R) \oplus Soc(T_R) = S \oplus Soc(T_T) \leq_e S \oplus T = R_R$ because $Soc(T_T) \leq_e T$ by (3). ■

Recall that a ring R is said to be *indecomposable* if R has no ring decompositions with more than one term.

Lemma 4.3.4. *[ELS11, Lemma 3] Let R be a ring with essential singular socle. If R has no middle class, then R is an indecomposable ring.*

PROOF. Assume that $R = R_1 \oplus R_2$ with two non-zero ideals R_1, R_2. Then every right ideal $I \leq R_2$ is R_1-injective because $Hom(X, I) = 0$ for every $X \leq R_1$. In particular, $Soc(R_2)$ is R_1-injective. Since R has no middle class, either $Soc(R_2)$ is injective or R_1 is semisimple. It follows that R always has a simple direct summand V, which contradict the hypothesis that $Soc(R_R)$ is singular. This proves that R is an indecomposable ring. ■

Lemma 4.3.5. *[ELS11, Lemma 6] Let R be a ring with singular right socle. If R has no middle class, then R is right noetherian.*

PROOF. Case 1 : R is right semiartinian. Then $Soc(R_R)$ is non-zero, so that there is a simple right ideal S of R. Since R has singular right socle, S is singular, which implies that S can not be a direct summand of R. It follows that S is not injective, that is, $S \neq E(S)$. As R is right semiartinian, $Soc(\frac{E(S)}{S} \neq 0$, and hence we can find a submodule S' of $E(S)$ such that $\frac{S'}{S}$ is simple. It is clear that S' is a module of length 2, so that S' is a non-semisimple noetherian module. Let $\{E_i | i \in I\}$ be any family of injective modules. Because S' is noetherian, then $\oplus_{i\in I}E_i$ is S'-injective. Since R has no middle class and S' is non-semisimple, we obtain that $\oplus_{i\in I}E_I$ is injective. This proves that R is a right noetherian ring.

Case 2 : R is not right semiartian. Let I be the union of the right socle series of R. Then $\frac{R}{I}$ is a non-zero ring with zero right socle. By 4.3.2, $\frac{R}{I}$ has no middle class. Applying 4.3.3(3), we obtain that $\frac{R}{I}$ is a right SI-ring. Then, by 1.6.5, $\frac{R}{I}$ is a right noetherian ring. Now R has a non-semisimple noetherian module $\frac{R}{I}$. By an argument similar to the argument in case 1, we conclude that R is a right noetherian ring ■

Lemma 4.3.6. *[ELS11, Lemma 7] Let R be a ring with non-zero singular socle. If R has no middle class, then R is right artinian.*

PROOF. By 4.3.5, in order to prove that R is right artinian it is enough to show that R is right semiartinian. Assume that R is not right semiartnian. Let I be the union of the right socle series of R and set $\overline{R} = \frac{R}{I}$. Then $Soc(\overline{R}) = 0$ and $\overline{R} \neq 0$. Assume that $\overline{R}_R$ is injective. Then $\overline{R}$ is a QF-ring because R is right noetherian, so that $\overline{R}$ is right artnian. It follows that $Soc(\overline{R}) \neq 0$, a contradiction. Therefore $\overline{R}_R$ is poor.

Let Z be an arbitrary non-semiartinian cyclic R-module and D be the union of the socle series of Z. Set $\overline{Z} = \frac{Z}{D}$. Then $Soc(\overline{Z}) = 0$ and $\overline{Z} \neq 0$. Now we claim that $\overline{Z}$ has a non-zero submodule $\overline{W} = \frac{W}{D}$ such that $\frac{\overline{Z}}{\overline{W}} \cong \frac{Z}{W}$. Assume the contrary. Then every factor $\frac{\overline{Z}}{X}$ with respect to a non-zero submodule $X \leq \overline{Z}$ is semiartinian. Note that $Soc(E(\overline{R}) \cap \overline{R} = Soc(\overline{R}) = 0$ implies that $Soc(E(\overline{R}) = 0$. Combining this with assumption that $\frac{\overline{Z}}{X}$ is semiartinian, we obtain that $Hom(\frac{\overline{Z}}{X}, E(\overline{R})) = 0$. Hence $\overline{R}$ is $\frac{\overline{Z}}{X}$, so that $\frac{\overline{Z}}{X}$ is semisimple because $\overline{R}$ is poor. Since R has non-zero singular right socle, there is a simple singular right ideal V of R, so that V can not be a direct summand of R. It follows that V is not injective, and hence V is poor because R has no middle class. Let G be an arbitrary submodule of $\overline{Z}$ and f be a non-zero morphism from G to V. Since $Soc(G) \leq Soc(\overline{Z}) = 0$, $\operatorname{Ker} f \neq 0$, so that $\frac{\overline{Z}}{\operatorname{Ker} f}$ is semisimple. Now we can write $\frac{\overline{Z}}{\operatorname{Ker} f} = \frac{G}{\operatorname{Ker} f} \oplus \frac{U}{\operatorname{Ker} f}$, for some submodule U of $\overline{Z}$. Let $g_1 : \overline{Z} \to \frac{\overline{Z}}{\operatorname{Ker} f}, g_2 : \frac{G}{\operatorname{Ker} f} \oplus \frac{U}{\operatorname{Ker} f} \to \frac{G}{\operatorname{Ker} f}$ be the canonical projections and $\overline{f} : \frac{G}{\operatorname{Ker} f} \to V$ be an isomorphism induced by f. Then $\overline{f}g_2g_1 : \overline{Z} \to V$ extends f. Thus V is $\overline{Z}$-injective, so that $\overline{Z}$ is semisimple, a contradiction. This proves the claim.

Taking $Z = R$, we obtain a non-zero right ideal A_0 of R with $\frac{R}{A_1}$ is non-semiartinian. Repeating this argument with $Z = \frac{R}{A_0}$ and so on, we have a strictly ascending chain $\{A_i | i \in \mathbb{N}\}$. This contradicts the fact that R is a right noetherian ring. Therefore R is right semiartinian. This completes the proof. ■

Definition 4.3.7. 1. A ring R is a QF-ring if R is right artinian and right self-injective.

2. A ring R is a right PCI ring if each proper cyclic right R-module is injective.

Proposition 4.3.8. *[ELS11, Proposition 3] Let R be a ring with no middle class. If R is not a right SI-ring, then R is the ring direct sum of a semisimple artinian ring S and a ring T is an indecomposable right artinian ring satisfying the following conditions:*

(a) $soc(T_T) = Z(T_T) = J(T)$,

(b) T has homogeneous right socle, and

(c) there is a unique non-injective simple right T-module up to isomorphism.

Moreover, T is either a QF-ring with $J(T)^2 = 0$, or T_T is poor.

Proof. By 4.3.3(3), we have a ring decomposition $R = S \oplus T$, where S is a semisimple artinian ring and T has essential socle with $Z(T_T) = Soc(T_T)$. Without loss of generality, we may assume that $T \neq 0$. By 4.3.2, T has no middle class, and it is an indecomposable ring by 4.3.4. Moreover R is a right artinian ring by 4.3.6.

Let E be an injective T-module. Because $f(Rad(E)) \leq Rad(E)$ for every $f \in End(E)$, $Rad(E)$ is a fully invariant submodule of its injective envelope, so that $Rad(E)$ is quasi-injective. Since R is right artinian, $Rad(E)$ is superfluous in E. Hence $Rad(E)$ is semisimple because R has no middle class. In particular, $Rad(E(T))$ is semisimple, so that $J(T) \leq Rad(E(T))$ is semisimple, that is, $J(T) \leq Soc(T_T)$. As $Z(T_T) = Soc(T_T)$, every simple right ideal is singular, and belongs to all maximal right ideals of R. This means $Soc(T_T) \leq J(T)$. Therefore $Z(T_T) = Soc(T_T) = J(T)$.

Now let S_1 be a simple right ideal of T. Since S_1 is singular, it is not a direct summand of T_T, and hence it can not be injective. Let S_2 be any non-injective simple T-module. Since T is right artinian, $Soc(\frac{E(S_2)}{S_2}) \neq 0$, so that we can find a submodule S_2' of $E(S_2)$ such that S_2 is maximal in S_2'. Since S_1 is non-injective and T has no middle class, S_1 is poor. Hence S_1 is not S_2'-injective, which implies that there exists a morphism $f : S_2' \to E(S_1)$ such that $f(S_2')$ is not contained in S_1. It follows that S_1 is properly contained in $f(S_2')$. Thus, the length of $f(S_2')$ is greater than 1. Moreover, S_2' has length 2. Hence $f(S_2')$ has length 2, so that f is a monomorphism. This gives that $S_2' \cong f(S_2')$, so that $S_2 = Soc(S_2') \cong Soc(f(S_2')) = S_1$. Therefore T has a unique non-injective simple module up to isomorphism. In particular, $Soc(T_T)$ is homogeneous. The last statement is now clear. ■

Lemma 4.3.9. *[ELS11, Lemma 8] Let R be a right nonsingular with no middle class. Then there is a ring direct sum $R = S \oplus T$ where S is a semisimple artinian and T is a ring with homogeneous (possibly zero) socle.*

PROOF. Assume that $Soc(R_R) = A \oplus B$ where A, B are infinitely generated orthogonal submodules. Then A and B are non-injective. Otherwise, either A or B is injective, so that one of them is a direct summand of R. This contradicts assumption that A and B are infinitely generated. Let f be a morphism from $E(A) \to E(B)$. Since A is orthogonal to B, B must be contained in $\operatorname{Ker} f$, so that Imf is singular. Since A is nonsingular, $E(A)$ is nonsingular, and hence Imf is nonsingular. Thus $Imf = 0$, that is, $f = 0$. Therefore A is $E(B)$-injective. Because R has no middle class and $E(B)$ is non-semisimple, A is injective, a contradiction. Similarly, we also obtain the two following facts: First, for any two non-isomorphic simple right ideals S_1 and S_2, at least one of them must be injective, because, by the same argument as above, each S_i is $E(S_j)$ for $i \neq j$ and $i, j \in \{1, 2\}$ if both of them are non-injective. And next, a simple right ideal S which is orthogonal to an infinitely generated semisimple right ideal I is injective, since S is $E(I)$-injective and $E(I)$ is not semisimple. Therefore $Soc(R_R)$ can only have finitely many homogeneous components, at most one of which may possibly be infinitely generated, in which case the rest of the homogeneous components will be injective. Now we have a ring decomposition $R = S \oplus T$ where S is a semisimple artinian ring and T is a ring with homogeneous (possibly zero) right socle. ■

Lemma 4.3.10. *[ELS11, Lemma 9] Let R be a right semiartinian. If R has no middle class, then R is either a right V-ring or a right artinian ring.*

PROOF. Assume R is not a right V-ring. We wish to prove that R is right artinan ring. In order to do this it is sufficient to show that R is right noetherian. Since R is not a right V-ring, then there is a non-injective simple module. Note that $Soc(\frac{E(S)}{S}) \neq 0$ because R is right semiartinian. Hence we can find a submodule S' of $E(S)$ such that $\frac{S'}{S}$ is simple. It is clear that S' is a module of length 2, so that S' is a non-semisimple noetherian module. Let $\{E_i | i \in I\}$ be any family of injective modules. Because S' is noetherian, then $\oplus_{i \in I} E_i$ is S'-injective. Since R has no middle class and S' is non-semisimple, we obtain that $\oplus_{i \in I} E_I$ is injective. This proves that R is right noetherian. ■

Lemma 4.3.11. *[ELS11, Lemma 10] Let R be a (non-semisimple) right SI-ring with $\frac{R}{Soc(R_R)}$ semisimple. If R has no middle class, then R has a unique simple singular R-module.*

PROOF. Since $\frac{R}{Soc(R_R)}$ is semisimple, $Soc_2(R_R) = Soc(R_R)$, so that R is right semiartinian. Hence $Soc(R_R)$ is essential in R_R. Now we can write $\frac{R}{Soc(R_R)} = \oplus_{i=1}^n \frac{B_i}{Soc(R_R)}$ for some right ideals B_i of R such that each $\frac{B_i}{Soc(R_R)}$ is simple. As R is not semisimple, then there exists a non-projective simple module V. It is clear that V is singular. Fix a non-zero element of V. Then $V = vR$ and $ann(v)$ contains $Soc(R_R)$ because it is essential in R_R. Thus $V = vR = \sum_{i=1}^n vB_i$ implies that $V = vB_i$ for some $i \in \{1, \ldots, n\}$. It follows that $V \cong \frac{B_i}{Soc(R_R)}$. Similarly, we can prove that every simple singular module is isomorphic to some $\frac{B_i}{Soc(R_R)}$. Let $i \neq j$. Then $B_j = aR + Soc(R_R)$ for some $a \in B_j$. Note that for all k, $E(R_R) = E(B_k)$ because B_k contains essential right socle of R_R. Since $Tr_{E(B_i)}(B_i)$ is quasi-injective, it is B_i-injective. Hence $Tr_{E(B_i)}(B_i)$ is not poor because B_i is not semisimple. This gives that $Tr_{E(B_i)}(B_i) =$

$E(B_i) = E(R_R)$. Thus there is an epimorphism $f : B_i^\Gamma \to E(R_R)$ for some index set Γ, so that there exists an element $x \in B_i^\Gamma$ such that $f(x) = a$. Therefore there exists a positive integer t and a morphism $\phi : B_i^t \to E(R_R)$ such that $aR \subseteq Im(\phi)$. Set $C = \phi^{-1}(aR)$. Since R is right SI, it is right hereditary by 1.6.4, so that aR is projective. It follows that aR is a direct summand of C, and hence aR can be embedded in C, whence in B_i^t as well. Without loss of generality, we may assume that $a = (b_1, \ldots, b_t)$, with $b_k \in B_k$ $(k = 1, \ldots, t)$. Since aR is non semisimple, there exists some $u \in \{1, \ldots, t\}$ with b_u not contained in $Soc(R_R)$. Hence $b_uR + Soc(R_R) = B_i$. Since $ann(a) \subset ann(b_u)$, we can defined an epimorphism $\pi : aR \to b_uR$ via $\pi(ax) = b_ux$. Because b_uR is projective, then we have $aR = \operatorname{Ker}\pi \oplus L$ for some $L \leq aR$. Note that $\frac{\operatorname{Ker}\pi}{Soc(\operatorname{Ker}\pi)} \oplus \frac{L}{Soc(L)} \cong \frac{\operatorname{Ker}\pi \oplus L}{Soc(\operatorname{Ker}\pi) \oplus Soc(L)} = \frac{aR}{Soc(aR)} \cong \frac{aR+Soc(R_R)}{Soc(R_R)} = \frac{B_j}{Soc(R_R)}$, and $L \cong b_uR$ is not semisimple. As $\frac{B_j}{Soc(R_R)}$ is simple, $\frac{\operatorname{Ker}\pi}{Soc(\operatorname{Ker}\pi)} = 0$, that is, $\operatorname{Ker}\pi$ is semisimple. Since $L \nsubseteq Soc(R_R)$, we have $L + Soc(R_R) = B_j$. Hence $\frac{B_j}{Soc(R_R)} \cong \frac{L}{Soc(L)} \cong \frac{b_uR}{Soc(b_uR)} \cong \frac{B_i}{Soc(R_R)}$. This completes the proof. ∎

Proposition 4.3.12. *[ELS11, Proposition 4] Let R be a right SI-ring with no middle class. Then $R = S \oplus T$ where S is semisimple artinian and either T is Morita equivalent to a right PCI-domain or T is an indecomposable right SI-ring satisfying the following conditions:*

(a) T is either a right artinian or a right V-ring,

(b) T has a homogeneous essential right socle, and

(c) there is a unique simple singular right T-module up to isomorphism, or

PROOF. By 4.3.9, we get that $R = S \oplus T$ where S is a semisimple artinian ring and T is a ring with homogeneous socle which may be zero. By 4.3.2, T has no middle class. We claim that T can not decompose into two non-semisimple artinian rings. If $T = T_1 \oplus T_2$ where T_i are ideals of T, and one of T_i is non-semisimple, say T_1, then every right ideal of T_2 is T_1-injective as a T-module. It follows that every right ideal of T_2 is injective, which splits in T_2. Hence T_2 is a semisimple artinian ring. This proves the claim.

Since R is right SI, so is T. Then, by 1.6.6 and since T can not decompose into two non-semisimple artinian rings, without loss of generality, we may assume that T is either Morita equivalent to a right PCI-domain, or $\frac{T}{Soc(T_T)}$ is a semisimple artinian ring with $Soc(T_T)$ essential in T_T. Now it remains to show that if $\frac{T}{Soc(T_T)}$ is a semisimple artinian ring with $Soc(T_T)$ essential in T_T, then T satisfies as in 4.3.13 (2). Now assume that $\frac{T}{Soc(T_T)}$ is a semisimple artinian ring with $Soc(T_T)$ essential in T_T. Then T is right semiartinian. Note that T is an indecomposable ring: Assume that $T = T_1 \oplus T_2$ where T_i are non-zero ideals of T. Since $Soc(T_T)$ is essential in $T_T, Soc(T_i)$ are essential in T_i. Hence, for each $i = 1, 2$, there is a simple right ideal V_i of T in T_i. But then $V_1T_2 = 0$ and $V_2T_2 = V_2$, which contradicts the fact that $Soc(T_T)$ is homogeneous. Moreover, by 4.3.10, T is either a right V-ring or a right artinian ring. To avoid triviality we may assume that T is non-semisimple artinian. Then, by 4.3.11, T has a unique simple singular module. ∎

Theorem 4.3.13. *[ELS11, Theorem 2] Let R be a ring with no middle class. Then $R \cong S \times T$, where S is a semisimple artinian ring and T is zero or it belongs to one of the three following classes:*

1. *T is Morita equivalent to a right PCI-domain, or*
2. *T is an indecomposable right SI-ring satisfying the following conditions:*
   (a) *T is either a right artinian or a right V-ring,*
   (b) *T has a homogeneous essential right socle, and*
   (c) *there is a unique simple singular right T-module up to isomorphism, or*
3. *T is an indecomposable right artinian ring satisfying the following conditions:*
   (a) *$soc(T_T) = Z(T_T) = J(T)$,*
   (b) *T has homogeneous right socle, and*
   (c) *there is a unique non-injective simple right T-module up to isomorphism.*

In the third case, T is either a QF-ring with $J(T)^2 = 0$, or T_T is poor.

PROOF. It follows from 4.3.8 and 4.3.12. ∎

Proposition 4.3.14. *[ELS11, Proposition 5] Let R be a ring which is Morita equivalent to a right PCI-domain T, then R has no middle class.*

PROOF. We claim that T has no middle class. Let A be an arbitrary T-module. Assume that A is non-injective and A is B-injective where B is cyclic. Because A is non-injective, every submodule of B is not isomorphic to R_R. Hence every submodule of B is injective, which splits in B. This gives that B is semisimple. Therefore A is poor. This proves the claim.

Since R is Morita equivalent to T, there is a category equivalence $F : \text{Mod-}R \to \text{Mod-}T$. Let M be an arbitrary R-module. Assume that M is not poor. Then there exists a non-semisimple N such that M is N-injective. Hence $F(M)$ is $F(N)$-injective as a T-module and $F(N)$ is non-semisimple. Because T has no middle class, $F(M)$ is injective, so that M is injective. ∎

Proposition 4.3.15. *[ELS11, Proposition 6] Let R be a right artinian right SI-ring with homogeneous right socle and a unique local module of length 2 up to isomorphism. Then R has no middle class.*

PROOF. Since R is right artinian, we have a decomposition $R_R = e_1R \oplus \cdots \oplus e_kR \oplus f_1R \cdots \oplus f_nR$ where e_iR are isomorphic simple right ideals by the hypothesis that R has homogeneous socle, and f_jR are local modules of length ≥ 2. Because R is right artinian right SI, for each $t \in \{1, \ldots, n\}$, $Soc(f_tR)$ is essential in f_tR and is contained in $Rad(f_tR) = f_tJ(R)$. Thus $\frac{f_tJ(R)}{Soc(f_tR)}$ is singular, so that it is injective and splits in $\frac{f_tR}{Soc(f_tR)}$. This gives that $Soc(f_tR) = f_tJ(R)$. Now, for any $t, t' \in \{1, \dots, n\}$, we can find two right ideals $A_t \leq f_tR$ and $A_{t'} \leq f_{t'}R$ such that $\frac{f_tR}{A_t}$ and $fracf_{t'}RA_{t'}$ are modules of length 2. Then, by assumption, $\frac{f_tR}{A_t} \cong \frac{f_{t'}R}{A_{t'}}$, which implies that $\frac{f_tR}{f_tJ(R)} \cong \frac{f_{t'}R}{f_{t'}J(R)}$. This means that $f_tR \cong f_{t'}R$.

Now let M be an arbitrary module. Assume that M is not poor. Then M is A-injective for some non-semisimple cyclic module A. We will show that M is injective. Since A is cyclic, then there is a epimorphism φ from R to A. Then there is an index $i \in \{1, \dots, n\}$ such that $A' = \varphi(f_iR) \neq 0$. It is clear that A' is local. Now, as in the preceding paragraph, we can find a factor B of A' such that B has length 2. Assume that $Soc(f_1R) = S_1 \oplus \cdots \oplus S_l$ for some simple right ideals S_i. For each i, set $V_i = \oplus_{t \in \{1,\dots,l\} \setminus \{i\}} S_t$ (if $l = 1$, set $V_1 = 0$). Then $\cap_{i=1}^l V_i = 0$ and $\frac{f_1R}{V_i}$ has length 2 for every $i \in \{1, \dots, l\}$ because $Soc(f_1R) = f_1J(R)$ and f_1R is local. By assumption, we have $\frac{f_1R}{V_i} \cong B$ for every $i = 1, \dots, l$. Note that M is B-injective thanks to the fact that M is A-injective. It follows that M is $\oplus_{i=1}^l \frac{f_1R}{V_i}$ because $\oplus_{i=1}^l \frac{f_1R}{V_i} \cong B^l$. Moreover, f_1R can be embedded into $\oplus_{i=1}^l \frac{f_1R}{V_i}$. Hence M is f_1R-injective. Since $f_1R \cong f_tR$ for every $t \in \{1, \dots, n\}$ and e_iR $(i = 1, \dots, k)$ are simple, M is R-injective, that is, M is injective. This completes the proof. ∎

Proposition 4.3.16. *[ELS11, Proposition 7] Let R be a right artinian ring with unique (up to isomorphism) local module of length 2, and homogeneous $Soc(R_R) = J(R)$. Then R has no middle class. In particular, R is a ring of 4.3.13 (3).*

Proof. By the same way as in the proof of the previous one, we also conclude that R has no middle class. It remains to show that the last statement. By 4.3.13, there is a ring direct sum $R = S \oplus T$ where S and T are as described in 4.3.13. Assume that R and T are non-zero. Then R has two simple right ideals (one in S and one in T) with distinct annihilators, which contradicts the homogeneous socle assumption. Moreover, R is not semisimple artinian because there exists a local module of length 2. Hence $R = T$, and T is not semisimple artinian, so that T can not be Morita equivalent to a domain since T is right artinian but not semisimple. Since R is right artinian and $Soc(R_R) = J(R)$, every right maximal ideal of R is essential in R_R. It follows that every simple module is singular, and hence $Z(R_R) \neq 0$. This means that R is not right SI-ring. Therefore R must be as in 4.3.13 (3). ∎

Proposition 4.3.17. *[ELS11, Proposition 9] If R is a non-semisimple QF-ring with homogeneous right socle and $J(R)^2 = 0$, then R has no right middle class.*

Proof. Since R is a QF ring and $J^2(R) = 0$, R is right artinian and $J(R)$ is semisimple. Now we have $R = \oplus_{i=1}^n e_iR$ with e_iR local for every $i = 1, \dots, n$. Because R is right self injective, e_iR is injective, so that e_iR is uniform for every $i = 1, \dots, n$. Hence, for each $i \in \{1, \dots, n\}$, the socle of e_iR is an essential simple submodule of e_iR. Since R has homogeneous right socle, $Soc(e_iR) \cong Soc(e_jR)$ for every $i, j \in \{1, \dots, n\}$, so that e_iR $(i = 1, \dots, n)$ are isomorphic modules. If $Rad(e_iR) = e_iJ(R) = 0$ for some $i \in \{1, \dots, n\}$, then e_iR is simple, which implies that R is semisimple artinian, a contradiction. Therefore $Rad(e_iR) \neq 0$ for every $i \in \{1, \dots, n\}$.

As $J(R) = \oplus_{i=1}^{n} Rad(e_iR)$ is semisimple, $0 \neq Rad(e_iR) \leq Soc(e_iR)$ for every $i = 1, \ldots, n$. Hence $Rad(e_iR) = Soc(e_iR)$ thanks to the fact that $Soc(e_iR)$ is simple. It follows that e_iR is an uniserial of length 2 for every $i = 1, \ldots, n$. This gives that R is an artinan serial ring.

Let M be an arbitrary module. Then $M = \oplus_{i \in I} M_i$ where M_i are cyclic uniserial. Each M_i is isomorphic to either e_tR or $Soc(e_tR)$ for some $t = 1, \ldots, n$. Assume that M is not injective. Then M_i is non-injective for some $i \in I$. We wish to show that M is poor. In order to prove that M is poor it suffices to show that M_i is poor. Suppose that M is not poor. Then there is a non-semisimple cyclic module N such that M_i is N-injective. We can write $N = \oplus_{k=1}^{m} N_k$, where N_k are uniserial modules, each isomorphic to e_tR or $Soc(e_tR)$. If $N_k \cong e_tR$, then M_i is e_tR-injective, which implies that M_i is injective, a contradiction. Therefore M is poor. ■

4.4 Rings over which every non-zero cyclic module is poor.

The results in this section are from the unpublished paper [ELT], which is presently being prepared for submission.

Lemma 4.4.1. *Let N be an essential submodule of a poor module M. Then N is poor.*

Proof. Since N is essential in M, $E(N)$ is also an injective envelope of M, we may assume that E is an injective envelope of M and N. Let xR be a cyclic module in the injectivity domain of N. Hence $\varphi(xR) \subseteq N$ for every $\varphi \in \mathrm{Hom}(xR, E(N))$. Therefore, $\varphi(xR) \subseteq N \subseteq M$ for every $\varphi \in \mathrm{Hom}(xR, E)$. It follows that xR belongs to the injectivity domain of M and thus xR is semisimple. ■

(P) stands for the property that every non-zero cyclic module is poor.

Proposition 4.4.2. *Let R be a ring with (P) and M be a cyclic R-module. Then every nonzero submodule of M is poor.*

Proof. Let K be a submodule of M. Then there is an R-module N such that $\frac{K \oplus N}{N}$ is essential in $\frac{M}{N}$. Since M is cyclic, so is $\frac{M}{N}$. By 4.4.1, we obtain that K is poor. ■

Lemma 4.4.3. *Let R be a ring. The following conditions are equivalent*

1. *There exists a nonzero nonsingular module.*
2. *$Z(R)$ is not essential in R.*
3. *There exists a nonzero module M such that $Z(M)$ is not essential in M.*

Proof. (1) $\Rightarrow$ (2): Assume M is a nonzero nonsingular module. Let $x \neq 0$ and $x \in M$. Thus, $\mathrm{Ann}(x)$ is not essential in R. Therefore, there exists a nonzero right ideal U of R such that $\mathrm{Ann}(x) \cap U = 0$. Since $U \cong \frac{U \oplus \mathrm{Ann}(x)}{\mathrm{Ann}(x)}$ and $xR \cong \frac{R}{\mathrm{Ann}(x)}$, U is nonzero nonsingular. It follows that $Z(R) \cap U = Z(U) = 0$ and hence, $Z(R)$ is not essential in R.

(2) $\Rightarrow$ (3): It is trivial.

(3) $\Rightarrow$ (1): Let M be as given in the Proposition. Since $Z(M)$ is not essential in M, there exists a nonzero submodule U such that $Z(M) \cap U = 0$. Hence U is non zero nonsingular because $Z(U) = Z(M) \cap U = 0$. ■

Lemma 4.4.4. *If R is a right SI-ring satisfying (P), then R is semisimple artinian.*

Proof. Assume that R is not semisimple artinian. Then there is a proper essential right ideal I of R_R. Hence, $\frac{R}{I}$ is a nonzero singular cyclic module. Therefore, it is an injective poor module so that R is semisimple artinian, a contradiction. ■

Theorem 4.4.5. *Let R be a ring with (P).Then either*

1. *R is semisimple artinian, or*
2. *R satisfies the following conditions:*
   a) *$Z(R_R)$ is essential in R.*
   b) *Every noetherian right module is artinian.*

Proof. Case 1 : $Z(R_R)$ is not essential in R_R. Assume that R is not semisimple artinian. Then R is not right SI-ring, so that there is a singular noninjective module S. By hypothesis, there exists a nonzero right ideal I such that $I \cap Z(R_R) = 0$, that is, I is nonsingular. Let $0 \neq x \in I$. Then $xR \leq I$ is a nonsingular poor module. Since S is singular and xR is nonsingular, xR is $E(S)$-injective, from which it follows that $E(S)$ is semisiple. Hence, $S = E(S)$, that is, S is injective, a contradiction. This proves that R is semisimple artinian.

Case 2 : $Z(R_R)$ is essential in R_R. Let M be a noetherian right module. In order to prove that M is artinian it suffices to show that M is semiartinian. We claim that every cyclic submodule of M is semiartinian. Let N be a cyclic submodule of M. Then N is noetherian. Assume that N is a non-semiartinian and I be the union of the socle series of N. Set $\bar{N} = \frac{N}{I}$. Then $\bar{N} \neq 0$ and $soc(\bar{N}) = 0$. We will show that there is a non-zero proper non-semiartinian factor K of $\bar{N}$. Note that K is isomorphic to a factor of N. Assume the contrary. Then every non-zero proper factor of $\bar{N}$ is semiartinian. Let $f \in Hom(K, E(\bar{N}))$ where K is a proper factor of $\bar{N}$. Hence $Imf \cong \frac{K}{\operatorname{Ker} f}$. Since $0 = soc(\bar{N}) = soc(E(\bar{N})) \cap \bar{N}$, we infer that $soc(E(\bar{N})) = 0$. It follows that $soc(Imf) = 0$. Thus $soc(\frac{K}{\operatorname{Ker} f}) = 0$. Because $\frac{K}{\operatorname{Ker} f}$ is a factor of $\bar{N}$, we get that $\frac{K}{\operatorname{Ker} f} = 0$ so that $f = 0$. Therefore $\bar{N}$ is K-injective, which implies that K is semisimple. Let V be a simple module. Then V is poor. Let N' be a submodule of $\bar{N}$ and $f : N' \to V$ be a non-zero morphism. Since $soc(N') = 0$, $\operatorname{Ker} f \neq 0$. As $\frac{\bar{N}}{\operatorname{Ker} f}$ is semisimple by the above argument, $\frac{\bar{N}}{\operatorname{Ker} f} = \frac{N'}{\operatorname{Ker} f} \oplus \frac{U}{\operatorname{Ker} f}$ for some submodule U of $\bar{N}$. Now let $g_1 : \bar{N} \to \frac{\bar{N}}{\operatorname{Ker} f}$, $g_2 : \frac{N'}{\operatorname{Ker} f} \oplus \frac{U}{\operatorname{Ker} f} \to \frac{N'}{\operatorname{Ker} f}$ be the canonical projections. Moreover, f induces an isomorphism $\bar{f} : \frac{N'}{\operatorname{Ker} f} \to V$. Then the morphism $\bar{f} g_2 g_1 : \bar{N} \to V$ extends f. Hence V is $\bar{N}$-injective so that $\bar{N}$ is non-zero semisimple, which contradicts the fact that $soc(\bar{N}) = 0$. This proves that there is a non-zero proper non-semiartinian factor K of $\bar{N}$. Note that K is isomorphic to $\frac{N}{K_1}$ where K_1 is a non-zero proper submodule of N. Repeating this argument with $\frac{N}{K_1}$ and so on, we have a strictly ascending chain $\{K_i/i \in \mathbb{N}\}$ of submodules of N, which contradicts the fact that N is noetherian. This proves the claim. Now it remains to show that M is semiartinian. Let $\frac{M'}{M}$ be a non-zero factor of M and $x + M'$ be a non-zero element of M/M'. Since $\frac{xR+M'}{M'} \cong \frac{xR}{M' \cap xR}$ and xR is semiartinian, we get that $soc(\frac{xR+M'}{M'}) \neq 0$. Hence $soc(\frac{M}{M'}) \neq 0$. It follows that M is semiartinian. This completes the proof. ■

Corollary 4.4.6. *If R is a simple ring with (P), then R is right artinian.*

Corollary 4.4.7. *A right noetherian ring R with (P) is right artinian.*

Proposition 4.4.8. *Let R be a ring with property (P). Then, either*

(i) R is right semiartinian, or

(ii) The only semiartinian (right) modules are the semisimple ones. In this case, $soc(\frac{M}{soc(M)}) = 0$ for any (right) module M, and the ring $\frac{R}{soc(R_R)}$ contains its right singular ideal essentially.

Proof. If R is not right semiartinian then there is a nonzero cyclic module A with $soc(A) = 0$. Then $soc(E(A)) = soc(A) = 0$. Let B be an arbitrary semiartinian module and $f \in Hom(B, E(A))$. Assume $f \neq 0$. Then $\frac{B}{\operatorname{Ker} f} \cong Imf$. Hence $soc(Imf) \neq 0$ because B is semiartinian. But this contradicts the fact that $soc(Imf) = soc(E(A)) \cap Imf = 0$. This gives $Hom(B, E(A)) = 0$. Therefore A is B-injective, so that B is semisimple artinian. It follows that every semiartinian module is semisimple. In this situation, since the second socle of any module M is semiartinian, we get $soc(\frac{M}{soc(M)}) = 0$, as desired. Taking $M = R$ in the preceding argument, we get that $\frac{R}{soc(R_R)}$ is not right semiartinian; 4.4.5 then yields the last part of (ii). ■

Proposition 4.4.9. *If R is a right semiartinian but nonsemisimple ring that satisfies (P), then R has a unique simple right R-module.*

Proof. Assume that A and B are nonisomorphic simple modules. If B were injective, it would be poor injective. Hence R would be semisimple artinian, a contradiction. Therefore B is not injective. By the semiartinianness assumption, there exists some $K \subseteq E(B)$ such that B is maximal in K. A is clearly K-injective because $Hom(B, A) = 0$ and K has only three submodules $0, B$ and K. Then K is semisimple, but then $B = K$ since $B \leq_e K \leq_e E(B)$. This contradicts the fact that B is maximal in K. This concludes the proof. ■

Proposition 4.4.10. *Let R be a ring satisfying (P) that is not right semiartinian. If M is a nonsemisimple module, then for any simple module S, there exists a sequence*

$$T_1 \subsetneq B_1 \subsetneq T_2 \subsetneq B_2 \subsetneq \ldots \subsetneq M$$

such that, for each $k \in \mathbb{N}$, $\frac{B_k}{T_k} \cong S$.

PROOF. Let M be as in the statement of the proposition. By Proposition 4.4.8, semiartinian right modules are semisimple and, in particular, $soc(\frac{M}{soc(M)}) = 0$. So, without loss of generality, we may assume that $soc(M) = 0$. Since S is poor, we can not have $Hom(D, S) = 0$ for all $D \subseteq M$. Thus, there exist modules $T_1 \subseteq B_1$ and a nonzero homomorphism $f : B_1 \to S$ such that $Ker(f) = T_1$, $B_1 \subseteq M$ and f can not be extended to any element of $Hom(M, S)$. This implies that the simple module $\frac{B_1}{T_1}$ is not a direct summand of $\frac{M}{T_1}$. Thus, $\frac{B_1}{T_1}$ is a superfluous submodule of $\frac{M}{T_1}$ and the latter is not a semisimple module. Moreover, $soc(\frac{\frac{M}{T_1}}{soc(\frac{M}{T_1})}) = 0$, whereas the module in the numerator is itself nonzero. Note that $\frac{B_1}{T_1} \subseteq soc(\frac{M}{T_1})$. So, we iterate the same argument as above for $\frac{\frac{M}{T_1}}{soc(\frac{M}{T_1})}$ and obtain some T_2 and B_2 such that $B_1 \subsetneq T_2 \subsetneq B_2 \subsetneq M$ and $\frac{B_2}{T_2} \cong S$. Continuing in this manner we build the sequence in the statement of this proposition.

4.4.8 and 4.4.10 show that a non-semiartinian ring with (P) is as far away from being semiartinian and Noetherian as possible. ■

Corollary 4.4.11. *If R is a non-right-semiartinian ring satisfying (P), then for any module M the following are equivalent:*

(i) M is noetherian,

(ii) M is finitely generated semiartinian (or artinian),

(iii) M is finitely generated semisimple.

The following simple lemma has some interesting consequences:

Lemma 4.4.12. *Let R be a ring satisfying (P). Then, for any ideal I of R such that $\frac{R}{I}$ is not semisimple Artinian and any nonzero right module B, there exists some $0 \neq C \subseteq B$ annihilated by I (i.e. $CI = 0$).*

PROOF. Since B is non-zero, there exists a non-zero cyclic submodule D of B. Then D is poor, so that D is not $\frac{R}{I}$-injective. Hence there exists an R-submodule $\frac{X}{I} \subseteq \frac{R}{I}$ and an R-homomorphism $f : \frac{X}{I} \to B$ that cannot be extended to any map $\frac{R}{I} \to B$. Then $C = f(\frac{X}{I})$ is the desired submodule of B. ■

Let R be a ring. A proper ideal I of R is called prime if for each $a, b \in R, aRb \subseteq I$ implies that $a \in I$ or $b \in I$, if and only if $AB \nsubseteq I$ whenever A and B are ideals of R not contained in I. The *prime radical* of R is the intersection of all prime ideals of R. In what follows, J and N will denote the Jacobson radical and the prime radical, respectively.

Proposition 4.4.13. Let R be a ring satisfying (P). Then the following hold:

(i) If $I_1, I_2, ..., I_n$ are ideals of R with $\frac{R}{I_k}$ nonsemisimple for each $k \in \{1, 2, ..., n\}$, then any nonzero module B has a nonzero submodule C such that $C(I_1 + I_2 + ... + I_n) = 0$.

(ii) If I_1, I_2, ..., I_n are ideals of R with $I_1 + I_2 + ... + I_n = R$, then $\frac{R}{I_k}$ is semisimple Artinian for some $k \in \{1, 2, ..., n\}$.

(iii) For any proper right ideal A and any (two-sided) ideal T of R with $\frac{R}{T}$ nonsemisimple, there is a right ideal B properly containing A such that $BT \subseteq A$.

(iv) Every ideal T for which $\frac{R}{T}$ is not semisimple Artinian is contained in the prime radical N of R.

(v) $J = N$ or R is semilocal. In particular, if R is semiprime, then either $J = 0$ or R is semilocal with $J = J^n$ for all $n \in \mathbb{N}$.

(vi) If R is not right semiartinian and T is the union of the socle series of R_R, then $soc(R_R) = T \subseteq N$.

(vii) If A and B are proper ideals with $A + B = R$, then $J \subseteq A, B$.

(viii) Every ideal of R is either below N or above J.

Proof. (i) By repeated application of Lemma 4.4.12, we can find a finite sequence $0 \neq C_n \subseteq C_{n-1} \subseteq ... \subseteq C_1 \subseteq B$ such that $C_k I_k = 0$ for each $k \in \{1, ..., n\}$. Then, $C_n I_k = 0$ for all $k \in \{1, ..., n\}$, implying that $C_n(I_1 + ... + I_k) = C_n R = 0$, a contradiction.

(ii) Assume the contrary. Then, by (i), there exists a nonzero C such that $C(I_1 + \cdots + I_n) = 0$. Hence $C = CR = C(I_1 + \cdots + I_n) = 0$, a contradiction.

(iii) Applying Lemma 4.4.12 to $\frac{R}{A}$, we get that there exists a right ideal B of R such that $0 \neq \frac{B}{A}$ and $(\frac{B}{A})T = 0$. It follows that B contains properly A and $BT \subseteq A$.

(iv) Let A be an arbitray prime ideal of R. Applying (iii) to A, we get that there exists a right ideal B containing properly A such that $BT \subseteq A$. This gives $T \subseteq A$ because A is prime and B contains properly A. Hence T is contained in the prime radical N of R.

(v) If R is not a semilocal ring, then $J \subseteq N$ by (iv). Hence $J = N$. In particular, if R is semiprime, then $N = 0$. Therefore either $J = 0$ or R is semilocal with $J \neq 0$. In the latter case, because R is semiprime and $J \neq 0$, we get that $J^n \neq 0$ for every $n \in \mathbb{N}$, so that $J^n \nsubseteq N$. By (iv), R/J^n must be semisimple artinian, which implies that $J \subseteq J^n$. It follows that $J = J^n$ for every $n \in \mathbb{N}$.

(vi) It follows from 4.4.8 and (iv).

(vii) Assume that $\frac{R}{A}$ is not semisimple artinian. Then, by (iv), the proper ideal A is contained in $N \subseteq J$, so that A is superfluous in R. Hence $A + B = R$ implies that $B = R$, a contradiction. This proves that $\frac{R}{A}$ is semisimple artinian. Therefore $J \subseteq A$. Similarly, $J \subseteq B$.

$(viii)$ Let A be an arbitrary ideal of R. If $\frac{R}{A}$ is not semisimple, then A is below N by (iv). Otherwise, $\frac{R}{A}$ is semisimple artinian, which implies that A is above J. ■

Lemma 4.4.14. *The property (P) is inherited by factor rings.*

Proof. Let M be a nonzero cyclic R/I-module. Then there is a right ideal K of R containing I such that M is isomorphic to R/K as R/I-module. We have that R/K is a poor module as R-module so that it is also a poor module as R/I-module. This completes the proof. ∎

Recall that an R-module M is said to be *uniserial* if for any submodules A and B of M we have $A \subseteq B$ or $B \subseteq A$. A ring R is a *right chain* in case R_R is uniserial. A *left chain* ring is defined similarly. A ring R is said to be a chain ring if it is both a right and a left chain ring.

Proposition 4.4.15. *If R is right noetherian with (P), then R is right artinian. Moreover, R is either semisimple artinian or isomorphic to a matrix ring over a local right artinian ring which is not a chain ring.*

Proof. By 4.4.11, R is right artinian. Hence $R_R = \oplus_{i=1}^{n} e_i R$ with $e_i (i = 1...n)$ local idempotents. Assume that R is not semisimple Artinian. Then, by 4.4.9, $e_j R \cong e_i R$. Hence $R \cong \mathrm{End}(R_R) \cong End((e_i R)^n) \cong M_n(End(e_i R))$. Set $S = End(e_i R)$. S is a local right artinian ring because e_i is a local idempotent and R is right Artinian which is Morita equivalent to S. If S is a chain ring, then S is a QF-ring, which implies that $R = M_n(S)$ is a QF-ring. Therefore R is semisimple Artinian because R has a poor injective module, namely R, a contradiction. ∎

Theorem 4.4.16. *Let R be a nonsemisimple ring satisfying the property (P). Then, R is an indecomposable ring such that*

(i) $Z(R_R)$ is essential in R_R, every Noetherian right R-module is Artinian, and

(ii) (a) $\frac{R}{J}$ is a simple Artinian ring and

(a_1) R is a right semiartinian but not Artinian ring, or

(a_2) $R \cong M_n(S)$, where S is a (nonuniserial) local right Artinian ring,

or

(b) R is not right semiartinian and the following conditions are equivalent for a right R-module M

(b_1) M is Noetherian,

(b_2) M is finitely generated semiartinian,

(b_3) M is Artinian,

(b_4) M is finitely generated semisimple,

and

(iii) Every ideal of R is either below the prime radical N or above the Jacobson radical J.

Proof. If $R = A \oplus B$, where A and B are non-zero ideals, then A and B are relatively injective cyclic R-modules, which by assumption of the condition (P), implies that R is semisimple artinian, a contradiction. Therefore, R must be indecomposable as a ring.

(i) follows from Theorem 4.4.5. Now assume R is right semiartinian. Then there is a unique simple R-module by Proposition 4.4.9. Hence there is a unique right primitive ideal, thus a unique maximal ideal, an dthey all coincide with $J(R)$. Therefore, $R/J(R)$ is a simple ring with the condition (P) by Lemma 4.4.14. It follows that $R/J(R)$ is a simple artinian by Corollary 4.4.6.

Now, if R is not right artinian, this yields part (a_1). Else, if it is right artinian, by uniqueness of the simple R-module, $R \cong (eR)^n$ for some primitive idempotent e and some $n \in \mathbb{N}$. This immediately yields that R is isomorphic to a matrix ring over a local right artinian ring, namely $M_n(eRe)$, yielding (a_2). With this, we have established $(ii)(a)$.

Part $(ii)(b)$ follows from Corollary 4.4.11, and part (iii) from Lemma 4.4.13. The proof of the theorem is now complete. ■

Proposition 4.4.17. *If R is a commutative Noetherian ring satisfying (P), then R is isomorphic to a finite direct product of fields.*

PROOF. To prove this proposition it is sufficient to show that R is semisimple Artinian. Assume that R is not semisimple Artinian. Then R is a commutative local Artinian ring thanks to the last proposition and the commutivity of R. Hence there is a right ideal A of R such that R/A is a local module of composition length 2. Note that $soc(R/A)$ is the only nonzero proper submodule of R/A. Let $f : soc(R/A) \to R/A$ be a nonzero morphism and $0 \neq x \in soc(R/A)$. Then $f(soc(R/A)) = soc(R/A)$ and there is an element $r \in R$ such that $f(x) = xr$. Since R is commutative, f extends to a morphism $\bar{f} : R/A \to R/A$ defined by $\bar{f}(y) = ry = yr$. Therefore R/A is quasi-injective, so that R/A is semisimple, a contradiction. ■

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