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Modular sheaves of de Rham classes on Hilbert formal modular schemes for unramified primes

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Sommario Adattiamo la nozione di fibrati vettoriali formali con sezioni marcate su schemi modulari di Hilbert e li usiamo per costruire fasci modulari con una connessione meromorfa integrabile che, in grado 0, interpola p-adicamente la filtrazione di Hodge usuale. Definiamo su tale fascio un operatore U_p e mostriamo il legame con il fascio delle forme modulari di Hilbert surconvergenti.

Abstract We define formal vector bundles with marked sections on Hilbert modular schemes and we show how to use them to construct modular sheaves with an integrable meromorphic connection and a filtration which, in degree 0, gives to us a p-adic interpolation of the usual Hodge filtration. We define an U_p-operator on this sheaf and relate it with the sheaf of overconvergent Hilbert modular forms.

1 Introduction

A quick historical recap

The story of geometric modular forms started with the work of Katz in [ModIII], aimed principally to create a unified geometric framework for various notions of modular forms (e.g. Serre's and overconvergent modular forms) of integral weight. The main idea was to construct compactifications of suitable moduli spaces for elliptic curves and use forms on the universal object to define a sheaf whose sections are indeed modular forms.

Coleman, in his paper [Col], addressed a conjecture of Gouvea predicting that everyoverconvergent modular form of integral weight and sufficently small slope is classical and the main tool in his research was the introduction of the sheaves

$$\mathcal{H}_k = \mathrm{Sym}^k_{\mathcal{O}_X} \mathbf{R}^1 \pi_* \Omega^\bullet_{E/X} \langle C \rangle$$

that is the degree k-th part of the symmetric algebra constructed on the first algebraic de Rham cohomology sheaf with log poles at the fibers above the cusps, where $\pi : E \to X$ is the generalized universal elliptic curve, toghether with the natural filtration induced by the Hodge filtration. Taking duals this definition is easily extended to all classical weights $k \in \mathbb{Z}$ thus providing a triple $(\mathcal{H}_k, \nabla_k, F^\bullet \mathcal{H}_k)$ for each $k \in \mathbb{Z}$, where ∇_κ is the Gauss-Manin connection on $\mathcal{H}_k$. He then accomplishes his task by means of a careful study of the cohomology of such triples.

In [AI] the authors generalized the construction of Coleman and defined p-adic families of de Rham cohomology classes, having as a main motivation the extension of the construction of triple product p-adic L-functions to the more general case of finite slope families of modular forms. Roughly, the idea is to p-adically interpolate the sheaves $\mathcal{H}_k$ where now k is a p-adic weight, that is a continuous character of $\mathbb{Z}_p^\times$. More precisely they introduce special open subsets $\mathcal{W}_I$ of the adic weight space and, using the theory of the canonical subgroup, define formal models $\mathfrak{X}_{r,I}$ of strict open neighbourhoods $\mathcal{X}_{r,I}$ of the ordinary locus in $X^{\mathrm{an}}_{\mathbb{Q}_p} \times \mathcal{W}_I$ and finite coverings $\mathfrak{IG}_{n,r,I} \to \mathfrak{X}_{r,I}$ over which the dual of the canonical subgroup of level n has a canonical generator. Using these data they construct a subsheaf $H^\#$ of $H^1_{\mathrm{dR}} (\mathfrak{E}/\mathfrak{IG}_{n,r,I})$ and a marked section (in the sense of [AI, Section 2]). The machinery of formal vector bundles with marked sections provides a sheaf $\mathbb{W}_\kappa$ in Banach modules over $\mathcal{X}_{r,I}$, together with a natural filtration $F^\bullet \mathbb{W}_\kappa$ in locally free coherent modules with the property that, for $k \in \mathbb{Z}$ (which means $k(x) = x^k$) we have an equality on the rigid analytic space

$$F^k \mathbb{W}_k = \mathcal{H}_k$$

and an integrable connection ∇ induced by the Gauss-Manin connection on H^1_{dR}. The interest in this construction is that ∇ can be p-adically interpolated: working with q-expansions, if s is a continuous character of $\mathbb{Z}_p^\times$ over a complete $\mathbb{Z}_p$-algebra R, define $d^s : R[[q]]^{U=0} \to R[[q]]^{U=0}$ as

$$d^s \left(\sum_{\substack{n \geq 1 \\ p \nmid n}} a_n q^n \right) = \sum_{\substack{n \geq 1 \\ p \nmid n}} s(n) a_n q^n.$$

Then, given another continuous character κ of $\mathbb{Z}_p^\times$, under some mild assumptions on κ and s, for every ω local section of $\mathbb{W}_\kappa$ they define

$$\nabla^s_\kappa (\omega) \in \mathbb{W}_{\kappa + 2s}$$

such that on q-expansions

$$\nabla_\kappa^s(\omega)(q) = d^s(\omega(q)). \tag{1.1}$$

This turns out to be the crucial point in the construction of the triple product p-adic L-functions for forms of finite slope.

Hilbert modular forms

Hilbert modular forms can be seen as a higher dimensional generalisation of (elliptic) modular forms, where the role of $\mathrm{GL}_2(\mathbb{Q})$ is played by $\mathrm{GL}_2(L)$ for a totally real number field L. From the geometric point of view this means that we need to consider the so-called *abelian varieties with real multiplication by* $\mathcal{O}_L$ instead of ellitic curves. In [TiXi] the authors extended the work of Coleman in the case of Hilbert modular forms, again with a classicity result in mind, and the purpose of this thesis is to build up the technical apparatus needed to extend the work of Andreatta-Iovita to the case of Hilbert modular forms. More precisely fix a totally real number field L, say of degree g over $\mathbb{Q}$, and let $N \geq 4$ and p be two coprime natural numbers with p prime. Finally let $\mathfrak{X}$ be the (p-adic formal scheme associated with a) smooth toroidal compactification of the moduli space of abelian schemes with real multiplication by $\mathcal{O}_L$ and μ_N-level structure. The advantage of working with toroidal compactifications lies in the fact that we have an universal semi-abelian object $\pi : \mathbf{A} \to \mathfrak{X}$ with an action of $\mathcal{O}_L$. Weights, in this setting, are locally analytic characters of the group $(\mathbb{Z}_p \otimes_\mathbb{Z} \mathcal{O}_L)^\times$ (we can actually consider all of them at once taking a universal character κ).

First we consider a slight variation of the construction of formal vector bundles with sections given in [AI] to keep track of the action of $\mathcal{O}_L$: pick a Galois closure L^{Gal} of L and let $d_L \in \mathbb{Z}$ be the discriminant of L, the main idea in this construction is that, if R is an $\mathcal{O}_{L^{\mathrm{Gal}}}\left[d_L^{-1}\right]$-algebra, then we have a ring isomorphism

$$\mathcal{O}_L \otimes_\mathbb{Z} R \to \prod_{\sigma \in \mathfrak{G}} R_\sigma,$$

given by $x \otimes 1 \mapsto (\sigma(x))_\sigma$, where $\mathfrak{G}$ is the set of embeddings $L \to L^{\mathrm{Gal}}$ and $R_\sigma = R$. Therefore for every $\mathcal{O}_L \otimes_\mathbb{Z} R$-module M we have a canonical decomposition as R-modules

$$M = \prod_{\sigma \in \mathfrak{G}} M(\sigma)$$

and we can construct formal vector bundles with $\mathcal{O}_L$-action by working for each σ separatedly.

Using the theory of canonical subgroups developped in [AIP2] we define formal models $\mathfrak{X}_r$ of the overconvergent rigid analytic neighborhoods as open subsets of particular admissible blow-ups of $\mathfrak{X}$ and finite coverings $\mathfrak{IG}_{n,r,I} \to \mathfrak{X}_{r,I}$ parametrising bases of the dual of the n-th canonical subgroup. Over these schemes we have locally free coherent $\mathcal{O}_{\mathfrak{IG}_{n,r,I}} \otimes_\mathbb{Z} \mathcal{O}_L$-modules $\Omega_\mathbf{A} \subseteq H^\#$ with a common marked section, that allow us, using the machinery explained above, to define sheaves of $\mathcal{O}_{\mathfrak{X}_r} \otimes_\mathbb{Z} \mathcal{O}_L$-modules $\mathbb{W}_\kappa$ and $\mathfrak{w}^\kappa$ such that (see Theorem 8.24 for a precise statement)

Theorem.

1. *The sheaf* $\mathbb{W}_\kappa$ *comes with a natual increasing filtration* $F^\bullet\mathbb{W}_\kappa$ *by locally free coherent* $\mathcal{O}_{\mathfrak{X}_r} \otimes_\mathbb{Z} \mathcal{O}_L$*-modules*

2. *$\mathbb{W}_\kappa$ is isomorphic to the completed limit $\widehat{\varinjlim} F^h \mathbb{W}_\kappa$ and the graded pieces are*

$$\mathrm{Gr}^h F^\bullet \mathbb{W}_\kappa \simeq \mathfrak{w}^\kappa \otimes_{\mathcal{O}_{\mathfrak{X}}} \underline{\omega}_{\mathbf{A}}^{-2h};$$

where $\underline{\omega}_{\mathbf{A}}$ is pullback of the universal object $\mathbf{A} \to \mathfrak{X}_r$ along the zero section;

3. *$F^0 \mathbb{W}_\kappa \simeq \mathfrak{w}^\kappa$ and its sections over the rigid analytic fibre $\mathcal{X}_r$ of $\mathfrak{X}_r$ are the overconvergent Hilbert modular forms as in [TiXi].*

Since the sheaf $\mathbb{W}_\kappa$ is constructed by means of the first de Rham cohomology of the morphism π, the theory of formal vector bundles with marked sections provides an integrable connection ∇_κ over $\mathfrak{IG}_{n,r,I}$, descending to a meromorphic integrable connection on $\mathfrak{X}_r$. However, on the analytic space $\mathcal{X}_r$ we have (see Proposition 8.26 for a precise statement)

Theorem. *On $\mathcal{X}_r$ the induced map on graded pieces*

$$\mathrm{Gr}^h F^\bullet \nabla_\kappa : \mathrm{Gr}^h F^\bullet \mathbb{W}_\kappa \to \mathrm{Gr}^{h+1} F^\bullet \mathbb{W}_\kappa \widehat{\otimes}_{\mathcal{O}_{\mathcal{X}_{r,I}}} \Omega^1_{\mathcal{X}_{r,I}}$$

is well-defined and it is the composition of an isomorphism and the product by an explicit element depending only on κ and h.

The sheaves $\mathbb{W}_\kappa$ on $\mathcal{X}_r$ come with a compact operator U_p on $H^i(\mathcal{X}_r, \mathbb{W}_\kappa)$. Using finite slope submodules taken with respect to U_p it is possibile to relate the cohomology modules $H^i(\mathcal{X}_r, \mathbb{W}_\kappa)$ to the simpler modules $H^i(\mathcal{X}_r, F^n \mathbb{W}_\kappa)$. More precisely (see Corollary 9.5)

Theorem 1.1. *Let $h \in \mathbb{Q}_{>0}$, then for m large enough we have*

$$H^i(\mathcal{X}_{r,I}, F^m \mathbb{W}_\kappa)^{(h)} = H^i(\mathcal{X}_{r,I}, \mathbb{W}_\kappa)^{(h)}$$

for every i.

Future perspectives

As said above, the goal of this thesis is to provide the technical tools necessary to extend the work of Andreatta-Iovita [AI] to the case of Hilbert modular forms, hence let me briefly explain two important applications.

De Rham cohomology and cusp forms

Using the results in Section 8.4 it is possible to construct the de Rham complex

$$\mathbb{W}_\kappa^\bullet : 0 \to \mathbb{W}_\kappa \to \mathbb{W}_\kappa \widehat{\otimes}_{\mathcal{O}_{\mathcal{X}_r}} \Omega^1_{\mathcal{X}_r} \to \cdots \to \mathbb{W}_\kappa \widehat{\otimes}_{\mathcal{O}_{\mathcal{X}_r}} \Omega^g_{\mathcal{X}_r} \to 0.$$

- In the elliptic case [AI, Corollary 3.35, pag. 42] it is shown, using sheaf cohomology arguments, that over an open subset $\mathcal{U}$ of the weight space given by removing a finite number of *classical* points there is an isomorphism

$$H^1_{\mathrm{dR}}(\mathcal{X}_r, \mathbb{W}_k^\bullet)^{(h)} \otimes \mathcal{O}(\mathcal{U}) \simeq H^0\left(\mathcal{X}_r, \mathfrak{w}^{k+2}\right)^{(h)} \otimes \mathcal{O}(\mathcal{U})$$

  where $\mathcal{U}$ depends on h and the slope decomposition is taken with respect to the action of U_p.

- In the Hilbert case, in [TiXi, Theorem 3.5, pag. 95]it shown that there is a sujective map

$$S_{\mathbf{k}}^{\dagger} \to H_{\mathrm{rig}}^{g}\left(\mathcal{X}_r^{\mathrm{ord}}, D, \mathcal{F}_{\mathbf{k}}\right)$$

  whose kernel can be described by means of a certain differential operator Θ, where $\mathbf{k}$ is a cohomological classical multiweight, $S^{\dagger}$ denotes the module of overconvergent cusp forms, D is the boundary divisor for a fixed toroidal compactification and $\mathcal{F}_{\mathbf{k}}$ is a sheaf similat to our $\mathbb{W}_{\mathbf{k}}$.

These two results suggest that it will be possible to relate the de Rham cohomology $H_{\mathrm{dR}}^{g}(\mathcal{X}_r, \mathbb{W}_k)$, or some finite slope cusp submodule thereof, with the space of Hilbert cusp forms. The initial step for this task will be, following [TiXi], the study of the dual BGG complex of $\mathbb{W}_\kappa$ in the spirit of [TiXi, Section 2.15, pag. 90]. Theorem 1.1 will also be of great help in reducing the problem to the study of the coherent locally free subcomplex

$$F^n\mathbb{W}_\kappa^\bullet : 0 \to F^n\mathbb{W}_\kappa \to F^{n+1}\mathbb{W}_\kappa \widehat{\otimes}_{\mathcal{O}_{\mathcal{X}_r}} \Omega^1_{\mathcal{X}_r} \to \cdots \to F^{n+g}\mathbb{W}_\kappa \widehat{\otimes}_{\mathcal{O}_{\mathcal{X}_r}} \Omega^g_{\mathcal{X}_r} \to 0$$

of $\mathbb{W}_\kappa^\bullet$.

Iteration of the Gauss-Manin connection and construction of triple product p-adic L-functions for finite slope Hilbert modular forms for unramified primes

The main result in [AI], as explained above, is the construction of triple product p-adic L-functions in the more general case of finite slopes elliptic modular forms instead of ordinary ones, and the extension of this result to the case of Hilbert modular forms is the natural application of the work carried out in this thesis. The main technical obstacle is the construction of iterations of the Gauss-Manin connection introduced in Section 8.4, that is of sections as in (1.1) under suitable conditions on the weights s and k.

This work will require the definition and the study of the notion of *nearly overconvergent Hilbert modular forms* rely either on computations using q-expansions ([AI]) or in terms of Serre-Tate coordinates ([Fan] or a recent unpublished work of S. Molina).

Contents

2 Hilbert modular schemes

Fix a totally real number field L with $[L:\mathbb{Q}] = g > 1$, let $\mathcal{O}_L$ be its ring of integers, $\mathcal{D}_L$ be its different ideal, where

$$\mathcal{D}_L^{-1} = \left\{ x \in L \mid \operatorname{Tr}_{L/\mathbb{Q}}(xy) \in \mathbb{Z}\ \forall y \in \mathcal{O}_L \right\},$$

and $d_L = \mathrm{N}_{L|\mathbb{Q}}\left(\mathcal{D}_L^{-1}\right)$ its discriminant. Denote

$$\Sigma_L = \{\text{embeddings } \sigma : L \to \mathbb{R}\}$$

If $S \subseteq L$ is any subset we denote

$$S^+ = \{s \in S \mid \sigma(s) > 0\ \ \forall \sigma \in \Sigma_L\}$$

and $\mathrm{Cl}(L)^+$ as the group of fractional ideals of L modulo the relation

$$I \sim J \quad \text{if and only if} \quad I = \lambda J \text{ for some } \lambda \in L^+.$$

We have an exact equence

$$1 \to \frac{L^\times}{\mathcal{O}_L^\times L^+} \to \mathrm{Cl}(L)^+ \to \mathrm{Cl}(L) \to 1.$$

Definition 2.1. An oriented $\mathcal{O}_L$-module is an invertible $\mathcal{O}_L$-module M together with an orientation of $M \otimes_\sigma \mathbb{R}$ for every $\sigma \in \Sigma_L$ (hence g choices), hence a choice of a "positive side". When such a notion is given, we denote with $M^+ \subseteq M$ the subset of totally positive elements. A morphism $\phi : (M, M^+) \to (N, N^+)$ of such objects is an $\mathcal{O}_L$-linear map $\phi : M \to N$ with $\phi(M^+) \subseteq N^+$. Let $\mathrm{Pos}_{\mathcal{O}_L}$ be the category just defined.

Example 2.2. The most immediate example is given by fractional ideals of L: if $\mathfrak{a}$ is such an ideal, it is a projective $\mathcal{O}_L$-module, necessarily of rank 1 and $\mathfrak{a}^+$ is the actual set of totally positive elements as defined above.

Lemma 2.3. *The tensor product $\otimes_{\mathcal{O}_L}$ makes the skeleton category of $\mathrm{Pol}_{\mathcal{O}_L}$ into an abelian group, still denoted $\mathrm{Pol}_{\mathcal{O}_L}$. Moreover the usual isomorphism between the group of invertible $\mathcal{O}_L$-modules and that of fractional ideal induces $\mathrm{Pol}_{\mathcal{O}_L} \simeq \mathrm{Cl}(L)^+$.*

Proof. See the argument sketched in [Gor, pag. 50] □

Example 2.4. This is the key example of an oriented $\mathcal{O}_L$-module we'll be dealing with. Let $A \to S$ be an abelian scheme. Suppose we have an injective ring map

$$[\bullet] : \mathcal{O}_L \hookrightarrow \mathrm{End}_S(A)$$

denoted $r \mapsto [r]$. Then we can see A as a functor from S-schemes to $\mathcal{O}_L$-modules. By duality[1] we have an action

$$[\bullet]^\vee : \mathcal{O}_L \hookrightarrow \mathrm{End}_S(A^\vee)$$

[1] The dual exists. Skim the section *Digressive Discussion About Representability of the Picard Functor of an Abelian Scheme A/S* in [FaCh, Chapter 1, pag. 2]

where

$$[r]^{\vee}(\mathcal{L}) = [r]^{*}\mathcal{L}$$

for every line bundle $\mathcal{L}$ on A and $r \in \mathcal{O}_L$. It follows that it makes sense to talk about $\mathcal{O}_L$-linear maps $\lambda : A \to A^{\vee}$ hence we can define

$$\mathcal{M}_A = \{\lambda : A \to A^{\vee} \mid \lambda \text{ is symmetric and } \mathcal{O}_L\text{-linear}\}.$$

Of course, for an S-scheme T, polarisations $\lambda : A \times_S T \to A^{\vee} \times_S T$ provide elements of $\mathcal{M}_A(T)$. It can be shown (see [Rap, Proposition 1.10, pag. 6]) that as long as T is a the spectrum of a field, the set $\mathcal{M}_A(T)$ always contains at least one polarisation. Even more, the functor $\mathcal{M}_A$ is an étale sheaf in $\mathcal{O}_L$-modules on S which is locally constant having as value an object of $\mathrm{Pos}_{\mathcal{O}_L}$ where the totally positive elements $\mathcal{M}_A^{+}$ are given by the polarisations. If moreover S is normal and connected, then $\mathcal{M}_A$ is actually constant. These are [Rap, Proposition 1.17, Variante 1.18]. Note also that we have a natural map

$$A \otimes_{\mathcal{O}_L} \mathcal{M}_A \to A^{\vee} \tag{2.1}$$

given on points by $x \otimes \lambda \mapsto \lambda(x)$.

2.1 Abelian schemes with real multiplication

Definition 2.5. Let $\pi : A \to S$ be an abelian scheme of relative dimension g and fix a fractional ideal $\mathfrak{c}$ of L

1. We say that π has real multiplication by $\mathcal{O}_L$ (we'll say that π is an $\mathrm{RM}_{\mathcal{O}_L}$.abelian scheme for short, or even an RM-scheme when $\mathcal{O}_L$ is clear from the context) if there exists an injective ring homomorphism $\mathcal{O}_L \to \mathrm{End}_S(A)$. For $r \in \mathcal{O}_L$ we'll usually denote the corresponding endomorphism with $[r]$.

2. If π is an RM-scheme, we say that it satisfies condition (R) if the conormal sheaf $\omega_{A/S}$, which is a locally free $\mathcal{O}_S$-module of rank g, is locally free as on $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathcal{O}_S$-module with rank 1.

3. If π is an RM-scheme, we say that it satisfies condition (DP) if the map (2.1) is an isomorphism.

It is natural to guess the relations between these notions.

Lemma 2.6. *Let k be a field and let $R = \mathcal{O}_L \otimes_{\mathbb{Z}} k$. Let M be an R-module and suppose that M has dimension g as a k-vector space. Then M is a free R-module (necessarily of rank 1) if and only if it is faithful. In particular M is a free R-module if and only if $M \otimes_k \overline{k}$ is a free $R \otimes_k \overline{k}$-module.*

Proof. This is the content of the proof of [Rap, Proposition 1.4, pag. 5]. □

Proposition 2.7. *An* RM-*scheme over a field of characteristic 0 always satisfies condition (R).*

Proof. By Lemma 2.6 we can suppose the field is alebraically closed and by the Lefschetz principle we can suppose it is the field of complex numbers, where the statement is [Gor, Corollary 2.6, pag. 53]. □

Theorem 2.8. *Let $\pi : A \to S$ be an* RM-*scheme. If π satisfies condition (R), then it also satisfies condition (DP).*

Proof. This is the first part of [Gor, Lemma 5.5, pag. 99]. The trick here is to show that for every prime ℓ, π has an isogeny of degree prime to ℓ. $\square$

Theorem 2.9. *Let $\pi : A \to S$ be an* RM*-scheme. If π satisfies condition (DP) and d_L is invertible on S, then it also satisfies condition (R).*

Proof. Suppose (DP) holds for π. Since we know that $\omega_{A/S}$ is a locally free $\mathcal{O}_S$-module, we reduce to the case when $S = \operatorname{Spec}(k)$ with k a field. The case when k has characteristic 0 is trivial in view of Lemma 2.7. Hence suppose k has characteristic $p > 0$. Since d_L is invertible on S we have that the ring $k \otimes_{\mathbb{Z}} \mathcal{O}_L$ is semi-simple and this implies the group $K_0 (k \otimes_{\mathbb{Z}} \mathcal{O}_L)$ is free generated by the classes of simple modules (this is also called *Devissage Theorem for* K_0, see for example [Ros, Theorem 3.1.8, pag. 117]), hence we just need to see that $\omega_{A/k}$ has the same class as $k \otimes_{\mathbb{Z}} \mathcal{O}_L$ in $K_0 (k \otimes_{\mathbb{Z}} \mathcal{O}_L)$. Looking at the proof of [DePa, Proposition 2.7, pag. 65], one sees that it works verbatim with H^1_{dR} instead of H_1^{dR}, showing that

$$\left[H^1_{\mathrm{dR}} (A/k)\right] = 2 \left[H^0 \left(A, \Omega^1_{A/k}\right)\right]$$

and we conclude since, in view of [Rap, Lemme 1.3, pag. 4], the $k \otimes_{\mathbb{Z}} \mathcal{O}_L$-module $H^1_{\mathrm{dR}} (A/k)$ is free of rank 2 and the group $K_0 (k \otimes_{\mathbb{Z}} \mathcal{O}_L)$ is free, hence it has no 2-torsion. $\square$

An important fact is the following

Theorem 2.10. *Let k be a field of characteristic $p > 0$ and let $A_{/k}$ be an ordinary abelian variety with faithful multiplication by $\mathcal{O}_L$. Then $A_{/k}$ satisfies (R).*

Proof. In view of Lemma 2.6, we can suppose that k is algebraically closed. Let $\mathfrak{A}_{/W(k)}$ be its canonical lift: it inherits a faithful multiplication by $\mathcal{O}_L$ as well as its general fibre $\mathfrak{A}_K = \mathfrak{A} \otimes_{W(k)} K$, so that we conclude in view of Lemma 2.7. $\square$

In order to deal with better moduli schemes we need to introduce some more constraints:

Definition 2.11. Let N be a positive integer, $\mathfrak{c}$ be a fractional ideal of L and let $\pi : A \to S$ be an RM-scheme.

1. A $\mathfrak{c}$-polarisation on π is an isomorphism of étale sheaves in oriented $\mathcal{O}_L$-modules on S
$$\lambda : \mathfrak{c} \to \mathcal{M}_A.$$
2. A μ_N-level structure on π (also called a $\Gamma_{00}(N)$-level structure) is the datum of an $\mathcal{O}_L$-linear embedding of S-group schemes
$$i : \mathcal{D}_L^{-1} \otimes_{\mathbb{Z}} \mu_N \hookrightarrow A,$$
where
$$\left(\mathcal{D}_L^{-1} \otimes_{\mathbb{Z}} \mu_N\right)(T) = \mathcal{D}_L^{-1} \otimes_{\mathbb{Z}} \mu_N (T).$$

Theorem 2.12. *Let $N \geq 4$ be an integer and $\mathfrak{c}$ be a fractional ideal of L. Consider the functor $\mathbf{M}(\mu_N, \mathfrak{c})$ which to any scheme S associates the set of* RM*-schemes that satisfy condition (DP), with $\mathfrak{c}$-polarisation and μ_N-level structure, modulo isomorphism. Then $\mathbf{M}(\mu_N, \mathfrak{c})$ is finely represented by a scheme $\mathbf{M}(\mu_N, \mathfrak{c}) = \mathbf{M}_N$ which is flat and relative complete intersection over $\mathbb{Z}$ and it is smooth over* $\operatorname{Spec}\left(\mathbb{Z}\left[d_L^{-1}\right]\right)$. *Let $\mathbf{A}_N = \mathbf{A}(\mu_N, \mathfrak{c}) \to \mathbf{M}_N$ be the universal object. If $p | d_L$ then the singular locus of $\mathbf{M}(\mu_N, \mathfrak{c}) \otimes \mathbb{F}_p$ has codimension 2, in particular $\mathbf{M}_N \to \operatorname{Spec}(\mathbb{Z})$ has normal fibres.*

Proof. The existence of $\mathbf{M}(\mu_N, \mathfrak{c}) \to \mathrm{Spec}(\mathbb{Z})$ as an algebraic stack follows from Artin's criterion as in [Rap, Théorème 1.20, pag. 11] (note that the level structure does not play any role there). One sees that $\mathbf{M}(\mu_N, \mathfrak{c}) \to \mathrm{Spec}(\mathbb{Z})$ is an algebraic space since the presence of the level structure rigidifies the moduli problem, see [Gor, Lemma 3.1, pag. 124], while the fact that it is a scheme comes from geometric invariant theory. The remaining statements are [DePa, Théorème 2.2, pag. 64] and [DePa, Corollaire 2.3, pag. 64]. $\square$

Corollary 2.13. *Let $N \geq 4$ and let k be a field of characteristic $p > 0$ with $p|N$. If*

$$s : \mathrm{Spec}(k) \to \mathbf{M}(\mu_N, \mathfrak{c})$$

is a k-rational point, then

$$\mathbf{A}_{N,s} \to \mathrm{Spec}(k)$$

is ordinary.

Proof. We have an isomorphism as group schemes

$$\mathcal{D}_L^{-1} \otimes \mu_N \simeq \mu_N^g$$

and hence, when $p|N$, the group scheme μ_p^g is a subgroup of $\mathcal{D}_L^{-1} \otimes \mu_N$. The scheme $\mathbf{A}_{N,s} \to \mathrm{Spec}(k)$ is an RM-scheme, hence it has dimension g. Note that

$$\mathrm{Hom}(\mu_p, \alpha_p) = \mathrm{Hom}\left(\mu_p, \underline{(\mathbb{Z}/p)}\right) = 0$$

hence the abelian scheme $\mathbf{A}_{N,s} \to \mathrm{Spec}(k)$ is ordinary. $\square$

Definition 2.14. Let $e_N : \mathbf{M}(\mu_N, \mathfrak{c}) \to \mathbf{A}_N$ be the unit section, define

$$\underline{\omega}(\mu_N, \mathfrak{c}) = \underline{\omega}_N = \underline{\omega}$$

as $e^*\Omega^1_{\mathbf{A}_N/\mathbf{M}(\mu_N,\mathfrak{c})}$.

The sheaf $\underline{\omega}_N$ is a locally free $\mathcal{O}_{\mathbf{M}_N}$-module of rank g and the property of being locally free as an $\mathcal{O}_{\mathbf{M}_N} \otimes_{\mathbb{Z}} \mathcal{O}_L$-module is open ([Rap, Remarque 1.2, ii, pag. 3]) hence the following definition makes sense.

Definition 2.15. Let $\mathbf{M}_N^R \subseteq \mathbf{M}_N$ be the open subscheme of points that correspond to RM-schemes with condition (R). This is called the Rapoport locus.

Remark 2.16. In view of Theorem 2.9, for every scheme $S \to \mathrm{Spec}\left(\mathbb{Z}\left[d_L^{-1}\right]\right)$, the map

$$\mathbf{M}_N^R \times_{\mathrm{Spec}(\mathbb{Z})} S \to \mathbf{M}_N \times_{\mathrm{Spec}(\mathbb{Z})} S$$

is an isomorphism, this holds in particular for $S = \mathrm{Spec}(\mathbb{Q})$. Moreover the morphism $\mathbf{M}_N^R \to \mathrm{Spec}(\mathbb{Z})$ is smooth.

2.2 Compactifications

2.2.1 Torus embeddings

Fix for this section a base scheme S. One can suppose $S = \mathrm{Spec}\,(\mathbb{Z})$, since this is the case in which we'll apply the construction.

Definition 2.17. Let G be a commutative S-group scheme. An action of G on an S-scheme T is a morphism

$$\rho : G \times_S T \to T$$

that satisfies the usual conditions for a group action, once suitably expressed in terms of commutative diagrams.

In particular the group multiplication $G \times_S G \to G$ gives an action of G on itself, called the translation action.

Definition 2.18. A torus embedding over S is a T-equivariant open immersion $T \hookrightarrow Z$ over S, where

- T is a split torus over S, i.e. isomorphic to $\mathbb{G}^n_{m|S}$ for some n;
- $Z \to S$ is a separated S-scheme with an action of T;

subject to the following conditions:

1. the T-action on Z extends the translation action of T on itself via the open embedding above;
2. for every point $s \in S$, the image of T_s in Z_s is dense (and open).

We can define morphisms between torus embeddings over S in the obvious way and hence end up with a category $\mathrm{TE}_{/S}$.

Let M be an oriented $\mathcal{O}_L$-module: recall that it is a free abelian group of rank g and let $M^\vee$ be the $\mathbb{Z}$-linear dual $\mathrm{Hom}_{\mathbb{Z}}\,(M, \mathbb{Z})$. The orientation on M induces an orientation on $M^\vee$ given by the orientation preserving $\mathbb{R}$-linear maps $M_{\mathbb{R}} \to \mathbb{R}$.

Definition 2.19. Let M be a free finitely generated abelian group. A rational polyhedral cone (r.p.c.) in $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ is a subset of the form

$$\sigma = \mathbb{R}_{>0} m_1 + \cdots + \mathbb{R}_{>0} m_t \quad m_i \in M \subseteq M \otimes_{\mathbb{Z}} \mathbb{R}.$$

A face of σ is a subset $\overline{\sigma} \cap H$ where H is an hyperplane in $M_{\mathbb{R}}$ which has empty intersection with σ. We say that σ is smooth if such m_i's can be taken to be part of a $\mathbb{Z}$-basis of M. We define the dual cone as

$$\sigma^\vee = \{ f \in M^\vee_{\mathbb{R}} \mid f\,(m) \geq 0 \ \forall m \in \sigma \}\,.$$

An r.p.c. decomposition in $M_{\mathbb{R}}$ is a mutually disjoint collection $C = \{\sigma_\alpha\}_\alpha$ of r.p.c.'s in $M_{\mathbb{R}}$ such that any face of σ_α is in C. The r.p.c. decomposition $C = \{\sigma_\alpha\}_\alpha$ is said to be generated by σ if it is given by σ and all its faces.

Lemma 2.20. *Let M be an oriented $\mathcal{O}_L$-module, then there exists a smooth r.p.c. decomposition $C = \{\sigma_\alpha\}_\alpha$ of $(M^\vee_{\mathbb{R}})^+ \cup \{0\}$ such that*

1. *C is invariant under the action of U_N^2, where*

$$U_N = \ker\left(\mathcal{O}_L^\times \to \left(\frac{\mathcal{O}_L}{N\mathcal{O}_L}\right)^\times\right)$$

and such action is free;

2. *the set C/U_N^2 is finite.*

Proof. This is [Rap, Lemme 4.2, pag. 45]. ☐

Proposition 2.21. *Let M be an oriented $\mathcal{O}_L$-module and let $C = \{\sigma_\alpha\}_\alpha$ be the r.p.c. decomposition generated by σin $M \otimes_{\mathbb{Z}} \mathbb{R}$. Set*

$$T = M^\vee \otimes_{\mathbb{Z}} \mathbb{G}_{m|S} = \mathbf{Spec}_S\left(\mathcal{O}_S\left[M^\vee\right]\right)$$

$$Z_\sigma = \mathbf{Spec}_S\left(\mathcal{O}_S\left[M^\vee \cap \sigma^\vee\right]\right),$$

then

1. *T is a g-dimensional split torus over S;*

2. *the map $T \to Z$ is a torus embedding, called the torus embeding associated with σ.*

Moreover, if τ is a face of σ, then Z_τ is naturally identified with an open subscheme of Z_σ and hence, if $D = \{\varsigma_\beta\}$ is any r.p.c. decomposition in $M_{\mathbb{R}}$, then we have a torus embedding $T \to Z(D)$ where $Z(D)$ is the S-scheme obtained by patching the Z_{ς_β} along the open subschemes corresponding to the faces.

Proof. This is part of [FaCh, Theorem 2.5, pag.100]. ☐

2.2.2 Toroidal and minimal compactifications

Fix an oriented invertible $\mathcal{O}_L$-module $\mathfrak{c}$, let $\mathfrak{c}^+$ denote the resulting totally positive elements.

Definition 2.22. A cusp C of $\mathbf{M}_N$ is the data of

1. a pair of invertible $\mathcal{O}_L$-modules $(\mathfrak{a}_C, \mathfrak{b}_C)$;

2. an isomorphism $\beta_C : \mathfrak{b}_C^{-1}\mathfrak{a}_C \to \mathfrak{c}$;

3. an exact sequence of $\mathcal{O}_L/N$-modules

$$\mathcal{E}_C : \quad 0 \to \mathcal{D}_L^{-1}\mathfrak{a}_C^{-1} \otimes_{\mathbb{Z}} \mu_N \to H_C \to \frac{N^{-1}\mathfrak{b}_C}{\mathfrak{b}_C} \to 0;$$

4. an injective $\mathbb{Z}$-linear map
$$\gamma_C : \mathcal{D}_L^{-1} \otimes_{\mathbb{Z}} \mu_N \to H_C.$$

Here $\mu_N = \mu_N(\mathbb{C})$. There is an obvious notion of isomorphism between two such sets (isomorphisms of ordered pairs respecting the additional data) and we identify two cusps which correspond under isomorphism.

Lemma 2.23. *Let M be an invertible $\mathcal{O}_L$-module, then there is a natural isomorphism*

$$\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z}) \simeq \mathrm{Hom}_{\mathcal{O}_L}\left(M,\mathcal{D}_L^{-1}\right).$$

In particular, if C is a cusp as in Definition 2.22 and $M = \mathfrak{a}_C\mathfrak{b}_C$, we have

$$M^\vee := \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z}) \simeq \mathfrak{c}\mathfrak{a}_C^{-2}\mathcal{D}_L^{-1}.$$

Fix now a cusp C and let $M = \mathfrak{a}_C\mathfrak{b}_C$. Set

$$R_N^0 = \mathbb{Z}\left[N^{-1}\cdot M\right] = \frac{\mathbb{Z}\left[(X_m)_{m\in N^{-1}\cdot M}\right]}{I_{N^{-1}\cdot M}}$$

where $I_{N^{-1}\cdot M}$ is the ideal generated by $X_0 - 1$ and

$$\left\{X_m\cdot X_n - X_{m+n} \mid m,n\in N^{-1}\cdot M\right\},$$

and $S_N^0 = \mathrm{Spec}\left(R_N^0\right)$. This is the split $\mathbb{Z}$-torus having $N^{-1}\cdot M$ as group of characters. Fix now a smooth r.p.c. decomposition $\underline{\sigma} = \{\sigma_\alpha\}_\alpha$ of $(M_{\mathbb{R}}^\vee)^+ \cup \{0\}$ as in Lemma 2.20. In view of Proposition 2.21 we get a torus embedding $S_N^0 \hookrightarrow S_N(\underline{\sigma})$ corresponding to such decomposition. Let $\widehat{S_N}(\underline{\sigma})$ be the completion of $S_N(\underline{\sigma})$ along the (reduced) closed subscheme $S_N^\infty = S_N(\underline{\sigma})\setminus S_N^0$. Denote with $S_{\sigma_\alpha}^\wedge = \mathrm{Spf}(R_{\sigma_\alpha})$ the induced formal affine covering and $\overline{S}_{\sigma_\alpha} = \mathrm{Spec}(R_{\sigma_\alpha})$. The action of U_N^2 on $\underline{\sigma}$ induces an action on $\widehat{S_N}(\underline{\sigma})$. Since the action is free, the resulting map

$$\widehat{S_N}(\underline{\sigma}) \to \frac{\widehat{S_N}(\underline{\sigma})}{U_N^2}$$

is formally étale and the finiteness of $\underline{\sigma}/U_N^2$ ensures that the support of $\widehat{S_N}(\underline{\sigma})/U_N^2$ is of finite type over $\mathbb{Z}$ [Rap, Lemme 4.5, pag. 47] and since $\underline{\sigma}$ is smooth one sees that R_{σ_α} is formally smooth over $\mathbb{Z}$ [Rap, Corollaire 5.3, pag. 69].

We have a canonical way to construct a semi-abelian scheme over $\mathrm{Spec}(R_{\sigma_\alpha})$ from these data: we have a $\mathbb{Z}$-linear map

$$\begin{aligned} M &\to R_{\sigma_\alpha}^\times = \mathbb{G}_m(\mathrm{Spec}(R_{\sigma_\alpha})) \\ m &\mapsto X_m \end{aligned}$$

which, in view of Lemma 2.23 gives an $\mathcal{O}_L$-linear map $M \to \mathbb{G}_m \otimes \mathcal{D}_L^{-1}$ and hence

$$q_{\sigma_\alpha} : \mathfrak{b}_C \to \mathbb{G}_m \otimes \mathcal{D}_L^{-1}\mathfrak{a}_C^{-1}.$$

Note again that $\mathbb{G}_m \otimes \mathcal{D}_L^{-1}\mathfrak{a}_C^{-1}$ is the split torus over $\mathrm{Spec}(R_{\sigma_\alpha})$ having $\mathcal{D}_L^{-1}\mathfrak{a}_C^{-1}$ as group of characters. In [Mum] the quotient semi-abelian scheme

$$\mathrm{Tate}_C^{\sigma_\alpha} = \frac{\mathbb{G}_m \otimes \mathcal{D}_L^{-1}\mathfrak{a}_C^{-1}}{q_{\sigma_\alpha}(\mathfrak{b}_C)}$$

is constructed. Note that it comes with a natural action of $\mathcal{O}_L$. One can show the following

Proposition 2.24. *Let $R^0_{\sigma_\alpha}$ be the quotient of R_{σ_α} by its topologically nilpotent elements, then* $\mathrm{Tate}^{\sigma_\alpha}_C \otimes_{R_{\sigma_\alpha}} R^0_{\sigma_\alpha}$ *is an* RM*-scheme that satisfies condition (R) and that comes with a natural $\mathfrak{c}$-polarisation. Moreover there is an exact sequence of finite flat group schemes over* $\mathrm{Spec}\left(R^0_{\sigma_\alpha}\right)$

$$0 \to \mathcal{D}_L^{-1}\mathfrak{a}_C^{-1} \otimes \mu_N \to \mathrm{Tate}^{\sigma_\alpha}_C[N] \to \frac{N^{-1}\mathfrak{b}_C}{\mathfrak{b}_C} \to 0.$$

In particular, through γ_C, every choice of an isomorphism $\mathcal{O}_L/N\mathcal{O}_L \simeq N^{-1}\mathfrak{a}_C^{-1}/\mathfrak{a}_C^{-1}$ gives a μ_N-level structure on $\mathrm{Tate}^{\sigma_\alpha}_C \otimes_{R_{\sigma_\alpha}} R^0_{\sigma_\alpha}$.

Proof. This is [Rap, Section 4]. See also [Kat, Chapter 1]. □

As we did before, we set

Theorem 2.25. *With notations as above, for every cusp C of $\mathbf{M}_N$ let $\underline{\sigma}^C = \left\{\sigma^C_\alpha\right\}_\alpha$ be a smooth r.p.c. decomposition of $\left((\mathfrak{a}_C\mathfrak{b}_C)^\vee_{\mathbb{R}}\right)^+ \cup \{0\}$ as in Lemma 2.20 such that the assignment $C \mapsto \underline{\sigma}^C$ is compatible with isomorphism of cusps, then*

1. *for every cusp C the above construction gives a morphism*

$$\theta_{\underline{\sigma}^C} : \coprod_\alpha \left(\mathrm{Spec}\left(R^0_{\sigma_\alpha}\right) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\xi_N, \frac{1}{N}\right]\right) \to \mathbf{M}_N \otimes_{\mathbb{Z}} \mathbb{Z}\left[\xi_N, \frac{1}{N}\right];$$

2. *there exists a proper and smooth scheme*

$$\mathbf{M}^{\mathrm{tor}}_N\left(\left(\underline{\sigma}^C\right)_C\right) \to \mathrm{Spec}\left(\mathbb{Z}\left[\frac{1}{N}\right]\right)$$

gotten by "attaching" the schemes Tate_C *to $\mathbf{M}_N \otimes_{\mathbb{Z}} \mathbb{Z}\left[\xi_N, \frac{1}{N}\right]$ via the maps $\theta_{\underline{\sigma}^C}$ and then descending from $\mathbb{Z}[\xi_N, 1/N]$ to $\mathbb{Z}[1/N]$;*

3. *there exists an open immersion*

$$j : \mathbf{M}_N \otimes_{\mathbb{Z}} \mathbb{Z}[1/N] \to \mathbf{M}^{\mathrm{tor}}_N\left(\left(\underline{\sigma}^C\right)_C\right)$$

and a semi-abelian scheme $\mathbf{A}^{\mathrm{tor}}_N \to \mathbf{M}^{\mathrm{tor}}_N\left(\left(\underline{\sigma}^C\right)_C\right)$ with $\mathcal{O}_L$-action such that

   (a) *the boundary $\mathbf{M}^{\mathrm{tor}}_N\left(\left(\underline{\sigma}^C\right)_C\right) \setminus j\left(\mathbf{M}_N \otimes_{\mathbb{Z}} \mathbb{Z}[1/N]\right)$ is a relative divisor with normal crossings;*

   (b) *the pullback of $\mathbf{A}^{\mathrm{tor}}_N$ via j is the universal family $\mathbf{A}_N \otimes_{\mathbb{Z}} \mathbb{Z}[1/N] \to \mathbf{M}_N \otimes_{\mathbb{Z}} \mathbb{Z}[1/N]$;*

   (c) *there exists an isomorphism*

$$\varphi : \coprod_{\text{classes of cusps } C} \left(\frac{\widehat{S_N\left(\underline{\sigma}^C\right)}}{U_N^2} \otimes_{\mathbb{Z}} \mathbb{Z}[1/N]\right) \to \mathbf{M}^{\mathrm{tor}}_N\left(\left(\underline{\sigma}^C\right)_C\right)^\wedge$$

   where $\mathbf{M}^{\mathrm{tor}}_N\left(\left(\underline{\sigma}^C\right)_C\right)^\wedge$ is the formal completion of $\mathbf{M}_N \otimes_{\mathbb{Z}} \mathbb{Z}[1/N]$ along its boundar and φ is compatible with the $\theta_{\underline{\sigma}^C}$'s.

Proof. The first part is clear in view of Proposition 2.24 and the rest follows verbatim as in [Rap, Théorème 5.1, pag. 65], [Rap, Proposition 5.2, pag. 68] and [Rap, Corollaire 5.3, pag. 69] over $\mathbb{Z}[\xi_N, 1/N]$. The descent from $\mathbb{Z}[\xi_N, 1/N]$ to $\mathbb{Z}[1/N]$ is described in [KiFL, 1.6.6, pag. 742]. See also the introduction of [DePa] and [KiFL, 1.7.1, pag. 743]. □

Definition 2.26. We call the morphism $\mathbf{M}_N^{\text{tor}} \to \operatorname{Spec}(\mathbb{Z}[1/N])$, together with the universal semi-abelian family $\mathbf{A}_N^{\text{tor}} \to \mathbf{M}_N^{\text{tor}}$a toroidal compactification of $\mathbf{A}_N \to \mathbf{M}_N$ (note that it does depend on the choice of the family $\left(\underline{\sigma}^C\right)_C$, even if we do not make it appearent through the notations). If e^{tor} is the unit section, we let

$$\underline{\omega}_{\mathbf{M}_N^{\text{tor}}} = \left(e^{\text{tor}}\right)^* \Omega^1_{\mathbf{A}_N^{\text{tor}}/\mathbf{M}_N^{\text{tor}}}.$$

Remark 2.27. The sheaf $\underline{\omega}_{\mathbf{M}_N^{\text{tor}}}$ is a locally free $\mathcal{O}_{\mathbf{M}_N^{\text{tor}}}$-module of rank g and we can recover the Rapoport locus as the open subset over which it is an invertible $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbf{M}_N^{\text{tor}}}$-module.

Proposition 2.28. *Let $G \to S$ be a semi-stable commutative group scheme which is abelian over an open dense subset of S and with S normal, excellent and noetherian. Let e denote its unit section and $\underline{\omega} = \det\left(e^*\Omega^1_{G/S}\right)$, then there exists m such that $\underline{\omega}^{\otimes m}$ is generated by global sections.*

Proof. This is [MoBa, Theorem 2.1, pag. 208]. □

Proposition 2.29. *Fix a smooth toroidal compactification $\mathbf{M}_N^{\text{tor}}$ of $\mathbf{M}_N$, then $\det\left(\underline{\omega}_{\mathbf{M}_N^{\text{tor}}}\right)^{\otimes m}$ is generated by global sections for some $m \geq 1$.*

Proof. The scheme $\mathbf{M}_N^{\text{tor}}$ is proper over $\operatorname{Spec}(\mathbb{Z}[1/N])$, in particular it's given by finitely generated $\mathbb{Z}$-algebras hence it is excellent and noetherian. It is normal since it is smooth over a regular scheme, hence $\mathbf{M}_N^{\text{tor}}$ itself is regular[2]. The rest is clear, since $\mathbf{A}_N^{\text{tor}}$ is abelian on $\mathbf{M}_N$ hence Proposition 2.28 applies. □

Note 2.30. In view of Proposition 2.29 we have a morphism $\phi_m : \mathbf{M}_N^{\text{tor}} \to \mathbb{P}^n$ induced by $\det\left(\underline{\omega}_{\mathbf{M}^{\text{tor}}}\right)^{\otimes m}$ and Stein factorisation gives a commutative diagram

$$\begin{array}{ccc} \mathbf{M}_N^{\text{tor}} & \xrightarrow{\ \phi_m\ } & \mathbb{P}^n_{\mathbb{Z}[1/N]} \\ & {\scriptstyle \overline{\pi}} \searrow \quad \nearrow {\scriptstyle \nu} & \\ & \mathbf{M}_N^{\min} & \end{array}$$

where $\overline{\pi}$ is proper with connected fibres, $\overline{\pi}_*\mathcal{O}_{\mathbf{M}_N^{\text{tor}}} = \mathcal{O}_{\mathbf{M}_N^{\min}}$ and ν is finite. One can show that there exists a very ample invertible sheaf L on $\mathbf{M}_N^{\min}$ such that $\overline{\pi}^*L = \det\left(\underline{\omega}_{\mathbf{M}^{\text{tor}}}\right)^{\otimes km}$ for some k, in particular

$$\mathbf{M}_N^{\min} = \operatorname{Proj}\left(\bigoplus_{k\geq 0} \Gamma\left(\mathbf{M}_N^{\text{tor}}, \det\left(\underline{\omega}_{\mathbf{M}_N^{\text{tor}}}\right)^{\otimes km}\right)\right)$$

can be shown to have the following properties

[2]This is contained in
Q.Liu, *Algebraic geometry and arithmetic curves* - Theorem 3.36, pag. 142

1. it does not depend on the smooth toroidal compactification $\mathbf{M}_N^{\mathrm{tor}}$ used to construct it: more precisely the restrictions

$$\Gamma\left(\mathbf{M}_N^{\mathrm{tor}}, \det\left(\underline{\omega}_{\mathbf{M}_N^{\mathrm{tor}}}\right)^{\otimes t}\right) \to \Gamma\left(\mathbf{M}_N, \det\left(\underline{\omega}_{\mathbf{M}_N}\right)^{\otimes t}\right)$$

are bijective (this is called Koecher principle, see [Rap, Proposition 4.9, pag. 49]). The proof of the mere independence from the decomposition is a more or less formal consequence of the properties of the toroidal compactification, see [FaCh, Remark 1.2 (b), pag. 137];

2. it is a projective normal scheme over Spec $([1/N])$ and the map $\overline{\pi}$ is surjective: see [Cha, Main Theorem (v), pag. 549];

3. there exists an open dense subscheme $\mathbf{M}_N^{\mathrm{min},*} \subseteq \mathbf{M}_N^{\mathrm{min}}$ such that

    (a) the restriction $\overline{\pi}_{|\mathbf{M}_N}$ factors through $\mathbf{M}_N^{\mathrm{min},*}$ (see [Cha, Main Theorem (vi), pag. 549]);

    (b) the reduced structure on $\mathbf{M}_N^{\mathrm{min}} \backslash \mathbf{M}_N^{\mathrm{min},*}$ is isomorphic to a disjoint union of copies of Spec $(\mathbb{Z}[1/N])$ naturaly in one-to-one correspondence with the cusps of $\mathbf{M}_N$, in particular it is finite étale over Spec $(\mathbb{Z}[1/N])$ (see [Cha, Main Theorem (vi), pag. 549] and [Cha, Main Theorem (vii), pag. 549].

Definition 2.31. For any toroidal compactification $\mathbf{M}_N^{\mathrm{tor}}$ of $\mathbf{M}_N$, we cal the minimal (or Satake, or Bailly-Borel-Satake) compactification of $\mathbf{M}_N$ the scheme $\mathbf{M}_N^{\mathrm{min}}$ as in Note 2.30 together with the canonical map $\overline{\pi} : \mathbf{M}_N^{\mathrm{tor}} \to \mathbf{M}_N^{\mathrm{min}}$.

3 Gauss-Manin connection

3.1 Connections

Let $X_{/S}$ be a scheme and $\mathcal{F}$ be an $\mathcal{O}_X$-module. An integrable connection on $\mathcal{F}$ (relative to S) is a map of abelian sheaves

$$\nabla : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$$

such that

- satisfies *Leibniz' rule*: that is for every open $U \subseteq X$ we have

$$\nabla(sf) = s\nabla(f) + f \otimes_{\mathcal{O}_X(U)} ds$$

  for $s \in \mathcal{O}_X(U)$ and $f \in \mathcal{F}(U)$. In particular ∇ is $\mathcal{O}_S$-linear;

- *integrability*: define

$$\nabla^{(i+1)} : \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^i_{X/S} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^{i+1}_{X/S}$$

  over an open $U \subseteq X$ by the formula

$$\nabla^{(i+1)}\left(f \otimes_{\mathcal{O}_X(U)} \omega\right) = (-1)^i \nabla(f) \wedge \omega + f \otimes_{\mathcal{O}_X(U)} d\omega$$

  where of course we denote with $\nabla(f) \wedge \omega$ the image of $\nabla(f) \otimes_{\mathcal{O}_X(U)} \omega$ via the exterior product map $\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_{X/S} \otimes_{\mathcal{O}_X} \Omega^i_{X/S} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^{i+1}_{X/S}$. Then we say that ∇ is integrable if $\nabla^{(i+1)} \circ \nabla^{(i)} = 0$ for every i. It is equivalent to the single condition $\nabla^{(2)} \circ \nabla = 0$.

Note 3.1. Given a scheme S and $\mathcal{O}_S$-modules $M_1, \ldots, M_n$ with connections ∇_i relative to some fixed morphism $S \to T$, define

$$\nabla_1 \otimes_{\mathcal{O}_S} \ldots \otimes_{\mathcal{O}_S} \nabla_n = \sum_{i=1}^{n} M_1 \otimes_{\mathcal{O}_S} \ldots \otimes_{\mathcal{O}_S} \nabla_i \otimes_{\mathcal{O}_S} \ldots \otimes_{\mathcal{O}_S} M_n$$

and, with a slight abuse of notation, consider it as a map

$$M_1 \otimes_{\mathcal{O}_S} \ldots \otimes_{\mathcal{O}_S} M_n \to M_1 \otimes_{\mathcal{O}_S} \ldots \otimes_{\mathcal{O}_S} M_n \otimes_{\mathcal{O}_S} \Omega^1_{S/T}.$$

Lemma 3.2. *Let $M_1, \ldots, M_n$ be $\mathcal{O}_S$-modules with connections $\nabla_1, \ldots, , \nabla_n$ relative to T. If all the ∇_i are integrable, then also $\nabla_1 \otimes \ldots \otimes \nabla_n$ is.*

Proof. It suffices to see it for $n = 2$, so consider two $\mathcal{O}_S$-modules M, N with integrable connections ∇_M, ∇_N. Write

$$\nabla_M(m) = \sum m_i \otimes \omega_i$$
$$\nabla_N(n) = \sum n_i \otimes \nu_i,$$

where we note that we can take the same set of indices in both summations. Integrability reads

$$(\nabla_M \cdot \nabla_M)(m) = \sum [\nabla_M(m_i) \otimes \omega_i - m_i \otimes d\omega_i] = 0$$

and the same for ∇_N. We compute

$$\begin{aligned}
[(\nabla_M \otimes \nabla_N) \cdot (\nabla_M \otimes \nabla_N)](m \otimes n) &= (\nabla_M \otimes \nabla_N)\left(\sum m_i \otimes n \otimes \omega_i + m \otimes n_i \otimes \nu_i\right) \\
&= \sum ((\nabla_M \otimes \nabla_N)(m_i \otimes n) \otimes \omega_i - m_i \otimes n \otimes d\omega_i \\
&\quad + (\nabla_M \otimes \nabla_N)(m \otimes n_i) \otimes \nu_i - m \otimes n_i \otimes d\nu_i) \\
&= \sum (\nabla_M(m_i) \otimes n \otimes \omega_i + m_i \otimes \nabla_N(n) \otimes \omega_i \\
&\quad - m_i \otimes n \otimes d\omega_i + \nabla_M(m) \otimes n_i \otimes \nu_i \\
&\quad + m \otimes \nabla_N(n_i) \otimes \nu_i - m \otimes n_i \otimes d\nu_i) \\
&= \sum (m_i \otimes \nabla_N(n) \otimes \omega_i + \nabla_M(m) \otimes n_i \otimes \nu_i) \\
&= \sum (m_i \otimes n_i \otimes \nu_i \wedge \omega_i + m_i \otimes n_i \otimes \omega_i \wedge \nu_i) \\
&= 0.
\end{aligned}$$

□

3.2 Grothendieck on connections

Here we briefly review Grothendieck's point of view about integrable connections. For more details see [**?**, Section 2].

Let $\mathcal{A} = \mathcal{A}_{X/S}$ be the $\mathcal{O}_X$-algebra $\mathcal{O}_X \oplus \Omega^1_{X/S}$ where the product is (locally) given by

$$(s, \omega) \cdot (t, \tau) = (st, s\tau + t\omega).$$

Note that $\Omega^1_{X/S} \subseteq \mathcal{A}$ is a square-zero ideal. We have two obvious ring maps $j_1, j_2 : \mathcal{O}_X \to \mathcal{A}$ given by

$$j_1(s) = (s, 0) \qquad j_2(s) = (s, ds)$$

and the natural quotient $\Delta : \mathcal{A} \to \mathcal{O}_X$.

Theorem 3.3. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module, then the data of an integrable connection ∇ on $\mathcal{F}$ (relative to S) is equivalent to the data of an $\mathcal{A}$-linear isomorphism*

$$\epsilon : \mathcal{F} \otimes_{j_2} \mathcal{A} \xrightarrow{\sim} \mathcal{F} \otimes_{j_1} \mathcal{A}$$

such that $\epsilon \otimes_\Delta \mathcal{O}_X = \mathrm{id}_{\mathcal{F}}$ and that satisfies a suitable cocycle condition (corresponding to integrability). The relation between these two notions is given by

$$\nabla(x) = \epsilon(x \otimes (1,0)) - x \otimes (1,0) \in \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}, \tag{3.1}$$

where we mean the image under the natural map $\mathcal{F} \otimes_{j_1} \mathcal{A} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$.

3.3 The Gauss-Manin connection

We briefly recall how the Gauss-Manin connection is defined in a general setting, see [KaOd]. Let

$$X \xrightarrow{f} Y \to S$$

be morphisms of schemes with f smooth, then we have an exact sequence

$$0 \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0$$

giving a filtration to the deRham complex $\Omega^\bullet_{X/S}$

$$F^p\Omega^\bullet_{X/S} = \mathrm{Im}\left(f^*\Omega^p_{Y/S} \otimes_{\mathcal{O}_X} \Omega^{\bullet-p}_{X/S} \to \Omega^\bullet_{X/S}\right).$$

Let $E^{\bullet,\bullet}_\bullet = E^{\bullet,\bullet}_\bullet\left(\Omega^\bullet_f\right)$ be the associated spectral sequence.

1. Define the Kodaira-Spencer map as the connecting map

$$\mathrm{KS}_f : f_*\Omega^1_{X/Y} \to R^1 f_* f^*\Omega^1_{Y/S} = R^1 f_*\mathcal{O}_X \otimes_{\mathcal{O}_X} \Omega^1_{Y/S}.$$

2. Define the Gauss-Manin connection ∇_{GM} as the boundary map

$$d_1^{0,q} : E_1^{0,q} = \mathbf{R}^q f_*\Omega^\bullet_{X/Y} \to \Omega^1_{Y/S} \otimes \mathbf{R}^q f_*\Omega^\bullet_{X/Y}.$$

   We denote the hypercohomology

$$H^q_{\mathrm{dR}}(X/Y) = \mathbf{R}^q f_*\Omega^\bullet_{X/Y}$$

   and call it the de Rham cohomology of the Y-scheme X.

Proposition 3.4. *The connection ∇_{GM} is integrable.*

Proof. See [KaOd]. □

4 Hilbert modular forms of classical weight

This section is essentially taken from [Kat]. Let L be a totally real number field, say with $[L:\mathbb{Q}] = g > 1$ and let p be a prime which is unramified in L, with $p\mathcal{O}_L = \mathfrak{p}_1 \dots \mathfrak{p}_d$.

Lemma 4.1. *There exist finite extension $\mathbb{Q}_p \subseteq F_i$ for $i = 1, \dots, d$ and a natural ring homomorphism*

$$\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p = \prod_{i=1}^{d} \mathcal{O}_{F_i}.$$

Moreover the extensions $\mathbb{Q}_p \subseteq F_i$ are finite and unramified, hence Galois. Let $K = F_1 \dots F_d$ denote the compositum, then $\mathbb{Q}_p \subseteq K$ is also finite and unramified, hence Galois.

Proof. The first statement is clear, where F_i is the $\mathfrak{p}_i$-adic completion of L. The fact that $\mathbb{Q}_p \subseteq F_i$ are unramified follows from the chain of natural isomorphisms

$$\begin{aligned}
\frac{\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p}{p \cdot \mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p} &= \frac{\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p}{\mathfrak{p}_1 \dots \mathfrak{p}_d \otimes_{\mathbb{Z}} \mathbb{Z}_p} \\
&= \frac{\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p}{\prod \mathfrak{p}_i \otimes_{\mathbb{Z}} \mathbb{Z}_p} \\
&= \prod \frac{\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p}{\mathfrak{p}_i \otimes_{\mathbb{Z}} \mathbb{Z}_p} \\
&= \prod \frac{\mathcal{O}_L}{\mathfrak{p}_i} \otimes_{\mathbb{Z}} \mathbb{Z}_p \\
&= \prod \frac{\mathcal{O}_L}{\mathfrak{p}_i}
\end{aligned}$$

and the fact that $\mathbb{Q}_p \subseteq K$ is unramified follows from [Neu, Proposition 7.2, pag. 153]. The next well known Lemma shows that they're also Galois. □

Lemma 4.2. *Let $\mathbb{Q}_p \subseteq K$ be a finite unramified extension, then it is Galois.*

Proof. Pick α such that $\mathcal{O}_K = \mathbb{Z}_p[\alpha]$ with irreducible polynomial $f \in \mathbb{Z}_p[X]$. We have $\mathbb{F}_K = \mathbb{F}_p(\overline{\alpha})$ and this is Galois with $[\mathbb{F}_K : \mathbb{F}_p] = [K:\mathbb{Q}_p]$, so that $\overline{f}$ splits completely in $\mathbb{F}_K$ and , by Hensel's Lemma, f splits completely in K. □

Let $N \geq 4$ be an integer not divisible by p and suppose that the fractional ideal $\mathfrak{c}$ is also coprime with p, then the base change $\mathbf{M} = \mathbf{M}_N \times \operatorname{Spec}(\mathcal{O}_K) \to \operatorname{Spec}(\mathcal{O}_K)$ is defined and

$$\mathcal{O}_L \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbf{M}} = \prod \mathcal{O}_{F_i} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbf{M}}.$$

Over $\mathbf{M}$ we have an exact sequence

$$0 \to \underline{\omega} \to H^1_{\mathrm{dR}} \to \underline{\omega}^\vee \to 0$$

of locally free $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbf{M}}$-modules, where $\underline{\omega}$ and $\underline{\omega}^\vee$ are invertible in view of Theorem 2.9. Since the sequence above is $\mathcal{O}_L \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbf{M}}$-linear, it splits as a product of sequences of $\mathcal{O}_{F_i} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbf{M}}$-modules

$$0 \to \underline{\omega}_i \to H^1_{\mathrm{dR},i} \to \underline{\omega}_i^\vee \to 0.$$

Fix one F_i as above and let $\mathfrak{G}_i$ be its Galois group. It is cyclic of order $\dim_{\mathbb{Q}_p} F_i$.

Lemma 4.3. *Let R be an $\mathcal{O}_K$-algebra and M be an $\mathcal{O}_{F_i} \otimes_{\mathbb{Z}_p} R$-module, and see the elements $\sigma \in \mathfrak{G}_i$ as maps $\mathcal{O}_{F_i} \otimes_{\mathbb{Z}_p} R \to R$. Then*

1. *there is a canonical decomposition*

$$M = \bigoplus_{\sigma \in \mathfrak{G}_i} M(\sigma)$$

where

$$M(\sigma) = \{m \in M \mid am = \sigma(a) m \text{ for every } a \in \mathcal{O}_{F_i}\}.$$

2. *Suppose M is invertible, then the above decomposition induces*

$$M^{\otimes k} = \bigoplus_{\sigma \in \mathfrak{G}_i} M\left(\sigma^k\right)$$

where the action of the multiplicative monoid $\mathcal{O}_{F_i} \otimes_{\mathbb{Z}_p} R$ is diagonal on $M^{\otimes k}$. Moreover

$$\mathrm{Sym}^k_R M = \bigoplus_{\substack{\chi \geq 0 \\ |\chi| = k}} M(\chi)$$

where, for $\chi = \prod \sigma^{k_\sigma}$ we define $|\chi| = \sum k_\sigma$ and $\chi \geq 0$ if $k_\sigma \geq 0$ for every σ.

Proof. This is [Kat, Lemma 2.0.8, pag. 227]. □

Let $\nabla : H^1_{\mathrm{dR}} \to H^1_{\mathrm{dR}} \otimes_{\mathcal{O}_{\mathbf{M}}} \Omega^1_{\mathbf{M}/R}$ be the Gauss-Manin connection and D be a local section of $T_{\mathbf{M}/R} = \mathrm{Der}(\mathcal{O}_{\mathbf{M}}, \mathcal{O}_{\mathbf{M}})$, then taking the composite map

$$\underline{\omega} \to H^1_{\mathrm{dR}} \xrightarrow{\nabla_D} H^1_{\mathrm{dR}} \to \underline{\omega}^\vee$$

defines the Kodaira-Spencer morphism

$$T_{\mathbf{M}/R} \to \mathrm{Hom}_{\mathcal{O}_{F_i} \otimes \mathcal{O}_{\mathbf{M}}}\left(\underline{\omega}, \underline{\omega}^\vee\right) = \mathrm{Hom}_{\mathcal{O}_{F_i} \otimes \mathcal{O}_{\mathbf{M}}}\left(\underline{\omega}^{\otimes 2}, \mathcal{O}_{F_i \otimes \mathcal{O}_{\mathbf{M}}}\right)$$

which is an isomorphism, in other terms it defines an isomorphism of $\mathcal{O}_{\mathbf{M}}$-modules

$$\Omega^1_{\mathbf{M}/R} \simeq \underline{\omega}^{\otimes 2}$$

where $\underline{\omega}^{\otimes 2}$ means tensor product as $\mathcal{O}_{F_i} \otimes \mathcal{O}_{\mathbf{M}}$-modules.

Note 4.4. Let $\nabla : H^1_{\mathrm{dR}} \to H^1_{\mathrm{dR}} \otimes_{\mathcal{O}_{\mathbf{M}}} \Omega^1_{\mathbf{M}/R}$ be the Gauss-Manin connection, the $\mathcal{O}_{\mathbf{M}}$-module $H^1_{\mathrm{dR}} \otimes_{\mathcal{O}_{\mathbf{M}}} \Omega^1_{\mathbf{M}/R}$ has two structures of $\mathcal{O}_{F_i}$-module: a left one coming from H^1_{dR} and a right one coming from $\Omega^1_{\mathbf{M}/R}$, induced by Kodaira-Spencer isomorphism above. We will consider the right one. Then we see that

$$\nabla\left(H^1_{\mathrm{dR},\sigma}\right) \subseteq H^1_{\mathrm{dR}} \otimes_{\mathcal{O}_{\mathbf{M}}} \Omega^1_{\mathbf{M}/R,\sigma}$$

therefore we can write ∇ as a sum of $\mathcal{O}_{F_i} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbf{M}}$-linear maps $\nabla = \sum \nabla_\sigma$ where ∇_σ is the composition

$$H^1_{\mathrm{dR}} \xrightarrow{\nabla} H^1_{\mathrm{dR}} \otimes_{\mathcal{O}_{\mathbf{M}}} \Omega^1_{\mathbf{M}/R} \to H^1_{\mathrm{dR}} \otimes_{\mathcal{O}_{\mathbf{M}}} \Omega^1_{\mathbf{M}/R.\sigma}.$$

More generally, for $k \geq 1$ define $\tilde{D}_\sigma$ as the composition

$$\mathrm{Sym}^k_{\mathcal{O}_\mathbf{M}} H^1_{\mathrm{dR}} \xrightarrow{\nabla} \mathrm{Sym}^k_{\mathcal{O}_\mathbf{M}} H^1_{\mathrm{dR}} \otimes_{\mathcal{O}_\mathbf{M}} \Omega^1_{\mathbf{M}/R} \simeq \bigoplus_{\tau \in \mathfrak{G}_i} \mathrm{Sym}^k_{\mathcal{O}_\mathbf{M}} H^1_{\mathrm{dR}} \otimes_{\mathcal{O}_\mathbf{M}} \underline{\omega}\left(\tau^2\right) \to \mathrm{Sym}^k_{\mathcal{O}_\mathbf{M}} H^1_{\mathrm{dR}} \otimes_{\mathcal{O}_\mathbf{M}} \underline{\omega}\left(\sigma^2\right).$$

In view of Lemma 4.3 we have an inclusion

$$\underline{\omega}\left(\sigma^2\right) \subseteq \mathrm{Sym}^2_{\mathcal{O}_\mathbf{M}} H^1_{\mathrm{dR}}.$$

Finally let

$$D_\sigma : \mathrm{Sym}^k_{\mathcal{O}_\mathbf{M}} H^1_{\mathrm{dR}} \to \mathrm{Sym}^{k+2}_{\mathcal{O}_\mathbf{M}} H^1_{\mathrm{dR}}$$

be the composition of $\tilde{D}_\sigma$ with the product

$$\mathrm{Sym}^k_{\mathcal{O}_\mathbf{M}} H^1_{\mathrm{dR}} \otimes_{\mathcal{O}_\mathbf{M}} \mathrm{Sym}^2_{\mathcal{O}_\mathbf{M}} H^1_{\mathrm{dR}} \to \mathrm{Sym}^{k+2}_{\mathcal{O}_\mathbf{M}} H^1_{\mathrm{dR}}.$$

Lemma 4.5. *The D_σ's mutually commute.*

Proof. This is [Kat, Lemma 2.1.14, pag. 229]. □

Classical weights Let $\mathbb{T} = \mathrm{Res}_{\mathcal{O}_{F_i}/\mathbb{Z}_p} \mathbb{G}_m$, namely the scheme representing the functor

$$R \to \left(\mathcal{O}_{F_i} \otimes_{\mathbb{Z}_p} R\right)^\times$$

for all $\mathcal{O}_{F_i}$-algebras R. Then for every $\mathcal{O}_{F_i}$-algebra R we have an isomorphism of R-algebras

$$\begin{aligned} \mathcal{O}_{F_i} \otimes_{\mathbb{Z}_p} R &\to \prod_{\sigma \in \mathfrak{G}_i} R \\ a \otimes 1 &\mapsto (\sigma(a))_{\sigma \in \mathfrak{G}_i} \end{aligned} \tag{4.1}$$

which induces a splitting $\mathbb{T}_{\mathcal{O}_{F_i}} \simeq \prod_{\sigma \in \mathfrak{G}_i} \mathbb{G}_{m,\mathcal{O}_{F_i}}$, in particular the character group $X\left(\mathbb{T}_{\mathcal{O}_{F_i}}\right)$ is a free abelian group with a basis given by the projections $\chi_\sigma : \prod_{\sigma \in \mathfrak{G}_i} \mathbb{G}_{m,\mathcal{O}_{F_i}} \to \mathbb{G}_{m,\mathcal{O}_{F_i}}$. We will identify $X\left(\mathbb{T}_{\mathcal{O}_{F_i}}\right)$ with $\mathbb{Z}^{\mathfrak{G}_i}$ via

$$\prod_{\sigma \in \mathfrak{G}_i} \chi_\sigma^{k_\sigma} \mapsto (k_\sigma)_{\sigma \in \mathfrak{G}_i}$$

and call its elements classical weights.

Modular forms of classical weights Let $N \geq 4$ be an integer, $\mathfrak{c}$ a fractional ideal of L, both coprime with p and $\mathbf{A}^{\mathrm{t}} \to \mathbf{M}^{\mathrm{t}}$ a fixed toroidal compactification as in Definition 2.26. Note that the $\mathcal{O}_{\mathbf{M}^{\mathrm{t}}} \otimes_{\mathbb{Z}_p} \mathcal{O}_{F_i}$-module $\underline{\omega}$ is invertible. Let R be an $\mathcal{O}_{F_i}$-algebra. For $\mathbf{k} = (k_\sigma)_{\sigma \in \mathfrak{G}_i}$ we set $\underline{\omega}^{\mathbf{k}} = \bigotimes_{\sigma \in \mathfrak{G}_i} \underline{\omega}(\sigma)^{k_\sigma}$.

Definition 4.6. Let $\mathbf{k} \in \mathbb{Z}^g$ be a classical weight and R be an $\mathcal{O}_{F_i}$-algebra, a Hilbert modular form of weight $\mathbf{k}$ and level N over R (relative to the ideal $\mathfrak{c}$) is an element of

$$\mathcal{M}(R, \mathbf{k}, \mu_N) = H^0\left(\mathbf{M}^{\mathrm{t}} \otimes R, \underline{\omega}^{\mathbf{k}}_R\right).$$

De Rham cohomology Let $D = \mathbf{M}^{\mathrm{t}} \backslash \mathbf{M}$ be the boundary of the toroidal compactification, then the Gauss-Manin connection ∇ extends to an integrable connection with logarithmic poles

$$\nabla : H^1_{\mathrm{dR}} \to H^1_{\mathrm{dR}} \otimes_{\mathcal{O}_{\mathbf{M}^{\mathrm{t}}}} \Omega^1_{\mathbf{M}^{\mathrm{t}}/R} \langle D \rangle$$

For $k \in \mathbb{Z}$ and $\mathfrak{G}_i$ we set

$$W^k_\sigma = \begin{cases} \mathrm{Sym}^k_{\mathcal{O}_{\mathbf{M}^{\mathrm{t}}}} H^1_{\mathrm{dR},\sigma} & k \geq 0 \\ \mathrm{Sym}^{-k}_{\mathcal{O}_{\mathbf{M}^t}} H^{1,\vee}_{\mathrm{dR},\sigma} & k < 0 \end{cases}$$

and

$$W^{\mathbf{k}} = \bigotimes_{\sigma \in \mathfrak{G}_i} W^{k_\sigma}_\sigma.$$

We end up with a de Rham complex

$$\mathrm{DR}^\bullet(\mathbf{k}) \; : \quad 0 \to W^{\mathbf{k}} \to W^{\mathbf{k}} \otimes_{\mathcal{O}_{\mathbf{M}^{\mathrm{t}}}} \Omega^1_{\mathbf{M}^{\mathrm{t}}/R} \langle D \rangle \to \cdots \to W^{\mathbf{k}} \otimes_{\mathcal{O}_{\mathbf{M}^{\mathrm{t}}}} \Omega^g_{\mathbf{M}^{\mathrm{t}}/R} \langle D \rangle \to 0.$$

Note 4.7. Given a scheme X and an exact sequence of $\mathcal{O}_X$-modules

$$0 \to M_1 \to M \to M_2 \to 0,$$

for every $k \geq 0$ we have an exact sequence

$$M_1 \otimes_{\mathcal{O}_X} \mathrm{Sym}^{k-1}_{\mathcal{O}_X} M \to \mathrm{Sym}^k_{\mathcal{O}_X} M \to \mathrm{Sym}^k_{\mathcal{O}_X} M_2 \to 0$$

that suggests the following non-negative filtration on $\mathrm{Sym}^k_{\mathcal{O}_X} M$: for $0 \leq n \leq k$

$$F^n \mathrm{Sym}^k_{\mathcal{O}_X} M = \begin{cases} \mathrm{Im}\left(\bigoplus_{i=n}^k \left(M_1^{\otimes i} \otimes_{\mathcal{O}_X} \mathrm{Sym}^{k-i}_{\mathcal{O}_X} M \right) \to \mathrm{Sym}^k_{\mathcal{O}_X} M \right) & 0 \leq n \leq k \\ 0 & n \geq k+1 \end{cases}.$$

In view of Note 4.7 we have a natural filtration on the $W^{k_\sigma}_\sigma$'s and hence, for $\mathbf{n} = (n_\sigma)_{\sigma \in \mathfrak{G}_i} \in \mathbb{N}^{\mathfrak{G}_i}$ and $\mathbf{k} = (k_\sigma)_{\sigma \in \mathfrak{G}_i} \in \mathbb{Z}^{\mathfrak{G}_i}$

$$F^{\mathbf{n}} W^{\mathbf{k}} = \bigotimes_{\sigma \in \mathfrak{G}_i} F^{n_\sigma} W^{k_\sigma}_\sigma.$$

5 Formal $\mathcal{O}_F$-module bundles with marked sections

In this section we let $\mathbb{Q}_p \subseteq F$ be a finite unramified and $\mathfrak{G}$ be the set of embeddings $\sigma : F \to F$. Let $\mathfrak{p}$ be the maximal ideal of $\mathcal{O}_F$. Let moreover $\mathfrak{X} \to S = \mathrm{Spf}(\mathcal{O}_F)$ be an admissible scheme with an invertible ideal of definition $\mathcal{I} = \alpha \mathcal{O}_{\mathfrak{X}}$ where $\alpha \in \mathfrak{p} \backslash \{0\} \subseteq \mathcal{O}_F$, say with $\alpha \mathcal{O}_F = \mathfrak{p}^n$. We will denote with $\overline{\mathfrak{X}}$ the reduction modulo $\mathcal{I}$.

5.1 Vector bundles with enhanced linearity

Lemma 5.1. *Let $\mathcal{E}$ be a coherent locally free $\mathcal{O}_F \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathfrak{X}}$-module. There exists an admissible morphism*

$$\pi : \mathbb{V}_{\mathcal{O}_F}(\mathcal{E}) \to \mathfrak{X}$$

representing the functor in $\mathcal{O}_F \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathfrak{X}}$-modules

$$\mathbb{V}_{\mathcal{O}_F}(\mathcal{E}) : (t : \mathfrak{Y}/\mathfrak{X}) \mapsto \mathrm{Hom}_{\mathcal{O}_{\mathfrak{Y}}}(t^*\mathcal{E}, \mathcal{O}_{\mathfrak{Y}}) = \mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{E}, t_*\mathcal{O}_{\mathfrak{Y}}),$$

For $\mathrm{Spf}(R) \subseteq \mathfrak{X}$ *we have*

$$\pi_*\mathcal{O}_{\mathbb{V}_{\mathcal{O}_F}(\mathcal{E})}(\mathrm{Spf}(R)) = \widehat{\bigotimes_{\sigma \in \mathfrak{G}}} \mathrm{Sym}_R\left(\mathcal{E}_{|R}(\sigma)\right).$$

Moreover we have a commutative diagram

$$\begin{array}{ccc} \mathbb{V}_{\mathcal{O}_F}(\mathcal{E}) & \xrightarrow{\pi} & \mathfrak{X} \\ & \searrow \quad \nearrow & \\ & \mathfrak{X} \times_{\mathrm{Spf}(\mathbb{Z}_p)} \mathrm{Spf}(\mathcal{O}_F) & \end{array}$$

Proof. Recall that the $\mathcal{O}_F$-linear structure on $\mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{E}, t_*\mathcal{O}_{\mathfrak{Y}})$ is defined as

$$(x \cdot f)(e) = f((x \otimes 1)e) \quad x \in \mathcal{O}_F,\ e \in \mathcal{E}$$

Let $\mathrm{Spf}(R) \subseteq \mathfrak{X}$ be an open affine and suppose $\mathcal{E}_{|\mathrm{Spf}(R)}$ corresponds to a finitely generated projective $\mathcal{O}_F \otimes_{\mathbb{Z}_p} R$-module M. Then for every admissible R-algebra A

$$\mathrm{Hom}_A(M \otimes_R A, A) = \prod_{\sigma \in \mathfrak{G}} \mathrm{Hom}_A(M(\sigma) \otimes_R A, A).$$

The functor

$$\mathbb{V}_\sigma(M) : A \mapsto \mathrm{Hom}_A(M(\sigma) \otimes_R A, A)$$

is represented by a formal scheme $\mathbb{V}_\sigma(M) = \mathrm{Spf}\left(\widehat{\mathrm{Sym}_R(M(\sigma))}\right)$ (see for example [AI, Section 2.1, pag. 7]), therefore

$$\mathbb{V}_{\mathcal{O}_F}(M) = \mathrm{Spf}\left(\widehat{\bigotimes_{\sigma \in \mathfrak{G}}} \mathrm{Sym}_R(M(\sigma))\right).$$

Finally, for $f = (f_\sigma)_\sigma \in \mathrm{Hom}_A(M \otimes_R A, A)$ and $x \in \mathcal{O}_F$ we have

$$x \cdot f = (x_\sigma \cdot f_\sigma)_\sigma = (\sigma(x) f_\sigma)_\sigma,$$

hence the $\mathcal{O}_F \otimes_{\mathbb{Z}_p} R$-algebra structure on $\widehat{\bigotimes}_{\sigma \in \mathfrak{G}} \mathrm{Sym}_R(M(\sigma))$ is given by

$$\mathcal{O}_F \otimes_{\mathbb{Z}_p} R \to \widehat{\bigotimes_{\sigma \in \mathfrak{G}}} \mathrm{Sym}_R(M(\sigma))$$
$$x \otimes 1 \mapsto \bigotimes_{\sigma \in \mathfrak{G}} \sigma(x)$$

□

Note 5.2. Let $\mathcal{E}$ be a coherent locally free $\mathcal{O}_F \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathfrak{X}}$-module and consider the formal scheme $\pi : \mathbb{V}_{\mathcal{O}_F}(\mathcal{E}) \to \mathfrak{X}$. For $\mathrm{Spf}(R) \subseteq \mathfrak{X}$ such that $\mathcal{E}_{|\mathrm{Spf}(R)}$ is free, say with a basis $\left\{ e_{i,\sigma} \mid \begin{array}{c} i = 1, \dots, n \\ \sigma \in \mathfrak{G} \end{array} \right\}$ we define[3]

$$F^h \pi_* \mathcal{O}_{\mathbb{V}_{\mathcal{O}_F}(\mathcal{E})}(\mathrm{Spf}(R)) = \bigotimes_{\sigma \in \mathfrak{G}} R[X_{1,\sigma}, \dots, X_{n,\sigma}]_{\leq h}.$$

We see that picking another basis of $\mathcal{E}(\sigma)$ doesn't affect the R-module $R[X_{1,\sigma}, \dots, X_{n,\sigma}]_{\leq h}$, therefore the modules $F^h \pi_* \mathcal{O}_{\mathbb{V}_{\mathcal{O}_F}(\mathcal{E})}$ define an increasing filtration of $\pi_* \mathcal{O}_{\mathbb{V}_{\mathcal{O}_F}(\mathcal{E})}$, moreover we have

$$\pi_* \mathcal{O}_{\mathbb{V}_{\mathcal{O}_F}(\mathcal{E})} = \widehat{\varinjlim_h} F^h \pi_* \mathcal{O}_{\mathbb{V}_{\mathcal{O}_F}(\mathcal{E})}$$

and

$$\mathrm{Gr}^h F^\bullet \pi_* \mathcal{O}_{\mathbb{V}_{\mathcal{O}_F}(\mathcal{E})} = \bigotimes_{\sigma \in \mathfrak{G}} R[X_{1,\sigma}, \dots, X_{n,\sigma}]_h.$$

5.2 Marked sections

Definition 5.3. With setting and notation as above, an $\mathrm{MS}_{\mathcal{O}_F}$-datum is a pair $(\mathcal{E}, s)$ with

1. $\mathcal{E}$ a locally free coherent $\mathcal{O}_F \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathfrak{X}}$-module;
2. $s \in H^0\left(\overline{\mathfrak{X}}, \overline{\mathcal{E}}\right) \setminus \{0\}$ a section such that the map
$$s \cdot \mathcal{O}_F \otimes_{\mathbb{Z}_p} \mathcal{O}_{\overline{\mathfrak{X}}} \to \overline{\mathcal{E}}$$
is injective and locally split;
3. the $\mathcal{O}_F \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathfrak{X}}$-submodule$\mathcal{F} \subseteq \mathcal{E}$ generated by lifts of $s$is locally a direct summand.

A morphism $f : (\mathcal{E}, s) \to (\mathcal{H}, t)$ of $\mathrm{MS}_{\mathcal{O}_F}$-data is an $\mathcal{O}_F \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathfrak{X}}$-linear map $f : \mathcal{E} \to \mathcal{H}$ such that $\overline{f}(s) = t$.

To an $\mathrm{MS}_{\mathcal{O}_F}$-datum $(\mathcal{E}, s)$ we associate a sheaf on the category of admissible $\mathfrak{X}$-formal schemes as

$$\mathbb{V}_0(\mathcal{E}, s) : (t : \mathfrak{Y}/\mathfrak{X}) \mapsto \left\{ f \in \mathrm{Hom}_{\mathcal{O}_{\mathfrak{Y}}}(t^*\mathcal{E}, \mathcal{O}_{\mathfrak{Y}}) \mid (f \mod t^*\mathcal{I})(t^*s) \in \underline{(\mathcal{O}_F/\mathfrak{p}^n)^\times} \right\}$$

where $\underline{(\mathcal{O}_F/\mathfrak{p}^n)^\times}$ denotes the constant sheaf and with $(f \mod t^*\mathcal{I})(t^*s) \in \underline{(\mathcal{O}_F/\mathfrak{p}^n)^\times}$ we mean the following: given

$$f = (f_\sigma)_\sigma \in \mathrm{Hom}_{\mathcal{O}_{\mathfrak{Y}}}(t^*\mathcal{E}, \mathcal{O}_{\mathfrak{Y}}) = \prod_{\sigma \in \mathfrak{G}} \mathrm{Hom}_{\mathcal{O}_{\mathfrak{Y}}}(t^*\mathcal{E}(\sigma), \mathcal{O}_{\mathfrak{Y}})$$

the condition on f is that

$$\overline{f_\sigma}(t^*(s_\sigma)) \in \mathrm{Im}\left(\overline{\sigma} : \underline{(\mathcal{O}_F/\mathfrak{p}^n)^\times} \to \overline{\mathcal{O}_{\mathfrak{Y}}}^\times\right).$$

[3]There many other possibilities leading to the same statements, for example if $\mathbf{k} = (k_\sigma)_\sigma \in \mathbb{N}^{\mathfrak{G}}$ set $|\mathbf{k}| = \sum k_i$, then one could define

$$F^{\mathbf{k}} \pi_* \mathcal{O}_{\mathbb{V}_{\mathcal{O}_F}(\mathcal{E})}(\mathrm{Spf}(R)) = \bigotimes_{\substack{\sigma \in \mathfrak{G} \\ |\mathbf{k}| = h}} R[X_{1,\sigma}, \dots, X_{n,\sigma}]_{\leq k_\sigma}$$

However we picked the one indexed by a totally ordered set.

Proposition 5.4. *The functor* $\mathbb{V}_0(\mathcal{E}, s)$ *is represented by an open subset of an admissible blow-up of* $\mathbb{V}_{\mathcal{O}_F}(\mathcal{E})$.

Proof. Let $\sigma \in \mathfrak{G}$ and let $S \subseteq \mathcal{O}_F^\times$ be a faithful set of lifts of the elements of $(\mathcal{O}_F/\mathfrak{p}^n)^\times$ and fix $u \in S$. We have an epimorphism $\overline{\mathcal{E}(\sigma)} \to Q_\sigma$ where Q_σ is a locally free sheaf given by modding out by s_σ. Let

$$\begin{aligned} \xi : \mathrm{Sym}_{\mathcal{O}_{\overline{\mathfrak{X}}}}\left(\overline{\mathcal{E}(\sigma)}\right) &\to \mathrm{Sym}_{\mathcal{O}_{\overline{\mathfrak{X}}}}(Q_\sigma) \\ s_\sigma &\mapsto \overline{\sigma(u)} \end{aligned}$$

and let $\overline{J}_{u,\sigma} = \ker(\xi) = \left(s_\sigma - \overline{\sigma(u)}\right)$. Note that we have $\overline{J}_{u,\sigma} = \overline{J}_{v,\sigma}$ for $u \equiv v \mod \mathfrak{p}^n$. Let $J_{u,\sigma} \subseteq \mathcal{O}_{\mathbb{V}_\sigma(\mathcal{E})}$ be the inverse image of $\overline{J}_{u,\sigma}$ and define $\mathbb{V}_{0,\sigma}^u(\mathcal{E}, s)$ as the open subset of $\widehat{\mathrm{Bl}_{\mathcal{J}_{u,\sigma}} \mathbb{V}_\sigma}(\mathcal{E})$ over which $J_{u,\sigma}$ is generated by α and

$$\mathbb{V}_0^u(\mathcal{E}, s) = \prod_{\sigma \in \mathfrak{G}} \mathbb{V}_{0,\sigma}^u(\mathcal{E}, s).$$

We claim that $\mathbb{V}_0^u(\mathcal{E}, s)$ represents the functor on admissible $\mathfrak{X}$-formal schemes

$$(t : \mathfrak{Y}/\mathfrak{X}) \mapsto \prod_{\sigma \in \mathfrak{G}} \left\{ f \in \mathbb{V}_\sigma(\mathcal{E})(t : \mathfrak{Y}/\mathfrak{X}) \mid \overline{f}(s_\sigma) = \overline{\sigma(u)} \in \overline{\mathcal{O}_{\mathfrak{Y},\sigma}}^\times \right\}$$

for every $\sigma \in \mathfrak{G}$. If $f \in \mathbb{V}_\sigma(\mathcal{E})(t : \mathfrak{Y}/\mathfrak{X})$ with $\overline{f}(s_\sigma) = \overline{\sigma(u)} \in \underline{(\mathcal{O}_F/\mathfrak{p}^n)^\times}$ then we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{Y} & \xrightarrow{\Lambda_f} & \mathbb{V}_\sigma(\mathcal{E}) \\ & {\scriptstyle t}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & \mathfrak{X} & \end{array}$$

and, if $\mathrm{Spf}(R) \subseteq \mathfrak{X}$ has $\mathcal{E}(\sigma)_{|\mathrm{Spf}(R)}$ free with basis $\{e_{1,\sigma}, \ldots, e_{n,\sigma}\}$ and $\overline{e_{1,\sigma}} = s_\sigma$, pick $\mathrm{Spf}(A) \subseteq \mathfrak{Y}$ admissible over $\mathrm{Spf}(R)$. Then

$$\Lambda_f^\# : R\langle X_{1,\sigma}, \ldots, X_{n,\sigma}\rangle \to A$$

with $\Lambda_f^\#(X_{1,\sigma}) - \sigma(u) \in \alpha A$, therefore $\Lambda_f^* J_{u,\sigma}$ is generated by α. On the other hand, if $\Lambda_f^* J_{u,\sigma}$ is generated by α then $\Lambda_f^\#(X_{1,\sigma}) - \sigma(u) \in \alpha A$ hence $\overline{f}(s_\sigma) = \overline{\sigma(u)}$.

Let $\mathrm{Spf}(R) \subseteq \mathfrak{X}$ an open subset over which $\mathcal{E}$ is free, say with basis $\{e_{1,\sigma}, \ldots, e_{n,\sigma}\}_{\sigma \in \mathfrak{G}}$ where $\overline{e_{1,\sigma}} = s_\sigma$. Then

$$\begin{aligned} \mathfrak{f}_* \mathcal{O}_{\mathbb{V}_0^u(\mathcal{E},s)}(\mathrm{Spf}(R)) &= \widehat{\bigotimes}_{\sigma \in \mathfrak{G}} \frac{R\langle X_{1,\sigma,u}, \ldots, X_{n,\sigma,u}, Z_{\sigma,u}\rangle}{(\alpha Z_\sigma - X_{1,\sigma} + \sigma(u))} \\ &= R\langle Z_{\sigma,u}, X_{2,\sigma,u}, \ldots, X_{n,\sigma,u} \mid \sigma \in \mathfrak{G}\rangle. \end{aligned}$$

The formal scheme representing $\mathbb{V}_0(\mathcal{E}, s)$ then is the disjoint union of the schemes $\mathbb{V}_0^u(\mathcal{E}, s)$ index by $u \in S$, and hence

$$\mathfrak{f}_* \mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)}(\mathrm{Spf}(R)) = \prod_{u \in S} R\langle Z_{\sigma,u}, X_{2,\sigma,u}, \ldots, X_{n,\sigma,u} \mid \sigma \in \mathfrak{G}\rangle.$$

□

Call $\mathfrak{f}: \mathbb{V}_0(\mathcal{E}, s) \to \mathfrak{X}$ the composition

$$\mathbb{V}_0(\mathcal{E}, s) \xrightarrow{\xi} \mathbb{V}_{\mathcal{O}_F}(\mathcal{E}) \xrightarrow{\pi} \mathfrak{X},$$

then the map

$$\xi^{\#}: \pi_* \mathcal{O}_{\mathbb{V}_{\mathcal{O}_F}(\mathcal{E})} \to \mathfrak{f}_* \mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)}$$

locally looks like

$$R\langle X_{1,\sigma}, X_{2,\sigma}, \dots, X_{n,\sigma} \mid \sigma \in \mathfrak{G}\rangle \to \prod_{u \in S} R\langle Z_{\sigma,u}, X_{2,\sigma,u}, \dots, X_{n,\sigma,u} \mid \sigma \in \mathfrak{G}\rangle$$
$$X_{i,\sigma} \mapsto \begin{cases} (\alpha Z_{\sigma,u} + \sigma(u))_{u \in S} & i = 1 \\ (X_{i,\sigma,u})_{u \in S} & i \neq 1 \end{cases}$$

5.3 Filtrations on the sheaf of functions

Let $\mathcal{F} \subseteq \mathcal{E}$ be the invertible local direct summand of lifts of the marked section $s \in H^0\left(\overline{\mathfrak{X}}, \overline{\mathcal{E}}\right)$. Note that $(\mathcal{F}, s)$ is also an $\mathrm{MS}_{\mathcal{O}_F}$-datum. With the basis we picked in the previous section, locally we have

$$\mathcal{F}_{|\mathrm{Spf}(R)} = \langle e_{1,\sigma} \mid \sigma \in \mathfrak{G}\rangle .$$

In this case we have a natural increasing filtration $\mathrm{Fil}^h \mathfrak{f}_* \mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)}$ for $h \geq 0$ such that

1. $\mathfrak{f}_* \mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)} = \widehat{\varinjlim_h} F^h \mathfrak{f}_* \mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)}$;
2. $\mathrm{Gr}^h F^\bullet \mathfrak{f}_* \mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)} = \mathfrak{f}_* \mathcal{O}_{\mathbb{V}_0(\mathcal{F},s)} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathrm{Gr}^h F^\bullet \pi_* \mathcal{O}_{\mathbb{V}_{\mathcal{O}_F}\left(\frac{\mathcal{E}}{\mathcal{F}}\right)}$

where $\mathrm{Gr}^h F^\bullet \pi_* \mathcal{O}_{\mathbb{V}_{\mathcal{O}_F}\left(\frac{\mathcal{E}}{\mathcal{F}}\right)}$ is defined as in Note 4.7. Indeed it follows from the functorial description that

$$\mathbb{V}_0(\mathcal{E}, s) = \mathbb{V}_0(\mathcal{F}, s) \times_{\mathbb{V}_{\mathcal{O}_F}(\mathcal{F})} \mathbb{V}_{\mathcal{O}_F}(\mathcal{E}),$$

then

$$\begin{aligned} \mathfrak{f}_* \mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)} &= \mathfrak{f}_* \mathcal{O}_{\mathbb{V}_0(\mathcal{F},s)} \widehat{\otimes}_{\pi_* \mathcal{O}_{\mathbb{V}_{\mathcal{O}_F}(\mathcal{F})}} \pi_* \mathcal{O}_{\mathbb{V}_{\mathcal{O}_F}(\mathcal{E})} \\ &= \mathfrak{f}_* \mathcal{O}_{\mathbb{V}_0(\mathcal{F},s)} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \pi_* \mathcal{O}_{\mathbb{V}_{\mathcal{O}_F}\left(\frac{\mathcal{E}}{\mathcal{F}}\right)}. \end{aligned}$$

and we apply the construction in Note 4.7 to $\pi_* \mathcal{O}_{\mathbb{V}_{\mathcal{O}_F}\left(\frac{\mathcal{E}}{\mathcal{F}}\right)}$ and get

$$F^h \mathfrak{f}_* \mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)} = \mathfrak{f}_* \mathcal{O}_{\mathbb{V}_0(\mathcal{F},s)} \otimes_{\mathcal{O}_{\mathfrak{X}}} F^h \pi_* \mathcal{O}_{\mathbb{V}_{\mathcal{O}_F}\left(\frac{\mathcal{E}}{\mathcal{F}}\right)}$$

Locally we have

$$\mathrm{Fil}^h \mathfrak{f}_* \mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)}(\mathrm{Spf}(R)) = \prod_{u \in S} R\langle Z_{\sigma,u} \mid \sigma \in \mathfrak{G}\rangle \left[X_{i,\sigma,u} \middle| \begin{array}{c} i = 2, \dots, n \\ \sigma \in \mathfrak{G} \end{array} \right]_{\leq h}.$$

5.4 Connections on the sheaf of functions

Let $(\mathcal{E}, s)$ be an $\mathrm{MS}_{\mathcal{O}_F}$-datum and $\nabla : \mathcal{E} \to \mathcal{E} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/S}$ be an integrable connection. We say that it is an $\mathrm{MS}_{\mathcal{O}_F}$-connection if

- it is compatible with the $\mathcal{O}_F$-structure, that is
$$\nabla (\mathcal{E}(\sigma)) \subseteq \mathcal{E}(\sigma) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/S}.$$
In this case the restriction $\nabla_\sigma = \nabla_{|\mathcal{E}(\sigma)}$ is again an integrable connection and
- s is horizontal for $\overline{\nabla}$, that is $\overline{\nabla}(s) = 0$.

In this case for every σ we have an $\mathcal{A}$-linear isomoprhism
$$\epsilon_\sigma : \mathcal{E}(\sigma) \otimes_{j_2} \mathcal{A} \to \mathcal{E}(\sigma) \otimes_{j_1} \mathcal{A}$$
such that $\overline{\epsilon_\sigma}(s_\sigma \otimes 1) = s_\sigma \otimes 1$ and $\epsilon_\sigma \otimes_\Delta \mathcal{O}_{\mathfrak{X}} = \mathrm{id}_{\mathcal{E}(\sigma)}$ plus a cocycle condition translating integrability. In particular ϵ_σ is an isomorphism of $\mathrm{MS}_{\mathcal{O}_F}$-data over $\mathcal{A}$ and by functoriality we have an isomorphism of $\underline{\mathrm{Spf}}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{A})$-formal schemes
$$\mathbb{V}_0(\mathcal{E}, s) \times_{j_2} \underline{\mathrm{Spf}}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{A}) \to \mathbb{V}_0(\mathcal{E}, s) \times_{j_1} \underline{\mathrm{Spf}}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{A})$$
giving an isomorphism
$$\mathfrak{f}_* \mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{A} \xrightarrow{\epsilon_0} \mathfrak{f}_* \mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{A}.$$
In view again of Grothendieck's formalism we have a commutative diagram where the vertical arrows are integrable connections
$$\begin{array}{ccccc} \mathcal{E} & \longrightarrow & \pi_* \mathcal{O}_{\mathbb{V}_{\mathcal{O}_F}(\mathcal{E})} & \longrightarrow & \mathfrak{f}_* \mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)} \\ \downarrow \nabla & & \downarrow \nabla_{\mathcal{O}_F} & & \downarrow \nabla_0 \\ \mathcal{E} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/S} & \longrightarrow & \pi_* \mathcal{O}_{\mathbb{V}_{\mathcal{O}_F}(\mathcal{E})} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/S} & \longrightarrow & \mathfrak{f}_* \mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/S} \end{array}$$
This allows to give a local description of the connections. If $\mathrm{Spf}(R) \subseteq \mathfrak{X}$ is such that $\mathcal{E}_{|\mathrm{Spf}(R)}$ is free with R-basis
$$\{e_{1,\sigma}, \ldots, e_{n,\sigma} \mid \sigma \in \mathfrak{G}\} \quad \text{with} \quad \overline{e_{1,\sigma}} = s_\sigma,$$
suppose
$$\nabla(e_{i,\sigma}) = \begin{cases} \sum_j \alpha r_{1,j,\sigma} e_{j,\sigma} \otimes \omega_{1,j,\sigma} & i = 1 \\ \sum_j r_{i,j,\sigma} e_{j,\sigma} \otimes \omega_{i,j,\sigma} & \text{otherwise.} \end{cases},$$
then
$$\nabla_{\mathcal{O}_F}(X_{i,\sigma}) = \begin{cases} \sum_j \alpha r_{1,j,\sigma} X_{j,\sigma} \otimes \omega_{1,j,\sigma} & i = 1 \\ \sum_j r_{i,j,\sigma} X_{j,\sigma} \otimes \omega_{i,j,\sigma} & \text{otherwise.} \end{cases}$$
We have $\nabla_0(\alpha Z_{\sigma,u}) = \nabla_{\mathcal{O}_F}(X_{1,\sigma})$ so that
$$\nabla_0(Z_{\sigma,u}) = \sum_j r_{1,j,\sigma} X_{j,\sigma,u} \otimes \omega_{1,j,\sigma} \quad \text{for every } \sigma \in \mathfrak{G}.$$

Summing up we have, in view of the local descriptions and Leibniz' rule

Proposition 5.5. *Let $(\mathcal{E}, s)$ be an $\mathrm{MS}_{\mathcal{O}_F}$-datum over $\mathfrak{X}$ and let $\nabla : \mathcal{E} \to \mathcal{E}\widehat{\otimes}_{\mathcal{O}_\mathfrak{X}}\Omega^1_{\mathfrak{X}/S}$ be an integrable $\mathrm{MS}_{\mathcal{O}_F}$-connection. Then the connection ∇_0 on $\mathfrak{f}_*\mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)}$ is integrable, satisfies Griffith's transversality, that is*

$$\nabla_0\left(\mathrm{Fil}^h\mathfrak{f}_*\mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)}\right) \subseteq \mathrm{Fil}^{h+1}\mathfrak{f}_*\mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)}\widehat{\otimes}_{\mathcal{O}_\mathfrak{X}}\Omega^1_{\mathfrak{X}/S}$$

and the induced map

$$\mathrm{Gr}^h(\nabla_0) : \mathrm{Gr}^h\left(\mathfrak{f}_*\mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)}\right) \to \mathrm{Gr}^{h+1}\left(\mathfrak{f}_*\mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)}\right)\widehat{\otimes}_{\mathcal{O}_\mathfrak{X}}\Omega^1_{\mathfrak{X}/S}$$

is $\mathcal{O}_\mathfrak{X}$-linear.

6 The weight space

This chapter is essentially borrowed from [AIP1, Section 2].

6.1 The Iwasawa algebra

Let $\mathbf{T} = \mathrm{Res}_{\mathcal{O}_L/\mathbb{Z}}\mathbb{G}_m$, that is $\mathbf{T}(R) = (R \otimes_\mathbb{Z} \mathcal{O}_L)^\times$ for every commutative ring R.

Definition 6.1. Denote with $\Lambda = \Lambda_L$ the completed group algebra

$$\Lambda = \mathbb{Z}_p[[\mathbf{T}(\mathbb{Z}_p)]]$$

and with $\kappa^\mathrm{u} : \mathbf{T}(\mathbb{Z}_p) \to \Lambda^\times$ the natural inclusion, which we will refer to as the universal character.

Remark 6.2. If $\mathrm{Tors}\mathbf{T}(\mathbb{Z}_p) \subseteq \mathbf{T}(\mathbb{Z}_p)$ denotes the torsion subgroup, we have a split exact sequence

$$0 \to \mathrm{Tors}\mathbf{T}(\mathbb{Z}_p) \to \mathbf{T}(\mathbb{Z}_p) \to \mathbf{T}(\mathbb{Z}_p)_\mathrm{tf} \to 0$$

(here the subscript tf stands for torsion-free). We consider on $\mathbb{Z}_p \otimes_\mathbb{Z} \mathcal{O}_L$ is the product topology, induced by one (hence every) $\mathbb{Z}_p$-module isomorphism with $\mathbb{Z}_p^g$. This makes $\mathbb{Z}_p \otimes_\mathbb{Z} \mathcal{O}_L$ a compact and separated $\mathbb{Z}_p$-algebra, from which $\mathbf{T}(\mathbb{Z}_p)$ acquires a structure of topological group, in particular the finite group $\mathrm{Tors}\mathbf{T}(\mathbb{Z}_p)$ is discrete. Moreover $\mathbf{T}(\mathbb{Z}_p)_\mathrm{tf}$ is naturally a compact (hence complete and separated) topological group. In particular the sequence is split in the category of topological abelian groups.

Definition 6.3. Let $\Lambda^0 = \mathbb{Z}_p\left[\left[\mathbf{T}(\mathbb{Z}_p)_\mathrm{tf}\right]\right]$ be the completed group algebra over $\mathbb{Z}_p$ corresponding to the torsion-free quotient $\mathbf{T}(\mathbb{Z}_p)_\mathrm{tf}$.

Remark 6.4. We have an isomorphism

$$\Lambda \simeq \Lambda^0[\mathrm{Tors}\mathbf{T}(\mathbb{Z}_p)],$$

making Λ finite free over Λ^0 and we can define the composition

$$\kappa : \mathbf{T}(\mathbb{Z}_p) \xrightarrow{\kappa^\mathrm{u}} \Lambda^\times \to \left(\Lambda^0\right)^\times$$

where the last map is the projection induced by the above isomorphism.

Proposition 6.5. *The ring Λ^0 is a regular local ring of Krull dimension $g+1$. Let $\gamma_1, \dots, \gamma_g$ be any topological basis of $\mathbf{T}(\mathbb{Z}_p)_\mathrm{tf}$ and let $\mathfrak{m} \subseteq \Lambda^0$ be the ideal generated by $p, \gamma_1 - 1, \dots, \gamma_g - 1$, then $\mathfrak{m}$ is the maximal ideal of Λ^0 and Λ^0 is complete with respect to the $\mathfrak{m}$-adic topology.*

6.2 The adic weight space

Let $\mathfrak{W} = \mathrm{Spf}(\Lambda)$ and $\mathfrak{W}^0 = \mathrm{Spf}(\Lambda^0)$. We consider the admissible (formal) blow-up $t : \mathrm{Bl}_{\mathfrak{m}}\mathfrak{W} \to \mathfrak{W}$ along the ideal $\mathfrak{m}$. In the same way we define $t^0 : \mathrm{Bl}_{\mathfrak{m}}\mathfrak{W}^0 \to \mathfrak{W}^0$. In view of Proposition B.7 we have a finite locally free natural map $\mathrm{Bl}_{\mathfrak{m}}\mathfrak{W} \to \mathrm{Bl}_{\mathfrak{m}}\mathfrak{W}^0$.

We will work with the adic spaces associated with such formal schemes.

Definition 6.6. Define $\mathcal{W}^0 = \left(\mathfrak{W}^0\right)^{\mathrm{an}}$ and for every $\alpha \in \mathfrak{m}$ let $\mathcal{W}^0_\alpha = \left(\mathfrak{W}^0_\alpha\right)^{\mathrm{an}}$.

Remark 6.7. Note that, as $\mathfrak{W}^0$ is a formal scheme over $\mathrm{Spf}(\mathbb{Z}_p)$, then $\mathcal{W}^0$ is an analytic adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$.

Proposition 6.8. *The space $\mathcal{W}^0$ is isomorphic to a finite disjoint union of open polydiscs of dimension g, moreover, for every complete Huber pair (B, B^+) over $(\mathbb{Q}_p, \mathbb{Z}_p)$, we have a natural bijection*

$$\mathrm{Hom}_{\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)}\left(\mathrm{Spa}\left(B, B^+\right), \mathcal{W}^0\right) \to \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{Z}}\left(\mathbf{T}\left(\mathbb{Z}_p\right), B^\times\right).$$

Proof. This is a restatement of the universal property of the Iwasawa algebra. □

Definition 6.9. Let $\frac{r}{s} \in \mathbb{Q}_{>0}$ be a reduced fraction, define the following subsets of $\mathcal{W}^0$:

- $\mathcal{W}^0_{\le \frac{r}{s}} = \left\{x \in \mathcal{W}^0 \mid |\alpha^r|_x \le |p^s|_x \neq 0 \quad \forall \alpha \in \mathfrak{m}\right\}$
- $\mathcal{W}^0_{\ge \frac{r}{s}} = \left\{x \in \mathcal{W}^0 \mid 0 \neq |\alpha^r|_x \ge |p^s|_x \quad \forall \alpha \in \mathfrak{m}\right\}$
- $\mathcal{W}^0_{\ge 0} = \mathcal{W}^0_{\le \infty} = \mathcal{W}^0$
- for $a, b \in \mathbb{Q}_{>0} \cup \{\infty\}$ and $I = [a, b]$ set $\mathcal{W}^0_I = \mathcal{W}^0_{\le b} \cap \mathcal{W}^0_{\ge a}$
- for $\alpha \in \mathfrak{m}$ we let $\mathcal{W}^0_{\alpha,I} = \mathcal{W}^0_I \cap \mathcal{W}^0_\alpha$.

We introduce formal models for these spaces: fix an interval $I = [a, b] \subseteq \mathbb{Q}_{>0} \cup \{\infty\}$. For $\alpha \in \mathfrak{m}$ let $\mathfrak{W}^0_\alpha = \mathrm{Spf}(B_\alpha)$, set $B^0_{\alpha,I} = H^0\left(\mathcal{W}^0_{\alpha,I}, \mathcal{O}^+_{\mathcal{W}^0_{\alpha,I}}\right)$ and $\mathfrak{W}^0_{\alpha,I} = \mathrm{Spf}\left(B^0_{\alpha,I}\right)$. It is clear from Remark C.5 that the analytic fibre of $\mathfrak{W}^0_{\alpha,I}$ is $\mathcal{W}^0_{\alpha,I}$ and that they give an affine cover of a locally noetherian formal scheme $\mathfrak{W}^0_I$ whose analytic fibre is $\mathcal{W}^0_I$.

Lemma 6.10. *Let $I \subseteq \mathbb{Q}_{>0}$ be a closed interval, then the $\mathfrak{m}$-adic topology and the p-adic topology coincide on $\mathfrak{W}^0_I$.*

Proof. Suppose $\max I = r/s$, we just need to check it on the various $B^0_{\alpha,I}$, but there the ideal $\mathfrak{m}$ is generated by α and

$$\alpha^r = \frac{\alpha^r}{p^s} \cdot p^s \in p^s \cdot B^0_{\alpha,I}.$$

□

Definition 6.11. For each closed interval $I \subseteq [0, \infty)$ we let

$$\kappa_I : T\left(\mathbb{Z}_p\right) \to \left(\mathcal{O}^+_{\mathcal{W}_I}\right)^\times$$

be the natural map.

6.3 The universal character

From now on we let $q = p$ if $p > 2$ and $q = 4$ if $p = 2$. Set moreover $H = \mathrm{Tors}\mathbf{T}(\mathbb{Z}_p)$. Note that, for the moment, we do not need to impose any condition on the ramification of p.

Lemma 6.12. *There exist $\mathbb{Q}_p \subseteq K_1, \ldots, K_t$ be extensions such that*

$$\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \mathcal{O}_{K_1} \times \cdots \times \mathcal{O}_{K_t}$$

and let $\mathfrak{p}_i \subseteq \mathcal{O}_{K_i}$ be the maximal ideal. Then $\mathfrak{p}_1 \times \cdots \times \mathfrak{p}_t$ corresponds to $\mathfrak{r}_p \cdot (\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p)$, where $\mathfrak{r}_\mathfrak{p}$ denotes the radical of $p\mathcal{O}_L$ in $\mathcal{O}_L$.

Definition 6.13. Let $\mathfrak{m} \subseteq \Lambda^0$ be its maximal ideal. For $n \geq 0$ define

$$\mathfrak{m}_n = \begin{cases} \mathfrak{m} & n = 0, 1 \\ \left(x^{p^{n-1}}, px^{p^{n-2}}, \ldots, p^{n-1}x \mid x \in \mathfrak{m}\right) & n \geq 1 \end{cases}$$

Lemma 6.14. *Let $\mathbb{Q}_p \subseteq K$ be a finite extension and let $n \geq 1$ be an integer. Then*

$$1 + p^n \mathcal{O}_K \subseteq \left(\mathcal{O}_K^\times\right)^{p^{n-1}}.$$

Proof. We'll make use of the following version of Hensel's Lemma

Theorem 6.15 (Hensel's Lemma)**.** *Let K be a complete non-archimedean field, let $\mathcal{O}_K$ be its ring of integers. Let $f(X) \in \mathcal{O}_K[X]$ and suppose there exists $c \in \mathcal{O}_K$ with*

$$|f(c)| < |f'(c)|^2.$$

Then there exists a unique $\alpha \in \mathcal{O}_K$ with $f(\alpha) = 0$ and $|\alpha - c| < |f'(c)|$.

Back to the proof, let $a \in \mathcal{O}_K$ and set $f(X) = X^{p^{n-1}} - 1 - p^n a$, note that

$$\begin{aligned} \left|\sum_{i=0}^{p^{n-1}} \binom{p^{n-1}}{i} p^{ni} a^i - 1 - p^n a\right| &= \left|\sum_{i=1}^{p^{n-1}} \binom{p^{n-1}}{i} p^{ni} a^i - p^n a\right| \\ &\leq \max\left\{|p^n a|, \left|\binom{p^{n-1}}{i} p^{ni} a^i\right| \mid i = 1, \ldots, p^{n-1}\right\} \\ &\leq \max\left\{|p^n|, \left|\binom{p^{n-1}}{i} p^{ni}\right| \mid i = 1, \ldots, p^{n-1}\right\} \leq |p|^n, \end{aligned}$$

this computation shows that we can apply Hensel's Lemma to $f(X)$ above with $c = 1 + p^n a$. ☐

Lemma 6.16. *Let $n \geq 1$, then $\kappa\left(\mathbf{T}(\mathbb{Z}_p)_{\mathrm{tf}}^{p^{n-1}}\right) \subseteq 1 + \mathfrak{m}_n$, in particular*

$$\kappa\left(1 + qp^{n-1} \cdot \mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p\right) \subseteq 1 + \mathfrak{m}_n$$

and $\kappa(\mathbf{T}(\mathbb{Z}_p)) \subseteq 1 + \mathfrak{m}_0$.

Proof. First of all note that, being defined via the projection $\Lambda \to \Lambda^0$, the character κ is trivial on H and that, tautologically, $\kappa(\mathbf{T}(\mathbb{Z}_p)) \subseteq 1 + \mathfrak{m}_0$. Let $\gamma_1, \dots, \gamma_g$ be a topological basis of $\mathbf{T}(\mathbb{Z}_p)_{\mathrm{tf}}$, we just need to see that $\kappa\left(\gamma_i^{p^{n-1}}\right) - 1 \in \mathfrak{m}_n$ for every i. Note that $\kappa(\gamma_i) \in \left(\Lambda^0\right)^\times$ has $\kappa(\gamma_i) - 1 \in \mathfrak{m}$ by definition, hence

$$\begin{aligned}\kappa\left(\gamma_i^{p^{n-1}}\right) - 1 &= \kappa(\gamma_i)^{p^{n-1}} - 1 \\ &= ((\kappa(\gamma_i) - 1) + 1)^{p^{n-1}} - 1 \\ &= \sum_{s=1}^{p^{n-1}} \binom{p^{n-1}}{s} (\kappa(\gamma_i) - 1)^s \in \mathfrak{m}_n.\end{aligned}$$

Since $1 + qp^{n-1} \cdot \mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ does not contain any torsion element of $\mathbf{T}(\mathbb{Z}_p)$, we just need to see that

$$1 + qp^{n-1} \cdot \mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p \subseteq \mathbf{T}(\mathbb{Z}_p)_{\mathrm{tf}}^{p^{n-1}},$$

that is, every element in $1 + qp^{n-1} \cdot \mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ has a p^{n-1}-th root. In view of Lemma 6.12 we reduce to show that for K a finite extension of $\mathbb{Q}_p$ we have $1 + p^n \mathcal{O}_K \subseteq \left(\mathcal{O}_K^\times\right)^{p^{n-1}}$, which is Lemma 6.14. □

Proposition 6.17 (Analyticity of the universal character)**.** *Let $n, m \geq 0$ be integers and let $I \subseteq [0, p^n] \cap \mathbb{Q}$ be a closed interval. Set*

$$\epsilon = \begin{cases} 1 & p \neq 2 \\ 3 & \text{otherwise} \end{cases},$$

then κ induces a pairing

$$\kappa_{\mathrm{p}} : \mathfrak{W}_I^0 \times \mathbf{T}(\mathbb{Z}_p) \cdot \left(1 + p^{n+\epsilon} \cdot \mathrm{Res}_{\mathcal{O}_L/\mathbb{Z}}(\mathbb{G}_a)\right) \to \mathbb{G}_m$$

on the category of p-adic formal schemes on $\mathfrak{W}_I^0$ that restricts to

$$\kappa_{\mathrm{p}} : \mathfrak{W}_I^0 \times \left(1 + p^{n+m+\epsilon} \cdot \mathrm{Res}_{\mathcal{O}_L/\mathbb{Z}}(\mathbb{G}_a)\right) \to 1 + qp^m \mathbb{G}_a.$$

Proof. We prove it for $p \neq 2$, the case $p = 2$ being analogous. Let $\mathrm{Spf}\left(B_{\alpha,I}^0\right) \subseteq \mathfrak{W}_I^0$, in view of Lemma 6.16 it is clear that

$$\kappa\left(\mathbf{T}(\mathbb{Z}_p) \cdot \left(1 + p^{n+1} \cdot \mathcal{O}_L \otimes_{\mathbb{Z}} B_{\alpha,I}^0\right)\right) \subseteq \left(B_{\alpha,I}^0\right)^\times.$$

Let us note that, for every $x \in \mathcal{W}_{\alpha,I}^0$, we have $\left|\alpha^{p^n}/p\right|_x \leq 1$ hence $\alpha^{p^n}/p \in B_{\alpha,I}^0$. It follows that for every $m \geq 0$ we have $\alpha^{p^{n+m}} \in p^{m+1} B_{\alpha,I}^0$, in particular $\mathfrak{m}_{n+m+1} B_{\alpha,I}^0 \subseteq p^{m+1} B_{\alpha,I}^0$. The Proposition now follows since, again in view of Lemma 6.16

$$\kappa\left(1 + p^{m+n+1} \cdot \mathcal{O}_L \otimes_{\mathbb{Z}} B_{\alpha,I}^0\right) \subseteq 1 + \mathfrak{m}_{n+m+1} B_{\alpha,I}^0 \subseteq 1 + p^{m+1} B_{\alpha,I}^0.$$

□

Note 6.18. Note that, for a p-adically complete and separated ring A with ideal of definiton I, for n large enough the exponential power series is convergent on I^n (indeed one can take n such that $I^n \subseteq pA$) , it follows that there exists an element $u_I \in p^{1-n}$ such that $k_I(t) = \exp(u_I \cdot \log(t))$ for $t \in 1 + p^n \mathcal{O}_L \otimes_{\mathbb{Z}} A$.

7 The Igusa tower

7.1 The Hasse invariant and the Hodge ideal

Let A be a p-adically complete and separated $\mathbb{Z}_p$-algebra and let $S \to \operatorname{Spec}(A)$ be a scheme with S locally given by p-adically complete rings. Let $\pi : X \to S$ be a semi-abelian scheme of relative dimension g with identity section $e : S \to X$. Let $S = \operatorname{Spec}(R)$ be such that the R-module $\underline{\omega}_{X/S} = H^0\left(S, e^*\Omega^1_{X/S}\right)$ is free of rank g. Let

$$\operatorname{Ver}^* : \underline{\omega}_{\overline{X}/\overline{S}} \to \underline{\omega}_{\overline{X}^{(p)}/\overline{S}}$$

be the pullback along the Verschiebung on the reductions modulo p.

Definition 7.1. With setting and notations as above, the element

$$\operatorname{Ha}\left(\overline{X}_{/\overline{S}}\right) := \det\left(\operatorname{Ver}^*\right) \in \frac{R}{pR}$$

is called the Hasse invariant of $\overline{X} \to \operatorname{Spec}(R/p)$. Define $\operatorname{Hdg}\left(X_{/S}\right) \subseteq R$ as the preimage of the ideal $\operatorname{Ha}\left(\overline{X}_{/\overline{S}}\right) \cdot \overline{R}$ along the quotient $R \to \overline{R}$, this is called the Hodge ideal of $X \to \operatorname{Spec}(R)$.

Lemma 7.2. *The following hold*

1. *The ideal* $\operatorname{Hdg}\left(X_{/S}\right)$ *is Zariski-locally generated by two elements;*
2. *if* $p \in \operatorname{Hdg}\left(X_{/S}\right)^2$*, then* $\operatorname{Hdg}\left(X_{/S}\right)$ *is an invertible* $\mathcal{O}_S$*-module locally generated by any lift of* $\operatorname{Ha}\left(\overline{X}_{/\overline{S}}\right)$.

Proof. The first claim is obvious, for the second consider a local lift h of $\operatorname{Ha}\left(\overline{X}_{/\overline{R}}\right)$ on some small open $\operatorname{Spec}(R) \subseteq X$, then $\operatorname{Hdg}\left(X_{/S}\right)_{|\operatorname{Spec}(R)} = (p, h)$ and in view of our assumptions there exist $a, b \in R$ with $p = ah + bp^2$ giving

$$p\,(1 - pb) = ah$$

and we conclude since $1 - pb$ is invertible. □

Proposition 7.3. *Let k be an algebraically closed field of characteristic $p > 0$ and let $A_{/k}$ be an abelian variety. Then $A_{/k}$ is ordinary if and only if* $\operatorname{Ha}\left(A_{/k}\right) \in k^\times$.

Proof. It follows from the definition that $\operatorname{Ha}\left(A_{/k}\right) \neq 0$ if and only if the Verschiebung induces an isomorphism between the tangent spaces and this means that it is finite étale. In view of the proof of [Sil, Theorem III.6.2.(e), pag. 86], we see that $\#A[p] = \deg_{\mathrm{sep}}(\operatorname{Ver})$, in particular $\#A[p] = \deg(\operatorname{Ver})$ since Ver is étale (hence separable). We conclude since by duality

$$\deg(\operatorname{Ver}) = \deg(\operatorname{Fr}) = p^{\dim A}.$$

□

7.2 Canonical subgroups

Let $N \geq 4$ be an integer and fix an invertible ideal $\mathfrak{c}$ of L. Let A be an integral $\mathbb{Z}_p$-algebra, $\alpha \in A \setminus \{0\}$ be such that A is the α-adic completion of a $\mathbb{Z}_p$-algebra of finite type with $p \in \alpha A$.

Definition 7.4. Let $\mathfrak{Y}^{\mathrm{tor}} \to \mathrm{Spf}\,(A)$ be the α-adic formal scheme associated with $\mathbf{M}_N^{\mathrm{tor}} \otimes_{\mathbb{Z}_p} A$. For $r \geq 0$ an integer, let $\mathfrak{Y}_r^{\mathrm{tor}}$ be the functor associating to any α-adically complete A-algebra R with no α-torsion the subset of

$$\left\{(f, \eta) \mid f : \mathrm{Spf}\,(R) \to \mathfrak{Y}^{\mathrm{tor}}, \quad \eta \in H^0\left(\mathrm{Spf}\,(R)\,, f^* \det\left(\underline{\omega}\right)^{\otimes(1-p)p^{r+1}}\right)\right\}$$

given by the pairs that satisfy

$$\mathrm{Ha}^{p^{r+1}} \cdot \eta = \alpha \qquad \mathrm{mod}\ p^2 \tag{7.1}$$

for any lift Ha of the Hasse invariant, modulo the relation

$$(f, \eta) \sim (f, \nu) \qquad \Longleftrightarrow \qquad \exists u \in R,\, \nu = \eta\left(1 + \frac{p^2}{\alpha} u\right).$$

Lemma 7.5. *The functor $\mathfrak{Y}_r^{\mathrm{tor}}$ in Definition 7.4 is well defined.*

Proof. First of all let us work out the equivalence relation: the only thing that may not be appearent is its reflexivity. Recall that R is suposed to have no α-torsion and that $p \in \alpha R$, from which α^2 divides p^2 and the elements of the form $1 + up^2\alpha^{-1}$ lie in $1 + \alpha R$. They are therefore invertible since αR is contained in the Jacobson radical of R. We're left with checking that equation (7.1) is insensitive to the choice of the lift Ha. Any other choice would be of the form $\mathrm{Ha} + px$ for some $x \in R$ and we conclude in view of next Lemma 7.6. □

Lemma 7.6. *Let R be any commutative ring, and $x, y \in R$. Then*

$$(y + px)^p - y^p \in p^2 R.$$

Proof. Explicitely

$$\begin{aligned}
(y + px)^p - y^p &= \sum_{i=0}^{p-1} \binom{p}{i} y^i\, (px)^{p-i} \\
&\equiv \binom{p}{p-1} y^{p-1} px \qquad \mathrm{mod}\ p^2 \\
&= p^2 y^{p-1} x \in p^2 R.
\end{aligned}$$

□

Proposition 7.7. *The functor in Definition 7.4 is representable by an open subset $\mathfrak{Y}_r^{\mathrm{tor}}$ of the admissibile blow-up of $\mathfrak{Y}^{\mathrm{tor}}$ along the ideal $\left(\mathrm{Hdg}^{p^{r+1}}, \alpha\right)$.*

Proof. Let $\pi : \tilde{\mathfrak{Y}}_r \to \mathfrak{Y}^{\mathrm{tor}}$ be the blow-up along the ideal $\left(\mathrm{Hdg}^{p^{r+1}}, \alpha\right)$. Note that such an ideal is admissible in view of our assumptions on the ring A. Let R be an α-adically complete A-algebra and $(f, \eta) \in \mathfrak{Y}_r^{\mathrm{tor}}\,(R)$. On R we have $\mathrm{Ha}^{p^{r+1}} \cdot \eta - \alpha \in p^2 R$, whence $\alpha \in \mathrm{Hdg}^{p^{r+1}} \subseteq \mathrm{Hdg}^2$. In view of Lemma 7.2, taking an affine open of $\mathrm{Spf}(R)$ if needed, we see that $f^*\left(\mathrm{Hdg}^{p^{r+1}},\alpha\right)$ is principal generated by any lift of $\mathrm{Ha}^{p^{r+1}}$. This shows that we have a map $\tilde{f}:\mathrm{Spf}(R)\to\tilde{\mathfrak{Y}}_r$ making the evident diagram commute. Note that, again by the universal property, $\tilde{f}$ factors through the open subset $\mathfrak{Y}_r^{\mathrm{tor}}\subseteq\tilde{\mathfrak{Y}}_r$ over which the ideal $\left(\mathrm{Hdg}^{p^{r+1}},\alpha\right)$ is generated by elements of $\mathrm{Hdg}^{p^{r+1}}$ (because this holds on R), but this means again that $\alpha\in\mathrm{Hdg}^{p^{r+1}}\subseteq\mathrm{Hdg}^2$ and hence $\mathfrak{Y}_r^{\mathrm{tor}}$ is the open subset defined by the condition that $\left(\mathrm{Hdg}^{p^{r+1}},\alpha\right)$ is generated by one, and hence any in view of Lemma 7.6, lift of Ha. □

Corollary 7.8. *The ideal* Hdg *is locally free on* $\mathfrak{Y}_r^{\mathrm{tor}}$ *with* $\alpha\in\mathrm{Hdg}$*.*

Definition 7.9. With notations as above, we denote $\mathfrak{Y}_r^R\subseteq\mathfrak{Y}_r^{\mathrm{tor}}$ the open subscheme defined by condition (R).

We recall here the main result about the existence of the canonical subgroup in its full generality (for the definition of the Hodge ideal within the framework of Barsotti-Tate groups see [AIP2, Appendix A.1]). For this we need to recall

Proposition 7.10. *Let R be an α-adically complete, separated A-algebra and with no α-torsion and G be a Barsotti-Tate group over R with dimension d, height h and level 1. Then there exists a canonical ideal $\underline{\delta}_G\subseteq R$ with $\underline{\delta}_G^{p-1}=\mathrm{Hdg}(G)$.*

Proof. This is [AIP2, Proposition A.3, pag. 43]. □

Theorem 7.11. *Let R be an α-adically complete, separated A-algebra and with no α-torsion and G be a Barsotti-Tate group over R with dimension d and height h. Let $r\geq 0$ be an integer with $p\in\mathrm{Hdg}(G)^{p^{r+1}}$, then*

1. *G has a canonical subgroup H_n of level n for every $n\leq r$ and $H_n\subseteq H_{n+1}$;*
2. *H_n is locally free of rank p^{nd} and we have*
$$H_n\equiv\ker(\mathrm{Fr}^n)\qquad\mathrm{mod}\ \frac{p}{\mathrm{Hdg}(G)^{\frac{p^n-1}{p-1}}};$$
3. *We have*
$$\mathrm{Hdg}\left(\frac{G}{H_n}\right)=\mathrm{Hdg}(G)^{p^n}$$
and G/H_n has a canonical subgroup H'_{r-n} sitting in an exact sequence
$$0\to H_n\to H_r\to H'_{r-n}\to 0;$$
4. *We have*
$$\mathrm{Hdg}(G)^{\frac{p^n-1}{p-1}}\cdot\underline{\omega}_{G[p^n]/H_n}=0$$
and
$$\det\left(\underline{\omega}_{\frac{G[p^n]}{H_n}}\right)\simeq\frac{\det\left(\underline{\omega}_{G[p^n]}\right)}{\mathrm{Hdg}(G)^{\frac{p^n-1}{p-1}}};$$

5. *We have* $\mathrm{Hdg}(G) = \mathrm{Hdg}\left(G^D\right)$ *and for every* $n \leq r$ *the pairing*

$$G[p^n] \times G[p^n]^D \to \mu_{p^n}$$

induces an isomorphism $H_n(G) \simeq H_n\left(G^D\right)^{\perp}$*;*

6. *If* $\alpha \in \mathrm{Hdg}(G)$*, then* $G[p^n]/H_n$ *is étale over* $\mathrm{Spec}\left(R\left[\alpha^{-1}\right]\right)$ *and it is locally constant isomorphic to* $\underline{(\mathbb{Z}/p^r)^{h-d}}$.

Proof. This is [AIP2, Corollaire A.2, pag. 40]. □

Corollary 7.12 ([AIP1, Proposition 3.2, pag. 11])**.** *Let* $p \in \alpha^{p^k} A$*. Then for every integer* $1 \leq n \leq r+k$ *one has a canonical sub-group scheme* $H_n \subseteq \mathbf{A}_N[p^n]$ *over* $\mathfrak{Y}_r^{\mathrm{tor}}$ *and* H_n *modulo* $pHdg^{-\frac{p^n-1}{p-1}}$ *lifts the kernel of the* n*-th power of Frobenius. Moreover* H_n *is finite flat and locally of rank* p^{ng}*, it is stable under the action of* $\mathcal{O}_L$*, and the Cartier dual* H_n^D *is étale locally over* $A\left[\alpha^{-1}\right]$ *isomorphic to* $\mathcal{O}_L/p^n$ *as an* $\mathcal{O}_L$*-module.*

Proof. Everything follows directly from Theorem 7.11 but the claims about the $\mathcal{O}_L$-structure. In view of the explicit description of H_n given in [AIP2, Corollaire A.1, pag. 40], we only need to check that, $\mod p$, the kernel of the Frobenius is stable under $\mathcal{O}_L$. This is clear since, in terms of the structure sheaf, $\mathcal{O}_L$ acts by ring endomorphisms which therefore commute with raising to the p-th power, that is, with the Frobenius. □

7.3 The partial Igusa tower

From now on we fix a totally real extension $L/\mathbb{Q}$ and a rational prime p which we suppose unramified in L, say with

$$p\mathcal{O}_L = \mathfrak{p}_1 \dots \mathfrak{p}_d$$

and let F_i be the completion of L at $\mathfrak{p}_i$. From Lemma 4.1 the extension $F_i/\mathbb{Q}$ is Galois and its Galois group is cyclic. We'll denote with e_i the corresponding idempotent in

$$\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p = \prod_{i=1}^{d} \mathcal{O}_{F_i}. \tag{7.2}$$

Note 7.13. Let L^{Gal} be a fixed Galois closure of L. Note that each extension $F_i/\mathbb{Q}_p$ is Galois and the map $\mathcal{O}_L \to \mathcal{O}_{F_i}$ is injective, so that L is naturally contained in each F_i. Let $L = \mathbb{Q}(\alpha)$ and let $f(x) \in \mathbb{Q}[x]$ be its minimal polynomial, then the extension $F_i|L\mathbb{Q}_p = \mathbb{Q}_p(\alpha)$ is Galois hence F_i contains all the roots of f seen as a polynomial in $\mathbb{Q}_p[x]$. These roots generated the field $L^{\mathrm{Gal}}\mathbb{Q}_p$ therefore we have an embedding $L^{\mathrm{Gal}} \subseteq F_i$. Moreover we have a natual bijection

$$\mathrm{Gal}(F_i/\mathbb{Q}_p) \to \left\{\sigma \in \mathrm{Hom}_{\mathbb{Q}}\left(L, L^{\mathrm{Gal}}\right) \middle| \sigma(\mathfrak{p}_i) = \mathfrak{p}_i\right\} =: \mathfrak{G}_i$$

induced by the identification

$$\mathrm{Hom}_{\mathbb{Q}}\left(L, L^{\mathrm{Gal}}\right) = \mathrm{Gal}\left(L^{\mathrm{Gal}}/\mathbb{Q}\right).$$

Hence we can see the $\mathbb{Q}_p$-embeddings $\sigma : F \to F$ as a subset of the embeddings $L \to L^{\mathrm{Gal}}$.

Let $\mathcal{S} = \mathrm{Spa}\left(A\left[\alpha^{-1}\right], A^+\right)$, where $A^+ \subseteq A\left[\alpha^{-1}\right]$ is the normalisation of A. Note that $(\mathfrak{Y}_r^{\mathrm{tor}})^{\mathrm{an}}$ is given by $(\mathfrak{Y}_r^{\mathrm{tor}})^{\mathrm{ad}} \times_{\mathrm{Spa}(A)} \mathcal{S}$ in view of Lemma C.15. Let $p \in \alpha^{p^k} A$ and $1 \leq n \leq r+k$, then it follows from Corollary 7.12 that we have an $\mathcal{O}_L$-module object H_n with p^n-torsion $(\mathfrak{Y}_r^{\mathrm{tor}})^{\mathrm{an}}$ which admits therefore a decomposition

$$H_n \simeq \prod_{i=1}^{d} e_i H_n = \prod_{i=1}^{d} H_n^{(i)}$$

in view of (7.2). In view of the same Corollary we have that $H_n^{(i),D}$ is étale-locally isomorphic to $\underline{(\mathcal{O}_{F_i}/\mathfrak{p}_i^n)}$. We can consider the $\underline{(\mathcal{O}_{F_i}/\mathfrak{p}_i^n)^\times}$-torsor $\underline{\mathrm{Isom}}\left(\underline{(\mathcal{O}_{F_i}/\mathfrak{p}_i^n)}, H_n^{(i),D}\right)$ on $(\mathfrak{Y}_r^{\mathrm{tor}})^{\mathrm{an}}$ which is representable in view of [Mil1, Theorem 4.3.(a), pag. 121]. Call $h^{(i),n} : \mathcal{IG}_{n,r}^{(i)} \to (\mathfrak{Y}_r^{\mathrm{tor}})^{\mathrm{an}}$ the resulting object.

Proposition 7.14. *The morphism $h^{(i),n}$ is finite étale and Galois with group $(\mathcal{O}_{F_i}/\mathfrak{p}_i^n)^\times$.*

Proof. The only non-trivial thing to check is that $h^{(i),n}$ is étale, but this is essentially a formal consequence of the étalness of H_n^D, see for example Section 8.1.1, pag. 330 in

- H. Hida, *p-adic automorphic forms on Shimura varieties*, Springer Monographs in Mathematics, (2004)

□

Now Theorem C.22 provides us with a finite morphism

$$\mathfrak{h}^{(\mathrm{i}),\mathfrak{n}} : \mathfrak{IG}_{n,r}^{(i)} \to \mathfrak{Y}_r^{\mathrm{tor}}$$

such that

1. since $h^{(i),n}$ is finite étale from Proposition 7.14, in view of Corollary C.23 we see that $\mathfrak{h}^{(\mathrm{i}),\mathfrak{n}}$ provides a formal model for $h^{(i),n}$, i.e. $\left(\mathfrak{h}^{(\mathrm{i}),\mathfrak{n}}\right)^{\mathrm{an}} = h^{(i),n}$;

2. in view of Property 3. in Theorem C.22 the map $\mathfrak{h}^{(\mathrm{i}),\mathfrak{n}}$ comes endowed with a natural action of $(\mathcal{O}_{F_i}/\mathfrak{p}_i^n)^\times$ compatible with the one on $h^{(i),n}$;

3. the natural map of étale sheaves

$$\underline{\mathrm{Isom}}\left(\underline{\mathcal{O}_{F_i}/\mathfrak{p}^n}, H_n^{(i),D}\right) \to \underline{\mathrm{Isom}}\left(\underline{\mathcal{O}_{F_i}/\mathfrak{p}_i^{n-1}}, H_{n-1}^{(i),D}\right)$$

   induces a morphism

$$\mathfrak{h}^{(i)} = \mathfrak{h}_{n-1}^{(i),n} : \mathfrak{IG}_{n,r}^{(i)} \to \mathfrak{IG}_{n-1,r}^{(i)}$$

   over $\mathfrak{Y}_r^{\mathrm{tor}}$ which is finite and invariant under the action of

$$\ker\left(\left(\frac{\mathcal{O}_{F_i}}{\mathfrak{p}_i^n}\right)^\times \to \left(\frac{\mathcal{O}_{F_i}}{\mathfrak{p}_i^{n-1}}\right)^\times\right).$$

Lemma 7.15. *We have*

$$\mathrm{Hdg}^{p^{n-1}}\mathcal{O}_{\mathfrak{IG}^{(i)}_{n-1,r}} \subseteq \mathrm{Tr}_{\mathfrak{h}^{(i)}}\left(\mathfrak{h}^{(i)}_*\mathcal{O}_{\mathfrak{IG}^{(i)}_{n,r}}\right) \quad \text{for } 1 \le n \le r+k$$

$$\mathrm{Tr}_{\mathfrak{h}^{(i)}}\left(\mathfrak{h}^{(i)}_*\mathcal{O}_{\mathfrak{IG}^{(i)}_{1,r}}\right) = \mathcal{O}_{\mathfrak{Y}^{\mathrm{tor}}_r}.$$

Proof. First note that the map $\mathrm{Tr}_{\mathfrak{h}^{(i)}} : \mathfrak{h}^{(i)}_*\mathcal{O}_{\mathfrak{IG}^{(i)}_{1,r}} \to \mathcal{O}_{\mathfrak{y}^{\mathrm{tor}}_r}$ is surjective having the map $\mathfrak{IG}^{(i)}_{1,r} \to \mathfrak{Y}^{\mathrm{tor}}_r$ degree $\#\left(\mathcal{O}_{F_i}/\mathfrak{p}^n_i\right)^\times$ prime with p. Let $\phi_n : \mathfrak{IG}^{(i)}_{n,r} \to H^{(i),D}_n$ be the map induced by

$$\begin{aligned}\underline{\mathrm{Isom}}\left(\underline{\frac{\mathcal{O}_{F_i}}{\mathfrak{p}^n_i}}, H^{(i),D}_n\right) &\to H^{(i),D}_n \\ \xi &\mapsto \xi(1),\end{aligned}$$

then we have a cartesian diagram

$$\begin{array}{ccc} \mathfrak{IG}^{(i)}_{n,r} & \xrightarrow{\phi_r} & H^{(i),D}_n \\ \downarrow{\scriptstyle \mathfrak{h}^{(i)}} & & \downarrow{\scriptstyle q} \\ \mathfrak{IG}^{(i)}_{n-1,r} & \xrightarrow[\phi_{r-1}]{} & H^{(i),D}_{n-1} \end{array}$$

using which we reduce to showing that

$$\mathrm{Hdg}^{p^{n-1}}\mathcal{O}_{H^{(i),D}_{n-1}} \subseteq \mathrm{Tr}_q\left(q_*\mathcal{O}_{H^{(i),D}_n}\right).$$

Is $\mathrm{Spf}(R) \subseteq \mathfrak{Y}^{\mathrm{tor}}_r$ is an open affine and $\mathrm{Spf}(B) \subseteq \left(H^{(i)}_n/H^{(i)}_{n-1}\right)^D$ its inverse images with different ideals $\mathcal{D}(A_n/A_{n-1})$ and $\mathcal{D}(B/R)$. Note that q is a principal homogeneous space under $\left(H^{(i)}_n/H^{(i)}_{n-1}\right)^D$ thus giving

$$\mathcal{D}(A_n/A_{n-1}) \otimes_{A_{n-1}} A_n = \mathcal{D}(B/R) \otimes_R A_n$$

as $A_n \otimes_{A_{n-1}} A_n = B \otimes_R A_n$-modules, therefore $\mathcal{D}(B/R) = \mathrm{Hdg}^{p^{n-1}} B$ and by faithfully flat descent

$$\mathcal{D}(A_n/A_{n-1}) = \mathrm{Hdg}^{p^{n-1}} B_n. \tag{7.3}$$

We can suppose A_n is free over A_{n-1}, then the map

$$\begin{aligned}\mathcal{D}(A_n/A_{n-1})^{-1} &\to \mathrm{Hom}_{A_{n-1}}(A_n, A_{n-1}) \\ x &\mapsto \left(y \mapsto \mathrm{Tr}_{A_n/A_{n-1}}(xy)\right)\end{aligned}$$

is an isomorphism. The existence of an A_{n-1}-linear surjective map $A_n \to A_{n-1}$ gives that

$$\mathrm{Tr}_{A_n/A_{n-1}} : \mathcal{D}(A_n/A_{n-1})^{-1} \to A_{n-1}$$

and in view of (7.3) we see that

$$\mathrm{Tr}_{A_n/A_{n-1}}(A_n) = \mathrm{Hdg}^{p^{n-1}} A_{n-1}.$$

□

Remark 7.16. Let β_n denote the ideal $p^n \mathrm{Hdg}^{-\frac{p^n-1}{p-1}} \subseteq \mathcal{O}_{\mathfrak{IG}_n^{(i)}}$. This makes sense in view of [AIP1, Lemma 4.2, pag. 16].

Define

$$\mathfrak{IG}_{n,r} = \prod_{i=1}^{d} \mathfrak{IG}_{n,r}^{(i)} \xrightarrow{\mathfrak{h}^n} \mathfrak{Y}_r^{\mathrm{tor}}.$$

Proposition 7.17. *Let $r \geq 2$ be an integer and let $n \leq (r-1)+k$ where $p \in \alpha^{p^k} A_0$, then the isogeny $\mathbf{A} \to \mathbf{A}/H_1(\mathbf{A})$ induces a commutative diagram*

$$\begin{array}{ccc} \mathfrak{IG}_{n,r} & \xrightarrow{\Phi_r} & \mathfrak{IG}_{n,r-1} \\ \downarrow & & \downarrow \\ \mathfrak{Y}_r & \xrightarrow[\phi_r]{} & \mathfrak{Y}_{r-1} \end{array}$$

where

1. *ϕ_r is a finite morphism whose restriction ϕ_r^R to the Rapoport locus defines a finite flat morphism $\phi_r^R : \mathfrak{Y}_r^R \to \mathfrak{Y}_{r-1}^R$ of degree p^g;*
2. *Φ_r is $(\mathcal{O}_L/p^n)^{\times}$-equivariant and restricts to morphisms*

$$\Phi_r^{(i)} : \mathfrak{IG}_{n,r}^{(i)} \to \mathfrak{IG}_{n,r}^{(i)}.$$

Proof. These two statements are [AIP1, Proposition 3.3, pag. 11] and [AIP1, Proposition 3.6, pag. 13]. We only sketch the construction of Φ_r. Let $\underline{\mathcal{O}_L/p^n} \to H_n^D$ be a local isomorphism on $\mathcal{Y}_r$, and note that $H_n' = H_{n+1}/H_1$ is the level n canonical subgroup of $\mathbf{A}/H_1(\mathbf{A})$ and multiplication by p gives an isomorphism $H_{n+1}/H_1 \to H_n$, whence $\underline{\mathcal{O}_L/p^n} \xrightarrow{\sim} (H_n')^D$. In particular all the maps involved are $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p$-linear and hence respect the product with the idempotents e_i. It follows that the restrictions $\Phi_r^{(i)}$ are well defined. □

Corollary 7.18. *Let $I' \subseteq I$ intervals, $r' \geq r$ and $n' \geq n$ integers that satisfy the conditions in Proposition 7.17. The we have a commutative diagram*

$$\begin{array}{ccc} \mathfrak{IG}_{n',r',I'} & \xrightarrow{\Phi} & \mathfrak{IG}_{n,r,I} \\ \downarrow & & \downarrow \\ \mathfrak{Y}_{r',I'} & \xrightarrow[\phi]{} & \mathfrak{Y}_{r,I} \end{array}$$

and Φ restricts to morphisms

$$\Phi^{(i)} : \mathfrak{IG}_{n',r',I'}^{(i)} \to \mathfrak{IG}_{n,r,I}^{(i)}.$$

Remark 7.19. We can apply the same method to define $\mathcal{IG}_{n,r,I}'^{(i)} \to \mathcal{IG}_{n,r,I}$ as the space that classifies isomorphisms $e_i A^D[p^n] \simeq (\mathcal{O}_{F_i}/\mathfrak{p}_i^n)^2$ which are compatible with $H_n^{(i),D}(A) \simeq \mathcal{O}_{F_i}/\mathfrak{p}_i^n$. The map $\mathcal{IG}_{n,r,I}'^{(i)} \to \mathcal{X}_{r,I}$ is finite étale and Galois. We call $\mathfrak{IG}_{n,r,I}'^{(i)} \to \mathfrak{IG}_{n,r,I}^{(i)}$ its normalisation and $\mathfrak{IG}_{n,r,I}' = \prod_{i=1}^{d} \mathfrak{IG}_{n,r,I}'^{(i)}$.

7.4 A digression on polarisations

Here we follows [AnGo1, Section 3]. Let $A_{/S}$ be an abelian scheme with real multiplication by $\mathcal{O}_L$ and assume that it satisfies condition (DP), that is, the natural map

$$\begin{aligned} \phi_A : A \otimes_{\mathcal{O}_L} \mathcal{M}_A &\to A^\vee \\ x \otimes \lambda &\mapsto \lambda(x) \end{aligned}$$

is an isomorphism as étale sheaves over S. Let $t \in \mathbb{N}$ be positive integer. Locally in the étale site of S we have an $\mathcal{O}_L$-module isomorphism

$$\eta : \frac{\mathcal{O}_L}{t\mathcal{O}_L} \to \frac{\mathcal{M}_A}{t\mathcal{M}_A}$$

and let $\lambda_t \in \mathcal{M}_A$ be any lift of $\eta(1)$. Looking at the commutative diagram

$$\begin{array}{ccc} A[t] & \longrightarrow & A \otimes_{\mathcal{O}_L} \frac{\mathcal{M}_A}{t\mathcal{M}_A} \\ \downarrow & & \uparrow \\ A[t] \otimes_{\mathcal{O}_L} \frac{\mathcal{O}_L}{t\mathcal{O}_L} & \longrightarrow & A[t] \otimes_{\mathcal{O}_L} \frac{\mathcal{M}_A}{t\mathcal{M}_A} \end{array}$$

we conclude that the map

$$\begin{aligned} A[t] &\to A \otimes_{\mathcal{O}_L} \frac{\mathcal{M}_A}{t\mathcal{M}_A} \\ x &\mapsto x \otimes \overline{\lambda_t} \end{aligned}$$

is an isomorphism. Define ξ_t using the diagram of isomorphisms

$$\begin{array}{ccc} A[t] & \xrightarrow{\xi_t} & A^\vee[t] \\ \downarrow & & \uparrow \\ A \otimes_{\mathcal{O}_L} \frac{\mathcal{M}_A}{t\mathcal{M}_A} & \longleftarrow & (A \otimes_{\mathcal{O}_L} \mathcal{M}_A)[t] \end{array}$$

getting that

$$\lambda_{t|A[t]} : A[t] \simeq A^\vee[t]$$

is an isomorphism. It follows from Sylow's firts Theorem that $\ker(\lambda_t)$ has order coprime with t and hence we proved

Proposition 7.20. *Let $A_{/S}$ be an abelian scheme with* RM *that satisfies condition (DP). Then for every integer $t \neq 0$ it admits a polarisation of degree prime to t.*

This fact is interesting because it allows to assume our p-divisible groups are principally polarised. Indeed let $A \to S = \operatorname{Spec}(R)$ be a g-dimensional abelian scheme over a p-adically complete ring R and let $f : A \to A^\vee$ be an isogeny whose degree, say k, is prime to p with kernel K. Let $U \to S$ be an $fppf$ open over which the sequence

$$0 \to K(U) \to A(U) \xrightarrow{f_U} A^\vee(U) \to 0$$

is an exact sequence in abelian groups (or modules over a fixed commutative ring $\mathcal{O}$). For every integer $n \geq 1$ note that the p^n-torsion functor $(\bullet)[p^n] = \mathrm{Hom}_{\mathcal{O}}\left(\frac{\mathcal{O}}{p^n}, \bullet\right)$, then we have the associated long exact sequence

$$0 \to K(U)[p^n] \to A(U)[p^n] \xrightarrow{f_U} A^\vee(U)[p^n] \to \mathrm{Ext}^1_{\mathcal{O}}\left(\frac{\mathcal{O}}{p^n}, K(U)\right),$$

but

$$K(U)[p^n] = \mathrm{Ext}^1_{\mathcal{O}}\left(\frac{\mathcal{O}}{p^n}, K(U)\right) = 0$$

since p^n acts both as 0 and as an invertible element (being k prime to p). In conclusion

$$f_p : A[p^\infty] \to A^\vee[p^\infty]$$

is an isomorphism if p-divisible $\mathcal{O}$-modules.

Note 7.21. In our case this remark applies as follows: let $A \to S$ be an object parametrised by $\mathrm{M}(\mu_N, \mathfrak{c})$, in particular $\mathfrak{c} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ has rank 1 as an $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p$-module and every element $x \in \mathfrak{c}$ such that $x \otimes_{\mathbb{Z}} 1$ is a generator gives rise to a polarisation $A \to A^D$ whose degree is necessarily prime to p. Suppose that A admits a canonical subgroup $H_n = H_n(A)$ of level n, then $A/H_n(A)$ gives a point in $\mathrm{M}(\mu_N, p^n\mathfrak{c})$ (cfr. [?, pag. 88]) therefore we can induce a prime-to-p polarisation by considering the element $p^n x$. Hence we fix such an element $x \in \mathfrak{c}$ once an for all. In view of the previous discussion we conclude that all the p-divisible $\mathcal{O}_L$-modules $\frac{A}{H_n(A)}[p^\infty]$ come with compatible isomorphisms (principal $\mathcal{O}_L$-polarisations)

$$\frac{A}{H_n(A)}[p^\infty] \to \left(\frac{A}{H_n(A)}\right)^\vee [p^\infty].$$

Note 7.22. Let $G = \mathbf{A}[p^\infty]$ be the p-divisible group associated to the universal object $\mathbf{A} \to \mathfrak{IG}_{n,r,I}$ and let $\lambda : G \to G'$ be the quotient by the canonical subgroup. Then $\mathrm{Hdg}(G') = \mathrm{Hdg}(G)^p$ and in particular $\delta^p_{G'} = \delta^{p^2}_G \subseteq \delta^p_G$. Note that $\lambda^\vee$ is Verschiebung modulo $p\mathrm{Hdg}_G^{-1}$ hence by definition of the Hasse invariant we have $(\lambda^\vee)^* \omega_G \subseteq \mathrm{Hdg}_G \omega_{G'}$. On the other hand $(\lambda^\vee)^*$ is an $\mathcal{O}_L$-linear map, being the $\mathcal{O}_L$-structure on H_1 compatible with the one on G, therefore for every $\sigma \in \Sigma_L$ we have

$$\mathrm{Hdg}_G \omega_{G', \mathrm{Fr}^{-1}\sigma} \subseteq (\lambda^\vee)^* \omega_{G,\sigma}$$

modulo $p\mathrm{Hdg}_G^{-1}$ since these components are now invertible $\mathcal{O}_{\overline{\mathfrak{IG}_{n,r,I}}}$-modules and finally

$$(\lambda^\vee)^* \omega_G = \mathrm{Hdg}_G \omega_{G'}.$$

modulo $p\mathrm{Hdg}_G^{-1}$. Note that $p \in \mathrm{Hdg}_G^{p^{n+1}}$ so that this means

$$(\lambda^\vee)^* \omega_G + p\mathrm{Hdg}_G^{-1} \omega_{G'} = \mathrm{Hdg}_G \omega_{G'},$$

in view of Nakayama's Lemma we conclude that

$$(\lambda^\vee)^* \omega_G = \mathrm{Hdg}_G \omega_{G'}.$$

Taking linear duals

$$\begin{aligned}(\lambda^*)^{\vee}\left(\delta_G^p\omega_G^{\vee}\right) &= p\mathrm{Hdg}_G^{-1}\delta_G^p\omega_{G'}^{\vee}\\ &= p\mathrm{Hdg}_G^{-p}\delta_G^{p-p^2}\cdot\delta_{G'}^p\omega_{G'}^{\vee}\\ &= p\mathrm{Hdg}_G^{-p-1}\cdot\delta_{G'}^p\omega_{G'}^{\vee}\\ &= \tau_G\cdot\delta_{G'}^p\omega_{G'}^{\vee}.\end{aligned}$$

where we set $\tau_G = p\mathrm{Hdg}_G^{-p-1}$. Recall that

$$p \in (\alpha) \subseteq \mathrm{Hdg}_G^{p^2} \tag{7.4}$$

(as follows from the construction in Proposition 7.7), then the inclusion (7.4) entails

$$\tau_G^{pm} \subseteq \left(\frac{\alpha}{\mathrm{Hdg}_G^{p^2}}\right)^{pm} \subseteq \alpha^m\left(\frac{\alpha}{\mathrm{Hdg}^{p^3}}\right)^m \subseteq (\alpha)^m.$$

8 The sheaves for p unramified

Recall that we introduced the formal scheme $\mathrm{Bl}_{\mathfrak{m}}\mathfrak{W}^0 \to \mathrm{Spf}\left(\mathbb{Z}_p\right)$ as a formal model for the adic weight space in 6.2. For $\alpha \in \mathfrak{m}\backslash\mathfrak{m}^2$ and an interval $I \subseteq \mathbb{Q}_{\geq 0}$, then the rings $B^0_{\alpha,I}$ satisfy the conditions of 7.2, hence we end up with formal schemes

$$\mathfrak{IG}^{(i)}_{n,r,I} \to \mathfrak{X}_{r,I} \to \mathrm{Spf}\left(B^0_{\alpha,I}\right).$$

Fix an integer $r \geq 1$, an interval $I = \left[p^k, p^s\right]$ for two integers $s \geq k \geq 0$ and $n \leq r+k$. We let $\mathfrak{X}$ denote $\mathfrak{X}_{r,I}$, $\mathfrak{IG}^{(i)}_n$ denote $\mathfrak{IG}^{(i)}_{n,r,I}$ etc... We let $\mathbf{A} = \mathbf{A}_{r,I}$ be the universal semi-abelian scheme over $\mathfrak{X}$ and $\omega_{\mathbf{A}}$ the corresponding sheaf of invariant differentials.

Remark 8.1. Under these assumptions the level-n canonical subgroup $H_n \subseteq \mathbf{A}\left[p^n\right]$ is defined (Theorem 7.11) and it comes with an $\mathcal{O}_L$-linear structure (Corollary 7.12) compatible with that of $\mathbf{A}$. Since H_n is p^n-torsion, the map $\omega_{\mathbf{A}} \to \omega_{H_n}$ factors through $\omega_{\mathbf{A}}/p^n\omega_{\mathbf{A}}$ and by 4. in Theorem 7.11 we see that the kernel of this last map is anihilated by $\mathrm{Hdg}^{\frac{p^n-1}{p-1}}$, in particular we have a sequence of epimorphisms

$$\omega_{\mathbf{A}} \to \omega_{H_n} \to \frac{\omega_{\mathbf{A}}}{\beta_n\underline{\omega}_{\mathbf{A}}}$$

as *fppf* sheaves in $\mathcal{O}_L\otimes_{\mathbb{Z}}\mathcal{O}_{\mathfrak{X}}$-modules over $\mathfrak{X}$.

Note 8.2. We quickly recall the constuction of the map dlog: given a finite flat commutative group $G \to S$, let $f : U = \mathrm{Spec}\,(R) \to S$ be an fppf morphism. An R-point $x \in G^D(R)$ is a morphism

$$x : G_{/U} \to \mathbb{G}_{m/U}$$

hence the section

$$x^*\left(\frac{dT}{T}\right) \in f^*\omega_{G/U}$$

is defined, where dT/T denotes the invariant differential differential on $\mathbb{G}_{m/U}$. This rule defines a morphism of fppf abelian sheaves

$$\mathrm{dlog}_G : G^D \to \omega_G.$$

Proposition 8.3. *Consider the morphism*

$$\mathrm{dlog}_{H_n} : H_n^D \to \omega_{H_n}$$

of abelian fppf sheaves on $\mathfrak{IG}_n$ *and let* P_n *be the universal generator of* H_n^D. *Denote* $\Omega_{\mathbf{A}}$ *the sub-*$\mathcal{O}_{\mathfrak{IG}_n} \otimes_{\mathbb{Z}} \mathcal{O}_L$*-module of* $\omega_{\mathbf{A}}$ *generated by the lifts of*

$$s = \mathrm{dlog}_{H_n}(P_n) \in \frac{\omega_{\mathbf{A}}}{p^n \mathrm{Hdg}^{-\frac{p^n-1}{p-1}} \omega_{\mathbf{A}}},$$

then $\Omega_{\mathbf{A}}$ *is a locally free* $\mathcal{O}_{\mathfrak{IG}_n} \otimes_{\mathbb{Z}} \mathcal{O}_L$*-module of rank 1 and the map* dlog_{H_n} *induces an* $\mathcal{O}_{\mathfrak{IG}_n} \otimes_{\mathbb{Z}} \mathcal{O}_L$*-module isomorphism*

$$\mathcal{O}_L \otimes_{\mathbb{Z}} \frac{\mathcal{O}_{\mathfrak{IG}_n}}{p^n \mathrm{Hdg}^{-\frac{p^n}{p-1}} \mathcal{O}_{\mathfrak{IG}_n}} \simeq \frac{\Omega_{\mathbf{A}}}{p^n \mathrm{Hdg}^{-\frac{p^n}{p-1}} \Omega_{\mathbf{A}}}.$$

Proof. This is [AIP1, Proposition 4.1, pag. 15]. □

8.1 The Gauss-Manin connection

Note 8.4. Let $X \to \mathrm{Spec}\mathcal{O}_{L^{\mathrm{Gal}}}\left[d_L^{-1}\right] = \mathrm{Spec}\,(A)$ be a scheme of finite type and let M be a coherent locally free $\mathcal{O}_X$-module with a connection $\nabla : M \to M \otimes_{\mathcal{O}_X} \Omega^1_{X/A}$. Consider the base-change

$$\nabla_{\mathbb{Q}} : M \otimes_{\mathbb{Z}} \mathbb{Q} \to (M \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathbb{Q}} \Omega^1_{X_{\mathbb{Q}}/L^{\mathrm{Gal}}},$$

where we used the fact that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$, and suppose that, for every $\gamma : L \to L^{\mathrm{Gal}}$ we have

$$\nabla_{\mathbb{Q}}\left((M \otimes_{\mathbb{Z}} \mathbb{Q})(\gamma)\right) \subseteq (M \otimes_{\mathbb{Z}} \mathbb{Q})(\gamma) \otimes_{\mathcal{O}_{X_{\mathbb{Q}}}} \Omega^1_{X_{\mathbb{Q}}/L^{\mathrm{Gal}}}.$$

Note that, since localisation is an additive functor, we have $(M \otimes_{\mathbb{Z}} \mathbb{Q})(\gamma) = M(\gamma) \otimes_{\mathbb{Z}} \mathbb{Q}$, hence[4]

$$\begin{aligned}(M \otimes_{\mathbb{Z}} \mathbb{Q})(\gamma) \otimes_{\mathcal{O}_{X_{\mathbb{Q}}}} \Omega^1_{X_{\mathbb{Q}}/L^{\mathrm{Gal}}} &= (M(\gamma) \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathcal{O}_{X_{\mathbb{Q}}}} \left(\Omega^1_{X/A} \otimes_{\mathbb{Z}} \mathbb{Q}\right) \\ &= \left(M(\gamma) \otimes_{\mathcal{O}_X} \Omega^1_{X/A}\right) \otimes_{\mathbb{Z}} \mathbb{Q}.\end{aligned}$$

Therefore

$$\nabla(M(\gamma)) \subseteq \left(M \otimes_{\mathcal{O}_X} \Omega^1_{X/A}\right) \cap \left[\left(M(\gamma) \otimes_{\mathcal{O}_X} \Omega^1_{X/A}\right) \otimes_{\mathbb{Z}} \mathbb{Q}\right] \subseteq M(\gamma) \otimes_{\mathcal{O}_X} \Omega^1_{X/A}.$$

Lemma 8.5. *Let* $\mathbf{M} = \mathbf{M}_N \to \mathrm{Spec}\left(\mathcal{O}_{L^{\mathrm{Gal}}}\left[d_L^{-1}\right]\right)$ *be the base change of the scheme defined in Theorem 2.12 and let*

$$\nabla : H^1_{\mathrm{dR}} = H^1_{\mathrm{dR}}(\mathbf{A}/\mathbf{M}) \to H^1_{\mathrm{dR}} \otimes_{\mathcal{O}_{\mathbf{M}}} \Omega^1_{\mathbf{M}/\mathcal{O}_{L^{\mathrm{Gal}}}\left[d_L^{-1}\right]}$$

be the Gauss-Manin connection. Then for every $\gamma : L \to L^{\mathrm{Gal}}$ *we have*

$$\nabla\left(H^1_{\mathrm{dR}}(\gamma)\right) \subseteq H^1_{\mathrm{dR}}(\gamma) \otimes_{\mathcal{O}_{\mathbf{M}}} \Omega^1_{\mathbf{M}/\mathcal{O}_{L^{\mathrm{Gal}}}\left[d_L^{-1}\right]}.$$

[4] Recall that given two $\mathbb{Q}$-vector spaces V, W we have

$$V \otimes_{\mathbb{Z}} W = V \otimes_{\mathbb{Q}} W.$$

Proof. We follow [Kat, Lemma 2.1.14, pag 229]. Let $\Sigma = \mathrm{Hom}_{\mathbb{Q}}\left(L, L^{\mathrm{Gal}}\right)$. Embed $L^{\mathrm{Gal}} \subseteq \mathbb{C}$ and consider the base change $\mathbf{M}_{\mathbb{C}} \to \mathrm{Spec}\,(\mathbb{C})$, whose associated analytic variety has uniformisation by $\mathfrak{h}\,(L) = \mathfrak{h}^{\Sigma}$, where $\mathfrak{h}$ is the usual complex half plane ([Kat, Section 1.4, pag. 213]). Over $\mathfrak{h}\,(L)$ we have a horizontal isomorphism between H^1_{dR} and a constant sheaf

$$V \otimes_{\mathbb{C}} \mathcal{O}_{\mathfrak{h}(L)} \simeq H^1_{\mathrm{dR}}$$

with the trivial connection $\mathrm{id}_V \otimes_{\mathbb{C}} d$. For every $\gamma \in \Sigma$ let $\{X_\gamma, Y_\gamma\}$ be a basis of horizontal sections of $H^1_{\mathrm{dR}}\,(\gamma)$. Then Kodaira-Spencer isomorphism reads

$$\Omega^1_{\mathfrak{h}(L)/\mathbb{C}} \to \underline{\omega}^{\otimes 2} = \bigoplus_{\sigma} (X_\sigma - \tau_\sigma Y_\sigma)^2 \cdot \mathcal{O}_{\mathfrak{h}(L)}$$
$$2\pi i\, d\tau_\sigma \mapsto (X_\sigma - \tau_\sigma Y_\sigma)^2$$

Let $\tilde{D}_\sigma : H^1_{\mathrm{dR}} \to H^1_{\mathrm{dR}} \otimes \omega\left(\sigma^2\right)$ as in Note 4.4, then

$$\tilde{D}_\sigma = \frac{1}{2\pi i} (X_\sigma - \tau_\sigma Y_\sigma)^2 \frac{\partial}{\partial \tau_\sigma},$$

hence if $\xi = f X_\gamma + g Y_\gamma \in H^1_{\mathrm{dR}}\,(\gamma)$ we have

$$\tilde{D}_\sigma\,(\xi) = \frac{1}{2\pi i} \left(\frac{\partial f}{\partial \tau_\sigma} X_\gamma + \frac{\partial g}{\partial \tau_\sigma} Y_\gamma\right) \otimes_{\mathbb{C}} (X_\sigma - \tau_\sigma Y_\sigma)^2 \in H^1_{\mathrm{dR}}\,(\gamma) \otimes \omega\left(\sigma^2\right).$$

Since have $\nabla = \sum_\sigma \tilde{D}_\sigma$ we conclude that the statements holds on $\mathbf{M}_L$ and in view of Note 8.4 that it holds over $\mathbf{M}$. □

Back to our setting

Proposition 8.6. *Let $\nabla : H^1_{\mathrm{dR}} \to H^1_{\mathrm{dR}} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/S}$ be the Gauss-Manin connection. Then*

1. *∇ restricts to an integrable connection*

$$\nabla^{(i)} : H^{1,(i)}_{\mathrm{dR}} \to H^{1,(i)}_{\mathrm{dR}} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/S}$$

for every i;

2. *the connection $\nabla^{(i)}$ respects the respects the $\mathcal{O}_{F_i}$-structure.*

Proof. In view of Lemma 8.5 we know that for every $\gamma : L \to L^{\mathrm{Gal}}$ we have

$$\nabla\left(H^1_{\mathrm{dR}}\,(\gamma)\right) \subseteq H^1_{\mathrm{dR}}\,(\gamma) \otimes_{\mathcal{O}_{\mathbf{M}}} \Omega^1_{\mathbf{M}/\mathcal{O}_{L^{\mathrm{Gal}}}\left[d_L^{-1}\right]}.$$

Note that, being H^1_{dR} an $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathcal{O}_{\mathfrak{X}}$-module we have

$$H^{1,(i)}_{\mathrm{dR}} = \bigoplus_{\sigma \in \mathfrak{G}_i} H^1_{\mathrm{dR}}\,(\sigma)$$

Consider the base change $\mathbf{M}_{\mathcal{O}_{F_i}} = \mathbf{M} \otimes_{\mathcal{O}_{L^{\mathrm{Gal}}}\left[d_L^{-1}\right]} \mathcal{O}_{F_i}$, then

$$\nabla\left(H^1_{\mathrm{dR}}\,(\gamma)\right) \subseteq H^1_{\mathrm{dR}}\,(\gamma) \otimes_{\mathcal{O}_{\mathbf{M}_{\mathcal{O}_{F_i}}}} \Omega^1_{\mathbf{M}_{\mathcal{O}_{F_i}}/\mathcal{O}_{F_i}}$$

for every $\gamma \in \mathfrak{G}_i$ since, in view of Remark 7.13 these automorphisms identify with a subset of the embeddings $L \to L^{\mathrm{Gal}}$. □

8.2 Descending to $\mathfrak{IG}_n$

Define the formal group $\mathfrak{T} = 1 + \beta_n \cdot \mathrm{Res}_{\mathcal{O}_L/\mathbb{Z}}(\mathbb{G}_a)$, hence over $\mathfrak{IG}_n^{(i)}$ the universal character gives a morphism $\kappa^{(i)} : \mathfrak{T}^{(i)} = 1 + \beta_n \mathrm{Res}_{\mathcal{O}_{F_i}|\mathbb{Z}_p}(\mathbb{G}_a) \to \mathbb{G}_{m|\mathfrak{IG}_n^{(i)}}$ for every i.

Proposition 8.7. *Let $(\mathcal{E}, s)$ be an $\mathrm{MS}_{\mathcal{O}_{F_i}}$-datum of rank m on $\mathfrak{IG}_n^{(i)}$ and denote with $\pi : \mathbb{V}_0(\mathcal{E}, s) \to \mathfrak{IG}_n^{(i)}$ the formal $\mathcal{O}_{F_i}$-module bundle with marked sections associated to it, then*

1. *the map π has a natural action of $\mathfrak{T}^{(i)}$;*
2. *let $\mathrm{Spf}(R) \subseteq \mathfrak{IG}_n^{(i)}$ be an open subset over which $\mathcal{E}$ and β_n are free, then*

$$\pi_* \mathcal{O}_{\mathbb{V}_{0,\sigma}(\mathcal{E},s)}\left[\kappa^{(i)}\right]_{|\mathrm{Spf}(R)} = \prod_{u \in S} R\langle V_{2,\sigma,u}, \dots, V_{m,\sigma,u}\rangle \cdot \kappa^{(i)}(\sigma(u) + \beta_n Z_{\sigma,u})$$

where $V_{k,\sigma,u} = (\sigma(u) + \beta_n Z_{\sigma,u})^{-1} X_{k,\sigma,u}$.

Proof.

1. The group $\mathfrak{T}^{(i)}$ acts naturally on $\mathcal{E}$ since it is an $\mathcal{O}_{F_i} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathfrak{IG}_n^{(i)}}$-module, moreover it is the trivial group modulo β_n, therefore it respects the marked section s and acts by automorphism of $\mathrm{MS}_{\mathcal{O}_{F_i}}$-datum on $(\mathcal{E}, s)$. The action on π is the one induced by functoriality.
2. The formal torus $\mathfrak{T}^{(i)}$ acts on $\mathbb{V}(\mathcal{E})$ via $\mathbb{G}_m$, hence for $t = (1 + \beta_n r_\sigma)_\sigma \in \mathfrak{T}^{(i)}(R)$ we have

$$t * X_{k,\sigma,u} = t X_{k,\sigma,u} = (1 + \beta_n r_\sigma) X_{k,\sigma,u}$$

for every k, σ, u. Requiring equivariance we get $t_\sigma * (\sigma(u) + \beta_n Z_\sigma) = (1 + \beta_n r_\sigma)(\sigma(u) + \beta_n Z_\sigma)$ hence

$$t * Z_{\sigma,u} = t_\sigma * Z_{\sigma,u} = \sigma(u) \frac{t_\sigma - 1}{\beta_n} + t_\sigma Z_{\sigma,u}.$$

In view of the local analitycity of κ, and hence of $\kappa^{(i)}$, we have

$$\begin{aligned} t * \kappa^{(i)}(\sigma(u) + \beta_n Z_{\sigma,u}) &= \kappa^{(i)}(t * (\sigma(u) + \beta_n Z_{\sigma,u})) \\ &= \kappa^{(i)}(t \cdot (\sigma(u) + \beta_n Z_{\sigma,u})). \end{aligned}$$

and hence $t * \kappa^{(i)}(\sigma(u) + \beta_n Z_{\sigma,u}) = \kappa^{(i)}(t)\, \kappa^{(i)}(\sigma(u) + \beta_n Z_{\sigma,u})$, that is $\kappa^{(i)}(\sigma(u) + \beta_n Z_{\sigma,u}) \in R\langle Z_{\sigma,u}, X_{2,\sigma,u} \dots, X_{m,\sigma,u}\rangle \left[\kappa^{(i)}\right]$ and

$$R\langle V_{2,\sigma,u}, \dots, V_{m,\sigma,u}\rangle \cdot \kappa^{(i)}(\sigma(u) + \beta_n Z_{\sigma,u}) \subseteq R\langle Z_{\sigma,u}, X_{2,\sigma,u} \dots, X_{m,\sigma,u}\rangle \left[\kappa^{(i)}\right].$$

To conclude we just need to show that

$$R\langle Z_{\sigma,u}, X_{2,\sigma,u}, \dots, X_{m,\sigma,u}\rangle^{\mathfrak{T}^{(i)}(R)} = R\langle V_{2,\sigma,u}, \dots, V_{m,\sigma,u}\rangle.$$

Note that $V_i \in R\langle Z_{\sigma,u}, X_{2,\sigma,u}, \dots, X_{m,\sigma,u}\rangle^{\mathfrak{T}^L(R)}$ for every i and $R\langle Z_{\sigma,u}, X_{2,\sigma,u}, \dots, X_{m,\sigma,u}\rangle = R\langle Z_{\sigma,u}, V_{2,\sigma,u}, \dots, V_{m,\sigma,u}\rangle$. Let

$$f = \sum_{|\gamma| \geq 0} c_\gamma Z_{\sigma,u}^{\gamma_1} V_{2,\sigma,u}^{\gamma_2} \cdots V_{m,\sigma,u}^{\gamma_m} \in R\langle Z_{\sigma,u}, X_{2,\sigma,u}, \dots, X_{m,\sigma,u}\rangle^{\mathfrak{T}^{(i)}(R)},$$

in view of the formulas above

$$f=\sum_{|\boldsymbol{\gamma}|\geq 0} c_{\boldsymbol{\gamma}}\left(\sigma\left(u\right)\frac{t_\sigma-1}{\beta_n}+t_\sigma Z_{\sigma,u}\right)^{\gamma_1} V_{2,\sigma,u}^{\gamma_2}\dots V_{m,\sigma,u}^{\gamma_m}$$

for every $t=\left(1+\beta_n r_\sigma\right)_\sigma\in\mathfrak{T}^{(i)}\left(R\right)$. We see it as an equality in $R\left\langle Z_{\sigma,u}\right\rangle\left[\left[V_{2,\sigma,u},\dots,V_{m,\sigma,u}\right]\right]$ giving

$$\sum_{|\boldsymbol{\gamma}|\geq 0} c_{\boldsymbol{\gamma}} Z_{\sigma,u}^{\gamma_1}=\sum_{|\boldsymbol{\gamma}|\geq 0} c_{\boldsymbol{\gamma}}\left(\sigma\left(u\right)\frac{t_\sigma-1}{\beta_n}+t_\sigma Z_{\sigma,u}\right)^{\gamma_1}.$$

Setting $Z_{\sigma,u}=0$ we see that, for every $r\in R$ the relation

$$\sum_{|\boldsymbol{\gamma}|\geq 0} c_{\boldsymbol{\gamma}}=\sum_{|\boldsymbol{\gamma}|\geq 0} c_{\boldsymbol{\gamma}} r^{\gamma_1}$$

and Weiertrass preparation says that $c_{\boldsymbol{\gamma}}=0$ for $\gamma_1\neq 0$. Finally we conclude since for $f\in R\left\langle Z_{\sigma,u},X_{2,\sigma,u}\dots,X_{m,\sigma,u}\right\rangle\left[\kappa^{(i)}\right]$ we have

$$f\left(Z_{\sigma,u},X_{2,\sigma,u}\dots,X_{m,\sigma,u}\right)\cdot\kappa^{(i)}\left(\sigma\left(u\right)+\beta_n Z_{\sigma,u}\right)^{-1}\in R\left\langle Z_{\sigma,u},X_{2,\sigma,u}\dots,X_{m,\sigma,u}\right\rangle^{\mathfrak{T}^{(i)}(R)}=R\left\langle V_{2,\sigma,u},\dots,V_{m,\sigma,u}\right\rangle.$$

□

Note that, since $\Omega_{\mathbf{A}}$ is an invertible $\mathcal{O}_L\otimes_{\mathbb{Z}}\mathcal{O}_{\mathfrak{IG}_n}$-module, $\Omega_{\mathbf{A}}^{(i)}$ is an invertible $\mathcal{O}_{F_i}\otimes_{\mathbb{Z}_p}\mathcal{O}_{\mathfrak{IG}_n^{(i)}}$-module.

Definition 8.8. Define $\varpi^{\kappa^{(i)}}=\varpi_{n,r,I}^{\kappa^{(i)}}=\left(\pi_*\mathcal{O}_{\mathbb{V}_0\left(\Omega_{\mathbf{A}}^{(i)},\mathbf{s}\right)}\right)\left[\kappa^{(i)}\right]$, that is the subsheaf of $\pi_*\mathcal{O}_{\mathbb{V}_0\left(\Omega_{\mathbf{A}}^{(i)},\mathbf{s}\right)}$ consisting of sections transforming according to the character $\kappa^{(i)}$ under the action of $\mathfrak{T}^{(i)}$.

Remark 8.9. It follows from Proposition 8.7 that $\varpi^{\kappa^{(i)}}$ is a locally free $\mathcal{O}_{\mathfrak{IG}_n^{(i)}}$-module of rank $\#S$.

What we did here for the character κ holds verbatim when κ is replaced by any locally analytic character $\chi:\mathfrak{T}\to\mathbb{G}_m$.

Corollary 8.10. *With setting and notations as in Proposition 8.7 , let $\mathcal{E}$ be an invertible $\mathcal{O}_{F_i}\otimes_{\mathbb{Z}_p}\mathcal{O}_{\mathfrak{IG}_n^{(i)}}$-module. Then for every integer k we have*

$$\pi_*\mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)}\left[k\right]\simeq\prod_{u\in S}\mathcal{E}^{\otimes k}.$$

Proof. By construction we have an embedding $\mathcal{E}\to\mathbb{V}_{\mathcal{O}_{F_i}}\left(\mathcal{E}\right)$ as the homogeneous component of degree 1. Since $\mathcal{E}$ is invertible, we have that $\mathcal{E}^{\otimes k}\subseteq\pi_*\mathcal{O}_{\mathbb{V}_{\mathcal{O}_F}(\mathcal{E})}$ is the submodule locally with basis $\left\{X_{1,\sigma}^k\right\}_\sigma$ therefore

$$\mathcal{E}^{\otimes k}=\pi_*\mathcal{O}_{\mathbb{V}_{\mathcal{O}_F}(\mathcal{E})}\left[k\right]$$

since $\mathfrak{T}^{(i)}$ acts on $\mathbb{V}_{\mathcal{O}_{F_i}}\left(\mathcal{E}\right)$ via $\mathbb{G}_m$. The map $\xi_u:\pi_*\mathcal{O}_{\mathbb{V}_{\mathcal{O}_{F_i}}(\mathcal{E})}\to\pi_*\mathcal{O}_{\mathbb{V}_0^u(\mathcal{E},s)}$ is $\mathfrak{T}^{(i)}$-equivariant hence it gives

$$\xi_u:\pi_*\mathcal{O}_{\mathbb{V}_{\mathcal{O}_{F_i}}(\mathcal{E})}\left[k\right]\to\pi_*\mathcal{O}_{\mathbb{V}_0^u(\mathcal{E},s)}\left[k\right].$$

In view of Proposition 8.7 it is an isomorphism because locally gives

$$\xi_u:X_{1,\sigma}^k\mapsto\left(\sigma\left(u\right)+\beta_n Z_{\sigma,u}\right)^k.$$

□

Note 8.11. It is shown in [AIP2, Proposition A.3, pag. 43] that there exists an invertible ideal $\underline{\delta} \subseteq \mathcal{O}_{\mathfrak{I}\mathfrak{G}_1}$ with the property that $\underline{\delta}^{p-1} = \mathrm{Hdg}$ and $\underline{\delta} \cdot \underline{\omega}_{\mathbf{A}} = \Omega_{\mathbf{A}}$.

Consider now the Hodge filtration on $\mathbb{H}^1_{\mathrm{dR}}(\mathbf{A}/\mathfrak{I}\mathfrak{G}_n)$, this is given by the exat sequence

$$\mathcal{H}^\bullet_{\mathbf{A}} \quad = \quad 0 \to \underline{\omega}_{\mathbf{A}} \to \mathbb{H}^1_{\mathrm{dR}}(\mathbf{A}/\mathfrak{I}\mathfrak{G}_n) \to \underline{\omega}^\vee_{\mathbf{A}^D} \to 0$$

and $\mathbb{H}^1_{\mathrm{dR}}(\mathbf{A}/\mathfrak{I}\mathfrak{G}_n)$ comes with the Gauss-Manin connection.

Definition 8.12. We let $H^\#_{\mathbf{A}}$ be the pushout of the diagram

$$\begin{array}{ccc} \underline{\delta}^p \cdot \underline{\omega}_{\mathbf{A}} & \longrightarrow & \underline{\delta}^p \cdot \mathbb{H}^1_{\mathrm{dR}}(\mathbf{A}/\mathfrak{I}\mathfrak{G}_n) \\ \downarrow & & \downarrow \\ \Omega_{\mathbf{A}} & \dashrightarrow & H^\#_{\mathbf{A}} \end{array} \tag{8.1}$$

Lemma 8.13. *We have*

$$\underline{\delta}^p \cdot \mathbb{H}^1_{\mathrm{dR}}(\mathbf{A}/\mathfrak{I}\mathfrak{G}_n) \cap \Omega_{\mathbf{A}} = \underline{\delta}^p \cdot \underline{\omega}_{\mathbf{A}}$$

as subsheaves of $\mathbb{H}^1_{\mathrm{dR}}(\mathbf{A}/\mathfrak{I}\mathfrak{G}_n)$*, so in particular*

$$H^\#_{\mathbf{A}} = \underline{\delta}^p \cdot \mathbb{H}^1_{\mathrm{dR}}(\mathbf{A}/\mathfrak{I}\mathfrak{G}_n) + \Omega_{\mathbf{A}} \subseteq \mathbb{H}^1_{\mathrm{dR}}(\mathbf{A}/\mathfrak{I}\mathfrak{G}_n).$$

Proof. We have an injective map

$$\underline{\delta}^p \cdot \underline{\omega}_{\mathbf{A}} \to \underline{\delta}^p \cdot \mathbb{H}^1_{\mathrm{dR}}(\mathbf{A}/\mathfrak{I}\mathfrak{G}_n) \cap \Omega_{\mathbf{A}} = \underline{\delta}^p \cdot \mathbb{H}^1_{\mathrm{dR}}(\mathbf{A}/\mathfrak{I}\mathfrak{G}_n) \cap \underline{\delta} \cdot \underline{\omega}_{\mathbf{A}},$$

whose surjectivity is checked locally. We have locally

$$\begin{aligned} \underline{\delta}^p \cdot \mathbb{H}^1_{\mathrm{dR}}(\mathbf{A}/\mathfrak{I}\mathfrak{G}_n) \cap \Omega_{\mathbf{A}} &= (\underline{\delta}^p \cdot \underline{\omega}_{\mathbf{A}} \oplus \underline{\delta}^p \cdot \underline{\omega}^\vee_{\mathbf{A}^D}) \cap \underline{\delta} \cdot \underline{\omega}_{\mathbf{A}} \\ &= \underline{\delta}^p \cdot \underline{\omega}_{\mathbf{A}} \cap \underline{\delta} \cdot \underline{\omega}_{\mathbf{A}} = \underline{\delta}^p \cdot \underline{\omega}_{\mathbf{A}} \end{aligned}$$

hence the map above is an isomorphism. This shows that to give a pair of morphisms

$$\underline{\delta}^p \cdot \mathbb{H}^1_{\mathrm{dR}}(\mathbf{A}/\mathfrak{I}\mathfrak{G}_n) \to \mathcal{F} \leftarrow \Omega_{\mathbf{A}}$$

that coincide on $\underline{\delta}^p \cdot \underline{\omega}_{\mathbf{A}}$ is the same as a morphism $\underline{\delta}^p \cdot \mathbb{H}^1_{\mathrm{dR}}(\mathbf{A}/\mathfrak{I}\mathfrak{G}_n) + \Omega_{\mathbf{A}} \to \mathcal{F}$, but this is the universal property of the pushout. □

Proposition 8.14 ([AI, Proposition 6.2, pag. 64])**.** *The following hold:*

1. *We have an exact sequence of* $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathcal{O}_{\mathfrak{I}\mathfrak{G}_n}$*-modules*

$$\left(\mathcal{H}^\#_{\mathbf{A}}\right)^\bullet \quad = \quad 0 \to \Omega_{\mathbf{A}} \to H^\#_{\mathbf{A}} \to \underline{\delta}^p \cdot \underline{\omega}^\vee_{\mathbf{A}^D} \to 0,$$

   in particular $H^\#_{\mathbf{A}}$ *is a locally free* $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathcal{O}_{\mathfrak{I}\mathfrak{G}_n}$*-module of rank 2.*

2. *There exists a natural monomorphism* $j^\bullet : \left(\mathcal{H}^\#_{\mathbf{A}}\right)^\bullet \to \mathcal{H}^\bullet_{\mathbf{A}}$.

3. *Let* $P_n \in H_n^{(i),D}\left(\mathfrak{IG}_n^{(i)}\right)$ *be an* $\mathcal{O}_{F_i}/\mathfrak{p}_i^n$*- basis and let* $s_i = \mathrm{dlog}\,(P_n)$*. Then the exact sequence* $\left(\mathcal{H}_{\mathbf{A}}^{\#,(i)}\right)^{\bullet}$ *realises* $\left(\Omega_{\mathbf{A}}^{(i)}, s_i\right)$ *as an* $\mathrm{MS}_{\mathcal{O}_{F_i}}$*-subdatum of* $\left(H_{\mathbf{A}}^{\#,(i)}, s_i\right)$ *with respect to the ideal* β_n*. Moreover the Gauss-Manin connection* $\nabla^{(i)}$ *on* $\mathbb{H}_{\mathrm{dR}}^1\,(\mathbf{A}/\mathfrak{IG}_n)^{(i)}$ *induces a connection*

$$\nabla^{\#,(i)} : H_{\mathbf{A}}^{\#,(i)} \to H_{\mathbf{A}}^{\#,(i)} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{IG}_n^{(i)}}} \underline{\delta}^{(i),-1} \cdot \Omega^1_{\mathfrak{IG}_n^{(i)}/\mathfrak{X}}.$$

Proof.

1. We use the description

$$H_{\mathbf{A}}^{\#} = \underline{\delta}^p \cdot \mathbb{H}_{\mathrm{dR}}^1\,(\mathbf{A}/\mathfrak{IG}_n) + \Omega_{\mathbf{A}}$$

of Lemma 8.13: clearly we have an injection $\Omega_{\mathbf{A}} \to H_{\mathbf{A}}^{\#}$ whose cokernel is

$$\frac{\underline{\delta}^p \cdot \mathbb{H}_{\mathrm{dR}}^1\,(\mathbf{A}/\mathfrak{IG}_n) + \Omega_{\mathbf{A}}}{\Omega_{\mathbf{A}}} = \frac{\underline{\delta}^p \cdot \mathbb{H}_{\mathrm{dR}}^1\,(\mathbf{A}/\mathfrak{IG}_n)}{\underline{\delta}^p \cdot \mathbb{H}_{\mathrm{dR}}^1\,(\mathbf{A}/\mathfrak{IG}_n) \cap \Omega_{\mathbf{A}}} = \frac{\underline{\delta}^p \cdot \mathbb{H}_{\mathrm{dR}}^1\,(\mathbf{A}/\mathfrak{IG}_n)}{\underline{\delta}^p \cdot \underline{\omega}_{\mathbf{A}}}$$

but in view of the Hodge filtration we conclude that

$$\mathrm{coker}\left(\Omega_{\mathbf{A}} \to H_{\mathbf{A}}^{\#}\right) = \underline{\delta}^p \cdot \underline{\omega}_{\mathbf{A}^D}^{\vee}.$$

2. Take j^0 and j^2 to be the natural incusions, then the claim follows in view of the commutative diagram

$$\begin{array}{ccccc} \underline{\delta}^p \cdot \underline{\omega}_{\mathbf{A}} & & \longrightarrow & & \underline{\delta}^p \cdot \mathbb{H}_{\mathrm{dR}}^1\,(\mathbf{A}/\mathfrak{IG}_n) \\ \downarrow & & & & \downarrow \\ \Omega_{\mathbf{A}} & \longrightarrow & \underline{\omega}_{\mathbf{A}} & \longrightarrow & \mathbb{H}_{\mathrm{dR}}^1\,(\mathbf{A}/\mathfrak{IG}_n) \end{array}$$

and the definition of pushout.

3. The first statement is a direct consequence of the fact the sequence $\left(\mathcal{H}_{\mathbf{A}}^{\#}\right)^{\bullet}$ is locally split, since $\underline{\delta}^p \cdot \underline{\omega}_{\mathbf{A}^D}^{\vee}$ is locally free. The second statement follows from the proof of [AI, Proposition 6.3, pag. 64].

□

Definition 8.15. In view of Proposition 8.14 we can consider $\pi^{(i)} : \mathbb{V}^{(i)} := \mathbb{V}_0\left(H_{\mathbf{A}}^{\#,(i)}, s_i\right) \to \mathfrak{IG}_n^{(i)}$. We define $\mathfrak{f}^{(i)}$ to be the composition

$$\mathfrak{f}^{(\mathrm{i})} : \mathbb{V}^{(i)} \xrightarrow{\pi^{(i)}} \mathfrak{IG}_n^{(i)} \xrightarrow{\mathfrak{h}^{(i),n}} \mathfrak{X}.$$

Remark 8.16. It follows from Proposition 8.7 that the $\mathcal{O}_{\mathfrak{IG}_n^{(i)}}$-module $\tilde{\mathbb{W}}_{\kappa^{(i)}} := \pi_*^{(i)}\mathcal{O}_{\mathbb{V}^{(i)}}\left[\kappa^{(i)}\right]$ has locally the form

$$\tilde{\mathbb{W}}_{\kappa^{(i)}|\mathrm{Spf}(R)} = \prod_{u \in S} R\left\langle V_{\sigma,u}\right\rangle \cdot \kappa^{(i)}\left(\sigma\left(u\right) + \beta_n Z_{\sigma,u}\right)$$

(see the Proposition for the notation). Moreover we can endow the $\mathcal{O}_{\mathfrak{IG}_n^{(i)}}$-module $\tilde{\mathbb{W}}_{\kappa^{(i)}}$ with a natural increasing filtration

$$F^\bullet \tilde{\mathbb{W}}_{\kappa^{(i)}} = \pi_*^{(i)} F^\bullet \mathcal{O}_{\mathbb{V}^{(i)}} \left[\kappa^{(i)}\right]$$

induced by $\left(\Omega_{\mathbf{A}}^{(i)}, s_i\right)$ with the property that

$$\mathrm{Gr}^h F^\bullet \tilde{\mathbb{W}}_{\kappa^{(i)}} = \pi_* \mathcal{O}_{\mathbb{V}_0\left(\Omega_{\mathbf{A}}^{(i)}, s_i\right)} \left[\kappa^{(i)}\right] \otimes \mathrm{Gr}^h F^\bullet \pi_*^{(i)} \mathcal{O}_{\mathbb{V}_{\mathcal{O}_{F_i}}\left(\left(\underline{\delta}^p \cdot \underline{\omega}_{\mathbf{A}^D}^\vee\right)^{(i)}\right)}.$$

Proposition 8.17. *The following hold:*

1. $F^h \tilde{\mathbb{W}}_{\kappa^{(i)}}$ *is a locally free coherent* $\mathcal{O}_{\mathfrak{IG}_n^{(i)}}$*-module for every* $h \geq 0$*;*
2. $\tilde{\mathbb{W}}_{\kappa^{(i)}}$ *is isomorphism to the completed limit* $\widehat{\varinjlim} F^h \tilde{\mathbb{W}}_{\kappa^{(i)}}$*;*
3. $F^0 \tilde{\mathbb{W}}_{\kappa^{(i)}} \simeq \varpi^{\kappa^{(i)}}$ *and* $\mathrm{Gr}^h F^\bullet \tilde{\mathbb{W}}_{\kappa^{(i)}} \simeq \varpi^{\kappa^{(i)}} \otimes_{\mathfrak{IG}_n^{(i)}} \left(\underline{\omega}_{\mathbf{A}}^{(i)}\right)^{-h} \otimes_{\mathfrak{IG}_n^{(i)}} \left(\underline{\omega}_{\mathbf{A}^D}^{(i)}\right)^{-h}$*. In particular, we have locally*

$$F^h \tilde{\mathbb{W}}_{\kappa^{(i)}} (\mathrm{Spf}(R)) = \varpi^{\kappa^{(i)}} \otimes_R \left(\bigoplus_{0 \leq k \leq h} \left(\underline{\omega}_{\mathbf{A}}^{(i)}\right)^{-k} \otimes_{\mathfrak{IG}_n^{(i)}} \left(\underline{\omega}_{\mathbf{A}^D}^{(i)}\right)^{-k} \right).$$

Proof. From the explicit description of the filtration we have locally on $\mathrm{Spf}(R) \subseteq \mathfrak{IG}_n^{(i)}$

$$F^h \tilde{\mathbb{W}}_{\kappa^{(i)}|\mathrm{Spf}(R)} = \prod_{u \in S} R\left[V_{\sigma,u}\right]_{\leq h} \cdot \kappa^{(i)} \left(\sigma(u) + \beta_n Z_{\sigma,u}\right)$$

from which points 1. and 2., while the first part of point 3. comes from Proposition 8.7. To conclude, in view again of Proposition 8.7, we can write locally

$$\begin{aligned}
\mathrm{Gr}^h F^\bullet \tilde{\mathbb{W}}_{\kappa^{(i)}} (\mathrm{Spf}(R)) &= \prod_{u \in S} \bigotimes_{\sigma \in \mathfrak{G}} R \cdot \kappa^{(i)} \left(\sigma(u) + \beta_n Z_{\sigma,u}\right) V_{\sigma,u}^h \\
&= \prod_{u \in S} \bigotimes_{\sigma \in \mathfrak{G}} R \cdot X_{2,\sigma,u}^h \kappa^{(i)} \left(\sigma(u) + \beta_n Z_{\sigma,u}\right) \left(\sigma(u) + \beta_n Z_{\sigma,u}\right)^{-h} \\
&= \prod_{u \in S} R \cdot \left(\prod_{\sigma \in \mathfrak{G}} \kappa^{(i)} \left(\sigma(u) + \beta_n Z_{\sigma,u}\right) \right) \left(\prod_{\sigma \in \mathfrak{G}} \left(\sigma(u) + \beta_n Z_{\sigma,u}\right)^{-h} \right) \left(\prod_{\sigma \in \mathfrak{G}} X_{2,\sigma,u}^h \right) \\
&= \left(\varpi^{\kappa^{(i)}} \otimes_{\mathfrak{IG}_n^{(i)}} \left(\underline{\omega}_{\mathbf{A}}^{(i)}\right)^{-h} \otimes_{\mathfrak{IG}_n^{(i)}} \left(\underline{\omega}_{\mathbf{A}^D}^{(i)}\right)^{-h} \right) (\mathrm{Spf}(R)).
\end{aligned}$$

□

8.3 Descending to $\mathfrak{X}$

Define the formal group $\mathfrak{T}^{\mathrm{ext}} = \mathbf{T}(\mathbb{Z}_p) \cdot \mathfrak{T}$. Over $\mathfrak{X}$ it decomposes as

$$\prod_{i=1}^d \mathfrak{T}^{\mathrm{ext},(i)} = \prod_{i=1}^d \mathcal{O}_{F_i}^\times \cdot \left(1 + \beta_n \mathrm{Res}_{\mathcal{O}_{F_i}|\mathbb{Z}_p} \mathbb{G}_a\right)$$

The group $\mathfrak{T}^{\text{ext},(i)}$ comes with a natural action on $\mathfrak{f}^{(i)} : \mathbb{V}_0\left(\Omega^{(i)}_{\mathbf{A}}, s_i\right) \to \mathfrak{X}$ that we now describe. Let $u : S \to \mathfrak{X}$ be a morphism of formal schemes, then a point of $\mathbb{V}_0\left(\Omega^{(i)}_{\mathbf{A}}, s_i\right)(u)$ is a pair $\left(\rho^{(i)}, v\right)$ where $\rho^{(i)} : S \to \mathfrak{IG}^{(i)}_n$ lifts u and $v \in \mathbb{V}_0\left(\Omega^{(i)}_{\mathbf{A}}, s_i\right)\left(\rho^{(i)}\right)$. In the same fashion a point of $\mathbb{V}_0\left(H^{\#,(i)}_{\mathbf{A}}, s_i\right)(u)$ is a pair $\left(\rho^{(i)}, w\right)$ where $\rho^{(i)} : S \to \mathfrak{IG}^{(i)}_n$ lifts u and $w \in \mathbb{V}_0\left(H^{\#,(i)}_{\mathbf{A}}, s_i\right)\left(\rho^{(i)}\right)$. Let $\overline{\lambda} \in \left(\mathcal{O}_{F_i}/\mathfrak{p}_i^n\right)^\times$ be seen as an element of the Galois group of the adic generic fibre $\mathcal{IG}^{(i)}_n \to \mathcal{X}$ and, by functoriality, as an $\mathfrak{X}$-automorphism of $\mathfrak{IG}^{(i)}_n$. Denote

$$\overline{\lambda}^* : \Omega^{(i)}_{\mathbf{A}} \to \Omega^{(i)}_{\mathbf{A}}, \quad H^{\#,(i)}_{\mathbf{A}} \to H^{\#,(i)}_{\mathbf{A}}$$

the isomorphisms it induces, which are characterised $\mod \beta_n$ by $\overline{\lambda}^*(s_i) = \overline{\lambda} s_i$. For a point $\left(\rho^{(i)}, v\right) \in \mathbb{V}_0\left(\Omega^{(i)}_{\mathbf{A}}, s_i\right)(u)$ we define

$$\lambda * \left(\rho^{(i)}, v\right) = \left(\overline{\lambda} \circ \rho^{(i)}, \lambda^{-1} v\right) \tag{8.2}$$

and for a point $\left(\rho^{(i)}, w\right) \in \mathbb{V}_0\left(H^{\#,(i)}_{\mathbf{A}}, s_i\right)(u)$ we define

$$\lambda * \left(\rho^{(i)}, w\right) = \left(\overline{\lambda} \circ \rho^{(i)}, \lambda^{-1} w\right) \tag{8.3}$$

Lemma 8.18. *Formula (8.2) defines an action of $\mathbf{T}(\mathbb{Z}_p)$ on $\mathfrak{f}^{(i)} : \mathbb{V}_0\left(\Omega^{(i)}_{\mathbf{A}}, s_i\right) \to \mathfrak{X}$ which is compatible with that of $\mathfrak{T}$ on $\pi^{(i)}$, thus giving an action of $\mathfrak{T}^{\text{ext},(i)}$ on $\mathfrak{f}^{(i)}$.*

Proof. The same proof of [AIP2, Lemme 5.1, pag. 21] works in this case. □

Definition 8.19. Let $\mathfrak{w}^{\kappa^{(i)}} = \left(\mathfrak{f}^{(i)}_* \mathcal{O}_{\mathbb{V}_0\left(\Omega^{(i)}_{\mathbf{A}}, s_i\right)}\right)\left[\kappa^{(i)}\right]$ that is, $\mathfrak{w}^{\kappa^{(i)}}$ is the $\mathcal{O}_{\mathfrak{X}}$-submodule of $\mathfrak{f}^{(i)}_* \mathcal{O}_{\mathbb{V}_0\left(\Omega^{(i)}_{\mathbf{A}}, s_i\right)}$ given by sections that tranforms via $\kappa^{(i)}$ under the action of $\mathfrak{T}^{\text{ext},(i)}$.

Note 8.20. In [AIP1, Section 4.1, pag. 15] the $\mathfrak{T}$-torsor $f : \mathfrak{F}_n \to \mathfrak{IG}_n$ is considered, defined by

$$\mathfrak{F}_n(R) = \left\{\omega \in \Omega_{\mathbf{A}}(R) \,|\, \overline{\omega} = s\right\}.$$

Let $\mathrm{Spf}(R) \subseteq \mathfrak{IG}^{(i)}_n$ be connected, then for every $\omega^{(i)}_\sigma \in \mathfrak{F}^{(i)}_{n,\sigma}(R)$ we can consider the map $\Omega^{(i)}_{\mathbf{A}}(\sigma) \to R$ defined by $f_\sigma\left(\omega^{(i)}_\sigma\right) = \sigma(u)$. This gives a map

$$\phi^{(i)}_{n,\sigma} : \mathfrak{F}^{(i)}_{n,\sigma} \to \mathbb{V}^u_{0,\sigma}\left(\Omega^{(i)}_{\mathbf{A}}, s_i\right)$$

over $\mathfrak{IG}^{(i)}_n$. On the other hand, given a section $f_\sigma \in \mathbb{V}^u_{0,\sigma}\left(\Omega^{(i)}_{\mathbf{A}}, s_i\right)(R)$ then $\sigma(u) f^\vee_\sigma \in \Omega^{(i)}_{\mathbf{A}}(\sigma)$ lifts s (since $\overline{f}_\sigma\left(\overline{\sigma(u) \cdot f^\vee_\sigma}\right) = \overline{f}_\sigma(s)$) and this gives a map

$$\mathbb{V}^u_{0,\sigma}\left(\Omega^{(i)}_{\mathbf{A}}, s_i\right) \to \mathfrak{F}^{(i)}_{n,\sigma}$$

again over $\mathfrak{IG}_n^{(i)}$ and it is clear from the functorial description that this is indeed the inverse of $\phi_{n,\sigma}^{(i)}$. If $h \in \mathfrak{T}^{(i)}(R)$, then $\phi_{n,\sigma}^{(i)}\left(h\omega_\sigma^{(i)}\right)$ is defined by $h\omega_\sigma^{(i)} \mapsto \sigma(u)$, hence we conclude that

$$\phi_{n,\sigma}^{(i)}\left(h\omega_\sigma^{(i)}\right) = h^{-1}\phi_{n,\sigma}^{(i)}\left(\omega_\sigma^{(i)}\right).$$

We proved that for every $u \in S$ there is a natural isomorphism

$$\phi_n : \mathfrak{F}_n \to \mathbb{V}_0^u(\Omega_{\mathbf{A}}, s)$$

over $\mathfrak{IG}_n$ that inverts the action of $\mathfrak{T}$. The importance of this isomorphism lies in the fact that from [AIP1, Section 4.2, pag. 16] the sheaf of overconvergent forms is

$$\mathfrak{w}_{n,r,I}^{\text{over}} := (\mathfrak{h}^n \circ f)_* \mathcal{O}_{\mathfrak{F}_n}\left[\kappa^{-1}\right]$$

over $\mathfrak{X}_{r,I}$. Hence we obtain that

$$\mathfrak{w}_{n,r,I}^{\kappa} = \prod_{u \in S} \mathfrak{w}_{n,r,I}^{\text{over}}.$$

Lemma 8.21. *The elements of $\kappa(\mathbf{T}(\mathbb{Z}_p)) - 1 \subseteq \mathcal{O}_{\mathfrak{X}}$ are topologically nilpotent.*

Proof. This is local on $\mathfrak{X}$, take $\operatorname{Spf}(R) \subseteq \mathfrak{X}$, it maps to the open $\mathfrak{W}_\alpha$ of $\tilde{\mathfrak{W}}_\alpha$ defined by the element $\alpha \in \mathfrak{m} \backslash \mathfrak{m}^2$ hence $\kappa(\mathbf{T}(\mathbb{Z}_p)) - 1 \subseteq \alpha B_\alpha^0$ and we conclude since $\alpha \mathrm{Hdg}^{-1}$ is made of topologically nilpotent elements of $\mathcal{O}_{\mathfrak{X}}$ in view of Proposition 7.7. $\square$

Note 8.22. Let $\operatorname{Spf}(A) \subseteq \mathfrak{X}$ be an open affine over which Hdg is free, say generated by $\tilde{\mathrm{Ha}}$, and let $\operatorname{Spf}\left(R_n^{(i)}\right) \subseteq \mathfrak{IG}_n^{(i)}$ be its inverse image. In view of Lemma 7.15, for $j = 0, \dots, n$, there exist elements $c_j \in \tilde{\mathrm{Ha}}^{-1} R_n^{(i)}$ such that

1. $c_j \in \tilde{\mathrm{Ha}}^{-\left(p^j - p\right)/(p-1)} R_n^{(i)}$;
2. $\mathrm{Tr}_{R_j/R_{j-1}}(c_j) = c_{j-1}$ (here we see R_{j-1} as a subring of R_j) and $c_0 = 1$.

Pick a generator $\kappa^{(i)}(\sigma(u) + \beta_n Z_{\sigma,u})$ of $\varpi_{\sigma,u}^{\kappa^{(i)}}$ over $R_n^{(i)}$ (compare with Proposition 8.7) and a lift $\tilde{\gamma} \in \mathcal{O}_{F_i}^\times$ for every $\gamma \in (\mathcal{O}_{F_i}/\mathfrak{p}_n^n)^\times$. Note that the element

$$b_{\sigma,u} = \sum_{\gamma \in \left(\mathcal{O}_{F_i}/\mathfrak{p}_n^n\right)^\times} \kappa^{(i)}(\tilde{\gamma})\, \sigma * \left(\kappa^{(i)}(\sigma(u) + \beta_n Z_{\sigma,u})\, c_n\right)$$

a priori lies in $\tilde{\mathrm{Ha}}^{-1} R_n^{(i)} \langle Z_{\sigma,u} \rangle$. In view of Proposition 6.17 we can write $\kappa^{(i)}(\sigma(u) + \beta_n Z_{\sigma,u}) = \kappa^{(i)}(\sigma(u)) + qh$ for some $h \in R_n^{(i)} \langle Z_{\sigma,u} \rangle$ hence

$$b_{\sigma,u} = \kappa^{(i)}(\sigma(u)) \sum_{\gamma \in \left(\mathcal{O}_{F_i}/\mathfrak{p}_n^n\right)^\times} \kappa^{(i)}(\tilde{\gamma})\, \gamma * c_n + q \sum_{\gamma \in \left(\mathcal{O}_{F_i}/\mathfrak{p}_n^n\right)^\times} \kappa^{(i)}(\tilde{\gamma})\, \gamma * (hc_n).$$

The term $q \sum_{\gamma \in \left(\mathcal{O}_{F_i}/\mathfrak{p}_n^n\right)^\times} \kappa^{(i)}(\tilde{\gamma})\, \gamma * (hc_n)$ lives in $q\tilde{\mathrm{Ha}}^{-(p^n - p)/(p-1)} R_n^{(i)} \langle Z_{\sigma,u} \rangle$, but $n \leq r + k$ implies that $\tilde{\mathrm{Ha}}^{(p^n - p)/(p-1)} | \tilde{\mathrm{Ha}}^{p^{r+k}}$, but $p\alpha^{-p^k} \in B_{\alpha,I}^0$ and $\alpha \mathrm{Hdg}^{-p^{r+1}} \subseteq R_n^{(i)}$ and it follows that

$\tilde{\mathrm{Ha}}^{p^{r+k+1}}|p$. Therefore, denoting $\left(R_n^{(i)}\right)^{\circ\circ}$ the ideal of topologically nilpotent elements in $R_n^{(i)}$, we have

$$q\sum_{\gamma\in\left(\mathcal{O}_{F_i}/\mathfrak{p}_n^n\right)^\times}\kappa^{(i)}\left(\tilde{\gamma}\right)\gamma*\left(hc_n\right)\in\left(R_n^{(i)}\right)^{\circ\circ}\cdot R_n^{(i)}\left\langle Z\right\rangle.$$

Consider now $\sum_{\gamma\in\left(\mathcal{O}_{F_i}/\mathfrak{p}_n^n\right)^\times}\kappa^{(i)}\left(\tilde{\gamma}\right)\gamma*c_n$: first note that in view of the choice of the sequence c_j and, since $\kappa^{(i)}\left(\tilde{\gamma}\right)\in 1+\left(R_n^{(i)}\right)^{\circ\circ}$ in view of Lemma 8.21, we have

$$\begin{aligned}\sum_{\gamma\in 1+\mathfrak{p}_i^{t-1}\mathcal{O}_{F_i}/\mathfrak{p}_i^t\mathcal{O}_{F_i}}\kappa^{(i)}\left(\tilde{\gamma}\right)\gamma*c_t &\equiv \sum_{\gamma\in 1+\mathfrak{p}_i^{t-1}\mathcal{O}_{F_i}/\mathfrak{p}_i^t\mathcal{O}_{F_i}}\gamma*c_t \mod \left(R_n^{(i)}\right)^{\circ\circ}\\ &= c_{t-1}\end{aligned}$$

therefore

$$\sum_{\gamma\in\left(\mathcal{O}_{F_i}/\mathfrak{p}_n^n\right)^\times}\kappa^{(i)}\left(\tilde{\gamma}\right)\gamma*c_n\equiv 1 \mod \left(R_n^{(i)}\right)^{\circ\circ}$$

and $b_{\sigma,u}$ is a well-defined element of $\mathfrak{w}_{\sigma,u}^{\kappa^{(i)}}=\mathfrak{f}_*^{(i)}\mathcal{O}_{\mathbb{V}_{0,\sigma}^u\left(\Omega_{\mathbf{A}}^{(i)},s_i\right)}\left[\kappa^{(i)}\right]$ over $\mathrm{Spf}\left(A\right)$. From topological Nakayama's Lemma [Mat2, Theorem 8.4, pag. 58] we conclude that $\mathfrak{w}_u^{\kappa^{(i)}}=\bigotimes_\sigma\mathfrak{w}_{\sigma,u}^{\kappa^{(i)}}$ is free over A with basis $\bigotimes_\sigma b_{\sigma,u}$.

Definition 8.23. Set $\mathbb{W}_{\kappa^{(i)}}=\mathfrak{f}_*^{(i)}\mathcal{O}_{\mathbb{V}^{(i)}}\left[\kappa^{(i)}\right]$.

Theorem 8.24. *With setting and notations as in Note 8.22, the action of* $\mathfrak{T}^{\mathrm{ext},(i)}$ *on* $\mathfrak{f}_*^{(i)}\mathcal{O}_{\mathbb{V}^{(i)}}$ *preserves the filtration* $F^\bullet\mathfrak{f}_*^{(i)}\mathcal{O}_{\mathbb{V}^{(i)}}$ *induced by the* $\mathrm{MS}_{\mathcal{O}_{F_i}}$*-subdatum* $\left(\Omega_{\mathbf{A}}^{(i)},s_i\right)$ *of* $\left(H_{\mathbf{A}}^{\#,(i)},s_i\right)$*. Set*

$$F^\bullet\mathbb{W}_{\kappa^{(i)}}=\mathfrak{f}_*^{(i)}F^\bullet\mathcal{O}_{\mathbb{V}^{(i)}}\left[\kappa^{(i)}\right],$$

then

1. $F^h\mathbb{W}_{\kappa^{(i)}}$ *is a locally free coherent* $\mathcal{O}_{\mathfrak{X}}$*-module;*
2. $\mathbb{W}_{\kappa^{(i)}}$ *is isomorphic to the completed limit* $\widehat{\varinjlim} F^h\mathbb{W}_{\kappa^{(i)}}$*, in particular* $\mathbb{W}_{\kappa^{(i)}}$ *is a flat* $\mathcal{O}_{\mathfrak{X}}$*-module;*
3. $F^0\mathbb{W}_{\kappa^{(i)}}\simeq\mathfrak{w}^{\kappa^{(i)}}$ *and* $\mathrm{Gr}^hF^\bullet\mathbb{W}_{\kappa^{(i)}}\simeq\mathfrak{w}^{\kappa^{(i)}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\underline{\omega}_{\mathbf{A}}^{-h}\otimes_{\mathcal{O}_{\mathfrak{X}}}\underline{\omega}_{\mathbf{A}^D}^{-h}$*. In particular, we have locally*

$$F^h\mathbb{W}_{\kappa^{(i)}}\left(\mathrm{Spf}\left(A\right)\right)\simeq\mathfrak{w}^{\kappa^{(i)}}\otimes_A\mathrm{Sym}_A^{\leq h}\left(\underline{\omega}_{\mathbf{A}}^{-1}\otimes_A\underline{\omega}_{\mathbf{A}^D}^{-1}\right).$$

Proof. Note that, in view of Proposition 8.17, points 1. and 2. follow from point 3. The isomorphism $F^0\mathbb{W}_{\kappa^{(i)}}\simeq\mathfrak{w}^{\kappa^{(i)}}$ comes from the very definition of the filtration

$$\begin{aligned}F^0\mathbb{W}_{\kappa^{(i)}}&=F^0\mathfrak{f}_*^{(i)}\mathcal{O}_{\mathbb{V}^{(i)}}\left[\kappa^{(i)}\right]\\ &=\mathfrak{f}_*^{(i)}\mathcal{O}_{\mathbb{V}_0\left(\Omega_{\mathbf{A}}^{(i)},s_i\right)}\left[\kappa^{(i)}\right]\\ &=\mathfrak{w}^{\kappa^{(i)}}.\end{aligned}$$

Finally, in view of Proposition 8.17, since $\omega_{\mathbf{A}}$ is defined on $\mathfrak{X}$, we see that

$$\begin{aligned}\mathrm{Gr}^h F^\bullet \mathbb{W}_{\kappa^{(i)}} &= \mathfrak{f}_*^{(i)} \mathcal{O}_{\mathbb{V}_0\left(\Omega_{\mathbf{A}}^{(i)}, s_i\right)} \left[\kappa^{(i)}\right] \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathrm{Gr}^h F^\bullet \mathcal{O}_{\mathbb{V}_{\mathcal{O}_{F_i}}\left(\left(\underline{\delta}^p \cdot \underline{\omega}_{\mathbf{A}^D}^\vee\right)^{(i)}\right)} \\ &= \mathfrak{w}^{\kappa^{(i)}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \underline{\omega}_{\mathbf{A}}^{-h} \otimes_{\mathcal{O}_{\mathfrak{X}}} \underline{\omega}_{\mathbf{A}^D}^{-h}.\end{aligned}$$

□

8.4 The Gauss-Manin connection

Lemma 8.25. *The Gauss-Manin connection* $\nabla_{\mathrm{GM}} : H^1_{\mathrm{dR}}(\mathbf{A}/\mathfrak{X}) \to H^1_{\mathrm{dR}}(\mathbf{A}/\mathfrak{X}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/B^0_{\alpha,I}}$ *induces an integrable* $\mathrm{MS}_{\mathcal{O}_{F_i}}$*-connection*

$$\nabla^{\#,(i)} : H_{\mathbf{A}}^{\#,(i)} \to H_{\mathbf{A}}^{\#,(i)} \otimes_{\mathcal{O}_{\mathfrak{IG}_n'^{(i)}}} \Omega^1_{\mathfrak{IG}_n'^{(i)}/B^0_{\alpha,I}}.$$

Proof. In view of Proposition 8.6 we have

$$\nabla\left(H^1_{\mathrm{dR}}(\gamma)\right) \subseteq H^1_{\mathrm{dR}}(\gamma) \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/B^0_{\alpha,I}}$$

on $\mathfrak{X}$ for every γ. It follows that

$$\nabla^{(i)}\left(H^{1,(i)}_{\mathrm{dR}}(\sigma)\right) \subseteq H^{1,(i)}_{\mathrm{dR}}(\sigma) \otimes_{\mathcal{O}_{\mathfrak{IG}'^{(i)}}} \Omega^1_{\mathfrak{IG}'^{(i)}/B^0_{\alpha,I}}$$

for every $\sigma \in \mathfrak{G}_i$. In view of [AI, Proposition 6.3, pag. 66] we see that the restriction $\nabla^{(i)}_{|H_{\mathbf{A}}^{\#,(i)}}$ is a well defined integrable connection with $\overline{\nabla^{(i)}}(s_i) = 0$ and we conclude since

$$H_{\mathbf{A}}^{\#,(i)}(\sigma) = H_{\mathbf{A}}^{\#,(i)} \cap H^{1,(i)}_{\mathrm{dR}}(\sigma) \quad \text{for every } \sigma \in \mathfrak{G}_i.$$

□

In view of Lemma 8.25 we have an integrable connection

$$\nabla^{(i)}_{\sigma,u} : \pi_* \mathcal{O}_{\mathbb{V}^u_{0,\sigma}\left(H_{\mathbf{A}}^{\#,(i)}, s_i\right)} \to \pi_* \mathcal{O}_{\mathbb{V}^u_{0,\sigma}\left(H_{\mathbf{A}}^{\#,(i)}, s_i\right)} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{IG}'_n{}^{(i)}}} \Omega^1_{\mathfrak{IG}'_n{}^{(i)}/B^0_{\alpha,i}}.$$

We want to see how the connection $\nabla^{(i)}_{\sigma,u}$ descends to $\mathfrak{IG}_n^{(i)}$. Recall that locally we have

$$H_{\mathbf{A}}^{\#} = \Omega_{\mathbf{A}} + \delta^p H^1_{\mathrm{dR}}(\mathbf{A}/\mathfrak{IG}_n) \subseteq H^1_{\mathrm{dR}}(\mathbf{A}/\mathfrak{IG}_n)$$

so let us check that

$$\nabla\left(\delta^p H^1_{\mathrm{dR}}(\mathbf{A}/\mathfrak{IG}_n)\right) \subseteq \delta^p H^1_{\mathrm{dR}}(\mathbf{A}/\mathfrak{IG}_n) \otimes_{\mathcal{O}_{\mathfrak{IG}_n}} \Omega^1_{\mathfrak{IG}_n/B^0_{\alpha,I}}.$$

For $x \in H^1_{\mathrm{dR}}(\mathbf{A}/\mathfrak{IG}_n)$ we have

$$\nabla(\delta^p x) = \delta^p \nabla(x) + p\delta^{p-1} x \otimes d\delta,$$

but by its very construction we have $p \in (\delta)$, whence $p\delta^{p-1} \subseteq (\delta)^p$. A problem arises when we take $\Omega_{\mathbf{A}}$ into account: recall that $\delta \cdot \underline{\omega}_{\mathbf{A}} = \Omega_{\mathbf{A}}$. Let $\mathrm{Spf}\,(R) \subseteq \mathfrak{IG}_n$ be an open affine subset with preimage $\mathrm{Spf}\,(R') \subseteq \mathfrak{IG}'_n$ and such that its image in $\mathfrak{X}$ is contained in an open affine $\mathrm{Spf}\,(A)$ over which the sequence $\mathcal{H}^\bullet_{\mathbf{A}}$ is split. Let ω, η, δ be bases of $\underline{\omega}_{\mathbf{A}|R}, \underline{\omega}^{-1}_{\mathbf{A}|R}$ and $\underline{\delta}_R$ respectively, then $\{\omega, \eta\}$ is a basis for $H^1_{\mathrm{dR}}\left(\mathbf{A}/\mathfrak{IG}_n\right)_{|R}$ and $\{\delta\omega, \delta^p\eta\}$ is a basis for $H^\#_{\mathbf{A}|R}$. We have a Kodaira-Spencer isomorphism

$$\mathrm{KS}: \underline{\omega}_{\mathbf{A}} \to \underline{\omega}^{-1}_{\mathbf{A}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/B^0_{\alpha,I}}$$

where we recall that the $\mathcal{O}_L$-structure is the one induced by this isomorphism, namely

$$\Omega^1_{\mathfrak{X}/B^0_{\alpha,I}} \simeq \underline{\omega}^{\otimes 2}_{\mathbf{A}}$$

and a generator $\theta^{(i)}_\sigma$ of $\Omega^{1,(i)}_{\mathfrak{X}/B^0_{\alpha,I}}(\sigma)$ over A characterised by the property that

$$\mathrm{KS}\left(\omega^{(i)}_\sigma\right) = \eta^{(i)}_\sigma \otimes \theta^{(i)}_\sigma.$$

We have elements $m^{(i)}_\sigma, t^{(i)}_\sigma, s^{(i)}_\sigma \in A$ such that

$$\nabla^{(i)}: \begin{cases} \omega^{(i)}_\sigma \mapsto m^{(i)}_\sigma \omega^{(i)}_\sigma \otimes \theta^{(i)}_\sigma + \eta^{(i)}_\sigma \otimes \theta^{(i)}_\sigma \\ \eta^{(i)}_\sigma \mapsto t^{(i)}_\sigma \omega^{(i)}_\sigma \otimes \theta^{(i)}_\sigma + s^{(i)}_\sigma \eta^{(i)}_\sigma \otimes \theta^{(i)}_\sigma \end{cases}$$

hence

$$\begin{aligned} \nabla^\#\left(\delta^{(i)}\omega^{(i)}_\sigma\right) &= \delta^{(i)}\nabla^\#\left(\omega^{(i)}_\sigma\right) + \omega^{(i)}_\sigma \otimes d\delta^{(i)} \\ &= m^{(i)}_\sigma \delta^{(i)}\omega^{(i)}_\sigma \otimes \theta^{(i)}_\sigma + \delta^{(i)}\eta^{(i)}_\sigma \otimes \theta^{(i)}_\sigma + \omega^{(i)}_\sigma \otimes d\delta^{(i)} \\ \nabla^\#\left(\left(\delta^{(i)}\right)^p \eta^{(i)}_\sigma\right) &= \left(\delta^{(i)}\right)^p \nabla^\#\left(\eta^{(i)}_\sigma\right) + \eta^{(i)}_\sigma \otimes p\left(\delta^{(i)}\right)^{p-1} d\delta^{(i)} \\ &= t^{(i)}_\sigma \left(\delta^{(i)}\right)^p \omega^{(i)}_\sigma \otimes \theta^{(i)}_\sigma + s^{(i)}_\sigma \left(\delta^{(i)}\right)^p \eta^{(i)}_\sigma \otimes \theta^{(i)}_\sigma + \left(\delta^{(i)}\right)^p \eta^{(i)}_\sigma \otimes p\mathrm{dlog}\delta^{(i)}, \end{aligned} \tag{8.4}$$

In view of the term $\mathrm{dlog}\delta^{(i)}$ we conclude that the connection $\nabla^{\#,(i)}$ descends to an integrable $\mathrm{MS}_{\mathcal{O}_{F_i}}$-connection

$$\nabla^{\#,(i)}: H^{\#,(i)}_{\mathbf{A}} \to H^{\#,(i)}_{\mathbf{A}} \otimes_{\mathcal{O}_{\mathfrak{IG}^{(i)}_n}} \mathrm{Hdg}^{-1} \cdot \Omega^1_{\mathfrak{IG}^{(i)}_n/B^0_{\alpha,I}}. \tag{8.5}$$

Proposition 8.26. *The integrable* $\mathrm{MS}_{\mathcal{O}_{F_i}}$*-connection*

$$\nabla^{\#,(i)}_\sigma: H^{\#,(i)}_{\mathbf{A}}(\sigma) \to H^{\#,(i)}_{\mathbf{A}}(\sigma) \otimes_{\mathcal{O}_{\mathfrak{IG}^{(i)}_n}} \mathrm{Hdg}^{-1} \cdot \Omega^1_{\mathfrak{IG}^{(i)}_n/B^0_{\alpha,I}}$$

induces integrable connections

$$\nabla_{\kappa^{(i)}}: \tilde{\mathbb{W}}_{\kappa^{(i)},\sigma,u} \to \tilde{\mathbb{W}}_{\kappa^{(i)},\sigma,u} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{IG}^{(i)}_n}} \mathrm{Hdg}^{-1} \cdot \Omega^1_{\mathfrak{IG}^{(i)}_n/B^0_{\alpha,I}}$$

and

$$\nabla_{\kappa^{(i)}}: \mathbb{W}_{\kappa^{(i)},\sigma,u} \to \mathbb{W}_{\kappa^{(i)},\sigma,u} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathrm{Hdg}^{-1} \cdot \Omega^1_{\mathfrak{X}/B^0_{\alpha,I}}$$

that respects Griffith transversality property for the filtration $F^\bullet\tilde{\mathbb{W}}_{\kappa^{(i)},\sigma,u}$. Moreover the induced map on graded pieces

$$\mathrm{Gr}^h F^\bullet\nabla_{\kappa^{(i)}} : \alpha^{-1}\mathrm{Gr}^h F^\bullet\mathbb{W}_{\kappa^{(i)},\sigma,u} \to \alpha^{-1}\mathrm{Gr}^{h+1} F^\bullet\mathbb{W}_{\kappa^{(i)},\sigma,u}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}^L_{r,I}}}\Omega^1_{\mathfrak{X}^L/\left(B^0_{\alpha,I}\otimes\mathcal{O}_L\right)}$$

is the composition of an isomorphism and the product by $u_I^{(i)} - h$ (cfr. Note 6.18).

Proof. Note that the map $\mathfrak{IG}_n^{(i)} \to \mathfrak{X}^{(i)}$ is finite étale after inverting Hdg with Galois group $(\mathcal{O}_{F_i}/\mathfrak{p}_i^n)^\times$(see Proposition 7.14), therefore faithfully flat descent applies to $\nabla_{\kappa^{(i)}}$ and we reduce to the proof of the existence of the connection over $\mathfrak{IG}_n^{(i)}$. Let $\mathrm{Spf}\,(R) \subseteq \mathfrak{IG}_n^{(i)}$ be an open affine over which $H_{\mathbf{A}}^{\#,(i)}$ is free, say with basis $f_\sigma^{(i)} = \delta^{(i)}\omega_\sigma^{(i)}, e_\sigma^{(i)} = \left(\delta^{(i)}\right)^p \eta_\sigma^{(i)}$, and let $\mathrm{Spf}\,(R') \subseteq \mathfrak{IG}'^{(i)}_n$ be its inverse image. Using notations as in 3.2 we have an $\mathcal{A}_{R'/A}$-linear isomorphism

$$\epsilon_\sigma^{(i)} : H_{\mathbf{A}}^{\#,(i)}(\sigma) \otimes_{j_2} \mathcal{A}_{R'/A} \to H_{\mathbf{A}}^{\#,(i)}(\sigma) \otimes_{j_1} \mathcal{A}_{R'/A}$$

which corresponds to a matrix

$$\begin{pmatrix} a_\sigma^{(i)} & b_\sigma^{(i)} \\ c_\sigma^{(i)} & d_\sigma^{(i)} \end{pmatrix} \in \mathrm{GL}_2\left(\mathcal{A}_{R'/A}\right)$$

in terms of the basis we picked. The condition that $\epsilon_\sigma^{(i)} \otimes_\Delta R' = \mathrm{id}_{H_{\mathbf{A}}^{\#,(i)}(\sigma)}$ translates into an equality

$$\begin{pmatrix} a_\sigma^{(i)} & b_\sigma^{(i)} \\ c_\sigma^{(i)} & d_\sigma^{(i)} \end{pmatrix} = \begin{pmatrix} \left(1, \omega_{a_\sigma^{(i)}}\right) & \left(0, \omega_{b_\sigma^{(i)}}\right) \\ \left(0, \omega_{c_\sigma^{(i)}}\right) & \left(1, \omega_{d_\sigma^{(i)}}\right) \end{pmatrix}.$$

On the one hand we can use formula 3.1 to compute

$$\nabla^\#\left(f_\sigma^{(i)}\right) = f_\sigma^{(i)} \otimes \left(0, \omega_{a_\sigma^{(i)}}\right) + e_\sigma^{(i)} \otimes \left(0, \omega_{c_\sigma^{(i)}}\right),$$

on the other hand we have formula 8.4 telling that

$$\nabla^\#\left(f_\sigma^{(i)}\right) = m_\sigma^{(i)} f_\sigma^{(i)} \otimes \theta_\sigma^{(i)} + \delta^{(i)}\eta_\sigma^{(i)} \otimes \theta_\sigma^{(i)} + \omega_\sigma^{(i)} \otimes d\delta^{(i)}.$$

Since $\omega_\sigma^{(i)}$and $\eta_\sigma^{(i)}$ are linearly independent we conclude that

$$\theta_\sigma^{(i)} = \left(\delta^{(i)}\right)^{p-1} \omega_{c_\sigma^{(i)}}$$

and note that this equality is over A (that is, over $\mathfrak{X}$) since $\left(\delta^{(i)}\right)^{p-1}$ is in Hdg (it is a generator indeed). This shows that $\left(\delta^{(i)}\right)^{p-1}\omega_{c_\sigma^{(i)}}$ is a local generator of $\Omega^1_{\mathfrak{IG}_n^{(i)}/B^0_{\alpha,I}}(\sigma)$. Consider the local sections $X_{1,\sigma}, X_{2,\sigma}$ of $\mathbb{V}_{\mathcal{O}_{F_i}}\left(H_{\mathbf{A}}^{\#,(i)}(\sigma)\right)$ obtained from $f_\sigma^{(i)}$ and $e_\sigma^{(i)}$. The action of $\epsilon_\sigma^{(i)}$ on them is given by

$$\epsilon_\sigma^{(i)}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} a_\sigma^{(i)} X + b_\sigma^{(i)} Y \\ c_\sigma^{(i)} X + d_\sigma^{(i)} Y \end{pmatrix},$$

then, setting $V = X_{2,\sigma} X_{1,\sigma}^{-1}$ and keeping in mind that the ideal $\Omega^1_{\mathfrak{X}/B^0_{\alpha,I}} \subseteq \mathcal{A}$ is a square-zero ideal, we compute

$$\begin{aligned}
\epsilon_\sigma^{(i)} \left(\kappa^{(i)} (X_{1,\sigma}) V_\sigma^h \right) &= \kappa^{(i)} \left(\epsilon_\sigma^{(i)} (X_{1,\sigma}) \right) \left(\frac{c_\sigma^{(i)} X_{1,\sigma} + d_\sigma^{(i)} X_{2,\sigma}}{a_\sigma^{(i)} X_{1,\sigma} + b_\sigma^{(i)} X_{2,\sigma}} \right)^h \\
&= \kappa^{(i)} (X_{1,\sigma}) \kappa^{(i)} \left(a_\sigma^{(i)} + b_\sigma^{(i)} V_\sigma \right) \left(c_\sigma^{(i)} + d_\sigma^{(i)} V_\sigma \right)^h \left(a_\sigma^{(i)} + b_\sigma^{(i)} V_\sigma \right)^{-h} \\
&= \kappa^{(i)} (X_{1,\sigma}) \exp \left(u_I^{(i)} \log \left((1,0) + \left(\left(0, \omega_{a_\sigma^{(i)}} \right) + \left(0, \omega_{b_\sigma^{(i)}} \right) V_\sigma \right) \right) \right) \\
&\quad \left(\left(0, \omega_{c_\sigma^{(i)}} \right) + \left(1, \omega_{d_\sigma^{(i)}} \right) V_\sigma \right)^h \left(\left(1, \omega_{a_\sigma^{(i)}} \right) + \left(0, \omega_{b_\sigma^{(i)}} \right) V_\sigma \right)^{-h} \\
&= \kappa^{(i)} (X_{1,\sigma}) \left((1,0) + u_I \left(\left(0, \omega_{a_\sigma^{(i)}} \right) + \left(0, \omega_{b_\sigma^{(i)}} \right) V_\sigma \right) \right) \\
&\quad \left((1, h\omega_d) V_\sigma^h + (0, h\omega_c) V_\sigma^{h-1} \right) \left((1,0) - h \left((0, \omega_a) + \left(0, \omega_{b_\sigma^{(i)}} \right) V_\sigma \right) \right) \\
&= \kappa^{(i)} (X_{1,\sigma}) \left(V^h + \left(0, h\omega_{d_\sigma^{(i)}} \right) V_\sigma^h - \left(0, h\omega_{a_\sigma^{(i)}} \right) V_\sigma^h - (0, h\omega_c) V_\sigma^{h+1} \right. \\
&\quad \left. + (0, h\omega_c) V_\sigma^{h-1} + (0, u\omega_a) V_\sigma^h + (0, u\omega_c) V_\sigma^{h+1} \right),
\end{aligned}$$

hence

$$\nabla^\# \left(\kappa^{(i)} (X_{1,\sigma}) V_\sigma^h \right) = \epsilon \left(\kappa^{(i)} (X_{1,\sigma}) V_\sigma^h \right) - \kappa^{(i)} (X_{1,\sigma}) V_\sigma^h \equiv \left(u_I^{(i)} - h \right) V_\sigma^{h+1} \otimes \omega_{c_\sigma^{(i)}} \in \mathrm{Gr}^{h+1} F^\bullet \mathbb{W}_{\kappa^{(i)},\sigma,u} \widehat{\otimes}_{\mathcal{O}_\mathfrak{X}} \Omega^1_{\mathfrak{X}/B^0_{\alpha,I}}. \tag{8.6}$$

The form $\omega_{c_\sigma^{(i)}}$ generates $\alpha^{-1} \Omega^1_{\mathfrak{X}/B^0_{\alpha,I}}$, hence, after pulling back to $\mathbb{V}^u_{0,\sigma} \left(H_\mathbf{A}^{\#,(i)}, s_i \right)$ (corresponding to $X_{1,\sigma} \mapsto \sigma(u) + \beta_n Z_{\sigma,u}$) we deduce that $\mathrm{Gr}^h F^\bullet \nabla_{\kappa^{(i)}}$ gives an isomorphism

$$\alpha^{-1} \mathrm{Gr}^h F^\bullet \mathbb{W}_{\kappa^{(i)},\sigma,u} \simeq \alpha^{-1} \left(u_I^{(i)} - h \right) \mathrm{Gr}^{h+1} F^\bullet \mathbb{W}_{\kappa^{(i)},\sigma,u} \widehat{\otimes}_{\mathcal{O}_\mathfrak{X}} \Omega^1_{\mathfrak{X}/B^0_{\alpha,I}}.$$

□

9 Sheaf cohomology and the U_p-operator

From now on, in order to have autoduality at our disposal (cfr. Note 7.21) we will work with p-divisible $\mathcal{O}_L$-modules rather then $\mathcal{O}_L$-module schemes. Let

$$\Phi : \mathfrak{IG}_{n,r+1,I} \to \mathfrak{IG}_{n,r,I}$$

be the map described in Proposition 7.17 and let

$$v_{n+1} : \mathfrak{IG}_{n+1,r+1,I} \to \mathfrak{IG}_{n,r+1,I}$$

be the forgetful map. Denote with $t_1 = \Phi \circ v_{n+1}$. Let $\mathfrak{G} = \mathbf{A}[p^\infty]$ be the universal p-divisible $\mathcal{O}_L$-module over $\mathfrak{IG}_{n+1,r+1,I}$ and let $\mathfrak{G}_1 = \frac{\mathbf{A}}{H_1(\mathbf{A})}[p^\infty]$ with quotient isogeny $f : \mathfrak{G} \to \mathfrak{G}_1$, then

1. f lifts the Frobenius morphism, this means that $f^* : \Omega_{\mathfrak{G}_1} \to \Omega_\mathfrak{G}$ lifts multiplication by p hence, being the Ω's invertible modules, it induces an isomorphism

$$f^* : \Omega_{\mathfrak{G}_1} \to p\Omega_\mathfrak{G}$$

in view of topological NAK [Mat2, Theorem 8.4, pag. 58]. In particular $p^{-1}f^*$ gives a map

$$\mathbb{V}_0\left(\Omega_{\mathfrak{G}}, s\right) \to \mathbb{V}_0\left(\Omega_{\mathfrak{G}_1}, s\right) = t_1^*\mathbb{V}_0\left(\Omega_{\mathfrak{G}}, s\right);$$

2. Φ is the quotient by the canonical subgroup, hence 7.11 tells that $\Phi^*(\mathrm{Hdg}) = \mathrm{Hdg}^p$, so

$$t_1^*\left(\underline{\beta}_n\right) = p^n \mathrm{Hdg}^{-\frac{p^{n+1}}{p-1}} = p^{-1}\underline{\beta}_{n+1}.$$

In particular we have a map

$$v_{n+1}^*\mathbb{V}_0\left(\Omega_{\mathfrak{G}}, s\right) \to \mathbb{V}_0\left(\Omega_{\mathfrak{G}}, s\right).$$

Composing the two morphisms described above we obtain a $\mathbf{T}^{\mathrm{ext}}$-equivariant commutative diagram

$$\begin{array}{ccccc}
v_{n+1}^*\mathbb{V}_0\left(\Omega_{\mathfrak{G}}, s\right) & \searrow & & & \\
\downarrow & & \mathfrak{IG}_{n+1,r+1,I} & \longrightarrow & \mathfrak{X}_{r+1,I} \\
t_1^*\mathbb{V}_0\left(\Omega_{\mathfrak{G}}, s\right) & \nearrow & & &
\end{array}$$

Moreover the dual isogeny f^D can be seen as a morphism $\mathfrak{G}_1 \to \mathfrak{G}$ hence inducing $\mathbf{T}^{\mathrm{ext}}$-equivariant maps

$$t_1^*\mathbb{V}_0\left(H_{\mathfrak{G}}^{\#}, s\right) \to \mathbb{V}_0\left(H_{\mathfrak{G}}^{\#}, s\right)$$
$$t_1^*\mathbb{V}_0\left(\Omega_{\mathfrak{G}}, s\right) \to \mathbb{V}_0\left(\Omega_{\mathfrak{G}}, s\right)$$

and $\phi : \mathfrak{X}_{r+1,I} \to \mathfrak{X}_{r,I}$ (Proposition 7.17) gives $\mathbb{V}_0\left(H_{\mathfrak{G}}^{\#}, s\right) \to \mathbb{V}_0\left(H_{\mathfrak{G}_1}^{\#}, s\right)$, hence

$$\phi^*\mathfrak{f}_*\mathcal{O}_{\mathbb{V}_0\left(H_{\mathfrak{G}}^{\#}, s\right)} \to \mathfrak{f}_*\mathcal{O}_{\mathbb{V}_0\left(H_{\mathfrak{G}_1}^{\#}, s\right)}$$

that gives an isomorphism

$$\phi^*\mathbb{W}_\kappa \to \mathfrak{f}_*\mathcal{O}_{\mathbb{V}\left(H_{\mathfrak{G}_1}^{\#}, s\right)}[\kappa].$$

Composing with the map induced by $t_1^*\mathbb{V}_0\left(H_{\mathfrak{G}}^{\#}, s\right) \to \mathbb{V}_0\left(H_{\mathfrak{G}}^{\#}, s\right)$ we have

$$\mathcal{U} : \mathbb{W}_\kappa = \mathfrak{f}_*\mathcal{O}_{\mathbb{V}_0\left(H_{\mathfrak{G}}^{\#}, s\right)}[\kappa] \to \mathfrak{f}_*\mathcal{O}_{\mathbb{V}_0\left(H_{\mathfrak{G}_1}^{\#}, s\right)}[\kappa] \to \phi^*\mathbb{W}_\kappa$$

which is compatible with the filtrations since all the maps involved are.

Definition 9.1. We define the uperator U_p on $H^i\left(\mathcal{X}_{r+1,I}, \mathbb{W}_\kappa\right)$ as the composition

$$H^i\left(\mathcal{X}_{r+1,I}, \mathbb{W}_\kappa\right) \to H^i\left(\mathcal{X}_{r+1,I}, \phi^*\mathbb{W}_\kappa\right) \xrightarrow{p^{-1}\mathrm{Tr}_\phi} H^i\left(\mathcal{X}_{r,I}, \mathbb{W}_\kappa\right) \to H^i\left(\mathcal{X}_{r+1,I}, \mathbb{W}_\kappa\right).$$

As an immediate consequence of Note 7.22 is that

Proposition 9.2. *The graded piece*

$$\mathrm{Gr}^m\mathcal{U} : \mathrm{Gr}^m F^\bullet \mathbb{W}_\kappa \to \mathrm{Gr}^m \phi^* F^\bullet \mathbb{W}_\kappa$$

has image contained in $\tau^m \mathrm{Gr}^m \phi^* F^\bullet \mathbb{W}_\kappa$.

Corollary 9.3. *Let* $h \in \mathbb{Q}_{>0}$*, then, for* m *large enough we have*

$$H^i\left(\mathcal{X}_r, \mathfrak{w}^{\kappa-2m}\right)^{(h)} = 0$$

for every i*, where the slopes are taken with respect to* U_p.

Proof. This follows from Proposition 9.2 since $\mathfrak{w}^\kappa = F^0\mathbb{W}_\kappa = \mathrm{Gr}^0 F^\bullet \mathbb{W}_\kappa$. □

Corollary 9.4. *The induced morphism*

$$\mathcal{U} : \frac{\mathbb{W}_\kappa}{F^i\mathbb{W}_\kappa} \to \phi^* \frac{\mathbb{W}_\kappa}{F^i\mathbb{W}_\kappa}$$

has image contained in $\tau^i \phi^* \frac{\mathbb{W}_\kappa}{F^i\mathbb{W}_\kappa}$.

Proof. For every i, consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \frac{F^{i+1}\mathbb{W}_\kappa}{F^i\mathbb{W}_\kappa} & \longrightarrow & \frac{F^{i+2}\mathbb{W}_\kappa}{F^i\mathbb{W}_\kappa} & \longrightarrow & \frac{F^{i+2}\mathbb{W}_\kappa}{F^{i+1}\mathbb{W}_\kappa} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathcal{U}} & & \downarrow{\scriptstyle \mathcal{U}} & & \downarrow{\scriptstyle \mathcal{U}} & & \\
0 & \longrightarrow & \phi^*\frac{F^{i+1}\mathbb{W}_\kappa}{F^i\mathbb{W}_\kappa} & \longrightarrow & \phi^*\frac{F^{i+2}\mathbb{W}_\kappa}{F^i\mathbb{W}_\kappa} & \longrightarrow & \phi^*\frac{F^{i+2}\mathbb{W}_\kappa}{F^{i+1}\mathbb{W}_\kappa} & \longrightarrow & 0
\end{array}$$

In view of Proposition 9.2 the first and the last vertical morphisms are 0 modulo τ^i, so does the middle one. We conclude since in view of Theorem 8.24 the module $\mathbb{W}_\kappa/F^i\mathbb{W}_\kappa$ is the α-adic completion of

$$\varinjlim_n \frac{F^{i+n}\mathbb{W}_\kappa}{F^i\mathbb{W}_\kappa}.$$

□

Corollary 9.5. *Let* $h \in \mathbb{Q}_{>0}$*, then for* m *large enough we have*

$$H^i\left(\mathcal{X}_{r,I}, F^m\mathbb{W}_\kappa\right)^{(h)} = H^i\left(\mathcal{X}_{r,I}, \mathbb{W}_\kappa\right)^{(h)}$$

for every i.

A Abelian varieties

A.1 Dual abelian varieties and polarisations

We recollect here, mostly without proof, the basic facts about the dual of an abelian variety and the notion of polarisation. For a more complete treatise see [Mil].

Fix a field k.

Proposition A.1. *Let $A_{/k}$ be an abelian variety and let $L \in \mathrm{Pic}(A)$. Then the following are equivalent:*

1. *L is translation invariant, that is for every $a \in A(\overline{k})$ we have $t_a^* L \simeq L$ on $A \otimes_k \overline{k}$;*
2. *$m^* L \simeq p^* L \otimes q^* L$ where $m : A \times_k A \to A$ is the product and $p, q : A \times_k A \to A$ are the projections;*
3. *there exist a connected k-variety T, two points $t_0, t_1 \in T(k)$ and an invertible $\mathcal{O}_{A \times_k T}$-module $\mathcal{F}$ such that*

$$\mathcal{F}_{|A \times_k \{t_0\}} \simeq \mathcal{O}_{A \times_k \{t_0\}}$$
$$\mathcal{F}_{|A \times_k \{t_1\}} \simeq L$$

   where in the last isomorphism we see L on $A \times_k \{t_1\}$ via the obvious pullback.

Definition A.2. Define $\mathrm{Pic}^0(A) \subseteq \mathrm{Pic}(A)$ as the subset of isomorphism classes of invertible sheaves that satisfy the conditions of Proposition A.1.

One can check, using the Theorem of the square, that Pic^0 is indeed a subgroup of Pic and, using for example condition 2 in Proposition A.1, that this definition gives a subfunctor of Pic.

Definition A.3. Let $A_{/k}$ be an abelian variety. The dual abelian variety is a pair $(A^\vee, \mathcal{P})$ where

1. $A^\vee_{/k}$ is an abelian variety and $\mathcal{P} \in \mathrm{Pic}(A \times_k A^\vee)$;
2. $\mathcal{P}_{|\{0\} \times_k A^\vee}$ is trivial and $\mathcal{P}_{|A \times_k \{a\}} \in \mathrm{Pic}^0(A_{\kappa(a)})$ for every $a \in A^\vee$;
3. for every k-scheme T and invertible sheaf L on $A \times_k T$ with $L_{|\{0\} \times_k T}$ trivial and $L_{|A \times_k \{t\}} \in \mathrm{Pic}^0(A_{\kappa(t)})$ for every t, there exists a unique morphism $f : T \to A^\vee$ such that

$$(A \times_k f)^* \mathcal{P} \simeq L.$$

Remark A.4. It follows that a dual abelian variety, when it exists, is unique up to a unique isomorphism. Moreover, applying the universal property to $T = \mathrm{Spec}(K)$ for an extension $k \subseteq K$, we see that there is a canonical isomorphism

$$A^\vee(K) \simeq \mathrm{Pic}^0(A_K).$$

Theorem A.5. *Let $A_{/k}$ be an abelian variety, then the dual abelian variety exists.*

Let now L be an invertible sheaf on an abelian variety $A_{/k}$, then we have a group homomorphism

$$\varphi_L : A(k) \to \mathrm{Pic}(A)$$
$$a \mapsto t_a^* L \otimes L^{-1}$$

whose image can be shown to be contained in $\mathrm{Pic}^0(A)$. If moreover L is ample and k is algebraically closed, then $\mathrm{Im}(\varphi_L) = \mathrm{Pic}^0(A)$.

Definition A.6. A polarisation on an abelian variety $A_{/k}$ is an isogeny $\lambda : A \to A^{\vee}$ such that $\lambda_{\overline{k}} = \varphi_L$ for some ample invertible sheaf L on $A_{/\overline{k}}$. The degree of a polarisation is its degree as an isogeny. An abelian variety together with a polarisation is called a polarised abelian variety; there is an obvious notion of a morphism of polarised abelian varieties. If λ has degree 1, then it is said to be a principal polarisation.

B Formal schemes

Here we quickly review the theory of formal schemes that we need and of coherent modules over them. All rings are supposed to be noetherian (and commutative).

Definition B.1. Let A be a ring and let $I \subseteq A$ be an ideal, the I-adic topology on A is the ring topology having $\{I^n\}_{n\geq 0}$ as a basis of open neighborhood of 0. If the natural map

$$A \to \widehat{A} := \varprojlim_n \frac{A}{I^n}$$

is injective, we say that A is I-adically separated (i.e. the topology is Hausdorff), we say that A is I-adically complete if it is surjective. With an abuse of terminology, I-adically complete will always mean I-adically complete and separated.

Given an I-adic ring A we define $\operatorname{Spf}(A)$ as the subspace of $\operatorname{Spec}(A)$ consisting of prime ideals containing I (that is the open ones), together with the sheaf of topological rings

$$\mathcal{O}_{\operatorname{Spf}(A)} : D(f) \cap \operatorname{Spf}(A) \mapsto A\left\langle f^{-1}\right\rangle := \varprojlim_n \frac{A}{I^n}\left[f^{-1}\right].$$

Such a ringed space will be referred to as an affine formal scheme. Note that, even if A is not I-adically complete, the sheaf $\mathcal{O}_{\operatorname{Spf}(A)}$ is a sheaf if I-adically complete A-algebras.

Remark B.2. Most of the basic operations between schemes carry over to this setting:

- a morphism $\operatorname{Spf}(A) \to \operatorname{Spf}(R)$ of locally ringed spaces is the same as a continuous morphism $\widehat{R} \to \widehat{A}$;
- given to morphisms $\operatorname{Spf}(A) \to \operatorname{Spf}(R) \leftarrow \operatorname{Spf}(S)$, where the topologies are given by $I \subseteq A$ and $J \subseteq S$, then

$$\operatorname{Spf}(A) \times_{\operatorname{Spf}(R)} \operatorname{Spf}(S) = \operatorname{Spf}\left(A \widehat{\otimes}_R S\right) := \operatorname{Spf}\left(\varprojlim_{a,b}\left(\frac{A}{I^a} \otimes_R \frac{S}{J^b}\right)\right);$$

- given an I-adic ring A and an A-module M, we can define a sheaf of $\mathcal{O}_{\operatorname{Spf}(A)}$-modules $\tilde{M}$ as

$$\tilde{M} : D(f) \cap \operatorname{Spf}(A) \mapsto \varprojlim_n \left(M \otimes_A \frac{A}{I^n}\left[f^{-1}\right]\right).$$

  Note that, if M is finitely generated, then $\varprojlim_n \left(M \otimes_A \frac{A}{I^n}\left[f^{-1}\right]\right) = M \otimes_A A\left\langle f^{-1}\right\rangle$. A sheaf of $\mathcal{O}_{\operatorname{Spf}(A)}$-modules $\mathcal{F}$ is said to be coherent if it is of the form $\tilde{M}$ for a finitely generated module M.

Definition B.3. A formal scheme is a locally topologically ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ which is locally isomorphic to an affine formal scheme. A sheaf of topological $\mathcal{O}_{\mathfrak{X}}$-modules $\mathcal{F}$ is said to be coherent if there exists an affine open covering $\{\mathfrak{U}_i\}_i$ of $\mathfrak{X}$ such that $\mathcal{F}_{|\mathfrak{U}_i}$ is a coherent $\mathcal{O}_{\mathfrak{U}_i}$-module in the sense of Remark B.2.

Remark B.4. As for schemes, we can define the fibre product $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ of a pair of morphisms $\mathfrak{X} \to \mathfrak{S} \leftarrow \mathfrak{Y}$ by taking affine converings and gluing the affine fibres products as defined in Remark B.2.

B.1 Admissible formal blow-ups

Definition B.5. Let A be an I-adically complete ring, we say that an A-module M does not have I-torsion if

$$I^n m = 0 \text{ for some } n \implies m = 0.$$

We say that an A-algebra R is admissible if it is isomorphic as a topological ring to a quotient of $A\langle \xi_1, \ldots, \xi_n \rangle$ for some n and it does not have I-torsion. A formal scheme $\mathfrak{X} \to \mathrm{Spf}\,(A)$ is admissible is $\mathfrak{X}$ has an open covering of the form $\{\mathrm{Spf}\,(R_i)\}_i$ with R_i admissible A-algebras.

Let now $\mathfrak{X} \to \mathrm{Spf}\,(A)$ be an admissible formal scheme and $\mathcal{J} \subseteq \mathcal{O}_{\mathfrak{X}}$ a coherent open ideal. We want to define a formal scheme $\pi : \mathrm{Bl}_{\mathcal{J}}\mathfrak{X} \to \mathfrak{X}$ over $\mathfrak{X}$ such that

1. the map π is admissible;
2. for every morphism $\phi : \mathfrak{Y} \to \mathfrak{X}$ over $\mathrm{Spf}\,(A)$ such that $\mathcal{J}\mathcal{O}_{\mathfrak{Y}}$ is an invertible ideal, there exists a unique morphism $\tilde{\phi} : \mathfrak{Y} \to \mathrm{Bl}_{\mathcal{J}}\mathfrak{X}$ over $\mathrm{Spf}\,(A)$ making the diagram

$$\begin{array}{ccc} \mathfrak{Y} & \xrightarrow{\phi} & \mathfrak{X} \\ & {\scriptstyle\tilde{\phi}}\searrow \quad \nearrow{\scriptstyle\pi} & \\ & \mathrm{Bl}_{\mathcal{J}}\mathfrak{X} & \end{array}$$

   commutative.

We describe the construction, all the details can be found in [Bos, Section 8.2]. Set

$$\mathrm{Bl}_{\mathcal{J}}\mathfrak{X} = \varinjlim_{n \geq 0} \mathrm{Proj}\left(\bigoplus_{d \geq 0} \frac{\mathcal{J}^d}{I^n \mathcal{J}^d} \right)$$

together with its natural map to $\mathfrak{X}$.

Lemma B.6. *Suppose $\mathfrak{X} = \mathrm{Spf}\,(R)$ is affine, then $\mathrm{Bl}_{\mathcal{J}}\mathfrak{X}$ is the I-adic completion of the scheme-theoretic blow-up $\mathrm{Bl}_{\mathcal{J}}\mathrm{Spec}\,(R)$ along the ideal $\mathcal{J} \subseteq R$.*

Proof. This is [Bos, Proposition 8.2.6, pag. 186]. Note that this is true under our convention that rings are noetherian. □

Proposition B.7. *The morphism $\pi : \mathrm{Bl}_{\mathcal{J}}\mathfrak{X} \to \mathfrak{X}$ satisfies the conditions above, moreover*

1. *let* $\mathrm{Spf}(R) \subseteq \mathfrak{X}$ *be an open affine over which* $\mathcal{J}$ *is generated by* $(r_1, \ldots, r_n)$*, let*
$$R_i = R\left\langle \frac{r_j}{r_i} \mid j \neq i \right\rangle \quad and \quad U_i = \mathrm{Spf}\left(\frac{R_i}{I - \mathrm{torsion}}\right).$$
Then $\{U_i\}_i$ *is an open covering of* $\mathrm{Bl}_{\mathcal{J}}\mathrm{Spf}(R)$*;*

2. *the open* U_i *is characterised by the property that* $\mathcal{J}$ *is generated by* r_i*.*

Proof. This is [Bos, Proposition 8.2.9 pag. 188] and [Bos, Proposition 8.2.7 pag. 186]. □

C Adic spaces

C.1 Analytic points

The main reference here is [Hub].

Definition C.1. Let $X = \mathrm{Spa}(A, A^+)$ be an affinoid adic space. A point $x \in X$ is called analytic if $\mathrm{Supp}(x) \in \mathrm{Spec}(A)$ is not open. If X is not affine, then $x \in X$ is called analytic if there exists an open neighborhood $x \in U$ such that $\mathcal{O}_X(U)$ contains a topologically nilpotent unit.

Remark C.2. It is obvious from the definition that the analytic points form an open subset.

Theorem C.3. *Let* $\mathfrak{X}$ *be a locally noetherian formal scheme, then there exists an adic space* $\mathfrak{X}^{\mathrm{ad}}$ *and a morphism of locally topologically ringed spaces*
$$\pi = \pi_{\mathfrak{X}} : \left(\mathfrak{X}^{\mathrm{ad}}, \mathcal{O}^+_{\mathfrak{X}^{\mathrm{ad}}}\right) \to (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$$
with the following universal property: let $f : \left(Y, \mathcal{O}^+_Y\right) \to (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ *is a morphism of locally topogically ringed spaces with* Y *an adic space, then* f *factors uniquely through* π *and the resulting map* $Y \to \mathfrak{X}^{\mathrm{ad}}$ *is a map of adic spaces. If* $\mathfrak{X} = \mathrm{Spf}(A)$*, then* $\mathfrak{X}^{\mathrm{ad}} = \mathrm{Spa}(A, A)$*. Moreover, for a morphism* $f : \mathfrak{X} \to \mathfrak{Y}$ *of locally noetherian formal schemes, let* f^{ad} *be the map of adic spaces induced by the universal property above: the resulting functor* $(\bullet)^{\mathrm{ad}}$ *from the category of locally noetherian formal schemes to the category of adic spaces is fully faithful and* f *is adic if and only if, for every analytic point* $x \in \mathfrak{X}^{\mathrm{ad}}$*, the point* $f^{\mathrm{ad}}(x)$ *is analytic.*

Proof. This is [Hub, Proposition 4.1, pag. 539] and [Hub, Proposition 4.2 (i), pag. 540]. □

Definition C.4. Let $\mathfrak{X}$ be a locally noetherian formal scheme, we call the adic space $\mathfrak{X}^{\mathrm{ad}}$ in Theorem C.3 the adic space associated to $\mathfrak{X}$. The space of analytic points $\mathfrak{X}^{\mathrm{an}} \subseteq \mathfrak{X}^{\mathrm{ad}}$ is called the analytic (adic) space associated to $\mathfrak{X}$.

Remark C.5. Suppose A is a topological noetherian ring with an invertible ideal of definition αA. Set $\mathfrak{X} = \mathrm{Spf}(A)$, then the analytic points of $\mathfrak{X}^{\mathrm{ad}}$ are given by the adic space
$$\mathfrak{X}^{\mathrm{an}} = \mathrm{Spa}\left(A\left[\frac{1}{\alpha}\right], A^+\right),$$
where A^+ denotes the integral closure of A in $A\left[\alpha^{-1}\right]$.

The next Lemma is not surprising if one thinks about adic spaces as an enhanced category of rigid analytic spaces.

Proposition C.6. *Let* A *be a normal*[5] *noetherian ring and suppose it is complete with respect to*

[5]That is, integrally closed in its total ring of fractions

the topology induced by an invertible ideal $\mathfrak{a}$. *Let* $\mathfrak{X} = \mathrm{Spf}\,(A)$ *and let* $t : \mathrm{Bl}_{\mathfrak{a}}\mathfrak{X} \to \mathfrak{X}$ *be the admissible formal blow-up along* $\mathfrak{a}$. *Then* t *is an adic morphism and the corresponding map*

$$t^{\mathrm{an}} : (\mathrm{Bl}_{\mathfrak{a}}\mathfrak{X})^{\mathrm{an}} \to \mathfrak{X}^{\mathrm{an}}$$

on the analytic locus is an isomorphism.

Proof. For $c \in \mathfrak{a}\backslash\mathfrak{a}^2$, let $\mathfrak{X}_c = \mathrm{Spf}\,(B_c)$ be the open subset of $\mathrm{Bl}_{\mathfrak{a}}\mathfrak{X}$ over which the ideal $\mathfrak{a}\mathcal{O}_{\mathrm{Bl}_{\mathfrak{a}}\mathfrak{X}}$ is generated by c. Note that the topology on B_c is then c-adic, moreover A being normal, we have that B_c is integrally closed in $B_c\left[c^{-1}\right]$ hence

$$\mathfrak{X}_c^{\mathrm{an}} = \mathrm{Spa}\left(B_c\left[c^{-1}\right], B_c\right)$$

in view of Remark C.5. We conclude since t^{an} is the restriction of the map on the general adic spaces, whence

$$t^{\mathrm{an}}\left(\mathfrak{X}_c^{\mathrm{an}}\right) = \{x \in \mathfrak{X}^{\mathrm{an}} \mid 0 \neq |c|_x \geq |a|_x \; \forall a \in I\}$$

and we conclude as these subsets cover $\mathfrak{X}^{\mathrm{ad}}$. □

C.2 Relative normalisation for formal schemes

We give here a slight generalisation of the construction of the normalisation of a formal scheme along a finite extension of its rigid analytic fibre as performed in [FGL, Appendix A] to the case of the analytic adic fibre. This allows more general bases.

Definition C.7. Fix a noetherian ring A which is complete for the I-adic topology, $I \subseteq A$ being an ideal. A topological A-algebra R is said to be topologically of finite type is there exists a continuous surjective homomorphism

$$A\langle X_1, \ldots, X_n\rangle \to R,$$

we say that it is admissible if, moreover, R does not have I-torsion, i.e. if

$$\{r \in R \mid I^n r = 0 \text{ for some } n\}$$

is the zero ideal.

Remark C.8. Here's a couple of algebraic remarks around this definition

1. Being $A\langle X_1, \ldots, X_n\rangle$ the I-adic completion of $A[X_1, \ldots, X_n]$, it is flat over A;
2. Every admissible A-algebra is automatically noetherian in view of [Bos, Remark 1, pag. 162] and I-adically complete and separated;
3. Let R be a I-adically complete and separated A-algebra. For all $n \geq 0$, set $A_n = A/I^{n+1}$, then R is topologically of finite type over A if and only if $R \otimes_A A_0$ is of finite type over A_0 (see [Bos, Proposition 10, pag. 166]);

Definition C.9. Let $\mathfrak{X} \to \mathrm{Spf}\,(A)$ be a formal scheme, we say that it is locally of topologically finite type (resp. admissible) if it has an affine open cover with the corresponding property.

Remark C.10. For a formal scheme over A locally of topologically finite type, quasi-separateness is automatic since it is locally a noetherian topological space.

We recall two basic results about excellent complete rings

Theorem C.11. *We have the following*

1. *Let* $(A, \mathfrak{m})$ *be a complete equicharacteristic local ring, then the ring* $A\langle X_1, \ldots, X_n\rangle$ *is excellent for every* $n \geq 0$*;*

2. *Let* C *be an excellent ring of characteristic 0 and Krull dimension 1, then for every* C*-algebra* A *of finite type and every ideal* $I \subseteq A$*, the* I*-adic completion* $\widehat{A}$ *is excellent.*

Proof. For the first part, see [Val], for the second see [Val1]. □

Notation C.12. From now we fix a ring A_0 and suppose that it is topologically of finite type over a complete DVR of mixed characteristic and we suppose that A_0 is complete with respect to the α-adic topology for $\alpha \in A_0$ a regular element. Moreover denote $A = A_0\left[\frac{1}{\alpha}\right]$ and let $A^+ \subseteq A$ be the integral closure of A_0. This assumptions on A_0 will have the effect that every A_0-algebra topologically of finite type will be automatically excellent in view of Theorem C.11. Of course the Theorem also allows rings of equal characteristic.

Definition C.13. Let $\mathfrak{X} \to \mathrm{Spf}\,(A_0)$ be an admissible and reduced formal scheme. We say that it is normal if, for every connected affine open $U \subseteq \mathfrak{X}$, the ring $\mathcal{O}_{\mathfrak{X}}(U)$ is integrally closed in its field of fractions.

Remark C.14. For an admissible and reduced formal scheme $\mathfrak{X}$ over A_0 the following are equivalent:

1. $\mathfrak{X}$ is normal;

2. There exists an affine open covering $\{U_i\}_i$ of $\mathfrak{X}$ such that, for every i, the ring $\mathcal{O}_{\mathfrak{X}}(U_i)$ is normal;

3. For all $x \in \mathfrak{X}$ the ring $\mathcal{O}_{\mathfrak{X},x}$ is a domain which is integrally closed inside its field of fractions.

See [FGL, Fait A.2.1, pag. 51] for a sketch of the proof.

For a ring R we use the shorthand $\mathrm{Spa}\,(R)$ to denote $\mathrm{Spa}\,(R, R)$.

Lemma C.15. *Let* $\mathfrak{X} \to \mathrm{Spf}\,(A_0)$ *be a formal scheme and suppose it is locally of topologically finite type. Then the fibre product*

$$\tilde{\mathfrak{X}} = \mathfrak{X}^{\mathrm{ad}} \times_{\mathrm{Spa}(A_0)} \mathrm{Spa}\left(A, A^+\right)$$

exists in the category of adic spaces. If $\mathrm{Spf}\,(R) \subseteq \mathfrak{X}$ *is an open affine, then its preimage in* $\tilde{\mathfrak{X}}$ *is*

$$\mathrm{Spa}\left(R\left[\frac{1}{\alpha}\right], R^+\right) = \mathrm{Spa}\,(R) \times_{\mathrm{Spa}(A_0)} \mathrm{Spa}\left(A, A^+\right),$$

where R^+ *is the integral closure of* R *in* $R\left[\frac{1}{\alpha}\right]$*, in particular* $\tilde{\mathfrak{X}}$ *is naturally identified with the space* $\mathfrak{X}^{\mathrm{an}}$ *of analytic points of* $\mathfrak{X}^{\mathrm{ad}}$*.*

Proof. Clearly the morphism $\mathrm{Spa}\,(A, A^+) \to \mathrm{Spa}\,(A_0)$ is adic and, in view of [Hub, Proposition 4.2, pag. 540] we see that $\mathfrak{X}^{\mathrm{ad}} \to \mathrm{Spa}\,(A_0)$ is locally of finite type. Moreover the ring A has a notherian ring of definition, so that we can apply [Hub, Proposition 3.7, pag. 535] to deduce that the fibre product $\mathfrak{X}^{\mathrm{ad}} \times_{\mathrm{Spa}(A_0)} \mathrm{Spa}\,(A, A^+)$ exists in the category of adic spaces. It follows from the very construction of the functor $(\bullet)^{\mathrm{ad}}$ that the inverse image of $\mathrm{Spf}\,(R)$ in $\mathfrak{X}^{\mathrm{ad}}$ is $\mathrm{Spa}\,(R)$, hence we need to show that

$$\mathrm{Spa}\left(R\left[\frac{1}{\alpha}\right], R^+\right) = \mathrm{Spa}\,(R) \times_{\mathrm{Spa}(A_0)} \mathrm{Spa}\left(A, A^+\right).$$

Let X be an adic space with a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\lambda_A} & \mathrm{Spa}\,(A, A^+)\ , \\ {\scriptstyle \lambda_R}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spa}\,(R) & \xrightarrow[g]{} & \mathrm{Spa}\,(A_0) \end{array}$$

this corresponds to morphisms

$$\lambda_A^{\#} : \left(A, A^+\right) \to \left(\mathcal{O}_X\,(X)\,, \mathcal{O}_X^+\,(X)\right)$$
$$\lambda_R^{\#} : (R, R) \to \left(\mathcal{O}_X\,(X)\,, \mathcal{O}_X^+\,(X)\right)$$

of Huber pairs. Setting $\lambda_R^{\#}\left(\alpha^{-1}\right) = \lambda_A^{\#}\left(\alpha^{-1}\right)$, we have a unique extension

$$\left(R\left[\frac{1}{\alpha}\right], R^+\right) \to \left(\mathcal{O}_X\,(X)\,, \mathcal{O}_X^+\,(X)\right)$$

of $\lambda_R^{\#}$. □

Lemma C.16. *Let*

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle f}\downarrow & & \downarrow \\ C & \longrightarrow & B \otimes_A C \end{array}$$

be a commutative diagram of commutative rings and suppose f is smooth. Let $\overline{A} \subseteq B$ be the integral closure of A, then $\overline{A} \otimes_A C \subseteq B \otimes_A C$ is the integal closure of C.

Proof. This is [*Stack project*, Lemma 03GG] □

Lemma C.17. *Let R be an admissible A_0-algebra and let $\Phi : \mathrm{Spa}\,(B, B^+) \to \mathrm{Spa}\left(R\left[\frac{1}{\alpha}\right], R^+\right)$ be a finite morphism, then $B^+\left[\frac{1}{\alpha}\right] = B$, in particular $\mathrm{Spf}\,(B^+)^{\mathrm{an}} = \mathrm{Spa}\,(B, B^+)$.*

Proof. First let us see that $B^+\left[\frac{1}{\alpha}\right]$ is integrally closed in B: this follows from Lemma C.16 applied to the diagram

$$\begin{array}{ccc} B^+ & \longrightarrow & B \\ \downarrow & & \| \\ B^+\left[\frac{1}{\alpha}\right] & \longrightarrow & B \end{array}$$

Moreover B is a finite $R\left[\frac{1}{\alpha}\right]$-module, hence in particular it is a finite $B^+\left[\frac{1}{\alpha}\right]$-module from which we conclude that $B = B^+\left[\frac{1}{\alpha}\right]$. The last statement follows (cfr. the computations in Lemma C.15). $\square$

Proposition C.18. *Let R be an excellent ring and let $\mathfrak{p} \in \mathrm{Spec}(R)$. Then for every finite field extension $\mathrm{Frac}\left(\frac{R}{\mathfrak{p}}\right) \subseteq L$ the integral closure of $\frac{R}{\mathfrak{p}}$ in L is a finitely generated $\frac{R}{\mathfrak{p}}$-module.*

Note C.19. Rings which satisfy the statements of Proposition C.18 are called Nagata rings. See [Mat, Section 31] for generalities about Nagata rings and [Mat, Theorem 78, pag. 257] for a proof of the fact that "quasi-excellent implies Nagata" (and hence of Proposition C.18).

Lemma C.20. *Let R be an admissible A_0-algebra and let $\Phi : \mathrm{Spa}(B, B^+) \to \mathrm{Spa}\left(R\left[\frac{1}{\alpha}\right], R^+\right)$ be a finite morphism, then the reduced ring B^+_{red} is fnitely generated as an R-module.*

Proof. This is a general fact: let $\mathfrak{q}_1, \dots, \mathfrak{q}_n$ be the minimal primes of B_{red}, then $(B_{\mathrm{red}})_{\mathfrak{q}_1} \times \cdots \times (B_{\mathrm{red}})_{\mathfrak{q}_m}$ is product of fields and by noetherianity it suffices to prove that the integral closure of R in each of them is finite. Let $K = (B_{\mathrm{red}})_{\mathfrak{q}_i}$ and let $\mathfrak{p} \subseteq R$ be the kernel of $R \to K$, then the extension $\mathrm{Frac}\left(\frac{R}{\mathfrak{p}}\right) \to K$ is finitely generated, from which we see that the algebraic closure

$$\mathrm{Frac}\left(\frac{R}{\mathfrak{p}}\right) \subseteq \overline{\mathrm{Frac}\left(\frac{R}{\mathfrak{p}}\right)} \subseteq K$$

is finite over $\mathrm{Frac}\left(\frac{R}{\mathfrak{p}}\right)$. In view of Proposition C.18 we see that the integral closure $B^+_{\mathrm{red}} = \frac{B^+}{\mathcal{N}_B \cap B^+}$ of R in B_{red} is finitely generated as an R-module, where $\mathcal{N}_B \subseteq B$ denotes the nilradical. $\square$

Lemma C.21. *Let R be an admissible A_0-algebra and let $\Phi : \mathrm{Spa}(B, B^+) \to \mathrm{Spa}\left(R\left[\frac{1}{\alpha}\right], R^+\right)$ be a finite morphism. For $f \in R$, the ring $B^+_{\mathrm{red}}\left\langle\frac{1}{f}\right\rangle$ is the integral closure of $R\left\langle\frac{1}{f}\right\rangle$ in $B_{\mathrm{red}}\left\langle\frac{1}{f}\right\rangle$.*

Proof. In view of Lemma C.16 we see that $B^+_{\mathrm{red}}\left[\frac{1}{f}\right]$ is integrally closed in $B_{\mathrm{red}}\left[\frac{1}{f}\right]$In view of Lemma C.20, the ring B^+_{red} is finite as an R-module, moreover the completion morphism $R\left[\frac{1}{f}\right] \to R\left\langle\frac{1}{f}\right\rangle$ is flat. It follows that

$$B^+_{\mathrm{red}} \otimes_R R\left\langle\frac{1}{f}\right\rangle \to B_{\mathrm{red}} \otimes_R R\left\langle\frac{1}{f}\right\rangle$$

is injective and, by finiteness, that $B^+_{\mathrm{red}} \otimes_R R\left\langle\frac{1}{f}\right\rangle \simeq B^+_{\mathrm{red}}\left\langle\frac{1}{f}\right\rangle$ and $B_{\mathrm{red}} \otimes_R R\left\langle\frac{1}{f}\right\rangle \simeq B_{\mathrm{red}}\left\langle\frac{1}{f}\right\rangle$. It follows that $B^+_{\mathrm{red}}\left\langle\frac{1}{f}\right\rangle$ is finite over $R\left\langle\frac{1}{f}\right\rangle$ and, in view of Theorem C.11 both $R\left\langle\frac{1}{f}\right\rangle$ and $B^+_{\mathrm{red}}\left\langle\frac{1}{f}\right\rangle$ are excellent, therefore they have the same integral closure in $B_{\mathrm{red}}\left\langle\frac{1}{f}\right\rangle$. We conclude since $\left(B_{\mathrm{red}}\left[\frac{1}{f}\right], B^+_{\mathrm{red}}\left[\frac{1}{f}\right]\right)$ is naturally an affinoid ring and it follows that $B^+_{\mathrm{red}}\left\langle\frac{1}{f}\right\rangle$ is integrally closed in $B_{\mathrm{red}}\left\langle\frac{1}{f}\right\rangle$. $\square$

Theorem C.22. *Let $\mathfrak{X} \to \mathrm{Spf}(A_0)$ be an admissible formal scheme and $\Phi : Y \to \mathfrak{X}^{\mathrm{an}}$ a finite morphism of adic spaces. Then there exists a finite morphism $\nu : \tilde{\mathfrak{X}} \to \mathfrak{X}$ such that*

1. *$\tilde{\mathfrak{X}}$ is reduced,*

2. Φ *factors as*

$$Y \xrightarrow{\Phi'} \tilde{\mathfrak{X}}^{\mathrm{an}} \xrightarrow{\nu^{\mathrm{an}}} \mathfrak{X}^{\mathrm{an}};$$

3. for every finite morphism $f : \mathfrak{Z} \to \mathfrak{X}$ of formal schemes with $\mathfrak{Z}$ reduced and normal such that there exists a commutative diagram of adic spaces

$$\begin{array}{ccc} \mathfrak{Z}^{\mathrm{an}} & \xrightarrow{g} & Y \\ & {\scriptstyle f^{\mathrm{an}}}\searrow \quad \swarrow{\scriptstyle \Phi} & \\ & \mathfrak{X}^{\mathrm{an}} & \end{array}$$

there exists a unique $g_0 : \mathfrak{Z} \to \tilde{\mathfrak{X}}$ making the diagram

$$\begin{array}{ccc} \mathfrak{Z} & \xrightarrow{g_0} & \tilde{\mathfrak{X}} \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \nu} & \\ & \mathfrak{X} & \end{array}$$

commute.

Proof. Let $\mathrm{Spf}\,(R) \subseteq \mathfrak{X}$ be an open affine, in view of Lemma C.15 the map Φ is given, above $\mathrm{Spf}\,(R)$, by a finite morphism $\Phi : \mathrm{Spa}\,(B, B^+) \to \mathrm{Spa}\left(R\left[\frac{1}{\alpha}\right], R^+\right)$. Let moreover $\mathrm{Spf}\,(C) \subseteq f^{-1}\left(\mathrm{Spf}\,(R)\right)$ be an open affine which we can suppose to be connected. Then g induces a morphism of affinoid rings

$$g^{\#}_{\mathrm{red}} : \left(B_{\mathrm{red}}, B^+_{\mathrm{red}}\right) \to \left(C\left[\frac{1}{\alpha}\right], C^+\right).$$

The open subset of $\tilde{\mathfrak{X}}$ over $\mathrm{Spf}\,(R)$ is defined as $\mathrm{Spf}\left(B^+_{\mathrm{red}}\right)$. It follows from Lemma C.20 that the map $\mathrm{Spf}\left(B^+_{\mathrm{red}}\right) \to \mathrm{Spf}\,(R)$ is finite, moreover Lemma C.21 tells that these affine pieces glue to a morphism of formal schemes $\nu : \tilde{\mathfrak{X}} \to \mathfrak{X}$.To conclude we only need to see that $g^{\#}_{\mathrm{red}}\left(B^+_{\mathrm{red}}\right) \subseteq C$, but C is normal by assumption, hence $C^+ = C$. □

Corollary C.23. *With notations and setting as in Theorem C.22, suppose that Y is reduced (e.g. when the map $\Phi : Y \to \mathfrak{X}^{\mathrm{an}}$ is finite étale with $\mathfrak{X}$ reduced). Then $Y = \tilde{\mathfrak{X}}^{\mathrm{an}}$.*

Proof. Just apply Lemma C.17. □

D Slope decompositions

In this section we briefly review the theory of slope decompositions as developped in [AsSt], following [Han]. Let us fix a prime p and a complete, non-archimedean normed ring A. For $a \in A \setminus \{0\}$ we set $v(a) = -\log_p |a|$.

Definition D.1. Let $f = a_d X^d + \cdots + a_1 X + a_0 \in A[X]$, we say that f is multiplicative if $a_d \in A^\times$ and we set

$$f^*(X) = X^{\deg(f)} f\left(\frac{1}{X}\right).$$

The Newton polygon $\mathrm{N}(f)$ of f is the convex hull of the set

$$\left\{(n, v(a_n)) \in \mathbb{R}^2 \mid a_n \neq 0\right\}.$$

For $h > 0$, we say that f has slope $\leq h$ if every edge of $\mathrm{N}(f)$ has slope $\leq h$. Write $A^{(h)}[X]$ for the set of such polynomials.

Example D.2. Suppose that A is noetherian domain and f is a monic polynomial with coefficients in A, then there exists an integral etension $A \subseteq \overline{A}$ such that f splits in $\overline{A}$. Let $\mu_1 \leq \cdots \leq \mu_r$ be the slopes of $\mathrm{N}(f)$, corresponding to the points $(i_1, v(a_{i_1})), \ldots, (i_r, v(a_{i_r}))$. Then, if $i_0 = 0$, for every $j = 1, \ldots, r$, the polynomial f over $\overline{A}$ has exactly $i_j - i_{j-1}$ roots with multiplicity $-\mu_j$.

Definition D.3. Let M be an $A[X]$-module, and let $h > 0$. We say that an element $m \in M$ has slope $\leq h$ if there exists $f \in A^{(h)}[X]$ with $f^*(X) \cdot m = 0$. Call $M^{(h)}$ the subset of M consisting of the elements of slope $\leq h$.

Lemma D.4. *$M^{(h)} \subseteq M$ is an A-submodule.*

Example D.5. Let V be a finite dimensional vector space over a complete non-archimedean field K. Then an element $v \in V$ has slope K is there exists a non-zero matrix M whose minimal polynomial over K has slope $\leq h$.

Definition D.6. We say that an $A[X]$-module M has a slope $\leq h$ decomposition (or simply an h-decomposition) if

1. $M^{(h)}$ is a finitely generated A-module;
2. the exact sequence
$$0 \to M^{(h)} \to M \to M_{(h)} := \frac{M}{M^{(h)}} \to 0$$
splits over $A[X]$;
3. for every $f \in A^{(h)}[X]$ the element $f^*(X)$ is invertible on $M_{(h)} := \frac{M}{M^{(h)}}$.

The two main results we need about slope decompositions in this setting are the following

Theorem D.7. *The assignment*

$$\begin{aligned} \mathcal{S}_h : \mathcal{M}_{A[X]} &\to \mathcal{M}_A \\ M &\mapsto M^{(h)} \end{aligned}$$

defines an exact additive functor with respect to restriction of morphisms. In particular

1. *given a short exact sequence in $\mathcal{M}_{A[X]}$, if two modules in the sequence have h-decomposition, then so does the third;*
2. *S_h commutes with cohomology of complexes over $\mathcal{M}_{A[X]}$;*
3. *Given a first quadrant spectral sequence $E_\bullet^{\bullet,\bullet}$ over $\mathcal{M}_{A[X]}$, if for some r all the modules in the page $E_r^{\bullet,\bullet}$ have h-decomposition, the so does its limit $H^\bullet$.*

Proof. The first item is a consequence of [AsSt], in particular [AsSt, Proposition 4.1.2, pag. 38]. The other two items follow. □

Theorem D.8. *Let A be a reduced affinoid algebra over a non-archimedean field k and let $C^\bullet$ be a bounded complex of orthonormalisable Banach $A[X]$-modules and suppose that the action of X is compact on the total A-module $\oplus C^n$. Let $x \in \mathrm{Spm}(A)$ and $h > 0$, then there exists an affinoid subdomain $\mathrm{Spm}(B) \subseteq \mathrm{Spm}(A)$ such that with the property that*

1. *the complex $B \widehat{\otimes}_A C^\bullet$ of $B[X]$-modules admits h-decomposition;*
2. *the modules $\left(B \widehat{\otimes}_A C^n\right)^{(h)}$ are finitely generated and flat over B.*

Proof. This is a restatement of [AsSt, Theorem 4.5.1, pag. 45], in view of Theorem D.7. □

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